MECHANICS OF VIBRATION

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H. M. HANSEN
Professor of Engineering Mechanics

PAUL F. CHENEA
Associate Professor of Engineering Mechanics

Both at the University of Michigan

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Part 1

Systems of One Degree of Freedom

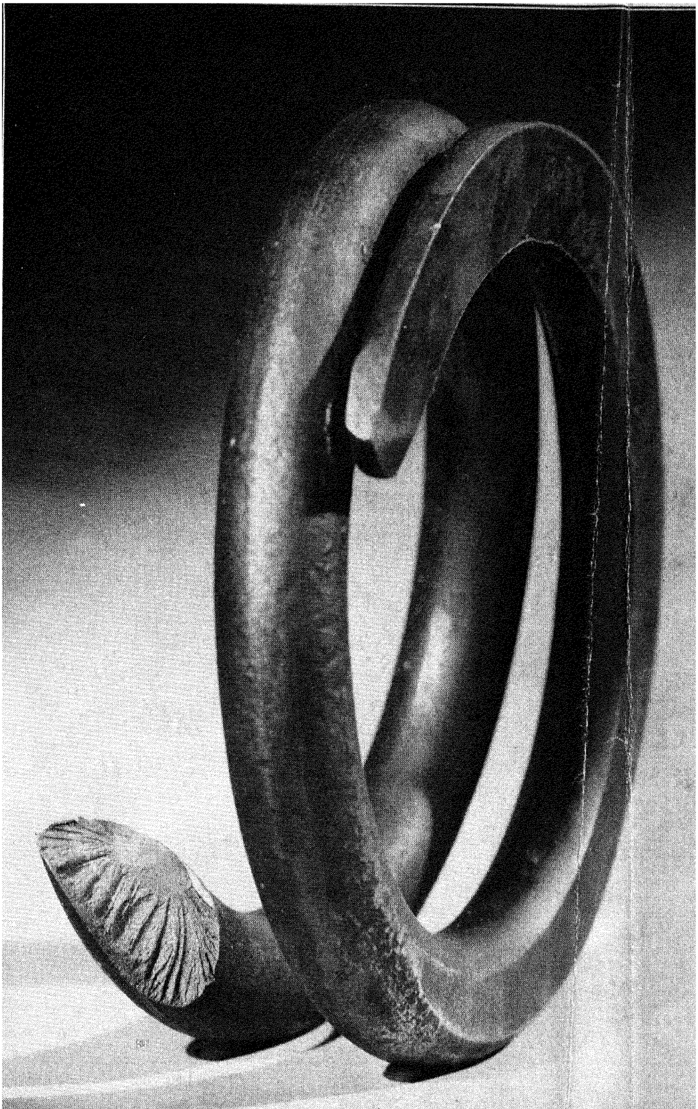

FIG. 1-1. Typical Fatigue Failure Due to Vibration.

PREFACE

This book has arisen out of a course in vibration analysis which has been offered for several years by the Department of Engineering Mechanics of the University of Michigan. This course is the first of a series which treat the theory of mechanical vibrations and its application to engineering problems. The aim of this first course has been to present the fundamentals and basic theory in a manner readily understood by the undergraduate, and yet, at the same time, on a plane acceptable to the graduate student.

Many excellent treatises are available which treat the theory of vibrations and its applications to engineering problems, and they serve as invaluable references for a student of this subject. However, with the increased demand by industry for engineers trained in the technique of vibration analysis and the resulting increase in the number of universities offering training in this subject, there appears to be a need for a treatment of vibrations designed primarily as a textbook; particularly a textbook that covers the all-important basic principles in a thorough fashion and yet is suitable for a student who has had nothing more than an elementary course in dynamics and the standard instruction in mathematics offered to undergraduate engineering students today. It is our sincere hope that this book will help to fill the need for a suitable textbook in this expanding field of applied mechanics.

A conscious effort has been made to present the theory in such a manner that it can be extended with ease to all the various and diverse vibration problems which the practicing engineer has come to know. An equally sincere effort has been made to avoid the extension in detail to specific problems, which is counter to the purpose of this book. Specific applications therefore have been considered only as a vehicle in demonstrating the theory and general technique. Considerable effort has been made to insure that the content of the book will be as broad as space permits. The necessity of keeping a textbook within reasonable size has forced us to select the material carefully. The inevitable decision as to the methods and topics to be included and those to be omitted has been made as judiciously as our experience in teaching will permit.

We firmly believed that it is essential for the student to obtain some confidence in his ability to set up a problem from its basic elements as well as to know how to solve the equations that arise in vibration analysis. The confidence on the part of the student that "he can get started," more than anything else, removes the "mystery" from vibration problems. Since the student is frequently able to picture the results better through a sketch or graph than by the manipulation of an equation, rather extensive use has been made of figures. The student is urged to portray his results graphically whenever it is appropriate. Many problems have been included on the theory that the student learns best through application. The problems have been graduated in difficulty to increase their value as a vehicle with which to master the theory.

The book is divided into three parts. The first deals with steady-state vibrations of systems of one degree of freedom. As these systems are of fundamental significance in most forms of vibration, a considerable amount of space has been devoted to their treatment. The second part extends the theory to systems of several degrees of freedom. The theory has been discussed from the classical standpoint, although emphasis has been placed on the extremely useful "mobility" concept. The third part consists of an introduction to special topics which are an essential part in a more general and more refined analysis of vibration problems. Although a thorough discussion of these topics is beyond the scope of this book, we believe that an introduction to these subjects is desirable to form a link between the idealized and more precise theories. These topics include nonlinear systems, systems with distributed physical characteristics, and systems subjected to transient motions.

A textbook of an elementary nature cannot be expected to include any new theories or specialized applications; however, the following items have been treated in a manner which is either new or considered to be an improvement over the usual presentation.

1. The concept of relaxation frequency and its physical meaning has been introduced and the concept used throughout.

2. The use of equivalent springs, dampers, and masses as well as dimensionless parameters has been stressed in the analysis of complex problems.

3. Energy methods in general and Rayleigh's method in particular have been discussed and used extensively in certain applications.

4. The mobility method has received extensive treatment. The use of velocity as a parameter, preferable, when dealing with problems of sound transmission, fluid flow, and electric currents, has been replaced

by a displacement parameter, which has a more direct significance in mechanical vibrations.

5. The solution of the frequency equation has been given considerable attention.

In the course of the development of this book we have inevitably been influenced by the classic works of Timoshenko, den Hartog, and others. These works and the teachings of these pioneers in vibration analysis have furnished the prime motivation for the present volume. A particular tribute is due to Mr. F. A. Firestone, whose paper, "Mobility Method," in the *Journal of Applied Physics*, June, 1938, represents the initial incentive for the development and adaptation of this method in this book. We are especially thankful to Mr. T. A. Hunter, who read the manuscript and made many valuable suggestions. Much credit is also due to Professor E. L. Eriksen, who lent constant encouragement. We are further indebted to Mr. R. E. Peterson of the Westinghouse Electric Corporation for the frontispiece.

H. M. Hansen
Paul F. Chenea

Ann Arbor, Michigan
February 1952

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1 General Concepts

PROBLEMS

TABLE 1

Symbol	*Quantity*	*Dimensions* FLT	*Dimensions* MLT
		FLT	*MLT*
A	Area	L^2	L^2
A	Amplitude	L	L
a	Acceleration	LT^{-2}	LT^{-2}
a	Amplitude per unit force	$F^{-1}L$	$M^{-1}T^2$
a	Wave velocity	LT^{-1}	LT^{-1}
c	Damping constant in translation	$FL^{-1}T$	MT^{-1}
c	Damping constant in rotation	FLT	ML^2T^{-1}
c	Speed of sound	LT^{-1}	LT^{-1}
c	Distance to centroid	L	L
c_{cr}	Critical damping constant	*See* Damping constant	
D	Diameter	L	L
d	Diameter	L	L
E	Young's modulus	FL^{-2}	$ML^{-1}T^{-2}$
E	Energy	FL	ML^2T^{-2}
e	Base of natural logarithms = 2.71828	Dimensionless	
e	Eccentricity	L	L
F	Dissipation function	FLT^{-1}	ML^2T^{-3}
F	Force	F	MLT^{-2}
F	Elliptic integral of the first kind	—	—
f	Frequency	T^{-1}	T^{-1}
G	Shear modulus	FL^{-2}	$ML^{-1}T^{-2}$
g	Acceleration gravity = 386 inches per second²	LT^{-2}	LT^{-2}
$\mathcal{H}$	Dynamic product of inertia (cross product)	FL	ML^2T^{-2}
h	Height or depth	L	L
I	Moment of inertia (mass)	FLT^2	ML^2
I	Moment of inertia (area)	L^4	L^4
$\mathcal{I}$	Dynamic moment of inertia	FL	ML^2T^{-2}
$\mathcal{I}$	Impulse	FT	MLT^{-1}
i	Index number	Dimensionless	
J	Moment of inertia (mass)	FLT^2	ML^2
J	Polar moment of inertia (area)	L^4	L^4
j	Imaginary unit $j^2 = -1$	Dimensionless	
K	Complete elliptic integral of first kind	—	—
k	Spring constant in translation	FL^{-1}	MT^{-2}
k	Spring constant in rotation	FL	ML^2T^{-2}
L	Length	L	L
l	Length	L	L
M	Mass	$FL^{-1}T^2$	M
M	Moment of a force	FL	ML^2T^{-2}
$\mathcal{M}$	Momentum	FT	MLT^{-1}
m	Mass	$FL^{-1}T^2$	M
n	Index number	Dimensionless	
n	Gear ratio	Dimensionless	
P	Force	F	MLT^{-2}
p	Natural circular frequency	T^{-1}	T^{-1}
p	Pressure	FL^{-2}	$ML^{-1}T^{-2}$
Q	Force	F	MLT^{-2}
Q	Discharge	L^3T^{-1}	L^3T^{-1}
q	Relaxation frequency	T^{-1}	T^{-1}
q_i	Generalized coordinates	—	—
R	Reaction	F	MLT^{-2}
R	Radius	L	L
r	Radius	L	L
$\bar{r}$	Radius of gyration	L	L
S	Tension	F	MLT^{-2}
T	Kinetic energy	FL	ML^2T^{-2}
T	Torque	FL	ML^2T^{-2}
T	Time	T	T
T	Tension	F	MLT^{-2}
t	Time	T	T
U	Energy	FL	ML^2T^{-2}

TABLE 1 (*Continued*)

Symbol	*Quantity*	*Dimensions*	
u	Displacement	L	L
V	Potential energy	FL	ML^2T^{-2}
V	Velocity	LT^{-1}	LT^{-1}
V	Volume	L^3	L^3
V	Shear force	F	MLT^{-2}
v	Velocity	LT^{-1}	LT^{-1}
W	Total complex displacement	—	—
W	Work	FL	ML^2T^{-2}
$W\sim$	Work per cycle	FL	ML^2T^{-2}
W	Weight	F	MLT^{-2}
$\dot{W}$	Power	FLT^{-1}	ML^2T^{-3}
w	Specific weight	FL^{-3}	$ML^{-2}T^{-2}$
w	Complex displacement per unit force	—	—
X	Force in x direction	F	MLT^{-2}
x	Coordinate	L	L
x	Displacement in x direction	L	L
Y	Force in y direction	F	MLT^{-2}
y	Coordinate	L	L
y	Displacement in y direction	L	L
Z	Total impedance	—	—
Z	Force in z direction	F	MLT^{-2}
z	Coordinate	L	L
z	Displacement in z direction	L	L
z	Impedance	—	—
α	Angular acceleration	T^{-2}	T^{-2}
α	Angle	Dimensionless	
α	Frequency ratio	Dimensionless	
β	Angle	Dimensionless	
γ	Specific gravity	Dimensionless	
γ	Angle	Dimensionless	
Δ	Frequency function	—	—
Δ	Determinant	—	—
δ	Displacement	L	L
δ_{st}	Static deformation	L	L
ϵ	Strain	Dimensionless	
Θ	Angular amplitude	Dimensionless	
θ	Angle	Dimensionless	
θ	Angular amplitude per unit torque	$F^{-1}L^{-1}$	$M^{-1}L^{-2}T^{2}$
μ	Coefficient of friction	Dimensionless	
μ	Absolute viscosity	$FL^{-2}T$	$ML^{-1}T^{-1}$
ν	Poisson's ratio	Dimensionless	
π	3.14159	Dimensionless	
ρ	Density	$FL^{-4}T^2$	ML^{-3}
σ	Stress	FL^{-2}	$ML^{-1}T^{-2}$
τ	Period	T	T
τ	Time	T	T
Φ	Angular amplitude	Dimensionless	
φ	Friction angle	Dimensionless	
φ	Angle	Dimensionless	
Ψ	Angular amplitude	Dimensionless	
ψ	Angle	Dimensionless	
Ω	Angular velocity	T^{-1}	T^{-1}
ω	Angular velocity	T^{-1}	T^{-1}
ω	Circular frequency	T^{-1}	T^{-1}

Chapter 1

GENERAL CONCEPTS

1.1 Introduction

Vibratory motions occur to some degree in practically every structural and mechanical device known to man, such as vehicles, machinery, bridges, and buildings. In the great majority of these instances, the vibration is of too small a magnitude to cause any concern. There are, however, numerous examples where vibrations are sufficiently dangerous to cause failure of structures which otherwise would have operated satisfactorily. Examples of such failures are found in broken crankshafts, failure of turbine blades, fracture of springs (Fig. 1–1), destruction of buildings due to earthquakes, and destruction of bridges due to vibrations induced by the wind.

Vibration is sometimes objectionable because of its effect on human comfort or its interference with the operation of delicate instruments. In these cases, structural failure may never occur, but the human discomfort or the inability of instruments to function properly make it mandatory that some form of vibration control be utilized.

In still other instances, vibrations may be essential to the operation of machines, shaking devices such as grain separators, or musical instruments. Such machines and instruments must be designed so that the finished product has the proper periodic motion. Not infrequently, the design problem is one of eliminating a particular vibratory motion while another is amplified. Many instruments designed to measure frequency are based upon properly tuned vibratory motions, and in other machines, such as fatigue-testing apparatus, vibration is employed to produce stress reversals in the test specimen, with a minimum of power input.

In all of these mechanical systems, regardless of whether the vibration is to be eliminated because of its objectionable stresses or human discomfort, or whether it is to be merely altered as a desirable feature, the first step is an analysis of the vibratory motion and an understanding of its characteristics. Only after an investigation of the pertinent properties of the vibration can an intelligent procedure be formulated to accomplish the desired change. It is the analysis of these fundamental properties of vibratory motion with which this book is concerned.

1.2 Examples of Vibratory Systems

As in other engineering topics, the introduction to a new subject is most easily made through the study of idealized elementary examples. This is a justifiable procedure because many of the more complex problems of vibration analysis can be replaced by a combination of elementary systems without appreciable sacrifice in accuracy. A vibratory system is usually understood to be a combination of elements which, either by interaction, or through the action of external forces, are able to sustain a periodic oscillating motion. In mechanical vibration these elements can be divided into three characteristic types which may be referred to as masses or inertia elements, springs or elastic elements, and resistors or damping elements. Of these, the first two, masses and springs or their equivalents, are able, by interaction, to produce and sustain oscillation while the third type, the damping elements, act as a deterent on the motion. In reality, all these characteristics are present in all mechanical parts, as any such part possesses mass which is to some degree elastic and which will in addition absorb some energy by being deformed. However, it is usually possible in practice to deal with a mechanical vibratory system as if it were made up of a number of idealized elements, each of which represents only one of the characteristic types which effect the energy distribution of the system in a specific manner.

The three characteristic elements as used in vibration analysis may be defined as follows:

Mass element

The mass element is assumed to be an inelastic solid or an incompressible and non-viscous fluid. It is therefore able to act only as an inertia and as such can gain or lose kinetic energy according to the manner in which its velocity is changed.

Spring element

The spring or elastic element is assumed to be without inertia and to resist deformation or displacement in such a manner that the work done in producing the deformation or displacement is conserved by the element in the form of potential energy until the work stored is recovered by a return to the initial shape or position.

The mass and spring elements together constitute in this idealized form a conservative system in which any energy stored in the mass due to its state of motion (kinetic energy) and any energy contained in the spring element or its equivalent due to its deformation or displacement (potential energy) can be completely recovered. The constant interchange of kinetic energy from the mass element to potential energy in the spring element is fundamental in most vibratory systems.

The spring may take the form of any elastic body. Common examples are a bent beam, a coiled spring, a twisted shaft, and an air or rubber cushion. The pull of gravity or the buoyancy exerted by a fluid on a floating body are likewise equivalent to a spring.

Damper

The third element common to all vibratory systems is a damper. A damper is any device that dissipates energy from the vibratory system. The damper gives rise to a force called the damping force which at all times resists the motion. The damping force dissipates mechanical energy which is usually converted to heat, thus depleting the mechanical energy of the system. Dampers that employ dry friction are called "friction dampers" and dampers that employ fluid friction are generally denoted as "viscous dampers."

The object of much modern engineering is to decrease frictional resistance in machines and instruments, as this resistance wastes energy and hinders performance. This same frictional resistance is the most common form of damping, and therefore many machines have extremely small damping forces associated with their operation. For this reason, the damping is frequently of small effect, and the vibration analysis is simplified by neglecting damping forces.

With these three basic elements in mind, all vibratory systems may be constructed. Consider the simple case of a mass suspended by a common coil spring, as shown in Fig. 1–2. The spring and mass in this system are easily recognized. The damping forces arise from the resistance of the air to the motion of the mass and spring as well as the internal resistance in the spring called hysteresis damping. The damping forces are usually small in this system.

As a second example, consider the simple or mathematical pendulum, as shown in Fig. 1–3. The ball constitutes the mass element whereas the equivalent of the spring in this example is the pull of gravity. The action of the force of gravity is always such as to restore the mass to the equilibrium position in which the pendulum hangs vertical. The damping forces are produced by the resistance of the air to the motion of the mass and string.

Another example is a vessel bobbing up and down in the water (Fig. 1–4). The mass is that of the vessel and the spring equivalent is

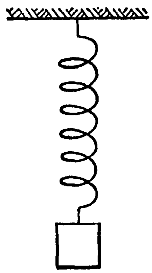


FIG. 1–2

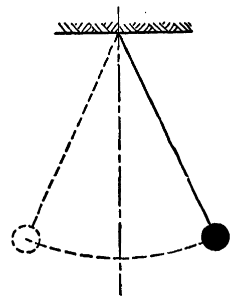


FIG. 1–3

FIG. 1–4

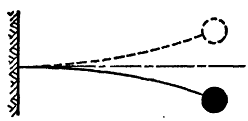


FIG. 1–5

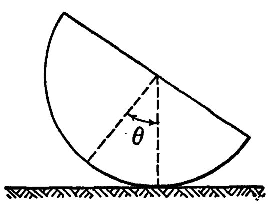


FIG. 1–6

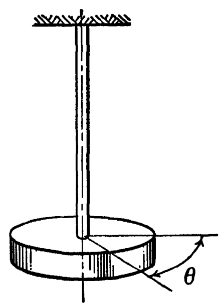


FIG. 1–7

the buoyancy of the water and the force of gravity acting together. The damping is partly air resistance and partly fluid friction.

Other simple systems are shown in Figs. 1–5, 1–6, and 1–7. The systems shown in Figs. 1–8 and 1–9 involve a combination of springs and dampers in the form of dashpots. By suitably combining masses or inertias with springs and dampers, complex systems may be constructed which, to a high degree of approximation, represent the actual mechanical system to be analyzed. It is through the use of these idealized systems that many engineering problems are solved. In those instances where idealized systems do not yield results to a sufficient degree of accuracy, more advanced methods are required. Some of these advanced methods are discussed in Part 3 of this book.

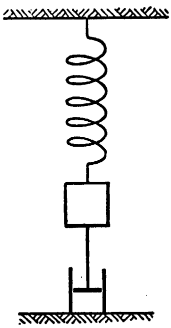


FIG. 1–8

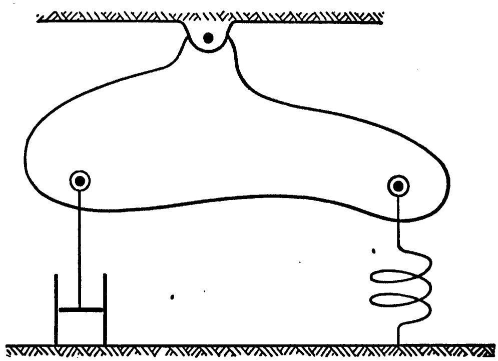


FIG. 1–9

1.3. Fundamental Definitions

As with any well-cultivated subject, there is a large number of special terms and phrases used in discussing vibration analysis which have been given precise meanings. It is essential that the meanings of the more important words and phrases be known well as they will be used throughout the text. Some of the more common terms together with their definitions follow.

Vibration

A vibration or a vibratory motion may be defined as a motion that is periodic. The motion consists of an oscillation about an equilibrium position in such a manner that it repeats itself in definite intervals of time. The graph of such a motion is shown in Fig. 1–10.

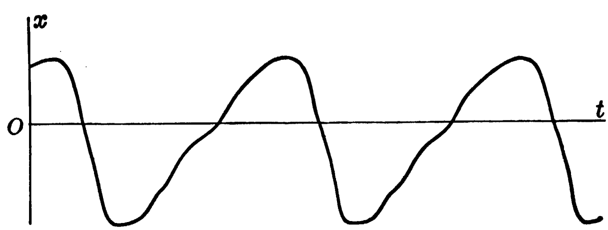


FIG. 1–10

Free vibration

A vibration that is independent of outside forces is said to be free. Free vibrations are frequently called natural vibrations.

Forced vibration

Vibrations that are caused and maintained by a periodic disturbing force are called forced vibrations.

One degree of freedom

A body is said to vibrate with one degree of freedom when its position may be completely defined by only one coordinate at any given instant.

Two or more degrees of freedom

A body or a system of bodies that requires two or more coordinates to define completely its configuration is said to have two or more degrees of freedom. In general, the number of degrees of freedom of a body or system of bodies is equal to the minimum number of coordinates required to define completely the configuration of the body, or the system of bodies, at any given time.

Amplitude

The amplitude of a vibration is the maximum linear or rotational displacement from the equilibrium position that occurs during a complete cycle of the motion.

Period

The period of a vibration is the time required to execute one complete cycle or oscillation.

Frequency

The reciprocal of the period is called the frequency. The frequency represents the number of cycles completed in a unit of time.

These are the basic terms of vibration analysis. Other terms which have specific meanings will appear, but these more complex terms are best-defined when they enter into the development of the theory.

1.4. Dimensions and Units

In the study of the physical sciences a so-called absolute system of units is employed. This system is based upon the fundamental dimensions of length L, mass m, and time T. The engineer finds it somewhat more convenient to use what is known as the gravitational system of units which is based on the fundamental dimensions of length L, force F, and time T. In each system all of the other quantities that arise in the study of mechanics may be expressed in terms of the three basic dimensions. The two systems of units are related through Newton's second law of motion,

$$F = ma$$

where a is the acceleration. This may be written dimensionally as

$$F = m\frac{L}{T^2} = mLT^{-2} \tag{1.4-1}$$

This basic law of mechanics gives another fundamental dimensional equality when solved for m,

$$m = \frac{FT^2}{L} = FL^{-1}T^2 \tag{1.4-2}$$

When the "engineer's" or gravitational system of units is employed, the pound is taken as the unit of force, the foot as the unit of length, and the second as the unit of time. The unit of mass becomes pounds second^2 per foot. It is natural to express the unit of mass in terms of the weight W of the body through the relation

$$W = mg$$

where g is the acceleration of gravity, whence

$$m = \frac{W}{g}$$

Since the acceleration of gravity has a value of 32.2 ft per sec^2, the unit mass is a body weighing 32.2 lb. This unit of mass has been denoted as the "slug." More often the gravitational acceleration is stated in terms of an inch unit of length giving $g = 386$ in. per sec^2. The corresponding unit of mass has a weight of 386 lb and may be called the "inch slug."

Certain significant quantities are involved in vibration analysis. These quantities are in part characteristics of the vibratory motion and in part characteristics of the elements of the particular vibrating system. The main characteristics of the vibratory motion are defined in the previous section. The two principal characteristics of the elements of a vibrating system other than the mass are the spring constant and the damping constant.

The spring constant is a measure of the stiffness of the spring. If the spring is designed to be deformed by changing its length, the spring constant k is defined as the force required for a unit change in length. Thus

$$k = \frac{F}{\delta} \approx \frac{F}{L} = FL^{-1} \tag{1.4-3}$$

where F is the force applied to the spring and δ is the change in length. If the spring is designed to be deformed by torsion the spring constant is defined as

$$k = \frac{M}{\theta} \approx \frac{FL}{L/L} = FL \tag{1.4-4}$$

where M is the moment or torque applied to the spring and θ is the corresponding angle of twist.

Certain types of springs do not have a constant ratio F/δ or M/θ, and these springs are said to be non-linear.

The damping constant c is defined as the force required to give a unit relative velocity to the damper elements. Thus

$$c = \frac{F}{V} \approx \frac{F}{L/T} = FL^{-1}T \tag{1.4-5}$$

for a linear or viscous damper designed for translation. For a linear torsional damper,

$$c = \frac{M}{\dot{\theta}} \approx FLT \tag{1.4-6}$$

The notation for the more common and repeatedly used quantities of vibration analysis is indicated in Table I. The dimensions of each quantity in both the "engineer's" and the absolute system of units has been included for reference. The importance of a complete understanding of the dimensions and units that occur in this branch as well as in any branch of mechanics cannot be overemphasized. It is essential that every term in each expression have the same dimensions and units, that is, be dimensionally homogeneous. An equation that does not check dimensionally is meaningless; in fact, it is not an equation.

An analysis of a system based upon the requirement of dimensional homogeniety presents in many cases a partial solution of the problem. As an example, a simple system, such as that shown in Fig. 1–2, is uniquely determined when the spring constant k and the mass m are specified. Therefore these are the only quantities that could possibly enter into an expression for the frequency of this system. Thus it may be expected that the frequency is some function of k and m. The only expression that is dimensionally correct is given by

$$f = C\sqrt{\frac{k}{m}} \approx \sqrt{\frac{F}{L}\left(\frac{L}{FT^2}\right)} = \frac{1}{T}$$

where C is some dimensionless constant. Later it will be shown that this result which was obtained only by dimensional reasoning is correct. Other combinations of physical constants, which have the same dimensions as f, may be obtained by dimensional considerations, such as $\sqrt{g/L}$ and k/c. Each of these will be shown later to be the frequency of a particular vibrating system.

1.5. Kinematics of Simple Harmonic Motion

A vibration has been defined as a periodic motion. Every periodic motion has two fundamental characteristics. The first of these characteristics is the period of the vibration, and the second is the amplitude. The period furnishes a measure of the length of time between repetitions of the same motion. Often, it is convenient to utilize the reciprocal concept of frequency, which indicates the number of cycles completed in a unit of time. The relation between the period τ and the frequency f is therefore expressed by the following equation:

$$\tau = \frac{1}{f} \tag{1.5–1}$$

The amplitude, which is the second basic characteristic of a periodic motion, indicates the magnitude of the largest displacement during a complete cycle. The amplitude, therefore, is a measure of the size of the motion.

In certain examples of periodic motion, as shown in Fig. 1–10, the maximum displacement may not be of the same magnitude on both sides of the origin, or equilibrium position. Furthermore, the form of the motion is not completely determined by the amplitude and period. These factors tend to complicate the study of a general periodic motion. Fortunately, most of the periodic motions that occur in the fundamental types of vibration problems are of one simple form. This most common type of periodic motion is called simple harmonic motion. Vibrations that are characterized by simple harmonic motions have been designated linear vibrations for reasons that will be apparent later. Vibrations that cannot be expressed as a simple harmonic motion or the sum of a finite number of simple harmonic motions are referred to as non-linear vibrations. This latter type of vibration is not easily analyzed, and its theory offers considerable difficulty. Since most of the basic engineering vibration problems can be treated as simple harmonic motions with a high degree of accuracy, it is important that the kinematics of this type of motion be thoroughly understood.

The relationship between the time t and the displacement x which characterizes a simple harmonic motion can be expressed in the form

$$x = A \cos \omega t \tag{1.5-2}$$

The quantity ω is the circular frequency. The largest possible displacement occurs when $\cos \omega t$ reaches its largest value, therefore A is the amplitude of the motion. A graphic presentation of displacement against time as related by equation (1.5-2) is shown in Fig. 1-11.

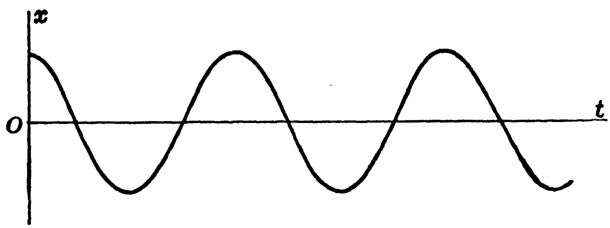


FIG. 1-11

The amplitude is the same on both sides of the equilibrium position, and the motion is a simple cosine wave. The motion repeats itself each time the angle ωt is increased by 2π rad. The period τ can then be found from the relation,

$$\omega\tau = 2\pi \quad \text{or} \quad \tau = \frac{2\pi}{\omega} \tag{1.5-3}$$

The relationship between the frequency f and the circular frequency ω is

$$f = \frac{\omega}{2\pi} \quad \text{or} \quad \omega = 2\pi f \tag{1.5-4}$$

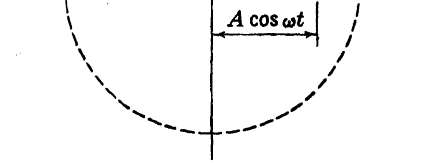


FIG. 1-12

These relationships between the factors involved in simple harmonic motion are most easily represented by a rotating vector. Let a vector of length A (Fig. 1-12) rotate about point O with a uniform angular velocity ω. Further, let the position of a particle P be determined by the projection of the vector upon the x axis. Then the angle between the vector and the x axis at any time t is ωt and the projection of the vector **A** upon the x axis is $A \cos \omega t$.

The displacement-time relation for the particle P is then

$$x = A \cos \omega t$$

The circular frequency ω is equivalent to the angular velocity of the radius vector **A**. The velocity of the particle P along the x axis is found by differentiating equation 1.5–2 with respect to time t, whence

$$\dot{x} = -A\omega \sin \omega t = A\omega \cos\left(\omega t + \frac{\pi}{2}\right) \tag{1.5–5}$$

The maximum velocity or amplitude of the velocity is therefore $A\omega$. The term $\pi/2$ indicates that the angle between the velocity vector and the displacement vector is $+\pi/2$ rad. By differentiating again the acceleration of P is obtained.

$$\ddot{x} = -A\omega^2 \cos \omega t = A\omega^2 \cos(\omega t + \pi) \tag{1.5–6}$$

where π is the angle between the acceleration and displacement vectors. This shows that the maximum acceleration or amplitude of the acceleration is

$$A\omega^2$$

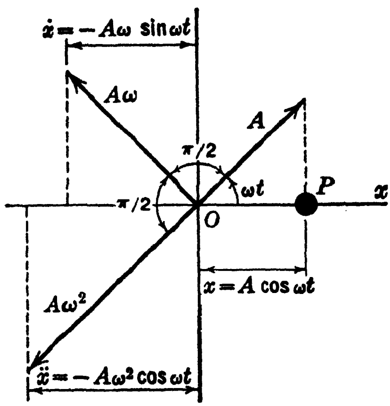


FIG. 1–13

The velocity and acceleration are represented by the rotating vectors indicated in Fig. 1–13. The magnitude of the projection of these vectors as a function of time is also indicated. It may be seen from Fig. 1–13 that the acceleration is always in the opposite direction to the displacement; in fact, the acceleration may be expressed in terms of the displacement by eliminating $\cos \omega t$ between equations 1.5–2 and 1.5–6 and thus may be obtained

$$\ddot{x} = -\omega^2 x \quad \text{or} \quad \ddot{x} + \omega^2 x = 0 \tag{1.5–7}$$

This relationship between the displacement and the acceleration, which is true at any time t, is of considerable importance. Another useful relationship exists between the absolute values of the three amplitudes of displacement, velocity, and acceleration: A, $A\omega$ and

$A\omega^2$. It will be noted that

$$(A\omega)^2 = A(A\omega^2)$$

showing that the amplitude of velocity is the geometric mean of the displacement amplitude and the acceleration amplitude. The geometric meaning of this is shown in Fig. 1–14. The terminals of the three vectors in Fig. 1–14 will always form a right triangle.

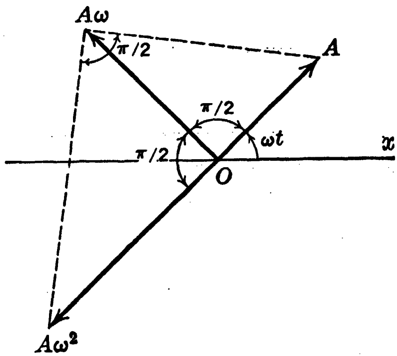


FIG. 1–14

The motion characterized by the equation

$$x = B \sin \omega t = B \cos \left(\omega t - \frac{\pi}{2}\right) \tag{1.5–8}$$

is also a simple harmonic motion. Vector representation of this motion is similar in every respect to that of equation 1.5–2, except that all vectors are rotated through an angle of $-\pi/2$ rad.

By suitably combining the two motions shown by equation 1.5–2 and 1.5–8, any simple harmonic motion may be described. Consider the simple harmonic motion defined by the rotation of vector **C** in Fig. 1–15, which "lags" vector **A** by an angle ϕ. The displacement of this motion is given by the equation

$$x = C \cos (\omega t - \phi) \tag{1.5–9}$$

The angle ϕ is called the phase angle of **C** with respect to **A**. Considering a vector **B** at right angles to **A**, the vector **C** may be said to lead **B** by a phase angle $\pi/2 - \phi$. It is always possible to resolve the motion described by **C** into a combination of the motions defined by the simultaneous rotation of **A** and **B**. This may be accomplished by writing equation 1.5–9 as

$$x = C(\cos \phi \cos \omega t + \sin \phi \sin \omega t) \tag{1.5–10}$$

It is evident from Fig. 1–15 that

$$\tan \phi = \frac{B}{A}, \quad \sin \phi = \frac{B}{C}, \quad \text{and} \quad \cos \phi = \frac{A}{C} \tag{1.5–11}$$

Substitution of these expressions into equation 1.5–10 gives

$$x = A \cos \omega t + B \sin \omega t \tag{1.5–12}$$

The geometric meaning of this resolution is pictured with the aid of

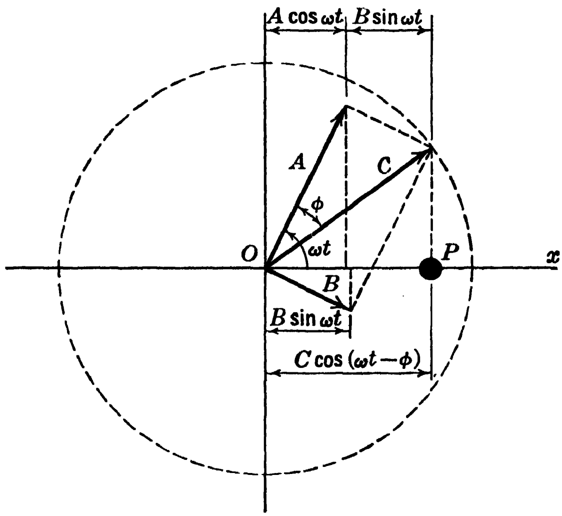


Fig. 1–15

Fig. 1–15. Vector **C** is the sum of **A** and **B**, and consequently the motion described by the projection of **C** on the x axis is identical with motion defined by the sum of the projections of **A** and **B** on this same axis.

A further useful relation to be noted is

$$C^2 = A^2 + B^2$$

which is apparent from either equations 1.5–11 or the geometry of Fig. 1–15. In a reverse manner, the motions defined by two rotating vectors **A** and **B** may be combined into a single motion defined by a vector **C**. It is not essential that **A** and **B** be perpendicular, and the only requisite is that the addition be accomplished vectorially. This addition can be extended to any number of vectors at various phase angles to some reference vector. The resultant of these vectors completely describes the total motion.

It is most important to note that all simple harmonic motions so combined must have the same frequency. There is no way in which two or more simple harmonic motions of different frequencies can be combined into a single simple harmonic motion. The converse is also true in that a single simple harmonic motion cannot be resolved into two or more separate and distinct simple harmonic motions of different frequencies.

Chapter 2

FREE VIBRATION WITHOUT DAMPING

2.1 Derivation of equations of motion for translation and rotation

The analysis of a vibrating system may be accomplished in three parts:

1. The derivation of the equations of motion.
2. The solution of the equations of motion.
3. The interpretation of the solution.

The derivation of the equations of motion is accomplished by applying the fundamental laws of dynamics. Consider, for example, the mass m supported on rollers and attached to a fixed wall by means of a spring, as shown in Fig. 2–1. Neglecting friction forces, the mass

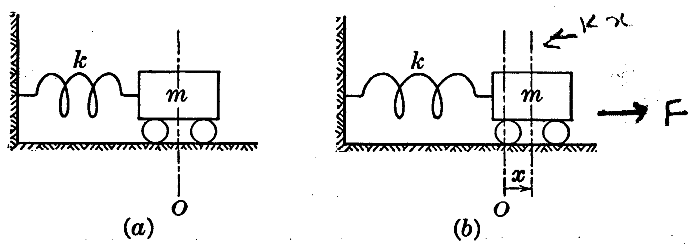


FIG. 2–1

will assume an equilibrium position (Fig. 2–1a) if no outside forces are acting on the system. In this position the force in the spring k will be zero. If the mass is set in motion and at some later instant has a displacement x, as shown in Fig. 2–1b, the force F in the spring will be acting in a direction opposite to the displacement, that is,

$$F = -kx$$

Assuming the displacement shown in Fig. 2–1b to be positive, Newton's second law may be used to write the equation of the motion,

$$m\ddot{x} = F = -kx$$

This equation may be written as

$$m\ddot{x} + kx = 0 \tag{2.1-1}$$

This equation describes the motion of the mass m when no external forces are acting on the system. In particular, it furnishes a relation between the acceleration and the displacement which is true for all values of the displacement x. It is a differential equation since it involves not only x but also the second derivative of x with respect to time. Equation 2.1-1 is conveniently written as

$$\ddot{x} + \frac{k}{m}x = 0 \quad \text{or} \quad \ddot{x} + p^2x = 0 \tag{2.1-2}$$

where the notation

$$p^2 = \frac{k}{m} = \frac{kg}{W}$$

has been introduced. The solution of equation 2.1-2 will then furnish the description of the motion of the mass m.

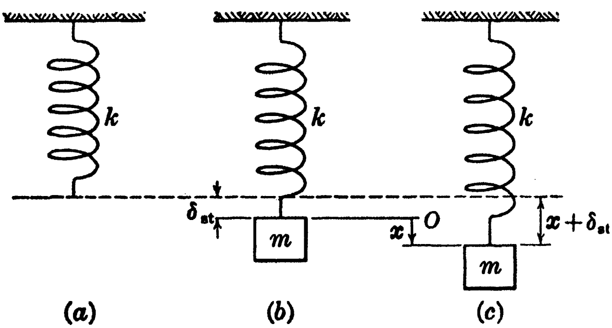


FIG. 2-2

Before solving this equation, it is instructive to consider some similar systems. The spring mass system shown in Fig. 2-2 offers another example. The weightless spring before the application of the mass m is indicated in Fig. 2-2a. Owing to the attraction of gravity, the spring will be elongated when the mass is attached, as shown in Fig. 2-2b. This elongation δ_{st}, the static deformation, is related to the mass m of the body and the spring constant k as follows:

$$\delta_{st} = \frac{mg}{k} = \frac{W}{k}$$

The tension in the spring is $F = k\delta_{st}$ which is equal to the force of gravity acting on the mass m. This configuration is the equilibrium position, and displacements are conveniently measured from here. Now let the mass m be given an arbitrary downward displacement x, considered as positive, from the equilibrium position. In this new position (Fig. 2–2c) the tension in the spring is

$$F = -k(x + \delta_{st}) \tag{2.1–3}$$

Utilizing Newton's second law, the equation of the motion may be written as follows:

$$m\ddot{x} = F + W$$

Substituting for F from equation 2.1–3 permits this to be expressed as

$$m\ddot{x} = -k(x + \delta_{st}) + W = -kx - k\delta_{st} + W$$

However,

$$W = k\delta_{st}$$

thus

$$m\ddot{x} + kx = 0$$

which is the same as equation 2.1–1. The systems shown in Fig. 2–1 and 2–2 are seen to be equivalent and hence will have the same solution.

Another system which yields a similar differential equation of motion

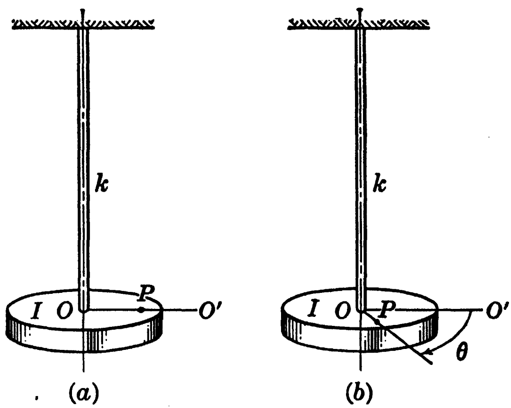


FIG. 2–3

is the torsional pendulum as shown in Fig. 2–3. This system consists of a disk which has a moment of inertia I about a vertical axis through its centroid. This axis coincides with the axis of the vertical supporting shaft. If the shaft is twisted through an angle θ, it will exert a torque T acting in a sense opposite to the displacement, that is,

$$T = -k\theta$$

where k is the torsional spring constant of the shaft. Let the equilibrium position of the disk be designated by a line OO'. In this position, as shown in Fig. 2–3a, both the torque and the angular displacement are zero. If the disk is displaced, as shown in Fig. 2–3b, through an arbitrary angle θ, a restoring torque T is created in the twisted shaft. The equation of motion may be established as above with the aid of Newton's second law.

$$I\ddot{\theta} = T$$

Substituting for the torque T gives

$$I\ddot{\theta} = -k\theta \quad \text{or} \quad \ddot{\theta} + \frac{k}{I}\theta = 0 \qquad (2.1\text{–}4)$$

This is the differential equation of motion for this system. It will take the same form as equation 2.1–2 if the quantity p^2 is defined in this instance by

$$p^2 = \frac{k}{I}$$

The equation of motion is then

$$\ddot{\theta} + p^2\theta = 0 \qquad (2.1\text{–}5)$$

The solution to this problem is therefore similar to the solution of the previous two. In fact, the differential equation for all three may be solved at the same time. Once this solution is found, it only remains to adapt it to the particular system at hand. So many vibration problems can be reduced to equation 2.1–2 that its solution is of major importance.

2.2 Solution of the Equation of Motion

It will be noted that the equation 2.1–2,

$$\ddot{x} + p^2x = 0$$

is the same as that developed for a simple harmonic motion (equation 1.5–7) where p in that instance was the circular frequency ω. It may therefore be expected that the general solution can be found in the form[1] of the general equation of simple harmonic motion (equation 1.5–12).

$$x = A\cos pt + B\sin pt \qquad (2.2\text{–}1)$$

This expression is readily verified to be the solution by substituting from equation 2.2–1 into the equation of motion (equation 2.1–2).

[1] For mathematical treatment of the solution of this equation, see any elementary text on differential equations.

The constants A and B determine the amplitude of the motion. Consider, for example, the system discussed above and shown in Fig. 2–1. This system will perform a simple harmonic motion, as defined by equation 2.2–1 about the equilibrium position. The velocity at any time t will be found by differentiating equation 2.2–1.

$$\dot{x} = -Ap \sin pt + Bp \cos pt \tag{2.2–2}$$

The constants A and B may be determined from the initial conditions, that is, from the displacement x_0 and the velocity $\dot{x}_0$ at the instant $t = 0$. For $t = 0$ equation 2.2–1 reduces to

$$x_0 = A$$

Similarly, equation 2.2–2 at the instant $t = 0$ reduces to

$$\dot{x}_0 = Bp$$

From these two expressions, the constants A and B are found in terms of the initial displacement and initial velocity. The final form of the solution is

$$x = x_0 \cos pt + \frac{\dot{x}_0}{p} \sin pt \tag{2.2–3}$$

Before studying the general solution as given by this equation it is helpful to consider two special cases, which arise when either $\dot{x}_0$ or x_0 are zero.

(*a*) $\dot{x}_0 = 0$. This condition exists when the time measurement is started at the instant of maximum displacement or if the mass of Fig. 2–1 is released at a distance x_0 from the equilibrium position. Equation 2.2–3 is then reduced to

$$x = x_0 \cos pt \tag{2.2–4}$$

The character of the motion is indicated in Fig. 2–4*a*. The radius vector has a magnitude x_0 and an angular velocity of p rad per sec. The velocity and displacement are found by differentiating equation 2.2–4.

$$\dot{x} = -x_0 p \sin pt$$

$$\ddot{x} = -x_0 p^2 \cos pt$$

It will be noted that the velocity and acceleration amplitudes are x_0p and x_0p^2, respectively. The period and frequency depend only on p.

$$\tau = \frac{2\pi}{p} = 2\pi\sqrt{\frac{W}{kg}}; \qquad f = \frac{p}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{kg}{W}} \tag{2.2-5}$$

The motion of the mass m is of the type described by equation 1.5–2.

(*b*) $x_0 = 0$. This condition occurs when the mass receives an instantaneous impulse while at rest in the equilibrium position, or

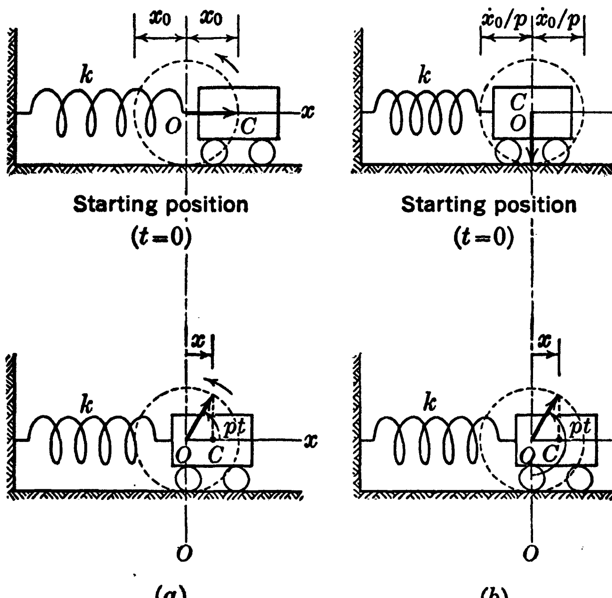


FIG. 2–4

when time measurements are started at the moment the mass moves past the point of equilibrium. In this special case, equation 2.2–3 reduces to

$$x = \frac{\dot{x}_0}{p}\sin pt = \frac{\dot{x}_0}{p}\cos\left(\omega t - \frac{\pi}{2}\right)$$

The motion may be described by a rotating radius vector as shown in Fig. 2–4*b*. The magnitude of the vector is $\dot{x}_0/p$ whereas its angular velocity is p rad per sec as in the previous case. The velocity and acceleration of the mass m are given by

$$\dot{x} = \dot{x}_0 \cos pt$$

$$\ddot{x} = -\dot{x}_0 p \sin pt \tag{2.2–6}$$

The velocity and acceleration amplitudes are $\dot{x}_0$ and $\dot{x}_0 p$, respectively, and the motion is as described by equation 1.5–8. The period and frequency depend only on p, and therefore they are also given by equations 2.2–5.

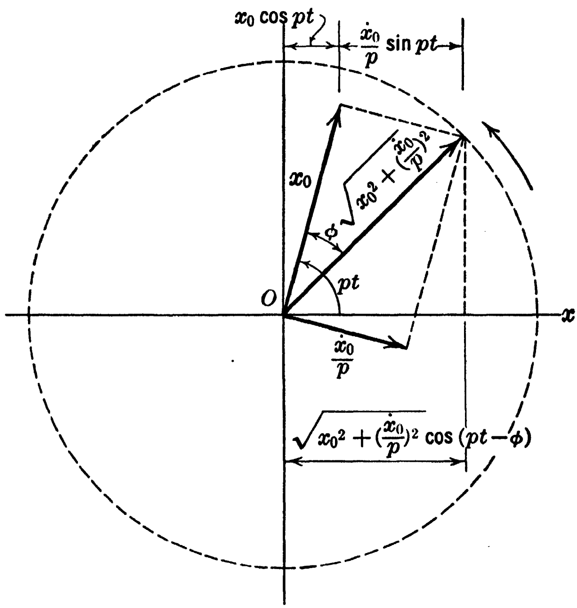


FIG. 2–5

The periodic motion indicated by the general solution, equation 2.2–3, represents the combination of the two special cases discussed above. The mass of Fig. 2–1 is displaced initially a distance x_0 from the equilibrium position and has an initial velocity $\dot{x}_0$; then the character of the motion is given by the rotating vector of Fig. 2–5. It will also be noted that the velocity and acceleration are a combination of the two previous special cases.

$$\dot{x} = -x_0 p \sin pt + \dot{x}_0 \cos pt$$

$$\ddot{x} = -x_0 p^2 \cos pt - \dot{x}_0 p \sin pt \tag{2.2–7}$$

Their vector representation is as indicated in Fig. 2–6, and the period and frequency of this motion is given by equations 2.2–5. These two

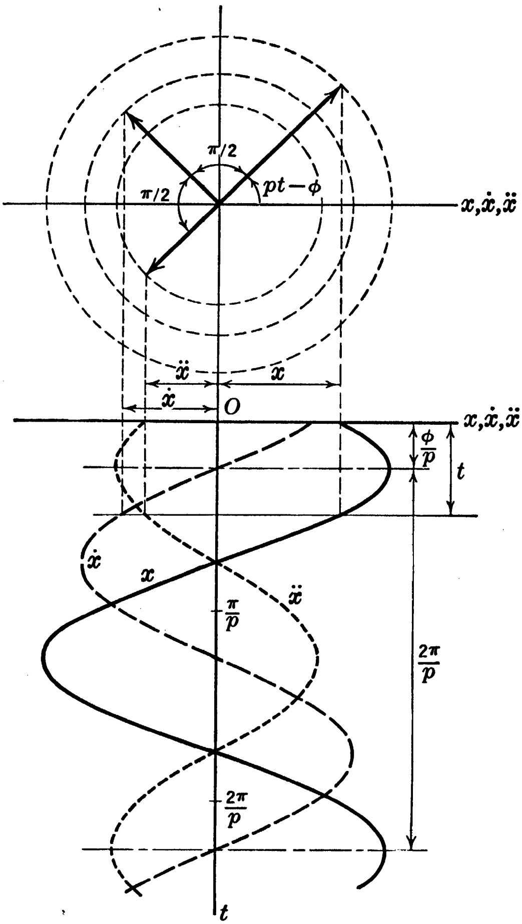


Fig. 2–6

equations, together with equation 2.2–3, may be written in the manner of section 1.5 to give

$$
\left.
\begin{aligned}
x &= \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{p}\right)^2} \cos (pt - \phi) \\
&= \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{p}\right)^2} \cos p\left(t - \frac{\phi}{p}\right) \\
\dot{x} &= -p \sqrt{x_0^2 + \left(\frac{\dot{x}}{p}\right)^2} \sin (pt - \phi) \quad \text{and} \\
\ddot{x} &= -p^2 \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{p}\right)^2} \cos (pt - \phi)
\end{aligned}
\right\} \quad (2.2\text{–}8)
$$

The amplitudes of the displacement, velocity, and acceleration can readily be obtained from these equations, and they are found to be

$$\sqrt{x_0^2 + \left(\frac{\dot{x}_0}{p}\right)^2}, \quad p\sqrt{x_0^2 + \left(\frac{\dot{x}_0}{p}\right)^2}, \quad \text{and} \quad p^2\sqrt{x_0^2 + \left(\frac{\dot{x}_0}{p}\right)^2}$$

respectively. The phase angle ϕ by which the total motion lags that part of the motion due to the initial displacement is found from Fig. 2–5, which shows that

$$\tan \phi = \frac{\dot{x}_0}{p x_0} \tag{2.2–9}$$

The general solution may also be adapted to the motion of the system shown in Fig. 2–2. The initial conditions will permit the proper evaluation of the coefficients in equation 2.2–3. The remaining interpretation of the solution is analogous to that for the system of Fig. 2–1. We may, however, in this instance, express the frequency and period in terms of the static deflection δ_{st}. From

$$p^2 = \frac{kg}{W} = \frac{g}{\delta_{st}}$$

it follows that

$$\tau = \frac{2\pi}{p} = 2\pi\sqrt{\frac{\delta_{st}}{g}} \quad \text{and} \quad f = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}}$$

These relations offer an extremely useful method of determining the period and the frequency of a complex system. Frequently, the static deflection can be determined experimentally, thus making possible the direct determination of the period and frequency by a measurement of length without determining the physical magnitudes of either W and k.

The solution for the torsional pendulum is found in a similar manner. The displacements in this instance are angles. The solution has the form

$$\theta = \theta_0 \cos pt + \frac{\dot{\theta}_0}{p} \sin pt$$

where θ_0 and $\dot{\theta}_0$ are the initial angular displacement and initial angular velocity, respectively. The angular velocity and acceleration are determined in the same way as shown for the previous systems. The period and frequency are found to be

$$\tau = \frac{2\pi}{p} = 2\pi\sqrt{\frac{I}{k}} \quad \text{and} \quad f = \frac{1}{2\pi}\sqrt{\frac{k}{I}}$$

It is frequently convenient to rewrite these expressions, utilizing the radius of gyration $\bar{r}$ of the rotating mass. The moment of inertia can then be expressed in terms of the mass m or the weight W by

$$I = m\bar{r}^2 = \frac{W}{g}\bar{r}^2$$

whence

$$\tau = 2\pi\sqrt{\frac{W\bar{r}^2}{kg}}; \qquad f = \frac{1}{2\pi}\sqrt{\frac{kg}{W\bar{r}^2}}$$

2.3. Equivalent Springs in Torsion and Translation

The spring element in an oscillating system may consist of one or several elastic units. If the individual spring rate or spring constant is

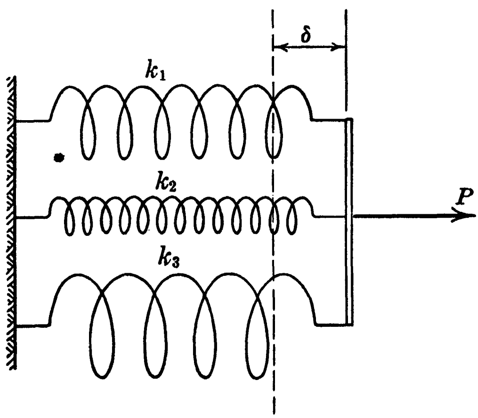


FIG. 2–7

known for each of these units, it is usually possible to determine the resultant or equivalent spring constant through two fundamental relationships and combinations thereof.

Springs in parallel

If several springs with spring constants k_1, k_2, and k_3 (Fig. 2–7), are connected in such a manner that the displacements δ for all the springs are the same, the springs are said to be in parallel. The displacement δ will be

$$\delta = \frac{P_1}{k_1} = \frac{P_2}{k_2} = \frac{P_3}{k_3} = \frac{P_1 + P_2 + P_3}{k_1 + k_2 + k_3} = \frac{P}{k_e}$$

where P_1, P_2, and P_3 are the forces in the individual springs. The resultant or equivalent spring constant k_e will therefore be

$$k_e = \frac{P}{\delta} = k_1 + k_2 + k_3$$

or in general the equivalent spring constant for springs in parallel will be equal to the sum of the individual spring constants,

$$k_{e\ \text{parallel}} = \Sigma k \tag{2.3-1}$$

Springs in series

If several springs are arranged in such a manner that they all are subjected to the same force, the springs are said to be in series. A

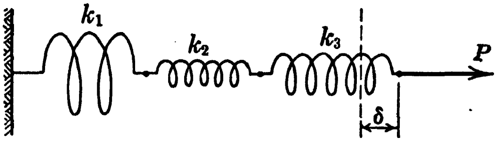


FIG. 2–8

particular example is that of springs connected end to end, as shown in Fig. 2–8. If a force P is applied at the end of these springs, the total elongation δ will be the sum of the individual elongations δ_1, δ_2, and δ_3 of the springs k_1, k_2, and k_3, respectively. From this follows that

$$\delta = \delta_1 + \delta_2 + \delta_3 = \frac{P}{k_e}$$

where

$$\delta_1 = \frac{P}{k_1}, \quad \delta_2 = \frac{P}{k_2}, \quad \text{and} \quad \delta_3 = \frac{P}{k_3}$$

Thus

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

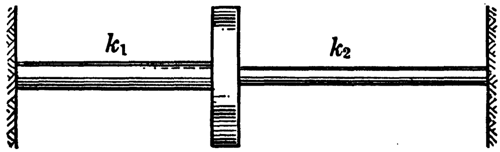
For springs in series, therefore, the reciprocal of the equivalent spring constant is equal to the sum of the reciprocals of the individual spring constants or

$$\frac{1}{k_e} = \sum \frac{1}{k}$$

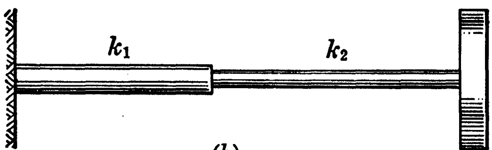
from which

$$k_{e\ \text{series}} = \frac{1}{\sum \frac{1}{k}} \tag{2.3-2}$$

The above results apply equally well to torsional springs. In the system shown in Fig. 2–9a, the torsional springs k_1 and k_2 are combined in parallel since the angular twists of the two shafts are equal, whereas



(*a*)



(*b*)

FIG. 2–9

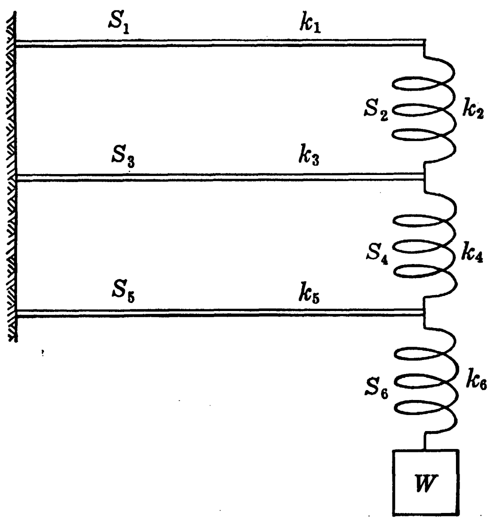


FIG. 2-10

in Fig. 2–9b they are combined in series because the torques are equal in the two shafts.

A complex spring element such as that shown in Fig. 2–10 involves spring units and spring combinations in series and in parallel. The resultant spring constant may be found by considering the spring units individually.

Referring to Fig. 2–10, k_1 represents the spring constant for the cantilever S_1 when the force is applied at the end where the coil spring S_2 is attached. A force applied at the lower end of S_2 will act through this spring on S_1, and the two springs are therefore in series, which makes the resultant spring constant,

$$k_a = \frac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2}}$$

The force applied at the end of the cantilever S_3 must produce the same displacement for this spring as for the lower end of the combination of S_1 and S_2. The combination of S_3 and the first two combined is therefore a system in parallel from which the equivalent spring constant,

$$k_b = k_3 + k_a$$

In a similar manner adding S_4 to the three previous springs gives

$$k_c = \frac{1}{\dfrac{1}{k_4} + \dfrac{1}{k_b}}$$

Adding the effect of S_5 gives

$$k_d = k_5 + k_c$$

and the joint combination with S_6 yields

$$k_e = \frac{1}{\dfrac{1}{k_6} + \dfrac{1}{k_d}}$$

By substitution the equivalent spring constant for the entire combination of springs may be written as

$$k_e = \cfrac{1}{\cfrac{1}{k_6} + \cfrac{1}{k_5 + \cfrac{1}{\cfrac{1}{k_4} + \cfrac{1}{k_3 + \cfrac{1}{\cfrac{1}{k_2} + \cfrac{1}{k_1}}}}}} \tag{2.3–3}$$

Systems involving equivalent spring elements which are in the form of buoyancy forces or gravity forces may be combined in a similar manner. Examples of such systems will be treated in the following section.

2.4. Applications

In the application of the theory, it is convenient to classify the vibratory systems by the type of restoring force or equivalent spring element involved. Oscillating systems are therefore designated as elastic, gravity, or buoyant. Systems containing two or more types of spring elements are called combined systems. Since there are some differences in the manner of treating these different problems, a typical example of each class is treated in this section.

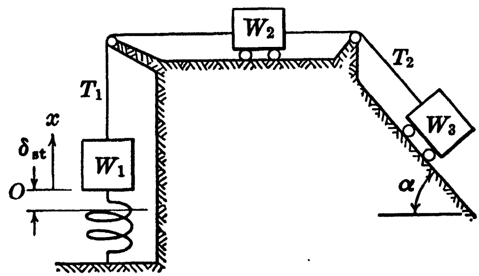


FIG. 2–11

Typical elastic system

Consider the system shown in Fig. 2–11. The position of all three bodies may be determined from the position of any one of the three. The position of any one of the bodies is defined by one coordinate (assuming tension in the cables between the masses and the spring in any position). Therefore, the whole system has only one degree of freedom. The displacements which elongate the spring may be considered as positive. Let it be required to find the natural frequency of the system and the tensions T_1 and T_2 in the cords at any time if the initial displacement is x_0. Further, let it also be required to find the maximum value of the initial displacement x_0 for which the solution is valid.

The position of the system in static equilibrium will, in general, be associated with an elongation or compression of the spring, which may be designated as δ_{st}. The displacement of the system is conveniently measured from this equilibrium position. With the aid of D'Alemberts principle, the equation of motion for the complete system is found to be

$$-k(\delta_{st} + x) - W_1 - (W_1 + W_2 + W_3)\frac{\ddot{x}}{g} + W_3 \sin \alpha = 0$$

or

$$(W_1 + W_2 + W_3)\frac{\ddot{x}}{g} + kx = -W_1 + W_3 \sin\alpha - k\delta_{st}$$

However,

$$k\delta_{st} = W_3 \sin\alpha - W_1$$

Therefore, the equation of motion may be written in the usual form,

$$\ddot{x} + p^2x = 0 \qquad (2.4\text{–}1)$$

where

$$p = \sqrt{\frac{kg}{W_1 + W_2 + W_3}}$$

is the circular frequency of the system. The frequency of the system is then

$$f = \frac{p}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{kg}{W_1 + W_2 + W_3}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$$

where

$$\delta = \frac{W_1 + W_2 + W_3}{k}; \qquad p^2 = \frac{g}{\delta}$$

and δ, in this illustration, is the deflection of the spring that would occur if the total weight of the system was applied directly to the spring. It is important to note that, in all cases of a single degree of freedom, the expression for frequency in terms of the so-called static deflection refers to the static deflection produced by the total weight of the system applied directly to the spring, regardless of the actual static deformation of the spring in the equilibrium position.

The tensions T_1 and T_2 in the cords may be obtained by considering the equation of motion of the individual bodies. Thus,

$$\frac{W_2}{g}\ddot{x} = T_2 - T_1 \quad \text{and} \quad \frac{W_3}{g}\ddot{x} = W_3 \sin\alpha - T_2 \qquad (2.4\text{–}2)$$

The acceleration is determined from the solution of equation 2.4–1 which may be taken as

$$x = x_0 \cos pt$$

and the acceleration at any time is

$$\ddot{x} = -x_0p^2 \cos pt = -\frac{x_0}{\delta} g \cos pt$$

Substituting this value into equations 2.4–2 yields

$$T_2 = W_3\left(\sin\alpha + \frac{x_0}{\delta}\sin pt\right) \qquad (2.4\text{–}3)$$

and

$$T_1 = W_3 \sin \alpha + \frac{x_0}{\delta} (W_2 + W_3) \sin pt \qquad (2.4\text{–}4)$$

It is to be remembered that this solution is based upon the assumption of a single degree of freedom which, in turn, requires that no relative movement exists between the masses. Therefore, a compression in the cord cannot be permitted, that is,

$$T_1 \geqq 0 \quad \text{and} \quad T_2 \geqq 0$$

These conditions may be restated from equations 2.4–3 and 2.4–4 as follows:

$$\sin \alpha \geqq \frac{x_0}{\delta}$$

and

$$\sin \alpha \geqq \frac{x_0}{\delta}\left(1 + \frac{W_2}{W_3}\right)$$

The first condition is automatically satisfied when the second is true since W_2/W_3 is always positive. The maximum value of the initial displacement for which the solution is valid is therefore

$$x_{0\,\max} = \frac{\delta \sin \alpha}{1 + \dfrac{W_2}{W_3}}$$

Limitations on the range of the displacement for which the solutions are valid are common in engineering problems. These limitations are more frequently due to the inelasticity of the spring since the force exerted by the spring is proportional to the displacement only so long as the displacements do not cause the spring material to exceed its elastic limit.

Another example of an elastic system is shown in Fig. 2–12. This system may be treated as a problem in rotation about the point O. The bar is assumed to be weightless and horizontal in its equilibrium position. The moment of inertia of the mass m about an axis through O is $I_0 = mb^2$. The restoring force resulting from the elongation of the spring is proportional to its elongation. Thus, if the bar and mass is rotated about point O through an angle θ, the spring will be elongated an amount $a\theta$ from its static position. The static balancing torque of the spring is readily seen to be

$$k(a\theta_{st})a = Wb$$

The total restoring torque will therefore be

$$ka(\theta + \theta_{st})a$$

and the equation of motion about O becomes

$$I_0\ddot{\theta} = Wb - ka^2(\theta + \theta_{st}) = -ka^2\theta$$

Substituting

$$I_0 = mb^2$$

and rearranging gives

$$\ddot{\theta} + \frac{ka^2}{mb^2}\theta = 0$$

from which the circular frequency is

$$p = \frac{a}{b}\sqrt{\frac{k}{m}}$$

or the period is

$$\tau = \frac{2\pi}{p} = 2\pi\frac{b}{a}\sqrt{\frac{m}{k}}$$

The period may be adjusted in this device by changing the length ratio b/a. The use of levers to give a desired frequency or period is

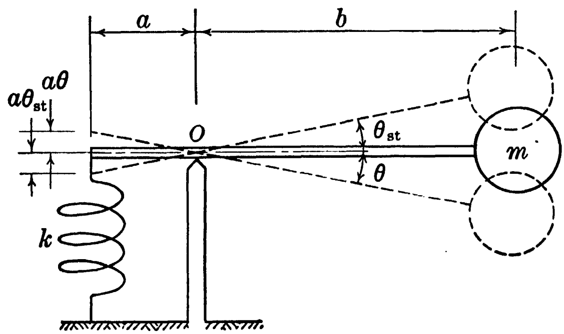


FIG. 2–12

common in practice. In particular, certain seismographs are designed with very large periods by the use of a large ratio b/a in a manner similar to that employed in this example.

It will be noted that the static displacement $(a\theta_{st})$ has no effect on the resulting equations of motion in this case of rotation. The same result has already been shown for translation (Fig. 2–2). In general each set of statically balanced forces in a vibrating system will cancel each other in the equation of motion, and such static displacements and their corresponding forces may therefore be disregarded in the frequency analysis.

The system of Fig. 2–13 is of the same nature as the previous problem in which the axis of rotation and the centroid of mass is in a horizontal plane in the position of equilibrium. It will oscillate in rotation about its axis O if displaced from its static position. The equation of motion may therefore be established in a manner identical with the preceding example. The moment of inertia about the point O may be written as $I_0 = m\bar{r}_0^2$ where $\bar{r}_0$ is the radius of gyration about point O. By eliminating the balanced forces the restoring torque due

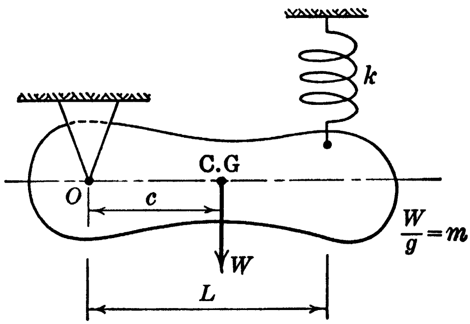


FIG. 2–13

to a small rotation θ about O is given by $kL^2\theta$. The equation of motion is therefore

$$I_0\ddot{\theta} = -kL^2\theta$$

or

$$\ddot{\theta} + \left(\frac{L}{\bar{r}_0}\right)^2 \frac{k}{m}\,\theta = 0$$

from which the period is seen to be

$$\tau = 2\pi\frac{\bar{r}_0}{L}\sqrt{\frac{m}{k}}$$

Further, it is of interest to note that the equivalent torsional spring constant is

$$k_e = kL^2$$

Typical gravity systems

The pendulum is representative of the systems that depend upon gravity for a restoring force. Such a pendulum consists in general of a mass pivoted about a horizontal axis at O, not its centroid, as shown in Fig. 2–14. In Fig. 2–14a is shown the so-called mathematical pendulum which consists of a concentrated mass suspended from a weightless cord, whereas Fig. 2–14*b* shows a pendulum consisting of a distributed mass. This is called a compound pendulum.

The equation of motion for both of these pendulums may be found in the same manner. If they are displaced through an angle θ from the position of equilibrium, the restoring moment for the two pendu-

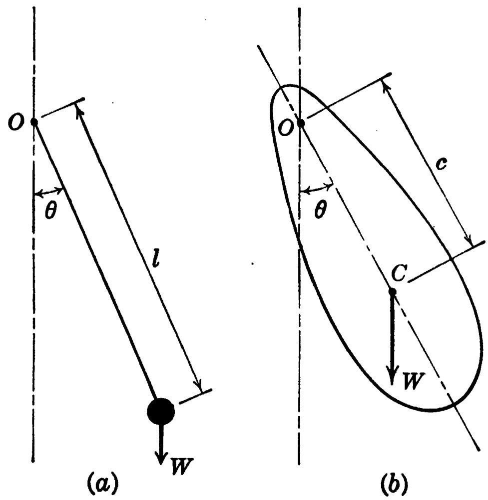


FIG. 2–14

lums about their axes of suspension will be

$$-Wl \sin \theta \quad \text{and} \quad -Wc \sin \theta$$

respectively. If the respective moments of inertia about the axes of suspension are designated by I_0, the equations of motion take the form,

$$I_0\ddot{\theta} = -Wl \sin \theta \qquad \text{for Fig. 2–14}a$$

and

$$I_0\ddot{\theta} = -Wc \sin \theta \qquad \text{for Fig. 2–14}b$$

If θ is small, $\sin \theta$ may be replaced by θ, and the equations can be written as

$$\left.\begin{aligned} \ddot{\theta} + \frac{Wl}{I_0}\theta &= 0 \\ \text{and} \qquad \ddot{\theta} + \frac{Wc}{I_0}\theta &= 0 \end{aligned}\right\} \qquad (2.4\text{–}5)$$

respectively.

Comparing these equations with equation 2.1–4 it will be noted that

Wl and Wc are equivalent to the torsional spring constant k. For an oscillating gravity system of this type, the equivalent torsional spring constant can therefore be written in general as

$$k_e = Wc$$

where c is the distance between axis of suspension and the centroid of the weight W.

In case of the mathematical pendulum (Fig. 2–14a),

$$I_0 = \frac{W}{g}\bar{r}_0^{\,2} = \frac{W}{g}l^2$$

and the circular frequency is

$$p = \sqrt{\frac{Wl}{I_0}} = \sqrt{\frac{Wl}{Wl^2}g} = \sqrt{\frac{g}{l}}$$

from which the period is

$$\tau = \frac{2\pi}{p} = 2\pi\sqrt{\frac{l}{g}}$$

For the compound pendulum (Fig. 2–14b),

$$I_0 = \frac{W}{g}\bar{r}_0^{\,2} = \frac{W}{g}(c^2 + \bar{r}_c^{\,2})$$

where $\bar{r}_c$ is the radius of gyration with respect to an axis parallel to the axis of suspension and passing through the centroid of the mass. The circular frequency becomes

$$p = \sqrt{\frac{Wc}{I_0}} = \sqrt{\frac{cg}{c^2 + \bar{r}_c^{\,2}}} = \sqrt{\frac{g}{c + \bar{r}_c^{\,2}/c}}$$

and the period is

$$\tau = 2\pi\sqrt{\frac{c + \bar{r}_c^{\,2}/c}{g}} = 2\pi\sqrt{\frac{l_e}{g}}$$

The equivalent length $l_e = c + \bar{r}_c^{\,2}/c$ corresponds to the length l of a mathematical pendulum that will have the same period as the compound pendulum.

It is useful to have a geometric understanding of the relationships of the lengths c, $\bar{r}_c$, $\bar{r}_0$ and l_e. As previously indicated the radius of gyration with respect to the axis of rotation can be expressed as

$$\bar{r}_0 = \sqrt{c^2 + \bar{r}_c^{\,2}}$$

where c is the distance to the centroid and $\bar{r}_c$ is the radius of gyration with respect to the axis through the centroid. The radius of gyration $\bar{r}_0$ is therefore the hypotenuse in a right-angled triangle with c and $\bar{r}_c$ as the other two sides. The length $\overline{OR}$, as shown in Fig. 2–15, is $\bar{r}_0$. If $\overline{RO'}$ is drawn perpendicular to $\overline{OR}$, the similarity of triangles

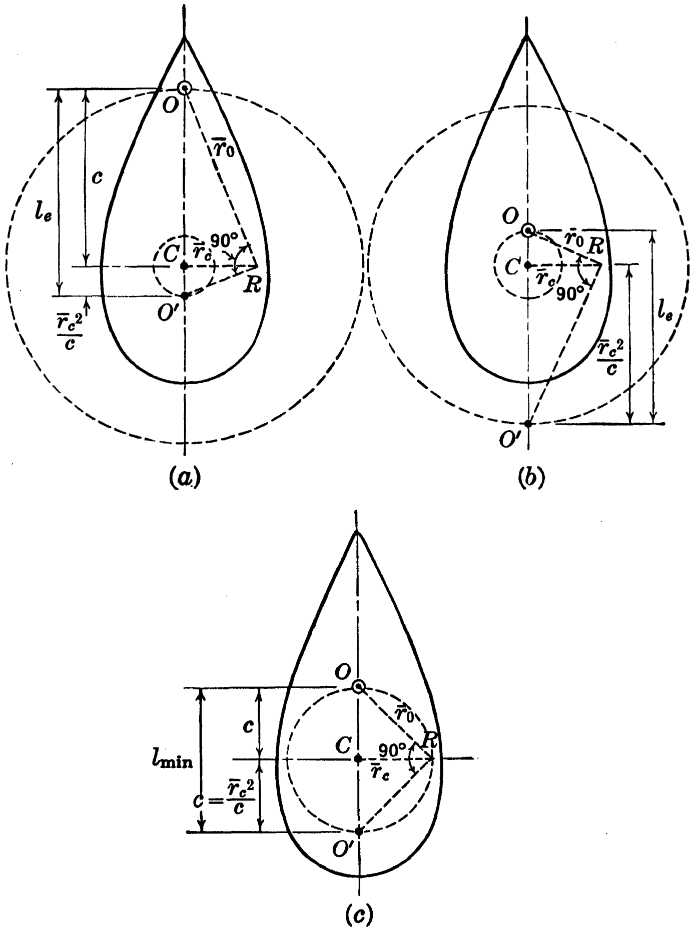


FIG. 2–15

$O'CR$ and RCO will show that $\overline{O'C} = \bar{r}_c^2/c$, which can also be realized from the fact that $\bar{r}_c$ is the geometric mean of c and $\bar{r}_c^2/c$. In this manner,

$$\overline{OO'} = c + \frac{\bar{r}_c^2}{c} = l_e$$

For other, mutually parallel axes of suspension the equivalent mathematical pendulum lengths may be determined geometrically, and it will be realized that points O and O' represent conjugate axes of suspension for which the periods of oscillation are the same. Furthermore, the loci for all the axes of suspension for which the period will be the same can easily be verified as the two conjugate circles shown in the Figs. 2–15*a*, *b*, and *c*. Figure 2–15*c* shows the particular symmetric case for which the period becomes a minimum. From the geometry, this will be found to be

$$\tau_{\min} = 2\pi\sqrt{\frac{2\bar{r}_c}{g}}$$

An additional example is a pendulum that oscillates in a centrifugal field rather than a gravitational field. A pendulum of this type is shown in Fig. 2–16. Essentially it consists of a pendulum that is attached at point B to a rigid body which is rotating at a constant angular velocity Ω. The centrifugal force which acts on the pendulum mass is

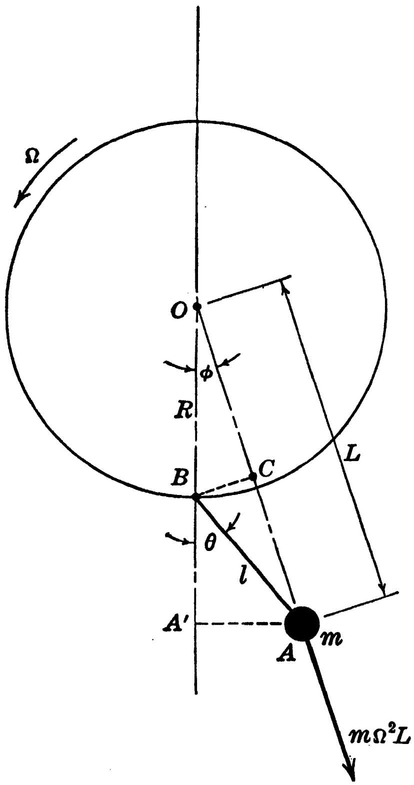


Fig. 2–16

$$m\Omega^2 L$$

for small oscillations. The restoring moment about the point B will then be

$$m\Omega^2 L(\overline{CB}) = m\Omega^2 LR \sin\phi$$

However it is seen that

$$L \sin\phi = \overline{AA'} = l \sin\theta$$

Thus, for small oscillations,

$$LR \sin\phi = Rl \sin\theta \cong Rl\theta$$

The equation of motion for rotation of the pendulum mass about B may now be written as

$$I_B\ddot{\theta} = -m\Omega^2 lR\theta$$

or

$$ml^2\ddot{\theta} + m\Omega^2 lR\theta = 0$$

which reduces to

$$\ddot{\theta} + \Omega^2 \frac{R}{l} \theta = 0 \qquad (2.4\text{–}6)$$

The natural circular frequency is seen to be

$$p = \Omega \sqrt{\frac{R}{l}}$$

and the frequency and period are therefore

$$f = \frac{p}{2\pi} = \frac{\Omega}{2\pi}\sqrt{\frac{R}{l}}; \qquad \tau = \frac{2\pi}{p} = \frac{2\pi}{\Omega}\sqrt{\frac{l}{R}}$$

It is of interest to note that this result coincides with the solution for the ordinary gravity pendulum for a value of

$$\Omega^2 R = g$$

The assumption of small amplitudes which was made in establishing equations 2.4–5 and 2.4–6 is usually sufficiently accurate for engineering purposes. Such a "linearized" solution is an approximation to the exact solution for which the frequency depends upon the amplitude of the motion. Comparison with the exact solution for a system of this type (Chapter 10) shows an error in the frequency of approximately 2% when the amplitude is as large as 30°. The linearized solution is therefore a good representation of the motion for reasonable amplitudes.

Typical buoyant system

Typical of the systems relying on buoyancy for restoring forces is a wall-sided vessel floating in a liquid of specific weight w (Fig. 2–17).

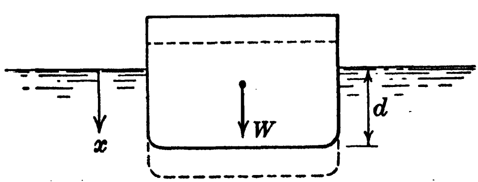


FIG. 2–17

The vessel will float in the equilibrium position as shown. From Archimedes' principle, the weight W of the vessel is equal to the weight of the displaced fluid. Therefore, it follows that

$$W = Vw = mg \qquad (2.4\text{–}7)$$

where V is the displaced volume of the vessel in equilibrium. Considering a downward displacement as positive, the equation of motion may be written as follows:

$$\frac{W}{g}\ddot{x} = W - w(V + xA)$$

where A is the cross section at the water line. Use of equation 2.4–7 permits this to be reduced to standard form,

$$\ddot{x} + \frac{Aw}{m}x = 0$$

The circular and natural frequency are, respectively,

$$p = \sqrt{\frac{k_e}{m}} = \sqrt{\frac{Aw}{m}} \quad \text{and} \quad f = \frac{1}{2\pi}\sqrt{\frac{Aw}{m}}$$

The equivalent spring constant is

$$k_e = Aw$$

Typical combined system

The system shown in Fig. 2–18 combines the spring elements due to gravity, buoyancy, and elasticity. This problem can conveniently be treated for small oscillations by referring the motion of all the elements of the system to the angular motion of the compound pendulum W_1. The total moment of inertia of the system about O is

$$I_0 = \frac{W_1}{g}(\bar{r}_c^{\,2} + c^2) + \frac{W_2}{g}l^2$$

where $\bar{r}_c$ is the radius of gyration of W_1 about C. The equivalent torsional spring constant is found from a consideration of the restoring torques acting on the system due to an angular displacement θ. The sum of the restoring torques is

$$\begin{aligned} T &= (kx + A_2wx)l + W_1c\theta \\ &= [(k + A_2w)l^2 + W_1c]\theta \end{aligned}$$

The equivalent torsional spring constant is therefore

$$k_e = \frac{T}{\theta} = (k + A_2w)l^2 + W_1c$$

Using Newton's second law, the equation of motion is

$$I_0\ddot{\theta} + k_e\theta = 0$$

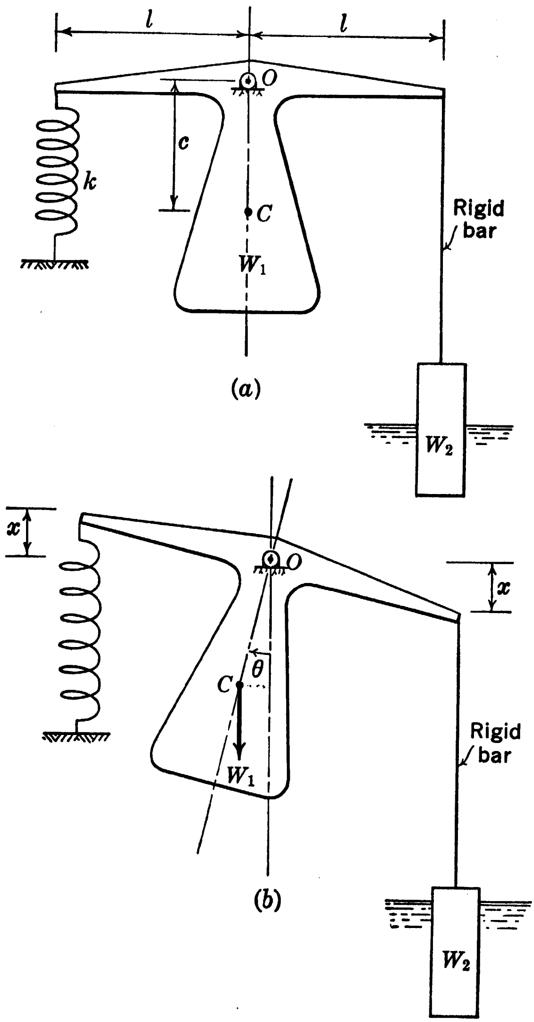


FIG. 2–18

The natural circular frequency is then

$$p = \sqrt{\frac{k_e}{I_0}}$$

and the period of the motion is

$$\tau = \frac{2\pi}{p} = 2\pi \sqrt{\frac{W_1(\bar{r}_c{}^2 + c^2) + W_2 l^2}{g[(k + A_2 w)l^2 + W_1 c]}}$$

As a second example of a combined system, consider the system of Fig. 2–19 where W_1 and W_2 are geared together. The moment of

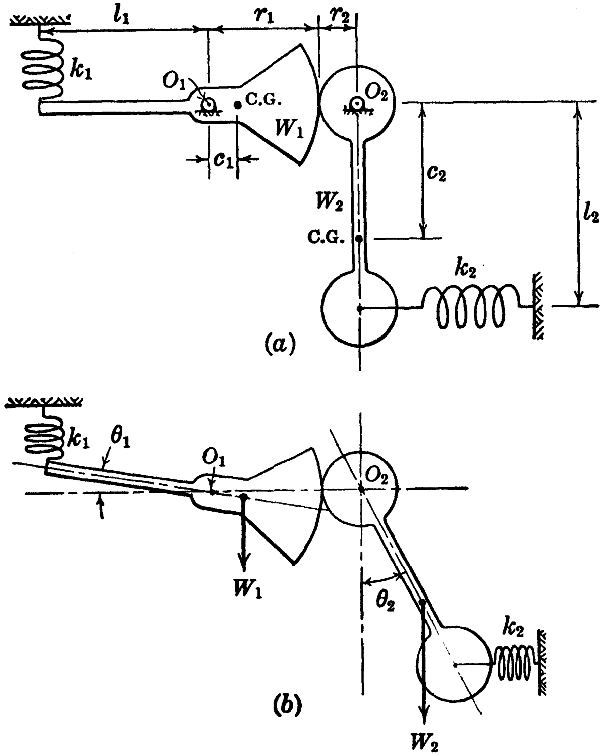


Fig. 2–19

inertia of W_1, about O_1, can be expressed as

$$I_1 = \frac{W_1}{g}(\bar{r}_{c1}^2 + c_1^2)$$

and the moment of inertia of W_2 about O_2 in the same manner is

$$I_2 = \frac{W_2}{g}(\bar{r}_{c2}^2 + c_2^2)$$

The displacement ratio n may be defined as

$$n = \frac{\theta_2}{\theta_1} = \frac{r_1}{r_2}; \quad \text{thus} \quad \theta_2 = n\theta_1 \tag{2.4–8}$$

Since the accelerations are proportional to the displacements, the ratio between the angular accelerations $\ddot{\theta}_1$ and $\ddot{\theta}_2$ of W_1 and W_2 is also n; thus

$$\ddot{\theta}_2 = n\ddot{\theta}_1 \tag{2.4–9}$$

In this manner all motion can be referred to W_1. For simplicity the equilibrium position is assumed as indicated in Fig. 2–19*a*. The equations of motion for the unbalanced position, as shown in Fig. 2–19*b*, may be written as follows:

$$I_1\ddot{\theta}_1 = -Fr_1 - k_1l_1^2\theta_1 \tag{2.4–10}$$

and

$$I_2\ddot{\theta}_2 = Fr_2 - k_2l_2^2\theta_2 - W_2c_2\theta_2 \tag{2.4–11}$$

where F is the interacting force at the gear contact. Substitution of equation 2.4–9 into equation 2.4–11 gives

$$I_2n\ddot{\theta}_1 = Fr_2 - k_2l_2^2n\theta_1 - W_2c_2n\theta_1$$

Multiplication of this equation by $r_1/r_2 = n$ gives

$$I_2n^2\ddot{\theta}_1 = Fr_1 - n^2(k_2l_2^2 + W_2c_2)\theta_1 \tag{2.4–12}$$

Adding equations 2.4–10 and 2.4–12 and transposing yields

$$(I_1 + n^2I_2)\ddot{\theta}_1 + [k_1l_1^2 + n^2(k_2l_2^2 + W_2c_2)]\theta_1 = 0$$

from which the equivalent spring constant and moment of inertia are seen to be

$$k_e = k_1l_1^2 + n^2(k_2l_2^2 + W_2c_2)$$

and

$$I_e = I_1 + n^2I_2$$

Thus the frequency is

$$f = \frac{p}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k_e}{I_e}} = \frac{1}{2\pi}\sqrt{\frac{k_1l_1^2 + n^2(k_2l_2^2 + W_2c_2)}{I_1 + n^2I_2}}$$

In this problem the displacements were expressed in terms of the motion of W_1. It will be noted that the equivalent spring constant k_e is made up of the sum of the torsional spring constants, $k_1l_1^2$ and the torsional spring constant, $k_2l_2^2 + W_2c_2$, multiplied by the square of the displacement ratio for W_2. (The springs and gravity equivalent are acting in parallel.) Similarly the equivalent moment of inertia I_e is the sum of I_1 and the square of the displacement ratio times I_2. This is true for any combination of gears or levers and leads in general to a very useful equivalent vibratory system in which the effect of gears or levers is eliminated by substituting equivalent masses and springs. These equivalents are obtained by multiplying each mass or spring by the square of the ratio of its displacement to the displacement of the mass to which all the motions are referred.

2.5. Energy Method

The law of conservation of energy offers an extremely useful method of determining the natural frequency of a vibrating system. It is instructive to consider first the distribution of energy contained

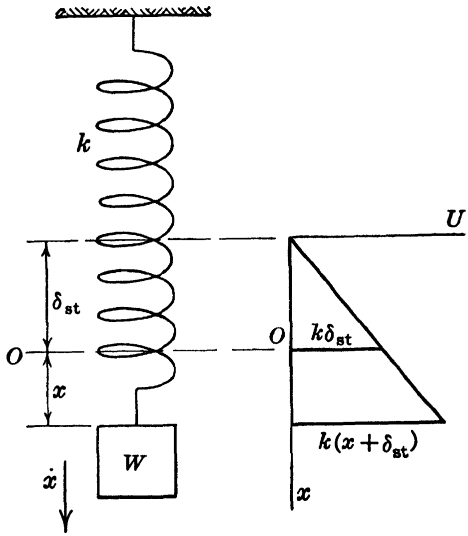


FIG. 2–20

in an oscillating system. For example, the system shown in Fig. 2–20 will in general contain both potential and kinetic energy. The potential energy is stored in the spring when the spring is deformed, whereas the kinetic energy is contained in the mass by virtue of its velocity.

The potential energy stored in the spring in the equilibrium position is $\frac{1}{2}k\delta_{st}^2$, as shown by the area of the small triangle in Fig. 2–20. If at some later time the displacement of the weight W from the equilibrium position is x, the change in potential energy due to this displacement consists of the increase in potential energy stored in the spring less the change in the potential energy of the weight due to the lowering of its position. The net change in potential energy is then

$$U = [\tfrac{1}{2}k(\delta_{st} + x)^2 - \tfrac{1}{2}k\delta_{st}^2] - Wx$$

This expression reduces to

$$U = k\delta_{st}x + \tfrac{1}{2}kx^2 - Wx$$

However,

$$k\delta_{st} = W$$

therefore, the net change in potential energy is

$$U = \tfrac{1}{2}kx^2$$

The potential energy U represents the work done on the system in displacing the system a distance x. If the velocity of the weight W at this same instant is $\dot{x}$, the kinetic energy in the system will be

$$T = \frac{1}{2}\frac{W}{g}\dot{x}^2$$

Systems performing free vibrations are by definition free of external sources of energy, and hence the total energy contained in the system is constant. This is expressed by the relation

$$U + T = \tfrac{1}{2}kx^2 + \frac{1}{2}\frac{W}{g}\dot{x}^2 = \text{constant} \qquad (2.5\text{–}1)$$

Note that equation 2.5–1 is the first integral of the equation of motion established in section 2.1 for this same system. This may be seen by differentiating equation 2.5–1 with respect to time, which leads to

$$\frac{W}{g}\ddot{x} + kx = 0$$

The principal advantage of the energy method is not that the equation of motion may be established by energy considerations but lies in the ease with which the frequency may be obtained from the energy equation. This is of particular advantage in complex systems and is illustrated in the following example.

When the weight W passes through the equilibrium position, the displacement x is zero and the velocity $\dot{x}$ is a maximum. In this position, the total energy E in the system is seen to be

$$E = \frac{1}{2}\frac{W}{g}\dot{x}^2_{max}$$

Similarly, at maximum amplitude, the velocity is zero and the total energy is

$$E = \tfrac{1}{2}kx^2_{max}$$

It therefore follows that the maximum potential energy of the system is equal to the maximum kinetic energy, or

$$\frac{1}{2}\frac{W}{g}\dot{x}^2_{\max} = \tfrac{1}{2}kx^2_{\max} \qquad (2.5\text{–}2)$$

The solution to the equation of motion for this system was established in section 2.2 as

$$x = x_0 \cos pt + \frac{\dot{x}_0}{p}\sin pt$$

and the maximum displacement and velocity were found to be, respectively,

$$x_{\max} = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{p}\right)^2}; \qquad \dot{x}_{\max} = p\sqrt{x_0^2 + \left(\frac{\dot{x}_0}{p}\right)^2}$$

Substituting these values into equation 2.5–2 yields

$$\frac{W}{g}p^2 = k \quad \text{or} \quad p^2 = \frac{kg}{W}$$

which agrees with the previous determination of the circular frequency. Thus the natural frequency may be found from the energy equation by simply substituting the maximum values of displacement and velocity, assuming that the system performs simple harmonic motion.

This method may readily be applied to more complex systems. Consider the system of Fig. 2–19. The total energy contained in the system is the sum of the potential and kinetic energies. For any arbitrary displacement, θ_1 and θ_2, and arbitrary angular velocities, $\dot{\theta}_1$ and $\dot{\theta}_2$, the total energy is

$$\tfrac{1}{2}(I_1\dot{\theta}_1^2 + I_2\dot{\theta}_2^2) + \tfrac{1}{2}k_1(\theta_1 l_1)^2 + \tfrac{1}{2}k_2(\theta_2 l_2)^2 + W_2c_2(1 - \cos\theta_2) = \text{constant}$$

where, for small values of θ_2,

$$(1 - \cos\theta_2) = \frac{\theta_2^2}{2}$$

Equating the energy at maximum displacement to the energy of the system at the instant it passes through the neutral position gives

$$\tfrac{1}{2}(I_1\dot{\theta}^2_{1\max} + I_2\dot{\theta}^2_{2\max}) = \tfrac{1}{2}(k_1l_1^2\theta^2_{1\max} + k_2l_2^2\theta^2_{2\max} + W_2c_2\theta^2_{2\max})$$

However,

$$\theta_{2\max} = n\theta_{1\max} \quad \text{and} \quad \dot{\theta}_{2\max} = n\dot{\theta}_{1\max}$$

and, for simple harmonic motion in general,

$$\dot{\theta}^2{}_{\max} = p^2\theta^2{}_{\max}$$

The equation reduces to

$$p^2(I_1 + n^2I_2) = k_1l_1{}^2 + n^2k_2l_2{}^2 + n^2W_2c_2$$

or

$$p^2 = \frac{k_1l_1{}^2 + n^2(k_2l_2{}^2 + W_2c_2)}{I_1 + n^2I_2}$$

which agrees with the previous result. This method may be applied advantageously to other systems.

Chapter 3

FORCED VIBRATIONS WITHOUT DAMPING

3.1. The Nature of Forced Vibrations

In contrast to free vibrations, which occur in a self-contained system, forced vibrations are directly due to periodic forces originating outside the system. As stated in the previous chapter, both a mass and a spring, or their equivalent, are essential components of a system performing free vibrations. This requirement is no longer necessary when forced vibrations are considered; in fact, a mass or a spring may be caused to oscillate by itself with the application of the proper external periodic force. The periodic external force that is responsible for forced vibrations may be of a complex nature. However, just as in the case of periodic motions, we confine our attention for the present to the fundamental case, in which the force is related to time by the following expression:

$$F = P \cos \omega t \tag{3.1-1}$$

To illustrate the essential differences between free and forced vibrations, two basic examples will be considered.

Periodic force and spring

The simplest type of forced vibration occurs when an oscillating force acts directly on a spring as shown in Fig. 3–1. Since the system is assumed to be without mass, the equation of motion can be written directly from the requirements of static equilibrium.

$$kx_k = P_k \cos \omega t$$

whence

$$x_k = \frac{P_k}{k} \cos \omega t \tag{3.1-2}$$

The motion of the end of the spring is seen to be simple harmonic with the same frequency as the exciting force. This system has the

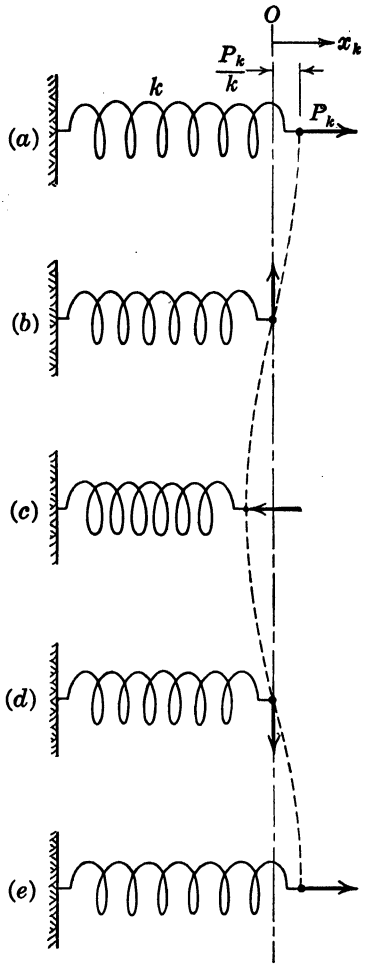


FIG. 3–1

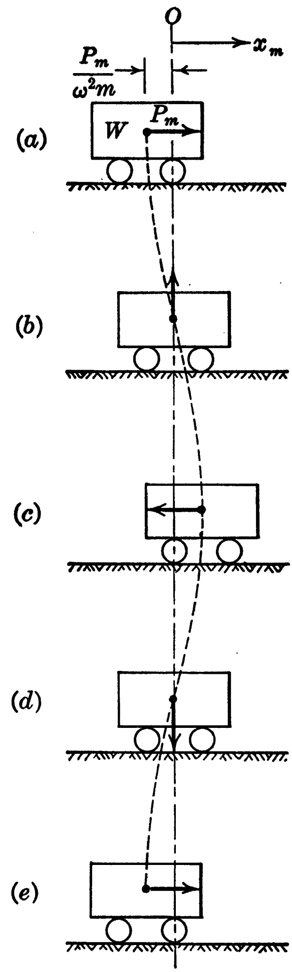


FIG. 3–2

capacity to store only potential energy. The total energy content of this system is

$$E_k = \tfrac{1}{2}kx_k^2 = \frac{1}{2}\frac{P_k^2}{k}\cos^2 \omega t$$

$$= \frac{1}{4}\frac{P_k^2}{k}(1 + \cos 2\omega t) \qquad (3.1\text{–}3)$$

As the energy stored in the system is a function of time, and therefore not constant, this is a non-conservative system.

Periodic force and a mass

A system that consists solely of a movable mass and an oscillating force is shown in Fig. 3–2. With the aid of Newton's second law, the equation of motion is established as

$$\frac{W}{g}\ddot{x}_m = P_m \cos \omega t \tag{3.1–4}$$

Integration of this equation leads to

$$\dot{x}_m = \frac{P_m g}{W\omega}\sin \omega t + C_1 \tag{3.1–5}$$

and

$$x_m = -\frac{P_m g}{W\omega^2}\cos \omega t + C_1 t + C_2 \tag{3.1–6}$$

The initial conditions are arbitrary, but for convenience in the following development these conditions are selected in such a manner that $C_1 = C_2 = 0$, which requires that the mass oscillate about the origin. This it will be found to be equivalent to the initial conditions,

$$\dot{x}_0 = 0 \quad \text{and} \quad x_0 = -\frac{P_m g}{W\omega^2}$$

Equation 3.1–6 may now be written as

$$x_m = -\frac{P_m g}{W\omega^2}\cos \omega t \tag{3.1–7}$$

The mass is seen to perform a simple harmonic motion. The negative sign indicates that the force always opposes the motion, as shown in Fig. 3–2, and therefore the phase angle between the force and the displacement is π rad. In this instance, the system can contain only kinetic energy. The total energy of the system at any time is

$$E_m = \frac{1}{2}\frac{W}{g}\dot{x}_m^{\,2} = \frac{1}{2}\frac{P_m^{\,2}g}{W\omega^2}\sin^2 \omega t$$

$$= \frac{1}{4}\frac{P_m^{\,2}g}{W\omega^2}(1 - \cos 2\omega t) \tag{3.1–8}$$

Since the energy stored in the system is a function of time, this is also a non-conservative system.

Combined system

A system containing both spring and mass elements is shown in Fig. 3–3. The motion of this system due to an exterior periodic force

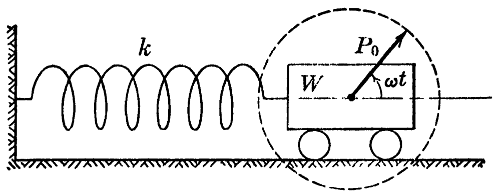


FIG. 3–3

$P_0 \cos \omega t$ may be found by a linear combination of the two previous solutions. For the combined case,

$$x = x_k = x_m; \qquad P_0 = P_k + P_m \tag{3.1–9}$$

From equations 3.1–2 and 3.1–7, the following are obtained:

$$\left.\begin{aligned} P_k &= \frac{kx}{\cos \omega t} \\ P_m &= -\frac{W\omega^2 x}{g \cos \omega t} \end{aligned}\right\} \tag{3.1–10}$$

Substitution from equation 3.1–10 into equation 3.1–9 yields

$$P_0 = \left(k - \frac{W\omega^2}{g}\right)\frac{x}{\cos \omega t} = k\left(1 - \frac{\omega^2}{(kg/W)}\right)\frac{x}{\cos \omega t}$$

from which

$$x = \frac{P_0/k}{1 - (\omega/p)^2} \cos \omega t \tag{3.1–11}$$

where, as before,

$$p^2 = \frac{kg}{W}$$

is the natural circular frequency of the system. The amplitude A is found to be

$$A = \frac{\delta_P}{1 - \left(\dfrac{\omega}{p}\right)^2} \tag{3.1–12}$$

where $\delta_P = P_0/k$ is the static deflection of the spring due to the force

P_0. The energy stored in this system is partially kinetic and partially potential in character. The total energy is

$$E = \tfrac{1}{2}kx^2 + \frac{1}{2}\frac{W}{g}\dot{x}^2$$

$$= \tfrac{1}{2}A^2\left(k\cos^2\omega t + \frac{W}{g}\omega^2\sin^2\omega t\right)$$

$$= \tfrac{1}{2}kA^2\left[\cos^2\omega t + \left(\frac{\omega}{p}\right)^2\sin^2\omega t\right] \qquad (3.1\text{–}13)$$

which is a function of time, and the system is seen to be non-conservative when $\omega \neq p$. The special situation that arises when $\omega = p$ is called resonance and will be discussed in detail in a later section.

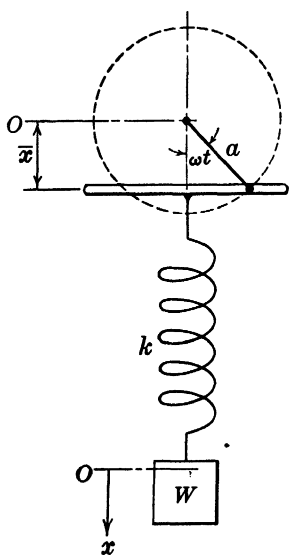


FIG. 3–4

A further problem which is of interest is shown in Fig. 3–4, in which one end of the spring is subjected to a simple harmonic motion. This problem is similar to that of a periodic force and a mass as previously treated. The force acting on the mass at any time is a function of the amount of compression of the spring, that is,

$$F = k(\bar{x} - x) = k(a\cos\omega t - x)$$

From equation 3.1–7 it is noticed that the motion of the mass is

$$x = -\frac{g}{W\omega^2}F$$

where $F = P\cos\omega t$ is the force that acts on the system. In this illustration, therefore,

$$x = -\frac{g}{W\omega^2}k(a\cos\omega t - x)$$

Solving for x gives

$$x = \frac{a}{1 - \left(\frac{\omega}{p}\right)^2}\cos\omega t \qquad (3.1\text{–}14)$$

The motion of the mass is simple harmonic, and the solution is similar to that of the previous problem as is seen from equation 3.1–11. The amplitude of the motion is

$$A = \frac{a}{1 - \left(\frac{\omega}{p}\right)^2} \tag{3.1–15}$$

The amplitude due to forced vibration is of sufficient importance in the field of mechanics to justify an intimate understanding of its physical meaning. To enhance this understanding there follows an additional way of evaluating this amplitude through another line of reasoning.

A mass m and a spring with a spring constant k comprise a conservative system with a circular frequency $p = \sqrt{k/m}$. When this system is subjected to an external oscillating force which has a circular frequency $\omega \neq p$, work is done periodically on the system by this force, and therefore the system as a whole is non-conservative. However it is in general possible to divide the system into two parts such that one part is a conservative system of natural frequency ω and the other part consists of an elementary non-conservative system which absorbs the total effect of the external force. From this point of view, the amplitude of the complete system is determined by the non-conservative part of the system and the conservative part merely "rides along," since it is self-contained and requires no external effort to maintain any arbitrary amplitude.

The actual division of the system in the above manner may be accomplished in several ways. For example, if $p < \omega$ the system may be said to have too much mass or too much flexibility (to little stiffness). Either the extra mass or the extra flexibility may be treated as the non-conservative part of the system.

To illustrate the above contention, it will be applied to the system of Fig. 3-5, which is essentially the same as that previously discussed and shown in Fig. 3–4. The spring length l can be changed to conform with the forced frequency ω by decreasing the length of the spring for $\omega > p$ or increasing the length of the spring for $\omega < p$.

1. For $\omega > p$, let it be assumed that a spring constant k_1 corresponding to a spring length $l_1 < l$ will make $p_1 = \sqrt{k_1/m} = \omega$. This new system will then have the same natural frequency as the forced motion at O, and it will oscillate as though the spring was suspended from a fixed point N_1 (the node). Each point in the spring will oscillate with an amplitude that will be in direct proportion to its distance from N_1. Since the point at O has an amplitude a, it follows that the

amplitude of the mass can be found from the relationship

$$\frac{a_1}{a} = \frac{l_1}{l - l_1} = -\frac{1}{1 - \dfrac{l}{l_1}}$$

Because the spring constants vary inversely as the length of the springs,

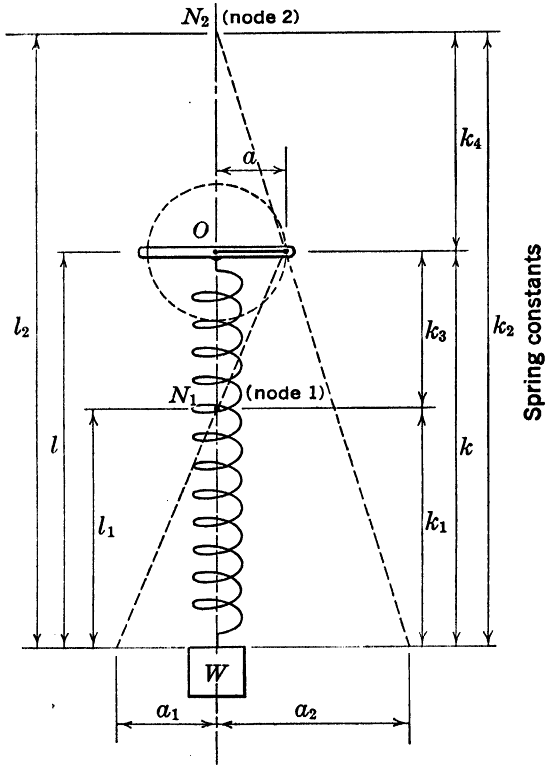


FIG. 3–5

and since

$$p^2 = \frac{k}{m} \quad \text{and} \quad \omega^2 = \frac{k_1}{m}$$

it follows that

$$\left(\frac{\omega}{p}\right)^2 = \frac{k_1}{k} = \frac{l}{l_1}$$

and thus

$$a_1 = \frac{-a}{1 - \left(\dfrac{\omega}{p}\right)^2} \tag{3.1–16}$$

which agrees with the previous result (equation 3.1–15). The negative sign in this equation means that the motion of W for $\omega > p$ is opposite to the forced motion at O. This is illustrated geometrically in Fig. 3–5 by locating a and a_1 on opposite sides of the line of motion.

The node at N_1 for $\omega > p$ is a point in the spring which theoretically has no motion. The combination of the spring k_1 and the mass m represents the conservative part of this system when the forced frequency is ω. The system composed of the spring ON_1 of length l–l_1 is the non-conservative part which "absorbs" the forced motion at O. The force necessary to produce the amplitude a in the non-conservative part of the system (i.e., the spring ON_1 of stiffness k_3) is $P = ak_3$. The spring constant k_3 is found from the relation

$$\frac{k_3}{k} = \frac{l}{l - l_1}$$

whence

$$k_3 = \frac{k}{1 - \dfrac{l_1}{l}} = \frac{k}{1 - \left(\dfrac{p}{\omega}\right)^2}$$

and thus

$$P = \frac{ak}{1 - \left(\dfrac{p}{\omega}\right)^2}$$

The amplitude of this force must be equal and opposite to the amplitude of the force in the conservative part of the system (i.e., the spring k_1 and the mass m). The amplitude a_1 of the mass has the value given by equation 3.1–16, and the amplitude of the force acting on the mass is

$$P_1 = a_1 k_1$$

However,

$$\frac{k_1}{k} = \frac{l}{l_1} \quad \text{or} \quad k_1 = \left(\frac{\omega}{p}\right)^2 k$$

which gives

$$P_1 = -\frac{a}{1 - \left(\dfrac{\omega}{p}\right)^2}\left(\frac{\omega}{p}\right)^2 k = \frac{ak}{1 - \left(\dfrac{p}{\omega}\right)^2} = P$$

showing that the two forces are equal.

It is well to realize that, although the node point is stationary for an ideal undamped system, this point will have some motion if damping is present as it is in all real systems. This motion will, as is shown later, be out of phase with the rest of the system.

2. By similar reasoning it may be shown that the forced amplitude a_2 for $\omega < p$ will be

$$a_2 = a\frac{l_2}{l_2 - l} = \frac{a}{1 - \frac{l}{l_2}} = \frac{a}{1 - \left(\frac{\omega}{p}\right)^2} \tag{3.1–17}$$

The amplitudes a and a_2 have the same sign, indicating that they are in phase for $\omega < p$. This is illustrated in Fig. 3–5 by locating a and a_2 on the same side of the line of motion. The node N_2 in this latter example is imaginary in that it lies in an extension of the spring. The analysis is the same as in 1 however, and similar results may be obtained.

If the oscillating force is applied directly to the mass, as shown in Fig. 3–6, the motion can be evaluated by assuming a division of the mass in such a manner that the part m_ω of the total mass, together with the spring k, will have the same natural frequency as the force. This system will oscillate with any amplitude of this frequency without any external excitation. The force P is only required to act on the remaining mass $m = m_p - m_\omega$. From equation 3.1–7 the amplitude of this system can be written as

$$a = -\frac{P}{m\omega^2} = -\frac{P}{(m_p - m_\omega)\omega^2}$$

Since $p^2 = k/m_p$ and $\omega^2 = k/m_\omega$, it follows that

$$a = -\frac{P}{\left(\frac{k}{p^2} - \frac{k}{\omega^2}\right)\omega^2} = \frac{P/k}{1 - \left(\frac{\omega}{p}\right)^2}$$

which is the same as found previously.

Figure 3–6*b* shows the case for $\omega > p$. From the relationship $\omega^2/p^2 = m_p/m_\omega$, it follows that $m_p > m_\omega$, and so $(m_p - m_\omega)$ is a positive quantity. An imaginary link connecting the two masses m_ω and $(m_p - m_\omega)$ will allow the two bodies to move together without any interacting force, and the motion will be similar to the one shown in Fig. 3–2. This means that the motion of the mass is opposite to the direction of the force.

Figure 3–6*c* shows the case of $\omega < p$. This requires that $m_\omega > m_p$,

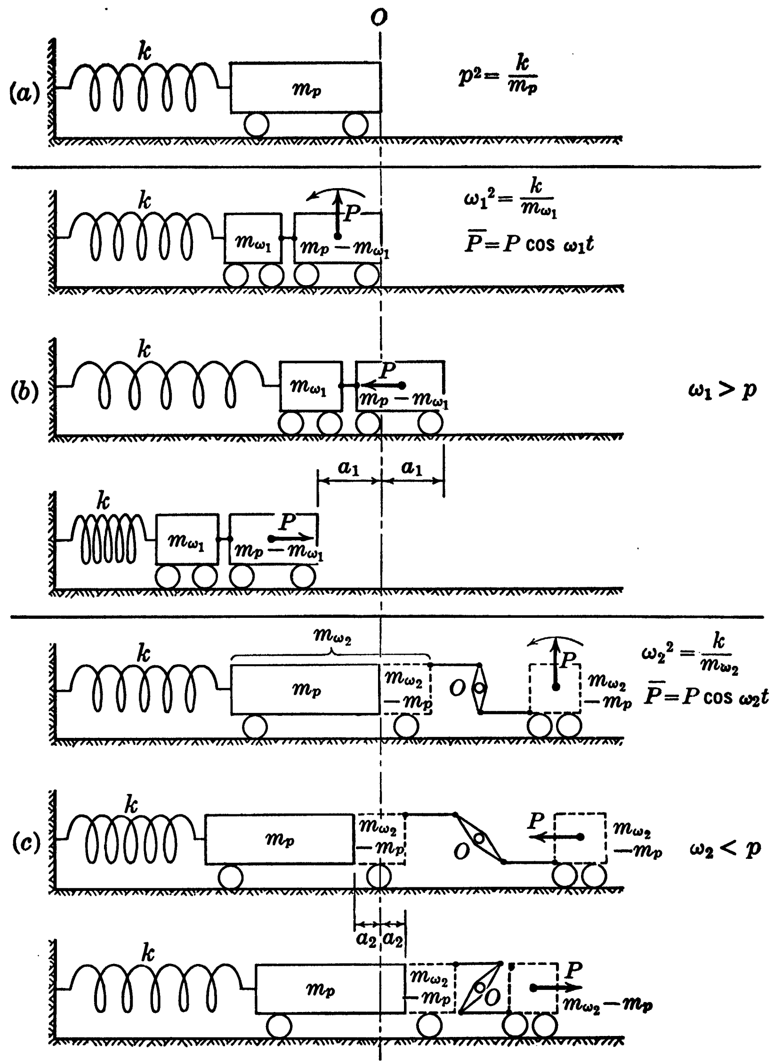


FIG. 3-6

and $(m_p - m_\omega)$ is a negative quantity. To interpret this it may be imagined that a mass $(m_p - m_\omega)$ is added and then "subtracted" through a lever connection as indicated. Such a physical arrangement, demanding no interacting forces through the connection, would produce the proper motion of the mass m_p. This means that the mass will move in the same direction as the force; that is, the exciting force and the displacement are in phase.

3.2. Derivation of the Equation of Motion

In the previous section, the nature of forced vibrations has been investigated as linear combinations of the motion resulting from a periodic force acting on a mass and on a spring. In this section, the motion will be studied by starting with the combined system. The advantages of this latter method lie in the directness of the treatment.

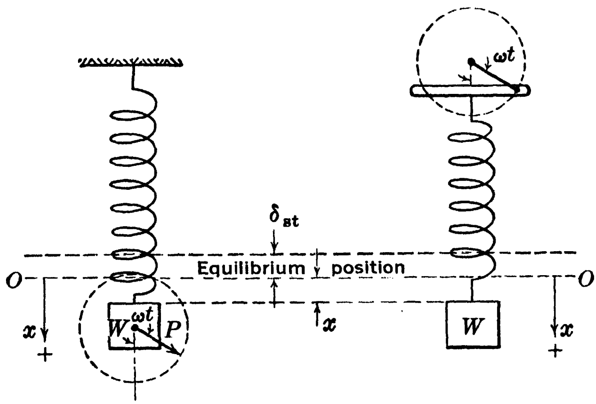


Fig. 3–7 Fig. 3–8

Consider the system shown in Fig. 3–7. The equation of motion may be written directly with the aid of Newton's second law as follows:

$$\frac{W}{g}\ddot{x} = W - k(\delta_{st} + x) + P\cos\omega t$$

Since

$$W = k\delta_{st}$$

this will be reduced to

$$\frac{W}{g}\ddot{x} + kx = P\cos\omega t \tag{3.2–1}$$

or

$$\ddot{x} + \frac{kg}{W}x = \frac{Pg}{W}\cos\omega t$$

The introduction of $p^2 = kg/W$ and $\delta_P = P/k$ yields

$$\ddot{x} + p^2x = \delta_P p^2\cos\omega t \tag{3.2–2}$$

Before proceeding with the solution of equation 3.2–2, it is convenient to establish the equation of motion of the similiar system of

Fig. 3–8. As above, this may be written directly as

$$\frac{W}{g}\ddot{x} = W - k(\delta_{st} + x - a \cos \omega t)$$

which reduces to

$$\frac{W}{g}\ddot{x} + kx = ak \cos \omega t$$

and finally

$$\ddot{x} + p^2x = ap^2 \cos \omega t \qquad (3.2\text{–}3)$$

It will be noted that equations 3.2–2 and 3.2–3 are identical in form, the only difference being in the constants δ_P and a. It is therefore to be expected that the motion of these two systems will be similar. This was seen to be true in the previous section, and so the solution to equations 3.2–3 and 3.2–3 may be treated simultaneously.

3.3. Solution of the Equations of Motion

The differential equations of motion 3.2–3 and 3.2–3 as developed for the forced vibration of a system of one degree of freedom, have the common form,

$$\ddot{x} + p^2x = A \cos \omega t \qquad (3.3\text{–}1)$$

where the constant A is a function of the exciting force or the forced amplitude, as the case may be. The general solution of this equation is composed of two parts.[1] The first part is the solution of the homogeneous equation which is obtained by equating the left-hand side to zero.

$$\ddot{x} + p^2x = 0 \qquad (3.3\text{–}2)$$

This homogeneous equation is identical with the equation of motion for free vibration, and its solution, called the complimentary solution, was found in Chapter 2 to be

$$x_c = C_1 \cos pt + C_2 \sin pt \qquad (3.3\text{–}3)$$

where C_1 and C_2 are arbitrary constants depending on the initial conditions.

The second part of the solution is known as the particular solution, and it must satisfy the complete differential equation. The particular solutions for two similar examples have been found in section 3.1. The particular solution may be expected to have the form

$$x_p = C_3 \cos \omega t + C_4 \sin \omega t \qquad (3.3\text{–}4)$$

[1] See any standard text on elementary differential equations.

where C_3 and C_4 are constants depending on the forced amplitude or exciting force and the physical constants of the system. The values of C_3 and C_4 may be found by substituting from equation 3.3–4 into the differential equation. Equation 3.3–1 then becomes

$$-C_3\omega^2 \cos \omega t - C_4\omega^2 \sin \omega t + C_3p^2 \cos \omega t + C_4p^2 \sin \omega t$$
$$= C_3(p^2 - \omega^2) \cos \omega t + C_4(p^2 - \omega^2) \sin \omega t = A \cos \omega t$$

This equation can only be satisfied for an arbitrary value of t and $\omega \neq p$ if

$$C_3 = \frac{A}{p^2 - \omega^2} \quad \text{and} \quad C_4 = 0$$

as may be seen by equating coefficients of $\sin \omega t$ and $\cos \omega t$ to zero. The particular solution to the differential equation of motion is therefore

$$x_p = \frac{A}{p^2 - \omega^2} \cos \omega t \tag{3.3–5}$$

The complete solution is the sum of the complimentary and the particular solutions, as follows:

$$x = x_c + x_p = C_1 \cos pt + C_2 \sin pt + \frac{A}{p^2 - \omega^2} \cos \omega t \tag{3.3–6}$$

It will be noted that the first two terms, which constitute the complimentary solution, represent a vibration having the same frequency as the natural frequency of the system and having an amplitude that depends on the initial conditions. This part of the motion is independent of the forced amplitude or of the exciting force. The physical meaning of this motion is simple. It may be interpreted as a free vibration initiated at or before the time the forced motion started. As may be seen from equation 3.3–6, the free vibration may be eliminated by requiring that the motion start with suitable initial conditions. This requirement is seldom met in nature, and the free vibration is usually present; however, in all physical systems, a certain amount of damping is usually present which eventually damps out the free vibration. For this reason, the free vibration is usually of a temporary character. Such motions are called transient vibrations, and these motions will be discussed more fully in Chapter 9.

The last term of the solution (equation 3.3–6) represents a motion that has the same frequency as the forced amplitude or exciting force. This motion will persist as long as the system is excited. For this reason, this latter part of the solution is called the steady-state solution, as it is the only part of the motion that survives in a damped system after a prolonged period of time.

The solution for the motion of the system shown in Fig. 3–7 may now be written directly from equation 3.3–6 by introducing

$$A = p^2\delta_P$$

This complete solution is

$$x = C_1 \cos pt + C_2 \sin pt + \frac{\delta_P}{1 - \left(\frac{\omega}{p}\right)^2} \cos \omega t$$

Similarly, the complete solution for the motion of the system of Fig. 3–8 may be established. In this instance,

$$A = p^2 a$$

and the complete solution is

$$x = C_1 \cos pt + C_2 \sin pt + \frac{a}{1 - \left(\frac{\omega}{p}\right)^2} \cos \omega t$$

The steady-state motion for these two systems is given, respectively, by

$$x = \frac{\delta_P}{1 - \left(\frac{\omega}{p}\right)^2} \cos \omega t \tag{3.3–7}$$

and

$$x = \frac{a}{1 - \left(\frac{\omega}{p}\right)^2} \cos \omega t \tag{3.3–8}$$

The amplitudes of the persistent motion, which are of primary interest in forced-vibration problems, are obtained from these equations.

3.4. Resonance

The particular solutions for the systems of Figs. 3–7 and 3–8 have an amplitude proportional to the "resonance" factor,

$$\frac{1}{1 - \left(\frac{\omega}{p}\right)^2} \tag{3.4–1}$$

It is instructive to picture the magnitude of this factor graphically, as shown in Fig. 3–9. The significance of the change in sign, which occurs when the ratio ω/p exceeds one was pointed out in section 3.1. For

$\omega/p < 1$, the factor is positive, and the motion of the mass is in the same direction as the exciting force or exciting amplitude. When $\omega/p > 1$, the resonance factor is negative, and the motion of the mass is opposed to the direction of the exciting force or forced amplitude. It is conventional however to plot it as positive as shown by the dotted line in Fig. 3–9.

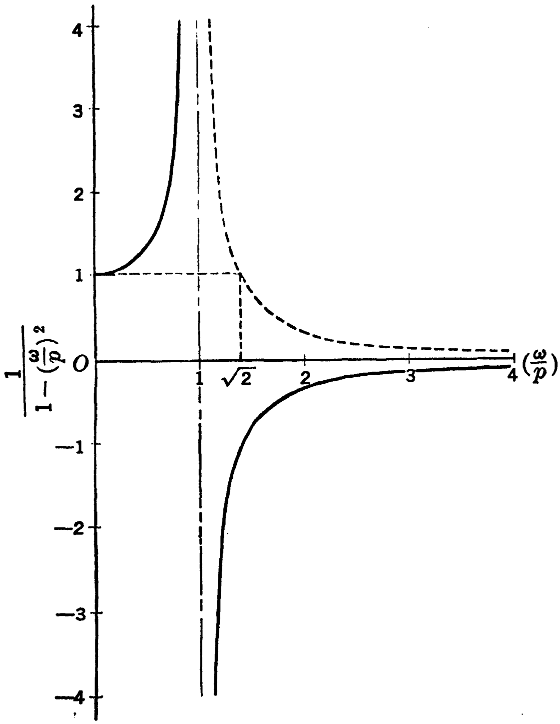


FIG. 3–9

When ω/p approaches 1, the amplitude is seen to take on infinitely large values. The condition existing when $\omega/p = 1$ is called resonance. At this point, the natural frequency of the system coincides with the forced frequency. The solution for the steady-state motion, as given by equations 3.3–7 and 3.3–8 is not valid in this instance, and there is no steady-state motion corresponding to $\omega = p$.

A solution, however, may be obtained from the original equation 3.3–1 by substituting p for ω, resulting in

$$\ddot{x} + p^2 x = A \cos pt \tag{3.4–2}$$

When the motion is initiated, the amplitude is zero. The indefinitely large amplitude is therefore the result of the motion growing with time. It is reasonable, then, to try a solution that contains a factor proportional to time in addition to the usual sine and cosine terms. The solution of this equation for the resonant case may therefore be expected to have the form

$$x_p = (C_3 \cos pt + C_4 \sin pt)t \tag{3.4–3}$$

The first and second derivatives of the trial solution are

$$\dot{x}_p = C_3 \cos pt + C_4 \sin pt - pt(C_3 \sin pt - C_4 \cos pt)$$

$$\ddot{x}_p = 2p(C_4 \cos pt - C_3 \sin pt) - p^2 x_p$$

Substitution into the differential equation 3.4–2 gives

$$2p(C_4 \cos pt - C_3 \sin pt) = A \cos pt$$

from which

$$C_3 = 0; \qquad C_4 = \frac{A}{2p}$$

The particular solution to equation 3.4–2 is then

$$x_p = \frac{At}{2p} \sin pt \tag{3.4–4}$$

If A is replaced by ap^2, $\delta_P p^2$, or in general δp^2, then this may be expressed as

$$x_p = \tfrac{1}{2}\delta pt \sin pt$$

The complete solution for the resonant case is obtained by adding this particular solution to the already known complimentary solution, as follows:

$$x = C_1 \cos pt + C_2 \sin pt + \tfrac{1}{2}\delta pt \sin pt$$

$$= C_1 \cos pt + (C_2 + \tfrac{1}{2}\delta pt) \sin pt \tag{3.4–5}$$

The amplitude of the motion is seen to grow in a linear fashion from the initial amplitude as given by the initial conditions to an indefinitely large value. The motion of a simple system at resonance is shown in Fig. 3–10. From a practical standpoint, there is a limit to the magnitude of the motion. This limit is reached when the motion is checked by a stop or when the spring element is overstressed and fails. One of the principal objects of vibration analysis is usually to avoid resonance.

A phenomenon of importance called beating occurs when the forced frequency is very near the natural frequency but not exactly at resonance. The forced vibration and the free vibration will then reinforce each other during the period they are approximately in phase and then nullify each other during the time when they are approximately 180° out of phase. The resulting amplitude which is

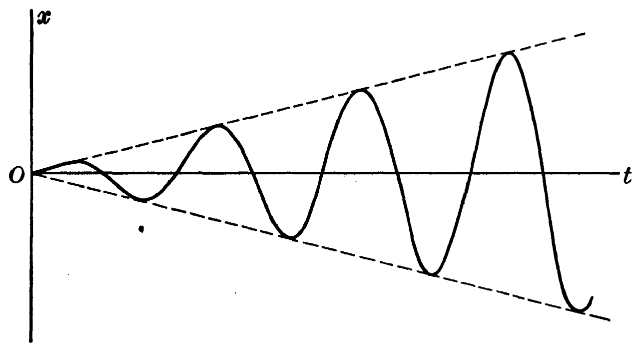


Fig. 3–10

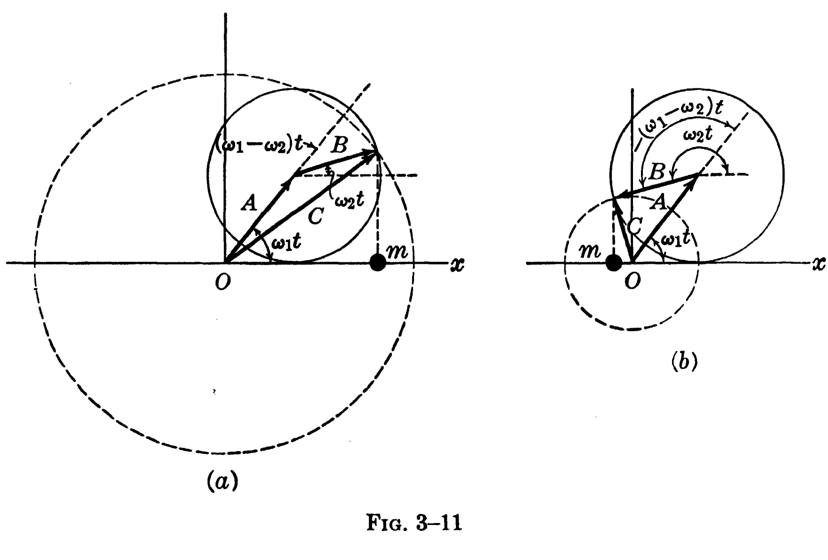


Fig. 3–11

the sum of these two motions will become relatively large for the period the motions are in phase and then almost disappear when they are opposed. This is particularly noticeable when the individual amplitudes of the two motions are of about the same magnitude. The same phenomenon may be observed when two exciting forces with frequencies that differ by a small amount are acting on the same system.

To be specific, we may consider two motions of amplitudes A and B

with circular frequencies ω_1 and ω_2 where ω_1 is very near ω_2. The combined amplitude C is the vector sum of these motions, as shown in Fig. 3–11. The magnitude of C will be found from the figure to be

$$C^2 = A^2 + B^2 + 2AB \cos (\omega_1 - \omega_2)t \tag{3.4–6}$$

for any time t. The beat circular frequency is

$$\omega_b = \omega_1 - \omega_2 \tag{3.4–7}$$

The combined motion is shown in Fig. 3–12.

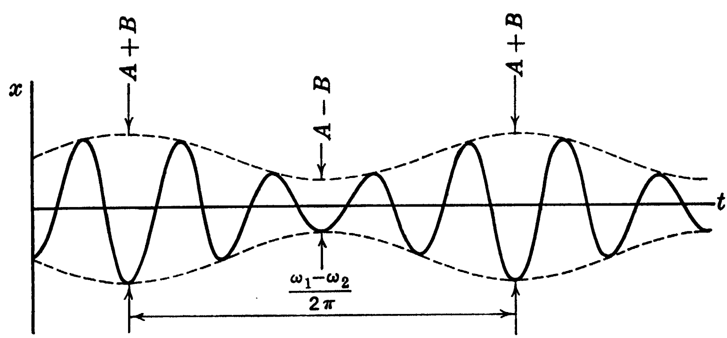


FIG. 3–12

3.5. Motion-Measuring Instruments

An immediate application of the theory of forced vibrations can be found in mechanical motion-measuring instruments. Two such instruments are the accelerometer and the vibrometer. A brief presentation of the theory of these instruments follows.

A typical idealized motion-measuring instrument is shown schematically in Fig. 3–13. This instrument is designed to record the motion of the mechanical part to which it is connected by recording the motion of a spring suspended weight W on a revolving drum. The motion of the weight W due to the forced amplitude a of circular frequency ω is given by the steady-state solution (equation 3.3–8).

$$x = \frac{a}{1 - \left(\frac{\omega}{p}\right)^2} \cos \omega t \tag{3.5–1}$$

Since the frame of the instrument is attached rigidly to the vibrating machine part, the recording drum will also have a motion of amplitude a. The record then is a graphic picture of the difference between the two motions; that is, the record indicates the relative motion between the weight W and the revolving drum. The amplitude of the record may thus be written as

$$A = \frac{a}{1 - \left(\frac{\omega}{p}\right)^2} - a = \frac{a\omega^2}{p^2 - \omega^2} \tag{3.5-2}$$

Assuming the motion of the forced vibration to be a simple harmonic, the acceleration amplitude of the forced motion is $a\omega^2$. Since the

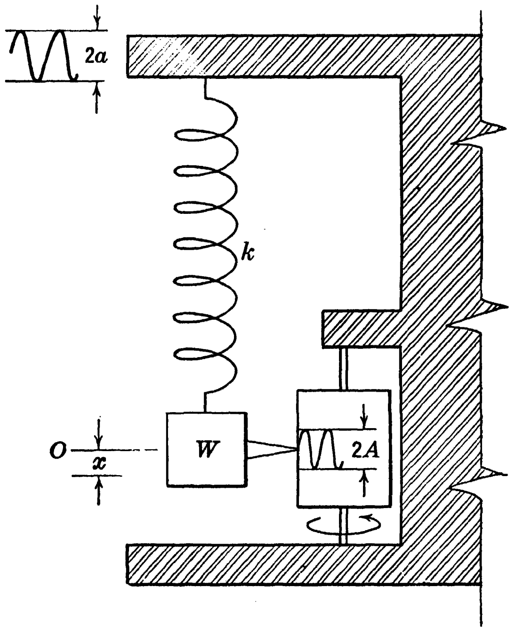


FIG. 3-13

natural circular frequency p is a characteristic of the instrument, the record may be made to register either accelerations or amplitudes by suitably adjusting the value of p. Consider two possible cases.

(a) $p >> \omega$. This condition may be achieved by using a very stiff spring and a small weight W. In this instance,

$$A \sim \frac{a\omega^2}{p^2} \tag{3.5-3}$$

and the record will be proportional to the acceleration of the machine to which the instrument is attached. Such an instrument is called an accelerometer.

(b) $p << \omega$. An instrument having this characteristic would involve a soft spring and a relatively large weight W. For this case, equation 3.5–2 may be written as

$$A \sim -\frac{a\omega^2}{\omega^2} = -a \tag{3.5–4}$$

An instrument of this design is called a vibrometer, and the record is a measure of the amplitude of the forced motion.

The record obtained through the use of either of the two instruments is approximate. The accuracy of this approximation depends to a large degree on the magnitude of the terms that are neglected in obtaining relations 3.5–3 and 3.5–4. In the design of a practical instrument, a small amount of damping is necessary to insure that the free vibration has been damped out and that only the steady-state motion is present.

3.6. Forced Oscillation of a Pendulum

The nature of free oscillation of a pendulum was treated in Chapter 2. For a mathematical pendulum of length l, the natural circular frequency and period were shown to be

$$p = \sqrt{\frac{g}{l}}; \qquad \tau = \frac{2\pi}{p} = 2\pi\sqrt{\frac{l}{g}} \tag{3.6–1}$$

If the point of suspension A (Fig. 3–14), is subjected to a forced oscillation $x_A = a \sin \omega t$, the motion of the mass for small amplitudes may be evaluated directly. The configuration of the motion for $p > \omega$ is shown in Fig. 3–14a and for $p < \omega$ in Fig. 3–14b. In both instances, the system will oscillate as though in free vibration about a virtual point of suspension N. The position of the virtual point of suspension or node may be found from the consideration of a pendulum of length l_ω having a natural frequency ω. From the geometry of Fig. 3–14,

$$\frac{a_B}{a} = \frac{l_\omega}{l_\omega - l} \quad \text{or} \quad a_B = \frac{a}{1 - \dfrac{l}{l_\omega}} \tag{3.6–2}$$

however,

$$\omega^2 = \frac{g}{l_\omega} \quad \text{and} \quad p^2 = \frac{g}{l} \quad \text{whence} \quad \frac{l}{l_\omega} = \left(\frac{\omega}{p}\right)^2$$

Equation 3.6–2 may therefore be written as

$$a_B = \frac{a}{1 - \left(\dfrac{\omega}{p}\right)^2} \tag{3.6–3a}$$

The force amplitude P necessary to maintain a steady-state amplitude a_B may be calculated from a consideration of the inertia force

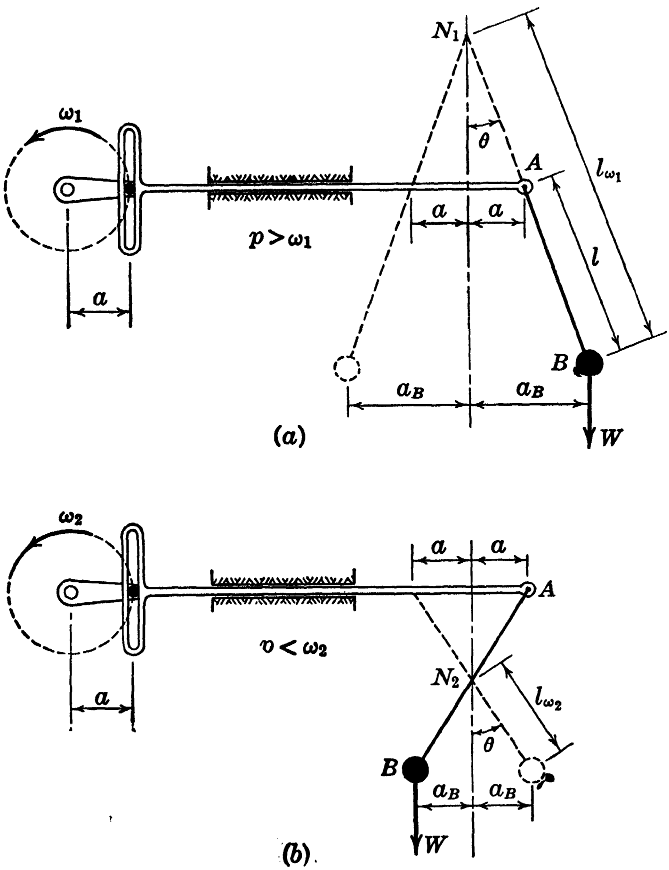


FIG. 3–14

of the mass m. The equilibrium of the pendulum requires that

$$P \sin \omega t - m\ddot{x}_B = 0$$

but, for small amplitudes, $x_B = a_B \sin \omega t$, from which $\ddot{x}_B = -\omega^2 a_B \sin \omega t$, whence

$$P = -m\omega^2 a_B = -m\omega^2 \frac{a}{1 - \left(\dfrac{\omega}{p}\right)^2} \tag{3.6–3b}$$

or

$$a = -\frac{P}{m\omega^2}\left[1 - \left(\frac{\omega}{p}\right)^2\right] \tag{3.6–4}$$

From equation 3.6–4, it is seen that a large exciting force P may be associated with a slight motion of the point of suspension A if the

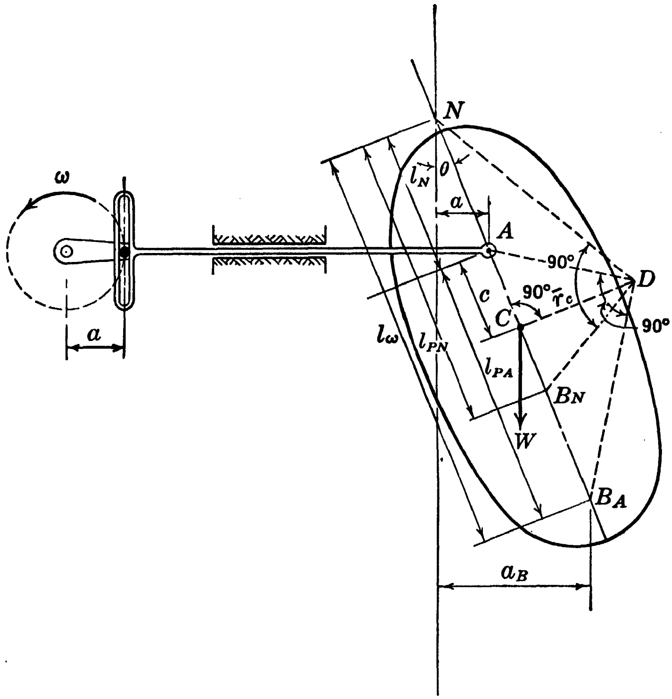


Fig. 3–15

pendulum is designed so that $p \sim \omega$, although a large amplitude of the pendulum mass may result. This is the principle of the pendulum damper, which will be discussed in greater detail in section 3.7.

The forced motion of a compound pendulum is indicated in Fig. 3–15. The treatment of this example is essentially the same as for the mathematical pendulum; however, to illustrate the analytical approach, this method will be used here. The displacement of the point of suspension A, as in the previous example, is

$$x = a \sin \omega t \tag{3.6–5}$$

whence

$$\ddot{x} = -a\omega^2 \sin \omega t$$

The force necessary to give the point of suspension the motion of equation 3.6–5 may be designated as some function of time,

$$P = P(t)$$

The total linear acceleration of the centroid of the pendulum is

$$\ddot{x} + c\ddot{\theta} = -a\omega^2 \sin \omega t + c\ddot{\theta}$$

The equation of motion for translation is therefore

$$\frac{W}{g} (\ddot{x} + c\ddot{\theta}) = P(t)$$

The equation of motion for rotation about the centroid is

$$\left(\frac{W}{g} \bar{r}_c^{\,2}\right) \ddot{\theta} = -Wc\theta - P(t)c$$

for small oscillations. Elimination of the unknown force $P(t)$ between these two equations gives

$$\frac{W}{g} (c^2 + \bar{r}_c^{\,2})\ddot{\theta} + Wc\theta = \frac{Wc}{g} a\omega^2 \sin \omega t$$

or

$$\ddot{\theta} + \frac{cg}{c^2 + \bar{r}_c^{\,2}} \theta = \frac{ca\omega^2}{c^2 + \bar{r}_c^{\,2}} \sin \omega t$$

Introducing the notation

$$p^2 = \frac{cg}{c^2 + \bar{r}_c^{\,2}} = \frac{g}{l_{pA}}$$

where $l_{pA} = c + \dfrac{\bar{r}_c^{\,2}}{c}$ is the equivalent mathematical pendulum length for forced oscillation, reduces the equation to

$$\ddot{\theta} + p^2\theta = p^2 \frac{a\omega^2}{g} \sin \omega t \qquad (3.6\text{–}6)$$

This equation has the form of equation 3.3–1, and the steady-state solution may hence be written as

$$\theta = \frac{p^2}{p^2 - \omega^2} \cdot \frac{a\omega^2}{g} \sin \omega t = \frac{a \sin \omega t}{\dfrac{g}{\omega^2} - \dfrac{g}{p^2}}$$

As $l_\omega = g/\omega^2$ is the equivalent mathematical pendulum length corresponding to the circular frequency ω, the angular amplitude becomes

$$\Theta = \frac{a}{l_\omega - l_{pA}} = \frac{a}{l_N} \tag{3.6-7}$$

The amplitude of the motion of the center of percussion B_A is

$$a_B = l_\omega \Theta = \frac{l_\omega a}{l_\omega - l_{pA}} = \frac{a}{1 - \dfrac{l_{pA}}{l_\omega}} = \frac{a}{1 - \left(\dfrac{\omega}{p}\right)^2}$$

which is equivalent to the result obtained for the mathematical pendulum, equation 3.6–3a. The magnitude of the force required for steady-state oscillations may be calculated from the equation of motion for translation,

$$P(t) = m(\ddot{x} + c\ddot{\theta}) = -m\left(a\omega^2 + c\frac{a}{l_N}\omega^2\right)$$

$$= -m\left(1 + \frac{c}{l_\omega - l_{pA}}\right)a\omega \sin \omega t$$

It will be noted that this value for $P(t)$ will revert to the value for P in equation 3.6–3b if $c = l_{pA}$, which is the case for a mathematical pendulum.

It should further be pointed out that a free oscillation about the node N as a point of suspension will produce a frequency different from ω because the equivalent pendulum length for this free oscillation is l_{pN} rather than l_ω as is readily seen from Fig. 3–15.

The forced oscillation of the centrifugal pendulum for small amplitudes is a similar problem. Consider the system of Fig. 3–16, in which the centrifugal pendulum is mounted on a disk which has an angular velocity Ω. The natural circular frequency of this system was previously found, in section 2.4, to be

$$p = \Omega\sqrt{\frac{R}{l}} \tag{3.6-8}$$

If now an oscillating motion $\psi = \Psi \sin \omega t$ is superimposed on the steady rotation of the disk, the pendulum will oscillate at the forced circular frequency ω. The node N will be relocated in such a way that the following relation is satisfied:

$$\omega = \Omega\sqrt{\frac{R_\omega}{l_\omega}} \tag{3.6-9}$$

It is convenient to introduce the ratio or order number n of the oscillatory frequencies,

$$n^2 = \frac{\omega^2}{\Omega^2} = \frac{R_\omega}{l_\omega}$$

The assumption of small oscillations permits the following relations

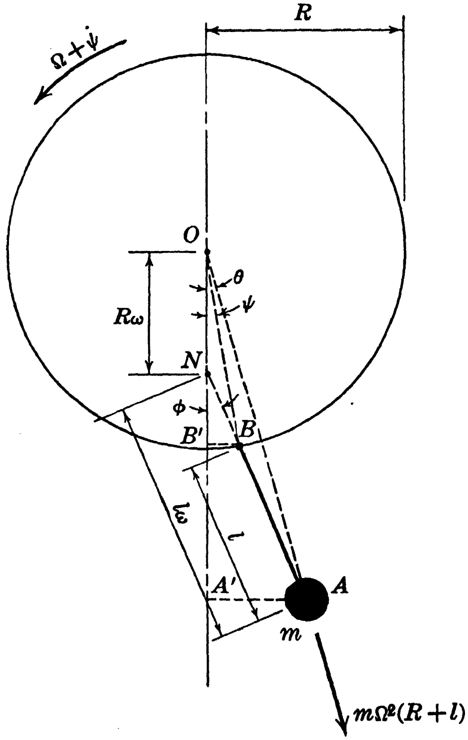


FIG. 3-16

to be derived from the geometry of Fig. 3-16:

$$\overline{OA} \cong R + l \cong R_\omega + l_\omega = l_\omega(1 + n^2)$$

Thus

$$l_\omega \cong \frac{R + l}{1 + n^2} \quad \text{and} \quad l_\omega - l \cong \frac{R - n^2 l}{1 + n^2}$$

from which

$$\frac{l_\omega}{l_\omega - l} \cong \frac{R + l}{R - n^2 l}$$

From the figure, it will also be seen that

$$\frac{\overline{AA'}}{\overline{BB'}} \cong \frac{(R + l)\theta}{R\psi} \cong \frac{\overline{NA}}{\overline{NB}} \cong \frac{l_\omega}{l_\omega - l} \cong \frac{R + l}{R - n^2 l}$$

or

$$\theta = \frac{R}{R - n^2 l}\psi = \frac{\psi}{1 - n^2 \dfrac{l}{R}} \tag{3.6–10}$$

By the use of equation 3.6–8, this may be written as

$$\theta = \frac{\psi}{1 - n^2 \dfrac{\Omega^2}{p^2}} = \frac{\psi}{1 - \left(\dfrac{\omega}{p}\right)^2}$$

It is instructive to compare the translatory amplitude ratio of the previous problem of the gravity pendulum with the angular amplitude ratio of the radial or centrifugal pendulum. For the gravity pendulum, equation 3.6–3 gives

$$\frac{a_B}{a} = \frac{1}{1 - \left(\dfrac{\omega}{p}\right)^2} \quad \text{where} \quad p^2 = \frac{g}{l}$$

whereas, for the centrifugal pendulum, equation 3.6–10 gives

$$\frac{\Theta}{\Psi} = \frac{1}{1 - \left(\dfrac{\omega}{p}\right)^2} \quad \text{where} \quad p^2 = \frac{\Omega^2 R}{l}$$

showing the similarity of the two problems.

In the design of centrifugal pendulums as vibration absorbers, it is convenient to replace the pendulum mass by an equivalent moment of inertia rotating about the axis at O. The inertia torque generated by the pendulum is given by

$$T = m(R + l)^2\ddot{\theta} = \frac{m(R + l)^2}{1 - \left(\dfrac{\omega}{p}\right)^2}\ddot{\psi} = I_{\text{eq}}\ddot{\psi}$$

Thus the equivalent moment of inertia is

$$I_{\text{eq}} = \frac{m(R + l)^2}{1 - \left(\dfrac{\omega}{p}\right)^2} = \frac{m(R + l)^2}{1 - n^2 \dfrac{l}{R}} \tag{3.6–11}$$

It is seen therefore that the centrifugal pendulum is equivalent in every way to a moment of inertia, rigidly attached to the disk and of a magnitude given by equation 3.6–11. I_{eq} varies with the frequency ratio ω/p and becomes infinite when $\omega/p = 1$. This situation that exists when the pendulum is "tuned" to the disturbance has the practical effect of stopping the motion of the pendulum carrier (i.e., the disk in this case) thus forcing a node into the system at this point.

It is, of course, possible to consider the resultant effect of the pendulum as an equivalent spring, in which case the equivalent spring constant is

$$k_{eq} = \frac{T}{\psi}$$

For simple harmonic motion (i.e., small amplitudes),

$$\ddot{\psi} = -\omega^2\psi$$

whence

$$k_{eq} = -\frac{\omega^2 m(R + l)^2}{1 - \left(\frac{\omega}{p}\right)^2} \tag{3.6–12}$$

From equations 3.6–11 and 3.6–12, it is seen that for $\omega/p < 1$. I_{eq} is positive whereas k_{eq} is negative. For $\omega/p > 1$ the reverse is true.

3.7. Critical Speeds of Rotating Shafts

One of the more important examples of forced vibrations is found in the lateral vibration of rotating shafts. An eccentric mass similar to

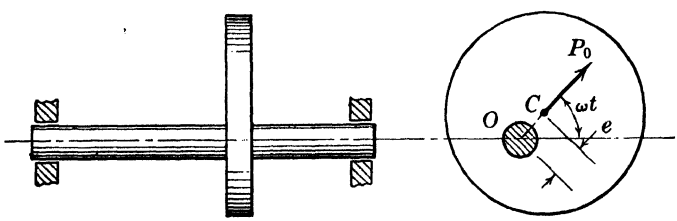


FIG. 3–17

that shown in Fig. 3–17 may induce lateral vibrations of large amplitude in the shaft if the center of gravity of the mass is not at the center of the shaft. As it is practically impossible to avoid some eccentricity, this type of vibration is not unusual in rotating machinery.

The effect of the eccentricity of the rotating mass on the shaft is to create a centrifugal force P which rotates with the disk. The force P has the magnitude

$$P = \frac{W}{g} \omega^2 e$$

where e is the eccentricity of the mass. The rotating force P will have projections on the x and y axis causing a simultaneous vibration in each of these directions. If the restoring force due to bending of the shaft and deflection of the bearings is the same in both directions, the motions will have the same frequency. In most practical cases this is not true, although the difference in stiffness may be small. Where the difference is appreciable, separate calculations for frequency must be made for the directions of maximum and minimum stiffness x and y.

Consider, for example, the motion of the disk in the x direction. Let the over-all spring constant associated with this direction be k. The component of the centrifugal force P acting on the system is

$$P \cos \omega t = \frac{W}{g} \omega^2 e \cos \omega t$$

The equation of motion is

$$\frac{W}{g} \ddot{x} = -kx + \frac{W}{g} \omega^2 e \cos \omega t$$

which may be written as

$$\ddot{x} + \frac{kg}{W} x = \omega^2 e \cos \omega t$$

or

$$\ddot{x} + p^2 x = \omega^2 e \cos \omega t$$

This is the usual equation for forced vibration. The steady-state solution for the case $\omega \neq p$ is

$$x_p = \frac{e}{\left(\frac{p}{\omega}\right)^2 - 1} \cos \omega t = \frac{e\left(\frac{\omega}{p}\right)^2}{1 - \left(\frac{\omega}{p}\right)^2} \cos \omega t$$

The critical condition at which resonance will occur arises when $\omega = p$. For this case, the amplitude of the steady-state motion may become very large, even though e is very small. It will be noted that, when $p > \omega$, the motion of the disk is in the same sense as the force

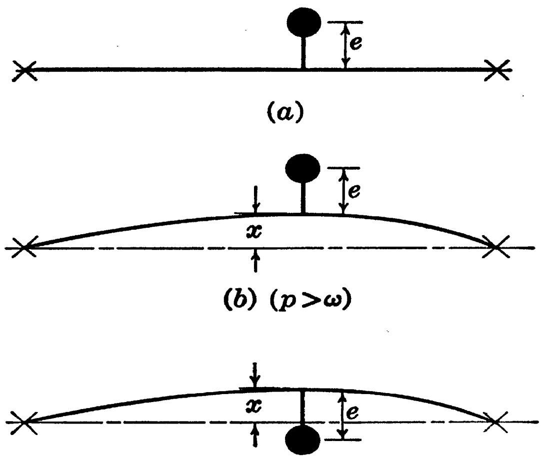


FIG. 3–18

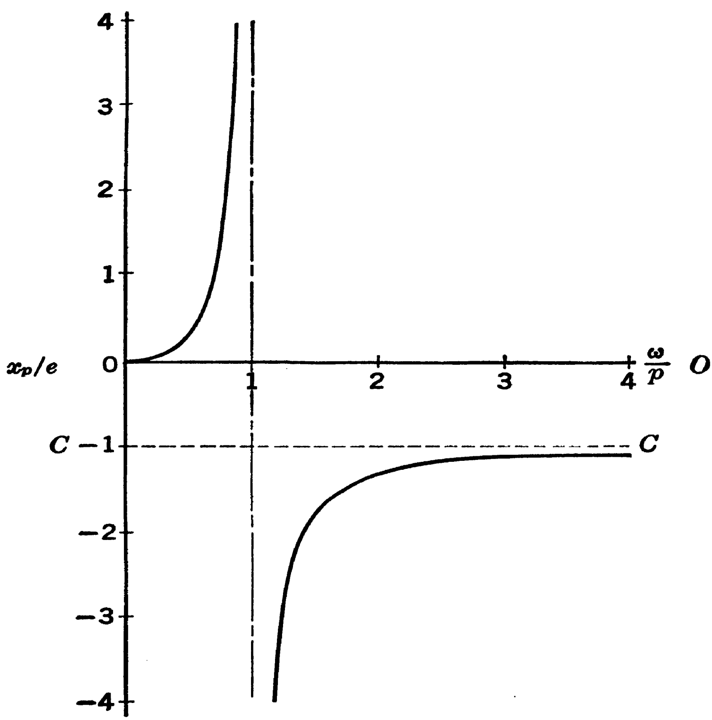


FIG. 3–19

$P \cos \omega t$, as shown in Fig. 3–18*b*. Above resonance, however, when $\omega > p$, the motion of the disk is opposite to the direction of the periodic force. For slow rotations $p >> \omega$ the center of rotation of the mass is almost stationary, and the centroid C moves in a circle of radius e. For large angular velocities $\omega >> p$, the centroid of the disk and the shaft axis are on opposite sides of the centerline, as shown in Fig. 3–18*b*. The amplitude of either O or C may be obtained directly from Fig. 3–19.

It should be noted that the motion of the mass is one of pure rotation when the spring constant is uniform in all directions. The deflection of the shaft may be obtained directly from static considerations; i.e., the deflection is the same as for a static load P_1 where

$$P_1 = \frac{W}{g} \omega^2(e + \delta)$$

where δ is the deflection of the shaft. The deflection may be calculated from

$$\delta = \frac{P_1}{k} = \frac{W}{kg} \omega^2(e + \delta) = \frac{\omega^2}{p^2} (e + \delta)$$

whence

$$\delta = \frac{\left(\frac{\omega}{p}\right)^2}{1 - \left(\frac{\omega}{p}\right)^2} e$$

The shaft center O moves in a circle of radius δ.

Chapter 4

FORCED VIBRATIONS WITH DAMPING

4.1. The Nature of Damping

All vibrations that occur in nature are associated with some damping or energy dissipation. The amount of damping is usually small in most well-designed machinery and, consequently, the loss of energy may be relatively small in each cycle. Some equipment and structures, however, contain elements capable of dissipating a large amount of energy per cycle. These systems are said to be heavily damped.

Damping or energy dissipation is usually undesirable, not only from the standpoint that it represents wasted energy and lower efficiency, but because the manner of dissipation is commonly a conversion of mechanical energy into heat. The heat thus generated causes hot bearings and an unequal expansion of the machine which in general impairs its functioning. Consequently, the designer, wherever feasible, attempts to eliminate damping in his machine.

There are occasions, however, when the engineer purposely introduces damping to dissipate unwanted energy which if allowed to accumulate would seriously hamper the operation or cause failure of the machine or structure. The most common form of unwanted energy is that which is stored or temporarily built up in a vibrating system. Many forms of vibration dampers and means for dissipating vibration energy have been devised to decrease or eliminate vibration. Dampers usually depend on friction forces, dry, viscous, or solid in nature. Dampers depending on the generation of electric current and its dissipation through resistance losses are also in use.

The theory of damped oscillatory motions is simplest when the resisting forces are proportional to the relative velocity of the damping elements. Such dampers are said to be "viscous" since this type of damping is very closely approximated by the sliding of well-lubricated surfaces. Other types of friction, such as dry friction and internal or solid friction, result in energy dissipations which are functions of displacement as well as velocity. Such dampers are complicated from a mathematical standpoint as their theory involves non-linear differential equations. For this reason, dry or solid friction dampers are frequently replaced in theory with equivalent viscous dampers. The equivalent viscous damper is determined so that the total energy dissipated per cycle is the same as for the non-linear dampers. In this way, many problems may be simplified and a very good approximation to the actual motion may be obtained.

Free vibrations of systems with damping elements are transient in nature. No real steady state exists, since the motion gradually dies away with time. Forced vibrations of systems with damping elements do result in a steady state, and this type of vibration is of considerable practical importance.

4.2. The Nature of Forced Vibrations of Linearly Damped Systems

As discussed in the previous chapter, forces that excite a system into motion may be of a complex nature. The most important fundamental case is the previously used periodic force of the form

$$F = P \cos \omega t \tag{4.2–1}$$

To gain a clear picture of the effect of damping in oscillating systems, excited by forces of this type, it is instructive to consider two basic examples.

Damper and a spring

Consider the system shown in Fig. 4–1. For viscous damping the force exerted by the damper c is proportional to the relative velocity of

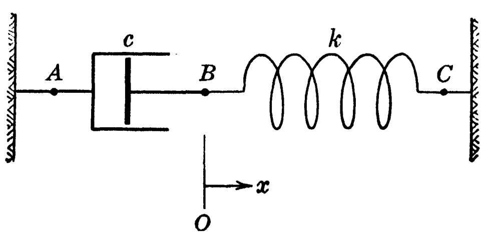


FIG. 4–1

its terminals A and B and is always opposed to the direction of the motion. Thus, for a relative velocity $\dot{x}$ in the positive direction the damping force is

$$F_c = -c\dot{x} \tag{4.2–2}$$

The spring force for a positive displacement is given by

$$F_k = -kx \tag{4.2-3}$$

where x is the relative displacement of the terminals B and C. The

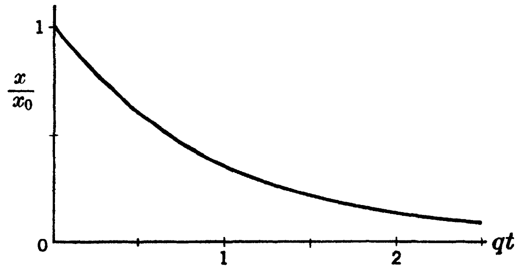


FIG. 4-2

equation of motion may be written directly by balancing the forces at B:

$$c\dot{x} + kx = 0 \tag{4.2-4}$$

or

$$\frac{dx}{dt} = -\frac{k}{c}x = -qx$$

where

$$q = \frac{k}{c} \tag{4.2-5}$$

The above expression is readily integrated to give

$$\log x = -qt + C_1$$

Introduction of the initial conditions,

$$x = x_0 \quad \text{when} \quad t = 0$$

permits the determination of the constant of integration,

$$C_1 = \log x_0$$

Substitution of this value into the above equation gives

$$\log \frac{x}{x_0} = -qt$$

or

$$x = x_0 e^{-qt} \tag{4.2-6}$$

The character of the motion is easily perceived from the graph of equation 4.2-6 shown in Fig. 4-2. If the system is initially displaced a distance x_0, it asymptotically approaches the equilibrium position, $x = 0$.

The parameter $q = k/c$ has the same dimensions as a circular frequency $(1/T)$ and is therefore dimensionally equivalent to the previously used terms ω and p. It shall be referred to as the *relaxation frequency*. Although q is not a periodic frequency in the same sense as ω and p, it will be shown (see Chapter 9) that q represents the maximum natural frequency that a given spring and damper element can reach with a corresponding mass in their most common combination of parallel motion. The concept of q as a frequency parameter will be found very convenient in both the formulation and interpretation of the theory of damped systems.

Damper and a periodic force

A system composed of a damper excited by a periodic force,

$$P_c \cos \omega t$$

is shown in Fig. 4–3. Combination of the exciting force with the damping force permits the equation of motion to be written as

$$-c\dot{x}_c + P_c \cos \omega t = 0 \tag{4.2–7}$$

from which

$$\dot{x}_c = \frac{P_c}{c} \cos \omega t \tag{4.2–8}$$

Integration of equation 4.2–8 yields

$$x_c = \frac{P_c}{\omega c} \sin \omega t + C_2 \tag{4.2–9}$$

The simplest form of equation 4.2–9 is obtained if the arbitrary constant C_2 is set equal to zero. The solution may in this case be written

$$x_c = \frac{P_c}{\omega c} \sin \omega t \tag{4.2–10}$$

from which the amplitude is seen to be

$$A = \frac{P_c}{\omega c}$$

The system is seen to perform simple harmonic motion with a displacement that lags the exciting force $P_c \cos \omega t$ by a phase angle of $\pi/2$ rad.

As used in the following applications, the exciting force may be advanced $\pi/2$ rad in phase to obtain a motion with an initial displacement. The exciting force is then

$$P_c \cos\left(\omega t + \frac{\pi}{2}\right) = -P_c \sin \omega t \quad (4.2\text{–}11)$$

and the resulting motion is given by

$$x_c = \frac{P_c}{\omega c} \cos \omega t \qquad (4.2\text{–}12)$$

A change in phase angle of an arbitrary amount ϕ is equivalent to changing the initial point from which time is measured by an increment of time $\Delta t = \phi/\omega$. The character of the motion is unchanged but shifted forward or backward in time according to the sign of ϕ. Increasing the phase angle by $\pi/2$ has the effect of moving the motion ahead in time one quarter of a cycle. The resultant motion is shown in Fig. 4–3 where $t = 0$ in Fig. 4–3a in accordance with equation 4.2–10. The motion for equation 4.2–12 corresponds to $t = 0$ in Fig. 4–3b.

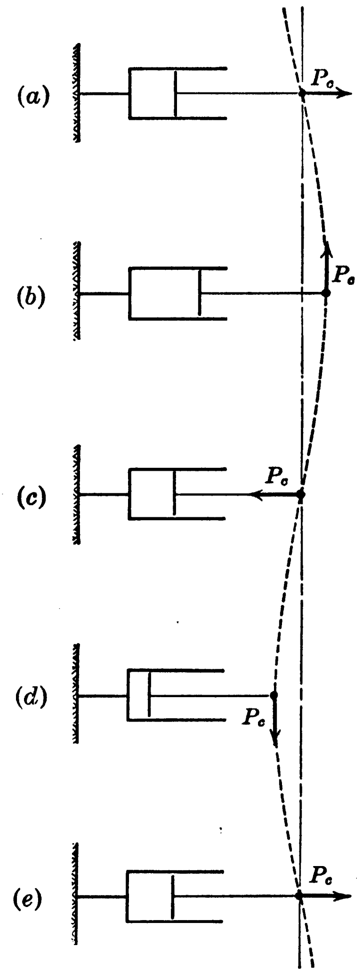


FIG. 4–3

Combined systems

There are many possible combinations of the mass, spring, and damping elements. A simple combination where all elements are in parallel is shown in Fig. 4–4. The displacements, amplitudes, and exciting forces have been previously determined for each of these elements.

Element	*Displacement*	*Amplitude*	*Exciting Force*
Spring	$x_k = \frac{P_k}{k} \cos \omega t$	$\frac{P_k}{k}$	$P_k \cos \omega t$
Damper	$x_c = \frac{P_c}{\omega c} \cos \omega t$	$\frac{P_c}{\omega c} = \frac{P_c}{k}\frac{q}{\omega}$	$P_c \cos\left(\omega t + \frac{\pi}{2}\right) = -P_c \sin \omega t$
Mass	$x_m = \frac{P_m}{\omega^2 m} \cos \omega t$	$\frac{P_m}{\omega^2 m} = \frac{P_m}{k}\left(\frac{p}{\omega}\right)^2$	$P_m \cos(\omega t + \pi) = -P_m \cos \omega t$

These elements may be combined by noting that the amplitudes of all elements shown in Fig. 4–4 must be the same. That is,

$$\frac{P_k}{k} = \frac{P_c}{\omega c} = \frac{P_m}{\omega^2 m}$$

This requires that the amplitudes of the exciting forces satisfy the

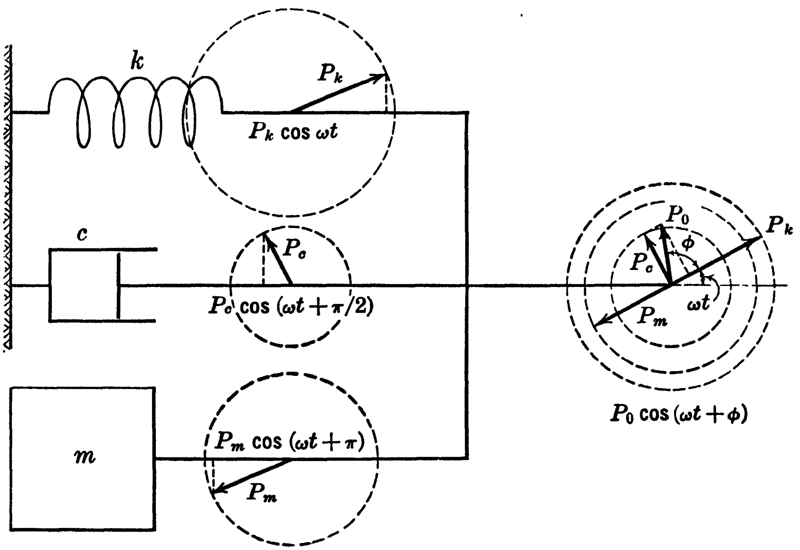


FIG. 4–4

following relations:

$$P_c = \frac{\omega c}{k} P_k = \left(\frac{\omega}{q}\right) P_k; \qquad P_m = \frac{\omega^2 m}{k} P_k = \left(\frac{\omega}{p}\right)^2 P_k \quad (4.2\text{–}13)$$

The resultant exciting force which acts on the system will be the vectorial sum of the forces acting on each element. Since the force acting on the damper is out of phase with the displacement, the resultant exciting force will make some phase angle ϕ with the displacement. The resultant exciting force will have the form

$$P_0 \cos (\omega t + \phi)$$

where

$$P_0 \cos (\omega t + \phi) = P_k \cos \omega t - P_c \sin \omega t - P_m \cos \omega t$$

Upon substitution from equation 4.2–13, this becomes

$$P_0 \cos (\omega t + \phi) = P_k \left[\cos \omega t - \frac{\omega}{q} \sin \omega t - \left(\frac{\omega}{p}\right)^2 \cos \omega t \right]$$

$$= P_k \left[\left(1 - \frac{\omega^2}{p^2}\right) \cos \omega t - \frac{\omega}{q} \sin \omega t \right]$$

From trigonometry, we have the identity

$$B \cos \omega t + C \sin \omega t = D \cos (\omega t + \phi)$$

where

$$D = \sqrt{B^2 + C^2} \quad \text{and} \quad \tan \phi = -\frac{C}{B}$$

Thus

$$P_0 \cos (\omega t + \phi) = P_k \sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \left(\frac{\omega}{q}\right)^2} \cos (\omega t + \phi)$$

and so it follows that

$$P_k = \frac{P_0}{\sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \left(\frac{\omega}{q}\right)^2}} \quad \text{and} \quad \tan \phi = \frac{\left(\frac{\omega}{q}\right)}{1 - \left(\frac{\omega}{p}\right)^2} = \frac{P_c}{P_k - P_m} \qquad (4.2\text{–}14)$$

Since the displacements of all elements are the same,

$$x = x_k = x_c = x_m = A \cos \omega t$$

the motion of the system is given by substituting for P_k in the expression for the displacement of the spring. This gives

$$x = x_k = \frac{P_0}{k} \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \left(\frac{\omega}{q}\right)^2}} \cos \omega t$$

$$= \frac{\delta_P}{\sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \left(\frac{\omega}{q}\right)^2}} \cos \omega t \qquad (4.2\text{–}15)$$

where the amplitude is

$$A = \frac{\delta_P}{\sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \left(\frac{\omega}{q}\right)^2}} \quad \text{and} \quad \delta_P = \frac{P_0}{k} \qquad (4.2\text{–}16)$$

Equation 4.2–15 is to be compared with equation 3.3–7 for the motion of a system without damping. The case of no damping can be obtained directly from equation 4.2–15 by noting that, as $c \to 0$, $q \to \infty$.

Another system of interest is that of Fig. 4–5, where the mass, spring, and damping elements are arranged in series. It is convenient in this case to reconsider the motion of each element when subjected to the same periodic force since this is the nature of a system in series.

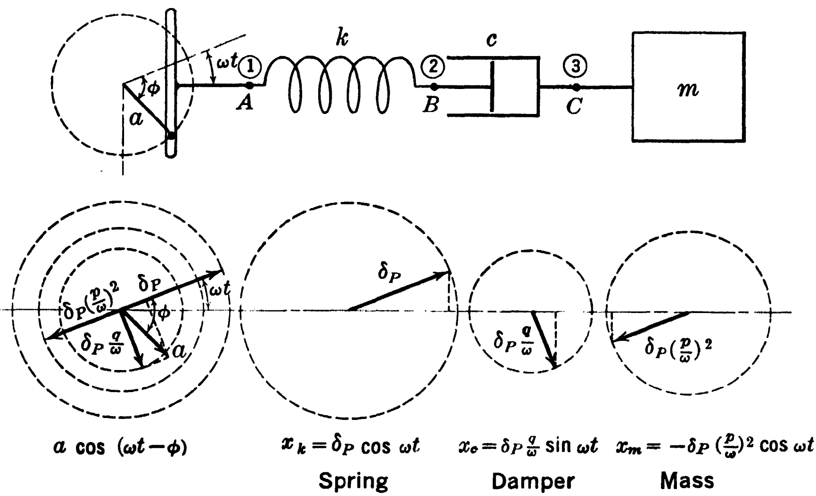


FIG. 4–5

The effect of a force $P_0 \cos \omega t$ acting on each element may be tabulated as follows:

Element	*Displacement*	*Amplitude*	*Exciting Force*
Spring	$x_k = \frac{P_0}{k} \cos \omega t$	$\frac{P_0}{k} = \delta_P$	$P_0 \cos \omega t$
Damper	$x_c = \frac{P_0}{\omega c} \cos \left(\omega t - \frac{\pi}{2}\right)$	$\frac{P_0}{k}\left(\frac{q}{\omega}\right) = \delta_P\left(\frac{q}{\omega}\right)$	$P_0 \cos \omega t$
	$= \frac{P_0}{k}\left(\frac{q}{\omega}\right) \sin \omega t$		
Mass	$x_m = \frac{P_0}{\omega^2 m} \cos(\omega t - \pi)$	$\frac{P_0}{k}\left(\frac{p}{\omega}\right)^2 = \delta_P\left(\frac{p}{\omega}\right)^2$	$P_0 \cos \omega t$
	$= -\frac{P_0}{k}\left(\frac{p}{\omega}\right)^2 \cos \omega t$		

It must be remembered that x_k and x_c are relative displacements between the two terminals of the spring and damper respectively, whereas x_m is the absolute displacement of the mass.

For the system in series, the difference in the motion of the mass and the forced displacement $a \cos(\omega t - \phi)$ is given by the sum of the displacements of the spring and damper. In general, the forced amplitude will not be in phase with any one of the elements. Thus,

$$a \cos (\omega t - \phi) - x_m = x_k + x_c$$

or

$$a \cos (\omega t - \phi) = x_k + x_c + x_m \tag{4.2–17}$$

Substitution from the above table gives

$$\begin{aligned} a \cos (\omega t - \phi) &= \frac{P_0}{k}\left[\left(1 - \frac{p^2}{\omega^2}\right) \cos \omega t + \frac{q}{\omega} \sin \omega t\right] \\ &= \delta_P \sqrt{\left(1 - \frac{p^2}{\omega^2}\right)^2 + \left(\frac{q}{\omega}\right)^2} \cos (\omega t - \phi) \end{aligned}$$

where

$$\tan \phi = \frac{\left(\frac{q}{\omega}\right)}{1 - \left(\frac{p}{\omega}\right)^2}$$

so

$$\delta_P = \frac{P_0}{k} = \frac{a}{\sqrt{\left(1 - \frac{p^2}{\omega^2}\right)^2 + \left(\frac{q}{\omega}\right)^2}} \tag{4.2–18}$$

Substitution from equation 4.2–18 into the expression in the table for the motion of the mass yields

$$x_m = -a \frac{\left(\frac{p}{\omega}\right)^2}{\sqrt{\left(1 - \frac{p^2}{\omega^2}\right)^2 + \left(\frac{q}{\omega}\right)^2}} \cos \omega t \tag{4.2–19}$$

Examination of equation 4.2–19 shows that, for $c \rightarrow 0$, $q \rightarrow \infty$, with the result that $x_m \rightarrow 0$. That is, for zero damping the mass has no motion since the system has become disconnected. For the other limiting case where $c \rightarrow \infty$, $q \rightarrow 0$ equation 4.2–19 reduces to equation 3.3–8. In this instance, the damper ceases to contribute to the motion as it becomes a rigid element in the system.

4.3. Derivation of the Equation of Motion

As in the case of undamped forced vibrations, a more direct treatment of the problem can be obtained by an analysis of the system as a whole. The advantage of the piecewise analysis used in the previous section where the elements were studied individually and then combined lies in the better insight into the internal forces and displacements that it offered. Although systems that involve many elements may always be "built up" by this method, the process may be long and laborious. The greater detail exposed to view tends to obscure the over-all picture. For more complex systems, it is usually more efficient to derive the differential equations of motion for

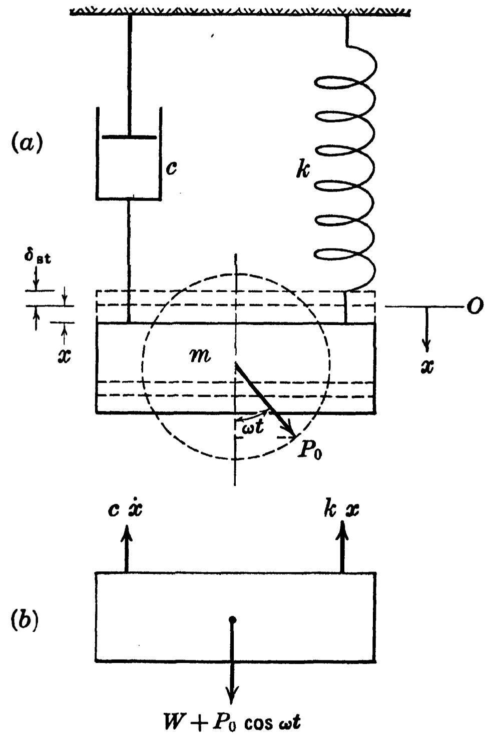


FIG. 4–6

the complete system and obtain their solution directly, or as an alternative to systematize the procedure of the previous approach. This latter alternative leads to the extremely useful method of solving vibration problems known as the mobility method.

The systems treated in the last section offer instructive examples. Consider the system of Fig. 4–6*a*, which is identical with that of Fig. 4.4. If the system is given a displacement x from the equilibrium position, the forces that act on the mass will be as shown in Fig. 4–6*b*. The equation of motion may be established with the aid of Newton's second law as

$$m\ddot{x} = -c\dot{x} - k(\delta_{st} + x) + W + P_0 \cos \omega t$$

Since $k\delta_{st} = W$, this reduces to

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{P_0}{m}\cos\omega t \tag{4.3–1}$$

Introduction of the previously established notation,

$$q = \frac{k}{c} \quad \text{and} \quad p^2 = \frac{k}{m}$$

permits the equation of motion to take the final form,

$$\ddot{x} + \frac{p^2}{q}\dot{x} + p^2x = \delta_P p^2 \cos\omega t \tag{4.3–2}$$

where

$$\delta_P = \frac{P_0}{k}$$

is the static deflection associated with the force P_0 acting directly on the spring.

4.4. Solution of the Equation of Motion

The solution of equation 4.3–2 consists of the complementary solution and the particular solution. The complementary solution is the solution of the homogeneous equation,

$$\ddot{x} + \frac{p^2}{q}\dot{x} + p^2x = 0 \tag{4.4–1}$$

and it may be found by the standard methods for equations of this type. The complementary solution is

$$x_c = e^{-(p^2/2q)t}(C_1 \cos p_1t + C_2 \sin p_1t) \tag{4.4–2}$$

where

$$p_1^2 = p^2\left[1 - \left(\frac{p}{2q}\right)^2\right] \tag{4.4–3}$$

and C_1 and C_2 are arbitrary constants which may be determined from the initial conditions. The natural circular frequency p_1 is less than the natural circular frequency without damping by an amount that depends on the relaxation frequency. The motion is seen to die away with time, and therefore is of a transient nature.

The second part of the solution, known as the particular solution, must satisfy the complete differential equation, i.e., equation (4.3–2). The particular solution for the damped case may be found in the same manner as was employed for the undamped system. A trial solution may be taken in the form

$$x_p = A \cos \omega t + B \sin \omega t$$

where the constants A and B are to be determined so that equation 4.3–2 will be satisfied. Substitution of the trial solution into equation 4.3–2 yields

$$-A\omega^2 \cos \omega t - B\omega^2 \sin \omega t - \frac{p^2}{q} A\omega \sin \omega t + \frac{p^2}{q} B\omega \cos \omega t + Ap^2 \cos \omega t + Bp^2 \sin \omega t = \delta_P p^2 \cos \omega t$$

Since this equation must be satisfied for all values of t, the coefficients of the sine and cosine terms must be identical for both sides of the equation. Equating these coefficients leads to the two equations,

$$A(p^2 - \omega^2) + B \frac{p^2\omega}{q} = \delta_P p^2$$

$$-A \frac{p^2\omega}{q} + B(p^2 - \omega^2) = 0$$

The solution of the two equations gives

$$A = \delta_P \frac{1 - \left(\frac{\omega}{p}\right)^2}{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \left(\frac{\omega}{q}\right)^2}$$

$$B = \delta_P \frac{\left(\frac{\omega}{q}\right)}{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \left(\frac{\omega}{q}\right)^2}$$

The particular solution may now be written as

$$x_p = \frac{\delta_P}{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \left(\frac{\omega}{q}\right)^2} \left[\left(1 - \frac{\omega^2}{p^2}\right) \cos \omega t + \frac{\omega}{q} \sin \omega t\right] \qquad (4.4\text{–}4)$$

Application of the trigonometric identity of section 4.2 permits equation 4.4–4 to be written as

$$x_p = \frac{\delta_P}{\sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \left(\frac{\omega}{q}\right)^2}} \cos (\omega t - \phi) \qquad (4.4\text{–}5)$$

where

$$\tan \phi = \frac{\dfrac{\omega}{q}}{1 - \left(\dfrac{\omega}{p}\right)^2} = \frac{\left(\dfrac{\omega}{p}\right)}{1 - \left(\dfrac{\omega}{p}\right)^2}\left(\frac{p}{q}\right) \tag{4.4–6}$$

The motion of the mass as given by equation 4.4–5 is a steady motion of constant amplitude. It will be noted that the motion of the mass lags behind the exciting force by a phase angle ϕ. This was also pointed out in section 4.2.

Equation 4.4–5 is the part of the motion that remains after the damped free vibration has subsided. The complete solution is the sum of the complimentary and the particular solutions.

$$x = x_c + x_p \tag{4.4–7}$$

4.5. Resonance

The steady-state solution for the system of Fig. 4–6 as given by equation 4.4–5 has an amplitude

$$\delta_P \left[\frac{1}{\sqrt{\left(1 - \dfrac{\omega^2}{p^2}\right)^2 + \left(\dfrac{\omega}{q}\right)^2}}\right] \tag{4.5–1}$$

The amplitude is seen to be proportional to the factor in the brackets. This factor is commonly called the resonance factor, and it is instructive to consider its magnitude as a function of the dimensionless frequency ratios ω/p and ω/q.

It is evident that the resonance factor can never become infinite as in the case of undamped vibrations. The maximum value will occur when

$$\left(1 - \frac{\omega^2}{p^2}\right)^2 + \left(\frac{\omega}{q}\right)^2 = \left(1 - \frac{\omega^2}{p^2}\right)^2 + \left(\frac{p}{q}\right)^2\left(\frac{\omega}{p}\right)^2$$

is a minimum. For a particular value of p and q, the proper value of ω/p to make this expression a minimum may be found by differentiation. For the minimum value,

$$\frac{d}{d\left(\dfrac{\omega}{p}\right)}\left[\left(1 - \frac{\omega^2}{p^2}\right)^2 + \left(\frac{p}{q}\right)^2\left(\frac{\omega}{p}\right)^2\right] = 0$$

thus,

$$-2\left(1 - \frac{\omega^2}{p^2}\right)2\left(\frac{\omega}{p}\right) + 2\left(\frac{p}{q}\right)^2\left(\frac{\omega}{p}\right) = 0$$

or

$$\left(\frac{\omega}{p}\right)^2 = 1 - \frac{1}{2}\left(\frac{p}{q}\right)^2 \tag{4.5-2}$$

A graph of the resonance factor for various values of ω/p and p/q is shown in Fig. 4–7. The value of this factor approaches the value of the undamped resonance factor discussed in the previous chapter as

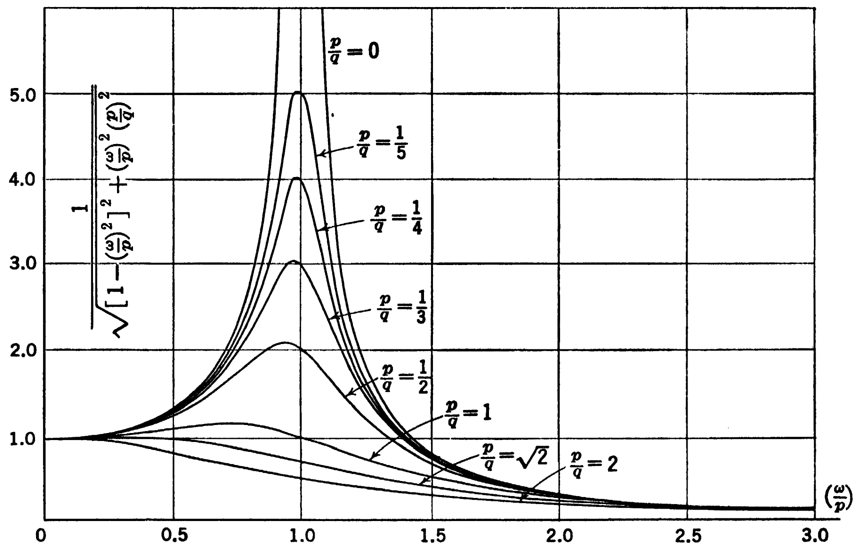


FIG. 4–7

$c \to 0$, i.e.: $p/q \to 0$. The value of the maximum amplitude obtained by subsituting from equation 4.5–2 into equation 4.5–1 becomes

$$A_{\max} = \delta_P \frac{q}{p} \frac{1}{\sqrt{1 - \left(\frac{p}{2q}\right)^2}} \tag{4.5-3}$$

When the damping is small, the maximum amplitude occurs near $\omega/p = 1$. Thus, for small damping, the maximum amplitude is given by

$$A_{\max} \cong \delta_P \frac{q}{p}$$

If $(p/q)^2 = 2$, then $A_{\max} = \delta_P$. This corresponds to $\omega = 0$, which may be seen from equation 4.5–2 and Fig. 4–7. For damping greater than this, there is no mathematical maximum amplitude, and the amplitude decreases with an increase in forced frequency. The above statement is illustrated graphically by Fig. 4–7. Resonance in the precise sense can therefore only occur when $(p/q)^2 > 2$.

The variation of the phase angle with respect to the frequency ratios,

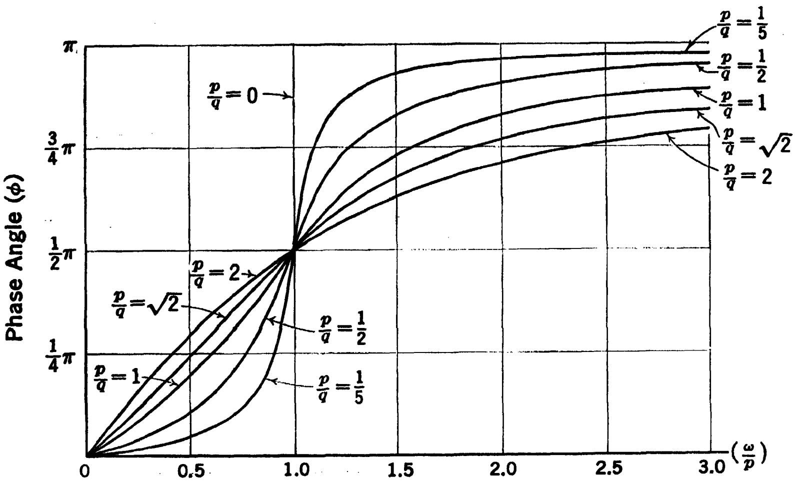


FIG. 4–8

ω/p and p/q, is also of interest. The phase angle is given by equation 4.4–6, whence

$$\phi = \arctan \frac{\dfrac{\omega}{q}}{1 - \left(\dfrac{\omega}{p}\right)^2} = \arctan \left(\frac{p}{q}\right) \frac{\left(\dfrac{\omega}{p}\right)}{1 - \left(\dfrac{\omega}{p}\right)^2}$$

For large values of damping, $q \to 0$, and

$$\phi \to \arctan \infty = \frac{\pi}{2}$$

For small damping, $q \to \infty$ and

$$\phi \to \arctan (0)$$

or

$$\phi = 0 \qquad \frac{\omega}{p} < 1$$

$$\phi = \pi \qquad \frac{\omega}{p} > 1$$

These are the conditions previously observed in systems without damping. At resonance ($\omega/p = 1$) undamped systems have an abrupt change in phase angle of π radians. In damped systems however, the change in phase angle is more gradual as shown by Fig. 4–8.

Graphical representation of the resonance factor and its phase angle

The amplitude of the resonance factor, as given by equation 4.5–1 and graphically portrayed in Fig. 4–7, offers a means of obtaining a comprehensive understanding of the changes in the forced amplitude due to variations in the frequency ratios, p/q and ω/p. The change in the phase angle due to variations in p/q and ω/p may be calculated from equation 4.4–6 or obtained directly from Fig. 4–8. The simultaneous changes in the resonance factor and the phase angle may easily be compared by a vectorial presentation of the amplitude as indicated in the following.

From equation 4.5–1, the resonance factor Δ can be written as

$$\Delta = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \left(\frac{\omega}{q}\right)^2}} \tag{a}$$

and, from equation 4.4–6, the phase angle is

$$\tan\phi = \frac{\frac{\omega}{q}}{1 - \left(\frac{\omega}{p}\right)^2} = \frac{\frac{\omega}{p}}{1 - \left(\frac{\omega}{p}\right)^2}\left(\frac{p}{q}\right) \tag{b}$$

Therefore

$$\Delta = \frac{1}{\left(1 - \frac{\omega^2}{p^2}\right)\sqrt{1 + \left(\frac{\omega/q}{1 - \omega^2/p^2}\right)^2}} = \frac{1}{\left(1 - \frac{\omega^2}{p^2}\right)\sqrt{1 + \tan^2\phi}}$$

$$= \frac{\cos\phi}{1 - \left(\frac{\omega}{p}\right)^2} \tag{c}$$

Introduction of the notation

$$x = \frac{\omega}{p}, \qquad y_0 = \frac{1}{1 - \left(\frac{\omega}{p}\right)^2} = \frac{1}{1 - x^2} \tag{d}$$

permits the resonance factor to take the form

$$\Delta = y_0 \cos\phi \tag{e}$$

A comprehensive concept of the relationship between the resonance factor Δ and its phase ϕ for varying values of the parameters p/q and ω/p may be had from a three-dimensional diagram as indicated isometrically in Figs. 4.9*a* and 4.10, where Δ is shown as a vector with its origin in the x axis and parallel to the yz plane. If $x = \omega/p$ is kept constant, equation *e* shows that the vector Δ for varying values of p/q will terminate in a circle of diameter y_0.

If p/q remains constant, Δ will terminate in a spiral-shaped curve, asymptotic to the x axis as shown in Fig. 4.10. A rotation of the vectors for this and similar curves into the xy (zero-phase) plane will result in the curves as shown in Fig. 4.7.

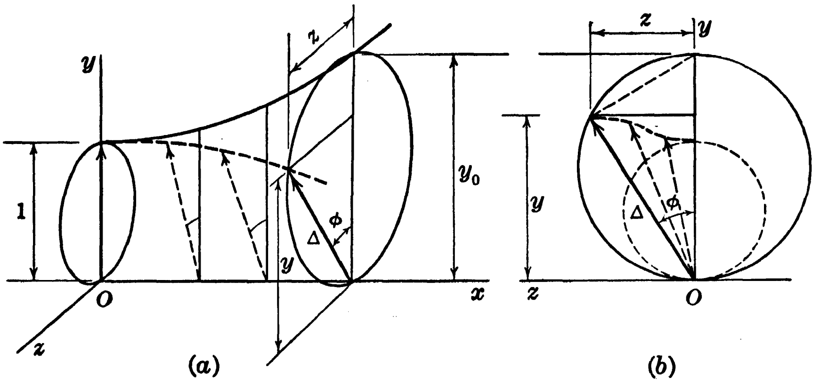


FIG. 4–9

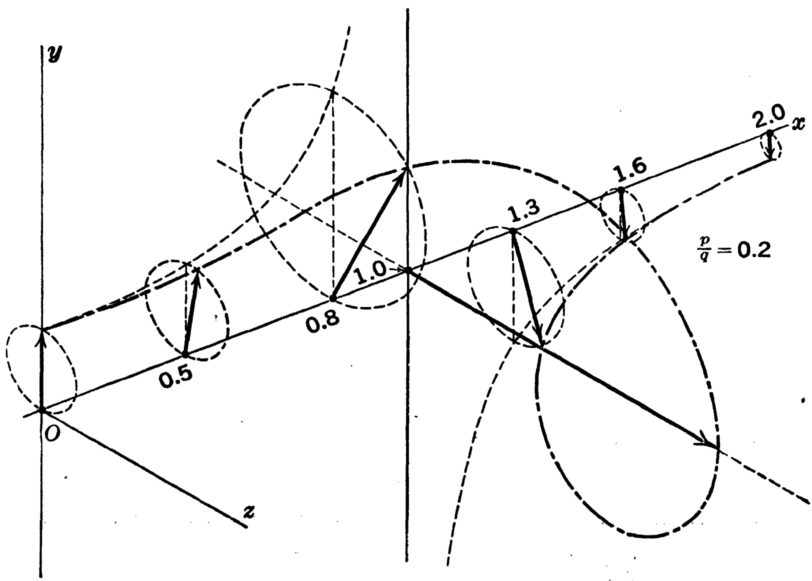


FIG. 4–10

It will be realized that the undamped curve for $p/q = 0$, as shown in Fig. 3.9, is only a two-dimensional curve, in the sense that the finite part of it can be shown as two separate plane curves with a phase difference of 180°. The configuration of this curve may in reality be conceived as a continuous curve having a circular loop with an infinite diameter in a plane normal to the x axis at $x = 1$. There will be a separate curve for each value of the parameter p/q, thus forming a surface which becomes the locus for the vector terminals. This double "horn-shaped" surface will have positive y coordinates for $x < 1$ and negative for $x > 1$. Cross sections of the surface parallel to the yz

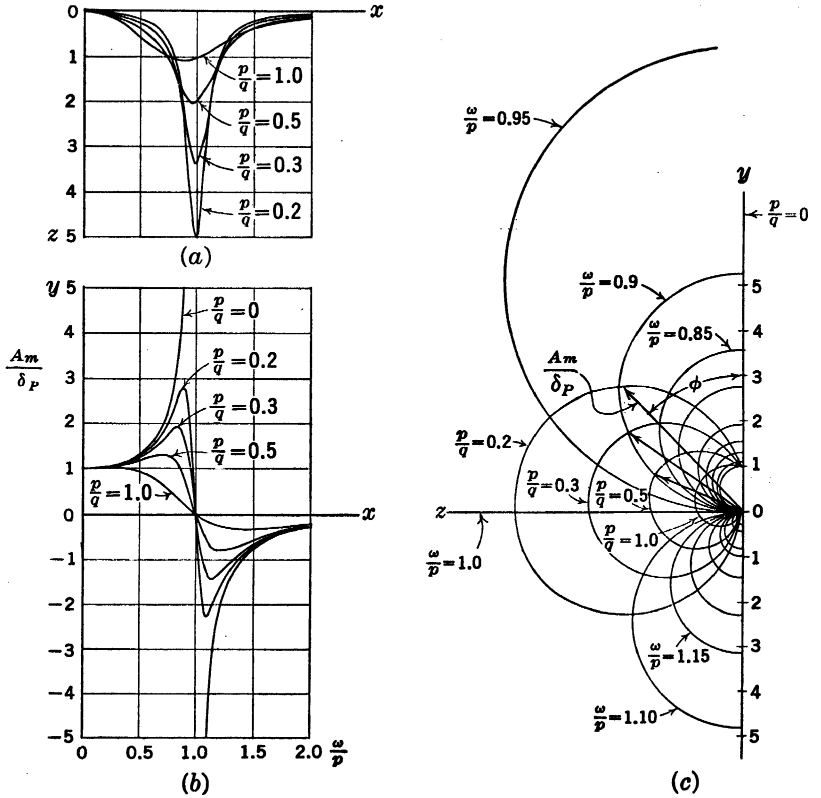


FIG. 4-11

plane are circles with diameters y_0, and the equation for this surface may be expressed in the form

$$\Delta^2 = y_0 y \quad \text{or} \quad y^2 + z^2 = \frac{y}{1 - x^2}$$

It is sufficient to consider only one half of this surface, say, the part on the positive side of the xy plane. Projections on the three coordinate planes of curves for various values of p/q are shown in Fig. 4-11. It is of particular interest to note that the projections of these curves on the

4.6. Energy Balance in Damped Systems

A damper in an oscillating system dissipates a certain amount of energy during each cycle. This energy must be replaced by the exciting force if the motion is to be steady (i.e. of constant amplitude).

Consider the system, previously discussed in section 4.2 which consists of a periodic force,

$$P_c \cos \omega t$$

acting directly on a damper. The displacement is given by equation 4.2–10 as

$$x = \frac{P_c}{\omega c} \sin \omega t$$

The work done during a small displacement dx is given by

$$dW = P_c \cos \omega t \, dx = P_c \cos \omega t \frac{dx}{dt} dt$$

$$= P_c \dot{x} \cos \omega t \, dt$$

where

$$\dot{x} = \frac{P_c}{c} \cos \omega t$$

Substitution for $\dot{x}$ gives

$$dW = \frac{P_c^2}{c} \cos^2 \omega t \, dt$$

The work W done per cycle is obtained by integrating over a time interval of one period, $\tau = 2\pi/\omega$, thus

$$W = \frac{P_c^2}{c} \int_0^\tau \cos^2 \omega t \, dt = \frac{P_c^2 \pi}{\omega c} \tag{4.6–1}$$

This is the energy dissipated in the damper by the work of viscous forces. In the common dampers this mechanical energy is changed into heat.

The requirement for energy balance may be utilized to determine the amplitude of a damped system. Consider the system of Fig. 4–6. The force operating on the system is

$$P_0 \cos \omega t$$

and the displacement of the system is of the form

$$x = A \cos (\omega t - \phi)$$

The velocity at any instant has the value

$$\dot{x} = -A\omega \sin (\omega t - \phi)$$

The work done by the exciting force during a small displacement dx is

$$\begin{aligned} dW &= P_0 \cos \omega t\, dx = P_0 \dot{x} \cos \omega t\, dt \\ &= -P_0 A\omega \sin (\omega t - \phi) \cos \omega t\, dt \\ &= -\tfrac{1}{2} P_0 A[\sin (2\omega t - \phi) - \sin \phi]\, d(\omega t) \end{aligned}$$

The work done per cycle by the exciting force is

$$\begin{aligned} W &= -\tfrac{1}{2} P_0 A \left[\int_0^{2\pi} \sin (2\omega t - \phi)\, d(\omega t) - \sin \phi \int_0^{2\pi} d(\omega t) \right] \\ &= \pi P_0 A \sin \phi \end{aligned} \tag{4.6–2}$$

The work per cycle is seen to vary directly as the amplitude and the sine of the phase angle ϕ.

The energy dissipated in the damper during a displacement dx is given by

$$\begin{aligned} dU &= c\dot{x}\, dx = c\dot{x}^2\, dt \\ &= cA^2\omega^2 \sin^2 (\omega t - \phi)\, dt \\ &= \tfrac{1}{2} A^2\omega[1 - \cos 2(\omega t - \phi)]\, d(\omega t) \end{aligned}$$

Integration of this expression over a complete cycle yields

$$U = \pi c A^2 \omega = \pi k A^2 \left(\frac{\omega}{q}\right) \tag{4.6–3}$$

Equating the work done on the system per cycle to the energy dissipated per cycle gives

$$U = W = \pi P_0 A \sin \phi = \pi k A^2 \left(\frac{\omega}{q}\right)$$

whence

$$A = \frac{P_0}{k} \frac{q}{\omega} \sin \phi = \delta_P \frac{q}{\omega} \sin \phi \tag{4.6–4}$$

From equation 4.4–6 the phase angle is given by

$$\tan \phi = \frac{\dfrac{\omega}{q}}{1 - \left(\dfrac{\omega}{p}\right)^2}$$

whence

$$\sin \phi = \frac{\dfrac{\omega}{q}}{\sqrt{\left(1 - \dfrac{\omega^2}{p^2}\right)^2 + \left(\dfrac{\omega}{q}\right)^2}}$$

Substitution into equation 4.6–4 yields

$$A = \frac{\delta_P}{\sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \left(\frac{\omega}{q}\right)^2}}$$

which coincides with the results of section 4.4. For small damping, resonance may be assumed to occur when $\omega/p = 1$. This corresponds to $\sin\phi = 1$. The maximum amplitude follows directly from equation 4.6–4 as

$$A_{\max} \sim \delta_P \frac{q}{p}$$

Equation 4.6–4 may be stated in the form,

$$\left.\begin{aligned} q &= \frac{kA\omega}{P_0 \sin\phi} = \frac{A\omega}{\delta_P \sin\phi} \\ \text{or} \qquad c &= \frac{P_0 \sin\phi}{A\omega} \end{aligned}\right\} \qquad (4.6\text{–}5)$$

Under conditions where the undamped resonance factor is not directly obtainable, equations 4.6–5 offer a means of determining the relaxation frequency and the damping constant if the damping is assumed to be linear in nature. If the damping is not linear, the relaxation frequency and the damping constant, so determined, correspond to the equivalent linear damping.

Part 2

Systems of Several Degrees of Freedom

Chapter 5

CLASSICAL METHOD

5.1. Degrees of Freedom

A system of springs, masses, and dampers that requires two or more independent coordinates to define completely its configuration at any time is referred to in general as a system of several degrees of freedom. Systems of two degrees of freedom occur in engineering practice almost as frequently as those of one degree of freedom. For this reason, the study of higher-order systems may best be initiated by a treatment of some typical examples involving only two degrees of freedom. Further, the application of the more general methods to a simple problem enhances the understanding of the fundamental theory.

5.2. Derivation of the Equations of Motion for Free Vibration

Of the many mechanical systems prevalent in engineering practice that possess two degrees of freedom, two typical examples indicative of the general nature of these systems are those of Figs. 5–1*a* and 5–2*a*.

Consider first the two-mass system of Fig. 5–1*a*. The motion of this system during free vibration may be obtained from a solution to the equations of motion for each of the masses m_1 and m_2. The equation of motion for mass m_1 is seen to be

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2)$$

or

$$\ddot{x}_1 + \frac{k_1 + k_2}{m_1}x_1 - \frac{k_2}{m_1}x_2 = 0$$

Introduction of the notation

$$p_{ij}{}^2 = \frac{k_i}{m_j}$$

permits this to be written as

$$\ddot{x}_1 + (p_{11}^2 + p_{21}^2)x_1 - p_{21}^2 x_2 = 0 \qquad (5.2\text{–}1)$$

The equation of motion for the second mass m_2 is

$$m_2\ddot{x}_2 = -k_3x_2 + k_2(x_1 - x_2)$$

which may be written in the form

$$\ddot{x}_2 + (p_{32}^2 + p_{22}^2)x_2 - p_{22}^2 x_1 = 0 \qquad (5.2\text{–}2)$$

The simultaneous solution of equations 5.2–1 and 5.2–2 will indicate the motion of the system.

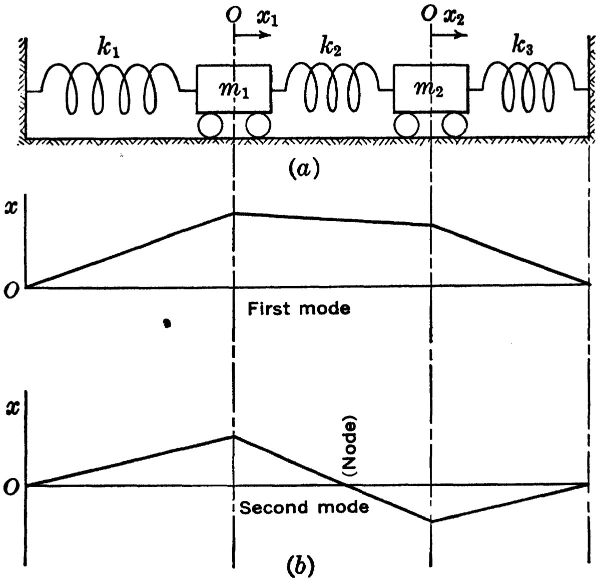


FIG. 5–1

Before proceeding with the solution for this system, it is instructive to consider a second example. This second system, shown in Fig. 5–2a, consists of three bodies connected by springs and supported so that they are free to rotate about the central axis.

From the previous statements regarding degrees of freedom, it would be expected that, since three coordinates θ_1, θ_2, and θ_3 are required to determine the configuration of this system, it should have three degrees of freedom. Although this is true in a general sense, it will be shown in the development to follow, that one of these degrees of freedom concerns itself with the rotation of the system as a rigid body and therefore is of no practical interest in a study of the vibrations of the system. The remaining two degrees of freedom are connected with modes of vibration, making the system, for the purposes of vibration analysis, essentially one of two degrees of freedom.

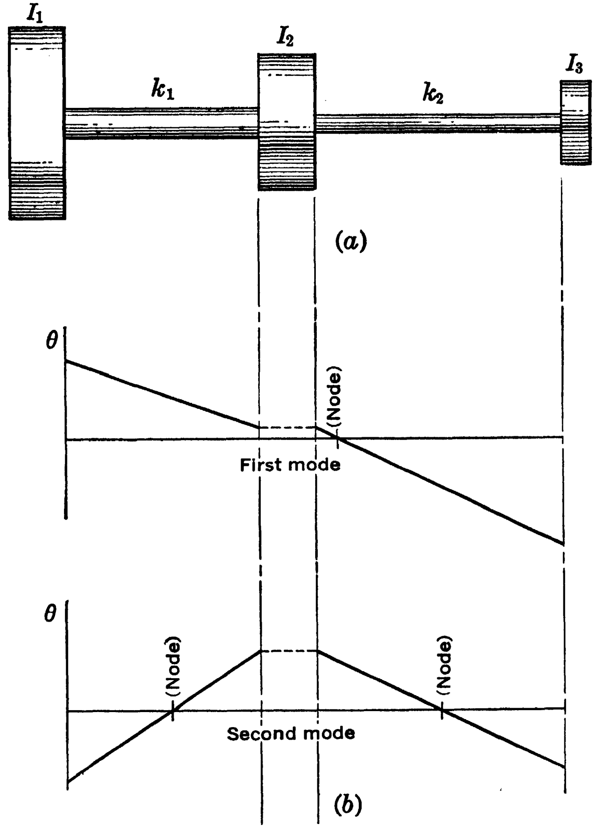


FIG. 5–2

The equations of motions for each body are established in the same manner as the preceding example, thus:

$$\left.\begin{aligned} \ddot{\theta}_1 + p_{11}^2\theta_1 - p_{11}^2\theta_2 &= 0 \\ \ddot{\theta}_2 + (p_{12}^2 + p_{22}^2)\theta_2 - p_{12}^2\theta_1 - p_{22}^2\theta_3 &= 0 \\ \ddot{\theta}_3 + p_{23}^2\theta_3 - p_{23}^2\theta_2 &= 0 \end{aligned}\right\} \qquad (5.2\text{–}3)$$

where the notation

$$p_{ij}^2 = \frac{k_i}{I_j}$$

has been introduced. The character of the motion of this system is obtained by a simultaneous solution of equations 5.2–3.

5.3. Solution of the Equations of Motion

The general approach to the solution of the equations of motion is the same for all linear systems. This method may be illustrated by the two examples of the preceding section.

Consider first the system of Fig. 5–1a. The previous study of systems involving one degree of freedom has shown that simple harmonic motion is to be associated with the motion of a linear vibrating mechanical system. It is therefore logical to try a solution in the form of a simple harmonic motion, for example,

$$\left.\begin{aligned} x_1 &= A_1 \cos pt \\ x_2 &= A_2 \cos pt \end{aligned}\right\} \tag{5.3–1}$$

where the amplitudes A_1 and A_2, as well as the natural circular frequency p, are to be determined. The assumption that equations 5.3–1 are the solution can only be verified by showing that they satisfy the equations of motion 5.2–1 and 5.2–2. If these equations are satisfied, then equations 5.3–1 are the solution.

Substitution from equations 5.3–1 into equations 5.2–1 and 5.2–2 gives

$$[-A_1p^2 + (p_{11}^2 + p_{21}^2)A_1 - p_{21}^2A_2] \cos pt = 0$$

$$[-A_2p^2 + (p_{32}^2 + p_{22}^2)A_2 - p_{22}^2A_1] \cos pt = 0$$

These equations must be true for all values of t; hence,

$$\left.\begin{aligned} (p_{11}^2 + p_{21}^2 - p^2)A_1 - p_{21}^2A_2 &= 0 \\ -p_{22}^2A_1 + (p_{32}^2 + p_{22}^2 - p^2)A_2 &= 0 \end{aligned}\right\} \tag{5.3–2}$$

Equations 5.3–2 may be written in the form

$$\frac{A_1}{A_2} = \frac{p_{21}^2}{p_{11}^2 + p_{21}^2 - p^2} = \frac{p_{32}^2 + p_{22}^2 - p^2}{p_{22}^2} \tag{5.3–3}$$

from which p may be determined and then the ratio A_1/A_2. Since there are only two equations and three unknowns, some indeterminancy is to be expected.

The equations 5.3–2 are a homogeneous pair of equations in the unknown A_1 and A_2. The determinant of this system must therefore vanish if the system is to have a non-trivial solution. That is,

$$\begin{vmatrix} (p_{11}^2 + p_{21}^2 - p^2) & (-p_{21}^2) \\ (-p_{22}^2) & (p_{32}^2 + p_{22}^2 - p^2) \end{vmatrix} = 0 \tag{5.3–4}$$

Equation 5.3–4 is called the frequency equation, and its solution will permit values to be found for the unknown frequency p. Expansion of equation 5.3–4 by usual methods gives

$$p^4 - (p_{11}^2 + p_{21}^2 + p_{22}^2 + p_{32}^2)p^2 + (p_{11}^2 p_{22}^2 + p_{21}^2 p_{32}^2 + p_{11}^2 p_{32}^2) = 0 \quad (5.3\text{–}5)$$

which also may be obtained from equation 5.3–3. This is a quadratic equation in p^2 and therefore furnishes two roots which may be designated by p_1^2 and p_2^2. These are the squares of the two natural circular frequencies of this system, and only for these particular values of p^2 can a solution be found in the form of equation 5.3–1. The values of p determined above may be substituted in equation 5.3–3 and the ratio of the amplitudes obtained.

To illustrate the above consider the specific system where

$$m_1 = m_2$$

$$k_1 = k_2 = k_3$$

Then

$$p_{11} = p_{21} = p_{22} = p_{32}$$

The frequency equation 5.3–5 is reduced to

$$p^4 - 4p_{11}^2 p^2 + 3p_{11}^4 = 0$$

whence

$$p^2 = 2p_{11}^2 \pm p_{11}^2$$

Thus the two natural circular frequencies are

$$p_1^2 = p_{11}^2 \quad \text{and} \quad p_2^2 = 3p_{11}^2$$

Each of these frequencies will give rise to a different configuration of the system known as a mode of motion. To determine the nature of these modes of motion, the natural frequencies are substituted into equations 5.3–3, which give, for the first mode ($p = p_1$),

$$A_1 = A_2$$

or

$$\frac{A_2}{A_1} = 1 \quad (5.3\text{–}6)$$

The absolute value of the amplitude is arbitrary for free vibration in this instance, just as for systems possessing only one degree of freedom. Equation 5.3–6 shows that the masses move in the same direction at the same time and have the same amplitude. The solution for this mode of motion is then

$$x_1 = x_2 = A_1 \cos p_{11}t$$

For the second mode of motion corresponding to the higher frequency ($p_2^2 = 3p_{11}^2$), equations 5.3–2 reduce to

$$-A_1 = A_2$$

or

$$\frac{A_2}{A_1} = -1$$

As for the first mode, the amplitudes of the two masses are the same; however, the masses now move in opposite directions; that is, they have a phase angle of π rad between their motions. The solution in this case is

$$x_1 = A_1 \cos 3p_{11}t$$

$$x_2 = -A_1 \cos 3p_{11}t = A_1 \cos (3p_{11}t + \pi)$$

The displacements of the systems in general are conveniently indicated graphically by Fig. 5–1*b*. In the second mode, the midpoint of the center spring k_2 does not move in this specific example. Such a point is called a node. A node in a system can be fixed or restrained without altering the character of the motion or its natural frequencies.

The solution to equations 5.2–3 may be obtained as in the previous case by assuming solutions:

$$\left.\begin{aligned} \theta_1 &= \Theta_1 \cos pt \\ \theta_2 &= \Theta_2 \cos pt \\ \theta_3 &= \Theta_3 \cos pt \end{aligned}\right\} \qquad (5.3\text{–}7)$$

Substitution of these trial solutions into equations 5.2–3 yields

$$\left.\begin{aligned} (p_{11}^2 - p^2)\Theta_1 - p_{11}^2\Theta_2 &= 0 \\ -p_{12}^2\Theta_1 + (p_{12}^2 + p_{22}^2 - p^2)\Theta_2 - p_{22}^2\Theta_3 &= 0 \\ -p_{23}^2\Theta_2 + (p_{23}^2 - p^2)\Theta_3 &= 0 \end{aligned}\right\} \qquad (5.3\text{–}8)$$

The frequency equation is obtained, as before, by the requirement that the determinant of this homogeneous set of equations vanish. Thus,

$$\begin{vmatrix} (p_{11}^2 - p^2) & (-p_{11}^2) & 0 \\ (-p_{12}^2) & (p_{12}^2 + p_{22}^2 - p^2) & (-p_{22}^2) \\ 0 & (-p_{23}^2) & (p_{23}^2 - p^2) \end{vmatrix} = 0$$

Expansion of this determinant yields

$$p^2[p^4 - (p_{11}^2 + p_{12}^2 + p_{22}^2 + p_{23}^2)p^2 + (p_{11}^2p_{22}^2 + p_{11}^2p_{23}^2 + p_{12}^2p_{23}^2)] = 0 \quad (5.3\text{–}9)$$

This is of the same general form as equation 5.3–5, except for the factor p^2. The root, $p = 0$, corresponding to this factor, is to be associated with a rigid body rotation of the whole system. The remaining two frequencies associated with vibratory motion may be found by obtaining a complete solution to equation 5.3–9.

For the specific simple system, where

$$I_1 = I_2 = I_3$$

$$k_1 = k_2$$

so that

$$p_{11} = p_{12} = p_{22} = p_{23}$$

equation 5.3–9 reduces to

$$p^4 - 4p_{11}^2p^2 + 3p_{11}^4 = 0$$

which has the roots,

$$p_1^2 = p_{11}^2 \quad \text{and} \quad p_2^2 = 3p_{11}^2$$

Substitution into equations 5.3–8 gives, for $p = p_1$,

$$\Theta_2 = 0$$

$$-\Theta_1 + \Theta_2 - \Theta_3 = 0$$

$$\Theta_2 = 0$$

from which it is seen that

$$\Theta_2 = 0; \qquad \frac{\Theta_3}{\Theta_1} = -1$$

Similarly for the second mode ($p_2^2 = 3p_{11}^2$), equations 5.3–8 reduce to

$$-2\Theta_1 - \Theta_2 = 0$$

$$-\Theta_1 - \Theta_2 - \Theta_3 = 0$$

$$-\Theta_2 - 2\Theta_3 = 0$$

from which is obtained

$$\frac{\Theta_2}{\Theta_1} = -2; \qquad \frac{\Theta_3}{\Theta_1} = 1$$

The form of the modes of motion associated with the two frequencies is shown graphically in Fig. 5–2*b*.

The analysis of the natural frequencies and their modes of motion for other systems of two degrees of freedom may be obtained in a similar manner. In general both modes of motion may be present at the same time. The resultant motion is then the sum of the individual motions. The free vibration of the system is conservative; however the distribution of the total energy in the system between the various modes may change with time. This, in turn, may produce variations in the total amplitudes of the elements in the system. These variations are periodic and this phenomenon is frequently referred to as "wandering energy."

5.4. Forced Vibration of Systems of Two Degrees of Freedom without Damping

The theory of forced vibrations of an undamped system of two degrees of freedom may be illustrated by an application to the system of Fig. 5-3. The periodic force which acts on the system is assumed to be of the simple form,

$$P = P_0 \cos \omega t$$

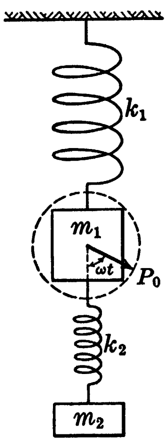


FIG. 5-3

and in this example the periodic exciting force has been applied to the mass m_1. The equations of motion are

$$\left.\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 - k_2(x_1 - x_2) + P_0 \cos \omega t \\ m_2\ddot{x}_2 &= k_2(x_1 - x_2) \end{aligned}\right\} \qquad (5.4\text{-}1)$$

To simplify these, it is convenient to introduce again the notation

$$p_{ij}{}^2 = \frac{k_i}{m_j} \quad \text{and} \quad \delta_P = \frac{P_0}{k}$$

Equations 5.4-1 may now be written as

$$\left.\begin{aligned} \ddot{x}_1 + p_{21}{}^2(x_1 - x_2) + p_{11}{}^2x_1 &= \delta_P p_{11}{}^2 \cos \omega t \\ \ddot{x}_2 + p_{22}{}^2(x_2 - x_1) &= 0 \end{aligned}\right\} \qquad (5.4\text{-}2)$$

The solution of these two equations, which will be discussed here, is the particular solution that represents the steady-state motion.

As in previous examples, it is to be expected that the motion of each mass will be simple harmonic in nature. It is natural then to try to find solutions in the form,

$$x_1 = A_1 \cos \omega t$$

$$x_2 = A_2 \cos \omega t$$

Substitution of these trial solutions into equations 5.4–1 gives

$$\left.\begin{aligned}[-\omega^2 A_1 + p_{21}^2(A_1 - A_2) + p_{11}^2 A_1]\cos\omega t &= \delta_P p_{11}^2 \cos\omega t\\ [-\omega^2 A_2 + p_{22}^2(A_2 - A_1)]\cos\omega t &= 0\end{aligned}\right\} \quad (5.4\text{–}3)$$

Since these equations must be satisfied for all values of t it follows that

$$\left.\begin{aligned}(p_{11}^2 + p_{21}^2 - \omega^2)A_1 - p_{21}^2 A_2 &= \delta_P p_{11}^2\\ -p_{22}^2 A_1 + (p_{22}^2 - \omega^2)A_2 &= 0\end{aligned}\right\} \quad (5.4\text{–}4)$$

The solution of these two equations gives the values of A_1 and A_2, and hence the motion is determined. The solution may be expressed by determinants. Denoting the determinant of the unknown coefficients of A_1 and A_2 by Δ, that is,

$$\Delta = \begin{vmatrix}(p_{11}^2 + p_{21}^2 - \omega^2) & (-p_{21}^2)\\ (-p_{22}^2) & (p_{22}^2 - \omega^2)\end{vmatrix} \quad (5.4\text{–}5)$$

and denoting by Δ_i the determinant obtained by substituting for the ith column the values on the right side of the equation, permits the amplitudes to be written symbolically as

$$A_1 = \frac{\Delta_1}{\Delta} \quad \text{and} \quad A_2 = \frac{\Delta_2}{\Delta} \quad (5.4\text{–}6)$$

It will be noted that the right sides of equations 5.4–4 will vanish if the exciting force vanishes. Examination of the method of obtaining Δ_i shows immediately that, for the system possessing only free vibrations,

$$\Delta_1 = \Delta_2 = 0$$

However, this requires that, as a minimum, $\Delta = 0$ if the amplitudes given by equations 5.4–5 are to be different from zero. The equation

$$\Delta = 0 \quad (5.4\text{–}7)$$

is called the frequency equation. The solution to equation 5.4–7 furnishes two values of ω which are the natural frequencies of the system in question.

To obtain these natural frequencies, consider the equation 5.4–7

$$\begin{aligned}\Delta = (p_{22}^2 - \omega^2)(p_{11}^2 + p_{21}^2 - \omega^2) - p_{11}^2 p_{21}^2&\\ = (\omega^2 - p_1^2)(\omega^2 - p_2^2) &= 0\end{aligned}$$

where p_1 and p_2 are the two natural frequencies of the system. Also

$$\Delta_1 = (p_{22}^2 - \omega^2)p_{11}^2\delta_P \quad \text{and} \quad \Delta_2 = p_{11}^2 p_{22}^2 \delta_P$$

whence

$$\left.\begin{aligned} A_1 &= \frac{(p_{22}^2 - \omega^2)p_{11}^2}{(\omega^2 - p_1^2)(\omega^2 - p_2^2)}\,\delta_P \\ A_2 &= \frac{p_{11}^2 p_{22}^2}{(\omega^2 - p_1^2)(\omega^2 - p_2^2)}\,\delta_P \end{aligned}\right\} \tag{5.4-8}$$

The amplitudes A_1 and A_2 are easily obtained by evaluating equations

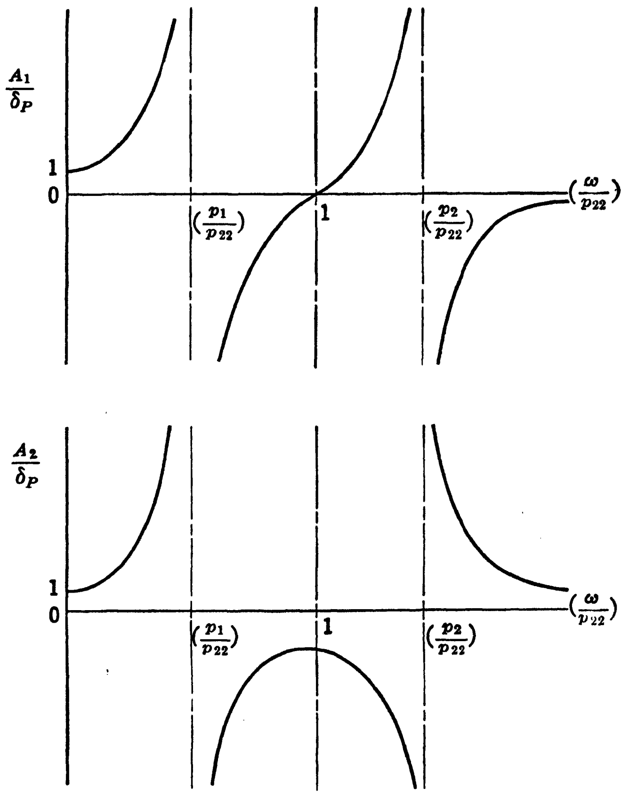


Fig. 5–4

5.4–8. It will be noted that A_1 will vanish if $\omega = p_{22}$. Since $p_{22}^2 = k_2/m_2$, it is seen that this is the natural frequency of the mass m_2 and spring k_2 considered as a separate system. It is therefore possible by suitably adjusting k_2 and m_2 to eliminate the motion of m_1. The elements k_2 and m_2 are known as a vibration absorber, and, when $p_{22} \doteq \omega$, the absorber is said to be tuned to the circular frequency ω. It is also apparent that there is no value of the spring constants and masses for which $A_2 = 0$. The infinite amplitudes which arise from

$\omega = p_1$ or $\omega = p_2$ are due to resonant conditions similar in every respect to those of a single degree of freedom systems. The resonance factors for the two masses are

$$\frac{A_1}{\delta_P} = \frac{(p_{22}^2 - \omega^2)p_{11}^2}{(\omega^2 - p_1^2)(\omega^2 - p_2^2)}; \qquad \frac{A_2}{\delta_P} = \frac{p_{11}^2 p_{22}^2}{(\omega^2 - p_1^2)(\omega^2 - p_2^2)} \tag{5.4–9}$$

A plot of these factors against the ratio ω/p_{22} for a typical system is shown in Fig. 5–4. It may be noted that the undamped absorber is rather sensitive to variations in ω/p_{22} near perfect tuning, and for this reason it is most effective as a vibration absorber where the forced frequency ω has small variations. For large variations in the frequency of the exciting force, and for applications involving large amplitudes of the absorber mass, it may be necessary to add a damping element to the system. This type of a system is known as the damped vibration absorber which is discussed in the next section.

The two resonant frequencies obtained in this instance are typical of a system with two degrees of freedom. A resonant frequency always accompanies each degree of freedom.

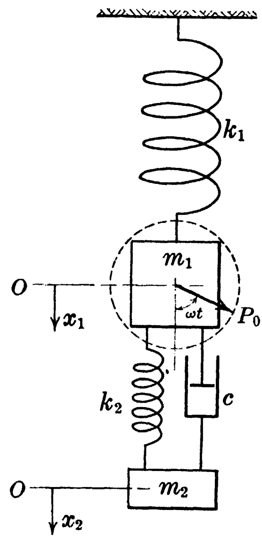


Fig. 5–5

5.5. Forced Vibration of Systems of Two Degrees of Freedom with Damping

To limit the resonant amplitudes and to increase its effective range as a vibration absorber, the system of Fig. 5–3 may be modified by the inclusion of a damper. Such a damped system is shown in Fig. 5–5. The equations of motion for this system may be established as

$$\left.\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 - k_2(x_1 - x_2) - c(\dot{x}_1 - \dot{x}_2) + P_0\cos\omega t \\ m_2\ddot{x}_2 &= k_2(x_1 - x_2) + c(\dot{x}_1 - \dot{x}_2) \end{aligned}\right\} \tag{5.5–1}$$

They may be written in the convenient form,

$$\left.\begin{aligned} \ddot{x}_1 + \frac{p_{21}^2}{q}(\dot{x}_1 - \dot{x}_2) + p_{21}^2(x_1 - x_2) + p_{11}^2x_1 &= \delta_P p_{11}^2\cos\omega t \\ \ddot{x}_2 - \frac{p_{22}^2}{q}(\dot{x}_1 - \dot{x}_2) - p_{22}^2(x_1 - x_2) \qquad &= 0 \end{aligned}\right\} \tag{5.5–2}$$

where $q = k_2/c$ and the remaining notation is the same as that introduced in the previous section. The straightforward solution of these two equations, assuming simple harmonic trial solutions of the form,

$$\left.\begin{aligned} x_1 &= B_1 \cos \omega t + C_1 \sin \omega t \\ x_2 &= B_2 \cos \omega t + C_2 \sin \omega t \end{aligned}\right\} \tag{5.5-3}$$

is possible but somewhat lengthy and tedious. The actual procedure parallels the development of section 5.4.

To simplify the procedure it is expedient to represent the motion in terms of complex variables, where $j = \sqrt{-1}$ is the imaginary unit. The displacements x_1 and x_2, as well as the exciting force, may be treated as real parts of complex displacements and a complex exciting force, that is:

$$\left.\begin{aligned} \mathbf{w}_1 = x_1 + jy_1, \quad \mathbf{w}_2 = x_2 + jy_2, \quad \mathbf{P} &= P_x + jP_y \\ &= P_0 e^{j\omega t} \end{aligned}\right\} \tag{5.5-4}$$

The underlying idea of this representation arises from the vectorial nature of the displacements and exciting force. This vectorial character has been demonstrated previously for one degree of freedom and will be further emphasized in Chapter 6.

The complex form of the equations of motion is then

$$\left.\begin{aligned} \ddot{\mathbf{w}}_1 + \frac{p_{21}^2}{q}(\dot{\mathbf{w}}_1 - \dot{\mathbf{w}}_2) + p_{21}^2(\mathbf{w}_1 - \mathbf{w}_2) + p_{11}^2\mathbf{w}_1 &= \delta_P p_{11}^2 e^{j\omega t} \\ \ddot{\mathbf{w}}_2 - \frac{p_{22}^2}{q}(\dot{\mathbf{w}}_1 - \dot{\mathbf{w}}_2) - p_{22}^2(\mathbf{w}_1 - \mathbf{w}_2) &= 0 \end{aligned}\right\} \tag{5.5-5}$$

Equations 5.5-2 are the real part of equations 5.5-5. Trial solutions to these equations may also be formulated in complex form as

$$\left.\begin{aligned} \mathbf{w}_1 &= A_1 e^{j\omega t} \\ \mathbf{w}_2 &= A_2 e^{j\omega t} \end{aligned}\right\} \tag{5.5-6}$$

Substitution into equations 5.5-5 gives

$$\left.\begin{aligned} \left(p_{11}^2 + p_{21}^2 - \omega^2 + j\frac{\omega}{q}p_{21}^2\right)\mathbf{A}_1 - \left(p_{21}^2 + j\frac{\omega}{q}p_{21}^2\right)\mathbf{A}_2 &= \delta_P p_{11}^2 \\ -\left(p_{22}^2 + j\frac{\omega}{q}p_{22}^2\right)\mathbf{A}_1 + \left(p_{22}^2 - \omega^2 + j\frac{\omega}{q}p_{22}^2\right)\mathbf{A}_2 &= 0 \end{aligned}\right\} \tag{5.5-7}$$

These two equations may be solved for the amplitudes $\mathbf{A}_1$ and $\mathbf{A}_2$. Thus, for example,

$$\mathbf{A}_1 = \left[\frac{\left(p_{22}^2 - \omega^2 + j\frac{\omega}{q} p_{22}^2\right) p_{11}^2}{(p_{11}^2 - \omega^2)(p_{22}^2 - \omega^2) - p_{21}^2\omega^2 + j\frac{\omega}{q}[p_{22}^2 p_{11}^2 - \omega^2(p_{21}^2 + p_{22}^2)]} \right] \delta_P \tag{5.5–8}$$

This is a complex quantity of the form

$$\mathbf{A}_1 = \frac{\alpha_1 + j\beta_1}{\alpha_2 + j\beta_2} \delta_P$$

where α_1, α_2, β_1, β_2, and δ_P are real numbers. The absolute value of this complex number is the amplitude of the motion of the mass m_1. The fact that δ_P and $\mathbf{A}_1$ are related through a complex factor is significant, and indicates that they are not in phase.

The absolute value of the complex factor may be found conveniently by writing the numerator and denominator in polar form. Thus

$$\frac{\alpha_1 + j\beta_1}{\alpha_2 + j\beta_1} = \frac{\sqrt{\alpha_1^2 + \beta_1^2} e^{j\phi_1}}{\sqrt{\alpha_2^2 + \beta_2^2} e^{j\phi_2}} = \sqrt{\frac{\alpha_1^2 + \beta_1^2}{\alpha_2^2 + \beta_2^2}} e^{j(\phi_1 - \phi\ )}$$

where the absolute value is

$$\sqrt{\frac{\alpha_1^2 + \beta_1^2}{\alpha_2^2 + \beta_2^2}} \quad \text{and} \quad \tan \phi_1 = \frac{\beta_1}{\alpha_1}, \quad \tan \phi_2 = \frac{\beta_2}{\alpha_2}$$

Substitution of the actual values for α_1, α_2, β_1, and β_2 permits a solution for the ratio of the actual amplitude to the deflection δ_P that occurs if P_0 acts statically on the system. This ratio may be expressed as

$$\left(\frac{A_1}{\delta_P}\right)^2 = \left[(p_{22}^2 - \omega^2)^2 + \left(\frac{\omega}{q}\right)^2 p_{22}^4\right] p_{11}^4 \div \Big\{[(p_{11}^2 - \omega^2)$$
$$(p_{22}^2 - \omega^2) - p_{21}^2\omega^2]^2 + \left(\frac{\omega}{q}\right)^2 [p_{22}^2 p_{11}^2 - \omega^2(p_{21}^2 + p_{22}^2)]^2\Big\} \tag{5.5–9}$$

It is to be noted that, for $c = 0$, $q = \infty$, this expression reduces to that previously obtained for the undamped absorber.

The angle $\phi_1 - \phi_2$ is the phase angle between the motion of m_1 and the exciting force P_0. It may be calculated from the expression

$$\phi_1 - \phi_2 = \arctan \frac{\beta_1}{\alpha_1} - \arctan \frac{\beta_2}{\alpha_2}$$

The denominator of the resonance factor for m_1 (equation 5.5–9) is always positive and can never be zero unless $q = \infty$ or $c = 0$. Thus the mass m. damped in this way, will always have some motion, but it may be limited to a reasonable magnitude by proper design.[1]

5.6 Orthogonality of the Principal Modes of Vibration

The orthogonality principle is an important and interesting property of vibrating systems having two or more degree of freedom. The system of Fig. 5–6 furnishes an example of this principle. To simplify the development

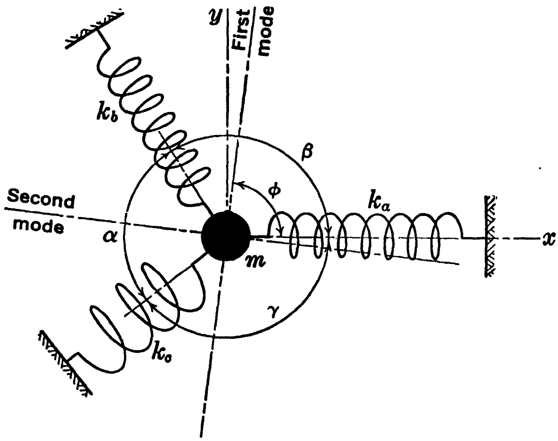


FIG. 5–6

the motion will be restricted to the xy plane, and hence the system has only two degrees of freedom. The orthogonality principle in this example requires that the two principal modes of vibration correspond to motion along two straight lines which are at right angles to each other. These axis will be found to represent the directions of the maximum and minimum spring resistance of the combined spring system.

The equations of motion may be referred to an arbitrary set of coordinate axes as indicated in Fig. 5–6. The equation of motion in the direction parallel to the x axis is

$$m\ddot{x} = -(k_a + k_b \cos^2 \beta + k_c \cos^2 \gamma)x - (k_b \sin \beta \cos \beta + k_c \sin \gamma \cos \gamma)y$$

This may conveniently be written in the form

$$\ddot{x} + (p_a{}^2 + p_b{}^2 \cos^2 \beta + p_c{}^2 \cos^2 \gamma)x + (p_b{}^2 \sin \beta \cos \beta + p_c{}^2 \sin \gamma \cos \gamma)y = 0 \quad (5.6\text{–}1)$$

where the notation,

$$p_i{}^2 = \frac{k_i}{m}$$

[1] For a detailed analysis of the design problem, see J. P. Den Hartog, *Mechanical Vibrations*, 3d Edition, McGraw-Hill, p. 119.

has been introduced. The equation of motion for the y direction is similarly found to be

$$m\ddot{y} = -(k_b \cos\beta \sin\beta + k_c \cos\gamma \sin\gamma)x - (k_b \sin^2\beta + k_c \sin^2\gamma)y$$

which may be written as

$$\ddot{y} + (p_b{}^2 \sin^2\beta + p_c{}^2 \sin^2\gamma)y + (p_b{}^2 \sin\beta\cos\beta + p_c{}^2 \sin\gamma\cos\gamma)x = 0 \tag{5.6–2}$$

Solutions of these equations may be expected to take the form

$$\left.\begin{aligned} x &= C_1 \cos pt \\ y &= C_2 \cos pt \end{aligned}\right\} \tag{5.6–3}$$

Substitution of these trial solutions into equations 5.6–1 and 5.6–2 gives

$$(p_a{}^2 + p_b{}^2 \cos^2\beta + p_c{}^2 \cos^2\gamma - p^2)C_1 + (p_b{}^2 \sin\beta \cos\beta - p_c{}^2 \sin\gamma\cos\gamma)C_2 = 0$$

$$(p_b{}^2 \sin\beta\cos\beta + p_c{}^2 \sin\gamma\cos\gamma)C_1 + (p_b{}^2 \sin^2\beta + p_c{}^2 \sin^2\gamma - p^2)C_2 = 0$$

from which

$$\frac{C_2}{C_1} = -\frac{p_a{}^2 + p_b{}^2\cos^2\beta + p_c{}^2\cos^2\gamma - p^2}{p_b{}^2 \sin\beta\cos\beta + p_c{}^2\sin\gamma\cos\gamma} = -\frac{p_b{}^2\sin\beta\cos\beta + p_c{}^2\sin\gamma\cos\gamma}{p_b{}^2\sin^2\beta + p_c{}^2\sin^2\gamma - p^2} \tag{5.6–4}$$

or

$$p^4 - p^2(p_a{}^2 + p_b{}^2 + p_c{}^2) + p_a{}^2(p_b{}^2\sin^2\beta + p_c{}^2\sin^2\gamma) + p_b{}^2p_c{}^2(\cos\beta\sin\gamma - \cos\gamma\sin\beta)^2 = 0 \tag{5.6–5}$$

The solution of this quadratic equation in p^2 gives the two natural circular frequencies p_1 and p_2. It will suffice for the present to note that the coefficient of p^2 in equation 5.6–5 must be the sum of the two roots, that is,

$$p_1{}^2 + p_2{}^2 = p_a{}^2 + p_b{}^2 + p_c{}^2 \tag{5.6–6}$$

The solutions for the two principal modes of motion may now be written as

$$\left.\begin{aligned} x_1 &= A_1 \cos p_1 t \\ y_1 &= A_2 \cos p_1 t \end{aligned}\right\} \quad \text{First mode}$$

$$\left.\begin{aligned} x_2 &= B_1 \cos p_2 t \\ y_2 &= B_2 \cos p_2 t \end{aligned}\right\} \quad \text{Second mode}$$

Each of these modes corresponds to motion on a straight line through the equilibrium position. The slopes of these straight lines are given, respectively, by

$$\tan\phi_1 = \frac{A_2}{A_1} \quad \text{and} \quad \tan\phi_2 = \frac{B_2}{B_1}$$

where the amplitude components A_1, A_2, B_1, B_2 are to be found from equations 5.6–4. Thus,

$$\left.\begin{aligned} \tan\phi_1 &= \frac{A_2}{A_1} = -\frac{p_a^2 + p_b^2\cos^2\beta + p_c^2\cos^2\gamma - p_1^2}{p_b^2\sin\beta\cos\beta + p_c^2\sin\gamma\cos\gamma} \\ \tan\phi_2 &= \frac{B_2}{B_1} = -\frac{p_b^2\sin\beta\cos\beta + p_c^2\sin\gamma\cos\gamma}{p_b^2\sin^2\beta + p_c^2\sin^2\gamma - p_2^2} \end{aligned}\right\} \tag{5.6–7}$$

In each of the above equations, two expressions are possible, owing to the presence of the two equations 5.6–4. However, both lead to identical results, and equations 5.6–7 are expedient in this instance. It is seen that

$$\begin{aligned} \tan\phi_1 \tan\phi_2 & \\ &= \frac{A_2B_2}{A_1B_1} = \frac{p_a^2 + p_b^2\cos^2\beta + p_c^2\cos^2\gamma - p^2}{p_b^2\sin^2\beta + p_c^2\sin^2\gamma - p_2^2} \\ &= \frac{(p_a^2 + p_b^2 + p_c^2 - p_1^2 - p_2^2) - (p_b^2\sin^2\beta + p_c^2\sin^2\gamma - p_2^2)}{p_b^2\sin^2\beta + p_c^2\sin^2\gamma - p_2^2} \\ &= -1 \end{aligned} \tag{5.6–8}$$

since the first group of terms in the numerator vanishes by equation 5.6–6. This establishes the orthogonality of the two paths of motion.

Equation 5.6–8 may also be written as

$$A_1B_1 + A_2B_2 = 0$$

This relation may be generalized. For a general vibrating system of n degree of freedom, the orthogonality principal has the form

$$\sum_{i=1}^{n} m_iA_iB_i = 0 \tag{5.6–9}$$

where the A_i and B_i are the components of the amplitudes referred to an arbitrary set of independent coordinate axes, and the m_i are the masses of the oscillating bodies. Any two sets of amplitudes coresponding to two of the n principal modes of vibration will satisfy equation 5.6–9.

5.7. Systems of Many Degrees of Freedom

The analysis of systems of more than two degrees of freedom is similar in principle to that of two degrees of freedom. The complexity of the calculations, however, increases very rapidly with more degrees of freedom. The problem of damped vibration in a system of many degrees of freedom involves even more laborious calculations. For this reason, the effect of damping is often neglected if it is of small magnitude. To illustrate the nature of the problem, an example common in practice involving rotating masses in series will be considered.

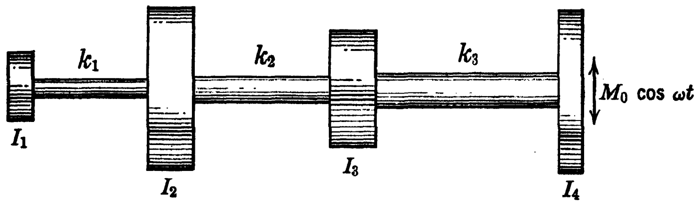


FIG. 5–7

A system in series of the rotating type is shown in Fig. 5–7. The equations of motion may be written directly as follows:

$$\left.\begin{aligned} I_1\ddot{\theta}_1 &= k_1(\theta_1 - \theta_2) \\ I_2\ddot{\theta}_2 &= k_1(\theta_1 - \theta_2) - k_2(\theta_2 - \theta_3) \\ I_3\ddot{\theta}_3 &= k_2(\theta_2 - \theta_3) - k_3(\theta_3 - \theta_4) \\ I_4\ddot{\theta}_4 &= k_3(\theta_3 - \theta_4) + M_0 \cos \omega t \end{aligned}\right\} \qquad (5.7\text{–}1)$$

where an oscillating torque $M_0 \cos \omega t$ is applied to inertia I_4.

Solutions to these equations may be expected to have the form:

$$\left.\begin{aligned} \theta_1 &= \Theta_1 \cos \omega t \\ \theta_2 &= \Theta_2 \cos \omega' \\ \theta_3 &= \Theta_3 \cos \omega t \\ \theta_4 &= \Theta_4 \cos \omega t \end{aligned}\right\} \qquad (5.7\text{–}2)$$

Substitution of these trial solutions into the equations of motion yield the four equations:

$$\left.\begin{aligned} (k_1 - \omega^2 I_1)\Theta_1 - k_2\Theta_2 \qquad\qquad &= 0 \\ -k_1\Theta_1 + (k_1 + k_2 - \omega^2 I_2)\Theta_2 - k_2\Theta_3 \qquad &= 0 \\ -k_2\Theta_2 + (k_2 + k_3 - \omega^2 I_3)\Theta_3 - k_3\Theta_4 &= 0 \\ -k_3\Theta_3 + (k_3 - \omega^2 I_4)\Theta_4 &= M_0 \end{aligned}\right\} \qquad (5.7\text{–}3)$$

The natural frequencies of the system are obtained from the solution to these equations when $M_0 = 0$. In this case, let $\omega = p$, and the frequency equation becomes

$$\Delta = \begin{vmatrix} (k_1 - p^2 I_1) & (-k_1) & 0 & 0 \\ (-k_1) & (k_1 + k_2 - p^2 I_2) & (-k_2) & 0 \\ 0 & (-k_2) & (k_2 + k_3 - p^2 I_3) & (-k_3) \\ 0 & 0 & -k_3 & (k_3 - p^2 I_4) \end{vmatrix} = 0 \tag{5.7-4}$$

This equation is a third-degree equation in p^2 after the root $p^2 = 0$, corresponding to rigid body motion of the system, is eliminated. The solution of a third-degree equation may be obtained directly. The general solution of a frequency equation of the fourth degree for a system involving an additional mass, although theoretically possible, is questionable from a practical standpoint as being too time-consuming. Equations of higher degrees have, as a rule, no general solutions. The numerical solution of such higher-order equations is treated in Chapter 7.

The frequency equation 5.7–4 may be expanded to the form

$$p^6 - Pp^4 + Qp^2 - R = 0 \tag{5.7-5}$$

where

$$P = p_{11}^2 + p_{12}^2 + p_{22}^2 + p_{23}^2 + p_{33}^2 + p_{34}^2$$

$$Q = p_{11}^2 p_{22}^2 + p_{11}^2 p_{23}^2 + p_{11}^2 p_{33}^2 + p_{11}^2 p_{34}^2 + p_{12}^2 p_{23}^2 + p_{12}^2 p_{33}^2 + p_{12}^2 p_{34}^2 + p_{22}^2 p_{33}^2 + p_{22}^2 p_{34}^2 + p_{23}^2 p_{34}^2$$

$$R = p_{11}^2 p_{22}^2 p_{33}^2 + p_{11}^2 p_{22}^2 p_{34}^2 + p_{11}^2 p_{23}^2 p_{34}^2 + p_{12}^2 p_{23}^2 p_{34}^2$$

and the notation

$$p_{ij}^2 = \frac{k_i}{I_j}$$

has been introduced. Owing to the fact that equation 5.7–5 must have three real and positive roots, the solution can always be written as

$$p^2 = \tfrac{1}{3}P - \tfrac{2}{3}\sqrt{P^2 - 3Q}\cos\tfrac{1}{3}(\phi + 2\pi n) \qquad (n = -1, 0, 1)$$

where

$$\cos\phi = \frac{\dfrac{PQ - 9R}{P^2 - 3Q} - \tfrac{2}{3}P}{\tfrac{2}{3}\sqrt{P^2 - 3Q}} \tag{5.7-6}$$

The roots to the frequency equation 5.7–5 may also conveniently be represented graphically as shown in Fig. 5–8.

Frequently, the upper and lower limits of the natural frequencies are all that are required to establish that none of the natural fre-

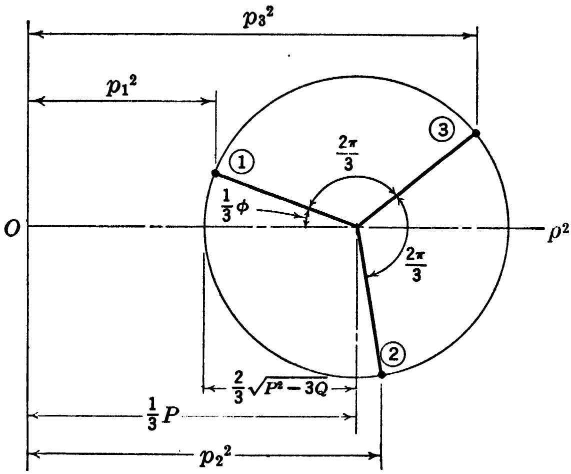


FIG. 5–8

quencies is in a range of the excitation frequencies that would cause resonance. These upper and lower limits are found to be

$$\tfrac{1}{3}P - \tfrac{2}{3}\sqrt{P^2 - 3Q} \leqq p^2 \leqq \tfrac{1}{3}P + \tfrac{2}{3}\sqrt{P^2 - 3Q}$$

The limits may be found as soon as the coefficients in equation 5.7–5 have been computed. The computation of the coefficients in equation 5.7–5 is the most time-consuming calculation involved in the whole procedure. Methods of determining the natural frequencies that do not require the calculation of the coefficients of the frequency equation have been devised (see Chapter 7); however they have the disadvantage of being somewhat indirect. The above method is usually advisable on systems with three non-zero natural frequencies. More complex systems with more degrees of freedom, however, require that the methods of Chapter 7 be employed.

When the external excitation M_0 is not zero, the system will oscillate with the forced frequency ω of the excitation. Equations 5.7–3 may be solved to determine the amplitudes of the forced vibration. Let the determinant obtained by replacing the ith column of the determinant Δ by the right-hand side of equations 5.7–3 be denoted by

Δ_i, as before. The solution for the amplitudes is given symbolically by the equations,

$$\Theta_1 = \frac{\Delta_1}{\Delta}; \quad \Theta_2 = \frac{\Delta_2}{\Delta}; \quad \Theta_3 = \frac{\Delta_3}{\Delta}; \quad \Theta_4 = \frac{\Delta_4}{\Delta}$$

The denominator of these fractions may be written as

$$\Delta = I_1 I_2 I_3 I_4 (\omega^2 - p_1^2)(\omega^2 - p_2^2)(\omega^2 - p_3^2)\omega^2$$

where p_1, p_2, and p_3 are the natural circular frequencies of the system. Expansion of the numerator determinants will show that the amplitudes may be written in the form:

$$\left.\begin{aligned}
\Theta_1 &= \frac{p_{11}^2 p_{22}^2 p_{33}^2}{(\omega^2 - p_1^2)(\omega^2 - p_2^2)(\omega^2 - p_3^2)} \left(\frac{M_0}{\omega^2 I_4}\right) \\
\Theta_2 &= \frac{(p_{11}^2 - \omega^2) p_{22}^2 p_{33}^2}{(\omega^2 - p_1^2)(\omega^2 - p_2^2)(\omega^2 - p_3^2)} \left(\frac{M_0}{\omega^2 I_4}\right) \\
\Theta_3 &= \frac{[(p_{11}^2 - \omega^2)(p_{22}^2 - \omega^2) - p_{12}^2\omega^2] p_{33}^2}{(\omega^2 - p_1^2)(\omega^2 - p_2^2)(\omega^2 - p_3^2)} \left(\frac{M_0}{\omega^2 I_4}\right) \\
\Theta_4 &= \left[\frac{\begin{matrix}(p_{11}^2 - \omega^2)(p_{22}^2 - \omega^2)(p_{33}^2 - \omega^2) \\ - [(p_{11}^2 - \omega^2) p_{23}^2 + (p_{33}^2 - \omega^2) p_{12}^2 + p_{12}^2 p_{23}^2]\omega^2\end{matrix}}{(\omega^2 - p_1^2)(\omega^2 - p_2^2)(\omega^2 - p_3^2)}\right] \left(\frac{M_0}{\omega^2 I_4}\right)
\end{aligned}\right\} \qquad (5.7\text{–}7)$$

It is possible to select frequencies that will cause the numerators to vanish while the denominator remains finite in the expressions for Θ_2, Θ_3, and Θ_4. In fact, there is one frequency at which Θ_2 is zero, two frequencies at which Θ_3 is zero, and three frequencies at which Θ_4 is zero. However, Θ_1 can never vanish. The frequencies at which the masses remain motionless may be determined by equating the numerators to zero and solving the resultant equation. Equations 5.7–7 may be plotted to obtain the resonance diagram for each body. These diagrams are shown in Fig. 5–9 for a particular example.

$$I_1 = 1, \quad I_2 = 4, \quad I_3 = 2, \quad I_4 = 3 \qquad \text{lb in. sec}^2$$

$$k_1 = 1, \quad k_2 = 2, \quad k_3 = 3 \qquad \text{lb in.}$$

Systems involving more degrees of freedom may be treated in an analogous fashion utilizing numerical methods to determine the natural frequencies. The inclusion of damping presents no difficulties in the theory, although the computations lengthen rapidly as the system is made more complex.

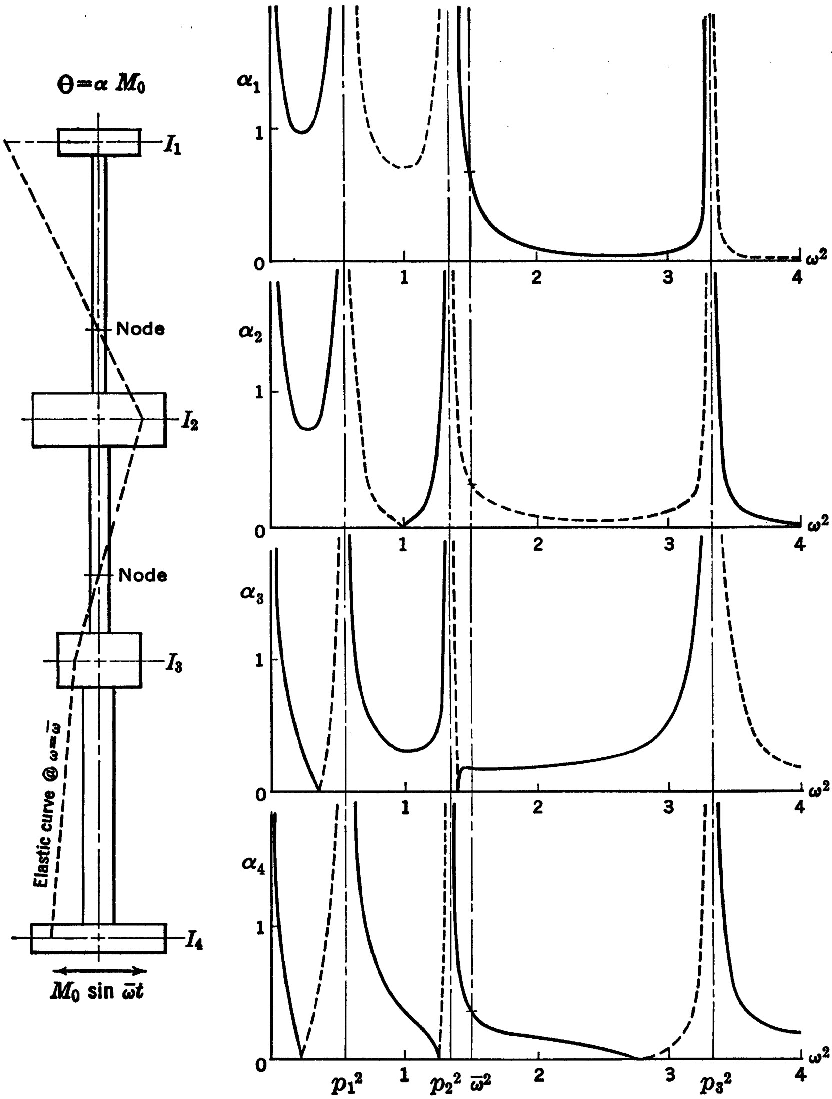


FIG. 5–9

5.8. Oscillation of Constrained Systems and Lagrange's Equations

In the previous examples of this chapter, systems of n degrees of freedom have been treated. In each instance the location of each body in the system could be specified by n independent coordinates. These coordinates were chosen as angles or the usual Cartesian coordinates, whichever appeared most convenient. The motion of those systems was completely free of any constraints.

If the system is constrained to move in a certain manner, the number of degrees of freedom will be decreased. To be specific, consider the example of Fig. 5–10. Four Cartesian coordinates are required to specify completely the configuration of the system; however, these coordinates cannot be chosen at random. The motion is constrained

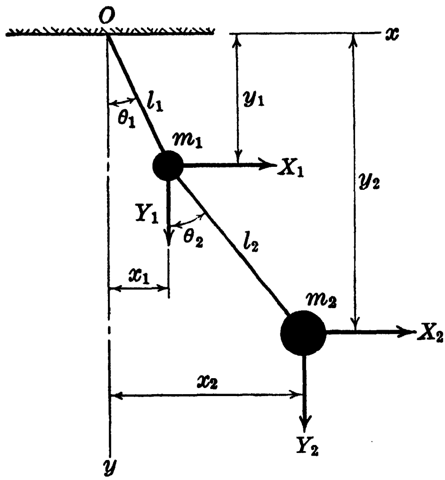


FIG. 5–10

to move in such a manner that the following equations are always satisfied:

$$x_1^2 + y_1^2 = l_1^2; \qquad (x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2 \quad (5.8\text{–}1)$$

Equations 5.8–1 are called the conditions of restraint. These equations may be used to eliminate two coordinates from the equations of motion, thus reducing the problem to two coordinates. The number of degrees of freedom in general is equal to the number of coordinates used to specify the location of the system, minus the number of conditions of restraint; thus, in this example there are $4 - 2 = 2$ degrees of freedom. The elimination of coordinates by direct methods for systems of many degrees of freedom frequently leads to complicated equations. This elimination may be accomplished in a general manner by methods first proposed by Lagrange.

Lagrange's method utilizes as coordinates independent parameters which completely specify the location of the system and yet are independent of any constraints. In the previous example, the angles θ_1 and θ_2 may be used as the independent parameters. These two angles will completely determine the configuration of the system, and they may have all possible values independent of each other. Such independent parameters may be angles, distances, or even areas and are usually referred to as generalized coordinates.

To simplify the following treatment, a system of two degrees of freedom such as shown in Fig. 5–10 and completely defined by two generalized coordinates q_1 and q_2 will be considered. The cartesian coordinates can always be expressed in terms of the independent or generalized coordinates. Thus:

$$\left.\begin{aligned} x_1 &= x_1(q_1, q_2) \qquad & y_1 &= y_1(q_1, q_2) \\ x_2 &= x_2(q_1, q_2) \qquad & y_2 &= y_2(q_1, q_2) \end{aligned}\right\} \tag{5.8–2}$$

Equations 5.8–2 immediately lead to the following expressions between the respective coordinates and their derivatives:

$$\left.\begin{aligned} \dot{x}_1 &= \frac{\partial x_1}{\partial q_1}\dot{q}_1 + \frac{\partial x_1}{\partial q_2}\dot{q}_2 \qquad & \dot{y}_1 &= \frac{\partial y_1}{\partial q_1}\dot{q}_1 + \frac{\partial y_1}{\partial q_2}\dot{q}_2 \\ &\text{and} \\ \frac{\partial \dot{x}_1}{\partial \dot{q}_1} &= \frac{\partial x_1}{\partial q_1} & \frac{\partial \dot{y}_1}{\partial \dot{q}_1} &= \frac{\partial y_1}{\partial q_1} \\ \frac{\partial \dot{x}_1}{\partial \dot{q}_2} &= \frac{\partial x_1}{\partial q_2} & \frac{\partial \dot{y}_1}{\partial \dot{q}_2} &= \frac{\partial y_1}{\partial q_2} \end{aligned}\right\} \tag{5.8–3}$$

The kinetic energy of the system in Cartesian coordinates is especially simple, and it is given by

$$T = \tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$$

From this, the rate of change of kinetic energy with respect to $\dot{q}_1$ and q_1 may be calculated as follows:

$$\frac{\partial T}{\partial \dot{q}_1} = m_1\left(\dot{x}_1\frac{\partial \dot{x}_1}{\partial \dot{q}_1} + \dot{y}_1\frac{\partial \dot{y}_1}{\partial \dot{q}_1}\right) + m_2\left(\dot{x}_2\frac{\partial \dot{x}_2}{\partial \dot{q}_1} + \dot{y}_2\frac{\partial \dot{y}_2}{\partial \dot{q}_1}\right) \tag{5.8–4}$$

and

$$\frac{\partial T}{\partial \dot{q}_1} = m_1\left(\dot{x}_1\frac{\partial \dot{x}_1}{\partial q_1} + \dot{y}_1\frac{\partial \dot{y}_1}{\partial q_1}\right) + m_2\left(\dot{x}_2\frac{\partial \dot{x}_2}{\partial q_1} + \dot{y}_2\frac{\partial \dot{y}_2}{\partial q_1}\right) \tag{5.8–5}$$

The equations of motion for the masses m_1 and m_2 are:

$$\begin{aligned} m_1\ddot{x}_1 &= X_1 \qquad & m_1\ddot{y}_1 &= Y_1 \\ m_2\ddot{x}_2 &= X_2 & m_2\ddot{y}_2 &= Y_2 \end{aligned}$$

where X_1, X_2, Y_1, and Y_2 are the components of the resultant of all the forces that act on the bodies. Multiplication of these four equations by

$$\frac{\partial x_1}{\partial q_1}, \quad \frac{\partial y_1}{\partial q_1}, \quad \frac{\partial x_2}{\partial q_1}, \quad \text{and} \quad \frac{\partial y_2}{\partial q_1}, \quad \text{respectively}$$

and addition gives

$$m_1\left(\ddot{x}_1 \frac{\partial x_1}{\partial q_1} + \ddot{y}_1 \frac{\partial y_1}{\partial q_1}\right) + m_2\left(\ddot{x}_2 \frac{\partial x_2}{\partial q_1} + \ddot{y}_2 \frac{\partial y_2}{\partial q_1}\right)$$
$$= X_1 \frac{\partial x_1}{\partial q_1} + Y_1 \frac{\partial y_1}{\partial q_1} + X_2 \frac{\partial x_2}{\partial q_1} + Y_2 \frac{\partial y_2}{\partial q_1} \qquad (5.8\text{–}6)$$

The left-hand side of equation 5.8–6 may be rewritten. From calculus, it is shown that

$$\frac{d}{dt}\left(\dot{x}_1 \frac{\partial x_1}{\partial q_1}\right) = \ddot{x}_1 \frac{\partial x_1}{\partial q_1} + \dot{x}_1 \frac{d}{dt}\left(\frac{\partial x_1}{\partial q_1}\right) = \ddot{x}_1 \frac{\partial x_1}{\partial q_1} + \dot{x}_1 \frac{\partial \dot{x}_1}{\partial q_1}$$

or

$$\ddot{x}_1 \frac{\partial x_1}{\partial q_1} = \frac{d}{dt}\left(\dot{x}_1 \frac{\partial x_1}{\partial q_1}\right) - \dot{x}_1 \frac{\partial \dot{x}_1}{\partial q_1}$$

Using equations 5.8–3 permits this to take the form

$$\ddot{x}_1 \frac{\partial x_1}{\partial q_1} = \frac{d}{dt}\left(\dot{x}_1 \frac{\partial \dot{x}_1}{\partial \dot{q}_1}\right) - \dot{x}_1 \frac{\partial \dot{x}_1}{\partial q_1}$$

and, similarly, for the other terms on the left side of equation 5.8–6. Substitution of these expressions into equation 5.8–6 gives, for the left side,

$$\frac{d}{dt}\left[m_1\left(\dot{x}_1 \frac{\partial \dot{x}_1}{\partial \dot{q}_1} + \dot{y}_1 \frac{\partial \dot{y}_1}{\partial \dot{q}_1}\right) + m_2\left(\dot{x}_2 \frac{\partial \dot{x}_2}{\partial q_1} + \dot{y}_2 \frac{\partial \dot{y}_2}{\partial \dot{q}_1}\right)\right]$$
$$- \left[m_1\left(\dot{x}_1 \frac{\partial \dot{x}_1}{\partial q_1} + \dot{y}_1 \frac{\partial \dot{y}_1}{\partial q_1}\right) + m_2\left(\dot{x}_2 \frac{\partial \dot{x}_2}{\partial q_1} + \dot{y}_2 \frac{\partial \dot{y}_2}{\partial q_1}\right)\right] \qquad (5.8\text{–}7)$$

Comparison of equation 5.8–7 with equations 5.8–4 and 5.8–5 shows that the left-hand side of equation 5.8–6 may be written in final form as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} \qquad (5.8\text{–}8)$$

It is also convenient to rewrite the right-hand side of equation 5.8–6. If the system undergoes small displacements, the original coordinates will be changed by amounts that can be calculated directly from equations 5.8–2, thus:

$$\delta x_1 = \frac{\partial x_1}{\partial q_1}\,\delta q_1 \qquad \delta y_1 = \frac{\partial y_1}{\partial q_1}\,\delta q_1$$

$$\delta x_2 = \frac{\partial x_2}{\partial q_2}\,\delta q_1 \qquad \delta y_2 = \frac{\partial y_2}{\partial q_1}\,\delta q_1$$

The work done by the forces X_1, Y_1, X_2, and Y_2 due to these small displacements will be

$$\begin{aligned}\delta W &= X_1\delta x_1 + Y_1\,\delta y_1 + X_2\,\delta x_2 + Y_2\,\delta y_2 \\ &= \left(X_1\frac{\partial x_1}{\partial q_1} + Y_1\frac{\partial y_1}{\partial q_1} + X_2\frac{\partial x_2}{\partial q_1} + Y_2\frac{\partial y_2}{\partial q_1}\right)\delta q_1\end{aligned}$$

The expression in the parenthesis is the effective force which is associated with the generalized coordinate q_1. For this reason, it is usually called a total generalized force, and it will be denoted by P_1. Thus

$$P_1 = X_1\frac{\partial x_1}{\partial q_1} + Y_1\frac{\partial y_1}{\partial q_1} + X_2\frac{\partial x_2}{\partial q_1} + Y_2\frac{\partial y_2}{\partial q_1} \tag{5.8–9}$$

Equation 5.8–6 may now be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} = P_1 \tag{5.8–10}$$

A similar development for the coordinate q_2 gives a second equation,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} = P_2 \tag{5.8–11}$$

The development given previously has been based on 2 degrees of freedom for convenience and brevity. A more general development for a system of n degrees of freedom and n generalized coordinates will be found similar to the foregoing treatment and results in n equations of the form,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = P_i \qquad (i = 1, 2 \cdots n) \tag{5.8–12}$$

These n equations are known as Lagrange's equations of motion.

It is instructive to apply Lagrange's equations to the example of Fig. 5–10. The generalized coordinates are

$$q_1 = \theta_1 \quad \text{and} \quad q_2 = \theta_2$$

Equations 5.8–2 have the form:

$$x_1 = l_1 \sin\theta_1 \qquad y_1 = l_1\cos\theta_1$$

$$x_2 = l_1\sin\theta_1 + l_2\sin\theta_2 \qquad y_2 = l_1\cos\theta_1 + l_2\cos\theta_2$$

The velocities of the masses may be obtained by differentiating with respect to time. Thus:

$$\dot{x}_1 = l_1 \cos \theta_1 \dot{\theta}_1 \qquad \dot{y}_1 = -l_1 \sin \theta_1 \dot{\theta}_1$$

$$\dot{x}_2 = l_1 \cos \theta_1 \dot{\theta}_1 + l_2 \cos \theta_2 \dot{\theta}_2 \qquad \dot{y}_2 = -l_1 \sin \theta_1 \dot{\theta}_1 - l_2 \sin \theta_2 \dot{\theta}_2$$

The kinetic energy of the system is then

$$\begin{aligned} T &= \tfrac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \\ &= \tfrac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 \\ &\qquad + \tfrac{1}{2} m_2 [l_1^2 \dot{\theta}_1^2 + 2 l_1 l_2 (\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) \dot{\theta}_1 \dot{\theta}_2 + l_2^2 \dot{\theta}_2^2] \\ &= \tfrac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \tfrac{1}{2} m_2 [l_1^2 \dot{\theta}_1^2 + 2 l_1 l_2 \cos (\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2 + l_2^2 \dot{\theta}_2^2] \end{aligned}$$

whence

$$\frac{\partial T}{\partial \dot{\theta}_1} = (m_1 + m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos (\theta_1 - \theta_2)$$

$$\frac{\partial T}{\partial \dot{\theta}_2} = m_2 l_1 l_2 \dot{\theta}_1 \cos (\theta_1 - \theta_2) + m_2 l_2^2 \dot{\theta}_2$$

$$\frac{\partial T}{\partial \theta_1} = -\frac{\partial T}{\partial \theta_2} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin (\theta_1 - \theta_2)$$

The equations of motion may be obtained by substituting the above expressions into equations 5.8–12. Thus,

$$\begin{aligned} (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos (\theta_1 - \theta_2) & \\ + m_2 l_1 l_2 \dot{\theta}_2^2 \sin (\theta_1 - \theta_2) &= P_1 \end{aligned}$$

$$m_2 l_1 l_2 \ddot{\theta}_1 \cos (\theta_1 - \theta_2) + m_2 l_2^2 \ddot{\theta}_2 - m_2 l_1 l_2 \dot{\theta}_1^2 \sin (\theta_1 - \theta_2) = P_2$$

The total generalized forces may be calculated directly from equation 5.8–9. Thus, if only the force of gravity acts on the system, then:

$$X_1 = 0 \qquad Y_1 = m_1 g$$

$$X_2 = 0 \qquad Y_2 = m_2 g$$

The relations between the coordinates show that

$$\frac{\partial y_1}{\partial \theta_1} = -l_1 \sin \theta_1 \qquad \frac{\partial y_2}{\partial \theta_1} = -l_1 \sin \theta_1$$

$$\frac{\partial y_1}{\partial \theta_2} = 0 \qquad \frac{\partial y_2}{\partial \theta_2} = -l_2 \sin \theta_2$$

hence the total generalized forces are found to be

$$P_1 = -m_1 g l_1 \sin\theta_1 - m_2 g l_1 \sin\theta_1$$
$$P_2 = -m_2 g l_2 \sin\theta_2$$

The equations of motion in final form are, then,

$$(m_1 + m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin\theta_1 = 0$$

$$m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) + m_2 l_2^2 \ddot{\theta}_2 - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2 = 0$$

It is most important to note that the assumption of small displacements has not been introduced. The above equations of motion apply for displacements of any magnitude. The problem, as formulated by these equations, is not a simple one. However, for small oscillations:

$$\sin\theta_1 \sim \theta_1, \qquad \sin\theta_2 \sim \theta_2, \qquad \cos(\theta_1 - \theta_2) \sim 1$$

The terms containing the velocities squared will be very small and can be neglected. Thus, for small oscillations, the equations of motion become

$$(m_1 + m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 + (m_1 + m_2) g l_1 \theta_1 = 0$$
$$m_1 l_1 l_2 \ddot{\theta}_1 + m_2 l_2^2 \ddot{\theta}_2 + m_2 g l_2 \theta_2 = 0$$

These equations are linear and are easily solved by the methods used in the previous sections. Assuming a solution in the form,

$$\theta_1 = \Theta_1 \cos pt \qquad \theta_2 = \Theta_2 \cos pt$$

and substituting into the above equations yields

$$(m_1 + m_2)l_1^2\left(p^2 - \frac{g}{l_1}\right)\Theta_1 + m_2 l_1 l_2 p^2 \Theta_2 = 0$$
$$m_2 l_1 l_2 p^2 \Theta_1 + m_2 l_2^2 \left(p^2 - \frac{g}{l_2}\right)\Theta_2 = 0$$

The frequency equation is then

$$(m_1 + m_2)\left(p^2 - \frac{g}{l_1}\right)\left(p^2 - \frac{g}{l_2}\right) - m_2 p^4 = 0$$

or

$$p^4 - p^2\left(1 + \frac{m_2}{m_1}\right)\left(\frac{g}{l_1} + \frac{g}{l_2}\right) + \left(1 + \frac{m_2}{m_1}\right)\frac{g^2}{l_1 l_2} = 0$$

from which the frequencies of the two natural modes of vibration are obtainable.

5.9. Potential Energy and the Dissipation Function

The generalized force, as defined in the last section, included both internal forces acting through the springs and dampers, as well as the external forces acting on the masses. It is convenient to treat these three types of forces separately. In general, the forces that act on the body m_1 in the direction of the x axis will be of the form

$$X_1 = -k(x_1 - x_2) - c_1(\dot{x}_1 - \dot{x}_2) + X_1' \qquad (5.9\text{–}1)$$

where the terms represent the elastic or spring forces, the damping or dissipative forces and the external forces, respectively. In complex systems there will be a similar term for each spring, damper, and external force that produces a force component in the direction of the x axis. The elastic forces will involve, in general, relative displacements, and the damping forces will depend on relative velocities of the terminals of the spring and damping elements. The external force X_1' may be a function of time. The corresponding terms in equation 5.8–9 due to X_1 will then be

$$X_1 \frac{\partial x_1}{\partial q_1} = -k_1(x_1 - x_2)\frac{\partial x_1}{\partial q_1} - c_1(\dot{x}_1 - \dot{x}_2)\frac{\partial x_1}{\partial q_1} + X_1'\frac{\partial x_1}{\partial q_1} \qquad (5.9\text{–}2)$$

It is convenient to consider the potential energy stored in the elastic elements. For spring k_1 subjected to a relative displacement $x_1 - x_2$, the potential energy is

$$V_{k1} = \tfrac{1}{2}k_1(x_1 - x_2)^2$$

and it is noted that

$$\frac{\partial V_{k1}}{\partial q_1} = k_1(x_1 - x_2)\frac{\partial x_1}{\partial q_1} \qquad (5.9\text{–}3)$$

Also the energy expended by the damper c_1 is

$$F_{c1} = \tfrac{1}{2}c_1(\dot{x}_1 - \dot{x}_2)^2$$

whence

$$\frac{\partial F_{c1}}{\partial \dot{q}_1} = c_1(\dot{x}_1 - \dot{x}_2)\frac{\partial \dot{x}_1}{\partial \dot{q}_1}$$

The last equation may be rewritten by making use of equations 5.8–3. Thus

$$\frac{\partial F_{c1}}{\partial \dot{q}_1} = c_1(\dot{x}_1 - \dot{x}_2)\frac{\partial x_1}{\partial q_1} \qquad (5.9\text{–}4)$$

With the aid of equations 5.9–3 and 5.9–4, equation 5.9–2 takes the form

$$X_1 \frac{\partial x_1}{\partial q_1} = -\frac{\partial V_{k1}}{\partial q_1} - \frac{\partial F_{c1}}{\partial \dot{q}_1} + X_1' \frac{\partial x_1}{\partial q_1}$$

Any additional terms of equation 5.9–1 may be treated in a similar fashion giving

$$P_1 = -\frac{\partial V}{\partial q_1} - \frac{\partial F}{\partial \dot{q}_1} + Q_1 \tag{5.9–5}$$

where V includes the total potential energy stored in all springs of the system as well as any potential energy produced by change in elevation of masses, and F is the energy dissipated by all dampers of the system. The quantity Q_1 is called the generalized force and includes the external forces which are a function of time. It should be noted that Q_1 will be a moment if q_1 is an angle. All external forces or moments that are a function of displacement or constant contribute to the potential energy and are included in the calculation of V. Gravity forces are a specific example of the type of external force that effects the potential energy. All external forces or moments that are a function of velocity will dissipate energy and are to be included in the computation of F. The quantity F is frequently called the dissipation function. Using equation 5.9–5 permits equation 5.8–10 to take the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} + \frac{\partial V}{\partial q_1} + \frac{\partial F}{\partial \dot{q}_1} = Q_1$$

or, in the case of n degrees of freedom,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial F}{\partial \dot{q}_i} = Q_i \qquad (i = 1, 2 \cdots n) \tag{5.9–6}$$

These n equations are the complete Lagrangian equations of motion.

One of the principle advantages of the use of Lagrange's equations, in addition to the automatic elimination of constraints, is the ease with which complex systems may be treated without recourse to detailed kinematic analysis. The displacements and the velocities which are usually easily obtained suffice for the computation of T, V, and F. These quantities are always positive and the question of signs, often troublesome, is avoided. The determination of the absolute accelerations, required if Newton's laws are applied directly, is avoided completely. These advantages help to make Lagrange's equations one of the most powerful tools available to the engineer in the analysis of complicated vibrating systems.

5.10. Application of Lagrange's Equation to a Centrifugal Pendulum

The system shown in Fig. 5–11 is called a compound centrifugal pendulum. Pendulums of this type are frequently used as torsional-vibration suppressors. The analysis of this system using Lagrange's equations is particularly simple and direct. The inertia I_1 has a

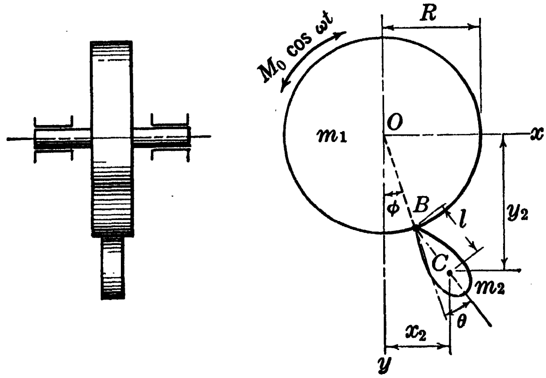


FIG. 5–11

uniform rotation Ω rad per sec plus an oscillation due to the impressed torque. The generalized coördinate for I_1 may be taken as ϕ where

$$q_1 = \phi = \Omega t + \psi$$

and ψ is the oscillatory motion. The angle θ may be selected as the generalized coordinate q_2 for the pendulum. It is seen that ϕ and θ completely determine the configuration of the system and that there are no conditions of restraint between these coordinates.

Equations 5.8–2, for this specific example, take the form:

$$x_1 = 0 \qquad\qquad y_1 = 0$$

$$x_2 = R \sin \phi + l \sin (\phi + \theta) \qquad y_2 = R \cos \phi + l \cos (\phi + \theta)$$

The velocity of the centroid O of the mass m_1 is zero, and the components of the velocity of the centroid C of the pendulum are

$$\dot{x}_2 = R\phi \cos \phi + l(\phi + \dot{\theta}) \cos (\phi + \theta)$$

$$\dot{y}_2 = -R\phi \sin \phi - l(\phi + \dot{\theta}) \sin (\phi + \theta)$$

The angular velocities of the masses m_1 and m_2 are ϕ and $\phi + \dot{\theta}$, respectively; hence the kinetic energy of the system may be expressed as

$$T = \tfrac{1}{2}I_1\phi^2 + \tfrac{1}{2}I_2(\phi + \dot{\theta})^2 + \tfrac{1}{2}m_2(\dot{x}_2{}^2 + \dot{y}_2{}^2)$$

where I_1 and I_2 are the moments of inertia of the masses m_1 and m_2 about their respective centroids. These may be conveniently expressed in terms of the corresponding radii of gyration, thus,

$$I_1 = m_1\bar{r}_1^{\,2} \qquad I_2 = m_2\bar{r}_2^{\,2}$$

Substitution for the moments of inertia and the Cartesian coordinates leads to

$$T = \tfrac{1}{2}m_1\bar{r}_1^{\,2}\dot{\phi}^2 + \tfrac{1}{2}m_2\bar{r}_2^{\,2}(\dot{\phi} + \dot{\theta})^2 + \tfrac{1}{2}m_2[R^2\dot{\phi}^2 + l^2(\dot{\phi} + \dot{\theta})^2 + 2Rl\dot{\phi}(\dot{\phi} + \dot{\theta}) \cos \theta]$$

whence

$$\frac{\partial T}{\partial \dot{\phi}} = m_1\bar{r}_1^{\,2}\dot{\phi} + m_2\bar{r}_2^{\,2}(\dot{\phi} + \dot{\theta}) + m_2[R^2\dot{\phi} + l^2(\dot{\phi} + \dot{\theta}) + Rl(2\dot{\phi} + \dot{\theta}) \cos \theta]$$

$$\frac{\partial T}{\partial \dot{\theta}} = m_2\bar{r}_2^{\,2}(\dot{\phi} + \dot{\theta}) + m_2[l^2(\dot{\phi} + \dot{\theta}) + Rl\dot{\phi} \cos \theta]$$

$$\frac{\partial T}{\partial \phi} = 0$$

$$\frac{\partial T}{\partial \theta} = -m_2Rl\dot{\phi}(\dot{\phi} + \dot{\theta}) \sin \theta$$

The only external force or moment that acts on the system is $M_0 \cos \omega t$; hence

$$V = 0$$

Assuming the damping to be negligible, it also follows that

$$F = 0$$

Since $\phi = \Omega t + \psi$, it follows that

$$\dot{\phi} = \Omega + \dot{\psi} \quad \text{and} \quad \ddot{\phi} = \ddot{\psi}$$

The equation of motion for m_1 is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = M_0 \cos \omega t$$

or

$$\begin{aligned} &m_1\bar{r}_1^{\,2}\ddot{\psi} + m_2\bar{r}_2^{\,2}(\ddot{\psi} + \ddot{\theta}) \\ &+ m_2[R^2\ddot{\psi} + l^2(\ddot{\psi} + \ddot{\theta}) + Rl(2\ddot{\psi} + \ddot{\theta}) \cos \theta - Rl(2\Omega + 2\dot{\psi} + \dot{\theta})\dot{\theta} \sin \theta] \\ &= \{m_1\bar{r}_1^{\,2} + m_2[R^2 + l^2 + 2Rl \cos \theta + \bar{r}_2^{\,2}]\}\ddot{\psi} \\ &\qquad + [m_2(\bar{r}_2^{\,2} + l^2 + Rl \cos \theta)]\ddot{\theta} - m_2Rl(2\Omega + 2\dot{\psi} + \dot{\theta})\dot{\theta} \sin \theta \\ &= M_0 \cos \omega t \end{aligned}$$

The equation of motion for m_2 is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = 0$$

or

$$\begin{aligned} m_2\bar{r}_2^2(\ddot{\psi} + \ddot{\theta}) + m_2[l^2(\ddot{\psi} + \ddot{\theta}) + Rl\ddot{\psi}\cos\theta - Rl(\Omega + \dot{\psi})\dot{\theta}\sin\theta] \\ + m_2Rl(\Omega + \dot{\psi})(\Omega + \dot{\psi} + \dot{\theta})\sin\theta \\ = [m_2(\bar{r}_2^2 + l^2 + Rl\cos\theta)]\ddot{\psi} + [m_2(\bar{r}_2^2 + l^2)]\ddot{\theta} \\ + m_2Rl(\Omega^2 + 2\Omega\dot{\psi} + \dot{\psi}^2)\sin\theta = 0 \end{aligned}$$

These equations are valid for motion through large values of ψ and θ; however their solution is not readily obtainable by elementary methods.[1] If the problem is restricted to small oscillations, that is, if

$$\sin\theta \sim \theta, \quad \cos\theta \sim 1, \quad \text{and} \quad \dot{\theta}^2,\ \dot{\theta}\dot{\psi},\ \dot{\theta}\theta,\ \dot{\psi}^2,\ \dot{\psi}\theta \sim 0$$

then the equations of motion reduce to

$$\{m_1\bar{r}_1^2 + m_2[(R + l)^2 + \bar{r}_2^2]\}\ddot{\psi} + [m_2(\bar{r}_2^2 + l^2 + Rl)]\ddot{\theta} = M_0\cos\omega t$$

$$[m_2(\bar{r}_2^2 + l^2 + Rl)]\ddot{\psi} + [m_2(\bar{r}_2^2 + l^2)]\ddot{\theta} + m_2Rl\Omega^2\theta = 0$$

Assuming that the solutions may be written in the form,

$$\psi = \Psi\cos\omega t; \qquad \theta = \Theta\cos\omega t$$

and substituting into the two equations of motion gives the algebraic equations,

$$-\{m_1\bar{r}_1^2 + m_2[(R + l)^2 + \bar{r}_2^2]\}\omega^2\Psi - [m_2(\bar{r}_2^2 + l^2 + Rl)]\omega^2\Theta = M_0$$

$$-[m_2(\bar{r}_2^2 + l^2 + Rl)]\omega^2\Psi + [-m_2(\bar{r}_2^2 + l^2)\omega^2 + m_2Rl\Omega^2]\Theta = 0$$

To obtain the natural frequencies of the system, the exciting torque may be set equal to zero and $\omega = p$, whence

$$\begin{aligned} \frac{\Delta}{m_2^2} = \left\{\left[\frac{m_1}{m_2}\bar{r}_1^2 + (R + l)^2 + \bar{r}_2^2\right](\bar{r}_2^2 + l^2) - (\bar{r}_2^2 + l^2 + Rl)^2\right\}p^4 \\ - \left\{\left[\frac{m_1}{m_2}\bar{r}_1^2 + (R + l)^2 + \bar{r}_2^2\right]Rl\Omega^2\right\}p^2 = 0 \end{aligned}$$

or

$$p_1^2 = 0$$

[1] For a solution valid for large amplitudes see J. P. Den Hartog, *Stephen Timoshenko, 60th Anniversary Volume*, Macmillan, and *Evaluation of Effects of Torsional Vibration*, SAE War Engineering Board, 1945.

and

$$p_2^2 = \frac{Rl\Omega^2\left[\frac{m_1}{m_2}\bar{r}_1^2 + (R+l)^2 + \bar{r}_2^2\right]}{\left[\frac{m_1}{m_2}\bar{r}_1^2 + (R+l)^2 + \bar{r}_2^2\right](\bar{r}_2^2 + l^2) - (\bar{r}_2^2 + l^2 + Rl)^2}$$

For $\bar{r}_2 = 0$ and $m_1 = \infty$ this reduces to the result of section 3.6. The value $p_1^2 = 0$ corresponds to a rigid body rotation and is of no interest here.

For the forced-vibration problem, $M_0 \neq 0$, the amplitudes are given by

$$\Psi = m_2[(\bar{r}_2^2 + l^2)\omega^2 - Rl\Omega^2]\frac{M_0}{\Delta}$$

$$\Theta = m_2(\bar{r}_2^2 + l^2 + Rl)\omega^2\frac{M_0}{\Delta}$$

It should be noted that, when

$$\frac{R}{l + \frac{\bar{r}_2^2}{l}} = \frac{R}{l_e} = \left(\frac{\omega}{\Omega}\right)^2$$

then $\Psi = 0$, and the pendulum is said to be "tuned" to suppress the motion of mass m_1.

5.11 Application of Lagrange's Equations to a Dissipative System

As an example of a dissipative system, the bell and clapper of Fig. 5–12 may be considered. The bearings for the bell and the clapper at O_1 and O_2 will give rise to viscous friction if they are lubricated. The activation of the bell is accomplished by an oscillating torque about the axis through O_1. The generalized coordinates may be selected as θ_1 and θ_2, since they are independent of any constraints and completely define the configuration of the system, thus,

$$q_1 = \theta_1 \quad \text{and} \quad q_2 = \theta_2$$

The coordinate axes are conveniently selected as shown. The relations between the Cartesian coordinates x and y and the generalized coordinates for the centroids of both the bell and clapper are seen to be:

$$x_1 = d_1 \sin\theta_1 \qquad y_1 = d_1 \cos\theta_1$$

$$x_2 = -a\sin\theta_1 + d_2\sin\theta_2 \qquad y_2 = -a\cos\theta_1 + d_2\cos\theta_2$$

The velocities of the centroids are, therefore:

$$\dot{x}_1 = d_1\cos\theta_1\dot{\theta}_1 \qquad \dot{y}_1 = -d_1\sin\theta_1\dot{\theta}_1$$

$$\dot{x}_2 = -a\cos\theta_1\dot{\theta}_1 + d_2\cos\theta_2\dot{\theta}_2 \qquad \dot{y}_2 = a\sin\theta_1\dot{\theta}_1 - d_2\sin\theta_2\dot{\theta}_2$$

The kinetic energy of the system may be expressed as

$$T = \tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) + \tfrac{1}{2}I_1\dot{\theta}_1^2 + \tfrac{1}{2}I_2\dot{\theta}^2$$

where I_1 and I_2 are the moments of inertia of the two masses about their cen-

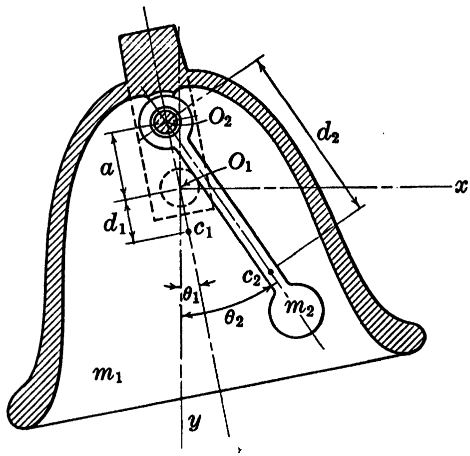


FIG. 5–12

troids. These can be expressed in terms of the radii of gyration as

$$I_1 = m_1\bar{r}_1^2 \qquad I_2 = m_2\bar{r}_2^2$$

Substitution for the Cartesian coordinates gives

$$T = \tfrac{1}{2}m_1(d_1^2 + \bar{r}_1^2)\dot{\theta}_1^2 + \tfrac{1}{2}m_2[a^2\dot{\theta}_1^2 - 2ad_2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 + (d_2^2 + \bar{r}_2^2)\dot{\theta}_2^2]$$

whence

$$\frac{\partial T}{\partial \dot{\theta}_1} = m_1(d_1^2 + \bar{r}_1^2)\dot{\theta}_1 + m_2[a^2\dot{\theta}_1 - ad_2\cos(\theta_1 - \theta_2)\dot{\theta}_2]$$

$$\frac{\partial T}{\partial \dot{\theta}_2} = m_2[-ad_2\cos(\theta_1 - \theta_2)\dot{\theta}_1 + (d_2^2 + \bar{r}_2^2)\dot{\theta}_1^2]$$

$$\frac{\partial T}{\partial \theta_1} = -\frac{\partial T}{\partial \theta_2} = m_2ad_2\sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2$$

The potential energy is due entirely to the change in elevation of the centroids from the equilibrium position. Thus

$$\begin{aligned} V &= m_1g(d_1 - y_1) + m_2g[(d_2 - a) - y_2] \\ &= m_1gd_1(1 - \cos\theta_1) + m_2g[-a(1 - \cos\theta_1) + d_2(1 - \cos\theta_2)] \end{aligned}$$

The damping is due to differences in the angular velocities, and so the dissipation function is seen to be

$$F = \tfrac{1}{2}c_1\dot{\theta}_1^2 + \tfrac{1}{2}c_2(\dot{\theta}_1 - \dot{\theta}_2)^2$$

where c_1 and c_2 are the angular damping constants about the axis O_1 and O_2, respectively. The generalized force is that which activates the bell and may be expressed as

$$Q_1 = M_0 \cos \omega t$$

The clapper is assumed to be free of external forces which are a function of time, and hence

$$Q_2 = 0$$

The equations of motion are obtained by substituting the above expressions for T, V, F, and Q_1 into equations 5.9–5. Thus:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = m_1(d_1^2 + \bar{r}_1^2)\ddot{\theta}_1 + m_2[a^2\ddot{\theta}_1 + ad_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\dot{\theta}_2 - ad_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2]$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = m_2[ad_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)\dot{\theta}_1 + ad_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 + (d_2^2 + \bar{r}_2^2)\ddot{\theta}_2]$$

$$\frac{\partial V}{\partial \theta_1} = m_1 g d_1 \sin \theta_1 - m_2 g a \sin \theta_1$$

$$\frac{\partial V}{\partial \theta_2} = m_2 g d_2 \sin \theta_2$$

$$\frac{\partial F}{\partial \dot{\theta}_1} = c_1\dot{\theta}_1 + c_2(\dot{\theta}_1 - \dot{\theta}_2) \qquad \frac{\partial F}{\partial \dot{\theta}_2} = -c_2(\dot{\theta}_1 - \dot{\theta}_2)$$

Substitution of the above into equation 5.9–5 gives, after cancelation,

$$[m_1(d_1^2 + \bar{r}_1^2) + m_2a^2]\ddot{\theta}_1 - m_2ad_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 + (c_1 + c_2)\dot{\theta}_1 - c_2\dot{\theta}_2 - m_2ad_2 \sin(\theta_1 - \theta_2)\dot{\theta}_2^2 + (m_1d_1 - m_2a)g \sin \theta_1 = M_0 \cos \omega t$$

$$-m_2ad_2 \cos(\theta_1 - \theta_1)\ddot{\theta}_1 + m_2(d_2^2 + \bar{r}_2^2)\ddot{\theta}_2 + m_2ad_2 \sin(\theta_1 - \dot{\theta}_2)\dot{\theta}_1^2 - c_2\dot{\theta}_1 + c_2\dot{\theta}_2 + mgd_2 \sin \theta_2 = 0$$

These equations are independent of any assumptions as to the magnitude of the oscillation of the bell or clapper. Their solution, however, is not readily obtainable. For oscillations of small magnitude, an approximate solution may be obtained by recognizing the following:

$$\sin \theta_1 \sim \theta_1 \qquad \cos(\theta_1 - \theta_2) \sim 1 \qquad \sin \theta_2 \sim \theta_2$$

and

$$\dot{\theta}_1^2 \sim 0 \qquad \dot{\theta}_2^2 \sim 0$$

Thus, for small oscillations the equations of motion take the form

$$[m_1(d^2 + \bar{r}_1^2) + m_2a^2]\ddot{\theta}_1 - m_2ad_2\ddot{\theta}_2 + (c_1 + c_2)\dot{\theta}_1 - c_2\dot{\theta}_2 + (m_1d_1 - m_2a)g\theta_1 = M_0 \cos \omega t$$

$$-m_2ad_2\ddot{\theta}_1 + m_2(d_2^2 + \bar{r}_2^2)\ddot{\theta}_2 - c_2(\dot{\theta}_1 - \dot{\theta}_2) + m_2gd_2\theta_2 = 0$$

These equations may be treated most easily by rewriting them in the complex form, introducing

$$\mathbf{z}_1 = \theta_1 + j\psi_1 \qquad \mathbf{z}_2 = \theta_2 + j\psi_2$$

and

$$M_0 e^{j\omega t} = M_0(\cos \omega t + j \sin \omega t)$$

Thus

$$[m_1(d_1^2 + \bar{r}_1^2) + m_2 a^2]\ddot{\mathbf{z}}_1 - m_2 a d_2 \ddot{\mathbf{z}}_2 + (c_1 + c_2)\dot{\mathbf{z}}_1 - c_2\dot{\mathbf{z}}_2 + (m_1 d_1 - m_2 a) g \mathbf{z}_1 = M_0 e^{j\omega t}$$

$$-m_2 a d_2 \ddot{\mathbf{z}}_1 + m_2(d_2^2 + \bar{r}_2^2)\ddot{\mathbf{z}}_2 - c_2\dot{\mathbf{z}}_1 + c_2\dot{\mathbf{z}}_2 + m_2 g d_2 \mathbf{z}_2 = 0$$

Assuming as in previous examples that the solutions may be taken as

$$\mathbf{z}_1 = \Theta_1 e^{j\omega t}$$

$$\mathbf{z}_2 = \Theta_2 e^{j\omega t}$$

leads to the algebraic equations,

$$\{(m_1 d_1 - m_2 a)g - [m_1(d_1^2 + \bar{r}_1^2) + m_2 a^2]\omega^2 + j\omega(c_1 + c_2)\}\Theta_1 + (m_2 a d_2 \omega^2 - j\omega c_2)\Theta_2 = M_0$$

$$(m_2 a d_2 \omega^2 - j\omega c_2)\Theta_1 + [m_2 g d_2 - m_2(d_2^2 + \bar{r}_2^2)\,\omega^2 + j\omega c_2]\Theta_2 = 0$$

Introduction of the determinant of these equations,

$$\Delta = \{(m_1 d_1 - m_2 a)g - [m_1(d_1^2 + \bar{r}_1^2) + m_2 a^2]\omega^2 + j\omega(c_1 + c_2)\} \times [m_2 g d_2 - m_2(d_2^2 + \bar{r}_2^2) + j\omega c_2] - (m_2 a d_2 \omega^2 - j\omega c_2)^2$$

permits the complex amplitude of the bell to be written as

$$\Theta_1 = [m_2 g d_2 - m_2(d_2^2 + \bar{r}_2^2)\omega^2 + j\omega c_2]\,\frac{M_0}{\Delta}$$

and of the clapper

$$\Theta_2 = -(m_2 a d_2 \omega^2 - j\omega c_2)\,\frac{M_0}{\Delta}$$

The difference in the complex amplitudes or the relative complex amplitude between the bell and clapper is seen to be

$$\Theta_1 - \Theta_2 = m_2[(a\omega^2 + g)d_2 - (d_2^2 + \bar{r}_2^2)\omega^2]\,\frac{M_0}{\Delta}$$

Thus, when

$$\omega^2 = \frac{d_2}{d_2^2 + \bar{r}_2^2 - a d_2}\,g$$

there will be no relative motion and the bell will not ring. In fact, there will be a range of frequencies over whieh M_0 must be very large to insure functioning of the bell. This critical range throughout which the bell is "dead" is determined by the equation $\Theta_1 - \Theta_2 = \alpha < \alpha_0$ where α is the relative displacement between the bell and clapper and α_0 is the value of α at contact. In general, α is complex, indicating a phase angle between the bell and clapper. The absolute magnitude of α may be computed in a manner identical with that used for the damped vibration absorber in section 5.5. The relationship between ω and M_0 for each value of the absolute magnitude α may then be calculated.

It is well to realize that the method of solution used in this example depends on small amplitudes. This assumption gives excellent results when the amplitudes are in fact small. For larger amplitudes the solution may still give a good approximation to the motion. A measure of the size of the amplitude that may be permitted and sufficient accuracy still be retained for practical purposes is furnished in the problem of the pendulum at large amplitudes. This problem is discussed in detail in Chapter 10.

It must not be supposed that the solution of this section is valid if there is impact between the bell and clapper even though at small amplitudes. Impact between the bell and clapper gives rise to interacting forces not contemplated in the above analysis.

Chapter 6

THE MOBILITY METHOD AND USE OF THE COMPLEX VARIABLE

6.1. The Vector and Its Complex Representation

The representation of simple harmonic motion by a rotating vector was indicated in Chapter 1. In the treatment that followed, it was shown that systems of one or more degrees of freedom vibrate with a simple harmonic motion. This is always true whenever

(*a*) The restoring forces are proportional to the displacement.
(*b*) The damping forces are proportional to the velocity.
(*c*) The masses or moments of inertia are constant.

Systems of this character are linear systems and inherently involve vector quantities. Because this type of system is so common, more expedient methods have been found to treat these vector properties. Such a method is offered by the elementary theory of complex variables.

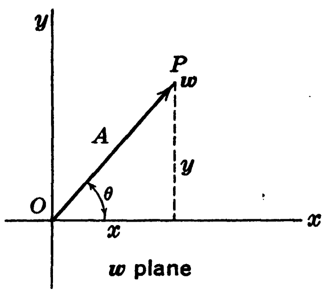


FIG. 6–1

A vector may be represented in the complex plane as shown in Fig. 6–1. The coordinates of the point P can be represented by a single complex number **w**,

$$\mathbf{w} = x + jy \qquad (6.1\text{–}1)$$

where j is the imaginary unit ($j^2 = -1$). The imaginary part of the complex number **w** is plotted as an ordinate and the real part of **w** as an abscissa. The point P with coordinate $\mathbf{w} = x + jy$ completely defines the vector. In fact, it is common to speak of the vector **w** as the vector that extends from the origin to the point P. Thus the unit 1 is a unit vector in the direction of the positive real axis and j is a unit vector in the direction of the positive so-called imaginary axis.[1]

[1] Vectors in general will be denoted by boldface type. Exceptions to this rule are the unit vectors 1 and j.

It is frequently more convenient to represent the vector **w** in the form

$$\mathbf{w} = A(\cos \theta + j \sin \theta) \tag{6.1–2}$$

where

$$x = A \cos \theta \quad \text{and} \quad y = A \sin \theta$$

Introduction of Euler's formula from calculus,

$$e^{j\theta} = \cos \theta + j \sin \theta$$

permits equation 6.1–2 to be written as

$$\mathbf{w} = Ae^{j\theta} \tag{6.1–3}$$

This is known as the polar form of **w** where A is the amplitude and θ is the argument of **w**. A vector written in the Cartesian form of equation 6.1–1 may always be transformed to the polar form by means of the relationships,

$$A = \sqrt{x^2 + y^2}$$

and

$$\tan \theta = \frac{y}{x}$$

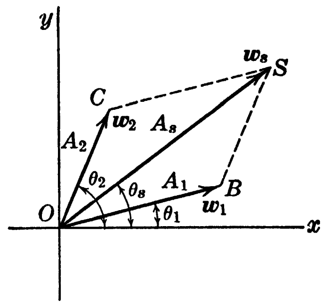


FIG. 6–2

Vectors are added and subtracted by the direct addition and subtraction of the complex numbers by which they are represented. In Fig. 6–2 let the vector $\overline{OB} = \mathbf{w}_1$, $\overline{OC} = \mathbf{w}_2$, and $\overline{OS} = \mathbf{w}_s$. Then

$$\mathbf{w}_1 + \mathbf{w}_2 = (x_1 + x_2) + j(y_1 + y_2) = \mathbf{w}_s = x_s + jy_s$$

$\mathbf{w}_s$ may also be expressed in polar form as

$$\mathbf{w}_1 + \mathbf{w}_2 = A_1e^{j\theta_1} + A_2e^{j\theta_2} = A_se^{j\theta_s} \equiv A_s\underline{/\theta_s}$$

where

$$A_s = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} = \sqrt{x_s^2 + y_s^2}$$

$$= \sqrt{A_1^2 + A_2^2 + 2A_1A_2 \cos (\theta_1 - \theta_2)}$$

$$\theta_s = \tan^{-1} \frac{y_1 + y_2}{x_1 + x_2} = \tan^{-1} \frac{A_1 \sin \theta_1 + A_2 \sin \theta_2}{A_1 \cos \theta_1 + A_2 \cos \theta_2}$$

Subtraction may be accomplished in a similar manner. In Fig. 6–3 let $\overline{OB} = \mathrm{w}_1$, $\overline{OC} = \mathrm{w}_2$, and $\overline{OD} = \mathrm{w}_d$.

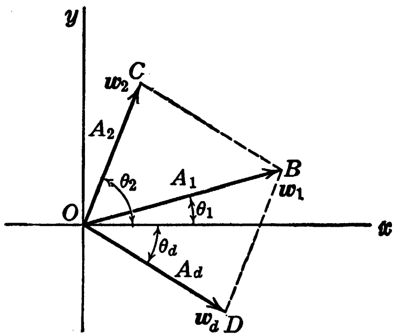


FIG. 6–3

Then

$$\mathrm{w}_1 - \mathrm{w}_2 = (x_1 - x_2) + j(y_1 - y_2) = \mathrm{w}_d = x_d + jy_d$$

or, in polar form,

$$\mathrm{w}_1 - \mathrm{w}_2 = A_1 e^{j\theta_1} - A_2 e^{j\theta_2} = \mathrm{w}_d = A_d e^{j\theta_d} \equiv A_d \underline{/\theta_d}$$

where

$$A_d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{x_d^2 + y_d^2}$$

$$= \sqrt{A_1^2 + A_2^2 - 2A_1A_2 \cos(\theta_1 - \theta_2)}$$

$$\theta_d = \tan^{-1} \frac{y_1 - y_2}{x_1 - x_2} = \tan^{-1} \frac{A_1 \sin \theta_1 - A_2 \sin \theta_2}{A_1 \cos \theta_1 - A_2 \cos \theta_2}$$

If several vectors are added and subtracted such as

$$\mathrm{w} = \mathrm{w}_1 + \mathrm{w}_2 - \mathrm{w}_3$$

then, in the Cartesian form,

$$\mathrm{w} = (x_1 + x_2 - x_3) + j(y_1 + y_2 - y_3)$$

whereas, in the polar form,

$$\mathrm{w} = \sqrt{(x_1 + x_2 - x_3)^2 + (y_1 + y_2 - y_3)^2}\, e^{j \tan^{-1} \frac{y_1 + y_2 - y_3}{x_1 + x_2 - x_3}}$$

The multiplication of vectors in Cartesian form is carried out directly. Thus, with reference to Fig. 6–4, the product is

$$\mathrm{w}_p = \mathrm{w}_1\mathrm{w}_2 = (x_1 + jy_1)(x_2 + jy_2) = (x_1x_2 - y_1y_2) + j(x_2y_1 + x_1y_2)$$

$$= x_p + jy_p$$

In polar form, this product becomes

$$\mathbf{w}_p = \mathbf{w}_1\mathbf{w}_2 = A_1e^{j\theta_1}A_2e^{j\theta_2} = A_1A_2e^{j(\theta_1+\theta_2)} = A_pe^{j\theta_p} \equiv A_p\underline{/\theta_p}$$

where

$$A_p = \sqrt{x_p^2 + y_p^2} = \sqrt{(x_1x_2 - y_1y_2)^2 + (x_2y_1 + x_1y_2)^2}$$
$$= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} = A_1A_2$$

and

$$\tan\theta_p = \frac{x_2y_1 + x_1y_2}{x_1x_2 - y_1y_2} = \frac{\dfrac{y_1}{x_1} + \dfrac{y_2}{x_2}}{1 - \dfrac{y_1}{x_1}\dfrac{y_2}{x_2}}$$
$$= \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1\tan\theta_2} = \tan(\theta_1 + \theta_2)$$

The multiplication of vectors is accomplished therefore, by multiplying their amplitudes and adding their arguments.

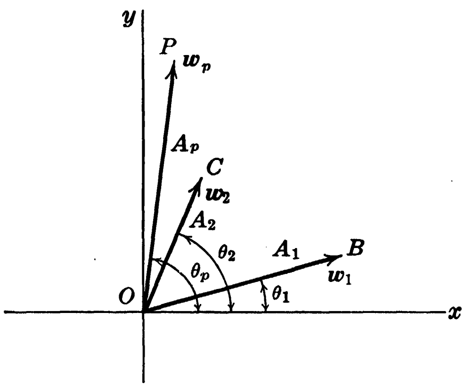


FIG. 6–4

Division of vectors is accomplished in a similar fashion. Thus the quotient (Fig. 6–5) is

$$\mathbf{w}_q = \frac{\mathbf{w}_1}{\mathbf{w}_2} = \frac{x_1 + jy_1}{x_2 + jy_2} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{(x_2 + jy_2)(x_2 - jy_2)}$$
$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + j\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2} = x_q + jy_q$$

or, in polar form,

$$\mathbf{w}_q = \frac{\mathbf{w}_1}{\mathbf{w}_2} = \frac{A_1e^{j\theta_1}}{A_2e^{j\theta_2}} = \frac{A_1}{A_2}e^{j(\theta_1-\theta_2)} = A_qe^{j\theta_q} \equiv A_q\underline{/\theta_q}$$

where

$$A_q = \sqrt{x_q^2 + y_q^2} = \sqrt{\frac{(x_1x_2 + y_1y_2)^2 + (x_2y_1 - x_1y_2)^2}{(x_2^2 + y_2^2)^2}}$$

$$= \sqrt{\frac{x_1^2 + y_1^2}{x_2^2 + y_2^2}} = \frac{A_1}{A_2}$$

and

$$\tan\theta_q = \frac{x_2y_1 - x_1y_2}{x_1x_2 + y_1y_2} = \frac{\dfrac{y_1}{x_1} - \dfrac{y_2}{x_2}}{1 + \dfrac{y_1}{x_1}\dfrac{y_2}{x_2}} = \tan(\theta_1 - \theta_2)$$

The division of vectors is accomplished, therefore, by obtaining the

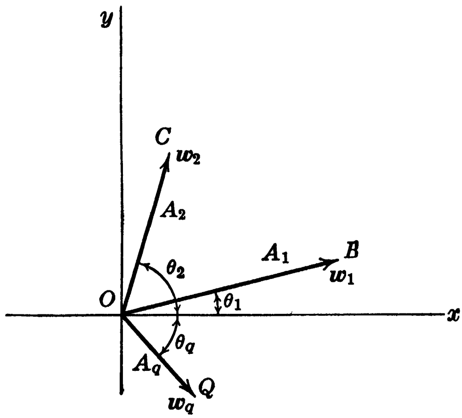


FIG. 6–5

ratio of the amplitudes and the difference of the arguments. It will be noted that addition and subtraction of vectors are accomplished most readily in the Cartesian form, whereas the polar form is the most convenient for division and multiplication.

In complex notation, a vector which rotates at a uniform angular velocity ω, has the form

$$\mathbf{w} = Ae^{j\omega t} \equiv A\underline{/\omega t} \qquad (6.1\text{–}4)$$

The real part of the complex number $\mathbf{w}$ is the projection of the vector on the x axis. It is seen that

$$x = A\cos\omega t \qquad (6.1\text{–}5)$$

and thus the real part of $\mathbf{w}$ may be used to represent a simple harmonic motion. It is not necessary to write the equation for the real part only, although it is the real part that has direct physical interpretation in the following treatment. It is much more convenient to indicate simple harmonic motion in the complete vector form (equation 6.1–4) than to base the analysis on the real part only (equation 6.1–5). The complex vector **w** is the complex displacement, and A is the amplitude (a real number) of the motion whereas the angle ωt is the argument of **w**.

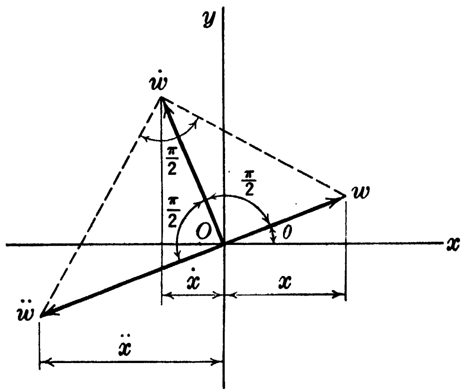


FIG. 6–6

The complex velocity is the time derivative of **w**,

$$\begin{aligned}\dot{\mathbf{w}} &= \dot{x} + j\dot{y} = Aj\omega e^{j\omega t}\\ &= Aj\omega(\cos\omega t + j\sin\omega t)\\ &= A\omega(j\cos\omega t - \sin\omega t)\end{aligned} \qquad (6.1\text{–}6)$$

The actual velocity is the real part of $\dot{\mathbf{w}}$, that is,

$$\dot{x} = -A\omega\sin\omega t = A\omega\cos\left(\omega t + \frac{\pi}{2}\right)$$

as is shown graphically in Fig. 6–6. The complex acceleration is obtained by a second differentiation with respect to time,

$$\begin{aligned}\ddot{\mathbf{w}} &= \ddot{x} + j\ddot{y} = -A\omega^2 e^{j\omega t}\\ &= -A\omega^2(\cos\omega t + j\sin\omega t)\end{aligned} \qquad (6.1\text{–}7)$$

The real part is the acceleration of a simple harmonic motion, thus

$$\ddot{x} = -A\omega^2\cos\omega t = A\omega^2\cos(\omega t + \pi)$$

as shown in Fig. 6–6.

For example an oscillating force may be written in complex form as

$$\mathbf{P} = \mathbf{P}_0 e^{j\omega t} = P_0 e^{j(\omega t+\phi)}$$

where $\mathbf{P}_0$ is the complex amplitude of the force vector $\mathbf{P}$, P_0 is the magnitude of the force $\mathbf{P}$, and ϕ is the phase angle between the force $\mathbf{P}$ and the rotating vector $e^{j\omega t}$, that is,

$$\mathbf{P}_0 = P_0 e^{j\phi}$$

Other examples of the complex representation of physical quantities will be introduced later.

The great advantage of this procedure is that it leads to a concise analysis and reduces the solution of the differential equations to a simple mechanical process.

6.2. Impedance of the Elements of Vibrating Systems

It is important to characterize the effect of an oscillating force on the three fundamental elements in a vibratory system. Each of these idealized elements, as defined in section 1.2, will be considered individually.

Spring element

Figure 6–7 shows an oscillating force acting directly on a spring.

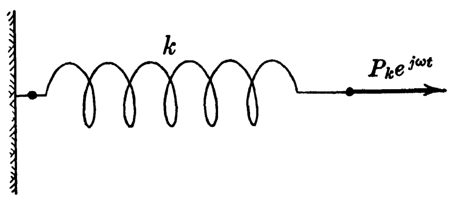


FIG. 6–7

The equation of motion for the displacement $\mathbf{W}_k$ may be written in complex form as

$$k\mathbf{W}_k = P_k e^{j\omega t}$$

or

$$\mathbf{W}_k = \frac{P_k}{k} e^{j\omega t}$$

The complex displacement $\mathbf{w}_k$ due to an oscillating force of unit amplitude has the form

$$\mathbf{w}_k = \frac{1}{k} e^{j\omega t} = \mathbf{a}_k e^{j\omega t} \qquad (6.2\text{–}1)$$

where $\mathbf{a}_k$ is the amplitude across the spring between terminals for a unit force.

The velocity and acceleration are, respectively,

$$\dot{\mathbf{w}}_k = \frac{j\omega}{k}\, e^{j\omega t} \quad \text{and} \quad \ddot{\mathbf{w}}_k = -\frac{\omega^2}{k}\, e^{j\omega t} \tag{6.2–2}$$

The reciprocal of the amplitude due to a unit oscillating force is denoted by **z**. Thus, for a simple spring element,

$$\mathbf{z}_k = \frac{1}{\mathbf{a}_k} = k \tag{6.2–3}$$

is referred to as the dynamic stiffness or the impedance which is the amplitude of the exciting force per unit displacement amplitude.

Viscous damper

Figure 6–8 indicates a damper subjected to an oscillating force.

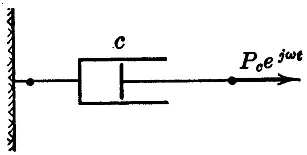


FIG. 6–8

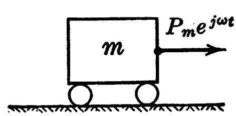


FIG. 6–9

The equation of motion may be written as

$$c\dot{\mathbf{W}}_c = P_c e^{j\omega t}$$

or, in terms of unit force,

$$\dot{\mathbf{w}}_c = \frac{1}{c}\, e^{j\omega t} \tag{6.2–4}$$

To obtain the displacement, equation 6.2–4 must be integrated to give

$$\mathbf{w}_c = \frac{1}{j\omega c}\, e^{j\omega t} = \mathbf{a}_c e^{j\omega t} \tag{6.2–5}$$

The impedance across the damper is therefore

$$\mathbf{z}_c = \frac{1}{\mathbf{a}_c} = j\omega c \tag{6.2–6}$$

The acceleration may be obtained by differentiation of equation 6.2–4

$$\ddot{\mathbf{w}}_c = \frac{j\omega}{c}\, e^{j\omega t} \tag{6.2–7}$$

Mass

An oscillating force acting on a mass is pictured in Fig. 6–9. The equation of motion has the form

$$m\ddot{\mathbf{W}}_m = P_m e^{j\omega t}$$

which, for a unit oscillating force, becomes

$$\ddot{\mathbf{w}}_m = \frac{1}{m} e^{j\omega t} \tag{6.2–8}$$

Integration gives the velocity as

$$\dot{\mathbf{w}}_m = \frac{1}{j\omega m} e^{j\omega t} \tag{6.2–9}$$

A second integration yields the complex displacement as

$$\mathbf{w}_m = \frac{1}{-\omega^2 m} e^{j\omega t} = \mathbf{a}_m e^{j\omega t} \tag{6.2–10}$$

The impedance of a mass m is then

$$\mathbf{z}_m = \frac{1}{\mathbf{a}_m} = -\omega^2 m \tag{6.2–11}$$

The results obtained in the above paragraphs are conveniently summarized in tabular and graphical form in Table 2. It must be recognized that the retention of the imaginary unit j which appears in the table of amplitudes is needed to indicate its phase relationship to the force. This will be seen from the identity

$$j = e^{j(\pi/2)}$$

and the fact that the magnitude of the vector j is unity. The information contained in Table 2 is basic to the development that follows. A thorough understanding of the table and its graphical representation is essential if full advantage is to be had from the use of the complex variable in vibration analysis.

It will be necessary to establish a clear-cut and convenient notation to distinguish between amplitudes per unit force or moment and total amplitudes of the various elements and combinations thereof. To this end the notation of this chapter is summarized below:

$\mathbf{A}$ = total translatory amplitude

$\boldsymbol{\Theta}$ = total torsional amplitude

$$\frac{1}{\mathbf{z}} = \begin{cases} \mathbf{a} = \text{translatory amplitude per unit force} \\ \boldsymbol{\theta} = \text{torsional amplitude per unit torque} \end{cases}$$

whence

$$\mathbf{A} = \mathbf{P}\mathbf{a} \qquad \boldsymbol{\Theta} = \mathbf{M}\boldsymbol{\theta} \tag{6.2–12}$$

$$\mathbf{P} = \mathbf{A}\mathbf{z} \qquad \mathbf{M} = \boldsymbol{\Theta}\mathbf{z} \tag{6.2–13}$$

$A_m, a_m, \Theta_l, \theta_l$ = absolute amplitudes of a mass element
$A_k, a_k, \Theta_k, \theta_k$ = relative amplitudes across (between terminals) of a spring element
$A_c, a_c, \Theta_c, \theta_c$ = relative amplitudes across (between terminals) of a damping element
$A_r, a_r, \Theta_r, \theta_r$ = absolute amplitude of terminal r
$A_{rs}, a_{rs}, \Theta_{rs}, \theta_{rs}$ = relative amplitude across (between) terminals r and s.

The evaluation of the unit displacement amplitude **a** and the unit impedance **z** of series and parallel systems of elements follows.

Systems in series

Consider the system in series shown in Fig. 6–10. This system is similar to the one shown in Fig. 4–5. To obtain the amplitude of point O, due to a unit oscillating force applied at O the amplitudes per unit force of each element are summed vectorially, as indicated in Fig. 6–10. Thus

$$\mathbf{a}_0 = \mathbf{a}_{k1} + \mathbf{a}_c + \mathbf{a}_{k2} + \mathbf{a}_m$$

or, in general for any system in series,

$$\mathbf{a}_0 = \Sigma\mathbf{a} \qquad (6.2\text{–}14)$$

The amplitude at O due to the oscillating force,

$$\mathbf{P}_0 e^{j\omega t}$$

is therefore given by

$$\mathbf{A}_0 = \mathbf{P}_0\mathbf{a}_0 = \mathbf{P}_0\Sigma\mathbf{a} \qquad (6.2\text{–}15)$$

Thus

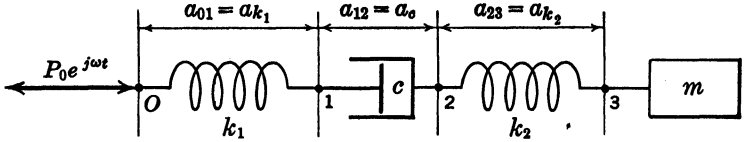
$$\mathbf{A}_0 = \mathbf{P}_0\Sigma\mathbf{a} = \mathbf{P}_0\sum\frac{1}{\mathbf{z}} = \frac{\mathbf{P}_0}{\mathbf{z}_0} \qquad (6.2\text{–}16)$$

where

$$\mathbf{z}_0 = \frac{1}{\sum\frac{1}{\mathbf{z}}} = \frac{1}{\Sigma\mathbf{a}} \qquad (6.2\text{–}17)$$

is called the impedance or dynamic stiffness of the system.

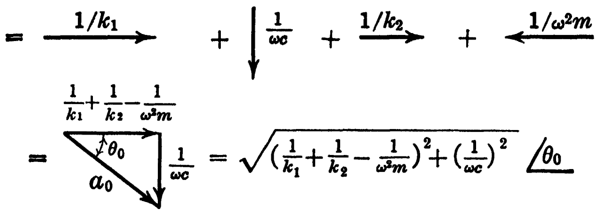
Equations 6.2–14, 6.2–16, and 6.2–17 are completely general, the only stipulation being that the elements of the system be arranged in series.



$a_0 = (a_{k_1} + a_1) \quad a_1 = (a_c + a_2) \quad a_2 = (a_{k_2} + a_3) \quad a_3 = a_m$

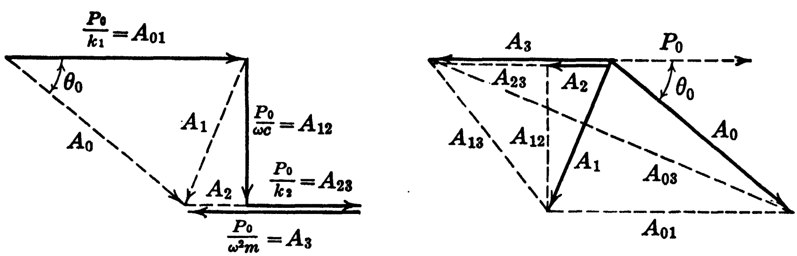
$P_0 = P_{k_1} = P_c = P_{k_2} = P_m$

$$a_0 = a_{k_1} + a_c + a_{k_2} + a_m = \frac{1}{k_1} + \frac{1}{j\omega c} + \frac{1}{k_2} + \frac{1}{-\omega^2 m}$$



$$= \sqrt{\left(\frac{1}{k_1} + \frac{1}{k_2} - \frac{1}{\omega^2 m}\right)^2 + \left(\frac{1}{\omega c}\right)^2} \;\angle\theta_0$$

$$\tan\theta_0 = \frac{-\frac{1}{\omega c}}{\frac{1}{k_1} + \frac{1}{k_2} - \frac{1}{\omega^2 m}}$$



$P_0 = P_{k_1} = P_c = P_{k_2} = P_m \qquad a_0 = \Sigma a$

$A_0 = A_{k_1} + A_c + A_{k_2} + A_m \qquad \frac{1}{z_0} = \Sigma\frac{1}{z}$

System in Series

FIG. 6–10

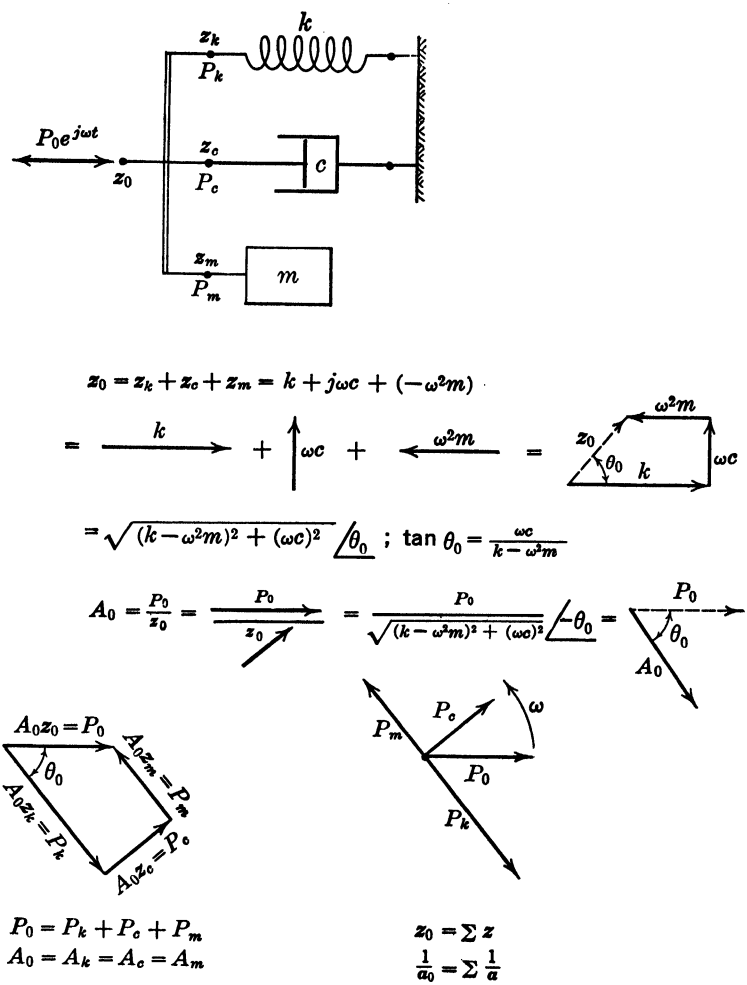
Systems in parallel

Figure 6–11 is an example of a system in parallel. The application of an oscillating force,

$$\mathbf{P} = \mathbf{P}_0 e^{j\omega t}$$

to point O will result in an amplitude at O of amount $\mathbf{A}_0$. It will be noted that the amplitude of the point O in this case is the same as the amplitudes of each of the elements of the system. Thus

$$\mathbf{A}_0 = \mathbf{A}_k = \mathbf{A}_m = \mathbf{A}_c \tag{6.2–18}$$



System in Parallel

FIG. 6–11

Let the amplitude of the force that acts on the spring be $\mathbf{P}_k$, on the mass $\mathbf{P}_m$, and on the damper $\mathbf{P}_c$. The relation between the unit amplitudes and the total amplitudes of the elements is then given by the equations:

$$\mathbf{A}_k = \mathbf{P}_k\mathbf{a}_k; \qquad \mathbf{A}_m = \mathbf{P}_m\mathbf{a}_m; \qquad \mathbf{A}_c = \mathbf{P}_c\mathbf{a}_c$$

However, from equation 6.2–18, it is seen that

$$\mathbf{A}_0 = \mathbf{P}_0\mathbf{a}_0 = \mathbf{P}_k\mathbf{a}_k = \mathbf{P}_m\mathbf{a}_m = \mathbf{P}_c\mathbf{a}_c$$

These equations give, in terms of impedances,

$$\mathbf{A}_0 = \frac{\mathbf{P}_0}{\mathbf{z}_0} = \frac{\mathbf{P}_k}{\mathbf{z}_k} = \frac{\mathbf{P}_m}{\mathbf{z}_m} = \frac{\mathbf{P}_c}{\mathbf{z}_c}$$

$$= \frac{\mathbf{P}_k + \mathbf{P}_m + \mathbf{P}_c}{\Sigma \mathbf{z}} = \frac{\mathbf{P}_0}{\Sigma \mathbf{z}} \tag{6.2–19}$$

since $\mathbf{P}_k + \mathbf{P}_m + \mathbf{P}_c = \mathbf{P}_0$

Thus the total impedance of the system is expressed by

$$\mathbf{z}_0 = \Sigma\, \mathbf{z} = \sum \frac{1}{\mathbf{a}} \tag{6.2–20}$$

Equation 6.2–20 is completely general and applies to all parallel systems. For forced vibrations, $\mathbf{P}_0 \neq 0$, and the amplitude of point O may be calculated from equation 6.2–19.

It must be emphasized that the additions indicated in equations 6.2–16 and 6.2–19 are vector additions. The unit amplitudes as well as total amplitudes are vectorial quantities in general and must be treated in the manner outlined in section 6.1 for all algebraic manipulations.

From either equation 6.2–16 or equation 6.2–19, it may be seen that free vibration of the system,

$$\mathbf{P}_0 = 0$$

must be associated with

$$\mathbf{z}_0(\omega) = 0 \tag{6.2–21}$$

The roots of this equation are the natural circular frequencies of the system under consideration.

6.3. Applications of the Mobility Method

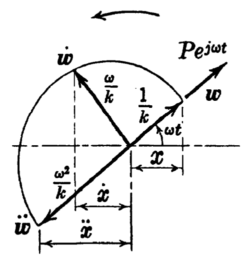
As a preliminary application of the foregoing theory, the system shown in Fig. 6–12*a* may be analyzed. This is a parallel system as evidenced by the equivalent system Fig. 6–12*b*. The total impedance of this system can be established, by the use of equation 6.2–20 and Table 2, as

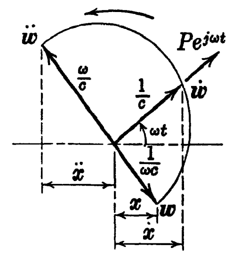
$$\mathbf{z}_0 = \Sigma\, \mathbf{z} = \mathbf{z}_k + \mathbf{z}_m = \frac{1}{\mathbf{a}_k} + \frac{1}{\mathbf{a}_m} = k - \omega^2 m$$

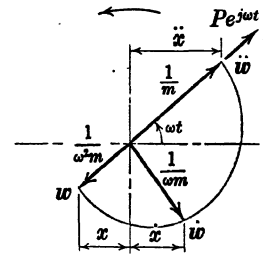
The amplitude of the mass is given by

$$\mathbf{A}_0 = \mathbf{A}_m = \frac{\mathbf{P}_0}{\mathbf{z}_0} = \frac{\mathbf{P}_0}{k^2 - \omega^2 m} = \frac{\mathbf{P}_0}{k} \cdot \frac{1}{1 - \left(\frac{\omega}{p}\right)^2}$$

TABLE 2







Characteristics of the Elements of Vibrating Systems in Translation				
Element		Spring	Damper	Mass
Symbol		k	c	m
Dimensions		FL^{-1}	$FL^{-1}T$	$FL^{-1}T^2$
Force in phase with		Displacement, w	Velocity, $\dot{w}$	Acceleration, $\ddot{w}$
Motion		Relative, between terminals	Relative, between terminals	Absolute
Displacement		$w=\frac{1}{k}e^{j\omega t}$	$w=\frac{1}{j\omega c}e^{j\omega t}$	$w=-\frac{1}{\omega^2 m}e^{j\omega t}$
Velocity		$\dot{w}=\frac{j\omega}{k}e^{j\omega t}$	$\dot{w}=\frac{1}{c}e^{j\omega t}$	$\dot{w}=\frac{1}{j\omega m}e^{j\omega t}$
Acceleration		$\ddot{w}=\frac{-\omega^2}{k}e^{j\omega t}$	$\ddot{w}=\frac{jw}{c}e^{j\omega t}$	$\ddot{w}=\frac{1}{m}e^{j\omega t}$
Amplitude per unit force	w	$\frac{1}{k}$	$\frac{1}{j\omega c}$	$-\frac{1}{\omega^2 m}$
	$\dot{w}$	$\frac{j\omega}{k}$	$\frac{1}{c}$	$\frac{1}{j\omega m}$
	$\ddot{w}$	$-\frac{\omega^2}{k}$	$\frac{j\omega}{c}$	$\frac{1}{m}$
Dynamic Stiffness or Impedance, z		k	$j\omega c$	$-\omega^2 m$

which is the familiar result. The frequency equation is obtained by equating the total impedance to zero, which gives

$$\mathbf{z}_0(\omega) = \mathbf{z}_0(p) = k - p^2m = 0 \quad \text{or} \quad p^2 = \frac{k}{m}$$

also a previously established result.

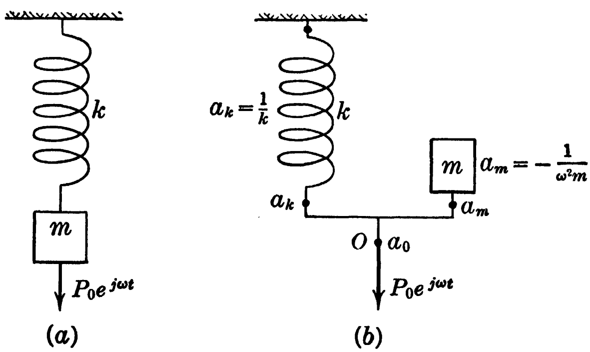


FIG. 6–12

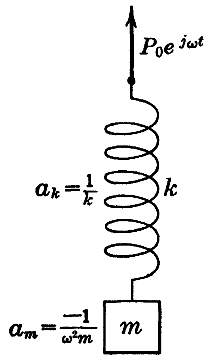


FIG. 6–13

Next, it is instructive to consider the simple system of Fig. 6–13. This is a system in series. The total amplitude of point O is given by

$$\mathbf{A}_0 = \mathbf{P}_0\mathbf{a}_0 = \mathbf{P}_0\Sigma\mathbf{a} = \mathbf{P}_0(\mathbf{a}_k + \mathbf{a}_m)$$

Then the force associated with an amplitude $\mathbf{A}_0$ is

$$\mathbf{P}_0 = \frac{\mathbf{A}_0}{\mathbf{a}_k + \mathbf{a}_m},$$

and the amplitude of the mass due $\mathbf{P}_0 = \mathbf{P}_m$ is

$$\mathbf{A}_m = \mathbf{P}_m\mathbf{a}_m = \mathbf{P}_0\mathbf{a}_m = \frac{\mathbf{A}_0}{1 + \dfrac{\mathbf{a}_k}{\mathbf{a}_m}} = \frac{\mathbf{A}_0}{1 - \left(\dfrac{\omega^2 m}{k}\right)} = \frac{\mathbf{A}_0}{1 - \left(\dfrac{\omega}{p}\right)^2}$$

which is the same as previously determined.

As a more complex system, consider the damped vibration absorber previously discussed by the classical method. The system of the damped vibration absorber is shown in Fig. 6–14*a*. The system is most conveniently arranged for the present purposes in the form shown in Fig. 6–14*b*. The amplitude across the system, which consists of the spring k_2 and the damper c in parallel, may be designated by $\mathbf{a}_2$, and the amplitude obtained by combining this system in series with

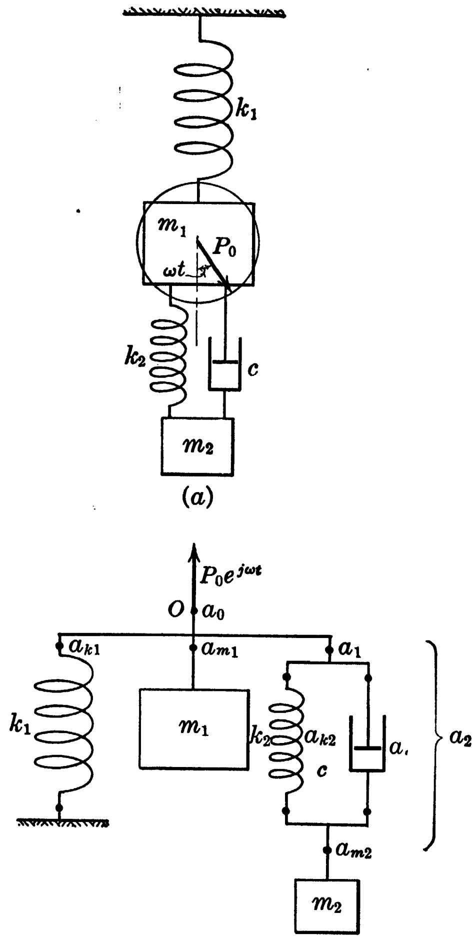


Fig. 6–14

the mass m_2 will be designated by $\mathbf{a}_1$. Thus

$$\mathbf{a}_1 = \mathbf{a}_2 + \mathbf{a}_{m2} = \frac{1}{\dfrac{1}{\mathbf{a}_{k2}} + \dfrac{1}{\mathbf{a}_c}} + \mathbf{a}_{m2}$$

$$= \frac{1}{k_2 - j\omega c} - \frac{1}{\omega^2 m_2} = -\frac{k_2 - \omega^2 m_2 + j\omega c}{\omega^2 m_2(k_2 - j\omega c)}$$

The amplitude of point O is a parallel combination of $\mathbf{a}_{k1}$, $\mathbf{a}_{m1}$, and $\mathbf{a}_1$, that is,

$$\mathbf{a}_0 = \frac{1}{\mathbf{z}_0} = \frac{1}{\sum \frac{1}{\mathbf{a}}} = \frac{1}{\frac{1}{\mathbf{a}_{k1}} + \frac{1}{\mathbf{a}_{m1}} + \frac{1}{\mathbf{a}_1}}$$

$$= \frac{1}{k_1 - \omega^2 m_1 - \frac{\omega^2 m_2 (k_2 - j\omega c)}{k_2 - \omega^2 m_2 + j\omega c}}$$

$$= \frac{k_2 - \omega^2 m_2 + j\omega c}{(k_1 - \omega^2 m_1)(k_2 - \omega^2 m_2) - \omega^2 k_2 m_2 + j\omega c[k_1 - \omega^2(m_1 + m_2)]}$$

Division of both the numerator and the denominator by $m_1 m_2$ and introduction of the notation

$$p_{ij}^2 = \frac{k_i}{m_j}, \quad q = \frac{k_2}{c}, \quad \text{and} \quad \delta_P = \frac{\mathbf{P}_0}{k_1}$$

shows that

$$\mathbf{A}_0 = \mathbf{P}_0 \mathbf{a}_0 = \frac{\mathbf{P}_0}{\mathbf{z}_0}$$

$$= \left[\frac{\left(p_{22}^2 - \omega^2 + j\frac{\omega}{q} p_{22}^2 \right) p_{11}^2}{(p_{11}^2 - \omega^2)(p_{22}^2 - \omega^2) - p_{21}^2 \omega^2 + j\frac{\omega}{q}[p_{11}^2 p_{22}^2 - \omega^2(p_{21}^2 + p_{22}^2)]} \right] \delta_P$$

which coincides with the result obtained in the previous chapter.

It is to be noted that the frequency equation for this system is

$$\mathbf{z}_0(p) = (p_{11}^2 - p^2)(p_{22}^2 - p^2) - p_{21}^2 p^2 + j\frac{p}{q}[p_{11}^2 p_{22}^2 - p^2(p_{21}^2 + p_{22}^2)] = 0$$

This may be written as

$$p^4 - j\frac{p_{21}^2 + p_{22}^2}{q} p^3 - (p_{11}^2 + p_{21}^2 + p_{22}^2)p^2 + j\frac{p_{11}^2 p_{22}^2}{q} p + p_{11}^2 p_{22}^2 = 0$$

The four values of p, which are roots of this equation, consist of two conjugate pairs and are of the form,

$$p = j\alpha_1 \pm p_1' \quad \text{and} \quad p = j\alpha_2 \pm p_2'$$

Each of these pairs of roots corresponds to a natural mode of vibration.

The complete solution for the free vibration of the mass m_1 has the forms

$$w_1 = e^{-\alpha_1 t}(\mathbf{C}_1' e^{jp_1't} + \mathbf{D}_1' e^{-jp_1't}) + e^{-\alpha_2 t}(\mathbf{C}_2' e^{jp_2't} + \mathbf{D}_2' e^{-jp_2't})$$

$$x_1 = e^{-\alpha_1 t}(C_1 \cos p_1't + D_1 \sin p_1't) + e^{-\alpha_2 t}(C_2 \cos p_2't + D_2 \sin p_2't)$$

where $\mathbf{C}_1'$, $\mathbf{D}_1'$, $\mathbf{C}_2'$, $\mathbf{D}_2'$, C_1, D_1, C_2, and D_2 are constants. The motion is seen to decrease with time and is therefore transient in character.

6.4. Branched Systems

Branched systems occur frequently in engineering problems. The method of treatment is essentially the same as for the preceding examples; however, certain differences are present. As a typical branched system, the one shown in Fig. 6–15 will be considered. The approach used here may be extended to the general case with r branches composed of r springs and r masses connected to a single mass of moment of inertia I_{r+1}. Most frequently, these connections are made through gears or levers, thus introducing different gear ratios into the system. Such gear or lever ratios can be treated as outlined in Chapter 2.

In the example of Fig. 6–15, the motion may be referred to the shafts k_1' and k_3', which have equal rigid body displacements. Let the gear ratio be n; then the effective moments of inertia are given by

$$I_1 = I_1' \qquad I_2 = n^2 I_2' \qquad I_3 = I_3' \qquad I_4 = n^2 I_4'$$

$$I_{r+1} = I_5' + I_7' + n^2(I_6' + I_8')$$

The effective springs constants have the values:

$$k_1 = k_1' \qquad k_2 = n^2 k_2' \qquad k_3 = k_3' \qquad k_4 = n^2 k_4'$$

The amplitude of the ith branch due to a unit oscillating force is then

$$\boldsymbol{\theta}_i = \boldsymbol{\theta}_{ki} + \boldsymbol{\theta}_{Ii} = \frac{1}{k_i} - \frac{1}{\omega^2 I_i} = \frac{\omega^2 - p_{ii}^2}{k_i \omega^2} = \frac{1}{\mathbf{z}_i}$$

where the notation

$$p_{ij}^2 = \frac{k_i}{I_j}$$

has been introduced.

The amplitude per unit torque of point O is given by

$$\boldsymbol{\theta}_0 = \frac{1}{\sum_{i=1}^{r} \mathbf{z}_i + \mathbf{z}_{Ir+1}} = \frac{1}{\sum_{i=1}^{r} \frac{k_i \omega^2}{\omega^2 - p_{ii}^2} - \omega^2 I_{r+1}}$$

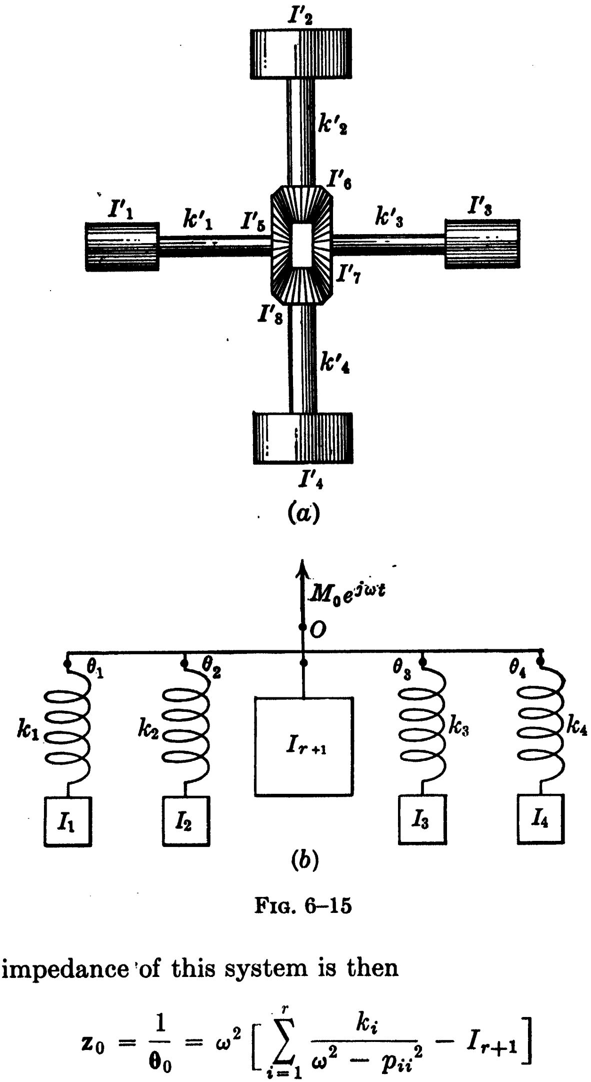


FIG. 6–15

The total impedance of this system is then

$$\mathbf{z}_0 = \frac{1}{\boldsymbol{\theta}_0} = \omega^2 \left[\sum_{i=1}^{r} \frac{k_i}{\omega^2 - p_{ii}^2} - I_{r+1} \right]$$

and so the frequency equation has the form ($\omega = p$)

$$I_{r+1} = \sum_{i=1}^{r} \frac{k_i}{p^2 - p_{ii}^2} \tag{6.4–1}$$

In this form it is seen that the frequency equation may be established for a simple branched system similar to Fig. 6–15. If the branches are unsymmetrical and involve many elements, then $\boldsymbol{\theta}_i$ will require special consideration for each branch. The frequency equation, however, still has the form

$$I_{r+1} = \frac{1}{p^2} \sum_{i=1}^{r} z_i \tag{6.4-2}$$

It is instructive to consider the special case which arises when

$$p_{ii}^2 = p_0^2$$

that is, all branches are so proportioned that

$$p_0^2 = \frac{k_i}{I_i} = \text{constant}$$

The frequency equation then becomes

$$I_{r+1} = \sum_{i=1}^{r} \frac{k_i}{p^2 - p_0^2} = \frac{\sum_{i=1}^{r} k_i}{p^2 - p_0^2}$$

whence

$$p^2 = \frac{\sum_{i=1}^{r} k_i}{I_{r+1}} + p_0^2 \tag{6.4-3}$$

This, however, is only one of the frequencies. The complete frequency equation has the form

$$(p^2 - p_0^2)^r I_{r+1} = (p^2 - p_0^2)^{r-1} \sum_{i=1}^{r} k_i \tag{6.4-4}$$

from which it is seen that there are $(r - 1)$ equal frequencies of the form

$$p^2 = p_0^2$$

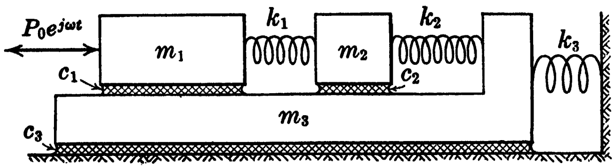
and only one of the form

$$p^2 = \frac{\sum_{i=1}^{r} k_i}{I_{r+1}} + p_0^2$$

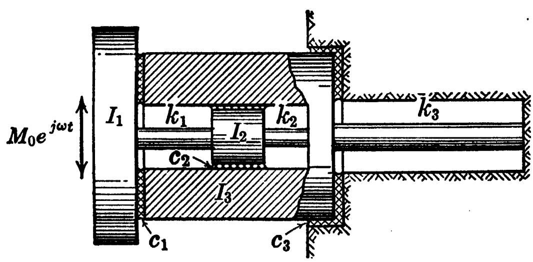
Equal roots to the frequency equation are only possible in a branched system although other systems may have roots nearly equal. It is important to appreciate this at an early stage in the analysis, especially when recourse is had to methods such as Graeffes' for solving the frequency equation.

6.5. Further Application of the Mobility Method

To illustrate the application of the mobility method in detail a complete treatment of the system of Fig. 6–16*a* is contained in this section. The system of Fig. 6–16*a* is a three-mass damped system actuated by an oscillating force acting on the mass m_1. As a matter of convenience the system is indicated here as translatory in character. An equivalent system with torsional motion is shown in Fig. 6–16*b*.



(*a*)



(*b*)

FIG. 6–16

The schematic diagrams Figs. 6–17*a* and Fig. 6–17*b*, showing the manner in which the elements of these systems are connected, i.e., parallel or series, are most readily indicated as translatory systems for both torsion and translation in the actual system. It will be noticed that the two diagrams differ only in the type of elements and corresponding notation.

The schematic diagrams, such as Fig. 6–17*a*, which serve to clarify the manner in which the system is constructed from series and parallel combinations of elements, is most readily obtained from Fig. 6–16*a* by starting with the elements on which the external oscillating force is acting. In this case the force P_0 is acting in part on the mass m_1 and in part on the remainder of the system through the spring k_1 and the damper c_1. The mass m_1 and the terminals 1 and 6 of the spring k_1 and the damper c_1, respectively, are therefore in parallel as indicated in Fig. 6–17*a*. The force $\mathbf{P}_{k1}$ acting through the spring k_1 is again

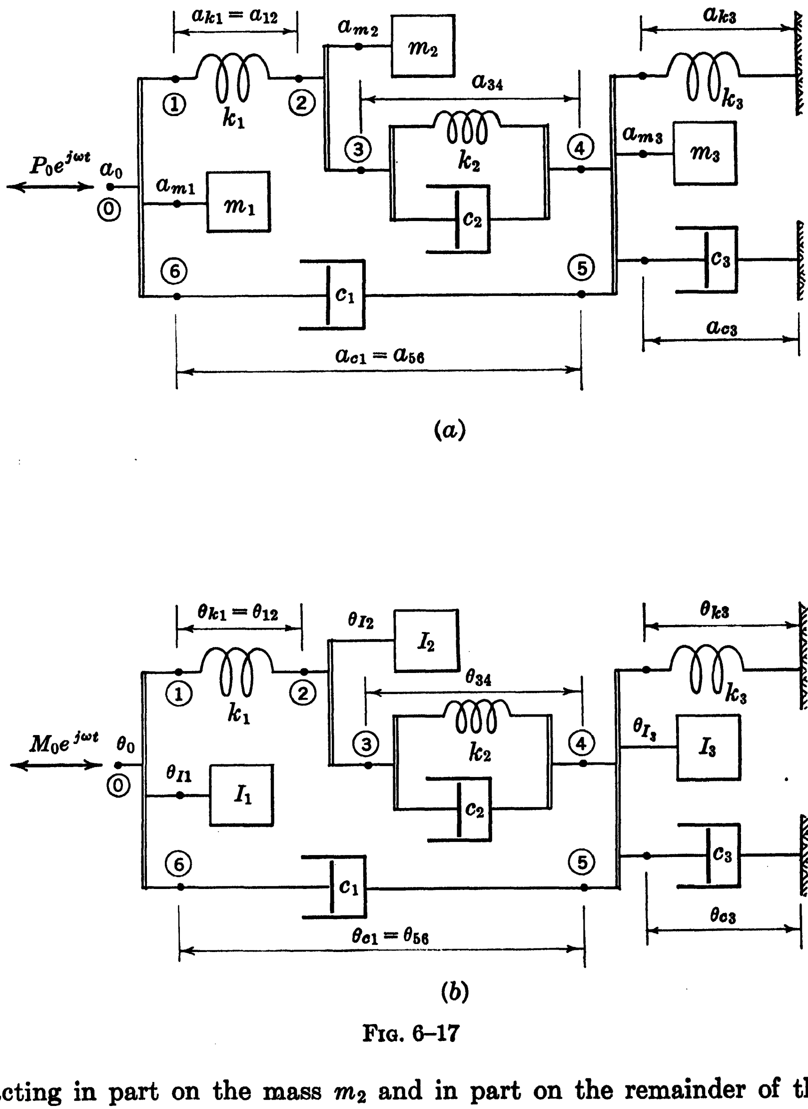


FIG. 6–17

acting in part on the mass m_2 and in part on the remainder of the system through the spring k_2 and the damper c_2. The two elements k_2 and c_2 are in parallel since both terminate at the same two masses m_2 and m_3. If, as in this example, several springs and dampers are acting in parallel, it will in general be found helpful in analyzing the system to combine these into a single equivalent element with a common set of terminals as indicated here by terminals 3 and 4.

One end or terminal of both the spring k_3 and damper c_3 is stationary or grounded. The displacements across these two elements therefore are the same as the displacement of mass m_3; hence the three elements are in parallel. The elements k_3 and c_3 could be shown with common terminals like k_2 and c_2, but, when the elements are fixed or grounded at one end, no particular advantage in clarity of analysis is gained by such an arrangement, and it is a matter of indifference which scheme is used.

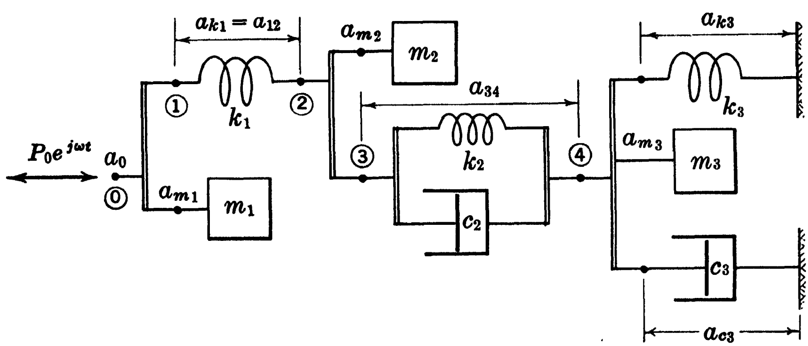


FIG. 6–18

The motion of m_3, k_3, and c_3 is produced in part by forces acting through k_2 and c_2 and in part by a force acting through c_1. This particular type of system in which the forces act through the terminals 1, 2, 3, 4, 5, and 6 as a closed circuit will be referred to as a closed system. If on the other hand the damper c_1 is reduced to zero, the schematic diagram reduces to Fig. 6–18, and the effect of the disturbing force P_0 is only carried through one channel 1, 2 to m_2 and from there through 3, 4 to m_3. This type of system will be referred to as an open system. The analysis of the two types of systems is different in the initial stages in that the amplitudes of the open system can be determined directly whereas the closed system requires the evaluation of the amplitudes through simultaneous equations.

The open system

For the particular value of $c_1 = 0$, the schematic diagram Fig. 6–17*a* reduces to Fig. 6–18, and the calculations may be based directly on this latter figure. It is expedient to determine the amplitudes a per unit force first and then to evaluate the total forces and amplitudes from them. The following treatment proceeds in this manner.

Amplitudes per unit force. The amplitude per unit force $\mathbf{a}_0$ at the point of application of the force $\mathbf{P}_0$ is obtained from the amplitude per unit force $\mathbf{a}_{m1}$ of the mass m_1 and the amplitudes per unit force $\mathbf{a}_1$ of the terminal 1. These two have been shown to act in parallel, and, therefore,

$$\mathbf{a}_0 = \frac{1}{\dfrac{1}{\mathbf{a}_{m1}} + \dfrac{1}{\mathbf{a}_1}} \quad \text{or} \quad \mathbf{z}_0 = \mathbf{z}_{m1} + \mathbf{z}_1 \tag{6.5–1}$$

where $\mathbf{z}_0$, $\mathbf{z}_{m1}$, and $\mathbf{z}_1$ are the corresponding impedances or forces per unit amplitude. The amplitude $\mathbf{a}_1$ is made up of the amplitude across the spring k_1 and the amplitude $\mathbf{a}_2$ acting in series, whence

$$\mathbf{a}_1 = \mathbf{a}_{k1} + \mathbf{a}_2 \quad \text{or} \quad \mathbf{z}_1 = \frac{1}{\dfrac{1}{\mathbf{z}_{k1}} + \dfrac{1}{\mathbf{z}_2}} \tag{6.5–2}$$

The amplitude of the mass m_2 and the amplitude of terminal 3 which act in parallel gives

$$\mathbf{a}_2 = \frac{1}{\dfrac{1}{\mathbf{a}_{m2}} + \dfrac{1}{\mathbf{a}_3}} = \frac{1}{\mathbf{z}_{m3} + \mathbf{z}_3} \quad \text{or} \quad \mathbf{z}_2 = \mathbf{z}_{m3} + \mathbf{z}_3 \tag{6.5–3}$$

The amplitude $\mathbf{a}_{34}$ across the spring k_2 and damper c_2 is in series with the amplitude of terminal 4, so that

$$\mathbf{a}_3 = \mathbf{a}_{34} + \mathbf{a}_4 \quad \text{or} \quad \mathbf{z}_3 = \frac{1}{\dfrac{1}{\mathbf{z}_{34}} + \dfrac{1}{\mathbf{z}_4}} \tag{6.5–4}$$

However, the spring k_2 and damper c_2 in parallel give

$$\mathbf{a}_{34} = \frac{1}{\dfrac{1}{\mathbf{a}_{k2}} + \dfrac{1}{\mathbf{a}_{c2}}} = \frac{1}{\mathbf{z}_{k2} + \mathbf{z}_{c2}} \quad \text{or} \quad \mathbf{z}_{34} = \mathbf{z}_{k2} + \mathbf{z}_{c2} \tag{6.5–5}$$

Similarly $\mathbf{a}_4$ is the amplitude of the elements m_3, k_3 and c_3 acting in parallel whence

$$\mathbf{a}_4 = \frac{1}{\dfrac{1}{\mathbf{a}_{k3}} + \dfrac{1}{\mathbf{a}_{m3}} + \dfrac{1}{\mathbf{a}_{c3}}} \quad \text{or} \quad \mathbf{z}_4 = \mathbf{z}_{k3} + \mathbf{z}_{m3} + \mathbf{z}_{c3} \tag{6.5–6}$$

It is therefore possible to express either the amplitude per unit force or force per unit amplitude at any terminal in terms of the known values of the elements of the system. By substitution it will be found that

$$\mathbf{a}_0 = \cfrac{1}{\cfrac{1}{\mathbf{a}_{m1}} + \cfrac{1}{\mathbf{a}_{k1} + \cfrac{1}{\cfrac{1}{\mathbf{a}_{m2}} + \cfrac{1}{\cfrac{1}{\cfrac{1}{\mathbf{a}_{k2}} + \cfrac{1}{\mathbf{a}_{c2}}} + \cfrac{1}{\cfrac{1}{\mathbf{a}_{k3}} + \cfrac{1}{\mathbf{a}_{m3}} + \cfrac{1}{\mathbf{a}_{c3}}}}}}} \tag{6.5-7}$$

or

$$\mathbf{z}_0 = \mathbf{z}_{m1} + \cfrac{1}{\cfrac{1}{\mathbf{z}_{k1}} + \cfrac{1}{\mathbf{z}_{m2} + \cfrac{1}{\cfrac{1}{\mathbf{z}_{k2} + \mathbf{z}_{c2}} + \cfrac{1}{\mathbf{z}_{k3} + \mathbf{z}_{m3} + \mathbf{z}_{c3}}}}} \tag{6.5-8}$$

Forces and total amplitudes. By definition the total amplitude of any given terminal or element is the total force acting through the terminal or element times the amplitude per unit force, i.e.;

$$\mathbf{A} = \mathbf{Pa}$$

From Fig. 6–18, it is seen that the terminals 0, 1, and m_1 are in parallel, and hence they have the same total amplitude. Therefore it follows that

$$\left.\begin{aligned} \mathbf{A}_0 &= \mathbf{A}_{m1} = \mathbf{A}_1 \\ \text{or} \quad \mathbf{A}_0 &= \mathbf{P}_0\mathbf{a}_0 = \mathbf{P}_{m1}\mathbf{a}_{m1} = \mathbf{P}_1\mathbf{a}_1 \end{aligned}\right\} \tag{6.5-9}$$

whence

$$\left.\begin{aligned} \mathbf{P}_{m1} &= \frac{\mathbf{a}_0}{\mathbf{a}_{m1}}\mathbf{P}_0 = \mathbf{P}_0\mathbf{a}_0\mathbf{z}_{m1} = \mathbf{A}_0\mathbf{z}_{m1} \\ \mathbf{P}_1 &= \frac{\mathbf{a}_0}{\mathbf{a}_1}\mathbf{P}_0 = \mathbf{P}_0\mathbf{a}_0\mathbf{z}_1 = \mathbf{A}_0\mathbf{z}_1 \end{aligned}\right\} \tag{6.5-10}$$

The force $\mathbf{P}_1$ acts at the terminal 1 through the spring k_1 on terminal 2, and therefore

$$\mathbf{P}_1 = \mathbf{P}_{k1} = \mathbf{P}_2$$

and, again by reasoning similar to that used above,

$$\left.\begin{aligned} &\mathbf{A}_2 = \mathbf{A}_{m2} = \mathbf{A}_3 \\ \text{or} \quad &\mathbf{A}_2 = \mathbf{P}_2\mathbf{a}_2 = \mathbf{P}_1\mathbf{a}_2 = \mathbf{P}_{m2}\mathbf{a}_{m2} = \mathbf{P}_3\mathbf{a}_3 \end{aligned}\right\} \qquad (6.5\text{–}11)$$

whence

$$\left.\begin{aligned} \mathbf{P}_{m2} &= \mathbf{P}_1 \frac{\mathbf{a}_2}{\mathbf{a}_{m2}} = \mathbf{P}_0 \frac{\mathbf{a}_0}{\mathbf{a}_1} \frac{\mathbf{a}_2}{\mathbf{a}_{m2}} \\ \mathbf{P}_3 &= \mathbf{P}_1 \frac{\mathbf{a}_2}{\mathbf{a}_3} = \mathbf{P}_0 \frac{\mathbf{a}_0}{\mathbf{a}_1} \frac{\mathbf{a}_2}{\mathbf{a}_3} \end{aligned}\right\} \qquad (6.5\text{–}12)$$

The force $\mathbf{P}_3$ acts at the terminal 3 through the spring k_2 and damper c_2, on terminal 4 from which

$$\mathbf{P}_3 = \mathbf{P}_{34} = \mathbf{P}_4$$

The forces $\mathbf{P}_{k2}$ and $\mathbf{P}_{c2}$ are acting through the spring k_2 and the damper c_2, respectively, and can be found from the relationship

$$\left.\begin{aligned} &\mathbf{A}_{34} = \mathbf{P}_3\mathbf{a}_{34} = \mathbf{P}_{k2}\mathbf{a}_{k2} = \mathbf{P}_{c2}\mathbf{a}_{c2} \\ &\text{whence} \\ &\mathbf{P}_{k2} = \mathbf{P}_3 \frac{\mathbf{a}_{34}}{\mathbf{a}_{k2}} = \frac{\mathbf{P}_3}{\mathbf{a}_{k2}} \cdot \frac{1}{\dfrac{1}{\mathbf{a}_{k2}} + \dfrac{1}{\mathbf{a}_{c2}}} = \frac{\mathbf{P}_3}{1 + \dfrac{\mathbf{a}_{k2}}{\mathbf{a}_{c2}}} \\ &\text{and similarly} \\ &\mathbf{P}_{c2} = \mathbf{P}_3 \frac{\mathbf{a}_{34}}{\mathbf{a}_{c2}} = \frac{\mathbf{P}_3}{1 + \dfrac{\mathbf{a}_{c2}}{\mathbf{a}_{k2}}} \end{aligned}\right\} \qquad (6.5\text{–}13)$$

Proceeding in the same manner, the forces acting on the elements k_3, m_3 and c_3 are found from

$$\left.\begin{aligned} &\mathbf{A}_4 = \mathbf{A}_{k3} = \mathbf{A}_{m3} = \mathbf{A}_{c3} \\ &\quad = \mathbf{P}_4\mathbf{a}_4 = \mathbf{P}_3\mathbf{a}_3 = \mathbf{P}_{k3}\mathbf{a}_{k3} = \mathbf{P}_{m3}\mathbf{a}_{m3} = \mathbf{P}_{c3}\mathbf{a}_{c3} \\ &\text{from which} \\ &\mathbf{P}_{k3} = \mathbf{P}_3 \frac{\mathbf{a}_4}{\mathbf{a}_{k3}}, \qquad \mathbf{P}_{m3} = \mathbf{P}_3 \frac{\mathbf{a}_4}{\mathbf{a}_{m3}}, \qquad \mathbf{P}_{c3} = \mathbf{P}_3 \frac{\mathbf{a}_4}{\mathbf{a}_{c3}} \end{aligned}\right\} \qquad (6.5\text{–}14)$$

The amplitudes of the three masses m_1, m_2, and m_3 are the same as the amplitudes of the three terminals 0, 2, and 4, respectively, hence:

$$\left.\begin{aligned} \mathbf{A}_{m1} &= \mathbf{A}_0 = \mathbf{P}_0\mathbf{a}_0 \\ \mathbf{A}_{m2} &= \mathbf{A}_2 = \mathbf{P}_0 \frac{\mathbf{a}_0}{\mathbf{a}_1} \mathbf{a}_2 \\ \mathbf{A}_{m3} &= \mathbf{A}_4 = \mathbf{P}_0 \frac{\mathbf{a}_0}{\mathbf{a}_1} \frac{\mathbf{a}_2}{\mathbf{a}_3} \mathbf{a}_4 \end{aligned}\right\} \qquad (6.5\text{–}15)$$

The relative amplitudes across the elements having two terminals, (springs and dampers) are:

$$
\left.\begin{aligned}
\mathbf{A}_{k1} &= \mathbf{A}_1 - \mathbf{A}_2 = \mathbf{A}_{m1} - \mathbf{A}_{m2} = \mathbf{P}_0 \frac{\mathbf{a}_0}{\mathbf{a}_1} \mathbf{a}_{k1} \\
&= \mathbf{P}_0 \mathbf{a}_0 \left(1 - \frac{\mathbf{a}_2}{\mathbf{a}_1}\right) \\
\mathbf{A}_{34} &= \mathbf{A}_3 - \mathbf{A}_4 = \mathbf{A}_{m2} - \mathbf{A}_{m3} = \mathbf{P}_0 \frac{\mathbf{a}_0}{\mathbf{a}_1} \frac{\mathbf{a}_2}{\mathbf{a}_3} \mathbf{a}_{34} \\
&= \mathbf{P}_0 \frac{\mathbf{a}_0}{\mathbf{a}_1} \mathbf{a}_2 \left(1 - \frac{\mathbf{a}_4}{\mathbf{a}_3}\right) \\
\mathbf{A}_{k3} &= \mathbf{A}_{c3} = \mathbf{A}_{m3} - 0 = \mathbf{P}_0 \frac{\mathbf{a}_0}{\mathbf{a}_1} \frac{\mathbf{a}_2}{\mathbf{a}_3} \mathbf{a}_4
\end{aligned}\right\} \quad (6.5\text{–}16)
$$

Equations 6.5–9 through 6.5–16 represent a general evaluation of all amplitudes and forces in the system.

Numerical calculations of forces and amplitudes. To illustrate the manner of carrying out the calculations in an actual numerical problem, the data of Table 3 will be used.

TABLE 3

$\omega = 100$ rad per sec										$\mathbf{P}_0 = 10$ lb	
Lb per in.	$\mathbf{a}_k$	$\mathbf{z}_k$	Lb		$\mathbf{a}_m$	$\mathbf{z}_m$	Lb sec per in.			$\mathbf{a}_c$	$\mathbf{z}_c$
k	$\frac{1}{k}$	k	W		$-\frac{1}{\omega^2 m}$	$-\omega^2 m$	c			$-\frac{j}{\omega c}$	$j\omega c$
k_1	0.04	25	W_1	1.5	−0.0257	− 38.9	c_1	I*	0	∞	0
								II	0.50	−0.02j	50j
k_2	0.0133	75	W_2	2	−0.0193	− 51.8	c_2		0.25	−0.04j	25j
k_3	0 0133	75	W_3	4	−0.0096	−103.6	c_3		0.50	−0.02j	50j

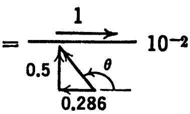
* I—Open system.
II—Closed system.

The calculations are most readily carried out by a successive evaluation of the amplitudes per unit force, starting with the elements furthest from the point of application of the applied force, in this case $\mathbf{a}_4$. This particular amplitude per unit force will be evaluated here in detail as a typical calculation of this kind. It should be noted that most vibratory amplitudes in practical applications are relatively small and can usually be measured in fractions of an inch.

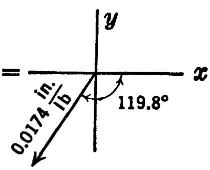
From equation 6.5–6 and Table 3:

$$a_4 = \frac{1}{z_{k3} + z_{m3} + z_{c3}} = \frac{1}{75 + (-103.6) + 50j} = \frac{10^{-2}}{0.75 + (-1.036) + 0.5j}$$

$$= \frac{\underrightarrow{1}}{\underrightarrow{0.75} + \underleftarrow{1.036} + \uparrow 0.5}\,10^{-2} = \frac{1}{-0.286 + 0.5j}\,10^{-2} =$$



$$= \frac{e^{j-\theta}}{\sqrt{(0.286)^2 + (0.5)^2}\,e^{j-\theta}}\,10^{-2} = \frac{e^{j(-\theta)}}{0.576}\,10^{-2} = 1.74 \times 10^{-2}\ \underline{/-119.8^\circ}$$



$$\text{where } \tan\theta = \frac{0.5}{-0.286}$$

The value of $\mathbf{a}_4 = x_4 + jy_4$ in its Cartesian form can be evaluated from the above polar form since $x_4 = 0.0174 \cos\theta$ and $y_4 = 0.0174 \sin\theta$. The Cartesian form of $\mathbf{a}_4$ however may be more readily obtained directly from the original data by rationalizing the denominator, thus

$$10^2\mathbf{a}_4 = \frac{1}{-0.286 + 0.5j} \cdot \frac{-0.286 - 0.5j}{-0.286 - 0.5j} = \frac{-0.286 - 0.5j}{(0.286)^2 + (0.5)^2}$$

$$= -\frac{0.286 + 0.5j}{0.332} = -(0.862 + 1.507j)$$

$$= \underleftarrow{0.862} + \downarrow 1.507$$

When both forms are needed, it will usually be expedient to obtain the Cartesian values first and then to compute the polar values from these, i.e.,

$$10^2\mathbf{a}_4 = \sqrt{(0.862)^2 + (1.507)^2}\,e^{j\tan^{-1}\frac{-1.507}{-0.862}} = 1.74\underline{/-119.8^\circ}$$

The calculation of the remainder of the amplitudes per unit force follows the same pattern. Equation 6.5–5 and Table 3 give

$$10^2\mathbf{a}_{34} = \frac{1}{\mathbf{z}_{k2} + \mathbf{z}_{c2}} = \frac{1}{0.75 + 0.25j} = \frac{4}{3 + j} = \frac{4}{(3)^2 + (1)^2}(3 - j)$$

$$= 1.2 - 0.4j = \sqrt{(1.2)^2 + (0.4)^2}\,e^{j\tan^{-1}\frac{-0.4}{1.2}}$$

$$= 1.26\underline{/-18.4^\circ}$$

Equation (6.5–4) and Table 3 yield

$$10^2 \mathbf{a}_3 = (\mathbf{a}_{34} + \mathbf{a}_4)10^2 = (1.2 - 0.862) + j(-0.4 - 1.507)$$

$$= 0.338 - 1.907j = \sqrt{(0.338)^2 + (1.907)^2}\, e^{j\tan^{-1}\frac{1.907}{0.338}}$$

$$= 1.94\underline{/-80.0^\circ}$$

From equation 6.5–3 and Table 3, there is obtained

$$10^2 \mathbf{a}_2 = \frac{1}{-0.518 + \dfrac{1}{0.338 - 1.907j}} = \frac{0.338 - 1.907j}{0.825 + 0.988j}$$

$$= \frac{(0.338 - 1.907j)(0.825 - 0.988j)}{(0.825)^2 + (0.988)^2} = \frac{-1.605 - 1.907j}{1.657}$$

$$= -0.969 - 1.151j = \sqrt{(0.969)^2 + (1.151)^2}\, e^{j\tan^{-1}\frac{-1.151}{-0.969}}$$

$$= 1.51\underline{/-130.1^\circ}$$

and equation 6.5–2 and Table 3 gives

$$10^2 \mathbf{a}_1 = 4.0 - 0.969 - 1.151j = 3.031 - 1.151j$$

$$= \sqrt{(3.031)^2 + (1.151)^2}\, e^{j\tan^{-1}\frac{-1.151}{3.031}}$$

$$= 3.242\underline{/-20.8^\circ}$$

Finally, from equation 6.5–1 and Table 3

$$10^2 \mathbf{a}_0 = \frac{1}{-0.389 + \dfrac{1}{3.031 - 1.151j}} = \frac{3.031 - 1.151j}{-0.179 + 0.448j}$$

$$= \frac{(3.031 - 1.151j)(-0.179 - 0.448j)}{(0.179)^2 + (0.448)^2} = \frac{-1.058 - 1.152j}{0.233}$$

$$= -4.549 - 4.949j = \sqrt{(4.549)^2 + (4.949)^2}\, e^{-j\tan^{-1}\frac{-4.949}{-4.549}}$$

$$= 6.72\underline{/-132.6^\circ}$$

The amplitudes per unit force calculated previously are summarized in Table 4.

TABLE 4

	Cartesian Form		*Polar Form*	
$10^2 \times$	$10^2 a_x$	$10^2 a_y j$	$10^2 a$	θ
$\mathbf{a}_0$	−4.549	−4.949j	6.72	−132.6°
$\mathbf{a}_1$	+3.031	−1.151j	3.24	− 20.8°
$\mathbf{a}_2$	−0.969	−1.151j	1.51	−130.1°
$\mathbf{a}_3$	+0.338	−1.907j	1.94	− 80.0°
$\mathbf{a}_4$	−0.862	−1.507j	1.74	−119.8°
$\mathbf{a}_{34}$	+1.200	−0.400j	1.26	− 18.4°

The numerical values of the forces acting in the system may be calculated with the aid of equations 6.5–10 through 6.5–14. Thus:

$$\mathbf{P}_{m1} = \mathbf{P}_0\mathbf{a}_0\mathbf{z}_{m1} = (10)(0.0672\underline{/-132.6^\circ})(38.9\underline{/180^\circ}) = 26.1 \text{ lb } \underline{/47.4^\circ}$$

$$\mathbf{P}_1 = \mathbf{P}_0\mathbf{a}_0\mathbf{z}_1 = \mathbf{P}_0\frac{\mathbf{a}_0}{\mathbf{a}_1} = (10)\frac{6.72\underline{/-132.6^\circ}}{3.24\underline{/-20.8^\circ}} = 20.7 \text{ lb}\underline{/-111.8^\circ} = \mathbf{P}_{k1} = \mathbf{P}_2$$

$$\mathbf{P}_{m2} = \mathbf{P}_1\frac{\mathbf{a}_2}{\mathbf{a}_{m2}} = \mathbf{P}_1\mathbf{a}_2\mathbf{z}_{m2}$$

$$= (20.7\underline{/-111.8^\circ})(0.0151\underline{/-130.1^\circ})(51.8\underline{/180^\circ}) = 16.2 \text{ lb}\underline{/-61.9^\circ}$$

and in a similar fashion all other forces may be calculated.

It will be noted that $\mathbf{P}_{m2}$ may be expressed as

$$\mathbf{P}_{m2} = \mathbf{A}_2\mathbf{z}_{m2} = \mathbf{A}_{m2}\mathbf{z}_{m2}$$

which reduces the calculations to the product of two factors in place of three. For this reason it will usually be expedient to calculate the total amplitudes of the masses first and the forces acting on the masses afterwards.

Numerical values of the amplitudes of the masses m_1, m_2, and m_3 are obtained from equations 6.5–15 and Table 4. Thus:

$$\mathbf{A}_{m1} = \mathbf{A}_0 = \mathbf{P}_0\mathbf{a}_0 = (10)(0.0672\underline{/-132.6^\circ}) = 0.672 \text{ in. } \underline{/-132.6^\circ}$$

$$\mathbf{A}_{m2} = \mathbf{A}_2 = \mathbf{P}_0\frac{\mathbf{a}_0}{\mathbf{a}_1}\mathbf{a}_2$$

$$= (10)\left(\frac{6.72}{3.24}\right)(0.0151)\underline{/(-132.6^\circ) - (-20.8^\circ) + (-130.1^\circ)}$$

$$= 0.312 \text{ in. } \underline{/118.1^\circ}$$

$$\mathbf{A}_{m3} = \mathbf{A}_4 = \mathbf{P}_0\frac{\mathbf{a}_0\mathbf{a}_2}{\mathbf{a}_1\mathbf{a}_3}\mathbf{a}_4 = \mathbf{A}_{m2}\frac{\mathbf{a}_4}{\mathbf{a}_3}$$

$$= (0.312)\left(\frac{1.74}{1.94}\right)\underline{/(118.1^\circ) - (-80.0^\circ) + (-119.8^\circ)} = 0.280 \text{ in. } \underline{/78.3^\circ}$$

The amplitudes across the spring k_1 and the combined spring k_2 and damper c_2 are obtained from equations 6.5–16 in a similar manner. Figures 6–21 and 6–22 show the vectorial values of the forces and amplitudes, respectively, for the complete system for $c_1 = 0$. For completeness the analytical values are given in Table 5.

TABLE 5

OPEN SYSTEM

A	*Cartesian*		*Polar*	
	X	*Y*	*Amplitude*	*Phase*
$\mathbf{A}_{m1}$	−0.45490	−0.49486*j*	0.672 in.	−132.6°
$\mathbf{A}_{m2}$	−0.14707	+0.27514*j*	0.312 in.	+118.1°
$\mathbf{A}_{m3}$	+0.05669	+0.27386*j*	0.280 in.	+ 78.3°
$\mathbf{A}_{m1} - \mathbf{A}_{m2}$	−0.30783	−0.77000*j*	0.829 in.	−111.8°
$\mathbf{A}_{m2} - \mathbf{A}_{m3}$	−0.20376	+0.00128*j*	0.204 in.	+179.6°

Absolute Amplitudes

$\mathbf{A}_0 = \mathbf{A}_1 = \mathbf{A}_{m1}$; $\mathbf{A}_2 = \mathbf{A}_3 = \mathbf{A}_{m2}$; $\mathbf{A}_4 = \mathbf{A}_{k3} = \mathbf{A}_{c3} = \mathbf{A}_{m3}$

Relative Amplitudes

$\mathbf{A}_{12} = \mathbf{A}_{k1} = \mathbf{A}_{m1} - \mathbf{A}_{m2}$; $\mathbf{A}_{34} = \mathbf{A}_{k2} = \mathbf{A}_{c2} = \mathbf{A}_{m2} - \mathbf{A}_{m3}$

P	*Cartesian*		*Polar*	
	P_x	P_y	*Force Amplitude*	*Phase*
$\mathbf{P}_{k3}$	+ 4.252	+20.540*j*	21.0 lb	+ 78.3°
$\mathbf{P}_{c3}$	−13.693	+ 2.834*j*	14.0 lb	+168.3°
$\mathbf{P}_{m3}$	− 5.873	−28.372*j*	29.0 lb	−101.7°
$\mathbf{P}_{m2}$	+ 7.618	−14.252*j*	16.2 lb	− 61.9°
$\mathbf{P}_{m1}$	+17.696	+19.250*j*	26.1 lb	+ 47.4°
$\Sigma\mathbf{P} = \mathbf{P}_0$	+10.000	0.000*j*	10.0 lb	0.0°
$\mathbf{P}_{k1}$	− 7.696	−19.250*j*	20.7 lb	−111.8°
$\mathbf{P}_{k2}$	−15.282	+ 0.096*j*	15.3 lb	+179.6°
$\mathbf{P}_{c2}$	− 0.032	− 5.096*j*	5.1 lb	− 90.4°

External and Inertia Forces

$\mathbf{P}_0 - (\mathbf{P}_{k3} + \mathbf{P}_{c3}) = \mathbf{P}_{m3} + \mathbf{P}_{m2} + \mathbf{P}_{m1}$

Internal Forces

$\mathbf{P}_1 = \mathbf{P}_{k1} = \mathbf{P}_2 = \mathbf{z}_{k1}(\mathbf{A}_{m1} - \mathbf{A}_{m2})$;

$\mathbf{P}_3 = \mathbf{P}_{k2} + \mathbf{P}_{c2} = \mathbf{P}_4 = (\mathbf{z}_{k2} + \mathbf{z}_{c2})(\mathbf{A}_{m2} - \mathbf{A}_{m3})$

Work and power dissipation. The work per cycle dissipated in a damping element was calculated in section 4.6. It is convenient to develop a more general expression for the work done at any point in the system where the amplitudes and effective forces may have any arbitrary phase angles. Let the force acting at any point in the system be

$$P_x = P\cos(\omega t + \phi_P) \tag{6.5–17}$$

and the displacement of the point of application of the force be

$$x = A\cos(\omega t + \phi_A) \tag{6.5–18}$$

From this, the velocity is seen to be

$$\dot{x} = -A\omega\sin(\omega t + \phi_A) \tag{6.5–19}$$

The increment of work done during the time dt by force P_x is

$$\begin{aligned} dW &= P_x\,dx = P_x\frac{dx}{dt}\,dt = P_x\dot{x}\,dt \\ &= -\omega PA\cos(\omega t + \phi_P)\sin(\omega t + \phi_A)\,dt \\ &= -\tfrac{1}{2}\omega PA[\sin(\phi_A - \phi_P) + \sin(2\omega t + \phi_A + \phi_P)]\,dt \end{aligned}$$

The power or the rate at which work is done is

$$\dot{W} = \frac{dW}{dt} = \tfrac{1}{2}\omega PA[\sin(\phi_P - \phi_A) - \sin(2\omega t + \phi_P + \phi_A)] \tag{6.5–20}$$

The variation of the values of P, x, $\dot{x}$ and $\dot{W}$ with time are shown in Fig. 6–19. For convenience the time scale has been measured in terms of ωt, and the diagram corresponds to one cycle of the motion.

It is evident from the figure that the maximum and minimum values of the power $\dot{W}$ will be

$$\tfrac{1}{2}\omega PA[\sin(\phi_P - \phi_A) \pm 1] \tag{6.5–21}$$

whereas the average power, which may be obtained by an integration over a cycle or directly from the figure, is

$$\dot{W}_{\text{avg}} = \tfrac{1}{2}\omega PA\sin(\phi_P - \phi_A) \tag{6.5–22}$$

The work per cycle is then

$$W_{\sim} = \tau\dot{W}_{\text{avg}} = \frac{2\pi}{\omega}\dot{W}_{\text{avg}} = \pi PA\sin(\phi_P - \phi_A) \tag{6.5–23}$$

It is seen that both the power and the work per cycle are proportional to the quantity $PA\sin(\phi_P - \phi_A)$ and, depending on the sign of $\sin(\phi_P - \phi_A)$, can have both positive and negative values. The positive value indicates the work done on the system whereas the negative value indicates work done by the system.

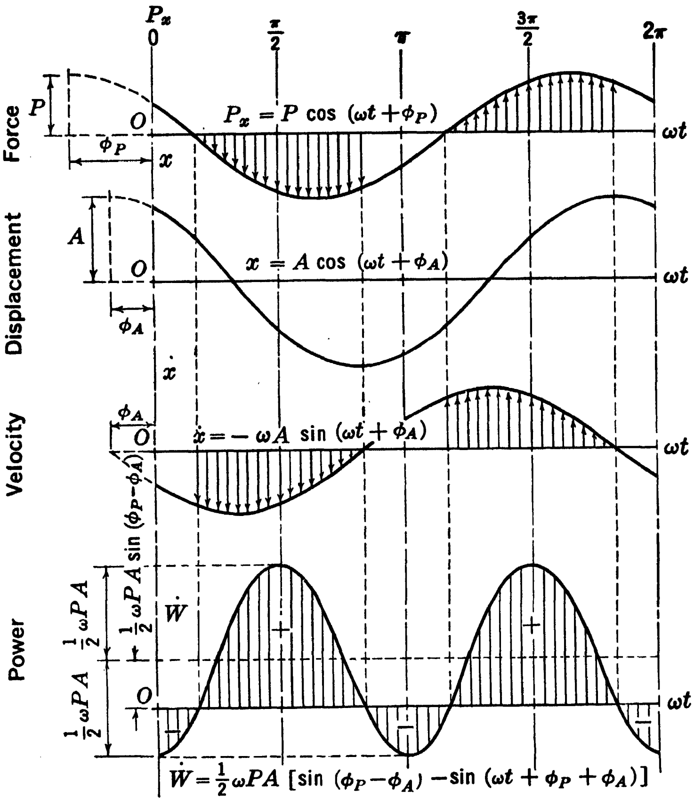


FIG. 6–19

The previous treatment may be discussed in terms of complex forces and displacements. Let the complex force and amplitude be

$$\left.\begin{aligned} \mathbf{P} &= P_x + jP_y \quad \text{and} \quad \mathbf{A} = X + jY \\ P &= \sqrt{P_x^2 + P_y^2} \quad \text{and} \quad A = \sqrt{X^2 + Y^2} \end{aligned}\right\} \qquad (6.5\text{–}24)$$

where

Then

$$\sin(\phi_P - \phi_A) = \sin\phi_P\cos\phi_A - \cos\phi_P\sin\phi_A$$

$$= \frac{P_y}{P}\cdot\frac{X}{A} - \frac{P_x}{P}\cdot\frac{Y}{A} = \frac{P_yX - P_xY}{PA}$$

and the power becomes

$$\dot{W}_{avg} = \tfrac{1}{2}\omega PA\sin(\phi_P - \phi_A) = \tfrac{1}{2}\omega PA\,\frac{P_yX - P_xY}{PA}$$

$$= \tfrac{1}{2}\omega(P_yX - P_xY) \qquad (6.5\text{–}25)$$

The work done per cycle is

$$W_{\sim} = \pi PA \sin(\phi_P - \phi_A) = \pi(P_y X - P_x Y) \qquad (6.5\text{–}26)$$

These expressions may be simplified for "open" systems where it is always possible to express the amplitude as a function of a single force, i.e.,

$$\mathbf{A} = \mathbf{Pa}$$

Writing $\mathbf{a} = a_x + ja_y$ permits this to take the form

$$\mathbf{A} = (P_x + jP_y)(a_x + ja_y) = P_x a_x - P_y a_y + j(P_x a_y + P_y a_x)$$
$$= X + jY$$

thus,

$$X = P_x a_x - P_y a_y \quad \text{and} \quad Y = P_x a_y + P_y a_x \qquad (6.5\text{–}27)$$

The power $\dot{W}$ may now be written as

$$\dot{W}_{\text{avg}} = \tfrac{1}{2}\omega[P_y(P_x a_x - P_y a_y) - P_x(P_x a_y + P_y a_x)]$$
$$= -\tfrac{1}{2}\omega(P_x^2 + P_y^2)a_y = -\tfrac{1}{2}\omega P^2 a_y \qquad (6.5\text{–}28)$$

for the open system. The work per cycle may similiarly be expressed as

$$W_{\sim} = -\pi P^2 a_y \qquad (6.5\text{–}29)$$

Equations 6.5–28 and 6.5–29 may be deduced in another fashion since the phase angle of the amplitude per unit force **a** is simply the relative phase angle between **A** and **P.** Thus

$$\sin(\phi_A - \phi_P) = -\sin(\phi_P - \phi_A) = \frac{a_y}{a}$$

whence equation 6.5–26 becomes

$$W_{\sim} = -\pi PA\,\frac{a_y}{a} = -\pi P^2 a_y$$

and similiarly for equation 6.5–28. It will be noted that only negative values of a_y will produce positive work. This is equivalent to the statement that the force vector must lead the displacement vector by a phase angle less than 180° if positive work is to be done.

The corresponding expressions for power and work per cycle for systems in rotation can be obtained from the expressions for translation by substitution of torque for force and angular displacement for linear displacement. Thus

$$M_x = M\cos(\omega t + \phi_M) \quad \text{and} \quad \Theta_x = \Theta\cos(\omega t + \phi_\Theta)$$

and the power becomes

$$\dot{W} = \tfrac{1}{2}\omega M\Theta[\sin(\phi_M - \phi_\Theta) - \sin(2\omega t + \phi_M + \phi_\Theta)]$$

The average value of the power is

$$\dot{W}_{avg} = \tfrac{1}{2}\omega M\Theta \sin(\phi_M - \phi_\Theta)$$

and its maximum and minimum values are

$$\tfrac{1}{2}\omega M\Theta[\sin(\phi_M - \phi_\Theta) \pm 1]$$

The work done per cycle may be expressed as

$$W_\sim = \pi M\Theta \sin(\phi_M - \phi_\Theta)$$

The torque and the amplitude may be expressed in the form,

$$\mathbf{M} = M_x + jM_y \quad \text{and} \quad \mathbf{\Theta} = \Theta_x + j\Theta_y$$

and the angular amplitude per unit force may be written as

$$\boldsymbol{\theta} = \theta_x + j\theta_y$$

In this notation the power and work may be expressed in the form of equations 6.5–25 and 6.5–26. For an open system, the expressions for power and work have forms similiar to equations 6.5–28 and 6.5–29, namely,

$$\dot{W}_{avg} = -\tfrac{1}{2}\omega M^2\theta_y$$

$$W_\sim = -\pi M^2\theta_y$$

Numerical calculation of work and power dissipation. The numerical values of the average power dissipated in the open system may be found now by the use of equation 6.5–28 and Tables 4 and 5.

The total power transmitted through terminal 0 is

$$\dot{W}_0 = -\tfrac{1}{2}\omega P_0{}^2 a_{0y} = -\tfrac{1}{2}(100)(10)^2(-4.949 \times 10^{-2})$$

$$= 247.4 \text{ in. lb per sec}$$

or, in terms of horsepower,

$$\dot{W}_0 = \frac{247.4}{(12)(550)} = 0.0375 \text{ hp}$$

The power transmitted through terminal 1 is

$$\dot{W}_1 = -\tfrac{1}{2}\omega P_1{}^2 a_{1y} = -\tfrac{1}{2}(100)(20.7)^2(-1.151 \times 10^{-2}) = 247.4 \text{ in. lb per sec}$$

or the same as terminal 0. A calculation of the power transmitted through terminal 3 will give the same result, since the work per cycle done on the mass m_1 and on the spring k_1 is zero. Work or power can be dissipated only at a damping element; therefore the power transmitted through terminal 3 will still be found to be the same, i.e.,

$$\begin{aligned}\dot{W}_3 &= -\tfrac{1}{2}\omega P_3{}^2 a_{3y} = -\tfrac{1}{2}(100)(16.11)^2(-1.907 \times 10^{-2}) \\ &= 247.4 \text{ in. lb per sec}\end{aligned}$$

The power that is transmitted through terminal 4, however, is

$$\begin{aligned}\dot{W}_4 &= -\tfrac{1}{2}\omega P_4{}^2 a_{4y} = -\tfrac{1}{2}(100)(16.11)^2(-1.507 \times 10^{-2}) \\ &= 195.5 \text{ in. lb per sec}\end{aligned}$$

The difference in power transmitted through terminals 3 and 4 is therefore

$$247.4 - 195.5 = 51.9 \text{ in. lb per sec}$$

This difference in power is the power that is dissipated in the damper c_2, and it can be found directly by computing the power dissipated either by the combined system of k_2 and c_2 or by the damping element c_2 itself. Thus,

$$\dot{W}_{34} = -\tfrac{1}{2}\omega P_{34}{}^2 a_{34y} = -\tfrac{1}{2}(100)(16.11)^2(-0.4 \times 10^{-2}) = 51.9 \text{ in. lb per sec}$$

or

$$\begin{aligned}\dot{W}_{c2} &= -\tfrac{1}{2}\omega P_{c2} a_{c2y} = -\tfrac{1}{2}\omega(z_{c2}A_{c2})^2 a_{c2y} = -\tfrac{1}{2}\omega(z_{c2}A_{34})^2 a_{c2y} \\ &= -\tfrac{1}{2}(100)(25 \times 0.2038)^2(-0.04) = 51.9 \text{ in. lb per sec}\end{aligned}$$

In a similar manner the power transmitted through terminal 4 must be dissipated by the damping element c_3, i.e.:

$$\begin{aligned}\dot{W}_{c3} &= -\tfrac{1}{2}\omega P_{c3}{}^2 a_{c3y} = -\tfrac{1}{2}\omega(z_{c3}A_4)^2 a_{c3y} \\ &= -\tfrac{1}{2}(100)(50 \times 0.280)^2(-0.02) = 196 \text{ in. lb per sec}\end{aligned}$$

which checks the above value. In general the different ways of obtaining the same value of the power transmitted or dissipated are a practical help in checking the computations.

The energy dissipated in the damping elements c_2 and c_3 is transmitted through the masses m_1 and m_2, as may be seen from Fig. 6–16a. The energy dissipated in c_3 is evidently transmitted entirely through m_3, whereas the energy dissipated in c_2 may be transmitted from either m_2 or m_3 or both. It is of interest to determine the "flow" of the energy or the power through the various elements.

It is clear that, in this particular problem, the work or energy must all "flow" through the spring k_1 into the mass m_2. From here it "flows" into the damper c_2 and through the spring k_2. The energy which "flows" through the spring k_2 into the mass m_3 in this example will be dissipated in part in the damper c_2 and the damper c_3. The actual power distribution may be calculated with the aid of equation 6.5–25 and Table 5. At terminal 3, the power distribution is as follows:

$$\begin{aligned} \dot{W}_{c2} &= \tfrac{1}{2}\omega(P_{c2y}X_{m2} - P_{c2x}Y_{m2}) = -\tfrac{1}{2}\omega P_{c2}A_{m2}\sin(\phi_{Pc2} - \phi_{Am2}) \\ &= \tfrac{1}{2}(100)[(-5.096)(-0.14707) - (-0.032)(0.27514)] \\ &= 37.9 \text{ in. lb per sec} \end{aligned}$$

and

$$\begin{aligned} \dot{W}_{k2} &= \tfrac{1}{2}\omega(P_{k2y}X_{m2} - P_{k2x}Y_{m2}) = \tfrac{1}{2}\omega P_{k2}A_{m2}\sin(\phi_{Pk2} - \phi_{Am2}) \\ &= \tfrac{1}{2}(100)[(0.096)(-0.14707) - (-15.282)(0.27514)] \\ &= 209.5 \text{ in. lb per sec} \end{aligned}$$

whereas the power distribution at terminal 4 is

$$\begin{aligned} \dot{W}_{c2} &= \tfrac{1}{2}\omega(P_{c2y}X_{m3} - P_{c2x}Y_{m3}) = \tfrac{1}{2}\omega P_{c2}A_{m3}\sin(\phi_{Pc2} - \phi_{Am2}) \\ &= \tfrac{1}{2}(100)[(-5.096)(0.05669) - (-0.032)(0.27386)] \\ &= -14.0 \text{ in. lb per sec} \end{aligned}$$

and

$$\begin{aligned} \dot{W}_{k2} &= \tfrac{1}{2}\omega(P_{k2}X_{m3} - P_{k2x}Y_{m3}) = \tfrac{1}{2}\omega P_{k2}A_{m3}\sin(\phi_{Pk2} - \phi_{Am2}) \\ &= \tfrac{1}{2}(100)[(0.096)(0.05669) - (-15.282)(0.27386)] \\ &= 209.5 \text{ in. lb per sec} \end{aligned}$$

which agrees with the result calculated above.

The net energy which "flows" into m_3 is therefore 209.5 in. lb per sec of which 14.0 in. lb per sec is dissipated in damper c_2 and 195.5 in. lb per sec in damper c_3. The total power dissipated in damper c_2 is

$$37.9 + 14.0 = 51.9 \text{ in. lb per sec}$$

and the total power dissipated in both dampers is

$$51.9 + 195.5 = 247.4 \text{ in. lb per sec}$$

which checks with the power input into the system. The "flow" of energy is shown diagrammatically in terms of percentages in Fig. 6–23.

The closed system

The system shown diagrammatically in Fig. 6–17*a* is, as previously mentioned, a closed system. Such a system consists, in general, of mass elements which are interconnected through springs and dampers in more than one way. A disturbing force or torque acting on the system at one point will therefore affect the mass elements through more than one "channel" as exemplified by Figs. 6–17*a* and 6–17*b*. For instance, the disturbing force $\mathbf{P}_0$ will act on the mass m_2 in part through the spring k_1 and in part through the elements c_1, m_3, k_2, and c_2. The removal of either the damper c_1 or the spring k_1 makes this

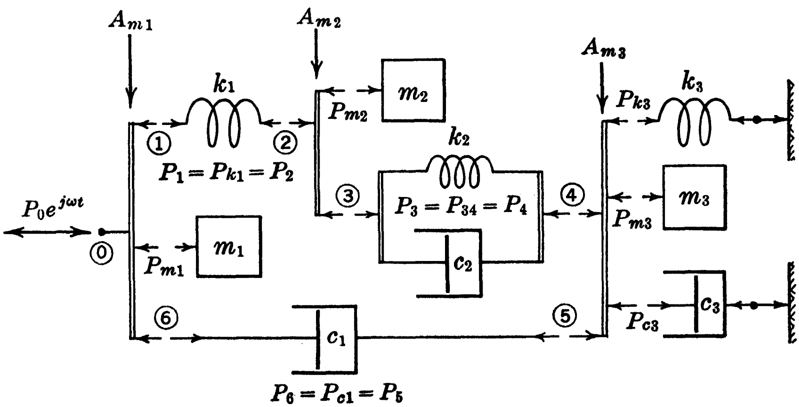


FIG. 6–20

system an open system, and the force $\mathbf{P}_0$ will then have only one "channel" through which to affect the masses m_2 and m_3.

In the open system, it is possible to determine the amplitudes per unit force at each terminal in terms of the characteristics of the elements on the opposite side of the terminal from that on which the disturbing force acts. For a closed system, this cannot be done in general, and the unit amplitudes at each terminal will involve the whole system. Owing to this interdependence, the analysis of the motion of the system must be accomplished by the solution of simultaneous equations.

Forces and total amplitudes. For the specific example shown in Fig. 6–17*a*, a "free-body" diagram indicating the forces acting through the amplitudes $\mathbf{A}_{m1}$, $\mathbf{A}_{m2}$, and $\mathbf{A}_{m3}$ of the respective masses is shown in Fig. 6–20. The vectorial sum of these forces must balance, that is:

$$\mathbf{P}_0 = \mathbf{P}_{m1} + \mathbf{P}_1 + \mathbf{P}_6 \tag{6.5–30}$$

$$\mathbf{P}_2 = \mathbf{P}_{m2} + \mathbf{P}_3 \tag{6.5–31}$$

$$\mathbf{P}_4 + \mathbf{P}_5 = \mathbf{P}_{m3} + \mathbf{P}_{k3} + \mathbf{P}_{c3} \tag{6.5–32}$$

but

$$\left.\begin{aligned}
\mathbf{P}_1 &= \mathbf{P}_2 = \mathbf{P}_{k1} = \mathbf{z}_{k1}\mathbf{A}_{k1} = \mathbf{z}_{k1}(\mathbf{A}_1 - \mathbf{A}_2) \\
&\qquad = \mathbf{z}_{k1}(\mathbf{A}_{m1} - \mathbf{A}_{m2}) \\
\mathbf{P}_3 &= \mathbf{P}_4 = \mathbf{P}_{34} = \mathbf{z}_{34}\mathbf{A}_{34} = (\mathbf{z}_{k2} + \mathbf{z}_{c2})(\mathbf{A}_3 - \mathbf{A}_4) \\
&\qquad = (\mathbf{z}_{k2} + \mathbf{z}_{c2})(\mathbf{A}_{m2} - \mathbf{A}_{m3}) \\
\mathbf{P}_5 &= \mathbf{P}_6 = \mathbf{P}_{c1} = \mathbf{z}_{c1}\mathbf{A}_{c1} = \mathbf{z}_{c1}(\mathbf{A}_6 - \mathbf{A}_5) \\
&\qquad = \mathbf{z}_{c1}(\mathbf{A}_{m1} - \mathbf{A}_{m3}) \\
&\text{and} \\
\mathbf{P}_{m1} &= \mathbf{z}_{m1}\mathbf{A}_{m1}, \quad \mathbf{P}_{m2} = \mathbf{z}_{m2}\mathbf{A}_{m2}, \quad \mathbf{P}_{m3} = \mathbf{z}_{m3}\mathbf{A}_{m3} \\
&\qquad \mathbf{P}_{k3} = \mathbf{z}_{k3}\mathbf{A}_{m3}, \quad \mathbf{P}_{c3} = \mathbf{z}_{c3}\mathbf{A}_{m3}
\end{aligned}\right\} \quad (6.5\text{–}33)$$

Substitution of these values into the above equations gives

$$\left.\begin{aligned}
\mathbf{P}_0 &= \mathbf{z}_{m1}\mathbf{A}_{m1} + \mathbf{z}_{k1}(\mathbf{A}_{m1} - \mathbf{A}_{m2}) + \mathbf{z}_{c1}(\mathbf{A}_{m1} - \mathbf{A}_{m3}) \\
\mathbf{z}_{k1}(\mathbf{A}_{m1} - \mathbf{A}_{m2}) &= \mathbf{z}_{m2}\mathbf{A}_{m2} + (\mathbf{z}_{k2} + \mathbf{z}_{c2})(\mathbf{A}_{m2} - \mathbf{A}_{m3}) \\
(\mathbf{z}_{k2} + \mathbf{z}_{c2})(\mathbf{A}_{m2} - \mathbf{A}_{m3}) &+ \mathbf{z}_{c1}(\mathbf{A}_{m1} - \mathbf{A}_{m2}) \\
&= (\mathbf{z}_{m3} + \mathbf{z}_{k3} + \mathbf{z}_{c3})\mathbf{A}_{m3}
\end{aligned}\right\} \quad (6.5\text{–}34)$$

Addition of these equations gives

$$\begin{aligned}
\mathbf{P}_0 &= \mathbf{z}_{m1}\mathbf{A}_{m1} + \mathbf{z}_{m2}\mathbf{A}_{m2} + (\mathbf{z}_{m3} + \mathbf{z}_{k3} + \mathbf{z}_{c3})\mathbf{A}_{m3} \\
&= \mathbf{P}_{m1} + \mathbf{P}_{m2} + \mathbf{P}_{m3} + \mathbf{P}_{k3} + \mathbf{P}_{c3}
\end{aligned}$$

which shows that the "external forces" acting on the system and the inertia forces of the system must balance. The addition of the equations in this manner provides a convenient check on the work and may be made at any time during the analysis on both the open and closed systems.

The rearrangement of the equations in terms of amplitudes permits them to be rewritten as

$$\begin{aligned}
(\mathbf{z}_{k1}+\mathbf{z}_{m1}+\mathbf{z}_{c1})\mathbf{A}_{m1} \quad -\mathbf{z}_{k1}\mathbf{A}_{m2} \quad -\mathbf{z}_{c1}\mathbf{A}_{m3} &= \mathbf{P}_0 \\
-\mathbf{z}_{k1}\mathbf{A}_{m1} + (\mathbf{z}_{k1}+\mathbf{z}_{k2}+\mathbf{z}_{m2}+\mathbf{z}_{c2})\mathbf{A}_{m2} \quad -(\mathbf{z}_{k2}+\mathbf{z}_{c2})\mathbf{A}_{m3} &= 0 \\
-\mathbf{z}_{c1}\mathbf{A}_{m1} \quad -(\mathbf{z}_{k2}+\mathbf{z}_{c2})\mathbf{A}_{m2} + (\mathbf{z}_{k2}+\mathbf{z}_{k3}+\mathbf{z}_{m3}+\mathbf{z}_{c1}+\mathbf{z}_{c2}+\mathbf{z}_{c3})\mathbf{A}_{m3} &= 0
\end{aligned} \quad (6.5\text{–}35)$$

The amplitudes may be evaluated from these equations as

$$\mathbf{A}_{m1} = \frac{\Delta_1}{\Delta}\mathbf{P}_0, \quad \mathbf{A}_{m2} = \frac{\Delta_2}{\Delta}\mathbf{P}_0, \quad \text{and} \quad \mathbf{A}_{m3} = \frac{\Delta_3}{\Delta}\mathbf{P}_0$$

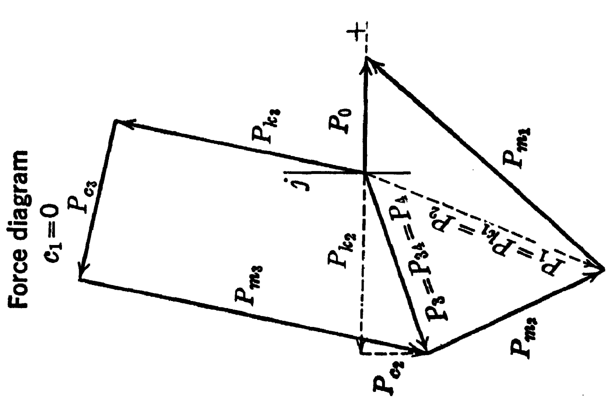


FIG. 6–21

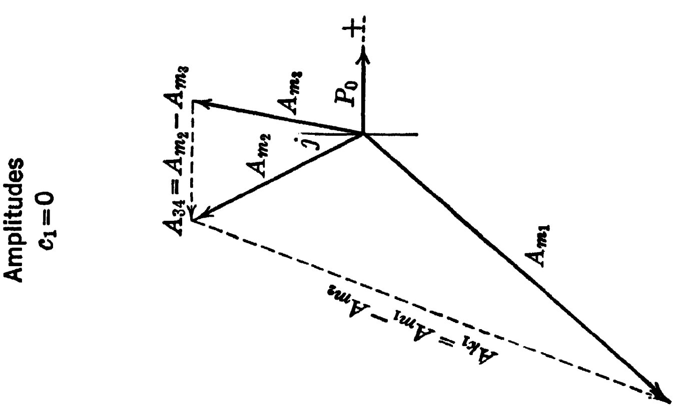


FIG. 6–22

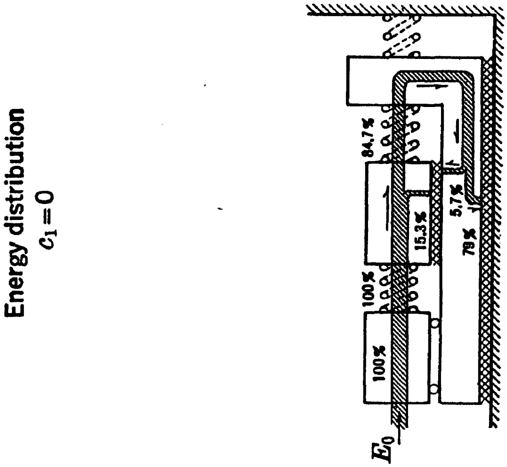


FIG. 6–23

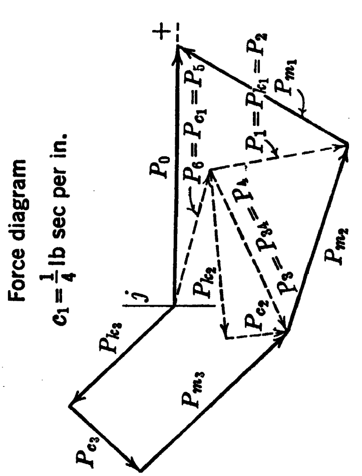


FIG. 6–24

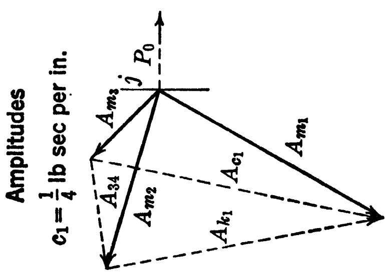


FIG. 6–25

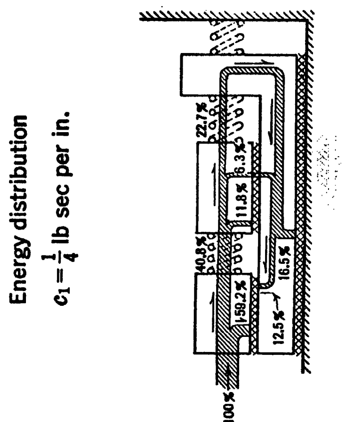


FIG. 6–26

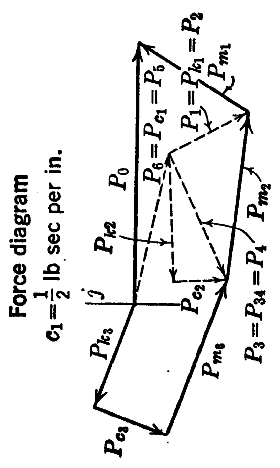


FIG. 6–27

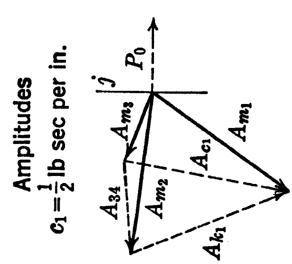


FIG. 6–28

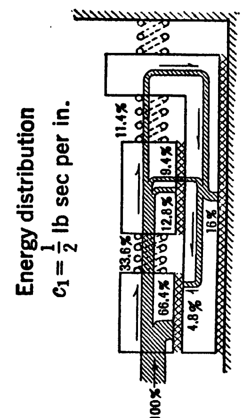


FIG. 6–29

Force diagram
$c_1 = 1$ lb sec per in.

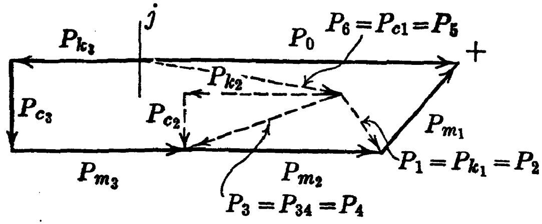


FIG. 6–30

Amplitudes
$c_1 = 1$ lb sec per in.

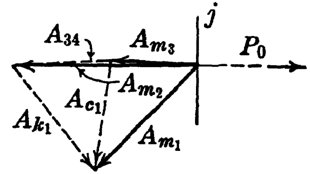


FIG. 6–31

Energy distribution
$c_1 = 1$ lb sec per in.

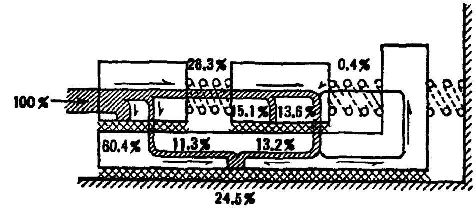


FIG. 6–32

Force diagram
$c_1 = \infty$

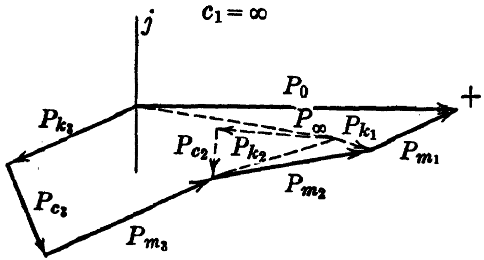


FIG. 6–33

Amplitudes
$c_1 = \infty$

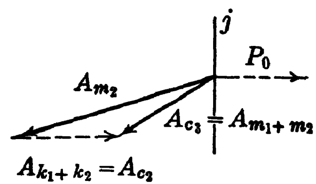


FIG. 6–34

Energy distribution
$c_1 = \infty$

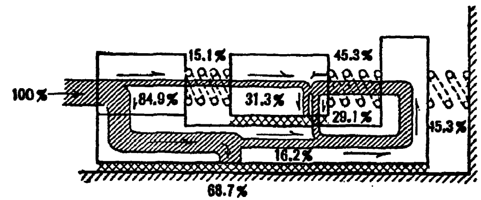


FIG. 6–35

where Δ_1, Δ_2, Δ_3 and Δ are given by

$$\Delta = \begin{vmatrix} (z_{k1}+z_{m1}+z_{c1}) & -z_{k1} & -z_{c1} \\ -z_{k1} & (z_{k1}+z_{k2}+z_{m2}+z_{c2}) & -(z_{k2}+z_{c2}) \\ -z_{c1} & -(z_{k2}+z_{c2}) & (z_{k2}+z_{k3}+z_{m3}+z_{c1}+z_{c2}+z_{c3}) \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} 1 & -z_{k1} & -z_{c1} \\ 0 & (z_{k1}+z_{k2}+z_{m2}+z_{c2}) & -(z_{k2}+z_{c2}) \\ 0 & -(z_{k2}+z_{c2}) & (z_{k2}+z_{k3}+z_{m3}+z_{c1}+z_{c2}+z_{c3}) \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} (z_{k1}+z_{m1}+z_{c1}) & 1 & -z_{c1} \\ -z_{k1} & 0 & -(z_{k2}+z_{c2}) \\ -z_{c1} & 0 & (z_{k2}+z_{k3}+z_{m3}+z_{c1}+z_{c2}+z_{c3}) \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} (z_{k1}+z_{m1}+z_{c1}) & -z_{k1} & 1 \\ -z_{k1} & (z_{k1}+z_{k2}+z_{m2}+z_{c2}) & 0 \\ -z_{c1} & -(z_{k2}+z_{c2}) & 0 \end{vmatrix}$$

It should be noted that

$$\Delta = 0 \tag{6.5–36}$$

is the frequency equation. The roots of this equation occur in conjugate pairs. The real part of each root denotes the natural circular frequency, whereas the imaginary part indicates the rate at which the motion is damped out. See section 7.3 for the determination of these roots by numerical methods. The solution for the open system can be found from these equations by setting $z_{c1} = 0$; however the previous treatment for the open system will usually be found to be less cumbersome. The four determinates may be expanded by the usual methods to

$$\begin{aligned} \Delta = {} & (z_{k1} + z_{m1} + z_{c1})(z_{k1} + z_{k2} + z_{m2} + z_{c2}) \times \\ & \qquad (z_{k2} + z_{k3} + z_{m3} + z_{c1} + z_{c2} + z_{c3}) \\ & - 2z_{k1}z_{c1}(z_{k2} + z_{c2}) - z_{c1}^2(z_{k1} + z_{k2} + z_{m2} + z_{c2}) \\ & - (z_{k2} + z_{c2})^2(z_{k1} + z_{m1} + z_{c1}) \\ & \qquad - z_{k1}^2(z_{k2} + z_{k3} + z_{m3} + z_{c1} + z_{c2} + z_{c3}) \end{aligned}$$

$$\begin{aligned} \Delta_1 = {} & (z_{k1} + z_{k2} + z_{m2} + z_{c2})(z_{k2} + z_{k3} + z_{m3} \\ & \qquad + z_{c1} + z_{c2} + z_{c3}) - (z_{k2} + z_{c2})^2 \end{aligned}$$

$$\Delta_2 = z_{c1}(z_{k2} + z_{c2}) + z_{k1}(z_{k2} + z_{k3} + z_{m3} + z_{c1} + z_{c2} + z_{c3})$$

$$\Delta_3 = z_{k1}(z_{k2} + z_{c2}) + z_{c1}(z_{k1} + z_{k2} + z_{m2} + z_{c2})$$

Numerical calculation of forces and total amplitudes. The numerical values of the total amplitudes are obtained by direct substitution from Table 3.

Thus

$$\Delta = (-13.9 + 50j)(48.2 + 25j)(46.4 + 125j) - 2(25)(50j)(75 + 25j) + (50)^2(48.2 + 25j) - (75 + 25j)^2(-13.9 + 50j) - (25)^2(46.4 + 125j)$$

As a matter of expediency, for these particular numerical values, it will be found convenient to reduce both sides of the equation by the factor $(25)^{-3}$. Thus

$$\begin{aligned}(25)^{-3}\Delta &= (-0.556 + 2j)(1.928 + j)(1.856 + 5j) - 4j(3 + j) \\ &\quad + 4(1.928 + j) - (3 + j)^2(0.556 + 2j) - (1.856 + 5j) \\ &= 4.1024 - 34.894j\end{aligned}$$

Similarly:

$$(25)^{-2}\Delta_1 = (1.928 + j)(1.856 + 5j) + (3 + j)^2 = -9.4126 + 5.496j$$

$$(25)^{-2}\Delta_2 = 2j(3 + j) + (1.856 + 5j) = -0.1440 + 11.000j$$

$$(25)^{-2}\Delta_3 = (3 + j) + 2j(1.928 + j) = 1.000 + 4.856j$$

From these values, the amplitudes may be obtained as

$$\mathbf{A}_{m1} = \frac{\Delta_1}{\Delta}\mathbf{P}_0 = \frac{-9.4126 + 5.496j}{4.1024 - 34.894j} \cdot \frac{10}{25}$$

$$= -0.07466 - 0.09921j = 0.124 \text{ in. } \underline{/-127.0^\circ}$$

Similarly,

$$\mathbf{A}_{m2} = -0.12455 + 0.01299j = 0.125 \text{ in. } \underline{/179.0^\circ}$$

and

$$\mathbf{A}_{m3} = -0.05357 + 0.01776j = 0.056 \text{ in. } \underline{/161.8^\circ}$$

The evaluation of the forces in the system may now be made in the same manner as for the open system. The results of these calculations are tabulated in Table 6 and shown vectorially in Figs. 6–27 and 6–28.

Numerical calculation of the work and power dissipation. The "flow" of energy in the closed system may be found in the same manner as in the open system. The total power input is found from equation 6.5–25 and Tables 3 and 6 as

$$\dot{W}_0 = \tfrac{1}{2}\omega(P_{0y}X_{m1} - P_{0x}Y_{m1})$$

$$= \tfrac{1}{2}(100)[(0)(0.07466) - (10)(-0.09921)] = 49.6 \text{ in. lb per sec}$$

The "flow" of this energy through the system may be determined as follows. The energy transmitted through terminal 6 is

$$\dot{W}_6 = \tfrac{1}{2}\omega(P_{c1y}X_{m1} - P_{c1x}Y_{m1})$$

$$= \tfrac{1}{2}(100)[(-1.0545)(-0.07466) - (5.8485)(-0.09921)]$$

$$= 32.9 \text{ in. lb per sec}$$

TABLE 6
CLOSED SYSTEM

A	Cartesian		Polar	
	X	*Y*	*Amplitude*	*Phase*
$\mathbf{A}_{m1}$	−0.07466	−0.09921*j*	0.124 in.	−127.0°
$\mathbf{A}_{m2}$	−0.12455	+0.01299*j*	0.125 in.	+174.0°
$\mathbf{A}_{m3}$	−0.05357	+0.01776*j*	0.056 in.	+161.7°
$\mathbf{A}_{m1} - \mathbf{A}_{m2}$	+0.04989	−0.11220*j*	0.123 in.	− 66.0°
$\mathbf{A}_{m2} - \mathbf{A}_{m3}$	−0.07098	−0.00477*j*	0.071 in.	−176.1°
$\mathbf{A}_{m1} - \mathbf{A}_{m3}$	−0.02109	−0.11697*j*	0.119 in.	− 99.2°

Absolute Amplitudes

$\mathbf{A}_0 = \mathbf{A}_1 = \mathbf{A}_6 = \mathbf{A}_{m1}$; $\mathbf{A}_2 = \mathbf{A}_3 = \mathbf{A}_{m2}$; $\mathbf{A}_4 = \mathbf{A}_5 = \mathbf{A}_{k3} = \mathbf{A}_{c3} = \mathbf{A}_{m3}$

Relative Amplitudes

$\mathbf{A}_{12} = \mathbf{A}_{k1} = \mathbf{A}_{m1} - \mathbf{A}_{m2}$; $\mathbf{A}_{34} = \mathbf{A}_{k2} = \mathbf{A}_{c2} = \mathbf{A}_{m2} - \mathbf{A}_{m3}$;
$\mathbf{A}_{65} = \mathbf{A}_{c1} = \mathbf{A}_{m1} - \mathbf{A}_{m3}$

P	Cartesian		Polar	
	$\mathbf{P}_x$	$\mathbf{P}_y$	*Force Amplitude*	*Phase*
$\mathbf{P}_{k3}$	−4.0178	+1.3320*j*	4.23 lb	+161.7°
$\mathbf{P}_{c3}$	−0.8880	−2.6785*j*	2.82 lb	−108.3°
$\mathbf{P}_{m3}$	+5.5498	−1.8399*j*	5.85 lb	− 18.3°
$\mathbf{P}_{m2}$	+6.4517	−0.6729*j*	6.48 lb	− 6.0°
$\mathbf{P}_{m1}$	+2.9043	+3.8593*j*	4.83 lb	+ 53.0°
$\Sigma\mathbf{P} = \mathbf{P}_0$	+10.0000	0.0000*j*	10.00 lb	0.0°
$\mathbf{P}_{k1}$	+1.2472	−2.8050*j*	3.07 lb	− 66.0°
$\mathbf{P}_{c1}$	+5.8485	−1.0545*j*	5.94 lb	− 10.2°
$\mathbf{P}_{k2}$	−5.3235	−0.3578*j*	5.34 lb	−176.1°
$\mathbf{P}_{c2}$	+0.1193	−1.7745*j*	1.78 lb	− 86.1°

External and Inertia Forces

$$\mathbf{P}_0 - (\mathbf{P}_{k3} + \mathbf{P}_{c3}) = \mathbf{P}_{m3} + \mathbf{P}_{m2} + \mathbf{P}_{m1}$$

Internal Forces

$$\mathbf{P}_1 = \mathbf{P}_{k1} = \mathbf{P}_2 = \mathbf{z}_{k1}(\mathbf{A}_{m1} - \mathbf{A}_{m2}); \quad \mathbf{P}_3 = \mathbf{P}_{k2} + \mathbf{P}_{c2} = \mathbf{P}_4 = (\mathbf{z}_{k2} + \mathbf{z}_{c2})(\mathbf{A}_{m2} - \mathbf{A}_{m3})$$

$$\mathbf{P}_6 = \mathbf{P}_{c1} = \mathbf{P}_5 = \mathbf{z}_{c1}(\mathbf{A}_{m1} - \mathbf{A}_{m3})$$

The difference

$$49.6 - 32.9 = 16.7 \text{ in. lb per sec}$$

is transmitted through the spring k_1 into the mass m_2. The energy "flow" into the damper c_2 at terminal 3 is

$$\begin{aligned}\dot{W}_{c2} &= \tfrac{1}{2}\omega(P_{c2y}X_{m2} - P_{c2x}Y_{m2}) \\ &= \tfrac{1}{2}(100)[(-1.7745)(-0.12455) - (0.1193)(0.01299)] \\ &= 11.0 \text{ in. lb per sec}\end{aligned}$$

Again, the difference

$$16.7 - 11.0 = 5.7 \text{ in. lb per sec}$$

is transmitted through the spring k_2 to the mass m_3. The energy transmitted through the damper c_2 into the mass m_3 is

$$\begin{aligned}&\tfrac{1}{2}\omega(P_{c2y}X_{m3} + P_{c2x}Y_{m3}) \\ &\qquad = \tfrac{1}{2}(100)[(-1.7745)(-0.05357) - (0.1193)(0.01776)] \\ &\qquad = 4.7 \text{ in. lb per sec}\end{aligned}$$

The difference

$$11.0 - 4.7 = 6.3 \text{ in. lb per sec}$$

is the energy dissipated in the damper c_2. The effect of the damper c_1 on m_1 at terminal 5 is found to be

$$\begin{aligned}&\tfrac{1}{2}\omega(P_{c1y}X_{m3} - P_{c1x}Y_{m3}) \\ &\qquad = \tfrac{1}{2}(100)[(-1.0545)(-0.05357) - (5.8485)(0.01776)] \\ &\qquad = -2.4 \text{ in. lb per sec}\end{aligned}$$

The negative sign shows that work is done by the mass m_3 on the damper c_1, and so the total power dissipated by c_1 is

$$32.9 + 2.4 = 35.3 \text{ in. lb per sec}$$

The amount of power dissipated by c_3 is

$$\begin{aligned}\dot{W}_{c3} &= \tfrac{1}{2}\omega(P_{c3y}X_{m3} - P_{c3x}Y_{m3}) \\ &= \tfrac{1}{2}(100)[(-2.6785)(-0.05357) - (0.888)(0.01776)] \\ &= 8.0 \text{ in. lb per sec}\end{aligned}$$

The "flow" of energy for this closed system is indicated in Fig. 6–29 in terms of percent of the total input to the system.

The amplitudes, forces, and the power distribution changes with variations in c_1: For $c_1 = 1/4$, 1, and ∞, the results are shown in Figs. 6–24 to 6–26 and 6–30 to 6–35. It will be noted that the "flow" of energy for values of $c_1 > 1$ is reversed through the spring k_2 and a certain amount of energy is "circulating" or "trapped" in the system. This may be explained by realizing that the present analysis is based on a steady-state motion with a balanced energy flow. The energy "trapped" in the system is necessary for this balance and is built up in the system during the initial transient state.

6.6. Transmissibility

The effect of a vibrating system on its surroundings depends on how much of the disturbing force is transmitted through the vibrating system to the surrounding elements. The smaller the force that is

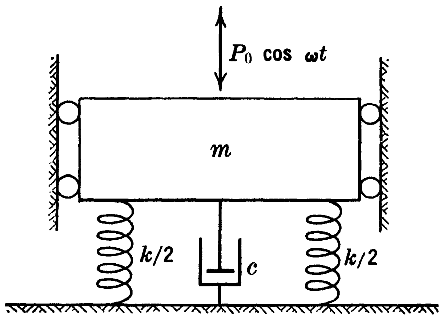


FIG. 6–36

transmitted from the vibrating system to the surroundings, the better isolated the system is said to be.

In general the transmissibility is defined as the ratio of the resultant amplitude of the forces or torques transmitted to the surroundings and the amplitude of the disturbing force or torque which acts directly on the system.

As a fundamental example, the transmissibility of the system indicated in Fig. 6–36 will be evaluated. This system has one degree of freedom, and its solution for forced vibration was obtained in section 4.2. The amplitude of the motion was given by equation 4.2–16 as

$$A_m = \frac{P_0/k}{\sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \left(\frac{\omega}{q}\right)^2}} \tag{6.6–1}$$

The resultant force transmitted to the supports is the vector sum of the spring force and the damping force, i.e.,

$$\mathbf{P}_T = \mathbf{P}_k + \mathbf{P}_c = \mathbf{A}_k \mathbf{z}_k + \mathbf{A}_c \mathbf{z}_c = \mathbf{A}_m(k + j\omega c)$$

$$= \mathbf{A}_m k\left(1 + j\frac{\omega}{q}\right)$$

since $\mathbf{A}_k = \mathbf{A}_c = \mathbf{A}_m$

The magnitude of the force $\mathbf{P}_T$ is

$$P_T = A_m k \sqrt{1 + \frac{\omega^2}{q^2}} = \frac{P_0 \sqrt{1 + \omega^2/q^2}}{\sqrt{(1 - \omega^2/p^2)^2 + \omega^2/q^2}} \tag{6.6–2}$$

from which the transmissibility of an oscillating force of constant amplitude is

$$\underset{\text{(constant force)}}{\text{Transmissibility}} = \frac{P_T}{P_0} = \sqrt{\frac{1 + \omega^2/q^2}{(1 - \omega^2/p^2)^2 + \omega^2/q^2}} \tag{6.6–3}$$

If the amplitude P_0 of the disturbing force $\mathbf{P}_0$ varies with the impressed frequency ω (inertia excitation), as in the case of a reciprocating piston or an unbalanced rotating mass, then P_0 may be expressed as

$$P_0 = m_0 e \omega^2$$

where m_0 is the mass of the piston or the unbalanced mass and e is the relative piston displacement or the equivalent eccentricity of the unbalanced mass. It will be convenient to write

$$P_0 = m_0 e p^2 \left(\frac{\omega}{p}\right)^2 = P_r \left(\frac{\omega}{p}\right)^2$$

where

$$P_r = m_0 e p^2 = \frac{m_0}{m} ek$$

is a constant representing the magnitude of the exciting force at undamped resonance. If the transmissibility is expressed as the ratio of the value of the force transmitted to the supports, to the value of P_r, then

$$\underset{\text{(inertia force)}}{\text{Transmissibility}} = \frac{P_T}{P_r} = \left(\frac{\omega}{p}\right)^2 \frac{P_T}{P_0} = \left(\frac{\omega}{p}\right)^2 \sqrt{\frac{1 + \omega^2/q^2}{(1 - \omega^2/p^2)^2 + (\omega/q)^2}} \tag{6.6–4}$$

The transmissibilities for constant-amplitude exciting force and inertia exciting force are shown in Fig. 6–37 and 6–38 respectively, for different amounts of damping, i.e., ratios p/q. It will be noted from these curves that the transmissibility, for this example, possessing one degree of freedom, is not affected by damping for $\omega/p = \sqrt{2}$. The transmissiblity is decreased by damping for $\omega/p < \sqrt{2}$ and increased by damping for $\omega/p > \sqrt{2}$.

In the more general problem of several degrees of freedom, as exemplified by Fig. 6–39, the resultant forces acting on each element may be evaluated from the amplitudes $\mathbf{A}_1$ and $\mathbf{A}_2$ of the masses m_1 and

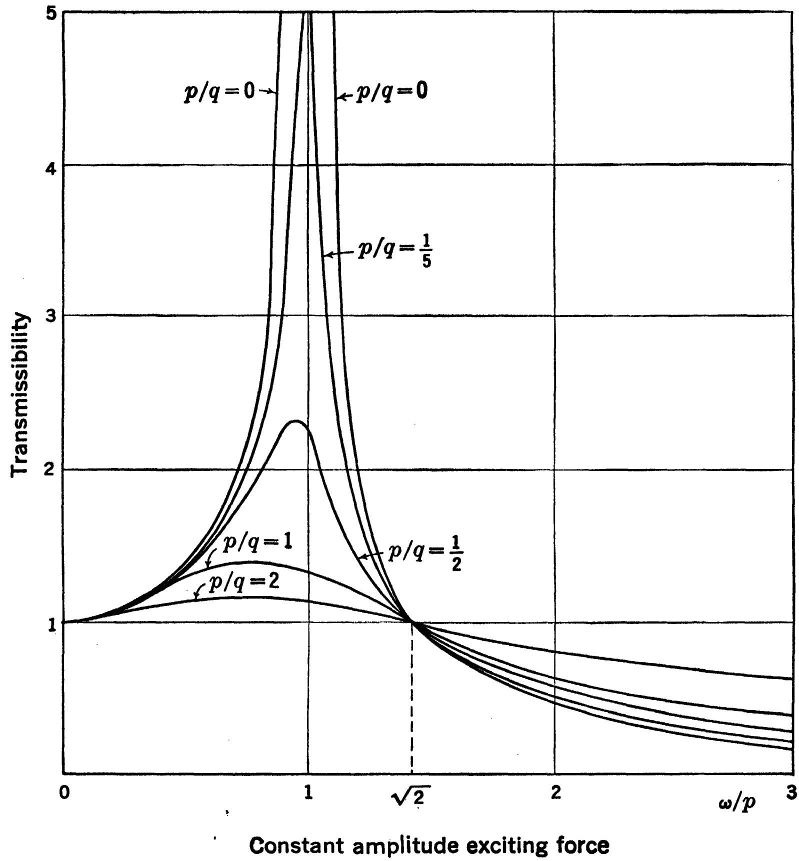


Fig. 6–37

m_2. The resultant force transmitted to the support is

$$\mathbf{P}_{k1} + \mathbf{P}_{c1} = \mathbf{P}_T = \mathbf{A}_1 k_1 + \mathbf{A}_2 j\omega c_1 \tag{6.6–5}$$

The resultant force acting on m_2 is

$$\mathbf{P}_{k2} + \mathbf{P}_{c2} - \mathbf{P}_{c1} = \mathbf{P}_{m2} = (\mathbf{A}_1 - \mathbf{A}_2)(k_2 + j\omega c_2) - \mathbf{A}_2 j\omega c_1 \tag{6.6–6}$$

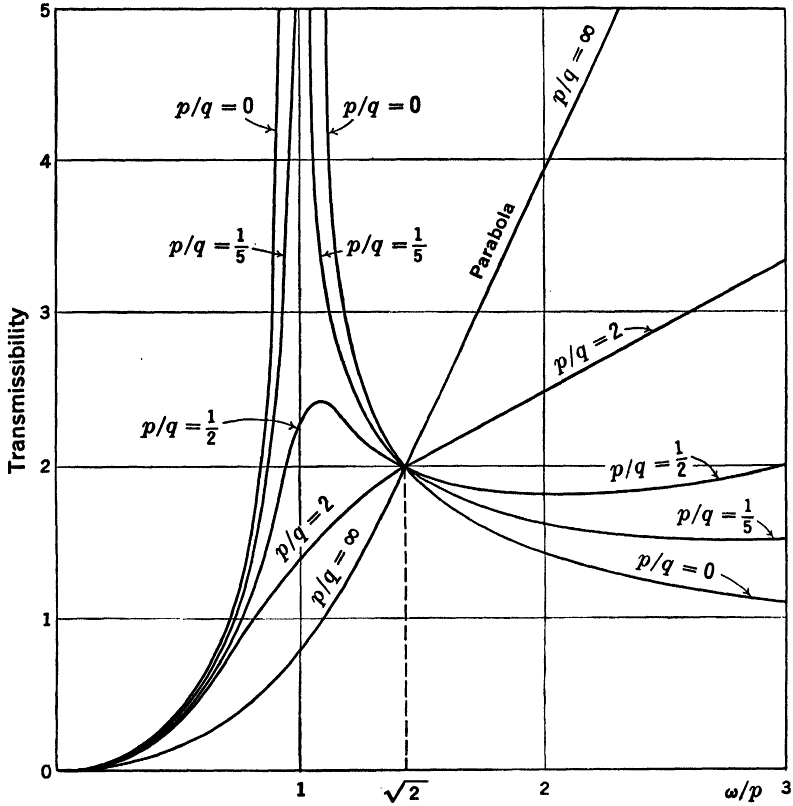
Similarly, the force that acts on m_1 is

$$\mathbf{P}_0 - \mathbf{P}_{k2} - \mathbf{P}_{c2} - \mathbf{P}_{k1} = \mathbf{P}_{m1} = \mathbf{P}_0 - (\mathbf{A}_1 - \mathbf{A}_2)(k_2 + j\omega c_2) - \mathbf{A}_1 k_1 \tag{6.6–7}$$

As an over-all check, it will be noted that the sum of these equations gives

$$\mathbf{P}_T + \mathbf{P}_{m1} + \mathbf{P}_{m2} = \mathbf{P}_0$$

The amplitudes $\mathbf{A}_1$ and $\mathbf{A}_2$ may be found by the methods of chapter 5



Inertia exciting force

FIG. 6–38

or 6. They are

$$\mathbf{A}_1 = \left[\frac{k_2 - \omega^2 m_2 + j\omega(c_1 + c_2)}{(k_1 - \omega^2 m_1)[k_2 - \omega^2 m_2 + j\omega(c_1 + c_2)] + (k_2 + j\omega c_2)(-\omega^2 m_2 + j\omega c_1)}\right]\mathbf{P}_0$$

and

$$\mathbf{A}_2 = \left[\frac{k_2 + j\omega c_2}{(k_1 - \omega^2 m_1)[k_2 - \omega^2 m_2 + j\omega(c_1 + c_2)] + (k_2 + j\omega c_2)(-\omega^2 m_2 + j\omega c_1)}\right]\mathbf{P}_0$$

By subsitution of these values into equation 6.6–5, the transmissibility is obtained as

$$\text{Transmissibility} = \frac{\mathbf{P}_T}{\mathbf{P}_0}$$

$$= \frac{k_1[k_2 - \omega^2 m_2 + j\omega(c_1 + c_2)] + j\omega c_1(k_2 + j\omega c_2)}{(k_1 - \omega^2 m_1)(k_2 - \omega^2 m_2) - \omega^2 k_2 m_2 - \omega^2 c_1 c_2 + j\omega[k_1(c_1 + c_2) + k_2 c_1 - \omega^2(m_1 c_1 + m_1 c_2 + m_2 c_2)]}$$

The absolute value of the transmissibility is therefore

$$\frac{P_T}{P_0} = \sqrt{\frac{[k_1(k_2 - \omega^2 m_2) - \omega^2 c_1 c_2]^2 + \omega^2[k_1(c_1 + c_2) + k_2 c_1]^2}{[(k_1 - \omega^2 m_1)(k_2 - \omega^2 m_2) - \omega^2(k_2 m_2 + c_1 c_2)]^2 + \omega^2[k_1(c_1 + c_2) + k_2 c_1 - \omega^2(m_1 c_1 + m_1 c_2 + m_2 c_2)]^2}} \qquad (6.6\text{–}8)$$

The resultant forces acting on the masses m_1 and m_2 may be obtained

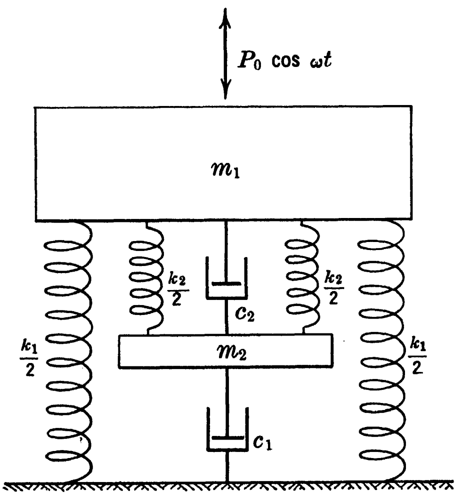


FIG. 6–39

in the same manner. For the particular values $c_1 = c_2 = 0$, there is obtained

$$\frac{P_T}{P_0} = \frac{k_1(k_2 - \omega^2 m_2)}{(k_1 - \omega^2 m_1)(k_2 - \omega^2 m_2) - \omega^2 m_2 k_2}$$

$$\frac{P_{m2}}{P_0} = \frac{-\omega^2 m_2 k_2}{(k_1 - \omega^2 m_1)(k_2 - \omega^2 m_2) - \omega^2 m_2 k_2}$$

$$\frac{P_{m1}}{P_0} = \frac{-\omega^2 m_1(k_2 - \omega^2 m_2)}{(k_1 - \omega^2 m_1)(k_2 - \omega^2 m_2) - \omega^2 m_2 k_2}$$

For this undamped example, the problem is reduced to the undamped vibration absorber treated in section 5.4. It will be seen that, for resonance of the absorber, i.e., $k_2 - \omega^2 m_2 = 0$,

$$\frac{P_{m1}}{P_0} = \frac{P_T}{P_0} = 0$$

and

$$\frac{P_{m2}}{P_0} = 1$$

This shows that the resultant force transmitted to the mass m_1 and the base is zero, whereas the total disturbing force is transmitted to the absorber mass m_2.

6.7. Two-Dimensional Systems Involving Plane Motion

The system shown in Fig. 6–40 is a combination of masses, springs,

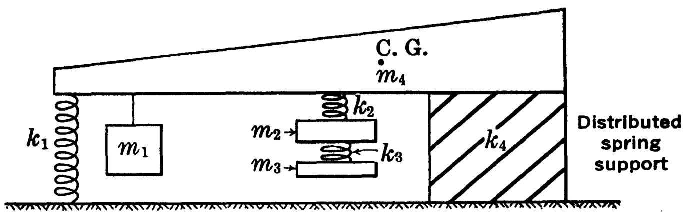


FIG. 6–40

and dampers capable of performing plane motion. The lever (the distributed mass m_4) is permitted to rotate in the plane and to translate vertically. The elements attached to the lever are assumed to move in a vertical line. Since the motion of the whole system is governed by the motion of the lever, it is expedient to use the lever as the reference mass. The system actually possesses two degrees of freedom for the lever and one additional degree for each additional elastically suspended mass. The analysis however is not greatly affected by the additional suspended masses, hence the treatment is not essentially different from that of a system of two degrees of freedom.

Lever systems without damping

If an oscillating torque M_0 is acting on the lever m_4, the system will oscillate in general in such a fashion that the lever rotates about some point O on its centroidal axis normal to the direction of motion, as shown in Fig. 6–41. It is convenient to consider the motion as one of rotation about a point C on the axis of the lever plus a translatory vibration of the point C of amplitude A_c. This point C, about which the sum of the moments of all of the dynamic forces present in the system during true translation is zero, will be referred to as the frequency center or centroid. It is so named because its position varies with the frequency of the oscillations of the system. The frequency centroid plays the same role in the oscillations of an elastic system as

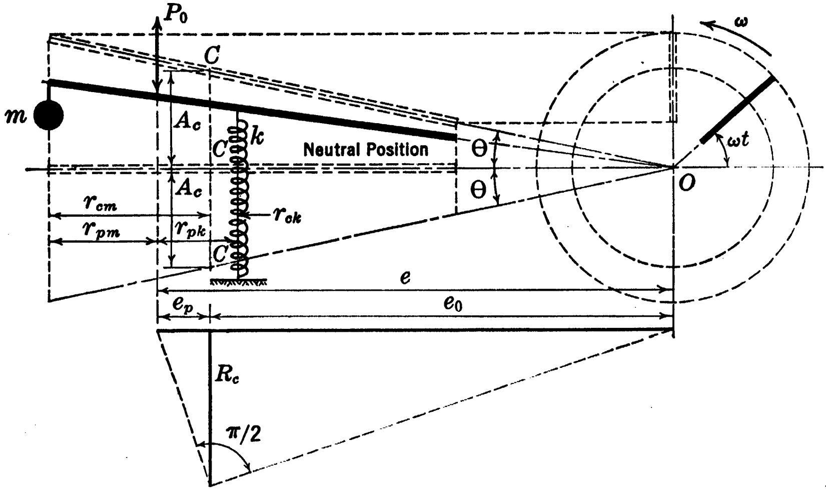


FIG. 6–41

the mass centroid plays in the plane motion of a rigid body.

From the definition of the frequency center, it follows that

$$\Sigma(\mathbf{z}A_c)r_c = A_c\Sigma\mathbf{z}r_c = 0$$

where $\mathbf{z}A_c$ is the amplitude of the dynamic force in the element whose amplitude for pure translation is A_c and whose impedance is $\mathbf{z}$; r_c is the corresponding lever arm, i.e. (r_{cm}, r_{ck}, etc.), as shown in Fig. 6–41. The summation is to extend over all masses and springs that affect the motion of the system. In the distributed masses or springs, the lever arm r_c is to be measured from the corresponding mass or spring centroid to the frequency center. The frequency center, therefore, is so located that

$$\Sigma\mathbf{z}r_c = 0 \qquad (6.7\text{–}1)$$

If r_P represents the distance from the point of application of the force $\mathbf{P} \doteq P_0e^{j\omega t}$ to the individual elements of the system (i.e., r_{Pm}, r_{Pk}, etc.), as shown in Fig. 6–41, and if e_P is the distance between the point of application of **P** and the frequency centroid C, then, since $r_c = r_P - e_P$, equation 6.7–1 takes the form

$$\Sigma \mathbf{z}(r_P - e_P) = 0 = \Sigma \mathbf{z} r_P - \Sigma \mathbf{z} e_P = \Sigma \mathbf{z} r_P - e_P \Sigma \mathbf{z}$$

from which

$$e_P = \frac{\Sigma \mathbf{z} r_P}{\Sigma \mathbf{z}} \tag{6.7–2}$$

This permits the frequency centroid to be located in a manner similar to that used in locating the centroid of masses and areas in statics.

Since a force applied to the system at the frequency center produces only translation of the system, it is convenient to replace the force **P** by another force **P** acting at the frequency centroid and a couple of magnitude $\mathbf{P}_0 e_P$. In this manner the motion consists of a pure translation, due to the force **P** acting at C, and a rotational motion about C due to the couple. The aplitude A_c of the translatory motion is obtained from

$$\Sigma \mathbf{z} A_c = P_0 = A_c \Sigma \mathbf{z}$$

or

$$A_c = \frac{P_0}{\Sigma \mathbf{z}} \tag{6.7–3}$$

The amplitude of each dynamic force caused by the rotational motion is of the form $\mathbf{z} r_c \Theta$, where Θ is the angular amplitude of the motion. Thus

$$\Sigma(\mathbf{z} r_c \Theta) r_c = P_0 e_P = \Theta \Sigma \mathbf{z} r_c^2$$

or, in general,

$$\Theta = \frac{P_0 e_P}{\Sigma \mathbf{z} \bar{r}_c^2} \tag{6.7–4}$$

Since some of the masses or springs may be distributed, the term $\bar{r}_c$ is introduced. It represents the equivalent radius of gyration of these elements about C. Thus in general $\bar{r}_c^2 = r_c^2 + \bar{r}_z^2$ where $\bar{r}_z$ is the equivalent radius of gyration of the distributed elements **z** with respect to its own centroid. For concentrated elements $\bar{r}_z = 0$.

The distance e_0 (Fig. 6–41) from the frequency center to the oscillation center or node O is

$$e_0 = \frac{A_c}{\Theta}$$

Substitution from equations 6.7–2, 6.7–3, and 6.7–4 into the above gives

$$e_0 = \frac{P_0}{\Sigma \mathbf{z}} \cdot \frac{\Sigma \mathbf{z} \bar{r}_c^2}{P_0 e_P} = \frac{\Sigma \mathbf{z} \bar{r}_c^2}{e_P \Sigma \mathbf{z}} = \frac{\Sigma \mathbf{z} \bar{r}_c^2}{\Sigma \mathbf{z} r_P} \tag{6.7–5}$$

also

$$e_0 e_P = \frac{\Sigma z \bar{r}_c^{\,2}}{\Sigma z} = \frac{\bar{R}_c^{\,2} \Sigma z}{\Sigma z} = \bar{R}_c^{\,2} \tag{6.7–6}$$

where $\bar{R}_c$ is the equivalent radius of gyration of the whole system. This relation indicates that $\bar{R}_c$ is the geometric mean of e_0 and e_P. The system may therefore be compared to a solid mass pivoted about a point, which corresponds to the oscillation center or node O. The point of application of the force **P** corresponds to the center of percussion of the solid mass, whereas the frequency centroid corresponds to the mass center of gravity.

The frequency equation for rotation of the system is obtained from equation 6.7–4, where $P_0 = 0$, and consequently $\Sigma z \bar{r}_c^{\,2} = 0$ if there is to be any rotary motion ($\Theta \neq 0$). Thus the frequency equation is

$$\Sigma z \bar{r}_c^{\,2} = 0 \tag{6.7–7}$$

If the point of application of the force **P**, for example, is taken as the reference point or origin, then the frequency equation is best rewritten as

$$\Sigma z \bar{r}_c^{\,2} = \Sigma z (r_P - e_P)^2 = \Sigma z \bar{r}_P^{\,2} - 2e_P \Sigma z r_P + e_P^{\,2} \Sigma z$$

Substitution of

$$e_P = \frac{\Sigma z r_P}{\Sigma z}$$

gives

$$\sum z \bar{r}_c^{\,2} = \sum z \bar{r}_P^{\,2} - \frac{2(\Sigma z r_P)^2}{\Sigma z} + \frac{(\Sigma z r_P)^2}{\Sigma z}$$

$$= \sum z \bar{r}_P^{\,2} - \frac{(\Sigma z r_P)^2}{\Sigma z}$$

and thus the frequency equation for rotation may be written as

$$\Sigma z \Sigma z \bar{r}_P^{\,2} - (\Sigma z r_P)^2 = 0 \tag{6.7–8}$$

In a similar manner, the angular amplitude may be expressed as

$$\Theta = \frac{\Sigma z r_P}{\Sigma z \Sigma z \bar{r}_P^{\,2} - (\Sigma z r_P)^2} P_0 \tag{6.7–9}$$

From equations 6.7–2 and 6.7–5, it is seen that

$$e = e_P + e_0 = \frac{\Sigma z r_P}{\Sigma z} + \frac{\Sigma z \bar{r}_c^{\,2}}{\Sigma z r_P} = \frac{\Sigma z r_P}{\Sigma z} + \frac{\Sigma z \Sigma z \bar{r}_P^{\,2} - (\Sigma z r_P)^2}{\Sigma z \Sigma z r_P}$$

whence

$$e = \frac{\Sigma z \bar{r}_P^2}{\Sigma z r_P} \tag{6.7–10}$$

The amplitude of the point of application of the exciting force is

$$A_P = e\Theta = \frac{\Sigma z \bar{r}_P^2}{\Sigma z \Sigma z \bar{r}_P^2 - (\Sigma z r_P)^2} P_0 \tag{6.7–11}$$

and the amplitude at any point a distance r_{Pn} from the point of application of the force P is

$$A_n = (e - r_{Pn})\Theta = \frac{\Sigma z \bar{r}_P^2 - r_{Pn}\Sigma z r_P}{\Sigma z \Sigma z \bar{r}_P^2 - (\Sigma z r_P)^2} P_0 \tag{6.7–12}$$

Illustrative example. As an example of the application of the theory presented above, the system of Fig. 6–42 will be considered. The system consists

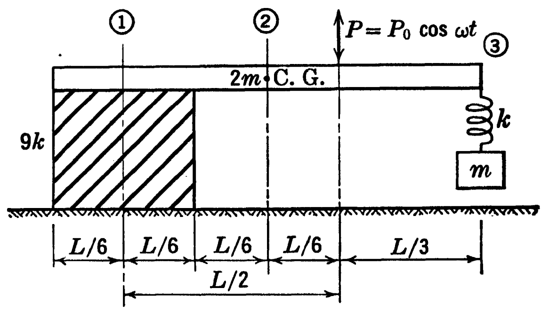


FIG. 6–42

of a bar, of uniform cross section and mass supported on an elastic support which is distributed over one third of the length. Such a support might be supplied by a rubber block bonded to the bar. In addition, a mass m is suspended from the other end of the bar by a spring of stiffness k. The distributed elastic support is assumed to have a spring constant equal to $9k$ in translation, and it is assumed to resist rotation as a bar of rectangular cross section in bending.

Let an oscillating force P with circular frequency ω be applied at a point $L/3$ from the right end of the bar as shown. Taking this point of application as the origin and the axis positive to the right and introducing the parameters $p^2 = k/m$ and $\alpha^2 = \omega^2/p^2$ permits the values of the impedances $\mathbf{z}$ of the individual elements, $\Sigma \mathbf{z}$, $\Sigma \mathbf{z} r_P$ and $\Sigma \mathbf{z} \bar{r}_P^2$ to be determined. Thus,

$$\mathbf{z}_1 = 9k, \quad \mathbf{z}_2 = -2\omega^2 m, \quad \text{and} \quad \mathbf{z}_3 = \frac{1}{\dfrac{1}{\mathbf{z}_k} + \dfrac{1}{\mathbf{z}_m}} = -\frac{\omega^2 k m}{k - \omega^2 m}$$

or

$$\mathbf{z}_3 = -\frac{\omega^2 m}{1 - (\omega/p)^2} = -\frac{\omega^2 m}{1 - \alpha^2}$$

from which

$$\sum z = 9k - 2\omega^2 m - \frac{\omega^2 m}{1-\alpha^2} = k\left(9 - 2\alpha^2 - \frac{\alpha^2}{1-\alpha^2}\right)$$
$$= \frac{k}{1-\alpha^2}(9 - 12\alpha^2 + 2\alpha^4)$$

$$\sum z r_P = \left[(-\tfrac{1}{2}L)9k - (-\tfrac{1}{6}L)2\omega^2 m - (\tfrac{1}{3}L)\frac{\omega^2 m}{1-\alpha^2}\right]$$
$$= -\frac{kL}{6(1-\alpha^2)}(27 - 27\alpha^2 + 2\alpha^4)$$

$$\sum z\bar{r}_P{}^2 = \left[\left(-\frac{L}{2}\right)^2 + \frac{1}{12}\left(\frac{L}{3}\right)^2\right]9k - \left[\left(-\frac{L}{6}\right)^2 + \frac{L^2}{12}\right]2\omega^2 m$$
$$- \left(\frac{L}{3}\right)^2 \frac{\omega^2 m}{1-\alpha^2}$$
$$= \left(\frac{L}{3}\right)^2 \frac{k}{1-\alpha^2}(21 - 24\alpha^2 + 2\alpha^4)$$

Then the frequency equation 6.7–8 yields

$$\sum z \sum z\bar{r}_P{}^2 - \left(\sum z r_P\right)^2 = \frac{1}{12}\frac{(kL)^2}{1-\alpha^2}(9 - 129\alpha^2 + 56\alpha^4 - 4\alpha^6) = 0$$

The distance from the force P_0 to center of rotation or the node O is

$$e = \frac{\Sigma z\bar{r}_P{}^2}{\Sigma z r_P} = -\frac{2}{3}\cdot\frac{21 - 24\alpha^2 + 2\alpha^4}{27 - 27\alpha^2 + 2\alpha^4}L$$

while, from equation 6.7–9,

$$\Theta = -\frac{2(27 - 27\alpha + 2\alpha^2)}{9 - 129\alpha^2 + 56\alpha^4 - 4\alpha^6}\cdot\frac{P_0}{kL}$$

and, from equation 6.7–11,

$$A_P = \frac{4}{3}\frac{21 - 24\alpha^2 + 2\alpha^4}{9 - 129\alpha^2 + 56\alpha^4 - 4\alpha^6}\cdot\frac{P_0}{k}$$

The amplitude at the end is found from equation 6.7–12,

$$A_3 = \frac{2(23 - 2\alpha^2)(1 - \alpha^2)}{9 - 129\alpha^2 + 56\alpha^4 - 4\alpha^6}\cdot\frac{P_0}{k}$$

The amplitude of the force at the end is

$$P_3 = z_3 A_3 = -\frac{2(23 - 2\alpha^2)(1 - \alpha^2)}{9 - 129\alpha^2 + 56\alpha^4 - 4\alpha^6}\cdot\frac{P_0}{k}\cdot\frac{\omega^2 m}{1-\alpha^2}$$
$$= -\frac{2(23 - 2\alpha^2)\alpha^2}{9 - 129\alpha^2 + 56\alpha^4 - 4\alpha^6}P_0$$

and the amplitude of mass m is

$$A_m = P_m a_m = -P_3 \frac{1}{\omega^2 m} = \frac{2(23 - 2\alpha^2)\alpha^2}{9 - 129\alpha^2 + 56\alpha^4 - 4\alpha^6} \cdot \frac{P_0}{\omega^2 m}$$

$$= \frac{2(23 - 2\alpha^2)}{9 - 129\alpha^2 + 56\alpha^4 - 4\alpha^6} \cdot \frac{P_0}{k}$$

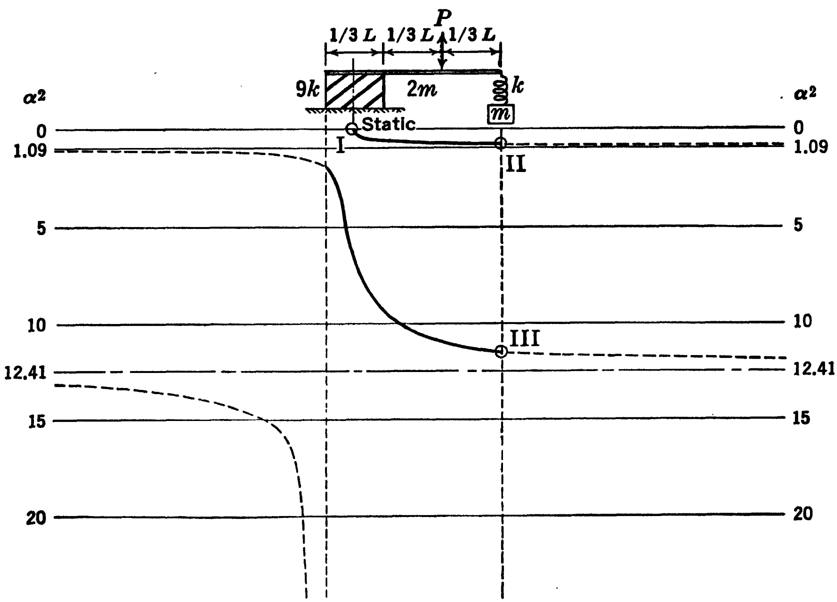
For the specific case $\alpha^2 = \omega^2/p^2 = 0$ which corresponds to a static loading by the force P_0,

$$e = -\tfrac{14}{27}L$$

which corresponds to point I in Fig. 6–43. For

$$\alpha^2 = 1 \quad \text{or} \quad \omega^2 = p^2 = k/m, \quad e = \tfrac{1}{3}L$$

corresponding to point II in Fig. 6–43, the node is at point 3 (Fig. 6–42). But this is the point at which the mass m is attached to the beam through the



Node position versus frequency ratio α^2

FIG. 6–43

spring k, and, since it is in resonance with the impressed frequency ω, it acts as a dynamic absorber, and, as previously shown in Chapter 5, it will cause a node at this point. The force acting at 3 through the spring k and the mass m, as well as the amplitude of m are found to be

$$P_3 = +\tfrac{21}{34}P_0$$

and

$$A_m = -\frac{21}{34}\frac{P_0}{k}$$

This shows that the amplitude of the mass m is 180° out of phase with the impressed force P.

The natural frequencies of the system are found from equation 6.7–8 which becomes

$$\alpha^6 - 14\alpha^4 + 32.25\alpha^2 - 2.25 = 0$$

The evaluation of α^2, as indicated in equation 5.7–6, gives the three values $\alpha_1^2 = 0.07$, $\alpha_2^2 = 2.81$, and $\alpha_3^2 = 11.12$. The location of the node or center of rotation for the different values of α^2 is shown in Fig. 6–43. The curves plotted in this manner are obtained from equation 6.7–10 which in this particular example may be written conveniently as

$$e = \left[\frac{2(2 - \alpha^2)}{27 - 27\alpha^2 + 2\alpha^4} - \frac{2}{3}\right] L$$

Pure translation occurs when $e = \infty$ or $\Theta = 0$. This condition requires that

$$27 - 27\alpha^2 + 2\alpha^4 = 0$$

or

$$\alpha_4^2 = 1.09 \quad \text{and} \quad \alpha_5^2 = 12.41$$

as indicated by the asymptotes in Fig. 6–43.

It is instructive to examine the particular conditions for a few of the nodes. It may be seen that any node outside the vibrating structure (dotted parts of the curves in Fig. 6–43 is "free," meaning that the node is not maintained by any shearing or translatory force. On the other hand, a node inside the structure may in general be expected to require a transverse force to maintain it. This has already been shown for the node at the end of the bar when $\alpha^2 = 1$. From equation 6.7–12, it will be seen that $A_3 = 0$ or a node again is formed at 3 when

$$23 - 2\alpha^2 = 0 \quad \text{or} \quad \alpha_6^2 = 11.5$$

(point III in Fig. 6–43). In this event P_3 and therefore also A_m will be zero. This then means that no sustaining force is required at the node for this frequency ratio.

Lever system with damping

Before considering a general case of springs, masses, and dampers attached in parallel to a lever, a simple example will be analyzed. Figure 6–44 shows a bar which is assumed to be weightless, with a spring attached at one end and a damper at the other. The equilibrium position of the system is along the line NN. The bar is subjected to an oscillating force as shown. In this particular problem the spring force, the damping force, and the disturbing force must all reach their maxima at the same time. This means that the spring will reach its maximum displacement at the same time the damper reaches its maximum velocity. The amplitudes $\mathbf{A}_k$ and $\mathbf{A}_c$ of the spring and damper, respectively, are therefore 90° out of phase, as shown in Fig. 6–44*a*. For any point in the bar the amplitudes and phase angles

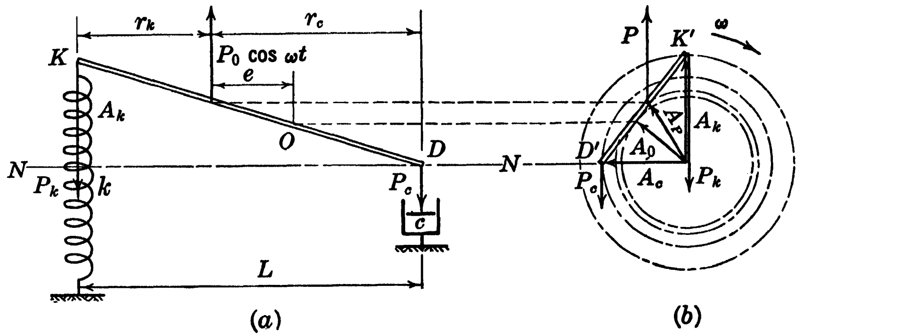


FIG. 6–44

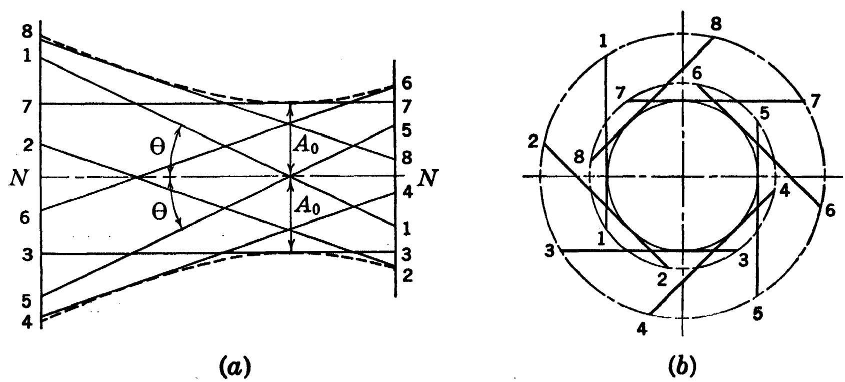


FIG. 6–45

with respect to the force are represented as vectors extending from the equilibrium position NN to the line $K'D'$ in Fig. 6–44*b*.

The motion of the bar can be visualized by rotating the vector system (Fig. 6–44*b*). The line $K'D'$ connecting the terminals of $\mathbf{A}_k$ and $\mathbf{A}_c$ will describe a hyperboloid of revolution (Fig. 6–45). This shows that the bar KD in its plane motion will "roll" on a hyperbola whose asymptotes 1–1 and 5–5 in Fig. 6–45 indicate the angular amplitude Θ, whereas the apex of the hyperbola O is the location of the least amplitude $\mathbf{A}_0$ of the bar. This point corresponds to the node in the undamped system. It will be noted from the configuration that the bar "rolls" on the "upper" part of the hyperbola from the angular amplitude Θ, or asymptotic position 1–1, to the other asymptotic position 5–5 but returns on the "lower" hyperbola. This is the same type of motion noticed when a floating bar is subjected to the motion of waves which are longer than the bar.

From the above consideration it may be noted that the motion of the bar is completely determined by the angular amplitude Θ, the "throat" or "node" amplitude A_0, and its location. The values of these parameters are in the general case obtained by reference to the position of the angular amplitude Θ, as shown in Fig. 6–46. It has

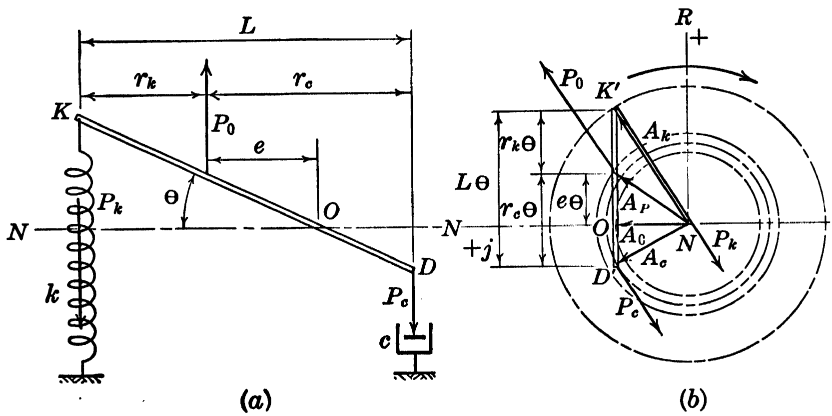


Fig. 6–46

already been concluded that the force **P** is in phase with the spring-displacement amplitude and the velocity amplitude of the damper. The amplitude of the forces will therefore be related as shown in Fig. 6–46*b*.

The summation of these forces give

$$\mathbf{P}_k + \mathbf{P}_c = \mathbf{P}_0$$

whereas the moment with respect to the point of application of force $\mathbf{P}_0$ yields

$$\mathbf{P}_k r_k = \mathbf{P}_c r_c$$

The scalar values of P_k and P_c are kA_k and ωcA_c, respectively, from which $kA_k + \omega cA_c = P_0$ and

$$kA_k r_k = \omega cA_c r_c \quad \text{or} \quad \frac{A_k}{A_c} = \frac{\omega c}{k} \cdot \frac{r_c}{r_k}$$

$$A_c = \frac{r_k}{L} \cdot \frac{P_0}{\omega c} \quad \text{and} \quad A_k = \frac{r_c}{L} \cdot \frac{P_0}{k}$$

The angular amplitude Θ is obtained from the geometry of Fig. 6–46, where $(L\Theta)^2 = A_k^2 + A_c^2$, or

$$\Theta = \frac{A_c}{L}\sqrt{1 + \left(\frac{A_k}{A_c}\right)^2} \tag{6.7–13}$$

Likewise,

$$\frac{A_0}{A_c} = \frac{A_k}{\sqrt{A_k^2 + A_c^2}} \quad \text{or} \quad A_0 = \frac{A_k}{\sqrt{1 + \left(\frac{A_k}{A_c}\right)^2}} \tag{6.7–14}$$

The location of the oscillating "node" O is obtained from the relationship $(r_k + e)\Theta/A_0 = A_k/A_c$, which, by substitution, gives

$$r_k + e = \frac{A_k^2}{A_k^2 + A_c^2} L = \frac{L}{1 + (A_c/A_k)^2} \tag{6.7–15}$$

The phase angle varies along the bar in this type of problem and is most readily referred to a characteristic position of the bar, such as that shown in Fig. 6–46. There the actual plane of motion of the bar, as shown in Fig. 6–46*a*, is indicated by *NR* in Fig. 6–46*b*, where *NR* is taken as the real axis and *NO* is referred to as the imaginary axis. If the point of application of the force $\mathbf{P}_0$ is taken as the origin for measurements along the bar (positive to the right), the amplitudes along the bar may be expressed vectorially as

$$\mathbf{A} = (e - r)\Theta + jA_0$$

If A_0 is expressed in terms of Θ or $A_0 = d\Theta$ where d is defined by the ratio $d = A_0/\Theta$, the amplitude can be written as

$$\mathbf{A} = (e - r + jd)\Theta$$

The amplitude of the point of application of the force **P** will then be

$$\mathbf{A}_P = (e + jd)\Theta$$

and the force **P**, as previously found, is in phase with the spring amplitude. The spring amplitude takes the form

$$\mathbf{A}_k = [e - (-r_k) + jd]\Theta$$
$$= (e + r_k + jd)\Theta$$

The phase angle ϕ for $\mathbf{A}_k$ as well as **P** is therefore obtained from

$$\tan \phi = \frac{d}{e + r_k}$$

In this manner the motion is completely determined.

For the general case of a lever which is attached in a parallel manner to a combination of springs, dampers, and masses, the problem becomes somewhat more involved because the phase angle between the oscillating force and the amplitude of the various dynamic forces is not directly obtainable or self-evident as in the previous example. The analysis of this general problem, however, may be carried out as follows.

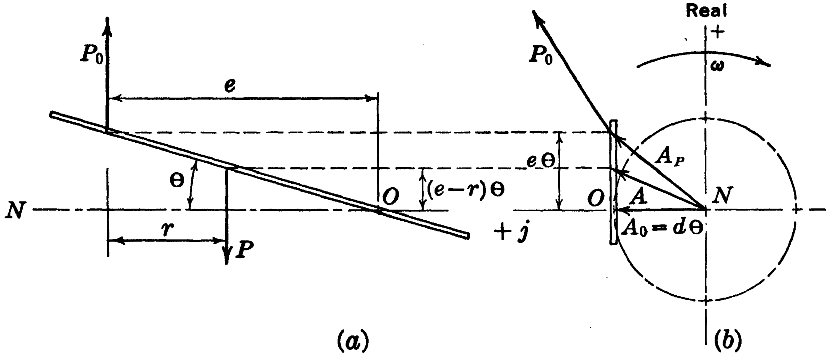


FIG. 6–47

Assuming the same reference and coordinate system as indicated in the previous problem, a dynamic force **P** with an amplitude as shown in Fig. 6–47 may be written as

$$\mathbf{P} = \mathbf{zA}$$

If the effective impedance **z** is expressed as

$$\mathbf{z} = a + jb$$

and **A**, as previously shown, may be written as

$$\mathbf{A} = (e - r + jd)\Theta \tag{6.7–16}$$

the dynamic force takes the form

$$\mathbf{P} = (a + jb)(e - r + jd)\Theta \tag{6.7–17}$$

and similiarly

$$\mathbf{P}_0 = F_1 + jF_2 \tag{6.7–18}$$

The equations of dynamic equilibrium require that

$$\Sigma\mathbf{P} = \mathbf{P}_0 \tag{6.7–19}$$

and

$$\Sigma\mathbf{P}r = 0 \tag{6.7–20}$$

Substitution for **P** and $\mathbf{P}_0$ from above gives

$$\Theta\Sigma(a + jb)(e - r + jd) = F_1 + jF_2 \qquad (6.7\text{–}21)$$

and

$$\Theta\Sigma(a + jb)(e - r + jd)r = 0 \qquad (6.7\text{–}22)$$

The evaluation of equations 6.7–21 and 6.7–22 yields

$$e\Sigma a - \Sigma ar - d\Sigma b + j(e\Sigma b - \Sigma br + d\Sigma a) = F_1/\Theta + jF_2/\Theta$$

and

$$e\Sigma ar - \Sigma a\bar{r}^2 - d\Sigma br + j(e\Sigma br - \Sigma b\bar{r}^2 + d\Sigma ar) = 0; \qquad \Theta \neq 0$$

This requires that

$$\left.\begin{aligned} e\Sigma a - \Sigma ar - d\Sigma b &= F_1/\Theta = f_1 \\ e\Sigma b - \Sigma br + d\Sigma a &= F_2/\Theta = f_2 \\ e\Sigma ar - \Sigma a\bar{r}^2 - d\Sigma br &= 0 \\ e\Sigma br - \Sigma b\bar{r}^2 + d\Sigma ar &= 0 \end{aligned}\right\} \qquad (6.7\text{–}23)$$

In the case of distributed elements, the distance r from the origin represents the distance to the centroid of the element, while $\bar{r}$ represents the radius of gyration of this element with respect to the origin in the same manner, as previously shown for no damping.

The last two equations 6.7–23 yield

$$\left.\begin{aligned} e &= \frac{\Sigma ar\Sigma a\bar{r}^2 + \Sigma br\Sigma b\bar{r}^2}{(\Sigma ar)^2 + (\Sigma br)^2} \\ d &= \frac{\Sigma ar\Sigma b\bar{r}^2 - \Sigma a\bar{r}^2\Sigma br}{(\Sigma ar)^2 + (\Sigma br)^2} \\ e^2 + d^2 &= \frac{(\Sigma a\bar{r}^2)^2 + (\Sigma b\bar{r}^2)^2}{(\Sigma ar)^2 + (\Sigma br)^2} \end{aligned}\right\} \qquad (6.7\text{–}24)$$

Substitution from equations 6.7–24 into the first two equations 6.7–23 gives the values of f_1 and f_2 from which finally Θ may be obtained from

$$f_1^2 + f_2^2 = \left(\frac{\mathbf{P}_0}{\Theta}\right)^2$$

or

$$\Theta = \frac{P_0}{\sqrt{f_1^2 + f_2^2}}$$

The phase of the force **P**, namely ϕ_P, with respect to the angular amplitude Θ, is found from

$$\tan \phi_P = \frac{f_2}{f_1} = \frac{e\Sigma a - \Sigma ar - d\Sigma b}{e\Sigma b - \Sigma br + d\Sigma a}$$

The respective values of e, d, and Θ may now be used in equations 6.7–16 and 6.7–17 to determine the amplitudes and dynamic forces at any point in the system.

Illustrative example. Figure 6–48 shows a system similar to the undamped system of Fig. 6–42. The elastic mounting or support with a distributed spring constant $9k$ is assumed to have a distributed coefficient of viscous

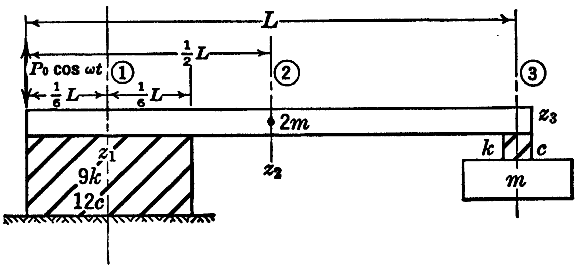


FIG. 6–48

damping $12c$, while the spring at point 3 has a spring constant k and a damping coefficient c. The masses are assumed to be $2m$ and m, as indicated in Fig. 6–42. The oscillating force P_0 has been applied to the left end of the bar in contrast to that shown in Fig. 6–42 in order to emphasize the damping effect.

In evaluating the dynamic forces and their first and second moments, it will be expedient to use the notation,

$$p^2 = \frac{k}{m}, \qquad \alpha^2 = \frac{\omega^2}{p^2}$$

as well as

$$q = \frac{k}{c} \quad \text{and} \quad \beta = \frac{\omega}{q}$$

From Fig. 6–48, it is apparent that

$$z_1 = 9k + 12j\omega c \qquad = 3(3 + 4j\beta)k$$

$$z_1 r = (9k + 12j\omega c)\frac{L}{6} \qquad = \tfrac{1}{2}(3 + 4j\beta)kL$$

$$z_1 \bar{r}^2 = (9k + 12j\omega c)\left[\left(\frac{L}{6}\right)^2 + \frac{1}{12}\left(\frac{L}{3}\right)^2\right] = \tfrac{1}{9}(3 + 4j\beta)kL^2$$

$$z_2 = -2\omega^2 m \qquad = -2\alpha^2 k$$

$$z_2 r = (-2\omega^2 m)\left(\frac{L}{2}\right) \qquad = -\alpha^2 kL$$

$$\mathbf{z}_2\bar{r}^2 = (-2\omega^2 m)\left[\left(\frac{L}{2}\right)^2 + \frac{L^2}{12}\right] \qquad = -\tfrac{2}{3}\alpha^2 kL^2$$

$$\mathbf{z}_3 = \frac{1}{\dfrac{1}{\mathbf{z}_k + \mathbf{z}_c} + \dfrac{1}{\mathbf{z}_m}} = \frac{1}{\dfrac{1}{k + j\omega c} - \dfrac{1}{\omega^2 m}} = \frac{k}{\dfrac{1}{1 + j\beta} - \dfrac{1}{\alpha^2}}$$

$$= -\frac{\alpha^2}{(1-\alpha^2)^2 + \beta^2}(1 - \alpha^2 + \beta^2 - j\alpha^2\beta)k$$

$$\mathbf{z}_3 r = \mathbf{z}_3 L$$

$$\mathbf{z}_3\bar{r}^2 = \mathbf{z}_3 L^2$$

To simplify the problem, some numerical values for α and β will be assumed. The elastic mounting of the bar at 1 as well as the elastic spring at 3, may be assumed to be rubber. For a case of heavy damping, it will not be unreasonable to base the damping coefficient on 1/4 of critical damping at 3. Let

$$c = \gamma c_{cr} = \gamma \cdot 2\sqrt{km}$$

where γ is a constant; then

$$q = \frac{k}{c} = \frac{k}{2\gamma\sqrt{km}} = \frac{1}{2\gamma}\sqrt{\frac{k}{m}} = \frac{p}{2\gamma}$$

or

$$\beta = \frac{\omega}{q} = 2\gamma\frac{\omega}{p} = 2\gamma\alpha$$

Thus $\beta = \frac{1}{2}\alpha$ for $\gamma = \frac{1}{4}$. If it is assumed further than $\alpha^2 = \omega^2/p^2 = 1$, then, corresponding to the points 1, 2 and 3, the following quantities may be expressed as

1	2	3
$\mathbf{z}_1 = 3(3 + 2j)k$	$\mathbf{z}_2 = -2k$	$\mathbf{z}_3 = (-1 + 2j)k$
$\mathbf{z}_1 r = \frac{1}{2}(3 + 2j)kL$	$\mathbf{z}_2 r = -kL$	$\mathbf{z}_3 r = (-1 + 2j)kL$
$\mathbf{z}_1\bar{r}^2 = \frac{1}{9}(3 + 2j)kL^2$	$\mathbf{z}_2\bar{r}^2 = -\frac{2}{3}kL^2$	$\mathbf{z}_3\bar{r}^2 = (-1 + 2j)kL^2$

Thus summation of the components gives

$$\Sigma a = (9 - 2 - 1)k = 6k \qquad \Sigma b = (6 + 2)k = 8k$$

$$\Sigma ar = (\tfrac{3}{2} - 1 - 1)kL = -\tfrac{1}{2}kL \qquad \Sigma br = (1 + 2)kL = 3kL$$

$$\Sigma a\bar{r}^2 = (\tfrac{1}{3} - \tfrac{2}{3} - 1)kL^2 = -\tfrac{4}{3}kL^2 \quad \Sigma b\bar{r}^2 = (\tfrac{2}{9} + 2)kL^2 = \tfrac{20}{9}kL^2$$

which finally leads to

$$e = \frac{\Sigma ar\Sigma a\bar{r}^2 + \Sigma br\Sigma b\bar{r}^2}{(\Sigma ar)^2 + (\Sigma br)^2} = \frac{(-\frac{1}{2})(-\frac{4}{3}) + (3)(\frac{20}{9})}{(-\frac{1}{2})^2 + (3)^2}L = \tfrac{264}{333}L$$

and

$$d = \frac{\Sigma ar\Sigma b\bar{r}^2 - \Sigma a\bar{r}^2\Sigma br}{(\Sigma ar)^2 + (\Sigma br)^2} = \frac{(-\frac{1}{2})(\frac{20}{9}) - (-\frac{4}{3})(3)}{(-\frac{1}{2})^2 + (3)^2}L = \tfrac{104}{333}L$$

Further

$$f_1 = L\Sigma a - \Sigma ar - d\Sigma b = [\tfrac{264}{333}(6) - (-\tfrac{1}{2}) - \tfrac{104}{333}(8)]kL = \tfrac{1837}{666}kL$$

$$f_2 = L\Sigma b - \Sigma br + d\Sigma a = [\tfrac{264}{333}(8) - 3 + \tfrac{104}{333}(8)]kL = \tfrac{3474}{666}kL$$

from which

$$\Theta = \frac{P_0}{\sqrt{f_1{}^2 + f_2{}^2}} = \frac{666}{\sqrt{(1837)^2 + (3474)^2}} \cdot \frac{P_0}{kL} = \frac{666}{3930}\frac{P_0}{kL} = 0.1695\frac{P_0}{kL}$$

The amplitude at any point in the bar can now be evaluated from equation 6.7–16. The amplitude at the "node" O becomes

$$\mathbf{A}_0 = j\,d\Theta = j\frac{104}{333}\cdot\frac{666}{3930}\frac{P_0}{k} = 208j\Delta$$

where $\Delta = P_0/3930k$. In the same manner, the amplitude at 1 is

$$\mathbf{A}_1 = (e + r_1 + jd)\Theta$$

or

$$\mathbf{A}_1 = \left(\frac{264}{333} - \frac{1}{6} + \frac{104}{333}j\right)\frac{666}{3930}\frac{P_0}{k} = (417 + 208j)\Delta$$

also

$$\mathbf{A}_2 = \left(\frac{264}{333} - \frac{1}{2} + \frac{104}{333}j\right)\frac{666}{3930}\frac{P_0}{k} = (195 + 208j)\Delta$$

and

$$\mathbf{A}_3 = \left(\frac{264}{333} - 1 + \frac{104}{333}j\right)\frac{666}{3930}\frac{P_0}{k} = (-138 + 208j)\Delta$$

The dynamic forces may be determined from the relationship

$$\mathbf{P} = \mathbf{zA}$$

which gives

$$\mathbf{P}_{9k} = \mathbf{z}_{9k}\mathbf{A}_1 = 9k(417 + 208j)\Delta = (3753 + 1872j)\,\Delta k$$

$$\mathbf{P}_{12c} = \mathbf{z}_{12c}\mathbf{A}_1 = 12j\omega c(417 + 208j)\,\Delta = (-1248 + 2502j)\Delta k$$

$$\mathbf{P}_{9k} + \mathbf{P}_{12c} = \mathbf{P}_1 = \mathbf{z}_1\mathbf{A}_1 = 3(3 + 2j)k(417 + 208j)\Delta = (2505 + 4374j)\,\Delta k$$

$$\mathbf{P}_2 = \mathbf{z}_2\mathbf{A}_2 = -2k(195 + 208j)\,\Delta = (-390 - 416j)\,\Delta k$$

$$\mathbf{P}_3 = \mathbf{z}_3\mathbf{A}_3 = (-1 + 2j)k(-138 + 208j)\,\Delta = (-278 - 484j)\,\Delta k$$

$$\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 = \mathbf{P} = (1837 + 3474j)\,\Delta k$$

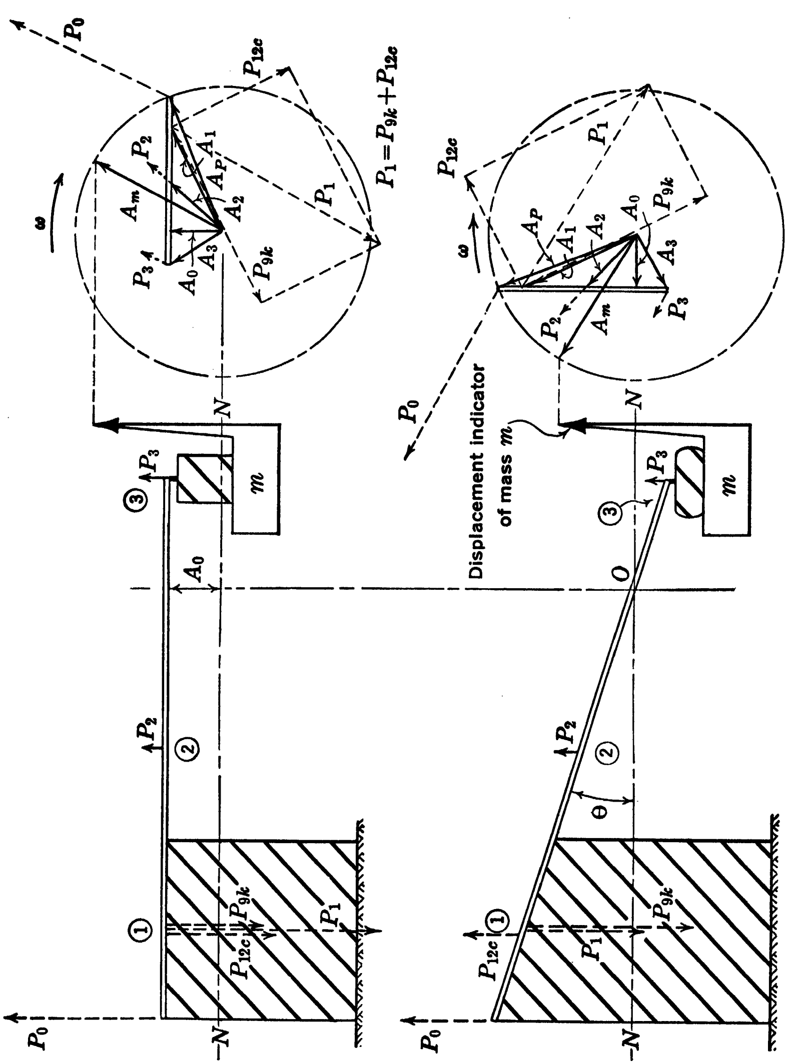

P0
P12c
P1 = P9k + P12c
P1
P2
A1
AP
A2
Am
A0
A3
P3
P9k
ω
N
m
Displacement indicator
of mass m
O
θ
N

FIG. 6–49

The latter offers a convenient check, since

$$\mathbf{P} = (f_1 + jf_2)\Theta = \left(\frac{1837}{666} + \frac{3474}{666}j\right) kL \left(\frac{666}{3930}\right) \frac{P}{kL}$$
$$= (1837 + 3474j)\,\Delta k$$

The amplitude of the mass m is determined from

$$\mathbf{A}_m = \mathbf{A}_3 \frac{\mathbf{z}_3}{\mathbf{z}_m} = \mathbf{A}_3 \frac{(-1 + 2j)}{-\omega^2 m} k = A_3(1 - 2j)$$

or

$$\mathbf{A}_m = (-138 + 208j)(1 - 2j)\Delta = (278 + 484j)\,\Delta$$

The amplitudes of the displacements and the dynamic forces are shown in their vectorial relationship in Fig. 6–49. The lever system is indicated in its zero angular displacement as well as at full angular amplitude.

6.8. Vibration of a Spring-Mounted Platform

The vibration analysis of a system capable of plane motion, as outlined in the previous section, can be extended readily to the third dimension by including the rotation about a second axis in a plane normal to the axis of translation. An important example of this type of system is that of a spring-mounted platform or a spring-suspended car body.

The development will be confined here to those problems in which the damping is so small that its effect can be neglected. The system under consideration consists in principle of a rigid plane platform normal to the direction of translation. To this platform is attached distributed or concentrated masses and spring elements, as well as combinations of these, similar to the two-dimensional problem treated in the previous section and shown in Fig. 6–42.

An oscillating force normal to the plane of the platform will in general produce a rotation about an axis (the nodal axis) in the fixed horizontal reference plane. All elements in the plane of the platform will therefore rotate through the same angle Θ during vibration. For this reason the problem can be treated in the same manner as the problem in elasticity involving the stress analysis of an elastic body subjected to an eccentric loading. The various analytic and graphic methods developed in mechanics to facilitate the solution of this well-known problem in statics can be employed readily in this problem in dynamics.

An analytic solution can be obtained by realizing that the amplitude A for any point x, y on the platform can be expressed as

$$A(x, y) = a + bx + cy \tag{6.8–1}$$

where a, b, and c are determined from the dynamic characteristics of the system. From Fig. 6–51, which represents the fixed-reference

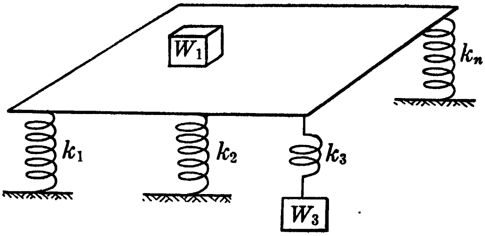


FIG. 6–50

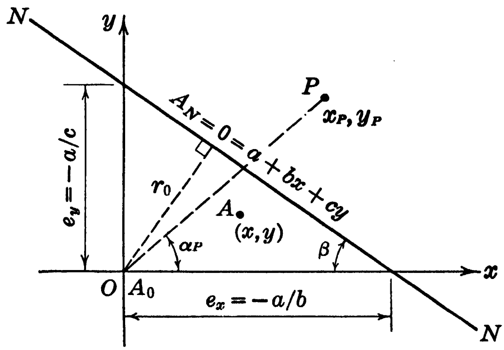


FIG. 6–51

plane, it will be seen that the nodal axis NN has the equation,

$$A_{NN} = 0 = a + bx + cy \tag{6.8–2}$$

whereas the intercepts on the x and y axes, respectively, are

$$e_x = -\frac{a}{b} \quad \text{and} \quad e_y = -\frac{a}{c} \tag{6.8–3}$$

The slope of the nodal axis is given by

$$\tan\beta = \frac{e_y}{e_x} = \frac{b}{c} \tag{6.8–4}$$

The angular amplitude can be evaluated from $\Theta = A_0/r_0$, where $A_0 = a$ and $r_0 = e_y \cos\beta = \dfrac{-a/c}{\sqrt{1 + \tan^2\beta}} = -\dfrac{a}{\sqrt{b^2 + c^2}}$. Thus

$$\Theta = \sqrt{b^2 + c^2} \tag{6.8–5}$$

The above relationships are of general application; however the values of a, b, and c will depend on the specific system and the nature of the exciting force or torque.

System excited by a single normal oscillating force $P_0 \cos wt$ acting at (x_P, y_P)

In a manner similar to that of the previous section, the equations of motion may be written in terms of the impedances of the elements of the system and the amplitude, as given by equation 6.8–1. Thus:

$$\left.\begin{aligned} P_0 &= \Sigma A\mathrm{z} = a\Sigma\mathrm{z} + b\Sigma\mathrm{z}x + c\Sigma\mathrm{z}y \\ &= a\Sigma\mathrm{z} + b\bar{x}\Sigma\mathrm{z} + c\bar{y}\Sigma\mathrm{z} \\ P_0x_P &= \Sigma(A\mathrm{z})x = a\Sigma\mathrm{z}x + b\Sigma\mathrm{z}x^2 + c\Sigma\mathrm{z}xy \\ &= a\bar{x}\Sigma\mathrm{z} + b\mathcal{I}_y + c\mathcal{H}_{xy} \\ P_0y_P &= \Sigma(A\mathrm{z})y = a\Sigma\mathrm{z}y + b\Sigma\mathrm{z}xy + c\Sigma\mathrm{z}y^2 \\ &= a\bar{y}\Sigma\mathrm{z} + b\mathcal{H}_{xy} + c\mathcal{I}_x \end{aligned}\right\} \qquad (6.8\text{–}6)$$

where

$$\bar{x} = \frac{\Sigma\mathrm{z}x}{\Sigma\mathrm{z}}, \qquad \bar{y} = \frac{\Sigma\mathrm{z}y}{\Sigma\mathrm{z}}$$

are the coordinates of the frequency center defined as in the previous section, and

$$\mathcal{I}_x = \Sigma\mathrm{z}y^2, \qquad \mathcal{I}_y = \Sigma\mathrm{z}x^2$$

represent the equivalent dynamic second moment (moment of inertia) of the elements or systems of elements attached to the platform. Similarly $\mathcal{H}_{xy} = \Sigma\mathrm{z}xy$ represents the dynamic cross product (product of inertia) of these elements.

Evaluating a, b, and c from equations 6.8–6 gives

$$\left.\begin{aligned} a &= \frac{P_0}{\Delta}\begin{vmatrix} 1 & \Sigma\mathrm{z}x & \Sigma\mathrm{z}y \\ x_P & \Sigma\mathrm{z}x^2 & \Sigma\mathrm{z}xy \\ y_P & \Sigma\mathrm{z}xy & \Sigma\mathrm{z}y^2 \end{vmatrix} = \frac{P_0}{\Delta}\begin{vmatrix} 1 & \bar{x}\Sigma\mathrm{z} & \bar{y}\Sigma\mathrm{z} \\ x_P & \mathcal{I}_y & \mathcal{H}_{xy} \\ y_P & \mathcal{H}_{xy} & \mathcal{I}_x \end{vmatrix} \\ b &= \frac{P_0}{\Delta}\begin{vmatrix} \Sigma\mathrm{z} & 1 & \Sigma\mathrm{z}y \\ \Sigma\mathrm{z}x & x_P & \Sigma\mathrm{z}xy \\ \Sigma\mathrm{z}y & y_P & \Sigma\mathrm{z}y^2 \end{vmatrix} = \frac{P_0}{\Delta}\Sigma\mathrm{z}\begin{vmatrix} 1 & 1 & \bar{y}\Sigma\mathrm{z} \\ \bar{x} & x_P & \mathcal{H}_{xy} \\ \bar{y} & y_P & \mathcal{I}_x \end{vmatrix} \\ c &= \frac{P_0}{\Delta}\begin{vmatrix} \Sigma\mathrm{z} & \Sigma\mathrm{z}x & 1 \\ \Sigma\mathrm{z}x & \Sigma\mathrm{z}x^2 & x_P \\ \Sigma\mathrm{z}y & \Sigma\mathrm{z}xy & y_P \end{vmatrix} = \frac{P_0}{\Delta}\Sigma\mathrm{z}\begin{vmatrix} 1 & \bar{x}\Sigma\mathrm{z} & 1 \\ \bar{x} & \mathcal{I}_y & x_P \\ \bar{y} & \mathcal{H}_{xy} & y_P \end{vmatrix} \end{aligned}\right\} \qquad (6.8\text{–}7)$$

where

$$\Delta = \begin{vmatrix} \Sigma\mathrm{z} & \Sigma\mathrm{z}x & \Sigma\mathrm{z}y \\ \Sigma\mathrm{z}x & \Sigma\mathrm{z}x^2 & \Sigma\mathrm{z}xy \\ \Sigma\mathrm{z}y & \Sigma\mathrm{z}xy & \Sigma\mathrm{z}y^2 \end{vmatrix} = \Sigma\mathrm{z}\begin{vmatrix} 1 & \bar{x}\Sigma\mathrm{z} & \bar{y}\Sigma\mathrm{z} \\ \bar{x} & \mathcal{I}_y & \mathcal{H}_{xy} \\ \bar{y} & \mathcal{H}_{xy} & \mathcal{I}_x \end{vmatrix}$$

Expanding the determinant above gives

$$\begin{aligned}
\Delta &= \Sigma z[\Sigma zx^2\Sigma zy^2 - (\Sigma zxy)^2] - \Sigma zx(\Sigma zx\Sigma zy^2 - \Sigma zy\Sigma zxy) \\
&\qquad - \Sigma zy(\Sigma zy\Sigma zx^2 - \Sigma zx\Sigma zxy) \\
&= \frac{1}{\Sigma z}\{[\Sigma z\Sigma zy^2 - (\Sigma zy)^2][\Sigma z\Sigma zx^2 - (\Sigma zx)^2] \\
&\qquad - (\Sigma z\Sigma zxy - \Sigma zx\Sigma zy)^2\} \\
&= \Sigma z[(\mathcal{I}_x - \bar{y}^2\Sigma z)(\mathcal{I}_y - \bar{x}^2\Sigma z) - (\mathcal{H}_{xy} - \bar{x}\bar{y}\Sigma z)^2] \\
&= \Sigma z[\mathcal{I}_{xc}\mathcal{I}_{yc} - \mathcal{H}^2{}_{xcyc}] = \Sigma z\mathcal{I}_{\text{I}c}\mathcal{I}_{\text{II}c}
\end{aligned} \tag{6.8–8}$$

where x_c and y_c are the coordinate axes at the frequency centroid parallel to the x and y axes, Fig. (6–52), and $\mathcal{I}_{\text{I}c}$ and $\mathcal{I}_{\text{II}c}$ are the principal second moments with respect to point c. The relationships used above are readily established from the fundamental characteristics of second moments.

In a similar manner the expanded values of a, b, and c may be shown to be:

$$\left.\begin{aligned}
a &= \frac{P_0}{\Delta}[\Sigma zx^2\Sigma zy^2 - (\Sigma zxy)^2 - x_P(\Sigma zx\Sigma zy^2 - \Sigma zy\Sigma zxy) \\
&\qquad - y_P(\Sigma zy\Sigma zx^2 - \Sigma zx\Sigma zxy)] \\
&= \frac{P_0}{\Delta}[\mathcal{I}_x\mathcal{I}_y - \mathcal{H}_{xy}{}^2 - x_P(\bar{x}\mathcal{I}_x - \bar{y}\mathcal{H}_{xy})\Sigma z \\
&\qquad - y_P(\bar{y}\mathcal{I}_y - \bar{x}\mathcal{H}_{xy})\Sigma z] \\
b &= -\frac{P_0}{\Delta}[\Sigma zx\Sigma zy^2 - \Sigma zy\Sigma zxy - x_P(\Sigma z\Sigma zy^2 - (\Sigma zy)^2 \\
&\qquad + y_P(\Sigma z\Sigma zxy - \Sigma zx\Sigma zy)] \\
&= -\frac{P_0}{\Delta}\Sigma z[\bar{x}\mathcal{I}_x\bar{y}\mathcal{H}_{xy} - x_P(\mathcal{I}_x - \bar{y}^2\Sigma z) \\
&\qquad + y_P(\mathcal{H}_{xy} - \bar{x}\bar{y}\Sigma z)] \\
c &= -\frac{P_0}{\Delta}[\Sigma zy\Sigma zx^2 - \Sigma zx\Sigma zxy \\
&\qquad + x_P(\Sigma z\Sigma zxy - \Sigma zx\Sigma zy) - y_P(\Sigma z\Sigma zx^2 - (\Sigma zx)^2)] \\
&= -\frac{P_0}{\Delta}\Sigma z[\bar{y}\mathcal{I}_y - \bar{x}\mathcal{H}_{xy} + x_P(\mathcal{H}_{xy} - \bar{x}\bar{y}\Sigma z) \\
&\qquad - y_P(\mathcal{I}_y - \bar{x}^2\Sigma z)]
\end{aligned}\right\} \tag{6.8–9}$$

These general expressions for a, b, and c may be reduced by selecting the origin at the point of application of the force, in which event

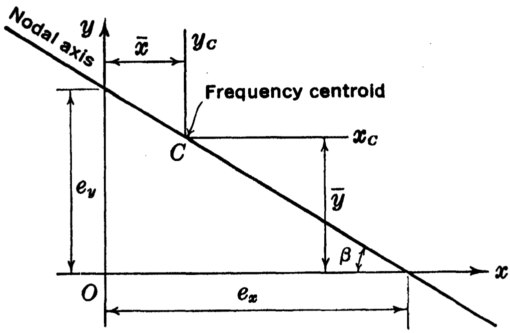


FIG. 6–52

$x_P = y_P = 0$ and

$$
\left.
\begin{aligned}
a &= \frac{P_0}{\Delta}[\Sigma z x^2 \Sigma z y^2 - (\Sigma z x y)^2] = \frac{P_0}{\Delta}(\mathcal{I}_x \mathcal{I}_y - \mathcal{K}_{xy}{}^2] \\
b &= -\frac{P_0}{\Delta}[\Sigma z x \Sigma z y^2 - \Sigma z y \Sigma z x y) = -\frac{P_0}{\Delta}\Sigma z(\bar{x}\mathcal{I}_x - \bar{y}\mathcal{K}_{xy}) \\
c &= -\frac{P_0}{\Delta}(\Sigma z y \Sigma z x^2 - \Sigma z x \Sigma z x y) \\
&= -\frac{P_0}{\Delta}\Sigma z(\bar{y}\mathcal{I}_y - \bar{x}\mathcal{K}_{xy})
\end{aligned}
\right\} \quad (6.8\text{–}10)
$$

from which

$$
A(x, y) = \frac{P_0}{\Delta}[\mathcal{I}_x \mathcal{I}_y - \mathcal{K}_{xy}{}^2 - x\Sigma z(\bar{x}\mathcal{I}_x - \bar{y}\mathcal{K}_{xy}) - y\Sigma z(\bar{y}\mathcal{I}_y - \bar{x}\mathcal{K}_{xy})] \quad (6.8\text{–}11)
$$

The intercepts become

$$
e_{xP} = -\frac{a}{b} = \frac{\mathcal{I}_x \mathcal{I}_y - \mathcal{K}_{xy}{}^2}{(\bar{x}\mathcal{I}_x - \bar{y}\mathcal{K}_{xy})\Sigma z}
$$

and

$$
e_{yP} = -\frac{a}{c} = \frac{\mathcal{I}_x \mathcal{I}_y - \mathcal{K}_{xy}{}^2}{(\bar{y}\mathcal{I}_y - \bar{x}\mathcal{K}_{xy})\Sigma z} \quad (6.8\text{–}12)
$$

The angular amplitude has the form

$$
\Theta = \sqrt{b^2 + c^2} = \frac{P_0}{\Delta}\Sigma z \sqrt{(\bar{x}\mathcal{I}_x - \bar{y}\mathcal{K}_{xy})^2 + (\bar{y}\mathcal{I}_y - \bar{x}\mathcal{K}_{xy})^2} \quad (6.8\text{–}13)
$$

System excited by a single oscillating torque

If the components of the torque about the x and y axes are M_x and M_y, respectively, then the equilibrium equation may be written in terms of the impedances of the elements of the system and the amplitude $A(x, y)$ as

$$\left.\begin{aligned} 0 &= \Sigma Az = a\Sigma z + b\Sigma zx + c\Sigma zy = a\Sigma z + b\bar{x}\Sigma z + c\bar{y}\Sigma z \\ M_y &= \Sigma(Az)x = a\Sigma zx + b\Sigma zx^2 + c\Sigma zxy = a\bar{x}\Sigma z + b\mathcal{I}_y + c\mathcal{K}_{xy} \\ M_x &= \Sigma(Az)y = a\Sigma zy + b\Sigma zxy + c\Sigma zy^2 = a\bar{y}\Sigma z + b\mathcal{K}_{xy} + c\mathcal{I}_x \end{aligned}\right\} \quad (6.8\text{–}14)$$

Evaluating a, b, and c from these equations gives:

$$\left.\begin{aligned} a &= \frac{1}{\Delta}\begin{vmatrix} 0 & \Sigma zx & \Sigma zy \\ M_y & \Sigma zx^2 & \Sigma zxy \\ M_x & \Sigma zxy & \Sigma zy^2 \end{vmatrix} = \frac{1}{\Delta}\begin{vmatrix} 0 & \bar{x}\Sigma z & \bar{y}\Sigma z \\ M_y & \mathcal{I}_y & \mathcal{K}_{xy} \\ M_x & \mathcal{K}_{xy} & \mathcal{I}_x \end{vmatrix} \\ &= \frac{1}{\Delta}[(\Sigma zy\Sigma zxy - \Sigma zx\Sigma zy^2)M_y \\ &\qquad\qquad + (\Sigma zx\Sigma zxy - \Sigma zy\Sigma zx^2)M_x] \\ &= \frac{\Sigma z}{\Delta}[(\bar{y}\mathcal{K}_{xy} - \bar{x}\mathcal{I}_x)M_y + (\bar{x}\mathcal{K}_{xy} - \bar{y}\mathcal{I}_y)M_x] \\ b &= \frac{1}{\Delta}\begin{vmatrix} \Sigma z & 0 & \Sigma zy \\ \Sigma zx & M_y & \Sigma zxy \\ \Sigma zy & M_x & \Sigma zy^2 \end{vmatrix} = \frac{\Sigma z}{\Delta}\begin{vmatrix} 1 & 0 & \bar{y}\Sigma z \\ \bar{x} & M_x & \mathcal{K}_{xy} \\ \bar{y} & M_y & \mathcal{I}_x \end{vmatrix} \\ &= \frac{1}{\Delta}\{[\Sigma z\Sigma zy^2 - (\Sigma zy)^2]M_y - (\Sigma z\Sigma zxy - \Sigma zx\Sigma zy)M_x\} \\ &= \frac{\Sigma z}{\Delta}[(\mathcal{I}_x - \bar{y}^2\Sigma z)M_y - (\mathcal{K}_{xy} - \bar{x}\bar{y}\Sigma z)M_x] \\ c &= \frac{1}{\Delta}\begin{vmatrix} \Sigma z & \Sigma zx & 0 \\ \Sigma zx & \Sigma zx^2 & M_y \\ \Sigma zy & \Sigma zxy & M_x \end{vmatrix} = \frac{\Sigma z}{\Delta}\begin{vmatrix} 1 & \bar{x}\Sigma z & 0 \\ \bar{x} & \mathcal{I}_y & M_y \\ \bar{y} & \mathcal{K}_{xy} & M_x \end{vmatrix} \\ &= \frac{1}{\Delta}\{[\Sigma z\Sigma zx^2 - (\Sigma zx)^2]M_x - (\Sigma z\Sigma zxy - \Sigma zx\Sigma zy)M_y\} \\ &= \frac{\Sigma z}{\Delta}[(\mathcal{I}_y - \bar{x}^2\Sigma z)M_x - (\mathcal{K}_{xy} - \bar{x}\bar{y}\Sigma z)M_y] \end{aligned}\right\} \quad (6.8\text{–}15)$$

where Δ has the same value as previously found in equation 6.8–8.

From the values of a, b, and c obtained from the above expressions, the amplitudes and the position of the nodal axis may be found.

The above general form may be simplified without loss in generality by selecting one of the axes, say the y axis, normal to the plane of the exciting torque, in which event $M_x = 0$ and $M_y = M_0$. The intercepts and amplitudes will then have the form:

$$\left.\begin{aligned} e_x &= -\frac{a}{b} = \frac{\bar{x}\mathcal{J}_x - \bar{y}\mathcal{K}_{xy}}{\mathcal{J}_x - \bar{y}^2\Sigma \mathrm{z}} \qquad e_y = -\frac{a}{c} = \frac{\bar{x}\mathcal{J}_x - \bar{y}\mathcal{K}_{xy}}{\bar{x}\bar{y}\Sigma \mathrm{z} - \mathcal{K}_{xy}} \\ \Theta &= \sqrt{a^2 + b^2} = \frac{M_0\Sigma \mathrm{z}}{\Delta}\sqrt{(\mathcal{J}_x - \bar{y}^2\Sigma \mathrm{z})^2 + (\bar{x}\bar{y}\Sigma \mathrm{z} - \mathcal{K}_{xy})^2} \\ &\text{and} \\ A(x, y) &= \frac{M_0\Sigma \mathrm{z}}{\Delta}[\bar{y}\mathcal{K}_{xy} - \bar{x}\mathcal{J}_x + x(\mathcal{J}_x - \bar{y}^2\Sigma \mathrm{z}) \\ &\qquad + y(\bar{x}\bar{y}\Sigma \mathrm{z} - \mathcal{K}_{xy})] \end{aligned}\right\} \quad (6.8\text{–}16)$$

It is important to note, in all of the analysis presented above, that the quantity Δ is dependent only on the characteristics of the elements of the system and completely independent of the location or type of the exciting force or torque. Δ, however, is dependent on the frequency of the vibration since the individual impedances of the mass elements are functions of the frequency.

The frequency equation for this system is

$$\Delta = 0 \qquad (6.8\text{–}17)$$

Systems of this type have a minimum of three degrees of freedom, and the frequency equation has at least three real roots representing the squares of the natural circular frequencies. The nodal axis for each natural frequency can be determined by substituting the frequency into equations 6.8–2 or 6.8–3. One of these axes may be found to be at infinity ($e_x = e_y = \infty$), which means that one of the modes is a pure translatory motion.

Each of the nodal axes will contain the frequency centroid for that particular frequency or mode. The nodal axis is the axis about which the dynamic resisting moment is zero, and the maximum dynamic resistance occurs about an axis perpendicular to the nodal axis and through the frequency center. These axes and dynamic resisting moments compare with the principal axes and the corresponding minimum and maximum second moments (moments of inertia) in static mechanics. Since the frequency centroid, as well as the direction of the principal axes, will vary with the frequency, it is not to be expected that the relative location or direction of the nodal axis will be obvious or even simply represented in the general case.

If the system consists solely of a rigid solid body on elastic supports (three degrees of freedom), the frequency centroid will be confined to a straight line through the elastic center of the supporting springs and the centroid of the solid body. The nodal axes will be normal to each other *only* if the elastic system and the mass system have one principal axis in common. This common axis will then become one of the nodal axes and will contain the frequency centroid.

Since the frequency centroid is a function of the frequency, in general, it will be expedient to write the frequency equation in the form,

$$\left.\begin{aligned}&\Sigma z[\Sigma zx^2\Sigma zy^2 - (\Sigma zxy)^2] - \Sigma zx(\Sigma zx\Sigma zy^2 \\ &\qquad - \Sigma zy\Sigma zxy) - \Sigma zy(\Sigma zy\Sigma zx^2 - \Sigma zx\Sigma zxy) = 0 \\ &\text{or} \\ &\frac{1}{\Sigma z}\{[\Sigma z\Sigma zx^2 - (\Sigma zy)^2][\Sigma z\Sigma zy^2 - (\Sigma zx)^2] \\ &\qquad - (\Sigma z\Sigma zxy - \Sigma zx\Sigma zy)^2\} = 0\end{aligned}\right\} \quad (6.8\text{–}18)$$

For those particular circumstances where the location of the frequency centroid is independent of the frequency, this centroid is a convenient choice of origin of coordinates thereby reducing the frequency equation to

$$\Sigma z[\Sigma zx^2\Sigma zy^2 - (\Sigma zxy)^2] = 0 \qquad (6.8\text{–}19)$$

It will be noted that $\Sigma z = 0$ under these circumstances will always yield a natural frequency corresponding to a pure translatory mode.

After the frequencies have been evaluated from equation 6.8–18, the corresponding frequency centroids and resisting moments with respect to the selected coordinate system can be determined. If $\mathcal{I}_{xc}$, $\mathcal{I}_{yc}$, and $\mathcal{K}_{xcyc}$ represent resisting moments with respect to a coordinate system with the frequency center as origin, then the frequency equation 6.8–19 for the mode other than pure translation is

$$\left.\begin{aligned}&\mathcal{I}_{xc}\mathcal{I}_{yc} = \mathcal{K}^2_{xcyc} \\ \text{or}\qquad &\mathcal{K}_{xcyc} = \sqrt{\mathcal{I}_{xc}\mathcal{I}_{yc}}\end{aligned}\right\} \quad (6.8\text{–}20)$$

Substitution of this relationship into equations 6.8–3 and 6.8–4 gives the following values for the intercepts and slopes of the nodal axis for free vibration:

$$\left.\begin{aligned}&e_x = \bar{x} - \bar{y}\sqrt{\frac{\mathcal{I}_{yc}}{\mathcal{I}_{xc}}} \quad \text{and} \quad e_y = \bar{y} - \bar{x}\sqrt{\frac{\mathcal{I}_{xc}}{\mathcal{I}_{yc}}} \\ &\tan\beta = \frac{e_y}{e_x} = -\sqrt{\frac{\mathcal{I}_{xc}}{\mathcal{I}_{yc}}} = -\sqrt{\frac{\mathcal{I}_x - \bar{y}^2\Sigma z}{\mathcal{I}_y - \bar{x}^2\Sigma z}}\end{aligned}\right\} \quad (6.8\text{–}21)$$

The evaluation of the intercepts e_x and e_y can, of course, be equally well made by direct substitution in equations 6.8–16.

Illustrative example. The following example consists of a rectangular homogeneous plate M on a rectangular uniform distributed elastic support k_1 and has concentrated spring supports k_2 and k_3, as indicated in Fig. 6–53.

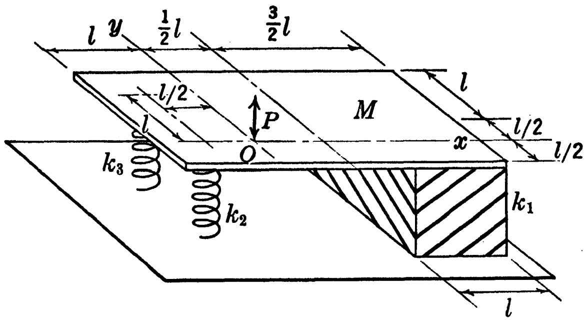


FIG. 6–53

The oscillating force P is applied at the origin of the coordinate system shown. It is required to determine the angular amplitude and nodal axis for this dynamic loading as well as to find the resonance frequencies and the corresponding nodal axes.

The dynamic resisting moment of the mass M, as well as the cross product with respect to the x and y axes, is readily seen to be:

$$\mathcal{J}_{xM} = z_M \bar{r}_x^{\,2} = -\omega^2 M[\tfrac{1}{12}(2l)^2 + (\tfrac{1}{2}l)^2] = -\tfrac{7}{12}\omega^2 M l^2$$

$$\mathcal{J}_{yM} = z_M \bar{r}_y^{\,2} = -\omega^2 M[\tfrac{1}{12}(3l)^2 + (\tfrac{1}{2}l)^2] = -\omega^2 M l^2$$

$$\mathcal{H}_{xyM} = z_M \bar{x}_M \bar{y}_M = -\omega^2 M(\tfrac{1}{2}l)(\tfrac{1}{2}l) = -\tfrac{1}{4}\omega^2 M l^2$$

The similar values for the springs are:

$$\mathcal{J}_{xk} = \Sigma z_k y^2 = k_1[\tfrac{1}{12}(2l)^2] + k_3 l^2 = (\tfrac{7}{12}k_1 + k_3)l^2$$

$$\mathcal{J}_{yk} = \Sigma z_k x^2 = k_1[\tfrac{1}{12}l^2 + (\tfrac{3}{2}l)^2] + (k_2 + k_3)(\tfrac{1}{2}l)^2 = [\tfrac{7}{3}k_1 + \tfrac{1}{4}(k_2 + k_3)]l^2$$

$$\mathcal{H}_{xyk} = \Sigma z_k xy = k_1(\tfrac{1}{2}l)(\tfrac{3}{2}l) + k_3(l)(-\tfrac{1}{2}l) = (\tfrac{3}{4}k_1 - \tfrac{1}{2}k_3)l^2$$

from which

$$\mathcal{J}_x = \Sigma z y^2 = (\tfrac{7}{12}k_1 + k_3 - \tfrac{7}{12}\omega^2 M)l^2$$

$$\mathcal{J}_y = \Sigma z x^2 = [\tfrac{7}{3}k_1 + \tfrac{1}{4}(k_2 + k_3) - \omega^2 M]l^2$$

$$\mathcal{H}_{xy} = \Sigma z xy = (\tfrac{3}{4}k_1 - \tfrac{1}{2}k_3 - \tfrac{1}{4}\omega^2 M)l^2$$

and also

$$\bar{x}\Sigma z = \Sigma zx = \left[\tfrac{3}{2}k_1 - \tfrac{1}{2}(k_2 + k_3) - \frac{\omega^2}{2} M\right] l$$

$$\bar{y}\Sigma z = \Sigma zy = (\tfrac{1}{2}k_1 + k_3 - \tfrac{1}{2}\omega^2 M)l$$

$$\Sigma z = k_1 + k_2 + k_3 - \omega^2 M$$

For simplicity let $k_1 = 3k$, $k_2 = 2k$, $k_3 = k$, and $M = 6m$. Also let

$$p^2 = \frac{k_1 + k_2 + k_3}{M} = \frac{6k}{6m} = \frac{k}{m}$$

and $\alpha^2 = \frac{\omega^2}{p^2}$. In this manner the above values can be expressed as

$$\mathcal{I}_x = \Sigma zy^2 = (11 - 14\alpha^2)\left(\frac{l}{2}\right)^2 k \qquad \bar{x}\Sigma z = \Sigma zx = (6 - 6\alpha^2)\left(\frac{l}{2}\right)k$$

$$\mathcal{I}_y = \Sigma zx^2 = (31 - 24\alpha^2)\left(\frac{l}{2}\right)^2 k \qquad \bar{y}\Sigma z = \Sigma zy = (5 - 6\alpha^2)\left(\frac{l}{2}\right)k$$

$$\mathcal{H}_{xy} = \Sigma zxy = (7 - 6\alpha^2)\left(\frac{l}{2}\right)^2 k \qquad \Sigma z = 6(1 - \alpha^2)k$$

The coordinates $\bar{x}$ and $\bar{y}$ for the frequency centroid become

$$\bar{x} = \frac{\Sigma zx}{\Sigma z} = \frac{6(1 - \alpha^2)\left(\frac{l}{2}\right)k}{6(1 - \alpha^2)k} = \frac{l}{2}$$

and

$$\bar{y} = \frac{\Sigma zy}{\Sigma z} = \frac{5 - 6\alpha^2}{6(1 - \alpha^2)}\,\frac{l}{2}$$

where $\bar{x}$ is shown to be independent of the frequency, and the centroid will, in this case, be confined to a line through the mass center parallel to the y axis.

The intercepts for the nodal axis can be found by substitution in equations 6.8–12, giving

$$e_x = \frac{\mathcal{I}_x\mathcal{I}_y - \mathcal{H}_{xy}^2}{(\bar{x}\mathcal{I}_x - \bar{y}\mathcal{H}_{xy})\Sigma z} = \frac{\Sigma zy^2\Sigma zx^2 - (\Sigma zxy)^2}{\Sigma zx\Sigma zy^2 - \Sigma zy\Sigma zxy}$$

$$= \frac{[(11 - 14\alpha^2)(31 - 24\alpha^2) - (7 - 6\alpha^2)^2]\left(\frac{l}{2}\right)^4 k^2}{[6(1 - \alpha^2)(11 - 14\alpha^2) - (5 - 6\alpha^2)(7 - 6\alpha^2)]\left(\frac{l}{2}\right)^3 k^2}$$

$$= \frac{146 - 307\alpha^2 + 150\alpha^4}{31 - 78\alpha^2 + 48\alpha^4}\cdot l$$

$$e_y = \frac{\mathcal{I}_x\mathcal{I}_y - \mathcal{K}_{xy}{}^2}{(\bar{y}\mathcal{I}_y - \bar{x}\mathcal{K}_{xy})\Sigma z} = \frac{\Sigma zy^2\Sigma zx^2 - (\Sigma zxy)^2}{\Sigma zy\Sigma zx^2 - \Sigma zx\Sigma zxy}$$

$$= \frac{[(11 - 14\alpha^2)(31 - 24\alpha^2) - (7 - 6\alpha^2)^2]\left(\frac{l}{2}\right)^4 k^2}{[(5 - 6\alpha^2)(31 - 24\alpha^2) - 6(1 - \alpha^2)(7 - 6\alpha^2)]\left(\frac{l}{2}\right)^3 k^2}$$

$$= \frac{146 - 307\alpha^2 + 150\alpha^4}{113 - 228\alpha^2 + 108\alpha^4} \cdot l$$

From equation 6.8–8, the value of Δ will be found as

$$\Delta = (1001 - 2964\alpha^2 + 2820\alpha^4 - 864\alpha^6)\left(\frac{l}{2}\right)^4 k^3$$

Equations 6.8–10 give

$$b = \frac{P_0}{\Delta}(\Sigma z\Sigma zy^2 - \Sigma zy\Sigma zxy) = \frac{P_0}{\Delta}(31 - 78\alpha^2 + 48\alpha^4)\left(\frac{l}{2}\right)^3 k^2$$

and

$$c = \frac{P_0}{\Delta}(\Sigma zy\Sigma zx^2 - \Sigma zx\Sigma zxy) = \frac{P_0}{\Delta}(113 - 228\alpha^2 + 108\alpha^4)\left(\frac{l}{2}\right)^3 k^2$$

Substituting these values in equation 6.8–13 gives the angular amplitude,

$$\Theta = \sqrt{b^2 + c^2} = 2\frac{\sqrt{(31 - 78\alpha^2 + 48\alpha^4)^2 + (113 - 228\alpha^2 + 108\alpha^4)^2}}{1001 - 2964\alpha^2 + 2820\alpha^4 - 864\alpha^6} \cdot \frac{P_0}{kl}$$

Specifically, if $\alpha^2 = \omega^2/p^2 = 1$, then,

$$\Theta = 2\frac{\sqrt{50}}{7} \cdot \frac{P_0}{kl}$$

and

$$e_x = 11l; \qquad e_y = \tfrac{11}{7}l$$

from which

$$\tan\beta = \tfrac{1}{7}; \qquad \text{also} \quad \bar{x} = l/2 \quad \text{and} \quad \bar{y} = \infty$$

The resonance frequencies are obtained from

$$\Delta = 0 \quad \text{or} \quad 1001 - 2964\alpha^2 + 2820\alpha^4 - 864\alpha^6 = 0$$

The solution of this cubic equation results in

$$\alpha^2 = \frac{\omega^2}{p^2} = \begin{cases} 0.7487 \\ 1.0729 \\ 1.4423 \end{cases}$$

Substituting these values for α^2 in the expression for the intercepts gives, for the first mode, where $\alpha^2 = 0.7487$,

$$e_{x1} = \frac{146 - 307\alpha^2 + 150\alpha^4}{31 - 78\alpha^2 + 48\alpha^4} l = \frac{0.2316}{-0.4921} l = -0.4706l$$

$$e_{y1} = \frac{146 - 307\alpha^2 + 150\alpha^4}{113 - 228\alpha^2 + 108\alpha^4} l = \frac{0.2316}{2.8358} l = 0.0817l$$

As the nodal axis at the resonance frequency contains the frequency centroid, this axis may also be obtained by evaluating the location of this centroid in

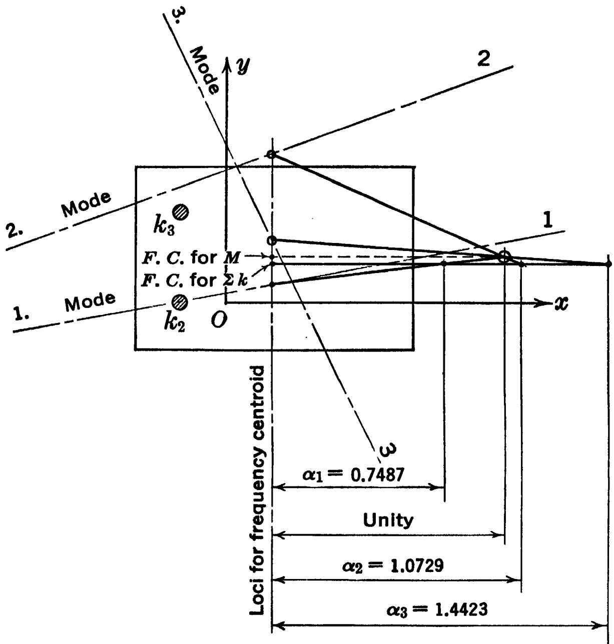


Fig. 6–54

place of one of the intercepts. For this first mode, the coordinates for the centroid will be

$$\bar{x}_1 = \frac{l}{2} \quad \text{and} \quad \bar{y}_1 = \frac{5 - 6\alpha^2}{6(1 - \alpha^2)} l = 0.1684l$$

In similar manner, the intercepts for the other two frequencies will be found to be:

$$e_{x2} = -4.1733l \qquad e_{y2} = 1.4674l \qquad \text{for second mode}$$

and

$$e_{x3} = 0.8309l \qquad e_{y3} = 1.7288l \qquad \text{for third mode}$$

The location of the nodal axes and corresponding frequency centroids are shown in Fig. 6–54.

A more comprehensive understanding of the variation in the location of the frequency centroid may be gained from a graphical interpretation thereof.

The resultant frequency centroid is obtained from the resultant of the dynamic spring force and the dynamic mass force through their respective centroids which are stationary for a single solid mass system. The resultant dynamic spring force or impedance K is constant whereas the dynamic mass force varies with the frequency in the manner $-w^2M$. If the spring force is taken as unity, the relative mass force becomes

$$-\frac{\omega^2 M}{K} = -\frac{\omega^2}{p^2} = -\alpha^2$$

The frequency centroid can therefore be obtained from the location of the resultant of two parallel forces of magnitude 1 and α^2, acting in opposite directions. The common graphic method of accomplishing this is shown in Fig. 6–54. It will be obvious from this that the centroid moves in a straight line through the two stationary centroids of the springs and the mass and that the frequency centroid cannot appear between the two fixed centroids. It also shows that the frequency centroid will move from the spring centroid toward infinity for $\alpha^2 \rightarrow 1$ and reaches infinity for $\alpha^2 = 1$, whereas it will be found on the side of the mass center for $\alpha^2 > 1$.

In the limiting case where the spring and mass centroids are at the same point, it can be deduced from the above that the frequency center will remain at this common centroid for any value of α^2 but $\alpha^2 = 1$.

Chapter 7

SOLUTION OF THE GENERAL FREQUENCY EQUATION

7.1. General Discussion

The solution of free-vibration problems with many degrees of freedom inherently involves the solution of an algebraic equation of higher order. This equation is called the frequency equation. In general a system possessing n modes of vibratory motions will have a frequency equation of the form

$$p^{2n} + a_{2n-1}p^{2n-1} + a_{2n-2}p^{2n-2} + \cdots + a_1p + a_0 = 0 \quad (7.1\text{–}1)$$

The a's in this equation are constants which depend on the physical constants of the system: i.e., the masses, springs, and dampers. The solution of this equation will yield the values of the natural frequencies of the system, as well as the dissipation rates for each mode of motion.

If the system is undamped, the frequency equation may be written as

$$p^{2n} + b_{2n-2}p^{2n-2} + \cdots + b_2p^2 + b_0 = 0 \quad (7.1\text{–}2)$$

where in general $a_m \neq b_m$. It is seen that the frequency equation for an undamped system may be considered as an nth degree equation in p^2, whereas the frequency equation for the damped system is of degree $2n$ in p. The roots p of the frequency equation for a damped system are, in general, complex. In contrast, the undamped system's frequency equation has real roots p^2. These two facts tend to make the solution of equation 7.1–1 somewhat more difficult than that of equation 7.1–2.

A moderate amount of damping has only a slight effect on the frequency, and the frequencies obtained by neglecting the damping will be good approximations to the true frequencies. Fortunately most practical problems are of this category, and the undamped solution may in such cases be taken as a very close estimate of the natural frequencies of the system. For this reason, the additional labor brought about by the inclusion of damping in the analysis can only be justified when the system is heavily damped.

Mechanical systems of two and three modes of vibratory motion give rise to frequency equations which are quadratic and cubic equations, respectively, in p^2. These equations are amenable to algebraic analysis, and their general solutions may be formulated readily. Equation 5.3–5 is an example of a quadratic frequency equation, and equation 5.7–5 illustrates the cubic type. The solution of the quadratic is elementary and needs no discussion here, and the solution of the cubic is given by equations 5.7–6. The quartic equation may also be formulated by exact algebraic methods, but the procedure is long and somewhat tedious. It is usually easier to resort to one of the approximate methods of solving higher-order equations for those of the fourth degree and higher. It is the solution of these higher-order equations with which this chapter is concerned.

There are several procedures available for approximating the natural frequencies of a given system. Broadly speaking they may be classified into two groups.

1. Those methods that stem directly from the equations of motion and do not require that the coefficients of the frequency equation be calculated.

2. Those methods that require that the frequency equation be deduced from the equations of motion.

In the former category fall the method of Holzer, matrix methods, and relaxation procedures, whereas the latter includes the methods of Graeffe, Newton, and Horner. The details of implementing these procedures form a large and replete literature in the engineering and mathematics journals. The mastering of all of these methods is a luxury which only the engineer who uses them every day can afford. Indeed, it is far better to be proficient in one or two methods than to have a broad but hazy notion of the multitude of these procedures that have been developed in the past century. It is idle to speculate on which is the "best" method as all methods have merit and each one has strong points in its favor. The "best" method is usually the one in which the "user" has the most proficiency and which he is most able to apply accurately and rapidly. Any other method, no matter how elegant, with which the user does not feel secure, is to be applied only as a check or an academic exercise until its procedure and particular advantages are mastered.

Of the many procedures available, only two have been chosen here for detailed treatment. These methods, namely those of Holzer and Graeffe, have been selected because they are easily applied and illustrate both groups of procedures. The following treatment is intended to be an exposition of the fundamentals rather than a complete presentation of the variations of these methods.

7.2. Holzer's Method

The method devised by H. Holzer[1] is particularly simple and direct. It has the advantage that a prededuction of the frequency equation is unnecessary and further that each operation is capable of physical interpretation. The disadvantages are that it is a method of successive approximations and that the frequency of only one mode of motion is determined at a time. This latter disadvantage is countered by the ability to evaluate each mode independent of the others.

One of the simplest applications of the Holzer method is to a system of n masses or inertias interconnected in series by $(n - 1)$ spring elements. Such a system is that of Fig. 7–1. The angular displacements of the inertias I_i may be denoted by a corresponding θ_i. The torque in the ith shaft is related to the angle of twist of the shaft by the relation

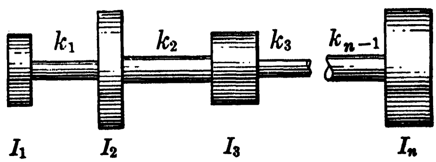


FIG. 7–1

$$T_i = k_i(\theta_{i+1} - \theta_i)$$

and the equation of motion for the ith mass is given by

$$T_i - T_{i-1} = I_i\ddot{\theta}_i$$

Starting at the left end of the system, the equations of motion for each inertia become:

$$\left.\begin{aligned}
T_1 &= I_1\ddot{\theta}_1 = k_1(\theta_2 - \theta_1) \\
T_2 - T_1 &= I_2\ddot{\theta}_2 = k_2(\theta_3 - \theta_2) - k_1(\theta_2 - \theta_1) \\
T_3 - T_2 &= I_3\ddot{\theta}_3 = k_3(\theta_4 - \theta_3) - k_2(\theta_3 - \theta_2) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
-T_{n-1} &= I_n\ddot{\theta}_n = \qquad\qquad - k_{n-1}(\theta_n - \theta_{n-1})
\end{aligned}\right\} \quad (7.2\text{–}1)$$

It is important to note that the addition of these equations results in

$$I_1\ddot{\theta}_1 + I_2\ddot{\theta}_2 + I_3\ddot{\theta}_3 + \cdots + I_n\ddot{\theta}_n = 0$$

[1] *Die Berechnung der Drehschwingungen,* H. Holzer, Berlin (1921).

or, on integration,

$$I_1\dot{\theta}_1 + I_2\dot{\theta}_2 + I_3\dot{\theta}_3 + \cdots + I_n\dot{\theta}_n = \text{constant} \qquad (7.2\text{–}2)$$

Physically, this shows that the total angular momentum of the system is constant.

Since the system is linear in character, solutions of the form

$$\theta_i = \Theta_i \cos pt \qquad (7.2\text{–}3)$$

may be expected. Substitution from equation 7.2–3 into equation 7.2–1 and division by the common factor cos pt gives:

$$\left.\begin{aligned}
-I_1p^2\Theta_1 &= k_1(\Theta_2 - \Theta_1)\\
-I_2p^2\Theta_2 &= k_2(\Theta_3 - \Theta_2) - k_1(\Theta_2 - \Theta_1)\\
-I_3p^2\Theta_3 &= k_3(\Theta_4 - \Theta_3) - k_2(\Theta_3 - \Theta_2)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
-I_np^2\Theta_n &= \qquad\qquad\qquad - k_{n-1}(\Theta_n - \Theta_{n-1})
\end{aligned}\right\} \qquad (7.2\text{–}4)$$

and equation 7.2–2 becomes

$$(I_1\Theta_1 + I_2\Theta_2 + I_3\Theta_3 + \cdots + I_n\Theta_n)p^2 = 0$$

The solutions of interest correspond to $p^2 \neq 0$, since $p = 0$ corresponds to a rigid body rotation; hence

$$\sum_{i=1}^{n} I_i\Theta_i = 0 \qquad (7.2\text{–}5)$$

It has been shown in previous chapters that the amplitude of free vibration is arbitrary and that only relative amplitudes may be determined; consequently there is no loss in generality if one of the amplitudes, say Θ_1, is given an arbitrary value. The equations 7.2–4 may then be looked upon as a set of n equations in the n unknowns $\Theta_2, \Theta_3, \Theta_4 \cdots \Theta_n$, and p^2. The unknowns $\Theta_2, \Theta_3 \cdots \Theta_n$ may be eliminated by solving the equations in succession. Thus, from the first,

$$\Theta_2 = \Theta_1 - \frac{I_1p^2\Theta_1}{k_1} \qquad (7.2\text{–}6)$$

From the sum of the first two,

$$\Theta_3 = \Theta_2 - \frac{p^2}{k_2}(I_1\Theta_1 + I_2\Theta_2)$$

From the sum of the first three,

$$\Theta_4 = \Theta_3 - \frac{p^2}{k_3}(I_1\Theta_1 + I_2\Theta_2 + I_3\Theta_3)$$

etc., giving for the sum of all except the last equation,

$$\Theta_n = \Theta_{n-1} - \frac{p^2}{k_{n-1}} \sum_{i=1}^{n-1} I_i\Theta_i$$

Thus we find Θ_2 from the first equation above in terms of Θ_1 and substitute its value into the next and solve for Θ_3, etc. The values of Θ_i so obtained may then be substituted into the equation 7.2–5 from which Θ_n may be factored out, leaving a single equation in the single unknown p^2. This is the frequency equation.

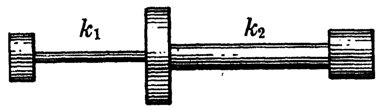


FIG. 7-2

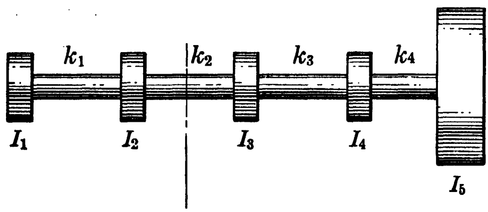
The above calculation is conveniently carried out in tabular form. Consider for simplicity the system of Fig. 7–2. The table used for this calculation follows.

1	2	3	4	5	6	7
I	p^2I	Θ	$p^2I\Theta$	$p^2\Sigma I\Theta$	k	$\frac{p^2}{k}\sum I\Theta$
I_1	p^2I_1	1	p^2I_1	p^2I_1	k_1	$\frac{p^2I_1}{k_1}$
I_2	p^2I_2	$1 - \frac{p^2I_1}{k_1}$	$p^2I_2\left(1 - \frac{p^2I_1}{k_1}\right)$	$p^2I_1 + p^2I_2 - \frac{p^4I_1I_2}{k_1}$	k_2	$\frac{p^2(I_1+I_2) - \frac{p^4I_1I_2}{k_1}}{k_2}$
I_3	p^2I_3	$1 - \frac{p^2I_1}{k_1} - \frac{p^2(I_1+I_2)}{k_2} + \frac{p^4I_1I_2}{k_1k_2}$	$p^2I_3 - \frac{p^4I_1I_3}{k_1} - \frac{p^4I_3(I_1+I_2)}{k_2} + \frac{p^6I_1I_2I_3}{k_1k_2}$	$p^2(I_1+I_2+I_3) - \frac{p^4}{k_1}I_1(I_2+I_3) - \frac{p^4}{k_2}I_3(I_1+I_2) + \frac{p^6}{k_1k_2}I_1I_2I_3$		

The frequency equation is

$$p^2\sum I\Theta = \frac{p^6I_1I_2I_3}{k_1k_2} - p^4\left[\frac{I_1}{k_1}(I_2+I_3) + \frac{I_3}{k_2}(I_1+I_2)\right] + p^2(I_1+I_2+I_3) = 0$$

In the first column, the values of the moments of inertia are arranged in sequence. In the second column, the product p^2I is entered as shown. In the third column, the amplitude of I_1 is entered, and, since it is arbitrary, it may be taken as unity. The fourth column contains the product of $p^2 I\Theta$, which, for the first row, is simply $p^2 I_1$. The sum of all the preceding values of $p^2 I$ is entered in the fifth column, and again, for the first row, this is simply $p^2 I$. In the sixth column, values of the spring constants are listed. The last column is obtained by dividing the entry in the fifth column by the entry in the sixth. Thus, for the first row, this is $p^2 I_1/k_1$. Turning our attention



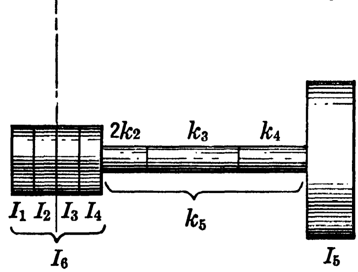


FIG. 7-3

FIG. 7-4

to equation 7.2-6, we see that, if this is subtracted from Θ_1, Θ_2 is obtained. This value is therefore entered in column three, row two.

The entries for the second row are then completed in the same manner as for the first row, then the third row entries are similarly completed. The final entry in the last row, when equated to zero, is the frequency equation in which only p is unknown. The physical meaning of this final entry is seen to be the residual torque of circular frequency p which must be supplied to the system if it is to vibrate with this circular frequency. The value of p which makes the residual torque vanish corresponds to the natural vibration of the system which occurs without any external exciting force.

As a second example of this procedure, the system of Fig. 7-3 may be treated. For simplicity, it is convenient to consider the special nondimensional case in which

$$I_1 = I_2 = I_3 = I_4 = \tfrac{1}{5} I_5 = 1$$

$$k_1 = k_2 = k_3 = \tfrac{1}{2} k_4 = 1$$

The reduction of the frequency equation for this system is shown in the following table.

Mass No.	I	p^2I	Θ	$p^2I\Theta$	$p^2\Sigma I\Theta$	k	$\frac{p^2}{k}\sum I\Theta$
1	1	p^2	1	p^2	p^2	1	p^2
2	1	p^2	$1 - p^2$	$p^2 - p^4$	$2p^2 - p^4$	1	$2p^2 - p^4$
3	1	p^2	$1 - 3p^2 + p^4$	$p^2 - 3p^4 + p^6$	$3p^2 - 4p^4 + p^6$	1	$3p^2 - 4p^4 + p^6$
4	1	p^2	$1 - 6p^2 + 5p^4 - p^6$	$p^2 - 6p^4 + 5p^6 - p^8$	$4p^2 - 10p^4 + 6p^6 - p^8$	2	$2p^2 - 5p^4 + 3p^6 - \frac{1}{2}p^8$
5	5	$5p^2$	$1 - 8p^2 + 10p^4 - 4p^6 + \frac{1}{2}p^8$	$5p^2 - 40p^4 + 50p^6 - 20p^8 + 2\frac{1}{2}p^{10}$	$9p^2 - 50p^4 + 56p^6 - 21p^8 + 2\frac{1}{2}p^{10}$		

The frequency equation is

$$p^2(p^8 - 8.4p^6 + 22.4p^4 - 20p^2 + 3.6) = 0$$

Although the use of the Holzer table to derive the frequency equations in the way outlined above is direct and useful, the main advantage of the Holzer table lies in its use as a means of calculating the frequencies p directly, without the deduction of the frequency equation. This may be accomplished by estimating the frequency p and then numerically calculating the residual torque corresponding to this frequency. After a few judicious trials, values of p^2 may be selected so that the residual torque is sufficiently small, thus permitting an approximate value of p^2 to be interpolated to the required accuracy.

As a guide in the first estimate of p^2 for this calculation, the actual system may be replaced by a two-mass system as shown in Fig. 7–4. The fundamental frequency of this simple system may be used as a first estimate. Thus concentration of the inertias I_1, I_2, I_3, and I_4 into I_6 and consideration of the shaft from I_5 to their center of gravity as k_5 gives

$$I_6 = \tfrac{4}{5}I_5 = 4$$

$$\frac{1}{k_5} = \frac{1}{k_4} + \frac{1}{k_3} + \frac{1}{2k_2} = \tfrac{1}{2} + 1 + \tfrac{1}{2} = 2$$

Thus $k_5 = \frac{1}{2}$.

Then, from the elementary theory for a two-mass system, the natural circular frequency p is given by

$$p^2 = k_5\left(\frac{1}{I_5} + \frac{1}{I_6}\right) = \tfrac{1}{2}(\tfrac{1}{5} + \tfrac{1}{4}) = \tfrac{9}{40} = 0.225 \text{ rad}^2 \text{ per sec}^2$$

This corresponds to the mode of vibration which has one node, namely the lowest mode.

Now that a rough estimate of the first node has been obtained, it is expedient to calculate systematically the residual torques at spaced intervals of p^2, starting with a value of p^2 which is surely less than that corresponding to the lowest natural frequency. In this specific case intervals of p^2 of about 0.5 appear to be appropriate, and so the values of the residual torque corresponding to $p^2 = 0.2, 0.5, 1.0, 1.5 \cdots$

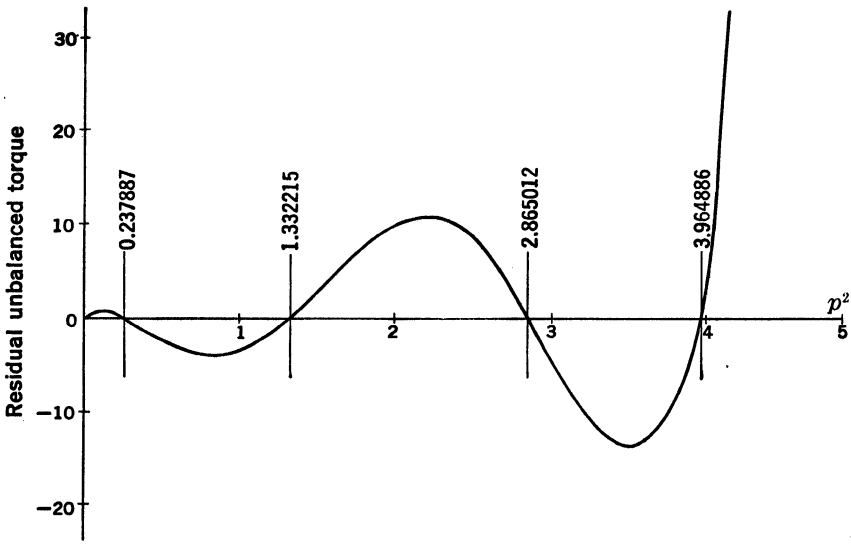


FIG. 7–5

have been calculated in the following tables. As these tables are calculated, a curve of the residual torque versus p^2 may be plotted as in Fig. 7–5.

Mass No.	I	p^2I	Θ	$p^2I\Theta$	$p^2\Sigma I\Theta$	k	$\frac{p^2}{k}\sum I\Theta$
				$p^2 = 0.2$			
1	1	0.2	+1	+0.2	+0.2	1	+0.2
2	1	0.2	+0.8	+0.16	+0.36	1	+0.36
3	1	0.2	+0.44	+0.088	+0.448	1	+0.448
4	1	0.2	−0.008	−0.0016	+0.4464	2	+0.2232
5	5	1.0	−0.2312	−0.2312	+0.2152		
				$p^2 = 0.5$			
1	1	0.5	+1	+0.5	+0.5	1	+0.5
2	1	0.5	+0.5	+0.25	+0.75	1	+0.75
3	1	0.5	−0.25	−0.125	+0.625	1	+0.625
4	1	0.5	−0.875	−0.4375	+0.1875	2	+0.0938
5	5	2.5	−0.9688	−2.4219	−2.344		

Mass No.	I	p^2I	Θ	$p^2I\Theta$	$p^2\Sigma I\Theta$	k	$\frac{p^2}{k}\sum I\Theta$
				$p^2 = 1.0$			
1	1	1	+1	+1	+1	1	+1
2	1	1	0	0	+1	1	+1
3	1	1	−1	−1	0	1	0
4	1	1	−1	−1	−1	2	−0.5
5	5	5	−0.5	−2.5	−3.5		
				$p^2 = 1.5$			
1	1	1.5	+1	+1.5	+1.5	1	+1.5
2	1	1.5	−0.50	−0.75	+0.75	1	+0.75
3	1	1.5	−1.25	−1.875	−1.125	1	−1.125
4	1	1.5	−0.125	−0.1875	−1.3125	2	−0.6563
5	5	7.5	+0.5313	+3.9844	+2.6719		
				$p^2 = 2.0$			
1	1	2	+1	+ 2	+ 2	1	+2
2	1	2	−1	− 2	0	1	0
3	1	2	−1	− 2	− 2	1	−2
4	1	2	+1	+ 2	0	2	0
5	5	10	+1	+10	+10		
				$p^2 = 2.5$			
1	1	2.5	+1	+2.5	+2.5	1	+2.5
2	1	2.5	−1.5	−3.75	−1.25	1	−1.25
3	1	2.5	−0.25	−0.625	−1.875	1	−1.875
4	1	2.5	+1.625	+4.0625	+2.1875	2	+1.0938
5	5	12.5	+0.5312	+6.6406	+8.8281		
				$p^2 = 3.0$			
1	1	3	+1	+3	+3	1	+3
2	1	3	−2	−6	−3	1	−3
3	1	3	+1	+3	0	1	0
4	1	3	+1	+3	+3	2	+1.5
5	5	15	−0.5	−7.5	−4.5		
				$p^2 = 3.5$			
1	1	3.5	+1	+ 3.5	+ 3.5	1	+3.5
2	1	3.5	−2.5	− 8.75	− 4.25	1	−4.25
3	1	3.5	+1.75	+ 6.125	+ 1.875	1	+1.875
4	1	3.5	−0.125	− 0.4375	+ 1.4375	2	+0.7188
5	5	17.5	−0.8438	−14.7656	−13.3281		
				$p^2 = 4$			
1	1	4	+1	+ 4	+ 4	1	+ 4
2	1	4	−3	−12	− 8	1	− 8
3	1	4	+5	+20	+12	1	+12
4	1	4	−7	−28	−16	2	− 8
5	5	20	+1	+20	+ 4		

Since this system possesses four degrees of freedom besides that of rigid body rotation, the calculations may be carried out until the four zero points of the unbalanced torque curve have been located. From this curve, the approximate values of p^2 for each mode may be estimated and refined by successive Holzer tables. The final results of these calculations in the form of the last Holzer table for each mode follows.

The sixth column which shows the total torque will at the same time indicate the relative position of the selected frequency with respect to the natural frequencies. In this manner the number of changes in sign in the total torque column will indicate the number of natural frequencies below the assumed p^2.

Mass No	I	p^2I	Θ	$p^2I\Theta$	$p^2\Sigma I\Theta$	k	$\frac{p^2}{k}\sum I\Theta$
				$p^2 = 3.964886$			
1	1	3.964886	+1.000000	+ 3.964886	+ 3.964886	1	+ 3.964886
2	1	3.964886	−2.964886	−11.755435	− 7.790549	1	− 7.790549
3	1	3.964886	+4.825663	+19.133204	+11.342655	1	+11.342655
4	1	3.964886	−6.516992	−25.839130	−14.496475	2	− 7.248238
5	5	19.824430	+0.731246	+14.496536	+ 0.000060		
				$p^2 = 2.865012$			
1	1	2.865012	+1.000000	+2.865012	+2.865012	1	+2.865012
2	1	2.865012	−1.865012	−5.343282	−2.478270	1	−2.478270
3	1	2.865012	+0.613258	+1.756992	−0.721278	1	−0.721278
4	1	2.865012	+1.334536	+3.823462	+3.102184	2	+1.551092
5	5	14.325060	−0.216558	−3.102178	+0.000006		
				$p^2 = 1.332215$			
1	1	1.332215	+1.000000	+1.332215	+1.332215	1	+1.332215
2	1	1.332215	−0.332215	−0.442582	+0.889633	1	+0.889633
3	1	1.332215	−1.221848	−1.627764	−0.738131	1	−0.738131
4	1	1.332215	−0.483717	−0.644415	−1.382546	2	−0.691273
5	5	6.661075	+0.207556	+1.382546	0.000000		
				$p^2 = 0.237887$			
1	1	0.237887	+1.000000	+0.237887	+0.237887	1	+0.237887
2	1	0.237887	+0.762113	+0.181297	+0.419184	1	+0.419184
3	1	0.237887	+0.342929	+0.081578	+0.500762	1	+0.500762
4	1	0.237887	−0.157833	−0.037546	+0.463216	2	+0.231608
5	5	1.189453	−0.389441	−0.463215	+0.000001		

The number of significant figures to which this calculation has been carried out is more than is usually required or justified by the accuracy to which the physical constants are known. The additional significant figures in this and later examples in this chapter are included as an aid to the student in checking the example.

It is not necessary to plot the residual torque curve in great detail. A few points to indicate clearly the regions of the "zeros" are all that is required. In some instances this will serve to show that there are no natural frequencies in the operating range and the analysis need not be carried further. In other examples one or two frequencies will be seen to be in the operating range and these frequencies may be located to the accuracy desired by successive Holzer tables. In this way the natural frequencies which are clearly not of interest are eliminated early in the analysis, and attention may be focused on only those frequencies that may be critical.

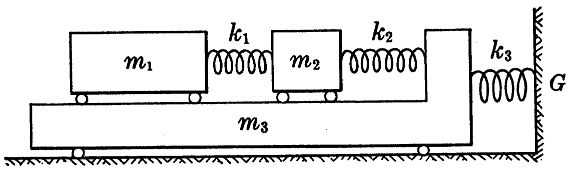


FIG. 7–6

As a further example of the use of the Holzer table in the determination of the natural frequencies, consider the system of Fig. 7–6. This system is identical with that treated in section 6.5 and shown in Fig. 6–16*a*, except that the damping elements have been eliminated. The system vibrates with a translatory motion, and hence the headings of the Holzer tables must be modified accordingly. Furthermore, since the system has a fixed point, namely the right end of spring k_3, it is convenient to start at this point G with the Holzer table. The amplitude of the mass m_3 may be established arbitrarily as -1. The Holzer table may then be worked backward and forward from this point, as shown in the following tables.

Mass No.	m	p^2m	A	p^2mA	$p^2\Sigma mA$	k	$\frac{p^2}{k}\sum mA$
			$p^2 = 1000$				
G	—	—	0	—	+75.00	75	+1.0000
3	0.01036	10.36	−1.0000	−10.36	+64.64	75	+0.8619
2	0.00518	5.18	−1.8619	− 9.64	+55.00	25	+2.2000
1	0.00389	3.89	−4.0619	−15.80	+39.20		
			$p^2 = 2500$				
G	—	—	0	—	+75.00	75	+1.0000
3	0.01036	25.90	−1.0000	−25.90	+49.10	75	+0.6547
2	0.00518	12.95	−1.6547	−21.43	+27.67	25	+1.1068
1	0.00389	9.73	−2.7615	−26.87	+ 0.80		
			$p^2 = 5000$				
G	—	—	0	—	+75.00	75	+1.0000
3	0.01036	51.80	−1.0000	−51.80	+23.20	75	+0.3093
2	0.00518	25.90	−1.3093	−33.91	−10.71	25	−0.4284
1	0.00389	19.45	−0.8809	−17.13	−27.84		

Mass No.	m	p^2m	A	p^2mA	$p^2\Sigma mA$	k	$\frac{p^2}{k}\sum mA$
			$p^2 = 10{,}000$				
G	—	—	0	—	+75.0	75	+1.0000
3	0.01036	103.6	−1.0000	−103.6	−28.6	75	−0.3813
2	0.00518	51.8	−0.6187	− 32.0	−60.6	25	−2.4240
1	0.00389	38.9	+1.8053	+ 70.2	+ 9.6		
			$p^2 = 20{,}000$				
G	—	—	0	—	+ 75.0	75	+1.0000
3	0.01036	207.2	−1.0000	−207.2	−132.2	75	−1.7627
2	0.00518	103.6	+0.7627	+ 79.2	− 53.0	25	−2.1200
1	0.00389	77.8	+2.8827	+224.3	+171.3		
			$p^2 = 30{,}000$				
G	—	—	0	—	+ 75.0	75	+1.0000
3	0.01036	310.8	−1.0000	−310.8	−235.8	75	−3.1440
2	0.00518	155.4	+2.1440	+333.2	+ 97.4	25	+3.8951
1	0.00389	116.7	−1.7511	−204.4	−107.0		

Mass No.	m	p^2m	A	p^2mA	$p^2\Sigma mA$	k	$\frac{p^2}{k}\sum mA$
			$p^2 = 2542.48$				
G	—	—	0	—	+ 75.000000	75	+1.000000
3	0.01036269	26.346932	−1.000000	− 26.346932	+ 48.653068	75	+0.648708
2	0.00518135	13.173479	−1.648708	− 21.719220	+ 26.933848	25	+1.077354
1	0.00388601	9.880103	−2.726062	− 26.933773	+ 0.000075		
			$p^2 = 9367.44$				
G	—	—	0	—	+ 75.000000	75	+1.000000
3	0.01036269	97.071877	−1.000000	− 97.071877	− 22.071877	75	−0.294292
2	0.00518135	48.535985	−0.705708	− 34.252233	− 56.324110	25	−2.252964
1	0.00388601	36.401966	+1.547256	+ 56.323160	− 0.000950		
			$p^2 = 28298.43$				
G	—	—	0	—	+ 75.000000	75	+1.000000
3	0.01036269	293.247858	−1.000000	−293.247858	−218.247858	75	−2.909971
2	0.00518135	146.624070	+1.909971	+280.047722	+ 61.799864	25	+2.471995
1	0.00388601	109.967982	−0.562024	− 61.804645	− 0.004781		

The first six of the preceding tables are for values of p^2 at spaced intervals designed to locate the regions in which the natural frequencies lie. The residual force curve, as constructed from these tables, is shown in Fig. 7–7. The last three Holzer tables shown are those approximating the natural frequencies. The mechanism of constructing the tables is similar to that of the preceding example.

The use of the Holzer table for calculating the natural frequencies of branched systems is not different in its essentials, although there are two differences in technique worthy of attention. Consider the rotational system shown in Fig. 7–8. This system consists of three branches joined at a central point. Systems of this type may contain many gear boxes and separate branches leading from these points.

It is convenient to consider some one combination of two branches as the main system and then think of the remaining branches as joining

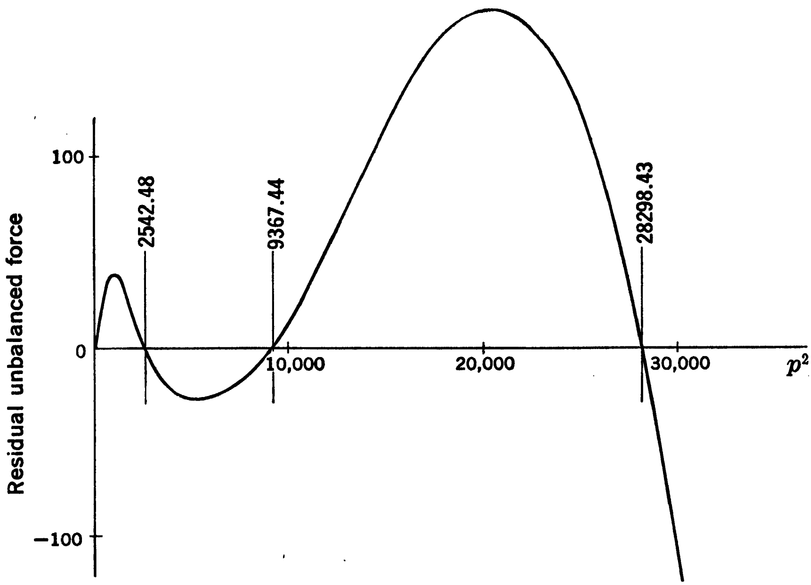


FIG. 7–7

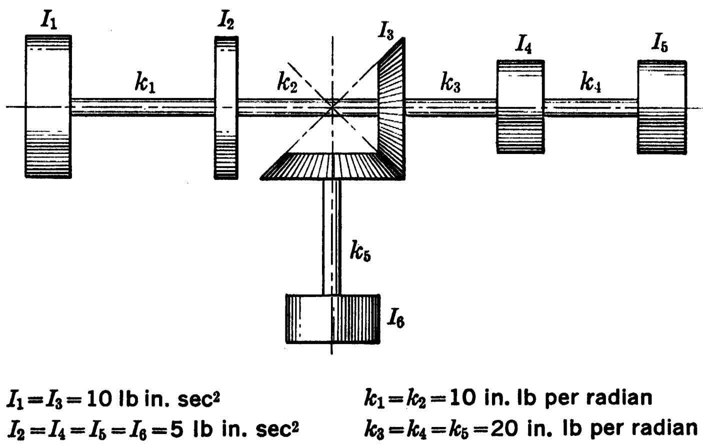


FIG. 7–8

this system. In the example of Fig. 7–8 the horizontal shaft with its attached masses is considered to be the main system. The inertia I_6 and spring k_5 form a branch to this system. It is immaterial in theory which part of the system is chosen as the main branch, although the selection may effect the simplicity of the actual calculations. The gear ratios that occur in the systems are considered next, and the system is usually reduced to the shaft speed of the main system. The corresponding modification of the spring constants and the moments of inertia are carried out in the usual manner, as discussed in Chapter 2.

The Holzer table is started at one end of the main system, say I_1, by assuming a unit amplitude and proceeding with the calculations up to the mass to which the first branch is connected. At this point the usual sequence of calculation is broken off with the calculation of the amplitude of this mass (I_3 in the example of Fig. 7–8). The calculations are now started anew by assuming the amplitude of the end mass of the branch I_6, and proceeding along the branch to the junction point I_3, thus arriving at another amplitude for the inertia at the junction point. Since this mass actually has only one definite amplitude, these two values for the amplitude must be equal. To make them equal, the assumed starting amplitude for the branch must be adjusted. Since the torque that is transmitted by the branch to the main system varies directly with the amplitude, it is convenient to adjust the branch torque directly as shown in the following tables.

Mass No.	I	p^2I	Θ	$p^2I\Theta$	$\Sigma p^2I\Theta$	k	$\frac{\Sigma p^2I\Theta}{k}$
				$p^2 = 3$			
1	10	30	1	30	30	10	3
2	5	15	−2	−30	0	10	0
3	10	30	−2	−60			
6	5	15	(1)	(15)	(15)	20	(0.75)
			(0.25)				
		$T =$	$(15)\left(\frac{-2}{0.25}\right)$	$= -120$			
3					−180	20	−9
4	5	15	7	10.5	−75	20	−3.75
5	5	15	10.75	161.25	86.25		
				$p^2 = 8$			
1	10	80	1	80	80	10	8
2	5	40	−7	−280	−200	10	−20
3	10	80	13	1040			
6	5	40	(1)	(40)	(40)	20	(2)
			(−1)				
		$T =$	$(40)\left(\frac{13}{-1}\right) =$	−520			
3					320	20	16
4	5	40	−3	−120	200	20	10
5	5	40	−13	−520	−320		

The adjusted branch torque may then be added to the torque accumulated in the main system and the calculation continued to the end of the main system and the residual torque obtained.

The computations as outlined and illustrated above fail in one instance, namely when the assumed circular frequency p is selected so that the branch has a node at the junction point. The adjusted

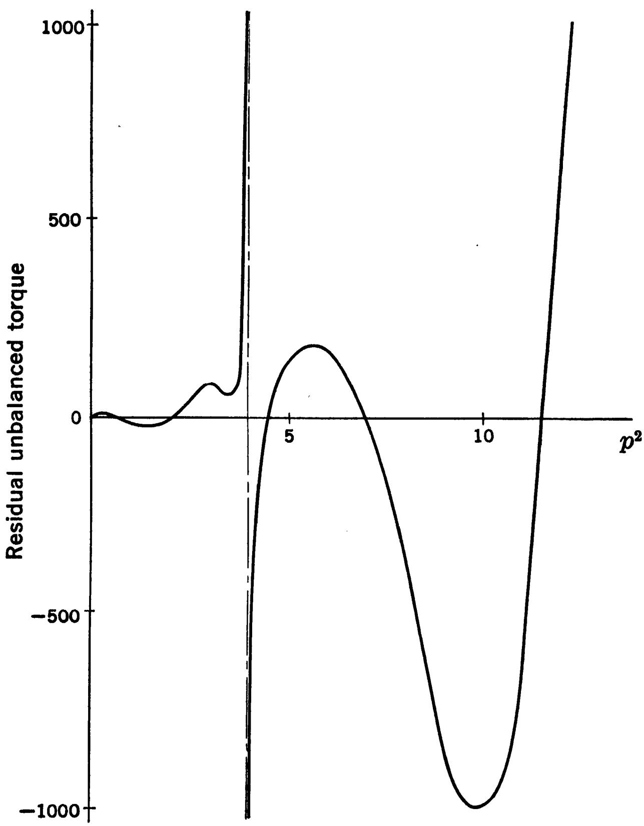


FIG. 7–9

branch torque then becomes infinite, and hence the residual torque is limitless. This peculiarity in no way hampers the application of the method to branched systems. The residual torque curve for the system of Fig. 7–8 is shown in Fig. 7–9. The infinite branch torque gives rise to a discontinuity in the residual torque curve. In the example considered, this occurs at the value of p^2 corresponding to a node in the branch at I_3. This is equivalent to a value of p^2 equal to the natural frequency of the branch considered separately, i.e.,

$$p^2 = \frac{k_5}{I_6} = \frac{20}{5} = 4 \text{ rad}^2 \text{ per sec}^2$$

The Holzer table may be applied in this way to systems with many branches by proceeding along the main system and including the adjusted branch torque at each junction point.

The Holzer table has been applied to damped systems by Den Hartog and Li[1] and has been reformulated and adapted to many specific problems. The method enjoys a great popularity among engineers because of the directness of its calculations and its great flexibility. It will undoubtedly continue as a standard tool in vibration analysis.

7.3. Graeffe's Method

Graeffe's method, for the determination of the roots of higher-order algebraic equations, belongs to the second group of methods discussed in section 7.1. The method presupposes the existence of the frequency equation, and it cannot be used until the frequency equation has been deduced. Unlike the method of Holzer, all of the frequencies are determined simultaneously. The method is easily applied to any system for which the frequency equation can be written in the form of an algebraic equation.

Graeffe's method is essentially the same for all types of algebraic equations, although the nature of the roots influences the procedure somewhat. It will be necessary to consider only the two most common types of problems here, namely those equations having real distinct roots (undamped vibrations) and those possessing roots in the form of conjugate pairs of complex numbers (damped vibrations).

1. All roots real and distinct

Consider the algebraic equation,

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0 \quad (7.3\text{–}1)$$

This equation may be factored, if the roots are known, into

$$f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \cdots (x - \alpha_n) = 0 \quad (7.3\text{–}2)$$

where α_i are the n roots to the equation.

The equation whose roots are $-\alpha_i^2$ may be formulated readily from equation 7.3–2 by multiplying it by

$$f(-x) = (-1)^n(x + \alpha_1)(x + \alpha_2)(x + \alpha_3) \cdots (x + \alpha_n)$$

whence there is obtained

$$(-1)^n(x^2 - \alpha_1^2)(x^2 - \alpha_2^2)(x^2 - \alpha_3^2) \cdots (x^2 - \alpha_n^2) = 0$$

[1] Li and Den Hartog, "Forced Torsional Vibrations with Damping: An Extension of Holzer's Method," *Trans. ASME*, 1946, p. A276.

or

$$(y + \alpha_1^2)(y + \alpha_2^2)(y + \alpha_3^2) \cdots (y + \alpha_n^2) = 0 \qquad (7.3\text{–}3)$$

where $y = -x^2$

The roots to the algebraic equation 7.3–3 in y are seen to be $-\alpha_i^2$. Equation 7.3–3 may also be expanded to

$$b_0 y^n + b_1 y^{n-1} + b_2 y^{n-2} + \cdots + b_n = 0 \qquad (7.3\text{–}4)$$

where the b's depend on the a's.

The coefficients b_i may be calculated by the direct multiplication of equation 7.3–1 by

$$f(-x) = a_0 x^n - a_1 x^{n-1} + a_2 x^{n-2} - a_3 x^{n-3} + \cdots + a_n$$

which gives

$$[a_0^2]x^{2n} - \begin{bmatrix} a_1^2 \\ -2a_0a_2 \end{bmatrix} x^{2n-2} + \begin{bmatrix} a_2^2 \\ -2a_1a_3 \\ +2a_0a_4 \end{bmatrix} x^{2n-4} - \begin{bmatrix} a_3^2 \\ -2a_2a_4 \\ +2a_1a_5 \\ -2a_0a_6 \end{bmatrix} x^{2n-6} + \cdots + a_n^2 = 0$$

Again letting $y = -x^2$ gives the equation

$$[a_0^2]y^n + \begin{bmatrix} a_1^2 \\ -2a_0a_2 \end{bmatrix} y^{n-1} + \begin{bmatrix} a_2^2 \\ -2a_1a_3 \\ +2a_0a_4 \end{bmatrix} y^{n-2} + \begin{bmatrix} a_3^2 \\ -2a_2a_4 \\ +2a_1a_5 \\ -2a_0a_6 \end{bmatrix} y^{n-3} + \cdots + a_n^2 = 0 \qquad (7.3\text{–}5)$$

whose roots are $-\alpha_i^2$. Comparison of equations 7.3–4 and 7.3–5 shows that

$$\begin{aligned} b_0 &= a_0^2 \\ b_1 &= a_1^2 - 2a_0a_2 \\ b_2 &= a_2^2 - 2a_1a_3 + 2a_0a_4 \\ b_3 &= a_3^2 - 2a_2a_4 + 2a_1a_5 - 2a_0a_6 \\ &\cdots\cdots\cdots\cdots \\ b_n &= a_n^2 \end{aligned} \qquad (7.3\text{–}6)$$

By repeating this process an equation may be formulated whose roots are $-(-\alpha_i^2)^2 = -\alpha_i^4$. Successive repetitions permit equations to be formulated whose roots have magnitudes that are $-\alpha^{2^m}$ where m is the number of repetitions. If the original roots are real and distinct, that is to say, real and different, their differences will be amplified by the successive squaring. For example, let the equation after m repetitions of the squaring process be

$$C_0 z^n + C_1 z^{n-1} + C_2 z^{n-2} + \cdots + C_n = 0 \tag{7.3–7}$$

and let the roots to this equation, in order of magnitude, be

$$\beta_1 = -\alpha_1^{2^m}, \quad \beta_2 = -\alpha_2^{2^m}, \quad \cdots, \quad \beta_n = -\alpha_n^{2^m}$$

It follows then for m sufficiently large that

$$\beta_1 >>> \beta_2 >>> \beta_3 >>> \cdots >>> \beta_n \tag{7.3–8}$$

In the elementary theory of algebraic equations[1] the following relations are established:

$$\left.\begin{aligned}
\frac{C_1}{C_0} &= \beta_1 + \beta_2 + \beta_3 + \cdots + \beta_n \\
\frac{C_2}{C_0} &= \beta_1\beta_2 + \beta_1\beta_3 + \cdots + \beta_{n-1}\beta_n \\
\frac{C_3}{C_0} &= \beta_1\beta_2\beta_3 + \beta_1\beta_2\beta_4 + \cdots \beta_{n-2}\beta_{n-1}\beta_n \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{C_n}{C_0} &= \beta_1\beta_2\beta_3 \cdots \beta_n
\end{aligned}\right\} \tag{7.3–9}$$

These relations may be simplified in view of the inequalities 7.3–8 to give

$$\frac{C_1}{C_0} \cong \beta_1, \qquad \frac{C_2}{C_0} \cong \beta_1\beta_2, \qquad \frac{C_3}{C_0} \cong \beta_1\beta_2\beta_3, \quad \text{etc.}$$

from which

$$\left.\begin{aligned}
\beta_1 &\cong \frac{C_1}{C_0} \\
\beta_2 &\cong \frac{1}{\beta_1}\frac{C_2}{C_0} = \frac{C_0}{C_1}\frac{C_2}{C_0} = \frac{C_2}{C_1} \\
\beta_3 &\cong \frac{1}{\beta_1\beta_2}\frac{C_3}{C_0} = \frac{C_0}{C_1}\frac{C_1}{C_2}\frac{C_3}{C_0} = \frac{C_3}{C_2}
\end{aligned}\right\} \tag{7.3–10}$$

etc.

Having the magnitude of the β's, we obtain the magnitude of the original roots by extracting the 2^mth root. The signs of the α's are

[1] See for instance, L. L. Smail, *College Algebra*, McGraw-Hill, 1931, or L. E. Dickson, *First Course in the Theory of Equations*, John Wiley, 1922.

not determined by this process, and they must be determined by substituting into the original equation 7.3–1.

To illustrate the procedure outlined above, the example of Fig. 7–4 will be treated. The frequency equation for this system was previously obtained in section 7.2 as

$$p^2(p^8 - 8.4p^6 + 22.4p^4 - 20p^2 + 3.6) = 0 \qquad (7.3\text{–}11)$$

The root $p^2 = 0$, which corresponds to rigid body rotation, may be discarded as it does not represent a vibratory motion. The remaining four roots furnish the four natural circular frequencies for the four modes of free vibration. To facilitate the squaring process, in accordance with the laws given by equation 7.3–5 or equations 7.3–6, a tabular form is expedient. This form is shown below.

TABLE FOR REPEATED "SQUARING" OF THE FREQUENCY EQUATION

m	p^8	p^6	p^4	p^2	p^0
0	a_0 +1.0000	a_1 −8.4000	a_2 +22.4000	a_3 −20.0000	a_4 +3.6000
	+1.0000	+70.5600 −44.8000	+501.7600 −336.0000 +7.2000	+400.0000 −161.2800	+12.9600
1	+1.0000	+25.7600	+172.9600	+238.7200	+12.9600
	+1.0000	+663.5776 −345.9200	+29915.16 −12298.85 +25.92	+56987.24 −4483.12	+167.9616
2	+1.0000	+317.6576	+17642.23	+52504.12	+167.9616
	+1.0000	$+1.009064 \times 10^5$ -0.352845×10^5	$+3.112483 \times 10^8$ -0.333567×10^8 $+0.000003 \times 10^8$	2.756683×10^9 -0.005926×10^9	$+2.821110 \times 10^4$
3	+1.0000	$+6.562190 \times 10^4$	$+2.778919 \times 10^8$	$+2.750757 \times 10^9$	$+2.821110 \times 10^4$
	+1.0000	$+4.306234 \times 10^9$ -0.555784×10^9	$+7.722391 \times 10^{16}$ -0.036102×10^{16}	$+7.566664 \times 10^{18}$ -0.000016×10^{18}	$+7.958662 \times 10^8$
4	+1.0000	$+3.750450 \times 10^9$	$+7.686289 \times 10^{16}$	$+7.566648 \times 10^{18}$	$+7.958662 \times 10^8$
	+1.0000	$+1.406588 \times 10^{19}$ -0.015373×10^{19}	$+5.907904 \times 10^{33}$ -0.000057×10^{33}	$+5.725416 \times 10^{37}$ —	$+6.334030 \times 10^{17}$
5	+1.0000	$+1.391215 \times 10^{19}$	$+5.907847 \times 10^{33}$	$+5.725416 \times 10^{37}$	$+6.334030 \times 10^{17}$
	+1.0000	$+1.935479 \times 10^{38}$ -0.000118×10^{38}	$+3.490266 \times 10^{67}$ —	$+3.278039 \times 10^{75}$	$+4.011994 \times 10^{35}$
6	+1.0000 c_0	$+1.935361 \times 10^{38}$ c_1	$+3.490266 \times 10^{67}$ c_2	$+3.278039 \times 10^{75}$ c_3	$+4.011994 \times 10^{35}$ c_4

The details of the table are self-explanatory as the calculations are carried out for each "squaring" in the same pattern as indicated in equation 7.3–5. The process is continued until the coefficients of the last equation, obtained by the "squaring" procedure, are the squares of the coefficients of the preceding equation to the accuracy desired. In other words, until the cross product, terms occurring in the coefficients of equation 7.3–5 do not affect the result to the last decimal place. In this particular example, six "squarings," $m = 6$, were required to obtain six-place accuracy. The seventh squaring, $m = 7$, is seen by inspection to lead to an equation whose coefficients are merely the squares of those of the equation corresponding to $m = 6$. The number of significant figures included in this example is larger than is usually justified in a practical problem, and they are included only as an aid in checking the table.

The roots of the squared equation may now be calculated by substituting into equations 7.3–10. Thus, for the last approximation ($m = 6$), there is obtained

$$\beta_1 = (p_1^2)^{2^m} = (p_1^2)^{64} = \frac{C_1}{C_0} = \frac{1.935361 \times 10^{38}}{1}$$

whence

$$p_1^2 = (1.935361 \times 10^{38})^{1/64} = 3.964886$$

Similarly:

$$\beta_2 = (p_2^2)^{2^m} = (p_2^2)^{64} = \frac{C_2}{C_1} = \frac{3.490266 \times 10^{67}}{1.935361 \times 10^{38}} = 1.803419 \times 10^{29}$$

$$p_2^2 = (1.803419 \times 10^{29})^{1/64} = 2.865012$$

$$\beta_3 = (p_3^2)^{2^m} = (p_3^2)^{64} = \frac{C_3}{C_2} = \frac{3.278039 \times 10^{75}}{3.490266 \times 10^{67}} = 9.391946 \times 10^{7}$$

$$p_3^2 = (9.391946 \times 10^{7})^{1/64} = 1.332215$$

$$\beta_4 = (p_4^2)^{2^m} = (p_4^2)^{64} = \frac{C_4}{C_3} = \frac{4.011994 \times 10^{35}}{3.278039 \times 10^{75}} = 1.223901 \times 10^{-40}$$

$$p_4^2 = (1.223901 \times 10^{-40})^{1/64} = 0.237887$$

These results are seen to agree with the solution to this problem previously obtained by the method of Holzer in section 7.2. The proper sign of the roots presents no problem in this case, since the system is undamped and hence it is known beforehand that the roots p^2 must be real and positive in accordance with the assumed periodic solution of the form

$$\theta_i = \Theta_i \cos pt$$

where p is the natural circular frequency.

It is of interest to investigate the rapidity of the convergence of Graeffe's method in this example. This may be accomplished by evaluating the roots after each "squaring" and comparing the results to the known solution. The successive values of the roots calculated on the basis of one, two, three, etc., "squarings" is shown below.

m	p_1^2	p_2^2	p_3^2	p_4^2	*Sum*	*Product*
0	8.400000	2.666667	0.892857	0.180000	12.139524	3.600000
1	5.075432	2.591194	1.174821	0.233001	9.074448	3.599999
2	4.221724	2.729912	1.313439	0.237823	8.502898	3.600001
3	4.000655	2.840230	1.331825	0.237887	8.410597	3.599999
4	3.966251	2.864027	1.332215	0.237887	8.400380	3.599999
5	3.964890	2.865010	1.332215	0.237887	8.400002	3.599998
6	3.964886	2.865012	1.332215	0.237887	8.400000	3.599999

It will be noted from this table that the convergence is quite good in this example. Further it will be seen that a comparison of the product of the roots to the constant term in the equation does not furnish a check. The nature of the calculation is such that this criterion is satisfied by all approximations. A better check is furnished by comparing the sum of the roots with the second coefficient a_1.

The application of Graeffe's method to more elaborate undamped problems offers no difficulties other than an increase in the amount of calculation involved. The length of the calculation is dependent to a high degree on the nearness of the respective roots to each other. In the limiting case, where two or more roots are actually equal to each other, the method must be modified slightly. Since this is so rare as to be a distinct oddity in actual engineering problems, this modification need not be treated here.[1]

[1] For a complete treatment of these special cases and Graeffe's theory as a whole, the student is referred to R. E. Doherty and Ernest G. Keller, *Mathematics of Modern Engineering*, Vol. I, John Wiley, or E. Bodewig, *Quart. Applied Math.*, Vol. IV, 1946, p. 177.

2. All roots complex and distinct

The frequency equation for a vibrating system possessing damping elements will have complex roots p which occur in conjugate pairs. Consider as an example of this type the following sixth-degree equation:

$$a_0x^6 + a_1x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6 = 0 \quad (7.3\text{–}12)$$

and let it be assumed that the roots are complex conjugate numbers which may be written in the polar form:

$$x_1 = r_1e^{j\phi_1}, \quad x_2 = r_1e^{-j\phi_1}, \quad x_3 = r_2e^{j\phi_2}, \quad \cdots, x_6 = r_3e^{-j\phi_3}$$

where

$$j^2 = -1 \quad \text{and} \quad r_1 > r_2 > r_3$$

Equation 7.3–12 may then be factored into

$$(x - r_1e^{j\phi_1})(x - r_1e^{-j\phi_1})(x - r_2e^{j\phi_2}) \cdots (x - r_3e^{-j\phi_3}) = 0 \quad (7.3\text{–}13)$$

Successive "squaring" of this equation m times in the same manner as outlined in the previous section for real roots will lead to the equation

$$(z - r_1{}^ne^{jn\phi_1})(z - r_1{}^ne^{-jn\phi_1})(z - r_2{}^ne^{jn\phi_2}) \cdots (z - r_3{}^ne^{-jn\phi_3}) = 0 \quad (7.3\text{–}14)$$

where

$$r_1{}^n >>> r_2{}^n >>> r_3{}^n, \quad z = x^n, \quad \text{and} \quad n = 2^m \quad (7.3\text{–}15)$$

This equation may also be expanded to

$$C_0z^6 + C_1z^5 + C_2z^4 + C_3z^3 + C_4z^2 + C_5z + C_6 = 0$$

where the coefficients are conveniently calculated from the a's in a tabular form in the same manner as the previous example. The relations between the roots and the coefficients of this equation are given by equations 7.3–9, namely

$$\frac{C_1}{C_0} = + (\beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6)$$

$$= - [r_1{}^n(e^{jn\phi_1} + e^{-jn\phi_1}) + r_2{}^n(e^{jn\phi_2} + e^{-jn\phi_2}) + r_3{}^n(e^{jn\phi_3} + e^{-jn\phi_3})]$$

Use of the identity

$$\cos n\phi = \tfrac{1}{2}(e^{jn\phi} + e^{-jn\phi})$$

permits this to be written as

$$\frac{C_1}{C_0} = -2(r_1{}^n \cos n\phi_1 + r_2{}^n \cos n\phi_2 + r_3{}^n \cos n\phi_3)$$

Similiarly:

$$\frac{C_2}{C_0} = +(\beta_1\beta_2 + \beta_1\beta_3 + \beta_1\beta_4 + \beta_1\beta_5 + \beta_1\beta_6$$

$$+ \beta_2\beta_3 + \beta_2\beta_4 + \beta_2\beta_5 + \beta_2\beta_6$$
$$+ \beta_3\beta_4 + \beta_3\beta_5 + \beta_3\beta_6$$
$$+ \beta_4\beta_5 + \beta_4\beta_6$$
$$+ \beta_5\beta_6)$$

$$= r_1^{2n} + r_2^{2n} + r_3^{2n} + 4[r_1^n r_2^n \cos n\phi_1 \cos n\phi_2$$
$$+ r_1^n r_3^n \cos n\phi_1 \cos n\phi_3 + r_2^n r_3^n \cos n\phi_2 \cos n\phi_3]$$

$$\frac{C_3}{C_0} = +\,(\beta_1\beta_2\beta_3 + \beta_1\beta_2\beta_4 + \beta_1\beta_2\beta_5 + \beta_1\beta_2\beta_6$$

$$+ \beta_1\beta_3\beta_4 + \beta_1\beta_3\beta_5 + \beta_1\beta_3\beta_6$$
$$+ \beta_1\beta_4\beta_5 + \beta_1\beta_4\beta_6$$
$$+ \beta_1\beta_5\beta_6$$
$$+ \beta_2\beta_3\beta_4 + \beta_2\beta_3\beta_5 + \beta_2\beta_3\beta_6$$
$$+ \beta_2\beta_4\beta_5 + \beta_2\beta_4\beta_6$$
$$+ \beta_2\beta_5\beta_6$$
$$+ \beta_3\beta_4\beta_5 + \beta_3\beta_4\beta_6$$
$$+ \beta_3\beta_5\beta_6$$
$$+ \beta_4\beta_5\beta_6)$$

$$= -\,2[r_1^n(r_2^{2n} + r_3^{2n}) \cos n\phi_1 + r_2^{2}(r_3^{2n} + r_1^{2n}) \cos n\phi_2$$
$$+ r_3^n(r_1^{2n} + r_2^{2n}) \cos n\phi_3$$
$$+ 4r_1^n r_2^n r_3^n \cos n\phi_1 \cos n\phi_2 \cos n\phi_3]$$

$$\frac{C_4}{C_0} = +\,(\beta_1\beta_2\beta_3\beta_4 + \beta_1\beta_2\beta_3\beta_5 + \beta_1\beta_2\beta_3\beta_6$$

$$+ \beta_1\beta_2\beta_4\beta_5 + \beta_1\beta_2\beta_4\beta_6$$
$$+ \beta_1\beta_2\beta_5\beta_6$$
$$+ \beta_1\beta_3\beta_4\beta_5 + \beta_1\beta_3\beta_4\beta_6$$
$$+ \beta_1\beta_3\beta_5\beta_6$$
$$+ \beta_1\beta_4\beta_5\beta_6$$
$$+ \beta_2\beta_3\beta_4\beta_5 + \beta_2\beta_3\beta_4\beta_6$$
$$+ \beta_2\beta_3\beta_5\beta_6$$
$$+ \beta_2\beta_4\beta_5\beta_6$$
$$+ \beta_3\beta_4\beta_5\beta_6)$$

$$= r_2^{2n} r_3^{2n} + r_3^{2n} r_1^{2n} + r_1^{2n} r_2^{2n}$$
$$+ 4r_1^n r_2^n r_3^n(r_3^n \cos n\phi_1 \cos n\phi_2 + r_2^n \cos n\phi_3 \cos n\phi_1$$
$$+ r_1^n \cos n\phi_2 \cos n\phi_3)$$

$$\frac{C_5}{C_0} = +(\beta_1\beta_2\beta_3\beta_4\beta_5 + \beta_1\beta_2\beta_3\beta_4\beta_6 + \beta_1\beta_2\beta_3\beta_5\beta_6 + \beta_1\beta_2\beta_4\beta_5\beta_6 + \beta_1\beta_3\beta_4\beta_5\beta_6 + \beta_2\beta_3\beta_4\beta_5\beta_6)$$
$$= -2r_1{}^n r_2{}^n r_3{}^n(r_1{}^n r_2{}^n \cos n\phi_3 + r_2{}^n r_3{}^n \cos n\phi_1 + r_3{}^n r_1{}^n \cos n\phi_2)$$

$$\frac{C_6}{C_0} = \beta_1\beta_2\beta_3\beta_4\beta_5\beta_6$$
$$= r_1{}^{2n} r_2{}^{2n} r_3{}^{2n}$$

In view of relations 7.3–15, the even coefficients may be approximated, for large values of n, by

$$\frac{C_2}{C_0} \cong r_1{}^{2n}$$

$$\frac{C_4}{C_0} \cong r_1{}^{2n} r_2{}^{2n}$$

$$\frac{C_6}{C_0} \cong r_1{}^{2n} r_2{}^{2n} r_3{}^{2n}$$

From these relations, it is seen that

$$\left.\begin{aligned} r_1 &\cong \left(\frac{C_2}{C_0}\right)^{\frac{1}{2n}} \\ r_2 &\cong \left(\frac{C_4}{C_2}\right)^{\frac{1}{2n}} \\ r_3 &\cong \left(\frac{C_6}{C_4}\right)^{\frac{1}{2n}} \end{aligned}\right\} \qquad (7.3\text{–}16)$$

It should be noted that the coefficients C_3 and C_5 will change in sign periodically as n is increased, and in fact this periodic change in sign in the "squaring" calculations may be taken as an indication that at least one of the roots is complex. It is therefore only necessary to carry out the successive squaring until the even coefficients in the cycle are the squares of those of the preceding cycle.

Equations 7.3–16 will furnish the absolute value of the complex roots r_1, r_2, and r_3. To obtain the real parts of the roots it is only necessary to note that the coefficients of the original equation are related to the roots by

$$\left.\begin{aligned}
\frac{a_1}{a_0} &= -2(r_1 \cos \phi_1 + r_2 \cos \phi_2 + r_3 \cos \phi_3) \\
\frac{a_2}{a_0} &= r_1^2 + r_2^2 + r_3^2 \\
&\quad + 4(r_1 r_2 \cos \phi_1 \cos \phi_2 + r_2 r_3 \cos \phi_2 \cos \phi_3 \\
&\quad + r_3 r_1 \cos \phi_3 \cos \phi_1) \\
\frac{a_3}{a_0} &= -2[r_1(r_2^2 + r_3^2) \cos \phi_1 + r_2(r_3^2 + r_1^2) \cos \phi_2 \\
&\quad + r_3(r_1^2 + r_2^2) \cos \phi_3 \\
&\quad + 4 r_1 r_2 r_3 \cos \phi_1 \cos \phi_2 \cos \phi_3] \\
\frac{a_4}{a_0} &= r_1^2 r_2^2 + r_2^2 r_3^2 + r_3^2 r_1^2 \\
&\quad + 4 r_1 r_2 r_3 (r_1 \cos \phi_2 \cos \phi_3 + r_2 \cos \phi_3 \cos \phi_1 \\
&\quad + r_3 \cos \phi_1 \cos \phi_2) \\
\frac{a_5}{a_0} &= -2 r_1 r_2 r_3 (r_1 r_2 \cos \phi_3 + r_2 r_3 \cos \phi_1 + r_3 r_1 \cos \phi_2)
\end{aligned}\right\} \qquad (7.3\text{–}17)$$

Any three of these equations may be solved simultaneously to obtain the values of $\cos \phi_1$, $\cos \phi_2$, and $\cos \phi_3$, from which the real and imaginary parts are easily calculated, i.e.,

$$x = r(\cos \phi \pm j \sin \phi)$$

To illustrate the application of the procedure outlined above, the problem of section 6.5 may be treated. The frequency equation for this system is given by equation 6.5–36, which is, for the open system ($c_3 = z_{c3} = 0$),

$$\begin{aligned}
\Delta = {} & (\mathbf{z}_{k1} + \mathbf{z}_{m1} + \mathbf{z}_{c1})(\mathbf{z}_{k1} + \mathbf{z}_{k2} + \mathbf{z}_{m2} + \mathbf{z}_{c2}) \\
& (\mathbf{z}_{k2} + \mathbf{z}_{k3} + \mathbf{z}_{m3} + \mathbf{z}_{c1} + \mathbf{z}_{c2}) \\
& - 2\mathbf{z}_{k1}\mathbf{z}_{c1}(\mathbf{z}_{k2} + \mathbf{z}_{c2}) - \mathbf{z}_{c1}^2(\mathbf{z}_{k1} + \mathbf{z}_{k2} + \mathbf{z}_{m2} + \mathbf{z}_{c2}) \\
& - (\mathbf{z}_{k2} + \mathbf{z}_{c2})^2(\mathbf{z}_{k1} + \mathbf{z}_{m1} + \mathbf{z}_{c1}) \\
& - \mathbf{z}_{k1}^2(\mathbf{z}_{k2} + \mathbf{z}_{k3} + \mathbf{z}_{m3} + \mathbf{z}_{c1} + \mathbf{z}_{c2}) = 0
\end{aligned}$$

Introducing the numerical values for the z's from Table 3, section 6.5, and the use of $\omega = p$ results in the frequency equation,

$$\begin{aligned}
-2.086500 \times 10^{-7} p^6 + 2.516847 \times 10^{-5} j p^5 + 8.875238 \times 10^{-3} p^4 \\
- 4.533679 \times 10^{-1} j p^3 - 78.416450 p^2 + 1406.25 j p + 140625 = 0
\end{aligned}$$

This equation may be simplified by introducing a new variable x where $p = -10^2 jx$, whence

$$x^6 + 1.206250x^5 + 4.253639x^4 + 2.172858x^3 + 3.758268x^2 + 0.673974x + 0.673974 = 0$$

The "squaring" calculation is carried out in the table given below.

m	x^6	x^5	10	x^4	10	x^3	10	x^2	10	x^1	10	x^0	10
0	a_0	a_1		a_2		a_3		a_4		a_5		a_6	
	+1	+ 1.206250		4.253639		+ 2.172858		+ 3.758268		+0.673974		+0.673974	
	+1	+ 1.455039		+18.093402		+ 4.721312		+14.124578		+0.454241		+0.454241	
		− 8.507278		− 5.242020		−31.972631		− 2.928900		−5.065950			
				+ 7.516536		+ 1.625962		+ 5.733684					
						− 1.347948							
1	+1	− 7.052239		+20.367918		−26.973305		+16.929362		−4.611709		+0.454241	
	+1	+ 4.973407	1	+ 4.148521	2	+ 7.275592	2	+ 2.866033	2	+2.126786	1	+2.063349	−1
		− 4.073548	1	− 3.804444	2	− 6.896317	2	− 2.487861	2	−1.538002	1		
				+ 0.338587	2	+ 0.650457	2	+ 0.185039	2				
						− 0.009085	2						
2	+1	+ 0.899823	1	+ 0.682664	2	+ 1.020647	2	+ 0.563211	2	+0.588784	1	+2.063349	−1
	+1	+ 8.096814	1	+ 4.660301	3	+ 1.041720	4	+ 3.172066	3	+3.466666	1	+4.257409	−2
		−13.653280	1	− 1.836803	3	− 0.768968	4	− 1.201881	3	−2.324202	1		
				+ 0.112642	3	+ 0.010596	4	+ 0.028171	3				
						− 0.000041	4						
3	+1	− 5.556466	1	+ 2.936140	3	+ 0.283307	4	+ 1.998356	3	+1.142464	1	+4.257409	−2
	+1	+ 3.087431	3	+ 8.620918	6	+ 0.802629	7	+ 3.993427	6	+1.305224	2	+1.812553	−3
		− 5.872280	3	+ 0.314837	6	− 1.173491	7	− 0.064734	6	−1.701564	2		
				+ 0.003997	6	− 0.000127	7	+ 0.000250	6				
4	+1	− 2.784849	3	+ 8.939752	6	− 0.370989	7	+ 3.928943	6	−0.396340	2	+1.812553	−3
	+1	+ 7.755384	6	+ 7.991917	13	+ 1.376328	13	+ 1.543659	13	+0.157085	4	+3.285348	−6
		−17.879504	6	− 0.002066	13	− 7.024755	13	− 0.000029	13	−1.424283	4		
5	+1	−10.124120	6	+ 7.989851	13	− 5.648427	13	+ 1.543630	13	−1.267198	4	+3.285348	−6
	c_0	c_1		c_2		c_3		c_4		c_5		c_6	

The odd columns 1, 3, and 5 are seen to alternate in sign as the calculation proceeds. The calculation is carried out until it can be seen that the next squaring will give coefficients in the even columns which are the squares of the preceding coefficients to the accuracy desired. In this example, the sixth squaring ($m = 6$) is seen to have little effect on these coefficients so the process is stopped at $m = 5$. The absolute values of the roots are calculated from equations 7.3–16 as follows:

$$m = 5 \qquad n = 2^m = 32$$

$$r_1 = \left(\frac{C_2}{C_0}\right)^{1/2n} = (7.989851 \times 10^{13})^{1/64} = 1.649020$$

$$r_2 = \left(\frac{C_4}{C_2}\right)^{1/2n} = \left(\frac{1.543630 \times 10^{13}}{7.989851 \times 10^{13}}\right)^{1/64} = 0.974639$$

$$r_3 = \left(\frac{C_6}{C_4}\right)^{1/2n} = \left(\frac{3.285348 \times 10^{-6}}{1.543630 \times 10^{13}}\right)^{1/64} = 0.510805$$

Substitution into the first, second, and last of equations 7.3–17 and the solution of these equations for cos ϕ_1, cos ϕ_2, and cos ϕ_3 gives two sets of roots since one of these is of the second degree. One of these sets are extraneous roots. Substitution of each of these sets into either of the remaining equations 7.3–17 will indicate the true roots. In this example these are found to be:

$$\cos \phi_1 = -0.261172$$

$$\cos \phi_2 = -0.117852$$

$$\cos \phi_3 = -0.112731$$

The real and imaginary parts of the roots are now easily calculated, giving

$$x_{1,2} = -0.430678 \pm 1.591786j$$

$$x_{3,4} = -0.114863 \pm 0.967848j$$

$$x_{5,6} = -0.057584 \pm 0.507548j$$

Since $p = -10^2jx$, there is obtained:

$$jp_{1,2} = -43.0678 \mp 159.1786j$$

$$jp_{3,4} = -11.4863 \mp 96.7848j$$

$$jp_{5,6} = -5.7584 \mp 50.7548j$$

It will be remembered that the solution assumed in establishing the mobility method was of the form e^{jpt}; thus the natural circular frequencies squared are:

$$(50.7548)^2 = 2576.05$$

$$(96.7848)^2 = 9367.28$$

$$(159.1786)^2 = 25337.83$$

These values are to be compared with the solution to this same problem without damping which was obtained in section 6.2. The real part of the solution is the rate at which the motion dies away with respect to time as pointed out in previous chapters.

The application of Graeffe's method to other problems of this nature is not different in principle. The calculations in complicated problems may be long, but a systematic approach and frequent check serves to minimize the labor. One of the disadvantages of Graeffe's method lies in the fact that it does not provide the amplitudes of the motion directly as in the Holzer table.

Part 3

Special Topics

Chapter 8

SYSTEMS WITH DISTRIBUTED PHYSICAL CONSTANTS

8.1. Introduction

The ideal systems involving separate and distinct masses, springs, and dampers, which are treated elsewhere in this text, are sufficiently representative of many problems confronting the engineer to offer all the accuracy usually required. There is, however, a large class of vibrating systems which cannot be idealized if the treatment is to be adequate. These systems usually involve distributed masses, springs, and dampers, or combinations of these elements. Frequently, both lumped constants and distributed constants are present. The analysis of systems with distributed constants rests on the same basic principles as those of previous treatment, although the actual analysis does possess some fundamental differences.

The introduction of a mass distributed in a continuous manner, such as a vibrating beam, introduces an infinite number of degrees of freedom; hence there are an infinite number of possible modes of motion in the free vibration of such a system. Not all of these modes may be present in any particular problem; however each or any combination of these is a possible motion. The fundamental mode is the mode with the lowest natural frequency. Higher modes correspond to higher frequencies, thus contributing an infinite number of natural frequencies to the system. This is an important consideration in problems involving the possibility of resonance.

Another point of difference introduced in continuous systems is the nature of the fundamental differential equation of motion. The equation of motion for continuous systems is a partial differential equation, and its solution must be obtained in a somewhat different manner from that of the previous examples. These basic differences will be brought out in the subsequent treatment of a few of the more important examples.

8.2. Derivation of the Equations of Motion

Four of the basic problems which are of fundamental importance to the engineer who is confronted with the vibration of continuous systems are:

(a) The lateral vibration of a taut string.

(b) The longitudinal vibration of an elastic bar.

(c) The torsional vibration of an elastic shaft.

(d) The lateral vibration of an elastic beam.

These problems have some similarity, and therefore the basic equation for each system will be derived in this section so that this similarity may be used to shorten the following development.

The equation of motion of the lateral vibration of a taut string

It is assumed that the string is uniform and of constant mass density ρ. The lateral vibration is assumed to be of small amplitude

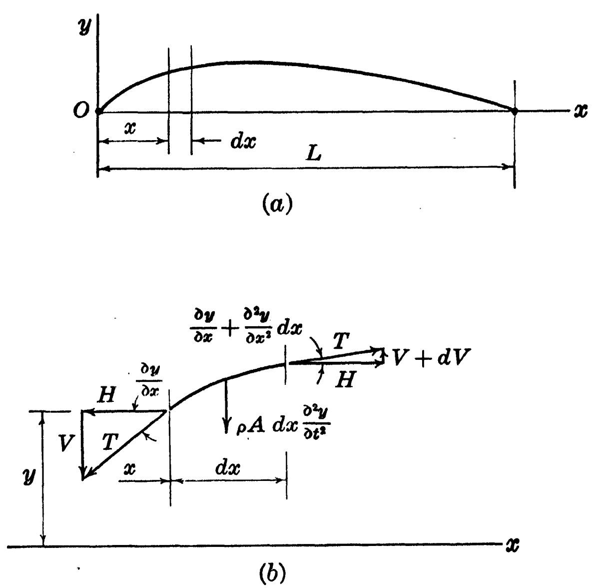


FIG. 8–1

so that the tension will have a constant value T. The equation of motion may be established directly from the application of Newton's second law to a small element of the string of length dx, as shown in Fig. 8–1b. The slope of the string at a distance x from the left support is given by $\partial y/\partial x$, and at a distance $x + dx$ the slope has changed by a small increment to $\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2}\,dx$ where the increment is the product of the rate of change of slope per unit length and the intervening distance dx between the two points. It follows at once that the vertical component of the tension is

$$V = T\frac{\partial y}{\partial x}$$

and

$$V + dV = T\left(\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2}\,dx\right)$$

or

$$dV = T\frac{\partial^2 y}{\partial x^2}\,dx$$

The mass of the element of the string is given by $\rho A\,dx$ where A is its cross-sectional area, assumed constant. The acceleration is $\partial^2 y/\partial t^2$. Newton's law of motion then gives

$$\rho A\frac{\partial y^2}{\partial t^2}\,dx = (V + dV) - V = dV = T\frac{\partial^2 y}{\partial x^2}\,dx$$

This is conveniently written as

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho A}\frac{\partial^2 y}{\partial x^2} \tag{8.2–1}$$

It will be shown later in this chapter that the speed c at which a longitudinal wave is propagated in an elastic medium (i.e., the speed of sound in the material) is given by

$$c = \sqrt{\frac{E}{\rho}} \tag{8.2–2}$$

where E is the modulus of elasticity of the material. The speed of sound depends only on the physical characteristics of the material, and for steel this speed has the approximate value $c = 2 \times 10^5$ in. per sec. The stress in the string is given by

$$\sigma = \frac{T}{A}$$

Thus

$$\frac{T}{\rho A} = c^2\frac{\sigma}{E} = c^2\epsilon \tag{8.2–3}$$

where ϵ is the unit strain in the string due to the tension T. Thus the basic equation for the lateral vibration of a string has the final form,

$$\frac{\partial^2 y}{\partial t^2} = c^2{}_\epsilon \frac{\partial^2 y}{\partial x^2} \tag{8.2-4}$$

The equation of motion for the longitudinal vibration of an elastic bar

The longitudinal vibration of an elastic bar of uniform cross section A and density ρ may be treated in a similar manner. Let the displacement of a section mn (Fig. 8–2) located a distance x from the left end be u. A section located an additional distance dx from the left end

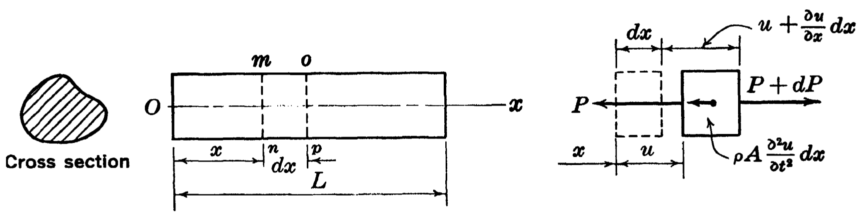


FIG. 8–2

will be displaced a distance $u + \frac{\partial u}{\partial x}\,dx$ where $\frac{\partial u}{\partial x}\,dx$ is the change in length of the element of length dx. The elongation per unit length of the element (strain) is given by $\epsilon = \frac{\partial u}{\partial x}$. The total force in the bar at section mn is therefore

$$P = EA\epsilon = EA\,\frac{\partial u}{\partial x}$$

In general, the total force in the bar at section op will differ from this by a small increment; thus the force at op is

$$P + dP = P + \frac{\partial P}{\partial x}\,dx = P + EA\,\frac{\partial^2 u}{\partial x^2}\,dx$$

The acceleration of the element of the bar shown in Fig. 8–2b is given by $\partial^2 u/\partial t^2$, and its mass is $\rho A\,dx$. Thus the equation of motion for this element of the bar is given by

$$\rho A\,\frac{\partial^2 u}{\partial t^2}\,dx = (P + dP) - P = dP = EA\,\frac{\partial^2 u}{\partial x^2}\,dx$$

This may be written as

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2} \tag{8.2-5}$$

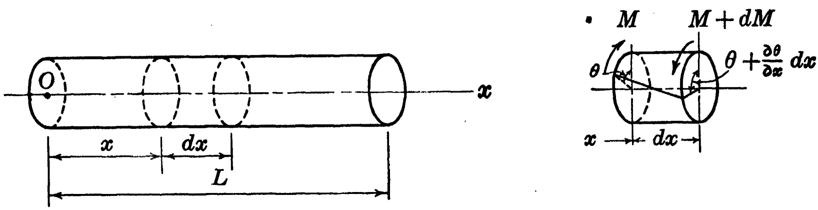
or, introducing the speed of sound,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{8.2-6}$$

which is the final form of the fundamental equation for the longitudinal vibration of an elastic bar.

The equation of motion for torsional vibration of an elastic shaft

An elastic shaft of uniform circular cross section is shown in Fig.



(a) (b)

FIG. 8–3

8–3. Let the angle of twist of a cross section a distance x from the left end be θ and at a distance $x + dx$ be $\theta + \dfrac{\partial\theta}{\partial x}dx$ where the angle of twist per unit length of the shaft is $\partial\theta/\partial x$. The torque M associated with this angle of twist is shown in elementary strength of materials to be

$$M = GJ\frac{\partial\theta}{\partial x}$$

where J is the polar moment of inertia of the cross section of the shaft, and G is the modulus of elasticity in shear.

The torque M will vary with x and the torque at $x + dx$ will be

$$M + dM = M + \frac{\partial M}{\partial x}dx = M + GJ\frac{\partial^2\theta}{\partial x^2}dx$$

The mass moment of inertia of the element of shaft is given by $\rho J\,dx$ while its angular acceleration is $\partial^2\theta/\partial t^2$. The equation of motion of the element of shaft is then

$$\rho J\frac{\partial^2\theta}{\partial t^2}dx = (M + dM) - M = dM = GJ\frac{\partial^2\theta}{\partial x^2}dx$$

This may be written as

$$\frac{\partial^2\theta}{\partial t^2} = \frac{G}{\rho}\frac{\partial^2\theta}{\partial x^2} \tag{8.2-7}$$

Introduction of the relations,

$$G = \frac{E}{2(1+\nu)}, \qquad c = \sqrt{\frac{E}{\rho}}$$

where ν is Poisson's ratio, permits this equation to be written in final form as

$$\frac{\partial^2\theta}{\partial t^2} = \frac{c^2}{2(1+\nu)}\frac{\partial^2\theta}{\partial x^2} \tag{8.2-8}$$

This is the basic equation for the torsional vibration of a uniform elastic shaft.

The equation of motion for the lateral vibration of an elastic beam

An elastic beam with a cross section of moment of inertia I is shown in Fig. 8–4a. As shown in strength of materials, the relation between

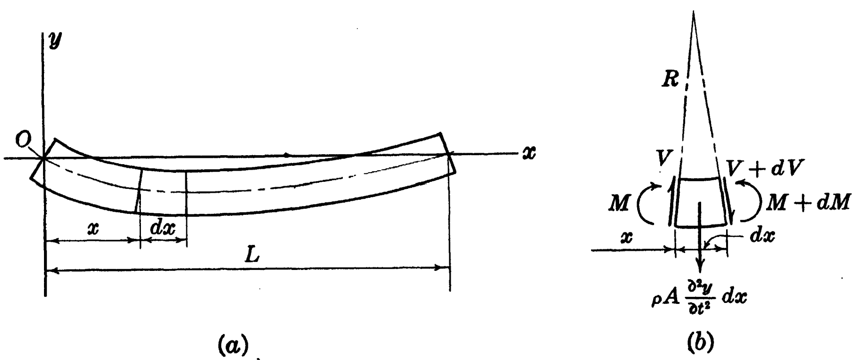


FIG. 8–4

the radius of curvature R of the beam and the bending moment M is given by the Bernoulli-Euler equation,

$$\frac{1}{R} = \frac{M}{EI} \tag{8.2-9}$$

The shear force V is related to the bending moment M by the expression

$$V = \frac{\partial M}{\partial x}; \quad \text{thus} \quad \frac{\partial V}{\partial x} = \frac{\partial^2 M}{\partial x^2}$$

The shear force V which acts on the beam will vary with x and the shear at $x + dx$ may be represented by

$$V + dV = V + \frac{\partial V}{\partial x} dx \tag{8.2–10}$$

The curvature of the beam can be expressed for small deflections as

$$\frac{1}{R} = \frac{\partial^2 y}{\partial x^2} \tag{8.2–11}$$

Thus equation 8.2–9 may be written as

$$EI \frac{\partial^2 y}{\partial x^2} = M$$

which, when differentiated twice with respect to x, takes the form

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2}\right) = \frac{\partial^2 M}{\partial x^2} = \frac{\partial V}{\partial x} \tag{8.2–12}$$

The mass of the element of the beam of length dx is $\rho A\, dx$ and its acceleration is $\partial^2 y/\partial t^2$. The equation of motion has the form

$$\rho A \frac{\partial^2 y}{\partial t^2} dx = V - (V + dV) = -dV$$

$$= -\frac{\partial V}{\partial x} dx = -\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2}\right) dx$$

or

$$\frac{\partial^2 y}{\partial t^2} = -\frac{1}{\rho A} \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2}\right) \tag{8.2–13}$$

If the stiffness EI of the beam is constant along its length, equation 8.2–13 may be written

$$\frac{\partial^2 y}{\partial t^2} = -\frac{EI}{\rho A} \frac{\partial^4 y}{\partial x^4} \tag{8.2–14}$$

Introduction of the speed of sound $c = \sqrt{E/\rho}$ and the radius of gyration $\bar{r} = \sqrt{I/A}$ permits the equation of motion for the lateral vibration of a uniform beam to take the form

$$\frac{\partial^2 y}{\partial t^2} = -c^2 \bar{r}^2 \frac{\partial^4 y}{\partial x^4} \tag{8.2–15}$$

The basic equations of motion of the first three problems considered above have a common form which may be represented as

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2} \tag{8.2–16}$$

where a is a constant depending on the physical properties of the vibrating element. The last problem, which gives rise to equation 8.2–15, is somewhat different in nature although the method of derivation has much in common with the previous problems. Other problems involving the vibration of plates, shells, or combinations of any or all of the elements discussed above may be attacked in a similar fashion, and the proper basic equations of motion may thus be established. The assumption of small oscillations is common to all such derivations. This assumption, necessary to insure a linear problem, is not a serious handicap in most cases. Large oscillations frequently destroy the physical properties by introducing inelastic behavior, and the solution is not valid in any event when this occurs.

8.3. Solution of the Equations of Motion

The solution of equations 8.2–15 and 8.2–16 of the previous section may be obtained by the classical method of separating the variables. This method of obtaining the solution is one of the most important of those available for the solution of boundary value problems involving partial differential equations. Fortunately, many of the fundamental problems met in this and other fields of engineering are susceptible to this attack, and the procedure illustrated below is intended to indicate the use of this tool as well as to obtain the solution of the immediate problems.

Solution of the wave equation

Equation 8.2–16 is common to many types of problems and is known as the wave equation. The solution of this equation will be a function of both independent variables, x and t. If the solution can be written as a product of two functions which individually depend on one variable only, then it may be expressed as

$$z = X(x)T(t) \tag{8.3–1}$$

Differentiation of this trial solution partially with respect to x and t and substitution into equation 8.2–16 gives

$$X\frac{d^2T}{dt^2} = a^2\frac{d^2X}{dx^2}T$$

Dividing both sides of this equation by XT yields

$$\frac{1}{T}\frac{d^2T}{dt^2} = \frac{a^2}{X}\frac{d^2X}{dx^2} \tag{8.3–2}$$

In this expression, the variables are separated; that is, the right side of the equation is a function of x only and does not change with variations in t. Similarly, the left side is a function of t only and thus independent of x. If this equation is to be valid for all possible values of x and t, it follows that both sides of equation 8.3–2 must be equal to a common constant which for convenience in the latter development will be designated by $-p^2$. The problem is then reduced to the solution of the two ordinary differential equations,

$$\frac{d^2X}{dx^2} + \frac{p^2}{a^2} X = 0 \tag{8.3–3}$$

$$\frac{d^2T}{dt^2} + p^2T = 0 \tag{8.3–4}$$

The solutions of these two equations, as previously found, are

$$X = A \cos \frac{p}{a} x + B \sin \frac{p}{a} x$$

$$T = C \cos pt + D \sin pt$$

A solution of the wave equation is therefore

$$z = XT = \left(A \cos \frac{p}{a} x + B \sin \frac{p}{a} x\right) (C \cos pt + D \sin pt) \tag{8.3–5}$$

The constants A, B, C, D, and p must be determined in such a manner that the boundary conditions are satisfied. The satisfaction of the boundary conditions and the determination of the arbitrary constants are discussed in the following sections.

Solution of the beam equation

The solution for equation 8.2–15 may be obtained in a similiar fashion. It is assumed that the variables can be separated and the solution can be written as

$$y = X(x)T(t) \tag{8.3–6}$$

Substitution of this trial solution into equation 8.2–15 yields

$$X \frac{d^2T}{dt^2} = -c^2\bar{r}^2 \frac{d^4X}{dx^4} T$$

which may be written as

$$\frac{1}{T} \frac{d^2T}{dt^2} = -\frac{c^2\bar{r}^2}{X} \frac{d^4X}{dx^4} \tag{8.3–7}$$

The variables are separated in this expression, the left side depending only on t and the right only on x. It therefore follows that these two expressions must be equal to some common constant, since x and t are independent variables. It will be expedient in the further development to denote this constant as before by $-p^2$. The problem of finding X and T is seen to rest upon the solution of the two ordinary differential equations,

$$\frac{d^4X}{dx^4} - \frac{p^2}{c^2\bar{r}^2}X = 0 \tag{8.3–8}$$

and

$$\frac{d^2T}{dt^2} + p^2T = 0 \tag{8.3–9}$$

The solution to equation 8.3–8 may be obtained by any of the standard methods of elementary differential equations. Its solution has the form

$$X = A \cos \beta x + B \sin \beta x + A' \cosh \beta x + B' \sinh \beta x$$

where

$$\beta^4 = \frac{p^2}{c^2\bar{r}^2}$$

The solution of equation 8.3–9, which is the same as equation 8.3–4, is

$$T = C \cos pt + D \sin pt$$

A solution of equation 8.2–15 may then be written as

$$y = XT = (A \cos \beta x + B \sin \beta x + A' \cosh \beta x + B' \sinh \beta x) \times (C \cos pt + D \sin pt) \tag{8.3–10}$$

where A, B, A', B', C, D, and p are constants which must be determined from the boundary conditions.

The slope of the beam,

$$\frac{\partial y}{\partial x} = \beta(-A \sin \beta x + B \cos \beta x + A' \sinh \beta x + B' \cosh \beta x) \times (C \cos pt + D \sin pt) \tag{8.3–11}$$

the bending moment,

$$EI\frac{\partial^2 y}{\partial x^2} = EI\beta^2(-A \cos \beta x - B \sin \beta x + A' \cosh \beta x + B' \sinh \beta x) \times (C \cos pt + D \sin pt) \tag{8.3–12}$$

and the shear,

$$EI\frac{\partial^3 y}{\partial x^3} = EI\beta^3(A \sin \beta x - B \cos \beta x + A' \sinh \beta x + B' \cosh \beta x)(C \cos pt + D \sin pt) \tag{8.3–13}$$

as given by this solution, will be needed in the applications to follow.

Some elementary examples of the four basic problems will be discussed in some detail to illustrate the determination of the arbitrary constants so that the boundary conditions may be satisfied. It should first be pointed out, however, that the constant p is not unique. In general, p has an infinite number of values giving rise to an infinity of solutions of the type of equation 8.3–5 or equation 8.3–10. Each of these possible solutions represents a possible mode of motion, and many or all of these modes may be present at the same time. For this reason, it is frequently necessary to combine all of these various solutions in a linear fashion to obtain a more general solution.

8.4. Lateral Vibration of a Taut String

The simplest problems of lateral vibration of a taut string is indicated in Fig. 8–1. In this example, the string is fixed at the ends $x = 0$ and $x = L$. The initial conditions on the string may vary, but, for the purposes of this illustrative example, it is assumed that the string starts from rest at the position given by the equation

$$y = f(x) \tag{8.4–1}$$

The boundary conditions may be stated in two parts:

$$y_{x=0} = 0 \qquad y_{x=L} = 0 \tag{8.4–2}$$

$$y_{t=0} = f(x) \qquad \dot{y}_{t=0} = \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \tag{8.4–3}$$

The boundary conditions for other specific examples may be established in a like manner.

The equation of motion for the string as previously established is

$$\frac{\partial^2 y}{\partial t^2} = c^2\epsilon \frac{\partial^2 y}{\partial x^2}$$

and a solution, as found in the previous section, is

$$y = \left(A \cos \frac{p}{a} x + B \sin \frac{p}{a} x\right) (C \cos pt + D \sin pt) \tag{8.4–4}$$

where

$$a^2 = c^2\epsilon$$

It is convenient to consider the boundary conditions 8.4–2 first. Substitution of these conditions into the solution gives

$$0 = A(C \cos pt + D \sin pt)$$

$$0 = \left(A \cos \frac{pL}{a} + B \sin \frac{pL}{a}\right) (C \cos pt + D \sin pt)$$

These two equations are satisfied when

$$A = 0 \quad \text{and} \quad B \sin \frac{pL}{a} = 0$$

Since $B = 0$ corresponds to the trivial solution $y = 0$, it must be required that $B \neq 0$ if a solution for the vibrating string is to be obtained. Consequently, the last equation must be written as

$$\sin \frac{pL}{a} = 0 \tag{8.4-5}$$

This is called the frequency equation, and its roots are

$$p = \frac{n\pi a}{L} \quad \text{where} \quad n = 0, 1, 2, \cdots$$

Thus, there is a solution for each value of n which has the form

$$y_n = \sin \frac{n\pi x}{L} \left(C_n \cos \frac{n\pi a}{L} t + D_n \sin \frac{n\pi a}{L} t \right) \tag{8.4-6}$$

where C_n and D_n are arbitrary constants which may be different for each value of n. The natural circular frequency of these solutions is seen to be

$$p = \frac{n\pi a}{L} = n\pi \frac{c\sqrt{\epsilon}}{L} \tag{8.4-7}$$

from which the period is

$$\tau = \frac{2\pi}{p} = \frac{2L}{na} = \frac{2L}{n\,c\sqrt{\epsilon}} \tag{8.4-8}$$

The boundary conditions 8.4-3 may now be considered. The velocity is found to be

$$\dot{y} = \sin \frac{n\pi x}{L} \left(\frac{n\pi a}{L} \right) \left(-C_n \sin \frac{n\pi a}{L} t + D_n \cos \frac{n\pi a}{L} t \right)$$

and so the last boundary condition 8.4-3 gives

$$0 = \sin \frac{n\pi x}{L} \left(\frac{n\pi a}{L} \right) D_n$$

whence it follows that

$$D_n = 0$$

The solution may now be reduced to

$$y_n = C_n \sin \frac{n\pi x}{L} \cos \frac{n\pi a}{L} t \tag{8.4-9}$$

and all boundary conditions except the first of equation 8.4–3 have been satisfied. Before proceeding with this last boundary condition which will determine the values of C_n, it is instructive to study equation 8.4–9. This may conveniently be written as

$$y_n = \tfrac{1}{2}C_n \left[\sin \frac{n\pi}{L}(x + at) + \sin \frac{n\pi}{L}(x - at) \right] \qquad (8.4\text{–}10)$$

The form of the string for each value of n is found from equation 8.4–9 to be a sine wave of varying amplitude. Equation 8.4–10 shows that this may also be considered as two sine waves moving in opposite directions with a velocity a. To see this, note from equation 8.4–10 that a particular point on the wave is given by

$$x \pm at = \text{constant} \quad \text{or} \quad x = \mp at + \text{constant}$$

The position of this point is seen to change at a rate given by a. The quantity a is the velocity at which a wave in the string is propagated. If it is recalled that

$$a = c\sqrt{\epsilon} = \sqrt{\frac{\sigma}{\rho}} \qquad (8.4\text{–}11)$$

it will be noted that this velocity depends only on the initial tension and the material of which the string is composed.

The mode corresponding to $n = 0$ represents a trivial solution, $y = 0$, of no interest here. The fundamental mode ($n = 1$) is given by

$$y_1 = C_1 \sin \frac{\pi x}{L} \cos \frac{\pi a}{L} t$$

and the fundamental frequency is

$$f_1 = \frac{p}{2\pi} = \frac{a}{2L} = \frac{C\sqrt{\epsilon}}{2L}$$

A plot of the first three modes of motion is shown in Fig. 8–5. Higher modes may be constructed immediately from equation 8.4–9. The fact that the higher modes have frequencies that are multiples of the fundamental is of importance in stringed instruments. These higher harmonics combine with the fundamental mode to produce pleasing musical tones. This is not a property of the vibrating membrane which accounts for the drum's lack of musical tones.

Another point of interest is that the tension is related to the fundamental frequency. It is easily shown that

$$T = 4L^2\rho A f_1^2$$

and it has long been a practice to estimate the tension in a wire by plucking it so as to excite the fundamental mode predominately and then note the resultant musical pitch.

The first of the boundary conditions 8.4–3 remains to be satisfied. It is seen to depend on the initial position of the string. Usually the engineering problem pertains to resonance and is not concerned with

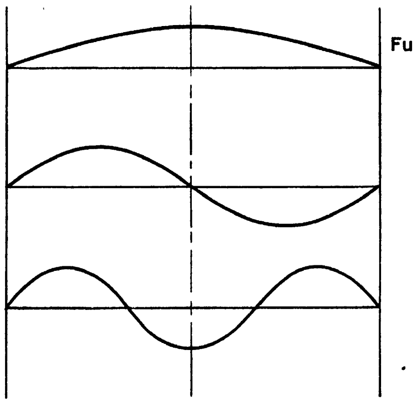


FIG. 8–5

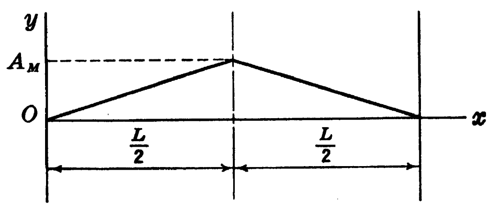


FIG. 8–6

the motion and configuration of the vibrating string. There is a class of design problems, however, in which this motion and a knowledge of the amplitudes of the several harmonics, which make up the total motion, is important.

As an example of this latter type of problem, the string plucked at the midpoint may be considered. Let the initial amplitude of the midpoint be A_M, as shown in Fig. 8–6. The first of the boundary conditions 8.4–3 then has the specific form:

$$\left.\begin{aligned} y_{0\,\underset{t=0}{=}} 2A_M \frac{x}{L} \qquad & 0 \leqq x \leqq \frac{L}{2} \\ y_0 = 2A_M\left(1 - \frac{x}{L}\right) \qquad & \frac{L}{2} \leqq x \leqq L \end{aligned}\right\} \tag{8.4–12}$$

It is immediately evident that none of the solutions 8.4–9 nor any finite combination of these solutions will satisfy this initial condition. Recourse must then be made to a combination of all of the infinite number of solutions in the form of an infinite sum or an infinite series such as

$$y_0 = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \cos \frac{n\pi a}{L} t \tag{8.4–13}$$

That this equation satisfies the fundamental differential equation, as well as the previously considered boundary conditions, rests on the linear character of the problem, and may easily be verified by direct substitution. For $t = 0$, equation 8.4–13 becomes

$$y_0 = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \tag{8.4–14}$$

It is evident that the boundary condition (equation 8.4–12) will be satisfied if it is possible to determine the C_n in equation 8.4–14 so that it agrees with equations 8.4–12. This is a standard problem in expanding a function into a Fourier series. The coefficients are given by

$$C_n = \frac{2}{L} \int_0^L y_0 \sin \frac{n\pi x}{L} dx \tag{8.4–15}$$

Substitution from equations 8.4–12 into equation 8.4–15 gives

$$C_n = \frac{4A_M}{L} \left[\int_0^{L/2} \frac{x}{L} \sin \frac{n\pi x}{L} dx + \int_{L/2}^{L} \left(1 - \frac{x}{L}\right) \sin \frac{n\pi x}{L} dx \right]$$

from which

$$C_n = \frac{8A_M}{n^2\pi^2} \sin \frac{n\pi}{2} \tag{8.4–16}$$

Thus the motion of the string is given by

$$y = \frac{8A_M}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{L} \cos \frac{n\pi a}{L} t \tag{8.4–17}$$

The values of C_n are the amplitudes of the respective modes which are present in the total motion. It will be noted that in this particular example the even modes ($n = 2, 4, 6, \cdots$) are absent. This may be seen more clearly by writing $n = 2m + 1$. Equation 8.4–17 may then be given the form

$$y = \frac{8A_M}{\pi^2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \sin \frac{(2m+1)\pi x}{L} \cos \frac{(2m+1)\pi a}{L} t$$

This may be explained physically from the fact that the motion of the string is symmetrical with respect to the center of the string, and the even modes, which lack this symmetry, are not present in the final solution.

8.5. Longitudinal Vibration of an Elastic Bar

As a specific illustrative example of the longitudinal vibration of an elastic bar, it is instructive to consider a bar that is initially free and falls vertically (Fig. 8–7) so as to strike a rigid table with a velocity

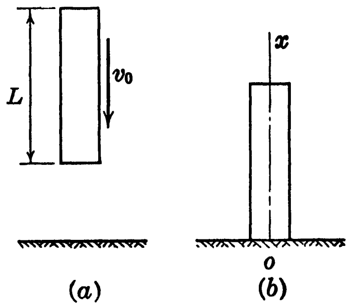


FIG. 8–7

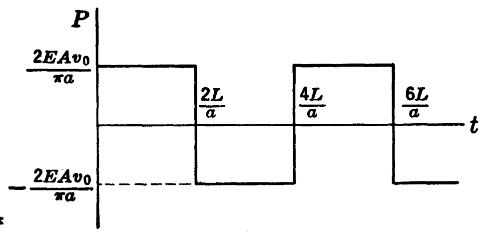


FIG. 8–8

v_0 and remain thereafter fixed to the table. The boundary conditions are then

$$\left.\begin{aligned} u_{x=0} &= 0 \\ P_{x=L} &= 0 \end{aligned}\right\} \qquad (8.5\text{–}1)$$

where

$$P = EA\frac{\partial u}{\partial x}$$

is the total longitudinal force in the bar, and

$$\left.\begin{aligned} u_{t=0} &= 0 \\ \dot{u}_{t=0} &= -v_0 \end{aligned}\right\} \qquad (8.5\text{–}2)$$

The fundamental equation of motion has been shown to be

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \qquad (8.2\text{–}6)$$

and its solution was found to be

$$u = \left(A\cos\frac{px}{a} + B\sin\frac{px}{a}\right)(C\cos pt + D\sin pt) \qquad (8.3\text{–}5)$$

where

$$a^2 = c^2$$

The boundary conditions given by equation 8.5–1 lead to

$$0 = A(C \cos pt + D \sin pt)$$

$$0 = EA\frac{p}{a}\left(-A \sin\frac{pL}{a} + B \cos\frac{pL}{a}\right)(C \cos pt + D \sin pt)$$

These equations are satisfied when

$$A = 0 \quad \text{and} \quad \cos\frac{pL}{a} = 0 \quad \text{or} \quad p_n = \frac{(2n+1)\pi a}{2L} \; (n = 0, 1, 2, \cdots)$$

where the latter equation is the frequency equation. These values give an infinite number of solutions, typified by

$$u_n = \sin\frac{(2n+1)\pi x}{2L}\left(C_n \cos\frac{(2n+1)\pi a}{2L}t + D_n \sin\frac{(2n+1)\pi a}{2L}t\right) \tag{8.5–3}$$

These solutions represent modes of motion in a bar fixed at one end and free at the other. The quantity a may be established as the velocity of longitudinal waves in the bar by an argument analogous to that of the proceeding section. Therefore, $a = c$ is the speed of sound in the bar. The natural frequency of the nth mode of motion of the bar is given by

$$f_n = \frac{p_n}{2\pi} = \frac{(2n+1)c}{4L} = \frac{2n+1}{4L}\sqrt{\frac{E}{\rho}} \tag{8.5–4}$$

the fundamental frequency being obtained when $n = 0$.

The first of the boundary conditions 8.5–2 may be substituted into equation 8.5–3 to give

$$0 = C_n \sin\frac{(2n+1)\pi x}{2L}$$

whence it is seen that

$$C_n = 0$$

Equation 8.5–3 then becomes

$$u_n = D_n \sin\frac{(2n+1)\pi x}{2L}\sin\frac{(2n+1)\pi a}{2L}t \tag{8.5–5}$$

and the velocity of the bar is

$$\dot{u}_n = \frac{\partial u_n}{\partial t} = \frac{(2n+1)\pi a}{2L}D_n \sin\frac{(2n+1)\pi x}{2L}\cos\frac{(2n+1)\pi a}{2L}t \tag{8.5–6}$$

It is apparent that none of the equations 8.5–6 or any finite combination of these solutions will be capable of satisfying the last of the boundary conditions 8.5–2. It remains then to try an infinite series of these solutions,

$$u = \sum_{n=1}^{\infty} D_n \sin \frac{(2n+1)\pi x}{2L} \sin \frac{(2n+1)\pi a}{2L} t \tag{8.5–7}$$

from which the velocity is seen to be

$$\dot{u} = \frac{\partial u}{\partial t} = \frac{\pi a}{2L} \sum_{n=1}^{\infty} (2n+1) D_n \sin \frac{(2n+1)\pi x}{2L} \cos \frac{(2n+1)\pi a}{2L} t$$

Substitution into the last of the boundary conditions 8.5–2 gives

$$-v_0 = \frac{\pi a}{2L} \sum_{n=1}^{\infty} (2n+1) D_n \sin \frac{(2n+1)\pi x}{2L} \tag{8.5–8}$$

The last boundary condition will be satisfied if the coefficients D_n can be selected so that equation 8.5–8 is valid for $0 \leqq x \leqq L$. This is the standard problem of expansion of a function into a Fourier series, and the coefficients are given by

$$\frac{\pi a(2n+1)}{2L} D_n = \frac{2}{L} \int_0^L - v_0 \sin \frac{(2n+1)\pi x}{2L} dx$$

Integration gives

$$D_n = -\frac{8v_0 L}{\pi^2 a(2n+1)^2} \tag{8.5–9}$$

The final solution is

$$u = -\frac{8v_0 L}{\pi^2 a} \sum_{n=1}^{\infty} \frac{1}{(2n+1)} \sin \frac{(2n+1)\pi x}{2L} \sin \frac{(2n+1)\pi a}{2L} t \tag{8.5–10}$$

The pressure between the end of the bar and the table is given by

$$P = -E\epsilon_0 A$$

where ϵ_0 is the strain at $x = 0$, i.e.,

$$\epsilon_0 = \left[\frac{\partial u}{\partial x}\right]_{x=0} = \frac{-2v_0}{\pi a} \sum_{n=1}^{\infty} \frac{1}{2n+1} \sin \frac{(2n+1)\pi a}{2L} t$$

Thus

$$P = \frac{2EAv_0}{\pi a} \sum_{n=1}^{\infty} \frac{1}{2n+1} \sin \frac{(2n+1)\pi a}{2L} t$$

A plot of the pressure as a function of time is shown in Fig. 8–8. The instantaneous change in P is due to the idealized assumptions usually made for perfectly elastic bodies, whereas, in an actual problem, the variation although extremely rapid is not instantaneous. It should be noted that the pressure becomes negative (tension) at time

$$t = \frac{2L}{a}$$

and, unless the bar is fastened, it will bounce off at this instant. The length of time in contact in this case is the time necessary for the pressure wave to travel up the bar and be reflected back to the table.

8.6. Torsional Vibration of an Elastic Shaft

A typical example of the torsional vibration of an elastic shaft is shown in Fig. 8-9 where the disks of moment of inertia I_1 and I_2 are

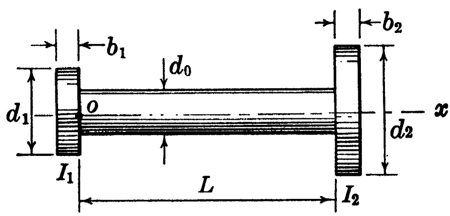


FIG. 8-9

assumed to be rigid. The boundary conditions will be given by

$$\left.\begin{aligned} GJ\left(\frac{\partial\theta}{\partial x}\right)_{x=0} &= +I_1\left(\frac{\partial^2\theta}{\partial t^2}\right)_{x=0} \\ GJ\left(\frac{\partial\theta}{\partial x}\right)_{x=L} &= -I_2\left(\frac{\partial^2\theta}{\partial t^2}\right)_{x=L} \end{aligned}\right\} \tag{8.6-1}$$

which require that the inertias I_1 and I_2 are in dynamic equilibrium; i.e., the applied torque equals the inertia torque, and

$$\left.\begin{aligned} \theta_{t=0} &= f_1(x) \\ \dot{\theta}_{t=0} &= f_2(x) \end{aligned}\right\} \tag{8.6-2}$$

The fundamental equation of the motion of the shaft has been shown to be

$$\frac{\partial^2\theta}{\partial t^2} = a^2\frac{\partial^2\theta}{\partial x^2} \quad \text{where} \quad a^2 = \frac{G}{\rho} = \frac{c^2}{2(1+\nu)}$$

The solution of this equation is

$$\theta = \left(A\cos\frac{px}{a} + B\sin\frac{px}{a}\right)(C\cos pt + D\sin pt) \tag{8.3-5}$$

from which there is obtained

$$\frac{\partial \theta}{\partial x} = \left(-A\frac{p}{a}\sin\frac{px}{a} + B\frac{p}{a}\cos\frac{px}{a}\right)(C\cos pt + D\sin pt)$$

and

$$\frac{\partial^2 \theta}{\partial t^2} = -\left(A\cos\frac{px}{a} + B\sin\frac{px}{a}\right)p^2(C\cos pt + D\sin pt) \quad (8.6\text{–}3)$$

Substitution of these expressions into the boundary conditions 8.6–1 yields the two equations,

$$GJB\frac{p}{a} = -I_1Ap^2$$

$$GJ\frac{p}{a}\left(-A\sin\frac{pL}{a} + B\cos\frac{pL}{a}\right) = I_2\left(A\cos\frac{pL}{a} + B\sin\frac{pL}{a}\right)$$

Elimination of A and B from these equations gives

$$\left(I_2p^2 - \frac{G^2J^2}{a^2I_1}\right)\tan\frac{pL}{a} = \frac{GJp}{a}\left(1 + \frac{I_2}{I_1}\right) \quad (8.6\text{–}4)$$

Introduction of the notation:

$$\beta = \frac{pL}{a} \qquad m = \frac{I_1}{I_0}$$

$$I_0 = \rho LJ = \frac{GJL}{a^2} \qquad n = \frac{I_2}{I_0}$$

where I_0 is the moment of inertia of the shaft as a rigid body, permits equation 8.6–4 to take the form

$$\tan\beta = \frac{\beta(m + n)}{mn\beta^2 - 1} \quad (8.6\text{–}5)$$

This is the frequency equation for the system and may readily be solved graphically. Each value of β obtained from this equation corresponds to a mode of motion. If, as is most often the case, only the frequencies of the system are required, equation 8.6–5 furnishes the complete solution, and there is no need to refine the boundary conditions 8.6–2. However a set of boundary conditions 8.6–2 may be specified and the solution for the motion obtained in a manner similar to that of previous problems.

The solution of equation 8.6–5 may be illustrated by a numerical example. Let

$d_0 = 2$ in. $\qquad L = 20$ in.

$d_1 = 4$ in. $\qquad b_1 = 1$ in.

$d_2 = 6$ in. $\qquad b_2 = 1$ in.

$G = 12 \times 10^6$ lb per sq in. $\qquad \rho = 0.735 \times 10^{-3}$ lb sec² per in.⁴

Then:

$$J = \frac{\pi}{2} r^4 \qquad I_1 = 0.01847 \text{ lb in. sec}^2$$

$$I_2 = 0.09352 \text{ lb in. sec}^2$$

$$I_0 = 0.02309 \text{ lb in. sec}^2$$

whence

$$m = 0.7999 \qquad n = 4.0502$$

The frequency equation for this example has the form

$$\tan \beta = \frac{\beta(m + n)}{mn\beta^2 - 1} = \frac{4.8501\beta}{3.2398\beta^2 - 1}$$

The graphical solution for the first two modes is shown in Fig. 8–10.

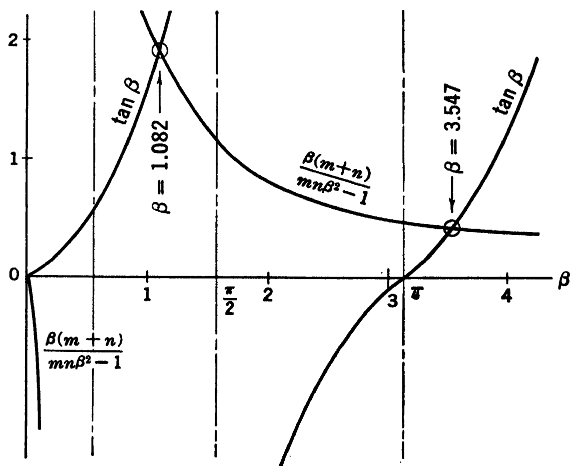


FIG. 8–10

The fundamental mode corresponds to $\beta = 1.082$, whence the exact solution is

$$p_1 = \frac{\beta a}{L} = \frac{\beta}{L}\sqrt{\frac{G}{\rho}} = 6.913 \times 10^3 \text{ rad per sec}$$

It is of interest to note that the fundamental frequency obtained by neglecting the mass of the shaft is given by

$$p = \sqrt{\frac{GJ}{L}\left(\frac{1}{I_1} + \frac{1}{I_2}\right)}$$

The elementary theory yields, for this particular example,

$$p = 7.817 \times 10^3 \text{ rad per sec}$$

which is 13% too high. Other approximations closer to the exact solution are discussed in section 8.8.

8.7. Lateral Vibration of an Elastic Beam

As an example, a beam, which is simply supported at each end as

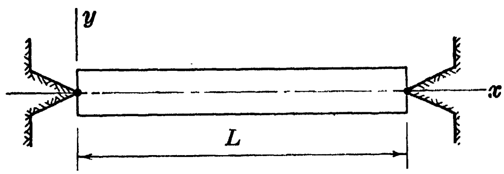


FIG. 8–11

shown in Fig. 8–11, will be considered. The fundamental equation of motion for this system is given by

$$\frac{\partial^2 y}{\partial t^2} = -c^2\bar{r}^2 \frac{\partial^4 y}{\partial x^4} \tag{8.2–15}$$

where

$$c^2 = \frac{E}{\rho} \quad \text{and} \quad \bar{r}^2 = \frac{I}{A}$$

The solution of this equation, as obtained in general, is

$$y = (A \cos \beta x + B \sin \beta x + A' \cosh \beta x + B' \sinh \beta x)(C \cos pt + D \sin pt) \tag{8.3–10}$$

where

$$\beta^4 = \frac{p^2}{c^2\bar{r}^2}$$

The boundary conditions are:

$$\left.\begin{aligned} y_{x=0} &= 0 \qquad & M_{x=0} &= EI\left(\frac{\partial^2 y}{\partial x^2}\right)_{x=0} = 0 \\ y_{x=L} &= 0 \qquad & M_{x=L} &= EI\left(\frac{\partial^2 y}{\partial x^2}\right)_{x=L} = 0 \end{aligned}\right\} \tag{8.7–1}$$

and

$$\left.\begin{aligned} y_{t=0} &= f_1(x) \\ \dot{y}_{t=0} &= f_2(x) \end{aligned}\right\} \tag{8.7-2}$$

Substituting into boundary conditions 8.7–1 and using the equation 8.3–12 gives

$$\frac{\partial^2 y}{\partial x^2} = \beta^2(-A \cos \beta x - B \sin \beta x + A' \cosh \beta x + B' \sinh \beta x)(C \cos pt + D \sin pt)$$

Thus, for $x = 0$,

$$A + A' = 0$$

$$-A + A' = 0$$

whence

$$A = A' = 0$$

From the conditions 8.7–1, for $x = L$, there are obtained

$$B \sin \beta L + B' \sinh \beta L = 0$$

$$-B \sin \beta L + B' \sinh \beta L = 0$$

whence

$$B \sin \beta L = 0 \quad \text{and} \quad B' \sinh \beta L = 0$$

but, since $\sinh \beta L \neq 0$, except for $\beta = 0$, it follows that

$$B' = 0 \quad \text{and} \quad \beta_n L = n\pi \qquad (n = 1, 2, 3 \cdots)$$

The particular solution satisfying boundary conditions 8.7–1 is therefore

$$y_n = \sin \frac{n\pi x}{L} (C_n \cos p_n t + D_n \sin p_n t) \tag{8.7-3}$$

where there is a distinct solution for each value of n. The frequency equation is obtained from $\beta^4 = p^2/c^2\bar{r}^2$; thus

$$p_n{}^2 = c^2\bar{r}^2\beta^4 = \frac{n^4\pi^4c^2\bar{r}^2}{L^4}$$

So

$$p_n = (n\pi)^2 \frac{\bar{r}}{L} \cdot \frac{c}{L} \tag{8.7-4}$$

Each of these circular frequencies corresponds to a particular mode of motion.

To determine the amplitudes of the respective modes of motion which are present in a given vibration, consideration must be given to the boundary conditions 8.7–2. As a specific illustrative example, the conditions as indicated in Fig. 8–12 will be treated. In this example, the beam is hinged permanently at one end and is initially free at the other. The beam is permitted to fall freely from rest through an angle θ_0, whereupon it strikes the support and is then simply supported at

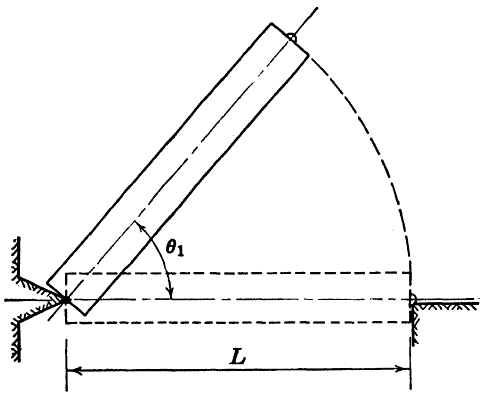


FIG. 8–12

both ends. The equation of motion during the period of free rotation is given by

$$I_A\ddot{\theta} + W\frac{L}{2}\cos\theta = 0$$

where $I_A = \dfrac{W}{g}\dfrac{L^2}{3}$ and W is the weight of the beam.

This equation may be integrated by multiplying through by $\dot{\theta}$, thus obtaining

$$\dot{\theta}\frac{d\dot{\theta}}{dt} + \frac{3g}{2L}\cos\theta\frac{d\theta}{dt} = 0$$

Integration yields

$$\dot{\theta}^2 + \frac{3g}{L}\sin\theta = C$$

Since the beam falls from rest, $\dot{\theta} = 0$ when $\theta = \theta_0$, and so

$$C = \frac{3g}{L}\sin\theta_0$$

The angular velocity when the beam contacts the support is

$$\dot{\theta}_0 = \sqrt{\frac{3g}{L}\sin\theta_0}$$

The vibration of the beam starts at this time, and the initial conditions 8.7–2 may now be stated as

$$\left.\begin{aligned} y_{t=0} &= 0 \\ \dot{y}_{t=0} &= -\dot{\theta}_0 x = -x\sqrt{\frac{3g}{L}\sin\theta_0} \end{aligned}\right\} \qquad (8.7\text{–}5)$$

Substituting the solution 8.7–3 into the first of these conditions gives

$$C_n \sin\frac{n\pi x}{L} = 0$$

thus

$$C_n = 0$$

The solution may now be written as

$$y_n = D_n \sin\frac{n\pi x}{L}\sin p_n t \qquad (8.7\text{–}6)$$

where a separate and distinct mode of motion is given for each value of n. The remaining boundary condition 8.7–5 cannot be satisfied by any finite combination of the solutions, indicating that all of the modes of motion are present and that the solution must be in the form

$$y = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{L}\sin p_n t$$

The velocity at any point becomes

$$\dot{y} = \frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} D_n p_n \sin\frac{n\pi x}{L}\cos p_n t$$

Substitution of this into the last boundary condition 8.7–5 gives

$$\dot{y}_{t=0} = -x\sqrt{\frac{3g}{L}\sin\theta_0} = \sum_{n=1}^{\infty} D_n p_n \sin\frac{n\pi x}{L}$$

The determination of the values of D_n so that this equation will be satisfied is a problem in the expansion of a given function into a Fourier series. The coefficients are given by

$$p_n D_n = \frac{2}{L}\int_0^L \dot{y}_{t=0}\sin\frac{n\pi x}{L}\,dx$$

The evaluation of this integral is accomplished by substituting for $\dot{y}_{t=0}$; thus

$$D_n = \frac{-2}{p_n L}\sqrt{\frac{3g}{L}}\sin\theta_0 \int_0^L x \sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2L^2}{n^2\pi^3 c\bar{r}}\sqrt{3gL}\sin\theta_0 \cos n\pi$$

The final solution may be written in the form

$$y = \frac{2L^2\sqrt{3gL}\sin\theta_0}{\pi^3 c\bar{r}} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin\frac{n\pi x}{L} \sin p_n t$$

Other initial conditions may be treated in a similar manner.

8.8. Application of Energy Method to Continuous Systems

The exact solution of vibration problems involving continuous systems is often laborious, and the required calculations for complex systems are frequently prohibitive. To reduce the labor involved in the solution of a complex system, it is necessary to take advantage of approximation methods. Since the most frequent vibration problems confronting the engineer are those concerning natural frequencies of systems, particularly the fundamental or lowest frequency, methods that will yield approximate fundamental frequencies with good accuracy are of great interest.

Of the many methods available for the determination of natural frequencies, Rayleigh's energy method is in most common use. The value of this method lies in its simplicity and the high degree of accuracy that may be obtained, particularly with reference to the fundamental mode. Rayleigh's method may be applied to higher modes; however its best application concerns the lowest mode, and it is for that purpose that its use will be demonstrated here.

The rudiments of the method are outlined in section 2.5, and with only slight modifications it may be adapted to continuous systems. In brief, the procedure consists of assuming a configuration of the system which closely approximates the position of the system at maximum amplitude for the fundamental mode. The assumed configuration should be compatible with all the boundary conditions if maximum accuracy is to be obtained, although in many instances sufficient accuracy may be obtained with assumed configurations which do not entirely meet this last requirement.

Based on the assumed configuration, the maximum value of the kinetic energy that is contained in the system during a cycle of vibration is computed. The maximum kinetic energy occurs at the instant of maximum velocity, that is, at the equilibrium position of the system. The gain in potential energy stored in the system, while passing from the position of maximum kinetic energy to the position of zero kinetic energy (i.e., maximum displacement), is likewise computed on the basis of the assumed configuration. The law of conservation of energy requires that these two quantities of energy must be equal in all undamped systems. The relationship obtained by equating the potential and kinetic energies computed in the above fashion may be solved for the natural frequency of the fundamental mode.

As the first illustration of the energy method, the fundamental frequency of the system shown in Fig. 8–13 may be treated. The system is composed of an elastic bar fixed at one end and rigidly attached to a concentrated mass at the other end. The bar is assumed to vibrate longitudinally with an amplitude at the mass equal to δ. As a first approximation to the fundamental mode, a point on the bar a distance x from the fixed end is assumed to have an amplitude $\frac{x}{L}\delta$ as shown in Fig. 8-13*b*. The equation of motion for an element of the bar and the end mass become, respectively,

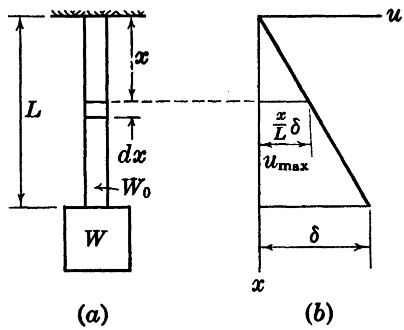


FIG. 8–13

$$u = \frac{x}{L}\delta \sin pt \quad \text{and} \quad u = \delta \sin pt \tag{8.8–1}$$

from which the maximum velocities are $\frac{x}{L}\delta p$ and δp, where p is the natural circular frequency. The maximum kinetic energy of the system is given by

$$V = \int_0^L \frac{\rho A}{2}\left(\frac{x}{L}\delta p\right)^2 dx + \frac{1}{2}\frac{W}{g}(\delta p)^2$$

$$= \tfrac{1}{2}\delta^2 p^2\left(\frac{W}{g} + \tfrac{1}{3}\rho AL\right) \tag{8.8–2}$$

The potential energy stored in the bar at maximum amplitude is

$$U = \frac{1}{2}\frac{EA}{L}\delta^2 \tag{8.8–3}$$

Equating expressions 8.8–2 and 8.8–3 and solving for p^2 gives

$$p^2 = \frac{EAg}{L(W + \frac{1}{3}W_0)} \tag{8.8–4}$$

where W_0 is the weight of the whole bar.

This shows that a more accurate evaluation of the fundamental frequency is obtained from the elementary theory by adding one third of the mass of the spring element to the attached mass, since the above problem is identical in form with that of a mass attached to a spring. The spring constant k is equivalent to EA/L in the above expression. Thus, for a heavy spring and mass, the fundamental frequency is approximated by

$$p^2 = \frac{kg}{W + \frac{1}{3}W_0} \tag{8.8–5}$$

Furthermore, a similar application of the energy method to the torsion pendulum (Fig. 2–3) yields

$$p^2 = \frac{GJ}{L(I + \frac{1}{3}\rho LJ)} \tag{8.8–6}$$

verification of which is left as an exercise.

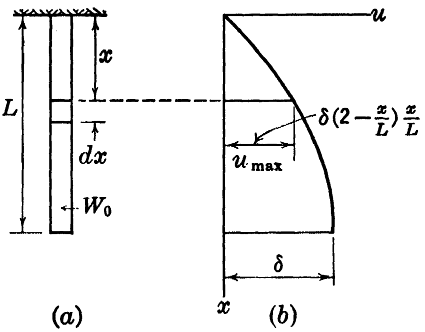


FIG. 8–14

As a second example a bar of length L will be investigated (Fig. 8–14). The configuration of the bar at maximum displacement may be approximated by the displacements due to the bar's own weight. In this instance, the elongation of a small length of the bar dx will be

$$du_{max} = \frac{\rho g(L - x)\,dx}{E}$$

which yields, upon integration,

$$u_{max} = \frac{\rho g}{E}\left(L - \frac{x}{2}\right)x = 2\delta\left(1 - \frac{x}{2L}\right)\frac{x}{L}$$

The equation of motion of the bar has the form

$$u = 2\delta\left(1 - \frac{x}{2L}\right)\frac{x}{L}\sin pt \tag{8.8–7}$$

The maximum kinetic energy is

$$V = \int_0^L 2\rho A \left(1 - \frac{x}{2L}\right)^2 \left(\frac{x}{L}\right)^2 \delta^2 p^2 \, dx = \tfrac{4}{15}\rho AL\delta^2 p^2$$

The maximum potential energy is

$$U = \int_0^L \tfrac{1}{2}EA \left(\frac{\partial u}{\partial x}\right)^2 dx = \frac{2}{3}\frac{EA}{L}\delta^2$$

Equating these two expressions gives

$$p^2 = \frac{5}{2}\frac{E}{\rho L^2} = \frac{5EAg}{2WL} \tag{8.8-8}$$

as compared to the exact solution (equation 8.5–4) which takes the form

$$p^2 = \frac{\pi^2}{4}\frac{E}{\rho L^2} = 2.4674\frac{E}{\rho L^2} \tag{8.8-9}$$

We may now obtain a better approximation to the previous problem of the uniform bar and the end mass by assuming the dynamic displacements to be proportional to both the static loading of the weight W and the weight of the bar, as shown in Fig. 8–15. The elongation of a small length of bar dx will be

$$du_{\max} = \frac{1}{AE}[\rho g A(L - x) + W]\,dx$$

whence, by integration,

$$u_{\max} = \frac{1}{AE}\left[\rho g A\left(L - \frac{x}{2}\right) + Wx\right]$$

$$= \delta \frac{W_0\left(1 - \dfrac{x}{2L}\right) + W}{W + \frac{1}{2}W_0} \cdot \frac{x}{L}$$

The equation of motion of the bar is

$$u = u_{\max} \sin pt \tag{8.8-10}$$

so the maximum kinetic energy is given by

$$V = \int_0^L \tfrac{1}{2}\rho A \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2}\frac{W}{g}\delta^2 p^2$$

$$= \frac{\delta^2 p^2}{2g}\left(W_0 \cdot \frac{\frac{1}{3}W^2 + \frac{5}{12}WW_0 + \frac{2}{15}W_0^2}{W^2 + WW_0 + \frac{1}{4}W_0^2} + W\right)$$

The potential energy is

$$U = \int_0^L \tfrac{1}{2}EA\left(\frac{\partial u}{\partial x}\right)^2 dx = \frac{1}{2}\frac{EA\delta^2}{L}\cdot\frac{W^2 + WW_0 + \frac{1}{3}W_0^2}{W^2 + WW_0 + \frac{1}{4}W_0^2}$$

Equating these two expressions gives

$$p^2 = \frac{EAg}{L}\cdot\frac{W^2 + WW_0 + \frac{1}{3}W_0^2}{W^3 + \frac{4}{3}W^2W_0 + \frac{2}{3}WW_0^2 + \frac{2}{15}W_0^3} \qquad (8.8\text{–}11)$$

When $W = 0$ and $W_0 \neq 0$, this reduces to equation 8.8–8, and, when

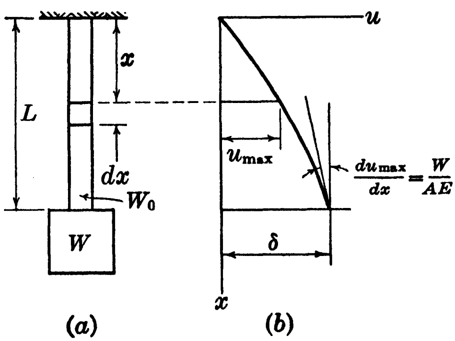


FIG. 8–15

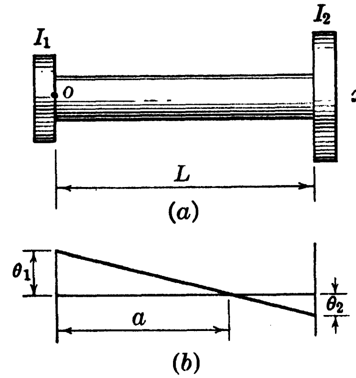


FIG. 8–16

$W \neq 0$ and $W_0 \to 0$, it reduces to equation 8.8–4. To see this, it is convenient to write equation 8.8–11 in the form

$$p^2 = \frac{kg}{W + W_0\left(\frac{1}{3} + \frac{\beta^2}{45} - \frac{\beta^3}{135} + \cdots\right)}$$

where

$$k = \frac{EA}{L} \quad \text{and} \quad \beta = \frac{W_0}{W}$$

As a further example of the use of the energy method, the problem of the two disks connected by an elastic shaft may be treated. The exact solution of this problem was found in section 8.6. This example may then be used to compare the relative accuracy of the various solutions.

The simplest configuration of the system that may be assumed as a basis for the application of the energy method is a straight line, as shown in Fig. 8–16. The location of the node may be obtained in two ways, either by equating the fundamental frequencies of the two systems as divided by the node or by equating the total inertia torque of the system to zero. In the first instance, making use of equation 8.8–6 permits a to be found from

$$p^2 = \frac{GJ}{a_1(I_1 + \frac{1}{3}\rho J a_1)} = \frac{GJ}{(L - a_1)[I_2 + \frac{1}{3}\rho J(L - a_1)]}$$

whence

$$a_1 = \frac{L}{1 + \beta_1} \quad \text{where} \quad \beta_1 = \frac{I_1 + \frac{1}{3}I_0}{I_2 + \frac{1}{3}I_0} = \frac{\theta_2}{\theta_1}$$

and

$$I_0 = \rho JL$$

Equating the total inertia torque for the whole system to zero, as a second alternative, gives

$$I_1 p^2 \theta_1 + \int_0^L \rho J p^2 \left[\theta_1 - (\theta_1 + \theta_2)\frac{x}{L}\right] dx - I_2 p^2 \theta_2 = 0$$

from which

$$a_2 = \frac{L}{1 + \beta_2} \quad \text{where} \quad \beta_2 = \frac{I_1 + \frac{1}{2}I_0}{I_2 + \frac{1}{2}I_0} = \frac{\theta_2}{\theta_1}$$

Application of Rayleigh's method to this system gives, for the maximum kinetic energy,

$$V = \tfrac{1}{2}I_1 p^2 \theta_1^2 + \tfrac{1}{2}I_2 p^2 (\beta\theta_1)^2 + \tfrac{1}{2}\int_0^L \rho J p^2 \left(1 - \frac{x}{a}\right)^2 \theta_1^2 \, dx$$

$$= \tfrac{1}{2}p^2\theta_1^2[I_1 + \beta^2 I_2 + \tfrac{1}{3}I_0(1 - \beta + \beta^2)]$$

The maximum potential energy is

$$U = \frac{1}{2}\frac{GJ}{L}(\theta_1 + \beta\theta_1)^2 = \frac{1}{2}\frac{GJ}{L}\theta_1^2(1 + \beta)^2$$

Equating these two expressions for the energy gives

$$p^2 = \frac{GJ}{L} \cdot \frac{(1 + \beta)^2}{I_1 + \beta^2 I_2 + \frac{1}{3}I_0(1 - \beta + \beta^2)}$$

The following table illustrates the accuracy that may be obtained by the various methods when applied to the numerical example of section 8.6.

Method	β	a/L	p	% *Error*
Elementary theory assuming $I_0 = 0$	0.1975	0.8351	7817	13.1
Energy method equating frequencies	0.2586	0.7946	6946	0.5
Energy method equating torques	0.2857	0.7778	6953	0.6
Exact solution..................	......	......	6913	0.0

The above values are indicative of the accuracies that are obtainable for the various methods but must not be interpreted as general. The elementary theory gives excellent results when the shaft has a moment of inertia that is small compared to that of the disks. On the other hand, less reasonable assumptions as to the configuration of the system will tend to increase the error in the use of the energy method. It is fortunate, however, that very satisfactory results can be obtained by the energy method, even though the assumed configuration is not a very good approximation to the exact form of the system. It is this fact that makes Rayleigh's method so useful. It should be noted that the use of the energy method gives frequencies that are in general higher than the exact solution.

8.9. Further Applications of the Energy Method

The energy method may also be used to determine the fundamental frequency of beams. To illustrate the procedure, the previously

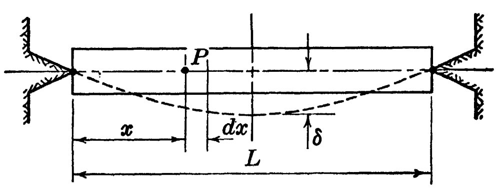


Fig. 8–17

treated example of the uniform beam simply supported at each end may be considered (see Fig. 8–17). As a first approximation, it will be assumed that the form of the beam for the fundamental mode at maximum displacement can be replaced by a parabola

$$y_{\max} = 4\delta\left(1 - \frac{x}{L}\right)\frac{x}{L}$$

The equation of motion of the beam will then have the form

$$y = 4\delta\left(1 - \frac{x}{L}\right)\frac{x}{L}\sin pt \tag{8.9–1}$$

where p is the natural circular frequency.

The velocity at a point P on the beam a distance x from the left end is given by

$$\dot{y}_x = \frac{\partial y}{\partial t} = 4\delta\left(1 - \frac{x}{L}\right)\frac{x}{L}p\cos pt \tag{8.9–2}$$

The maximum value of the velocity of P is

$$\dot{y}_x = 4\delta\left(1 - \frac{x}{L}\right)\frac{x}{L}p$$

and the maximum kinetic energy of an element of the beam of length dx is $\frac{\rho A}{2}(dx)\dot{y}_x^2$. The maximum kinetic energy of the whole beam may be found by integration. Thus

$$V = \int_0^L \frac{\rho A}{2}\left[4\delta\left(1 - \frac{x}{L}\right)\frac{x}{L}p\right]^2 dx = \frac{4\rho AL\,\delta^2 p^2}{15} \tag{8.9–3}$$

The potential energy stored in the beam is shown in strength of materials to be

$$U = \int_0^L \frac{M^2}{2EI}dx = \int_0^L \frac{EI}{2}\left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx$$

From equation 8.9–1, it is seen that

$$\frac{\partial^2 y}{\partial x^2} = -\frac{8\delta}{L^2}\sin pt$$

The maximum value of the potential energy, which occurs at maximum deflection, is therefore

$$U = \int_0^L \frac{EI}{2}\left(-\frac{8\delta}{L^2}\right)^2 dx = \frac{32EI\delta^2}{L^3} \tag{8.9–4}$$

Equating the maximum potential energy to the maximum kinetic energy gives

$$\frac{32EI\delta^2}{L^3} = \frac{32EA\bar{r}^2\delta^2}{L^3} = \frac{4\rho AL\,\delta^2 p^2}{15}$$

which yields the approximate circular frequency as

$$p = \sqrt{120\frac{E}{\rho}}\left(\frac{\bar{r}}{L^2}\right) = 10.95\frac{c\bar{r}}{L^2} \tag{8.9–5}$$

The exact solution as previously obtained (equation 8.7–4) is

$$p = \pi^2 \frac{c\bar{r}}{L^2}$$

from which it is seen that the approximate solution is about 11% high. It is to be noted that the approximate form of the beam, as given by equation 8.9–2, does not satisfy all the boundary conditions. (The bending moment is not zero at the ends.) The result obtained however is reasonably good for a first approximation. A better approximation may be obtained by using the static deflection curve, as obtained in strength of materials, for an approximate form of the first mode amplitudes. In this example this gives

$$y = \tfrac{16}{5}\delta \left[\frac{x}{L} - 2\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right)^4\right] \sin pt \qquad (8.9\text{–}6)$$

Using this expression in lieu of equation 8.9–1 and recalculating the natural circular frequency on this basis gives

$$p^2 = \frac{12}{L^2}\sqrt{\frac{21EI}{31\rho A}} = 9.8767 \frac{c\bar{r}}{L^2} \qquad (8.9\text{–}7)$$

which is less than one tenth of a percent in error.

The example treated above suggests that the configuration due to the static loading of the system with its own masses or weights may be an excellent choice for the calculation of the fundamental frequency by Rayleigh's method. This is true, and frequencies calculated on this basis are well within the accuracy required for most engineering purposes.

The use of the static deflection configuration also makes it possible to simplify the calculation of the maximum potential energy of the system. The potential energy stored in the system at maximum amplitude is equal to the work that must be done on the system to give it that configuration. If the maximum amplitude configuration is taken as the form of the system due to static loading by its own masses, then the work done on the system when loaded with its own weights is equal to the potential energy involved in Rayleigh's method. Thus

$$U = \tfrac{1}{2}\Sigma W_i \delta_i$$

where δ_i is the static deflection of the weight W_i. The maximum velocity of the weight W_i with amplitude δ_i is $p\delta_i$. The maximum kinetic energy of the system is therefore

$$V = \tfrac{1}{2}p^2 \sum \frac{W_i}{g} \delta_i^2$$

Equating the maximum values of the kinetic and potential energies gives, for the natural circular frequency,

$$p^2 = \frac{\Sigma W_i \delta_i}{\Sigma W_i \delta_i^2} g \qquad (8.9\text{–}8)$$

If the system contains a distributed mass as well as lumped masses,

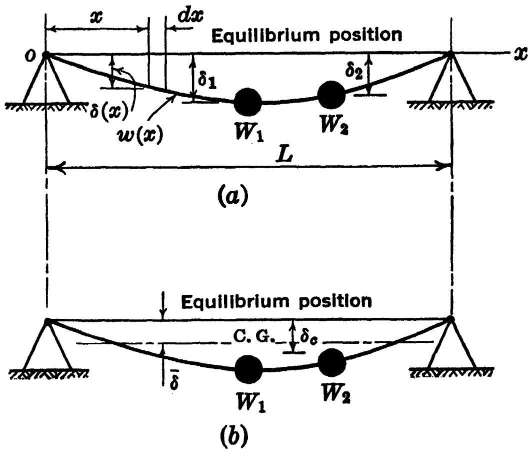


FIG. 8–18

then the potential energy may be expressed as

$$U = \tfrac{1}{2} \sum W_i \delta_i + \tfrac{1}{2} \int_0^L w(x)\, \delta(x)\, dx$$

and the kinetic energy becomes

$$V = \tfrac{1}{2} p^2 \sum \frac{W_i}{g} \delta_i + \tfrac{1}{2} p^2 \int_0^L \frac{w(x)}{g} \delta^2(x)\, dx$$

where $w(x)$ is the weight per unit length of the continuous member and $\delta(x)$ is its static deflection. The natural circular frequency is seen to be

$$p^2 = \frac{\sum W_i \delta_i + \int_0^L w(x)\, \delta(x)\, dx}{\sum W_i \delta_i^2 + \int_0^L w(x)\, \delta^2(x)\, dx} g \qquad (8.9\text{–}9)$$

Consider the system shown in Fig. 8–18. Equation 8.9–9 may be interpreted for this system as follows. The numerator of this expression is the first moment $\bar{M}$ of the weight of the oscillating beam and attached weights about the equilibrium position, whereas the denominator is the second moment or moment of inertia $\bar{I}$ of the mass of the beam and attached masses about the equilibrium position. Thus equation 8.9–9 may be written as

$$p^2 = \frac{\bar{M}}{\bar{I}} = \frac{\bar{W}\delta_c}{\frac{\bar{W}}{g}\bar{\delta}^2} = g\left(\frac{\delta_c}{\bar{\delta}^2}\right) = \frac{g}{\delta_{st}} \tag{8.9–10}$$

where $\bar{W}$ is the total weight of the system and δ_c is the displacement of the centroid of the system and $\bar{\delta}$ is the corresponding radius of gyration

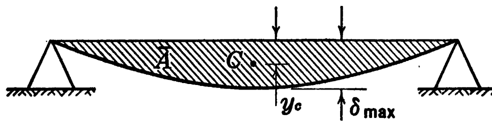


FIG. 8–19

of the system about the neutral position. See Fig. 8–18*b*.

For a uniform beam without concentrated loads, equation 8.9–9 may be written as

$$p^2 = \frac{\int_0^L \delta\, dx}{\int_0^L \delta^2\, dx} g = \frac{\int_0^L \delta\, dx}{\int_0^L \delta(\delta\, dx)} g = \frac{\bar{A}g}{2y_c\bar{A}} = \frac{g}{2y_c} \tag{8.9–11}$$

where y_c is the distance from the neutral axis to the centroid of the area $\bar{A}$ which is bounded by the equilibrium position and the position of maximum deflection of the beam as shown in Fig. 8–19. Equation 8.9–11 may be applied to the simply supported beam of previous treatment by assuming the form of the beam in the static position to be a parabola and determining the maximum deflection from the static position of the beam; hence,

$$y_c = \tfrac{2}{5}\delta_{\max} \quad \text{and} \quad \delta_{\max} = \frac{5}{384}\frac{wL^4}{EI}$$

From this there is obtained

$$p^2 = 96\frac{EIg}{wL^4} \tag{8.9–12}$$

Comparison with the exact solution (equation 8.7–4 for $n = 1$) shows this to give a value of p which is 0.7% too low.

A word of caution is in order in this connection, as equation 8.9–11 cannot be relied on to give values of the natural frequencies which are always too high if the static deflection curve is estimated as in the above example. If the correct static deflection curve is used, the frequency will always err on the high side, as the direct application of

Rayleigh's method may be shown to do in general. This deviation, when static deflection curves are estimated, does not impair the usefulness of equations 8.9–9, 8.9–10 and 8.9–11, and excellent results are obtained by the use of these expressions if reasonable assumptions are made concerning the configuration of the system. This latter statement must apply to the use of Rayleigh's method in any instance.

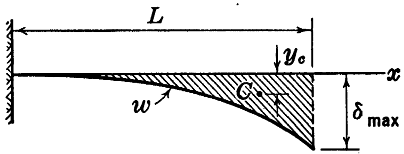


FIG. 8–20

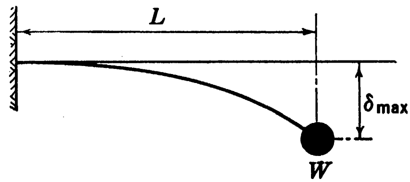


FIG. 8–21

The uniform cantilever beam, shown in Fig. 8–20, may be treated in this manner by assuming a parabolic form, from which the distance to the centroid of the enclosed area is found to be

$$y_c = \tfrac{3}{10}\delta_{\max}$$

The maximum static deflection in this case is

$$\delta_{\max} = \frac{wL^4}{8EI}$$

where w is the weight of the beam per unit length. Thus equation 8.9–11 gives, for the approximate natural circular frequency of the fundamental mode,

$$p = \sqrt{\frac{40}{3}\frac{EIg}{wL^4}} = 3.65\sqrt{\frac{EIg}{wL^4}} \tag{8.9–13}$$

The exact solution is

$$p = 3.55\sqrt{\frac{EIg}{wL^4}}$$

indicating an error of 3.9%.

The cantilever beam with a concentrated mass attached to the end may be treated similarly. Again assuming a parabola, and noting that the static deflection at the end (Fig. 8–21) is given by

$$\delta_{\max} = \frac{WL^3}{3EI} + \frac{wL^4}{8EI} = \frac{L^3}{24EI}(8W + 3W_0)$$

then

$$\bar{M} = (W + \tfrac{1}{4}W_0)\delta_{\max}$$

and

$$\bar{I} = \left(W + \tfrac{8}{5} \cdot \tfrac{1}{4} W_0\right) \frac{\delta^2{}_{\max}}{g}$$

whence, from equation 8.9–10,

$$\begin{aligned} p^2 &= \frac{3EIg}{L^3} \cdot \frac{8W + 3W_0}{8W^2 + \frac{21}{5} W W_0 + \frac{9}{20} W_0{}^2} \\ &= \frac{3EIg}{L^3} \cdot \frac{1}{W + \frac{11}{40} W_0 - \dfrac{W_0{}^2}{20(4W + W_0)}} \end{aligned} \qquad (8.9\text{–}14)$$

For $W = 0$, this reduces to equation 8.9–13, and, for $W_0 \rightarrow 0$, it shows that a good approximation may be obtained by adding $^{11}\!/_{40}$ of the weight of the beam to the concentrated weight at the end of the beam and treating the beam as a weightless spring.

Chapter 9

VIBRATIONS OF TRANSIENT CHARACTER

9.1. The Nature of Transient Motion

Transient motion may best be defined by contrast to steady-state motion. Steady-state motion is characterized by a periodic oscillation which does not vary with time either in form or in amplitude. Transient motion may then be defined as all motion other than steady-state oscillation. Transient motion may be oscillatory in nature, but it is not necessarily so, as is exemplified by the motion of overdamped systems.

From the engineering viewpoint, transient motion can be associated with a change in the external forces acting on the system. This change may be caused by the sudden application of an additional force or by the sudden variation in magnitude of an existing periodic force. Any such change in the external forces will induce transient motion into the system. The motion of a system that is acted on by a transient force is called the response of the system. In an undamped system steady-state motion may persist as either a forced vibration or a free vibration, whereas a damped system can only have a steady-state motion if it is forced. All free vibrations in damped systems are transient, and the amplitude of the motion dies away with time. Since free vibration with damping is the simplest type of transient motion, it will be discussed first.

9.2. Free Vibrations with Damping

A simple damped system of one degree of freedom is shown in Fig. 9–1. The equation of motion of this system is obtained by considering the forces acting on the mass in a displaced position. Application of Newton's law gives

$$m\ddot{x} = -c\dot{x} - kx \tag{9.2–1}$$

as the equation of motion. Introduction of the previously established notation,

$$\left.\begin{aligned} p^2 &= \frac{kg}{W} \\ q &= \frac{k}{c} \end{aligned}\right\} \tag{9.2–2}$$

where p is the natural circular frequency of the undamped system and

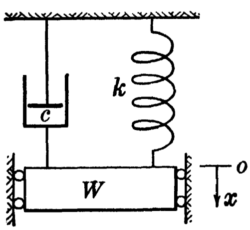


FIG. 9–1

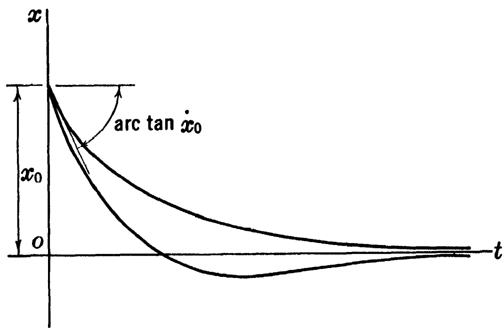


FIG. 9–2

q is the relaxation frequency, permits equation 9.2–1 to take the form

$$\ddot{x} + \frac{p^2}{q}\dot{x} + p^2 x = 0 \tag{9.2–3}$$

The solution of this equation can be obtained by any of the standard methods of elementary differential equations. Assume, for example, a solution of the form

$$x = Ae^{\lambda t}$$

Differentiation of this expression gives $\dot{x} = \lambda A e^{\lambda t}$ and $\ddot{x} = \lambda^2 A e^{\lambda t}$. Substitution of these values into equation 9.2–3 leads to

$$\lambda^2 + \frac{p^2}{q}\lambda + p^2 = 0$$

from which

$$\lambda = -\frac{p^2}{2q} \pm \sqrt{\left(\frac{p^2}{2q}\right)^2 - p^2} \tag{9.2–4}$$

There are three specific cases to be investigated:

Case

(I) $p = \dfrac{p^2}{2q}$; $p = 2q$; $\lambda = -p$ is a double root.

(II) $p > \dfrac{p^2}{2q}$; $p < 2q$; both roots are complex.

(III) $p < \dfrac{p^2}{2q}$; $p > 2q$; both roots are real.

Case (I) $p = 2q, \quad \lambda = -p,$ (critical damping)

In this instance, the solution has the form

$$x = Ae^{-pt} + Bte^{-pt} \tag{9.2–5}$$

This is a non-oscillatory motion which dies away with time. The constants A and B can be determined from the initial displacement x_0 and the initial velocity $\dot{x}_0$. Thus,

$$A = x_0 \quad \text{and} \quad B = x_0 p + \dot{x}_0$$

so

$$x = [x_0(1 + pt) + \dot{x}_0 t]e^{-pt} \tag{9.2–6}$$

In general the motion does not pass through the neutral position; however, for $\dot{x}_0$ sufficiently large and opposite in direction to x_0, the mass may be made to pass through the neutral position once, as shown in Fig. 9–2.

The value of the damping which makes $p = 2q$ is called critical damping, that is, by substitution,

$$\sqrt{\frac{k}{m}} = 2\frac{k}{c_{cr}} \tag{9.2–7}$$

or

$$c_{cr} = 2\sqrt{km}$$

Case (II) $p < 2q$ (underdamping)

For this condition, the values of λ are complex, and equation 9.2–4 may be written as

$$\lambda = \frac{p^2}{2q} \pm jp_1$$

where

$$j^2 = -1 \quad \text{and} \quad p_1{}^2 = p^2 - \left(\frac{p^2}{2q}\right)^2 \tag{9.2–8}$$

The solution can now be written as

$$x = e^{-(p^2/2q)t}(A'e^{jp_1t} + B'e^{-jp_1t})$$
$$= e^{-(p^2/2q)t}(A \cos p_1t + B \sin p_1t) \quad (9.2\text{-}9)$$

where A', B', A, and B are arbitrary constants to be determined from

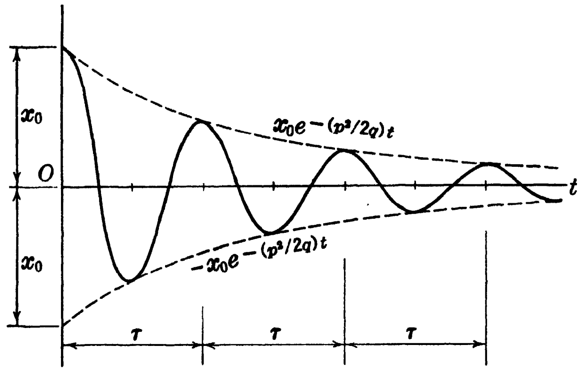


FIG. 9–3

the initial conditions. For an initial displacement x_0 and an initial velocity $\dot{x}_0$,

$$A = x_0 \quad \text{and} \quad B = \frac{\dot{x}_0 + \dfrac{p^2}{2q}x_0}{p_1}$$

hence equation 9.2–9 becomes

$$x = e^{-(p^2/2q)t}\left(x_0 \cos p_1t + \frac{\dot{x}_0 + \dfrac{p^2}{2q}x_0}{p_1} \sin p_1t\right) \quad (9.2\text{-}10)$$

It is seen that this is an oscillatory motion (see Fig. 9–3) of amplitude

$$C = e^{-(p^2/2q)t}\sqrt{x_0^2 + \left(\frac{\dot{x}_0 + \dfrac{p^2}{2q}x_0}{p_1}\right)^2} \quad (9.2\text{-}11)$$

The amplitude is seen to decrease with time.

Equation 9.2–10 may also be written

$$x \doteq e^{-(p^2/2q)t}\sqrt{x_0^2 + \frac{1}{p_1^2}\left(\dot{x}_0 + \frac{p^2}{2q}x_0\right)^2} \cos(p_1t - \alpha) \quad (9.2\text{-}12)$$

where

$$\tan \alpha = \frac{\dot{x}_0 + \dfrac{p^2}{2q} x_0}{p_1 x_0}$$

The circular frequency of this motion is p_1, which from equation 9.2–8 is seen to be less than that of the undamped system. The motion expressed by equation 9.2–12 may be represented by a rotating vector

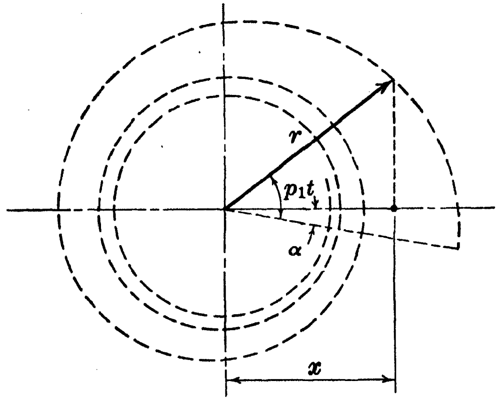


FIG. 9–4

of constantly decreasing length (see Fig. 9–4). The terminal of the vector travels along a logarithmic spiral whose polar equation is

$$r = \sqrt{x_0^2 + \frac{1}{p_1^2}\left(\dot{x}_0 + \frac{p^2}{2q} x_0\right)^2}\; e^{-(p^2/2qp_1)(\theta-\alpha)} \qquad (9.2\text{–}13)$$

The ratio of the length of the vector at the beginning and end of any complete cycle is

$$\frac{r_{t+\tau}}{r_t} = e^{-(\pi p^2/qp_1)}$$

The logarithmic decrement $\beta = \dfrac{\pi p^2}{qp_1}$ is a measure of the rate of decay of the oscillation. From equations 9.2–7 and 9.2–8,

$$\beta = \frac{\pi p^2}{qp_1} = \frac{2\pi}{\sqrt{\left(\dfrac{2q}{p}\right)^2 - 1}} = \frac{2\pi}{\sqrt{\left(\dfrac{c_{cr}}{c}\right)^2 - 1}}$$

The maximum value of the circular frequency p_1 that can be obtained with any given damper and spring is of significance in giving a physical meaning to the relaxation frequency q. To determine this maximum, $\partial p_1/\partial m$ may be set equal to zero. Thus, from equation 9.2–8,

$$p_1^2 = \frac{k}{m} - \left(\frac{k}{2qm}\right)^2$$

so

$$\frac{\partial p^2}{\partial m} = -\frac{k}{m^2} + 2\left(\frac{k}{2q}\right)^2 \frac{1}{m^3} = 0$$

from which

$$2q^2 = \frac{k}{m} \quad \text{or} \quad m = \frac{k}{2q^2} = \frac{c^2}{2k} \tag{9.2–14}$$

hence, from equation 9.2–8,

$$p_{1\max} = \frac{k}{c} = q \tag{9.2–15}$$

that is, the relaxation frequency q is the maximum possible frequency obtainable in a system with a given spring and damper.

The value of m to give critical damping is obtained from the relation

$$q = \tfrac{1}{2}p$$

whence

$$m_{cr} = \frac{c^2}{4k} \tag{9.2–16}$$

A comparison of equations 9.2–14 and 9.2–16 shows that the mass producing the highest frequency is twice the mass producing critical damping. Thus a convenient parameter by which the ratio p_1/q may be compared is

$$\left(\frac{2q}{p}\right)^2 = \frac{4km}{c^2} = \left(\frac{c_{cr}}{c}\right)^2 = \left(\frac{m}{m_{cr}}\right)$$

By making use of these parameters equation 9.2–8 may be written as

$$p_1^2 = p^2\left[1 - \left(\frac{p}{2q}\right)^2\right] = p^2\left[1 - \left(\frac{c}{c_{cr}}\right)^2\right] = p^2\left(1 - \frac{m_{cr}}{m}\right) \tag{9.2–17}$$

and

$$\frac{p_1}{q} = 2\frac{m_{cr}}{m}\sqrt{\frac{m}{m_{cr}} - 1}$$

The variation of p_1/q with $(2q/p)^2 = m/m_{cr}$ is shown in Fig. 9–5. The maximum frequency occurs at $m/m_{cr} = 2$ and critical damping corresponds to $(2q/p)^2 = m/m_{cr} = 1$. The value of the mass m_{cr} is extremely small in most practical problems; in fact the equivalent mass of the spring usually exceeds the critical value in practical applications. In certain instances however, where the spring is soft and the damping relatively large, it is possible to approach the value of the critical mass m_{cr}.

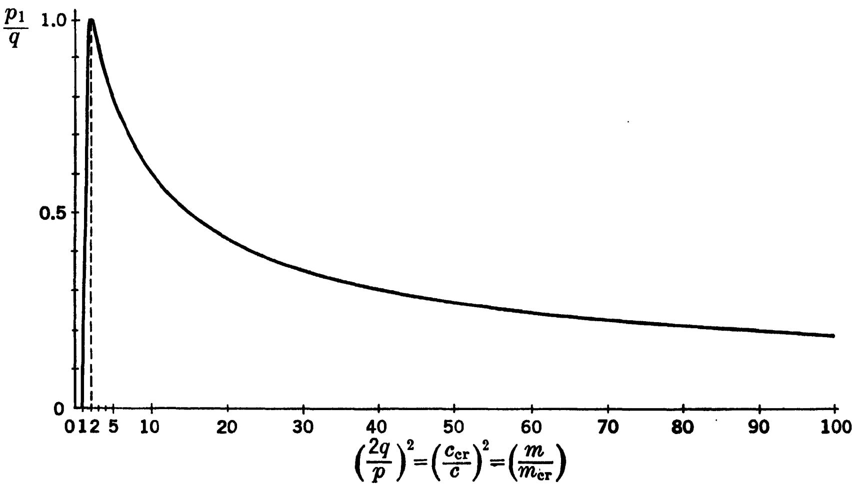


FIG. 9–5

Case III

$$p > 2q \qquad \text{(overdamping)}$$

In this instance the roots of equation 9.2–4 are real, and the solution may be written as

$$x = e^{-(p^2/2q)t}(A'e^{p_2t} + B'e^{-p_2t})$$

$$= e^{-(p^2/2q)t}(A \cosh p_2t + B \sinh p_2t) \qquad (9.2\text{–}18)$$

where

$$p_2^2 = \left(\frac{p^2}{2q}\right)^2 - p^2 = p^2\left[\left(\frac{p}{2q}\right)^2 - 1\right] \qquad (9.2\text{–}19)$$

This is a non-oscillatory motion which can cross the neutral axis but once for any values of the initial conditions. This so-called subsidence motion is of infrequent engineering interest and will not be treated further.

9.3. Response to a Step Function

One of the simple force functions which is transient in nature is the so-called step function. A plot of a step function is shown in Fig. 9–6*a*. The function is identically zero at any time $t < t_1$, and equal to F_0 (a constant force) at any time $t > t_1$. A convenient notation for this step function is $F_{t_1}(t)$. When $F_0 = 1$, the function is said to be a unit step function.

The motion of the system of Fig. 9–1, when acted on by a step function $F_{t_1}(t)$, is easily obtained. The system will be assumed at rest in

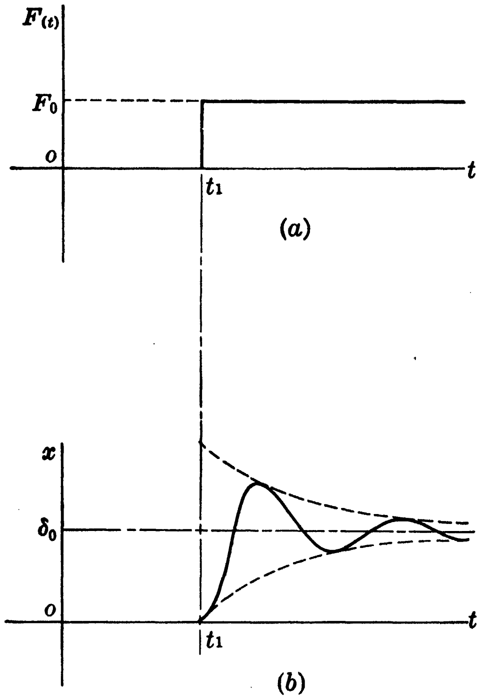


Fig. 9–6

the equilibrium position for $t < t_1$. The equation of motion for $t > t_1$ is

$$\ddot{x} + \frac{p^2}{q}\dot{x} + p^2 x = \frac{F_0}{m} = p^2 \frac{F_0}{k} = p^2 \delta_0 \qquad (9.3\text{–}1)$$

where δ_0 is the static displacement due to a static load F_0. The solution of equation 9.3–1 is composed of the complimentary solution plus the particular solution. The complimentary solution for the underdamped case, $2q/p > 1$, is given by equation 9.2–9 as

$$x_c = e^{-(p^2/2q)t}(A \cos p_1 t + B \sin p_1 t) \qquad (9.3\text{–}2)$$

where

$$p_1^2 = p^2\left[1 - \left(\frac{p}{2q}\right)^2\right]$$

A particular solution may be obtained by assuming it to be a constant, $x_p = c$, and substituting it into equation 9.3–1, whence

$$x_p = \delta_0 \tag{9.3–3}$$

The complete solution is therefore

$$x = e^{-(p^2/2q)t}(A \cos p_1 t + B \sin p_1 t) + \delta_0 \tag{9.3–4}$$

The constants A and B can be determined from the initial conditions,

$$\left.\begin{aligned} x &= 0 \\ \dot{x} &= 0 \end{aligned}\right\} \quad \text{when} \quad t = t_1$$

Substituting these conditions into equation 9.3–4 and solving for A and B, one gets

$$A = -\delta_0 e^{-(p^2/2q)t_1}\left(\cos p_1 t_1 - \frac{p^2}{2qp_1} \sin p_1 t_1\right)$$

$$B = -\delta_0 e^{-(p^2/2q)t_1}\left(\sin p_1 t_1 + \frac{p^2}{2qp_1} \cos p_1 t_1\right)$$

Equation 9.3–4 now becomes

$$x = \delta_0\left[1 - e^{-(p^2/2q)(t-t_1)}\left(\cos p_1(t - t_1) + \frac{p^2}{2qp_1} \sin p_1(t - t_1)\right)\right]$$

To stipulate the time relation, the equation is conventionally written

$$x = \delta_0 1_{t1}(t)\left[1 - e^{-(p^2/2q)(t-t_1)}\left(\cos p_1(t - t_1) + \frac{p^2}{2qp_1} \sin p_1(t - t_1)\right)\right] \tag{9.3–5}$$

where

$$\delta_0 1_{t1}(t) = \frac{F_0 1_{t1}(t)}{k} = \frac{F_{t1}(t)}{k}$$

and

$$1_{t1}(t) = \begin{cases} 0 & \text{when} \quad t < t_1 \\ 1 & \text{when} \quad t > t_1 \end{cases}$$

If the step function is translated to the origin, $t_1 = 0$, equation 9.3–5 may be written as

$$x = \delta_0 1_0(t)\left[1 - e^{-(p^2/2q)t}\left(\cos p_1 t + \frac{p^2}{2qp_1} \sin p_1 t\right)\right] \tag{9.3–6}$$

and, for the particular case of no damping $c = 0$ or $q = \infty$, this reduces further to

$$x = \delta_0 1_0(t)(1 - \cos pt) \tag{9.3–7}$$

A graph of equation 9.3–5 is shown in Fig. 9–6*b*. The net effect of applying a step force is to shift the position of equilibrium. The resulting free vibration oscillates about the new equilibrium position with steadily decreasing amplitude.

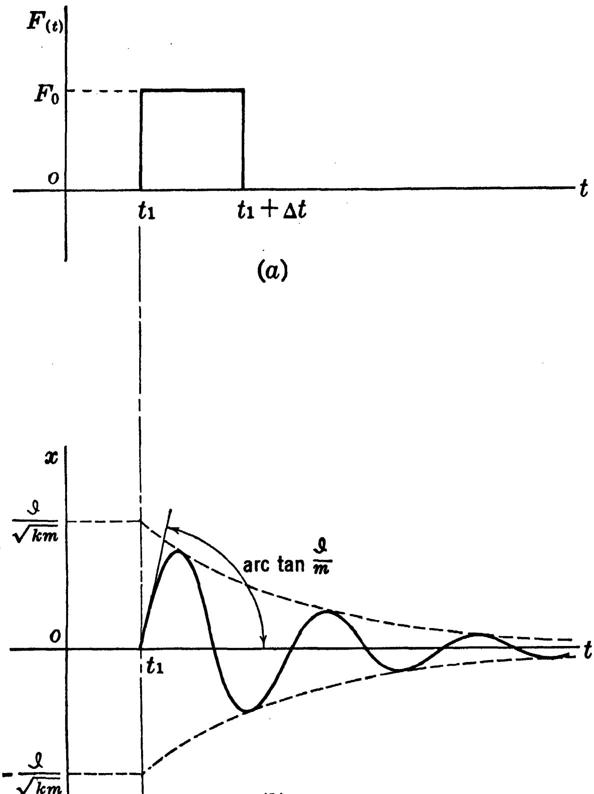


FIG. 9–7

When the force function is a unit step, that is, $F_0 = 1$, the response (equation 9.3–5, 9.3–6, or 9.3–7), is called the indicial admittance, a term that has been borrowed from electrical engineering. The indicial admittance is frequently denoted by $\mathcal{A}(t)$; thus equation 9.3–5 becomes, for $F_0 = 1$ and $\delta_0 = 1/k$,

$$x = A_{t1}(t) =$$
$$\frac{1_{t1}(t)}{k}\left[1 - e^{-(p^2/2q)(t-t_1)}\left(\cos p_1(t-t_1) + \frac{p^2}{2qp_1}\sin p_1(t-t_1)\right)\right] \quad (9.3\text{–}8)$$

9.4. Response to an Impulse

When a force is applied over a finite interval of time and then removed, an impulse is said to have been delivered to the system. A simple form of an impulse is shown in Fig. 9–7*a*, where a constant force F_0 is assumed to act over an interval of time Δt. The magnitude of the impulse is given by the area under the curve in Fig. 9–7*a*.

$$\mathcal{I} = F_0(\Delta t) \tag{9.4–1}$$

The response to an impulse of this nature may be obtained by superimposing the motion of the system due to two step functions, $F_{t1}(t)$ and $-F_{t_1+\Delta t}(t)$. The system is assumed to be that of Fig. 9–1. The system is motionless for $t < t_1$ while during the interval $t_1 < t < t_1 + \Delta t$ the solution is given by equation 9.3–5. For $t > t_1 + \Delta t$, the solution becomes

$$x = \delta_0 1_{t1}(t) \left\{1 - e^{-(p^2/2q)(t-t_1)} \left[\cos p_1(t - t_1) + \frac{p^2}{2qp_1} \sin p_1(t - t_1)\right]\right\}$$
$$- \delta_0 1_{t_1+\Delta t}(t) \left\{1 - e^{-(p^2/2q)(t-t_1-\Delta t)} \left[\cos p_1(t - t_1 - \Delta t) + \frac{p^2}{2qp_1} \sin p_1(t - t_1 - \Delta t)\right]\right\} \tag{9.4–2}$$

An important application is found in the fundamental problem of impact where Δt is permitted to approach zero while F_0 is increased such that

$$\mathcal{I} = F_0\,\Delta t = \text{constant}$$

This is the case of an instantaneous impulse of magnitude $\mathcal{I}$.

To obtain the limit as $\Delta t \to 0$, it is convenient to write equation 9.4–2 for $t > t_1 + \Delta t$ in the form

$$x = \delta_0 e^{-(p^2/2q)(t-t_1)} \times \left\{\left[e^{(p^2/2q)\Delta t}\left(\cos p_1\,\Delta t - \frac{p^2}{2qp_1}\sin p_1\,\Delta t\right) - 1\right]\cos p_1\,(t - t_1) + \left[e^{(p^2/2q)\Delta t}\left(\sin p_1\,\Delta t + \frac{p^2}{2qp_1}\cos p_1\,\Delta t\right) - \frac{p^2}{2qp_1}\right]\sin p_1\,(t - t_1)\right\}$$

Noting that $\delta_0 = F_0/k$ and $\mathcal{I} = F_0\,\Delta t$ permits the above to be written, upon expansion of the quantities in the brackets, as

$$x = \delta_0 e^{-(p^2/2q)(t-t_1)} \left\{ \left[\left(1 + \frac{p^2}{2q}\Delta t + \frac{1}{2}\left(\frac{p^2}{2q}\right)^2 (\Delta t)^2 + \cdots\right) \times \right.\right.$$

$$\left.\left(1 - \frac{p^2}{2q}\Delta t - \tfrac{1}{2}p_1^2(\Delta t)^2 - \cdots\right) - 1\right] \cos p_1(t - t_1)$$

$$+ \left[\left(1 + \frac{p^2}{2q}\Delta t + \frac{1}{2}\left(\frac{p^2}{2q}\right)^2 (\Delta t)^2 + \cdots\right) \times\right.$$

$$\left.\left.\left(\frac{p^2}{2qp_1} + p_1(\Delta t) - \frac{p^2 p_1}{4q}(\Delta t)^2 + \cdots\right) - \frac{p^2}{2qp_1}\right] \sin p_1(t - t_1)\right\}$$

$$= e^{-(p^2/2q)(t-t_1)} \left\{ \left[\left(-\tfrac{1}{2}p_1^2 - \frac{p^4}{8q^2}\right)\frac{(\Delta t)\mathcal{J}}{k} + \cdots\right] \cos p_1(t - t_1)\right.$$

$$+ \left[\left(p_1 + \frac{p^4}{4q^2p_1}\right)\frac{\mathcal{J}}{k} + \frac{p^2}{4q}\left(p_1 + \frac{p^4}{4q^2p_1}\right)\frac{(\Delta t)\mathcal{J}}{k} + \cdots\right] \times$$

$$\left. \sin p_1(t - t_1)\right\}$$

By letting $\Delta t \rightarrow 0$, the response to an instantaneous impulse is obtained at the limit as

$$x = \frac{\mathcal{J}}{k}\left(p_1 + \frac{p^4}{4q^2p_1}\right) e^{-(p^2/2q)(t-t_1)} \sin p_1(t - t_1) \qquad (9.4\text{–}3)$$

This response is shown graphically in Fig. 9–7*b*. For the case of no damping, $q = \infty$, $p_1 = p = \sqrt{k/m}$, this is reduced to

$$x = \frac{\mathcal{J}}{\sqrt{km}} \sin p(t - t_1) \qquad (9.4\text{–}4)$$

The application of an instantaneous impulse is seen to start a free vibration with zero initial displacement and a velocity at $t = t_1$, given by

$$\dot{x}_0 = \frac{\mathcal{J}}{k}\left(p_1 + \frac{p^4}{4q^2p_1}\right) p_1 = \frac{\mathcal{J}p^2}{k}$$

This may be rewritten as

$$\dot{x}_0 = \frac{\mathcal{J}}{m}$$

The response to a unit impulse may be denoted by $B(t)$; hence for

$\mathcal{g} = 1$, equation 9.4–3 becomes

$$x = B_{t1}(t) = \frac{1}{k}\left(p_1 + \frac{p^4}{4q^2p_1}\right) e^{-(p^2/2q)(t-t_1)} \sin p_1(t - t_1)$$

$$= \frac{1}{k}\frac{p^2}{p_1} e^{-(p^2/2q)(t-t_1)} \sin p_1(t - t_1) \quad \text{for} \quad t > t_1$$

where equation 9.2–8 has been used to simplify this expression. Then, for all values of t,

$$B_{t1}(t) = \frac{1_{t1}(t)}{mp_1} e^{-(p^2/2q)(t-t_1)} \sin p_1(t - t_1) \tag{9.4–5}$$

It should be noted that the time derivative of the indicial admittance equals the response to a unit impulse,

$$\dot{A}_{t1}(t) = B_{t1}(t) \tag{9.4–6}$$

This may be verified by direct differentiation of equation 9.3–8 and comparison of the result with equation 9.4–5.

9.5. Response to a General Force Function

A force which is an arbitrary function of time can be treated using the results of the previous sections. Such a force function is shown in Fig. 9–8a. This force may be replaced by a series of step functions of appropriate magnitude. The total response to $F(t)$ can be approximated by summarizing the responses to the series of step functions. Thus, at a time $t > \tau$ the response due to a small step force function $\Delta F 1_\tau(t)$, obtained from equation 9.3–8, is

$$\Delta x(t) = \Delta F A_\tau(t)$$

and the total response to the series of step functions is given by

$$x(t) = F(t_1)A_{t1}(t) + \sum_{\tau=t_1}^{t} \Delta F A_\tau(t) \tag{9.5–1}$$

This may be written as

$$x(t) = F(t_1)A_{t1}(t) + \sum_{\tau=t_1}^{t} \frac{\Delta F}{\Delta \tau} A_\tau(t)\, \Delta\tau$$

By proceeding to the limit, $\Delta\tau \to 0$, this becomes

$$x(t) = F(t_1)A_{t1}(t) + \int_{t_1}^{t} \frac{dF}{d\tau} A_\tau(t)\, d\tau \tag{9.5–2}$$

This is known as Duhamel's integral formula, and calculation of this integral permits the evaluation of the response at any time (t). Another form of this expression is obtained by integrating the integral in equation 9.5–2 by parts which gives

$$x(t) = F(t)A_t(t) + \int_{t_1}^{t} F(\tau)\dot{A}_\tau(t)\,d\tau$$

However, since $A_t(t) = 0$ and $\dot{A}_\tau(t) = B_\tau(t)$, (see equation 9.4–6),

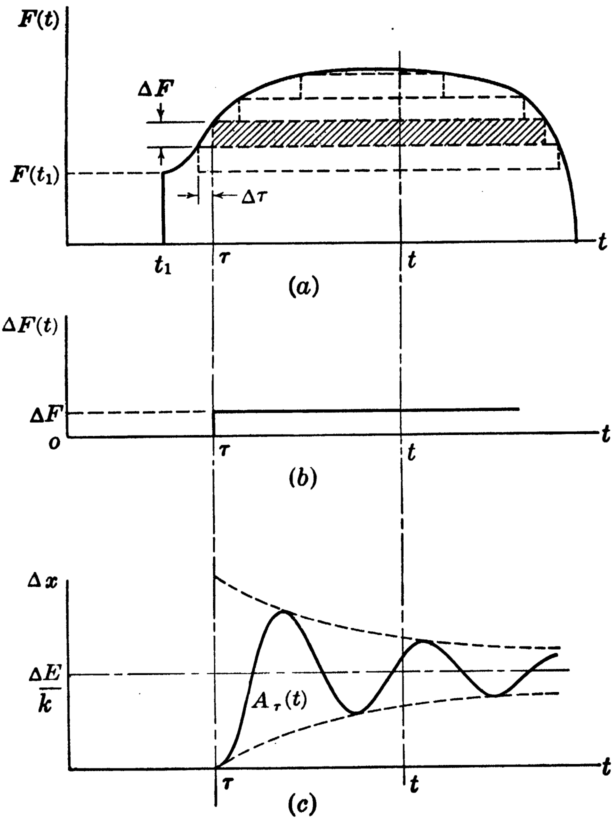


FIG. 9–8

this may be written as

$$x(t) = \int_{t_1}^{t} F(\tau)B_\tau(t)\,d\tau \tag{9.5–3}$$

This integral may be interpreted as a summation of the response to the infinitesimal impulses $F(\tau)\,d\tau$ which occur between $t = t_1$ and $t = t$. Frequently equation 9.5–3 is somewhat easier to apply than equation 9.5–2, owing to its simpler form.

As an example, the force function shown in Fig. 9–9a may be considered to act on the system of Fig. 9–1, for the case of zero damping. The force function $F(\tau)$ may be expressed as

$$F(\tau) = 2F_0\left(\frac{\tau}{T}\right) \qquad 0 < \tau < \tfrac{1}{2}T$$

$$= 2F_0\left(1 - \frac{\tau}{T}\right) \qquad \tfrac{1}{2}T < \tau < T$$

The response to a unit impulse for the undamped system can be

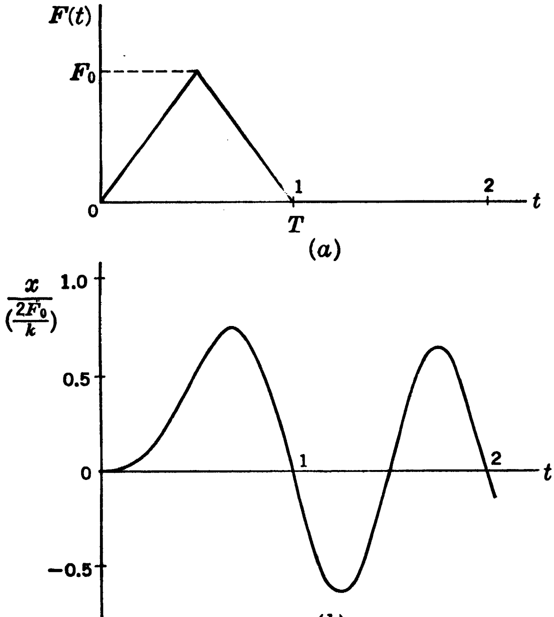


FIG. 9–9

obtained from equation 9.4–5, ($q = \infty$, $p_1 = p$),

$$B_\tau(t) = \frac{p}{k}\sin p(t - \tau) \qquad (t > \tau)$$

Application of Duhamel's integral formula, as given by equation 9.5–3, gives the following results:

For $0 < t < \tfrac{1}{2}T$:

$$x(t) = \frac{2F_0 p}{kT}\int_0^t \tau \sin p(t - \tau)\,d\tau = \frac{2F_0}{k}\left[\frac{t}{T} - \frac{\sin pt}{pT}\right]$$

For $\frac{1}{2}T < t < T$:

$$x(t) = \frac{2F_0 p}{kT} \int_0^{\frac{1}{2}T} \tau \sin p(t-\tau)\, d\tau + \frac{2F_0 p}{k} \int_{\frac{1}{2}T}^{t} \left(1 - \frac{\tau}{T}\right) \sin p(t-\tau)\, d\tau$$

$$= \frac{2F_0}{k}\left[1 - \frac{t}{T} + \frac{2 \sin p(t - \frac{1}{2}T) - \sin pt}{pT}\right]$$

For $t > T$:

$$x(t) = \frac{2F_0 p}{kT} \int_0^{\frac{1}{2}T} \tau \sin p(t-\tau)\, d\tau + \frac{2F_0 p}{k} \int_{\frac{1}{2}T}^{T} \left(1 - \frac{\tau}{T}\right) \sin p(t-\tau)\, d\tau$$

$$= \frac{2F_0}{kpT}\,[2 \sin p(t - \tfrac{1}{2}T) - \sin pt - \sin p(t - T)]$$

$$= \frac{4P_0}{k}\left(\frac{1 - \cos \frac{1}{2}pT}{pT}\right) \sin p(t - \tfrac{1}{2}T)$$

The response for the particular example $T = 1$ and $p = 2\pi$ is shown

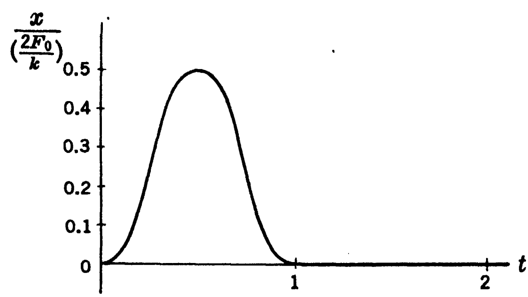


FIG. 9-10

in Fig. 9–9*b*. It is of some interest to note that, for certain values of T, the system is at rest for $t > T$. Such values of T are

$$T = \frac{4n\pi}{p}$$

A graph of the response for $T = 1$, $p = 4\pi$ is shown in Fig. 9–10, in which case the response is one cycle of a sine curve. An investigation of the work done on the system for these particular values of T shows that no net work is done on the system, and hence its kinetic energy at the end of the application of the force function has not been increased above its initial level.

As a second example, the force function shown in Fig. 9–11*a* will be assumed to act on the system of Fig. 9–1. This force function is not unlike that which results from certain types of explosions. The

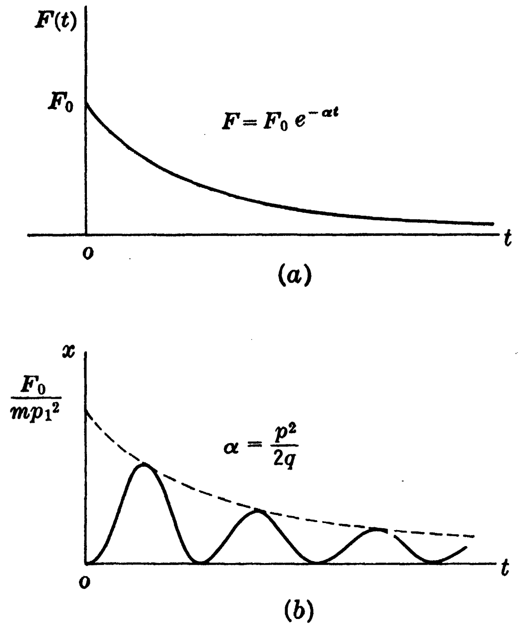


FIG. 9–11

force function can be represented to a good approximation by

$$F(\tau) = F_0 e^{-\alpha\tau} \qquad \tau > 0$$

The response to a unit impulse is given by equation 9.4–5,

$$B_T(t) = \frac{1}{mp_1} e^{-(p^2/2q)(t-\tau)} \sin p_1(t-\tau) \qquad t > \tau$$

Use of Duhamel's integral equation gives

$$x = \frac{F_0}{mp_1} e^{-(p^2/2q)t} \int_0^t e^{[(p^2/2q)-\alpha]\tau} \sin p_1(t-\tau)\, d\tau$$

$$= \frac{F_0}{m\left(p^2 - \dfrac{p^2}{q}\alpha + \alpha^2\right)} \left\{ e^{-\alpha t} - e^{-(p^2/2q)t} \left[\frac{1}{p_1}\left(\frac{p^2}{2q} - \alpha\right) \sin p_1 t + \cos p_1 t \right] \right\}$$

As a particular simple example, let $\alpha = p^2/2q$ whence the above reduces to

$$x = \frac{F_0}{mp_1^2} e^{-(p^2/2q)t}(1 - \cos p_1 t)$$

The response given by this last expression is plotted in Fig. 9–11*b*. Other special cases may be treated by using appropriate values of α. For $\alpha = 0$, the response to a step function is obtained as given by equation 9.3–5.

Chapter 10

VIBRATIONS OF NON-LINEAR CHARACTER

10.1. The Nature of Non-linear Systems

A system whose equations of motion involve the displacement or velocity to powers other than the first is referred to as a non-linear system, and the resulting differential equations are said to be non-linear.

The mathematical solution of non-linear problems is usually extremely difficult. The recourse in many non-linear problems is to approximate methods; however even these are frequently long and tedious. It is therefore most fortunate that a majority of the problems of engineering interest can be approximated by linear systems.

It is beyond the scope of this book to attempt an extensive discussion of the methods of treating non-linear problems; in fact many problems exist for which no exact solution is known, and the approximate methods yield results of doubtful value. There are certain problems, however, that cannot be represented to a sufficient accuracy by an idealized linear system, but for which the solution to the non-linear problem may readily be obtained. Still other problems may be treated by approximation methods with excellent accuracy. Some of these problems which involve free vibration and are of engineering interest will be discussed in this chapter.

10.2. Undamped Free Vibrations of Systems with a Non-linear Restoring Force

There are many practical problems that have a non-linear restoring force. These forces may result from the characteristics of the springs, or the general geometry of the motion. In general, the equation of motion for systems of a single degree of freedom will have the form,

$$m\ddot{x} + kf(x) = 0 \quad \text{or} \quad \ddot{x} + p^2 f(x) = 0 \qquad (10.2\text{–}1)$$

where $kf(x)$ represents the restoring force as a function of displacement x. The first integral of this equation may be obtained by multiplying through by $\dot{x}$ and integrating. Thus,

$$m\left(\frac{dx}{dt}\right)\frac{d}{dt}\left(\frac{dx}{dt}\right) + kf(x)\frac{dx}{dt} = 0$$

or

$$\tfrac{1}{2}md\left(\frac{dx}{dt}\right)^2 + kf(x)\,dx = 0$$

Integration gives

$$\tfrac{1}{2}m\left(\frac{dx}{dt}\right)^2 + k\int f(x)\,dx = \text{constant} \qquad (10.2\text{–}2)$$

The first term represents the kinetic energy of the system and the second term the potential energy stored in the system. Equation 10.–22 is simply a statement of the law of conservation of energy, and it may be written as

$$\left(\frac{dx}{dt}\right)^2 = -2p^2\int_{x_0}^{x} f(x)\,dx = 2p^2\int_{x}^{x_0} f(x)\,dx \qquad (10.2\text{–}3)$$

where x_0 is the amplitude of the motion and $p^2 = k/m$.

The second integration may be performed by first taking the square root of both sides, whence

$$\frac{dx}{dt} = p\sqrt{2\int_{x}^{x_0} f(x)\,dx}$$

This can be written as

$$dt = \frac{dx}{p\sqrt{2\int_{x}^{x_0} f(x)\,dx}}$$

whence

$$t = \frac{1}{p}\int\frac{dx}{\sqrt{2\int_{x}^{x_0} f(x)\,dx}} + \text{constant} \qquad (10.2\text{–}4)$$

The solution of equation 10.2–4, obtained by performing the required integrations, yields the displacement x as a function of time t.

As a simple example, the familiar linear case $f(x) = x$ will be considered for the initial conditions,

$$x_{t=0} = x_0 \quad \text{and} \quad \dot{x}_{t=0} = 0$$

It should be noted that equation 10.2–1 reduces to

$$\ddot{x} + p^2x = 0$$

in this instance. From equation 10.2–4,

$$pt = \int_{x_0}^{x} \frac{dx}{\sqrt{x_0^2 - x^2}} = \int_{x_0}^{x} \frac{d\left(\frac{x}{x_0}\right)}{\sqrt{1 - \left(\frac{x}{x_0}\right)^2}}$$

whence

$$pt = \cos^{-1}\left(\frac{x}{x_0}\right)$$

or

$$x = x_0 \cos pt$$

which checks with the previous results for linear systems.

As a second example, let

$$f(x) = x^{2n-1} \tag{10.2–5}$$

for which

$$pt = n^{1/2} \int_{x_0}^{x} \frac{dx}{\sqrt{x_0^{2n} - x^{2n}}}$$

This integral cannot be evaluated in general, but the period may be obtained as the time required to execute four quarter cycles. Thus

$$\tau = \frac{4n^{1/2}}{p x_0^{n-1}} \int_{x_0}^{0} \frac{d\left(\frac{x}{x_0}\right)}{\sqrt{1 - \left(\frac{x}{x_0}\right)^{2n}}} = \frac{4n^{1/2}}{p x_0^{n-1}} \int_{1}^{0} \frac{du}{\sqrt{1 - u^{2n}}} \tag{10.2–6}$$

This equation shows that the period τ will be a function of the amplitude x_0 for all values of n except $n = 1$ which corresponds to simple harmonic motion.

10.3. The Motion of a Pendulum for Large Amplitudes

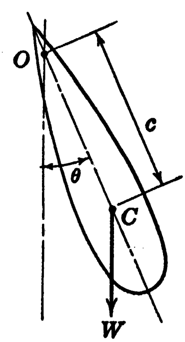


FIG. 10–1

The motion of a pendulum (Fig. 10–1) for small amplitudes is treated in section 2.4. The equation of motion is established as

$$I_0 \ddot{\theta} = -mgc \sin \theta \tag{10.3–1}$$

where I_0 is the moment of inertia of the pendulum mass about the pivot point O. Introduction of the natural circular frequency p for small oscillations (see section 2.4) where

$$p^2 = \frac{mgc}{I_0} = \frac{cg}{r_0^2} \tag{10.3–2}$$

permits equation 10.3–1 to assume the form

$$\ddot{\theta} + p^2 \sin \theta = 0 \tag{10.3–3}$$

Comparison of this equation with equation 10.2–1 shows that

$$f(\theta) = \sin \theta$$

Then from equation 10.2–4, the solution is seen to be

$$pt = \int_0^\theta \frac{d\theta}{\sqrt{2 \int_\theta^{\theta_0} f(\theta)\, d\theta}}$$

where

$$\int^{\theta_0} f(\theta)\, d\theta = \int_\theta^{\theta_0} \sin \theta\, d\theta = \cos \theta - \cos \theta_0$$

Thus

$$pt = \int_0^\theta \frac{d\theta}{\sqrt{2(\cos \theta - \cos \theta_0}} \tag{10.3–4}$$

for the initial conditions $\theta = 0$, which is convenient in this example. The integral in equation 10.3–4 is known as an elliptic integral of the first kind. It is convenient to introduce a new variable of integration ϕ defined by

$$\sin \phi = \frac{\sin \frac{1}{2}\theta}{\sin \frac{1}{2}\theta_0} \tag{10.3–5}$$

Then:

$$\sin \tfrac{1}{2}\theta = \sin \tfrac{1}{2}\theta_0 \sin \phi, \qquad \cos \tfrac{1}{2}\theta = \sqrt{1 - \sin^2 \tfrac{1}{2}\theta_0 \sin^2 \phi}$$

$$\cos \theta = 1 - 2 \sin^2 \tfrac{1}{2}\theta = 1 - 2 \sin^2 \tfrac{1}{2}\theta_0 \sin^2 \phi$$

$$d\theta = \frac{2 \sin \frac{1}{2}\theta_0 \cos \phi}{\cos \frac{1}{2}\theta}\, d\phi$$

and

$$\cos \theta_0 = 1 - 2 \sin^2 \tfrac{1}{2}\theta_0$$

Substituting from these equations into equation 10.3–4 yields

$$pt = \int_0^\phi \frac{2 \sin \frac{1}{2}\theta_0 \cos \phi\, d\phi}{\cos \frac{1}{2}\theta \sqrt{4 \sin^2 \frac{1}{2}\theta_0 (1 - \sin^2 \phi)}}$$
$$= \int_0^\phi \frac{d\phi}{\cos \frac{1}{2}\theta}$$

or

$$pt = \int_0^\phi \frac{d\phi}{\sqrt{1 - \sin^2 \frac{1}{2}\theta_0 \sin^2 \phi}} = F(k, \phi) \tag{10.3–6}$$

where

$$k^2 = \sin^2 \tfrac{1}{2}\theta_0$$

Equation 10.3–6 presents the standard form of the elliptic integral of the first kind, which is denoted by the symbol $F(k, \phi)$. The cycle is given by

$$\left.\begin{aligned} p\tau &= 4\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = 4K \\ \text{where} \quad & \\ K &= F\left(k, \frac{\pi}{2}\right) \end{aligned}\right\} \qquad (10.3\text{–}7)$$

and τ is the period. Values of K, which is known as the complete elliptic integral of the first kind can be obtained from tables for various values of k. For small values of θ_0, the motion approaches simple harmonic motion. For such small oscillations, the following relations are approximately true:

$$\sin \frac{\theta}{2} = \frac{\theta}{2}, \qquad \sin \frac{\theta_0}{2} = \frac{\theta_0}{2}, \qquad K = \frac{\pi}{2}$$

whence

$$\tau = \frac{2\pi}{p}$$

as previously obtained.

The nature of the elliptic integral of equation 10.3–6 is that of an inverse trigonometric function, as obtained for the linear example in section 10.2. It is convenient to redefine ϕ as a function of t by

$$\phi \equiv \text{amplitude of } pt \equiv \text{am } (pt) \qquad (10.3\text{–}8)$$

The sine of ϕ is denoted as

$$\sin \phi \equiv \sin [\text{am}(pt)] = \text{sn } (pt) \qquad (10.3\text{–}9)$$

where sn is called the elliptic sine. The displacement θ may now be obtained from equation 10.3–5 as

$$\sin \tfrac{1}{2}\theta = \sin \tfrac{1}{2}\theta_0 \text{ sn } (pt) \qquad (10.3\text{–}10)$$

The form of the elliptic sine is shown in Fig. 10–6.

It is important to compare the exact solution for the motion of a pendulum with the linear solution in order to gain an appreciation of the accuracy of the linear solution. The ratio of the period, when computed by the exact expression, equation 10.3–7, to that of the linear solution $\tau = 2\pi/p$ is shown in Fig. 10–2. The error in the linear solution for amplitudes up to 30° is found to be less than 2%, and, at $\theta_0 = 90°$, the error is only 18%, as shown in the figure. Since the error in the linear solution is small, the engineer is justified in "linearizing" most problems of a similar nature for reasonable amplitudes.

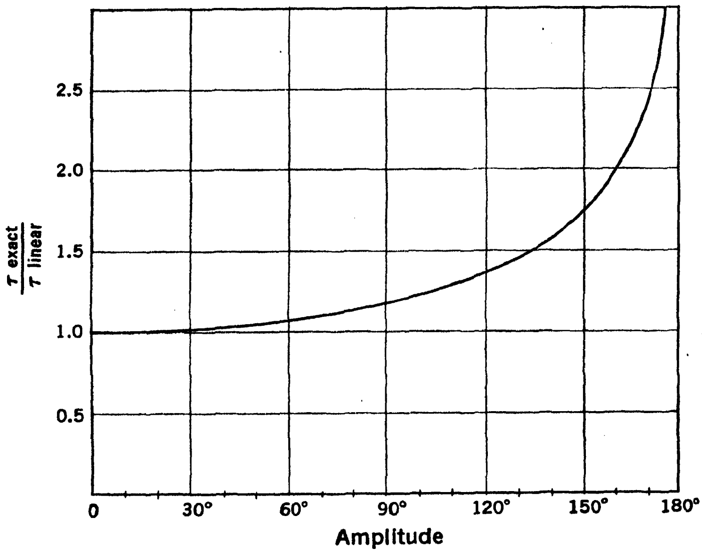


FIG. 10–2

10.4. Oscillations with a Restoring Force of the Form

$$k\left(1 \pm \frac{x^2}{a^2}\right)x$$

A large number of problems in practice lead to an equation of motion having the form

$$\ddot{x} + p^2\left(1 \pm \frac{x^2}{a^2}\right)x = 0 \qquad (10.4\text{–}1)$$

The negative sign denotes what is known as a "soft" system which is characterized by a spring function as shown in Fig. 10–3*a*. The positive sign denotes a "hard" system, and the corresponding spring function is shown in Fig. 10–3*b*.

An example of a soft system is the pendulum treated in the last section. The equation of motion for the pendulum was found to be

$$\ddot{\theta} + p^2 \sin\theta = 0$$

Expanding $\sin\theta$ into an infinite series and retaining only the first two terms gives

$$\ddot{\theta} = p^2\left(1 - \frac{\theta^2}{6}\right)\theta = 0 \qquad (10.4\text{–}2)$$

which denotes a soft system as defined previously.

A system that contains a hard spring is shown in Fig. 10–4. The

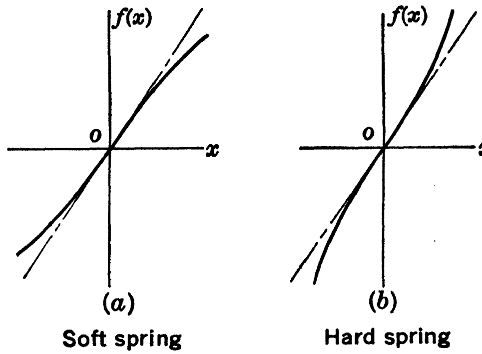


FIG. 10–3

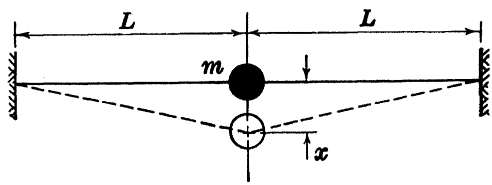


FIG. 10–4

mass m is attached to the midpoint of a taut wire of initial tension S_0. When the mass is displaced laterally a distance x, the wire is extended an amount δ where

$$x^2 + L^2 = (L + \delta)^2 = L^2 + 2\delta L + \delta^2$$

or

$$\delta^2 + 2\delta L - x^2 = 0$$

hence

$$\delta = L\left[1 - \sqrt{1 - \left(\frac{x}{L}\right)^2}\right] \sim \frac{x^2}{2L}$$

The tension in the wire at a displacement x is therefore

$$S = S_0 + AE\frac{\delta}{L} = S_0 + \tfrac{1}{2}AE\left(\frac{x}{L}\right)^2$$

The restoring force acting on the mass m is readily found to be

$$kf(x) = \frac{2x}{L+\delta} S \sim \frac{2x}{L}\left[S_0 + \tfrac{1}{2}AE\left(\frac{x}{L}\right)^2\right]$$

from which the equation of motion for the mass m is

$$m\ddot{x} + 2\frac{S_0}{L}\left(1 + \frac{1}{2}\frac{AE}{S_0L^2}x^2\right)x = 0 \qquad (10.4\text{–}3)$$

This, therefore, indicates a hard system. Similarly many other systems may be reduced to the form 10.4–1 as the first approximation beyond the linearized system.

The spring function $f(x)$ for this general class of problems has then the form

$$f(x) = \left(1 \pm \frac{x^2}{\alpha^2}\right)x$$

and

$$\int_x^{x_0} f(x)\,dx = \tfrac{1}{2}(x_0^2 - x^2) \pm \frac{1}{4\alpha^2}(x_0^4 - x^4) \qquad (10.4\text{–}4)$$

Substitution of this expression into equation 10.2–4 gives

$$t = \frac{1}{p}\int \frac{dx}{\sqrt{(x_0^2 - x^2) \pm \dfrac{1}{2\alpha^2}(x_0^4 - x^4)}} \qquad (10.4\text{–}5)$$

which is an elliptic integral of the first kind.

The expression under the radical is readily factored, whence, for the soft system,

$$pt = \int \frac{dx}{\sqrt{\dfrac{1}{2\alpha^2}(x^2 - 2\alpha^2 + x_0^2)(x^2 - x_0^2)}} \qquad (10.4\text{–}6)$$

and, for the hard system,

$$pt = \int \frac{dx}{\sqrt{-\dfrac{1}{2\alpha^2}(x^2 + 2\alpha^2 + x_0^2)(x^2 - x_0^2)}} \qquad (10.4\text{–}7)$$

Treating the soft system first and introducing the initial conditions $x_{t=0} = 0$ into equation 10.4–6 gives

$$pt = \int_0^x \frac{dx}{\sqrt{\dfrac{1}{2\alpha^2}(x^2 - 2\alpha^2 + x_0^2)(x^2 - x_0^2)}} \qquad (10.4\text{–}8)$$

It is convenient to introduce the variable ϕ defined by

$$\sin \phi = \frac{x}{x_0} \quad \text{or} \quad x = x_0 \sin \phi \tag{10.4–9}$$

Substitution from equation 10.4–9 into equation 10.4–8 gives

$$pt = \frac{1}{\sqrt{1 - \frac{1}{2}\left(\frac{x_0}{\alpha}\right)^2}} \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \frac{F(k, \phi)}{\sqrt{1 - \frac{1}{2}\left(\frac{x_0}{\alpha}\right)^2}} \tag{10.4–10}$$

where

$$k^2 = \frac{1}{2\left(\frac{\alpha}{x_0}\right)^2 - 1} \tag{10.4–11}$$

The cycle of the motion is given in terms of the complete elliptic integral as

$$p\tau = \frac{4F\left(k, \frac{\pi}{2}\right)}{\sqrt{1 - \frac{1}{2}\left(\frac{x_0}{\alpha}\right)^2}} = \frac{4K}{\sqrt{1 - \frac{1}{2}\left(\frac{x_0}{\alpha}\right)^2}}$$

The ratio of the non-linear period to the linear period is given by

$$\frac{\tau}{\tau_L} = \frac{\tau}{\left(\frac{2\pi}{p}\right)} = \frac{2K}{\pi\sqrt{1 - \frac{1}{2}\left(\frac{x_0}{\alpha}\right)^2}} \tag{10.4–12}$$

Equation 10.4–10 is more conveniently written as

$$\phi = \text{am}\left[\sqrt{1 - \frac{1}{2}\left(\frac{x_0}{\alpha}\right)^2}\, pt\right]$$

or

$$x = x_0 \sin \phi = x_0 \,\text{sn}\left[\sqrt{1 - \frac{1}{2}\left(\frac{x_0}{\alpha}\right)^2}\, pt\right] \tag{10.4–13}$$

It must be noted from expression 10.4–11 that x_0^2 must be less than α^2 if periodic solutions are to be obtained since $k^2 \leq 1$.

The hard system which is characterized by equation 10.4–7 can most easily be treated in conjunction with the initial conditions $x_{t=0} = x_0$, whence

$$pt = \int_{x_0}^{x} \frac{dx}{\sqrt{-\frac{1}{2\alpha^2}(x^2 + 2\alpha^2 + x_0^2)(x^2 - x_0^2)}} \tag{10.4–14}$$

In this instance, the substitution

$$\cos\phi = \frac{x}{x_0} \quad \text{or} \quad x = x_0 \cos\phi$$

reduces the integral to standard form,

$$pt = \frac{1}{\sqrt{1 + \left(\frac{x_0}{\alpha}\right)^2}} \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} = \frac{F(k, \phi)}{\sqrt{1 + \left(\frac{x_0}{\alpha}\right)^2}} \tag{10.4–15}$$

where

$$k^2 = \frac{1}{2\left(\frac{\alpha^2}{x_0^2} + 1\right)}$$

The cycle is therefore

$$p\tau = \frac{4F\left(k, \frac{\pi}{2}\right)}{\sqrt{1 + \left(\frac{x_0}{\alpha}\right)^2}} = \frac{4K}{\sqrt{1 + \left(\frac{x_0}{\alpha}\right)^2}}$$

The ratio of the non-linear period to the linear period (small oscillations) is given by

$$\frac{\tau}{\tau_L} = \frac{\tau}{\left(\frac{2\pi}{p}\right)} = \frac{2K}{\sqrt{1 + \left(\frac{x_0}{\alpha}\right)^2}} \tag{10.4–16}$$

Rewriting equation 10.4–15 in terms of the previous notation gives

$$\phi = \operatorname{am}\left[\sqrt{1 + \left(\frac{x_0}{\alpha}\right)^2}\, pt\right] \tag{10.4–17}$$

and

$$x = x_0 \cos\phi = x_0 \operatorname{cn}\left[\sqrt{1 + \left(\frac{x_0}{\alpha}\right)^2}\, pt\right] \tag{10.4–18}$$

where cn stands for the elliptic cosine.

Values of the complete elliptic integral,

$$F\left(k, \frac{\pi}{2}\right) = K(k)$$

as used in equations 10.3–7, 10.4–12, and 10.4–16, are plotted in Fig. 10–5. The form of solutions 10.4–13 and 10.4–18 are found from the graph of the elliptic functions sn and cn as functions of the argument divided by a quarter period as shown in Figs. 10–6 and 10–7. These curves can be plotted from tabulated values of $F(k, \phi)$.

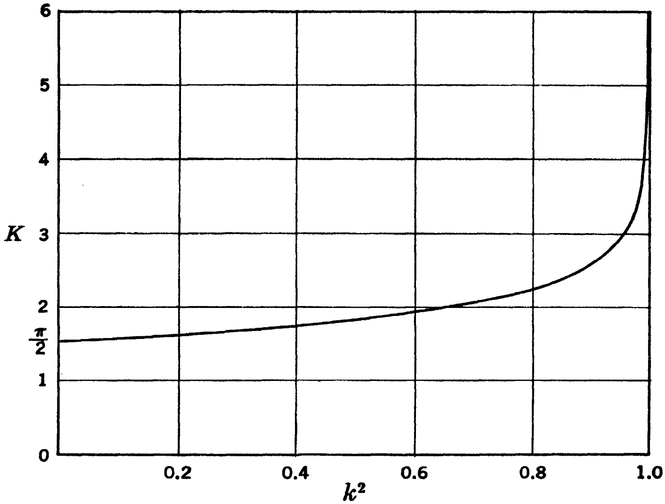


FIG. 10–5

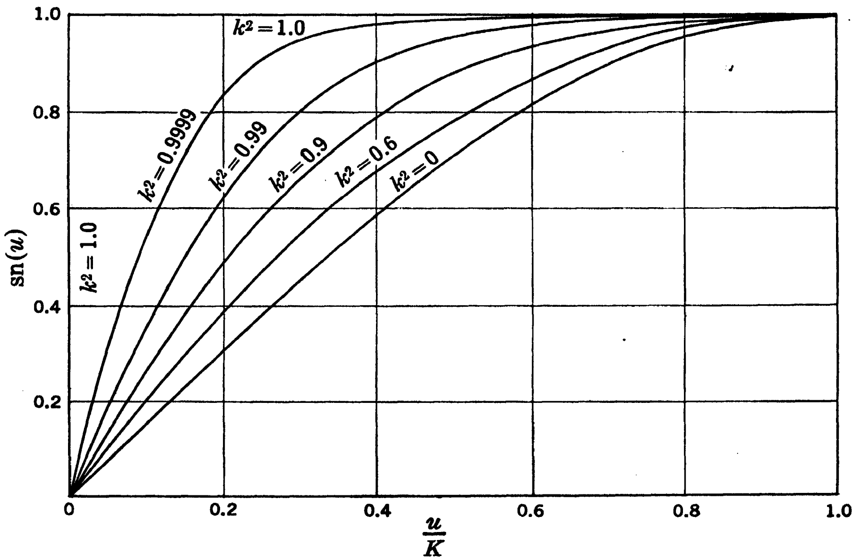


FIG. 10–6

Other examples of oscillating systems which may be treated by the method of this section are included in the exercises.

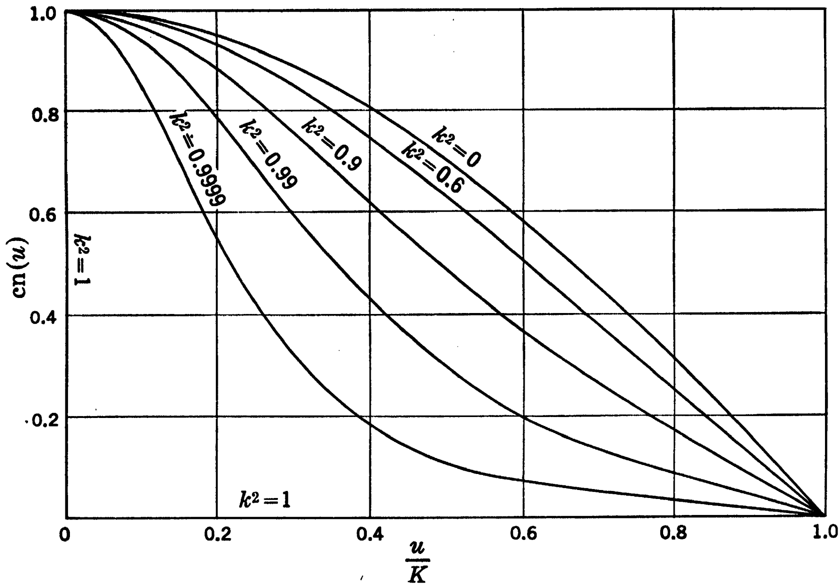


Fig. 10–7

10.5. Systems with a Restoring Force Linear in Segments

There are several systems of practical importance which have non-linear characteristics due to abrupt changes in the nature of the restoring force. Examples of systems with restoring forces of this type are shown in Fig. 10–8 and Fig. 10–9. The method of solution in these instances is one of piecing linear solutions together so that the displacements and velocities are compatible at the points of discontinuity.

The system shown in Fig. 10–8*a* is essentially a system representing the condition of backlash which is always more or less present in geared transmission systems. For simplicity, the initial conditions will be taken as

$$x_{t=0} = x_0 > \frac{\delta}{2} \quad \text{and} \quad \dot{x}_{t=0} = 0$$

The equation of motion over the range in which the mass is in contact with a spring is

$$m\ddot{x} + k\left(x - \frac{\delta}{2}\right) = 0$$

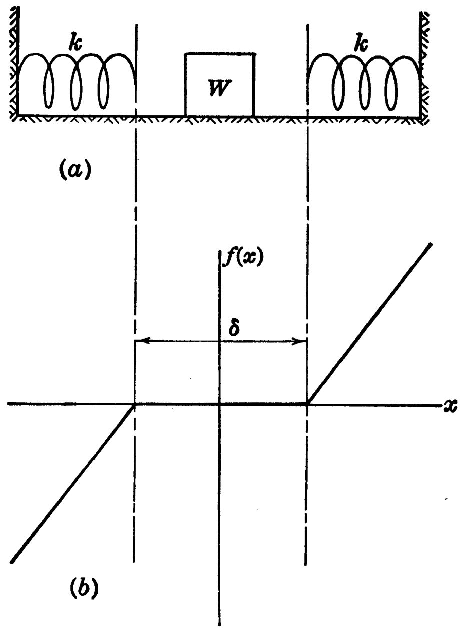


FIG. 10–8

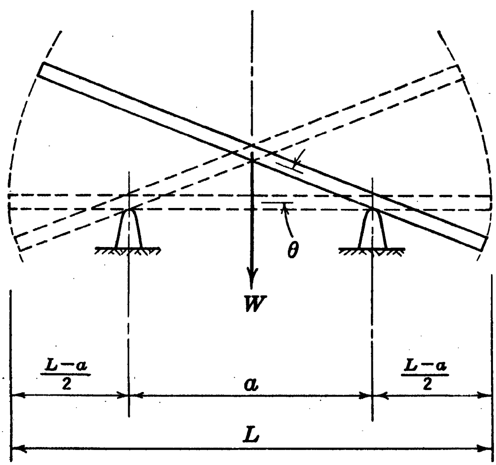


FIG. 10–9

or

$$\ddot{x} + p^2 x = p^2 \frac{\delta}{2} \qquad (10.5\text{–}1)$$

The solution to this equation is

$$x = C_1 \cos pt + C_2 \sin pt + \tfrac{1}{2}\delta$$

Substituting from the initial conditions shows that

$$C_1 = x_0 - \tfrac{1}{2}\delta, \qquad C_2 = 0$$

hence

$$x = (x_0 - \tfrac{1}{2}\delta) \cos pt + \tfrac{1}{2}\delta$$

The time required for the mass to move from the initial position $x = x_0$ to $x = \frac{1}{2}\delta$ where it leaves the spring is seen to be

$$t_1 = \frac{\pi}{2p}$$

The velocity of the mass as it crosses the gap between springs, neglecting friction, is given by

$$v = -(x_0 - \tfrac{1}{2}\delta)p \sin pt_1 = -(x_0 - \tfrac{1}{2}\delta)p$$

The time required to reach the center of the gap starting from $x = x_0$ is therefore

$$\frac{\tau}{4} = \frac{\pi}{2p} + \frac{\delta}{2|v|}$$

from which the period becomes

$$\tau = \frac{2\pi}{p}\left[1 + \frac{4}{\pi\left(2\frac{x_0}{\delta} - 1\right)}\right] \qquad (10.5\text{–}2)$$

The period of the motion is seen to increase as the ratio x_0/δ decreases. The effect of backlash in machinery is thus to decrease the natural frequency of the system.

A second example of a motion that can be handled in a manner similar to that in the previous problem is the rocking motion of a bar on two rigid supports, as shown in Fig. 10–9. For simplicity the oscillations may be assumed to have a small amplitude. Let the initial conditions be $\theta = \theta_0$ and $\dot{\theta} = 0$ when $t = 0$. The equation of motion for a slender bar with respect to one of the supports is

$$\frac{W}{g}\left[\frac{L^2}{12} + \left(\frac{a}{2}\right)^2\right]\ddot{\theta} + \frac{Wa}{2}\cos\theta = 0 \qquad (0 < \theta < \theta_0) \qquad (10.5\text{–}3)$$

The solution of this equation for $\cos\theta \sim 1$ is

$$\theta = -\frac{3ag}{L^2 + 3a^2}t^2 + C_1 t + C_2$$

The constants of integration may be found from the initial conditions, giving

$$\theta = \theta_0 - \frac{3ag}{L^2 + 3a^2}t^2$$

The time required to execute a quarter cycle (i.e. from $\theta = \theta_0$ to $\theta = 0$) is

$$\frac{\tau}{4} = \sqrt{\frac{L^2 + 3a^2}{3ag}\theta_0}$$

hence

$$\tau = 4\sqrt{\frac{L^2 + 3a^2}{3ag}\theta_0} \qquad (10.5\text{–}4)$$

The period is seen to vary as the square root of the amplitude. This motion is unique in the sense that it does not become nearly linear at small amplitudes; in fact, the period approaches zero as the amplitude decreases to zero. The above motion is usually very heavily damped in the sense that a considerable amount of energy is lost in impact during each cycle. The required correction on the period may be obtained by consideration of the proper coefficient of restitution for impact between the bar and the support.

10.6. Systems with Non-linear Damping

Damping forces that are not proportional to the velocity are said to be non-linear. One of the most common examples of non-linear damping is dry friction or Coulomb damping in which the damping forces are constant. Other examples are certain kinds of fluid damping in which the damping forces are proportional to the second and higher powers of the velocity, and solid friction or hysteresis damping which depends on powers of the amplitude, greater than one.

Coulomb damping gives rise to an equation of motion for the free vibration of a single-degree-of-freedom system (see Fig. 10–10) of the form

$$\frac{W}{g}\ddot{x} + kx = F = \mu W \qquad (10.6\text{–}1)$$

where μ is the coefficient of friction between W and the surface on which it slides. This may be written as

$$\ddot{x} + p^2x = \mu g \tag{10.6–2}$$

The solution to equation 10.6–2 is

$$x = C_1 \cos pt + C_2 \sin pt + \frac{\mu g}{p^2} \tag{10.6–3}$$

To simplify the solution, the initial conditions may be taken as

$$x_{t=0} = x_0 \quad \text{and} \quad \dot{x}_{t=0} = 0$$

whence

$$x = (x_0 - \delta_F) \cos pt + \delta_F \tag{10.6–4}$$

where

$$\delta_F = \frac{F}{k} = \frac{\mu W}{k} = \frac{\mu g}{p^2}$$

is the deflection that would result if the friction force was applied directly on the spring. It is important to note that equation 10.6–4

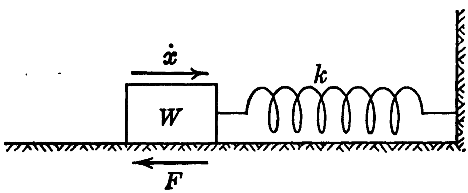


FIG. 10–10

applies only so long as the motion of W is in the same direction, that is, for a half cycle.

The velocity obtained by differentiating equation 10.6–4 with respect to time is

$$\dot{x} = -p(x_0 - \delta_F) \sin pt$$

from which it is seen that a half cycle occurs in a time

$$\frac{\tau}{2} = \frac{\pi}{p}$$

hence the period is

$$\tau = \frac{2\pi}{p} \tag{10.6–5}$$

It is apparent from this equation that the period is the same as for the undamped system. The reduction in amplitude for a half cycle is

$2\delta_F$, thus for each cycle the amplitude is reduced by an amount $4\delta_F$ which is independent of the initial amplitude. The form of the motion is obtained by joining successive solutions to each half cycle of the oscillation. A graphical representation of the complete motion is shown in Fig. 10–11.

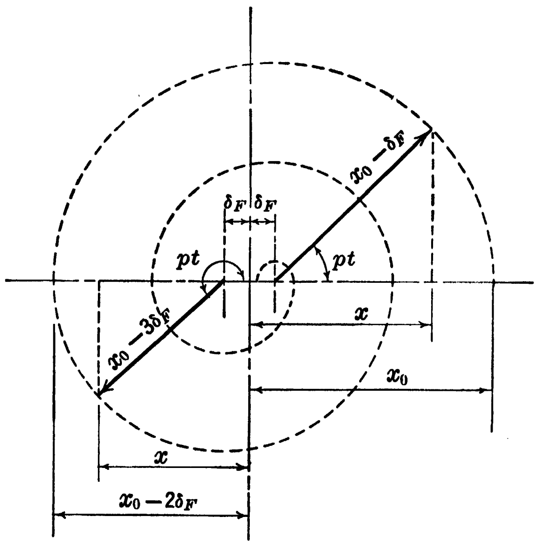


FIG. 10–11

The exact treatment of the free vibration of a system with general non-linear damping is apt to be so lengthy as to be impractical. An approximate solution to the problem for damping of any form may be obtained by replacing the non-linear damper with an equivalent linear damper. A linear damper is said to be equivalent to another damper if it dissipates the same amount of energy per cycle. Thus if $F(x, \dot{x})$ is the friction force in the non-linear damper and the form of the motion is represented approximately by

$$x = A \sin pt$$

then the energy dissipated per cycle will be

$$E = \int F(x, \dot{x})\, dx = \int F \frac{dx}{dt}\, dt$$

$$= \int_0^\tau F\dot{x}\, dt = Ap \int_0^\tau F \cos pt\, dt \qquad (10.6\text{–}6)$$

For linear damping,

$$F = c\dot{x}$$

whence

$$E = Ap \int_0^{\tau} c\dot{x} \cos pt \, dt = A^2p^2 \int_0^{\tau} \cos^2 pt \, dt$$

$$= A^2p^2c\frac{\tau}{2} = \pi A^2 pc \tag{10.6–7}$$

The equivalent linear damping constant c is obtained by equating 10.6–6 and 10.6–7 and solving for c, whence

$$c = \frac{1}{\pi A} \int_0^{\tau} F(x, \dot{x}) \cos pt \, dt \tag{10.6–8}$$

The decrease in amplitude may be estimated from the equivalent viscous damping calculated from equation 10.6–8, or it may be calculated directly as follows. The velocity of the mass is

$$\dot{x} = pA \cos pt$$

The loss in kinetic energy during the cycle from $t = 0$ to $t = \tau = 2\pi/p$ is given by

$$E = \tfrac{1}{2}m(\dot{x}_0^2 - \dot{x}_\tau^2)$$

$$= \tfrac{1}{2}mp^2(A_0^2 - A_\tau^2) = \tfrac{1}{2}mp^2[A_0^2 - (A_0 - \Delta A)^2]$$

where ΔA is the decrease in amplitude during the cycle. Thus

$$E = \tfrac{1}{2}mp^2(2A_0 - \Delta A)\,\Delta A \tag{10.6–9}$$

For small damping as is usually encountered in practice, ΔA is a small quantity which may be neglected in comparison to $2A_0$. This equation can therefore usually be written as

$$E = mp^2 A_0 \,\Delta A$$

or

$$\Delta A = \frac{E}{mp^2 A_0} = \frac{E}{kA_0} \tag{10.6–10}$$

By substitution from equation 10.6–6, the decrease in amplitude per cycle becomes

$$\Delta A = \frac{1}{mp} \int_0^{\tau} F(x, \dot{x}) \cos pt \, dt \tag{10.6–11}$$

As an example of the use of this equation, the case of Coulomb damping $F(x, \dot{x}) = F_0$ may be treated. To satisfy the requirement that F_0 always oppose the motion, it is necessary to integrate over a quarter cycle and multiply by four, thus

$$\Delta A = \frac{4}{mp}\int_0^{\pi/2p} F_0 \cos pt\, dt = \frac{4F_0}{mp^2}$$

$$= \frac{4F_0}{k} = 4\delta_F \qquad (10.6\text{–}12)$$

which agrees with the exact result obtained earlier.

The case of damping proportional to the square of the velocity is of considerable practical importance. In this instance $F(x, \dot{x}) = c_1\dot{x}^2$. The equivalent viscous damping is given by

$$c = \frac{4}{\pi A}\int_0^{\pi/2p} c_1\dot{x}^2 \cos pt\, dt$$

or, since $\dot{x} = pA \cos pt$,

$$c = \frac{4c_1Ap^2}{\pi}\int_0^{\pi/2p} \cos^3 pt\, dt$$

from which

$$c = \frac{8Ap}{3\pi} c_1 \qquad (10.6\text{–}13)$$

The decrease in amplitude per cycle for small damping may then be found by substituting in equation 10.6–11,

$$\Delta A = \frac{1}{mp}\int_0^{2\pi/p} c\dot{x} \cos pt\, dt = \frac{8A^2}{3m} c_1$$

Hysteresis damping or solid damping occurs internally in a stressed solid body. The nature of hysteresis damping is not well-known; however the energy dissipated per cycle may usually be represented as

$$E = c_2A^n \qquad (10.6\text{–}14)$$

where c_2 is a constant and the exponent n must be evaluated in each instance. For mild steel, $n = 2.3$ to 3.0 is in common use. The equivalent viscous damping obtained from equations 10.6–7 and 10.6–14 is

$$c = \frac{c_2A^{n-2}}{\pi p} \qquad (10.6\text{–}15)$$

The decrease in amplitude per cycle is obtained by substituting from equation 10.6–14 into 10.6–10, whence

$$\Delta A = \frac{c_2 A^{n-1}}{k}$$

Other types of nonlinear damping may be treated in a similar manner. Equivalent linear damping constants, so obtained, may be utilized in the linear theory as previously discussed.

PROBLEMS

Chapter 1. General Concepts

1–1. Determine the number of degrees of freedom of the systems shown in Figs. 1–9, 5–1, 6–15*a* and 6–16*b*, by inspection.

1–2. A vibratory motion has an amplitude of 0.1 in. and a period of 0.01 sec. What are the frequency, the maximum velocity, and the maximum acceleration?

1–3. The maximum velocity of an oscillating body is 10 in. per sec, and its frequency is 60 cycles per second. What are the amplitude, maximum acceleration, and period of the motion?

1–4. An accelerometer is used to measure a vibratory motion. The frequency and the maximum acceleration are found to be 3600 cycles per minute and 1000 in. per $\sec^2$, respectively. What are the period, amplitude, and maximum velocity of the motion?

1–5. A vibratory motion is represented by the equation

$$x = 10 \cos 2t$$

Plot the displacement, velocity, and acceleration versus time curves.

1–6. The vibratory motion of a body is governed by the equation for the velocity

$$\dot{x} = 10 \cos (\pi t + \pi/4)$$

Plot the displacement, velocity, and acceleration versus time curves.

1–7. A body oscillates according to the law

$$\ddot{x} = 10 \sin (2\pi t + \pi/6)$$

Plot the displacement, velocity, and acceleration versus time curves.

1–8. Calculate the ratio of the mean velocity to the maximum velocity for a half cycle of vibratory motion starting from rest.

1–9. Two vibratory motions have equal maximum accelerations, but their frequencies are in the ratio of 1 to 4. What is the ratio of their amplitudes?

1–10. If it is possible to measure the maximum acceleration of a certain vibratory motion with an error not exceeding 5 per cent and to determine the frequency to within 2 per cent, what is the maximum possible percentage error in the value of the amplitude calculated from such data?

1–11. The displacement of a particle is given by the equation

$$x = A \cos (\omega t + \varphi) + B \sin (\omega t + \theta)$$

Determine the amplitude of the motion of the particle.

1–12. The motion of problem 1–11 can be represented in the form

$$x = C \cos \omega t + D \sin \omega t$$

Determine the values of C and D.

1–13. The motion of problem 1–11 can be represented in the form

$$x = E \cos (\omega t + \psi)$$

Determine the phase angle ψ.

1–14. Solve problem 1–11 graphically for the specific values

$$A = 10 \text{ in.} \qquad \varphi = 60°$$
$$B = 5 \text{ in.} \qquad \theta = 30°$$

1–15. Solve problem 1–12 graphically for the specific values

$$A = 4 \text{ in.} \qquad \varphi = 45°$$
$$B = 6 \text{ in.} \qquad \theta = 30°$$

1–16. Solve problem 1–12 graphically for the specific values

$$A = 9 \text{ in.} \qquad \varphi = 75°$$
$$B = 6 \text{ in.} \qquad \theta = 15°$$

1–17. A vibratory motion has a displacement given by the equation

$$x = C \cos \omega t$$

Resolve this motion into two components, one of which leads the given motion by a phase angle φ and one of which lags the given motion by a phase angle θ.

1–18. An oscillating particle performs a motion given by

$$x = C \sin \omega t$$

This motion is the sum of two components, one of which is $A \sin (\omega t + \varphi)$. What is the other component?

1–19. A vibratory motion defined by

$$x = C \cos \omega t$$

is to be resolved into two components,

$$x = A \cos (\omega t + \varphi) + B \cos (\omega t - \theta)$$

where A and B are known and $A + B > C$. Determine the phase angles φ and θ.

1–20. A particle moves in a plane according to the equations

$$x = A \cos \omega t$$
$$y = B \cos 2\omega t$$

Plot the motion of the particle.

1-21. A particle moves in a plane according to the equations

$$x = A \sin \omega t$$

$$y = B \sin 2\omega t$$

Plot the motion of the particle.

1-22. A particle moves in a plane according to the equations

$$x = A \sin \omega t$$

$$y = B \sin \left(\omega t + \frac{\pi}{4}\right)$$

Plot the motion of the particle.

1-23. A balance wheel in a watch oscillates with a simple harmonic motion

$$\theta = \Theta \sin (\omega t + \varphi)$$

At time t_0 it receives a small angular impulse which increases its velocity in the direction of motion from $\dot{\theta}$ to $\dot{\theta} + \delta\dot{\theta}$. Show that the change in amplitude $\delta\Theta$ and phase angle $\delta\varphi$ are given by

$$\delta\Theta = \frac{\delta\dot{\theta}}{\omega\Theta} \cos (\omega t_0 + \varphi) \qquad \delta\varphi = \frac{-\delta\dot{\theta}}{\omega} \sin (\omega t_0 + \varphi)$$

1-24. A pendulum performs a simple harmonic motion given by $\theta = \Theta \sin \omega t$. A heavy plate is inserted in its path so that the pendulum ball strikes it normally when its displacement is $\alpha\Theta$, $(\alpha < 1)$. The pendulum bob rebounds, and the plate is withdrawn. If the coefficient of restitution for the impact is e, calculate the ratio of the amplitudes before and after impact and the phase angle introduced into the motion.

1-25. A horizontal table is made to oscillate vertically with a frequency of f cycles per second. What is the greatest amplitude it can have if objects that lie upon it are to remain in contact with the table?

Chapter 2. Free Vibration without Damping

2-1. The spring in the system of Fig. 2-2 elongates 0.1 in. when stretched by a 10-lb force. Calculate the static deflection and the period of vibration if the mass weighs 25 lb.

2-2. The system of Fig. 2-2 has a natural frequency of 20 cycles per second, and the mass has a weight of 5 lb. What is the spring constant of the spring?

2-3. The system of Fig. 2-2 is designed so that it has a period of 1 sec. What is the static deflection?

2-4. The system of Fig. 2-3 is constructed from a steel shaft 1 in. in diameter and 30 in. long. The circular steel disk has a radius of 12 in., and it is ½ in. thick. What is the natural frequency of the system?

2–5. The springs shown in Fig. 2–7 have spring constants $k_1 = 0.5$ lb per in., $k_2 = 1$ lb per in., and $k_3 = 0.1$ lb per in. Determine the equivalent spring constant.

2–6. The springs shown in Fig. 2–8 have spring constants $k_1 = 0.1$ lb per in., $k_2 = 1$ lb per in. and $k_3 = 0.5$ lb per in. Calculate the equivalent spring constant.

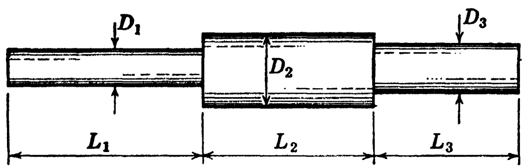
2–7. The springs S_1, S_3, and S_5 in Fig. 2–10 are round steel rods ½ in. in diameter and 10 in. long. The springs S_2, S_4, and S_6 have spring constants of 50 lb per in. Calculate the equivalent spring constant for this system.

2–8. A hollow shaft having an outside diameter D, an inside diameter d, and a length L has a torsional spring constant

$$k = \frac{\pi G}{32} \frac{D^4 - d^4}{L}$$

where G is the modulus of rigidity. Find the length L_e of a solid shaft whose diameter is D_e which has the same spring constant as the hollow shaft.

2–9. Find the spring constant of the shaft as indicated, and find the equivalent length L_e of a shaft with diameter D_e which has the same spring constant as the shaft shown.



PROB. 2–9

2–10. Show that a tapered conical shaft with end diameters d and D and length L has the same spring constant as a uniform shaft of diameter d and a length

$$L_e = \tfrac{1}{3}(1 + \alpha + \alpha^2)\alpha L \quad \text{where} \quad \alpha = \frac{d}{D}$$

2–11. Find the natural frequency of the shafts and solid disk as shown if the density of the disk is ρ and modulus of rigidity of the shaft is G. (Neglect inertia of shaft.)

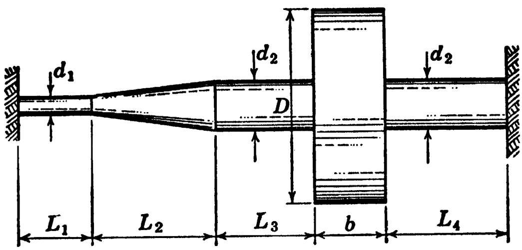
$L_1 = 2$ in. $\qquad d_1 = 1$ in.

$L_2 = 6$ in. $\qquad d_2 = 2$ in.

$L_3 = 4$ in. $\qquad D = 10$ in.

$L_4 = 8$ in.

$b = 4$ in.



PROB. 2-11

2-12. The circular frequency p may be expressed in the form $p = \sqrt{g/L}$, where the parameter L represents an equivalent pendulum length of an oscillating system. Show that this equivalent length, for a buoyant mass in a tank with a surface area n times the cross section area A at the liquid surface of the floating mass, can be expressed as $\frac{V}{A}\left(1 - \frac{1}{n}\right)$. It is assumed that the frequency is small and that the surface of the liquid remains level. V is the displaced volume of the floating mass. Show also that the equivalent spring constant is

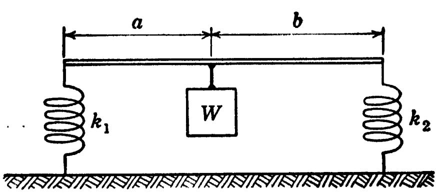
$$k_{eq} = \frac{wA}{1 - \frac{1}{n}}$$

where w is the specific weight of the fluid.

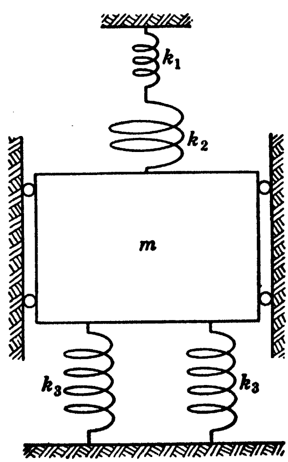
2-13. Find the natural frequency of the system shown if

(*a*) $a = b$
(*b*) $a = 2b$

Neglect the weight of the horizontal bar.



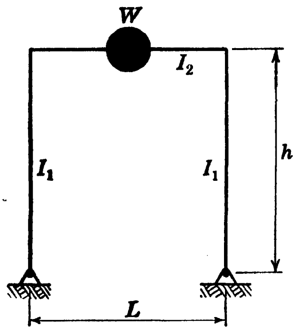
PROB. 2-13



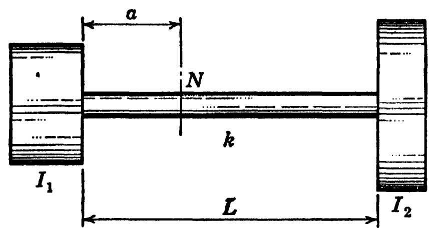
PROB. 2-14

2-14. Find the period of free vibration of the system shown.

2-15. The frame shown supports a single concentrated load W. Calculate the natural frequency of the system for vertical and horizontal vibration of the weight W. Neglect the mass of the frame.



PROB. 2-15

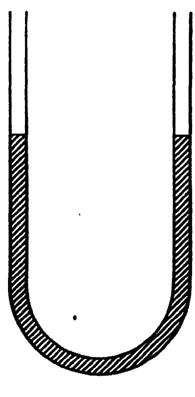


PROB. 2-16

2-16. A uniform shaft whose spring constant is k is rigidly attached to two bodies as shown. Locate the position of the node N by equating the frequencies of the system to the left and to the right side of the node. Calculate the frequency of the system.

2-17. Solve problem 2-16 by equating the total angular momentum of the system to zero thus locating the node.

2-18. A U tube containing a liquid column of length l oscillates under the influence of gravity. Determine the period of the motion.



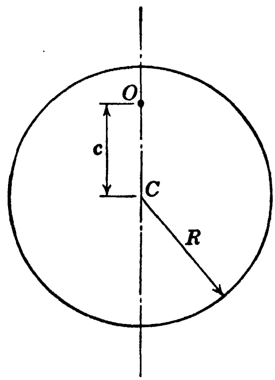
PROB. 2-18

2-19. A pendulum has a specific gravity of γ.

(a) Find the ratio between its periods when oscillating in a vacuum and in water, for small oscillations.

(b) For what specific gravity will the ratio, calculated above, be unity.

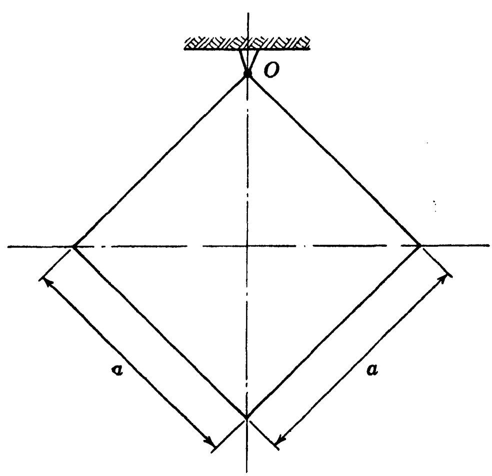
2–20. A circular disk of constant thickness is pivoted at point O as shown. Determine the natural frequency of the disk if $c = R/2$.



PROB. 2–20

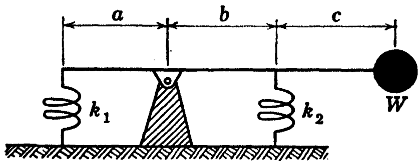
2–21. What value should c have, in problem 2–20, to make the frequency a maximum? What is this maximum frequency?

2–22. A thin square plate of uniform thickness is suspended at O as shown. Determine the periods of oscillation in the plane of the plate and normal to the plane of the plate.

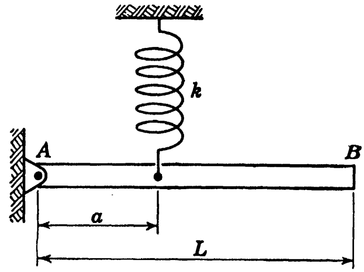


PROB. 2–22

2–23. Calculate the period of the system shown in the figure.



PROB. 2–23

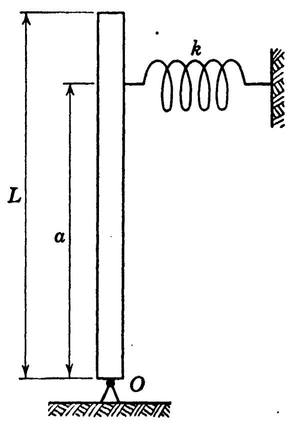


PROB. 2–24

2–24. A uniform thin rod AB of weight W is supported as shown. Determine the period of this system.

2–25. The spring in problem 2–24 was elongated an amount δ when the bar was attached. What is the spring constant k and the frequency of the system.

2–26. A uniform bar of weight W is pivoted at the lower end as shown. Under what condition will the bar oscillate about the vertical position? What is its natural frequency for this type of oscillation?



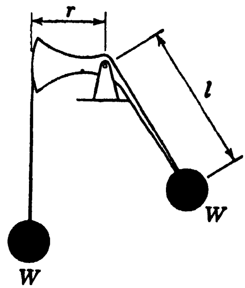
PROB. 2–26

2–27. The system of problem 2–26 is inclined so that the bar makes an angle α with the vertical. What is the natural frequency of the system in this position.

2–28. Show that the period of this oscillating system can be expressed as

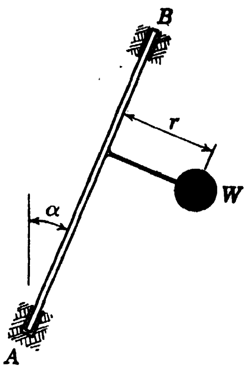
$$\tau = 2\pi \sqrt{\frac{l^2 + r^2}{g\sqrt{l^2 - r^2}}}$$

if the weight of the lever is neglected and the two equal weights W are assumed to be concentrated at a point. The system is in equilibrium in the position shown.



PROB. 2–28

2–29. A weight W is free to rotate about an axis AB on an arm of length r. If the axis AB makes an angle α with the vertical, what is the natural frequency?



PROB. 2–29

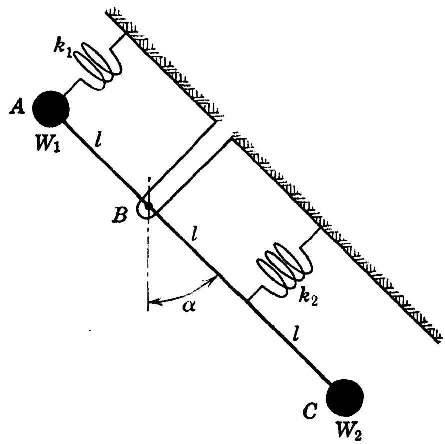
2–30. A circular disk of weight W and radius r is suspended from several equally spaced vertical strings, of length l, attached to its rim, so that its plane is horizontal. The observed period of oscillation about the vertical axes through the centroid of the disk is τ. What is the radius of gyration of the disk?

2–31. The bar ABC is pivoted at B. Attached to the bar are two weights W_1 and W_2, assumed concentrated, at A and C, and two springs k_1 and k_2 as shown. The bar makes an angle α with the vertical. Find the period:

(a) If $k_1 = k_2$ and $W_1 = W_2$.

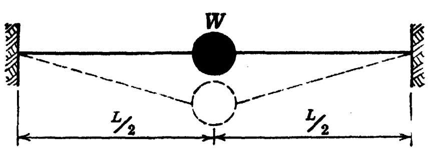
(b) If $k_1 = 2k_2$ and $W_1 = 2W_2$.

$\alpha = 60°$ and $\frac{W_2}{k_2} = \frac{1}{2}l$ for both cases.

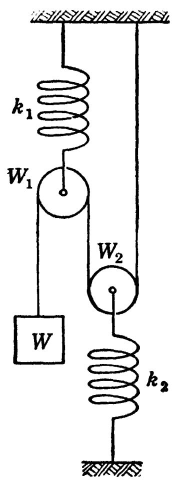


PROB. 2-31

2-32. A weight W is attached to a taut string of length L. If the tension on the string is T calculate the natural frequency for small oscillations.



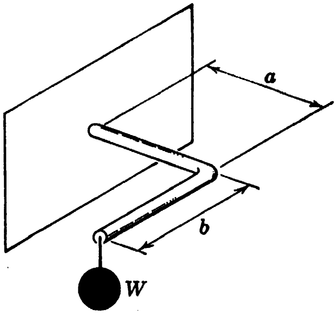
PROB. 2-32



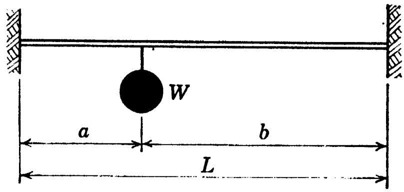
PROB. 2-33

2-33. Find the natural frequency of the system shown in the figure. Neglect the weight of the pulleys W_1 and W_2.

2-34. A round rod of radius r is bent into a right angle and used to support a weight W as shown. Determine the static deflection and the natural frequency of the system. Neglect the weight of the rod.



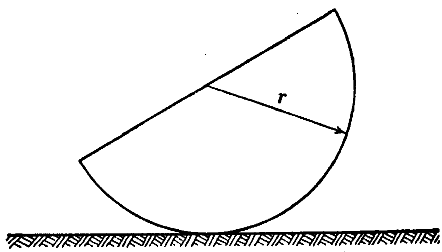
PROB. 2–34



PROB. 2–35

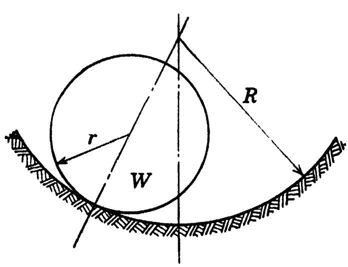
2–35. A weightless beam, built in at the ends, has a stiffness EI, and it supports a weight W as shown. Determine the period of the system.

2–36. A semicircular disk of weight W is free to rock on a horizontal plane surface. Determine its natural frequency for small oscillations.



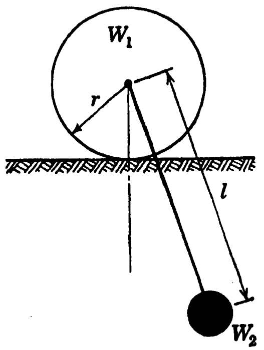
PROB. 2–36

2–37. A cylinder of radius r and weight W is free to roll in a circular groove of radius R. Determine the period of the cylinder for small oscillations if it rolls without slipping.

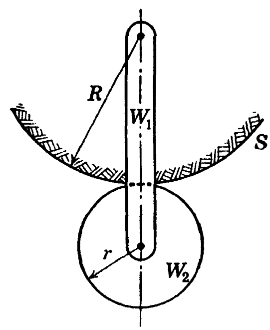


PROB. 2–37

2-38. The cylinder W_1 and the weight W_2 are rigidly connected by a rod. If the cylinder rolls without slipping, what is the period of the system for small oscillations.



PROB. 2-38

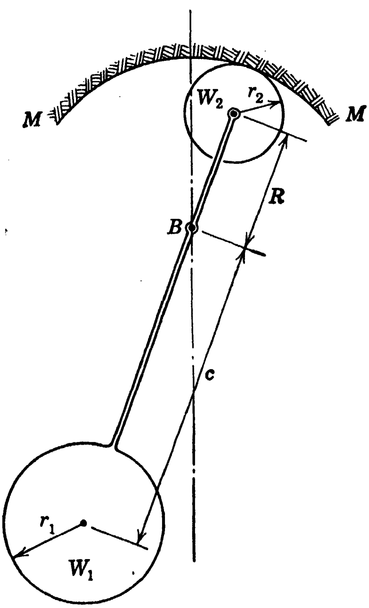


PROB. 2-39

2-39. The solid cylinder W_2 rolls without slipping on the surface S. Find the period of the system.

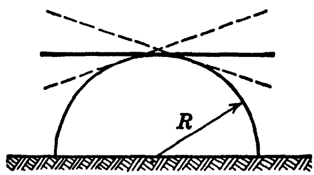
2-40. The pendulum suspended at B consists of a sphere W_1 and a cylinder W_2 which rolls without slipping on the track M–N. Find the period of the system.

$$(W_2R < W_1c)$$



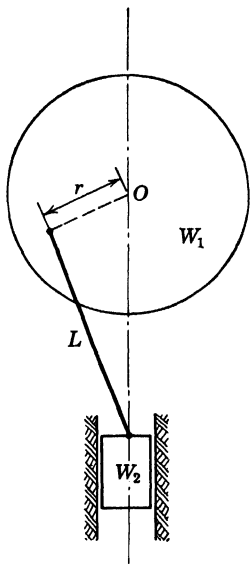
PROB. 2-40

2–41. A thin rod of length L rocks on a semicircle of radius R. Determine the natural frequency for small oscillations.

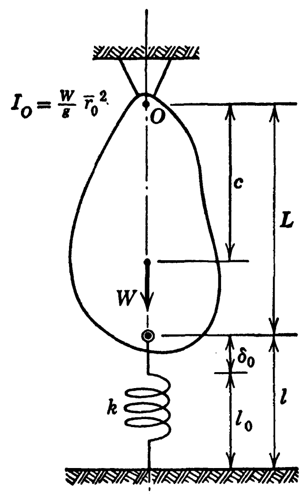


PROB. 2–41

2–42. A wheel of weight W_1 and radius of gyration $\bar{r}$ is attached to a piston of weight W_2 by means of a weightless connecting rod. Assuming that the system is free of friction, determine the natural frequency for small oscillations.



PROB. 2–42



PROB. 2–43

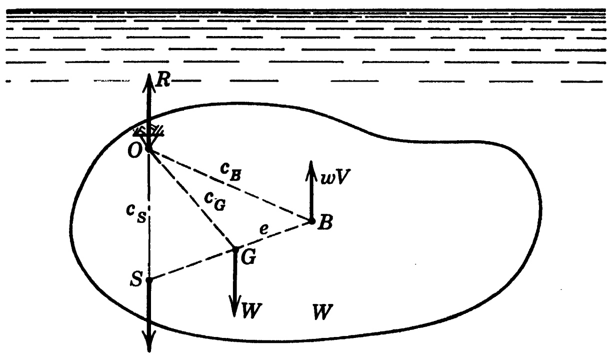
2–43. A spring with an unstrained length l_0 is attached to a pendulum as indicated. Find the frequency of the system for small oscillations.

2–44. A hydrometer of weight W floats upright in water. It has a displacement V, and its stem diameter is d. What is its period for small vertical oscillations? What will be its period when floating in another liquid of specific gravity γ.

2–45. A fluid is assumed to have a specific gravity that increases linearly with depth so that it is equal to 2 at depth l compared to 1 at the surface. A body of specific gravity $\gamma > 1$ is allowed to sink in the fluid until it comes to equilibrium at a depth z_0. Calculate z_0 and the equivalent pendulum length for small oscillations about the equilibrium position.

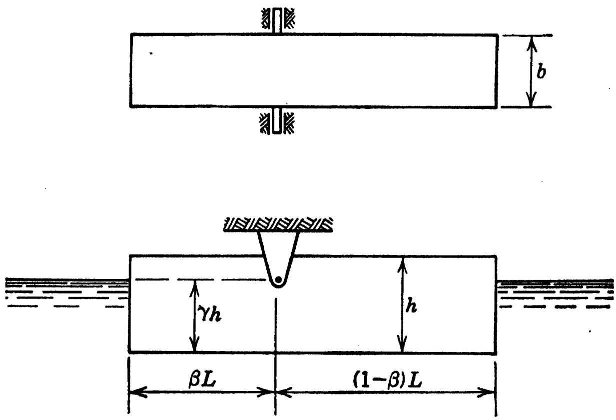
2-46. A balloon bag of weight W_B and volume V is filled with gas of specific weight w_g and held captive in air (sp wt w_a) by a cord. The radius of gyration of the balloon bag and gas is r_0, and the cord length is l. Find the period for small oscillations.

2-47. A non-homogeneous body of volume V is immersed in a fluid (liquid or gas) of specific weight w, and suspended from a horizontal pivot at O. If G is the center of mass of the body and B is the buoyancy centroid, while S is the specific centroid of the immersed body, show that the period for small oscillations can be expressed as $\tau = 2\pi \sqrt{I_0/Rc_S}$, where I_0 is the moment of inertia of the mass about the pivot axis O, and R is the static reaction at O. Show also that the period becomes $\tau = 2\pi \sqrt{I_0/Wl}$ for $R = 0$.



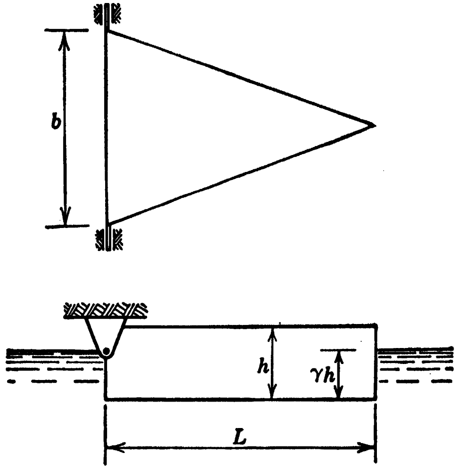
Prob. 2-47

2-48. A homogeneous rectangular floating bar of specific gravity γ is hinged in the water line as indicated. Find its natural frequency about the hinges.



Prob. 2-48

2–49. A triangular-shaped homogeneous slab floats in water. It is hinged along one edge in the water line as shown. Find its natural frequency about the hinges if its specific gravity is γ.

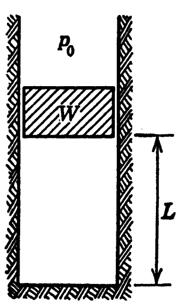


PROB. 2–49

2–50. A homogeneous rectangular bar of length l, height h, and width b floats in water with a draft d. Show that the period of small rolling oscillations about the longitudinal axis can be expressed as

$$\tau = 2\pi \sqrt{\frac{(h^2 + b^2)}{[b^2 - 6(h - d)d]} \left(\frac{d}{g}\right)}$$

2–51. A piston of weight W and area A is supported on a column of air in a cylinder as shown. The equilibrium position of the piston is obtained when the air column has a length L. What is the natural frequency of the system for small oscillations about the equilibrium position if the process is (*a*) isothermal? (*b*) adiabatic?



PROB. 2–51

2–52. A piston of weight W is free to slide in a cylinder. One end of the cylinder is opened to the atmosphere, and the other end is closed as shown.

The piston is in equilibrium in the center of the tube. If the tube is rotated at an angular velocity Ω about the open end, show that the new equilibrium position of the piston is at a distance δ from the middle of the cylinder and that the natural circular frequency for small oscillations about this point is

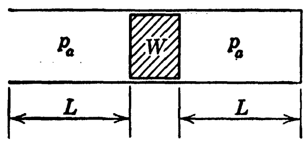
$$p = \sqrt{\frac{\gamma p_a g A}{LW(1 - \delta/L)^{\gamma+1}} - \Omega^2}$$

where

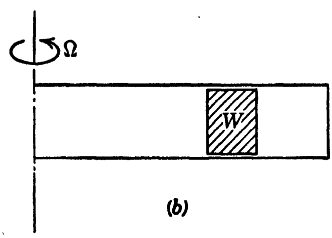
$$\delta = L\left[\sqrt{\left(\frac{p_a g A}{2W\Omega^2 L}\right)^2 + 1} - \frac{p_a g A}{2W\Omega^2 L}\right]$$

A is the piston area, γ is the ratio of the specific heats of the gas in the closed end of the cylinder, and p_a is atmospheric pressure.

Hint: The process by which the piston reaches its new equilibrium position is isothermal whereas the small expansions or compressions due to vibration will be adiabatic.



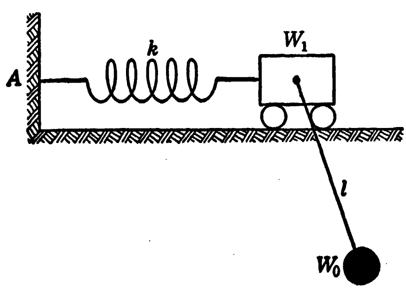
(a)



(b)

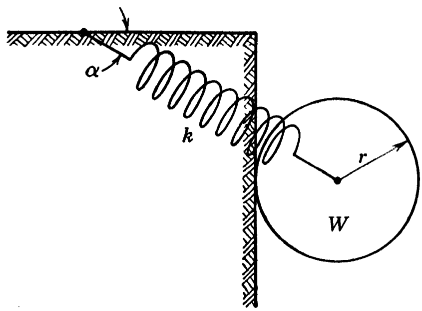
PROB. 2–52

2–53. Find the natural frequency of the system if the weight W_1 is neglected.



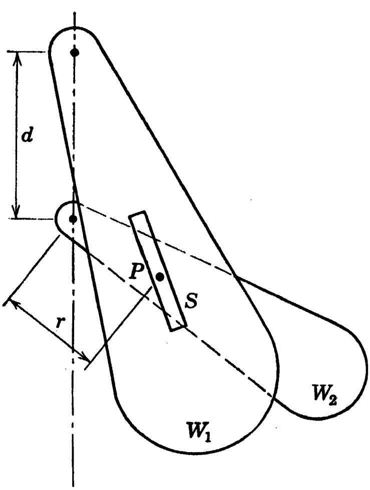
PROB. 2–53

2–54. A solid cylinder of weight W is held against a vertical wall by a spring k. Find the natural frequency of the system for small oscillations if the cylinder rolls without slipping.

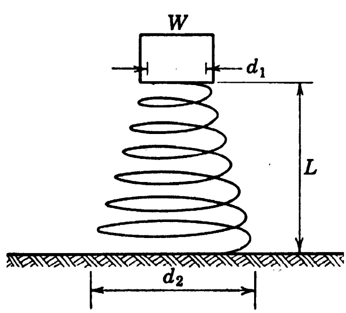


PROB. 2–54

2–55. Two pendulums are linked together by a pin P and a slot S as indicated. Symmetry is assumed for both pendulums with respect to the vertical static position. Find the natural frequency of the system.



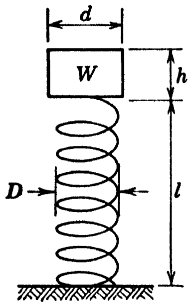
PROB. 2–55



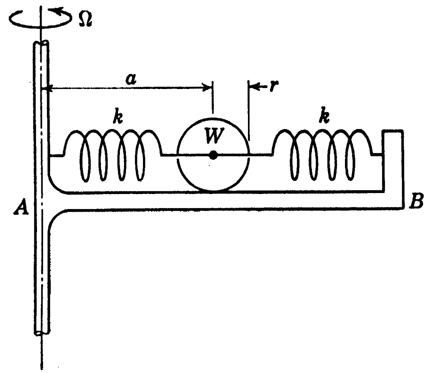
PROB. 2–56

2–56. A tapered weightless spring of wire of radius r, which has N coils uniformly spaced along its length, supports a weight W as shown. Calculate the spring constant and the natural frequency of the system for vertical oscillations.

2-57. Determine the criteria so that the natural frequencies for torsional, lateral, and vertical oscillations of the cylinder W will be the same.



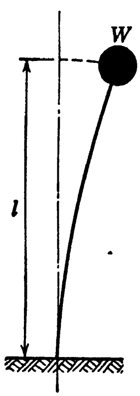
PROB. 2-57



PROB. 2-58

2-58. A solid cylindrical roller of weight W is mounted on an arm AB and held in position by two springs of spring constant k. If the arm revolves at a uniform angular velocity Ω, determine the change in the equilibrium position of the roller and its natural circular frequency for small oscillations about this position.

2-59. The vertical slender rod shown, has a length l and a stiffness EI. The rod supports a weight W at one end, and it is fixed at the other end. Show that the period of this system is



$$\tau = 2\pi\sqrt{\frac{\tan kl - kl}{kg}}$$

where
$$k^2 = \frac{W}{EI}$$

Show that for sufficiently small values of kl this result may be written as

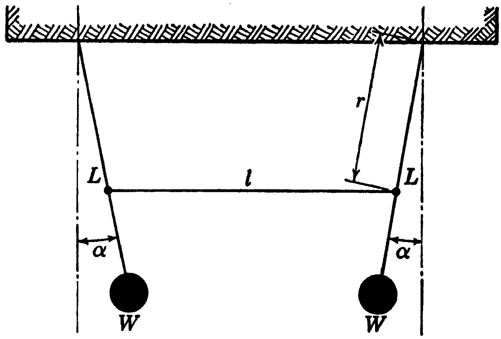
$$\tau = 2\pi\sqrt{\frac{Wl^3}{3gEI}}$$

PROB. 2-59

What is the meaning of $kl = \pi/2$ or $\tau = \infty$ in this problem?

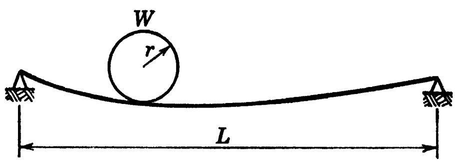
2-60. Solve problem 2-59 if the figure is inverted (weight hanging down).

2-61. Two simple pendulums are tied together by a weightless rod of length l. Calculate the natural circular frequency of this system for small oscillations.



PROB. 2–61

2–62. A flexible horizontal beam of stiffness EI and length L is simply supported as shown. A small cylinder of radius r and weight W rolls back and forth across the beam through small oscillations. Determine the period of the system. Neglect the weight of the beam.

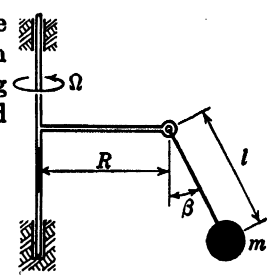


PROB. 2–62

2–63. Solve problem 2–62 if the beam is fixed at the ends.

2–64. Find the natural circular frequency of the mathematical pendulum constrained to oscillate in the rotating vertical plane as indicated, assuming small oscillations, and show that it may be expressed in terms of its dynamic equilibrium position as

$$p^2 = \frac{g}{l\cos\beta}\cdot\frac{R + l\sin^3\beta}{R + l\sin\beta} = \frac{\Omega^2(R + l\sin^3\beta)}{l\sin\beta}$$



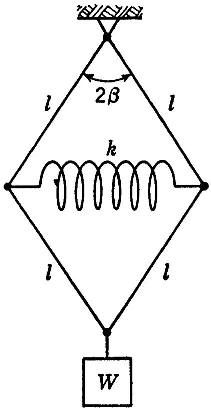
PROB. 2–64

For the particular case of $R = 0$, show that this reduces to

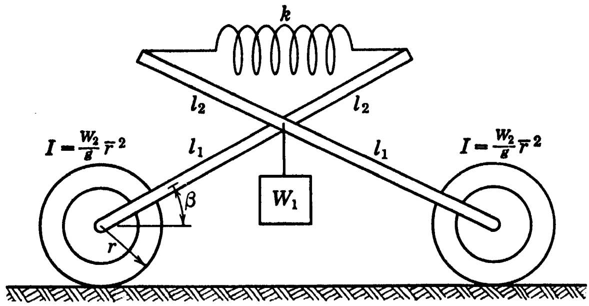
$$p^2 = \frac{g}{l} - \Omega^2 \quad \text{when} \quad \frac{g}{l} > \Omega^2$$

$$p^2 = \Omega^2 - \frac{g}{l} \quad \text{when} \quad \frac{g}{l} < \Omega^2$$

2–65. The system shown is constructed from four weightless links, a spring, and a weight W. Calculate the natural frequency for small vertical oscillations of the weight.



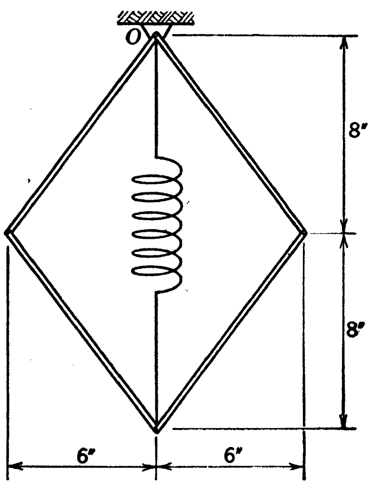
PROB. 2–65



PROB. 2–66

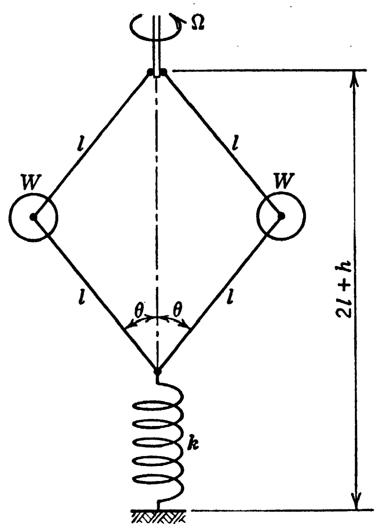
2–66. Find the natural frequency of the system shown if the weights of the frame and spring can be neglected.

2–67. Four equal slender links are connected by a diagonal spring k and suspended by a horizontal pin at O, as indicated in the static position of the system shown. Each of the links weigh 3 lb and are 10 in. long. The unstrained length of the spring is also 10 in. Find the natural frequencies of the system if the weight of the spring is neglected.



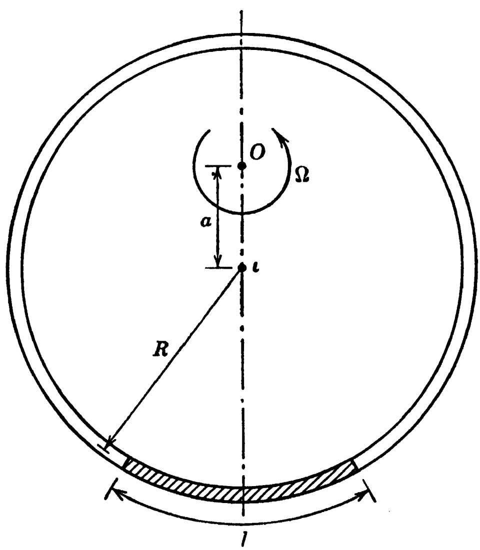
PROB. 2–67

2–68. The masses m are attached to a revolving shaft by means of four weightless arms as shown. The free length of the spring is h. Find the equilibrium position and the frequency for small oscillations about this position if the shaft speed is Ω.



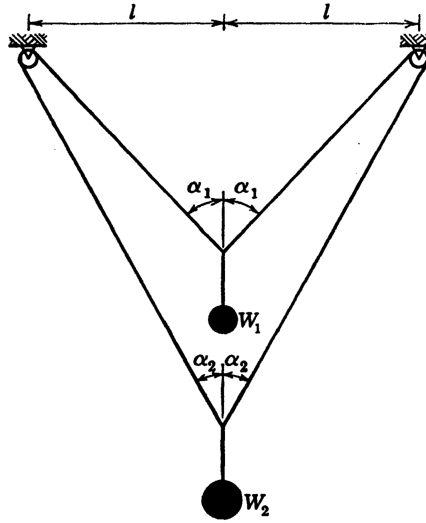
PROB. 2–68

2–69. A circular tube of radius R rotates in a horizontal plane about point O with an angular velocity Ω. The tube contains a column of liquid of length l. Calculate the frequency of the liquid column for small oscillations in the tube.



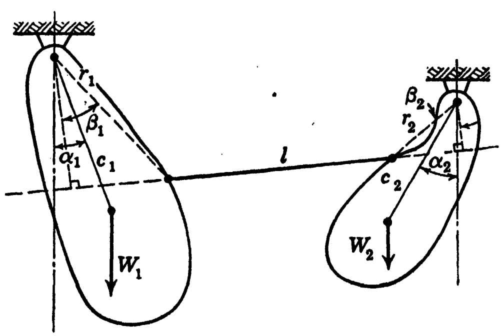
PROB. 2–69

2–70. The weights W_1 and W_2 are supported by a light inextensible cord which passes over two small pulleys at A and B. Determine the natural frequency of the system for small vertical oscillations of the weights. α_1 and α_2 are the angles between the cords and the vertical when the system is at rest.



PROB. 2–70

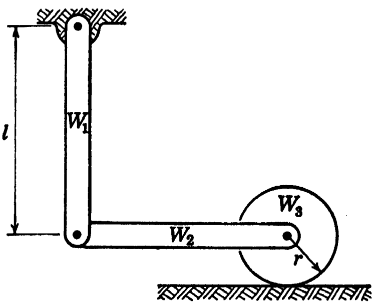
2–71. Two compound pendulums are connected by a link l, as shown in the figure. Find the natural frequency for small oscillations if the link is assumed weightless.



PROB. 2–71

2–72. Solve problem 2–24 by the energy method.
2–73. Solve problem 2–38 by the energy method.
2–74. Solve problem 2–54 by the energy method.
2–75. Solve problem 2–70 by the energy method.

2–76. The link system shown consists of two slender bars W_1 and W_2 and a solid rolling cylinder W_3. Find the natural frequency of this system.



PROB. 2–76

Chapter 3. Forced Vibrations without Damping

3–1. In the system shown in Fig. 3–3,

$$k = 10 \text{ lb per in.} \qquad W = 15 \text{ lb} \qquad P_0 = 20 \text{ lb}$$

Plot the amplitude A versus forced frequency ω for the range

$$0 \leq \omega \leq 50 \text{ rad per sec}$$

3–2. In the system of Fig. 3–4,

$$a = 1 \text{ in.} \qquad \omega = 50 \text{ rad per sec}$$

$$k = 20 \text{ lb per in.} \qquad W = 10 \text{ lb}$$

What is the maximum force in the spring?

3–3. In the previous problem let $\omega = 100$ rad per sec. Locate the node in the spring in terms of a fraction of the spring length above the weight W.

3–4. The motion of the mass in Fig. 3–4 is limited by stops set at a distance $b = 3a$ from the equilibrium position. Over what range of forced frequencies will the mass contact the stops.

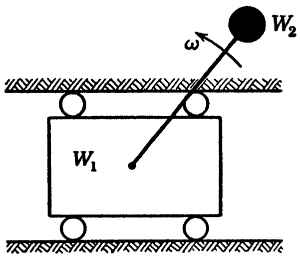
3–5. In the previous problem, what must the magnitude of the forced amplitude be to cause the mass to hit the stops if $\omega = 2p = 2\sqrt{k/m}$.

3–6. The system of Fig. 3–7 is forced to oscillate with an amplitude $A = 2\dfrac{P_0}{k}$. What excitation frequencies will give this amplitude if (*a*) $\omega < p$ and (*b*) $\omega > p$.

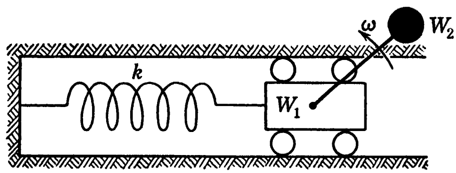
3–7. Show that, if two forced circular frequencies, ω_1 and ω_2, result in the same amplitude for the same system and magnitude of the exciting force, then they are related to the natural frequency of the system by

$$2p^2 = \omega_1^2 + \omega_2^2$$

3–8. A weight W_1 is supported on rollers so that it may move horizontally without friction. A weight W_2 is attached to W_1 by an arm of length r which rotates at and angular velocity ω rad per sec. Calculate the amplitude A of the mass W_1 and its maximum velocity.



Prob. 3–8

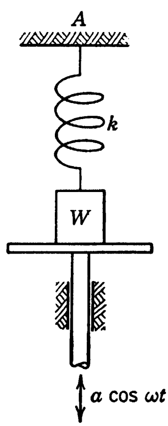


Prob. 3–9

3–9. A spring k is made to oscillate by the action of a rotating weight W_2 on an arm of length r. Calculate the amplitude of the motion.

3–10. An electric motor runs at 1750 rpm and has a total weight of 20 lb. The armature is unbalanced so that a periodic force is transmitted to its surroundings. It is desired to remount the motor on springs so that the force transmitted to the surroundings is reduced by 90 per cent. Determine the spring constant that the spring mount should have.

3–11. A weight W is suspended from a spring k as shown. The weight also rests on a table which oscillates in the vertical direction. When the table is in its mean position, the spring is compressed an amount δ. Over what range of frequencies will the weight remain in contact with the table at all times.



Prob. 3–11

3–12. Solve the previous problem if, in addition, the point of suspension of the spring A has a forced motion

$$x = b \cos \omega t$$

3–13. An accelerometer is used to measure the motion of a structure which vibrates at 10 cycles per minute. The static deflection of the accelerometer is 0.5 in. The record has an amplitude 0.2 in. What is the amplitude of the motion of the structure?

3–14. Find the limiting frequency for which the acceleration can be indicated within 1 per cent of accuracy by an accelerometer having a natural frequency f_a.

3–15. Find the limiting frequency for which the amplitudes can be indicated within η per cent of accuracy by a vibrometer having a natural frequency f_v.

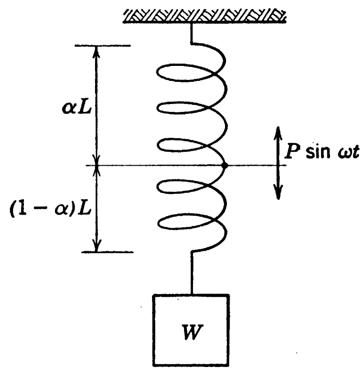
3–16. Find the static deflection of a vibrometer able to indicate within 1 per cent of accuracy the amplitude of systems vibrating with a frequency of 1200 cycles per minute or more.

3–17. The bob of a simple pendulum oscillates with an amplitude $2a$, which is caused by a forced horizontal motion $a \cos \omega t$ of the point of support. Find the length L of the pendulum.

3–18. A horizontal periodic force $P_0 \cos \omega t$ acts on the bob of a simple pendulum of length L and weight W. Determine the amplitude of the motion, assuming small angular displacements.

3–19. A solid cylinder, free to roll without slipping, rests on a horizontal platform which performs a simple harmonic motion in a horizontal plane, perpendicular to the axis of the cylinder. Show that the amplitude of the center of the cylinder is one-third the amplitude of the platform.

3–20. The simple mass-spring system is excited by an oscillating force $P \sin \omega t$, which acts at a point on the spring as shown. Calculate the amplitude of the mass if the spring constant for the whole spring is k.



PROB. 3–20

3–21. Show that the displacement of A may be expressed as

$$x_A = r(\cos \omega t + q_2 \cos 2\omega t + q_4 \cos 4\omega t + \cdots)$$

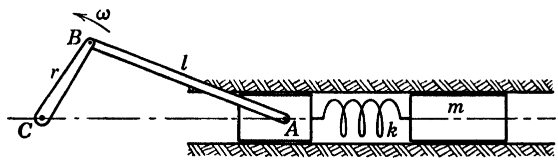
where

$$q_2 = \frac{1}{4}\frac{r}{l}\left[1 + \frac{1}{4}\left(\frac{r}{l}\right)^2 + \frac{15}{128}\left(\frac{r}{l}\right)^4 + \cdots\right]$$

and

$$q_4 = -\frac{1}{64}\left(\frac{r}{l}\right)^3\left[1 + \frac{3}{4}\left(\frac{r}{l}\right)^4 + \cdots\right]$$

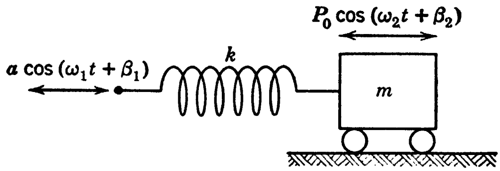
and use this to determine the spring constant k for which the amplitude of the first- and second-order oscillations are the same. Determine also the resultant



PROB. 3–21

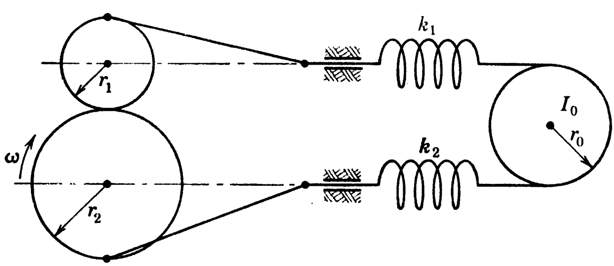
amplitude, and plot the combined steady-state motion of the mass m for these two orders (ω and 2ω).

3–22. Find the motion of the mass in the system shown.



PROB. 3–22

3–23. Determine the motion of the mass of inertia I_0. Consider only the first harmonic of the crank motion in each instance.

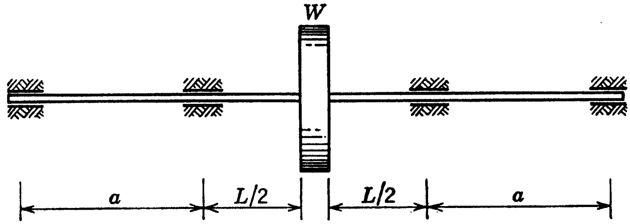


PROB. 3–23

3–24. The circular groove of problem 2–37 is made to oscillate horizontally in the direction normal to the groove and cylinder axis. If the motion of the groove is $a \cos \omega t$, what is the angular amplitude of the cylinder? What is the criterion to insure continuous contact between the cylinder and groove?

3–25. The track along which the cylinder W_1 rolls, in problem 2–38, is given a motion $a \sin \omega t$. Determine the angular amplitude of the system. If the coefficient of friction between the cylinder and the track is μ, find the value of μ so the cylinder rolls without slipping.

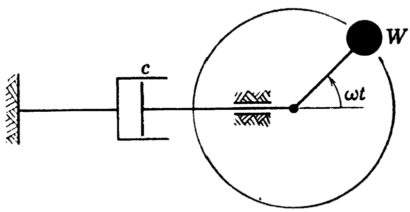
3–26. Calculate the critical speed for the system shown.



PROB. 3–26

Chapter 4. Forced Vibrations with Damping

4–1. A damper c is made to oscillate by the action of a rotating weight W on an arm of length r. Calculate the amplitude and phase of the motion.



PROB. 4–1

4–2. A mass m, subjected to an oscillating force $P \cos \omega t$, rests upon a lubricated surface whose effective coefficient of viscous damping is c. Find its amplitude and its phase referred to the exciting force.

4–3. A mass m rests on an oscillating table which has a motion $x = a \cos \omega t$. The effective damping coefficient between the mass and table is c. Find the amplitude and phase of the tangential force acting between the mass and table. Also determine the amplitude and phase of the mass m.

4–4. A piston oscillates with a harmonic motion

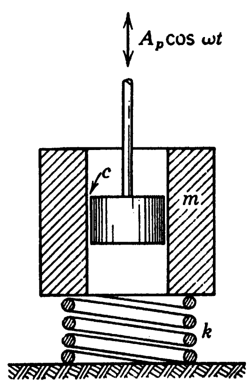
$$x_p = A_p \cos \omega t$$

in a cylinder of mass m which is supported by a spring having a spring constant k. The viscous damping coefficient between piston and cylinder wall is c.

(*a*) Find the motion of the cylinder m and its phase relationship to the piston.

(*b*) Determine the damping coefficient c if the amplitude of the cylinder is found to be A_m and the undamped natural frequency of cylinder and spring are known.

(*c*) With the same data find also the viscosity μ if the bore of the cylinder is D and the diameter and length of the piston are, respectively, d and l.

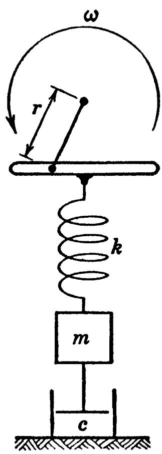


PROB. 4–4

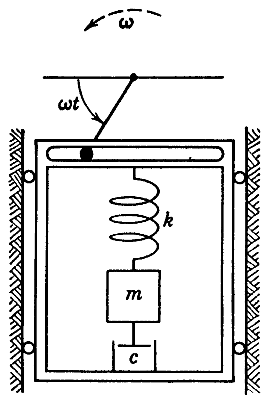
4-5. A Scotch yoke causes a spring, a mass, and a damper to oscillate as indicated.

(*a*) Find the motion of the mass and its phase relationship to the motion of the yoke; $x = r \sin \omega t$.

(*b*) Find the value of the damping coefficient c in terms of the amplitude of the mass A_m when the yoke is operated at the resonance frequency.



PROB. 4-5



PROB. 4-6

4-6. In the system shown, one terminal of the spring and damper, respectively, is tied to the oscillating frame.

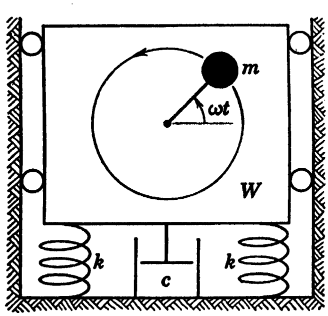
(*a*) Find the motion of the mass and its phase relationship to the motion of the frame

$$x_F = r \sin \omega t$$

(*b*) Find the relative amplitude between mass and frame at resonance.

(*c*) Determine the value of the coefficient of damping c from the measured absolute amplitude of the mass when the frame is oscillating at resonance.

4-7. An unbalanced mass m on an arm r is rotated at an angular velocity ω by a motor, the total weight of which is W. The motor rests on four springs,



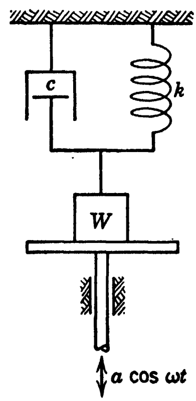
PROB. 4-7

each of which has a spring constant k. The damping is represented by the dashpot shown, which has a viscous damping coefficient c. Find the amplitude and phase of the weight W relative to rotating mass.

$mg = 3$ lb	$c = 25$ lb per in. per sec
$W = 190$ lb	$\omega = 200$ rad per sec
$k = 625$ lb per in.	$r = 1$ in.

4-8. In the previous problem, the amplitude at resonance ($\omega^2 = 4kg/W$) was measured and found to be A. What is the value of the damping coefficient in terms of A and the other constants of the system?

4-9. A weight W is suspended from a spring k and a damper c as shown. The weight also rests on a table which oscillates in the vertical direction. When the table is in its mean position, the spring is compressed an amount δ. Over what range of frequencies will the weight remain in contact with the table at all times? Determine the greatest value of a for which contact is maintained.

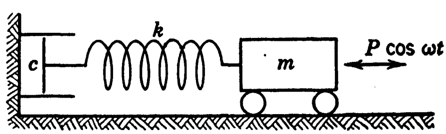


PROB. 4-9

4-10. The mass m is subjected to an oscillating force $P \cos \omega t$. Determine the resonance factor for the mass m if it is defined as

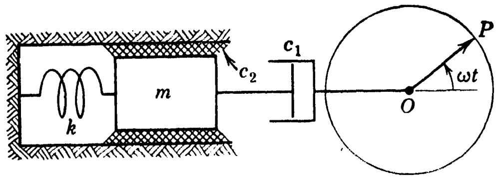
$$\text{Resonance factor} = \frac{kA}{P}$$

where A is the amplitude of the mass m. Check your result with the undamped resonance factor for the limiting case of $c \rightarrow \infty$.



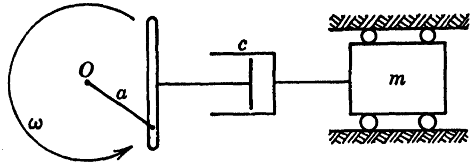
PROB. 4-10

4–11. Find the amplitude and phase of the mass m in the system shown. For what values of c_1 and c_2 will the mass be 90° out of phase with the exciting force?

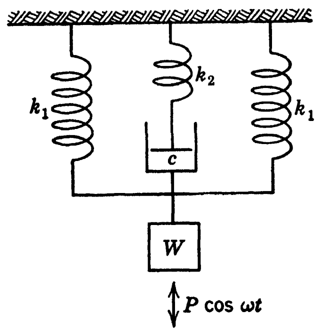


Prob. 4–11

4–12. Calculate the amplitude and phase of the mass m for the system shown.



Prob. 4–12



Prob. 4–13

4–13. Determine the amplitude and phase of the mass as shown.

4–14. In problem 3–23 the inertia I_0 has a viscous bearing friction with an equivalent torsional damping constant c. Calculate the maximum force in each spring.

4–15. If the amplitude of the system of Fig. 4–4 is the same for two different frequencies ω_1 and ω_2, determine the damping constant c.

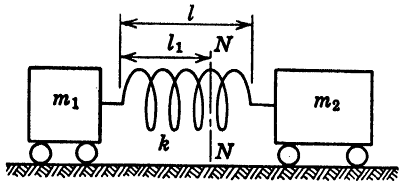
4–16. The accelerometer of problem 3–13 is damped by viscous friction between the weight W and the frame. Assuming the damping to be 10 per cent of critical damping or $(p/q = 0.2)$, calculate the amplitude of the structure. Use the same data as in problem 3–13.

4–17. Find the limiting frequency for which the amplitudes can be indicated to within η per cent of accuracy by a vibrometer having a natural frequency f_v and one-fifth critical damping. $(p/q = 0.40)$. Show that such damping is not significant in this respect.

Chapter 5. Classical Methods

Problems 5–74 to 5–79 in this section are especially selected as exercises in the use of Lagrange's equations, but other problems in this chapter may equally well be used as exercises in this method.

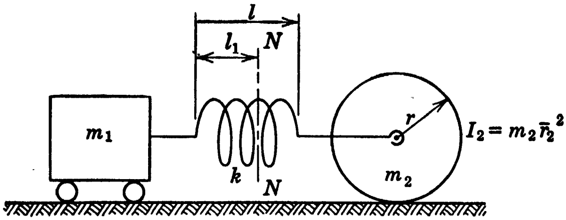
5–1. Find natural frequency, and locate the node in the system shown.



PROB. 5–1

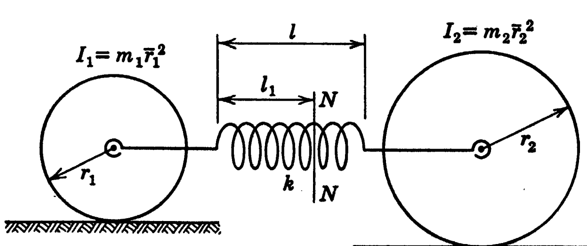
5–2. Plot the motion and velocity during a cycle of the two masses in roblem 5–1 if the mass m is given an initial impact velocity V_1. m_2 is initially at rest.

5–3. Find the natural frequency and node of the system shown, in which the roller with radius r and moment of inertia $I_2 = m_2\bar{r}_2^2$ rolls without slipping.



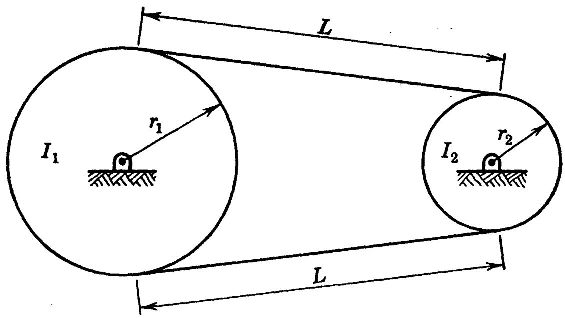
PROB. 5–3

5–4. Two rollers, connected by a spring, roll without slipping as indicated. Find the natural frequency and node of the system.



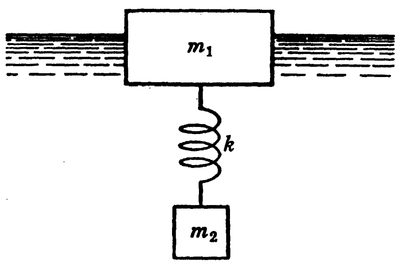
PROB. 5–4

5–5. Find the natural frequency of the pulleys connected by an elastic belt. The pulley moments of inertia are I_1 and I_2 with radii r_1 and r_2, respectively. The belt has an initial tension T, a cross-sectional area A, a modulus of elasticity E, and free lengths L, as indicated.



PROB. 5–5

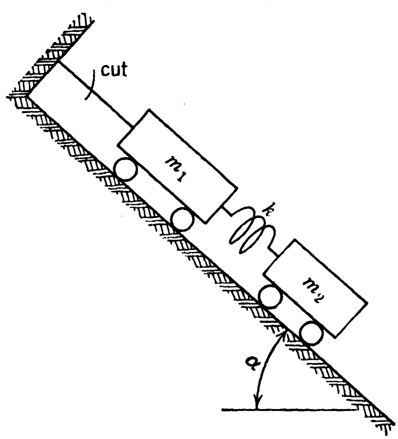
5–6. Find the natural frequencies and the relative amplitudes of the float and the suspended weight as indicated, if the surface area of the float is A and the specific weight of the fluid is w.



PROB. 5–6

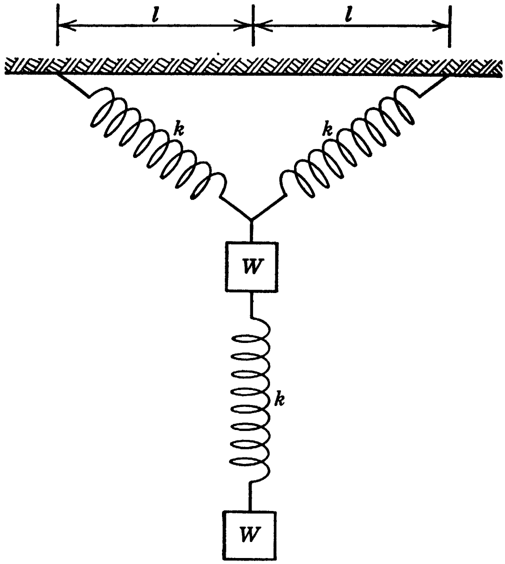
5–7. Find the natural frequencies of the system of problem 2–53 if the weight W_1 is considered.

5–8. Two masses connected by a spring, as indicated, are cut loose and roll without friction down the slope. Find the position of each mass after t sec.



PROB. 5–8

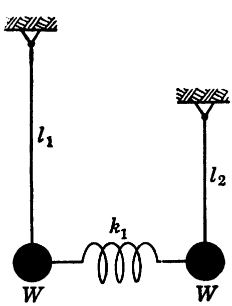
5-9. Two equal weights W are suspended by three equal springs of unstrained lengths l. The two upper springs are elongated $\frac{1}{4}l$ in their static position. Find the natural frequencies of the system if the weights are constrained to vertical motion, and determine the relative amplitudes of the masses.



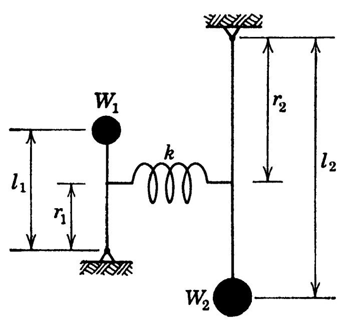
PROB. 5-9

5-10. Show that the ratio of amplitudes between the two equal pendulum weights W, constrained to plane motion, can be expressed as

$$\frac{1}{2}\frac{W}{k}\left(\frac{1}{l_2} - \frac{1}{l_1}\right) \pm 1$$



PROB. 5-10



PROB. 5-11

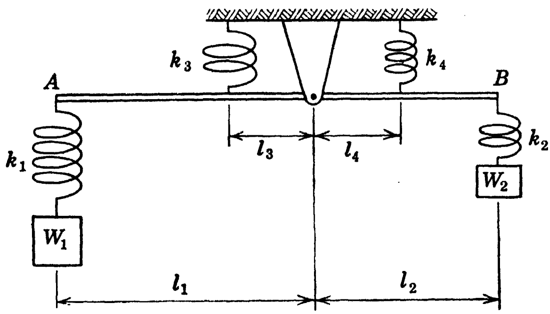
5-11. Find natural frequencies of the pendulum system, and determine the condition for stability of free oscillations about the position shown.

5–12. (*a*) Write the frequency equation for the system shown, neglecting the weight of the lever *AB*.

(*b*) Find the frequency for the particular case in which

$$k_1 = k_2 = k_3 = k_4$$
$$l_1 = 2l_2 = 4l_3 = 4l_4$$
$$W_2 = 2W_1$$

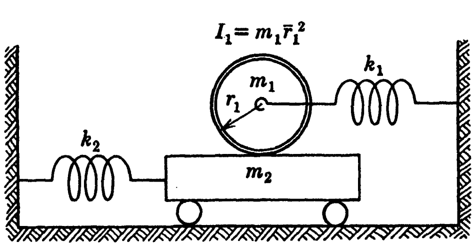
(*c*) Find for these particular values the ratios between the amplitude of W_1 and W_2 and the corresponding amplitude of the lever *AB*.



PROB. 5–12

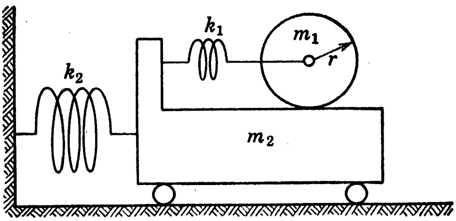
5–13. Find the natural frequencies of the system shown, and determine the relative amplitudes if no sliding occurs:

(*a*) With the spring k_1 inactive.
(*b*) With the spring k_2 inactive.
(*c*) With both springs k_1 and k_2 active.



PROB. 5–13

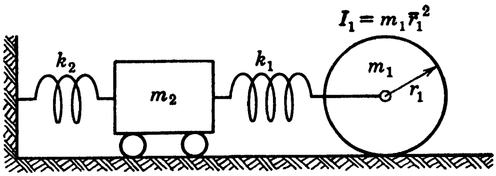
5–14. Find the natural frequencies and the ratio of the amplitudes for a solid homogeneous cylinder m_1, rolling without slipping on a carrier m_2 as indicated. Evaluate for $m_1 = m_2$ and $k_1 = k_2$.



PROB. 5–14

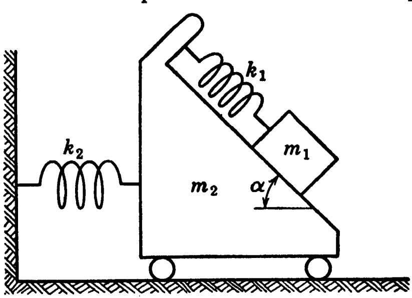
5–15. Find the minimum value of the coefficient of friction between m_1, and m_2 in Problem 5–14 to insure rolling without slipping.

5–16. Find the natural frequencies and relative amplitudes for the system shown.



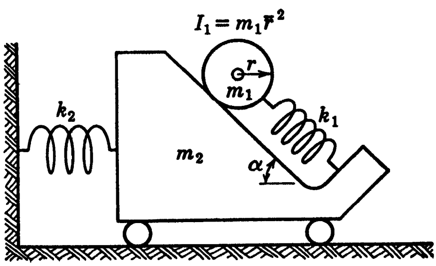
PROB. 5–16

5–17. A block slides without friction on an inclined plane of a carrier as indicated. Find the natural frequencies and relative amplitudes.



PROB. 5–17

5–18. A solid cylinder rolls without slipping on an inclined plane of a carrier as indicated. Find natural frequencies and relative amplitudes.

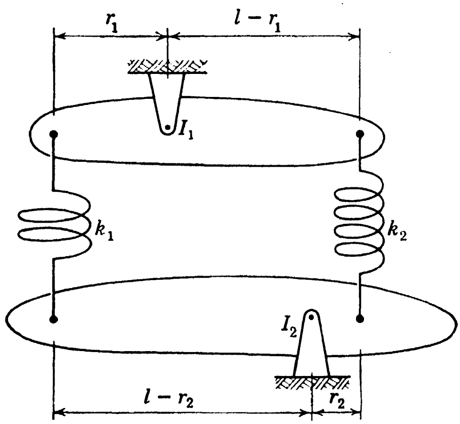


PROB. 5–18

5-19. Find the natural frequencies of a homogeneous rectangular plate supported at each corner by parallel springs normal to the plate of equal spring constant. Consider only those modes that do not involve motion in the plane of the plate.

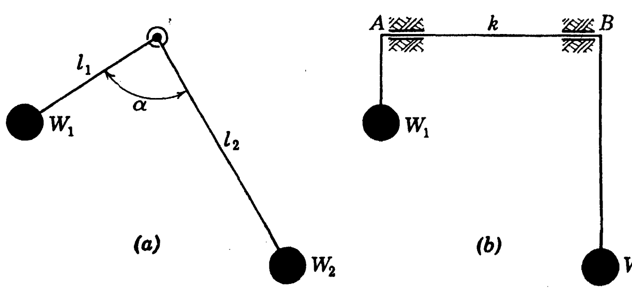
5-20. Determine the natural frequencies and nodal axes of a homogeneous triangular plate with sides 3:4:5 which is supported at each corner by springs of equal spring constant.

5-21. Write the frequency equation for the system, and evaluate the frequencies for $I_1 = I_2$, $r_1 = r_2 = \beta l$, and $k_1 = k_2$.



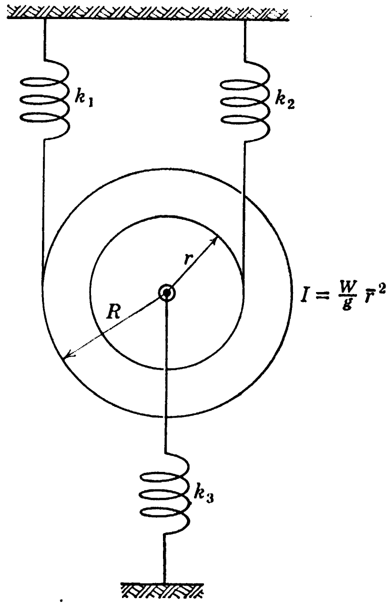
PROB. 5-21

5-22. Two concentrated weights, W_1 and W_2, are connected by a horizontal shaft AB whose torsional spring constant is k as indicated. Find the natural frequencies of the system and evaluate for $\alpha = 90°$, $l_2 = 2l_1 = l$, $W_1 = 3W$, and $W_2 = 2W$.



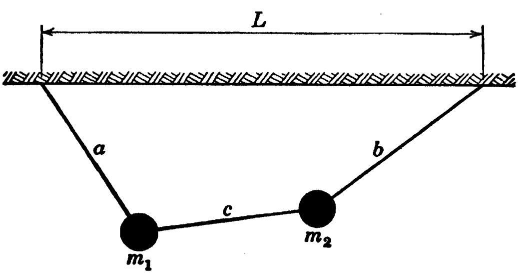
PROB. 5-22

5-23. Find natural frequencies and the relationship between the angular and translatory amplitudes of system shown.



PROB. 5–23

5–24. Find the natural frequencies and the relative amplitudes of the corresponding modes for the two concentrated masses, suspended by strings, as indicated if $a = b$ and $m_1 = m_2$.



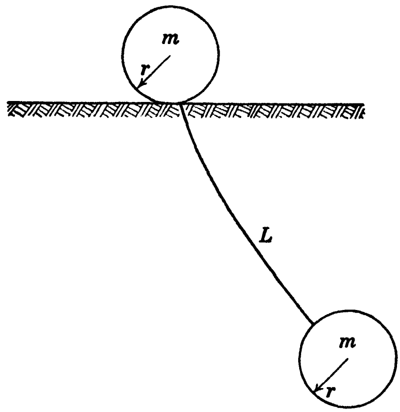
PROB. 5–24

5–25. Find the frequencies of the linked pendulums in Problem 2–61 if the connecting link of length l is replaced by a coil spring of the same length in its static position and has a spring constant k.

5–26. Write the frequency equations for the system in problem 2–71 if the link is replaced by a weightless coil spring having a spring constant k.

5–27. Find the natural frequencies of the mechanism in problem 2–42 if the connecting rod L is replaced by an elastic bar having a spring constant k.

5–28. Two cylinders of equal mass and radii are connected by a flexible leaf spring of stiffness EI and length L. The upper cylinder rolls without slipping on a horizontal track. Write the frequency determinant for this system.



PROB. 5–28

5–29. Find the amplitude of the two masses, m_1 and m_2, in problem 5–1 if m_1 is subjected to an oscillating force $P_0 \sin \omega t$.

5–30. Find the amplitudes of the masses m_1 and m_2 in problem 5–3 if

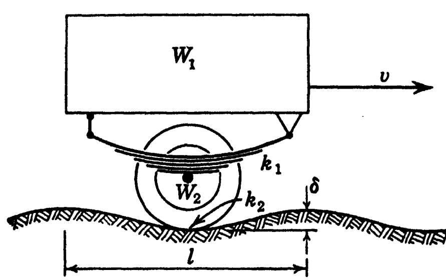
(*a*) a horizontal force $P_0 \cos \omega t$ is applied to m_1.

(*b*) a horizontal force $P_0 \cos \omega t$ is applied to the center of m_2.

5–31. Find the amplitudes of the rollers in problem 5–4 if a force $P_0 \sin \omega t$ acts at the center of m_1 and if no slipping takes place.

5–32. Find the amplitudes of the two weights W_0 and W_1 in problem 2–53 if an oscillating force $P_0 \cos \omega t$ acts at a point on the spring one third of the length of the spring from W_1. Evaluate for $W_1 = 3W_0$.

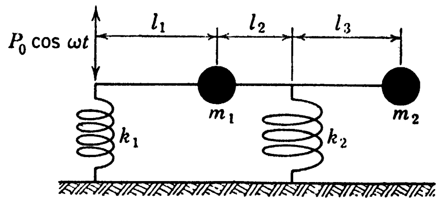
5–33. A two-wheeled trailer is towed over a sinusoidal surface at a speed v. The sprung weight is W_1, and the weight of the wheels and axle is W_2. The elliptic springs have a spring constant k_1, and the tires have a spring constant k_2. Calculate the critical speeds, and plot the amplitude versus speed curves for both masses.



PROB. 5–33

5-34. (*a*) Find the node and the angular amplitude of the system shown.

(*b*) Determine the natural frequencies and the corresponding node. Evaluate for $m_1 = 2m_2$, $k_1 = k_2$, and $l_1 = 3l_2 = l_3$ and $\omega^2 = \frac{k_z}{m_z}$.

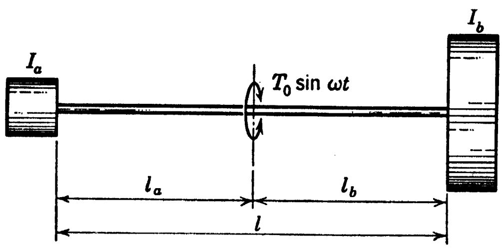


PROB. 5-34

5-35. The oscillating torque $T_0 \sin \omega t$ is applied at a point on a shaft between two masses, I_a and I_b, as indicated in the diagram.

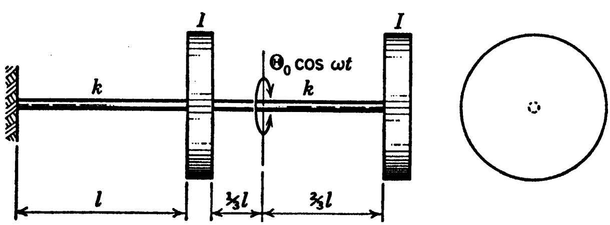
(*a*) Find the amplitude of the masses.

(*b*) Determine the amplitude of one of the masses if the amplitude of the other is zero, and find the corresponding torque frequency.



PROB. 5-35

5-36. A point on a shaft, $\frac{2}{3}l$ from the end, as shown, is subjected to a forced oscillation of $\Theta_0 \cos \omega t$. Find the amplitudes of the two masses, the required torque amplitude, and the resonance frequencies.

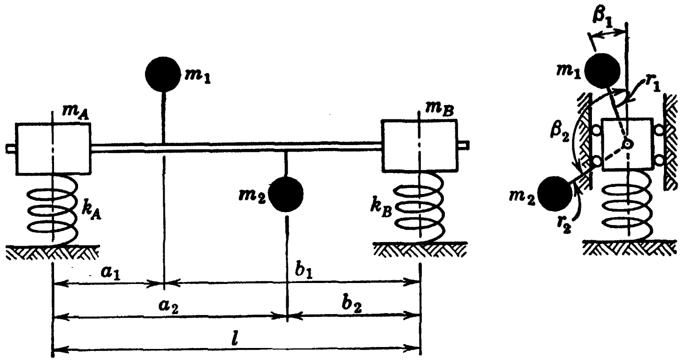


PROB. 5-36

5-37. A shaft with unbalanced masses m_1 and m_2 is elastically supported in the vertical plane as indicated. The shaft rotates at a constant angular velocity.

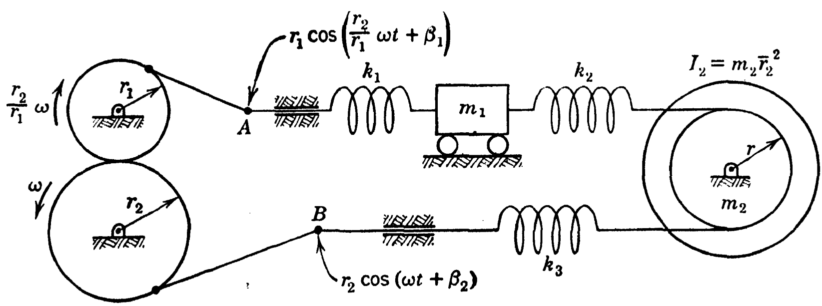
(*a*) Find the vertical component of the dynamic forces, P_A and P_B, at A and B and their phase relationship at any time t.

(*b*) Find the motion of the bearings and the angular amplitude of the shaft.



Prob. 5–37

5–38. Assuming simple harmonic motion of A and B as indicated in the diagram, find the amplitudes of the masses m_1 and m_2.



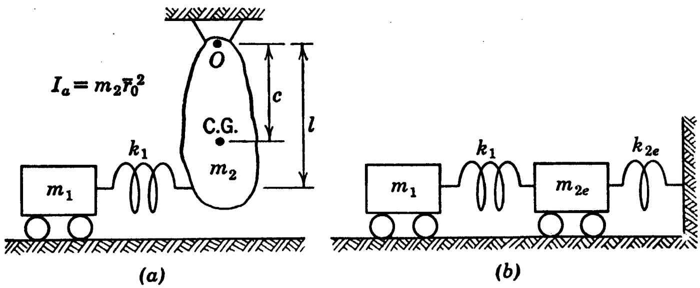
Prob. 5–38

5–39. If the surface of the fluid in problem 5–6 is subjected to a vertical wave motion $a \sin \omega t$, find the amplitude of the submerged mass.

5–40. Reduce the system of a translatory mass and a pendulum as shown in problem 2–53 to an equivalent system, of two translatory masses, which has the same frequencies, and determine the value of the equivalent spring constant.

5–41. Reduce the system in problem 5–40 to an equivalent torsional system, and determine the values of the equivalent torsional spring constants and moments of inertia.

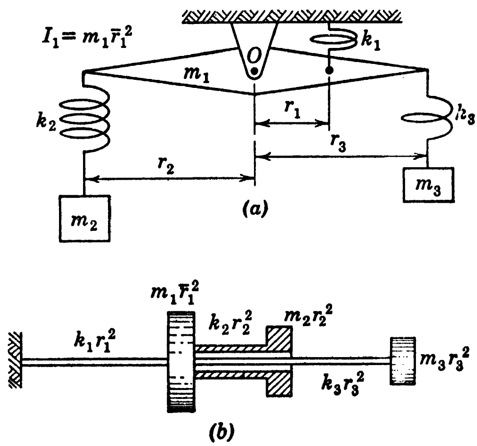
5–42. Reduce the system indicated in *a* to an equivalent translatory system, of the same frequencies, as shown in *b*. Determine the value of the equivalent springs and masses and also the equivalent force for an oscillating torque T_0 applied at O, in a.



PROB. 5–42

5–43. Reduce the system in problem 5–42 to an equivalent torsional system, and evaluate the equivalent torsional spring constants and moments of inertia.

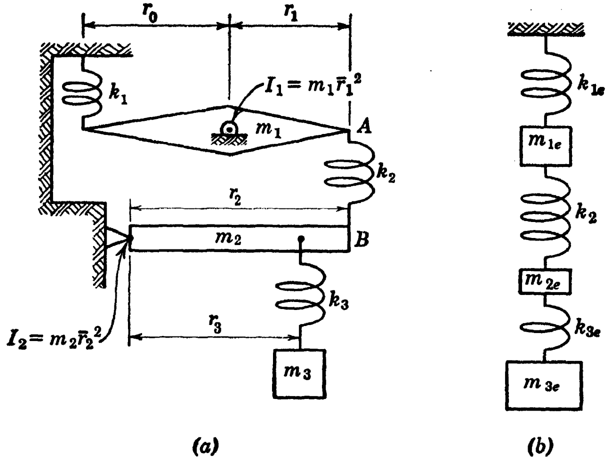
5–44. Show that the system indicated in a may be reduced to an equivalent torsional branched system as indicated in b.



PROB. 5–44

5–45. Replace the equivalent torsional system shown in problem 5–44 by an equivalent translatory system, and determine the values of the equivalent springs and masses.

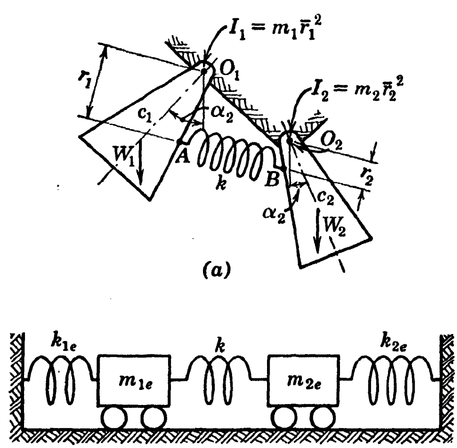
5–46. Find the equivalent values of the springs and masses in the translatory system b, which has the same frequencies as the original system a. Assume the spring k_2 to be the same in both systems.



PROB. 5–46

5–47. Replace the system in problem 5–46 by an equivalent rotary system, and determine the value of the equivalent torsional spring constants and moments of inertia. Let I_1 be the same as for the original system.

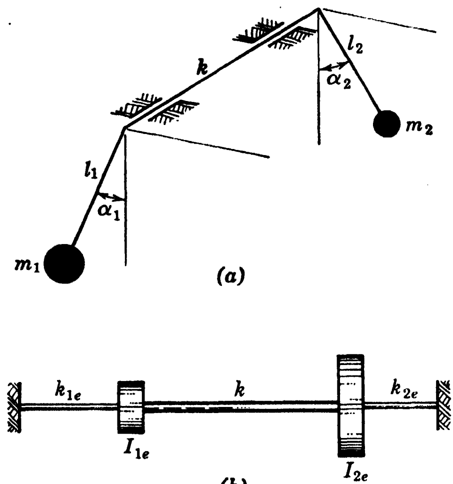
5–48. Determine the values of the springs and masses in the translatory system b which have the same frequencies as the system a. (O_1A and O_2B are assumed to be perpendicular to the spring axis AB.)



PROB. 5–48

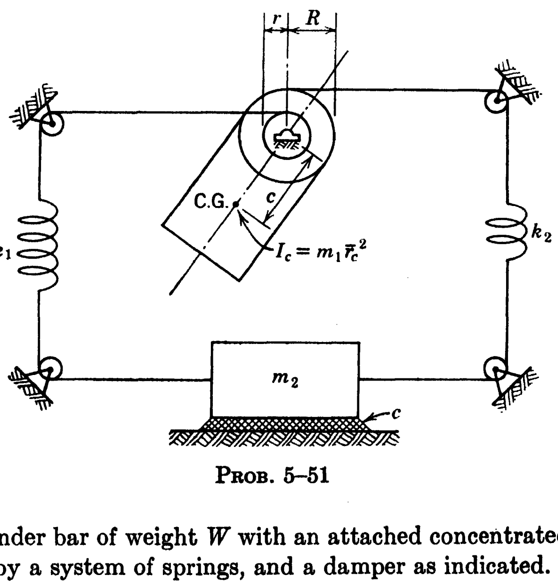
5–49. Replace the system in problem 5–48a by an torsional system, and determine the equivalent torsional spring constants and moments of inertia.

5–50. Determine the values of the torsional spring constants and moments of inertia in *b* which will make it have the same frequencies as the system of two pendulums connected by a horizontal elastic shaft as shown in *a*.



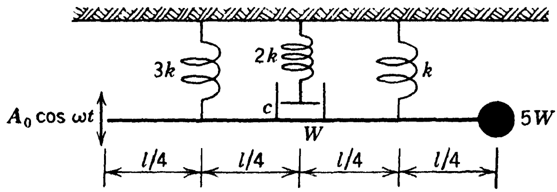
Prob. 5–50

5–51. Find the natural frequencies of the system shown if the damping coefficient $c = 0$.



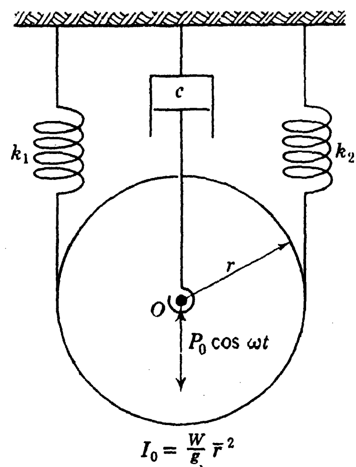
Prob. 5–51

5–52. A slender bar of weight W with an attached concentrated weight $5W$ is suspended by a system of springs, and a damper as indicated. One end of the bar is made to oscillate with a motion $A_0 \cos \omega t$. Find the amplitude of the concentrated mass and angular amplitude of the bar for the specific case for which $\omega = 4k/c = 2\sqrt{kg/W}$.

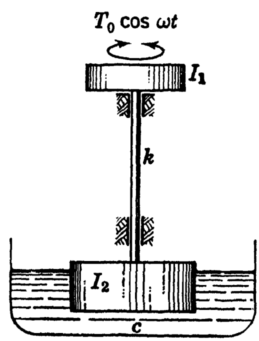


PROB. 5–52

5–53. Find angular and translatory amplitudes of the system shown.



PROB. 5–53



PROB. 5–54

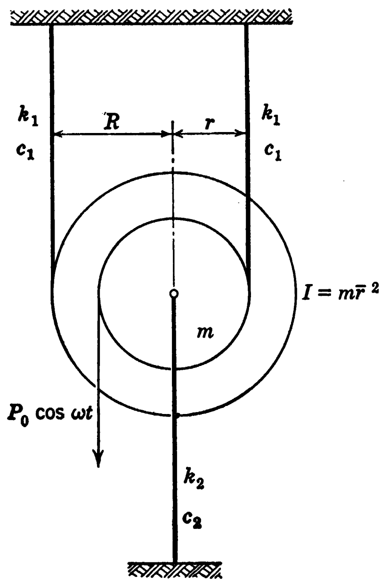
5–54. Find the angular amplitudes of I_1, I_2, and the shaft k, as well as the respective phase relationships with the torque $T_0 \cos \omega t$:

(a) Without damping ($c = 0$).

(b) With damping as shown if

$$I_2 = 4I_1 \quad \text{and} \quad \omega = 4\frac{k}{c} = \sqrt{\frac{k}{I_1}}$$

5–55. Find the translatory and angular amplitudes of the pulley suspended by rubber bands as indicated and subjected to an eccentric oscillating force $P_0 \cos \omega t$ as shown. The rubber bands are assumed to act as a spring and damper connected in parallel. Use $k_1 = k_2 = k_3$, $c_1 = c_2 = c_3$, $R = 2r = 2\bar{r}$ and $\omega = 2\frac{k_1}{c_1} = \sqrt{\frac{k_1}{m}}$.



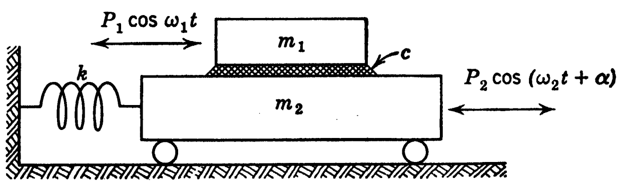
PROB. 5–55

5–56. Assuming a horizontal oscillating force $P_0 \cos \omega t$, acting on the mass m_2, in problem 5–17, and in addition an interacting viscous damping force between m_1 and m_2, find the amplitudes and phase of the masses relative to P_0. Evaluate these results for $k = k_1 = k_2$, $m = m_1 = m_2$, and

$$\omega = \frac{k}{c} = \sqrt{\frac{k}{m}}; \text{ for } \cos \alpha = 1 \text{ and } 4/5.$$

5–57. Find the amplitudes of m_1 and m_2 and the phase relationship with the disturbing forces in the system shown:

(*a*) for $P_1 = 0$.
(*b*) for $P_2 = 0$.
(*c*) with both forces, P_1 and P_2, acting.



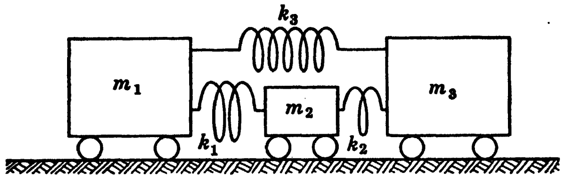
PROB. 5–57

5–58. Determine the number of degrees of freedom in the system shown and write the frequency equation in determinant form. Solve this equation for the particular case in which

$$m_1 = m_2 = m_3$$

and

$$k_1 = k_2 = k_3$$

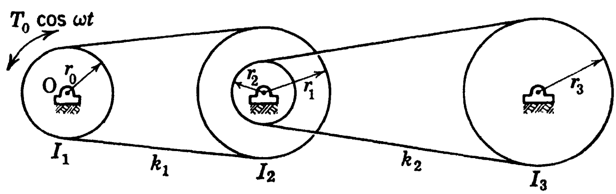


PROB. 5–58

5–59. Show that the frequency equation for this system has the form

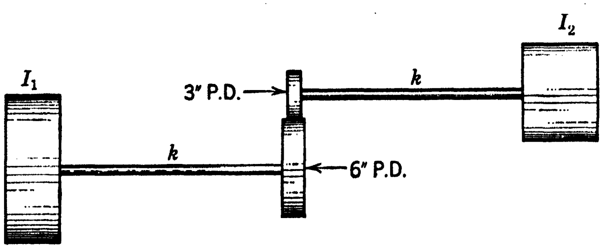
$$I_1 I_2 I_3 \omega^4 - [I_1 I_2 k_2 r_3^2 + I_1 I_3 (k_1 r_1^2 + k_2 r_2^2) + I_2 I_3 k_1 r_0^2]\omega^2 + k_1 k_2 [I_1 r_1^2 r_3^2 + I_2 r_0^2 r_3^2 + I_3 r_0^2 r_2^2] = 0$$

and find the forced amplitudes of the pulleys, if a torque $T_0 \cos \omega t$ is applied at O as shown. The spring constants k_1 and k_2 are the total for each belt.



PROB. 5–59

5–60. Two shafts of equal length and diameters connect two inertias, I_1 and I_2, through the gears as indicated. (The mass of shafts and gears is assumed to be negligible.) I_1 is subjected to a torque $T = T_{avg} + T_0 \sin \omega t$.



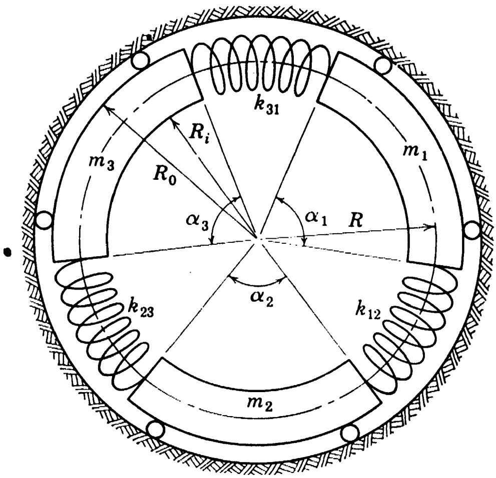
PROB. 5–60

Show that the amplitude of oscillating torque transmitted to I_2 can be expressed as

$$T_2 = \frac{T_0}{1 + \dfrac{I_1}{I_{2e}} - \left(\dfrac{\omega}{p_1}\right)^2}$$

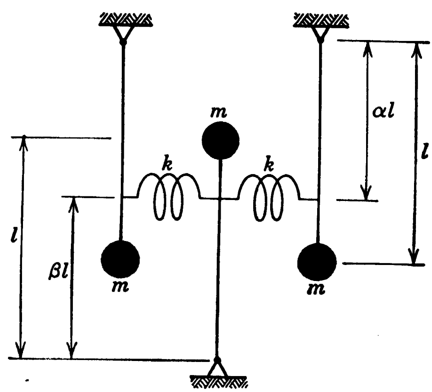
where I_{2e} is the moment of inertia of I_2 corrected to the speed of I_1, and p_1 is the natural circular frequency of I_1 and the shafting if I_2 is fixed. Find the amplitudes Θ_1 and Θ_2 of I_1 and I_2, respectively. Evaluate the answers obtained above for $I_1 = I_2$ and $\omega^2 = 2k/I_1$, where k is the torsional spring constant of each shaft.

5–61. Write the frequency equation for the system shown where each mass has a rectangular cross section.



PROB. 5–61

5–62. Find the frequencies of the system indicated, and determine the condition for stability of oscillations about the indicated static position, if $\alpha = \beta$.

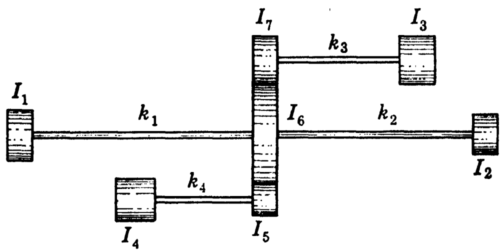


PROB. 5–62

5-63. Reduce the geared system shown to an equivalent ungeared system referred to the center shaft, using the following data:

$$\begin{aligned} k_1 &= 0.8 \times 10^5 \text{ in.-lb} & d_5 &= 4 \text{ in.} \\ k_2 &= 1 \times 10^5 \text{ in.-lb} & d_6 &= 12 \text{ in.} \\ k_3 &= 4 \times 10^5 \text{ in.-lb} & d_7 &= 6 \text{ in.} \\ k_4 &= 6 \times 10^5 \text{ in.-lb} \\ I_1 &= I_7 = 20 \text{ in.-lb sec}^2 \\ I_2 &= I_5 = 10 \text{ in.-lb sec}^2 \\ I_3 &= I_4 = 30 \text{ in.-lb sec}^2 \\ I_6 &= 60 \text{ in.-lb sec}^2 \end{aligned}$$

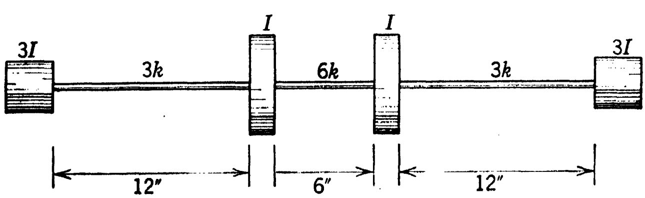
where d_5, d_6, and d_7 are the pitch diameters of the gears I_5, I_6, and I_7, respectively.



PROB. 5-63

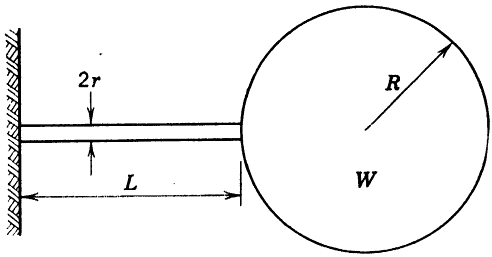
5-64. Use the numerical data from problem 5-63 to write the frequency equation as formulated in equation 6.4-1. Determine by inspection the range within which the natural frequencies of the system will be found. Estimate these frequencies by trial substitutions in the frequency equation, determine the nodes, and show the normal elastic curves of the system for each mode.

5-65. In the symmetric system indicated, determine the second mode by inspection and find the frequency and the location of nodes for this mode.



PROB. 5-65

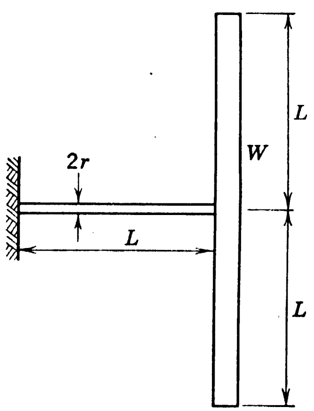
5-66. A solid sphere is attached to a solid circular shaft of length L. Find the natural frequencies if the weight of the shaft is neglected.



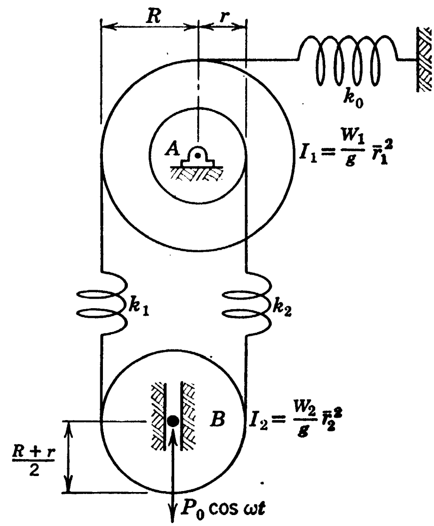
PROB. 5-66

5-67. Find the frequencies of the system shown in problem 2-33 if the inertia of the pulleys W_1 and W_2 are not neglected. Find the ratio of amplitudes of the two pulleys. Evaluate this result if the pulleys and springs are identical. Assume that the pulley radius is equal to its radius of gyration and that $W = W_1 = W_2$.

5-68. A rigid slender homogeneous bar of weight W is fastened to a solid circular shaft as indicated. Find the natural frequencies if the weight of the shaft is neglected.



PROB. 5-68



PROB. 5-69

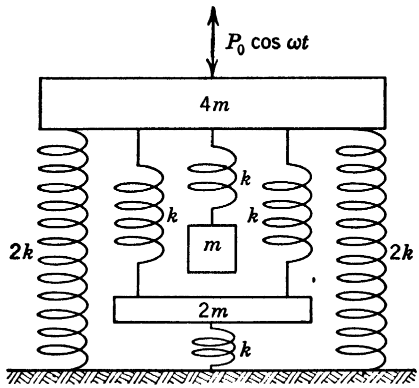
5-69. Find natural frequencies of system shown, and determine the relationship between the amplitudes of the springs k_1 and k_2. Evaluate for the particular case of $k_0 = \frac{2}{3}k_1 = \frac{2}{3}k_2$, $R = 3r$, and $I_1 = 4I_2 = \frac{W_1}{g}(2r)^2$, when $W_1 = W_2 = W$.

5-70. A three mass system, as shown, is subjected to a forced oscillation by the force $P_0 \cos \omega t$.

(*a*) Determine the amplitude of the masses.

(*b*) Find the frequencies ω for which a node is produced at a mass, and evaluate the amplitudes of the other masses under these conditions.

(*c*) find the resonance frequencies of the system.



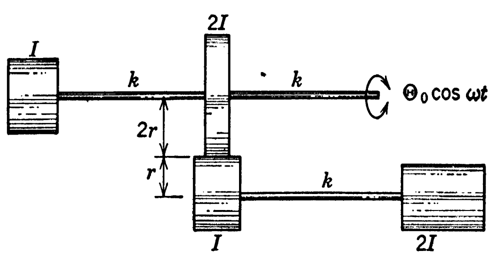
PROB. 5-70

5-71. In the branched geared system shown:

(*a*) find the amplitudes and nodes of the forced system.

(*b*) find the amplitude of the forcing torque T_0 corresponding to the forced amplitude Θ_0.

(*c*) find the natural frequencies and determine the normal elastic curves of the free system.



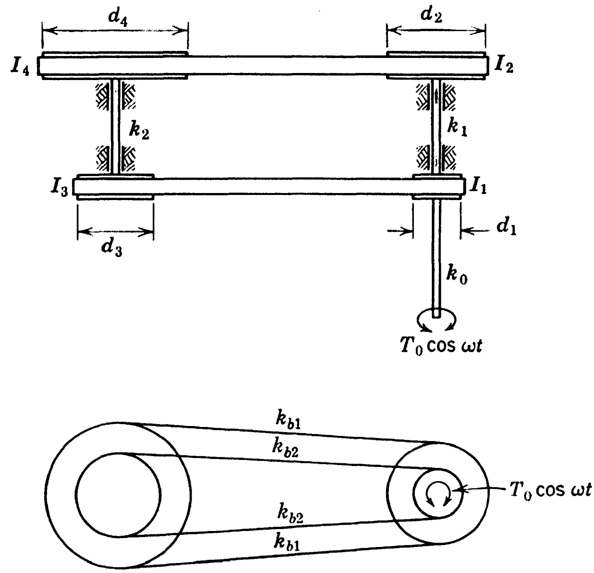
PROB. 5-71

5-72. In problem 5-69 an oscillating force $P_0 \cos \omega t$ acts on pulley B as shown. Find the amplitudes of the pulleys in translation and rotation. Consider the specific case for which $\omega = \sqrt{k_0 g/W}$, and use the specific values of the springs and masses given in problem 5-69.

5-73. Two shafts are connected by elastic belts through two sets of pulleys with the same diameter ratio.

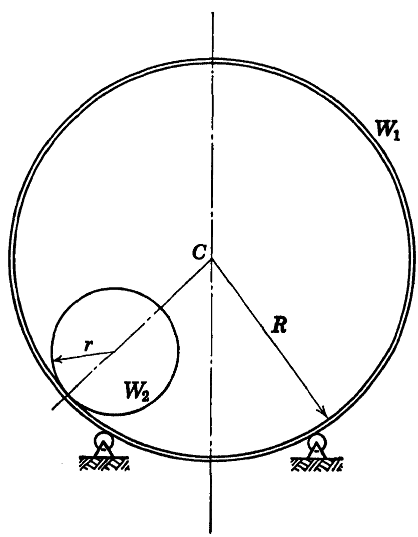
(*a*) Find amplitudes of pulleys if the system is subjected to an oscillating torque $T_0 \cos \omega t$ as indicated in the diagram.

(*b*) Write the frequency equation in determinant form.



PROB. 5–73

5–74. A thin cylindrical shell of radius R and weight W_1 is supported on frictionless rollers so that it can rotate about its longitudinal axis through C.

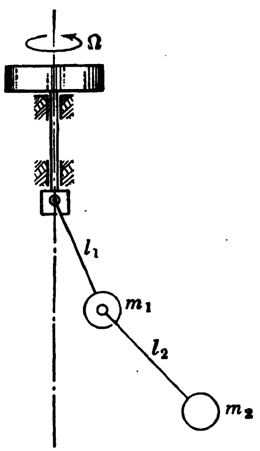


PROB. 5–74

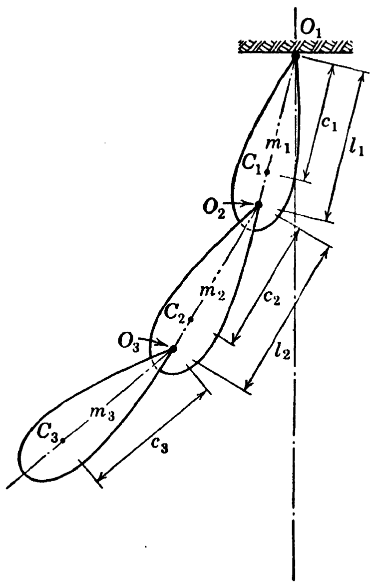
A small cylinder of radius r and weight W_2 is placed inside the cylindrical shell as shown. Determine the natural frequency of this system for small oscillations if there is no slipping between the cylinder W_2 and the shell W_1.

5-75. Solve problem 5-74 if the cylindrical shell W_2 is permitted to roll on a horizontal plane.

5-76. Find the natural frequencies of a double centrifugal pendulum, constrained to oscillate in a vertical rotating plane.



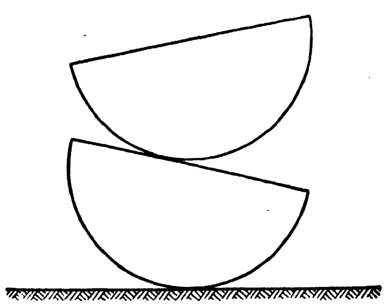
Prob. 5-76



Prob. 5-77

5-77. Determine the frequencies and corresponding modes of the triple pendulum shown. The radii of gyration of the masses m_1, m_2, and m_3 are $\bar{r}_1$, $\bar{r}_2$, and $\bar{r}_3$ with respect to their centroids.

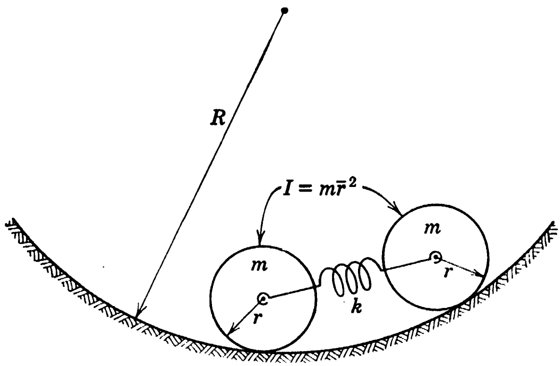
5-78. A cylinder is split into two halves and one placed on top of the other as indicated. Find the natural frequencies of this system for small oscillations.



Prob. 5-78

The static position is symmetrical about a vertical axis, and the system rolls without slipping. Locate the centroids of the half cylinders so that the system will just be stable, and calculate the frequencies for this condition.

5–79. Two rollers, connected by a spring, roll without slipping on a cylindrical surface as indicated. Find the natural frequencies for small amplitudes if the radii R through the centers of the rollers make an angle φ_0 with the vertical in the equilibrium positions.

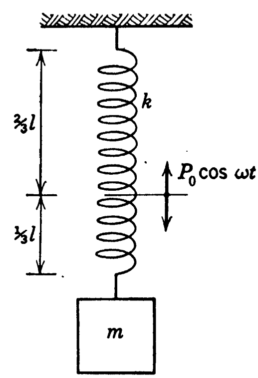


PROB. 5–79

Chapter 6. Mobility Method

The problems listed under this heading are especially selected as exercises in the use of this method. A schematic diagram or circuit showing the proper combination of the elements, with appropriate notation, should be made before the analysis of each problem. Of course, all problems listed under the previous chapters may be utilized as exercises for the mobility method.

6–1. The oscillating force $P_0 \cos \omega t$ acts at a point on the spring one third of its length from the mass m. Find the amplitude of the mass and of the point of application of the force.



PROB. 6–1

6–2. Solve problem 4–1 by the mobility method.

6–3. Solve problem 4–2 by the mobility method.

6–4. Solve problem 4–3 by the mobility method.

6–5. Solve problem 4–4 by the mobility method.

6–6. Solve problem 4–5 by the mobility method.

6–7. Solve problem 4–6 by the mobility method.

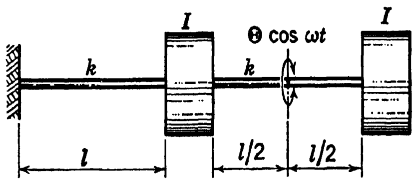
6–8. Solve problem 4–7 by the mobility method.

6–9. Solve problem 4–8 by the mobility method.

6–10. Solve problem 4–10 by the mobility method.

6–11. Solve problem 4–12 by the mobility method.

6–12. In the torsional system shown, the midpoint of one shaft is subjected to an oscillation $\Theta \cos \omega t$. Find the amplitudes of the two masses. Also find the torque amplitude, and write the frequency equation ($\mathbf{z} = 0$). Plot the normal elastic curve for the forced oscillation and for the two modes of free vibration.



PROB. 6–12

6–13. Find the amplitudes of the masses m_1 and m_2 in problem 5–42 if a torque $T_0 \cos \omega t$ is applied to m_2.

6–14. Find the amplitudes of the masses in problem 5–42 if a horizontal force $P_0 \cos (\omega t + \alpha)$ is applied to the mass m_1.

6–15. Find the amplitudes of the three masses, m_1, m_2, and m_3, in problem 5–44 if a torque $T_0 \cos \omega t$ is applied to the lever m_1. Write the frequency equation ($\mathbf{z} = 0$) for the free system.

6–16. Find the resultant amplitudes of the masses and their phase relationship if the torque T_0 and the force P_0 as stated in the previous problems 6–13 and 6–14 are acting simultaneously and the phase angle $\alpha = \pi/2$.

6–17. An oscillating force causes the mass m_3 in problem 5–44 to oscillate with a motion $A_3 \cos \omega t$. Find the amplitude of the other masses and of the disturbing force. Evaluate these results for $k_1 = k_2 = 2k_3$; $r_2 = r_3 = 2r_1 = 2\bar{r}_1$; $m_1 = 2m_2 = 2m_3$; and $\omega^2 = k_2/m_2$.

6–18. An oscillating torque applied at the point of suspension of the pendulum of length l_1 in problem 5–10 causes it to oscillate with an amplitude Θ_1 and a circular frequency ω. Find the amplitude of the torque and the angular amplitude Θ_2 of the other pendulum if $l_1 = 2l_2$ and $\omega^2 = g/l_1$.

6–19. Evaluate the force and displacement amplitudes for the system in Fig. 6–16a in the text, using the numerical data given in Table 3, assuming that $c_1 = c_3 = 0$ and m_2 is negligible.

6–20. Evaluate the force and displacement amplitudes for the system shown in Fig. 6–16*a*, using the numerical data given in Table 3, with the following variations of the damping coefficient c_1:

(*a*) Use $c_1 = \frac{1}{4}$ lb sec per in. (Figs. 6–24 and 6–25.)

(*b*) Use $c_1 = 1$ lb sec per in. (Figs. 6–30 and 6–31.)

(*c*) Use $c_1 = \infty$ lb sec per in. (Figs. 6–33 and 6–34.)

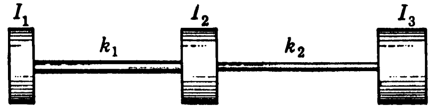
6–21. Using the data in problem 6–20, show that the "energy flow" will follow a distribution indicated in Figs. 6–26, 6–32, and 6–35.

6–22. Find the angular amplitudes of the three masses:

(*a*) For a torque $T_1e^{j\omega t}$ applied to I_1.

(*b*) For a torque $T_2e^{j\omega t}$ applied to I_2.

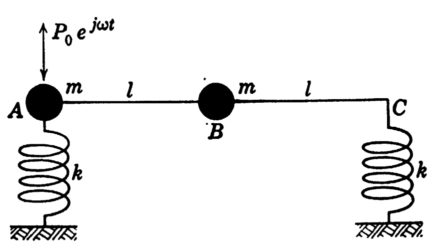
Evaluate these results for the particular case of $I_1 = I_2 = I_3 = I$ and $k_1 = k_2 = k$, $\omega^2 = 2k/I = 2p_0^2$.



PROB. 6–22

6–23. Find the resultant angular amplitudes of the masses in problem 6–22 if I_1 is subjected to a torque $T_1 = T_0e^{j\omega t}$, while I_2 at the same time is subjected to a torque $T_2 = 2T_0e^{j(\omega t+\beta)}$. Evaluate for (*a*) $\beta = 0$ and (*b*) $\beta = -\pi/2$. Assume that the masses and spring constants are the same as in problem 6–22.

6–24. Determine the angular amplitude, and locate the center of oscillation or node of the system shown. Evaluate these results for $\omega^2 = k/m$. Find the natural frequencies of the system, and locate the corresponding nodes. Determine also the forced frequencies for which a forced node will occur at the point of application of the force, as well as the frequency and corresponding amplitude for which the bar executes pure translation.



PROB. 6–24

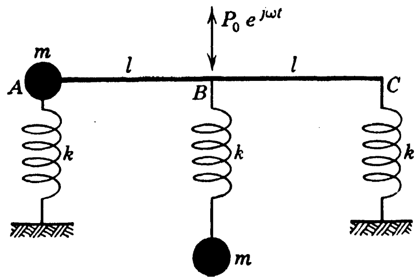
6–25. Solve problem 6–24 if the disturbing force $P_0e^{j\omega t}$ is applied at *B*.

6–26. Solve problem 6–24 if the disturbing force $P_0e^{j\omega t}$ is applied at *C*.

6–27. Find the angular amplitude and node for the system in problem 6–24 if the bar *ABC* is subjected to an oscillating moment $M_0e^{j\omega t} = P_0le^{j\omega t}$.

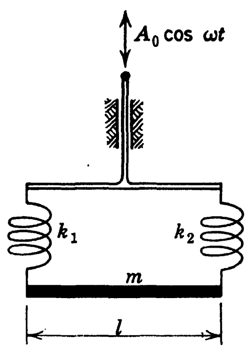
6–28. Find the angular amplitude of the bar and the location of the throat and its amplitude for the system in problem 6–24 if a force $P_Ae^{j\omega t}$ is applied at A while a force $P_C e^{j(\omega t+\beta)}$ is applied at C. Evaluate for $P_A = 2P_C$; $\omega^2 = k/m$, and $\beta = \pi/2$.

6-29. Determine the amplitudes of the masses and angular amplitude of the bar. Evaluate for $\omega^2 = 2k/m = 2p_0^2$. Find the natural frequencies and relative amplitudes of the masses as well as the location of the corresponding nodes.

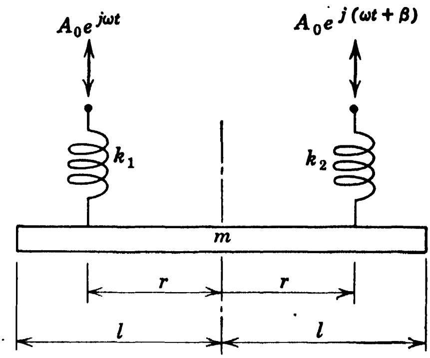


PROB. 6-29

6-30. A slender homogeneous beam is attached through springs k_1 and k_2 to a bar with a reciprocating motion $A_0 \cos \omega t$ as shown. Find the angular amplitude and node of the beam. Evaluate for $k_1 = 2k_2$ and $\omega^2 = k_2/m$.



PROB. 6-30



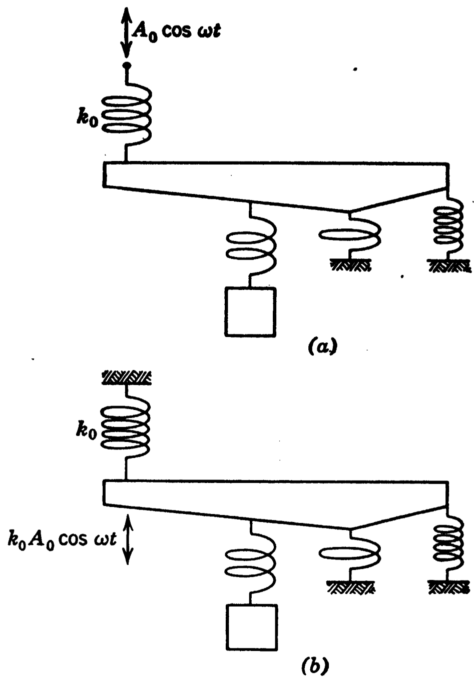
PROB. 6-31

6-31. The slender homogeneous beam vibrates, owing to the oscillations at the ends of the two supporting springs k_1 and k_2. The oscillations, as indicated, have the same amplitudes and frequencies but are operating with a phase angle β. Find the angular amplitude of the beam, the "throat" amplitude and its location, as well as the phase relationship with the force amplitude at k_1.

Evaluate (a) for $k_1 = k_2$, (b) for $k_1 = 2k_2$; $\beta = \pi/2$, $\omega^2 = k_2/m$, and $r = 2l/3$.

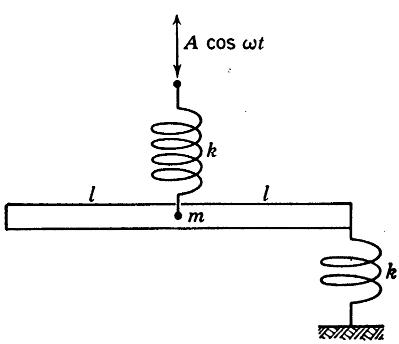
(Determine amplitude of the beam at the springs analytically, and evaluate from these, graphically, the angular amplitude, the throat amplitude, and its location.)

6–32. A system as indicated (*a*) is subjected to an oscillating disturbance $A_0 \cos \omega t$ through a spring k_0. Show that this system can be replaced by a system as indicated in (*b*) in which the spring k_0 is "grounded" and an oscillating force $k_0 A_0 \cos \omega t$ acts at the point of application of the spring.



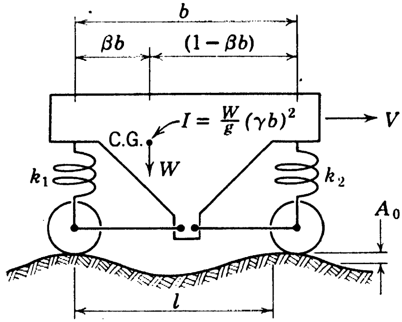
PROB. 6–32

6–33. A slender homogeneous bar is subjected to an oscillating motion through a spring attached at its midpoint while supported by a spring of equal stiffness at one end as indicated. Find its angular amplitude and the location of its node in terms of the parameter $\alpha^2 = \omega^2 m/k$.



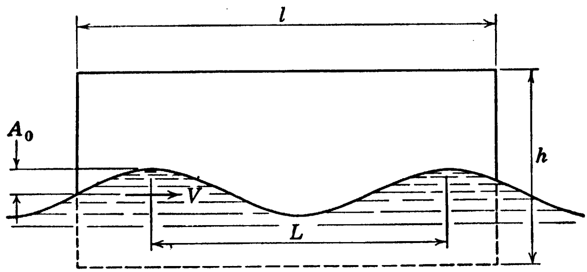
PROB. 6–33

6-34. A car, moves over a sinusoidal road surface with a constant velocity V. Find the amplitudes of the upper terminals of the springs, and determine the angular and throat amplitudes of the car and its location, if the weight of the wheels is neglected. Evaluate these results for $k_1 = k_2$; $b/l = 6\frac{3}{8}$; $\beta = \frac{1}{3}$; $\gamma^2 = \frac{1}{6}$; and $\left(\frac{V}{l}\right)^2 \frac{W}{k_1 g} = 6$.



PROB. 6-34

6-35. A rectangular homogeneous block with specific gravity $\gamma < 1$, a surface area $l \times b$, and a depth h floats in water and is subjected to a sinosoidal wave motion which is moving along the length of the block l with a relative velocity V. The wave length is L, and the wave amplitude A_0 as indicated.



PROB. 6-35

Show that the resultant buoyancy force due to the wave motion can be expressed as

$$P = k_e A_0 \frac{\sin \beta}{\beta} \cos \omega t$$

while the moment of the buoyant forces, due to the wave motion, about an axis through the centroid, may be expressed as

$$M = \frac{1}{2} k_e A_0 l \frac{1}{\beta^2} [\sin \beta - \beta \cos \beta] \sin \omega t$$

where $k_e = wlb$; $\beta = \pi \frac{l}{L}$ and $\omega = 2\pi \frac{V}{L}$.

Hint: The increment of buoyant force per unit width at any point along the length l, referred to a convenient origin, may be expressed as

$$dP = wA_0 \cos\left[\frac{2\pi}{L}(x - Vt)\right] dx$$

6–36. Find, in problem 6–35, the amplitudes of the two modes of motion of the floating block, and determine the corresponding critical frequencies of passing waves.

Hint: Restoring buoyant moment for an angular rotation θ of the block is

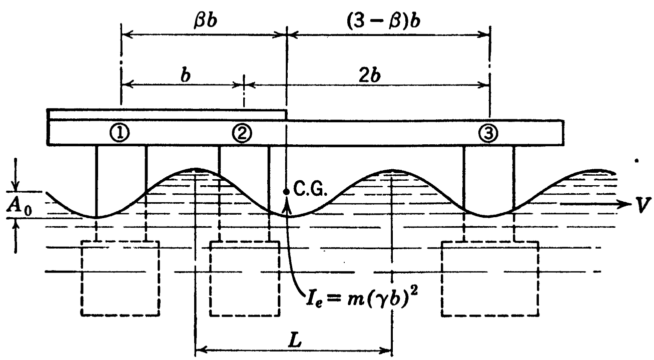
$$w[(1 + \tfrac{1}{2}\tan^2\theta)(\tfrac{1}{12}l^2)lb - \tfrac{1}{2}(1 - \gamma)h(\gamma hlb)]\sin\theta$$

where γ is the specific gravity of the block and w is the specific weight of the water. For small angular rotations this reduces to

$$\tfrac{1}{12}wbl^3\left[1 - 6\gamma(1 - \gamma)\left(\frac{h}{l}\right)^2\right]\theta$$

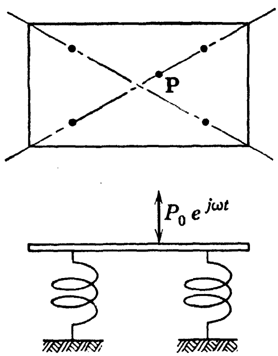
6–37. Find, in problem 6–35, the two lowest values of the ratio between the length l of the block and the wave length L for which the block will ride the waves without pitching (rotation), and show its limit value for large values of l/L.

6–38. A float, is carried by three sets of pontoons, each set having equal cross-sectional areas A at the water surface. Waves of length L move along the float with a relative velocity V. Assuming plane motion of the float, determine its motion. Obtain analytically the amplitudes of points 1, 2, and 3, and evaluate this result for $\dfrac{b}{l} = \dfrac{3}{8}$; $\beta = 1$, $\gamma = 2$, and $\left(\dfrac{V}{L}\right)^2 \dfrac{wA}{m} = 1$. Determine graphically the angular amplitude and the throat (node) amplitude and its location, as well as the phase relationship with respect to the wave amplitude at point 1.

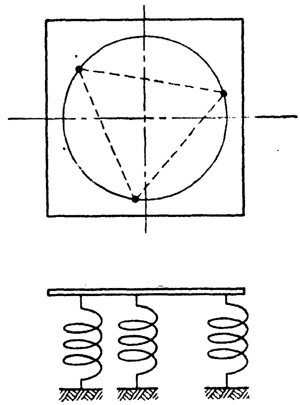


PROB. 6–38

6–39. Show that a homogeneous rectangular plate supported by four equal springs symmetrically attached at the diagonals as indicated will oscillate about a nodal axis which will be parallel to one diagonal if the oscillating force is acting on the other diagonal.



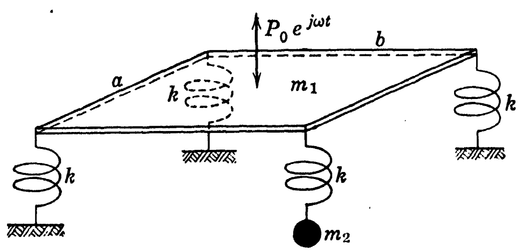
PROB. 6–39



PROB. 6–40

6–40. A homogeneous square plate is supported by three equal springs acting in the corners of an equilateral triangle which is concentric with the plate. Show that an oscillating couple acting on the plate in any plane normal to it will produce the same angular amplitude with a nodal axis normal to the plane of the couple.

6–41. A homogeneous rectangular plate is supported at three corners by springs. A mass m_2 is suspended from the fourth corner by a spring while a disturbing force $P_0e^{j\omega t}$ acts at the midpoint of the plate as indicated. Find the nodal axis and angular amplitude of the plate as well as the amplitude of the suspended mass m_2. Evaluate these results for $a = 3l$, $b = 4l$, $m_1 = 3m_2 = 3m$, all spring constants $= k$, and $\omega^2 = k/m$.



PROB. 6–41

6–42. Find the natural frequency and the corresponding nodal axes for the system in problem 6–41 without the suspended mass m_2.

6–43. The block as described in problem 6–35 is subjected to a sinosoidal wave motion which is moving with a relative wave velocity V in a direction making an angle α with the longitudinal axis, as indicated. Show that the resultant buoyant force due to the wave motion may be expressed as

$$P = k_e A_0 \frac{\sin \beta_l}{\beta_l} \times \frac{\sin \beta_b}{\beta_b} \cos \omega t$$

whereas the moment of this force with respect to the y and x axes may be expressed as

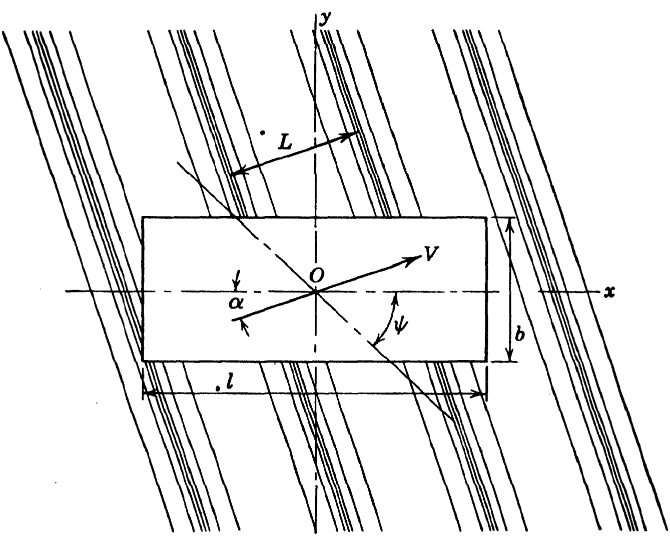
$$M_y = \frac{1}{2} k_e A_0 l \left[\frac{\sin \beta_l}{\beta_l^2} - \frac{\cos \beta_l}{\beta_l} \right] \frac{\sin \beta_b}{\beta_b} \sin \omega t$$

$$M_x = \frac{1}{2} k_e A_0 b \left[\frac{\sin \beta_b}{\beta_b^2} - \frac{\cos \beta_b}{\beta_b} \right] \frac{\sin \beta_l}{\beta_l} \sin \omega t$$

where $k_e = wlb$, $\beta_l = \pi \dfrac{l \cos \alpha}{L}$; $\beta_b = \pi \dfrac{b \sin \alpha}{L}$, and $\omega = 2\pi \dfrac{V}{L}$.

Hint: The increment of buoyant force acting at a point x, y, with respect to a convenient coordinate system, may be expressed as

$$dP = wA_0 \cos \left[\frac{2\pi}{L} (x \cos \alpha + y \sin \alpha - Vt) \right] dx\, dy$$

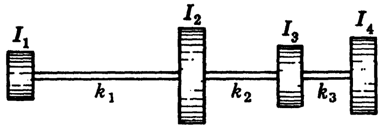


PROB. 6–43

6–44. Find, in problem 6–43, the amplitudes of the two modes of motion of the block, and find the angle ψ of the nodal (throat) axis.

Chapter 7. Solution of the General Frequency Equation

7–1. Find the natural frequencies of the torsional system shown using the Holzer method. Evaluate these results for $k_2 = 2k_1$, $k_3 = 3k_1$, $I_2 = 4I_1$, $I_3 = 2I_1$, and $I_4 = 3I_1$, when $k_1 = 2 \times 10^6$ lb in. per rad and $I_1 = 200$ lb in. sec^2.



PROB. 7–1

7–2. Reduce the system in Problem 7-1 to a non-dimensional system by substitution of spring ratios k_1/k_1, k_2/k_1, k_3/k_1, and inertia ratios I_1/I_1, I_2/I_1, I_3/I_1, and I_4/I_1. Show that the natural frequencies obtained from this equivalent non-dimensional system equal the corresponding frequencies in the dimensional system when multiplied by $\sqrt{k_1/I_1}$.

7–3. Find for the non-dimensional system, as obtained from the previous problems 7–1 and 7–2, the natural frequencies by solving the resulting cubic equation:

(*a*) As obtained by evaluating the coefficients P, Q, and R from the formulation of these as given with equation 5.7–5.

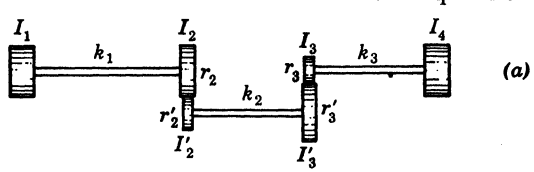
(*b*) As obtained through the tabulated evaluation by the Holzer table.

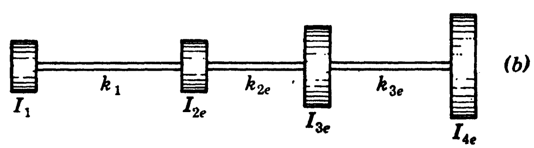
7–4. Use the Holzer tabulation to evaluate the torsional amplitudes of the masses for the natural frequencies of the non-dimensional system as obtained in Problem 7–3, and draw the respective normal elastic curves.

7–5. Solve Problem 7–1 if $I_4 = \infty$ (This is equivalent to the assumption that k_3 is fixed at I_4).

7–6. The system shown in Problem 7–1 is subjected to a forced frequency $f = 100\sqrt{3}/2\pi$ and the amplitude of the inertia I_2 is found to be 0.1 rad. Determine by the Holzer method the amplitude of the other masses, and draw the resulting elastic curve for the system.

7–7. Reduce the geared system indicated in *a* to an equivalent ungeared system as indicated in *b*. Determine the natural frequencies of this system,



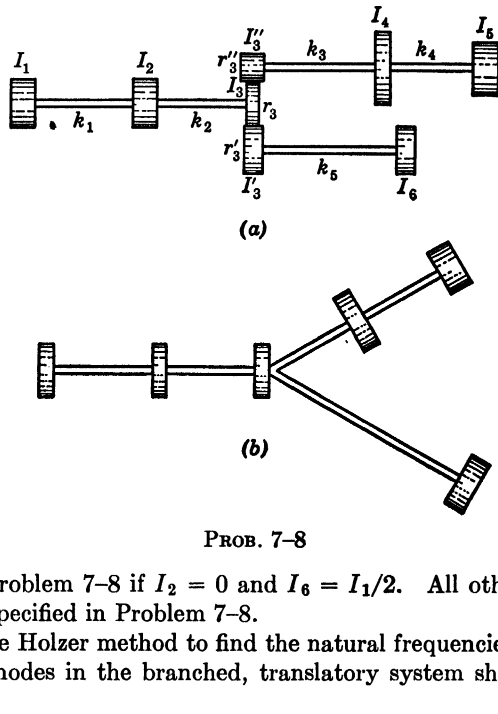


PROB. 7–7

and draw the normal elastic curves for the real system. Evaluate the results for $I_1 = I_2 = I_4 = 4I_3$, $I_2' = 2I_3$, $I_3' = 6I_3$, $k_1 = k_2 = k_3$, $r_2/r_2' = 2$, $r_3/r_3' = 1/3$, $\sqrt{k_3/I_3} = 10$ rad per sec.

7-8. Find the natural frequencies of the first and second mode and corresponding nodes of the branched system shown in *a*. Reduce the system to an equivalent non-geared and non-dimensional system as diagrammatically indicated in *b*. Evaluate the frequencies for

$$I_1 = I_2 = 2I_3 = 2I_3' = 4I_3'' = I_4 = 2I_5 = I_6 = 40 \text{ lb in. sec}^2$$
$$k_1 = k_2 = 2k_3 = 4k_4 = k_5 = 1{,}600{,}000 \text{ lb in. per rad}$$
$$r_3 = r_3' = 2r_3''$$

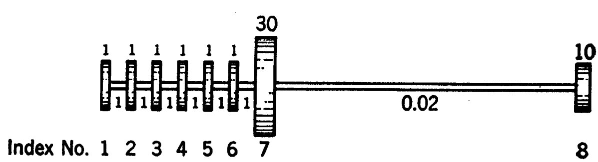


PROB. 7-8

7-9. Solve Problem 7-8 if $I_2 = 0$ and $I_6 = I_1/2$. All other values are related to I_1 as specified in Problem 7-8.

7-10. Use the Holzer method to find the natural frequencies and locate the corresponding nodes in the branched, translatory system shown in Problem 5-70.

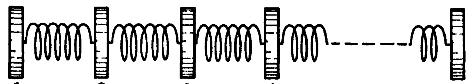
7-11. The diagram shown represents the equivalent non-dimensional torsional system of a six-cylinder marine Diesel engine with flywheel and propeller for which $\sqrt{k_1/I_1} = 500$ rad per sec. Find the natural frequencies for the first three modes, and draw the corresponding normal elastic curves.



PROB. 7-11

7–12. Find the natural frequencies of the first two modes, and draw the corresponding normal elastic curves for the system in Problem 7–11 if the propeller and propeller shaft are neglected.

7–13. In a torsional or translatory system consisting of n discrete equal masses and $n - 1$ equal springs as indicated, determine by inspection the exact or approximate location of the nodes for the various modes, and draw the normal elastic curve for the specific cases of $n = 2, 3, 4$, and 6.



PROB. 7–13

7–14. Use the Holzer tabulation to evaluate, successively, the residual torque T_n or force F_n needed to sustain a circular frequency ratio α in a homogeneous, non-dimensional, elastic system of n discrete masses as indicated in Problem 7–13.

Evaluate these torque or force frequency functions for systems from $n = 1$ to $n = 6$.

Show by inference from such tabulation that the dimensionless residual torque or force of the nth mass may be expressed as[1]

$$T_n \text{ or } F_n = (2 - \alpha^2)T_{n-1} - T_{n-2} = \frac{\sin nx}{\sin x}\alpha^2$$

where

$$\cos x = 1 - \tfrac{1}{2}\alpha^2$$

and that the frequency equation for such system takes the general form $\sin nx = 0$.

Show also that the amplitude (angular or linear) may be expressed in terms of the torques or forces as

$$\alpha^2\Theta_n = T_n - T_{n-1} = (1 - \alpha^2)T_{n-1} - T_{n-2} = \frac{\sin nx - \sin (n-1)x}{\sin x}\alpha^2$$

7–15. Show that the frequency equation $\sin nx = 0$ for a homogeneous elastic system as presented in Problem 7–14 leads to the resonance condition

$$\alpha^2 = \frac{\omega^2}{p^2} = 2\left(1 - \cos\frac{\gamma\pi}{n}\right)$$

where p is the circular frequency $\sqrt{k/I}$ or $\sqrt{k/m}$ of the elementary spring and masses while γ is an integer varying from 1 to $n - 1$.

Use this relationship to determine the natural frequencies of such homogeneous elastic system as specified in Problem 7–13, and draw the normal elastic curves.

[1] For a general development of this function see Biezeno-Grammel *Technische Dynamik,* Julius Springer, Berlin, 1939.

7–16. Use the residual torque and corresponding amplitudes as obtained in Problem 7–14 for $n = 6$ to write the frequency equation for the seven-mass torsional system in Problem 7–12.

7–17. Find by Graeffe's method the three roots of the frequency equation

$$p^6 - 6.9p^4 + 12.6p^2 - 4 = 0$$

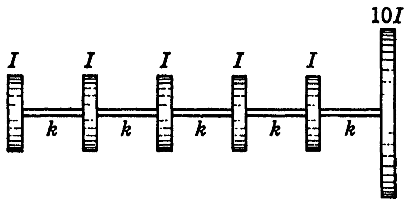
7–18. Find the three roots of the frequency equation

$$p^6 - 9.9p^4 + 30.5p^2 - 30 = 0$$

7–19. Find the four roots of the frequency equation

$$p^8 - 14p^6 + 67.75p^4 - 133.5p^2 + 90 = 0$$

7–20. Obtain the frequency equation of the torsional system shown by the Holzer method, and find the five natural frequencies by the Graffe method.



PROB. 7–20

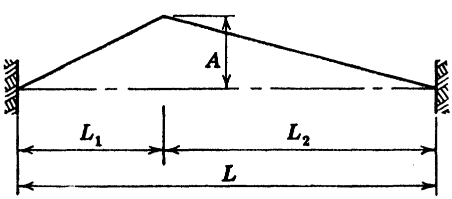
7–21. The two-mass system of Fig. 5–5 has physical constants with the following values:

$$m_1g = 10 \text{ lb}$$
$$m_2g = 5 \text{ lb}$$
$$k_1 = 20 \text{ lb per in.}$$
$$k_2 = 15 \text{ lb per in.}$$
$$c = \tfrac{1}{2}\sqrt{2k_2m_2}$$

Calculate the natural frequencies of the system by Graeffe's method.

Chapter 8. Systems with Distributed Physical Constants

8–1. A taut string similar to that discussed in section 8–4 is fixed at the ends and initially displaced in the manner shown. Determine the expression for the resultant motion of the string.



PROB. 8–1

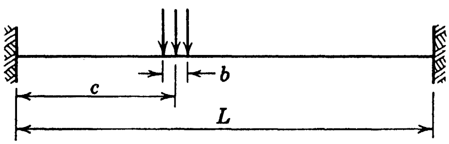
8–2. A taut string is initially straight. It has a constant lateral velocity v_0 at $t = 0$, at all points along the string. Show that the resultant motion is given by

$$y = \frac{4v_0L}{\pi^2 a} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^2} \sin\frac{n\pi at}{L} \sin\frac{n\pi x}{L}$$

8–3. A piano wire is struck with a hammer which rebounds after the blow. The blow results in an initial velocity v_0 of that portion of the string of length b that was contacted by the hammer, whereas the remainder of the wire is initially undisturbed. Show that the resultant motion of the wire is given by

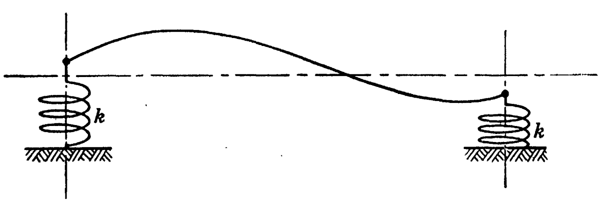
$$y = \frac{4v_0L}{\pi^2 a} \sum_{n=1,2,3\cdots}^{\infty} \frac{1}{n^2} \sin\frac{n\pi c}{L} \sin\frac{n\pi b}{2L} \sin\frac{n\pi x}{L} \sin\frac{n\pi at}{L}$$

What modes will be excited if the hammer falls in the middle? Where should the hammer fall to excite the third mode at maximum amplitude?



PROB. 8–3

8–4. The ends of a taut string may be displaced normal to the axis of the string. The ends are held in position by springs k. Deduce the frequency equation, and show that it reduces to the expression for fixed ends if $k \to \infty$.



PROB. 8–4

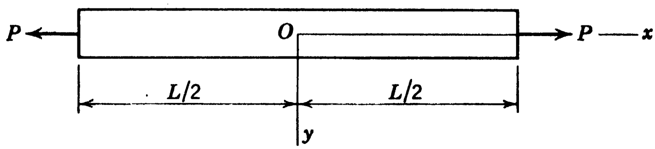
8–5. What is the effect on the calculated natural frequency if a 1 percent error is made in measuring the radius of a vibrating wire of circular cross section?

8–6. A prismatic bar is free at both ends. Determine the frequencies of the modes of longitudinal vibration.

8–7. A prismatic bar is fixed at each end. Determine the frequencies of the modes of longitudinal vibration.

8–8. What is the effect of a constant longitudinal force on the natural frequencies of a prismatic bar for longitudinal vibrations?

8–9. A prismatic bar of cross-sectional area A is stretched by a force P as shown. At time $t = 0$ the bar is released. Determine the displacements u thereafter. Assume the center section of the bar to remain stationary.



PROB. 8–9

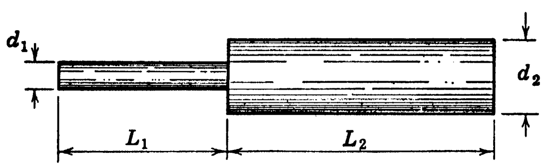
8–10. Show that the equation of motion for the longitudinal vibration of a tapered prismatic bar has the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \frac{\partial}{\partial x} (\log A) \right]$$

where $A(x)$ is the cross-sectional area.

8–11. Determine the torsional natural frequencies of a circular shaft with one end fixed and one end free. Check your result with equation 8.6–4 for $I_2 = 0$ and $I_1 = \infty$.

8–12. The shaft shown has two sections of different diameter and length. Determine the frequency equation for torsional vibration if the ends are free.



PROB. 8–12

8–13. Determine the natural frequencies of a cantilever beam.

8–14. A steel tuning fork is to be designed with a fundamental frequency of 256 cycles per second. If the effective length of the prongs of the fork are 5 in., what should be their thickness?

8–15. Determine the natural frequencies of a beam built in at each end.

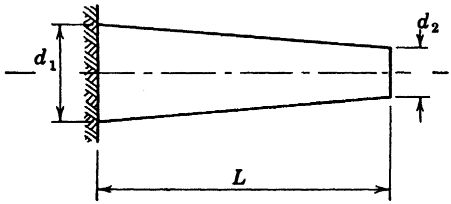
8–16. Determine the natural frequencies of a beam free at each end.

8–17. Show that the natural frequencies of the beam shown in Fig. 8–21 are given by the roots of the equation

$$\frac{\beta L(\cosh \beta L \sin \beta L - \sinh \beta L \cos \beta L)}{1 + \cosh \beta L \cos \beta L} = \frac{W_0}{W}$$

where W_0 is the total weight of the beam.

8–18. Calculate the fundamental natural frequency of a conical bar in longitudinal vibration if it is fixed at one end and free at the other end as shown (Use Rayleigh's method).



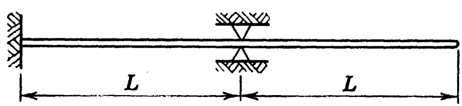
PROB. 8–18

8–19. Determine the lowest natural frequency of the conical bar of Problem 8–18 for torsional vibration, using Rayleigh's method.

8–20. Solve the problem of Figure 8–18 for a single concentrated load W at the center of the span, if the beam has a weight of w lb per ft.

8–21. Determine the lowest natural frequency for longitudinal vibrations for a conical spring of constant wire diameter if the number of coils per unit length is constant. Assume the ends to be free.

8–22. The beam shown has a stiffness EI and a weight w per unit length. Develop the frequency equation for this system.



PROB. 8–22

8–23. Determine the lowest natural frequency for Problem 8–22 by Rayleigh's method.

8–24. Show that the natural frequencies in bending of a cantilever beam clamped at one end and carrying a weight W at the free end can be obtained from the solution of

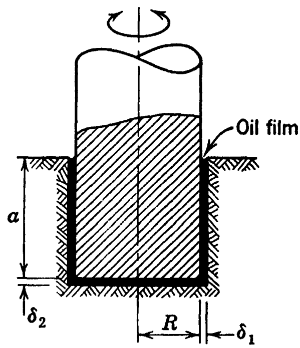
$$\frac{1 + \cosh \beta L \cos \beta L}{\cosh \beta L \sin \beta L - \sinh \beta L \cos \beta L} = \frac{W(\beta L)}{wL}$$

where w is the weight of beam per unit length.

Chapter 9. Vibrations of Transient Character

9–1. A mass m is supported on an oil film of thickness δ. The contact area is A and the oil viscosity is μ. Determine the damping coefficient for this system for small oscillations parallel to the plane of the oil film.

9–2. The pivot bearing shown has clearances δ_1 and δ_2 on the sides and end, respectively, of the shaft. If the oil viscosity is μ, find an approximate value of the torsional damping coefficient c for this bearing.



PROB. 9–2

9–3. A captive balloon, whose average inflated density is α times that of air ($\alpha < 1$), is held by a cord of length L. The equivalent viscous damping coefficient for the air friction is c. What is its period?

9–4. The motion of a ballistic galvanometer is damped by a resisting couple proportional to the angular velocity of swing. The scale is adjusted to give a zero reading with the galvanometer at rest. The moving coil is given a displacement of 100 scale divisions and then released. The maximum displacement in the same direction on the return swing is observed to be 25 divisions, and it occurs 2 sec after the instant of release. What is the time required for the first quarter swing and what is the displacement at the end of the second return swing?

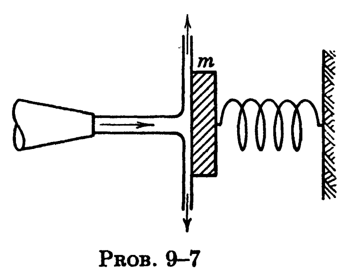
9–5. Calculate the appropriate damping constant for problem 2–18 if the motion of the liquid is laminar and its viscosity is μ.

9–6. Solve problem 2–69 including the effect of damping in the liquid motion. Assume that the motion of the liquid is laminar.

9–7. A jet of liquid impinges on a mass m as shown. The mass m is held in place by a spring k. The jet nozzle has a discharge Q at a velocity V. Determine the equilibrium position of the mass m, and show that its equation of motion for small oscillations can be written as

$$\ddot{x} + 2\frac{Q\rho}{m}\dot{x} + \frac{k}{m}x = \frac{Q\rho V}{m}$$

where ρ is the density of the liquid. What is the equivalent damping constant for this system?



PROB. 9–7

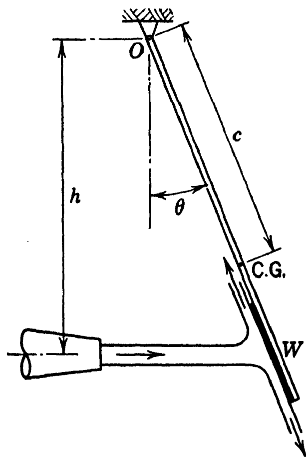
9–8. A pendulum of weight W is acted upon by a jet of discharge Q at a velocity V. Show that the equation of motion for small oscillations about the equilibrium position is

$$\ddot{\psi} + 2\frac{h^2 Q \rho g}{W(\bar{r}^2 + c^2)\cos^2\theta_0}\dot{\psi} + \frac{gc\cos\theta_0}{\bar{r}^2 + c^2}\psi = 0$$

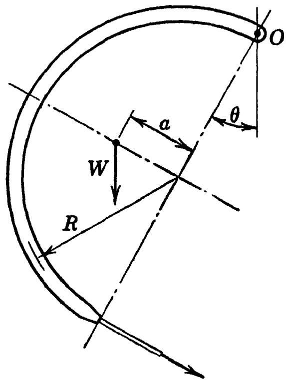
where

$$\sin\theta_0 = \frac{Q\rho V h}{Wc}$$

and θ_0 is the equilibrium angle, ψ is the angular position of the pendulum measured from the equilibrium position, and $\bar{r}$ is the radius of gyration of the pendulum about its centroid.



Prob. 9–8



Prob. 9–9

9–9. A small semicircular tube has a nozzle at one end, and it is suspended at O so that it can swing in a vertical plane. Fluid enters at O and is ejected through the nozzle at a rate Q and velocity V. The tube has a radius R and a total weight W, and its centroid is located as shown. Show that the equilibrium angle θ_0 is given by

$$\frac{a}{R}\cos\theta_0 + \sin\theta_0 = \frac{2Q\rho V}{W}$$

and that the equation of motion for small oscillations, about this position, has the form

$$\ddot{\psi} + \frac{2Q\rho g}{W}\dot{\psi} + \frac{g}{2R}\left(\cos\theta_0 - \frac{a}{R}\sin\theta_0\right)\psi = 0$$

9–10. A given impulse $\mathcal{I}$ is applied to a simple vibratory system without damping, of 1 degree of freedom, over a period of time T. Show that the amplitude of the resultant motion is

$$A = \frac{2\mathcal{F}}{kT} \sin \frac{pT}{2}$$

9–11. A small constant horizontal force acts on the bob of a pendulum of length l which is initially at rest. If the force acts for a short time T, show that the subsequent undamped oscillations have an amplitude

$$A = \frac{2Fl}{gm} \sin \frac{1}{2} \sqrt{\frac{g}{l}}\, T$$

9–12. A shaft of stiffness k has a flywheel of moment of inertia I at one end. The system is initially at rest. If the free end of the shaft is suddenly connected to a large machine which rotates the free end at a constant speed of Ω rad per sec, show that the maximum torque in the shaft is

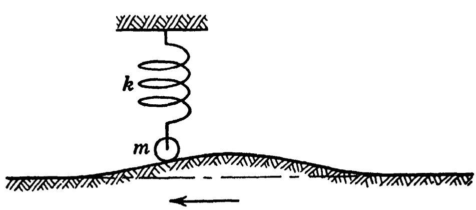
$$M = \Omega \sqrt{kI}$$

9–13. A mass m is suspended from a spring k. At time $t = 0$ an oscillating force $F = F_0 \sin pt$ begins to act on the mass. If $p^2 = k/m$, determine the displacement time relation for m.

9–14. A wheel of mass m is held against a cam by a spring k with an initial compression force P. The cam has a single rise whose profile can be represented by

$$y = \delta \left(1 - \cos \frac{2\pi t}{T}\right)$$

for $0 < t < T$ and a given cam speed. The vertical displacement of the wheel is zero for $t < 0$ and $t > T$. What is the minimum value of T that will permit the wheel to remain in contact with the cam?



PROB. 9–14

9–15. A two-wheeled trailer of weight W has a velocity V. It is suspended upon a set of springs whose resultant spring constant is k. The trailer travels a roadway which is level except for the single bump

$$y = \delta \left(1 - \cos \frac{2\pi x}{L}\right)$$

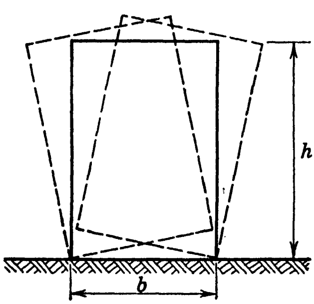
which is traversed by both wheels simultaneously. What is the amplitude of its motion after traversing the bump? (Neglect damping and the weight of the wheels and axle.)

Chapter 10. Vibrations of Non-Linear Character

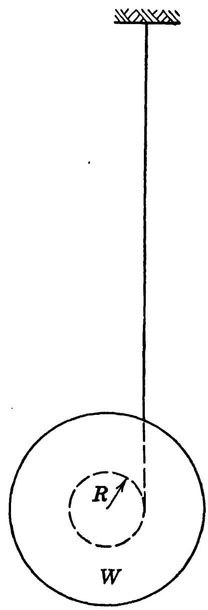
10–1. A solid rectangular block rocks from corner to corner as shown. If the impact losses are neglected show that the period is

$$\tau = 8\sqrt{\frac{\varphi_0}{3g}\left(b + \frac{h^2}{b}\right)}$$

where φ_0 is the angular amplitude, $\tan\varphi_0 < b/h$, and the oscillations are small. What is the expression for the period if the oscillations are not small but $\tan\varphi_0$ is still less than b/h.



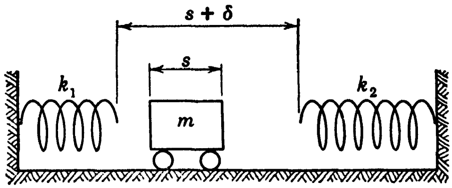
PROB. 10–1



PROB. 10–2

10–2. A spool is allowed to fall as a string of length L unwinds as shown. When the spool reaches the end of the string, it winds up the string in the other direction, causing the spool to rise again. If the total vertical motion of the spool is L, calculate the period of the oscillation. Let the radius of gyration of the spool be $\bar{r}$, and assume that $R << L$.

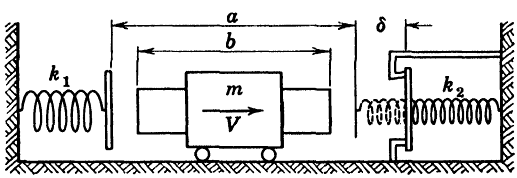
10–3. Assuming that the springs have negligible mass, find period and amplitude of oscillation of the mass m. Plot the acceleration, velocity, and displacement diagrams. The mass has a velocity v across the open gap δ.



Prob. 10–3

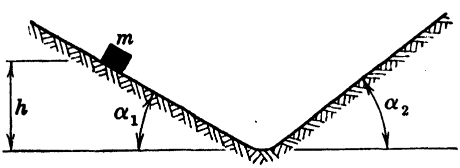
10–4. A mass m moves with a velocity V across the open space between springs k_1 and k_2. The distance between the unstrained springs is a. The spring k_2 has been prestrained a distance δ by the one-way stop indicated. Find the period of a cycle and the total amplitude. Evaluate for the particular relationships.

$$k_1 = 4k_2, \qquad a - b = \delta = V\sqrt{\frac{m}{k_2}}$$

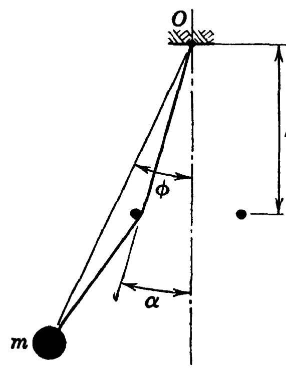


Prob. 10–4

10–5. The small block of mass m slides in a V groove as shown. The coefficient of friction for the block and plane is μ. Assuming that the block negotiates the sharp turn at the bottom of the groove without difficulty, calculate the period for one oscillation starting from the position shown.



Prob. 10–5



Prob. 10–6

10–6. A pendulum is constructed from a string of length L and a mass m. The pendulum is arranged so that the string contacts pegs during part of its swing. Calculate the period of the pendulum for an amplitude $\varphi_0 = 2\alpha$. $\varphi_0 << 1$.

10–7. A pendulum performs a simple harmonic motion given by

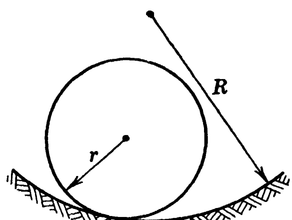
$$\theta = \Theta \sin \omega t$$

At a displacement θ_0 from the equilibrium position it is allowed to strike a plate normal to its path from which it rebounds. If the coefficient of restitution for the impact is ϵ, determine the ratio of the amplitudes of the pendulum before and after impact and the change in phase angle on impact.

10–8. A pendulum has a period of 2 sec when the amplitude is 5 degrees. How many cycles will it perform in an hour if its amplitude is 10 degrees? 30 degrees?

10–9. A solid cylindrical roller of radius r and mass m rolls in the bottom of a cylindrical groove of radius $R > r$. If the cylinder rolls without slipping, show that the period for large oscillations is the same as for a pendulum of length

$$l = \tfrac{3}{2}(R - r)$$



PROB. 10–9

10–10. A semicircular log of radius R floats in water. Develop the equations of motion for rolling, and show that it may be compared with a simple pendulum of length

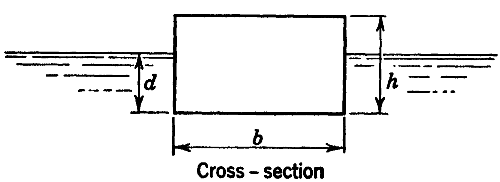
$$L = R\left(\frac{4}{3\pi} + \frac{3\pi}{8}\right)$$

for large oscillations.

10–11. A rectangular timber of specific gravity γ floats in water as shown. Show that its equation of motion for rolling oscillations about its longitudinal axis is

$$\ddot{\theta} + \frac{g}{d}\left[\frac{1 - 6\alpha^2\gamma(1-\gamma) + \frac{1}{2}\tan^2\theta}{1 + \alpha^2}\right]\sin\theta = 0$$

where d is the undisturbed draft and $\alpha = h/b$. Neglect wave action. Is this a "hard" or a "soft" system?

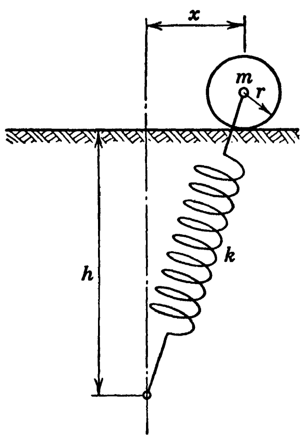


PROB. 10–11

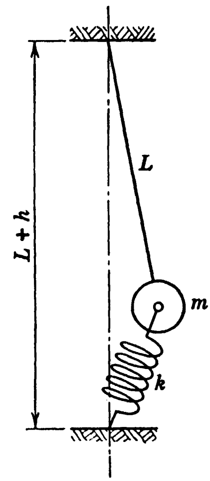
10–12. A solid cylinder of mass m and radius r rolls on a straight track under the influence of a spring k as shown. If the free length of the spring is $l < h + r$, show that the equation of motion of the roller is

$$\ddot{x} + \frac{2k}{3m}\left(1 - \frac{l}{h}\frac{1}{\sqrt{1 + \frac{x^2}{h^2}}}\right)x = 0$$

Obtain the solution for the first non-linear approximation.



PROB. 10–12



PROB. 10–13

10–13. The pendulum shown is constructed from a mass m and a rigid rod L. The spring k has a free length l. Determine the frequency for small oscillations and the frequency based upon the first non-linear approximation. Show that motion about the vertical position is unstable if

$$\frac{l}{h} > 1 + \frac{W}{k(L + h)}$$

10–14. A solid circular roller of mass m and radius r rolls on a simply supported flexible beam. If the beam deflections are small, obtain the period for small oscillations. Show that the solution for large amplitudes of the roller may be expressed as

$$x = x_0 \operatorname{sn} p_0 t$$

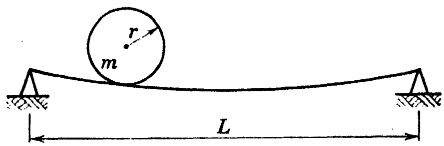
where x_0 is the amplitude of the motion,

$$p_0^2 = \frac{mg^2 l}{9EI}\left(1 - \frac{2x_0^2}{L^2}\right),$$

and k, the modulus of the elliptic function, is given by

$$k^2 = \frac{1}{\dfrac{L^2}{2x_0^2} - 1}$$

Neglect the weight of the beam.



PROB. 10–14

10–15. The equations of motion for the centrifugal pendulum were derived in section 5.10. Show that the equation of motion of the pendulum for free vibrations reduces to

$$m_2(\bar{r}_2^2 + l^2)\ddot{\theta} + m_2 R l \Omega^2 \sin\theta = 0$$

if the carrier I is forced to rotate at constant speed ($\dot{\psi} = 0$). Obtain the solution

$$\theta = 2 \arcsin\left[\sin\frac{\theta_0}{2} \operatorname{sn} pt\right]$$

where θ_0 is the amplitude and

$$p^2 = \frac{Rl\Omega^2}{\bar{r}_2^2 + l^2}$$

is the natural circular frequency for small amplitudes.

10–16. A thin rod of length L is suspended in the horizontal position by two vertical wires, of length L, connected to the ends of the rod. If the rod is made to oscillate about a vertical axis, show that the angle φ which the wires make with the vertical is given by the expression

$$pt = \int_0^{\varphi} \sqrt{\frac{1 + 3\sin^2\varphi}{2(\cos\varphi - \cos\varphi_0)}}\, d\varphi$$

where p is the natural circular frequency for small oscillations and φ_0 is the amplitude of the motion.

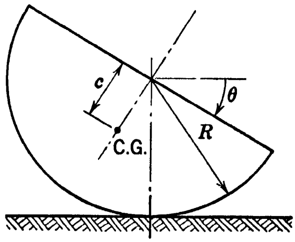
10–17. Show that the equation of motion for large oscillations of a semicircular rocker on a plane is

$$[I_c + m(R^2 - 2Rc\cos\theta + c^2)]\ddot{\theta} + mRc\sin\theta\,\dot{\theta}^2 + Wc\sin\theta = 0$$

and that the solution is given by

$$pt = \int_0^{\theta} \sqrt{\frac{1 + \dfrac{2p^2R}{g}(1 - \cos\theta)}{2(\cos\theta - \cos\theta_0)}}\, d\theta$$

where p is the natural circular frequency for small oscillations, θ_0 is the amplitude, and I_c is the moment of inertia of the rocker about its centroidal axis.



PROB. 10–17

10–18. A uniform thin rod of length L and mass m rocks without sliding on a fixed circular cylinder of radius R as in Problem 2–41. Develop the equation of motion, and obtain the solution

$$t = \int_0^\theta \sqrt{\frac{1 - \beta\theta^2}{\omega^2 + 2\beta \frac{g}{R}(1 - \cos\theta - \theta \sin\theta)}}\, d\theta$$

where

$$\beta = \frac{12R^2}{L^2}$$

and $\omega = \dot{\theta}$ for $\theta = 0$.

10–19. A bead oscillates on a circular wire under the action of gravity. If the coefficient of friction between the wire and the bead is μ, show that the equation of motion can be written as

$$\ddot{\theta} + \mu\dot{\theta}^2 + \frac{g}{r\cos\varphi}\sin(\theta \pm \varphi) = 0$$

where $\tan\varphi = \mu$. Integrate this equation to find the time required for a quarter cycle of the bead motion.

ANSWERS TO PROBLEMS

As a matter of expediency the circular frequency p rather than the natural frequency f has been given in numerous answers.

Chapter 1

1–1. 1, 2, 5, 3.

1–2. 100 cycles per sec, 62.83 in. per sec, 39,480 in. per sec^2.

1–3. 0.0265 in., 3766 in. per sec^2, 0.01667 sec.

1–4. 0.01667 sec, 0.0704 in., 26.53 in. per sec.

1–8. $2/\pi$.

1–9. 1 to 16.

1–10. 9%.

1–11. $A^2 + B^2 + 2AB \sin(\theta - \varphi)$.

1–12. $C = A \cos \varphi + B \sin \theta$.
$D = -A \sin \varphi + B \cos \theta$.

1–14. 8.66 in.

1–15. $C = 5.83$ in.
$D = 2.36$ in.

1–17. $x = \dfrac{C}{\sin(\varphi + \theta)} [\sin \theta \cos(\omega t + \varphi) + \sin \varphi \cos(\omega t - \theta)]$.

1–18. $B \sin\left[\omega t - \sin^{-1}\left(\dfrac{A \sin \varphi}{B}\right)\right]$

where
$B = \sqrt{A^2 + C^2 - 2AC \cos \varphi}$.

1–19. $\theta = \cos^{-1}\left(\dfrac{B^2 + C^2 - A^2}{2BC}\right)$.

$\varphi = \cos^{-1}\left(\dfrac{A^2 + C^2 - B^2}{2AC}\right)$.

1–25. $A \leq g/4\pi^2 f^2$.

Chapter 2

2–1. 0.25 in., 0.160 sec.

2–2. 204 lb per in.

2–3. $\delta_{st} = 9.79$ in.

2–4. 9.12 cycles per sec.

2–5. $k = 1.6$ lb per in.

2–8. $L_e = \dfrac{D_e^4}{D^4 - d^4} L$.

2–9. $L_e = D_e^4 \left(\dfrac{L_1}{D_1^4} + \dfrac{L_2}{D_2^4} + \dfrac{L_3}{D_3^4}\right)$.

2–11. $f = 149$ cycles per sec.

2–13. (a) $f = \dfrac{1}{2\pi} \sqrt{\dfrac{4g}{\left(\dfrac{1}{k_1} + \dfrac{1}{k_2}\right) W}}$.

(b) $f = \dfrac{1}{2\pi} \sqrt{\dfrac{9g}{\left(\dfrac{1}{k_1} + \dfrac{4}{k_2}\right) W}}$.

2–15. $f_{\text{HOR}} = \dfrac{1}{2\pi} \sqrt{\dfrac{6EI_1 g}{Wh^3 \left(1 + \dfrac{1}{2} \dfrac{l}{h} \dfrac{I_1}{I_2}\right)}}$

2–16. $f = \frac{1}{2\pi}\sqrt{k\left(\frac{1}{I_1} + \frac{1}{I_2}\right)}.$

2–18. $\tau = 2\pi\sqrt{l/2g}.$

2–20. $f = \frac{1}{2\pi}\sqrt{\frac{2g}{3R}}.$

2–23. $\tau = 2\pi\sqrt{\frac{W(b+c)^2}{(k_1a^2 + k_2b^2)g}}.$

2–24. $\tau = 2\pi\frac{L}{a}\sqrt{\frac{W}{3kg}}.$

2–26. $k > \frac{WL}{2a^2};$

$$f = \frac{1}{2\pi}\sqrt{\frac{3g}{2L}\left(\frac{2ka^2}{WL} - 1\right)}.$$

2–29. $f = \frac{1}{2\pi}\sqrt{\frac{g \sin \alpha}{r}}.$

2–32. $f = 1/2\pi\sqrt{4Tg/WL}.$

2–33. $f = \frac{1}{2\pi}\sqrt{\frac{g}{2W\left(\frac{1}{k_1} + \frac{1}{k_2}\right)}}.$

2–34. $\delta_{st} = \frac{32W}{\pi d^4}\left[\frac{2(a^3 + b^3)}{3E} + \frac{ab^2}{G}\right]$

2–37. $\tau = 2\pi\sqrt{3(R - r)/2g}.$

2–38. $f = \frac{1}{2\pi}\sqrt{\frac{W_2lg}{\frac{3}{2}W_1r^2 + W_2l^2}}.$

2–41. $f = \frac{1}{\pi}\sqrt{\frac{3Rg}{L^2}}.$

2–42. $f = \frac{1}{2\pi}\sqrt{\frac{W_2r\left(1 + \frac{r}{L}\right)}{W_1\bar{r}^2}g}.$

2–45. $z_0 = l(\gamma - 1).$
$l_{eq} = l\gamma.$

2–48. $f = \frac{1}{2\pi}\sqrt{\frac{1}{1 + \frac{1 - 3\gamma(1-\gamma)}{1 - 3\beta(1-\beta)}\left(\frac{h}{L}\right)^2}\cdot\frac{g}{\gamma h}}.$

2–51. $f_T = \frac{1}{2\pi}\sqrt{\left(\frac{p_0A}{W} + 1\right)\frac{g}{L}}.$

$$f_A = \frac{1}{2\pi}\sqrt{\left(\frac{p_0A}{W} + 1\right)\frac{\gamma g}{L}}$$

where $\gamma = \frac{c_p}{c_v}.$

2–53. $f = \frac{1}{2\pi}\sqrt{\frac{g}{l + \frac{W_0}{k}}}.$

2–55. $f = \frac{1}{2\pi}\sqrt{\frac{W_1c_1 + n^2W_2c_2}{W_1\bar{r}_1^2 + n^2W_2\bar{r}_2^2}\cdot g}$

where $n = 1 + \frac{d}{r}.$

2–57. $\frac{2}{D^2} = \frac{4(1+\nu)}{d^2} = \frac{1+\nu}{(2+\nu)\bar{e}^2}$
where

$$\bar{e}^2 = \left(\frac{l+h}{2}\right)^2 + \left(\frac{d^2}{16} + \frac{h^2}{12}\right)$$

and ν = Poisson's ratio.

2–58. $\delta = \frac{a}{\frac{2kg}{W\Omega^2} - 1}.$

$$p^2 = \frac{2}{3}\left(\frac{2kg}{W} - \Omega^2\right).$$

2–61. $p^2 = \frac{1 + 2\frac{r}{l}\sin^3\alpha}{\cos\alpha}\cdot\frac{g}{L}.$

2–66. $f = \frac{\tan\beta}{\pi}\frac{l_2}{l_1}\sqrt{\frac{kg}{W_1 + 2W_2\left(\frac{\bar{r}}{r}\tan\beta\right)^2}}.$

2-67. $f = \frac{3}{2\pi}\sqrt{\frac{g}{13}};\ \frac{1}{2\pi}\sqrt{\frac{6g}{73}}.$

2-69. $f = \frac{\Omega}{2\pi}\sqrt{\frac{2a}{l}\sin\frac{l}{2R}}.$

2-70. $f = \frac{1}{2\pi}\sqrt{\frac{\cos^2\alpha_1\sin^3\alpha_2 + \cos^2\alpha_2\sin^3\alpha_1}{\cos\alpha_1\cos\alpha_2(\cos\alpha_1 + \cos\alpha_2)}\cdot\frac{g}{l}}.$

2-71. $f = \frac{1}{2\pi}\sqrt{\frac{\gamma_{11}W_1c_1 + \gamma_{22}n^2W_2c_2}{W_1\bar{r}_1^2 + n^2W_2\bar{r}_2^2}g}$

where $\gamma_{11} = \cos\alpha_1\left[1 + \left(1 + \frac{r_1}{l}\sin\beta_1\right)\tan\beta_1\tan\alpha_1\right],$

$\gamma_{22} = \cos\alpha_2\left[1 + \left(1 + \frac{r_2}{l}\sin\beta_2\right)\tan\beta_2\tan\alpha_2\right],$

$n = \frac{r_1\cos\beta_1}{r_2\cos\beta_2}.$

2-76. $f = \frac{1}{2\pi}\sqrt{\frac{3(W_1 + W_2)}{2W_1 + 6W_2 + 9W_3}\cdot\frac{g}{l}}.$

Chapter 3

3-2. 28.9 lb.

3-3. 0.077.

3-6. $\omega = \sqrt{\frac{kg}{2W}},\ \sqrt{\frac{3kg}{2W}}.$

3-8. $A = \frac{W_2}{W_1 + W_2}r.$

3-9. $A = \frac{r}{1 - \frac{kg}{(W_1 + W_2)\omega^2}}$

3-11. $p^2\left(1 - \frac{\delta + \delta_{st}}{a}\right) < \omega^2 < p^2\left(1 + \frac{\delta + \delta_{st}}{a}\right).$

3-14. $f \leq \frac{1}{10}f_a.$

3-15. $f \geq \frac{10}{\sqrt{\eta}}f_v.$

3-18. $\Theta = \frac{P_0/W}{1 - \frac{\omega^2 L}{g}}.$

3-20. $A = \frac{P}{k}\frac{\alpha}{1 - (\omega/p)^2}$

where $p^2 = \frac{kg}{W}.$

3-22. $x = \frac{a(\omega_1/p)^2}{1 - (\omega_1/p)^2}\cos(\omega_1 t + \beta_1)$

$- \frac{\delta_P}{1 - (\omega_2/p)^2}\cos(\omega_2 t + \beta_2)$

where $p^2 = k/m$ and $\delta_P = P_0/k.$

3-24. $\Theta = \frac{a\omega^2}{g\left(1 - \frac{\omega^2}{p^2}\right)},$

$\frac{a\omega^2}{g} < \sqrt{\frac{\omega^2}{p^2} - 1}$ for $\omega > p.$

3-25. $\Theta = \frac{\omega^2 a}{g}\cdot\frac{\left(\frac{W_1}{W_2} + 1\right)\frac{r}{l} - 1}{\left(1 - \frac{\omega^2}{p^2}\right)}$

where $p^2 = \frac{g}{l}\cdot\frac{1}{\frac{3}{2}\left(\frac{W_1 r}{W_2 l}\right)^2 + \left(1 - \frac{r}{l}\right)^2}.$

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and

$$\mu \geq \frac{a\omega^2}{g}\left\{1 + \frac{l\omega^2}{g}\frac{\left[\left(\frac{W_1}{W_2}+1\right)\frac{r}{l}-1\right]^2}{\frac{W_1}{W_2}\left(1-\frac{\omega^2}{p^2}\right)}\right\}.$$

Chapter 4

4–1. $A = \dfrac{r}{\sqrt{1 + (c/\omega m)^2}}$, $\varphi = \pi + \tan^{-1}(c/\omega m)$.

4–2. $A = \dfrac{P}{\omega\sqrt{(m\omega)^2 + c^2}}$, $\varphi = \pi + \tan^{-1}(c/\omega m)$

4–4. (a) $x = \dfrac{\omega/q}{\sqrt{\left(1-\dfrac{\omega^2}{p^2}\right)^2 + \left(\dfrac{\omega}{q}\right)^2}} A_p \cos(\omega t + \beta)$ where $\tan\beta = \dfrac{1-(\omega/p)^2}{\omega/q}$.

(b) $c = \dfrac{kA_m}{\omega A_p}\dfrac{1-(\omega/p)^2}{\sqrt{1-(A_m/A_p)^2}}$. (c) $\mu = \dfrac{1}{\pi}\dfrac{D-d}{(D+d)l}c$.

4–5. (a) $x = \dfrac{r}{\sqrt{\left(1-\dfrac{\omega^2}{p^2}\right)^2 + \left(\dfrac{\omega}{q}\right)^2}} \sin(\omega t - \beta)$ where $\tan\beta = \dfrac{\omega/q}{1-(\omega/p)^2}$.

(b) $c = \dfrac{kr}{\omega A_m} = \dfrac{kr}{pA_m} = \dfrac{r}{A_m}\sqrt{km}$.

4–7. $A = 0.017$ in.; $\tan\beta = \dfrac{2}{-7}$.

4–8. $c = \dfrac{r}{A}\sqrt{\dfrac{km}{1+n}}$ where $n = \dfrac{W}{mg}$.

4–10. $\dfrac{kA}{P} = \sqrt{\dfrac{1+(q/\omega)^2}{\left(1-\dfrac{\omega^2}{p^2}\right)^2 + \left(\dfrac{q}{\omega}\right)^2\left(\dfrac{\omega}{p}\right)^4}}$ where $q = k/c$ and $p^2 = k/m$.

4–13. $A = \dfrac{P_0\sqrt{1+(\omega c/k_2)^2}}{\sqrt{(2k_1 - \omega^2 m)^2 + (\omega c/k_2)^2(k_2 + 2k_1 - \omega^2 m)^2}}$.

4–15. $c = \sqrt{km\left[2 - \dfrac{m}{k}(\omega_1^2 + \omega_2^2)\right]}$.

Chapter 5

5–1. $f = \dfrac{1}{2\pi}\sqrt{k\left(\dfrac{1}{m_1} + \dfrac{1}{m_2}\right)}$, $l_1 = \dfrac{m_2}{m_1 + m_2}l$.

5–3. $f = \dfrac{1}{2\pi}\sqrt{k\left(\dfrac{1}{m_1} + \dfrac{1}{m_2'}\right)}$ where $m_2' = \left(1 + \dfrac{\bar{r}^2}{r^2}\right)m_2$. $l_1 = \dfrac{m_2'}{m_1 + m_2'}l$.

5–5. $f = \dfrac{1}{2\pi}\sqrt{\dfrac{2AE}{L}r_1^2\left(\dfrac{1}{I_1} + \dfrac{1}{n^2 I_2}\right)}$ where $n = r_1/r_2$.

5-6. $p^4 - p^2\left(\frac{k + wA}{m_1} + \frac{k}{m_2}\right) + \frac{wAk}{m_1 m_2} = 0.$

$\frac{A_{m1}}{A_{m2}} = 1 - \frac{m_2}{k} p^2.$

5-7. $p^4 - p^2\left[\frac{g}{l} + \left(k + \frac{W_0}{l}\right)\frac{g}{W_1}\right] + \frac{kg^2}{W_1 l} = 0.$

5-8. $x_1 = \left(\frac{1}{2}t^2 - \frac{m_2}{m_1}\frac{\cos pt}{p^2}\right) g \sin \alpha,$

$x_2 = \left(\frac{1}{2}t^2 + \frac{\cos pt}{p^2}\right) g \sin \alpha,$

where $p^2 = k\left(\frac{1}{m_1} + \frac{1}{m_2}\right).$

5-11. $p^4 - p^2[p_1{}^2(\alpha_1 - 1)$
$+ p_2{}^2(\alpha_2 + 1)]$
$+ (\alpha_1 - \alpha_2 - 1)p_1{}^2 p_2{}^2 = 0,$
$\alpha_1 \geq \alpha_2 + 1$, where

$\alpha_1 = \frac{kr_1{}^2}{W_1 l_1},\ \alpha_2 = \frac{kr_2{}^2}{W_2 l_2};$

$p_1{}^2 = \frac{g}{l_1},\ p_2{}^2 = \frac{g}{l_2}.$

5-12. (c) $A_{w2}/A_{w1} = \frac{1}{16}(3 \pm \sqrt{137}).$
$\Theta l_4/A_{w1} = \frac{1}{176}(29 \mp \sqrt{137}).$

5-13. (c) $p^4\left[\left(\frac{\bar{r}_1}{r_1}\right)^2 m_1\right]^2$

$- p^2\left[k_2\left(1 + \frac{\bar{r}_1{}^2}{r_1{}^2}\right)m_1\right.$

$\left.+ k_1\left(m_1\frac{\bar{r}_1{}^2}{r_1{}^2} + m_2\right)\right] + k_1 k_2 = 0.$

5-17. $p^4(m_1 \cos \alpha)^2$
$- p^2[k_1(m_1 + m_2) + k_2 m_1]$
$+ k_1 k_2 = 0.$

5-18. $\frac{A_{m1}}{A_{m2}} = \frac{k_2 - p^2(m_1 + m_2)}{p^2 m_1 \cos \alpha}$

$= \frac{3p^2 m_1 \cos \alpha}{2k_1 - 3p^2 m_1}.$

5-20. $f_1 = \frac{1}{2} f_2 = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}}$

where δ_{st} is the static deflection of the plate.

5-21. $p^2 = \frac{k_1 l^2}{I_1}[1 - 2\beta(1 - \beta)$
$\pm 2\beta(1 - \beta)].$

5-22. $p^4 - p^2(\frac{11}{24}p_1{}^2 + p_2{}^2)$
$+ p_2{}^2(\frac{5}{24}p_1{}^2 + \frac{6}{25}p_2{}^2) = 0$
where $p_1{}^2 = kg/Wl^2$, $p_2{}^2 = g/l$.

5-23. $\frac{\Theta r}{A} = -\frac{k_1 + k_2 + k_3 - p^2 m}{nk_1 - k_2}$

$= -\frac{nk_1 - k_2}{n^2 k_1 + k_2 - p^2 m\left(\frac{\bar{r}}{r}\right)^2}$

where $n = R/r$.

5-24. $p^2 = \frac{g}{a}\left(1 + 2\frac{a}{c}\sin^3 \varphi\right),$

$\frac{g}{a \cos \varphi},\ \frac{g}{a \cos \varphi}\left(\frac{c}{a} + 2 \sin \varphi\right),$

where $\sin \varphi = \frac{L - c}{2a}.$

5-25. $p^2 = \left(1 + 2\frac{r}{l}\sin^3 \alpha\right)\frac{g}{L \cos \alpha},$

$\left(1 + \frac{2kr^2}{WL}\cos^3 \alpha\right)\frac{g}{L \cos \alpha}.$

5-26. $(\gamma_{11}W_1 c_1 + kr_1{}^2 \cos \beta_1{}^2 - p^2 I_1)(\gamma_{22}W_2 c_2 + kr_2{}^2 \cos^2 \beta_2 - p^2 I_2)$
$- (\gamma_{21}W_1 c_1 - kr_1 r_2 \cos \beta_1 \cos \beta_2)(\gamma_{12}W_2 c_2 - kr_1 r_2 \cos \beta_1 \cos \beta_2) = 0$
where

$\gamma_{11} = \cos \alpha_1\left[1 + \left(1 + \frac{r_1}{l}\sin \beta_1\right)\tan \beta_1 \tan \alpha_1\right],$

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$$\gamma_{22} = \cos \alpha_2 \left[1 + \left(1 + \frac{r_2}{l} \sin \beta_2 \right) \tan \beta_2 \tan \alpha_2 \right],$$

$$\gamma_{21} = \frac{r_1}{l} \sin \beta_1 \sin \beta_2 \tan \alpha_2, \qquad \gamma_{12} = \frac{r_2}{l} \sin \beta_1 \sin \beta_2 \tan \alpha_1.$$

5–30. (a) $A_{m1} = \dfrac{1 - (\omega/p_1)^2}{Z} \dfrac{P_0}{k}$, $\quad A_{m2} = \dfrac{1}{Z} \dfrac{P_0}{k}$.

(b) $A_{m1} = \dfrac{1}{Z} \dfrac{P_0}{k}$, $\quad A_{m2} = \dfrac{1 - (\omega/p_2)^2}{Z} \dfrac{P_0}{k}$,

where $p_1^2 = \dfrac{k}{m_1}$, $p_2^2 = \dfrac{2k}{3m_2}$, $\quad Z = \left(1 - \dfrac{\omega^2}{p_1^2}\right)\left(1 - \dfrac{\omega^2}{p_2^2}\right) - 1.$

5–32. $$A_{w1} = \frac{2}{3} \frac{1 - \dfrac{l\omega^2}{g}}{\left(1 - \dfrac{3W_0}{kg}\omega^2\right)\left(1 - \dfrac{l\omega^2}{g}\right) - \dfrac{W_0}{kg}\omega^2} \frac{P_0}{k}.$$

$$\frac{A_{w1}}{A_{w0}} = \frac{1}{1 - \dfrac{l\omega^2}{g}}.$$

5–34. (a) $e = \dfrac{17}{3} l_2$, $\Theta = \dfrac{3}{10} \dfrac{P_0}{k_2 l_2}$.

(b) $p^2 = (2.2114, 0.2261) \dfrac{k_2}{m_2}$,

$e = (5.34, 0.80) l_2$,

where e is nodal distance measured from the left end.

5–36. $\Theta_A = \dfrac{3\Theta_0}{3 - 2\alpha^2}$, $\Theta_B = \dfrac{3\Theta_0}{4 - \alpha^2}$,

$T = 9k\Theta_0 \dfrac{\alpha^4 - 3\alpha^2 + 1}{(3 - 2\alpha^2)(4 - \alpha^2)}$,

$p^2 = \dfrac{1}{2}(3 \pm \sqrt{5}) \dfrac{k}{I}$,

where $\alpha^2 = \dfrac{\omega^2 I}{k}$.

5–37. (a) $P_A = \dfrac{\omega^2}{l} \sqrt{(\Sigma X_A' b)^2 + (\Sigma Y_A' b)^2} \cos (\omega t + \beta_A)$,

$P_B = \dfrac{\omega^2}{l} \sqrt{(\Sigma X_B' a)^2 + (\Sigma Y_B' a)^2} \cos (\omega t + \beta_B)$,

where $\Sigma X_A' b = m_1 r_1 b_1 \cos \beta_1 + m_2 r_2 b_2 \cos \beta_2$,
$\Sigma Y_A' b = m_1 r_1 b_1 \sin \beta_1 + m_2 r_2 b_2 \sin \beta_2$,
$\tan \beta_A = \Sigma Y_A' b / \Sigma X_A' b$, etc.

(b) $x_A = \dfrac{P_A}{k_A} \cdot \dfrac{1}{1 - \dfrac{\omega^2}{p_A^2}} \cos (\omega t + \beta_A) = a_A \cos (\omega t + \beta_A)$, etc.,

where $p_A^2 = \dfrac{k_A}{m_A + \dfrac{1}{2} m_s + \dfrac{1}{l} \Sigma mb}$, etc., and m_s is the mass of the shaft.

$$\Theta = \frac{1}{l} \sqrt{a_A^2 + a_B^2 - 2a_A a_B \cos (\beta_A - \beta_B)}.$$

5–38. $x_1 = \dfrac{k_1[k_2 + k_3 - (n\omega)^2 m_2 (\bar{r}/r)^2]}{Z_2} r_1 \cos (n\omega t + \beta_1) + \dfrac{k_2 k_3}{Z_1} r_2 \cos (\omega t + \beta_2)$,

(continued on next page)

$$r\theta_2 = \frac{k_1k_2}{Z_2} r_1 \cos(n\omega t + \beta_1) + \frac{k_3(k_1 + k_2 - \omega^2 m_1)}{Z_1} r_2 \cos(\omega t + \beta_2),$$

where $Z_1 = k_1(k_2 + k_3) + k_2k_3 - \omega^2[(k_1 + k_2)m_2(\bar{r}/r)^2 + (k_2 + k_3)m_1] + \omega^4 m_1 m_2(\bar{r}/r)^2,$

$Z_2 = k_1(k_2 + k_3) + k_2k_3 - (n\omega)^2[(k_1 + k_2)m_2(\bar{r}/r)^2 + (k_2 + k_3)m_1] + (n\omega)^4 m_1 m_2(\bar{r}/r)^2.$

5–42. $m_{2e} = m_2\left(\frac{\bar{r}_0}{l}\right)^2, \ k_{2e} = m_2\frac{cg}{l^2}.$

5–43. $I_{1e} = m_1 l^2, \ I_{2e} = m_2\bar{r}_0^2;$
$k_{1e} = k_1 l^2, \ k_{2e} = mgc.$

5–45. $k_{1e} = k_1, \ k_{2e} = k_2(r_2/r_1)^2,$
$k_{3e} = k_3(r_2/r_1)^2;$
$m_{1e} = m_1(\bar{r}_1/r_1)^2,$
$m_{2e} = m_2(r_2/r_1)^2,$
$m_{3e} = m_3(r_3/r_1)^2.$

5–46. $k_{1e} = k_1(r_0/r_1)^2, \ k_{2e} = k_2,$
$k_{3e} = k_3(r_3/r_2)^2;$
$m_{1e} = m_1(\bar{r}_1/r_1)^2,$
$m_{2e} = m_2(\bar{r}_2/r_2)^2,$
$m_{3e} = m_3(r_3/r_2)^2.$

5–48. $k_{1e} = W_1\frac{c_1 \cos \alpha_1}{r_1^2},$
$k_{2e} = W_2\frac{c_2 \cos \alpha_2}{r_2^2};$
$m_{1e} = m_1(\bar{r}_1/r_1)^2;$
$m_{2e} = m_2(\bar{r}_2/r_2)^2.$

5–50. $k_{1e} = m_1 g l_1 \cos \alpha_1,$
$k_{3e} = m_2 g l_2 \cos \alpha_2;$
$I_{1e} = m_1 l_1^2, \ I_{2e} = m_2 l_2^2.$

5–52. $\Theta = \left|\frac{1}{1.065 + 0.013j}\frac{A_0}{l}\right| = 0.939\frac{A_0}{l}.$

$A_{5w} = \left|\frac{0.065 + 0.013j}{1.065 + 0.013j}A_0\right| = 0.0622A_0.$

5–53. $A_0 = \frac{1 + n}{2} a_1, \ \Theta = \frac{1 - n}{r} a_1,$

where $a_1 = \frac{P_0}{k_1 + nk_2 - \frac{1}{2}(1 + n)\omega^2 m + \frac{1}{2}(1 + n)j\omega c},$

$$n = \frac{\frac{k_1}{m} - \omega^2}{\frac{k_2}{m} - \omega^2}.$$

5–55. $A = \frac{1}{2\sqrt{10}}\frac{P_0}{k}, \ \Theta = \frac{1}{\sqrt{281}}\frac{P_0}{kr}.$

5–56. $A_1 = \frac{\cos \alpha}{\sqrt{1 + \cos^4 \alpha}}\frac{P_0}{k},$
$\varphi_1 = -\tan^{-1}\left(\frac{1}{\cos^2 \alpha}\right);$
$A_2 = \frac{1}{\sqrt{1 + \cos^4 \alpha}}\frac{P_0}{k},$
$\varphi_2 = \tan^{-1}(\cos^2 \alpha).$

5–61. $(k_{31} + k_{12} - p^2 m_1')(k_{12} + k_{23} - p^2 m_2')(k_{23} + k_{31} - p^2 m_3') - 2k_{12}k_{23}k_{31} - (k_{31} + k_{12} - p^2 m_1')k_{23}^2 - (k_{12} + k_{23} - p^2 m_2')k_{31}^2 - (k_{23} + k_{31} - p^2 m_3')k_{12}^2 = 0.$

where $m' = \frac{m}{2}\left[\frac{(R_0/R)^2 - (R_i/R_0)^2(R_i/R)^2}{1 - (R_i/R_0)^2}\right].$

5–62. $[p^4 - p^2(p_\alpha^2 + p_\beta^2) + p_l^2(p_\beta^2 - p_\alpha^2 - p_l^2)] \times (p^2 - p_\alpha^2 - p_l^2) = 0$
where $p_\alpha^2 = \alpha^2 k/m,$
$p_\beta^2 = 2\beta^2 k/m, \ p_l^2 = g/l.$
$\frac{k}{m}(2\beta^2 - \alpha^2) > \frac{g}{l}.$

5–64. $\frac{8 \times 10^4}{\omega^2 - 4000} + \frac{10 \times 10^4}{\omega^2 - 10{,}000} + \frac{16 \times 10^5}{\omega^2 - 13{,}333} + \frac{54 \times 10^5}{\omega^2 - 20{,}000} = 230.$

5–66. $p_{1,2}^2 = \dfrac{\pi}{4\beta^2}\left[5 + 15\beta + 21\beta^2 \pm \sqrt{(5 + 15\beta + 21\beta^2)^2 - 30\beta^2}\right]\dfrac{Er^4g}{WL^3},$

$$p_3^2 = \frac{5\pi}{8(1+\nu)} \cdot \frac{Er^4g}{WR^2L},$$

where $\beta = \dfrac{R}{L}$ and ν = Poisson's ratio.

5–67. $24(p/p_0)^4 - 16(p/p_0)^2 + 1 = 0$
where $p_0^2 = kg/W.$
$A_1/A = 1 \mp \sqrt{10}/4,$
$A_2/A = -\frac{1}{2} \pm \sqrt{10}/4.$

5–68. $p^2 = \dfrac{3\pi}{4(1+\nu)} \cdot \dfrac{Er^4}{WL^3}\, g,$

$$\frac{3\pi}{4}\,\frac{Er^4}{WL^3}\, g,$$

$$\frac{3\pi}{2}\,(2 \pm \sqrt{3})\,\frac{Er^4}{WL^3}\, g.$$

5–69. $p^2 = (0.8407, 2.661, 10.498)\,\dfrac{kg}{W}.$

$$\frac{A_{k_1}}{A_{k_2}} = 3 - \frac{2\alpha^2}{4+\alpha^2} \qquad \text{where } \alpha^2 = \frac{p^2W}{kg}.$$

5–70. (*a*) $A_m = \dfrac{3 - 2\alpha^2}{Z}\,\dfrac{P_0}{k},$

$$A_{2m} = \frac{2(1-\alpha^2)}{Z}\,\frac{P_0}{k},$$

$$A_{4m} = \frac{(1-\alpha^2)(3-2\alpha^2)}{Z}\,\frac{P_0}{k},$$

where $\alpha^2 = \dfrac{\omega^2W}{kg};$

$Z = 14 - 41\alpha^2 + 34\alpha^4 - 8\alpha^6.$

(*c*) $\alpha^2 = 0.590,\ 1.211,\ 2.449.$

5–74. $p^2 = \dfrac{g}{R - r}\left[\dfrac{1}{1 + \left(\dfrac{\bar{r}}{r}\right)^2 - \dfrac{(\bar{r}/r)^4}{\left(\dfrac{\bar{r}}{r}\right)^2 + \dfrac{W_1}{W_2}}}\right].$

5–75. $p^2 = \dfrac{2g}{3(R-r)} \cdot \dfrac{1}{1 - \dfrac{1}{\dfrac{M}{3m} + 1}}.$

5–76. $p^2 = \alpha \pm \sqrt{\alpha^2 - \beta^2}$ where $\alpha = \dfrac{1}{2}\left(1 + \dfrac{m_2}{m_1}\right)\left(\dfrac{g}{l_1} + \dfrac{g}{l_2} - 2\Omega^2\right) + \dfrac{m_2}{m_1}\,\Omega^2,$

$$\beta^2 = \left(1 + \frac{m_2}{m_1}\right)\left(\frac{g}{l_1} - \Omega^2\right)\left(\frac{g}{l_2} - \Omega^2\right) - \frac{m_2}{m_1}\,\Omega^4.$$

5–78. $p^2 = 0,\ \dfrac{(6\alpha - 2) + \beta^2(2\alpha + 1)}{(1-\alpha)^4 + \beta^2(6\alpha - 5) + \beta^4} \cdot \dfrac{g}{R},$
where $\alpha^2 + \alpha = 1,\ \alpha = c/R,\ \beta = \bar{r}_c/R.$

5–79. $p^2 = \dfrac{1}{1 + \dfrac{\bar{r}^2}{r^2}}\left[\dfrac{g}{(R-r)\cos\varphi_0} + \dfrac{k}{m}\,(1 \pm 1)\cos^2\varphi_0\right].$

Chapter 6

6–22. (a) $\Theta_1 = \Theta_2 = -\Theta_3 = -\frac{1}{2}\frac{T_1}{k}$.

(b) $\Theta_1 = -\Theta_2 = \Theta_3 = -\frac{1}{2}\frac{T_2}{k}$.

6–23. (a) $\Theta_1 = -3\Theta_2 = 3\Theta_3 = -\frac{3}{2}\frac{T_0}{k}$.

(b) $\Theta_1 = \frac{\sqrt{5}}{2}\frac{T_0}{k}\,\underline{/\frac{\pi}{2} + \gamma}$,

$\Theta_2 = \frac{\sqrt{5}}{2}\frac{T_0}{k}\,\underline{/-\left(\frac{\pi}{2} + \gamma\right)}$,

$\Theta_3 = \frac{\sqrt{5}}{2}\frac{T_0}{k}\,\underline{/\frac{\pi}{2} - \gamma}$,

where $\tan\gamma = \frac{1}{2}$.

6–24. $\Theta = -P_0/kl$,

$e = 3l$ to right of A;

$p^2 = (3 \pm \sqrt{5})\frac{k}{m}$,

$e = \frac{1}{2}(3 \mp \sqrt{5})l$ to right of A;

$\omega^2 = 2k/m$.

6–29. $A_{mA} = -\frac{3}{2}\frac{P_0}{k}$,

$A_{mB} = \frac{1}{2}\frac{P_0}{k}$, $\Theta = -\frac{P_0}{kl}$,

$p^2 = (1, 1 \pm \sqrt{\tfrac{1}{5}})\frac{k}{m}$,

$\frac{A_{mA}}{A_{mB}} = [\tfrac{1}{2}, \tfrac{1}{2}(1 \pm 3\sqrt{\tfrac{1}{5}})]$,

$e = [1, \tfrac{5}{2}(1 \mp \sqrt{\tfrac{1}{5}})]l$ to right of A.

6–30. $\Theta = \frac{6}{13}\frac{A_0}{l}$, $e = 3l$ to left of k_1

6–31. (a) $\Theta = 0.780\frac{A_0}{r}\,\underline{/45^\circ}$,

$A_d = 1.41A_0\,\underline{/-45^\circ}$.

(b) $\Theta = 0.775\frac{A_0}{r}\,\underline{/54.5^\circ}$,

$A_d = 1.12A_0\,\underline{/-35.5^\circ}$.

6–33. $\Theta = \frac{3}{3 - 5\alpha^2 + \alpha^4}\frac{A_0}{l}$,

$e = (1 - \frac{1}{3}\alpha^2)l$ to right of mid-point of beam.

6–34. $A_{k1} = \tfrac{1}{5}\sqrt{5 + 2\sqrt{2}}$

$A_0\,\underline{/\tan^{-1}[-\tfrac{1}{7}(2\sqrt{2} - 1)]}$

$= 0.56A_0\,\underline{/165.5^\circ}$,

$A_{k2} = \tfrac{1}{5}\sqrt{65 + 8\sqrt{2}}$

$A_0\,\underline{/\tan^{-1}[-\tfrac{1}{31}(32 - 4\sqrt{2})]}$

$= 1.75A_0\,\underline{/-40.3^\circ}$,

$A_d = 0.18A_0\,\underline{/-124^\circ}$,

$e = 0.237b$ to right of k_1,

$\Theta = 2.24\frac{A_0}{b}\,\underline{/-34^\circ}$.

6–36. $$A = \frac{\sin\beta}{\beta}\,\frac{1}{1 - \frac{\gamma h}{g}\left(\frac{2\pi V}{L}\right)^2}A_0,$$

$$\Theta = \frac{\sin\beta - \beta\cos\beta}{1 - 6\gamma(1 - \gamma)\left(\frac{h}{l}\right)^2 - \left(1 + \frac{h^2}{l^2}\right)\frac{\gamma h}{g}\left(\frac{2\pi V}{L}\right)^2}\cdot\frac{A_0}{l}.$$

where $\beta = \pi\frac{l}{L}$.

$$\underset{\text{Translation}}{V_{cr}} = \frac{L}{2\pi}\sqrt{\frac{g}{\gamma h}}, \quad \underset{\text{Rotation}}{V_{cr}} = \underset{\text{Translation}}{V_{cr}}\cdot\sqrt{\frac{1 - 6\gamma(1 - \gamma)\,(h/l)^2}{1 + (h/l)^2}}.$$

6–37. $l/L = 1.43$, $l/L = 2.46$, $l/L = n + \frac{1}{2}$, where n is an integer.

6-38. $A_{11} = 5A_0\underline{/0}$,
$A_{21} = 2A_0\underline{/0}$,
$A_{31} = -4A_0\underline{/0}$;
$A_{12} = 2A_0 \underline{/\frac{3\pi}{4}}$,
$A_{22} = A_0 \underline{/\frac{3\pi}{4}}$,
$A_{32} = -A_0 \underline{/\frac{3\pi}{4}}$;
$A_{13} = -4A_0 \underline{/\frac{\pi}{2}}$,
$A_{23} = -A_0 \underline{/\frac{\pi}{2}}$,
$A_{33} = 5A_0 \underline{/\frac{\pi}{2}}$;

where A_{mn} indicates the amplitude at m due to the wave effect at n.
$A_{①} = 4.42A_0\underline{/-35.8°}$,
$A_{③} = 5.41A_0\underline{/127.5°}$,
$A_d = 0.707A_0\underline{/45°}$,
$\Theta = 3.24\frac{A_0}{b}\underline{/-45°}$.
node A_d is $1.35b$ to right of point 1.

6-44. $$\Theta_y = \frac{6}{\beta_l^2} \cdot \frac{\sin\beta_l - \beta_l\cos\beta_l}{1 - 6\gamma(1-\gamma)\left(\frac{h}{l}\right)^2 - \left(1 + \frac{h^2}{l^2}\right)\left(\frac{\omega}{p}\right)^2} \times \frac{\sin\beta_b}{\beta_b}\cdot\frac{A_0}{l},$$
$$\Theta_x = \frac{6}{\beta_b^2} \cdot \frac{\sin\beta_b - \beta_b\cos\beta_b}{1 - 6\gamma(1-\gamma)\left(\frac{h}{b}\right)^2 - \left(1 + \frac{h^2}{b^2}\right)\left(\frac{\omega}{p}\right)^2} \times \frac{\sin\beta_l}{\beta_l}\cdot\frac{A_0}{b},$$
from which $\Theta = \sqrt{\Theta_x^2 + \Theta_y^2}$, and $\tan\psi = \Theta_y/\Theta_x$.
$$A = \frac{\sin\beta_l\sin\beta_b}{\beta_l\beta_b}\cdot\frac{A_0}{1-(\omega/p)^2} \text{ where } \omega = 2\pi\frac{V}{L} \text{ and } p^2 = \frac{g}{\gamma h}.$$

Chapter 7

7-1. $f = 11.25, 18.60, 29.08$ cycles per sec.

7-3. $8\alpha^6 - 42\alpha^4 + 57\alpha^2 - 20 = 0$ where $\alpha^2 = \frac{I_1}{k_1}p^2 = 100p^2$.

7-14. $T_0 = 0$,
$T_1 = \alpha^2$,
$T_2 = (2 - \alpha^2)\alpha^2$,
$T_3 = (3 - 4\alpha^2 + \alpha^4)\alpha^2$,
$T_4 = (4 - 10\alpha^2 + 6\alpha^4 - \alpha^6)\alpha^2$,
$T_5 = (5 - 20\alpha^2 + 21\alpha^4 - 8\alpha^6 + \alpha^8)\alpha^2$,
$T_6 = (6 - 35\alpha^2 + 56\alpha^4 - 36\alpha^6 + 10\alpha^8 - \alpha^{10})\alpha^2$,
where $p^2 = \alpha^2\frac{k}{I}$ or $\alpha^2\frac{k}{m}$.

7-7. $p^2 = (0.6820,\ 1.9221,\ 3.7292)\frac{k_1}{I_1}$.

7-8. $(\alpha^2 - 1)(8\alpha^8 - 50\alpha^6 + 96\alpha^4 - 60\alpha^2 + 11) = 0$ where $p^2 = \alpha^2\frac{k_1}{I_1}$.

7-16. $30\alpha^{12} - 331\alpha^{10} + 1360\alpha^8 - 2556\alpha^6 + 2156\alpha^4 - 665\alpha^2 + 36 = 0$.

7-17. $p^2 = 0.4, 2.5, 4.0$ rad² per sec².

7-18. $p^2 = 2.4, 2.5, 5.0$ rad² per sec².

7-19. $p^2 = 1.5, 2.5, 4.0, 6.0$ rad² per sec².

Chapter 8

8-1. $y = \frac{2AL^2}{\pi^3 L_1 L_2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi L_1}{L} \sin \frac{n\pi x}{L} \cos \frac{n\pi at}{L}.$

8-5. 2%.

8-6. $f = nc/2L, \quad n = 1, 2, 3 \cdots.$

8-7. $f = nc/2L, \quad n = 1, 2, 3 \cdots.$

8-8. None.

8-9. $u = + \frac{4PL}{\pi^2 AE} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^2} \sin \frac{n\pi x}{L} \cos \frac{n\pi at}{L}.$

8-11. $f = \frac{n + \frac{1}{2}}{2} \frac{c}{\sqrt{2(1+\nu)}}.$

8-13. $f = c\bar{r}\beta^2/2\pi$ where β is a root of $\cosh \beta L \cos \beta L = -1$.

8-15. $f = c\bar{r}\beta^2/2\pi$ where β is a root of $\cosh \beta L \cos \beta L = 1$.

8-16. Same as answer for Prob. 8-15.

8-20. $p^2 = \frac{48EI}{L^3(W + \frac{17}{35}wL)} g$

where the trial solution is similiar to that for the static deflection due to the weight W.

Chapter 9

9-1. $c = A\mu/\delta.$

9-5. $c = 8\pi\mu L.$

9-7. $c = 2Q\rho.$

9-15. $A = \frac{2\delta \sin \frac{pL}{2V_0}}{(pL/2\pi V_0)^2 - 1}$

if $p^2 \neq (2\pi V_0/L)^2$,

$A = pL\delta/2V_0$ if $p^2 = (2\pi V_0/L)^2$,

where $p^2 = kg/W$.

Chapter 10

10-2. $\tau = 2\sqrt{2\left(\frac{\bar{r}^2}{R^2} + 1\right)\frac{L}{g}}.$

10-4. $A = 2.91V\sqrt{m/k_2}.$

$\tau = 7.14\sqrt{m/k_2}.$

10-13. $f = \frac{1}{2\pi}\sqrt{\left[1 + \frac{kL}{W}\left(1 + \frac{h}{L}\right)\left(1 - \frac{l}{h}\right)\right]\frac{g}{L}}.$

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