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A JUNIOR GEOMETRY

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A JUNIOR GEOMETRY

BASED ON THE VARIOUS GEOMETRY BOOKS
BY GODFREY AND SIDDONS

By

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PREFACE

THIS book is based on the various Geometry books by Godfrey and Siddons. It is intended to be a first Geometry book for boys and girls beginning the subject between the ages of ten and thirteen. It covers all that is wanted in Preparatory Schools, and it meets all the requirements of many Girls' Schools, Junior Day and Evening Technical Schools, and Central Schools.

Most Geometry books hitherto have aimed at covering the whole school course for non-specialists up to, and even beyond, Leaving Certificate Standard; the result has been that the teacher of young pupils has had to omit parts of the books and amplify other parts, besides supplying easy examples and different explanations. The tendency has been to give the boy or girl a treatment more suitable for a later age, to suppress instead of to develop intuition, and to encourage the learning of proofs of theorems far too early. In fact the attempt to meet the requirements of both old and young pupils in a single book has tended to make such a book less suitable for both classes.

Throughout this book methods of experiment, intuition, and deduction are used, and pupils are trained to appreciate the true use of each. Logic appeals to quite young pupils in easy cases; in other cases it is experiment or intuition that appeals, but the pupil must be gradually trained to see that the ultimate aim should be a logical proof. On the other hand, teachers should remember that logical proof of the obvious does not appeal to young pupils; it is only at a later stage, after much ground has been covered, that the proof in itself appeals, and the desire comes to have a systematic treatment based on the smallest possible number of assumptions (see *Theoretical Geometry*, Chapter I, by the same authors).

Special points in this book are.—

1. The needs of the teacher are kept in the foreground throughout.
2. It gives a full course up to the age of 14 or 15 for the able child, and beyond that age for the slower child.

3. Ample explanation is given, and there are many *viva voce* questions to develop the pupil's powers of intuition.
4. There is a good supply of questions requiring calculation or accurate drawing.
5. Many examples are worked out in full to show the pupil how he should arrange his work.
6. There is a large number of easy riders to develop the logical aspect of the subject.
7. Logical proofs of suitable theorems are given and should be learned.
8. Throughout this book examples in three dimensions are introduced, and at the end of the book there is a chapter (which may be taken earlier) designed to help the pupil to make freehand drawings of solid figures and to understand such drawings.
9. The examples are carefully grouped so that the teacher can easily pick out the type of question he wants.
10. There are three sets of Revision Papers and three sets of Miscellaneous Exercises.
11. There is a very full index at the end of the book which will be found useful in finding particular pieces of work.

This book does not aim at giving a complete training in Practical Geometry, or a systematic course of Logical Geometry—that is left to *Practical Geometry* and *Theoretical Geometry* by the same authors.

We have to acknowledge the courtesy of H.M. Stationery Office, of the Oxford and Cambridge Joint Board, of the Oxford Local Examinations Delegacy, of the Cambridge Local Examinations Syndicate, of the Universities of London and Bristol, and of the Board for the Common Examination for Entrance to Public Schools for permission to include exercises from public examination papers.

Finally we have to thank many colleagues and friends for suggestions and criticisms.

A. W. S.
R. T. H.

HARROW,
March, 1926

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TO THE TEACHER

Exercises distinguished by a paragraph sign thus ¶Ex. 1 on p. 1, are intended for discussion in class.

Exercises of a theoretical character (riders) are marked thus †Ex. 17, on p. 21

Note the summaries at the ends of the chapters

Note the very full index at the end of the book.

Many suggestions will be found in “Notes and Answers ”

CHAPTER 1

MEASUREMENT OF LINES AND ANGLES

SOLIDS.

§ I If you look at a brick you find that it is bounded by six flat or **plane**, surfaces. These are called **faces**. Where two faces meet there is an **edge**. Where two edges meet there is a corner, or **vertex**. The brick is called a **solid**.

In other solids, e.g. a pencil, some of the bounding surfaces are curved, in a ball there is only one surface.

¶ Ex. 1. Fill up a table such as the following, using models of solids* .

	Number of faces	Number of edges	Number of vertices
Cube			
Cuboid (or rect- angular block)			
3-sided pyramid			
4-sided pyramid			
6-sided pyramid			
3-sided prism			
4-sided prism			
6-sided prism			
Cone			
Cylinder			
Sphere			
Hemisphere			

* See Notes and Answers

¶Ex. 2 Place (or hold) a cube so that you can only see one of its faces; make a sketch of it

¶Ex. 3 Place (or hold) a cube so that you can see two of its faces; make a sketch of it

¶Ex. 4. Place a cube so that you can see three of its faces, make a sketch of it.

¶Ex 5. Repeat Exx 2-4 for a 3-sided prism

¶Ex 6 Make sketches of a 6-sided prism in various positions. What is the greatest number of faces you can see at one time?

¶Ex 7. Repeat Ex 6 for (i) a 3-sided pyramid, (ii) a 4-sided pyramid

¶Ex 8 Make sketches of a cylinder and a cone

*Ex 9 Make a paper or cardboard box in the shape of a cuboid. You must first draw a pattern such as Fig 1

Ex 10 Make a paper or cardboard model of (i) a triangular prism (see fig 2), (ii) a square pyramid

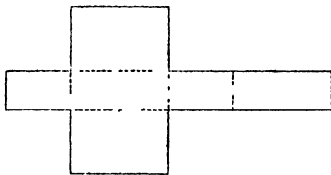


Fig 1

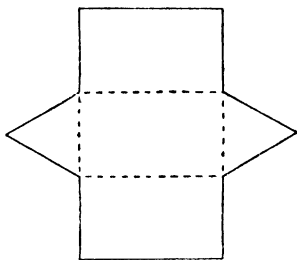


Fig 2

MEASUREMENT OF STRAIGHT LINES*

§2. In stating the length of an object always give the unit, the following abbreviations may be used — *in* for *inch*, *cm* for *centimetre*, *mm* for *millimetre* 2 inches may be written 2"

In Chaps 1-5*, all lengths measured in inches are to be given to the nearest tenth of an inch, all lengths measured in centimetres to the nearest millimetre.

Always give your answers in decimals.

* See Notes and Answers.

Ex. 11 Measure the lengths AB, CD, EF, GH in fig 3 (i) in inches, (ii) in centimetres.

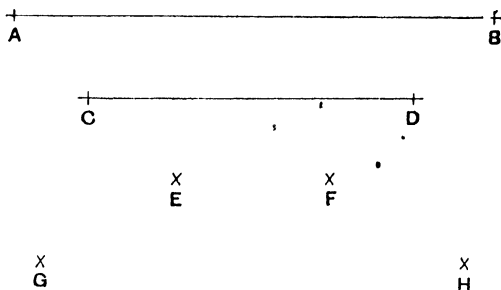


Fig 3

Ex 12 Measure in inches the lengths AB, BC, CD in fig. 4, arrange your results in tabular form and add them together.

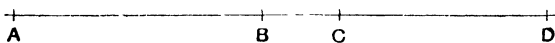


Fig 4

$$AB = \quad \text{in}$$

$$BC = \quad \text{in}$$

$$CD = \quad \text{in}$$

$$AB + BC + CD = \quad \text{in}$$

Check by measuring AD*

Ex 13 Repeat Ex 12, using centimetres instead of inches

Ex 14 Measure in centimetres the lengths AB, CB in fig 5, and find their difference, arrange your results in tabular form



Fig 5

$$AB = \quad \text{cm}$$

$$CB = \quad \text{cm}$$

$$AB - CB = \quad \text{cm}$$

Check by measuring AC

Ex 15 Repeat Ex 14, using inches instead of centimetres.

* See Notes and Answers.

Ex. 16. Measure in inches, and also in centimetres, the length of the paper you are using

[Your ruler is probably too short to measure directly; divide the length into two (or more) parts by making a pencil mark on the edge, and add these lengths together.]

Ex. 17 Measure the breadth of your paper in inches and also in centimetres

Ex. 18 Draw a straight line about 6 in. long and cut off a part $AB = 2$ in., a part $BC = 1.5$ in., and a part $CD = 1.8$ in., find the length of AD by adding these lengths, check by measuring AD . [Make a table as in Ex. 12]

Ex. 19 Repeat Ex. 18, with

- | | | |
|-----------------------|-----------------|----------------|
| (i) $AB = 2.7$ cm, | $BC = 9.6$ cm, | $CD = 1.3$ cm |
| (ii) $AB = 0.7$ in., | $BC = 2.6$ in., | $CD = 2.4$ in. |
| (iii) $AB = 1.8$ in., | $BC = 2.9$ in., | $CD = 0.6$ in. |

Ex. 20 A man walks 3.2 miles due north and then 1.5 miles due south, how far is he from his starting point? Draw a plan (1 mile being represented by 1 inch) and find the distance by measurement

Ex. 21 A man walks 5.4 miles due west and then 8.2 miles due east, how far is he from his starting point? Draw a plan (Represent 1 mile by 1 centimetre)

Ex. 22. A man walks 7.3 miles due south, then 12.7 miles due north, then 1.1 miles due south, how far is he from his starting point? (Represent 1 mile by 1 centimetre)

Ex. 23 Draw a straight line, guess its middle point and mark it by a short cross-line; test your guess by measuring the two parts

Ex. 24 Repeat Ex. 23 three times with lines of various lengths. Show by a table how far you are wrong

Ex. 25 Draw a straight line of 10.6 cm, bisect it by calculating the length of half the line and measuring off that length from one end of the line, then measure the remaining part

When told to draw a line of some given length, you should draw a line a little too long and cut off a part equal to the given length as in fig. 6. You should also write the length of the line against it, being careful to state the unit

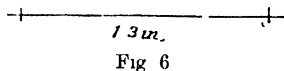


Fig 6

Ex. 26 Draw a straight line 3.2 in long; bisect it as in Ex. 25. Repeat with a straight line 2.7 in long.

Ex. 27 Draw straight lines of the following lengths, bisect each of them — (i) 7.6 cm, (ii) 10.5 cm, (iii) 4.1 in, (iv) 9 in

A good practical method of bisecting a straight line (AB) is as follows (see fig. 7) — measure off with compasses equal lengths

(AC, BD) from each end of the line (these lengths should be very nearly half the length of the line) and bisect the remaining portion (CD) by eye.

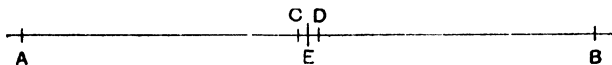


Fig 7.

Ex. 28 Draw three or four straight lines and bisect them with your compasses (as explained above); verify by measuring each part of the line (remember to write its length against each part)

Ex. 29 Open your compasses 1 cm, apply them to the inch scale and so find the number of inches in 1 centimetre

Ex. 30. Find the number of inches in 10 cm as in Ex 29, hence express 1 cm in inches. Arrange your results in tabular form

Ex. 31 Find the number of centimetres in 5 in as in Ex 29, hence find the number of centimetres in 1 inch

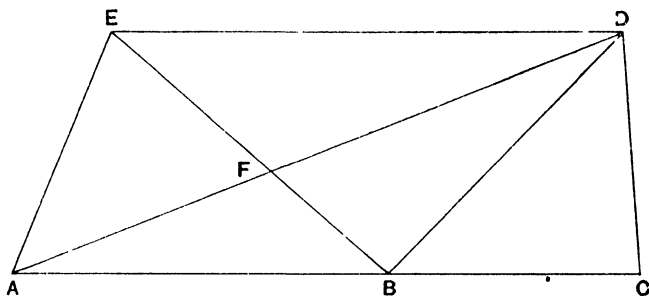


Fig 8

Ex. 32 Guess the lengths of the lines in fig 8 (i) in inches, (ii) in centimetres, verify by measurement. Make a table thus —

Line	AB	BC	
Guessed			
Measured			

DIRECTION*

§3. ¶**Ex. 33.** How would you test whether the edge of an open door is truly vertical?

¶**Ex. 34.** How would you test whether a line is truly horizontal?

* See Notes and Answers.

¶Ex 35. Place a cube with *one* edge vertical. How many edges do you now find to be vertical? how many horizontal?

¶Ex 36 Place a cube with one edge horizontal, and no edge vertical. How many edges are now horizontal?

¶Ex. 37. Place a cube with one edge vertical, and no edge horizontal

¶Ex 38 How many vertical lines can you have through a point? how many horizontal?

¶Ex 39 How many horizontal lines can you draw on a sloping desk through a point?

¶Ex 40. How many vertical lines can you draw on a sloping desk?

¶Ex 41 How many horizontal lines can you draw on the wall through a point? how many vertical?

¶Ex 42 How many horizontal lines can you draw on the floor through a point? how many vertical?

¶Ex 43. Show me two or three horizontal planes? Are they parallel? Show me a line at right angles to a horizontal plane. What is its direction called?

¶Ex 44. Show me two or three vertical planes. Are they parallel? Show me a line at right angles to a vertical plane. What is its direction called?

¶Ex. 45 Give some other instances of lines, not horizontal, in the same direction

¶Ex 46. Are all vertical lines in the room in the same direction?

¶Ex. 47 Are all horizontal lines in the room in the same direction?

¶Ex. 48 Two men are looking at the same star. Are they looking in the same direction?

¶Ex 49 Two men are looking at the same weather-cock. Are they looking in the same direction?

ANGLE*.

§4 Set the hands of a clock to 12 0, now move the large hand, it has turned through an **angle**. The angle through which the large hand turns between 12.0 and 12.15 is called a **right angle**.

* See Notes and Answers.

¶Ex. 50. What angle* do you turn through on receiving the following orders — (i) right turn, (ii) left turn, (iii) about turn, (iv) half right turn?

¶Ex. 51. What angle* is there between the following directions — (i) N. and N E, (ii) N and E, (iii) N and S E, (iv) N and S ?
[In each case draw the directions freehand]

¶Ex 52 What is the angle* between N E. and S W ?

¶Ex 53 How long does the earth take to turn through a right angle?

¶Ex 54 Through how many right angles does the hour hand of a clock turn in 3 hours, 1 hour, 6 hours, 9 hours?

¶Ex. 55 The wind shifts from W to N E. Through what angle* has it turned?

Ex. 56. A man walks one mile N, then turns through $\frac{1}{2}$ right angle to the right and walks another mile. Draw his path, guessing the angle. (Scale 2 inches to 1 mile)

Ex 57. Draw the following journey to scale, guessing the angles One mile N., then turn through $\frac{1}{2}$ right angle left and go 2 miles, then turn $1\frac{1}{2}$ right angles left and go 1 mile, then turn $\frac{1}{2}$ right angle left and go 2 miles (Scale 1 inch to 1 mile)

§5 If you hold one arm of your compasses firm and turn the other about the hinge, the two arms form an angle.

In the same way if two straight lines OA, OB are drawn from a point O, they form an angle at O. O is called the **vertex** of the angle, and OA, OB its **arms**.

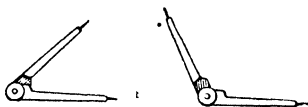


Fig 9

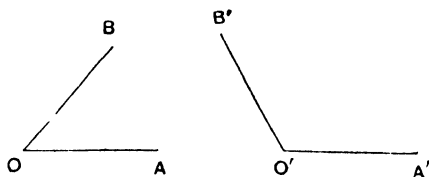


Fig 10

* Answer to be in right angles and fractions of a right angle

An angle may be denoted by three letters; thus we speak of the angle AOB , the middle letter denoting the vertex of the angle and the outside letters denoting points on its arms

If there is only one angle at a point O , we call it the angle O .

Sometimes an angle is denoted by a small letter placed in it, thus in fig. 11 we have two angles a and b

$\angle$ is the abbreviation for *angle*

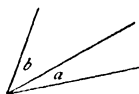


Fig 11.

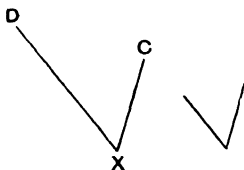


Fig 12

It is important to notice that *the size of an angle does not depend on the lengths of its arms*. The two angles in fig 12 are equal

Ex 58. Which is the greater angle in fig 13? Test by making on tracing paper an angle equal to one of the angles and fitting the trace on the other

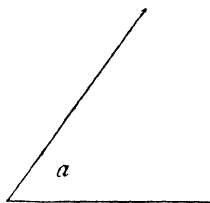


Fig. 13

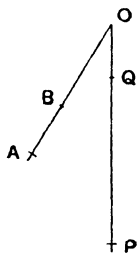


Fig 14

Ex. 59. Name the angle at O in fig. 14 in as many different ways as you can.

Ex. 60. Take a piece of paper and fold it; you will get something like fig. 15, fold it again so that the edge OB fits on the edge OA , now open the paper, you have four right angles made by the creases, as in fig. 16.

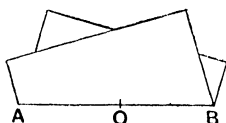
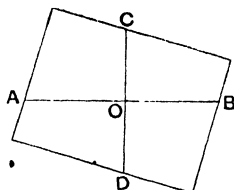


Fig 15



, Fig 16

¶Ex. 61. Make a right angle $\angle BOC$ as in Ex. 60, cut it out and fold so that OB falls on OC , call the crease OE . What fraction of a right angle is each of the $\angle$ s $\angle BOE$, $\angle EOC$?

MEASUREMENT OF ANGLES

§6. If a right angle is divided into 90 equal angles, each of these angles is called a **degree**

25° is the abbreviation for "25 degrees"

An angle less than a right angle is called an **acute** angle. An angle greater than a right angle is called an **obtuse** angle

¶Ex. 62 What fractions of a right angle are the angles between the hands of a clock at the following times —(i) 3 0, (ii) 1.0, (iii) 10.0, (iv) 5 0, (v) 8 0? State in each case whether the angle is acute, right, or obtuse

Ex 63 Find the number of degrees in each of the angles in Ex 62 [Use the results of that Ex]

Ex. 64. Fig 17 shows the points of the compass, what are the angles* between (i) N and E, (ii) W and S W, (iii) W and W N.W, (iv) E. and N.N.W, (v) N.E and N N W, (vi) S W and S E?

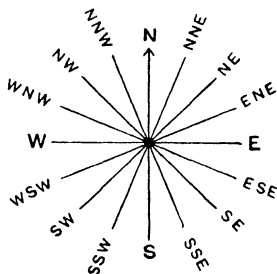


Fig 17

* Answers to be in degrees

Fig 18 represents a **protractor**, if each graduation on the edge were joined to C, we should get a set of angles at C each of which would be an angle of one degree

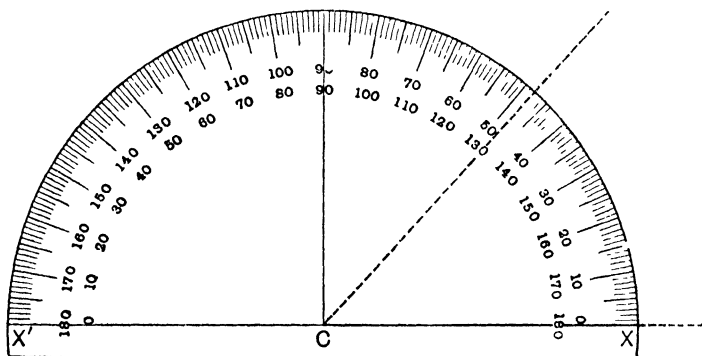


Fig. 18

§7 To measure an angle, place the protractor so that its centre C is at the vertex of the angle and its base, CX, along one arm of the angle, then note under which graduation the other arm passes, thus in fig 18, the angle = 48° .

In using a protractor such as that in fig 18, care must be taken to choose the right set of numbers, e.g. if the one arm of the angle to be measured lies along CX, the set of numbers to be used is obviously the one in which the numbers increase as the line turns round C from CX towards CX'

You should also check your measurement by noticing whether the angle is acute or obtuse

When you measure an angle in a figure that you have drawn (or make an angle to a given measure), always indicate in your figure the number of degrees, as in fig 19.

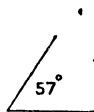


Fig 19

¶Ex 65. Cut out of paper a right angle, bisect it as in Ex 61, and measure the two angles thus formed.

¶Ex 66 Measure the angles of your set square (i) directly, (ii) by making a copy on paper and measuring the copy.

It is difficult to draw a straight line right to the corner of a set square; it is better to draw the lines to within half a centimetre of the corner and afterwards produce them (i.e. prolong them) with the ruler till they cross.

Ex. 67 Measure $\angle$ s AOB, BOC in fig 20, add; and check your result by measuring $\angle$ AOC (Arrange in tabular form.)

Ex 68 Measure $\angle$ s AOC, COD, AOD in fig 20 Check your results

Ex. 69 Measure $\angle$ s AOB, BOD, AOD in fig 20 Check your results.

Ex 70 Repeat the last three exercises for fig 21.

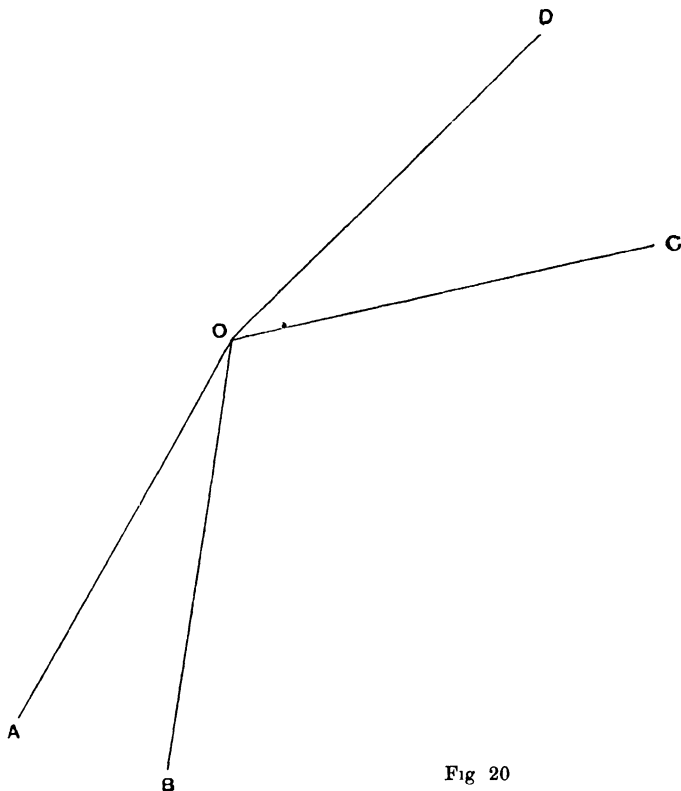


Fig 20

§8. To make an angle to a given measure. Suppose that you have a line AB and that at the point A you wish to make an angle of 73° . Place the protractor so that its centre is at A and its base along AB , mark the 73° graduation with the point of your compasses (only a small mark should be made), and join this point to A (Remember to write 73° in the angle)

Ex. 71. Make a copy of the smallest angle of your set square and bisect it as follows:—measure the angle with your protractor, calculate the number of degrees in half the angle, mark off this number (as explained above) and join to the vertex. Verify by measuring each half (This will be referred to as the method of bisecting an angle *by means of the protractor*.)

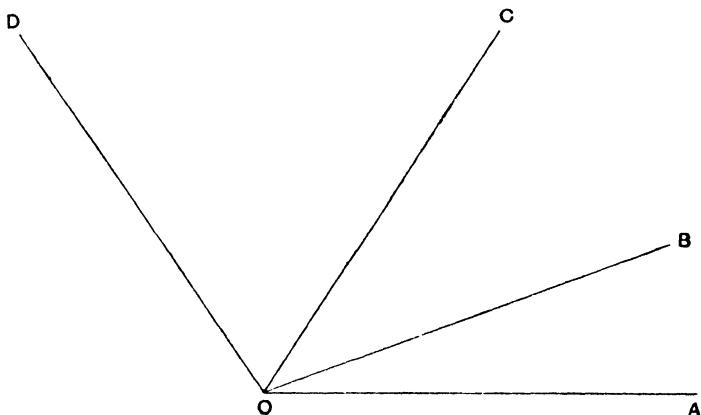


Fig 21

Ex 72. Make angles of 20° , 35° , 64° , 130° , 157° , 176° (let them point in different directions). State whether each one is acute, right, or obtuse

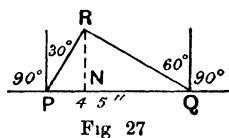
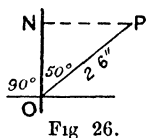
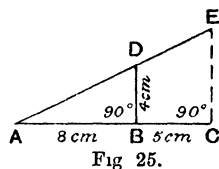
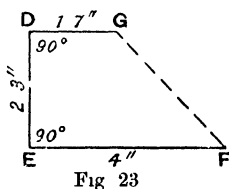
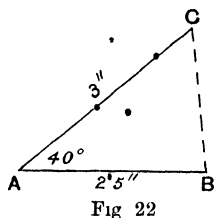
Ex 73 Make the following angles and bisect each by means of the protractor 24° , 78° , 152° , 65° , 111° . (Let them point in different directions)

CHAPTER 2

SCALE DRAWING I*

§1. In this chapter always use your set square for making a right angle

Ex 1. Copy the following figures to the given measurements. In each case measure in your drawing the dotted lines and the angles that are not given you.



§2. In scale drawing

(i) **Make a freehand sketch** of the required figure. This should be done on the same sheet as the finished drawing, it should be neat and fairly large, and should show all the facts given in the question

(ii) **State the scale** on which you propose to make your accurate drawing.

1 in. or 1 cm. to 1 mile will be a suitable scale in Exx 2-4.

(iii) **Make the accurate drawing**, referring to the freehand sketch for lengths and angles. Neatness and accuracy are essential

(iv) **State your answer** thus: The distance from Battersea to Liverpool Street is _____ miles

* See Notes and Answers. These exercises do not require the use of parallels or intersecting arcs

At the end of the chapter there is a note suggesting experimental work that may be taken side by side with the work of the chapter

* Ex 2 Liverpool Street Station is 3.5 miles E. of Marylebone, and Battersea is 3 miles S. of Marylebone. How far is it from Battersea to Liverpool Street?

* Ex 3 St Albans is 12.5 miles N. of Harrow, and Gerrard's Cross is 9.5 miles W. of Harrow. Find the distance from Gerrard's Cross to St Albans.

* Ex 4 King's Cross is 2.2 miles N. of Westminster and Holland Park is 3.4 miles W. of Westminster. What is the distance from Holland Park to King's Cross?

§3 In choosing suitable scales for drawing a figure, remember the following points —

(1) **Choose as large a scale** as possible, but not so large that any line in your figure is more than about 5 inches long.

(ii) **Scales which involve only division or multiplication by 10 or 100 are the simplest**; the next best are those that involve division or multiplication by 2, 20, 5, or 50. Avoid division or multiplication by 3, 6, 7, 8, etc.

¶ Ex 5. What scales would you choose if the longest line in the figure is to be 10 ft., 20 ft., 45 ft.? And what lengths would represent these in the scale drawing?

¶ Ex 6. Repeat Ex 5 for the following — (i) 78 miles, (ii) 15 miles, (iii) 1410 ft., (iv) 5278 metres.

‡ Ex 7. What lengths are represented by (i) 3.4 inches, when the scale is 1 inch to 1 mile, (ii) 7.4 cm., scale 1 cm. to 10 miles, (iii) 4.5 in., scale 2 in. to 1 mile, (iv) 8.3 cm., scale 1 cm. to 20 miles?

§4 Ex 8. In fig. 28, AB is a cricket pitch, the stumps being at A and B. Point stands at C. How far is he from A?

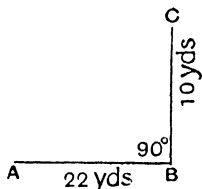


Fig 28

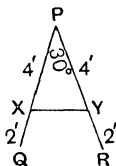


Fig 29

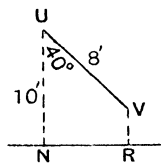


Fig 30

Ex 9. Fig 29 represents a step ladder. Find the length of the rope XY.

* In these and in many subsequent exercises it is often wise for the freehand sketch to be shown to the master before the accurate drawing is made.

‡ Many more examples like this may be necessary, but it is easy to make them up.

Ex 10 In fig 30, UV represents the ropes of a swing, U being the point from which the ropes hang, V the seat, NR the ground. How far is the seat from the ground when the ropes make an angle of 40° with the vertical line through U ? (Do not attempt to draw your lines broken, as in the above figure.)

Ex 11 Draw a plan of a football field, 100 ft long and 65 ft wide, find the distances between opposite corners.

Ex 12 Draw a plan of a tennis court 78 ft by 36 ft. What is the distance between the opposite corners?

Ex 13 The legs of a pair of compasses are 10 cm long. I open them to an angle of 35° . What is the distance between the compass points?

Ex 14 From G go 9 miles W to H , from H go 12 miles N to A , from A go 17 miles W to R . What is the distance from G to R ?

§5. The bearing* of a place A from a second place B is the point of the compass towards which a man at B would be facing if he were looking in the direction of A .

By " $N 10^\circ W$ " or " $10^\circ W$ of N " is meant the direction in which you would be looking if you first faced due north and then turned through an angle of 10° towards the west.

In bearing questions a N and S line must first be drawn through the points from which bearings are to be measured.

¶Ex 15 Express the following points of the compass in the form of bearings, reckoning from N or S according as N or S is the nearer (sailors never reckon from E or W). NE , SW , $NN E$, $SS W$, $NN W$, $WN W$.

‡Ex 16 Find the bearing and distance (in sea-miles and decimals of a sea-mile see scale marked on chart) from Ryde Pier (Middle Light) of (i) Stokes Bay Pier (S Light), (ii) Ryde Church, (iii) Fishbourne Flagstaff (E^S)

‡Ex 17 What is the bearing and distance of NE Middle Buoy from (i) Middle Light of Ryde Pier, (ii) S. Light of Stokes Bay Pier?

* It is now becoming customary, e.g. in the Army and the Navy, to give bearings by an angle only, viz. the angle turned through from the north in a clockwise direction (i.e. turning from N to E). Note that any west bearing must be greater than 180° . This method of giving bearings is not used in the exercises in this book.

‡ These Exx refer to the chart "The Solent to Spithead" published by the Cambridge University Press, Fetter Lane, London, and obtainable from this firm or from any bookseller (price 1d each, 8s per hundred). It will be found inserted at the end of this book.

|| Work from the mid-point of the base line of the figure representing the buoy.

* **Ex. 18.** What is the bearing of S.E. Middle Buoy from (1) Osborne House Flagstaff (on the most northerly point of Osborne House), (2) Fishbourne Flagstaff?

* **Ex. 19.** What is the bearing of W. Bramble Buoy from (1) Egypt Point, (2) East Cowes Coastguard Station (C.G.)?

* **Ex. 20.** What is the bearing from the head of Lee-on-Solent Pier of (1) E. Bramble Buoy, (2) N.E. Middle Buoy?

Ex. 21 A is 2.5 miles W. of B, and C is 4.5 miles S. of A. What is the distance from B to C? What is the bearing of B from C?

Ex. 22 G is 7.5 miles S. of H, and 10 miles W. of K. What is the distance and bearing of K from H?

Ex. 23 Brading is 17 miles due E. of Freshwater. Gosport is due N. of Brading and $N. 62^{\circ} E.$ of Freshwater. Find the distances of Gosport from Freshwater and Brading.

Ex. 24 Ringwood is 7.5 miles N. of Christchurch, Lymington bears $N. 80^{\circ} E.$ from Christchurch and $S. 57^{\circ} E.$ from Ringwood. Find the distances of Lymington from Christchurch and Ringwood.

Ex. 25 Ely is 15 miles $N. 21^{\circ} E.$ of Cambridge. Newmarket is 12 miles $N. 76^{\circ} E.$ of Cambridge. What is the distance of Newmarket from Ely?

Ex. 26 Rugby is 44 miles N. of Oxford, and Reading is 24 miles $S. 30^{\circ} E.$ of Oxford. Find the distance from Rugby to Reading.

Ex. 27 A ship observes a light bearing $N. 52^{\circ} E.$ at a distance of 5 miles. She then steams due S. 6 miles, and again observes the light. What should she find the bearing and distance of the light to be at the second observation?

§6 If a man who is looking at a tower through a telescope holds the telescope horizontally, and then raises (or "elevates") the end of it till he is looking at the top of the tower, the angle he has turned the telescope through is called the **angle of elevation** of the top of the tower.

The angle of elevation of the sun is called the **altitude of the sun**.

Ex. 28. A vertical flagstaff 50 feet high stands on a horizontal plane. Find the angles of elevation of the top and middle point of the flagstaff from a point on the horizontal plane 15 feet from the foot of the flagstaff.

These Exx. refer to the chart "The Solent to Spithead."

Ex. 29 The angle of elevation of the top of the spire of Salisbury Cathedral at a point 1410 feet from its base was found to be 16° . What is the height of the spire?

Ex. 30 The shadow of a tree is 30 feet long when the sun's altitude is 59° . Find the height of the tree.

Ex. 31 The elevation of a hill-top as observed from a station on the sea-shore is 12.5° . The horizontal distance, as shown on a map, between the station and the hill-top is 5278 metres. Find the height of the hill-top above the shore in metres.

Ex. 32 The angle of elevation of the top of a tower on level ground is read off on a theodolite. Find the height of the tower from the following data — Reading of theodolite = 15° . Distance of theodolite from foot of tower = 372 yards.

Ex. 33 A telegraph pole standing upright on level ground is 23.6 feet high and is partly supported by a wire attached to the top of the pole at one end and fixed to the ground at the other so that its inclination to the pole is 54° . Find the length of the wire.

Ex. 34 Boulogne is 26 miles S 24° E of Dover and Calais lies 21 miles S 67° E. of Dover. Find the distance from Boulogne to Calais in miles.

Ex. 35 Exeter is 48 miles W of Dorchester, and Barnstaple is 35 miles N W of Exeter. What is the distance and bearing of Barnstaple from Dorchester?

Ex. 36 At two points on opposite sides of a poplar the angles of elevation of its top are 39° and 48° . If the distance between the points is 150 feet, what is the height of the tree?

Ex. 37 A man observes the angle of elevation of the top of a spire to be 23° , he walks 40 yards towards it, and then finds the angle to be 39° . What is the height of the spire?

NOTE ON EXPERIMENTAL WORK

Scale drawing is far more interesting when it is based on measurements made by the pupils themselves. The opportunities for making such measurements and the method of making them depend so much on local conditions that it is impossible to do more than make suggestions.

Practical work, both indoors and out, must be carefully planned beforehand, otherwise the profit from it will be small considering the time spent on it.

When measurements are to be made, the pupil should first make a freehand sketch of the plan to be made and record the measurements on that.

The appropriate practical work should precede the corresponding scale drawing work given in the earlier part of the chapter. Its object should be to stimulate interest and to make the scale drawing that follows it more full of meaning.

Before starting the chapter

1. Use local ordnance maps (particularly the 1 in. to 1 mile map) to find distances between known landmarks

Use plans of the school (if available) to compare measurements of the building and plan

Before § 2

2 Draw a plan of the schoolroom

Measurements may be made with a measuring tape, or with a yard stick marked in feet and inches, or even with string

3. Draw a plan of the playground

Angles may be estimated, or measured with a rough large protractor

With § 5

4. Determination of bearings from a chart (see end of book) or a local map.

5. Determination of bearings from some view point with a compass.

With § 6

6 Estimation of the height of a flagstaff (or building) by measuring an angle of elevation and the distance of the point of observation from the foot of the flagstaff

7 Estimation of the angle of slope of a steep road

This may be done by hanging a plumb line from the centre of a large rough protractor, and holding the base of the protractor so that it points straight up the hill towards the eye of another boy standing on the hill.

CHAPTER 3

ANGLES AT A POINT

§ I You have learnt that, in carrying out the order “about turn,” you turn through two right angles. If you turn the hand of a clock from XII to VI, it turns through two right angles. You might first of all turn it from XII to II, through an acute angle, and then again from II to VI, through a further angle, an obtuse angle. These two angles together make up two right angles. This would be equally true if you broke the journey at some other place than II. Hence

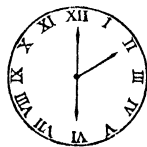


Fig 31

*** THEOREM 1** If a straight line stands on another straight line, the sum of the two angles so formed is equal to two right angles.

¶ Ex 1 If, in fig. 32, $\angle AOB = 57^\circ$, what is $\angle BOC$?

¶ Ex 2 (i) If, in fig. 32, $\angle BOC = 137^\circ$, what is $\angle AOB$?

(ii) „ „ $\angle BOC = 93^\circ$, „ „ $\angle AOB$?

(iii) „ „ $\angle AOB = 5^\circ$, „ „ $\angle BOC$?

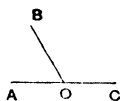


Fig 32

When the sum of two angles is equal to two right angles, each is called the **supplement** of the other, or is said to be **supplementary** to the other.

¶ Ex. 3 What are the supplements of the following angles — (i) 60° , (ii) 54° , (iii) 0° , (iv) 90° , (v) 130° , (vi) 162° , (vii) 180° , (viii) x° ?

¶ Ex. 4 Name the supplements of $\angle AOB$ and $\angle BOC$ in fig. 32.

¶ Ex. 5 State the above theorem, introducing the term “supplementary.”

¶ Ex. 6. From fig. 32, show how to obtain another supplement of $\angle AOB$.

¶ Ex. 7 Copy fig. 32 freehand and draw OP in the angle COB . What is the sum of $\angle COP$, $\angle POB$, $\angle BOA$?

* The numbers of Theorems in this book correspond to the numbers in *Theoretical Geometry* by the same authors.

§2 If a number of lines meet at a point, as in fig 33, a number of angles are formed at that point. If the clock-hand turns from CA to OB, then to OC, then to OD, and finally back to OA, how many separate angles does it turn through? What is the sum of these angles? How many degrees?

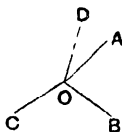


Fig 33

THEOREM 1 COR If any number of straight lines meet at a point, the sum of all the angles made by the lines taken consecutively is equal to four right angles.

¶Ex 8 In fig 33, produce BO to B'. Now there are three angles on the one side of BOB', two angles on the other side. What is the sum of these three angles? of these two angles?

¶Ex 9. If three angles in fig 33 are 30° , 140° , 110° , what is the fourth?

¶Ex 10. There are 12 equal angles at a point O. What is each?

¶Ex 11 A wheel has 20 spokes. What is the angle between any consecutive pair? If one is broken, what is the angle in the gap?

Ex 12 From a point draw 5 lines anyhow. Measure all the angles and find their sum.

Ex 13 (See fig. 33) Make $\angle AOD = 25^\circ$, $\angle DOC = 135^\circ$, $\angle AOB = 95^\circ$. Measure $\angle BOC$. Check your result by calculation.

§3 Lay two pencils along one another, now rotate one of them so as to form a letter X. do the two parts of the pencil turn through equal angles?

The opposite angles made by two intersecting straight lines are called **vertically opposite** angles (e.g. $\angle$ s AOB, COD in fig 34). *Vertically* is rather an unfortunate word, as it has nothing to do with vertical lines, the angles are *opposite* and have the same *vertex*; hence "vertically opposite."

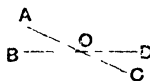


Fig 34

THEOREM 3. If two straight lines intersect, the vertically opposite angles are equal.

¶Ex. 14 Mention two pairs of vertically opposite angles in fig 34

Ex 16. Draw three straight lines at random, as in fig 35, number all the angles in the figure, using the same numbers for angles that are equal



Fig 35

†Ex 16 If two straight lines AOC, BOD intersect at O (see fig. 34), what is the sum of $\angle$ s AOB, BOC? What is the sum of $\angle$ s BOC, COD? Hence prove the above theorem

¶†Ex 17 In fig. 36 prove* that

- (i) if $\angle b = \angle f$, then $\angle c = \angle g$.
- (ii) if $\angle c = \angle f$, then $\angle d = \angle e$
- (iii) If $\angle d + \angle f = 2 \text{ rt } \angle$ s, then $\angle b = \angle g$
- (iv) if $\angle g = \angle e$, then $\angle d = \angle h$
- (v) if $\angle h = \angle a$, then $\angle e = \angle d$
- (vi) if $\angle a = \angle e$, then $\angle b = \angle g$

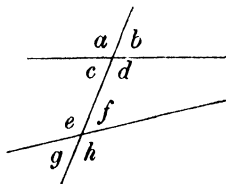


Fig 36

SUMMARY OF RESULTS

If a straight line stands on another straight line, the sum of the two angles so formed is equal to two right angles

If any number of straight lines meet at a point, the sum of all the angles made by consecutive lines is equal to four right angles

If two straight lines intersect, the vertically opposite angles are equal

† The dagger indicates an exercise of a theoretical character (a ridei)

* Each part should be put out separately, like the following

Data

$$\angle b = \angle f$$

To prove that

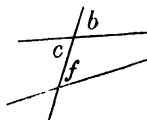
$$\angle c = \angle g$$

Proof

$$\angle b = \angle g \text{ (vert. opp.)},$$

$$\angle b = \angle f \text{ (data),}$$

$$\angle c = \angle g$$



CHAPTER 4*

PARALLEL STRAIGHT LINES

§ 1 What is the distance between two of the ruled lines on your paper? How do you place your ruler with respect to the lines to measure this distance? Is the distance between the two lines everywhere the same?

What is the distance between the two long edges of your paper? How do you place your ruler to measure this distance? Is this distance everywhere the same? Is the distance between the following everywhere the same?—

- (1) Two rails of the same tram line.
- (2) The sides of a picture frame
- (3) The upright edges of a door

Straight lines which are always the same distance apart are called parallel lines

Do parallel lines ever meet if produced (i.e. made longer)? How could you tell this? e.g. How could you tell whether the top and bottom edges of your paper would ever meet if produced indefinitely?

You conclude that the lines never meet, but you see how difficult it is really to settle this question

Lines in the same plane which never meet are parallel.

¶Ex. 1 Can there be lines in different planes which are not parallel and yet never meet?

¶Ex. 2. Give instances of lines (e.g. edges of a chalk box) which are not parallel and which never meet however far they are produced

We see then that parallel lines have the following properties —

- (1) They are everywhere equidistant
- (2) They never meet however far they are produced

We have noticed that we cannot be sure of property (1) above without getting lines crossing the parallels at right angles, also that we cannot easily test property (2)

CORRESPONDING ANGLES.

§2' We will now point out some other properties of parallel straight lines which are more fruitful in geometry.

¶Ex. 3. Draw with your ruler two straight lines as nearly parallel as you can judge, draw a straight line cutting them as in fig 37, notice the angles marked. These are called **corresponding** angles. What suggestion do you make about them?

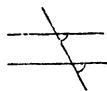


Fig 37

¶Ex. 4. Name some pairs of corresponding angles in figs 43, 44

¶Ex. 5. Place your set square so that one of its edges touches your ruler as at (i) in fig 38, hold the ruler firm and slide the set square along the ruler to position (ii). Draw the line AB for two positions of the set square, notice that the two straight lines point in the same direction, in fact, the two lines are parallel.

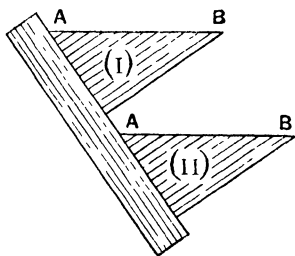


Fig 38

This method of drawing parallels suggests an explanation of the term *corresponding* angles.

This motion of a line AB from one position into another by sliding without any turning is called **translation**, as opposed to **rotation** which means turning about a point or axis.

¶Ex. 6. Draw a pair of parallel lines as in Ex. 5, draw a line cutting them as in fig 37. What do you notice about the angles in the figure?

¶Ex. 7. If in fig 37 one of the lines is turned so that the lines cease to be parallel, will the marked angles remain equal?

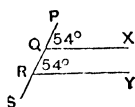


Fig. 39

¶Ex. 8. Draw fig. 39, first draw PQRS, then make the angles as shown. Are QX, RY parallel?

A man walks along a zig-zag road ABCD, AB being parallel to CD, so that he finishes up by walking in his original direction.

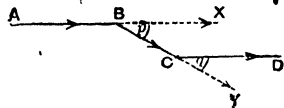


Fig. 40

At B he turns to the *right* through $\angle p$ (NB *not* through $\angle ABC$), and at C he turns to the *left* through $\angle q$ *

Since his last direction is the same as his first, his right and left turns must compensate, $\therefore \angle p = \angle q$

Hence we may see that BCY, meeting the parallel lines BX, CD, makes the corresponding angles p and q equal.

From the above and Exx 3-8 we arrive at a geometrical fact which we will state in two different ways

THEOREM 4 (CORRESP) When a straight line cuts two other straight lines, if a pair of corresponding angles are equal, then the two straight lines are parallel.

Conversely **THEOREM 5 (CORRESP)** When a straight line cuts two parallel straight lines, the corresponding angles are equal.

ALTERNATE ANGLES.

§3 ¶Ex. 9 Draw a pair of parallel lines (using set square and ruler as in Ex 5), draw a line cutting them as in fig 41. Look at the angles marked. These are called **alternate** angles. You may be tempted to make some statement about these angles but before you commit yourself draw a different figure with a similar construction.

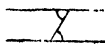


Fig 41

Alternate angles need not be acute the obtuse angles in fig 41 are alternate

¶Ex 10 Name some pairs of alternate angles in figs 43, 44

¶Ex 11 In fig 42 what do you call $\angle s b$ and c ? What do you call $\angle s a$ and c ? What does the above theorem say about $\angle s a$ and c ? What do you call $\angle s a$ and b ? What do you know about $\angle s a$ and b ? What then do you *infer* about $\angle s b$ and c ?

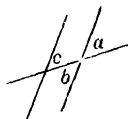


Fig 42

THEOREM 4 (ALT.). When a straight line cuts two other straight lines, if a pair of alternate angles are equal, then the two straight lines are parallel.

Conversely **THEOREM 5 (ALT.)** When a straight line cuts two parallel straight lines, the alternate angles are equal.

¶Ex 12 Copy fig 43 freehand. If CD, EF are parallel and $\angle AGD = 72^\circ$, find all the angles in the figure, giving your reasons, make a table of values

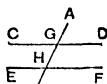


Fig 43

¶Ex 13 Repeat Ex 12 with $\angle FHB = 120^\circ$, CD, EF being parallel

¶+Ex 14 In fig 44 there are two pairs of parallel lines*, prove that the following pairs of angles are equal — (i) b, l , (ii) j, k , (iii) m, s , (iv) f, h , (v) r, l , (vi) s, h , (vii) s, q , (viii) s, k , (ix) s, a , (x) g, l

[State your reasons carefully.

e.g. WX, YZ are $\parallel$ and ST cuts them,

$\angle q = \text{cor } \angle f$]

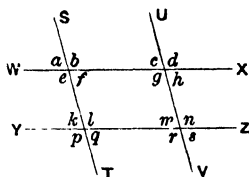


Fig 44

¶Ex. 15 The bearing of B from A is $N. 20^\circ E$. What is the bearing of A from B? See fig 45

¶Ex 16 A submarine is observed by a destroyer to bear $S. 30^\circ E$. What is the bearing of the destroyer from the submarine?



Fig 45

Ex 17 In fig 46 FG is parallel to BC , find all the angles in the figure, stating your reasons carefully

Ex 18 If, in fig 56, DAE is $\parallel$ to BC , $\angle B = 33^\circ$, $\angle C = 52^\circ$. Find all the angles in the figure, giving your reasons in each case

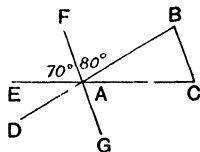


Fig 46

* In *vis a voce* work, and even in written work, it is often convenient to colour the one pair of lines red and the other blue and to give references thus —

$\angle q = \text{cor } \angle f$, blue lines parallel

There is less distraction in naming the colour than in searching for the letters that name the lines

Boys should be accustomed to label angles with one small letter whenever this can conveniently be done, the three letter method of referring to angles, though generally unavoidable, is wearisome to teachers, boys and examiners

INTERIOR ANGLES.

§4 ¶Ex. 19*. Draw a pair of parallel lines (using set square and ruler), draw a line cutting them as in fig 47. Measure the two interior angles on one side of the cutting line. What is their sum?

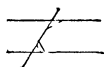


Fig 47.

¶Ex. 20. Measure the two interior angles on the other side of the cutting line. What is their sum?

¶Ex. 21. In fig. 48 what do you know about $\angle s c$ and b ? By which theorem? What do you know about $\angle s c$ and a ? What do you infer about $\angle s a$ and b ?

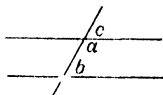


Fig 48

THEOREM 4 (INT.). When a straight line cuts two other straight lines, if a pair of interior angles on the same side of the cutting line are together equal to two right angles, then the two straight lines are parallel.

Conversely **THEOREM 5 (INT.)** When a straight line cuts two parallel straight lines, the interior angles on the same side of the cutting line are together equal to two right angles.

Ex. 22 What do you know about the sums of (i) $\angle s f, g$, (ii) $\angle s f, l$, (iii) $\angle s m, n$, in fig 44? Give your reasons

Ex. 23. Fig 49 shows two parallel lines and a third line cutting them. Write down the sizes of the angles denoted by letters, giving reasons very shortly.

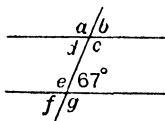


Fig 49

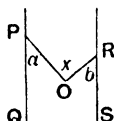


Fig 50

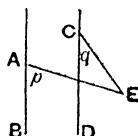


Fig. 51

Ex 24 In fig 50 PQ and RS are parallel. If $\angle a = 42^\circ$ and $\angle b = 47^\circ$, find $\angle x$ [Through O draw a parallel to PQ and RS]

Ex. 25. In fig 51 AB is parallel to CD. Find the $\angle$ at E in terms of p and q

Ex 26 In fig 52 BE is parallel to AF. BC bisects the angle ABE. $\angle A = 60^\circ$. Find all the other angles

Ex 27. In fig 53 AD is parallel to BC. Show that DC is parallel to AB.

Ex. 28. In fig 54 AD is parallel to BC and CD to BA . Find all the other angles, giving your reasons.*

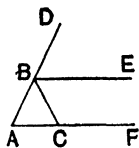


Fig. 52

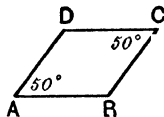


Fig. 53.

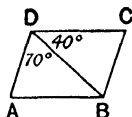


Fig. 54.

Ex. 29 Repeat Ex 28 for fig 55.

Ex 30. In fig 57, AC is $\parallel$ to BD and AB to CD . If $\angle B = 72^\circ$ and $\angle BAD = 78^\circ$, find the other angles.

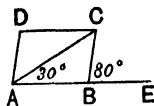


Fig. 55.

Ex 31 In fig 56, DAE is $\parallel$ to BC . Mention pairs of equal angles, giving reasons.

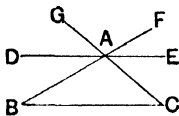


Fig. 56.

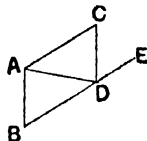


Fig. 57

Ex. 32 In fig. 57 AB is $\parallel$ to CD and AC is $\parallel$ to BD . Mention pairs of equal angles, giving reasons

†**Ex 33.** In fig 57 AB is $\parallel$ to CD and AC is $\parallel$ to BD ; prove that (i) $\angle B = \angle C$,
(ii) $\angle BAC = \angle CDB$

¶**Ex 34** What tests would you apply to determine whether the following were parallel?

- (1) the short edges of a piece of writing paper,
- (2) the long edges of a football field,
- (3) two side roads branching obliquely off a straight main road,
- (4) two flag-posts on a horizontal plane,
- (5) two edges of a wide plank,
- (6) the "creases" at each end of a cricket pitch

¶**Ex 35** Are "Parallels of Latitude" parallel straight lines?

PRACTICAL CONSTRUCTIONS

§ 5. To draw a parallel to a given line QR through a given point P by means of a set square and a straight edge.

It is important that the straight edge should not be bevelled (if it is bevelled the set square will slip over it), in the figures below, a ruler (with an unbevelled edge is shown), but the base of the protractor or the edge of another set square will do equally well

Place a set square so that one of its edges lies along the given line QR (as at (i)), hold it in that position and place the straight (unbevelled) edge in contact with it. now hold the straight edge firmly and slide the set square along it. The edge which originally lay along QR will always be parallel to QR. Slide the set square till this edge passes through P (as at (ii)), hold it firmly and rule the line

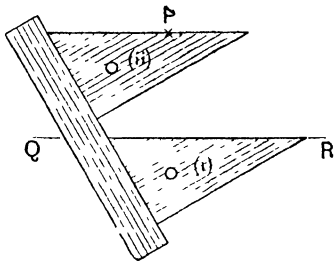


Fig 58

Ex 36 Draw a straight line QR and mark a point P, through P draw a parallel to QR

Ex 37 Repeat Ex. 36 several times, using the different edges of the set square (See fig 60)

Ex 38 Draw a parallelogram (i.e. a 4-sided figure with its opposite sides parallel)

Ex 39 Near the middle of your paper draw a triangle, through each vertex draw a line parallel to the opposite side

Ex. 40. Draw any triangle. From any point O draw three lines parallel to the sides

Ex 41 Draw three lines from a point forming angles of 120° and 150° . Draw any triangle with its sides parallel to these lines. Measure the angles of the triangle.

Ex. 42 Draw any five-sided figure with its sides produced in order (see fig 67). From any point O draw 5 lines parallel to the sides of the figure in the directions of the arrow-heads. What does this tell you about the angles of the figure?

§6 To draw a straight line parallel to a given straight line and at a given distance from it.

Take two points on the line (as far apart as possible), with these points as centres describe circles with the given distance as radius, then with the ruler draw a straight line to touch both circles

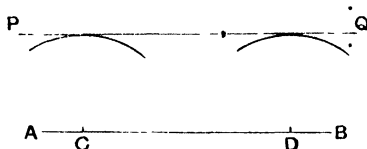


Fig 59

Ex 43 Draw any line Draw two lines, above and below, parallel to this line at a distance of 1 inch

Ex 44 Draw two long lines intersecting Draw parallels to each at a distance of 3 cm Join opposite points of intersection of these parallels

Ex 45 Draw any triangle At a distance of 1 in draw parallels to the sides to form another triangle outside Draw the lines joining corresponding vertices and produce them What fact do you notice?

Ex 46 Repeat the previous Ex with parallels inside (Draw a large triangle)

Ex 47 Repeat Exs 45 and 46 for any quadrilateral

NOTE A parallel to a given line QR through a point P may also be drawn by drawing any line through P to cut QR , and drawing another line through P , making an alternate (or a corresponding) angle equal to the angle formed by the cutting line with QR .

This is best done by ruler and compass construction See Ch 10, p 81

§7 To draw through a given point P a straight line perpendicular to a given straight line QR .

The difficulty of drawing a line right to the corner of a set square can be overcome as follows (see fig 60)

Place a set square so that one of the edges containing the right angle lies along the given line QR (as at (i)), place the straight edge in contact with the side opposite the right angle, now hold the straight edge firmly and slide the set square along it, the

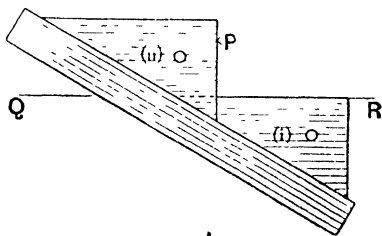


Fig. 60

edge which lay along **QR** will always be parallel to **QR** and the other edge containing the right angle will always be perpendicular to **QR**. Slide the set square till this other edge passes through **P**, then draw the perpendicular.

In drawing a perpendicular to a given line, make the perpendicular project slightly beyond the line, so that you may have a well-defined point of intersection.

Ex. 48 Through a given point in a straight line draw a perpendicular to that line

Ex. 49. Draw an acute-angled triangle, from each vertex draw a perpendicular to the opposite side. What fact is suggested?

Ex. 50 Repeat **Ex. 49** with an obtuse-angled triangle. (You will find it necessary to produce two of the sides)

Ex. 51. Draw an acute-angled triangle, from the middle point of each side draw a straight line at right angles to that side. What fact is suggested?

Ex. 52. Repeat **Ex. 51** with an obtuse-angled triangle

SUMMARY OF RESULTS

When a straight line cuts two other straight lines, if

(i) a pair of corresponding angles are equal,

or (ii) a pair of alternate angles are equal,

or (iii) a pair of interior angles on the same side of the cutting line are together equal to two right angles,

then the two straight lines are parallel.

Conversely If a straight line cuts two parallel straight lines,

(i) corresponding angles are equal,

(ii) alternate angles are equal,

(iii) the interior angles on the same side of the cutting line are together equal to two right angles.

CHAPTER 5

ANGLES OF A TRIANGLE ANGLES OF A POLYGON

ANGLES OF A TRIANGLE.

§ I. A plane figure bounded by three straight lines is called a **triangle**.

A plane figure means a figure drawn on a plane surface

¶ Ex 1 Draw any triangle, and measure its angles. What is the sum of its angles?

¶ Ex 2. Repeat Ex 1 for a triangle with one of its angles obtuse.

¶ Ex. 3 Cut out a paper triangle, mark its angles, tear off the corners and fit them together with their vertices at one point, as in fig. 61.

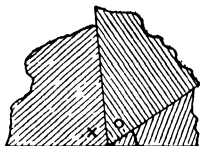


Fig 61

What relation between the angles of a triangle is suggested by this experiment?

Your inference may be proved **A proof** must not depend on any measurements by instruments, nor on experimental work. The result is arrived at by reasoning from facts which are known to be true or which can be taken as obvious. Particular attention should be paid to the style* in which a proof is written out

* *Style in writing out theorems and riders* The pupil should gradually be trained to write out his theorems and riders in the form given here; in no case should the "Data" and "To prove that" be omitted. In writing the data, care should be taken to make them as explicit as possible, e.g. instead of saying "ABCD is a 4-gon with its opp sides $\parallel$ " say "ABCD is a 4-gon with AB $\parallel$ to DC and AD $\parallel$ to BC". One of the chief difficulties pupils meet in solving riders is that of focussing their minds on the precise details of facts given. Hence the importance of stating the data explicitly.

THEOREM 8 The sum of the angles of a triangle is equal to two right angles.

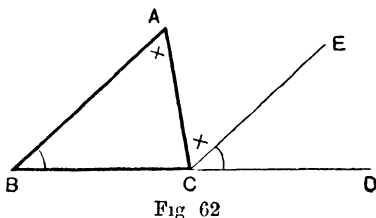


Fig 62

Data

ABC is a triangle

To prove that

$$\angle A + \angle B + \angle BCA = 2 \text{ rt } \angle s$$

Construction

Produce BC to D

Through C draw CE $\parallel$ to BA

Proof

[We shall prove that $\angle s$ of $\triangle = \angle s$ at C]

Since AC cuts the $\parallel s$ AB, CE,

$$\angle A = \text{alt } \angle ACE.$$

And since BC cuts the $\parallel s$ AB, CE,

$$\angle B = \text{corresp } \angle ECD,$$

$$\therefore \angle A + \angle B = \angle ACE + \angle ECD$$

Add $\angle ACB$ to each side of this equation,

$$\begin{aligned} \therefore \angle A + \angle B + \angle BCA &= \angle BCA + \angle ACE + \angle ECD \\ &= 2 \text{ rt } \angle s \quad (\text{for BCD is a st. line}), \\ \text{sum of } \angle s \text{ of } \triangle ABC &= 2 \text{ rt } \angle s \end{aligned}$$

¶Ex 4 Would it be possible to have triangles with angles of

$$(i) \ 90^\circ, 60^\circ, 30^\circ, \quad (ii) \ 77^\circ, 84^\circ, 20^\circ, \quad (iii) \ 59^\circ, 60^\circ, 61^\circ,$$

$$(iv) \ 135^\circ, 22^\circ, 22^\circ, \quad (v) \ 73^\circ, 73^\circ, 33^\circ, \quad (vi) \ 54^\circ, 54^\circ, 72^\circ?$$

¶Ex 5 (i) Give two sets of angles which would do for the angles of a triangle

(ii) Give two sets which would not do

¶Ex 6 What is the third angle of a triangle in which two angles are given to be (i) $54^\circ, 76^\circ$, (ii) $27^\circ, 117^\circ$, (iii) $23^\circ, 31^\circ$, (iv) $65^\circ, 132^\circ$, (v) a°, b° ?

¶Ex 7 If the angles of a triangle are all equal, what is the number of degrees in each?

¶Ex 8 Show that the greatest angle of a triangle must be at least 60°

¶Ex. 9 If one angle of a triangle is 36° , and the other two angles are equal, find the other two angles

¶Ex. 10 Repeat Ex. 9 with the given angle (i) 90° , (ii) 132° , (iii) 108° , (iv) x°

Ex 11 In fig 63, triangle ABC has $\angle A = 90^\circ$, AD is drawn perpendicular to BC . If $\angle B = 27^\circ$, find the angles marked x , y , z

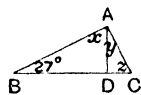


Fig 63

Ex 12 Repeat Ex. 11 with (i) $\angle B = 54^\circ$, (ii) $\angle B = 33^\circ$, (iii) $\angle B = 45^\circ$, (iv) $\angle B = a^\circ$

Ex 13 A triangle ABC has $\angle A = 75^\circ$, $\angle B = 36^\circ$, if AD is drawn perpendicular to BC , find each angle in the figure

†**Ex 14** Draw a triangle ABC , through A draw DAE parallel to BC . With this construction give another proof of Theorem 8

†**Ex 15** If one angle of a triangle is a right angle, the other two angles must be acute*

†**Ex 16** If one angle of a triangle is obtuse, the other two angles must be acute*

§ 2 A triangle which has one of its angles an obtuse angle is called an **obtuse-angled** triangle

A triangle which has one of its angles a right angle is called a **right-angled** triangle

The side opposite the right angle is called the **hypotenuse** (a Greek word which means "subtending" the right angle)

A triangle which has *all* its angles acute angles is called an **acute-angled** triangle

In Exx 15, 16 we have seen that **every triangle must have at least two of its angles acute.**

A triangle which has all its sides equal is called an **equilateral** triangle

A triangle which has two of its sides equal is called an **isosceles** triangle. (Isosceles is a Greek word meaning "equal-legged")

A triangle which has no two of its sides equal is called a **scalene** triangle (Scalene comes from a Greek word meaning "lumping.")

A triangle which has all its angles equal is said to be **equiangular.**

* This means that the truth of the statement is to be proved. Questions are often set in this form.

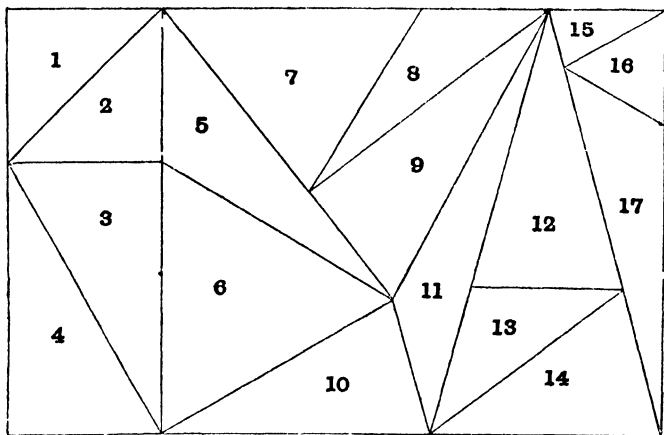


Fig 64

Ex 17 Make a table showing in column A whether the triangles in fig 64 are acute, right-, or obtuse-angled, and in column B whether they are equilateral, isosceles, or scalene

Triangle numbered	A	B
1		
2		
3		

§3 ¶Ex 18 Draw a triangle ABC, produce BC to D. Measure angles A, B, what is their sum? Measure $\angle ACD$. What do you infer?

THEOREM 8. COR 1 If one side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior opposite angles.

This corollary was proved in the course of the proof of Theorem 8

†**Ex 19.** Write out the proof of the above corollary

¶† **Ex 20.** Prove that an exterior angle of a triangle is greater than either of the interior opposite angles.

¶Ex 21 In fig 65 which are the interior angles opposite to (i) $\angle ABD$, (ii) $\angle ACE$? In fig. 66 which are the interior angles opposite to (i) $\angle BCY$, (ii) $\angle CBX$?

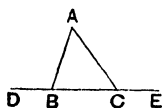


Fig 65.

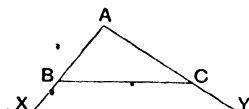


Fig 66.

Ex 22 In fig 65, $\angle A = 40^\circ$, $\angle ACE = 120^\circ$, find the remaining angles

Ex 23 In fig 66, $\angle A = 90^\circ$, prove that $\angle XBC + \angle YCB = 3 \text{ rt } \angle$ s

Ex 24 Two straight by-roads cross a straight main road at different points and make angles of 60° and 42° with it. At what angle do the by-roads meet?

¶Ex 25 Prove that, if the sides of a triangle are produced in order, the sum of the three exterior angles is four right angles

¶Ex 26 Draw a triangle ABC with AB greater than AC. D is a point in AB such that $\angle ADC = \angle ACD$. Prove that $\angle ACB$ is greater than the $\angle ABC$.

¶Ex 27 If O is any point inside a $\triangle ABC$, prove $\angle BOC$ greater than $\angle BAC$. [Produce BO to cut AC in D.]

¶Ex 28 A triangle has the two base angles equal, and a side is produced beyond the vertex. Prove that the exterior angle is twice a base angle.

ANGLES OF A POLYGON*

§4 ¶Ex 29. Draw any quadrilateral (a four-sided figure) and find the sum of its angles.

¶Ex 30 Cut out a paper quadrilateral and proceed as in Ex 3

¶Ex 31 Prove that the sum of the angles of a quadrilateral is equal to four right angles.

[Join two opposite corners.]

¶Ex 32. What is the sum of the angles of a pentagon (a five-sided figure)? [Join one vertex to the two opposite vertices.]

Ex. 33 In a quadrilateral ABCD, $\angle A = 77^\circ$, $\angle B = 88^\circ$, $\angle C = 99^\circ$, find $\angle D$ (Freehand)

Ex 34 In a quadrilateral ABCD, $\angle A = 37^\circ$, $\angle B = 111^\circ$, and $\angle C = \angle D$, find $\angle C$ and $\angle D$ (Freehand)

¶Ex 35 ABCD is any quadrilateral, BC is produced to P and CD to Q. Prove that the sum of the exterior angles PCD and QDA is equal to the sum of the interior angles at A and B.

* See Notes and Answers.

†Ex. 36. ABCD is a quadrilateral, $\angle A = \angle D$, $\angle B = \angle C$; prove $AD \parallel$ to BC

†Ex 37 ABCD is a quadrilateral, $\angle A = \angle C$, $\angle B = \angle D$; prove $AB \parallel$ to CD and $AD \parallel$ to BC .

†Ex. 38. If one pair of opposite angles of a quadrilateral are supplementary, the other pair must also be supplementary.

§ 5. A plane figure bounded by straight lines is called a **polygon**. (From two Greek words meaning "many" "angles or corners")

A polygon which has all its sides equal and all its angles equal is called a **regular polygon**.

A polygon of 3 sides is called a **triangle** ($\triangle$).

„	4	„	„	quadrilateral (4 gon)
„	5	„	„	pentagon (5-gon)
„	6	„	„	hexagon (6-gon).
„	7	„	„	heptagon (7-gon)
„	8	„	„	octagon (8-gon)

The corners of a triangle or polygon are called its **vertices**.

§ 6 ¶Ex. 39 A yacht sails from A round the pentagon BCDEF back to A (see fig 67) Sketch the figure freehand, and mark the angle the yacht turns through at B, C, D, E, F. When it gets back to A, it has headed towards every point of the compass, what then is the sum of the angles through which it has turned?

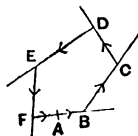


Fig. 67.

¶Ex 40. Draw a figure to show which angles a yacht turns through in sailing round a 6-sided course What is the sum of these angles?

¶Ex 41 Repeat Ex 40 for a triangular course

¶Ex 42. Produce the sides of a square* in order (see fig 68), what is the sum of the exterior angles?

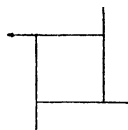


Fig 68

¶Ex 43 What fact about the exterior angles of a polygon do you infer from the above examples?

THEOREM 9 If the sides of a convex polygon are produced in order, the sum of the angles so formed is equal to four right angles.

* A square may be assumed to have (i) all its sides equal and (ii) all its angles right angles

The polygon in fig. 67 is **convex**, whereas such a polygon as that of fig. 69 is called **re-entrant**



Fig. 69

†Ex 44. Use this theorem to prove the theorem on p. 32

¶Ex 45. If two of the angles of a triangle are 67° and 79° , what is the third angle? What are the exterior angles, formed by producing the sides in order? What is their sum?

¶Ex 46. What is the size of each exterior angle of a regular octagon (8-gon)? Hence find the size of each interior angle

§7. To calculate the angle of a Regular Polygon: first find the exterior angle by dividing 360° by the number of sides, then get the interior angle by subtracting the result from 180° .

Ex 47. What are the exterior angles of regular polygons of 12, 10, 5, 3, n sides? Hence find the interior angles of these polygons

¶Ex 48. The exterior angle of a regular polygon is 60° , how many sides has the polygon?

Ex 49. How many sides have the regular polygons whose exterior angles are (i) 10° , (ii) 1° , (iii) $2\frac{1}{2}^\circ$?

Ex 50. Is it possible to have regular polygons whose exterior angles are (i) 15° , (ii) 7° , (iii) 11° , (iv) 6° , (v) 5° , (vi) 4° ?

¶Ex 51. Can there be regular polygons with exterior angles obtuse?

¶Ex 52. Is it possible to have regular polygons whose interior angles are (i) 108° , (ii) 120° , (iii) 130° , (iv) 114° , (v) 60° ? (Think of the exterior angles) In the cases which are possible, find the number of sides

Ex 53. Draw a regular pentagon having each side 2 in. long

[Calculate its angles, draw $AB = 2$ in., at B make $\angle ABC$ = the angle of the regular pentagon, cut off $BC = 2$ in., etc., etc.]

Ex 54. Draw a regular octagon having each side 2 in. long

Note. The construction of regular polygons is much simplified by using a circle for further examples on polygons see p. 144

SUMMARY OF RESULTS

The sum of the angles of a triangle is equal to two right angles.

If one side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior opposite angles.

If the sides of a convex polygon are produced in order, the sum of the angles so formed is equal to four right angles.

CHAPTER 6

SCALE DRAWING II*

MORE EXACT MEASUREMENT BY ESTIMATION

§ I. So far you have only measured to one place of decimals in inches or centimetres, but you will often need to measure more accurately. To do this you must *imagine* each tenth of an inch (or centimetre) divided again into 10 equal parts.

The line AB is more than 1.2 in. and less than 1.3 in.;



Fig 70

if its length is almost exactly half-

way between these measurements you will say it is 1.25 in.,

if it is a little more than half-way you will say it is 1.26 in.,

if it is about a third of the way you will say it is 1.23 in.,

if it is about two-thirds of the way you will say it is 1.27 in. and so on

With a little practice you ought to get this figure nearly accurate

In the same way you can measure angles to within less than a degree

¶Ex. 1. What fraction of an inch does a figure in the second place of decimals represent? What fraction of an inch is .03?

§ 2. If a man standing on the edge of a cliff looks through a horizontal telescope and then lowers (or “depresses”) the end of it till he is looking straight at a boat, the angle he has turned the telescope through is called the **angle of depression** of the boat

Remember that the angle of elevation and the angle of depression are always angles at the observer’s eye, and are always measured from the horizontal.

If A is an observer and X and Y two points (see fig. 66), the angle XAY is said to be the **angle subtended** at A by XY.

¶Ex. 2 In fig 63 name the angles subtended (i) by BD at A, (ii) by AD at B, (iii) by AC at B.

* These exercises do not require the use of intersecting arcs.

¶Ex 3 The two ends of a pier bear S. 16° E and S 18° E. from a picket boat What angle does the pier subtend at the picket boat?

¶Ex 4 Repeat Ex. 3, with bearings N. 3° E and N 2° W

In Chapter 2 some notes were given about scale drawing, for the sake of completeness these are repeated and amplified here.

(i) **Make a freehand sketch*** of the required figure. This should be made on the same sheet as the finished drawing, it should be neat and fairly large.

(ii) **Mark all the data** in the sketch. If this is done in ink, it is then distinguished from any deductions made from the data.

(iii) **Mark any deductions** that you can make that will help you in the drawing (do this in pencil), e.g. equal alternate angles, or the third angle of a triangle.

(iv) **State the scale** to be used. Choose this wisely (see p. 14).

(v) **Make the accurate drawing**, working from your free-hand sketch

(vi) **Do not rub out any construction lines.** They show how you made the figure

(vii) **State your answer**, which should be the actual length required and *not* the scaled length.

NOTE In the freehand sketch the actual lengths of the data should be shown, the scaled lengths may be shown in brackets

In the finished drawing the required length should be shown, the scaled length may be shown in brackets.

Notice that a scale drawing of a figure is similar to the figure it represents, the corresponding angles in the two are equal, and corresponding lengths are proportional or in the same ratio.

HEIGHTS AND DISTANCES.

Ex 5 A torpedo boat passes at a distance of 350 feet from a fort the guns of which are 100 feet above sea-level, to what angle should the guns be depressed so that they may point straight at the torpedo boat?

* It should be noted that in later work the freehand sketch is often all that is required. From the freehand sketch the required lengths and angles can be calculated without making an accurate drawing.

Ex 6 A look-out on the top of a vertical cliff 100 feet high sights a mine just awash and finds its depression to be 20° . Find the distance of the mine from the foot of the cliff below the look-out

Ex 7 From the top of a mountain, 2500 feet high, a ship at sea is observed at an angle of depression of 25° . What is the distance of the ship from the top of the mountain? Give your answer in feet.

Ex 8 Find, by drawing, the length of a man's shadow if the man is 6 feet high and the altitude of the sun is 37°

Ex 9 A man 6 feet high throws a shadow 8 feet long. By drawing a figure to scale, find the height of a man whose shadow is 7 feet long

Ex 10 A man standing on the bank of a river sees a tree on the far bank in a direction 20° W of N. He walks 200 yards along the bank and finds that its direction is now N E. If the river flows east and west, find its breadth

Ex 11 A and B are two points on a straight road which runs due east and west, from A a flagstaff F bears N 30° E, and from B it bears N 40° W. If the distance AB is 1125 yards, find the distance of F from the road

Ex 12. Fig 71 is a sketch of a road which consists of three straight parts. The length and direction of each part of the road were measured and found to be as follows —AB 6.4 miles, 23° east of north, BC 2.8 miles, 40° east of north, CD 5.7 miles, 74° east of north

Find the distance in miles direct from A to D, and the angle which AD makes with the northerly direction

Ex 13 X is 17.5 miles N.W. of Y, Y is 2.3 miles N.E. of Z. What is the distance and bearing of X from Z?

Ex 14 Southampton is 72 miles S 53° W of London, Gloucester is 75° W of N from London, and 29° W. of N from Southampton. Find the distance between Southampton and Gloucester

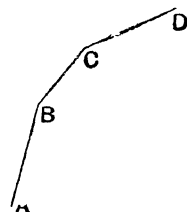


Fig 71

Ex 15 The position of a place X from A is N 35° E. From B, which is S.E. of A and 10 miles distant, X appears to be N 30° W.

Find, by drawing to scale, the distances of X from A and B

Ex 16 Brixham is 4.6 miles N.E. of Dartmouth, Torquay is 4 miles N. of Brixham, Totnes is 7.4 miles S 75° W of Torquay, what is the distance and bearing of Totnes from Dartmouth?

Ex 17. An observer takes the bearings etc. of two church spires A and B, and obtains the following data —A bears N. 22° W., and is distant from him 2000 yards, B bears S 78° W, and is distant from him 3000 yards. What is the distance and bearing of A from B?

Ex 18 From the top of a mast 70 feet high, two buoys are observed due N at angles of depression 57° and 37° , find the distance between the buoys to the nearest foot.

Ex 19 At a window 15 feet from the ground a flagstaff subtends an angle of 43° , if the angle of depression of its foot is 11° , find its height

Ex 20. An observer in an aeroplane observed the angle of depression of a mark on the ground to be 50° , after descending by means of a spiral to a point 1500 ft vertically below the previous point of observation he saw that the angle of depression of the same mark was now 38° . Find, by drawing to scale, how far above the ground the aeroplane was then.

Ex 21 Wishing to find the height of a cliff I fix two marks **A** and **B** on the same level in line with the foot of the cliff. From **A** the angle of elevation of the top of the cliff is 37° and from **B** the angle of elevation is $23^\circ 30'$. If **A** and **B** are 120 feet apart, find the height of the cliff.

CHART EXERCISES*

[N B One minute of latitude = one sea-mile $1^\circ = 60$ minutes (60')]

Ex 22 What is the angle subtended at N E Middle Buoy by the line joining Middle Light on Ryde Pier and S Light on Stokes Bay Pier?

Ex 23 What is the angle subtended by the line joining N E Middle Buoy and Stokes Bay Pier S Light at Ryde Pier Middle Light?

Ex 24 What angle does the line joining F S at Fishbourne and F S. on Osborne House subtend at the Red Buoy (R) on Peel Bank? What is the bearing of the buoy from Fishbourne F S?

Ex 25 A ship is observed to bear N 46° W from Ryde Pier (Middle Light) and S 54° W from Stokes Bay Pier (S Light). Mark its position on the chart, find its distance in sea-miles from the former light, and in cables from the Red Buoy on the Mother Bank.

Ex 26 A position is in Long $1^\circ 10' W$, find its latitude (i) if bearing of Bay House, Alverstoke, is N $13\frac{1}{2}^\circ E$, (ii) if bearing of flagstaff at Lee Point is N $41^\circ W$.

Ex 27 A position is in Long $1^\circ 15' W$, find its latitude (i) if bearing of Norris Castle Flagstaff is S $23' W$, (ii) if bearing of Lee-on-Solent Pier Light is N $59^\circ E$.

Ex 28 A position is in Lat $50^\circ 45' N$, find its longitude (i) if Wootton Coastguard Station bears S $46\frac{1}{2}^\circ W$, (ii) if Wootton Coastguard Station bears S. $42^\circ E$.

Ex 29 Find a position the bearing of which from R Y S' Club House is N $72\frac{1}{4}^\circ E$, and from F S at H M Hard is N $7^\circ W$. Give its latitude and longitude.

Ex 30 Find a position the bearing from which of Ryde Church is S $33^\circ E$, and of F S at Fishbourne is S $20^\circ W$. Give its latitude and longitude.

Ex 31 A ship approaching Cowes observes All Saints' Church, Ryde, bearing S $24^\circ E$, and F S of Alverstoke Coastguard Station bearing N $42^\circ E$. After 15 minutes she observes Trinity Church, Cowes, bearing S $85\frac{1}{4}^\circ W$, F S of H M Hard bearing S $18^\circ W$. Find her course and speed.

What is the bearing and distance of the Flagstaff of Browdown Camp from the first position?

* These Exs refer to the chart "The Solent to Spithead" See footnote ‡ on p 15

Ex. 32. A ship approaching Cowes observes Alverstoke Church bearing $N\ 34^\circ\ E$, and Ryde Church bearing due south. After 12 minutes she observes Fishbourne F.S. bearing $S.\ 10^\circ\ E$, and Osborne House F.S. bearing $S.\ 68^\circ\ W$. Find her course and speed.

MISCELLANEOUS.

Ex. 33. A plank 6 ft. long makes an angle of $27\frac{1}{2}^\circ$ with the horizontal ground. How far is the upper end of the plank above the ground?

Ex 34. A swimming bath is 50 ft. long and its depth is 6 ft. at one end and 3 ft at the other. What is the angle the bottom of the bath makes with the horizontal?

Ex 35 An upright ship's mast is braced fore and aft by two equal stays, whose combined length is 250 feet, each making an angle of 65° with the deck. Find (i) the height of the mast, (ii) the distance on the deck between the ends of the stays

Ex. 36 A haystack, of which the edge of the thatch is 15 ft vertically above the ground, overhangs its base by 3 ft. What length of ladder will be required so that it will just reach to the edge of the thatch if the foot of the ladder is placed on the ground 8 ft from the base of the haystack?

Ex 37. A ladder 10 ft long leans over a wall that is 6 ft high. If the foot of the ladder is 2 ft from the wall, how high is the top of the ladder above the ground?

Ex 38 A rod AB hangs vertically down from A . If it is now turned about A till it makes an angle of 40° with the vertical, through what vertical height has B risen?

Ex 39 The seat of a swing is supported by two parallel ropes each 10 ft. long. If the ropes swing through 30° on each side of the vertical, what is the greatest height of the seat from the ground if the least height is 2 ft?

Ex 40 A see-saw consists of a plank 10 ft long with its middle point supported at a point 2 ft above the ground. Through what angle can the see-saw swing?

What is the greatest height of an end from the ground?

Ex. 41 The end view of a garage shows two side-walls each 7 ft high and two equal sloping sides of the roof. If the width of the garage is 15 ft. and the height of the highest point of the roof $11\frac{1}{2}$ ft, what is the angle of slope of the roof and the length of the slope?

Ex 42 A rectangular drawing board 18" by 14" lies on a table with one corner on the edge of the table and the long side making an angle of 30° with the edge. How far are the other corners from the edge?

Ex 43 A horizontal trap door measures 2 ft 6 in. by 1 ft 9 in., the hinges being along the shorter edge. What is the height of the edge opposite the hinges above the horizontal when the door is opened through an angle of 50° ?

CHAPTER 7

CONSTRUCTION OF TRIANGLES AND POLYGONS

§ 1. A triangle has three sides and three angles, these sides and angles may be called the six parts of the triangle

¶Ex 1 How many parts has a quadrilateral?

To copy a triangle, how many parts must be copied?

In copying a triangle, you might expect that all 6 parts would have to be copied, but you will find that this would be a waste of labour

Suppose that you are making a copy of a triangle ABC. Put a piece of tracing-paper over the triangle and trace freehand first AB, then $\angle ABC$, then BC. Then take the tracing-paper away.

You can now complete the copy without tracing any more parts by joining BC.

You have thus made a copy of the triangle by copying *three only* of its parts. *The other three parts take care of themselves.*

If you are copying with tracing-paper, no doubt it is as easy to copy 6 parts as to copy 3. But if you have no tracing-paper, and have to copy the sides and angles by measuring with scale and protractor, to measure all 6 parts would be a waste of time.

¶Ex 2 Try how many parts of a quadrilateral you must copy in order to copy a quadrilateral. (Either use tracing-paper, or else copy the sides and angles freehand*.)

¶Ex 3 Repeat Ex 2 for a 5-sided figure (pentagon).

§ 2 ¶Ex 4 Try if you can copy the $\triangle PQR$ in fig. 72 by copying PQ, $\angle PQR$ and QR.

Hence we find that we can make a copy of a given triangle, that is to say a triangle equal in all respects to the given triangle, by making *two sides and the included angle* the same as in the given triangle.

¶Ex 5 Would *any* two sides and the included angle serve to fix the copy of a triangle?

In copying sides freehand, measure the lengths roughly along your pencil, copy angles by eye; mark on your figure the parts you have copied.

¶ Ex. 6. Try if you can copy $\triangle PQR$ freehand* by copying two sides PQ and PR , and an angle not included say $\angle Q$. You will find that there is a difficulty here. Remember that this case has not been dealt with properly.

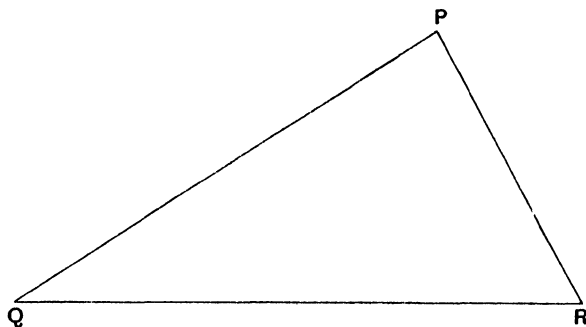


Fig. 72

‡ Ex. 7 Construct triangles to the following measurements, in each case measure the remaining side and angles —

- (i) $\angle ABC = 80^\circ$, $AB = 2.2$ in, $BC = 2.9$ in
- (ii) $\angle B = 28^\circ$, $AB = 7.3$ cm, $BC = 12.1$ cm
- (iii) $\angle B = 126^\circ$, $AB = 6.1$ cm, $BC = 6.1$ cm
- (iv) $\angle C = 90^\circ$, $BC = 3.9$ in, $CA = 2.8$ in
- (v) $CA = 2.6$ in, $AB = 3.3$ in, $\angle A = 162^\circ$
- (vi) $\angle C = 79^\circ$, $b = 4.7$ cm, $a = 6.1$ cm.
- (vii) $c = 4.6$ cm, $b = 8.7$ cm, $\angle A = 58^\circ$

Ex. 8 Two ladders 15 ft. and 20 ft. long lean against each other at their ends forming an angle of 80° . What is the distance between their feet?

Ex. 9 The lid of a box 2 ft. long is propped open at an angle of 58° by a stick. What must be the length of the stick?

Ex. 10 One aeroplane sets off due N at 80 m.p.h. and at the same time another sets off N.E. at 100 m.p.h. How far apart will they be after 1 hour and what bearing will the latter have from the former?

§ 3. ¶ Ex. 11 Try if you can copy a triangle (drawing freehand*) by copying a side and the two angles at the ends of the side.

* In copying sides freehand, measure the lengths roughly along your pencil, copy angles by eye, mark on your figure the parts you have copied.

‡ Note that the sides of a $\triangle ABC$ are often called a, b, c ; remember that a is the side opposite the angle A and so on.

¶ Ex 12 With ruler and protractor, make an accurate copy of $\triangle PQR$ in fig 72 by copying QR , $\angle Q$, and $\angle R$

¶ Ex. 13 Try to construct a triangle given one side and two angles *one of which is not at the end of the given side*. Say $QR = 2$ in, $\angle Q = 60^\circ$, $\angle P = 70^\circ$. If you fail at first, make a freehand sketch and think what must be the size of $\angle R$. Measure PQ when you have completed the copy.

Hence we find that we can make a triangle equal in all respects to a given triangle, by making *one side and any two corresponding angles* the same as in the given triangle

Ex 14 Construct triangles to the following measurements, in each case measure the remaining sides and angle —

(In case the construction is impossible with the given measurements, try to explain why it is impossible)

- (i) $AB = 8.3$ cm, $\angle A = 45^\circ$, $\angle B = 72^\circ$
- (ii) $\angle B = 90^\circ$, $BC = 7.2$ cm, $\angle C = 42^\circ$
- (iii) $\angle C = 116^\circ$, $CA = 1.8$ in, $\angle A = 78^\circ$
- (iv) $b = 9.2$ cm, $\angle C = 31^\circ$, $\angle A = 59^\circ$
- (v) $c = 2.8$ in, $\angle A = 50^\circ$, $\angle B = 130^\circ$.

Ex. 15 Construct triangles to the following measurements, in each case measure the remaining sides and angle —

(Here it is specially important to make a freehand sketch)

- (i) $BC = 8.0$ cm, $\angle A = 77^\circ$, $\angle B = 46^\circ$.
- (ii) $AB = 7.3$ cm., $\angle B = \angle C = 57^\circ$
- (iii) $\angle C = \angle A = 60^\circ$, $AB = 2.7$ in
- (iv) $BC = 1.1$ in, $\angle A = 14^\circ$, $\angle C = 52^\circ$.

Ex 16 The angle of elevation of the top of a flagstaff from a point in the horizontal plane 100 yards from the foot of the post is 30° . Find the height of the flagstaff

Ex 17 AB is a straight road 400 yards long running due N. The bearing of a church from A is 60° E of N and from B , 40° E of N. How far is the church from A ?

Ex 18. To find the distance of a point G from A , a straight line AB , 300 yards long, is measured out. The $\angle GAB$ is found to be 65° and $\angle GBA$, 70° . How far is G from A ?

Ex 19 From two points 100 yards apart in a straight line with the foot of a tower the angles of elevation of the top are 30° and 50° . Find the height of the tower.

§4. ¶Ex. 20. Try if *the lengths of three sides* are enough to fix a triangle, nothing being said about the angles. For this experiment, take three sticks. See if you can fit them together, end to end, in various ways so as to form different triangles, or if you always get the same triangle however you may arrange them.

¶Ex 21. Copy the triangle PQR in fig. 72 by first drawing QR of the required length and then finding the third point by means of circles (the whole circles need not be drawn, see fig 73, which is not drawn full size). Hence we find that we can make a triangle equal in all respects to a given triangle by making the *three sides* the same as in the given triangle.

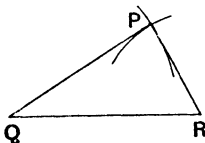


Fig. 73

Ex 22 Construct triangles to the following measurements, in each case measure the angles —

(It is advisable to draw the longest side first)

- (i) $BC = 8.9 \text{ cm}$, $CA = 8.3 \text{ cm}$, $AB = 6.7 \text{ cm}$
- (ii) $a = 5.3 \text{ cm}$, $b = 8.3 \text{ cm}$, $c = 2.5 \text{ cm}$
- (iii) $BC = 3.9 \text{ in}$, $CA = 2.5 \text{ in}$, $AB = 2.5 \text{ in}$
- (iv) $BC = 6.9 \text{ cm}$, $CA = 6.9 \text{ cm}$, $AB = 6.9 \text{ cm}$
- (v) $a = 6.5 \text{ cm}$, $b = 9.6 \text{ cm}$, $c = 7.2 \text{ cm}$
- (vi) $BC = 2.1 \text{ in}$, $CA = 1.1 \text{ in}$, $AB = 3.2 \text{ in}$
- (vii) $a = 4.1 \text{ in}$, $b = 4.1 \text{ in}$, $c = 4.1 \text{ in}$.

Ex 23. On a horizontal beam there are two hooks 4 ft apart. A weight hangs from the hooks by two strings of length 3 in. and 2.5 in. What angles do these strings make with the horizontal?

Ex 24. A is 5 miles due N. of B, C is 8 miles from A and 6 miles from B. What is the bearing of C from A and from B? What is its distance from the line AB?

Ex. 25 Two beams 20 ft. and 15 ft. long support a vertical wall by pressure at the same point. Their ends, 10 ft. apart, are in a line at right angles to the wall. What is the height of the point of support above the ground, assumed horizontal?

Ex 26. The Chesapeake Bay seaplane race-course is triangular, the sides being 10.3 miles, 8.9 miles, 11.8 miles. Through what angle does a machine have to turn at each corner?

THE ISOSCELES TRIANGLE*

§5 ¶Ex 27 Draw any angle Using compasses cut off $AB = AC$ along the arms. The triangle ABC is isosceles What should you expect to find about $\angle B$ and $\angle C$?

You will agree that if we make $AB = AC$ then $\angle B = \angle C$

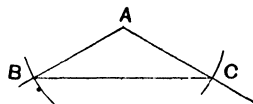


Fig 71

You will also readily agree that if we began by making the $\angle B = \angle C$ then the side AB would be equal to the side AC

The phrase "the sides" of an isosceles triangle is often used to mean the equal sides, "the base" to mean the other side, "the vertex" to mean the point at which the equal sides meet, and "the vertical angle" to mean the angle at the vertex

To construct an isosceles triangle on a base 2" with equal sides 3".

Draw base $BC = 2''$ With centres B and C and radius $3''$ draw arcs of circles intersecting at A Measure $\angle B$ and $\angle C$

Ex 28 Draw any line BC , at B and C make any two equal angles CBA, BCA Measure AB and AC .

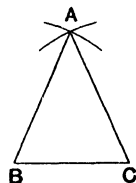


Fig 75.

THE EQUILATERAL TRIANGLE.

¶Ex. 29. Draw BC any length With centres B and C and radius BC describe circles† to intersect at A Then ABC is an equilateral triangle Measure the angles of the triangle

Ex 30 Construct isosceles triangles (i) on a base $2''$ with equal sides $1\frac{1}{2}''$, (ii) on a base 6 cm. with equal angles 70° .

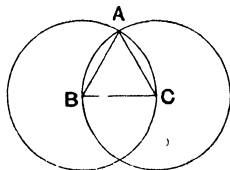


Fig. 76.

Ex. 31 Construct triangles to the following measurements, in each case measure the angles —

- (i) $BC = 3\frac{1}{2}$ in, $CA = AB = 2\frac{1}{2}$ in
- (ii) $a = b = 3\frac{1}{2}$ in, $c = 1\frac{1}{2}$ in
- (iii) $a = b = c = 6\frac{1}{2}$ cm

The properties of the Isosceles Triangle are mostly obvious to a beginner from intuitive ideas of symmetry. These can be developed by leading questions See Ch 11.

† It is not necessary to show the whole circles, but they have been shown here as a matter of historical interest, fig. 76 is the figure of Euclid's first proposition.

Ex 32 Construct an equilateral triangle with a side of 5 cm. On each side of this triangle and outside it construct other equilateral triangles.

Ex. 33 Draw any isosceles triangle ABC with $AB=AC$. From A draw AD perpendicular to BC . Measure BD and DC .

From the above exercises the following theorems should be quite obvious —

THEOREM 12. If two sides of a triangle are equal, the angles opposite to these sides are equal.

THEOREM 13 If two angles of a triangle are equal, the sides opposite to these angles are equal.

Ex. 34. In a triangle XYZ , $XY=XZ$, find the angles of the triangle in the following cases —(i) $\angle Y=74^\circ$, (ii) $\angle X=36^\circ$, (iii) $\angle X=142^\circ$, (iv) $\angle Y=13^\circ$, (v) $\angle Z=97^\circ$, (vi) $\angle Z=45^\circ$, (vii) $\angle X=a^\circ$, (viii) $\angle Y=b^\circ$

Ex 35 Find the angles of an isosceles triangle in which each of the base angles is half of the vertical angle.

Ex 36 Find the angles of an isosceles triangle in which each of the base angles is double of the vertical angle

†Ex 37 Prove that a triangle which is equilateral is also equiangular.

[If PQR is an equilateral triangle, $QP=QR$ $\angle^o=\angle^o$.]

Ex 38 In a triangle ABC , $AB=9.2$ cm, $\angle C=82^\circ$, $AC=9.2$ cm., AB, AC are produced to D, E respectively. Find all the angles in the figure

§6

THE RIGHT-ANGLED TRIANGLE

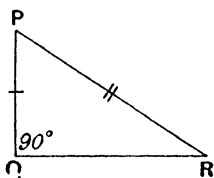


Fig 77

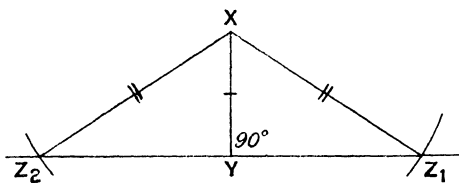


Fig 78

¶ Ex 39 Copy the right-angled triangle PQR as follows —draw $XY=PQ$, through Y draw a line at right angles to XY , with centre X and radius $=PR$ draw a circle to cut this line at Z . You will see that there are two possible positions for Z , but $\triangle s XYZ_1$ and XYZ_2 are exactly equal

Hence we find that we can make a triangle equal in all respects to a given *right-angled* triangle by making the *hypotenuse and one side* the same as in the given triangle. (Note that this special case is a case of two sides and the non-included angle.)

Ex 40 Construct right-angled triangles to the following measurements, in each case measure the remaining sides and angles —

(i) $AB = 8 \text{ cm}$, $AC = 6 \text{ cm}$, $\angle C = 90^\circ$

(ii) $BC = 2 \text{ in}$, $BA = 2.8 \text{ in}$, $\angle C = 90^\circ$

(iii) Hypotenuse = 6 cm ., one side = 5.2 cm .

(iv) $\angle A = 90^\circ$, $a = 4 \text{ in}$, $b = 2 \text{ in}$.

Ex 41 R is 12 m south of P, Q is due west of R, and PQ is 20 m. How far is Q from R, and what is the bearing of Q from P?

Ex 42 A ladder 20 ft. long rests against a vertical wall with the foot of the ladder 10 ft. from the base of the wall. To what height up the wall does the ladder reach?

Ex 43 A vertical mast 100 ft. high is secured by straight cables 120 ft. long. At what distance from the foot of the mast are the cables fixed in the ground?

Ex 44 To reach the top of a vertical tower 60 ft. high, trucks are dragged up a straight slope 200 ft. long. What is the horizontal distance between the foot of the tower and the foot of the slope?

§ 7 Construction of triangles—miscellaneous data.

Ex 45 Construct triangles to the following measurements, in each case measure the remaining parts —

- | | | |
|-----------------------------------|----------------------------|-------------------------|
| (i) $BC = 3.18 \text{ in}$, | $AB = 3.18 \text{ in}$, | $\angle B = 33.5^\circ$ |
| (ii) $a = 2.39 \text{ in}$, | $b = 2.44 \text{ in}$, | $\angle C = 63.5^\circ$ |
| (iii) $AB = 2.82 \text{ in}$, | $AC = 2.77 \text{ in}$, | $\angle A = 137^\circ$ |
| (iv) $c = 3.00 \text{ in}$, | $\angle A = 61^\circ$, | $\angle B = 59^\circ$ |
| (v) $BC = 3.52 \text{ in}$, | $\angle B = 25^\circ$, | $\angle C = 23^\circ$ |
| (vi) $AC = 10.65 \text{ cm}$., | $\angle A = 54.5^\circ$, | $\angle C = 36^\circ$ |
| (vii) $BC = 6.40 \text{ cm}$, | $CA = 9.05 \text{ cm}$, | $AB = 7.63 \text{ cm}$ |
| (viii) $a = 7.69 \text{ cm}$, | $b = 9.30 \text{ cm}$, | $c = 5.30 \text{ cm}$ |
| (ix) $BC = 4.53 \text{ in}$, | $CA = 2.68 \text{ in}$, | $AB = 2.02 \text{ in}$ |
| (x) $c = 2.71 \text{ in}$, | $\angle B = 55.5^\circ$, | $\angle C = 67.5^\circ$ |
| (xi) $\angle A = 24^\circ$, | $\angle C = 47.5^\circ$, | $BC = 3.04 \text{ cm}$ |
| (xii) $\angle A = 133^\circ$, | $a = 10.73 \text{ cm}$, | $\angle B = 23.5^\circ$ |
| (xiii) $\angle C = 90^\circ$, | $BC = 1.00 \text{ in}$, | $CA = 2.00 \text{ in}$ |
| (xiv) $BC = 4.09 \text{ cm}$, | $CA = 3.31 \text{ cm}$, | $AB = 7.54 \text{ cm}$ |
| (xv) $\angle A = 90.5^\circ$, | $\angle B = 78^\circ$, | $a = 3.54 \text{ in}$ |
| (xvi) $AB = 2.99 \text{ in}$, | $\angle B = 127.5^\circ$, | $\angle C = 53.5^\circ$ |
| (xvii) $c = 2.92 \text{ in}$, | $\angle B = 59^\circ$, | $b = 2.39 \text{ in}$ |
| (xviii) $\angle B = 33.5^\circ$, | $BC = 2.61 \text{ in}$, | $CA = 1.54 \text{ in}$ |
| (xix) $a = 2.16 \text{ in}$, | $b = 2.64 \text{ in}$, | $\angle B = 64.5^\circ$ |
| (xx) $\angle A = 24^\circ$, | $c = 7.76 \text{ cm}$, | $a = 2.87 \text{ cm}$ |

FRAMEWORKS

§ 8 Fig 79 represents a gate, but a bad gate. If anyone sat on the top bar, there would be a danger of the gate collapsing, and

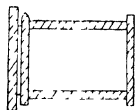


Fig 79

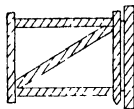


Fig 80

assuming this shape . To resist a strain of this nature, there is nothing but the stiffness of the corners, and to rely upon this is bad carpentry. To make this a good gate, an additional bar must be introduced as in fig 80. The fifth bar makes it a **rigid framework**, not relying upon the stiffness of the corners.

Take 4 strips of cardboard, of any lengths. Punch a hole at each end of each strip (with a paper-punch if possible) and joint the strips together with paper-fasteners, or preferably with bootmaker's eyelets*. The result is a quadrilateral with 4 sides given. The quadrilateral can be *deformed* into various shapes, if the hinges at the corners work freely.

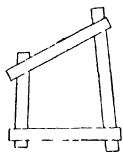


Fig 81

To convert this into a *rigid* framework, you may add a fifth strip along a diagonal.

This shows that a quadrilateral is fixed given 4 sides and 1 diagonal, 5 data in all.

But the original deformable framework might have been made rigid in other ways. One angle might be fixed by pinching or gluing one corner. Fixing one angle will fix all the angles. Here again 5 data fix the quadrilateral.



Fig 82

Notice that three strips jointed together at their ends form a rigid framework.

¶Ex. 46 Try if the deformable quadrilateral is made rigid by adding a 5th strip (i) joining points on adjacent sides, (ii) joining points on opposite sides.

¶Ex. 47 By freehand drawing, see if a quadrilateral is fixed if 3 sides are given, together with the 2 angles between these sides—5 data in all.

* Or Meccano strips might be used. See Notes and Answers.

¶Ex 48 Is a quadrilateral fixed by the following 5 data 4 angles and 1 side?

¶Ex 49. By experiment or otherwise, ascertain how many data are needed to fix a pentagon

§9. Construction of quadrilaterals, etc. from sufficient data.

Here it is especially important that, before beginning the construction, a freehand sketch should be made showing the given parts.

Note that the letters must be taken in order round the quadrilateral, e.g. the quadrilateral in fig 83 is called ABCD and not ABDC



Fig. 83

Ex 50 Construct quadrilaterals ABCD to the following measurements; measure the remaining sides —

- (i) $AB = 6.3 \text{ cm}$, $\angle B = 82^\circ$, $BC = 8.2 \text{ cm}$, $\angle C = 90^\circ$, $CD = 7.7 \text{ cm}$.
- (ii) $\angle B = 116^\circ$, $BC = 1.4 \text{ in}$, $\angle C = 99^\circ$, $CD = 1.9 \text{ in}$, $\angle D = 92^\circ$
- (iii) $\angle B = 122^\circ$, $\angle C = 130^\circ$, $\angle D = 130^\circ$, $BC = CD = 1.6 \text{ in}$
- (iv) $AC = 5.6 \text{ cm}$, $\angle BAC = 58^\circ$, $\angle DAC = 69^\circ$, $\angle BCA = 58^\circ$, $\angle DCA = 69^\circ$

Ex 51 Construct quadrilaterals ABCD to the following measurements, measure the angles —

- (i) $AB = 2.3 \text{ in}$, $BC = 2.1 \text{ in}$, $CD = 3.3 \text{ in}$, $DA = 1.5 \text{ in}$, $BD = 3.4 \text{ in}$
- (ii) $AB = CD = 6.4 \text{ cm}$, $BC = DA = 3.7 \text{ cm}$, $BD = 5.7 \text{ cm}$.
- (iii) $AB = BC = CD = DA = 5.1 \text{ cm}$, $AC = 9.2 \text{ cm}$
- (iv) $AB = 3.8 \text{ in}$, $BC = 1.7 \text{ in}$, $CD = 1.0 \text{ in}$, $DA = 4.9 \text{ in}$, $\angle B = 146^\circ$

Ex. 52. Construct pentagons ABCDE to the following measurements, measure the angles at D and E —

- (i) $AB = 2.0 \text{ in}$, $BC = 2.2 \text{ in}$, $CD = 1.7 \text{ in}$, $DE = 2.2 \text{ in}$, $EA = 2.5 \text{ in}$, $\angle B = 111^\circ$, $\angle C = 149^\circ$.
- (ii) $AB = 1.7 \text{ in}$, $BC = 1.0 \text{ in}$, $CD = 2.2 \text{ in}$, $DE = 3.4 \text{ in}$, $EA = 0.5 \text{ in}$, $\angle A = 126^\circ$, $\angle B = 137^\circ$
- (iii) $AB = BC = CD = DE = EA = 5.0 \text{ cm}$, $AC = BE = 8.1 \text{ cm}$

MISCELLANEOUS EXERCISES A

Ex 1. In fig. 84, RY is parallel to QP and QX is parallel to RP . Determine all the angles in the figure, giving your reasons

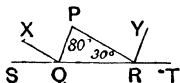


Fig 84.

Ex 2 In fig 85, $ABCD$ is a parallelogram, and FG is parallel to the diagonal DB . Copy the figure freehand and find all the angles in the figure. Give a brief explanation

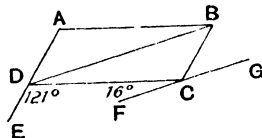


Fig 85

Ex 4 In fig 86, EB , AD , FC are all perpendicular to BC , given that $\angle EBA = 20^\circ$ and $\angle FCA = 50^\circ$, find the other angles of the figure. Give your reasons carefully

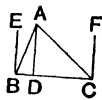


Fig 86

Ex 5 Draw a triangle ABC right-angled at A ; through A draw XAY parallel to BC , and draw AD perpendicular to BC . Assume that $\angle ACB = 30^\circ$ and find the number of degrees in the other angles of the figure. Give your reasons carefully.

Ex 6 In fig 87, $STXY$, $XYZW$ are two parallelograms. Make a freehand sketch of the figure, and write in each angle the number of degrees it contains. Explain briefly

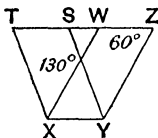


Fig 87.

Ex 7 Draw an acute-angled triangle XYZ , produce YX to P , through X draw a straight line QXR parallel to YZ . Assuming that $\angle XYZ = 40^\circ$ and $\angle XZY = 60^\circ$, calculate each angle in the figure. Give your reasons carefully.

Ex. 8 Draw freehand a triangle ABC having $BA = BC$, through B draw DBE parallel to AC . Now assuming $BA = BC$, and $\angle BAC = 42^\circ$, find the other angles in the figure. Give your reasons carefully

Ex 9. In fig 88, $AB = AC$, and CE is parallel to BD . Find all the angles and give a brief explanation.

Are there any equal lines in the figure? Give reasons for your answer.

†**Ex. 10.** Draw a triangle ABC , produce BC to D , draw CE the bisector of $\angle ACD$. Prove that, if CE is parallel to BA , $\triangle ABC$ is isosceles

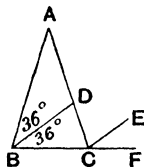


Fig 88

Ex. 11. In fig. 89 find the angle marked x , giving reasons
Find also the angle at which CN and AB would meet if produced.

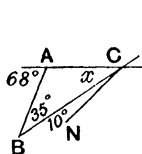


Fig. 89.



Fig 90

Ex 12 In fig 90, $\angle A = 40^\circ$, $\angle XCB = 30^\circ$, $AB = AC$, and XY is parallel to BC . Find the various angles of the figure, giving your reasons in each case. What pairs of lines in the figure must be equal?

Ex 13 In fig 91, O is the centre of the circle and OC is parallel to AB . $\angle OAB = 40^\circ$. Find all the angles of the figure. Explain briefly.



Fig 91

†**Ex. 14.** PQR , SQR are two isosceles triangles on the same side of their common base QR . Prove that $\angle PQS = \angle PRS$.

Ex 15. In fig 92, ABC is an equilateral triangle, and each of the marked angles is 20° . What is the magnitude of $\angle BPC$? (Give your reasons.) Prove that $\triangle PQR$ is equilateral.

Ex. 16 Make a freehand copy of the triangle in fig. 92, draw BP and produce it to X so that $BX = BA$. The figure now represents a framework of four equal rods. Assuming that $\angle XBC = 20^\circ$, find $\angle XCA$.

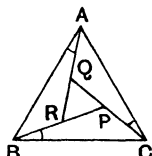


Fig 92

†**Ex 17** ABC is an isosceles triangle, the equal sides AB , AC are produced to D , E respectively, the bisectors of $\angle DBC$, $\angle ECB$ intersect at F . Prove that $\triangle FCB$ is isosceles.

†**Ex. 18** Draw a straight line XO , produce it to Y so that $OY = XO$, from O draw another line OZ and make it equal to OX . Prove that $\angle XZY$ is a right angle.

†**Ex 19** PS , the bisector of the $\angle P$ of a $\triangle PQR$, cuts QR at S . Through S , ST and SU are drawn parallel to PR and PQ , thus forming a quadrilateral $TPUS$. Prove that the sides of $TPUS$ are all equal to one another.

†**Ex 20** $\angle AOC$ is an acute angle and OB divides it into two unequal angles. From P , any point in OC , perpendiculars PN and PM are drawn to OA and OB . Prove that the angle between these perpendiculars is equal to the angle AOB .

Repeat the above for the case in which $\angle AOC$ is obtuse and each of the angles $\angle AOB$, $\angle BOC$ acute.

†Ex 21 In fig 93, $\angle CAD = \angle ABC$ and CPQ bisects $\angle ACB$ Prove $\angle AQP = \angle APQ$.

†Ex 22 The bisector of the angle A of a triangle ABC meets BC in D , and BC is produced to E Prove that

$$\angle ABC + \angle ACE = 2\angle ADC$$

†Ex 23 AB, CD are two parallel straight lines drawn in the same direction, and P is any point inside the quadrilateral $ABDC$ Prove that $\angle BPD = \angle ABP + \angle CDP$

Ex. 24. The gable end of a house is in the form of a pentagon, of which the three angles at the ridge and eaves are equal to each other. show that each of these angles is equal to 120°

Ex 25 How many diagonals can be drawn through one vertex of (i) a quadrilateral, (ii) a hexagon, (iii) a n -gon?

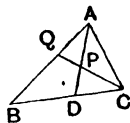


Fig 93

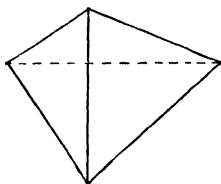


Fig 94

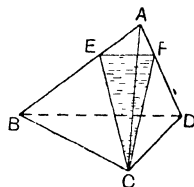


Fig 95

Ex 26 Fig 94 represents a tetrahedron (or pyramid on a triangular base). How many faces, edges, vertices has it?

Ex 27 If the tetrahedron in fig 94 has its top cut off by a plane parallel to the base, how many faces, edges, vertices would the resulting figure have?

Ex 28 Fig 95 shows a tetrahedron. Imagine it cut along the shaded section and the top removed. How many faces, edges, vertices would the resulting figure have?

Ex 29 Repeat Exs 26, 27 for a square pyramid

Ex 30 Fig 96 shows a solid formed by building a 4-sided pyramid upon a cube, the sides of the pyramid being equilateral triangles. How many faces (including the base) has the whole solid? how many edges? how many vertices? What is the sum, in degrees, of the angles at A , at B , and at O ?

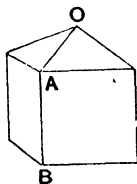


Fig 96

REVISION PAPERS **A**

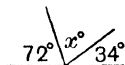
Paper 1.

1 Show by means of freehand figures and any necessary explanation that you understand the meaning of (i) vertically opposite angles, (ii) alternate angles, (iii) rectangle, (iv) isosceles triangle

2 How many edges has (i) a cube, (ii) a cuboid, (iii) a square pyramid? Draw a freehand sketch of each

3. Calculate the value of x in the figure.

4 What can you state about (i) the angles of any triangle, (ii) the angles of an equilateral triangle?



5 What is the sum of the angles of a quadrilateral? Prove your statement in two ways if you can

6 Through what angle does the large hand of a clock turn between 12.5 p.m. and 12.45 p.m.?

The large hand is 10 inches long and the small hand is 6 inches long. By drawing to scale, find the distance between the tips of the two hands at 4 p.m.

7 In the given figure AB is parallel to CD (i) Name a pair of alternate angles (ii) Name a pair of corresponding angles (iii) Assume $\angle FGD = 70^\circ$ and find all the other angles in the figure, give the reasons carefully



8. Draw a $\triangle ABC$ having $AB = 3.56$ in, $AC = 2.33$ in, $\angle A = 43^\circ$ Measure BC

9 Find, by drawing, the length of the shadow of a man 6 feet high when the altitude of the sun is 57°

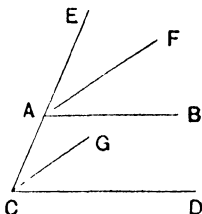
Paper 2.

1 Mention and explain the various adjectives that are used to describe different kinds of triangles

2 Two or more straight lines are drawn through a point forming angles Draw a figure and mention three facts connected with such angles

3 Draw (freehand) an irregular pentagon and find the sum of its angles.

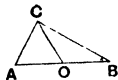
4 In the given figure AB is parallel to CD , and AF and CG are the bisectors of $\angle BAE$ and $\angle DCA$ Prove that AF and CG are parallel



5 I walk for two-thirds of a mile along a road. I then turn to the right at right angles and walk for half-a-mile. I then turn to the left at right angles and walk for a mile, all three parts of the road being straight. Draw a diagram and find by measurement my distance from my starting point

[See next page]

6 In the given figure $CA = CO$, $\angle COA = 58^\circ$, $\angle OCB = 29^\circ$
Find all the other angles. What other pairs of lines are equal and why?



7. Draw a $\triangle ABC$ having $AC = 9.45$ cm., $\angle A = 43^\circ$, $\angle C = 65^\circ$ Measure the distance of B from the mid-point of AC

8. A ship steaming N. 55° W at 18 knots sights a lighthouse bearing N 42° W, distant 3.5 miles, at noon. Find, by drawing, how near the ship will pass to the lighthouse, if she keeps on her course. Find also at what time (to the nearest minute) she will pass the lighthouse

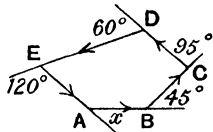
Paper 3.

1. If two straight lines are parallel and are cut by a straight line, what consequences follow?

2 Two angles of a triangle are 65° and 43° Find the third angle.

Three angles of a quadrilateral are 65° , 43° and 102° Find the fourth angle.

3 A man starts from A and walks along the roads $ABCDEA$ the angles through which he turns at the corners are shown in the figure. What is the number of degrees in the angle x through which he has to turn at A ?

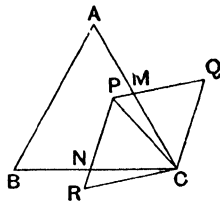


Show that AE and CD are parallel.

4. The side BC of the triangle ABC is produced to K . Being given $\angle ABC = 41^\circ$, and $\angle ACK = 73^\circ$, find $\angle BAC$

Quote the enunciation of the theorem you use

5 In the figure the triangles ABC , PQC , PRC are equilateral triangles. $\angle ACP$ is 12° Find all the angles of the triangles MQC and NRC



6. Draw freehand an acute-angled triangle ABC , draw AD to bisect $\angle BAC$ and cut BC at D , draw DE parallel to BA cutting CA at E . Now suppose that $\angle C = 40^\circ$ and $\angle B = 60^\circ$, determine all the angles in the figure and hence find any pairs of equal lines. Give reasons.

7. Draw a $\triangle ABC$ in which $a = 3.55$ in., $b = 3.36$ in., $c = 2.72$ in. Measure the perpendicular from A to BC

8 A captive balloon is observed from two positions A and B on a horizontal plane, A being due north of the balloon and B due south of it. A and B are 2 miles apart. From A the angle of elevation of the balloon is 27° and from B it is 18° . Find, by drawing, the height of the balloon.

Paper 4.

1. Explain the terms (i) angle of depression, (ii) corresponding angles

2. (i) How many faces has a prism on a six-sided base? (ii) How many edges has a pyramid on a base of 5 sides?

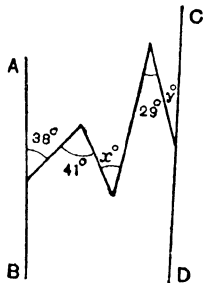
3 Mention four arguments any one of which may be used to prove that two straight lines are parallel

4. Quote the theorem about the exterior angles of a convex polygon Why convex? Show that the theorem about the interior angles of a triangle can be proved from the above.

†5 The triangle OAB has $\angle A = \angle B = \text{twice } \angle AOB$. Calculate the number of degrees in $\angle AOB$.

OB is produced to C so that $BC = BA$. Prove that $\angle OBA = \text{twice } \angle ACB$. Also prove that $AC = OB$.

6 In the figure, which is not drawn to scale, find y if AB and CD are parallel and x is 18, also prove that AB and CD will meet if produced towards B and D (whatever x may be), provided that $x + y$ is greater than 32



7. Draw a $\triangle ABC$ having $\angle B = 90^\circ$, $AB = 6.77$ cm, $AC = 9.33$ cm. Measure BC.

8 Three ships, A, B and C, start together from a port A proceeds due north, B, N E, and C, east. If the ships always keep in a line, A going 20 knots and B 12 knots, what is C's speed?

Paper 5.

1 Give careful definitions of the following, and draw simple figures to make your definitions more clear —Supplementary angles, angle of elevation

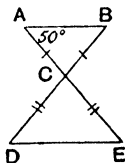
Explain with sketches the meaning of (i) prism, (ii) triangular pyramid

2 How can a plane quadrilateral framework with free joints be made rigid? How many facts must be given to enable you to draw a quadrilateral?

3. What is a regular polygon? What remark can you make about a regular polygon of 100 sides?

4 Quote in full two theorems in the enunciations of which the phrase "corresponding angles" occurs

5 In the given figure $AC = CB$, $CD = CE$ and $\angle A = 50^\circ$. Find the remaining angles, stating your reasons



†6. AB and CD are two parallel straight lines and a straight line is drawn to cut them in E and F. If B and D are both on the same side of EF, prove that the bisectors of $\angle BEF$ and $\angle DFE$ are at right angles to one another

7 Draw a triangle ABC having $AB = 3.64$ in., $AC = 2.48$ in., and $\angle ABC = 37^\circ$. Measure BC. Must all the triangles drawn to these measurements be alike?

8 Two points of land, A, B, on the shore are 2.8 miles apart, A being S. 71° W. of B. A ship at sea observes A to bear N. 17° E., and B to bear N. 42° E. Find the distance of the ship from A and from B

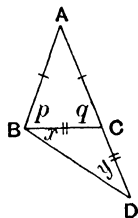
If the ship's course is N. 50° E., how near will she pass to B?

Paper 6.

- 1 What is the meaning of "subtend"?
- 2 Give the different sets of data which suffice to fix a triangle (except in position)
- 3 How many sides has a regular polygon the angle of which contains 162° ? Is it possible for a regular figure to have angles of 130° ?

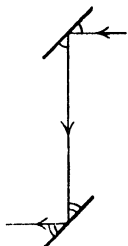
4. Draw a figure of a cuboid showing three of its faces. If you place a cuboid with one of its faces vertical, how many of its faces (i) must be vertical, (ii) might be vertical? If you place it with one of its edges horizontal, how many of its edges (i) must be horizontal, (ii) might be horizontal, (iii) must be vertical? How many diagonals has a cuboid?

5. ABC is an isosceles triangle ($AB = AC$) and AC is produced to D so that $CD = BC$



- (i) Mention two pairs of equal angles in the figure
- (ii) What relation exists between the angle q and the angles x and y ?
- (iii) Show that $\angle p = \text{twice } \angle x$
- (iv) What is the sum of the angle A and twice the angle p ?
- (v) If $\angle A = 40^\circ$, what is $\angle y$?

6 The figure shows two parallel mirrors such as are used in a trench periscope. A ray of light is shown, impinging on the upper mirror, reflected on to the lower and so out. A ray of light is reflected off a mirror so that the marked angles are equal. Show that the ray that emerges from the periscope is parallel to the ray that enters it



7 Draw a quadrilateral having one angle equal to 50° , and having each side 2 2 in long. Measure the longer diagonal

8. A tower, whose base is a circle of diameter 40 feet, is surmounted by a spire. The tip of the shadow of the spire is 33 feet from the nearest point of the base, and the altitude of the sun is found to be 60° . Draw a sketch to scale, and measure to the nearest foot the height of the top of the spire above the ground

CHAPTER 8*

CONGRUENT TRIANGLES, THE ISOSCELES TRIANGLE

CONGRUENT TRIANGLES.

§ I. If two figures when applied to one another can be made to **coincide** (i.e. fit exactly) they must be equal in all respects

Figures which are equal in all respects are said to be **congruent**.

The sign $\equiv$ is used to denote that figures are congruent

You have learnt that if you copy two sides and the contained (or included) angle of a triangle, you make a triangle congruent with the first. This fact may be stated as follows —

THEOREM 10 If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles contained by those sides equal, the triangles are congruent.

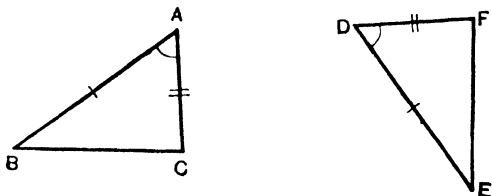


Fig 97

N B It must be noted carefully that the congruence of the triangles cannot be inferred unless the equal angles are the angles contained by the sides which are given equal

• **Ex. 1** Make a list of all the equal sides and angles in $\triangle$ s ABC and DEF of fig 97. Which were given equal?

¶ **Ex. 2.** Draw (frechand) two triangles PQR, XYZ having $QR = XY$, $RP = YZ$, and $\angle Q = \angle Z$. Would this prove the triangles congruent? Give two reasons

† **Ex. 3** D is the mid-point of the side BC of a $\triangle$ ABC, AD is produced to E so that $DE = AD$. Prove that $AB = EC$ and that AB, EC are parallel.

[First prove a pair of triangles congruent]

* This is a chapter of deductive geometry, with very young pupils it might well be postponed till further acquaintance with less formal work has been gained from the following chapters.

†Ex. 4 The bisector of the angle between the equal sides of an isosceles triangle is perpendicular to the base*.

†Ex. 5 W is the mid-point of a straight line YZ , WX is drawn at right angles to YZ . Prove that $XY=XZ$.

†Ex. 6 $ABCD$ is a square, E is the mid-point of AB , join CE and DE . Prove that $\triangle AED \cong \triangle BEC$.

†Ex. 7 Show that the distance between G and H , the opposite corners of a house, can be found as follows. At a point P set up a post, step off HP and an equal distance PN , taking care to keep in a straight line with the post and the corner H , step off GP and an equal distance PM , M being in the same straight line as G and P . Measure MN , this must be equal to GH .

Draw a ground plan and prove that $MN=GH$.

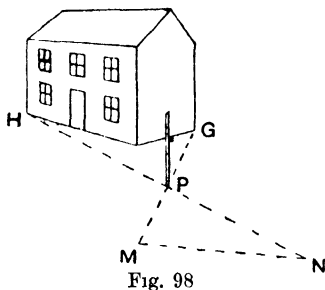


Fig. 98

§2 You have learnt that if you copy two angles and a side (not necessarily the side between the angles) of a triangle, you make a triangle congruent with the first. This fact may be stated as follows —

THEOREM 11. If two triangles have two angles of the one equal to two angles of the other, each to each, and also one side of the one equal to the corresponding side of the other, the triangles are congruent.

†Ex. 8 Draw two $\triangle$ s GHK , XYZ , and mark $GH=XY$, $\angle H=\angle Y$, and $\angle K=\angle X$, are the triangles congruent?

†Ex. 9. The following method may be used to find the breadth of a river. Choose a place where the river is straight, note some conspicuous object T (e.g. a tree) on the edge of the other bank, from a point O opposite T measure a distance OS along the bank, put a stick in the ground at S , walk straight on to a point P such that $SP=OS$; from P walk at right angles to the river till you are in the same straight line as S and T . PQ is equal to the breadth of the river. Prove this.

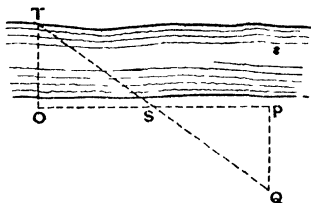


Fig. 99

* This means that the truth of the statement is to be proved. Questions are often set in this form.

†Ex. 10. If the bisector of an angle of a triangle cuts the opposite side at right angles, the triangle must be isosceles.

†Ex 11 P is a point on the bisector of an angle, and PM and PN are perpendiculars to the arms. Prove $PM = PN$

§3 You have learnt that if you copy three sides of a triangle, you make a triangle congruent with the first. This fact may be stated as follows —

THEOREM 14 If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles are congruent.

†Ex 12 If, in a quadrilateral ABCD, $AB = AD$, $CB = CD$, prove that AC bisects $\angle A$ and $\angle C$

†Ex 13 PQ and RS are two equal chords of a circle whose centre is O. Prove that $\angle POQ = \angle ROS$

†Ex. 14. P is a point equidistant from A and B. M is the middle point of AB. Prove that PM is perpendicular to AB.

†Ex 15 Prove that, if the opposite sides of a quadrilateral are equal, they are parallel

§4 You have learnt that if you copy the hypotenuse and one other side of a right-angled triangle you make a triangle congruent with the given triangle. This fact may be stated as follows —

THEOREM 15. If two right-angled triangles have their hypotenuses equal, and one side of the one equal to one side of the other, the triangles are congruent.

•†Ex 16 In a triangle XYZ, $XY = XZ$, and XW is drawn at right angles to YZ. prove that $\triangle XYW \equiv \triangle XZW$

†Ex 17. A and B are two points on a circle and C is the centre. Prove that the perpendicular from C to AB bisects AB

†Ex. 18 P is a point equidistant from the arms of an angle AOB. Prove that PO bisects the angle

†Ex. 19. ABCD is a quadrilateral with $AB = AD$ and the angles at B and D right angles. Prove that $CD = CB$.

§5 We will now *prove* the theorems discussed on pp. 47, 48.

THEOREM 12. **If two sides of a triangle are equal, the angles opposite to these sides are equal.**

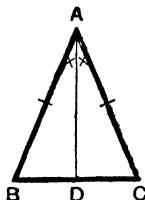


Fig 100.

Data ABC is a triangle which has $AB = AC$

To prove that $\angle C = \angle B$

Construction Draw AD to represent the bisector of $\angle BAC$

Let it cut BC at D

<i>Proof</i>	In the $\triangle$ s ABD, ACD	$\begin{cases} AB = AC, \\ AD \text{ is common,} \\ \angle BAD = \angle CAD, \end{cases}$	<i>Data</i>
	$\therefore \triangle ABD \equiv \triangle ACD,$		<i>Constr</i>
	$\angle B = \angle C$		<i>Th 10</i>

THEOREM 13 **If two angles of a triangle are equal, the sides opposite to these angles are equal.**

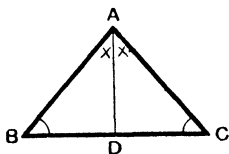


Fig 101

Data ABC is a triangle which has $\angle B = \angle C$.

To prove that $AC = AB$.

Construction Draw AD to represent the bisector of $\angle BAC$

Let it cut BC at D.

<i>Proof</i>	In the $\triangle$ s ABD, ACD	$\begin{cases} \angle B = \angle C, \\ \angle BAD = \angle CAD, \\ AD \text{ is common,} \end{cases}$	<i>Data.</i>
	$\triangle ABD \equiv \triangle ACD,$		<i>Constr.</i>
	$\therefore AB = AC.$		<i>Th. 11</i>

NOTE ON A THEOREM AND ITS CONVERSE

The enunciation of a theorem can generally be divided into two parts (i) the data, or hypothesis, (ii) the conclusion

For instance in Th 12 the hypothesis is that $AB = AC$, the conclusion is that $\angle B = \angle C$

If the hypothesis and conclusion are interchanged a second theorem is obtained, which is called the **converse** of the first theorem

Thus the converse of Th 12 asserts that, if $\angle B = \angle C$, then $AB = AC$ This is Th 13

It must not be assumed that the converses of all true theorems are true e.g. "if two angles are vertically opposite, they are equal" is a true theorem, but the converse "if two angles are equal, they are vertically opposite" is not a true theorem

¶Ex. 20 State the converses of the following are they true?

- (i) If a triangle has one of its angles a right angle, two of its angles are acute
- (ii) If a figure has four sides, the angles make up 4 rt $\angle$ s
- (iii) London Bridge is a stone bridge
- (iv) A nigger is a man with woolly hair
- (v) If I play football, I get hot.

RIDERS ON THE ISOSCELES TRIANGLE.

†Ex. 21 Prove that if a triangle PQR is equiangular, it must also be equilateral. [$\angle Q = \angle R$, side ? = side ?] State and prove the converse

¶Ex 22 Through the vertex P of an isosceles triangle PQR a straight line XPY is drawn parallel to QR, prove that $\angle QPX = \angle RPY$

Ex 23 In fig 102 there is an isosceles triangle with a line drawn parallel to the base Assuming the vertical angle to be 40° , find all the other angles in the figure Make a freehand sketch showing all the angles

†Ex 24 Prove that a straight line drawn parallel to the base of an isosceles triangle to cut the equal sides forms another isosceles triangle (See fig. 102.)



Fig 102

†Ex. 25. XYZ is an isosceles triangle, the bisectors of the equal angles (Y, Z) meet at O, prove that $\triangle OYZ$ is also isosceles.

†Ex 26 From Q and R, the extremities of the base of an isosceles triangle PQR, perpendiculars are drawn to the opposite sides. If these perpendiculars intersect at X, prove that $XQ = XR$

†Ex 27 From the mid-point O of a straight line AB a straight line OC is drawn; if $OC = OA$, prove that $\angle ACB$ is a right angle (See fig 103)

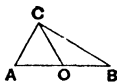


Fig 103

†Ex 28. If through any point in the bisector of an angle a line is drawn parallel to either of the arms of the angle, prove that the triangle thus formed is isosceles (See fig 104.)



Fig. 104

†Ex 29 Prove that the angle between a diagonal and a side of a square is 45°

†Ex 30 OA, OB are radii of a circle, AO is produced to P, prove that $\angle BOP = 2\angle BAP$ (See fig 105)

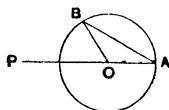


Fig 105

†Ex 31 If the $\angle$ s G, H of a triangle FGH are each double of $\angle$ F, and if the bisector of $\angle$ G meets FH in K, prove that $FK = GK$ and $GK = GH$

†Ex 32 ABC is a triangle, BA is produced to X so that $AX = AC$, CA is produced to Y so that $AY = AB$. Prove that BY is parallel to CX.

RIDERS ON CONGRUENCE.

Ex 33 ABC and DEF are two triangles. If the following facts hold, are the triangles congruent (give reasons for your answers) —

- (i) $AB = DE$, $AC = DF$, $\angle A = \angle D$,
- (ii) $AB = DF$, $AC = DE$, $\angle A = \angle D$,
- (iii) $AB = EF$, $AC = DF$, $\angle A = \angle F$,
- (iv) $BC = DE$, $\angle A = \angle F$, $\angle B = \angle D$,
- (v) $BC = DE$, $\angle A = \angle E$, $\angle B = \angle F$?

†Ex 34 ABCD is a quadrilateral with its opposite sides parallel (i.e. a parallelogram). Prove that —

- (i) $AB = CD$, $AD = BC$
- (ii) $\angle A = \angle C$ and $\angle B = \angle D$
- (iii) $AO = OC$, $BO = OD$, where O is the point of intersection of the diagonals

†Ex. 35 In fig 57, AB and CD are equal and parallel, prove that $AC = BD$

†Ex 36 Show that, if the diagonals of a quadrilateral bisect one another, opposite sides are parallel

†Ex. 37. The equal sides QP, RP of an isosceles triangle PQR are produced to S, T so that $PS = RT$; prove that $TQ = SR$.

†Ex. 38 The equal sides AB, AC of an isosceles triangle ABC are produced to X, Y so that $AX=AY$. Prove that $CX=BY$.

†Ex. 39. AB is equal and parallel to CD (see fig. 57). Show that AD bisects and is bisected by BC .

†Ex. 40 GHK is an equilateral triangle, in GH, HK, KG take points X, Y, Z so that $GX=HY=KZ$. Prove that the triangle XYZ is equilateral.

†Ex. 41. If the diagonal PR of a quadrilateral $PQRS$ bisects the angles at P and R , prove that the quadrilateral has two pairs of equal sides.

†Ex. 42 A triangle ABC has $AB=AC$, prove that the perpendiculars from B and C on the opposite sides are equal to one another.

†Ex. 43 In fig. 106, PM, QN are drawn perpendicular to the diameter AOB , O being the centre of the circle, show that, if $PM=QN$, then $\angle POM=\angle QON$.

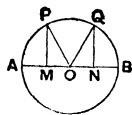


Fig 106.

†Ex. 44 A triangle XYZ has $\angle Y=\angle Z$; prove that the perpendiculars from the mid-point of YZ to XY and XZ are equal to one another.

†Ex. 45 If the perpendiculars from two corners of a triangle to the opposite sides are equal, the triangle is isosceles.

†Ex. 46 $ABCD$ is a square, E, F, G are the mid-points of AB, BC, CD respectively. Join EF and FG and prove them equal.

†Ex. 47 In fig. 107, OX, OY are a pair of axes at right angles to one another and $OPQR$ is a square. Prove that the distance of P from OY is equal to the distance of R from OX .

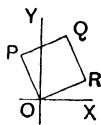


Fig 107

†Ex. 48 From a point O in a straight line XOX' two equal straight lines OP, OQ are drawn so that $\angle POQ$ is a right angle. PM and QN are drawn perpendicular to XX' . Prove that $PM=QN$.

†Ex. 49 On the sides AB, BC of a triangle ABC squares $ABFG, BCED$ are described, both outside the triangle. Prove that $\triangle CBF \cong \triangle DBA$.

†Ex. 50 Two squares $OABC, OPQR$ of different sizes have a common vertex O , the letters running round the squares in the same direction. Show that $AP=CR$.

†Ex. 51 ABC is an equilateral triangle and D is any point in AB , on the side of AD remote from C an equilateral triangle ADE is described, prove that $BE=CD$.

†Ex. 52. On the sides of any triangle ABC , equilateral triangles BCD, CAE, ABF are described (all pointing outwards), prove that AD, BE, CF are all equal [Prove $\triangle s ACD, ECB$ congruent].

†Ex. 53 In a quadrilateral $ABCD$, $AB=CD$ and $\angle B=\angle C$, prove that AD is parallel to BC .

†Ex. 54. Two triangles ABC, DCB stand on the same base BC and on the same side of it, prove that AD is parallel to BC if $AB=DC$ and $AC=DB$.

†Ex. 53 ABCD is a square, AC, BD meet at O. A circle is drawn with centre A and radius AO cutting AB at P and AD at Q. Prove that $PQ = AB$, by proving two triangles with PQ and AB as bases to be congruent

[You may assume that AO, BO, CO, DO are all equal and that the angles at O are right angles]

†Ex. 56 Fig. 108 shows two equal set squares. Prove that
(i) $CX = BY$, (ii) XB is parallel to CY

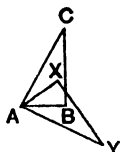


Fig 108

SUMMARY OF RESULTS

If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles contained by those sides equal, the triangles are congruent

If two triangles have two angles of the one equal to two angles of the other, each to each, and also one side of the one equal to the corresponding side of the other, the triangles are congruent

If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles are congruent.

If two right-angled triangles have their hypotenuses equal and one side of the one equal to one side of the other, the triangles are congruent

If two sides of a triangle are equal, the angles opposite to these sides are equal

If two angles of a triangle are equal, the sides opposite to these angles are equal.

The above may be put out in tabular form.

Sides	Angles	Are the triangles congruent?
3	0	Yes
2	1	Yes, if the angle is the included angle
1	2	Yes
0	3	No

Besides the above cases, there is the special case of right-angled triangles.

CHAPTER 9

THE PARALLELOGRAM

§ 1 DEF. A **parallelogram** is a four-sided figure with its opposite sides parallel.

Parallelograms are most easily drawn by drawing two pairs of parallel lines with a set square sliding along a ruler

¶Ex 1 Draw any parallelogram. What facts would you expect to find true about the sides? the angles? Test by measuring your figure

§ 2.

THEOREM 20.

(i) **The opposite sides of a parallelogram are equal.**

(ii) **The opposite angles of a parallelogram are equal.**

(iii) **Each diagonal bisects the parallelogram.**

Data ABCD is a parallelogram, and BD
one of its diagonals

To prove that (i) $AB = CD$, $AD = CB$,

(ii) $\angle A = \angle C$, $\angle B = \angle D$,

(iii) BD bisects the parallelogram.

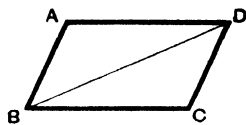


Fig 109.

Proof [$\triangle$ s ABD, CDB must be proved congruent]

Since AD is $\parallel$ to BC and BD meets them,

$\angle ADB = \text{alt } \angle CBD$

Th 5.

Since AB is $\parallel$ to CD and BD meets them,

$\angle ABD = \text{alt } \angle CDB$

Th 5

in $\triangle$ s ABD, CDB $\left\{ \begin{array}{l} \angle ADB = \angle CBD, \\ \angle ABD = \angle CDB, \\ \text{BD is common,} \end{array} \right.$

$\triangle ABD \equiv \triangle CDB$,

Th. 11

$AB = CD$, $AD = CB$

Also since

$\triangle ABD \equiv \triangle CDB$,

$\therefore \angle A = \angle C$.

Sim^{ly} $\angle B = \angle D$ (by joining AC)

And since $\triangle ABD \equiv \triangle CDB$, BD bisects the parallelogram

Sim^{ly} AC bisects the parallelogram.

(iv) **The diagonals of a parallelogram bisect one another.**

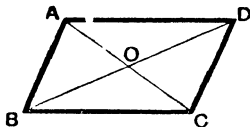


Fig 110

Data ABCD is a parallelogram, its diagonals AC, BD intersect at O.

To prove that $OA = OC$ and $OD = OB$

Proof [$\triangle s$ OAD, OCB must be proved congruent]

Since AD is $\parallel$ to BC and BD cuts them,

$$\angle ADO = \text{alt. } \angle CBO$$

Th. 5.

$$\therefore \text{ in } \triangle s \text{ OAD, OCB } \begin{cases} \angle ADO = \angle CBO, \\ \angle AOD = \text{vert. opp } \angle COB, \\ AD = CB, \end{cases}$$

Th. 20 (i)

$$\triangle OAD \equiv \triangle OCB,$$

Th. 11.

$$\therefore OA = OC \text{ and } OD = OB$$

COR 1 If a parallelogram has one of its angles a right angle, all its angles must be right angles.

COR 2 If one pair of adjacent sides of a parallelogram are equal, all its sides are equal.

†**Ex 2** Prove that opposite angles of a parallelogram are equal by considering the sums of the angles at A and B, and at B and C

THEOREM 21 [CONVERSES OF THEOREM 20] (1) A quadrilateral is a parallelogram if both pairs of opposite angles are equal.

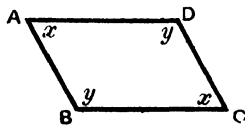


Fig. 111.

This fact also follows by adding the two lines marked *Th. 5.*

Data ABCD is a quadrilateral in which

$$\angle A = \angle C = \angle x \text{ (say) and } \angle B = \angle D = \angle y \text{ (say)}$$

To prove that ABCD is a parallelogram.

Proof The sum of the angles of a quadrilateral is equal to 4 rt $\angle$ s, *Ex 31, p 35*

$$\therefore 2\angle x + 2\angle y = 4 \text{ rt. } \angle \text{s,}$$

$$\angle x + \angle y = 2 \text{ rt. } \angle \text{s,}$$

$$\angle A + \angle B = 2 \text{ rt. } \angle \text{s,}$$

$$AD \text{ is } \parallel \text{ to } BC \quad \text{Th 4}$$

$$\text{Also } \angle A + \angle D = 2 \text{ rt. } \angle \text{s,}$$

$$AB \text{ is } \parallel \text{ to } DC, \quad \text{Th 4}$$

$$ABCD \text{ is a } \parallel^{\text{ogram}}$$

(ii) **A quadrilateral is a parallelogram if both pairs of opposite sides are equal.**

(Draw a diagonal and prove the two triangles congruent)

(iii) **A quadrilateral is a parallelogram if one pair of opposite sides are equal and parallel.**

(Draw a diagonal and prove the two triangles congruent)

(iv) **A quadrilateral is a parallelogram if its diagonals bisect one another.**

(Prove two opposite triangles congruent)

COR. If equal perpendiculars are erected on the same side of a straight line, the straight line joining their extremities is parallel to the given line.

†**Ex. 3.** Write out the proofs of (ii), (iii), (iv) and Cor. in the above theorem

¶**Ex 4** Of which part of Th 20 is the converse untrue?

NOTE ON DEFINITIONS

§3. Many definitions are mere explanations of terms and have no logical importance, e.g. median, chord

In other cases, a figure is defined by selecting certain of its properties, and it is necessary to prove the existence of other properties. The best example of this is the parallelogram

By definition a quadrilateral with its opposite sides parallel is called a parallelogram.

You are familiar with other properties of a parallelogram, e.g. that opposite sides are equal, these are not enumerated in the definition, for a definition should not contain anything superfluous or redundant, these further properties can be deduced from the above sufficient definition.

It would be quite reasonable to select another set of properties of a parallelogram for purpose of definition, e.g. we might have defined a parallelogram as a quadrilateral with one pair of opposite sides equal and parallel, or as a plane quadrilateral with opposite sides equal. But for a logical sequence it is necessary to choose one definition and adhere to it.

Some parallelograms have special properties which lead to figures to which special names are given. For example from Theorem 20, Cor. 2, a parallelogram with one pair of adjacent sides equal has all its sides equal, such a figure is called a *rhombus*. Also from Cor. 1, a parallelogram with one angle a right angle has four equal angles each right angles, such a parallelogram is called a *rectangle*. If in addition one pair of adjacent sides are equal it becomes a *square*.

But these explanations of a rhombus, rectangle and square are not necessarily the only definitions.

It must be noticed that all facts true for any parallelogram are true for these special cases, but owing to their additional properties certain things are true for them which are not true for every parallelogram.

DEF. A **Rhombus** is a parallelogram with one pair of adjacent sides equal (and therefore all its sides equal).

DEF. A **Rectangle** is a parallelogram with one angle a right angle (and therefore all its angles right angles).

DEF. A **Square** is a rectangle with one pair of adjacent sides equal (and therefore all its sides equal and all its angles right angles).

* Ex. 5. Prove that in a **Rhombus** the diagonals (i) are at right angles and (ii) bisect the angles of the rhombus.

* Ex. 6. Prove that in a **Rectangle** the diagonals are equal.

* Ex. 7. Prove that in a **Square** the diagonals (i) are at right angles, (ii) are equal, (iii) bisect the angles of the square.

* All these properties are suggested by Symmetry. See Ch 11, p 84

Ex 8 Copy the table given below, indicate for which figures the given properties are always true by inserting the words "yes" or "no" in the corresponding spaces

	Opposite sides and angles equal	Diagonals bisect one another	Angles at corners right angles	Diagonals equal	Diagonals at right angles	Adjacent sides equal	Diagonals bisect angles
Parallelogram							
Rectangle							
Square							
Rhombus							

†Ex 9. Fold a piece of paper and cut out two equal triangles by cutting through the double paper. Fit the triangles together so that they form a figure that looks like a parallelogram. Prove that it is a parallelogram.

RIDERS ON PARALLELOGRAMS.

†Ex 10. PQRS is a parallelogram, X is the mid-point of PQ, RX and SP are produced to meet at Y. Prove that $PY = PS$.

†Ex 11 ABCD is a parallelogram and O is the mid-point of BD, prove that O is also the mid-point of XY, where XY is any line through O meeting opposite sides in X and Y.

†Ex 12 The bisectors of two adjacent angles of a parallelogram are at right angles to one another.

†Ex 13 The bisectors of two opposite angles of a parallelogram are parallel.

†Ex 14. ABCD is a parallelogram, AB, CD are bisected at X, Y respectively, prove that BXDY is a parallelogram.

†Ex 15 If the diagonals of a parallelogram are equal, it must be a rectangle.

†Ex 16. In the diagonal AC of a parallelogram ABCD points P, Q are taken such that $AP = CQ$, prove that BPDQ is a parallelogram.

†Ex 17 ABCD, ABXY are two parallelograms on the same base and on the same side of it. Prove that CDYX is a parallelogram.

†Ex 18 E, F, G, H are points in the sides AB, BC, CD, DA respectively of a parallelogram ABCD, such that $AH = CF$ and $AE = CG$, show that EFGH is a parallelogram.

†Ex 19. Draw a parallelogram ABCD and a rectangle ABPQ such that CD and PQ are in the same line. Prove $\triangle ADQ \equiv \triangle BCP$.

†Ex. 20 Draw two parallelograms ABCD and ABPQ with CD and PQ in the same line. Prove $\triangle ADQ \equiv \triangle BCP$.

†Ex 21 What figure is formed by joining the mid-points of the sides of a rectangle?

†Ex 22 ABCD is a square, from A lines are drawn to the mid-points of BC, CD, from C lines are drawn to the mid-points of DA, AB. What figure is enclosed by these lines? Give a proof.

†Ex 23 In a triangle ABC, BE and CF are drawn to cut the opposite sides in E and F, prove that BE and CF cannot bisect one another.

THE MID-POINT THEOREMS.

§ 4. THEOREM 22 The straight line drawn through the middle point of one side of a triangle parallel to another side bisects the third side.

Data ABC is a triangle

Through D, the mid-pt. of AB,

DE is drawn $\parallel$ to BC to cut AC at E.

To prove that $AE = EC$

Construction Through E draw $EF \parallel$ to AB to cut BC at F.

Proof $EF = DB$ (DEFB is a $\parallel$ ogram by constr.)

$= AD$

Data

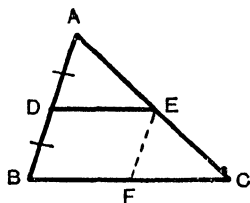


Fig. 112

In the $\triangle$ s ADE, EFC $\left\{ \begin{array}{l} AD = EF, \\ \angle AED = \text{cor. } \angle ECF \text{ (DE } \parallel \text{ to BC), } \text{Th. 5} \\ \angle EAD = \text{cor. } \angle CEF \text{ (EF } \parallel \text{ to AB), } \text{Th. 5.} \end{array} \right.$

$\therefore \triangle ADE \equiv \triangle EFC,$ Th. 11.

$AE = EC$

THEOREM 23. The straight line joining the middle points of two sides of a triangle is parallel to the third side and equal to half of it.

Data ABC is a triangle D, E are the mid-pts. of AB, AC respectively

To prove that DE is $\parallel$ to BC, and $DE = \frac{1}{2}BC$

Construction Through C, draw CF $\parallel$ to BA to cut DE produced at F.

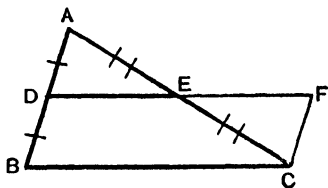


Fig. 113

Proof In the $\angle$ s ADE, CFE $\left\{ \begin{array}{l} AE = EC, \\ \angle AED = \text{vert. opp. } \angle CEF, \\ \angle ADE = \text{alt } \angle CFE \text{ (CF } \parallel \text{ to BA)}, \text{ Th. 5} \\ \therefore \triangle ADE \equiv \triangle CFE, \\ \therefore AD = CF \end{array} \right. \begin{array}{l} \text{Data} \\ \\ \\ \text{Th. 11.} \end{array}$

But $AD = DB$ (*data*),

$CF = BD$

Now CF is equal and $\parallel$ to DB,

DBCF is a $\parallel$ ogram,

Th 21

DE is $\parallel$ to BC.

Also $DE = EF$ (from the congruent $\triangle$ s).

$CE = \frac{1}{2}DF = \frac{1}{2}BC$ (DF, BC are opp sides of $\parallel$ ogram DBCF).

¶Ex 24. Show that the triangle formed by joining the mid-points of the sides of any triangle is equiangular with the given triangle. How do corresponding sides compare?

Ex 25. A step ladder rests on the ground with its feet 3' 6" apart. What is the length of the ropes joining the mid-points of the legs?

†Ex 26 AB is a line and C its middle point p, q are the lengths of the perpendiculars from A and B on to any other line. Prove that the perpendicular from C equals $\frac{1}{2}(p + q)$.

[Join A to foot of the perpendicular from B.]

Ex. 27. The back and front of a lean-to shed are 10 ft and 7' 6" high respectively. What is the height of a point in the middle of the roof?

†Ex 28 X, Y, Z are the mid-points of the sides BC, CA, AB of a $\triangle ABC$. Prove that AZXY is a parallelogram.

†Ex. 29 The straight lines joining the mid-points of the sides of a triangle divide it into four congruent triangles.

†Ex 30 Show how to construct a triangle, given the three mid-points of its sides. Give a proof

†Ex 31 If $AD = \frac{1}{2}AB$ and $AE = \frac{1}{2}AC$, prove that DE is parallel to BC and equal to a quarter of BC .

Ex 32 PQR is a triangle right-angled at P ; through O , the mid-point of QR , parallels OX, OY are drawn to PQ, PR to meet the sides in X, Y . Prove that $XY = OP = OQ = OR$

†Ex. 33. If the mid-points of the adjacent sides of a quadrilateral are joined, the figure thus formed is a parallelogram

[Draw a diagonal of the quadrilateral]

†Ex 34 The straight lines joining the mid-points of opposite sides of a quadrilateral bisect one another.

§ 5 Construction of Parallelograms (including the Rhombus, Rectangle and Square) to given measurements.

To construct a parallelogram with sides 3 in. and 2 in. including an angle of 50° .

Make $AB = 3$ in, $\angle BAD = 50^\circ$, $AD = 2$ in.

Through D by sliding a set square along a ruler (see p 28) draw DC parallel to AB Similarly through B draw BC parallel to AD

Alternative method.

With centre B and radius 2 in draw an arc

With centre D and radius 3 in draw an arc

The point of intersection of these arcs gives the fourth point C .

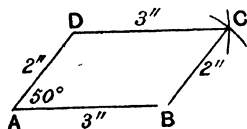


Fig 114

Ex. 35 (i) Draw a rectangle with sides 3 in. and 4 in.

Draw Parallelograms given

- (ii) sides of 5 cm. and 4 cm, included angle 50° . Measure both diagonals,
- (iii) sides of 3.5 in and 1 in., included angle 110° . Measure both diagonals,
- (iv) sides 3 in, 2 in, shorter diagonal 2.5 in. Measure the longer diagonal.

Draw Rhombuses given

- (v) sides 6 cm, angle 70° ,
- (vi) sides 2.75 in., diagonal 3 in.

Draw Squares given

- (vii) sides 2 in Measure the diagonals,
- (viii) diagonals 3 in. Measure the sides.

In Exx. 36-43 use ruler and compasses.

Ex 36. On a given straight line construct a rhombus having, one of its angles = 60°

Ex 37. Construct a parallelogram having given two sides 8 cm. and 5 cm and a diagonal 10 cm. Measure the angles.

Ex 38. Construct a rectangle given one side 3 in and a diagonal 5 in Measure the other side

Ex 39 Construct a parallelogram given the diagonals 3 in. and 4 in. and the angle between them 45° Measure the sides.

Ex. 40 Construct a rhombus given the lengths of the diagonals 10 cm and 8 cm. Measure the sides

Ex. 41 On a base 3 in long construct a parallelogram of height 1 2 in with an angle of 55° Measure the other side

Ex 42 Construct a rhombus each of whose sides is 7.3 cm., the distance between a pair of opposite sides being 5.6 cm. Measure its acute angle.

Ex 43. Draw a quadrilateral having one angle equal to 50° , and having each side 2 2 in long Measure the longer diagonal.

SUMMARY OF RESULTS

A parallelogram is a quadrilateral with its opposite sides parallel.

In a parallelogram

- (i) *opposite sides are equal,*
- (ii) *opposite angles are equal,*
- (iii) *a diagonal bisects the figure,*
- (iv) *the diagonals bisect each other.*

A quadrilateral is a parallelogram if

- (i) *both pairs of opposite angles are equal,*
- (ii) *both pairs of opposite sides are equal,*
- (iii) *one pair of opposite sides are equal and parallel,*
- (iv) *its diagonals bisect each other*

A rhombus is a parallelogram with one pair of adjacent sides equal.

In a rhombus, all sides are equal, the diagonals intersect at right angles and bisect the angles of the rhombus

A rectangle is a parallelogram with one angle a right angle (and therefore all its angles are right angles).

• *A square is a rectangle with one pair of adjacent sides equal (and therefore all its sides equal and all its angles right angles).*

CHAPTER 10

RULER AND COMPASS CONSTRUCTIONS

§ I. Euclid's attitude with regard to constructions was as follows. He asked to be allowed the use of two instruments only (1) a straight edge, *ungraduated*, (11) a pair of compasses which shut up as soon as they were lifted from the paper, they were not to be used for transferring distances

Hitherto we have constructed our figures mainly with the help of graduated instruments and set squares. We shall now make certain constructions with the aid of nothing but a straight edge (supposed not graduated) and a pair of compasses, but we shall use the compasses for transferring distances. These methods are usually more accurate than those that require graduated instruments, for there is always a chance of a small error in making and transferring a measurement. Constructions made with these restrictions are sometimes called Euclidean Constructions

When you are asked to **construct** a certain figure, this modified Euclidean method is to be used and no instruments except ruler and compass. When the question is "to **draw**," set square and protractor may be used. (This distinction is not always observed.)

In the exercises, when you are asked to construct a figure, you should generally explain your construction in words.

In figs 115-119 the given lines are drawn thick but you should not thicken the lines in your drawings

In making constructions, only the necessary parts of construction circles should be drawn even though "the circle" is spoken of

None of the construction lines should be rubbed out.

§ 2 Through a point O in a straight line OX to draw a straight line OY so that angle XOY may be equal to a given angle BAC .

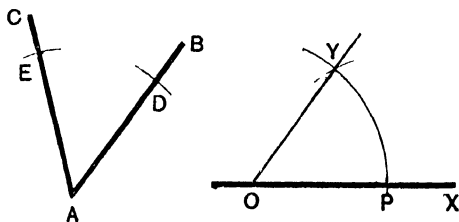


Fig 115

Construction With centre A and any radius describe a circle cutting AB , AC at D , E respectively

With centre O and the same radius describe a circle PY cutting OX at P .

With centre P and radius $= DE$ describe a circle cutting the circle PY at Y . Join OY

Then $\angle XOY = \angle BAC$.

Proof Join ED , PY (Show the Δ s congruent)

The only instruments to be used in Exs 1-6 are a ruler and a pair of compasses

Ex 1. Draw an acute angle and construct an equal angle.

Ex 2. Draw an obtuse angle and construct a copy of it

Ex 3 Draw a triangle ABC , at a point O construct angles in the manner of fig 61.

Ex 4 Draw a straight line EF and mark a point G (about 2 in. from the line), through G construct a line parallel to EF [Draw any line through G cutting EF at H , make $\angle HGC = \angle GHF$, see fig 43]

Ex 5. Draw a triangle ABC with $AB = 3$ in., $BC = 4$ in., $CA = 2$ in. Construct an equiangular triangle $A'B'C'$ with $B'C' = 3$ in. Measure $C'A'$ and $A'B'$.

Ex 6 Draw a large pentagon Construct an equiangular pentagon with its sides half as long Measure a pair of corresponding diagonals in each.

§3. To bisect a given angle.

Let $\angle BAC$ be the given angle

Construction From AB , AC cut off equal lengths AD , AE .

With centres D and E and any convenient radius describe equal circles intersecting at F . Join AF .

Then AF bisects $\angle BAC$.

Proof Join DF and EF .

In the $\triangle s$ ADF , AEF ,

$$\begin{cases} AD = AE \text{ (Constr.)}, \\ DF = EF \text{ (radii of equal circles)} \\ \text{and } AF \text{ is common.} \end{cases}$$

$\triangle ADF \equiv \triangle AEF$ and $\angle DAF = \angle EAF$, *Th 14.*

AF bisects $\angle BAC$

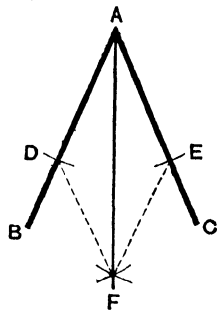


Fig 116

"Any convenient radius" If it is found that the equal circles do not intersect, the radius chosen is not convenient, for the construction breaks down; it is necessary to take a larger radius so that the circles may intersect

Ex 7 Draw two lines AOA' , BOB' intersecting in O . Bisect the four angles by lines OP , OQ , OR , OS (OP , OR in vertically opposite angles) What fact is suggested about (i) OP and OR , (ii) OP and OQ ?

Ex. 8 Draw a large triangle and bisect each of its angles. What fact is suggested?

Ex. 9. Draw a triangle ABC and produce the sides AB and AC . Bisect the $\angle A$ and the exterior angles at B and C (draw long lines) What fact is suggested?

Ex 10 Draw two parallel lines and a line crossing them. Bisect all four interior angles. What figure is formed by the bisectors?

§4 To draw the perpendicular bisector of a given straight line.

To bisect a given straight line.

Let AB be the given straight line

Construction With centres A and B and any convenient radius describe equal circles intersecting at C and D .

Join CD and let it cut AB at E

Then CD is the perpendicular bisector of AB , and E is the mid-point of AB

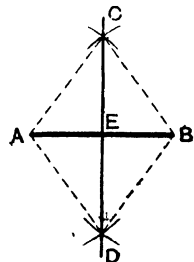


Fig. 117.

Proof Join AC, AD, BC, BD.

In the $\triangle$ s ACD, BCD $\begin{cases} AC = BC \text{ (radii of equal circles),} \\ AD = BD \text{ (radii of equal circles),} \\ CD \text{ is common,} \end{cases}$

$\triangle ACD \equiv \triangle BCD$, *Th 14*

$\angle ACD = \angle BCD$

In the $\triangle$ s ACE, BCE $\begin{cases} AC = BC \text{ (radii of equal circles),} \\ CE \text{ is common,} \\ \angle ACE = \angle BCE, \end{cases}$

$\triangle ACE \equiv \triangle BCE$, *Proved*
Th 10

$AE = BE$,

and $\angle$ s CEA, CEB are equal and are therefore rt. $\angle$ s,

CD bisects AB at right angles

Ex 11 Draw a large acute-angled triangle, draw the perpendicular bisectors of its three sides. What fact is suggested?

Ex. 12 Repeat Ex 11 for (i) a right-angled, (ii) an obtuse-angled triangle.

Ex. 13 Draw any three chords of a circle and the perpendicular bisector of each. What fact is suggested?

Ex 14 Describe a circle and on it take three points A, B, C, join BC, CA, AB. Bisect angle BAC and draw the perpendicular bisector of BC. Produce the two bisectors to meet

§ 5 **Ex 15** Draw a large obtuse angle (very nearly 180°) and bisect it

To draw a straight line perpendicular to a given straight line AB from a given point P in AB.

Construction From PA, PB cut off equal lengths PC, PD

With centres C and D and any convenient radius describe equal circles intersecting at E. Join PE.

Then PE is $\perp$ to AB.

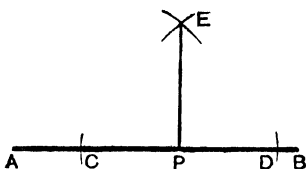


Fig 118

Proof (Show $\triangle$ s ECP $\equiv$ $\triangle$ EDP.)

Ex. 16 Draw a straight line, and construct a straight line at right angles to it. Test with set square.

Ex. 17 Construct a rectangle with sides 8 cm. and 5 cm.

Ex. 18 Construct a square on a side of 2.5 in

Ex. 19 AB is any line. At A erect a perpendicular AP 1.2 in long, at B erect a perpendicular BQ 2.4 in. Join PQ. At C, the mid-point of AB, erect a perpendicular to meet PQ in R. Measure CR.

Ex. 20 ABC is any triangle. Construct a perpendicular to AB from B and a perpendicular to AC from C. Let these lines meet in R. Verify by measuring that the angles BAC and BRC are equal or supplementary.

§6 To draw a straight line perpendicular to a given straight line AB from a given point P outside AB.

Construction With centre P and any convenient radius describe a circle cutting AB at X and Y.

With centres X and Y and any convenient radius describe equal circles intersecting at Q. Join PQ cutting AB at Z.

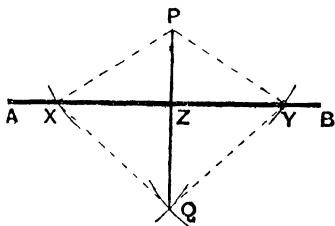


Fig 119

Then PZ is $\perp$ to AB.

Proof Join XP, XQ, YP, YQ.

In the $\triangle$ s PQX, PQY $\left\{ \begin{array}{l} PX = PY \text{ (radii of a } \odot), \\ QX = QY \text{ (radii of equal } \odot\text{s)} \\ PQ \text{ is common} \end{array} \right.$

$$\triangle PQX \equiv \triangle PQY$$

Th. 14.

$$\angle XPQ = \angle YPQ$$

In the $\triangle$ s PXZ, PYZ $\left\{ \begin{array}{l} PX = PY \text{ (radii of a } \odot), \\ PZ \text{ is common,} \\ \angle XPZ = \angle YPZ \end{array} \right.$

Proved

$$\triangle PXZ \equiv \triangle PYZ$$

Th. 10.

$$\angle PZX = \angle PZY$$

PZ is $\perp$ to AB.

Ex. 21 Draw a large acute-angled triangle, from each vertex draw a perpendicular to the opposite side. What fact is suggested?

Ex. 22. Repeat Ex. 21 with a right-angled triangle

Ex. 23 Repeat Ex. 21 with an obtuse-angled triangle

[You will have to produce two of the sides]

Ex. 24. From the centre of a circle drop a perpendicular on to a chord of the circle. What fact is suggested?

§ 7. Through a given point P to construct a straight line parallel to a given straight line AB.

Construction Through P draw a straight line PQ meeting AB in Q. With centre Q and any convenient radius draw a circle cutting QB and QP in D, E respectively. With centre P and the same radius draw a circle cutting PQ in F. With centre F and radius equal to DE draw a circle cutting the second circle in G. Join PG.

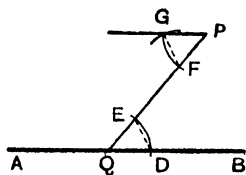


Fig 120.

Then PG is $\parallel$ to BA. Why?

Ex 25 Draw a parallel through a given point to a given straight line by making corresponding angles equal

CONSTRUCTION OF SPECIAL ANGLES

§ 8 To construct an angle of 60° .

Draw any line OX. With centre O and any radius OA draw a circle cutting OX at A.

With centre A and the same radius draw a circle. Let the circles cut at B.

The $\angle AOB$ is 60° .

[For by construction AOB is an equilateral triangle.]

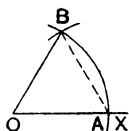


Fig. 121

To construct an angle of 30° .

Construct an angle of 60° and bisect it.

To construct an angle of 90° .

This is the same as drawing a perpendicular to a line at a point in it, see § 5.

Alternative Method

Draw a line OX. With centre O and any radius OA draw a circle cutting OX at A.

With centre A and the same radius draw an arc to cut the circle at B.

With centre B and the same radius draw an arc to cut the circle at C.

With centres B and C and the same radius draw arcs intersecting in P.

Then POX is 90° .

(For the $\angle$ s AOB and BOC are each 60° and $\angle BOP = \frac{1}{2} \angle BOC = 30^\circ$.)

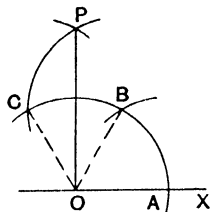


Fig 122

To construct an angle of 45° .

Construct a right angle and bisect it

Ex 26 Construct an angle of 120°

Ex 27 Construct angles of (i) 15° , (ii) 75° , (iii) 150°

Ex 28 Construct angles of (i) $22\frac{1}{2}^\circ$, (ii) $157\frac{1}{2}^\circ$

Ex 29. Trisect a right angle.

Ex 30 Trisect an angle of 45°

HOW TO COPY A GIVEN RECTILINEAR FIGURE

§9. A **rectilinear** figure is a figure made up of straight lines

An exact copy of a given rectilinear figure may be made in various ways, e.g. by pricking through the corners, or by means of tracing-paper, or as follows —

1st method Suppose that it is required to copy a pentagon $ABCDE$ (as in fig 123). First copy side AB , then $\angle ABC$, then side BC , then $\angle BCD$, etc. You will not find it necessary to copy *all* the sides and angles

Ex 31. Draw a good-sized quadrilateral, copy it by Method 1. If you have tracing-paper, make the copy on this, then see if it fits the original

Ex 32 Repeat Ex 31 with an (irregular) pentagon

2nd method—by intersecting arcs.

To copy $ABCDE$ by this method (see fig 123) Make $A'B' = AB$. With centre A' and radius equal to AC describe an arc of a circle. With centre B' and radius equal to BC describe an arc of a circle. Let these arcs intersect at C' . Then C' is the copy of C .

Similarly, fix D' by means of the distances $A'D'$ and $B'D'$, fix E' by means of the distances $A'E'$ and $B'E'$.

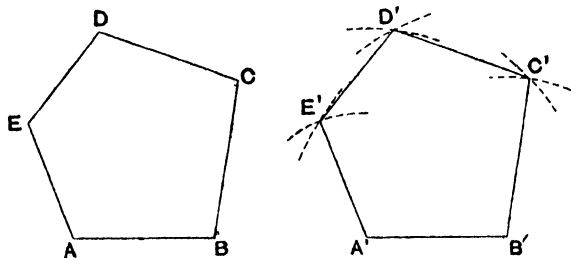


Fig 123.

The five vertices A' , B' , C' , D' , E' are now fixed, and the copy may be completed by joining up

In Exx 33-35 the copies should be made on **tracing-paper** if possible; the copies can then be tested by fitting on to the originals.

Ex 33 Draw, and copy (i) a quadrilateral, (ii) a pentagon, by the method of intersecting arcs. If tracing-paper is not used, the copy may be checked by comparing its angles with those of the original.

Ex 34 By intersecting arcs, copy fig 2

Ex 35 By intersecting arcs, copy fig 8

EXERCISES ON THE VARIOUS CONSTRUCTIONS.

In Exr 36-47 set squares and protractors are not to be used

Ex. 36 Construct an angle AOB of 60° . P is a point on the bisector of the angle 3 in from O . Find by construction and measurement the distance of P from each arm of the angle.

Ex 37 Verify by accurate construction and measurement that lines bisecting the angles of a parallelogram intersect so as to form a rectangle.

Ex 38 Draw a large triangle and the bisectors of two of its angles. From the point of intersection of the bisectors construct perpendiculars to each of the three sides of the triangle. Measure their lengths. State the fact suggested.

Ex. 39 Verify by accurate construction that in an equilateral triangle the bisector of an angle and the perpendicular bisector of the opposite side are the same line.

Ex 40 Construct a rectangle with diagonals 3.5" intersecting at an angle of 30° .

Ex 41 Construct a rhombus with diagonals 8 cm and 10 cm. Measure a side.

Ex 42 Construct a rhombus with an angle of 60° and the diagonal bisecting that angle 3" long.

Ex 43 Construct a parallelogram having $BC=6$ cm, $AB=5$ cm, and $\angle BAC=90^\circ$.

Ex. 44 Two posts 20 ft. and 15 ft. high are placed upright in the ground with their feet 12 ft. apart. Find from an accurate drawing the length of a straight wire joining their tops.

Ex 45 Construct any triangle with angles of 30° , 60° , 90° . How do the lengths of the sides including the 60° compare? Test your conclusion by drawing another triangle of different size but with the same angles.

Ex 46. At the centre of a circle draw two lines at right angles and bisect all the angles. Join, in order, the points where all the lines meet the circle. What figure is formed?

Ex 47 Divide the angle at the centre of a circle into six equal parts each 60° and join the points, in order, where the lines cut the circle. What figure is formed?

CHAPTER 11

SYMMETRY*

§ I You will notice that, in several of the foregoing ruler and compass constructions, the right and left sides of the figure correspond (figs 116, 117, 119). The figure can be folded about a central line so that the one half coincides with the other. Such figures are said to be **symmetrical about an axis** and the line about which the figure is folded is called the **axis of symmetry**.

Fold a piece of paper once, cut the folded sheet into any pattern you please, then open it out (see fig 124). The figure you obtain is symmetrical about the line of folding, which is the **axis of symmetry**.

If anything is written or drawn in ink on a sheet of paper, and the paper folded before the ink is quite dry so that it copies off on the other side of the fold, then the original together with the copy forms a pattern that has symmetry about the fold as axis. See fig 125.

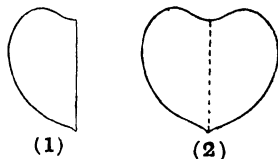


Fig 124

SYMMETRY YRTĒMMYƧ

Fig 125

Note that some of the letters in fig. 125 are unchanged in the transfer. These letters are themselves symmetrical.

* Symmetry is a fundamental geometrical property, but its existence is nowhere recognised by Euclid. It seems right to raise the principle of symmetry into a position worthy of its intrinsic interest and importance; and it follows logically that the principle should be used freely as a method of approaching subsequent propositions. The outcome should be a sensible lightening of the burden of work, and a more lively and intuitional appreciation of the properties of figures. It is not intended that the Principle should be used as a means of proof in any formal sequence of Deductive Geometry, but as a means of discovering properties of figures, as a help in construction, and as a means of summarising results.

The reversed writing is "looking-glass writing", it can be read in a mirror. If, sitting in the usual position for writing, you write a word on the under side of the table you will find that you have written in looking-glass writing.

¶Ex 1 Which block-capital letters are symmetrical about an axis?

¶Ex 2 Which Roman numerals (I to XII) are symmetrical about an axis?

Place two equal set squares together as in figs 126 and 127.

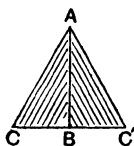


Fig 126

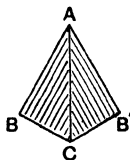


Fig 127.

What are the axes of symmetry in these two figures?

¶Ex. 3. About what axes are the following figures symmetrical.—(i) rectangle, (ii) semicircle, (iii) isosceles Δ , (iv) an open pair of scissors, (v) a two-foot rule partly opened at the hinge, (vi) two equal straight lines drawn from a point, (vii) a single finite straight line, (viii) a pair of parallel straight lines, (ix) a square, (x) a circle, (xi) a semicircular protractor?

Solids may have symmetry. The human body is more or less symmetrical about a plane. Consider the reflection in a mirror of the interior of a room. The objects in the room together with their reflections form a symmetrical whole, the surface of the mirror is the **plane of symmetry**.

¶Ex 4 Give four instances of solids possessing planes of symmetry.

§2 Fold a sheet of paper once. Prick a number of holes through the double paper, forming any pattern. On opening the paper you will find that the pin-holes have marked out a symmetrical figure.

Join **corresponding points** as in fig 128. Notice that when the figure was folded NA' fitted on to NA. This shows that $NA' = NA$. Also ANA' is $\perp$ to PQ. This fact may be stated as follows —

(1) **The line joining any pair of corresponding points, in a figure which is symmetrical about an axis, is bisected by and is perpendicular to the axis of symmetry.**

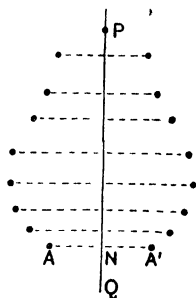


Fig 128

Corresponding points (e.g. A and A' in fig 128) are said to be **images** of one another in the axis of symmetry

Similarly a line has an image in an axis, in fig 130 $A'B'$ is the image of AB . It is obvious that a line and its image are equal

Also two lines forming an angle have as an image two other lines forming an equal angle

Again any area has an image in an axis of symmetry, and the area and its image will be equivalent, for the one will coincide with the other on folding about the axis

Ex 5 If a point P lies on the axis of symmetry, where is the image of P ?

Ex 6 What is the image of a semicircle in its bounding diameter?

Crease a sheet of paper and then open it out flat. Draw an ink line crossing the crease. fold the paper along the crease while the ink is still wet. on opening it you will find another line, the image of the first. Where do the line and its image intersect? See fig 129

Repeat this performance, but do not let the ink line meet the crease. The line and its image now do not appear to intersect, but in general they will intersect if produced. Where? See fig 130

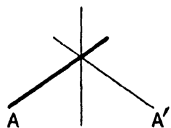


Fig 129

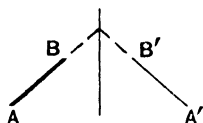


Fig 130

(ii) **A line and its image intersect on the axis of symmetry.**

(iii) **A line and its image make equal angles with the axis of symmetry.**

§3 The above facts about Symmetry about an axis enable us to discover properties of Figures. We will consider their applications to certain cases

ISOSCELES TRIANGLE

We all feel the reasonableness of the theorems about the isosceles triangle, this is due to our innate sense of symmetry, though we may not call it by that name. Let us see how the truth of these theorems might have been led up to from considerations of symmetry

Fig 126 has **AB** as an axis of symmetry

Note that **CBC'** is a straight line (why?), so that the whole figure is an isosceles triangle

Crease a sheet of paper and open out. Draw an ink line **PQ**, with **P** on the crease. Fold the paper along the crease, with the ink wet. open out again and letter the image of **PQ**, **PR**. Join **QR**.

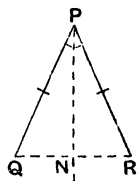


Fig 131.

Then **PQR** is a triangle in which $PQ = PR$ since they fitted on one another (see fig 131). The triangle is symmetrical about the crease, the bisector of angle **P**, and $\angle Q = \angle R$, since they fitted on one another when the figure was folded.

Hence if two sides of a triangle are equal the angles opposite these sides are equal.

A straight line **BC** has symmetry about its perpendicular bisector.

If at **B** and **C** we make equal angles, as in fig. 132, the resulting figure is still symmetrical about the same axis.

If the arms of these angles are produced to meet at **A**, the figure formed will be symmetrical, and **A** will be on the axis, and if folded **AC** would fit on **AB**.

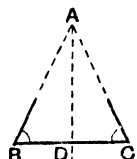


Fig 132

Hence if two angles of a triangle are equal, the sides opposite to these angles are equal.

KITE

Fold a piece of paper, and cut out a triangle with one side along the crease, cutting through both thicknesses. Open out, and you have a symmetrical quadrilateral, such as fig 133. (Two set squares may be used instead of paper.) This may be called a **kite**. The kite is symmetrical about one of its diagonals.

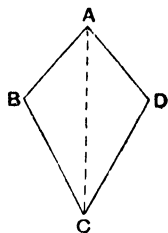


Fig. 133

¶Ex 7 Are any of the sides of a kite equal? What can be stated about the two diagonals? How would you describe the figures into which **BD** divides the kite?

RHOMBUS

Fold a piece of paper in four, so as to form a right angle at the corner. By a straight cut through all four thicknesses, cut off a right-angled triangle from the corner; unfold it, and you have a rhombus. See fig. 134

Letter this $ABCD$, with the diagonals AC , BD intersecting at O .

The rhombus is a symmetrical figure. What axes of symmetry has it?

It will be noticed that a rhombus is a particular kind of kite. How would it be necessary to modify the method given above of cutting out a kite in order to obtain a rhombus?

By considerations of symmetry, we see that in a rhombus (i) all four sides are equal, (ii) the diagonals bisect one another, (iii) the diagonals are at right angles, (iv) the diagonals bisect the angles

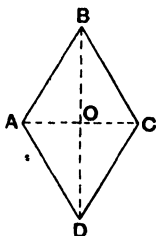


Fig. 134

CIRCLE

The circle is the best example of a symmetrical figure. By considerations of symmetry a large number of its properties may be discovered. This will be done later in Chapter 17.

SYMMETRY ABOUT A CENTRE

§4 The capital letter **S** has no axis of symmetry, if viewed in a mirror it becomes z . But though there is no axis of symmetry, this letter appears to possess symmetry of a kind. Thus, if through the middle point O any lines be drawn such as AOA' , BOB' , COC' , then each of these lines is bisected at O .

Cover the **S** in fig 135 with a piece of tracing-paper, and copy the outline of the letter. Then stick a pin through the point O of both. Now turn the traced **S** through 180° , and you will find that it again fits the original.

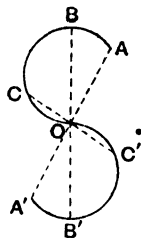


Fig. 135.

A plane figure which covers its original trace if turned in its own plane through 180° round the centre is said to be **symmetrical about a centre**.

The line joining any pair of corresponding points in a figure which is symmetrical about a centre is bisected by the centre of symmetry.

¶Ex. 8. Mention other objects, figures or letters that are symmetrical about a centre

¶Ex. 9. Has an equilateral triangle symmetry about a centre?

¶Ex. 10. Draw a circle and two diameters AOA' , BOB' intersecting at an angle of about 60° ; from A and A' draw (freehand) AN , $A'N'$ perpendicular to BOB' . What symmetry has the figure? What can you state about the triangles in the figure?

¶Ex. 11. D is the mid-point of the side BC of the triangle ABC , CE is drawn parallel to AB to cut AD produced at E . What part of the figure has central symmetry? Hence notice that $AB = EC$

¶Ex. 12. Has a parallelogram (i) symmetry about any axis, (ii) central symmetry about the intersection of its diagonals?

EXERCISES ON SYMMETRY

Ex. 13. Draw any equilateral triangle. Draw dotted lines to show all its axes of symmetry

Ex. 14. Draw any square and show by dotted lines all its axes of symmetry

Ex. 15. Draw two equal circles (i) cutting each other, (ii) not cutting. Indicate the axes of symmetry

Ex. 16. Repeat Ex. 15 for two unequal circles

Ex. 17. Sketch freehand figures showing triangles which are symmetrical about (i) the bisector of one angle, (ii) the perpendicular bisector of one side

Ex. 18. Sketch freehand figures showing quadrilaterals which are symmetrical about (i) one diagonal, (ii) both diagonals, (iii) the intersection of the two diagonals

Ex. 19. Sketch freehand the symmetrical figures of which the following figures are the halves, the axis being the dotted line

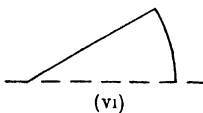
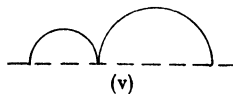
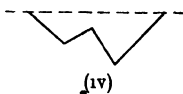
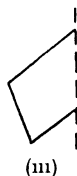
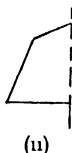


Fig. 136.

Ex. 2C. Copy the following figures freehand and indicate all axes of symmetry by dotted lines

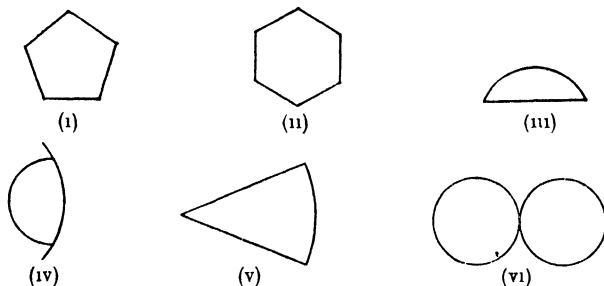


Fig. 137

Ex. 21 Sketch freehand the following figures and their images in the dotted line as axis of symmetry.

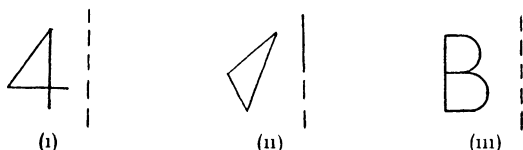


Fig. 138

Ex. 22. (Inch paper) Draw the line joining the points $(1, 1)$, $(2, 2)$. Draw its image (i) in the axis of x , (ii) in the axis of y

Ex. 23. (Inch paper) Draw the figures given by the following points and their images (i) in the axis of x , (ii) in the axis of y —

(i) $(0, 0)$, $(1, 2)$, $(3, 1)$, $(2, 0)$,

(ii) $(1, 1)$, $(2, 3)$, $(3, 2)$.

Ex. 24. (Inch paper) $ABCD$ is a quadrilateral, A being the point $(0, 0)$, B $(0, 2)$, C $(1, 3)$, D $(3, 3)$ Construct a figure symmetrical about AD as axis

Ex. 25 Repeat Ex. 24 with CB as axis

Ex. 26 Draw a triangle with sides 2 in, 3 in, 4 in Complete the figure so that it has central symmetry about the mid-point of the 4 in side

Ex. 27 Draw any triangle and mark any point outside the triangle Complete the figure so that it has central symmetry about the point marked

Ex. 28 Repeat Ex. 27, with the point on the perpendicular bisector of one side of the triangle

Ex. 29 Draw a quadrilateral $ABCD$ given $AB = 4$ cm, $AD = 5$ cm, $CD = 5$ cm, $AC = 6$ cm, $BC = 4$ cm Indicate the axis of symmetry

Ex. 30. Draw a right-angled triangle with sides 1.5 in, 2 in, 2.5 in This is half a symmetrical figure with the hypotenuse as axis Complete the figure

Ex. 31 Construct a quadrilateral $ABCD$ given $AB=2$ in., $BC=2\frac{1}{2}$ in., $CD=3$ in., $AD=4$ in., $AC=3\frac{1}{2}$ in. Using AD as axis, construct a symmetrical figure of which $ABCD$ is one-half.

Ex 32. Draw any triangle and a line parallel to the longest side, outside the triangle. Construct the image of the triangle in this parallel line as axis of symmetry.

Ex 33 ABC is a triangle, D, E the middle points of the sides. The upper triangle is folded about DE . Draw (freehand) the $\triangle ABC$ and show ADE in its new position in the following cases —(i) ABC equilateral, (ii) ABC isosceles, (iii) ABC scalene, (iv) the angle C obtuse, (v) the angle A a right angle, (vi) the angle C a right angle

Ex 34 Draw any rectangle $ABCD$ and the diagonal AC . Draw the new position of the $\triangle ADC$ when folded about AC

Ex 35 Draw any triangle and mark the line about which it must be folded to bring the vertex to the middle point of the base.

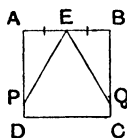


Fig 139

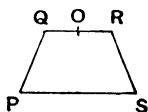


Fig 140

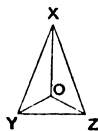


Fig 141.

Ex. 36 In fig 139 $ABCD$ is a square, E is the mid-point of AB , $AP=BQ$. About what line is the figure symmetrical? If the figure were folded about the axis of symmetry, what points would coincide? Make a list of the lines and angles that you see to be equal on account of the symmetry of the figure

Ex 37 Repeat Ex 36 for fig 140. O is the mid-point of QR , $PQ=SR$, $\angle Q=\angle R$. Join OP, OS, PR, QS . Where will PR, QS intersect?

Ex. 38 Repeat Ex 36 for fig 141. XYZ is an isosceles triangle ($XY=XZ$). XO, YO are the bisectors of $\angle s X, Y$

Ex 39. Repeat Ex 36 for fig 142. $AB=AC, BD=CE$

Ex 40 In the letter Z in fig 143 AB is equal and parallel to CD , the $\angle A=90^\circ$ and $BC=2AB$

Show by sketches how the two triangles may be placed so as to form (i) an equilateral triangle, (ii) an isosceles triangle with an angle of 120° , (iii) a figure like a kite symmetrical about a certain line

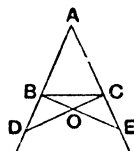


Fig 142

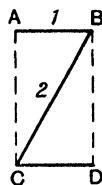


Fig 143

SUMMARY OF RESULTS

The line joining any pair of corresponding points, in a figure which is symmetrical about an axis, is bisected by and is perpendicular to the axis of symmetry.

A line and its image intersect on the axis of symmetry and make equal angles with the axis

The line joining any pair of corresponding points in a figure which is symmetrical about a centre is bisected by the centre of symmetry

CHAPTER 12

SCALE DRAWING III

Ex 1. The sides of a set square measure 3 in , 2 5 in , 1 5 in Find the size of its angles

Ex 2 A ladder 60 ft long leans against a vertical wall and the foot of the ladder is 20 ft from the wall How high is the top of the ladder above the ground?

Ex 3 P is 64 miles W of Q, R is due N. of Q, if PR is 72 miles, what is QR? What is the bearing of P from R?

Ex 4 A is 12 miles N. of H, D is 24 miles S of H, O is due W of A, and OH is 42 miles, find OD and OA

Ex 5 XT=19 miles, MX=11 miles, MT=17 5 miles, how far is M from the line XT?

Ex 6 Stafford is 27 miles from Derby and the same distance from Shrewsbury, and the three towns are in a straight line Birmingham is 40 miles from Shrewsbury and 35 from Derby. How far is Stafford from Birmingham?

Ex 7 A side-view of a camp-stool looks like the letter X with its two top corners joined, and the cross-pieces are hinged at their middle points How high is the top of the stool above the ground if the cross-pieces are each 21 in long and the top piece 14 in.?

Ex 8 A rod AB 4 ft long is hinged to a vertical wall at A, and B is supported by a string $3\frac{1}{2}$ ft long fastened to a point on the wall $4\frac{1}{2}$ ft above A What angle does the rod make with the wall and how far is B from the wall?

Ex 9 Find the vertical height of a cone from the following measurements Diameter of base = 12 in , Slant height = 14 in

Ex 10 A ladder 24' long, resting against a vertical wall with its foot 8' from the base of the wall, is kept from slipping by a rope fastened to the base of the wall and a point 6' from the foot of the ladder Find the length of the rope.

Ex 11. A donkey is tethered to a point 20 feet from a long straight hedge, he can reach a distance of 35 feet from the point to which he is tethered. How much of the hedge can he nibble?

Ex 12. A ferry-boat is moored by a rope 30 yards long to a point in the middle of a river. The rope is kept taut by the current What angle does it turn through as the boat crosses the river, whose width is 30 yards?

These exercises involve intersecting arcs.

Ex. 13 In the given figure $AB = 42$ ft, $AP = 23$ ft, $BP = 43$ ft., $AQ = 47$ ft., $QB = 32$ ft. Find the distance PQ and the angle it would make with AB if they were produced to meet

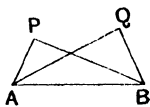


Fig. 144

Ex. 14. AB is a diameter of a circle and is 70 ft long, P and Q are two points on the circumference of the circle on the same side of AB , $AP = 30$ ft, $BQ = 40$ ft. Find the length of PQ and the angle it subtends at the centre of the circle

Ex. 15. Fig. 145 represents the side-view of a folding step ladder, HB is hinged at H and attached by a cord XY to a point X on the steps. Given that $HB = HC = 5$ ft, $AH = XC = YB = 1$ ft, and $XY = 2.5$ ft, draw the diagram to scale and find the height of A above the ground BC

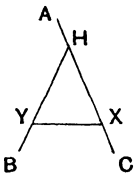


Fig. 145

Ex. 16 A ship is situated 3.7 miles from a straight shore. Two beacons on the shore are respectively 5.4 miles and 6.2 miles from the ship. Determine the distance between the beacons

Ex. 17 Draw a plan of a rectangular park 400 yards long and 250 yards wide with a circular pond 80 yards in diameter exactly in the centre. If a man walked from one corner directly towards the opposite corner, how far would he walk before he came to the edge of the pond?

Ex. 18 $ABCD$ is a field in which $AB = 115$ yards, $BC = 105$ yards, $CD = 165$ yards, $AD = 75$ yards. At B and D are gates connected by a straight path of length 170 yards. Another path runs between two stiles placed at the middle points of the sides AB and CD . Find its length

Ex. 19. A ship steaming $N. 30^\circ E$ is on a course that will bring it within 100 miles of a wireless station, it can receive signals from the station up to a distance of 300 miles. Find the bearing of the station from the ship when the signals are first heard

Ex. 20 $ABCD$ is a flexible wire. A is a fixed point and C a fixed pulley. B is a weight which slides freely along the wire, so that AB , CB are always equally inclined to the vertical. Find by drawing or otherwise how far B rises if D is pulled down 2 feet.

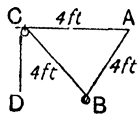


Fig. 146.

Ex. 21. A banner 10 ft. wide and 6 ft. deep is suspended above a street from the tops of houses each side of the street by two ropes each 35 ft. long. If the heights of the houses are each 50 ft. and the width of the street is 65 ft., by how much does the bottom of the banner clear the street?

Ex. 22. A rod 3 feet long is supported horizontally by two strings each 2 feet long fastened to the ends of the rod and to a point above it. By how much must each string be lengthened to lower the rod through 1 foot?

Ex. 23 A , B are two points 900 yards apart on a straight coastline. A sunken rock is in the sea 700 yards from A and 550 yards from B . A boat sails parallel to the coastline AB so as to pass outside the rock 100 yards from it. When nearest to the rock, how far is the boat from A and B ?

CHAPTER 13

LOCI

§ I. Mark two points A and B, say about 2 inches apart. Mark a point equidistant from A and B (i.e. at equal distances from A and B), judging the distance by eye.

In a similar way mark about 10 points equidistant from A and B, some above and some below AB.

Notice what pattern this set of points seems to form. Draw a line passing through all of them.

Find a point on AB equidistant from A and B, this belongs to the set of points.

The pattern formed by all possible points equidistant from two fixed points A and B is called the **locus** of points equidistant from A and B.

¶Ex. 1. What is the locus of points at a distance of 2 ft. from a fixed point O (i) in one plane, (ii) in space?

¶Ex. 2. Draw a straight line right across your paper. What is the locus of points distant 2 inches from this line? Remember that the distance is reckoned perpendicular to the line.

¶Ex. 3. A truck is running straight along a level track. What is the locus of any corner of the truck?

¶Ex. 4. What is the locus of a man's hand as he winds the starting handle of a motor car?

¶Ex. 5. What is the locus of a man's hand as he works the handle of a common pump?

¶Ex. 6. Sound travels about 1100 feet per second. A gun is fired; what is the locus of all the people who hear the sound after 1 sec.?

¶Ex. 7. A round ruler rolls down a sloping plank, what is the locus of the centre of one of the ends of the ruler?

¶Ex. 8. Two coins are placed on a table with their edges in contact. One is held firm, and the other rolls round the circumference of the fixed coin. What is the locus of the centre of the moving coin? Would the locus be the same if there were slipping at the point of contact?

¶Ex. 9. What is the locus of a door-handle as the door opens?

¶Ex. 10. Slide your set square round on your paper, so that one corner always remains at a fixed point. What are the loci of the other two vertices?

The above exercises suggest the following definition of a locus.

§ 2. DEF. If a point moves so as to satisfy certain conditions the **path** traced out by the point is called its **locus**

¶ **Ex. 11** (On squared paper) Draw a pair of lines at right angles (OX , OY), plot a series of points each of which is twice as far from OX as from OY . What is the locus? (Keep your figure for the next Ex.)

¶ **Ex. 12** Using the figure of Ex. 11, plot the locus of points 3 times as far from OX as from OY , also the locus of points half as far from OX as from OY .

¶ **Ex. 13** (On squared paper) Plot the locus of a point which moves so that the sum of its distances from two lines at right angles is always 5 inches.

¶ **Ex. 14** (On squared paper) Plot the locus of a point which moves so that the difference of its distances from two lines at right angles is always 1 inch.

Ex. 15. Draw a line, and mark a point O 3 inches distant from the line. Let P be a point moving along the line. Experimentally, plot the locus of the mid-point of OP

Ex. 16 A point O is 2 in from the centre of a circle of radius 2.5 in. Plot the locus of the mid-point of OP , when P moves round the circumference of the circle.

Ex. 17. A rod OP is freely hinged at O , what is the locus of P ? At P another rod PQ is freely hinged and always hangs vertically; what is the locus of Q ?

Ex. 18 Draw $ABCD$ a parallelogram. Now regard the figure as a framework freely jointed at A, B, C, D . If AB is fixed, what are the loci (i) of C , (ii) of D ?

Ex. 19 In the preceding question if the diagonals intersect in O what is the locus of O ?

Ex. 20 A square is rolled along a straight line, being always in the same vertical plane. Sketch the paths traced out by one vertex until the square has turned round once.

Ex. 21 Repeat Ex. 20 for an equilateral triangle.

Ex. 22. Repeat Ex. 20 for a scalene triangle.

Ex. 23. A packing case is rolled about one edge, sketch the locus of a point on the opposite edge.

The use of tracing-paper is recommended in Exs. 24-26

Ex. 24. A man stands on the middle rung of a ladder against a wall. The ladder slips down, find the locus of the man's feet

[Do this by drawing two straight lines at right angles to represent the wall and the ground, take a length of, say, 4 inches to represent the ladder; draw the ladder (the line of 4 inches) on tracing-paper; then bring the ends of the ladder on to the two lines of the paper below, and prick through the middle point. Or you may use the edge of your ruler.]

Ex 25. The ends of a rod slide on two wires which cross at right angles. Find the locus of a point on the rod. See fig 147

Ex 26 Two points A, B of a straight line move along two intersecting lines at right angles. Plot the locus of a point P, in AB produced.

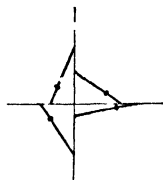
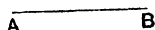


Fig 147

TWO IMPORTANT LOCI

§ 3 At the beginning of the chapter you marked by eye a set of points equidistant from two given points. Now do this accurately as follows —

Mark two points A and B about 2 inches apart and, by drawing pairs of equal circles with A and B as centres, mark a series of points equidistant from A and B. (See fig 148)



You will notice that all the points lie on a straight line, the perpendicular bisector of AB. See Chap 10, p 78, Chap 11, p 85

Again, no point off this line can be equidistant from A and B. A point to the left of the line is nearer to A, a point to the right of the line is nearer to B. Hence

Fig 148

THEOREM 34. The locus of points equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points.

¶Ex 27 What does the locus become if P is not necessarily in the plane of the paper?

§ 4 Draw two unlimited lines intersecting near the middle of your paper. By eye, mark a point equidistant from the two lines. Mark a number of such points, say 20, in various positions. What pattern is formed?

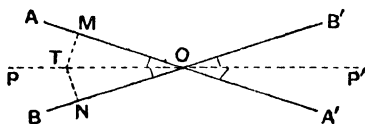


Fig 149.

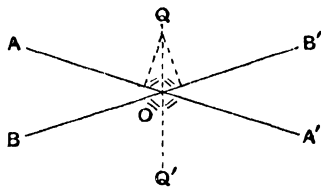


Fig 150.

If AA' and BB' , in figs. 149, 150, are the lines, you will find that the points either lie on the line PP' or the line QQ' , and you will notice that these lines appear to bisect the angles

There is no easy means of finding accurately a lot of these points, but you can draw the bisectors of the angles and test that all points on the bisectors are equidistant from the arms

Ex 28 Draw AOA' , BOB' and bisect the angles with lines POP' and QQQ'

Take a point on POP' and draw perpendiculars from it to AOA' and BOB'

Measure the two perpendiculars Repeat this for several other points on POP' and also for points on QQQ'

A point that lies on neither bisector must be nearer to the one line than to the other For instance, a point inside the angle AOP (fig 149) is nearer to AOA' , etc

Hence

THEOREM 35 **The locus of points equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines.**

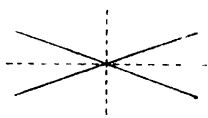


Fig 151.

INTERSECTION OF LOCI.

§5 *Find a point (or points) equidistant from two given points and 1 in from a given line*

A and B are the given points, l the given line

The locus of points equidistant from A and B is the $\perp$ bisector of AB

. Draw the $\perp$ bisector of AB

The locus of points 1 in. from the line l consists of the two parallels to l 1 in away from it.

$\therefore$ Draw the two parallels to l 1 in away from it

The points of intersection of the $\perp$ bisector and the parallels give the points required, namely P and Q

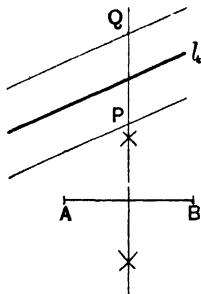


Fig 152

In Exx 29-41 the figure must be drawn accurately and an explanation given similar to the above.

Ex 29 Draw an unlimited straight line and mark a point O 2 inches from the line. Find a point (or points) 3 inches from O and 3 inches from the line. (What is the locus of points 3 inches from O ? What is the locus of points 3 inches from the line? Draw these loci) Measure the distance between the two points found

Ex. 30 In Ex 29 find as many points as you can distant 1 inch from both point and line

Ex 31. If A, B are two points 7 cm. apart, show how to find a point (or points) distant 9 cm from A and 5 cm from B .

Ex 32. Make an angle of 40° , on one of the arms mark a point A 6 cm. from the vertex of the angle Find a point (or points) equidistant from the arms of the angle, and 5 cm. from A Measure the distance between the two points found

Ex 33. Draw two unlimited straight lines Find all points 1 inch from one line and 1 5 inches from the other

Ex 34 Draw a triangle ABC , find a point O which is equidistant from B, C ; and also equidistant from C, A Test by drawing a circle with centre O to pass through A, B, C

Ex 35 Draw a triangle ABC Inside the triangle find a point P which is equidistant from AB and BC , and also equidistant from BC and CA . From P draw perpendiculars to the three sides; with P as centre and one of the perpendiculars as radius draw a circle

Ex 36 Draw a triangle ABC , find a point O which is equidistant from B and C and also equidistant from AB and AC

Ex 37 Two walls of a room are parallel and 20 ft. apart¹ A third wall crosses at an angle of 70° Draw a figure showing the position of a stool which is equidistant from the two parallel walls and the third wall. What is this distance?

Ex 38 Two straight railway tracks converge at an angle of 40° . A shed is to be erected 200 yards from the junction and equidistant from each track. Draw a figure showing its position and find its distance from each track.

Ex 39. A harbour is formed by two breakwaters AB and BC . AB , 600 yds. long, is perpendicular to the shore, and BC makes an angle of 70° with AB . A ship is anchored 150 yds from BC and 250 yds. from AB . How far is it from the shore?

Ex 40 A vessel is steering a course parallel to a straight shore and $\frac{1}{2}$ mile distant from it Over what distance would it be in danger from torpedoes fired from a point on shore if the effective range of a torpedo is 1 mile?

Ex. 41. A weight hanging by a string $3\frac{1}{2}$ ft. long from a point in a wall is pulled till it is 2 ft. away from the wall. What angle does the string now make with the wall?

Ex. 42. The position of hidden treasure in a triangular field is described as being equidistant from two sides and also equidistant from two corners. Could this statement be true?

USE OF LOCI IN CONSTRUCTION OF TRIANGLES

Construct a $\triangle ABC$ given base $BC = 10$ cm., height $= 6$ cm., and median to middle point of $BC = 7$ cm

Draw a freehand sketch (not shown here).

Draw base $BC = 10$ cm and take M its middle point

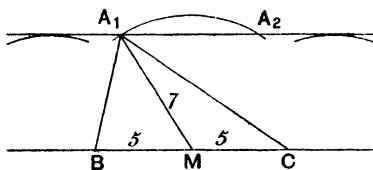


Fig 153

Since the height is 6 cm, A lies on a $\parallel$ to BC distant 6 cm.

Since the median from M is 7 cm, A lies on a circle centre M and radius 7 cm.

These two loci meet at A_1 and A_2 giving two positions of the triangle. Note that the two triangles are congruent.

Ex 43 Construct a triangle ABC given

- (i) base $BC = 3$ in, height $= 1.75$ in, $\angle B = 56^\circ$. Measure AB .
- (i) base $BC = 89$ mm, height $= 58$ mm., $AB = 72$ mm. Measure AC
- (iii) height $AD = 2.25$ in, $AB = 3.4$ in, $AC = 2.75$ in. (Draw height first.)
- (iv) base $BC = 3.6$ in, side $AB = 2.9$ in, median $AM = 2.4$ in

CO-ORDINATES

§6. Through a point O draw NS and EW lines. Mark the points you get by following the instructions below.—

- (i) From O go 2 in. E and then 3 in. N
- (ii) From O go 3 in. E. and 1 in. S.
- (iii) From O go 2 in. W. and 3 in. N
- (iv) From O go 1 in. W. and 2 in. S.

It is usual to indicate the point you obtained in (i) as the point (2, 3) and you are to understand that the first number tells you how far you are to travel East and the second number how far North.

If we regard distances to the East as positive, distances to the West are naturally regarded as negative, similarly for North and South. Hence point (ii) would be (3, -1), point (iii) (-2, 3) and point (iv) (-1, -2)

Ex 44. Mark (freehand) the following points —(3, 1), (-3, 1), (2, -3), (-2, -1)

When the position of a point is fixed by its distances from two fixed lines at right angles, the lines are called **axes**, their point of intersection the **origin** and the distances **co-ordinates**.

In general we do not call the axes the N.S line and E.W line but the y -axis and the x -axis. The y -axis is always taken to correspond with a N S line

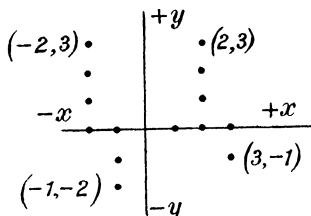


Fig 154

By saying that the co-ordinates of a point are (2, 3) we mean that $x=2$, $y=3$, that is to say the point is 2 units in the direction Ox to the right of the y -axis, and 3 units in the direction Oy above the x -axis

If the x co-ordinate is negative it is to the left of the y -axis, and if the y co-ordinate is negative it is below the x -axis

When ordinary paper is used the position of a point whose co-ordinates are given is best found by drawing parallels to the axes at the required distances, the required point is where these parallels intersect

¶Ex 45 What is the locus of all points (i) at which $x=2$, (ii) at which $y=3$?

Hence to find the point (2, 3) draw a parallel to the y -axis 2 units to the right, and a parallel to the x -axis 3 units above

¶Ex. 46 What lines would you draw to fix accurately the points (3, -1), (-2, 3), (-1, -2)?

Ex. 47 Using ordinary paper and taking as axes two long lines at right angles in the middle of your paper, find the position of the following points by drawing parallels. Use 1 inch as unit (i) (3, 4), (ii) (1, -2), (iii) (-1, 3), (iv) (-2, -2), (v) (3.5, 1.5), (vi) (2.5, 0), (vii) (0, -3).

Indicate the point by writing its co-ordinates quite near to it.

To fix the position of a point by drawing parallels takes some time and trouble; hence in practice squared paper is used. This gives us parallels already drawn and it is only necessary to count or measure along the lines.

Ex 48 Using inch paper and taking 0.1 inch as unit —

(i) Plot the points $(0, 0)$, $(2, 0)$, $(-2, 0)$, $(0, 13)$, $(1, -10)$, $(8, 6)$, $(-8, -6)$, $(-3, -5)$ (The constellation of Orion)

(ii) Plot the points $(-12, -2)$, $(-8, 0)$, $(-4, 0)$, $(0, 0)$, $(3, -2)$, $(7, 0)$, $(5, 4)$ (The Great Bear)

SUMMARY OF RESULTS

The locus of a point which is equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points.

The locus of a point which is equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines.

CHAPTER 14

AREA

§ I The fundamental method of finding the length of a given line is to take a unit of length (say 1 inch) and see how many times it can be fitted on to the given line

In the same way, the fundamental method of finding a given area is to take a unit of area and see how many times it can be fitted on to the given area

If we are measuring lengths in inches, we take as our unit of area a square inch. Similarly with feet or centimetres

Though the methods referred to above are fundamental, we do not use them in practice. Thus to measure the length of a line we place a scale graduated in inches against the line. In the same way, to measure a given area, we can fit on it a sheet of paper ruled off into square inches and count the number of square inches that fit on the given area but we shall see in the course of this chapter that we need not resort to this laborious method, except in the case of areas bounded by curved lines.

Area of a curvilinear figure. This cannot be found *exactly* by the method of counting squares the approximate value however is easily found as follows

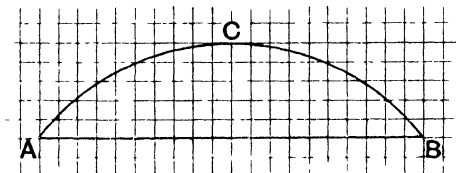


Fig. 155.

To find the area of the fig ACBA, notice that the curved boundary ACB *cuts through* various squares, in counting squares we have to decide what is to be done with these broken squares. The following

rule gives a useful approach to the true value — *If the broken square is more than half a complete square, count 1, if less than half a square, count 0.*

Counting up the squares in ACBA on this system, we find that the area is 72 squares. As each of the above squares is $\frac{1}{100}$ sq. inch (100 of these make a sq. in.) the area is 72 sq. inch.

Ex 1. On inch paper draw a circle of radius 1 inch, find its area as above, and reduce to square inches. (The counting can be shortened in various ways, e.g. by dividing the circle into 4 quarters by radii.)

Ex 2. Use transparent squared tracing-paper to find the area of (i) the protractor in fig. 18, p. 10, (ii) the area labelled "shoals" in fig. 242, p. 164.

§ 2.

AREA OF RECTANGLE

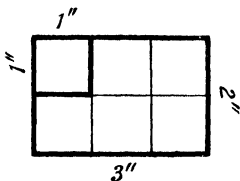


Fig. 156

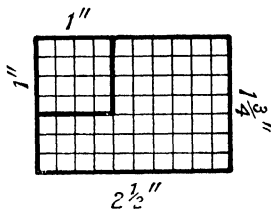


Fig. 157

Fig. 156 represents a rectangle, 3 inches long and 2 inches wide, divided up into square inches. By counting we see that the area is $3 \times 2 = 6$ square inches.

Fig. 157 represents a rectangle, $2\frac{1}{2}$ in. by $1\frac{3}{4}$ in., i.e. $\frac{5}{2}$ in. by $\frac{7}{4}$ in., divided up into squares whose sides are each $\frac{1}{4}$ in. The number of squares is obviously 10×7 and the number of squares in a square inch is 4×4 .

∴ the area is $\frac{10 \times 7}{4 \times 4}$ sq. in., i.e. $2\frac{1}{2} \times 1\frac{3}{4}$ sq. in.

Similarly, if the rectangle measures 2.15 in. by 1.67 in., it can be divided up into squares whose sides are each .01 ($=\frac{1}{100}$) inch long. The number of squares is obviously 215×167 and the number in a square inch 100×100 .

the area = $\frac{215 \times 167}{100 \times 100}$ sq. in. = 2.15×1.67 sq. in.

More generally if l in. be the length and b in. the breadth of a rectangle, its area is $l \times b$ sq. in.

Or still more generally

To find the number of square units in the area of a rectangle, multiply together the numbers of units in the length and breadth of the rectangle.

¶Ex 3 Draw a figure to show that if the side of one square is 3 times the side of another square, the area of the one square is 9 times the area of the other (*Freehand*)

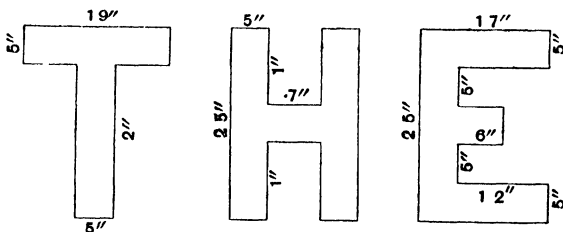


Fig 158

Ex 4 Make freehand sketches of the given figures (fig 158) In each case find the area

¶Ex 5 Find the other dimension of a rectangle, given that the area = 140 sq. ft., one dimension = 35 ft.

AREA OF RIGHT-ANGLED TRIANGLE

§3 By drawing a diagonal of a rectangle we divide the rectangle into two equal right-angled triangles. Hence the area of a right-angled triangle may be found by regarding it as half a certain rectangle

Ex 6 Find the areas of right-angled triangles in which the sides containing the right angle are (i) 2", 5", (ii) 3.5 cm, 2.9 cm, (iii) x' , c' , (iv) a'' , b''

AREA OF PARALLELOGRAM

§4 Take a sheet of paper (a rectangle) and call the corners P, B, C, Q, BC being one of the longer sides (fig 159). Mark a point A on the side PQ. Join BA, and cut (or tear) off the right-angled triangle PBA. You now have two pieces of paper, you will find that you can fit them together to make a parallelogram (ABCD in fig 159).

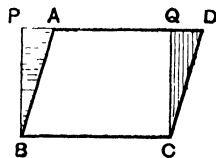


Fig 159.

Notice (1) that the rectangle you had at first and the parallelogram you have now made are composed of the same paper, and *therefore have the same area*. Figures which are equal in area are said to be **equivalent**.

(ii) that the rectangle and the parallelogram are on the same base BC, and both lie between the same pair of parallel lines BC and PAQD. Or, we may say that they have *the same height*.

Hence, a parallelogram is equivalent to a rectangle on the same base and between the same parallels (or, of the same height).

The area of the rectangle is measured by the product of the base and height, and as the parallelogram has the same height as the rectangle, we have

The area of a parallelogram is measured by the product of the base and the height.

Any side of a parallelogram may be taken as the base. The perpendicular distance between the base and the opposite (parallel) side is called the height, or **altitude**.

Thus in fig 160 if BC be taken as base, MN (which may be drawn from any point of the base) is the height (or altitude). If AB be taken as base, GH is the height.

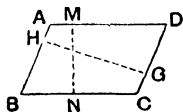


Fig 160

Ex 7 In fig. 160 what is the height if CD be taken as base? if AD be taken?

Ex 8 Find the area of a parallelogram (i) of sides 3.5 in and 4.5 in and of angle 70° , (ii) of sides 2 in and 3 in. and of angle 30° , (iii) of sides 7.85 cm and 8.75 cm and of angle 45° .

Ex. 9. Draw a rectangle on base 10 cm and of altitude 8 cm, on the same base construct an equivalent parallelogram of angle 60° , and measure its longer diagonal.

Ex 10. Construct a rhombus of side 3.5 in equal in area to a parallelogram of sides 3.5 in and 2.75 in. and of angle 70°

AREA OF TRIANGLE

§ 5. To find the area of a triangle, $\triangle ABC$ in fig 161, draw through A and C lines parallel to BC, BA respectively, forming a parallelogram. The triangle is half the parallelogram. But the area of the parallelogram is $BC \times AN$. Therefore the area of the triangle is $\frac{1}{2} BC \times AN$. Hence

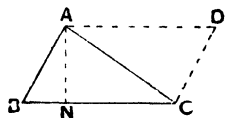


Fig 161

The area of a triangle is measured by half the product of the base and the height.

This fact may also be stated —

The area of a triangle is one-half the area of a rectangle of the same base and the same height. (See fig 162)

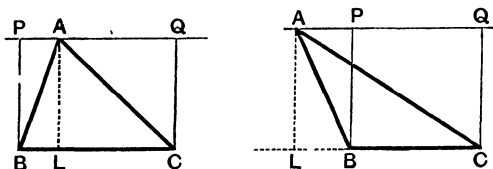


Fig 162

Ex. 11 Write out a proof of this fact, using fig 162.

Any side of a triangle may be taken as base. The line drawn perpendicular to the base from the opposite vertex is called the height, or altitude.

There will be three different altitudes according to the side which is taken as base

¶Ex 12 Draw an acute-angled triangle and draw the three altitudes (*Free-hand*)

¶Ex 13 Repeat Ex 12 for a right-angled triangle (*Freehand*)

¶Ex 14. Repeat Ex. 12 for an obtuse-angled triangle (*Freehand*)

¶Ex 15. In what case are two of the altitudes of a triangle equal?

¶Ex. 16 A triangle ABC has $AB = 3$ in and $BC = 4$ in, trace the changes in its area as $\angle ABC$ increases from 0° to 180°

¶Ex. 17 A parallelogram PQRS has $PQ = 10$ cm and $QR = 7$ cm, trace the changes in its area as $\angle PQR$ increases from 0° to 180°

Since any one of the three sides may be taken for base, there are three different ways of forming the product of a base and the corresponding altitude. Thus the area may be calculated in three different ways, and of course, theoretically, the result is the same in each case. Practically, none of the measurements will be quite exact, and the results will generally differ slightly. To get the best possible value for the area **take the average of the three results.**

Ex 18 Find, to three significant figures, the areas of the following triangles, taking the average of three results in each case —

(i) sides 3, 4, 4.5 in

(iv) sides 2, 3, 4.5 in.

(ii) sides 6, 8, 9 cm

(v) sides 3, 4 in, included $\angle 120^\circ$

(iii) sides 3, 4, 5 in

(vi) $BC = 7.2$ cm., $\angle B = 20^\circ$, $\angle C = 40^\circ$

Ex. 19. Find the area of a rhombus whose diagonals are 6" and 8".

Ex. 20 Find the area of a kite whose diagonals are 14" and 5"

Ex. 21 The section of a trough is triangular, the sides being 15" and the width 12" Find the sectional area

Ex. 22 Find the surface (i.e. the sum of the areas of all the faces) —

(i) of a tetrahedron (three-sided pyramid) each of whose edges is 4 inches.

(ii) of a square pyramid whose base edges are each 2 inches and whose slant edges are each 2.5 inches

(iii) of a regular 3-sided prism whose base is an equilateral triangle of side 2 inches and whose height is 3.5 inches

AREA OF TRAPEZIUM.

§6. A **trapezium** is a quadrilateral with two of its sides parallel

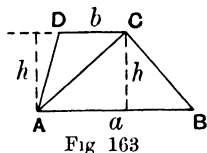
The parallel sides of the trapezium ABCD (fig. 163), are a and b , the height is h

The area of the trapezium

$$= \text{area of } \triangle ABC + \text{area of } \triangle ADC$$

$$= \frac{1}{2}ah + \frac{1}{2}bh$$

$$= \frac{1}{2}(a+b)h$$



Thus **the area of a trapezium is measured by the product of the height and the average of the two parallel sides.**

Ex. 23 Find the area of trapezia in which

(i) the parallel sides are 3 in. and 5 in., distance between them 4 in.,

(ii) „ „ „ „ 4 cm „ 12 cm „ „ „ 7 cm „

(iii) „ „ „ „ 1.5 in „ 4.5 in „ „ „ 3 in.,

(iv) „ „ „ „ 2.1 in „ 3.6 in „ „ „ 1.22 in

Ex. 24 Find the area of a trapezium ABCD given $AB=2.5$, $BC=5$, $AD=3$, and angles at A and B right angles

Ex. 25. ABC is a straight line, $AB=7$ in, $BC=5$ in. At A, B, C are erected perpendiculars $AP=10$ in, $BQ=14$ in, $CR=8$ in. Find the area of the pentagon ACRQP.

Ex. 26. (Inch paper) Find the area of the quadrilateral (1, 1), (3, 0), (3, 3), (1, 2)

Ex. 27 Draw a trapezium one of whose parallel sides is 3", the angles at the extremities of this side being 70° and 50° , and the height 2". Measure the parallel side and find the area of the trapezium

¶Ex. 28 If the parallel sides of a trapezium become equal, what does the trapezium become? What does the above rule lead to in this case?

¶Ex. 29. If one of the parallel sides is reduced to zero, what does the rule lead to?

Ex 30. Draw a trapezium on a doubled sheet of paper, cut it out and place the two parts together so as to form a parallelogram. Hence prove the rule for finding the area of a trapezium

Ex 31 Prove the rule for finding the area of a trapezium, by drawing in fig. 163 a parallel to AD through C

AREA OF ANY RECTILINEAR FIGURE

§7 There are three general ways of finding the area of any rectilinear figure

METHOD I By dividing up the figure into triangles

Ex 32 Construct the quadrilateral $ABCD$ and find its area (draw a diagonal) when

(i) $DA = 1 \text{ in.}$, $\angle A = 100^\circ$, $AB = 2 \text{ in.}$, $\angle B = 64^\circ$, $BC = 1 \text{ in.}$

(ii) $AB = 5 \text{ cm.}$, $BC = 5 \text{ cm.}$, $CD = 1 \text{ cm.}$, $DA = 3 \text{ cm.}$, $\angle A = 76^\circ$

Ex. 33 Construct a regular pentagon on a side of 2 in. and find its area [Join one vertex to two opposite vertices]

Ex 34 Repeat the previous example, using one triangle and a trapezium

METHOD II By the “base line and offsets” method In figs 164, 165, AB is the “base line” and the perpendiculars to AB are the “offsets.” Notice that the figures obtained are right-angled triangles or right-angled trapezia.

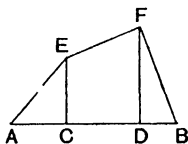


Fig. 164

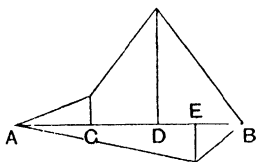


Fig 165

Ex. 35 In fig 164 $AC = 12 \text{ yd}$, $CD = 15 \text{ yd}$, $DB = 8 \text{ yd}$, $CE = 14 \text{ yd}$, $CF = 20 \text{ yd}$. Find the area

Ex 36 In fig 165 find the area in acres from the following data —

	C	D	E	B
Distance from A (in chains) *	4	7.5	9.5	12
Offset (in chains)	1.5	6	2	—

* $1 \text{ chain} = 22 \text{ yd}$ $1 \text{ acre} = 10 \text{ sq chains}$

METHOD III* It is often convenient to construct a single triangle equivalent to the given figure

¶Ex. 37. Show how to transform a given quadrilateral $ABCD$ into an equivalent quadrilateral $ABCD'$, so that the three vertices A, B, C may be unchanged, and $\angle BAD' = 170^\circ$ [Join AC]

¶Ex. 38. Repeat Ex. 37, making $\angle BAD' = 180^\circ$ What kind of figure is produced?

To construct a triangle equivalent to a given quadrilateral $ABCD$.

Construction Join CA . Through D draw $DD' \parallel CA$, meeting BA produced in D' .

Join CD'

Then $\triangle BCD' = \text{quadrilateral } ABCD$

Proof Since DD' is $\parallel$ to AC

$$\triangle ACD' = \triangle ACD$$

Th 26

Add to each $\triangle ACB$.

$$\triangle BCD' = \text{quadrilateral } ABCD.$$

In a similar way a pentagon may be reduced, first to an equivalent quadrilateral and then to an equivalent triangle. and so for figures of more sides. The area of the triangle can then be found as already explained. A convenient method of dealing with the pentagon is shown in fig 167

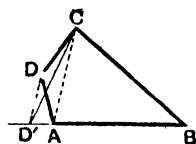


Fig 166

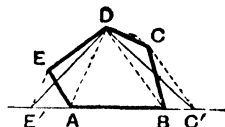


Fig 167

†Ex. 39 Explain the construction of fig. 167, and prove that

$$\triangle C'DE' = \text{figure } ABCDE$$

†Ex. 40 Given a quadrilateral $ABCD$, construct an equivalent triangle on base AB having $\angle A$ in common with the quadrilateral (*Freehand*)

Ex 41 Reduce the quadrilaterals in Ex 32 to triangles, and so find their areas

Ex 42. Find the area of a pentagon $ABCDE$, given $AB = 6.5 \text{ cm}$, $BC = 2.4 \text{ cm}$, $CD = DE = 4 \text{ cm}$, $EA = 2.5 \text{ cm}$, $\angle A = 80^\circ$, $\angle B = 133^\circ$. (Reduce the pentagon to a triangle)

* This should be postponed till after Theorem 26 and the exercises on it have been worked through. See p. 112

THEOREMS ON AREA.

§8. THEOREM 25 **Parallelograms on the same base and between the same parallels (or, of the same altitude) are equivalent.**

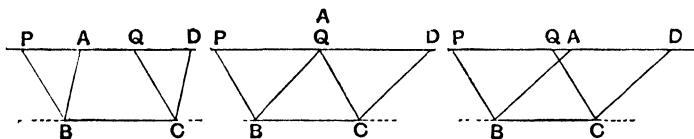


Fig 168

Data $ABCD$, $PBCQ$ are $\parallel$ ograms on the same base BC , and between the same parallels BC , PD

To prove that $ABCD$ and $PBCQ$ are equivalent.

[These figures are left if $\triangle$ s PBA , QCD are taken from $PBCD$]

Proof In the $\triangle$ s PBA , QCD ,

$$\begin{cases} \angle BAP = \text{corresp } \angle CDQ \text{ (} BA, CD \text{ are } \parallel), & \text{Th 5} \\ \angle BPA = \text{corresp } \angle CQD \text{ (} \because BP, CQ \text{ are } \parallel), & \text{Th 5} \\ BA = CD \text{ (opp sides of } \parallel\text{ogram } ABCD), & \text{Th 20} \end{cases}$$

$$\triangle PBA \equiv \triangle QCD. \quad \text{Th 11}$$

Now if $\triangle PBA$ is subtracted from figure $PBCD$, $\parallel$ ogram BQ is left, and if $\triangle QCD$ is subtracted from figure $PBCD$, $\parallel$ ogram BQ is left.

Hence the $\parallel$ ograms are equivalent

COR 1. The area of a parallelogram is measured by the product of the base and the altitude

(For the $\parallel$ ogram is equivalent to a rectangle on the same base and of the same altitude, whose area = base $\times$ altitude.)

COR 2 Parallelograms on equal bases and of the same altitude are equivalent

(For they can be so placed as to be on the same base and between the same parallels.)

THEOREM 26 **Triangles on the same base and between the same parallels (or, of the same altitude) are equivalent.**

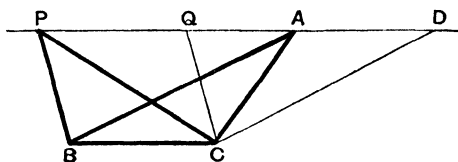


Fig 169

Data $\triangle ABC$, $\triangle PBC$ are $\triangle$ s on the same base BC , and between the same parallels BC , PA

To prove that $\triangle$ s ABC , PBC are equivalent.

Construction Complete the $\parallel^{\text{ogram}}$ s $ABCD$, $PBCQ$ by drawing CD , $CQ \parallel$ to BA , BP respectively, to meet PA (produced if necessary) in D , Q

Proof Then $\triangle ABC = \frac{1}{2} \parallel^{\text{ogram}} ABCD$, *Th. 20 (ii)*

and $\triangle PBC = \frac{1}{2} \parallel^{\text{ogram}} PBCQ$ *Th. 20 (ii)*

But $\parallel^{\text{ogram}}$ s $ABCD$, $PBCQ$ are equivalent, being on the same base and between the same parallels. *Th. 25*

$$\triangle ABC = \triangle PBC$$

COR 1 Triangles on equal bases and of the same altitude are equivalent. (For they can be so placed as to be on the same base and between the same parallels)

The converse theorem is also true, namely —

Equivalent triangles on the same base and on the same side of it are between the same parallels

EX 43 Draw a triangle PQR , mark X , Y the mid-points of PQ , PR join XY , QY , RX . By *Th. 23*, p. 73, we know that XY is $\parallel$ to QR . (i) Name two equivalent triangles on the base QR . (ii) Name two equivalent triangles on the base XY . (iii) Name two equivalent triangles with X as their common vertex. (iv) Name two equivalent triangles with Y as their common vertex. (v) Prove that $\triangle$ s PXR , PYQ are equivalent. (vi) If QY , RX intersect at O , prove that $\triangle$ s OQX , ORY are equivalent

†Ex. 44 In fig. 170, PA is parallel to BC . Prove that $\triangle POB = \triangle AOC$

†Ex. 45 A line parallel to the base BC of $\triangle ABC$ cuts the sides AB , AC in D , E respectively. Prove that $\triangle ABE = \triangle ACD$.

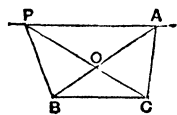


Fig 170

†Ex. 46. F is any point on the base BC of $\triangle ABC$. E is the mid-point of BC. ED is drawn parallel to AF. Prove that $\triangle DFC = \frac{1}{2} \triangle ABC$ (See fig 171.)



Fig 171.

†Ex. 47. Draw a line through a given point of a side of a triangle to bisect the area of the triangle. (See Ex. 46.)

†Ex. 48. O is a point inside a parallelogram ABCD, prove that

$$\triangle OAB + \triangle OCD = \frac{1}{2} \text{ABCD.}$$

¶Ex. 49. Draw a scalene triangle, and transform it into an equivalent isosceles triangle on the same base. (Keep the base fixed, where must the vertex be in order that the triangle may be isosceles? Where must the vertex be in order that the triangle may be equivalent to the given triangle?) (*Freehand*)

¶Ex. 50. Show how to transform a given triangle

(i) into an equivalent right-angled triangle

(ii) into an equivalent triangle on the same base, having one side of 2 inches. Is this always possible?

(iii) into an equivalent triangle having one angle equal to a given angle

(iv) into an equivalent right-angled triangle with one of the sides about the right angle equal to 5 cm. (First make one side 5 cm, then take this as base and make the triangle right-angled.)

(v) into an equivalent isosceles triangle with base equal to a given line

Ex. 51. Show how to construct, on a base of given length, a triangle equivalent to a given triangle

Ex. 52. Show how to construct a triangle having an altitude of given length and area equal to that of a given triangle

Ex. 53. Show how to transform a given rectangle into an equivalent rectangle having a side equal to a given length

THEOREM 28 If a triangle and a parallelogram stand on the same base and between the same parallels, the area of the triangle is half that of the parallelogram.

Data $\triangle EBC$ and $\parallel^{\text{ogram}} ABCD$ stand on the same base BC and between the same parallels BC, AE

To prove that

$$\triangle EBC = \frac{1}{2} \parallel^{\text{ogram}} ABCD$$

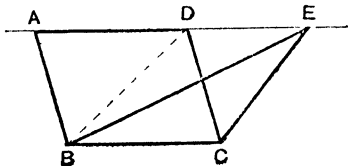


Fig 172

Construction Join BD

Proof Since AE is $\parallel$ to BC, $\therefore \triangle EBC = \triangle DBC$,

$$\text{and } \triangle DBC = \frac{1}{2} \parallel^{\text{ogram}} ABCD,$$

$$\therefore \triangle EBC = \frac{1}{2} \parallel^{\text{ogram}} ABCD.$$

Th. 26.

Th. 20 (iii)

†Ex 54 Show how to construct a rectangle equal to a given triangle Give a proof

†Ex. 55 F, E are the mid-points of the sides AD, BC of a parallelogram ABCD, P is any point in FE. Prove that $\triangle APB = \frac{1}{2}ABCD$

†Ex 56 P, Q are any points upon adjacent sides AB, BC of a parallelogram ABCD, prove that $\triangle CDP = \triangle ADQ$.

†Ex. 57 ABCD is a parallelogram, P, Q the mid-points of AB, AD Prove that $\triangle APQ = \frac{1}{8}$ of ABCD (Join PD, BD)

Ex 58 Draw a parallelogram of sides 8 cm and 5 cm. and an angle of 70° . Draw a rectangle of equal area on base 8 cm and find the area. Calculate the height of an equal triangle on base 8 cm

Ex 59 Draw two parallel lines 5 cm apart On one take three points A, B, C such that $AB=3$ cm, $BC=4$ cm Draw $\angle BAN$ equal to 65° , and draw BS parallel to AN, the points N and S being on the second of the parallel lines. Join SC

(i) Find the areas of the parallelogram ANSB and of the triangle SBC

(ii) Draw a parallelogram between the same parallels equal in area to their difference

MISCELLANEOUS RIDERS ON AREAS

†Ex 60 A straight line is drawn parallel to the base BC of a triangle ABC cutting AB at X and AC at Y, prove that triangles XBC, YBC are equivalent, also triangles ABY, ACX

†Ex 61 X and Y are points in the sides PQ and PR respectively of a triangle PQR If XY is parallel to QR and if QY, RX intersect at O, prove that the triangles OXQ, OYR are equal in area

†Ex 62 ABC is a triangle, AB is produced to X so that $BX=AB$ AC is produced to Y so that $CY=AC$ Prove that the triangles BCX, BCY are equivalent

†Ex 63 ABCD is a quadrilateral with AB parallel to CD, P is the mid-point of AB and Q the mid-point of CD Prove that PQ divides the quadrilateral into two equivalent areas

†Ex 64 Draw a triangle PQR, bisect QR at O, join OP and bisect it at X, join QX, RX Prove that the four triangles so formed are equal in area

†Ex 65 The diagonals of a quadrilateral ABCD intersect at X, BD is produced to Y so that $DY=BX$ Prove that $\triangle ACY$ is equal in area to the quadrilateral

†Ex 66 ABCD is any quadrilateral and X is the mid-point of BD, prove that the sum of the areas of the $\triangle s$ AXD and BXC is equal to half the area of the quadrilateral

†Ex 67 AB, CD are parallel sides of a trapezium ABCD, E is the mid-point of AD, prove that $\triangle BEC = \frac{1}{2}$ trapezium (Through E draw a line parallel to BC cutting the two parallel sides)

†Ex 68. Prove that the area of a rhombus is half the product of its diagonals.

†Ex 69. The base BC of $\triangle ABC$ is divided at D so that $BD = \frac{1}{3}BC$; prove that $\triangle ABD = \frac{1}{3}\triangle ABC$

†Ex 70. $ABCD$ is a quadrilateral and the diagonal AC bisects the diagonal BD . Prove that AC divides the quadrilateral into equivalent triangles (fig 173)

†Ex 71. If $ABCD$ is divided into equal areas by the diagonal AC , prove that BD is bisected by AC

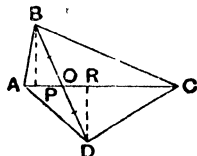


Fig 173

SUMMARY OF RESULTS

The area of a parallelogram is measured by the product of the base and the height

The area of a triangle is measured by half the product of the base and the height

The area of a trapezium is measured by the product of the height and the average of the two parallel sides.

Parallelograms on the same base and between the same parallels (or, of the same height) are equivalent

Triangles on the same base and between the same parallels (or, of the same height) are equivalent

CHAPTER 15

THE THEOREM OF PYTHAGORAS

§ I. Fig 174 represents an isosceles right-angled triangle with squares described upon each of the sides. The dotted lines divide up the squares into right-angled triangles, each of which is obviously equal to the original triangle. This sub-division shows that the square on the hypotenuse of the above right-angled triangle is equal to the sum of the squares on the sides containing the right angle. (Some tiled pavements show this fact very clearly.)

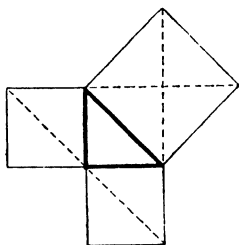


Fig 174

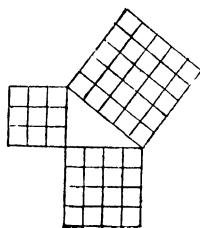


Fig 175

¶ Ex 1. Construct a right-angled triangle with sides of 3 cm and 4 cm containing the right angle. Measure the length of the hypotenuse. What relation is there between the square on the hypotenuse and the squares on the other two sides? See fig 175.

Ex. 2. Draw a good-sized scalene right-angled triangle ABC , right-angled at A . Measure the three sides and calculate the areas of the squares upon them. Add together the areas of the two smaller squares, and arrange your results like this—

$$\begin{aligned}
 AB &= \text{ cm , sq. on AB} = \text{ sq. cm ,} \\
 AC &= \text{ cm. , sq. on AC} = \text{ sq. cm. ,} \\
 \text{sum of sqq on AB, AC} &= \text{ sq. cm ,} \\
 BC &= \text{ cm. , sq on BC} = \text{ sq cm.}
 \end{aligned}$$

The above exercises lead up to the theorem of Pythagoras which is stated and proved below

This famous theorem was discovered by Pythagoras, a Greek who lived in Sicily, B.C. 570-500.

THEOREM 29 (PYTHAGORAS) In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the sides containing the right angle.

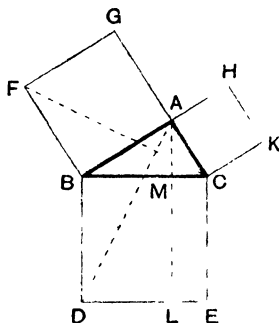


Fig 176.

Data ABC is a triangle, right-angled at A.

The figures BE, CH, AF are squares described upon BC, CA, AB respectively

To prove that $\text{sq. BE} = \text{sq. CH} + \text{sq. AF}$

Construction Through A draw AL || to BD (or CE).

Join CF, AD.

Proof [We shall prove that $\text{sq. AF, CH} = \text{rect. BL, CL}$]

$\triangle ABD \equiv \triangle FBC$	{	Rt. $\angle CBD = \text{rt. } \angle FBA$,
		add to each $\angle ABC$,
		$\angle ABD = \angle FBC$.
		Hence, in $\triangle s ABD, FBC$,
		$\left\{ \begin{array}{l} \angle ABD = \angle FBC, \\ AB = FB \text{ (sides of a square),} \\ BD = BC \text{ (" " ")}, \end{array} \right.$
		$\triangle ABD \equiv \triangle FBC$.

Th. 10.

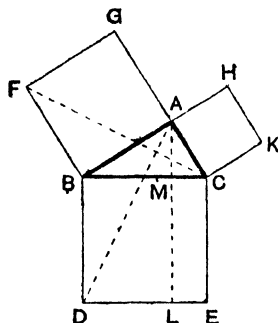


Fig 176

	Since each of the angles BAC, BAG is a right angle, $\therefore$ CAG is a st. line, <i>Th 2.</i> and this line is $\parallel$ to BF $\triangle FBC$ and sq. AF are on the same base BF, and between the same parallels BF, CG, $\therefore \triangle FBC = \frac{1}{2}$ sq. AF <i>Th 28</i> Again $\triangle ABD$ and rect. BL are on the same base BD and between the same parallels BD, AL, $\therefore \triangle ABD = \frac{1}{2}$ rect. BL. <i>Th 28</i>
$\triangle FBC = \frac{1}{2}$ sq. AF $\triangle ABD = \frac{1}{2}$ rect. BL	
But $\triangle FBC \equiv \triangle ABD$. <i>Proved</i> $\therefore$ sq. AF = rect. BL	
In a similar way, by joining BK, AE, it may be shown that sq. CH = rect. CL Hence sq. AF + sq. CH = rect. BL + rect. CL $=$ sq. BE	

†Ex 3 Alternative proofs of Pythagoras' theorem are suggested (i) by Fig 177, (ii) by Fig 178. Write out these proofs.

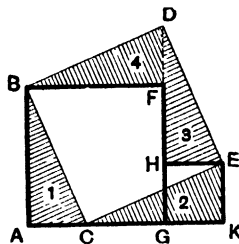


Fig 177.

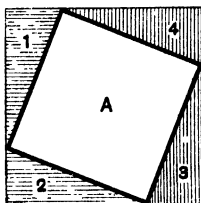
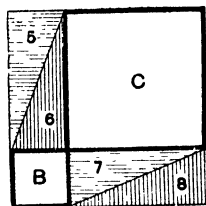


Fig 178



NOTE. Pythagoras' theorem may be expressed thus —

If $\angle BAC$ is a right angle, $BC^2 = CA^2 + AB^2$,
or $a^2 = b^2 + c^2$

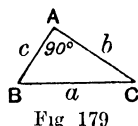


Fig 179

EXAMPLES OF THE USE OF PYTHAGORAS' THEOREM

*Calculate the height of an equilateral triangle of side 2 inches
Thence calculate the area*

ABC is the equilateral triangle

AD is $\perp$ to BC BD = DC

BD = 1 in

Let height AD = x in

Since the $\triangle ABD$ is right-angled, by Pythagoras' theorem

$$AB^2 = AD^2 + BD^2$$

$$2^2 = x^2 + 1$$

$$4 = x^2 + 1,$$

$$3 = x^2,$$

$$x = \sqrt{3},$$

the height = 1.73 in (to 2 places)

Area of triangle = $\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} BC \cdot AD = \frac{1}{2} \times 2 \times 1.73$ sq in
= 1.73 sq in. (to 2 places)

To find the ratio of the diagonal to the side of a square

ABCD is a square, side a (say)

Since the $\triangle ABC$ is right-angled, by Pythagoras' theorem

$$AC^2 = AB^2 + BC^2$$

$$= a^2 + a^2$$

$$= 2a^2$$

$$AC = \sqrt{2} a$$

$$\frac{\text{Diagonal}}{\text{Side}} = \frac{AC}{a} = \frac{\sqrt{2}}{1} = 1.41 \text{ to 2 places}$$

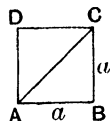


Fig 181.

Ex 4 What is the square on the hypotenuse of a right-angled triangle if the sides containing the right angle are 6 cm and 8 cm? Hence calculate the length of the hypotenuse

Ex 5 Find the hypotenuse of a right-angled triangle when the sides containing the right angle are (i) 5 cm, 12 cm, (ii) 7.8 cm, 9.4 cm., (iii) $4\frac{1}{2}$ miles, $5\frac{3}{4}$ miles, (iv) a mm, b mm.

Ex 6. Find the remaining side and the area of a right-angled triangle, given the hypotenuse and one side, as follows —(i) hyp = 15 cm., side = 12 cm, (ii) hyp = 8 in., side = 4 in., (iii) hyp = 160 mm, side = 100 mm, (iv) hyp = x in, side = y in

Ex 7. A flagstaff 40 ft high is held up by several 50 ft. ropes, each rope is fastened at one end to the top of the flagstaff, and at the other end to a peg in the ground. Find the distance between the peg and the foot of the flagstaff.

Ex 8 Find the diagonal of a rectangle whose sides are (i) 4 in. and 6 in., (ii) a cm and b cm

Ex 9 Find the remaining side and the area of a rectangle, given (i) diagonal = 10 cm., one side = 7 cm, (ii) diagonal = x'' , one side = y'' .

Ex 10 Find the diagonal of a square whose side is (i) 1'', (ii) 5'', (iii) x''

Ex 11 Find the side and area of a square whose diagonal is (i) 2 in, (ii) 14.14 cm., (iii) r feet

Ex 12 Find the side of a rhombus whose diagonals are (i) 16 cm, 12 cm, (ii) x in, y in.

Ex 13 Find the altitude of an isosceles triangle, given (i) base = 4 in, side = 5 in, (ii) base = 64 mm, side = 40 mm, (iii) base = $2x''$, side = y''

Ex 14 In fig 182, ABCD represents a square of side 3 in, AE = AH = CF = CG = 1 in. Prove that EFGH is a rectangle, find its perimeter and diagonal.

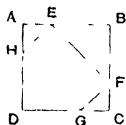


Fig 182

Ex 15 Find how far a traveller is from his starting-point after the following journeys —(i) first 5 miles N, then 4 miles E, (ii) first 4 miles E., then 5 miles N, (iii) 40 km S W and 30 km. S E, (iv) 14 miles S, 10 miles E., 4 miles N. (try to complete a right-angled triangle having the required line for hypotenuse), (v) 4 miles E, 6 miles N, 3 miles E, 1 mile N

Ex 16 (Inch paper.) If the coordinates of a point P are (1, 1) and of Q (2, 3), find the distance PQ (PQ is the diagonal of a certain rectangle)

Ex 17 (Inch paper) In each of the following cases find the distance between the pair of points whose coordinates are given —(i) (2, 1) and (1, 3), (ii) (0, 0) and (3, 1)

Ex 18 Newhaven is 90 miles N of Havre, and 50 miles E of Portsmouth. How far is it from Portsmouth to Havre?

Ex 19 St Albans is 32 miles N of Leatherhead, and Leatherhead is 52 miles from Oxford. Oxford is due W. of St Albans. How far is Oxford from St Albans?

Ex. 20. A ship's head is pointed N, and it is steaming at 15 miles per hour. At the same time it is being carried E. by a current at the rate of 4 miles per hour. How far does it actually go in an hour, and in what direction?

Ex. 21 A ladder 60 ft long is placed against a wall with its foot 20 ft from the foot of the wall. How high will the top of the ladder be?

Ex. 22 Find the distance between the summits of two columns, 60 and 40 ft high respectively, and 30 ft apart

Ex. 23 Fig. 183 represents the end view of a table. Show that the height of the table from the floor is about 25 inches

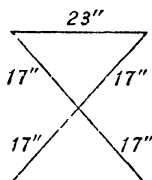


Fig. 183

Ex. 24 In fig. 184, ABCD is a square, AXYZ a rectangle. Determine the lengths XY, YZ, YC.

Ex. 25 The hands of a clock are 20 feet long and 17 feet long. Calculate, to within a tenth of a foot, the distance between the tips of the hands at 3 o'clock.

Ex. 26 The edge of the roof of a Swiss chalet is 5 metres vertically above the ground and projects 1 m beyond the wall. What length of ladder will be required so that it will just reach to the edge of the roof if placed at a point on the ground 4 m from the wall?

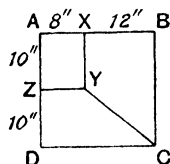


Fig. 184

Further examples on Pythagoras' theorem occur under chords, Ch 17, and tangents of circles, Ch 18

EXERCISES IN THREE DIMENSIONS

Calculate the length of a diagonal of a rectangular solid whose edges are 10", 8" and 7"

Fig 185 represents the solid, HB is the diagonal whose length is required. All the diagonals are equal.

In the $\triangle HDB$, $\angle D$ is a rt. $\angle$

by Pythagoras' theorem

$$HB^2 = HD^2 + DB^2 \quad (1)$$

In the $\triangle DAB$, $\angle A$ is a rt. $\angle$

by Pythagoras' theorem

$$DB^2 = DA^2 + AB^2 \quad (2)$$

Substituting in (1) from (2)

$$\begin{aligned} HB^2 &= HD^2 + DA^2 + AB^2 \\ &= 7^2 + 8^2 + 10^2 \\ &= 49 + 64 + 100 \\ &= 213. \end{aligned}$$

$$\therefore HB = \sqrt{213} = 14.6'' \text{ (to 1 place)}$$

i.e. length of diagonal is 14.6 in.

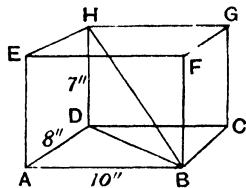


Fig. 185

Find the height of a cone whose slant side is 10" and base radius 3".

Fig 186 represents the cone, VC is the height required
(say h in)

In the $\triangle VCA$, $\angle C$ is a rt $\angle$

by Pythagoras' theorem

$$VA^2 = VC^2 + CA^2,$$

$$10^2 = h^2 + 3^2,$$

$$100 = h^2 + 9,$$

$$91 = h^2$$

$$h = \sqrt{91} = 9.54 \text{ (to 2 places)}$$

the height is 9.54 in

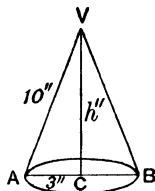


Fig 186

Ex 27 The edges of a certain cuboid (rectangular block) are 3", 4", 6", find the diagonals of the faces, and the diagonals of the cuboid

Ex 28 A box is 3 ft long, 2 ft 6 in wide, 2 ft. high Find the diagonals of the sides Find the diagonal of the box

Ex 29. Find the length of a stick which just reaches across a cubical box with edge three feet, from one corner of the bottom to the opposite corner of the top

Ex 30 Find the diagonal of the face of a cubic foot Also find the diagonal of the cube

Ex 31 Find the slant side of a cone of (i) height 5 cm , base-radius 8 cm , (ii) height 50 mm , base diameter 46 mm , (iii) height h feet, base-diameter $2r$ feet

Ex 32 Find the height of a cone of (i) slant side 10", base-radius 4", (ii) slant side 5.8 m , base-diameter 11 m , (iii) slant side l' , base-radius r'

Ex 33 Find the base-radius of a cone of (i) slant side 7 ft , height 5 ft , (ii) slant side l' , height h''

Ex 34 A pyramid stands on a base 10 ft square and is 30 ft high Calculate the length of a slant edge. (See fig 295)

Ex 35 The length of the edge of a cube is a Prove that the square on the diagonal of the cube is $3a^2$

Ex 36 The edges of a rectangular solid which meet in a point are of length a , b , c . Prove that the square on the diagonal of the solid is equal to $a^2 + b^2 + c^2$

§2 Pythagoras' theorem asserts that if a triangle is right-angled a certain relation holds between the lengths of its sides The converse theorem is also true.

THEOREM 30. If a triangle is such that the square on one side is equal to the sum of the squares on the other two sides, then the angle contained by these two sides is a right angle.

This may be stated algebraically thus:—

If the sides of a triangle are connected by the relation $a^2 = b^2 + c^2$, then the angle opposite to a is a right angle. Thus a triangle whose sides are 3, 4, 5 units is right-angled, for $5^2 = 3^2 + 4^2$

This leads to a *method of constructing a right angle* which was perhaps used by the ancient Egyptians as early as 2000 B.C., in constructing the ground plan of temples and pyramids. It is still the best method for marking out the right angles at the corners of a lawn tennis court and in much field work.

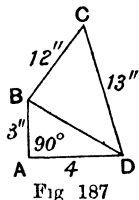
Suppose that the base-line has been marked. On a piece of rope mark off three lengths of 3, 4, 5 yards. Put in a peg at one end of the base-line, another peg on the base-line 4 yards from the corner. Stretch the rope round these two pegs, so that the ends of the 4 yard length coincide with the two pegs. Now bring the two ends of the rope together so that a triangle is formed with sides of 3, 4, 5 yards. This will give a right angle at the corner, and the side line can be marked out.

Ex 37. Are the triangles right-angled whose sides are

- (i) 8, 17, 15, (ii) 12, 36, 34, (iii) 25, 5, 25, 7, 3, 2,
(iv) $4n$, $4n^2 - 1$, $4n^2 + 1$, (v) $m^2 + n^2$, $m^2 - n^2$, $2mn$, (vi) a , b , $a + b$

Ex 38. Bristol is 71 miles due W of Reading, Reading is 55 miles from Northampton, Northampton is 92 miles from Bristol. Ascertain whether Northampton is due N of Reading.

Ex 39. Fig 187 is a freehand sketch (not to scale) of a quadrilateral with $\angle A = 90^\circ$ and the lengths shown. Prove that $\angle CBD = 90^\circ$.



Ex 40. ABC is a right angle and BD is perpendicular to AC. If $AD = 8$ in, $DC = \frac{1}{2}$ in, $BD = 6$ in, show that $\angle ABC$ is a right angle.

RIDERS ON PYTHAGORAS' THEOREM

ABC is a triangle and AD perpendicular to BC. Prove

$$AB^2 - AC^2 = BD^2 - CD^2.$$

Data AD at right angles to BC

To prove $AB^2 - AC^2 = BD^2 - CD^2$

Proof Since $\triangle ABD$ is right-angled,
by Pythagoras' theorem, $AB^2 = AD^2 + BD^2$

Since $\triangle ADC$ is right-angled,
by Pythagoras' theorem, $AC^2 = AD^2 + CD^2$

by subtraction

$$AB^2 - AC^2 = BD^2 - CD^2$$

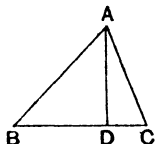


Fig 188

†Ex. 41. In fig. 189 the marked angles are right angles and the marked lines are equal. Show that $AD^2 = 3AB^2$. Give your reasons carefully.

†Ex. 42. HVQ is a triangle right-angled at V , HVT is a triangle on the opposite side of HV having $\angle THV$ a right angle; prove that the squares on HT , HQ are together equal to the squares on TV , VQ .

†Ex. 43. ABC is a triangle right-angled at B , D is the mid-point of BC . Prove that $AD^2 = AC^2 - 3BD^2$.

†Ex. 44. ABC is an equilateral triangle and AD is perpendicular to BC . Prove that $3AB^2 = 4AD^2$.

†Ex. 45. PQR is a triangle, right-angled at Q . On QR a point S is taken. Prove that $PS^2 + QR^2 = PR^2 + QS^2$.

†Ex. 46. The diagonals of a quadrilateral $ABCD$ intersect at right angles. Show that $AB^2 + CD^2 = BC^2 + DA^2$.

†Ex. 47. The sum of the squares on the diagonals of a rectangle is equal to the sum of the squares on its four sides.

†Ex. 48. The square described on the diagonal of a given square is double the given square.

†Ex. 49. The sum of the squares on the diagonals of a rhombus is equal to four times the square on one side.

†Ex. 50. $PQRS$ is a quadrilateral with $\angle P$ a right angle. Given that $SR^2 = SP^2 + PQ^2 + QR^2$, prove that $\angle SQR$ is a right angle.

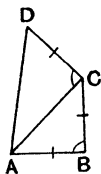


Fig. 189

Ex. 51. In fig. 188 let $AB = 3$ in., $AC = 2$ in., $BC = 3$ in. Calculate $BD^2 - CD^2$. Hence find $BD - CD$.

$$[BD^2 - CD^2 = (BD - CD)(BD + CD) = (BD - CD)BC]$$

Knowing $BD - CD$ and $BD + CD$, you may now find BD and CD . Hence find AD . Hence find area of $\triangle ABC$.

Ex. 52. Find the area of a triangle whose sides are 4 in., 5 in., 6 in. by the method of Ex. 51.

SUMMARY OF RESULTS

In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the sides containing the right angle.

If a triangle is such that the square on one side is equal to the sum of the squares on the other two sides, then the angle contained by these two sides is a right angle.

MISCELLANEOUS EXERCISES B

NUMERICAL AND PRACTICAL

Ex 1. Draw a triangle ABC having $\angle A = 40^\circ$, $\angle B = 82^\circ$. Construct the bisectors of the angle A and the perpendicular bisector of BC . Measure the angle between these lines. Verify your result by calculation.

Ex 2. Draw any circle and three lines touching it. Let the lines meet in A, B, C . Bisect the angles at A, B . Where do the bisectors intersect?

Ex 3. A and B are two points 3 in. apart. Show in an accurately constructed figure, the positions of all points equidistant from A and B which are not less than 2 in. from both A and B , and not more than 3.5 in. from A and B . Explain your work.

Ex 4. Construct a rhombus with sides 3 in. and an angle of 120° . Bisect each side and join the points of bisection in order. What figure is formed?

Ex 5. Mark any two points A and C . At A make an angle of 70° and at C an angle of 110° . Let the arms of the angles meet to form a quadrilateral $ABCD$. Construct the perpendicular bisectors of each side. What fact do you notice?

Ex 6. Draw an equilateral triangle ABC of side 2 in. Find all the points that are 1 in. from BC and also 3 in. from A . Explain how you find the points. What figure do they form? What symmetry does the whole of your drawing possess?

Ex 7. Construct a right-angled triangle, given hypotenuse = 10 cm., another side = 5 cm. Measure the smallest angle.

Ex 8. Construct a right-angled triangle ABC , given $\angle A = 90^\circ$, $AB = 7$ cm., distance of A from $BC = 2.5$ cm. Measure the smallest angle.

Ex 9. Construct a triangle, given height = 2 in., angles at the extremities of the base = 40° and 60° . Find length of base.

Ex 10. Construct a quadrilateral $ABCD$, given diagonal $AC = 9$ cm., diagonal $BD = 10$ cm., distances of B, D from AC 5 cm. and 4 cm. respectively, side $AB = 7$ cm. Measure CD .

Ex 11. Construct a trapezium $ABCD$, given base $AB = 10$ cm., height = 4 cm., $AD = 4.5$ cm., $BC = 4.2$ cm. Measure angles A and B . (There are 4 cases.)

Ex 12. A board 4 ft. square rests in a vertical plane on one corner with the lower edge at an angle of 20° with the horizontal. Find by drawing to scale the height of the top corner above the horizontal.

Ex 13. A table is 4 ft. square and stands on four legs at its corners each 2' 9" high. It is tilted on its two front legs through an angle of 20° . Find by drawing to scale what is now the height of the back edge above the floor.

Ex 14 Draw two parallel lines 2 in apart and mark any point P between them. Through P draw a line making angles of 50° with the parallels. Explain your method.

Ex 15. Draw two parallel lines 6 cm apart and mark any point P outside (or between) them. Through P draw a straight line so that the length of the part cut off by the parallel lines is 9 cm. State and give reasons for your method.

Ex. 16. Draw two lines OX, OY meeting at an angle of 50° and mark a point P between them (not close to O). Through P draw a line making equal angles with OX, OY . Justify your work.

Ex 17 Draw two lines OX, OY at an angle of 40° . Draw a line perpendicular to OX and ending on OY of length 1.5 in. Give reasons for your work.

Ex. 18 Draw any large triangle ABC with the angle B acute. Draw a line parallel to AC so that the length cut off between AB and BC is 5 cm.

Ex. 19 Draw two lines $AB=3$ in, $CD=4$ in intersecting at any angle. Join the ends to form a quadrilateral $ADBC$ and bisect the sides in P, Q, R, S . Measure the sides of $PQRS$. Will their lengths depend on how AB, CD are drawn? What is the figure $PQRS$?

Ex 20 Two pairs of parallel lines each 6 cm apart intersect to form a quadrilateral. What is this figure? Find its area when an angle of 50° is formed.

Ex. 21 Draw a rectangle of area 44 sq cm on a base of 8 cm. Construct an equivalent parallelogram on the same base with a diagonal 7.5 cm. Find the length of its side.

Ex 22 Construct an equilateral triangle with a side of 2.5" and produce one side a further .5". With the line 3" as base construct a triangle equal in area to the equilateral triangle with one angle common. What is the area of either triangle?

Ex 23 Construct a parallelogram with sides 3 in and 2 in at an angle of 60° . Construct a parallelogram equal in area to this with the angle of 60° common and one side 3.75 in. What is the length of the other side?

Ex. 24 Draw a trapezium having its parallel sides 8 cm and 6 cm, altitude 5 cm, and one of its acute angles 68° . Transform the trapezium into an equivalent triangle. Describe your construction briefly and show how the rule for finding the area of a trapezium follows directly from your new figure.

Ex 25 Construct an isosceles triangle with equal sides of 2.5 in containing an angle of 130° . On one equal side construct another isosceles triangle equal in area to the first. What is its height?

Ex. 26. A parallelogram is constructed on a base of 25 cm. with a height of 12 cm. What is the area of each of the four triangles into which it is divided by its diagonals?

Ex. 27. $ABCD$ is a vertical side of a rectangular tank, the vertical edge $AB=3$ ft 6 in., the bottom $BC=5$ ft. The tank contains water to a depth of 2 ft. If the tank is tilted about the other edge through B , what is the depth of the

water along CD when the water is just on the point of overflowing? Find, by drawing, the angle of tilt

Ex 28. A triangular board, base 30 in and height 18 in, is cut into two pieces along a line parallel to the base, bisecting the height. Find the area of the larger piece. An isosceles triangle on the same base is to be cut from this piece, how much more wood must be removed?

Ex 29 An upright pole 40 ft high stands on level ground, a rope is stretched from the top of the pole to a point on the ground 25 ft. from the bottom of the pole. Find by calculation the length of the rope

Ex 30. A picture cord 3 ft. long is attached to two rings 10 in. apart in the upper edge of a picture frame and the picture is hung from a nail. Find by calculation the vertical depth of the upper edge of the frame from the nail

Ex 31 ABCD is a parallelogram with $AB = 1''$, $BD = 4''$, $BC = 3''$. Calculate the distance between each pair of parallel sides, and the area of the parallelogram

Ex 32 A four-sided field ABCD has the sides $AB = 240$ yd, $BC = 156$ yd, $CD = 208$ yd, $DA = 100$ yd and $\angle A$ is a right angle. Is the angle C a right angle?

Calculate the distances of A and C from the diagonal BD

Ex 33 In fig 190 OX and OY are two lines at right angles and ABCD is a square. The positions of A and B are given as follows —

	A	B
Distance from OX	7.3 in	8.5 in
„ „ OY	8.0 in	9.6 in

Find the distances of C from OX and OY and the area of the square [An accurate figure is not required]

Ex 34 ABC is a triangle with $AB = 72$ mm, $CA = 48$ mm. BQ, CR are drawn perpendicular to AC, AB respectively, if $BQ = 36$ mm, calculate BC, CR

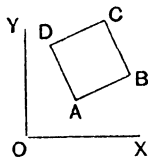


Fig 190

Ex 35 A rectangular piece of paper ABCD, $AB = 4$ in, $BC = 3$ in, is folded so that AB falls on CD. An isosceles triangle of base 2 in, height 1 in, with its base placed symmetrically along the creased edge, is then cut out. The paper is now unfolded and spread out flat. Draw the figure obtained

RIDERS.

†**Ex 36** ABCD is a parallelogram in which $\angle A = 60^\circ$. From AD cut off $AE = AB$, join BE and produce it to cut CD produced at F. Prove that $\triangle DEF$ is equilateral

†**Ex 37.** ABC is a triangle in which $\angle C$ is three times $\angle B$. Find a point D in AB such that $DB = DC$ and prove that in this figure $AD = AC$

†**Ex 38** X is the vertex of an isosceles right angled triangle XYZ; YR bisects $\angle Y$ and meets XZ at R, RN is drawn perpendicular to YZ. Prove that $NZ = XR$

†Ex. 39. ABC is an equilateral triangle, DBC is an isosceles triangle on the same base BC and on the same side of it, and $\angle BDC = \frac{1}{2}\angle BAC$. Prove that $AD = BC$

†Ex. 40. In a triangle ABC , the bisector of the angle A and the perpendicular bisector of BC intersect at a point D ; from D , DX , DY are drawn perpendicular to the sides AB , AC produced if necessary. Prove that $AX = AY$ and $BX = CY$. [Join BD , CD] Does it follow that $AB = AC$? (See W. W. R. Ball, *Mathematical Problems and Recreations*, p. 42.)

†Ex. 41. PQR is a triangle, and S is the mid-point of QR . From S , ST is drawn parallel to QP , meeting PR in T , and SU parallel to RP , meeting PQ in U . Prove $SU = RT$ and also $SU = TP$.

†Ex. 42. Draw an isosceles triangle ABC ($AB = AC$) with $\angle A$ less than 60° (make it roughly 30° say). Show how to find a point X in AB such that $AX = XC$. Determine the angles of the given isosceles triangle for the case in which AX , XC , BC are all equal.

†Ex. 43. $ABCD$ is an equilateral four-sided figure. A point P is such that $PA = PC$. Prove that P , B , D are in one and the same straight line.

†Ex. 44. PS , the bisector of the angle P of a triangle PQR , cuts QR at S , through S , ST and SU are drawn parallel to PR and PQ , thus forming a quadrilateral $TPUS$, prove that $TPUS$ is a rhombus.

†Ex. 45. $ABCDE$ is a regular pentagon. AC and BD intersect at O . Prove that the figure $AODE$ is a rhombus.

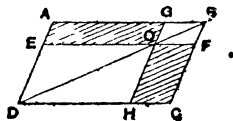
†Ex. 46. ABC is a triangle right-angled at A , with squares $ACKH$, $BCED$ drawn on AC , BC (as in the figure of Pythagoras' theorem); prove that triangles BCK , ECA are congruent.

†Ex. 47. $ABCD$ is a parallelogram whose diagonals intersect at O . Through O a straight line is drawn to cut BA produced and DC produced in P and Q respectively. Prove that $PD = QB$.

†Ex. 48. If the diagonal AC of a quadrilateral $ABCD$ divides it into two parts of equal area, prove that the diagonal BD is bisected by AC .

If, in addition, AC bisects the angle at A , prove that the triangles ACB , ACD are equal in all respects.

†Ex. 49. In the given figure O is any point on the diagonal BD of a parallelogram $ABCD$. EOF , GOH are parallel to AB , BC respectively. Prove that parallelogram $AO =$ parallelogram CO .



†Ex. 50. E is any point on the diagonal AC of a parallelogram $ABCD$. Prove that $\triangle ABE = \triangle ADE$.

†Ex. 51. Produce the median BD of a triangle ABC to E , making $DE = DB$. Prove that $\triangle EBC = \triangle ABC$.

†Ex. 52. L , M are the mid-points of the parallel sides AB , DC of a trapezium $ABCD$. Prove that LM bisects the trapezium.

†Ex. 53. In Ex. 52, O is the mid-point of LM , prove that any line through O which cuts AB , CD (not produced) bisects the trapezium.

†Ex. 54 The medians BD , CE of $\triangle ABC$ intersect at G , prove that quadrilateral $ADGE = \triangle BGC$. (Add to each a certain triangle.)

†Ex. 55 ABC is a triangle, any three parallel straight lines AP , BQ , CR meet BC , CA , AB produced where necessary at P , Q , R . Prove that the triangle AQR is equal in area to the triangle CPQ .

†Ex. 56 Show that if parallels are drawn to the diagonals through the vertices of a rhombus, they intersect to form a rectangle whose area is twice that of the rhombus.

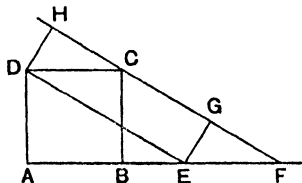
†Ex. 57 $ABCD$ is a parallelogram and P is a point in CD . Prove that the difference between the areas of triangles PAB , PAD is equal to the area of triangle PAC .

†Ex. 58 Draw a quadrilateral $ABCD$, from B and C draw BE and CF perpendicular to AD . Prove that the area of the quadrilateral $ABCD$ is equal to the sum of the areas of the triangles ABE and ECF .

[First consider the case in which angles A and D are both acute.]

Ex. 59 In the given figure $ABCD$ is a square and $DEGH$ is a rectangle. Prove that they are equal in area.

†Ex. 60 Draw an angle ABY , on AB , BY take points X , C , complete the parallelograms $ABCD$, $XEYZ$. Prove that, if the parallelograms are equal in area, XC is parallel to DZ .



†Ex. 61 $ABCD$ is a rectangle and P is a point of trisection of BC . Prove that $AP^2 + PD^2 = \frac{5}{3} AD^2$.

†Ex. 62 A straight line AB is produced to C , so that $AC = 3AB$, on BC an equilateral triangle BCD is described. Prove that the square on AD is seven times the square on AB .

†Ex. 63 If the diagonals of a quadrilateral intersect at right angles, prove that the sum of the squares on one pair of opposite sides is equal to the sum of the squares on the other pair of opposite sides.

REVISION PAPERS B

Paper 7.

1. What is the meaning of the prefixes penta-, hepta-, deca-, octa-?
2. Define a parallelogram and enumerate its principal properties
- †3. ABC is an isosceles triangle ($AB=AC$), through C is drawn CD at right angles to BC, CD cuts BA produced at D. Prove that ACD is an isosceles triangle.
- †4. From the ends of a line AB equal and parallel lines AM, BN are drawn in the same sense, the angle BAM being acute. AN, BM are drawn meeting at O. Prove the triangles AOM, NOB to be congruent.
5. In a given line a point A is taken, and B is a point outside the line. Find a point in the line equidistant from A and B. Can such a point always be found?
6. Draw a parallelogram having sides of 4 cm. and 6.5 cm. and an angle of 75° . Find its area.
- †7. Prove that the line joining the vertex of a triangle to the middle point of the base divides the triangle into two triangles of equal area. How could you divide a triangle into three parts of equal area?
8. Bristol is 90 miles due West of London, Brighton is 40 miles due South of London. Calculate the distance from Bristol to Brighton. (You may leave your answer in the form of a square root, but give a very rough approximation to its value.)

Paper 8.

1. What is meant by the phrase "ruler and compass constructions"? Why is such a construction likely to be peculiarly accurate?
2. Describe two types of symmetry. How many axes of symmetry has a regular pentagon? a regular hexagon?
- †3. PQR is an isosceles triangle having $PQ=PR$. A straight line is drawn perpendicular to QR and cuts PQ, PR (one of them produced) in X, Y. Prove that the triangle PXY is isosceles.
- †4. The side BC of an equilateral triangle ABC is produced to D so that $CD=BC$. Prove that the perpendiculars to AC drawn through B and C respectively trisect AD.
5. Construct a parallelogram ABCD whose sides AB, AD are 5.4" and 3" while the distance between AB and CD is 2.6". Measure the angle ADC. Find a point P in AB which is equidistant from D and C. Measure PB. State your construction.
6. Find the area of a triangle ABC in which $BC=10$ cm., $\angle B=30^\circ$, $\angle C=40^\circ$. Do this first taking BC as base, then taking AC as base. Tabulate your measurements carefully. Take the average of your two results.

7. A ship is situated 45 miles from a straight shore. Two beacons on the shore are respectively 6 miles and 89 miles from the ship. Calculate the distance between the beacons.

†8. Draw a triangle PQR (not isosceles); draw HK parallel to QR to cut PQ , PR at H , K , let HR and QK intersect at S . Name two triangles on the base QR which are equal in area. Also prove that triangles HSQ , KSR are equal in area.

Paper 9.

1. A rhombus and a rectangle have certain properties in common. State them.

2. In marking out a football ground, after having made the opposite sides equal, I was uncertain if it was rectangular. What test should I apply to find out if it is so?

†3. CX , the bisector of an exterior angle of $\triangle ABC$, which is not isosceles, meets AB in X . Prove that $\angle AXC$ is equal to half the difference between the angles A and B .

†4. $ABCD$ is a parallelogram, X , Y are the mid-points of AB , CD . Prove that $DX = BY$.

5. Draw XOX' , YOY' , two straight lines intersecting at O , so that $\angle XOY = 50^\circ$. Make $OX = 4$ in., $OY = 3$ in. Find a point P equidistant from X , Y , and at the same time equidistant from XOX' , YOY' . Find another such point, Q . Explain (in two or three lines) how these points are found. Measure OP , OQ in inches.

6. How is the area of a parallelogram measured?

Construct a rhombus whose area is 8 sq. in. and whose sides are each $3\frac{2}{3}$ long.

†7. Assuming that the medians of a triangle ABC pass through one point, prove that the six triangles into which they divide the triangle ABC are equal in area to one another.

8. A volcanic mountain is in the shape of a cone 4000 ft high. The base is a circle of 8000 ft radius. Calculate (to the nearest tenth of a mile) the length of a rack-and-pinion railway which takes the shortest way to the top.

Paper 10.

1. What is a rectilinear figure? Describe shortly how to copy such a figure by the method of intersecting arcs.

2. How is the area of any rectilinear figure determined?

†3. If the bisector of an exterior angle of a triangle is parallel to one of the sides, prove that the triangle is isosceles.

†4. $ABCD$ is a parallelogram, through A a straight line is drawn parallel to the diagonal BD to cut CD produced at E . Prove that $CD = DE$.

[See next page

5 A and B are two points on paper. What is the locus of the point C under the following conditions —*firstly*, when C is restricted to being in the plane of the paper, *secondly*, when C may be anywhere in space. —(i) C is equidistant from A and B. (ii) Angle CAB is 20° . (iii) C is always 1 inch from AB (which may be produced indefinitely in both directions)

6 The area of a triangle is 24 square inches, the altitude is 8", find the length of the base, and on it describe a parallelogram, the area of which shall be 48 square inches

7 Amiens is 70 miles due north of Paris, Dieppe is 54 miles due west of Amiens. What is the direct distance from Dieppe to Paris?

†8 XN is an altitude of a triangle XYZ, prove that $YN^2 - ZN^2 = XY^2 - XZ^2$. Take the triangle to be acute-angled

Also draw a figure in which $\angle XZY$ is an obtuse angle, does the same proof hold for this case?

Paper 11.

1 Is each of the following statements true for any parallelogram? If not, state in each case a particular kind of parallelogram for which it is true. No proofs are required. (i) The diagonals bisect one another. (ii) The diagonals bisect the angles. (iii) The opposite angles together make two right angles. (iv) The diagonals are equal.

2 What is a locus? Give two instances of loci.

†3 The side BA of $\triangle ABC$ is produced to D so that $AD = AC$. AE bisects $\angle BAC$. Prove that (i) $\angle BAC = \text{twice } \angle ADC$, (ii) AE is parallel to DC.

†4 ABCD is a parallelogram, DA and DC are produced to X and Y respectively so that $AX = DA$ and $CY = DC$, XB and BY are drawn. Prove that XBY is a straight line.

5 Draw a plan of a room which is 16 feet long and 12 feet wide. Mark the position of a point which is 6 feet from one of the long walls and 4 feet from one of the shorter walls. Calculate how far this point is from the nearest corner.

6 Define a "trapezium". How is its area found?

Find the area of the cross-section of an embankment 10 ft high, which is 20 ft wide at the top and 30 ft at the bottom.

†7 Call the corners of a rectangular sheet of paper A, B, C, D (AB being a long side of the rectangle), if it were folded along the diagonal AC, then AB and CD would cut at a point we will call O. Make a freehand sketch of the figure you would obtain and prove triangles BCO, ADO, (in the folded figure) equal in area. Show that DB in its new position will be parallel to AC.

8. Construct (without any calculation) a square whose area shall be equal to the difference between the areas of two squares of sides 7 and 4 cm. Measure the length of its side.

Paper 12.

1 In drawing a perpendicular to a line, why should the perpendicular be produced a short distance beyond the line? How is this done when the perpendicular is drawn with a set square?

2 Mention two theorems the truth of which is evident from considerations of symmetry

3. ABCD is a parallelogram, E is the mid-point of AB, CE and DA are produced to meet at F. What angles in the figure are equal to angles ECD and ECB? Give reasons. Also prove $AF = AD$

4 P, Q are two points 6 cm apart. PS is a straight line making an angle of 40° with PQ. Find a point (or points) equidistant from P and Q and 4 cm from PS. Measure the distance of your point (or points) from P. Is this problem always possible whatever the angle SPQ?

5. In a triangle ABC, $AB = AC = 13$ cm, $BC = 10$ cm. Find by calculation the perpendicular distance of A from BC.

6 ABCD is a square of side 4 in. P, Q are the mid-points of BC, CD respectively. Calculate the perimeter and area of the triangle PAQ

7 ABCD is a trapezium whose parallel sides AD, BC are of lengths a , b respectively and are at a distance h apart (b is greater than a)

(i) State without proof an expression in terms of a , b , h for the area of the trapezium

(ii) From A is drawn AS parallel to DC to cut BC at S, and from D is drawn DK parallel to AB to cut BC at K. Prove that the triangles ABS, DKC are equal in area

8 ABCD is a rectangle, through A a straight line XAY is drawn at right angles to AC, DE (x), BF (y) are drawn perpendicular to XY. Prove

$$EA = AF, \quad x + y = AC, \quad x^2 + y^2 = AC^2 - 2AF^2$$

Given $AC = 5$ cm, and $AF = 2.4$ cm, find the value of x and y , x to be less than y

Paper 13.

1 For how many years has it been known that there is a relation between the hypotenuse and the sides of a right-angled triangle? State the relation.

2 Draw freehand diagrams of (i) a quadrilateral with only two sides parallel which has equal diagonals, (ii) any other quadrilateral which has equal diagonals

What is the name of the quadrilateral (i)?

3. ABCD and ACPQ are a parallelogram and a rectangle on opposite sides of a straight line AB, join DQ, CP, prove that CDQP is a parallelogram.

[See next pag

4 (i) A shot is fired from an airship high overhead. Assuming that sound travels through the atmosphere at a uniform rate of 1100 feet per second, what is the solid-locus of points at which the report will be heard in one second? What is the locus of points on the ground at which the sound is heard simultaneously?

(ii) What is the space-locus of points equally distant from two given points?

5 A shelter trench of rectangular section is 3 ft wide and 4 ft. deep, the earth excavated is piled up in front as a rampart, if the vertical section is a right-angled isosceles triangle, how high is the rampart?

6. In a right-angled triangle the sides containing the right angle are 3 feet and 4 feet long; calculate the length of the hypotenuse, the area of the triangle, and the length of the altitude from the right angle to the hypotenuse.

+7. P and Q are points on opposite sides of a straight line AB. PX and QY are the perpendiculars from P and Q to AB. Prove that $PY^2 + YQ^2 = PX^2 + QX^2$.

Now suppose the triangle PXY were folded about XY till its plane was perpendicular to the plane of the triangle QXY; mention two things that would be equal to PQ^2 in this three-dimensional figure. (Give reasons.)

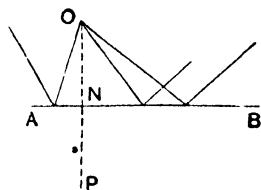
8. A paper pyramid on a square base is made as follows. On each side of a square of side 3 inches is constructed an isosceles triangle of height 5 inches, the triangles lying outside the square. A four-pointed star is thus formed, which is cut out of the paper. By folding the triangles upwards a pyramid is formed. Find its height, and the length of each of its sloping edges.

Paper 14.

1. Two roads cross in the middle of a town, one running N and S, the other E and W. The streets are parallel to these roads. By what measurements can the position of a house in the town be described?

2. Why is it impossible to *prove* a geometrical theorem by measurement, while it is possible to disprove a statement by measurement?

+3. Rays of light proceeding from a point O fall on a mirror AB and are reflected, making an equal angle with the mirror. Prove that the reflected rays, if produced backwards, would all be found to pass through a point P, the image of O in AB.



4. Treasure is known to be buried in a field 20 yards from a straight hedge, and 30 yards from a cairn, this being inside the field and 40 yards from the hedge. Show that it may be in either of two positions. Find the distance apart of these positions (i) by measurement; (ii) by calculation.

5. A and B are fixed centres, and AP, PQ, and BQ are stiff rods with pivots at A, B, P, and Q. The lengths are $AB=6$ cm., $AP=2$ cm., $BQ=6$ cm., and $PQ=8$ cm. Show that the rods can take up two positions in which AP and PQ are in the same straight line, and draw the mechanism in those positions.

In each case find the angles of the triangle AQB.

Also draw the mechanism with the angle PAB equal to 90° , and determine the angle ABQ in this case

†6. A straight line is drawn parallel to the base BC of a triangle ABC cutting AB at X and AC at Y, prove (i) that triangles XBC, YBC are equal in area, and (ii) that triangles ABY, ACX are equal in area

7. In a field in the form of a convex quadrilateral ABCD, B is due north of A and D is due east of A. Also $AB=7.5$ chains, $BC=8.4$ chains, $CD=5.3$ chains, $DA=4$ chains. Find the area of the field in acres

8. The two ends of a piece of string 17 in. long are attached to two points A, B on a board that are 13 in. apart, so that the string forms a loop. The point of a pencil is placed inside the loop and the string tightened so as to form two sides AP, PB of a triangle APB, the pencil point being at P. The pencil point can now be moved so as to draw an ellipse on the board, the string being kept taut. At a certain instant the triangle will be found to have a right angle at P. What are the lengths of AP, PB at this instant?

Paper 15.

1. The top of a mountain can be seen from a level plain. Describe fully what observations you would take, how you would take them, and how you would use them to find the height of the mountain

2. Draw a square ABCD with sides of 2 in. On BC describe an equilateral triangle PBC inside the square

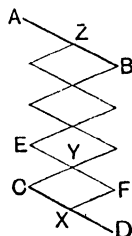
(i) How does the figure show you at once that the area of the triangle is less than half that of the square?

(ii) Find the area of the $\triangle PBC$

3. If two pairs of straight railway lines cross one another, what figure do they enclose? [You are to assume that the perpendicular distance between one pair of lines is the same as the perpendicular distance between the other pair of lines]

4. The figure gives a side-view of a collapsible periscope. It is jointed where the bars meet or cross, and all the parts such as XC, XF, CY, FY, YE, BZ are equal in length. AB and CD are the mirrors. Prove that any quadrilateral with its opposite sides equal must also have its opposite sides parallel, and deduce that for all positions of the framework of the periscope the mirrors AB, CD are parallel

Each of the lengths XC, XF, etc. is 3 inches. Calculate the distance of Z from X for the position in which the angle X is 90° .



[See next page

5. ABC is a triangle having a fixed base BC , 5 cm long, and a moveable vertex A . What is the locus of A (i) when ABC is isosceles ($AB=AC$)? (ii) when ABC has a fixed area ($=10$ sq cm)? (iii) when the median AM has a fixed length ($=6$ cm)?

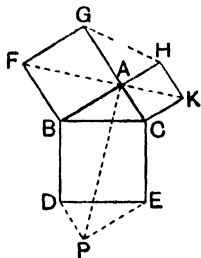
6. The lengths of the sides of a triangle are 5, 7.5 and 9 yards; determine whether one of the angles of the triangle is nearly a right angle or exactly a right angle.

7. In a four-sided field $ABCD$, $AB=470$ yd, $BC=238$ yd, $CD=327$ yd., $AC=525$ yd, and $DA=356$ yd. Draw a plan of the field to the scale of 100 yd to an inch and construct a triangle of equal area, giving a few words to explain your construction.

Find the area of the field by means of the triangle.

8. ABC is a triangle right-angled at A . On the sides BC , CA , AB are drawn squares $BCED$, $CAHK$, $ABFG$ as in the figure. EP , DP are drawn parallel to AB , AC .

Name two triangles congruent with ABC . Discuss the symmetry of the figures $FGHKCB$ and $ABDPEC$. (This leads to a proof of the theorem of Pythagoras.)



CHAPTER 16

THE CIRCLE. ARCS, CIRCUMFERENCE, AREA

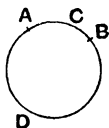
§ I A **circle** is a line, lying in a plane, such that all points in the line are equidistant from a certain fixed point, called the **centre** of the circle

In view of what has been said already about loci we may give the following alternative definition of a circle —

A **circle** is the locus of points in a plane that lie at a fixed distance from a fixed point (the centre) The fixed distance is called the **radius** of the circle

The word “circle” has been defined above to mean a certain kind of curved line The term is, however, often used to indicate the part of the plane inside this line. If any doubt exists as to the meaning, the line is called the **circumference** of the circle

To cut a circular hoop into two pieces needs two cuts. Similarly, if two points A and B (fig. 191) are taken on the circumference of



F.g 191

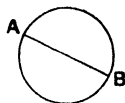


Fig 192

a circle, the circumference is divided into two **arcs** (Latin *arcus* = bow) ACB, ADB. To distinguish these, ACB is called the **minor** arc, ADB the **major** arc

But it may happen that the points A, B are so taken that the two arcs are equal. Each arc is then a **semicircle** (fig 192). The line joining the two extremities of a semicircle clearly passes through the centre it is called a **diameter**.

The term “semicircle” like the term “circle” is used in two different senses; sometimes in the sense of an arc (as in the definition), sometimes as the part of the plane bounded by a semi-circumference and the corresponding diameter.

The circumference of a semicircular protractor is graduated in equal arcs, 180 in number. Each of these equal arcs subtends at

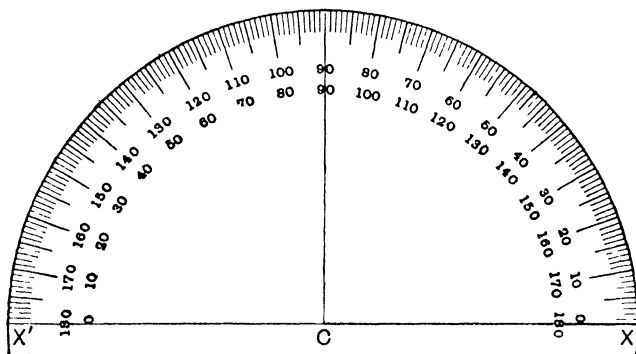


Fig. 193

the centre an angle of 1° . For this reason, the degree is commonly used as a measure of arc as well as of angle thus an arc that subtends 30° at the centre is called an **arc of 30°** . There is no danger of confusing these two meanings of the word "degree"

It is clear that **equal arcs of a circle subtend equal angles at the centre.**

In fact, **arcs of a circle are proportional to the angles which they subtend at the centre.**

Thus an arc subtending an angle of D° at the centre will be $\frac{D}{360}$ of the whole circumference.

§ 2 ¶ Ex 1 Draw a large circle and measure its diameter. Now measure its circumference by means of a piece of cotton*.

¶ Ex. 2. Measure the diameter of a cylinder. Measure its circumference by wrapping paper round it and picking through two thicknesses of paper

* The best way to do this is to mark a point on the circumference, lay the cotton along a short arc (the end of the cotton being at the point marked), hold that part of the cotton firm with one finger and fit more of the cotton along the next short arc, now hold that and free the first piece, continue the process until you have been round the whole circumference. This method is explained more fully in Notes and Answers.

¶ Ex 3 Use the results of Exx. 1 and 2 to find the ratio

$$\frac{\text{Circumference}}{\text{Diameter}}$$

The ratio $\frac{\text{Circumference}}{\text{Diameter}}$ is a constant, i.e. it is the same for all circles. This constant has been worked out to 700 places of decimals and begins thus

$$3\ 1415926535$$

For the sake of brevity this number is denoted by the Greek letter π , a useful approximation is $\frac{22}{7}$ or $3\frac{1}{7}$

$$\begin{aligned}\text{Thus circumference of circle} &= \text{diameter} \times \pi \\ &= \text{radius} \times 2\pi \\ &= 2\pi r, \text{ where } r \text{ is the radius.}\end{aligned}$$

Ex 4 Calculate the circumference of a circle whose radius is (i) 14 in., (ii) 21 miles. (Take $\pi = \frac{22}{7}$)

Ex 5 Calculate to three significant figures the circumference of the earth, measured round the equator, taking radius = 3963 miles

Ex 6 How far does a wheel roll in one revolution if its diameter is 7 in.?

Ex 7 In fig 194, AD is divided into three equal parts and all the arcs are semicircles, show that the four curved lines which connect A with D are of equal length

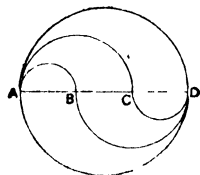


Fig 194

Ex 8. Find r the radius in terms of c the circumference of a circle

Ex 9 Calculate the radius of a circle of circumference (i) 44 ft., (ii) 60 cm (to three significant figures)

Ex 10 What is the diameter of a wheel which makes 36 turns in running over a distance of 66 ft.?

Ex. 11. What fractions of a circumference are arcs of 90° , 60° , 120° , 1° , 5° , 300° ?

Ex 12 Calculate the lengths of arcs of 20° and 150° in a circle of radius 12 in

Ex 13 The circumference of a circle is 12 in and the length of a certain arc is 4.5 in. What decimal of the circumference is the arc? What angle does the arc subtend at the centre?

Ex 14 Calculate the distance measured along a meridian of longitude from the equator to Lat 50° N. (take circumference of earth = 25000 miles)

Ex 15. Calculate the length of 1° of longitude at the equator

Ex. 16 A cog-wheel with 10 teeth, driving a larger one with 50 teeth, makes one revolution. Through what angle has the larger turned?

AREA OF A CIRCLE

§ 3. It can be proved* that the

$$\begin{aligned}\text{Area of a circle} &= \pi \times (\text{radius})^2 \\ &= \pi r^2\end{aligned}$$

In the following exercises it will generally be sufficient if answers are given correct to three significant figures

Ex 17 Calculate the area of a circle whose radius is (i) 2 m, (ii) 3½ ft., (iii) 15 cm.

Ex 18 The radius of one circle is twice the radius of another, how many times does the area of the greater contain the area of the smaller?

Ex 19. Find the ratio of the area of a circle to the area of the circumscribing square.

Ex 20 A circular pond of radius 100 ft. is surrounded by a gravel path 10 ft wide Find the area of the path

Ex 21. Two concentric circles have radii R, r What is the area between them?

Ex 22 The outer and inner diameters of a pipe are 14" and 12" What is the area of a cross-section?

Ex 23 Calculate the radius of a circle whose area is (i) 44 sq in, (ii) 143 sq. cm

Ex 24 A circular track is made round 10 acres of ground What is the diameter of the track? (1 acre = 4840 sq yd)

Ex 25 A circular disc has a circular hole drilled in it, a diameter of the hole being a radius of the disc What fraction of the original disc is now left? Make a freehand sketch of the figure

SUMMARY OF RESULTS

Equal arcs of a circle subtend equal angles at the centre.

Arcs of a circle are proportional to the angles which they subtend at the centre

The circumference of a circle = $2\pi r$, where r is the radius, and $\pi = 3\frac{1}{7}$ (approx.) or 3.14 (approx.).

The area of a circle = πr^2

* A useful experiment is to draw a large circle on squared paper and estimate its area by counting squares.

CHAPTER 17

THE CIRCLE CHORDS

§ I A straight line cannot cut a circle in more than two points. In fact, an unlimited straight line may

(i) cut a circle in two points, e g AB or CD in fig 195. In this case the part of the line which lies inside the circle is called a **chord** of the circle.

(ii) The line may meet the circle in one point only, thus EF meets the circle in T. In this case the line is said to **touch** the circle, it is called a **tangent**; T is called the **point of contact** of the tangent.

(iii) The line may lie entirely outside the circle, and have no point in common with the circle, e g GH in fig 195.

A chord may be said to be the straight line joining two points on a circle. If the chord passes through the centre it is a diameter, e g. AOB in fig. 195

The length of a diameter is twice the length of the radius, all the diameters of a circle are equal

A **segment** of a circle is the part of the plane bounded by an arc and its chord (fig 196)

A **sector** of a circle is the part of the plane bounded by two radii and the arc which they intercept (fig 196)

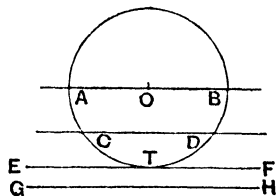


Fig. 195.

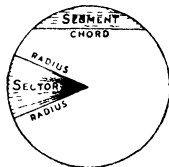


Fig 196

SYMMETRY OF THE CIRCLE CHORD AND CENTRE

§ 2 It will be seen that the circle is symmetrical about any diameter

¶Ex. 1 What axes of symmetry has (i) a sector, (ii) a segment, (iii) an arc, of a circle?

¶Ex. 2. Draw a circle of about 3 in. radius, draw freehand a set of parallel chords (about six), bisect each chord by eye. What is the locus of the mid-points of the chords?

¶Ex 3 Draw a circle and a diameter. This is an axis of symmetry. Mark freehand four pairs of corresponding points on the circle. Is there any case in which a pair of corresponding points coincide?

Ex. 4. AB is a chord of a circle. What is the locus of points equidistant from A and B? What conclusion does this lead to about the position of the centre?

From considerations of symmetry, then, we arrive at the following theorem, which may be proved by congruence of triangles.

THEOREM 36. A straight line, drawn from the centre of a circle to bisect a chord which is not a diameter, is at right angles to the chord;

Conversely, the perpendicular to a chord from the centre bisects the chord.

(i) *Data* O is the centre of $\odot ABC$, D is the mid-point of the chord AB

To prove that OD is $\perp$ to AB.

Construction Join OA, OB

Proof In the $\triangle$ s OAD, OBD,

$$\begin{cases} OA = OB \text{ (radii),} \\ OD \text{ is common,} \\ AD = BD \text{ (Data)} \end{cases}$$

$$\triangle OAD \equiv \triangle OBD,$$

$$\therefore \angle ODA = \angle ODB,$$

$$OD \text{ is } \perp \text{ to AB}$$

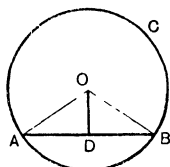


Fig. 197.

Th 14

(ii) **CONVERSE THEOREM**

Data O is the centre of $\odot ABC$, OD is the perpendicular on to the chord AB

To prove that AD = BD

Construction Join OA, OB

Proof Since the angles at D are right angles, the $\triangle$ s OAD, OBD are right-angled $\triangle$ s with hyp OA = hyp. OB (radii), and OD common

$$\triangle OAD \equiv \triangle OBD,$$

$$\therefore AD = BD$$

Th 15

COR. 1. The perpendicular bisector of a chord of a circle will, if produced, pass through the centre of the circle. (For only one perpendicular can be drawn to a given line at a given point in it)

COR. 2. The locus of the centres of circles through two given points is the perpendicular bisector of the line joining the points.

§3 To find the centre of a given circle.

Analysis and construction Let A, B be any two points on the circumference. The centre lies somewhere on the $\perp$ bisector of the chord AB, draw this line, EF (It is unnecessary to join AB)

Let C, D be another pair of points on the circumference, draw GH, the $\perp$ bisector of CD. The centre lies somewhere on GH.

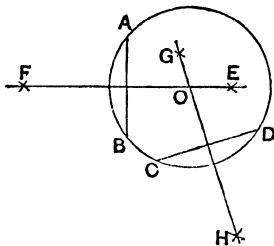


Fig 198

The centre therefore is the point of intersection of EF and GH, namely O

(If the points on the circumference were so chosen that AB and CD were parallel, this construction would fail. Why?)

To draw a circle passing through three given points not in the same straight line; in other words, To find a point equidistant from three given points.

This is done as in the preceding construction

¶Ex. 5 How would the construction fail if the three points were in a straight line?

Ex. 6 With a fine-pointed pencil trace round part of the edge of a penny, or better still the curved edge of your protractor, so as to obtain an arc of a circle. (Take care to keep the pencil perpendicular to the paper.) Complete the circle by finding the centre.

CIRCUMSCRIBED CIRCLE

§4. If a circle passes through all the vertices of a polygon, the circle is said to be **circumscribed** about the polygon (Latin *circum* = around), and the polygon is said to be **inscribed** in the circle (fig 199)

To circumscribe a circle about a given triangle.

This is the same problem as that of describing a circle to pass through three given points, namely the three vertices of the triangle

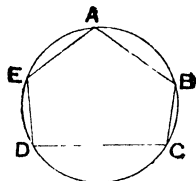


Fig 199

The centre of the circle circumscribed about a triangle is called the **circumcentre** of the triangle.

Notice that, though the perpendicular bisectors of all three sides pass through the circumcentre, yet it is not necessary to draw more than *two* of these bisectors in order to find the centre.

Ex. 7. Find the radius of a circle circumscribing a triangle whose sides are (i) 3", 4", 4", (ii) 3", 4", 5", (iii) 3", 4", 6"

Ex. 8. (Inch paper) Draw a circle to pass through the points (0, 3), (2, 0), (-1, 0), and measure its radius.

Ex. 9. (Inch paper) Draw the circumcircle of the triangle whose vertices are (0, 2), (4, 0), (-1, 0), and find its radius.

Ex. 10. (Inch paper) Find the circumradius, and the co-ordinates of the circumcentre of (0, 1), (3, 0), (-3, 0)

Ex. 11. A well is to be sunk equidistant from a farmhouse and two cottages. The cottages are $\frac{1}{2}$ mile apart and each is 300 yd from the farmhouse. What will be the distance of the well from each?

Ex. 12. The arch of a bridge is a portion of a circle. If the width of the bridge is 20 ft and its height 8 ft, find by drawing and measuring the radius of the circle.

Ex. 13. A rectangular field, 210 yd. by 140 yd., has trees at three of its corners. What is the radius of a circular track which just encloses the three trees?

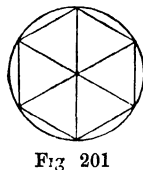
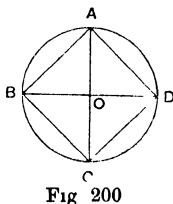
Ex. 14. A window is in the shape of a segment of a circle of width 2' 6" and greatest height 10". Draw the outline of the window, to scale, and give the radius of the circular arc.

Ex. 15. Test whether a circle can be circumscribed about (i) a square, (ii) a rectangle.

Ex. 16. Draw a parallelogram (not rectangular) and test whether a circle can be circumscribed about it.

TO INSCRIBE A REGULAR POLYGON IN A CIRCLE

§ 5 **Ex. 17.** Describe a circle of radius 5 cm. (see fig 200), at its centre O draw two lines at right angles to cut the circle at A, B, C, D. Join AB, BC, CD, DA. Is the figure ABCD regular? (See definition, there are two questions to be settled.)



Ex. 18. Describe a circle of radius 5 cm. (see fig 201), at its centre make a set of angles each equal to 60° (i.e. $\frac{360^\circ}{6}$), join the points where the arms cut the circle. Is the resulting figure regular?

Ex. 19. Calculate the angle at the centre for each of the following regular polygons —(i) 5-gon, (ii) 8-gon, (iii) 9-gon, (iv) triangle, (v) 10-gon, (vi) 16-gon, (vii) n -gon. Arrange your results in the form of a table

Number of sides	3	4	5	
Angle at centre	120°	90°		

Ex. 20 Inscribe a square in a circle of diameter 8.45 cm. Measure the sides

Ex. 21, Draw a circle radius 1.5" and inscribe in it a regular 5-gon. Measure the distance of a vertex from the opposite side.

Ex. 22 On a side of 2" construct a regular pentagon [Calculate the angle of the pentagon, draw two sides and then a circle]

Ex. 23 Find, by drawing, the lengths of the diagonals of an octagonal framework of side 2 ft

Ex. 24 In the same circle inscribe a regular 3-gon, and 6-gon with one vertex common. Find the ratio of their sides

To inscribe a regular hexagon in a circle.

Analysis and construction Suppose the figure drawn join two vertices A, B to the centre O. The 6 equal sides subtend 6 equal angles at the centre the sum of these angles is 360° , $\angle AOB = 60^\circ$

But $OA = OB$, $\angle x = \angle y$

And $\angle x + \angle y = 120^\circ$, $\angle x = \angle y = \angle O = 60^\circ$

$\triangle AOB$ is equilateral and $AB = OA$

Hence the following construction using compasses, place end to end in the circle six chords each equal to the radius.

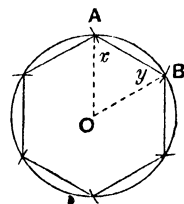


Fig. 202

To describe a regular hexagon of given side.

Describe a circle with radius equal to the given length, using compasses, place end to end in the circle six chords each equal to the radius.

Ex. 25. Construct regular hexagons side (i) 2", (ii) 5 cm.

Ex. 26 Find, by drawing, the area of a hexagonal nut of side 1 cm.

Ex. 27. Find the shortest distance through a hexagonal prism of side 10 cm

Ex. 28. Without assuming the value of π , prove that the circumference of a circle is $>$ three times the diameter, by inscribing a hexagon in the circle

¶ Ex. 29. Without assuming the value of π , prove that the circumference is $<$ four times the diameter by circumscribing a square about the circle. (See def. of a circumscribed polygon on p 152)

The problem of circumscribing a regular polygon about a given circle is dealt with in Chapter 18.

LENGTH OF CHORD.

§ 6. Note that the length of a chord of a circle is not proportional to the angle which it subtends at the centre. What is the connection between the *arc* and the angle?

The perpendicular from the centre upon a chord bisects the chord, therefore the radius, distance of chord, and half-length of chord are connected by Pythagoras' theorem (See fig. 203.)

Ex 30. Calculate the distances from the centre of a circle (radius 5 cm) of chords whose lengths are (i) 8 cm , (ii) 6 cm., (iii) $2a$ cm.

Ex. 31 Calculate the lengths of chords of a circle (radius 2.5 in) whose distances from the centre are (i) 2 in , (ii) 1 in , (iii) d in

Ex 32. The lengths of two parallel chords of a circle of radius 6 in are 10 in and 6 in respectively. Calculate the distance between the chords (There are two cases)

Ex 33 Calculate the radius of a circle, given that a chord 8 cm. long is 5 cm from the centre

Ex 34 What is the radius of a circle when a chord of length $2l$ is at distance d from the centre?

Ex 35 Given that a chord 6 in long is distant 1 in from the centre, calculate (i) the length of a chord distant 2 in from centre, (ii) the distance from the centre of a chord 3 in long.

Ex 36 ABC is a semicircle diameter AB=10 cm , CN is $\perp$ to AB. If NB=2 cm , calculate CN, CA, CB

Ex 37. What is the greatest height in a London Electric Tube given that the section is a circle of 13 ft. diameter and the width of the flat portion on which the rails rest is $5\frac{1}{2}$ ft ?

Ex. 38 A and B are two places 8 miles apart, as the crow flies, on an approximately circular lake of diameter 12 miles. If a boat steams direct from A to B, what is its greatest distance from the shore at any time?

Ex 39 The arch of a bridge is an arc of a circle of 30 ft radius. If the span of the bridge is 50 ft., what is the greatest height?

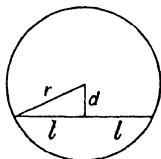


Fig 203.

A sphere of radius 6" is cut by a plane distant 3" from the centre. Calculate the radius of the circular section

A plane cuts a sphere in a circular section. Let ANB be the diameter of this section and N its centre, x'' its radius.

If C is the centre of the sphere, the plane CAB will cut the sphere in another circular section ABP whose radius is the radius of the sphere.

Also AB is a chord of the circle ABP and N is its middle point. $\angle CNA$ is a right angle.

Hence $\triangle CNA$ is a right-angled triangle

by Pythagoras' theorem

$$CA^2 = CN^2 + NA^2$$

$$6^2 = 3^2 + x^2$$

$$36 = 9 + x^2$$

$$27 = x^2$$

$$\therefore x = \sqrt{27} = 5.20 \text{ (to 2 places)}$$

the radius of the circular section = 5.20 in

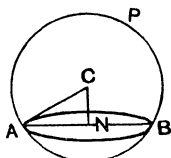


Fig 204.

Ex 40 A wooden sphere of 5" radius is planed down till there is a flat circular face of radius 2.5". If the sphere is now made to stand on the flat face, how high will it stand?

Ex. 41 A ball of radius 5 cm floats in water immersed to the depth of $\frac{4}{5}$ of its diameter. Calculate the circumference of the water-line circle.

Ex. 42 A hollow sphere has outside radius 6 in and inside radius $4\frac{1}{2}$ in. What thickness can be planed off without actually making a hole in the surface?

Ex 43 A cylinder height 20 cm and diameter 10 cm when upright on a table is just covered by a hemispherical bowl, the bowl also touching the table. Find the diameter of the bowl.

Ex 44 What is the radius of the parallel of latitude $45^\circ N$?

(Radius of earth = 4000 miles)

RIDERS ON CHORDS

†**Ex 45** Equal chords of a circle are equidistant from the centre.

†**Ex 46.** Chords of a circle that are equidistant from the centre are equal.

Ex. 47. Find the locus of the mid-points of chords 6 cm in length in a circle of radius 5 cm.

†**Ex. 48** Prove that the locus of the middle points of a set of equal chords of a circle is a concentric circle.

†**Ex. 49.** If two chords make equal angles with the diameter through their point of intersection, they are equal.

Ex. 50 A straight line is drawn cutting two equal circles and parallel to the line joining their centres; prove that the chords intercepted by the two circles are equal.

†**Ex. 51** AB, CD are two chords of a circle whose centre is O. OE, OF, the distances of the chords from the centre, are given to be equal. If the chords intersect at X, prove (i) that $XE = XF$, (ii) that AX is equal to DX (or CX)

†**Ex. 52** A straight line is drawn cutting two equal circles, and passing through the point midway between their centres. Prove that the chords intercepted by the two circles are equal.

†**Ex. 53** If a chord cuts two concentric circles in A, B; C, D, when $AC = BD$. [Draw the perpendicular from the centre on to the chord.]

†**Ex. 54** From a point O outside a circle two equal lines OP, OQ are drawn to the circumference. Prove that the bisector of $\angle POQ$ passes through the centre of the circle.

Ex. 55. O is a point 4 inches from the centre of a circle of radius 2 inches. Show how to construct with O as vertex an isosceles triangle having for base a chord of the circle, and a vertical angle of 50° . How many solutions are there? (*Freehand*)

†**Ex. 56** If one chord of a circle is greater than another, it is nearer to the centre. [Use Pythagoras' theorem]

†**Ex. 57.** The shortest chord that can be drawn through a point inside a circle is that which is perpendicular to the diameter through the point
[Prove that it is farthest from the centre]

†**Ex. 58** In a circle are placed, end to end, equal chords PQ, QR, RS, ST. Prove that $PR = QS = RT$

†**Ex. 59** **Equal chords of a circle cut off equal arcs.**

[Equal arcs subtend equal angles at the centre]

†**Ex. 60** Draw a chord of a circle and the angle it subtends at the centre. Prove that the chord which subtends half the angle at the centre is greater than half the original chord.

†**Ex. 61** **Two circles intersect at A, B; prove that the line of centres bisects AB (the common chord) at right angles.** [Use Theorem 34, p. 97]

SUMMARY OF RESULTS

A straight line, drawn from the centre of a circle to bisect a chord which is not a diameter, is at right angles to the chord.

The perpendicular to a chord from the centre bisects the chord.

The perpendicular bisector of a chord of a circle will, if produced, pass through the centre of the circle

CHAPTER 18

THE CIRCLE TANGENTS

§ I. The meaning of the term tangent has been explained on p. 141. Note that the line in fig. 205 is *not* a tangent and is *not* said to touch the circle, if produced it would cut it in two points



Think of a wheel with spokes. The roadway is a tangent to the circle, a spoke that happens to be vertical and downwards is the radius to the point of contact. The radius, then, is perpendicular to the tangent at its extremity.

Fig 205.

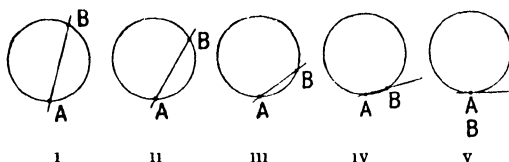


Fig 206

The tangent may also be viewed in another way suggested by the above figure. I — V show four positions of a chord AB (produced both ways). Looking at the figures from left to right, the chord is seen to be turning about the point A, as it turns, the second point of intersection, B, comes nearer and nearer to A until, in fig. V, B has coincided with A, and the chord has become the tangent at A.

A tangent therefore may be regarded as the **limiting position of a produced chord** whose two points of intersection with the circle have come to coincide.

Fig. 207 suggests another way in which the chord may approach its limiting position—the tangent.

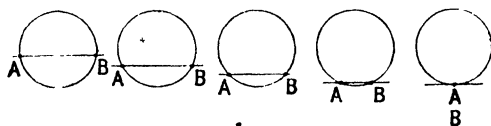


Fig. 207.

Ex 1 In fig. 207 what becomes of the theorem that “the perpendicular from the centre on a chord bisects the chord” when B comes to coincide with A?

In fig. 208, i, ii, the triangle OAB is isosceles,

$$\therefore \angle OAB = \angle OBA, \quad \therefore \angle OAP = \angle OBQ$$

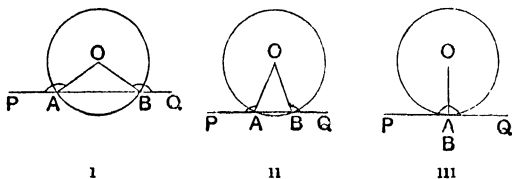


Fig 208.

When the chord attains its limiting position, the tangent, in fig iii, we are left with $\angle OAP = \angle OAQ$. Hence again

THEOREM 41 The tangent at any point of a circle and the radius through the point are perpendicular to one another.

A straight line drawn through the point of contact of a tangent at right angles to the tangent will, if produced, pass through the centre of the circle

§ 2. To draw the tangent to a circle at a given point on the circle, join the point to the centre, and draw a straight line through the point perpendicular to the radius

The tangent from an external point may be drawn with the ruler (by eye). It is not possible to distinguish the point of contact accurately without further construction, to find this point, drop a perpendicular upon the tangent from the centre, the foot of this perpendicular is the point of contact

This method is accurate enough for many purposes, another construction is given at the end of this chapter.

It is clear from symmetry that **the two tangents which can be drawn to a circle from a point A are (i) equal, (ii) equally inclined to AO; and (iii) AO bisects the chord of contact PQ at right angles.**

Ex. 2. Prove (i), (ii) and (iii) by Congruence of Triangles.

Ex 3. Prove (iii) by Th. 34, p 97

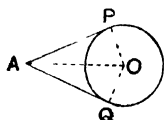


Fig. 209.

LENGTH OF A TANGENT

§3 In fig 209, since $\angle P$ is a right angle, $\triangle APO$ is a right-angled triangle. Hence, if AO and OP are known, the length of the tangent AP can always be found by Pythagoras' theorem.

Ex 4 A is 4 in distant from O , the centre of a circle of radius 3 in. From A draw a tangent with your ruler. Determine P , the point of contact, (i) by eye, (ii) by drawing a perpendicular from O .

Calculate AP , the length of the tangent. Verify by measurement.

Ex 5 Calculate the lengths of the tangents to a circle of radius r from a point distant d from the centre when (i) $r=3$ cm, $d=4$ cm, (ii) $r=2$ in., $d=4$ in.

Ex 6 At a point A of a circle (radius r , centre O) is drawn a tangent AP of length l , find OP .

Ex 7 At a point P on the circumference of a circle of radius 4 cm is drawn a tangent PT 3 cm in length. Find the locus of T as P moves round the circle.

Ex 8 Two circles, of radii 2 and 1 in, are concentric. Calculate the length of a chord of the outer circle which touches the inner.

Ex 9 Show that all chords 3 in long of a circle of radius 2 in touch a certain concentric circle, find its radius.

Ex 10 A cylindrical pipe of radius 10" lies on the ground, a plank, with one end A on the ground, rests on the pipe touching it at B . What is the distance of A from the centre of the pipe if AB is 2' 6"?

Ex 11 A cylindrical boiler 22" diameter stands upright between two walls at right angles, touching each. How far is the central axis of the boiler from the corner?

†Ex 12 If fig 2C were spun about OA , what figure would be generated (i) by the circle, (ii) by AP , (iii) by PQ ? Hence find the locus of the points of contact of tangents from a fixed point to a fixed sphere.

Ex 13 If P is a point 25 cm from the centre of a sphere of radius 20 cm, what is the length of a tangent line from P to the sphere? Are all the tangent lines from P equal?

Ex 14 A sphere radius 5 cm just fits into a hollow cone of slant side 8 cm touching round the base of the cone. What is the distance of the centre of the sphere from the vertex of the cone?

†Ex 15. All chords of a circle which touch an interior concentric circle are equal, and are bisected at the point of contact.

†Ex 16 A chord makes equal angles with the tangents at its extremities.

Ex 17 Each of the tangents, TA , TB , at the ends of a certain chord AB is equal to the chord, find the angle between the tangents, and the angle subtended at the centre by the chord.

†**Ex 18** In fig 209, the angles PAQ , POQ are supplementary.

Ex 19 Show how to draw a tangent to a given circle (i) parallel to a given line, (ii) perpendicular to a given line, (iii) making a given angle with a given line

Ex. 20 Show how to draw two tangents to a circle inclined to one another at a given angle.

†**Ex. 21** AOB is a diameter of a circle whose centre is O . At C on the circumference a tangent CE is drawn OE , parallel to AC , meets this tangent at E . Prove that EB touches the circle

†**Ex 22** A straight line PQ is drawn through a point P on the circumference of a circle, making a constant angle with the tangent at P Prove that, as P moves round the circumference, PQ touches a fixed circle

†**Ex. 23** Two parallel tangents meet a third tangent at U , V ; prove that UV subtends a right angle at the centre

†**Ex 24** The angles subtended at the centre of a circle by two opposite sides of a circumscribed quadrilateral are supplementary

Ex. 25 Draw a circle and a triangle ABC , the side BC touching the circle at D , CA touching at E , and AB touching at F

Mark on your figure all pairs of equal lines and prove that $AF + BC = \frac{1}{2}$ perimeter of triangle

†**Ex 26** $PQRS$ is a quadrilateral circumscribed about a circle Prove that $PQ + RS = QR + SP$. [Give symbols to all pairs of equal lines]

INSCRIBED CIRCLE

§ 4 If a circle touches all the sides of a polygon, the circle is said to be **inscribed** in the polygon, and the polygon is said to be **circumscribed** about the circle (fig 210)

¶**Ex 27.** What is the locus of the centres of circles touching two lines which cross at an angle? (Remember that two lines form four angles at a point) Draw a number of such circles

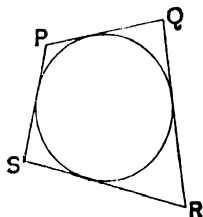


Fig. 210.

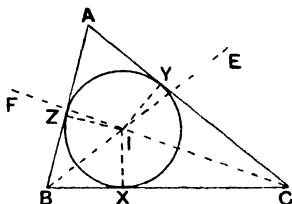
To inscribe a circle in a given triangle.

Fig 211.

Analysis and construction The centre is equidistant from the three sides

Since it is equidistant from BA and BC, and is obviously inside the triangle, it lies somewhere on the internal bisector of $\angle ABC$. Draw this line, BE

Similarly, it lies somewhere on CF, the internal bisector of $\angle ACB$

The centre therefore is the point of intersection of BE and CF, namely I.

To find the radius, draw $IX \perp$ to BC. X is the point of contact and IX is the radius. The circle can now be drawn

This circle is the **inscribed** circle of the triangle ABC.

Ex 28 The point of intersection of the internal bisectors of the angles of a triangle is equidistant from the three sides.

Ex 29 Draw the inscribed circle of a triangle whose sides are (i) 3, 4, 5 in, (ii) 10, 6, 10 cm. Measure the radii of the circles

Ex. 30 Draw any triangle ABC and produce the sides. By bisecting the exterior angles, draw a circle to touch BC and the other two sides produced (called an **escribed** circle) (See fig 212.)

Ex. 31 Draw any triangle ABC and the escribed circles opposite B and C

Ex 32 A point is to be found equidistant from the three edges of a triangular field whose sides are 150 yd, 230 yd and 215 yd. How far is the point from each side?

Ex. 33 What is the radius of the largest circle that can be cut out of a triangular piece of cardboard of sides 12, 10, 16 inches?

Ex 34 In any isosceles triangle draw two equal circles touching each other and each touching the base and a side.

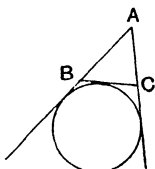


Fig. 212

Ex 35 Inscribe a circle in a quadrant* of a circle.

[Draw the tangent where the bisector of the angle meets the circle, and inscribe a circle in the triangle it forms with the radii]

Ex 36 Draw a sector of a circle radius 6 cm. with an angle of 30° . Find the diameters of the two circles which each touch the radii of the sector (or the radii produced) and the arc

Ex. 37 Two parallel lines are cut by a third line. Draw all the circles touching the three lines

Ex. 38 Inscribe a circle in a rhombus.

Ex 39 Draw a triangle sides 7 cm, 8.5 cm, 13 cm. Find the distance between the centres of the inscribed and circumscribed circles

Ex 40 Draw a rectangle with sides 10 cm and 7 cm. Find a point equidistant from three sides. What is its distance from the fourth side? (There are two cases)

Ex 41 What is the radius of the largest sphere which can be enclosed in a closed cone of height 3.5 in. and base radius 2 in.?

Ex 42 The section of a trough of depth 8" is an isosceles trapezium with parallel sides 16" and 9". Find the radius of the largest cylindrical log which can lie in the trough touching the two sides and the bottom

CIRCUMSCRIBED POLYGONS

§5. To circumscribe a regular polygon about a given circle.

The points of contact of the sides with the circle obviously form a regular polygon of the same number of sides inscribed in the circle

Hence the construction consists of finding the vertices of an inscribed polygon of the given number of sides and drawing the tangents at these points

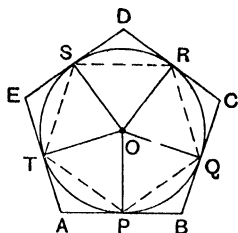


Fig 213

Ex 43. Draw an equilateral triangle whose sides touch a circle of radius 4 cm

Ex. 44. Circumscribe a square about a given circle

Ex 45 Circumscribe a hexagon about a circle of radius 1.5 in. Measure the side of the hexagon.

Ex. 46. A circular glass of 30" diameter is enclosed in a symmetrical octagonal frame each side of which touches the edge of the glass. Find the length of an edge of the frame

Ex. 47. Find the length of the side of a regular 9-gon circumscribed to a circle of radius 8 cm. [Do not attempt to draw all the sides]

* A sector whose angle is a right angle.

CONSTRUCTION OF TANGENTS TO CIRCLES.

§6. To construct tangents to a given circle ABC from a given point T outside the circle.

[We must find a point on the circle at which OT subtends 90°]

Construction Join T to O, the centre of $\odot ABC$. On OT as diameter describe a $\odot$ cutting the given circle in A, B.

Join TA, TB

These lines are tangents

Proof Join OA, OB.

Since OT is the diameter of $\odot OAT$,

$\angle OAT$ is a right angle (*Th 49**),

$\therefore$ AT, being $\perp$ to radius OA, is the tangent at A.

Sim^{ly} BT is the tangent at B

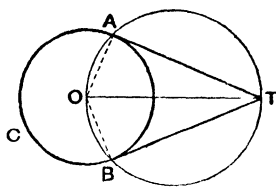


Fig. 214.

Ex 48. Construct the tangents to a circle of radius 1 5" from a point 3 6" from the centre. Measure their length

Ex. 49 Construct tangents to a circle from two points on a diameter produced which are equidistant from the centre. On what line do the tangents produced meet again? What figure do they form?

Ex. 50 Find by an accurate drawing the length of the straight portion of a rope which passes round a wheel of 5 ft radius and round a peg 8 ft. from the centre of the wheel

Ex. 51 A point source of light is placed 5 ft from the centre of a cylinder of radius 2 ft. Determine, by an accurate geometric construction, the width of the shadow at a distance of 3 ft. from the centre.

SUMMARY OF RESULTS

The tangent at any point of a circle and the radius through the point are perpendicular to one another

The two tangents drawn to a circle from a point are equal and equally inclined to the line joining the point to the centre.

* Theorem 49 has yet to be proved. See p 159

CHAPTER 19

THE CIRCLE ANGLE PROPERTIES

§ I. ¶Ex. 1. Draw a figure like fig. 215, making the radius of the circle about 2 in. Measure angles x and y . Notice that they are subtended* by the same arc.

¶Ex. 2. Do the same for figs. 216, 217. What relation do you notice between the angle x and the angle y in the three experiments?



Fig. 215

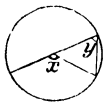


Fig. 216.



Fig. 217.

†Ex. 3. A side BA of an isosceles triangle ABC is produced, through the vertex A, to a point D. Prove that $\angle DAC = 2\angle ABC = 2\angle ACB$.

THEOREM 44. The angle which an arc of a circle subtends* at the centre is double that which it subtends at any point on the remaining part of the circumference.

Data The arc ACB of $\odot ACB$ subtends $\angle AOB$ at the centre O, and subtends $\angle APB$ at P, any point on the remaining part of the circumference. (See figs. 218, 219.)

To prove that $\angle AOB = 2\angle APB$

Construction Join PO, and produce it to Q.

Proof [We shall prove $\angle QOA = 2\angle OPA$ and $\angle QOB = 2\angle OPB$]

CASE I When the centre O is inside $\angle APB$

In $\triangle AOP$, $OA = OP$ (radii),

$\angle OAP = \angle OPA$.

Th 12

Now $\angle QOA$ is an exterior $\angle$ of $\triangle AOP$,

$\angle QOA = \angle OPA + \angle OAP = 2\angle OPA$ *Th 8, Cor. 1*

Similarly $\angle QOB = 2\angle OPB$.

$\therefore \angle QOA + \angle QOB = 2(\angle OPA + \angle OPB)$,

$\therefore \angle AOB = 2\angle APB.$

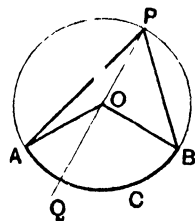


Fig. 218.

* For "subtend" see p. 38 and Index

CASE II When the centre O is outside $\angle APB$

As before, $\angle QOB = 2\angle OPB$,

and $\angle QOA = 2\angle OPA$,

$$\therefore \angle QOB - \angle QOA = 2(\angle OPB - \angle OPA),$$

$$\angle AOB = 2\angle APB$$

†Ex 4 Prove the above theorem for the case in which O lies on AP (see fig 216)

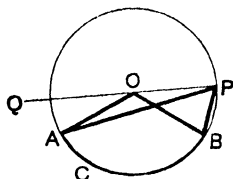


Fig 219.

§ 2 Reflex angles. Take your dividers and open them slowly. The angle between the legs is first an acute angle, then a right angle, then an obtuse angle. When the dividers are opened out flat, the angle has become two right angles (180°). If the dividers are opened still further the angle of opening is greater than 180° and is called a reflex angle.

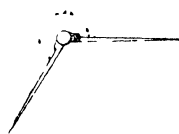


Fig. 220

A **reflex angle** is an angle greater than two right angles and less than four right angles. Fig 221 shows two straight lines OA , OB forming a reflex angle (marked), and also an obtuse angle (unmarked).

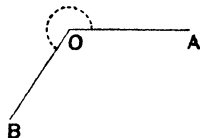


Fig. 221

†Ex. 5 Account for the necessity of the phrase “less than four right angles” in the above definition.

†Ex. 6 What is the sum of the reflex angle and the obtuse angle in fig. 221? If the obtuse angle = 136° , what is the reflex angle?

†Ex. 7 What kind of angle is subtended at the centre of a circle by a major arc?

†Ex. 8. Express the following bearings* by reflex angles —(i) S 17° W, (ii) N 40° W, (iii) S.S.W.

†Ex. 9 Draw a quadrilateral having one angle reflex. Prove that the sum of the four angles is 360° .

†Ex. 10. Is it possible for (i) a four-sided figure, (ii) a five-sided figure to have two of its angles reflex?

†Ex 11 Prove the above theorem for the case in which ACB is a major arc, and the angle subtended at the centre a reflex angle (see fig. 222). What kind of angle is $\angle APB$ in this case?

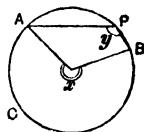


Fig. 222.

* See note *, p 15.

Ex. 12. Draw a freehand figure for the case of Theorem 44 in which arc ACB is a semicircle. What does $\angle AOB$ become in this case? What does $\angle APB$ become?

†**Ex. 13** Prove that in fig. 223 $\angle a = \angle b$

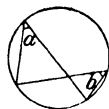
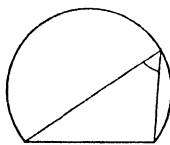


Fig. 223

ANGLES IN A SEGMENT

§ 3. A **segment** of a circle is the part of the plane bounded by an arc and its chord



MAJOR SEGMENT



SEMICIRCLE



MINOR SEGMENT

Fig. 224.

An **angle in a segment** of a circle is an angle subtended by the chord of the segment at a point on the arc (fig. 224).

A segment is called a **major** segment or a **minor** segment according as its arc is a major or a minor arc. It is obvious that a major segment of a circle is greater than the semicircle (considered as an area) and that a minor segment is less than the semicircle.

¶**Ex. 14** Draw a circle, place in it a chord AB , mark four points P, Q, R, S in the major arc. What relation do you notice between the angles subtended at these points by AB (angles in the same segment)?

¶**Ex. 15** In the figure of Ex. 14 mark three points X, Y, Z in the minor arc. What do you notice about the angles subtended at X, Y, Z by AB ?

THEOREM 45. Angles in the same segment of a circle are equal.



Fig. 225

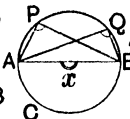


Fig. 226

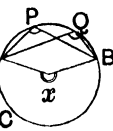


Fig. 227

Data $\angle s$ APB, AQB are two $\angle s$ in the same segment $APQB$ of $\odot APB$

(Three figures are drawn, for the three cases in which the segment $>, =$ or $<$ a semicircle.)

To prove that

$$\angle APB = \angle AQB$$

Construction

Join A, B to the centre.

Proof In each figure $\angle x = 2 \angle APB$, subtended by the same arc ACB.

Th. 44

$$\text{Similarly } \angle x = 2 \angle AQB, \therefore \angle APB = \angle AQB$$

NOTE Since all the angles in a segment are equal, we may in future speak of *the* angle in a segment when we mean the magnitude of any angle in the segment

THEOREM 49 The angle in a semicircle is a right angle.

For, in fig. 226, since AB is a diameter, $\angle x = 2 \text{ rt } \angle$ s

$$\angle APB = 1 \text{ rt } \angle$$

¶Ex 16 Prove Th. 49 without assuming the result in Th. 44

[Join the centre to the point on the semicircle]

¶Ex 17 Are $\angle$ s PAQ, PBQ in figs. 225–227 equal? Give a reason.

¶Ex 18 What can be said about the angle in a major segment? in a minor segment?

Ex. 19 Copy fig. 228 (freehand), join BC, BD. Find all the angles in your figure

¶Ex 20 AC, BD are perpendicular diameters of a circle, P is a point on the minor arc AB, prove that the arcs AD, DC, CB each subtend 45° at P

Ex. 21 The diagonals of a cyclic quadrilateral* PQRS intersect at X, $\angle SPR = 54^\circ$, $\angle RPQ = 33^\circ$, $\angle PSQ = 20^\circ$. Find all the angles of the figure and show them in a freehand sketch. Explain briefly.

Ex. 22 Draw a circle and a diameter AB; mark two points C, D on the circumference on opposite sides of AB, join CD, AD, AC, CB, BD. Assuming that $\angle BAC = 30^\circ$ and $\angle BAD = 40^\circ$, determine all the angles in the figure. Give your reasons carefully.

Ex. 23 From the given angles, find all the angles in fig. 229

Ex. 24 Repeat Ex. 23, taking $\angle B = 71^\circ$, $\angle BCO = 36^\circ$, $\angle AOD = 108^\circ$. Prove that in this case AD is $\parallel$ to BC

Ex. 25. Repeat Ex. 23, taking

$$\angle B = x^\circ, \angle BCO = y^\circ, \angle AOD = z^\circ$$

Ex. 26 Make a freehand sketch of fig. 230, and fill in the number of degrees in each angle, also explain how you have found the various angles

Ex. 27 In fig. 227 join AB and produce it to D. If $\angle APB = 107^\circ$, $\angle PAQ = 27^\circ$, $\angle QBD = 125^\circ$, find the remaining angles, giving reasons.

* A cyclic quadrilateral is one whose vertices lie on a circle.

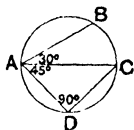


Fig. 228.

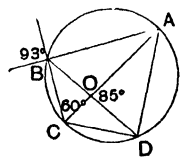


Fig. 229

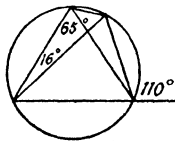


Fig. 230

CYCLIC QUADRILATERAL.

§ 4 Ex. 28. Find the magnitude of all the marked angles in fig. 231. What is the sum of the angles at the centre? of $\angle$ s ACB and ADB? of $\angle$ s CAD and CBD? What property of a quadrilateral inscribed in a circle does this suggest?

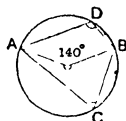


Fig 231

THEOREM 47 The opposite angles of any quadrilateral inscribed in a circle are supplementary.

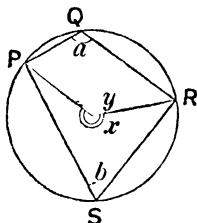


Fig 232.

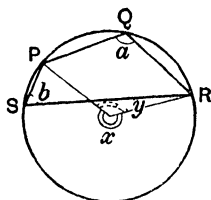


Fig 233.

Data PQRS is a quadrilateral inscribed in $\odot$ PQR*.

To prove that (i) $\angle$ PQR + $\angle$ PSR = 2 rt. $\angle$ s,
(ii) $\angle$ SPQ + $\angle$ SRQ = 2 rt. $\angle$ s

Construction Join P and R to the centre of the circle.

Proof $\angle a = \frac{1}{2} \angle x$, $\angle b = \frac{1}{2} \angle y$,
 $\angle a + \angle b = \frac{1}{2} (\angle x + \angle y)$

Th. 44

But $\angle x + \angle y = 4$ rt. $\angle$ s,
 $\angle a + \angle b = 2$ rt. $\angle$ s,

i.e. $\angle$ PQR + $\angle$ PSR = 2 rt. $\angle$ s

Similarly it may be shown that

$\angle$ SPQ + $\angle$ SRQ = 2 rt. $\angle$ s.

COR. If a side of a quadrilateral inscribed in a circle is produced, the exterior angle so formed is equal to the interior opposite angle of the quadrilateral.

¶ Ex. 29 Prove the corollary.

¶ Ex. 30. What is the relation between the angles subtended by a chord at a point in its minor arc, and at a point in its major arc?

* The two figures represent the two cases in which the centre is (i) inside, (ii) outside the quadrilateral. The same proof applies to both.

Ex. 31. In fig. 234, O is the centre of the circle and $\angle A$ is 70° , find as many of the remaining angles in the figure as you can, giving reasons in each case.

Ex. 32. A, B, C are three points on a circle such that $\angle ACB = 122^\circ$. O is the centre. Prove that $\angle OAB = 32^\circ$. At what angle will the tangents at A and B intersect? Prove your statement.

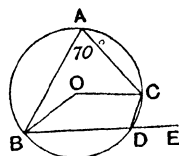


Fig. 234

Ex. 33. $ABCD$ is a quadrilateral with the angles at B and D right angles. Draw the diagonals AC and BD . Given that the whole angle at A is 60° and $\angle CBD = 40^\circ$, find the remaining angles in the figure.

Ex. 34. State the converse of Th. 47.

†Ex. 35. A parallelogram inscribed in a circle must be a rectangle.

†Ex. 36. If a trapezium can be inscribed in a circle, it must be isosceles.

†Ex. 37. The sides BA, CD of a quadrilateral $ABCD$, inscribed in a circle, are produced to meet at O , prove that $\triangle s OAD, OCB$ are equiangular.

†Ex. 38. $ABCD$ is a quadrilateral inscribed in a circle, having $\angle A = 60^\circ$, O is the centre of the circle. Prove that $\angle OBD + \angle ODB = \angle CBD + \angle CDB$.

†Ex. 39. ABC is an isosceles triangle, D is any point in the base BC . Prove that the circumcircles of the triangles ABD, ACD are equal.

Ex. 40. Show how to circumscribe about a given circle a triangle of given angles. (Suppose the figure drawn, and the points of contact joined to the centre: what are the angles at the centre?)

ANGLE IN ALTERNATE SEGMENT

§5. Looking at the tangent as the limit of a chord produced, you will see from fig. 235 that as B moves round towards A the constant

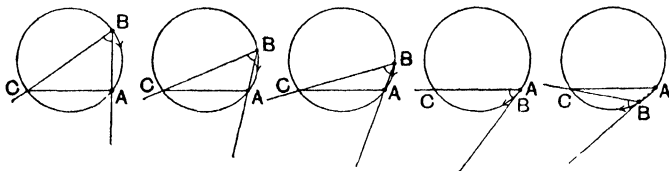


Fig. 235

angle CBA becomes, in the limit, the angle between the chord and the tangent at A . This suggests the following theorem.

THEOREM 51. If a straight line touch a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the angles in the alternate segments.

Data AB touches $\odot$ CDE at C, the chord CD is drawn through C, meeting the $\odot$ again in D

To prove that (i) $\angle BCD = \angle$ in alternate segment CED,
(ii) $\angle ACD = \angle$ in alternate segment CFD.

(i) *Construction* Through C draw CE $\perp$ to AB, meeting $\odot$ in E.

Join DE

Proof [We shall prove

$$\angle BCD = 90^\circ - \angle DCE = \angle CED,$$

an angle in segment CED]

Since CE is drawn $\perp$ to tangent AB, at its point of contact C,

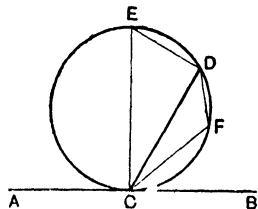


Fig 236

$\therefore$ CE passes through centre of $\odot$, and is a diameter,

$$\therefore \angle CDE \text{ is a rt } \angle,$$

Th. 49.

$$\therefore \text{ in } \triangle CDE, \angle CED + \angle DCE = 1 \text{ rt } \angle$$

Ln. 8

$$\text{Now } \angle BCD + \angle DCE = 1 \text{ rt } \angle$$

Constr.

$$\therefore \angle BCD + \angle DCE = \angle CED + \angle DCE,$$

$$\therefore \angle BCD = \angle CED.$$

$$\therefore \angle BCD = \text{any angle in the segment CED.}$$

Th. 45

(ii) *Construction* Take any point F in arc CFD, join CF, DF.

$$\text{Proof } \angle BCD + \angle ACD = 2 \text{ rt. } \angle s$$

Th. 1

Also, since CFDE is a quadrilateral inscribed in a circle,

$$\angle CED + \angle CFD = 2 \text{ rt } \angle s,$$

Th. 47

$$\therefore \angle BCD + \angle ACD = \angle CED + \angle CFD.$$

$$\text{But } \angle BCD = \angle CED,$$

Proved.

$$\therefore \angle ACD = \angle CFD.$$

¶Ex 41 In fig 236 point out an angle equal to $\angle BCF$

¶Ex 42 Taking CE as the chord (fig 236), what is the segment alternate to $\angle ACE$?

Ex 43 Find all the angles of fig. 236, supposing that $\angle BCD = 60^\circ$, and that $\angle FCD = 20^\circ$. What angles do the chords ED, CD, FC subtend at the centre?

Ex 44. Find all the angles of fig 237

Ex 45 Find all the angles of fig. 238

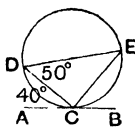


Fig 237

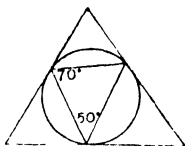


Fig 238.

Ex 46 If in fig 236 arc $CD = 2$ arc DE , what is $\angle BCD$?

Ex 47 If in fig. 236 arc ED were $\frac{1}{2}$ arc DC , what would be the magnitude of $\angle BCD$?

Ex. 48 Make a rough sketch of fig 239, and fill in the number of degrees in each angle; AB is a tangent to the circle Explain briefly the principles you use.

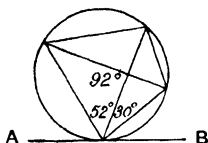


Fig 239

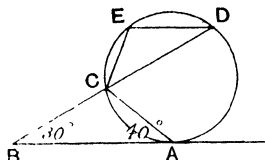


Fig 240

Ex 49 In fig 240, AB is the tangent at A and DE is parallel to AB Find, by reasoning, the magnitude of the angle ECA .

Ex 50 ACB is a triangle inscribed in a circle If the $\angle ACB = 55^\circ$, at what angle do the tangents at A and B intersect? If $\angle ACB = x^\circ$, what is the angle?

†Ex 51 A chord AB of a circle is produced to meet the tangent at a point T in P Show that the triangles PTA and PBT are equiangular

Ex 52 In fig. 236, if CF is drawn to bisect $\angle BCD$, prove that the arcs CF and FD are equal

†Ex 53. Two circles, centres A and B , intersect at C Prove that one angle between the tangents at C is equal to the angle ACB .

†Ex. 54 ABC is a triangle inscribed in a circle with $\angle B = 2\angle A$ Prove that the tangent at C is parallel to the bisector of the $\angle B$

Ex. 55. Show how to inscribe in a given circle a triangle of given angles [Supposing the figure drawn, and the tangent at one vertex also drawn, the three angles at this point are known.]

CONSTANT ANGLE LOCUS.

§ 6. ¶Ex 56 Stick two pins into the paper 2 in apart at A and B; place the set square on the paper so that the sides containing the 60° are in contact with the pins, mark the point where the vertex of the angle rests. Now slide the set square about, keeping the same two sides against the pins, and plot the locus of the 60° vertex. What is the locus? are A, B points on the locus?

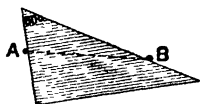


Fig 241

¶Ex 57. Repeat the experiment of Ex. 56 with (i) the 90° vertex, (ii) the 30° vertex, (iii) an angle of 120° cut out of paper.

¶Ex 58 Draw, on tracing-paper, two straight lines intersecting at P. On your drawing paper mark two points A, B. Move your tracing-paper about so that the one line may always pass through A, and the other through B. Plot the locus of P by pricking through.

The foregoing exercises will have prepared the reader for the following theorem —

The locus of points (on one side of a given straight line) at which the line subtends a constant angle is an arc of a circle, the given line being the chord of the arc.

¶Ex 59 What is the theorem on the converse of which this depends?

¶Ex 60 What kind of arc is obtained if the constant angle is (i) acute, (ii) a right angle, (iii) obtuse? On which side of the base is the centre in each case?

¶Ex 61 If the constant angle is 45° , what angle is subtended by the given line at the centre of the circle? Using this suggestion, show how to draw the locus of points at which a line of 5 cm. subtends 45° , without actually determining any of the points.

An important special case of this theorem is —

The locus of the vertices of right-angled triangles with the same hypotenuse is the circle on the hypotenuse as diameter.

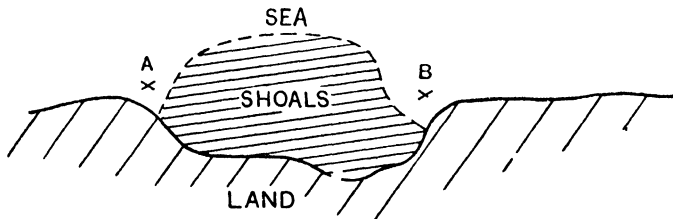


Fig. 242.

The constant angle locus is used in navigation to keep ships clear of dangerous areas. For example, in fig. 242 the navigator is instructed to take bearings of the fixed objects A and B and to take care that the angle subtended by AB never exceeds a certain angle (say 130°), by doing this he will keep outside a certain arc which encloses the shoal.

¶Ex. 62. If in fig. 242 a ship moves so that AB subtends at the ship an angle of 130° , what is the locus of the ship? If the ship is at a point outside the locus, will the angle subtended by AB at the ship be more or less than 130° ?

§7. To construct the constant angle locus.

I.e. On a given base to construct a segment of a circle to contain a given angle.

First Method

First make a freehand sketch of the required figure. The centre is the vertex of an isosceles triangle whose angles you can calculate. Consider the three cases in which the centre of the circle is (i) on the same side of the line as the locus, (ii) on the line, (iii) on the side of the line opposite to the locus.

On a line of 2" construct a segment of a circle to contain an angle of 44° .

Draw a freehand figure (see fig. 243)

$\angle$ at circumference = 44° ,

$\angle$ at centre O = 88° ,

$\therefore \angle OAB = \angle OBA = \frac{1}{2}(180 - 88) = 46^\circ$

Now make $AB = 2''$, $\angle OAB = 46^\circ$ and $\angle OBA = 46^\circ$

O is the centre of the circle.

¶Ex. 63. If the constant angle is x° ($x < 90$), what is the angle at the vertex of the isosceles triangle? What are the base angles of the triangle?

¶Ex. 64. Repeat Ex. 63 for the case in which $x > 90$

Second Method.

AB is the given base.

At A, below AB, make the $\angle BAP$ equal to the given angle

Draw $AO \perp$ to AP,

and OD the $\perp$ bisector of AB.

The circle, centre O and radius OA, gives the required segment.

¶Ex. 65. Prove this construction (see Theorem 51)

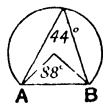


Fig. 243

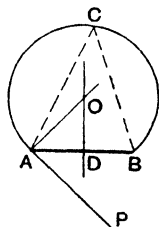


Fig. 244

†Ex. 66. On a chord of 3.5 in. construct a segment of a circle to contain an angle of 70° . Measure the radius. Use both methods.

Ex. 67. Repeat Ex. 66 with chord of 7.24 cm. and angle of 110° .

Ex. 68. A rhombus is made with 4 loosely jointed rods each 4 ft. long. One rod is fixed horizontally and the figure swings in a vertical plane. Construct the locus of the intersection of the diagonals.

Ex. 69. A, B, C are three landmarks on a straight shore, $AB = 10$ miles, $BC = 6$ miles. From a ship at sea AB subtends an angle of 54° and BC an angle of 68° . Find the distance of the ship from B.

Ex. 70. ABC is a triangle, $AB = 3.25''$, $BC = 4''$, $CA = 2.75''$. Find a point O within the triangle such that $\angle AOB = 106^\circ$, $\angle AOC = 120^\circ$. Measure $\angle BOC$.

Ex. 71. Construct a triangle having

- (i) base = 4 in., altitude = 1 in., vertical angle = 90°
- (ii) base = 10 cm., altitude = 2 cm., vertical angle = 120° .
- (iii) base = 8 cm., altitude = 5 cm., vertical angle = 90°
- (iv) base = 3.5 in., altitude = 1 in., vertical angle = 54° .

In each case measure the base angles.

†Ex. 72. On the same base and the same side of it are drawn three triangles of vertical angle 30° . Show that a circle can be drawn to circumscribe all three triangles and that its radius is equal to the base.

†Ex. 73. Prove that the locus of the mid-points of chords of a circle which are drawn through a fixed point is a circle.

†Ex. 74. Of all triangles of given base and vertical angle, the isosceles triangle has greatest area.

†Ex. 75. P is a variable point on an arc AB . AP is produced to Q so that $PQ = PB$. Prove that the locus of Q is a circular arc.

†Ex. 76. Show how to construct a triangle given (i) base, vertical angle, and median, (ii) base, vertical angle, and area.

†Ex. 77. Show how to construct a quadrilateral $ABCD$, given AB , AC , AD , $\angle BAD$, $\angle BCD$.

†Ex. 78. Show how to construct a quadrilateral $ABCD$, given AB , BC , CA , AD , $\angle BDC$.

†Ex. 79. Show how to construct a parallelogram given the base and height, and the angle (subtended by the base) at the intersection of the diagonals.

CONCYCLIC POINTS

§8. If three points are taken at random, a circle can be drawn to pass through them (except in one case, what is this case?)

Mark four points at random and try (freehand) if a circle can be drawn to pass through all four. A circle can as a rule be drawn through any three of them, is this bound to pass through the fourth? For instance, can a circle be circumscribed about a rhombus?

In general, a circle cannot be drawn to pass through four points in a plane, it can be so drawn only if, on joining the points, there are certain relations between the angles, either that (i) equal angles are obtained as in fig. 245 (a) or (ii) a quadrilateral with opposite angles supplementary is formed as in fig. 245 (b). If either of these conditions holds, a circle can be described to pass through the four points.



Fig 245

A set of points such that a circle can be described to pass through them are called **concyelic**. A quadrilateral through whose vertices a circle can be described is called **cyclic**.

†Ex 80 The two conditions stated above, for four points to be concyclic, depend on the converses of two theorems. State those two theorems.

†Ex 81 BE, CF are altitudes of the triangle ABC, prove that B, F, E, C are concyclic. Sketch in the circle, where is its centre?

†Ex 82 BE, CF, two altitudes of $\triangle ABC$, intersect at H. Prove that AEHF is a cyclic quadrilateral. Sketch in the circle, where is its centre?

†Ex 83 ABCD is a parallelogram. A circle drawn through A, B cuts AD, BC (produced if necessary) in E, F respectively. Prove that E, F, C, D are concyclic.

†Ex 84. ABCD is a quadrilateral inscribed in a circle. DA, CB are produced to meet at E, AB, DC to meet at F. (Produce the sides to meet before putting in the letters.) Prove that, if a circle can be drawn through AEFC, then EF is a diameter of this circle, and BD is the diameter of $\odot ABCD$.

†Ex 85 The straight lines bisecting the angles of any convex quadrilateral form a cyclic quadrilateral.

Ex 86 Show how to construct a cyclic quadrilateral ABCD, given AB, BC, CD, $\angle B$.

Ex. 87. Explain how to construct a cyclic quadrilateral using only a protractor and a ruler.

MISCELLANEOUS ANGLE PROPERTIES.

†Ex. 88 A, B, C, D are points on a circle; the diagonals of ABCD meet at X, prove that $\triangle s ABX, DCX$ are equiangular; as also $\triangle s BCX, ADX$.

†Ex 89 Through X, a point outside a circle, XAB, XCD are drawn to cut the circle in A, B, C, D. Prove that $\triangle s XAD, XCB$ are equiangular.

†Ex 90 Two circles intersect at P, Q. Through P diameters PA, PB of the two circles are drawn. Show that AQ, QB are in the same straight line (Join QP).

* A third condition for four points to be concyclic follows from Theorem 59, Chapter 23.

†Ex. 91. Prove the following construction for erecting a perpendicular to a given straight line AB at its extremity B . With centres A, B describe arcs of equal circles, cutting at C . With centre C and radius CA describe a circle. Produce AC to meet this circle again in D , then BD is $\perp$ to AB .

†Ex. 92. The circle described on a side of an isosceles triangle as diameter bisects the base.

†Ex. 93. The circles drawn on two sides of a triangle as diameters intersect on the base.

†Ex. 94. Through P, Q , the points of intersection of two circles, are drawn chords APB, CQD , cutting the one circle at A, C and the other at B, D , prove that AC is parallel to BD . [Join PQ]

What does this theorem become if A, C are made to coincide?

†Ex. 95. Two circles touch at P . Through P are drawn straight lines APB, CPD , cutting the one circle at A, C and the other at B, D . Prove that AC is parallel to BD . (Draw tangent at A . Compare Ex. 94.)

†Ex. 96. Two circles cut at P, Q . A, a point on the one circle, is joined to P, Q , and these lines are produced to meet the other circle in B, D . Prove that BD is parallel to the tangent at A . (Compare Ex. 94.)

†Ex. 97. Through P, Q , the points of intersection of two circles, are drawn parallel chords APB, CQD , prove that $AB = CD$.

†Ex. 98. If two opposite sides of a cyclic quadrilateral are equal, the other two are parallel.

†Ex. 99. Each of two equal circles passes through the centre of the other. AB is their common chord. Through A is drawn a line cutting the two circles again in C, D , prove that $\triangle BCD$ is equilateral.

†Ex. 100. ABC is an equilateral triangle inscribed in a circle, P is any point on the minor arc BC . Prove that $PA = PB + PC$. [Make $PX = PB$. Then prove $XA = PC$. See fig. 246.]

†Ex. 101. ABC is a triangle, BE, CF , perpendiculars to the opposite sides, meet in H . Prove that $\angle FHE$ is equal or supplementary to the angle A .

†Ex. 102. In Ex. 101, prove that $\triangle AFE$ and ABC are equiangular.

†Ex. 103. In Ex. 101, O is the mid-point of BC . Prove $OF = OE$.

†Ex. 104. ACB is a segment of a circle containing an angle of 60° . Prove that the tangents at A and B form with AB an equilateral triangle.

†Ex. 105. Two circles intersect in A, B . PAQ is a straight line cutting the circles in P, Q , the tangents at P, Q meet in T . Prove T, P, B, Q concyclic.

†Ex. 106. Tangents to two equal circles from a point on their common chord produced meet the circles in A, B and C, D . Prove that A, B, C, D are concyclic points.

†Ex. 107. A chord AB of a circle bisects the angle between the diameter through A , and the perpendicular from A upon the tangent at B .

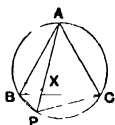


Fig. 246.

†Ex 108 AB, AC are two chords of a circle, BD is drawn parallel to the tangent at A, to meet AC in D, prove that $\angle ABD$ is equal or supplementary to $\angle BCD$

†Ex. 109 Two circles, whose centres are P, Q, intersect at A, B. The tangents at A to the two circles are drawn, and produced to cut the circumferences at D, C (see fig 247). Prove that (i) the triangles BAC, BDA are equiangular, (ii) the angles APC, AQD are equal

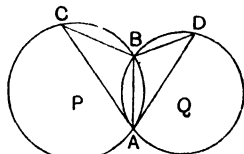


Fig. 247

In the following exercises (Exx 110–116) the student is advised to make use of “the angle subtended at the circumference.”

†Ex 110 Prove that equal arcs or chords of a circle subtend equal (or supplementary) angles at a point on the circumference. Make a sketch to illustrate the case of supplementary angles

Ex 111. In fig 248 what fractions of the circumference are the arcs AB, BC, CD, DA, BCD?

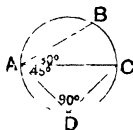


Fig 248

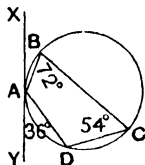


Fig 249

Ex 112 In fig 249 what fractions of the circumference are the arcs AB, BC, CD, DA?

Ex 113 The two tangents OA, OB from a point O are inclined at an angle of 48° . How many degrees are there in the minor and major arc AB respectively? What is the ratio of the major to the minor arc?

†Ex 114. AB, CD are parallel chords of a circle. Prove that arc AC = arc BD.

†Ex 115. On a circle are marked off equal arcs AC, BD. Prove that AD is parallel or equal to CB

†Ex 116 AOB, COD are two chords of a circle, intersecting at right angles. Show that arc AC + arc BD = arc CB + arc DA

SUMMARY OF RESULTS

The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

Angles in the same segment of a circle are equal

The angle in a semicircle is a right angle

The opposite angles of any quadrilateral inscribed in a circle are supplementary.

If a side of a quadrilateral inscribed in a circle is produced, the exterior angle so formed is equal to the interior opposite angle of the quadrilateral.

If a straight line touch a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the angles in the alternate segments.

The locus of points (on one side of a given straight line) at which the line subtends a constant angle is an arc of a circle, the given line being the chord of the arc

The locus of the vertices of right-angled triangles with the same hypotenuse is the circle on the hypotenuse as diameter

CHAPTER 20

CONTACT OF CIRCLES. COMMON TANGENTS

§ 1. The different relative positions which are possible for two circles are shown in fig 250

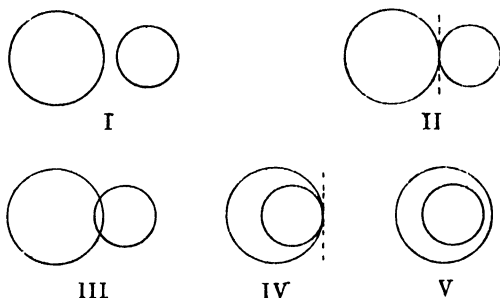


Fig 250

In II the circles are said to **touch externally**, in IV to **touch internally**. The formal definition of contact of circles is as follows —

If two circles touch the same line at the same point, they are said to touch one another

¶Ex. 1. Draw a figure showing the different relative positions possible for two equal circles.

¶Ex. 2 Describe in words each of the relative positions shown in fig 251

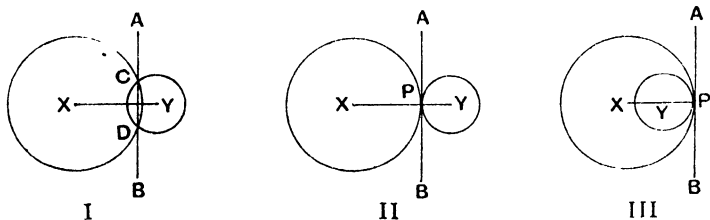


Fig. 251

Fig. 251 (I) shows two intersecting circles, their common chord produced and then centres X, Y. The whole figure is symmetrical about XY, hence XY bisects CD.

Let the small circle move to the *right* until the position of external contact is reached (I). The produced chord becomes the common tangent AB , the points C and D coincide at P , and XY which formerly bisected CD now passes through P .

Now let fig 251 (I) change by moving the small circle to the *left*, until the position of internal contact is reached (III). By similar reasoning we see that XY produced passes through P . Hence we infer Theorem 43, which is proved formally below.

THEOREM 43 **If two circles touch, the point of contact lies in the straight line through the centres.**

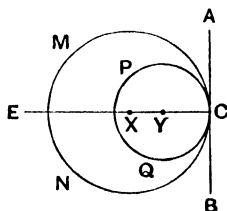


Fig. 252

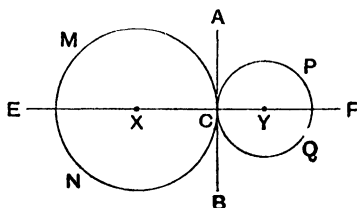


Fig. 253

CASE I When the circles touch internally (fig. 252)

Data The $\odot$ s CMN , CPQ touch internally at C , X and Y the centres.

To prove that XY produced passes through C

Construction Draw ACB the common tangent at C .

Join CX , CY .

Proof [CX , CY are not to be assumed to be the same straight line, we have to prove that they are]

Since CA is the tangent at C to $\odot CMN$ and CX the radius through C ,

$\angle XCA$ is a rt $\angle$ Th. 41.

Simly $\angle YCA$ is a rt $\angle$,

CX and CY must be the same straight line,

$\therefore XY$ produced will pass through C .

CASE II. When the circles touch externally (fig. 253).

Data The $\odot$ s CMN , CPQ touch externally at C , X and Y the centres

To prove that XY passes through C

Construction Draw ACB the common tangent at C .

Join CX , CY .

Proof [CX, CY are not to be assumed to be in one straight line; we have to prove that they are]

Since CA is the tangent at C to $\odot$ CMN and CX the radius through C,

$\therefore \angle XCA$ is a rt $\angle$ *Th. 41.*

Similarly $\angle YCA$ is a rt $\angle$,

$\angle XCA + \angle YCA = 2$ rt. $\angle$ s,

XCY is one straight line *Th. 2*

COR. If two circles touch internally, the distance between their centres is the difference of their radii; if externally, the distance between their centres is the sum of their radii.

¶Ex. 3 Describe the relative position of the two circles in each of the following cases (d is the distance between the centres, R and r are the radii). Do this, if you can, without drawing the circles

- (i) $d = 4.1$ cm, $R = 2.1$ cm, $r = 1.1$ cm
- (ii) $d = 0.7$ cm, $R = 2.2$ cm, $r = 1.2$ cm
- (iii) $d = 3.4$ cm, $R = 2.0$ cm, $r = 1.4$ cm.
- (iv) $d = 0.8$ cm, $R = 2.1$ cm., $r = 1.3$ cm
- (v) $d = 0$ cm, $R = 1.9$ cm, $r = 1.2$ cm
- (vi) $d = 1.5$ cm, $R = 2.0$ cm., $r = 1.5$ cm
- (vii) $d = 2.5$ cm, $R = 1.7$ cm., $r = 1.7$ cm

Ex 4 What is the distance between the centres of two circles of radii 12 and 10 in. (i) if they have external contact, (ii) if they have internal contact?

Ex 5 Show how to draw four circles having for centres the vertices of a square of side 3 in, so that each circle may touch two others externally.

Ex 6 Three circles, of radii 3, 4, 5 cm, touch externally, each circle touching the other two. What are the distances between the centres? Draw the circles.

Ex. 7 Draw two circles radii 2" and 1" with their centres 4" apart. (i) Draw circles of radii 1.5" to touch each externally (ii) Draw circles of radii 4.5" to enclose both

Ex 8 Draw circles of radii 8 cm and 3 cm. with their centres 4 cm. apart. Draw a circle of radius 3.5 cm. to touch one internally and the other externally.

Ex. 9 Draw circles of radii 4 cm. and 2.5 cm. touching externally. Find the radius of the smallest circle which touches and encloses both

COMMON TANGENTS TO TWO CIRCLES.

§2. A straight line which touches two circles is called a **common tangent** to the two circles

Fig. 254 shows that when the circles do not intersect there are four common tangents

If the two circles lie on the same side of a common tangent, it is called an **exterior common tangent**, thus AB, CD (fig 254) are exterior common tangents. If the two circles lie on opposite sides of a common tangent, it is called an **interior common tangent**, thus EF, GH are interior common tangents

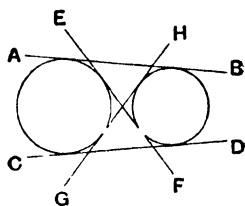


Fig 254

Ex 10 Draw sketches to show how many common tangents can be drawn in Cases II, III, IV, V, of fig 250, in each case state the number of exterior and of interior common tangents

For practical purposes, the common tangents may be drawn "by eye" with a ruler, this method will not however give the points of contact with any accuracy, and to obtain these, *perpendiculars must be drawn from the centre.*

To calculate the length of the common tangents. By symmetry it is obvious that the two exterior common tangents are equal, as are the two interior common tangents. If the radii (r , R) and the distance between the centres (l) are given, the lengths (i.e. the distance between the points of contact) are calculated as follows —

Exterior common tangents ST is a common tangent AS is the $\perp$ from A to ST, BT the $\perp$ from B. Draw BP $\perp$ to AS. Then BTSP is a rectangle. (Why?) $ST = PB$. But in the right-angled $\triangle APB$, we have $AB = l$ and $AP = R - r$. Hence BP may be calculated.

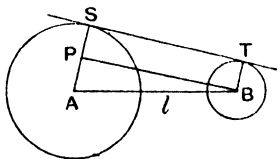


Fig 255

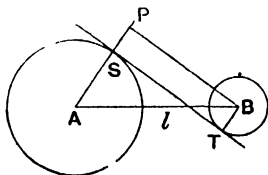


Fig. 256.

Interior common tangents are calculated by a similar method, the figure being as in fig. 256. Notice that $AP = R + r$.

Two circles of radii 5 in and 2 in have their centres 10 in apart. Calculate the length of (i) the exterior, (ii) the interior, common tangents.

- (i) *Construction* If ST is the common tangent and A, B the centres of the circles

Draw $AS \perp$ to ST , $BT \perp$ to ST and $BP \perp$ to AS

Then APB is a right-angled triangle, $AB = 10$, $AP = 5 - 2 = 3$. Also

$$BP = ST = t \text{ (say)}$$

By Pythagoras' theorem,

$$AB^2 = AP^2 + PB^2$$

$$100 = 9 + t^2, \quad t^2 = 91, \quad t = 9.54 \text{ (to 2 places)}$$

- (ii) *Construction* Draw $AS \perp$ to ST , $BT \perp$ to ST , $BP \perp$ to AS produced

Then APB is a right-angled triangle, $AB = 10$, $AP = 5 + 2 = 7$,

$$PB = ST = t \text{ (say)}$$

By Pythagoras' theorem,

$$AB^2 = AP^2 + PB^2$$

$$100 = 49 + t^2, \quad t^2 = 51, \quad t = 7.14 \text{ (to 2 places)}$$

Ex 11 Draw two circles of radii 4 cm and 2 cm, the centres 7 cm apart. Measure and calculate the length of the exterior common tangents.

Ex. 12 Draw the two circles of Ex. 11, and draw the interior common tangents. Measure and calculate the length of these tangents.

Ex 13 Two equal circles of radii 1 in have their centres 3 in apart. Calculate the length of their interior common tangents by considering the point where they cut the line of centres. What is the length of the exterior common tangents?

Ex 14 Two halfpennies (diameter 1 in) are held in contact, both lying flat on a table by a stretched elastic band. What is the stretched length of this band?

Ex 15 The diameters of the wheels of an old-fashioned bicycle are 2 ft and 1 ft, and the distance between the points where the wheels touch the ground is $2\frac{1}{2}$ ft. Calculate the distance between the centres of the wheels.

Ex 16 A thin endless cord is wrapped round two equal pulley wheels. Each wheel is 12 inches in diameter, and the distance between the centres is 20 inches. Find the length of the chord.

Ex 17. In Ex 16, if the cord crossed midway between the centres, what length of cord would now be required?

[Find the length of the portion touching the wheel by measuring the angle it subtends at the centre. See Chapter 16, p 138.]

Ex 18. A chain passes directly round two wheels of radii 4" and 10" with their centres 20" apart. (i) Calculate the length of the straight portions. (ii) By drawing, measurement and calculation, find the length of the portions touching the wheels. Hence find the whole length of the chain.

Ex. 19. Three equal cylindrical pipes of 14" diameter are bound together, each touching the other two. What length of cord is necessary?

Ex. 20. Two circles touch externally at A and PQ is a common tangent which meets the tangent at A in T. Prove that $TP = TQ$

Ex. 21. Two circles touch externally and the radius of one is three times the radius of the other. Prove that the common tangents make an angle of 30° with the line of centres

Ex. 22. Prove that a common tangent of two unequal circles is shorter than the distance between their centres.

SUMMARY OF RESULTS

If two circles touch externally, the point of contact lies in the straight line through the centres and the distance between the centres is the sum of the radii.

If two circles touch internally, the point of contact lies in the straight line through the centres and the distance between the centres is the difference of the radii.

CHAPTER 21

SIMILAR FIGURES

RATIO

§1 Consider a model of an engine and the engine itself. If the model is 3 ft. long and the engine 15 ft. long, we say that the model is to the scale of 3 to 15 or $\frac{3}{15}$. Also it is clear that if the width of the model is 1 ft., the width of the engine is 5 ft. In fact the ratio of any length in the model to the corresponding length in the engine is 1 : 5 or $\frac{1}{5}$.

§2 In the next chapter we shall develop the meaning of the word "ratio" more fully and in a slightly different way, but we may define the **ratio** of a length PQ to a length RS as the quotient

$$\frac{\text{measure of PQ}}{\text{measure of RS}},$$

the two measurements being made with respect to the same unit of length. Thus the ratio of 2 ft. to 9 in. is the ratio of 24 in. to 9 in. $= \frac{24}{9} = \frac{8}{3}$.

¶Ex 1 What are the ratios of the following lengths?

(i) 3 yd. to 2 yd., (ii) 4 yd. to 12 yd., (iii) 3 yd. to 10 ft., (iv) 3 ft. 6 in. to 5 ft., (v) 3 ft. to 1 ft. 6 in., (vi) 15 cm. to 5 cm., (vii) 1 metre to 1 cm., (viii) 6 miles to $1\frac{1}{2}$ miles.

In order to compare two ratios it is often necessary to express the vulgar fractions obtained in the form of decimals. Thus

the ratio of 2.78 in. to 3.59 in. $= \frac{2.78}{3.59} = 0.774$ (by division), and

the ratio of 64 ft. to 83 ft. $= \frac{64}{83} = 0.771$ (by division).

We see that these two ratios are nearly equal, the first ratio being slightly larger than the second.

Ex 2 Compare the following pairs of ratios by expressing each as a decimal —

(i) 1.4 in. to 4.5 in., 7 in. to 22 in., (ii) 25 cm., 2 cm., 64 cm., 5 cm., (iii) 7.65 in., 4.25 in., 5.7 in., 3.15 in.

* The ratio of a to b is written $\frac{a}{b}$, or a/b , or $a : b$, or $a - b$.

Ex. 3. Find the ratio $\frac{PQ}{RS}$ (fig. 257); measure (i) in inches, (ii) in centimetres. Work out the ratios to three significant figures. Why might you expect your results to differ?

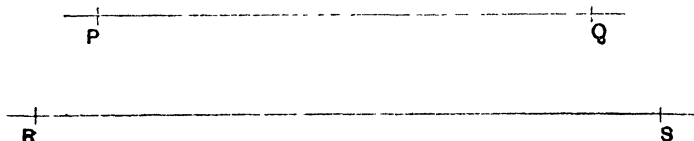


Fig. 257

The ratio of two magnitudes is independent of the unit chosen.

For example, the ratio of a length of 5 yd to a length of 2 yd is $5 : 2$, if these lengths are measured in feet the measures are 15 and 6, and the ratio is $15 : 6$. Now we know that $5 : 2 = 15 : 6$.

Ex. 4. Fill in the missing terms in the following —

$$(i) \frac{2}{3} = \frac{\quad}{6},$$

$$(iii) 7 : \quad = 3 : 11,$$

$$(v) \frac{\quad}{2p} = \frac{3}{2},$$

$$(ii) \frac{\quad}{5} = \frac{1}{2},$$

$$(iv) 5 : 2 = 7 : \quad,$$

$$(vi) \frac{a}{b} = \frac{\quad}{d}$$

SIMILAR FIGURES

§3 Look at two different-sized maps of England. Are they the same shape? Are they similar?

If two boys draw the same figure to scale but choose different scales, will the figures be of the same shape? Are they similar?

It is only possible to answer the above questions in a general way until the meaning of the word similar has been considered more fully.

Look at fig. 258 (i), (ii), are they the same shape? Or, in other words, are they similar? What tests must we apply? If AB were double $A'B'$, what could you say about BC and $B'C'$, about CD and $C'D'$, etc?

Measure the lengths of the lines in the figure and calculate the ratios $\frac{A'B'}{AB}$, $\frac{B'C'}{BC}$, etc. Are they all equal (to the degree of accuracy justified by your measurements)?

Is the equality of these ratios enough?

If the figures were made of jointed rods, could you alter the shape of the quadrilateral $A'B'C'E'$? Would any of the ratios $\frac{A'B'}{AB}$, $\frac{B'C'}{BC}$, etc

be altered? Would $A'B'C'E'$ still be similar to (of the same shape as) $ABCE$ *?

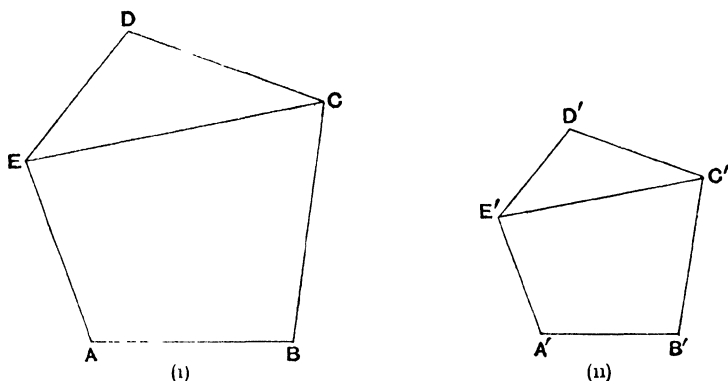


Fig 258

We see that **for figures to be similar** two conditions must be satisfied —

- (i) **the ratios of corresponding sides must be equal,**
- (ii) **corresponding angles must be equal.**

§ 4 Now go back to the scale drawings of the same figure made to two different scales — (i) will the ratios of corresponding lengths in the two drawings all be equal? (ii) will corresponding angles in the two drawings all be equal?

Take two different-sized maps of England. By measurement we can find the ratio of the lengths in the two maps that represent (i) London to Bristol, (ii) London to Liverpool, etc. Will the ratios † be equal?

Are the angles ‡ Birmingham—London—Bristol in the two maps equal to one another?

The last few paragraphs show that we have been using the idea of similar figures in all our scale drawing and whenever we have used a map or a plan.

* See Notes and Answers.

† If needs be actual measurements can be made, but intuition should be enough unless a boy doubts.

Again, if a small photograph be taken of a large picture, the photograph is similar to the picture. Notice that the lengths are altered in a certain ratio but the angles are unaltered.

Also, a kinematograph projects on the screen a large picture similar in every respect to the small picture on the film.

¶Ex 5 Give further instances* of practical processes in which similar figures occur

Similar figures are not necessarily flat, i.e. in two dimensions. They may be solid, i.e. in three dimensions. A model of a ship is a solid figure similar to the ship.

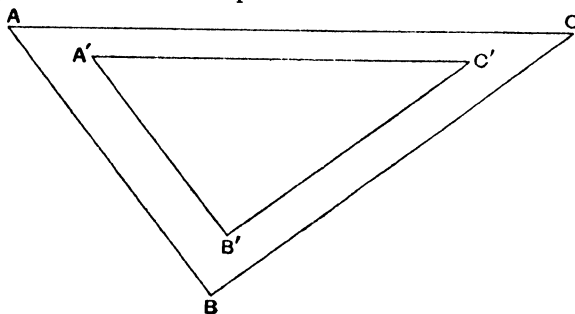


Fig. 259

¶Ex 6 Give further instances of similar solid figures

¶Ex. 7. Give instances of two similar curved figures, plane or solid.

¶Ex. 8 In fig. 259 (i) Are the angles equal? (ii) Measure the lengths of the sides in centimetres and calculate the ratios $\frac{A'B'}{AB}$, $\frac{B'C'}{BC}$, $\frac{C'A'}{CA}$

Are the triangles similar?

Ex. 9. In fig 260 the sides of $P'Q'R'S'T'$ are $\parallel$ to the sides of $PQRST$. (i) Show that corresponding angles of the two figures are equal (ii) Measure the sides of the two figures and find the ratios of each pair of corresponding sides. Are $PQRST$ and $P'Q'R'S'T'$ similar?

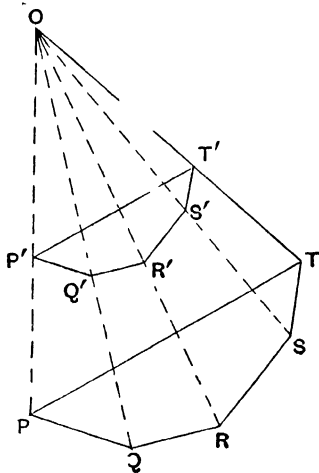


Fig 260

SIMILAR TRIANGLES

§5 In the case of two congruent triangles we know that corresponding angles are equal and also that corresponding sides are equal. But we also know that we need not be told all these six facts to be sure that the triangles are congruent.

If two triangles are similar, we know that they have their angles equal and their corresponding sides in the same ratio. But the question arises—can we argue that the triangles are similar if we are given less information than the above?

I. It will be remembered that to fix a triangle three data are needed and that three data are generally sufficient. But if the three data are *three angles* (adding up to 180° of course), then the triangle is not fixed. Why is this? It may be answered that two angles of a triangle give us the third, and to know three angles is really not more than to know two. But what do we really know if the three angles are given? We know the *shape*, but not the *size* of the triangle.

• Draw any triangle freehand, and then draw lines parallel to the sides, produce these parallels to form a second triangle. The two triangles are equiangular*, obviously they are of the same shape. If the base of one triangle is double that of the other, will the sides of the one triangle be doubles of the sides of the other?

In Ex 8 we had two equiangular triangles and we found their corresponding sides in the same ratio. This suggests

THEOREM 54 **If two triangles are equiangular, their corresponding sides are in the same ratio.**

In other words, if you can prove that two triangles have their angles equal, you may infer that the *corresponding* sides are in the same ratio.

Thus, if (fig 261) $\angle A = \angle P$, $\angle B = \angle Q$,

$\angle C = \angle R$, then $\frac{a}{p} = \frac{b}{q} = \frac{c}{r}$

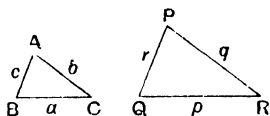


Fig 261

* Note the two uses of the word "equiangular." A triangle is equiangular when each of its angles is 60° . Two triangles are equiangular when the angles of the one are respectively equal to the angles of the other.

Though this may seem evident, it should be noticed that this theorem is not true of polygons with more than three sides. The square and the rectangle in fig. 262 are equiangular, but they are not similar. To prove that two quadrilaterals are similar, it is not sufficient to show that corresponding angles are equal.



Fig. 262.

Ex. 10 Test the accuracy of Theorem 54 by drawing two equiangular triangles ABC , PQR , measuring their sides and calculating the ratios $\frac{a}{p}$, $\frac{b}{q}$, $\frac{c}{r}$.

Ex. 11. ABC , PQR are two equiangular triangles

(i) If $a=2$, $b=3$, $c=4$, $p=12$, what are q , r ?

(ii) If $p=4$, $q=5$, $r=6$, $a=3$, what are b , c ?

[Example If $a=3$, $b=4$, $c=2$, $q=6$, find p .

Draw a figure and mark in the given lengths

Since $\triangle ABC$ is equiangular with $\triangle PQR$, $\frac{a}{p} = \frac{b}{q}$

$$\frac{3}{p} = \frac{4}{6}$$

Multiplying both sides by $6p$, $18 = 4p$.

$$p = \frac{18}{4} = 4\frac{1}{2}$$

Ex. 12 ABC is a triangle with sides $a=3''$, $b=5''$, $c=6''$. Calculate the sides of an equiangular triangle which is to have $10''$ for its longest side.

Ex. 13 Two right-angled triangles have each an angle of 53° . Their hypotenuses are 75 mm. and 45 mm. and the large has the other two sides 45 mm. and 60 mm. Calculate the remaining sides of the smaller.

§6. II. Construct a triangle of sides $1''$, $1\frac{1}{2}''$, $2''$. Now construct a second triangle with sides double those of the former. The two triangles have their sides in the same ratio. Are they similar? Observe that no attention has been paid to the angles in the drawing.

A triangle must exist similar to the first and on twice the scale. It must have sides of $2''$, $3''$, $4''$.

But three sides fix a triangle, and this is the triangle you have drawn. This suggests

THEOREM 55. If two triangles have their corresponding sides in the same ratio*, the triangles are equiangular.

Thus, if in fig. 263

$$\frac{a}{p} = \frac{b}{q} = \frac{c}{r}, \text{ then } \angle A = \angle P, \angle B = \angle Q, \angle C = \angle R.$$

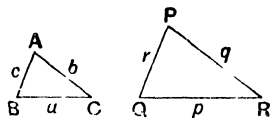


Fig. 263.

Here again is an evident property of triangles that is not true for polygons generally. The sides of the rhombus in fig. 264 are $\frac{2}{3}$ of the sides of the square, the sides of the two figures are proportional, but the figures are not similar.

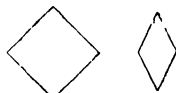


Fig. 264

Ex 14. Are the following pairs of triangles similar?

- (i) Sides 3, 6, 5 cm and 2, 4, $3\frac{1}{2}$ cm
- (ii) Sides 15, 2, 3 in and 3, 4, 6 in
- (iii) Sides 12, 20, 10 ft and 3, 5, 2.5 in.
- (iv) Sides 25, 45, 3 cm and 125, 2, 15 cm

§7‡ III. Draw a triangle ABC with an angle A of say 40° , the sides containing the angle being called x in. and y in. Now make a triangle A'B'C', having $\angle A' = \angle A$, the sides containing $\angle A'$ being $\frac{3}{4}x$ in. and $\frac{3}{4}y$ in.

Observe that nothing has been said about the base of $\triangle A'B'C'$, or the other two angles.

Now there must exist a triangle similar to $\triangle ABC$, and on a $\frac{3}{4}$ scale. And this similar triangle must have an angle of 40° and containing sides of $\frac{3}{4}x$ in., $\frac{3}{4}y$ in. But two sides and the included angle fix a triangle. The $\triangle A'B'C'$ must therefore be similar to $\triangle ABC$. This suggests

THEOREM 56 If two triangles have one angle of the one equal to one angle of the other and the sides about these equal angles in the same ratio, the triangles are similar.

* The pupil has not yet been introduced to the word "proportional"; hence the use of the phrase "in the same ratio." See note on p. 194

‡ This section may be postponed at a first reading

Ex 15. What is the ratio of $B'C'$ to BC in the triangle mentioned? What can be said about the angles B' , C' ?

Ex. 16 ABC is a rod, 12 cm long, moveable about the fixed point A . AB is 5 cm and BC 7 cm. long. The rod is turned to a new position $A'B'C'$. $AA' = 4$ cm, what is CC' ?

The action of **proportional compasses** depends on this theorem.

Fig 265 represents a pair of proportional compasses. $AB = AC$ and $AH = AK$,

$$\therefore \frac{AH}{AB} = \frac{AK}{AC}, \text{ and } \angle HAK = \angle BAC,$$

$\therefore \triangle s AHK, ABC$ are similar

Hence $\frac{HK}{BC} = \frac{AH}{AB}$, which is constant for any fixed position of the hinge. In fig 265 the hinge is adjusted so that $\frac{AH}{AB} = \frac{1}{2}$, thus, whatever the angle to which the compasses are opened, $HK = \frac{1}{2} BC$

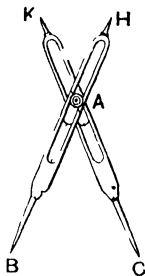


Fig 265

NUMERICAL EXAMPLES ON SIMILAR TRIANGLES

Ex 17. Draw a triangle sides 6, 8, 10 cm. Construct an equiangular triangle having its shortest side 4.8 cm. Measure the remaining sides and verify by calculation.

Ex. 18. ABC is a triangle having $BC = 3$ in, $CA = 4$ in, $AB = 5$ in. Draw an equiangular triangle DEF having $EF = 2.2$ in (EF corresponding to BC). Calculate DE , DF , and check by measurement.

Ex 19 Repeat Ex. 18 with $BC = 5.8$ cm, $CA = 7.7$ cm., $AB = 8.3$ cm., $EF = 3.8$ cm.

Ex. 20. The end of the roof of a house is a triangle with sides 20 ft, 15 ft and 12 ft. In a photograph the longest side is 1.5 in. What is the length of the other two sides?

Ex. 21 The shadow of a flagstaff is found to be 120 feet long and at the same moment the shadow of a man 6 feet high is 15 feet long. What is the height of the flagstaff?

Ex 22 A triangular face of a small crystal is viewed in a microscope and its sides appear to be 4.2 cm., 1.9 cm, 3.6 cm. If the longest side is actually 2 mm, what are the lengths of the other two sides?

ABC is a triangle, DE parallel to AC cuts AB, BC in D, E, if $AB = 20$ cm, $AC = 12$ cm., $BD = 8$ cm., calculate DE.

In the two Δ s $\hat{ABC}$, $\hat{DBE}$, since $DE \parallel$ to AC ,

$$\angle BAC = \text{cor. } \angle BDE,$$

$$\angle ACB = \text{cor. } \angle DEB,$$

$$\angle B \text{ is common}$$

the Δ s are equiangular

$$\frac{BA}{BD} = \frac{AC}{DE}$$

$$\text{Let } DE = x \text{ cm} \quad \frac{20}{8} = \frac{12}{x} \quad 20x = 96 \quad x = \frac{96}{20}$$

$$DE = 4.8 \text{ cm.}$$

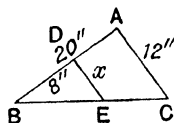


Fig. 266

Ex. 23. In a triangle ABC , DE is drawn parallel to AC , as in fig. 266.

(i) If $BE = 1''$, $EC = 3''$, $ED = 1.5''$, calculate AC

(ii) If $BC = 16''$, $CA = 9''$, $ED = 4''$, calculate BE

Ex. 24 A ladder 30 ft long is held with one end on the ground resting on a wall 8 ft high. The length along the ladder from the ground to the top of the wall is 10 ft. What height is the end of the ladder above the ground?

Ex. 25. A spot light is placed on the ground 3 ft in front of an upright board 4 ft. high. What will be the height of the shadow on a wall 6 ft behind the board?

Ex. 26 A spot light is 2 ft in front of the middle of a board 30 in wide suspended vertically. What is the width of the shadow 3' 6'' behind the board?

Ex. 27 A triangular wedge is 3'' long and $\frac{1}{2}''$ wide at its base. It is driven $\frac{3}{4}''$ into a block of wood. What is the width of the hole made?

Ex. 28 Two equal rods 11'' long are hinged together 5'' from one end of each, and are placed with the ends of their longer portions 10'' apart. What is the distance between their other ends?

Ex. 29 A man climbs 16 feet up a 20 foot ladder, placed against a wall with the top 16 feet from the ground. How far are his feet from the ground?

Ex. 30 In order to find the height of a tower a man sets up a pole 10 feet high 190 feet from the foot of the tower. He then walks away from the tower until he sees the top of the tower in line with the top of the pole, he is now 10 feet from the pole. His eye is 5 ft 6 in. from the ground. How high is the tower?

For further numerical exercises on Similar Triangles see p. 210.

RIDERS ON SIMILAR TRIANGLES*.

¶¶Ex 31. Make a freehand copy of the figure on p 129, let BC, DE intersect at X . Prove the following pairs of triangles to be equiangular and in each case write down the ratios of the three pairs of corresponding sides —

- (i) $\triangle s XBE, XCD$. (ii) $\triangle s EGF, CBF$ (iii) $\triangle s XCD, DHC$
 (iv) $\triangle s DAE, XBE$ (v) $\triangle s XBE, CBF$ (vi) $\triangle s XBE, EGF$

†Ex. 32 The altitudes BE, CF of an acute-angled triangle ABC intersect at H . Prove that $\triangle s HBF, HCE$ are equiangular and write down the three equal ratios of corresponding sides

†Ex 33. $PQRS$ is a quadrilateral inscribed in a circle whose diagonals intersect at X , prove that the $\triangle s XPS, XQR$ are equiangular Write down the three equal ratios of corresponding sides

†Ex 34 In the figure of Ex 33, prove that $\frac{PQ}{SR} = \frac{XP}{XS}$

[If you colour PQ, SR red, and XP, XS blue, you will see which two triangles you require]

†Ex 35 $XYZW$ is a cyclic quadrilateral, XY, WZ produced intersect at a point P outside the circle, prove that $\frac{PY}{PW} = \frac{PZ}{PX}$

†Ex 36 TP touches a circle at P , TQR cuts it at Q, R , prove that $\frac{TP}{TQ} = \frac{TR}{TP}$

†Ex. 37 ABC is a triangle right-angled at A , prove that the altitude AD divides the triangle into two triangles which are similar to $\triangle ABC$. Write down the ratio properties you obtain from the similarity of $\triangle s BDA, BAC$

†Ex. 38 The altitude QN of a triangle PQR right-angled at Q cuts RP in N , prove that $\frac{QN}{RN} = \frac{PN}{QN}$

[Find two equiangular triangles, colour the given lines, as in Ex 34]

†Ex 39 XYZ is a triangle inscribed in a circle, XN is an altitude of the triangle, and XD a diameter of the circle, prove that $XY \cdot XD = XN \cdot XZ$

†Ex 40 XYZ is a triangle inscribed in a circle, the bisector of $\angle X$ meets YZ in P , and the circle in Q , prove that $XY \cdot XQ = XP \cdot XZ$

†Ex 41 $PQRS$ is a quadrilateral inscribed in a circle, PT is drawn so that $\angle SPT = \angle QPR$ (see fig 267) Prove that

- (i) $SP \cdot PR = ST \cdot QR$, (ii) $SP \cdot PT = SR \cdot TQ$

†Ex 42. S is a point in the side PQ of $\triangle PQR$, ST is drawn parallel to QR and of such a length that

$$ST \cdot QR = PS \cdot PQ$$

Prove that T lies in PR

[Prove $\angle SPT = \angle QPR$]

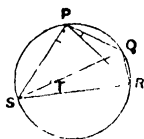


Fig. 267.

* For further riders on Similar Triangles see p. 215

†Ex. 43 In a triangle ABC, AD is drawn perpendicular to the base; if

$$BD \cdot DA = DA \cdot DC,$$

prove that $\triangle ABC$ is right-angled.

†Ex. 44 XYZ is a triangle inscribed in a circle, XN is an altitude of the triangle and XD a diameter of the circle, prove that $\frac{XY}{XN} = \frac{YD}{NZ}$. What rectangle property can be obtained from this by clearing of fractions?

†Ex. 45 With the same construction as in Ex. 44, prove that

$$XZ \cdot NY = XN \cdot ZD *$$

[You will have to pick out two equal ratios from two equiangular triangles. If you colour XZ, NY red and XN, ZD blue you will see which are the triangles.]

†Ex. 46 ABCD is a quadrilateral inscribed in a circle, its diagonals intersect at X. Prove that (i) $AX \cdot BC = AD \cdot BX$, (ii) $AX \cdot XC = BX \cdot XD$.

†Ex. 47 ABCD is a quadrilateral inscribed in a circle, AB, DC produced intersect at Y. Prove that

$$(i) YA \cdot BD = YD \cdot CA, \quad (ii) YA \cdot YB = YC \cdot YD.$$

†Ex. 48 The bisector of the angle A of $\triangle ABC$ meets the base in P and the circumcircle in Q. Prove that the rectangle contained by the sides AB, AC is equal to rect. AP · AQ.

†Ex. 49. In Ex. 41, prove that $PQ \cdot SR = PR \cdot TQ$.

THREE DIMENSIONS

The height of a cone is 20" and the radius of the base 8". Calculate the radius of a right section 7" from the vertex.

PRQ is the section, VR = 7 in., PR is the radius required, say x in. VC = 20 in.

The triangles VRP and VCA are similar,

$$\frac{VC}{VR} = \frac{CA}{RP},$$

$$\frac{20}{7} = \frac{8}{x}.$$

$$20x = 7 \times 8 \quad r = \frac{56}{20} = 2.8$$

radius of section = 2.8 in.

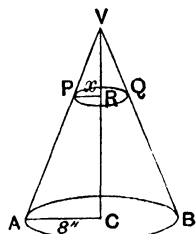


Fig. 268.

* Beware of muddling $XZ \cdot NY$ (which means $XZ \times NY$) and $XZ \cdot NY$ (which means $XZ \div NY$).

A right pyramid stands on a base 10 ft square and is 15 ft. in height. It is cut by a plane 10 ft. above the base and parallel to the base. Calculate the sides of the figure formed.

PQRS is the plane cutting the pyramid

OV = 15', ON = 10' NV = 5'

PQRS is a figure similar to ABCD, that is, a square. Let x' be its side

The $\triangle$ s VNP and VOA are similar,

$$\frac{AO}{PN} = \frac{OV}{NV} = \frac{15}{5}$$

The $\triangle$ s PNQ and AOB are similar,

$$\frac{AO}{PN} = \frac{AB}{PQ} = \frac{10}{x}$$

Hence
$$\frac{15}{5} = \frac{10}{x}$$

$$x = \frac{10}{3}$$

side of section = $3\frac{1}{3}$ ft.

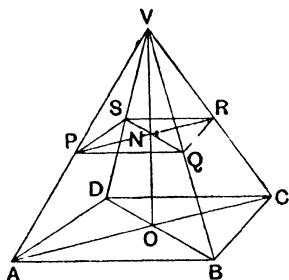


Fig. 269

Ex. 50 A cone has diameter of base $10\frac{1}{2}$ in. and slant side 9 in. A circular section cuts a slant side 4 in. from the vertex. Find the radius of the section.

Ex. 51 The radii of the base and a parallel section of a cone are 12 cm. and 5 cm., and the distance between their planes is 6 cm. Find the height of the cone.

Ex. 52 A right pyramid is cut by a plane parallel to the base. The base is a square side 18 cm. and the section a square side 10 cm. The whole height of the pyramid is 15 cm. Find the distance between the plane of section and the base.

Ex. 53 A beam of light issuing from a point source within a box emerges from a circular aperture of 2 in. diameter. If the source of light is 10 in. behind the aperture, what is the width of the beam 3 ft. from the aperture?

SUMMARY OF RESULTS

Similar figures must have

(i) *corresponding angles equal,*

(ii) *corresponding sides in the same ratio.*

If two triangles are equiangular, their sides are in the same ratio.

If two triangles have their sides in the same ratio they are equiangular, and therefore similar.

If two triangles have one angle of one equal to one angle of the other and the sides about these angles in the same ratio, they are similar.

CHAPTER 22

RATIO AND PROPORTION

DIVISION OF STRAIGHT LINES INTO ANY NUMBER OF EQUAL PARTS.

§ I Ordinary ruled paper is ruled in parallel equidistant lines. If a slanting line be drawn, at any angle, cutting across these lines, it is evidently divided into equal parts.

Ex 1 On the edge of a piece of paper, mark any length (say about 3 or 4 in, but do not measure it exactly), by laying this across the parallel lines of ruled paper, divide it into 7 equal parts. Mark off $\frac{3}{7}$ of the length.

Ex 2 As in Ex. 1, cut off $\frac{4}{9}$ of a given line

If you have no ruled paper or if you doubt the accuracy of the ruling, a set of equidistant parallels may be easily drawn as follows. Draw a line of convenient length, with compass step off a number of equal lengths on it, through the points of division draw a set of parallel lines. This set of parallels may now be used for sub-dividing any other length (provided that it is not too short).

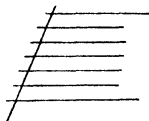


Fig 270

¶Ex. 3 What is the length of the shortest line that can be divided into 5 equal parts by means of the parallels in fig 270?

If the line to be divided is already in position on paper, the construction is performed as follows —

To divide a given straight line into five equal parts.

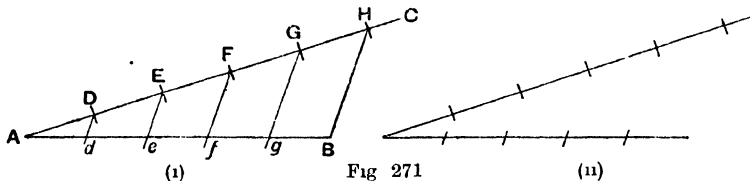


Fig 271

Let AB be the given straight line.

Construction Through A draw AC making any angle with AB

With a pair of compasses, on AC mark off 5 equal parts AD ,

DE , EF , FG , GH .

Join BH . Through D , E , F , G draw st. lines $\parallel$ to BH .

Then AB is divided into 5 equal parts

Note that the "auxiliary" line AC has *not* been sub-divided; all that has been done is to step off equal lengths along it

In practice, it is not necessary to show the whole length of the parallels, but only the bits where they cross AB (fig 271 ii) Also the short arcs cutting AC should be shown

Ex. 4. Trisect a given straight line by eye, check by making the construction.

Ex. 5. Divide a straight line of 10 cm into six equal parts; check

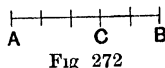
The principle underlying the above construction may be stated as a theorem, thus

THEOREM 24. **If there are three or more parallel straight lines, and the intercepts made by them on any one straight line that cuts them are equal, then the corresponding intercepts on any other straight line that cuts them are also equal.**

RATIO OF SEGMENTS OF A LINE

§2 If the line AB is divided into 5 equal parts and if AC contains 3 of those parts, we say that

$$AC = \frac{3}{5} AB, \text{ or } \frac{AC}{AB} = \frac{3}{5}$$



Again, if $AC = \frac{3}{5} AB$, it is clear that $CB = \frac{2}{5} AB$, then

$$AC : CB = 3 : 2$$

¶**Ex. 6** Draw a line 5 in long. Calculate the length of $\frac{3}{5}$ of the line and cut off this length from the line. In what ratio is the line now divided?

¶**Ex. 7.** If you want to divide a line in the ratio of 3 : 4, what portion of the line must you cut off?

¶**Ex. 8** By eye, divide lines in the ratios of 5 : 3, 4 : 3, etc

¶**Ex. 9.** A line 5 in long is divided in the ratio of 3 : 4. Calculate the lengths of the parts

¶**Ex. 10** A line l is divided in the ratio of $a : b$. What expressions give the lengths of the two parts?

Ex. 11 A line 12 cm long is divided at C in the ratio of 2 : 3, and at C' in the ratio of 3 : 5. What is the length CC'?

Ex. 12. The sides of a triangle are 6, 8, 7 in long. The side 7 in is divided in the ratio of the other two sides. What are the lengths of the parts?

Ex. 13 Two circles have radii 10 cm, 15 cm, and the distance between their centres is 30 cm. How far from the centre of the larger is a point which divides the distance between their centres in the ratio of their radii, the larger segment being measured from the centre of the larger circle?

Ex. 14. A line 27 ft. long is divided into 3 parts in the ratio of 2 : 3 : 4. What are the lengths of the segments?

To divide a given straight line in a given ratio

Suppose that the given ratio is 3 : 7

Call the given line AB. Divide AB into 10 equal parts. If AX contains 3 of those parts, then XB contains 7 of them, and $AX : XB = 3 : 7$

If the given ratio is $x : y$ where x and y are integers, we must divide the line into $x + y$ equal parts.

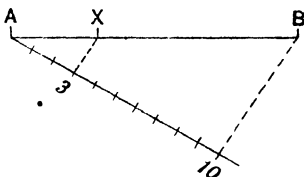


Fig. 273

Ex 15 From a given straight line cut off a part equal to $\frac{2}{9}$ of the whole line

Ex 16 Divide a straight line of 13.3 cm. in the ratio of 2 : 5

Ex 17. ABC is a triangle. From AB, AC cut off AP, AQ so that $AP = \frac{2}{3}AB$, $AQ = \frac{2}{3}AC$. Find by measurement the ratio of PQ : BC

Ex 18 Trisect two adjacent sides of any triangle and join corresponding points. Find by measurement the ratio of each joining line to the base of the triangle

Ex 19 Draw any two unequal parallel lines AB, CD. Divide each line in the ratio of 2 : 3 at the points P, Q. Join AC, BD, PQ and produce them till they intersect

Ex 20 Repeat Ex 19 with AB, CD not parallel

§3 In fig. 274 (which is part of fig. 271), XY is parallel to BH and it is clear that $\frac{AY}{AH} = \frac{2}{5} = \frac{AX}{AB}$, also $\frac{AY}{YH} = \frac{2}{3} = \frac{AX}{XB}$,

also $\frac{AH}{HY} = \frac{5}{3} = \frac{AB}{BX}$

In fact, the ratio of any two parts along AB is equal to the ratio of the corresponding parts along AH.

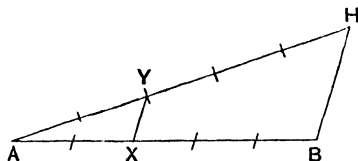


Fig. 274

In fig. 274, X, Y coincide with e, E of fig. 271. It is clear that the ratios would still be equal if X, Y had coincided with any pair of corresponding points of division, it is also clear that this would

b true if the lines, instead of being divided into 5 equal parts, had been divided into 7 (or any other number of) equal parts

Hence we see that, when two lines (such as AB , AH) are divided (at X , Y) in the same ratio, the ratio of *any* pair of parts on the one line is equal to the ratio of the corresponding parts on the other. The lines are then said to be **divided proportionally**.

From the above we also infer

THEOREM 52. If a line is drawn parallel to one side of a triangle, it divides the other two sides proportionally.

The converse theorem is also true, namely

THEOREM 53. If a line divides two sides of a triangle proportionally, it is parallel to the third side.

It should be noticed that the "mid-point" theorems (22, 23), proved on pp. 72, 73, are special cases of theorems 52, 53

NOTES ON EQUAL RATIOS.

1. It should be noted that, though we cannot have the ratio of two different kinds of quantity, we can deal with the ratio of two lengths measured in different units by changing to the same unit. For example the ratio

$$5 \text{ kilometres} : 10 \text{ miles} = 5 \times \frac{5}{8} \text{ miles} : 10 \text{ miles} \quad (\text{for } 1 \text{ km} = \frac{5}{8} \text{ m});$$

and we can test the equality of two ratios with mixed units as in the following example

Is it true that $15 \text{ in} : 25 \text{ cm} = 9 \text{ in} : 15 \text{ cm}$?

It is clear that $15 : 25 = 9 : 15$, but $15 \text{ in} : 25 \text{ cm}$ is not equal to $15 : 25$. Suppose $1 \text{ in} = x \text{ cm}$. Then $15 \text{ in} = 15x \text{ cm}$, and the question is equivalent to "Is it true that $15x \text{ cm} : 25 \text{ cm} = 9x \text{ cm} : 15 \text{ cm}$?" or "Is it true that $\frac{15x}{25} = \frac{9x}{15}$?" which is obviously true

2. If two ratios $\frac{a}{b}$, $\frac{c}{d}$ are equal, then we have the equation $\frac{a}{b} = \frac{c}{d}$

multiplying both sides by bd , $ad = bc$

The equality of the products is often a useful test of the equality of the ratios

Ex. 21 If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{a}{c} = \frac{b}{d}$. Verify by taking numerical values for a , b , c , d

Ex. 22 Find x if

$$(i) \frac{3}{5} = \frac{x}{20}, \quad (ii) 6'' : 13'' = 5 \text{ yd} : x \text{ yd}, \quad (iii) 30 \text{ cm} : x = 25 \text{ cm} : 45 \text{ in}.$$

Ex 23 Are the following true?

$$(i) 13 \text{ cm } 0.7 \text{ in.} = 32.5 \text{ cm } 1.75 \text{ in.}, \quad (ii) \frac{315}{276} = \frac{42}{368},$$

$$(iii) \frac{41}{90} = \frac{42}{91}, \quad (iv) \frac{a}{b} = \frac{a+1}{b+1}, \quad (v) \frac{a}{b} = \frac{ma}{mb}.$$

ABC is a triangle with $AB = 3''$, $AC = 2.5''$, $BC = 4''$. If $AF = 2.1''$ and FE is drawn parallel to BC meeting AC in E , calculate the length of AE .

Since $FE \parallel$ to BC ,

$$\frac{AF}{AB} = \frac{AE}{AC},$$

$$\frac{2.1}{3} = \frac{AE}{2.5}$$

Multiplying both sides by 3×2.5 ,

$$2.5 \times 2.1 = 3 \times AE,$$

$$AE = \frac{2.5 \times 2.1}{3} = 1.75''$$

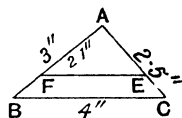


Fig. 275

Ex 24 In the triangle of fig. 275, if $F'E' \parallel$ to BC and $\frac{AF'}{F'B} = \frac{3}{2}$, calculate AF' , AE' , $E'C$

Ex 25 ABC is a triangle with $AB = 4''$, $BC = 5''$, $CA = 3.5''$, FE is drawn parallel to BC as in fig. 275 and $CE = \frac{3}{5}CA$, calculate BF , FA , AE

Ex. 26 If in the fig. of Ex 25 $\frac{BF}{BA} = \frac{2}{5}$, calculate FE , AE , EC

Ex 27 In a figure like fig. 275 with $FE \parallel$ to BC , $AB = 12 \text{ cm}$, $AC = 9 \text{ cm}$, $AE = 6 \text{ cm}$, calculate EC , AF , FB . Does the size of $\angle A$ affect the result?

Ex 28. ABC is a triangle with $AB = 6''$, $AC = 5''$. AB is produced to F so that $BF = 2''$. FE , parallel to BC , meets AC produced in E . Calculate CE

RIDERS.

†Ex 29 A variable line, drawn through a fixed point O , cuts two fixed parallel straight lines at P , Q , prove that $OP \cdot OQ$ is constant

†Ex 30 O is a fixed point and P moves along a fixed line. OP is divided at Q (internally or externally) in a fixed ratio. Find the locus of Q

†Ex 31 D is a point in the side AB of $\triangle ABC$, DE is drawn parallel to BC and cuts AC at E , EF is drawn parallel to AB and cuts BC at F . Prove that $AD \cdot DB = BF \cdot FC$

†Ex. 32 P is a point in the side AB of $\triangle ABC$; PQ is drawn parallel to BC and cuts AC at Q , CR is drawn parallel to QB and cuts AB produced at R . Prove that $AP \cdot AB = AB \cdot AR$.

†Ex 33 AD , BC are the parallel sides of a trapezium; prove that a line drawn parallel to these sides cuts the other sides proportionally.

†Ex. 34 From a point E in the common base AB of two triangles ACB , ADB , straight lines are drawn parallel to AC , AD , meeting BC , BD at F , G ; show that FG is parallel to CD .

§4 If $a:b=c:d$, the four magnitudes a, b, c, d are said to be **in proportion**.*

If x is such a magnitude that $\frac{a}{b} = \frac{c}{x}$ (or $a:b=c:x$), x is called the **fourth proportional** to the three magnitudes a, b, c .

To find the fourth proportional to three given straight lines.

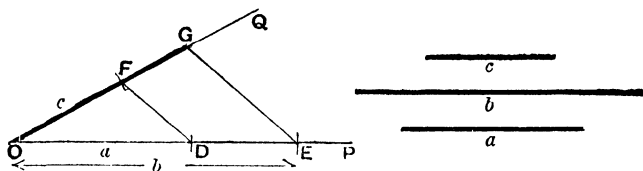


Fig 276

Let a, b, c be the three given straight lines.

Construction. Make an angle POQ . From OP cut off $OD = a$, and $OE = b$. From OQ cut off $OF = c$. Join DF . Through E draw $EG \parallel$ to DF , cutting OQ in G .

Then OG is the fourth proportional to a, b, c .

Ex. 35 Write out the proof of the above.

Ex. 36 Find graphically the fourth proportional to 3, 4, 5. Check by calculation.

Ex. 37. Find, both graphically and by calculation, the value of x in the following cases —

$$(i) \quad \frac{2}{3} = \frac{3}{x},$$

$$(ii) \quad \frac{0.8}{x} = \frac{1.25}{2.75},$$

$$(iii) \quad x : 2.6 = 5 : 2.8,$$

$$(iv) \quad 10 : 7.5 = x : 3$$

Solve graphically Ex. 38–44

Ex. 38 If 6 books cost 15s, what will 8 cost?

Ex. 39. If 1 kilogram = 2.2 lb, how many lb equal 2.6 kilograms?

Ex. 40 A motor car travels 60 m p.h. and a cycle 40 m p.h. By how much will the motor beat the cycle in a 5 mile race if both start together?

Ex. 41. If $\frac{y}{x} = \frac{3}{2}$, find x when $y = 5$.

* If a, b, c, p, q, r are such that $\frac{a}{p} = \frac{b}{q} = \frac{c}{r}$, then a, b, c are said to be proportional to p, q, r . Thus, instead of saying "corresponding sides of two triangles are in the same ratio," we might say "the sides of two triangles are proportional." See Theorems 54, 55, 56.

Ex. 42 Find geometrically the value of $\frac{3 \cdot 1 \times 2 \cdot 6}{2 \cdot 9}$.

Ex. 43. It is required to construct a rectangle on a side of 2.1" equal in area to a rectangle of sides 1.9" and 2.8" Find by a graphical construction the other side of the rectangle

Ex. 44. Construct a triangle on a base of 7 cm. equal in area to a triangle whose base is 9 cm. and height 5 cm.

MISCELLANEOUS RIDERS ON SIMILARITY.

††**Ex. 45.** AB is a straight line and C is the mid-point of AB. AX, BY, CZ are perpendiculars on any other straight line. Prove that $CZ = \frac{1}{2}(AX + BY)$.
[Join AY.]

††**Ex. 46.** AB is a straight line and C is such that $\frac{AC}{CB} = \frac{2}{1}$. p, q, r are the perpendiculars from A, B, C on any straight line. Prove that $r = \frac{1}{3}(2q + p)$
[Join A to foot of $\perp$ from B]

††**Ex. 47.** Prove that the ratio of the heights of two similar triangles is equal to the ratio of their bases, the bases being a pair of corresponding sides

†**Ex. 48.** AB, AC are any two lines and BN, CM are perpendiculars to AC and AB respectively. Prove $\frac{BN}{CM} = \frac{AB}{AC}$

†**Ex. 49** The diagonals of the trapezium ABCD meet in O. Prove that $\frac{AO}{OC} = \frac{AB}{CD}$, where AB and CD are the parallel sides.

†**Ex. 50.** ABCD is a rectangle and a line through C meets AB and AD, both produced, in P and Q. Prove that $QD \cdot BP = \text{area of rectangle}$

†**Ex. 51** The tangent at any point R of a circle cuts the tangents at the ends of any diameter in P and Q. Prove that $RP \cdot RQ = (\text{radius})^2$

†**Ex. 52** ABC is an isosceles triangle with $AB = AC$. A circle centre B and radius BC cuts AC at D. Prove that $AC \cdot CD = BC^2$

Ex. 53. Draw a straight line AB, on it take two points P, Q, draw another straight line CD, show how to divide CD similarly to AB.

Ex. 54 If $AD = \frac{1}{2}AB$ and $AE = \frac{1}{2}AC$, what can you state about the direction and length of DE?

Ex. 55 Draw a triangle ABC and show how to find points P, Q in AB, AC such that PQ is parallel to the base BC and $= \frac{1}{3}BC$.

†**Ex. 56** Any straight line drawn from the vertex to the base of a triangle is bisected by the line joining the mid-points of the sides.

†**Ex. 57.** The medians BE, CF of a triangle ABC intersect at G, GB, GC are bisected at H, K respectively. Prove that HKEF is a parallelogram. Hence prove that G is a point of trisection of BE and CF

†Ex. 58 T, V are the mid-points of the opposite sides PQ, RS of a parallelogram PQRS. Prove that ST, QV trisect PR.

†Ex. 59 Two straight lines BGE, CGF intersect at G so that $GE = \frac{1}{3} BE$ and $GF = \frac{1}{3} CF$; BF and CE are produced to meet at A, prove that $BF = FA$ and $CE = EA$.

SUMMARY OF RESULTS

If a line is drawn parallel to one side of a triangle, it divides the other two sides proportionally.

If a line divides two sides of a triangle proportionally, it is parallel to the third side.

CHAPTER 23

SOME APPLICATIONS OF SIMILAR TRIANGLES

INTERSECTING CHORDS OF A CIRCLE.

§ I. THEOREM 59 (1). **If AB, CD, two chords of a circle, intersect at a point P inside the circle, then**

$$PA \cdot PB = PC \cdot PD$$

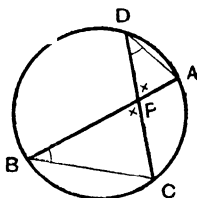


Fig 277

Construction

Join BC, AD

Proof ¹

In the $\triangle$ s PAD, PCB,

$\angle APD = \angle CPB$ (vert. opp.), and $\angle D = \angle B$ (in the same segment),

the third angles are equal, and the $\triangle$ s are equiangular,

$$\therefore \frac{PA}{PC} = \frac{PD}{PB},$$

$$\therefore PA \cdot PB = PC \cdot PD$$

To calculate the value of PA . PB

Suppose that EPF is the chord which is bisected at P. Let O be the centre of the circle

Then $PA \cdot PB = PE \cdot PF = r^2 - OP^2$. (Why?)

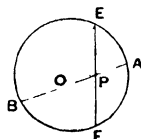


Fig. 278.

THEOREM 59 (11) If AB, CD, two chords of a circle, intersect at a point P outside the circle, then $PA \cdot PB = PC \cdot PD$.

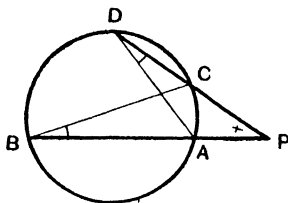


Fig. 279.

Construction

Join BC, AD.

Proof

In the $\triangle$ s PAD, PCB,

$\angle P$ is common, and $\angle D = \angle B$ (in the same segment),

the third angles are equal, and the $\triangle$ s are equiangular,

$$\frac{PA}{PC} = \frac{PD}{PB},$$

$$\therefore PA \cdot PB = PC \cdot PD.$$

PARTICULAR CASE If PT is a tangent to a circle and AB a chord of the circle passing through P, then $PT^2 = PA \cdot PB$. (See fig. 280)

NOTE The above theorems are really two different cases of the same theorem, notice that the proofs are nearly identical.

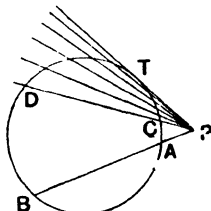


Fig. 280.

To calculate the value of $PA \cdot PB$ in fig. 280

Use the fact that $PA \cdot PB = PT^2 = OP^2 - OT^2$, where O is the centre of the circle.

NOTE. The product $PA \cdot PB$ (or $PC \cdot PD$) mentioned in the above theorems is the measure of the area of a rectangle whose sides are equal to PA and PB (or PC and PD). Euclid enunciated these theorems in the following form — “the rectangle contained by the segments of the one chord is equal to the rectangle contained by the segments of the other”

The pupil must beware of muddling $PA \cdot PB$ (which means $PA \times PB$) and $PA : PB$ (which means $PA - PB$).

Ex. 1. Calculate (and check graphically) the areas of the rectangles contained by the segments of chords passing through P when (i) $r=5$ in., $OP=3$ in., (ii) $r=5$ cm., $OP=13$ cm., (iii) $r=3\frac{1}{2}$ in., $OP=2\frac{1}{2}$ in., (iv) $r=2\frac{1}{2}$ in., $OP=3\frac{1}{2}$ in.

Ex. 2 Find expressions for the areas in Ex. 1, r being the radius, and d the distance OP (i) when $d < r$, (ii) when $d > r$. Explain fully.

Ex. 3. Draw two straight lines APB , CPD intersecting at P , make $PA=4$ cm., $PB=6$ cm., $PC=3$ cm. Describe a circle through ABC , cutting CP produced in D . Calculate PD , and check by measurement.

What would be the result if the exercise were repeated with the same lengths, but a different angle between APB , CPD ?

Ex. 4. From a point P draw two straight lines PAB , PCD ; make $PA=4$ cm., $PB=9$ cm., $PC=6$ cm. Describe a circle through ABC , let it cut PC again at D . Calculate PD , and check by measurement.

†**Ex. 5.** If the common chord of two intersecting circles be produced to any point T , the tangents to the circles from T are equal to one another.

†**Ex. 6** The common chord of two intersecting circles bisects their common tangents.

†**Ex. 7** The altitudes BE , CF of a triangle ABC intersect at H , prove that

(i) $BH \cdot HE = CH \cdot HF$, (ii) $AF \cdot AB = AE \cdot AC$, (iii) $BH \cdot BE = BF \cdot BA$.

†**Ex. 8** Two circles intersect at A , B , T is any point in AB , or AB produced; TCD , TEF are drawn cutting the one circle in C , D , the other in E , F . Prove that C , D , E , F are concyclic.

AREAS OF SIMILAR FIGURES.

§ 2. †**Ex. 9** If the side of a square is doubled, how is the area altered? Illustrate with a freehand sketch.

†**Ex. 10.** If the side of a square is trebled, how is the area altered? Illustrate with a freehand sketch.

†**Ex. 11.** If the side of a square is halved, how does the small square compare with the large? Illustrate with a freehand sketch.

†**Ex. 12.** If each side of a rectangle is doubled, how does the area of the new figure compare with that of the old? Illustrate with a freehand sketch.

Consider any figure, rectilinear or curvilinear, let it be divided up into small equal squares for purposes of counting the number of units of area. (See fig. 281.) Now suppose that the figure, squares and all, is magnified, say 3 times, we shall have a figure similar to the former, the linear dimensions of the two being in the ratio 3:1. The area of each of the small squares will be magnified 9 times, and as the

number of squares is unchanged, the area of the whole figure will be magnified 9 times. This suggests

The ratio of the areas of similar figures is equal to the square of the ratio of their linear dimensions.

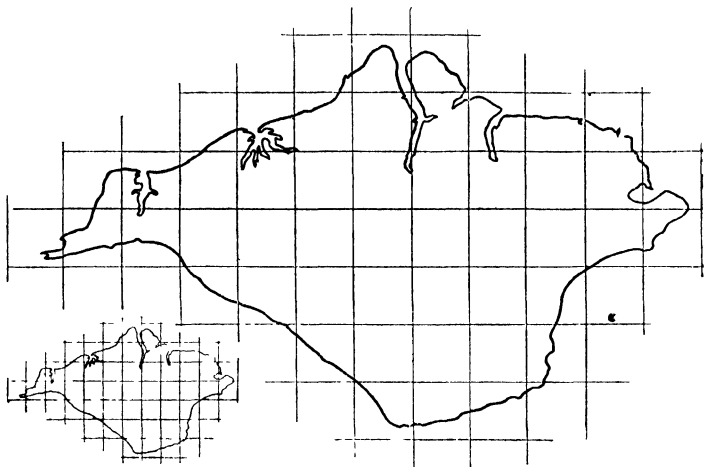


Fig 281

¶Ex. 13 What is the ratio of the areas of two similar triangles (i) on bases of 3 in and 4 in, (ii) on bases of a in and b in?

¶Ex. 14. The area of a triangle with a base of 12 cm is 60 sq cm, find the area of a similar triangle with a base of 9 cm

What is the area of a similar triangle on a base of 9 in?

¶Ex. 15 The areas of two similar triangles are 10^4 sq. cm. and 64 sq. cm the base of the greater is 7 cm., find the base of the smaller

¶Ex. 16 What is the ratio of the area of a room to the area by which it is represented on a plan whose scale is 1 in. to 1 ft?

¶Ex. 17 On a map whose scale is 1 mile to 1 in, a piece of land is represented by an area of 20 sq in, what is the area of the land?

¶Ex. 18 On a map whose scale is 2 miles to 1 in, a piece of land is represented by an area of 24 sq in, what is the area of the land?

Ex. 19. What is the acreage of a field which is represented by an area of 3 sq in on a map whose scale is 25 in to the mile? (640 acres = 1 sq mile)

Ex. 20. The height of the Great Pyramid is 149 metres, an exact model of the pyramid is made of height 1 49 metres, its side faces being triangles similar to the side faces of the pyramid. What is the ratio of the total slant surface of the pyramid to that of the model?

VOLUMES OF SIMILAR SOLIDS.

§3. ¶Ex. 21. What is the volume of a cube with edge 2 in.? How many times does it contain a cube with edge 1 in.?

¶Ex. 22. What is the volume of a cube with edge 3 in.? How many times does it contain a cube with edge 1 in.?

¶Ex. 23. If the edge of a cube is doubled, how is the volume altered?

¶Ex. 24. If the edge of a cube is trebled, how is the volume altered?

¶Ex. 25. If the edge of a cube is made n times as big, how is the volume altered?

¶Ex. 26. If the ratio of the edges of two cubes is 2 : 3, what is the ratio of their volumes?

Consider any solid, its volume is a certain number of cubic units, which number depends on the length, breadth, and height of the solid. If we have a similar solid, its length, breadth and height will each be altered in the same ratio. Hence if this ratio is $n : 1$ the volume will be increased in the ratio of $n^3 : 1$, i.e. the new volume will be n^3 times the original volume. If the ratios of the lengths, breadths and heights are each $\frac{a}{b}$, then the volume will be $\frac{a^3}{b^3}$ times the original volume. This suggests

The ratio of the volumes of similar solids is equal to the cube of the ratio of their linear dimensions.

Ex. 27. What is the ratio of the volumes of two similar solids of heights 2" and 5"?

Ex. 28. The volumes of two similar solids are 27 c.c. and 125 c.c. One side of the larger is 1.5 cm., what is the corresponding side in the other?

Ex. 29. A model of a tank is on a scale of 1 to 100. The model holds 1 pint of water. How many gallons will the tank hold?

Ex. 30. A wooden pyramid of height 3" weighs $\frac{1}{2}$ lb. What will a similar pyramid of height 14" weigh, if made of the same material?

Ex. 31. What would be the side of a cube whose volume is half that of a cube of side 1 foot?

§4 THEOREM 60, COR. 1. If ABC is a triangle right-angled at A , and AD is drawn perpendicular to BC , then

$$AD^2 = BD \cdot DC$$

Proof Since $\angle BAC$ is a right angle,

$$\angle BAD + \angle DAC = \text{a rt } \angle$$

Since $\angle ADC$ is a rt $\angle$ in $\triangle ADC$,

$$\therefore \angle ACD + \angle DAC = \text{a rt } \angle,$$

$$\angle BAD = \angle ACD$$

Hence, in the $\triangle s$ ABD , CAD ,

$$\angle BAD = \angle ACD, \angle ADB = \angle CDA \text{ (rt. } \angle s),$$

$\therefore$ the third angles are equal,

the triangles are equiangular,

$$\therefore \frac{BD}{AD} = \frac{AD}{CD},$$

$$\therefore AD^2 = BD \cdot DC.$$

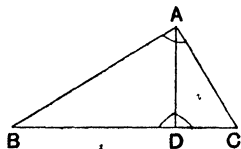


Fig 282.

†Ex 32. In fig. 282, prove that $\triangle s$ DBA , DAC are each similar to $\triangle ABC$. Hence show that $BA^2 = BD \cdot BC$ and $CA^2 = CD \cdot CB$, and deduce Pythagoras' theorem by adding these results.

MEAN PROPORTIONAL

§5 If x is such a quantity that $a : x = x : b$, then x is called the **mean proportional** between a and b

If $\frac{a}{x} = \frac{x}{b}$, $x^2 = ab$, and therefore $x = \sqrt{ab}$, thus the mean proportional between two numbers is the square root of their product.

¶Ex. 33. Find the mean proportional between

$$(i) \ 4 \text{ and } 9, \quad (ii) \ 1 \text{ and } 100, \quad (iii) \ \frac{1}{2} \text{ and } 2,$$

$$(iv) \ \frac{3}{4} \text{ and } \frac{4}{3}, \quad (v) \ 1 \text{ and } 2, \quad (vi) \ 2 \text{ and } 4$$

To find the mean proportional between two given straight lines.

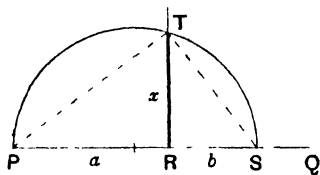
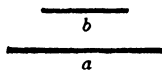


Fig 283.



Let a , b be the two given straight lines.

Construction Draw a straight line PQ

From PQ cut off $PR = a$, and $RS = b$

On PS as diameter describe a semicircle.

Through R draw $RT \perp$ to PS to cut the semicircle at T.

Then RT (x) is the mean proportional between a , b .

¶Ex 34 Prove the above by § 4

Ex 35. Find geometrically the mean proportional to 3 and 2. Verify by calculation.

Ex 36 Find the side of a square whose area is equal to a rectangle of sides 4 cm, 5.5 cm

Ex 37. Construct a square equal in area to a rectangle of area 6 sq in

Ex. 38 Bisect the area of any triangle by a line parallel to the base.

Ex. 39. Draw any triangle Construct a similar triangle twice its area

Ex 40 Find geometrically the value of $\sqrt{3}$, $\sqrt{15}$

SUMMARY OF RESULTS

If two chords AB, CD of a circle intersect at a point P inside or outside the circle, then $PA \cdot PB = PC \cdot PD$.

If PT is a tangent to a circle and AB a chord passing through P, then $PT^2 = PA \cdot PB$

If ABC is a triangle right-angled at A, and AD is perpendicular to BC, then $AD^2 = BD \cdot DC$

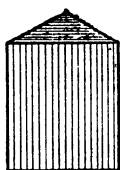
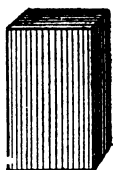
The ratio of the areas of similar figures is equal to the square of the ratio of their linear dimensions.

The ratio of the volumes of similar solids is equal to the cube of the ratio of their linear dimensions

CHAPTER 24

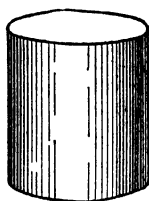
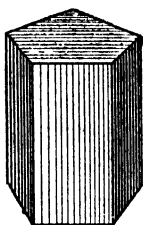
SOLID FIGURES

§ I. In many examples we have already dealt with solid figures. It is important to realise that much of the plane geometry that we have done applies to three dimensions, for it applies to any plane section of a solid figure.



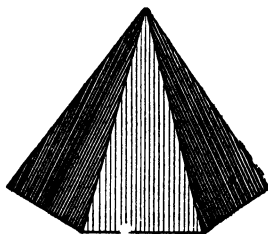
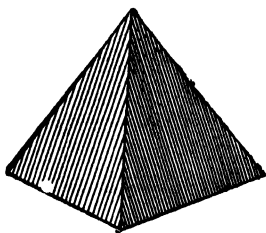
Right Prisms

Fig. 284



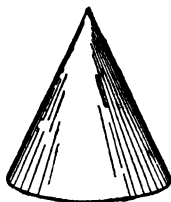
Right Cylinder

Fig. 285.



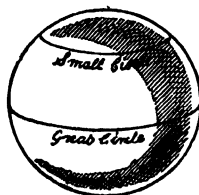
Pyramids

Fig. 286



Cone

Fig. 287.



Sphere

Fig. 288

§2 A little practice at interpreting a few simple drawings is very helpful

What does fig. 289 represent?

(i) It represents a cube with the middle corner in front

•(ii) It represents three planes at right angles to one another with the middle corner the furthest point.

Practise seeing the above and practise changing quickly from (i) to (ii) and back again

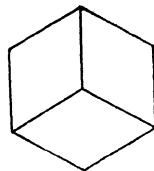


Fig. 289

Consider fig. 290. It represents (i) a cone with the vertex towards you, or (ii) a hollow cone with the vertex the furthest point. Practise changing from (i) to (ii) and back again

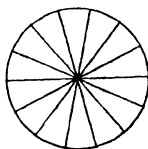


Fig. 290

Again, consider fig. 291. It represents either

(i) A framework of rods forming a tetrahedron (or a triangular pyramid) with the edge AC in front, or

(ii) A tetrahedron with the edge BD in front

To make (i) clearer rub out the parts of BD that are close to AC. Similarly to make (ii) clearer rub out the parts of AC that are close to BD

As soon as (i) and (ii) can easily be visualised, practise changing from (i) to (ii) and back again as quickly as possible

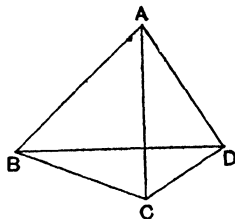


Fig. 291

These simple exercises will help you to realise a solid figure, even from a comparatively poor drawing

§3 After practice in seeing three dimensions in a two-dimensional drawing, the next thing is to learn to make simple drawings of solid figures.

* The first difficulty in teaching solid geometry is to get the boy to visualise a solid figure from a flat drawing. Of course a shaded figure, or a photograph, or a stereoscope make it easier for the boy but he has to learn to see three dimensions in a drawing that he has made himself, so that clever drawings made by the master, far from being a help, may be a hindrance.

„Ex. 1 Draw a figure of a box or brick as follows —

On squared paper draw a rectangle along the lines of the paper, draw an equal rectangle slightly to the right and slightly higher up on the paper, but well overlapping the first; then join corresponding corners.

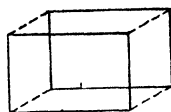


Fig. 292.

Repeat this on plain paper

Here are a few points that will help you in making drawings of solid figures

(i) All lines that are vertical in a solid figure should be represented by lines that are parallel to the side edges of your paper. Verify this by looking at any pictures

(ii) Note that horizontal lines are not necessarily parallel to the top and bottom edges of your paper. See fig. 293, which represents an E and W. road crossing a N and S. road with a vertical flagstaff OT at the crossing.

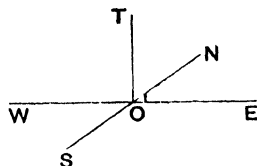


Fig. 293

(iii) Note also that in a drawing

A rectangle generally looks like a parallelogram,

A circle generally looks like an oval figure called an ellipse,

Right angles generally look like acute or obtuse angles

Ex. 2 Make a freehand sketch of a horizontal drawing-board, with a vertical line standing at the centre of the board. Show the angles of elevation of the top of the line from (i) each corner of the board, (ii) the middle of the nearest edge

Ex. 3. Make a freehand sketch of the following —

Two roads run, one E and W. and the other N and S, A is the point of intersection, B is a point two miles due E of A, C is a point one mile due N of B. At C there is a vertical tower CD. Mark the angles of elevation of the top of the tower from A and B. Also name all the right angles in the figure

Ex. 4 Consider fig. 294, which represents a regular tetrahedron. Each face is an equilateral triangle, VO is its altitude, the three lines through O are the altitudes of the triangle ABC. Make accurate drawings of

(i) Triangle ABC and the lines AD, BE, CF,

(ii) Triangle VAC, showing VE;

(iii) Triangle VBE, getting the lengths of BE and VE from (i) and (ii). Draw VO perpendicular to BE and check that $OE = \frac{1}{3}BE$. Without further reference to fig. 294, draw a general view of the tetrahedron like the figure. It is best first to draw the base and the various lines on it and then draw VO parallel to the edge of your paper.

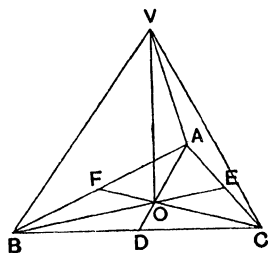


Fig. 294.

Ex. 5. In a pyramid $OABCD$, the base $ABCD$ is a square 6 cm. in the side (see fig. 295) N is the middle point of AB . The centre of the base is M , and the line OM is perpendicular to all lines lying in the base, such as MA and MN ; OM is 4 cm. long. Draw the square base; then draw the right-angled triangles OMN and OMA , and measure the lengths ON and OA . Then draw the face OAB and give its area.

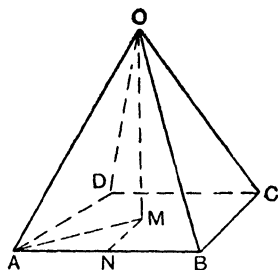


Fig 295

Draw fig. 295 without further reference to the figure. First draw the base and then draw MO parallel to the edge of your paper

Ex. 6. Draw a figure to represent a sphere and a plane section. On the plane section show two lines through its centre which are at right angles to one another. See fig. 204, p. 147.

Ex. 7 In fig 296, $ABCD$ is a regular tetrahedron on each edge of which measures 12 cm. $AE=AF=4$ cm. The shaded area represents a section through C, E, F . Draw the side-faces of the tetrahedron; then draw the triangle CEF and find its area.

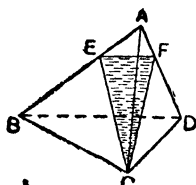


Fig. 296.

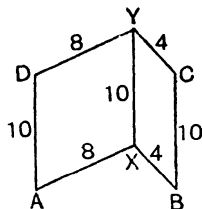


Fig 297.

Ex. 8. A rectangular sheet of paper 12 in. by 10 in. is folded along XY , a line 4 in. from the shorter side BC (see fig. 297). Find the length of AC (i) before folding, (ii) when the two parts of the sheet are at right angles.

Ex. 9 In the cuboid in fig. 298, AE is perp. to BC . By drawing various sections, find the length of DE .

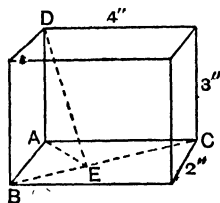


Fig. 298.

Ex. 10 In fig 299 suppose that A, C and H are three points on the level and B an observation balloon directly above H. You are to determine the distance CB and the height HB of the balloon by drawing. For the triangle BAC use the data $AC=850$ ft, $\angle BAC=48^\circ$, $\angle BCA=90^\circ$, scale 1 cm^s to 100 ft., and for the triangle BCH use the additional data $\angle BCH=32^\circ$, $\angle H=90^\circ$. State your construction

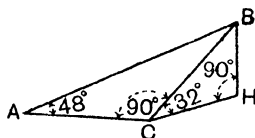


Fig. 299

Ex. 11. Cut out a paper triangle ABC having $AB=4''$, $\angle B=90^\circ$, $BC=2''$, on it mark D, E the mid-points of AB, AC, and fold about DE so that the two parts of the paper are at right angles.

Now make a drawing of the paper. Either by drawing or calculation, find the lengths of AE, EC, AC and determine whether angle AEC is a right angle.

Ex. 12 Fig 300 represents a coal-box. Find the distance from a back top corner to the opposite front bottom corner

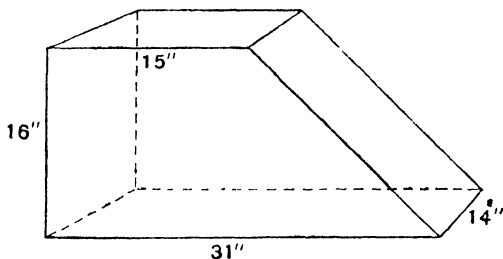


Fig. 300

MISCELLANEOUS EXERCISES C

THE CIRCLE (CALCULATIONS).

Ex. 1. If the radius of a ball is 4 inches, find the radius of a section by a plane distant 2 inches from the centre of the ball

Ex. 2. If AB is a tangent to a circle of radius 5", where A is any point on the circumference and B is 12" from A , find the locus of B as A moves round the circle.

Ex. 3 ABC is a triangle with $\angle A = 40^\circ$ and $\angle B = 82^\circ$. The bisector of $\angle A$ meets BC in D , BN is drawn perpendicular to AD and is produced to cut AC in X . Calculate the angles of the quadrilateral $XCDN$, giving reasons

Ex. 4. Two circles with centres A and B intersect in C . If $\angle ACB$ is 110° , find the angle between the tangents at C to the two circles. Give reasons.

Ex. 5 ABC is a triangle inscribed in a circle. The tangent at A meets BC produced in P . If $\angle B = 46^\circ$, $\angle C = 72^\circ$, find $\angle P$, stating your reasons clearly

Ex. 6. ABC is a tangent to a circle at B , BD is a diameter and BE , BF are chords such that $\angle ABE = 15^\circ$, $\angle CBF = 55^\circ$, DE , DF , EF are joined. Find all the angles of the figure

Ex. 7. Draw a quadrilateral $ABCD$ having $AB = 6.25$ cm, $BC = 8.4$ cm, $CD = 5.3$ cm., $\angle ABC = 95^\circ$, $\angle BCD = 105^\circ$. Measure AD and $\angle ADC$. Without further drawing, state whether D will be inside or outside the circle ABC . Give your reasons. Then test your conclusions by drawing the circle ABC .

Ex. 8 A square $ABCD$ rolls, without sliding, along a line turning about B , C , in succession. Draw the path of the point A , starting with AB on the line, until it again lies on the line. Calculate the length of this path.

Ex. 9 A small wheel with 10 cogs drives a larger wheel with 35 cogs. When the small wheel has made one revolution, through what angle has the larger turned?

Ex. 10. The length of a curved line is measured by rolling a disc of radius 1 cm. along it. If the disc makes $8\frac{1}{2}$ revolutions, what is the length of the line?

Ex. 11. Six tiles in the shape of equilateral triangles of side 5 in. are placed flat around a point with one vertex of each at the point. A thin circular ring is then placed around the tiles. Find the area of the space between the ring and the tiles.

Ex. 12 OA is the vertical line which is the junction between two walls of a room, OB and OC the horizontal lines running along the junction between the walls and the floor. What is the locus of the point P which moves about in the room under the following conditions — (i) so as to be always 5 ft. above the floor? (ii) so that the angle AOP is always 50° ? (iii) so as to be always equidistant from OB and OC ? (iv) so as to be 4 ft. from O and 2 ft. from the plane AOC ?

Ex. 13 Fig. 301 represents a bridge, whose span AB is 80 ft., supported on an arch in the form of an arc of a circle $AC=BD$. Let r ft. denote the radius of the circular arc and b ft. the height of the roadway AB above the highest point of the arch. Find an algebraic equation connecting r with b . Then use it to calculate (i) the value of b when $r=65$, (ii) the value of r when $b=1$.



Fig. 301.

Ex. 14 A penny and a halfpenny are placed on a table with their centres 4 cm. apart, and a ruler is laid on the table against them so as to touch each coin. If the diameter of a penny is 3.05 cm. and that of a halfpenny 2.54 cm., what is the distance between the points at which the ruler touches them?

Ex. 15 O is the centre of a circle of 2 in. radius, A is a point 3 in. from O , AP is a tangent from A . If OP is produced to Q so that $PQ=2AP$, prove that the circle whose centre is A and radius AQ will touch the given circle.

SIMILARITY (CALCULATIONS)

Ex. 16 Use similar triangles to find the relations between x and y in Figs. 302–305.

In Figs. 302, 303, $ABCD$ is a square.

In Fig. 304 the short lines are parallel.

In Fig. 305 the long lines are parallel.

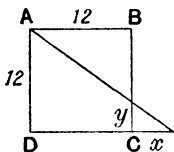


Fig. 302

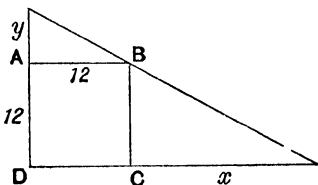


Fig. 303

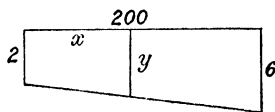


Fig. 304.

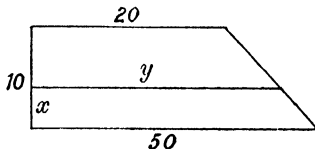


Fig. 305.

Ex. 17 A swimming bath 200 ft. long is 2 ft. deep at one end, 6 ft. at the other and the bottom slopes uniformly. Find the depth at a distance of 60 ft. from the shallow end.

Ex. 18 The sides of a rectangle are $\sqrt{2}$ in. and 1 in. Show that a parallel to the shorter side through the mid-points of the longer sides divides the rectangle into two other rectangles similar to the first.

Ex. 19. The diameter of the base of a cone is 20" and its height 15"; find the diameter of the section parallel to the base and 6" from it.

Ex 20. A small light above the centre of a rectangular table whose length is 5 ft. 6 in forms a shadow of the table on the floor whose length is 8 ft. The table is 2 ft 9 in. high. What is the height of the light above the floor?

Ex. 21 The parallel sides of a trapezium are of lengths 4 in. and 12 in.; the other two sides are each divided in the ratio of 3 : 5 by a line parallel to the parallel sides. Calculate the two possible values of the length of this line.

Ex. 22 A triangular framework ABC has AB = 3 ft., BC = 3 ft. 6 in., AC = 2 ft. 9 in. A rigid stick 2 ft. long is wedged between AB and AC so as to be parallel to BC. How far are its ends from A?

Ex. 23. The base of an isosceles triangle is 40 cm., a line 30 cm. long parallel to the base and 5 cm. distant from it can be drawn with its ends on the sides of the triangle. Calculate the height of the triangle.

Ex. 24 An isosceles trapezium with parallel sides 10 in. and 6 in. and slant sides 5 in. was cut from a piece of cardboard in the shape of an isosceles triangle whose base was 10 in. Calculate the length of the sides of the triangle.

Ex 25 A pyramid stands on a square base of side 16 in. and its slant edges are of length 24 in. A section is cut through points along each edge 5 in. from the vertex. Calculate the length of the sides of the section. Find also the depth of the section below the vertex. (See fig. 269)

Ex. 26 A rod DC, 2 ft. long, is suspended from two points A, B in the ceiling 4 ft. apart by two equal wires AD, BC each 6 ft. long. The rod DC is now twisted round through 180° so that the wires cross at O. Calculate the length of AO and OD.

The rod is now untwisted into its original position and the points which were at O are joined by a string. What is the length of this string?

Ex 27 A triangular set square with angles of 30° , 60° , 90° and hypotenuse 6 in. long stands in a vertical plane with the right angle on a table and the shorter side inclined at 30° to the table. Calculate the height of the other two vertices above the table.

Ex. 28 ABC is an isosceles triangle with base 4 in. and sides 6 in. A line is drawn from the middle point of the base perpendicular to one side. Find the distance of each vertex from this line.

Ex. 29. ABC is a right-angled triangle with hypotenuse AB = 15 cm. and AC = 6 cm. From a point D on BC, 9 cm. from B, DE is drawn perpendicular to AB. Calculate the length of DE.

Ex 30. A rectangular board 16 in. by 12 in. is sawn in two along a line perpendicular to a diagonal at a point 8 in. from one end of the diagonal. Calculate the length of the saw-cut.

Ex. 31. ABCD is a parallelogram with AB = 3", AD = 5". Through C a straight line is drawn cutting AB, AD produced in P, Q respectively. Prove that the product PB · QD is constant.

Ex. 32 A parallelogram ABCD has $AB=8$ in., $AD=5$ in. From P in AB, such that $AP=3$ in., PQ is drawn parallel to AD cutting the opposite side in Q and the diagonal BD in R. Calculate the lengths of PR, RQ.

Ex. 33. ABC is an isosceles triangle with $AB=AC=18$ cm, $BC=10$ cm. A circle radius 2 cm touches AB, AC. Find, by calculation, the distance of its centre from A.

Ex. 34. Draw a parallelogram ABCD with diagonals $AC=8$ cm, $BD=10$ cm. Describe a circle through ABC and let it cut BD in E. If O is the intersection of the diagonals, calculate OE.

Ex. 35. ABC is a triangle inscribed in a circle, the tangent at C meets BA produced in T. If $TA=5$ in., $AB=3$ in., $AC=4$ in., calculate BC.

Ex. 36. BAC is a semicircle on BC as diameter and AD is perpendicular to BC. If $AD=3$ cm, $BD=5$ cm, find CD and the radius of the semicircle.

CONSTRUCTIONS.

Ex. 37 Determine the length of the radius of the circle circumscribing a triangle whose sides are 3", 4", 6".

Ex. 38 Construct a triangle OPR, given $OP=6$ cm., $OR=5$ cm., angle $O=74^\circ$. Draw a circle to pass through O, P, and R, and measure its radius.

Ex. 39. Inscribe a regular octagon in a circle of 2 in radius. Find the amount by which its perimeter exceeds the perimeter of the square inscribed in the same circle.

†Ex. 40 Construct a triangle ABC in which $a=3''$, $c=4''$, and $B=29^\circ$. Draw its inscribed circle, and also the escribed circle which touches AC between A and C. Measure the radius of each circle, and show theoretically that the line joining their centres must pass through B.

Ex. 41. Draw any triangle and its inscribed circle. Join each point of contact to the centre. Now draw a circle of 1 in. radius and, working from your first figure, circumscribe about this circle a triangle similar to the first triangle.

Ex. 42. In a quadrilateral ABCD, AB is parallel to CD and the diagonals AC and BD, each 3 in long, cut at an angle of 60° at a point of trisection of each. Verify by drawing that A, B, C, D lie on a circle and give any reasons why this must be so.

Ex. 43. OY, OX are two straight lines at right angles. On OX two points A, B are marked so that $OA=1''$, $OB=3''$. By construction find a point (or points) on OY at which AB subtends an angle of 25° . Explain your construction and measure the distance of the point (or points) from O. Find the position of the point on OY at which AB subtends the greatest possible angle.

Ex. 44. State, without actually performing any construction, how you would solve the problem of drawing two tangents to a circle, which should include a given angle and intersect upon a given straight line. How many solutions of the problem would you expect to get?

Ex. 45 Draw a circle 3 in. in diameter and place in it a chord AB 2.5 in. in length; draw the diameter BC and produce BC to D so that D is 1 in. distant from the circle, and through D draw DE perpendicular to BD. Then draw a circle touching DE and also touching the former circle at A. State the steps of your construction.

Ex. 46 Draw two straight lines making an angle of 60° with one another and intersecting at O. On one of the lines take the two points X, Y on opposite sides of O such that $XO = 2$ in., $OY = 4.5$ in. Draw the circle through the points X and Y which will cut the other line in two points equidistant from O. Calculate the distance of each of these points from O.

Ex. 47 Draw a rectangle, sides 10 cm., 8 cm. Construct two similar rectangles each with one side 6 cm. long.

Ex. 48 In a circle of radius 2 in. inscribe a regular pentagon. Construct a similar pentagon whose sides shall be $\frac{3}{4}$ the sides of the first.

Ex. 49 Draw a regular hexagon of side 4 cm. Construct a similar hexagon whose area shall be $\frac{9}{4}$ times the area of the first.

Ex. 50 CD and EF are two given parallel straight lines, and A is a given point in CD. B is a given point on the side of EF remote from A. It is required to determine the position of a point P in CD, such that, when the straight line PB is drawn crossing EF at Q, then PQ may be equal to AP. Prove that the perpendicular distance of the line PB from the point A is equal to the distance between the parallel lines. Hence solve the problem, and show that there are in general two possible positions for the point P.

Draw the figure, making the perpendicular distance between the parallel lines equal to 3 cm., $AB = 8$ cm. and angle $DAB = 60^\circ$. Determine from your drawing the two possible lengths of PQ.

SCALE DRAWING.

Ex. 51 A weight on the end of a string 3 ft. long swings on each side of the vertical so that the greatest distance between its extreme positions is 2 ft. Through what angle does the string swing and what is the least vertical distance of the weight below the point of suspension?

Ex. 52 A position X lies 4000 yd. N. 68° E. of Y, whilst Z lies 3000 yd. due S. of Y. Find the distance and bearing from X of a position which is equidistant from the three positions X, Y and Z.

Ex. 53 Draw the plan and elevation of a cylindrical can 10" high and 6" diameter standing on its base. Thence find the length of the longest stick that can be placed in it without projecting over the edge.

Ex. 54 Draw the plan and elevation of a bucket whose two ends are circles of radii 6" and 4", the depth of the bucket being 14". From your figures find the radius of the largest sphere which will go into the bucket so as just to touch the bottom.

Ex. 55. A weight hangs by a string from the end B of a beam AB, 20 ft. long, and just rests on the ground when AB is inclined to the vertical at an angle of 42° (the end A being on the ground). If the beam is turned about A until AB is vertical, through what vertical distance is the weight raised?

Ex. 56. A, B, C are three landmarks. B is 200 yd. due E of A, and C is 200 yd. N. 26° E of B. An observer in a ship which is due N of B observes that AC subtends an angle of 90° at his eye. Find, by drawing, the distances of the ship from A, B, and C.

Ex. 57. AB is a breakwater 2000 yd long, B being due E of A. The breakwater subtends an angle of 50° at each of two ships, X and Y. If X bears N. 10° E. from A, and Y is 800 yd. to the eastward (not due E.) of X, find the distance of each ship from the breakwater.

CIRCLE (RIDERS)

†**Ex. 58.** Chords AP, BQ are drawn perpendicular to a chord AB at its extremities. Prove that $AP = BQ$.

†**Ex. 59** ABC is a triangle inscribed in a circle and BT is drawn parallel to AC cutting the tangent at A in T; prove $\angle ABC = \angle BTA$.

Ex. 60. A chord BA of a circle is produced to meet the tangent at T in P. The tangent at A meets PT in Q. If $\angle APT = x$, $\angle BAQ = y$, prove that $\angle PTA = \frac{1}{2}(y - x)$.

†**Ex. 61** QS, RT are altitudes of an acute-angled triangle PQR; prove that $\angle STR = \angle RQS$.

†**Ex. 62.** The triangle ABC is right-angled at A. If P is the centre of the square described on BC, prove that AP bisects $\angle BAC$.

Ex. 63. A mechanism consists of a straight rod ABC, 8 in. long, and a rod BD, 4 in. long, hinged to ABC at B, the mid-point of AC. Prove that, however the mechanism is placed, AD is always at right angles to DC.

If D is fixed and A moves along a straight line DX, what is the greatest and least distances of A from D if DC cannot exceed 3 in.

†**Ex. 64.** ABCD is a parallelogram. If the circle through the points A, B, C cuts CD in P, prove that $AP = AD$.

†**Ex. 65.** ABC is an equilateral triangle inscribed in a circle whose centre is O, AP is a diameter. Prove that the triangle BOP is equilateral, and that the square on AB is three times the square on AO.

†**Ex. 66.** A triangle ACE is inscribed in a circle ABCDEF, prove that the sum of the angles ABC, CDE, EFA is equal to four right angles.

†**Ex. 67** AB is a chord of a circle whose centre is O, and AB is parallel to the tangent at P. If the tangents at A and B intersect at T, prove that the angles POA, TAB are equal to one another.

†**Ex. 68.** A circle whose centre is O is touched internally at A by a circle of half its radius. A radius OQ of the former circle cuts the smaller circle at P. Prove that $\text{arc } AQ = \text{arc } AP$.

†Ex 69. In fig. 306, BC touches the circle ABD . Show that CE touches the circle ADE at E . (You may assume the converse of the "alternate segment" theorem)

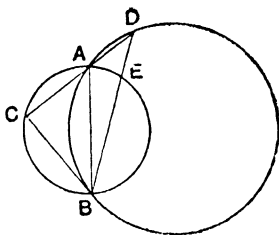


Fig 306

†Ex 71. $ABCD$ is a quadrilateral in a circle. One side BC is produced to E . Prove that the bisectors of the angles BAD , DCE meet on the circumference.

†Ex 72. To two circles, centres O and O' , an internal and an external common tangent are drawn, meeting in P . Prove that P lies on the circle on OO' as diameter.

†Ex 73. O, A, B, C, D are points on the circumference of a circle such that the angles AOB, BOC, COD are equal. Prove that the angle between the chords AC, OB is equal to that between AD, OC .

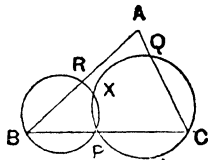


Fig 307.

†Ex 74. In fig 307 show that the four points A, Q, X, R lie on a circle.

†Ex 75. If a small circle is drawn to touch a large circle internally and to pass through the centre of the large circle, prove that any chord of the large circle which passes through the point of contact is bisected by the small circle.

SIMILARITY (RIDERS).

†Ex 76. AB, CD are the parallel sides of a trapezium whose diagonals intersect at O . Prove that $\triangle OAB, OCD$ are similar, and write down the three equal ratios of corresponding sides.

†Ex 77. From P , a point in the base YZ of a triangle XYZ , PQ, PR are drawn parallel to YX, ZX to meet XZ, XY in Q, R respectively. Prove that $\frac{YR}{PQ} = \frac{YP}{PZ}$.

†Ex 78. A, B, C, D, E are points on a straight line such that $AB = BC = CD = DE$;

O is a point outside the line. Through B a straight line is drawn parallel to OE to cut OA at X and OC produced at Y . Prove that $BY = \frac{1}{2}OE$ and $XB = \frac{1}{2}BY$.

†Ex 79. $ABCD$ is a parallelogram, and any straight line through C cuts AB and AD produced at X and Y respectively, prove that $AD \cdot BX = DY \cdot AB$.

How must the line XCY be drawn to make the $\triangle XBC$ and YDC congruent?

Ex 80. Show how to divide a triangle into four triangles each similar to the original triangle.

Ex. 81. O is a point inside a triangle ABC; OA, OB, OC are divided at A', B', C' so that $OA' : OA = OB' : OB = OC' : OC = 2 : 3$. Prove that Δs ABC, A'B'C' are similar.

†Ex. 82 The diagonals of a rectangle ABCD meet in O. Prove that, if OA, OB, OC, OD are all divided in the same ratio at P, Q, R, S, then PQRS is a rectangle similar to the first

†Ex. 83 ABC is an isosceles triangle, D the mid-point of the base BC; DE the perpendicular to the side AB. Prove that $DE = \frac{AD \cdot DC}{AC}$

†Ex. 84. ABCD is a parallelogram. From A a straight line is drawn cutting BC in E and BD in F. DE cuts AB produced in Q. Prove that $AF \cdot FC = AQ \cdot BQ$

†Ex. 85. From P any point in the base BC of a triangle ABC draw PQ parallel to CA meeting AB in Q. Join QC and draw AR parallel to QC meeting BC produced in R. Prove that $BP \cdot PC = BC \cdot CR$

†Ex. 86 ABCD is any quadrilateral whose diagonals intersect at O. From O lines are drawn parallel to DC, DA meeting BC, BA in P, Q respectively. Prove that PQ is parallel to AC

†Ex. 87 YAX is an acute angle. From P, a point outside the angle, perpendiculars PM, PN are drawn to AX, AY respectively. PK bisects the angle MPN meeting AY in K, and AX in T. Prove that the triangles PMT, PNK are similar and that $AT = AK$

†Ex. 88. Through a point O within a triangle ABC, POQ is drawn parallel to AB meeting BC in P, AC in Q, and ROS is drawn parallel to AC meeting BC in R and AB in S. Prove $AB \cdot AQ \cdot AC \cdot AS = BP \cdot CR$.

†Ex. 89 ABCD is a rectangle ($AB > 2AD$) and the circle on AB as diameter cuts CD at P. If AP, PB are joined, prove the three triangles APD, BAP, PBC all similar. Hence show that the rectangle AP · PB is equal to the given rectangle

†Ex. 90 The bisector of the angle A of a triangle ABC meets the base in D. A parallel to DA through C meets BA produced in E. Prove that $AE = AC$ and hence that $DB \cdot DC = AB \cdot AC$

†Ex. 91. The bisector of the exterior angle at A of a triangle ABC meets the base BC produced in F. Prove that $FB \cdot FC = AB \cdot AC$ [Draw a parallel to FA through C]

†Ex. 92 ABCD is a quadrilateral inscribed in a circle of which AC is a diameter, from any point P in AC, PQ and PR are drawn perpendicular to CD and AB respectively. Prove that $DQ \cdot PR = DC \cdot BC$.

†Ex. 93. A triangle PQR is inscribed in a circle and the tangent to the circle at the other end of the diameter through P cuts the sides PQ, PR produced at H, K respectively, prove that the Δs PKH, PQR are similar

†Ex. 94. The sides AC, BD of two triangles ABC, DCB on the same base BC and between the same parallels meet at E, prove that a parallel to BC through E, meeting AB, CD, is bisected at E

†Ex. 95. AB, AC are equal straight lines. D and E are points, D in AB E in AC produced, such that $DB=CE$; prove that DE is bisected by BC .

†Ex. 96. Through the vertex A , of a triangle ABC , DAE is drawn parallel to BC and AD is made equal to AE , CD cuts AB at X and BE cuts AC at Y , prove XY parallel to BC .

†Ex. 97. $ABCD$ is a rhombus, a straight line through C meets AB and AD , both produced, at P and Q respectively. Prove that $PB \cdot DQ = AP^2 \cdot AQ^2$.

†Ex. 98. A square $BCDE$ is described on the base BC of a triangle ABC , and on the side opposite to A . If AD, AE cut BC in F, G respectively, prove that FG is the base of a square inscribed in the triangle ABC .

†Ex. 99. The straight lines joining the mid-points of opposite sides of a quadrilateral bisect one another.

†Ex. 100. ABC is an equilateral triangle. From D , the mid-point of BC , DE is drawn perpendicular to AB . Prove that the triangle DEB is one-eighth the area of the triangle ABC .

Ex. 101. Through the vertices A, B, C of an equilateral triangle straight lines are drawn perpendicular to the sides AB, BC, CA respectively, so as to form another equilateral triangle. Compare the areas of the two triangles.

†Ex. 102. $ABCD$ is a rectangle and any line APQ meets DC in P and BC produced in Q . Prove that the rectangle $CD \cdot CQ$ is twice the area of the triangle PBQ .

Prove also that the rectangle $ABCD$ is twice the difference of the areas of the triangles DBQ and PBQ .

†Ex. 103. A is a point on a circle and XY is any line parallel to the tangent at A and not cutting the circle. Any two lines PAQ, RAS through A cut the circle at P, R and XY at Q, S . Prove $PA \cdot AQ = RA \cdot AS$.

†Ex. 104. Two tangents from a point T to a circle touch the circle at P and Q . PR is a chord parallel to TQ . Prove that $PR \cdot PT = PQ^2$.

REVISION PAPERS C

Paper 16.

1. On what property of a circle does the use of the semicircular protractor depend?

2. In a circle a chord 24 in. long is 5 in. distant from the centre. Calculate (i) the radius of the circle, (ii) the length of a chord which is 10 in. distant from the centre

3. Find the length of the tangent to a circle of radius 5 cm. drawn from a point 13 cm. from the centre of the circle.

4. AB is a diameter of a circle; through A and B parallel chords of the circle are drawn. Prove that these chords are equal

5. Draw a triangle the lengths of whose sides are 2, 2.2, 2.5 inches, state how you would find the centre of the circle through its vertices, construct the circle and measure its radius. You need not give a proof.

6. Circles of radii 3 and 7 cm. respectively have their centres 14 cm. apart. Calculate the length of the interior common tangents

7. Is a square equiangular with a rectangle? Are the figures similar?

8. Draw a straight line 4.7 inches long. Divide it into three equal parts *without measuring one-third of 4.7 inches*. Describe your construction, but do not give a proof

9. One of the parallel sides of a trapezium is double the other; show that the diagonals bisect one another.

Paper 17.

1. What do you know about the exterior angle formed by producing a side of a cyclic quadrilateral?

2. ABCD is a cyclic quadrilateral with its diagonals drawn. If $\angle DAC = 50^\circ$, $\angle CAB = 40^\circ$, $\angle ACD = 70^\circ$, find the other angles in the figure

3. What is the length of the tangent drawn to a circle of radius r'' from a point distant t'' from the centre?

4. Two circles touch at A and AT is their common tangent at A. BC touches the circles at B and C and cuts AT at T. Prove that BT = TC.

5. Construct a triangle ADE such that AD, DE, EA measure 5, 6.1, 9.7 cm. respectively. Construct the circumscribing circle and the circle escribed to DE. Measure the radii of the circles and the distance between their centres.

6 Draw an isosceles triangle whose base, BC , is 3 in., and vertex angle A is 36° . Construct a circle to pass through A and touch BC at C . Let the circle cut AB at P . Measure AP . Prove that $CP = CB$.

7. Have a square and a rhombus their corresponding sides proportional? Are they similar?

8 A straight rod AB 3 ft. 9 in. long is held under water, A being 2 ft. 6 in. and B 9 in. below the surface. Calculate the distance below the surface of a point C on the stick which is 12 in. from A .

9. In a triangle PQR , $PQ = PR = 2$ inches, and $QR = 1$ inch. In the side PQ a point S is taken such that $QS = \frac{1}{2}$ inch. Prove that the triangle QRS is isosceles.

Paper 18.

1. What is meant by "an angle in a segment of a circle"? Why is it legitimate to speak of the angle in a segment?

2 Describe a circle radius 7 cm. and take a point 3 cm. from the centre. Draw and measure the shortest possible chord through this point. Verify by calculation.

3. ABC is a tangent to a circle at B , BD is a diameter and BE , BF are chords such that $\angle ABE = 20^\circ$, $\angle CBF = 60^\circ$, DE , DF , EF are joined. Find all the angles of the figure.

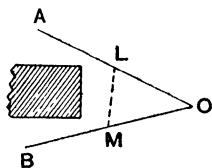
4. AB , CD are two chords of a circle whose centre is O . OE , OF , the distances of the chords from the centre, are given to be equal. If the chords intersect at X , prove (i) that $XE = XF$, (ii) that AX is equal to DX (or CX).

5. Describe a triangle with sides 4.5, 6, 7.5 cm.; find the centres of the inscribed and circumscribed circles, and measure the distance between them.

6 Two circles of radii 4 cm. and 7 cm. have their centres 9 cm. apart. Calculate the length of the common tangents to the two circles.

7 Draw a quadrilateral $ABCD$; draw a straight line parallel to CD to cut BC at P and AD at Q . Prove that $ABCD$, $ABPQ$ are equiangular. Are they similar?

8. Wishing to find the exact distance in a straight line between points A and B (the direct line between which is blocked by a house), I take a piece of string and mark out AL , LO , OM , MB all equal (ALO , BMO being straight lines). I then measure LM , which is 22 ft. 6 in. What is AB , and why?



9 A straight line AB is divided internally at C ; equilateral triangles ACD , CBE are described on the same side of AB ; DE and AB produced meet at F . Prove that $FB : BC = FC : CA$.

Paper 19.

1. Draw an angle of 73° with your protractor and bisect it by construction with ruler and compass. Explain in your own words and with the help of the figure you have drawn the meaning of the statement "the bisector of an angle is the locus of points equidistant from the arms of the angle"

2. Draw a quadrilateral ABCD and its diagonals. Now suppose that a circle could be drawn through the points A, B, C, D and that $\angle DAC = 48^\circ$, $\angle BDC = 52^\circ$ and $\angle BDA = 64^\circ$; calculate the other angles in the figure. Give your reasons carefully.

3. A small rectangular wooden platform measures 4 ft. by 6 ft.; the corners are rounded off into quarter-circles of 6 in. radius, the edge is bound all round with a brass strip. Draw a plan of the platform on squared paper.

Calculate the area of the platform in square feet to three significant figures, and the length of the brass strip, in feet and inches, to the nearest inch.

4. ABC is a triangle. Points X, Y are taken in AC, BC respectively such that the angle XYZ is equal to the angle BAC. Prove that the angle XYA = the angle ABX

(i) A circle passes through A and B, what do you know about its centre?

(ii) A circle touches BC at B, what do you know about its centre?

Draw a triangle ABC having $BC = 5$ cm, $AB = 4.5$ cm, $AC = 4$ cm, and construct a circle to pass through A and to touch BC at B

5. Draw a circle of radius 7 cm and a chord PQ distant 4 cm from the centre. Now draw a circle of radius 5 cm. to touch your first circle internally, and also to touch PQ. State your construction.

7. A rectangular picture frame is made of wood 1 in. wide, are the inside and outside of the frame similar rectangles?

8. In order to measure the distance away of a man who is 6 feet high, I hold a foot rule pointing vertically upwards at a distance of 3 feet from my eye, and observe the length on the scale which subtends the same angle at my eye as the man. If this length be x inches, give a formula for the distance d of the man in yards.

Calculate the value of x when the man is 20, 30, 40 yards away

9. ABCD is a parallelogram. From any point E in the diagonal AC, EH is drawn parallel to AD to meet DC at H, and EF parallel to DC to meet BC at F. Prove that the triangles ABD, EFH are similar

Paper 20.

1. A packing case is rolled about one edge, what is the locus of a point on the opposite edge? [Make a sketch]

2. A model boat sails in a straight line across a circular pond, towards a point 50 yd. away. The greatest distance across the pond is 70 yd. How near to the centre of the pond will the boat go? What will be the boat's least distance from the point on the pond's edge exactly opposite the starting point? [Both answers by calculation.]

3 A triangle ABC is inscribed in a circle (centre O , and radius 4").

If the angles of the triangle are $A=72^\circ$, $B=55^\circ$, $C=53^\circ$, what are the angles BOC , COA , AOB ? Hence find the sides of the triangle by drawing.

4 AB is an arc of a circle and C its middle point. Prove that the angle ABC is one-quarter of the angle which the arc AB subtends at the centre of the circle.

5 AB is a fixed line. Through A a line AC is drawn, of length 2.4 in., making an angle of 40° with AB . Draw the figure to full scale, and construct a circle to touch AB at A and to pass through C . Measure the radius of this circle.

6 The back and front wheels of a carriage have radii 2 ft 6 in. and 1 ft 6 in., and the distance between their centres is 5 ft 6 in. Find, by drawing, the distance between the points at which they touch the ground.

Verify your result by calculation.

7. Construct a straight line equal to $2\frac{1}{3}$ of a given straight line.

8 Show how to draw through a given point within a given angle a straight line to be terminated by the arms of the angle, and divided in a given ratio (say $\frac{2}{3}$) at the given point.

9 In any triangle ABC , P is a point in BC such that BP is one-third of BC . Join AP and take on it a point Q such that AQ is one-third of AP . Then prove that the area of the triangle ABQ is one-ninth of that of ABC .

What is the ratio of the areas of the triangles ABQ and ACQ ? Give your reason.

Paper 21.

1. A gun is fired at noon and the sound (which travels with known velocity) heard by two listening stations at observed times. How can the position of the gun be laid down on a chart?

2. (i) In a circle of radius 5 cm, a chord is drawn at a distance 3 cm. from the centre e ; find the length of the chord.

(ii) In a sphere of radius 5 cm, a plane is drawn at a distance 3 cm from the centre, find the radius of the circle in which this plane cuts the sphere.

3 A and B are two forts on a straight coast 5 miles apart. The effective range of A 's guns is $3\frac{1}{2}$ miles, and of B 's, 3 miles. Draw the circles bounding the area covered by both forts, and let C be one of the points of intersection of these circles. An enemy's ship comes to C so as to be able to bombard a town lying between A and B without being within the range of the guns from either fort. By measurement, find how far C is from the coast-line. Measure the angles CAB and CBA and hence calculate approximately the number of square miles covered by the zone of effective fire from both forts.

4 A quadrilateral is inscribed in a circle, and one of its sides is produced. Prove that the exterior angle so formed is equal to one of the interior angles.

Two circles intersect in P , Q . Chords AP , BQ of one are produced to meet the other in C , D . Prove that AB is parallel to CD .

[See next page]

6. O and P are points 1000 yd. apart, P being due east of O. At Q the line OP subtends an angle of 63° . Q is 450 yd. from the line OP. Draw a figure to scale, and find the distance and bearing of Q from O.

7. Draw a circle of 4 cm radius to touch two circles of radii 3 cm. and 2 cm respectively, whose centres are 6 cm. apart. The 3-cm. circle is to lie entirely inside the 4-cm. circle, and the 2-cm. circle is to lie entirely outside it.

8. Give the different sets of data which suffice to fix the *shape* of a triangle.

9. Construct a $\triangle ABC$ having $a=8$ cm., $b=5$ cm., $c=6$ cm. Find points P and Q in AB and AC respectively such that PQ is parallel to BC and of length 3.5 cm.

10. At two points A, B of a straight line perpendiculars AC, BD are erected and AD, BC meet in a point E; from E a perpendicular EF is drawn to AB. Prove that

$$\frac{EF}{AC} + \frac{EF}{BD} = 1$$

Paper 22.

1. State, without proof, (i) what is the locus of a point which moves so that it is always 1 foot away from a given point, (ii) what is the locus of a point which moves so that it is always 1 foot away from a given straight line, (iii) what is the locus of a point which moves so that it is always 1 foot away from a given circle.

2. In a circle of 2 inch radius place a chord of length 1.6 inches parallel to a given line. Explain your construction, and prove that it is correct.

3. PR is the diameter of a circle and PT a chord making 25° with PR, find the angles PTR, PRT, giving reasons.

If $TR=2.4$ cm., what is the length of the perpendicular from the centre of the circle to PT and why?

What is the angle which the tangent at T makes with TR and why?

4. O is the centre of a circle PQR, RT is the tangent to the circle at R; TPQ is a straight line cutting the circle at P and Q; OS is drawn perpendicular to PQ, OR, OT, SR are joined. Find a cyclic quadrilateral in the figure; why is it cyclic? Hence prove that $\angle OSR$ and $\angle QTR$ are equal or else together equal two right angles. Draw figures to illustrate both cases.

5. Draw two straight lines AB, AC enclosing an angle of 48° . Take a point D in AB such that $AD=2.6$ in. Construct a circle DEF to touch AB in D and also to touch AC.

Construct another circle to touch AB, AC and also to touch the circle DEF.

6. A lawn-mower has two cylindrical rollers in contact with the ground. The smaller has a diameter of 3 in. and the larger of 13 in. The lines in contact with the ground are 1 ft. apart. Calculate the distance between the axes of the rollers.

7. ABC , PQR are two triangles. Give two different sets of conditions either of which would be enough to prove the triangles similar

8 Show how to construct a triangle similar to and of double the area of a given triangle

†9 Two circles touch at A , one of the circles being inside the other. From B , a point on the tangent at A , a tangent BC is drawn to touch the outer circle at C . AC cuts the inner circle at D , BD produced cuts the inner circle at E , and AE produced cuts the outer circle at F . Prove that (i) ED is parallel to FC and (ii) $BC^2 = BD \cdot BE$

Paper 23.

1. Explain what you mean by "the constant-angle locus" Give two methods of constructing it, taking a case in which the constant angle is obtuse

2. $ABCD$ is a circle, AC , BD meet in X . Given that $\angle ABD = 33^\circ$, $\angle ADB = 27^\circ$, $\angle BAC = 45^\circ$, calculate the angles BXC , ACD , ABC , showing your reasoning clearly but shortly

3 P and Q are two points on the circumference of a circle, and the tangents to the circle at P and Q intersect at an angle of 56° . What fraction of the whole circumference is the minor arc PQ ? and what is the ratio of the major arc PQ to the minor arc PQ ?

†4 Two unequal circles with centres O and C touch externally at A

(i) What do you know about the points O , A , C (give no proof)?

(ii) Through A the line LAM is drawn cutting the circle whose centre is O at L and the other circle at M . Prove that OL , CM are parallel.

5 Draw a straight line $AB = 3$ in. Find the centre of a circle ABC which passes through A and B so that $\angle ACB = 50^\circ$. Measure the radius of the circle.

6 A and B are the doors of two houses and they are 52 ft. apart. A straight road is to be constructed so that A is 30 ft. from it and B is 50 ft. Using a scale of $\frac{1}{20}$ th inch to the foot, draw a figure showing how the road must be made

7. If a straight line is drawn parallel to the parallel sides of a trapezium, how does it divide the other two sides?

8 OA is 3 inches, AB at right angles to OA is 1.6 inches. Find, preferably by calculation, the length of OB .

Draw the circle centre O radius OB , and construct the tangent at B to this circle. If the tangent cuts OA produced at T , find, either by measurement or calculation, the length of AT .

†9. $EBDF$ is a rectangle. EB is produced to C so that $BC = BD$. A semi-circle is drawn on EC (on the side opposite to D) and DB is produced to meet the circumference at A . Prove that the square on AB is equal to the rectangle $EBDF$

Paper 24.

1. What is the locus of centres of (i) circles which touch a fixed line PQ at a fixed point P ; (ii) circles of radius $3''$ which touch a fixed line PQ ? Also of the following points —(iii) the points of contact of tangents drawn from a fixed point to a fixed sphere; (iv) points on the earth which are 3000 nautical miles N. of the equator?

2. A penny falls into a cup whose shape is an exact hemisphere of radius 5 cm. If the penny lies symmetrically at the bottom and its diameter is 3 cm., how far below the penny is the lowest point of the cup?

3. What would be the radius of a circle in which an arc $11''$ in length subtended an angle of $31\frac{1}{2}^\circ$ at the circumference?

4. On the same base and the same side of it are drawn three triangles of vertical angle 30° . Show that a circle can be drawn to circumscribe all three triangles, and that its radius is equal to the base.

5. Show that it is usually possible to draw two circles each of which touches two given sides of a given triangle (produced if necessary) and has its centre on the third side or that side produced, but that under certain circumstances only one such circle can be drawn.

6. P is a point on the circumference of a circle of centre O and radius $1\frac{1}{2}$ in. Q is taken so that $\angle POQ = 40^\circ$ and $OQ = 3$ in. Construct a circle to touch the given circle at P and to pass through Q . Measure the radius of this circle.

7. Explain the term "mean proportional."

8. A circle is circumscribed round a triangle ABC right-angled at A . Its radius is found to be $6''$. AN is drawn perpendicular to the base BC . O is the mid-point of BC and $ON = 2.4''$. Calculate the lengths of AN , AB , AC .

9. AB is a chord of a circle and AD the tangent at A . A chord QP is drawn parallel to AB , meeting the tangent AD at D . Prove that the triangles DPA and AQB are similar.

Paper 25.

1. Assuming that the line joining the centres of two intersecting circles bisects the common chord at right angles, examine what this theorem becomes in the limiting case when the circles touch.

2. A dirty football was found to leave a circle of mud, $11''$ in circumference, when bouncing, if the radius of the ball was $6\frac{1}{4}''$, find the depth to which it was squashed in by the impact.

3. $ABCD$ is a rectangle in which $AB = 4$ in, $BC = 6$ in. A circle with A as centre passes through the middle point of BC and cuts AD at F . Calculate the length of CF .

4. ABC is an acute-angled triangle, BE , CF are perpendiculars drawn to CA , AB respectively. Prove that the points B , C , E , F lie on a circle. Name two acute angles in the figure which are equal to one another. Where is the centre of the circle $BFEC$?

5 Draw a straight line AB of length 5 cm. Find a point P at which AP subtends an angle of 54° and such that AP is 4 cm. Measure the distance PB .

6 Two halfpennies (diameter 1 in.) lie on a sheet of paper with their circumferences touching. A semicircle is drawn to circumscribe the two halfpennies, so that its diameter and its circumference touches each coin. Make an accurate drawing of this arrangement, and determine the diameter of the semicircle.

7 What measurements would you have to make in order to compare as shortly as possible the areas of two similar figures?

8 From a point P , outside a circle of radius a , are drawn the tangent PQ (of length x) and the straight line PAB through the centre cutting the circle in points A and B , A being nearer to P and PA being of length y . Write down the relation connecting the lengths of the lines PQ , PA , PB , and express it in terms of x , y , a .

If the circle is taken as representing a section of the earth through its centre, PQ will be the range of vision of a person situated at a height y above the surface. Take $a = 4000$ miles, use the relation in the approximate form $2ay = x^2$, and find, in miles and in feet, to what height it is necessary to ascend in order to have a range of vision of 50 miles.

9 The hypotenuse AB of a right-angled triangle ABC has been measured and found to be p in. The angle A of the triangle has also been measured and found to be n° . Have sufficient measurements been taken to determine the triangle in size and shape? Give reasons for your answer.

A point D is taken in the hypotenuse AB such that the angle DCA is equal to the angle A . Prove that the two triangles DBC and DAC are both isosceles. Is it possible for these two isosceles triangles to have the same shape? If so, what must be the values of n or p or of both of these measurements? If the measurements are such that the two isosceles triangles have the same shape, do they also have the same size?

Paper 26.

1 The phrase "alternate segment" occurs in the enunciation of a theorem. Quote the theorem and explain the phrase with the aid of freehand figures.

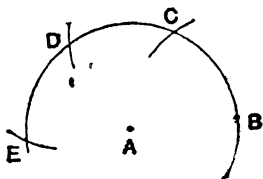
2 A, B, C, D are four points on a circle of which O is the centre. AC is a diameter and $\angle BAC = 35^\circ$, $\angle DBC = 40^\circ$. Find $\angle ODC$, $\angle ODB$, giving your reasons briefly.

3 A circle touches a straight line AA' at its middle point O . Chords OB, OC are drawn making angles of 36° and 72° respectively with the tangent OA , and chords OB', OC' making 36° and 72° with OA' .

State which of the arcs $OB, BC, OB', B'C', CC'$ are equal by reason of the symmetry of the curve about its diameter through O , also state which of them are equal for other reasons, giving proofs of their equality.

[See next page

Ex. 4 With centre A and radius AB a circle is drawn. With centre B and equal radius an arc is drawn intersecting the first circle at C. Similarly from centre C the point D is determined, and from centre D the point E. Prove that BAE is a straight line. (This shows how a point may be obtained on a straight line "produced" without the use of a straight edge.)



5. On a fixed line AB, 8 cm. long as base, construct a triangle ABC, whose area shall be 24 sq. cm., and vertical angle 63° . Measure (i) the smallest angle, (ii) the radius of the circumcircle of the triangle.

6. AB is a diameter of a circle of radius 5 cm. Draw a chord CD of the circle perpendicular to AB and 6 cm. in length. Also, through O, the point of intersection of AB and CD, draw a chord of the circle 8 cm. long. State the steps of your construction.

7. A model of a house is made on a scale of 1 inch to 1 foot. What are the ratios of (i) the floor spaces, (ii) the volumes, of the house and model?

8. Draw a circle of radius 5 cm. and take a point O at a distance of 10 cm. from its centre. From O draw a line cutting the circle in P and Q such that P is the middle point of OQ.

9. OAB, OC are the two arms of an angle (O, A, B being in one straight line). If OA = 2 inches, OB = 4.5 inches, OC = 3 inches, show that the circle circumscribing ABC touches OC.

Paper 27.

1. Explain what you mean by "conyclic points." Give two sets of conditions that would make four points conyclic.

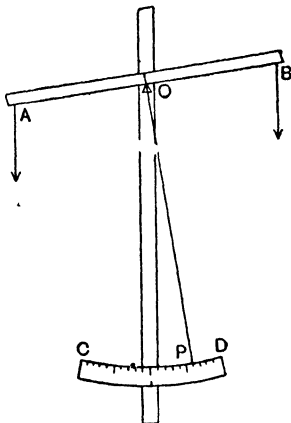
2. Two concentric circles have radii 5 cm. and 3 cm. Find by calculation the length of the tangent drawn from any point on the outer circle to the inner circle.

A straight line ABCD cuts the outer circle at A and D, and the inner circle at B and C. If BC = 3.6 cm., calculate the length of AD.

3. In the accompanying diagram of a balance the beam is supported on a knife-edge at O, the pointer OP is fixed to the beam at right angles, the tip of the pointer moves along the scale CD, the scale pans hang from A and B.

$$OA = OB = 4'', \quad OP = 7'', \quad \text{arc } CD = 3''$$

Calculate the angle through which the pointer swings as P moves from C to D; also find the vertical height through which each scale pan moves.



†4 A, B are points on the circumference of a circle whose centre is C and a circle is drawn through A, B, C. On the minor arc CB a point P is taken, and AP produced to cut the other circle in Q. Prove that

$$(i) \angle APB = 2 \angle AQB, \quad (ii) PQ = PB$$

5 Draw a straight line OBC, making $OB = 2.5 \text{ cm}$, $OC = 6.4 \text{ cm}$. Through O draw a line OA making the angle $AOB = 42^\circ$. Then draw a circle passing through B and C and touching OA. (Describe the steps of your construction.)

6 In playing with pennies a boy observed that he could arrange six coins round a centre one, each touching the centre one and two others. Show the possibility of this by drawing a careful figure in which each circle has a radius of 2 cm. State clearly how you determine the centres of the circles and what help you get in this construction from considerations of symmetry, then justify by general reasoning the method you have adopted.

7 Sketch plane figures for cutting out to form

- (i) a rectangular box, dimensions 2'', 3'', 4'',
- (ii) a prism 8 cm long whose section is a triangle of sides 4 cm., 5 cm., 6 cm.;
- (iii) a pyramid, base 2'' square, slant edges 3'',
- (iv) a cone, base radius 4 cm., height 10 cm.

8 OT is a tangent to a circle at T, and PQ is a diameter, OPQ being a straight line, $OP = 2b$ and the radius of the circle is $a - b$. Prove that

$$OT^2 = (a + b)^2 - (a - b)^2$$

Simplify this expression, and deduce that $OT^2 = OP \cdot OQ$.

†9. In the right-angled triangle ABC, BC is the hypotenuse and the side AB is double the side AC. A square is described on BC and is divided into two rectangles by a line through A perpendicular to BC. Prove that one rectangle is four times the other.

INDEX—LIST OF DEFINITIONS

Acute angle, obtuse angle, reflex angle. An angle less than a right angle is said to be acute, an angle greater than a right angle and less than two right angles is said to be obtuse, an angle greater than two and less than four right angles is said to be reflex pp 9, 157

Acute-angled triangle. A triangle which has all its angles acute is called an acute-angled triangle p 33

Adjacent angles. When three straight lines are drawn from a point, if one of them is regarded as lying between the other two, the angles which this line makes with the other two are called adjacent angles

Alternate angles. p 24

Alternate segment. p 161

Altitude. See **triangle, parallelogram.**

Altitude of sun. p 16

Angle. When two straight lines are drawn from a point, they are said to form, or contain, an angle The point is called the **vertex** of the angle, and the straight lines are called the **arms** of the angle p 6

Angle in a segment. An angle in a segment of a circle is the angle subtended by the chord of the segment at a point on the arc p 158

Angle in a semicircle. p 159

Angle of depression. p 38

Angle of elevation. p 16

Angle properties of circle. p 156.

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Base. See **triangle**, **parallelogram**.

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Bearing. p. 15.

Bisect. Divide into two equal parts

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Chain. p. 109

Chart exercises. pp. 15, 41, and chart at end of book

Chord of a circle p. 141

Circle. A circle is a line, lying in a plane, such that all points in the line are equidistant from a certain fixed point, called the **centre** of the circle. The fixed distance is called the **radius** of the circle. p. 137

Circumcentre. The centre of a circle circumscribed about a triangle is called the circumcentre of the triangle p. 143

Circumference of a circle pp. 137, 139

Circumscribed circle. p. 143

Circumscribed polygon. pp. 152, 154

Coincide. p. 59

Common tangents, exterior and interior. p. 174

Concyclic. Points which lie on the same circle are said to be concyclic.
p. 167

Cone. pp. 122, 187, 204

Congruent. Figures which are equal in all respects are said to be congruent
p. 59

Congruent triangles. p. 59

Constant angle locus. p. 164

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Constructions—angle equal to given angle. p. 77

bisection of lines and angles. pp. 77, 78

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Converse. p. 63

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Corresponding angles. p 23.

Cube, cuboid. pp. 1, 121, 206

Cyclic quadrilateral. A quadrilateral through whose vertices a circle can be described is called cyclic pp 160, 167.

Cylinder. p 204.

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Depression p 38

Diagonal. See quadrilateral.

Diameter. p. 137

Direction. p 5

Division of straight line. pp. 189, 191

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Elevation. p 16.

Equiangular. This word is used in two senses —(1) a figure is said to be equiangular when all its angles are equal (p. 33), (2) two figures are said to be equiangular when the angles of the one figure are equal respectively to the angles of the other figure taken in the same order

Equilateral triangle. A triangle which has all its sides equal is called an equilateral triangle. pp. 33, 47.

Equivalent. Figures which are equal in area are said to be equivalent. p. 106.

Escribed circles of a triangle. p. 153.

Experimental work. p. 17.

Exterior angle of cyclic quadrilateral. p. 160.

polygon. p. 37.

triangle. p. 34.

Fourth proportional. If x is such a magnitude that $a : b = c : x$, then x is called the fourth proportional to the three magnitudes a, b, c . p. 194.

Framework. p. 50.

Height. See **triangle, parallelogram.**

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Hypotenuse. See **right-angled triangle.**

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Isosceles trapezium. See **trapezium.**

Isosceles triangle. A triangle which has two of its sides equal is called an isosceles triangle. pp. 33, 47, 62, 86

Kite. p. 87

Length of chord. p. 146

tangent. p. 151

Limit. pp. 149, 161

Locus, Intersection of. p. 98

Locus. If a point moves so as to satisfy certain conditions, the path traced out by the point is called its locus. p. 95

Major arc, minor arc. p. 137

Major segment, minor segment. p. 158

Mean proportional. If x is such a magnitude that $a : x = x : b$, then x is called the mean proportional between a and b . p. 202

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Median. See **triangle.**

Mid-point theorems. p. 72

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Obtuse angle. See **acute angle**.

Obtuse-angled triangle. A triangle which has one of its angles an obtuse angle is called an obtuse-angled triangle. p. 33.

Octagon. See **pentagon**.

Offsets. p. 109

Origin. p. 101

Parallel straight lines are straight lines in the same plane, which do not meet however far they are produced in either direction p. 22

Parallelogram. A quadrilateral with its opposite sides parallel is called a parallelogram p. 67

Any side of a parallelogram may be taken as the **base**. The perpendicular distance between the base and the opposite (parallel) side is called the **height**, or **altitude**. p. 106.

Pentagon, hexagon, heptagon, octagon, etc.—a polygon of 5, 6, 7, 8, sides, 5-gon, 6-gon, 7-gon, 8-gon pp. 36, 37, 51, 144, 154

Perimeter. The perimeter of a figure is the sum of its sides.

Perpendicular. See **right angle**.

Polygon. A plane figure bounded by straight lines is called a polygon, or, a **rectilinear figure** pp. 36, 37, 51, 144, 154

Projection. The projection of a point on a line (or plane) is the foot of the perpendicular from the point to the line (or plane)

The projection of a line is the locus of the projection of the points of that line

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Proportion. p. 194

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Proportional division. p. 192

Protractor. pp. 10, 138.

Pyramid. pp. 188, 204–207

Pythagoras' theorem. p. 116

Quadrant of a circle is the part of the circle between two perpendicular radii

Quadrilateral. A plane figure bounded by four straight lines is called a quadrilateral pp. 36, 50, 51, 110, 160, 167.

The straight lines which join opposite corners of a quadrilateral are called its **diagonals**.

Radius. See **circle**. p. 137

Ratio. pp. 177, 190.

Note on equal. p. 192.

Rectangle. p. 70

Rectilinear figure. A figure bounded by straight lines. p. 82

Re-entrant. p. 37

Reflex angle. p. 157 See **acute angle**.

Regular polygon. A polygon which has all its sides equal and all its angles equal is called a regular polygon pp. 36, 144, 154.

Revision papers A. p. 55.

B. p. 130

C. p. 218

Rhombus. A parallelogram which has two adjacent sides equal is called a rhombus pp. 70, 88

Right angle, perpendicular. When one straight line stands on another straight line and makes the adjacent angles equal, each of the angles is called a right angle, and the two straight lines are said to be at right angles, or perpendicular to one another

Right-angled triangle. A triangle which has one of its angles a right angle is called a right-angled triangle

The side opposite the right angle is called the **hypotenuse**. pp. 33, 48

Rotation. p. 23

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Scale, Choice of. p. 14.

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Scalene triangle. A triangle which has no two of its sides equal is called a scalene triangle p. 33

Sector of a circle p. 141

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Semicircle. p. 137.

Similar figures. p. 178

Similar triangles. p. 181

Solid figures. pp. 121, 122, 147, 187, 188, 201, 204-208

Sphere. pp. 17, 204

Square. p. 70.

Subtend. If P is a point outside a line AB (curved or straight), $\angle APB$ is the angle which AB subtends at P p. 38

Supplementary angles. When the sum of two angles is equal to two right angles, each is called the supplement of the other, or is said to be supplementary to the other p. 19

Symmetry. p. 84

Tangent. A tangent to a circle is a straight line which, however far it may be produced, has one point, and one only, in common with the circle. pp. 141, 149

The tangent is said to **touch** the circle, the common point is called the **point of contact**.

Tangents, Common. p. 174.

Tetrahedron. A solid with four triangular faces pp. 204-207.

Theorems proved:—

8. **Angle sum of a triangle.** p. 32.

12, 13 **Isosceles triangle.** p. 62.

20, 21. **Parallelograms.** p. 67.

22, 23. **Mid-point theorems.** p. 72.

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36. **Perpendicular on a chord of a circle.** p. 142

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51 **Alternate segment theorem.** p. 162

59 **Rectangle properties of a circle.** p. 197

60 **Mean proportional.** p. 202

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Translation. p. 23

Trapezium. A quadrilateral which has only one pair of sides parallel is called a trapezium. A trapezium in which the sides that are not parallel are equal is called an **isosceles** trapezium p. 108

Triangle. A plane figure bounded by three straight lines is called a triangle p. 31

Any side of a triangle may be taken as **base**. The straight line drawn perpendicular to the base from the opposite vertex is called the **height**, or **altitude**.

The straight line joining a vertex of a triangle to the mid-point of the opposite side is called a **median**.

Vertically opposite angles. The opposite angles made by two intersecting straight lines are called vertically opposite angles (*vertically* opposite because they have the same vertex). p. 20.

Vertices. The corners of a triangle or polygon are called its vertices

Volume of similar figures. p. 201.

