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THE MATHEMATICAL THEORY OF INVESTMENT

BY

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REVISED EDITION



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PREFACE TO THE REVISED EDITION

The first edition of this book, published in 1913, was an experiment. After ten years the author feels that he has reason to be gratified with the result of the experiment and to believe that the material then presented for the first time in organized form has become and will remain an important part of our mathematical curricula.

While the present revision follows in the main the plan of the first edition, two important changes have been made.

The first is the addition of a short chapter on interpolation (Chapter III) and the more extensive use of the method of interpolation in the solution of the important problem of finding interest rates. The increasing emphasis placed upon statistical problems would seem to justify the reintroduction of this long-neglected and almost forgotten topic of the old algebras.

The second important change is the addition of a considerable amount of material on depreciation (Chapter XII). Perhaps the most important contribution to this subject made since 1913 is the chapter on depreciation contained in "The Final Report of the Special Committee to formulate Principles and Methods for the Valuation of Railroad Property and Other Public Utilities," presented to the American Society of Civil Engineers and published in the Society's *Transactions* in 1917. I have utilized freely the material contained in this excellent and comprehensive Report.

On one point I have departed from the terminology of the Report and of subsequent writers on the subject. The name "interest-on-investment method" seems to me to describe more fully the method which they have called the "compound-interest method."

I have tried also to bring out clearly the substantial identity of the interest-on-investment method and the sinking-fund method. This identity follows at once from a definition of "book

value" so framed as to bring out clearly the necessity for allowing depreciation charges; namely, that invested capital should not be impaired.

The author is indebted to many teachers of mathematics throughout the country for notices of typographic errors and for suggested improvements both in form and in content for the revised edition. Special thanks are due to Dr. Florence E. Allen, Mr. E. B. Miller, Professors H. W. March and Warren Weaver of the University of Wisconsin, and Mr. M. H. Ingraham, Fellow in mathematics in The University of Chicago, who have read carefully several chapters of the manuscript revision. I have also had material help from Professor G. R. Clements of the United States Naval Academy, Professor H. T. Davis of the University of Indiana, Professor O. H. Rechard of the University of Wyoming, and Professor T. M. Simpson of the University of Florida, who, as former instructors in the University of Wisconsin, have given me much valuable criticism based on actual classroom experience with the book.

ERNEST B. SKINNER

THE UNIVERSITY OF WISCONSIN

PREFACE TO THE FIRST EDITION

This book has been written for the use of students of business and public affairs. The fact that, with a few notable exceptions, colleges and universities have hitherto made no provision for courses in mathematics adapted to meet the needs of students trained for commercial careers and for the public service, is due in no small measure to the lack of a suitable textbook. In recent years the closer study of business methods and the extension of governmental control over many forms of industrial and financial activity have greatly emphasized the value, to students of finance, of a knowledge of what the Germans call "political arithmetic."

In an attempt to meet the need for a book covering this field, some of the more important topics relating to the theory of interest and its application to the larger affairs of modern everyday life have been brought together. The following pages contain, in somewhat elaborated form, the substance of a course of lectures given annually for the last five years to the students in the course in commerce in the University of Wisconsin. While I have tried to make the book practical throughout, I have sought to make it a book of first principles rather than a guide to detailed practice. I trust that the reader will find the treatment adapted to present-day conditions and needs. I shall be gratified if I have in any way contributed to the more efficient training, in principles of sound finance, of the splendid body of young men who are going out from our courses in commerce to become leaders in the business world.

In selecting topics for consideration, an effort has been made to avoid those that are still the subject of controversy. If this rule has been departed from in devoting some space to depreciation, it is on account of the great importance of the subject. The public-utility commission seeking to make an equitable adjustment of rates must make some sort of quantitative determination of depreciation, even though neither the exact meaning

of the term nor the method of treating the subject has been agreed upon. It is believed that the discussion in §§ 56-58 follows logically from the definitions therein proposed. It might be added that it is in substantial agreement with the practice of the Wisconsin Railroad Commission.

Much of the inspiration for the book has come from M. Cantor's admirable "Politische Arithmetik," while most of the material may be found in some form in Todhunter and King's "Institute of Actuaries' Text-Book." I have consulted freely Fuzet and Reclus, "Précis de mathématiques commerciales et financières"; Martini, "Aritmetica commerciale e politica"; Schlimbach, "Politische Arithmetik"; Wolff, "Inheritance Tax Computations"; Broggi, "Matematica attuariale"; Löwy, "Versicherungsmathematik"; Dawson, "Practical Lessons in Actuarial Science"; and Willey, "Principles and Practice of Life Insurance," revised by Moir.

Tables III-VII are based upon Spitzer's "Tabellen für Zinseszinsen- und Rentenrechnung," and have been carefully compared with Vintéjoux's "Nouvelles tables d'intérêts composés et annuités." Tables XI and XII have been taken, with the kind permission of The Spectator Company, from the books of Dawson and Willey.

In preparing the manuscript, I have had the benefit of the valued criticism of my colleagues, Professors E. B. Van Vleck and L. W. Dowling, and of Mr. L. A. Anderson, Actuary for the Wisconsin Insurance Department. Professor Arnold Dresden, Dr. Florence Allen, Dr. G. R. Clements, and Mr. T. M. Simpson, who have tried out a large part of the material in the classroom, have made many helpful suggestions in the selection of material and examples, and have rendered valuable assistance in verifying worked-out examples and in correcting proofs. I am also under obligations to Professor D. F. Campbell, of Armour Institute, for placing at my disposal notes and other material relating to the elements of the theory of life insurance.

ERNEST B. SKINNER

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THE MATHEMATICAL THEORY OF INVESTMENT

PART I. ALGEBRAIC INTRODUCTION

CHAPTER I

PROGRESSIONS

1. Definitions. An *arithmetic progression* is a succession of terms such that any term may be obtained from the preceding term by the addition of a constant number called the *common difference*. If the common difference is positive, the progression is said to be *increasing*; if it is negative, the progression is said to be *decreasing*.

ILLUSTRATIVE EXAMPLES. The progressions

2, 4, 6, 8

and

1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$

are increasing arithmetic progressions, the first with common difference 2 and the second with common difference $\frac{1}{2}$.

The progression

8, 6, 4, 2, 0, -2

is a decreasing arithmetic progression with common difference -2.

A *geometric progression* is a succession of terms such that any term may be obtained by multiplying the preceding term by a constant number. The constant multiplier by means of which any term is derived from the preceding term is called the *ratio*. If the ratio is numerically greater than 1, the progression is called an *increasing* geometric progression; if it is numerically less than 1, the progression is called a *decreasing* geometric progression.

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ILLUSTRATIVE EXAMPLES. The progression

$$1, 2, 4, 8, 16$$

is an increasing geometric progression with ratio 2, while the progression

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$$

is a decreasing geometric progression with ratio $\frac{1}{2}$.

When a progression is given, the student should be able to determine quickly whether it is arithmetic or geometric or whether it belongs to some other class. The test is supplied by the respective definitions.

A progression cannot be at the same time arithmetic and geometric, except in the trivial case where the common difference is zero or the ratio is 1. For example, the succession

$$2, 2, 2, 2$$

may be considered as an arithmetic progression with common difference zero, or as a geometric progression with ratio 1.

The important things to be considered in connection with progressions are the first term, the last term, the number of terms, the common difference if the progression be arithmetic, or the ratio if it be geometric, and the sum of all the terms. It will soon appear that any one of these five numbers may be expressed in terms of three others. In other words, if three of the five numbers are given, it is possible to find out all about the progression, provided it is known whether it is arithmetic or geometric.

Progressions play a very important part in the theory of investment.

2. Formulas for arithmetic progressions. The letters a , l , n , d , and s are used to denote the first term, the last term, the number of terms, the common difference, and the sum of all the terms of an arithmetic progression. The terms are then

$$a, a + d, a + 2d, a + 3d, \dots, a + (n - 1)d,$$

and the last, or n th, term is clearly given by the formula

$$l = a + (n - 1)d. \quad (A)$$

The sum of all the terms is

$$s = a + a + d + a + 2d + \dots + a + (n-1)d,$$

or, if the terms be written in reverse order,

$$s = l + l - d + l - 2d + \dots + l - (n-1)d.$$

If these two identities be added, the result will be

$$2s = (a + l) + (a + l) + \dots \text{to } n \text{ terms};$$

whence

$$s = \frac{n}{2}(a + l). \quad (B)$$

3. Formulas for geometric progressions. Similarly, the first term, the last term, the number of terms, the ratio, and the sum of all the terms of a geometric progression are denoted by the letters

$$a, \quad l, \quad n, \quad r, \quad \text{and} \quad s.$$

The terms of the series are then

$$a, \quad ar, \quad ar^2, \quad \dots, \quad ar^{n-1},$$

and the formula for the last, or n th, term is

$$l = ar^{n-1}. \quad (C)$$

The sum of n terms written out at length is

$$s = a + ar + ar^2 + \dots + ar^{n-1}.$$

To obtain a formula for computing the sum, this equation is first multiplied by r . The resulting equation is

$$rs = ar + ar^2 + \dots + ar^{n-1} + ar^n.$$

If now the equation for s be subtracted from the equation giving the value of rs , the result is

$$rs - s = ar^n - a,$$

and the value of s obtained from this equation is

$$s = \frac{ar^n - a}{r - 1} = \frac{a - ar^n}{1 - r}. \quad (D)$$

The expression ar^n is equivalent to $ar^{n-1} \cdot r$, and, by equation (C),

$$ar^{n-1} \cdot r = rl.$$

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If ar^n be replaced by its value rl , equation (D) takes the convenient form

$$s = \frac{rl - a}{r - 1}. \quad (D')$$

4. Solution of problems. By means of formulas (A) and (B), considered as a pair of simultaneous equations, any possible problem in arithmetic progressions may be solved when three of the numbers

$$a, \quad l, \quad n, \quad d, \quad \text{and} \quad s$$

are given. The resulting equations, when numbers are substituted for the three known quantities, will constitute either a linear system or a linear quadratic system, and a solution can be found by elementary algebra in every case.

In a similar fashion, problems in geometric progressions may be solved by means of equations (C) and (D) or (D'). In certain cases, however, the simultaneous system cannot be solved by elementary means. This is always true when n is unknown, and usually true when r is unknown. Methods for obtaining a solution when n is unknown will be given in § 28.

In the solution of the examples the student should pay particular attention to the setting up of the pair of simultaneous equations from which the solution is obtained. It is advisable for the beginner to write down both equations, even though only one is needed.

EXAMPLES

1. The sum of a number of terms of a progression whose first three terms are 32, 24, and 16 is 80. How many terms has the progression?

Solution. The progression is arithmetic with the common difference $d = -8$. Furthermore, $a = 32$, $s = 80$. If these values for a , d , and s are inserted in equations (A) and (B), the equations become

$$\begin{aligned} l &= 32 - (n - 1)8, \\ 80 &= \frac{n}{2}(32 + l). \end{aligned}$$

Eliminating l , which is not called for, we find the quadratic equation

$$n^2 - 9n + 20 = 0,$$

whose solution is given by $n = 4$ and $n = 5$. It is easy to see that the sum of four terms is the same as the sum of five terms, since the fifth term is 0.

2. The sum of eight terms of the progression of which three terms are 8, 16, 32 is 510. What is the first term?

Solution. In this problem the series is geometric and $n = 8$, $r = 2$, $s = 510$. These values, substituted in equations (C) and (D), give

$$l = a 2^7,$$

$$510 = \frac{a 2^8 - a}{2 - 1}.$$

In this case the value of the required number a may be found from the second equation alone, but if the equations (C) and (D) had been used, it would have been necessary to determine a by means of the two equations

$$l = a 2^7,$$

$$510 = \frac{2l - a}{2 - 1}.$$

The result is the same whichever way the solution is found : viz. $a = 2$.

3. The first and last terms of an arithmetic progression are 4 and 100 respectively, and the difference is 4. Find the number of terms and the sum of all the terms.

4. The first three terms of a progression are 3, 6, and 9. Find the number of terms when the sum of all the terms is 234.

5. Suppose that in Example 4 the sum were 225 and the remaining data the same. What would be the value of n ? What is the meaning of the result?

6. The first three terms of a progression are 3, 6, 12. Find the sum of ten terms.

7. An elastic ball is dropped from a height of 20 feet, and each time it strikes the ground it rebounds to one half the distance through which it fell. How far will it have traveled when it strikes the ground for the tenth time?

8. Find the sum of the multiples of 3 that lie between 200 and 400.

9. A body falling from rest falls approximately 16.1 feet the first second, 48.3 feet the second, 80.5 feet the third, and so on. How far will it fall in 10 seconds?

10. The value of the timber in a certain forest increases at the rate of 4% annually. If it was worth \$10,000 at the beginning of a five-year period, how much will it be worth at the end of the period?

11. Find the fifth term of the series

$$\frac{1}{\sqrt{3} + \sqrt{2}}, \quad 5 - 2\sqrt{6}, \quad 9\sqrt{3} - 11\sqrt{2}, \quad \dots$$

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12. Prove that three numbers in arithmetic progression cannot be in geometric progression unless the common difference is zero.

13. Prove that three numbers in geometric progression cannot be in arithmetic progression at the same time unless the ratio is $+1$.

5. **Geometric progressions in which the number of terms increases indefinitely.** In every problem in progressions so far considered the number of terms has been finite. There are, however, many important problems whose solution requires the concept of a decreasing geometric progression in which the number of terms is indefinitely increased. An example will help to make the concept clear. Suppose a stick 2 feet long is cut into two equal parts, and one of these is cut into two equal parts, and so on, one of the two parts into which a given piece is cut being bisected each time. In this way a series of shorter sticks is obtained whose lengths are $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, and so on, each fraction having for its denominator a power of 2. These terms form a geometric progression whose ratio is $\frac{1}{2}$ and whose n th term is given by

$$l = 1 \cdot \left(\frac{1}{2}\right)^{n-1} = \frac{1}{2^{n-1}}.$$

The sum of n terms will be, by formula (D'),

$$s = \frac{\frac{1}{2^n} - 1}{\frac{1}{2} - 1} = 2 - \frac{1}{2^{n-1}}.$$

Consider now what happens if the process of bisection is carried on indefinitely. Two things are clear from the nature of the problem: First, the length of the last piece becomes indefinitely small, or, to use a customary phrase, the length of the last piece *tends toward zero as a limit*. In the second place, however far the process is carried on, the sum of all the pieces cut off can never exceed 2, though it can be made to approach as near to 2 as we please. This is seen from the expression just found for s , viz.

$$s = 2 - \frac{1}{2^{n-1}}.$$

This relation may be written

$$2 - s = \frac{1}{2^{n-1}}$$

from which form it can be seen that the right member becomes small at will when n is increased, and consequently the difference $2 - s$ may be made as small as we please. In such a case it is customary to say that the limit of $2 - s$ is equal to 0, where the term *limit* is used to indicate that the variable s can be made to differ from the constant 2 by a number smaller than any number that may be assigned in advance. The relation is written

$$\lim_{n=\infty} (2 - s) = 0.$$

From this relation it follows that

$$\lim_{n=\infty} s = 2.$$

To generalize these notions for any geometric progression whose ratio is numerically less than unity, consider the sum

$$s_n = a + ar + ar^2 + \dots + ar^{n-1},$$

where the subscript n is used to emphasize the fact that the sum of n terms is under consideration. By (D) of § 3,

$$s_n = \frac{a - ar^n}{1 - r},$$

which may be written $s_n = \frac{a}{1 - r} - \frac{ar^n}{1 - r},$

or $\frac{a}{1 - r} - s_n = \frac{ar^n}{1 - r}.$

If in this equation r be taken *numerically less than unity*, r^n will be smaller than r itself, and as n increases, r^n , and consequently $\frac{ar^n}{1 - r}$, will tend toward zero as a limit. According to the notation used in the stick problem above,

$$\lim_{n=\infty} \left(\frac{a}{1 - r} - s_n \right) = \lim_{n=\infty} \frac{ar^n}{1 - r} = 0,$$

and, finally,

$$\lim_{n=\infty} s_n = \frac{a}{1 - r}. \quad (E)$$

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It is important to note carefully that equation (E) does not state that s_n is equal to $\frac{a}{1-r}$. The equation means just what it says; namely, that $\frac{a}{1-r}$ is the limit toward which s_n is tending, and that by taking n large enough the difference between s_n and $\frac{a}{1-r}$ can be made as small as we please. Its importance lies in the fact that it furnishes a direct means for the solution of all problems like the stick problem. In that problem $a=1$ and $r=\frac{1}{2}$. Substituting these values in equation (E),

$$\lim_{n=\infty} s_n = \frac{1}{1-\frac{1}{2}} = 2.$$

Another example is the determination of the limit toward which a repeating decimal like .333... tends. The repeating decimal may be written in the form

$$.3 + .03 + .003 + \dots,$$

from which it is readily seen that the terms are in geometric progression, with the first term $a=.3$ and the ratio $r=.1$. If these values be substituted in the right member of equation (E), the result is

$$\frac{.3}{1-.1} = \frac{1}{3},$$

which is, as we know, the common fraction that gives rise to the repeating decimal .333... In the same way we may find the common fraction which is the equivalent of any repeating decimal. For example, the repeating decimal .23459459... is equivalent to the decimal .23 increased by the geometric progression .

$$.00459 + .00000459 + \dots,$$

whose ratio is .001; consequently,

$$.23459459 \dots = .23 + \frac{.00459}{1-.001} = \frac{217}{925}.$$

A geometric progression in which the ratio is numerically less than unity and in which the number of terms is increased indefinitely is called an *infinite geometric progression*.

EXAMPLES

1. Find the limit of the sum for each of the following infinite geometric progressions :

$$(a) 2 + .5 + .125 + \dots,$$

$$(b) \sqrt{2} + 1 + \frac{1}{\sqrt{2}} + \dots,$$

$$(c) \sqrt{2} + 1 + 1 + \sqrt{2} - 1 + \dots$$

2. What common fraction is the equivalent of the repeating decimal .237237...?

3. Find the common fraction which is equivalent to the repeating decimal .235959....

4. Find the limit of the sum of the infinite geometric progression

$$\frac{1}{1.05} + \frac{1}{(1.05)^2} + \frac{1}{(1.05)^3} + \dots$$

6. **Means and averages.** The *arithmetic mean* or *simple average* of two or more numbers is the sum of the numbers divided by their number. In symbols the arithmetic mean A of the n numbers $a_1, a_2, \dots, a_n$ is

$$A = \frac{a_1 + a_2 + \dots + a_n}{n}. \quad (1)$$

For example, if a board is 10 inches wide at one end and 14 inches wide at the other, the width at the middle point is the arithmetic mean, or $\frac{10 + 14}{2} = 12$. Any term of an arithmetic progression is the arithmetic mean of the two terms adjacent to it. It is also the arithmetic mean of the r terms immediately preceding and the r terms immediately following.

The *geometric mean* of n numbers $a_1, a_2, \dots, a_n$ is the n th root of their product and may be written

$$G = \sqrt[n]{a_1 a_2 \dots a_n}. \quad (2)$$

Any term of a geometric progression is the geometric mean of the two terms adjacent to it.

A mean or average that figures largely in problems in statistics is the so-called *weighted average*. The weighted average of several

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numbers is the sum of the products formed by multiplying each number by a number called its weight, divided by the sum of the weights.

The formula for the weighted average of n numbers $a_1, a_2, \dots, a_n$, having weights $w_1, w_2, \dots, w_n$, is

$$W = \frac{w_1 a_1 + w_2 a_2 + \dots + w_n a_n}{w_1 + w_2 + \dots + w_n}. \quad (3)$$

For example, one pound of a mixture containing 5 pounds of a substance costing 30 cents a pound, 6 pounds of another costing 40 cents a pound, and 2 pounds of another costing \$1.50 a pound would cost

$$\frac{5 \times .30 + 6 \times .40 + 2 \times 1.50}{5 + 6 + 2} = .53\frac{1}{3}.$$

Formula (3) has an important physical interpretation that is frequently of use in statistical problems. If $w_1, w_2, \dots, w_n$ are weights of masses at distances $a_1, a_2, \dots, a_n$ from a point, the weighted mean W is the distance from the point to the *center of gravity* of the system of masses.

When the weights are all unity the weighted average reduces to the simple average.

If a set of numbers forms an arithmetic progression, their reciprocals form a *harmonic progression*. The *harmonic mean* of two numbers is defined as the reciprocal of the arithmetic mean of the reciprocals of the two numbers.

EXAMPLES

1. Prove that any term of an arithmetic progression is the arithmetic mean of the two adjacent terms.
2. Prove that any term of a geometric progression is the geometric mean of the two adjacent terms.
3. Prove that the harmonic mean of two numbers a and b is

$$H = \frac{2ab}{a+b}. \quad (4)$$

4. Express the harmonic mean of two numbers in terms of their arithmetic and geometric means.

5. Prove that the roots of a quadratic equation

$$ax^2 + 2bx + c = 0$$

- are (a) real when a , b , and c are in arithmetic progression;
(b) equal when a , b , and c are in geometric progression;
(c) imaginary when a , b , and c are in harmonic progression.

6. Prove that the logarithms of the terms of a geometric progression form an arithmetic progression.

7. The rainfall in inches at Madison, Wisconsin, for the years 1917, 1918, 1919, 1920, 1921 was 29.40, 27.27, 35.16, 27.16, 38.56, respectively. What was the average annual rainfall for the five-year period?

8. Three masses, weighing 10, 12, and 20 pounds, are located respectively 3 feet to the right, 4 feet to the left, and 5 feet to the right of a certain point. Where is the center of gravity of the system with respect to the point?

9. A community has 1 citizen with an income of \$20,000, 15 with incomes of \$5000, 40 with incomes of \$3000, 80 with incomes of \$2000, and 200 with incomes of \$1000. What is the average income for the citizens of this community?

10. A merchant increased his capital by 10% the first year, 15% the second year, and 20% the third year. What was the average rate for the three years?

HINT. If r be the average rate of increase, the capital at the end of three years would be the original capital multiplied by $(1 + r)^3$.

11. Two successive miles were traveled at the rate of 10 and 15 miles per hour respectively. What was the average rate?

12. Two successive miles were traveled at the rate of a and b miles per hour respectively. What sort of a mean expresses the average rate?

CHAPTER II

INFINITE SERIES

7. Infinite series with constant terms. The infinite geometric progression considered in § 5 is a special case of a very important class of expressions which are called *infinite series*.

Let $u_1, u_2, u_3, \dots, u_n, \dots$

be a sequence of terms formed according to some law. Further, suppose that the process of setting up terms of the sequence may be continued without limit. The expression

$$u_1 + u_2 + u_3 + \dots$$

is called an infinite series.

The following are examples of infinite series with constant terms:

$$(1) 1 + \frac{1}{2} + \frac{1}{4} + \dots$$

$$(2) 1 + 2 + 3 + 4 + \dots$$

$$(3) 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

$$(4) 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \dots$$

8. The sum of an infinite series; convergence. Clearly the sum of a definite number of terms of an infinite series with constant terms is a definite number. The sum of an infinite series has no meaning, at least for the present. In order to make use of an infinite series we form the sum of n terms and try to find out what happens if n is increased without limit.

$$\text{Let} \quad s_n = u_1 + u_2 + u_3 + \dots + u_n$$

be the sum of the first n terms. If n is increased without limit one of three things must happen:

(1) The sum s_n will approach a definite limit. For example, the limit of n terms of the series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \dots$$

of the problem considered in § 5 is 2.

(2) The sum s_n will increase without limit. This clearly happens in the case of the series

$$1 + 2 + 3 + \dots$$

(3) The sum s_n will oscillate between several values, as in the case of the series

$$1 + 1 - 1 - 1 + 1 + 1 - \dots,$$

where s_n takes the values 1, 2, 1, 0 in a never-ending repetition.

When limit s_n is a finite and definite number the series is said to be *convergent*; otherwise, it is *divergent*.

In ordinary work only convergent series can be utilized. We are therefore concerned with tests for convergence.

9. Convergence tests. No test for convergence applicable in every case exists. There are many special tests, of which two, the *direct-comparison test* and the *test-ratio test*, are the most useful. These two tests are given here without proof. For proofs the reader is referred to texts on algebra and analysis.

THE DIRECT-COMPARISON TEST. *If the terms of a series from a given term onward are less than, or at most equal to, the corresponding terms of a series known to be convergent, the series to be tested is convergent; if the terms from a given term onward are equal to or greater than the corresponding terms of a series known to be divergent, the series to be tested is divergent.*

As a comparison series we may take any infinite geometric progression whose ratio is less than 1. Another important convergent series is the series

$$1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \dots$$

One of the best-known divergent series is the so-called harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

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THE TEST-RATIO TEST. *Let*

$$u_1 + u_2 + u_3 + \dots$$

be an infinite series and let

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = t;$$

then, if $|t| < 1$, the series is convergent ;

if $|t| > 1$, the series is divergent ;

if $|t| = 1$, the character of the series is undetermined.

ILLUSTRATIVE EXAMPLE. It is required to determine the character of the series

$$1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \dots$$

The general term is $\frac{1}{1 \cdot 2 \cdot 3 \dots n}$, and the ratio $\frac{u_{n+1}}{u_n}$ is therefore

$$\frac{\frac{1}{1 \cdot 2 \cdot 3 \dots n \cdot (n+1)}}{\frac{1}{1 \cdot 2 \cdot 3 \dots n}} = \frac{1}{n+1}.$$

Consequently, $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = t = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0.$

Since $t = 0$, the series is convergent.

EXAMPLES

Determine the character of the following series :

1. $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \dots$

2. $1 + \frac{1}{1^2 \cdot 2^2} + \frac{1}{2^2 \cdot 3^2} + \frac{1}{3^2 \cdot 4^2} + \dots$

3. $\frac{3}{4} + \frac{3 \cdot 5}{4 \cdot 7} + \frac{3 \cdot 5 \cdot 7}{4 \cdot 7 \cdot 10} + \dots + \frac{3 \cdot 5 \cdot 7 \dots (2n+1)}{4 \cdot 7 \cdot 10 \dots (3n+1)} + \dots$

4. $\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \dots$

5. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots$

7. $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$

6. $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots$

8. $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots$

10. Power series. The most important series for practical work are those whose terms contain a variable, and of these the most important are the so-called *power series*. A power series is a series of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \dots, \quad (1)$$

i.e. a series whose terms are all composed of the product of a constant and a positive integral power of a variable and are arranged according to the ascending powers of the variable.

In the case of a power series the problem is usually to determine, not whether the series is convergent, but for what value of the variable the series is convergent. For this purpose the test-ratio test is usually the most effective. For example, we know that the geometric series

$$1 + x + x^2 + x^3 + \dots$$

is convergent for all values of the ratio x which satisfy the condition

$$-1 < x < 1,$$

and divergent for all other values of x except possibly for $x = -1$. The n th term is x^{n-1} and the $(n+1)$ st is x^n . By the test-ratio test,

$$\lim_{n \rightarrow \infty} \frac{x^n}{x^{n-1}} = \lim x$$

will be less in absolute value than 1 for $-1 < x < +1$, which agrees with the result already known.

The great value of the power series for problems in mathematics lies in the fact that they furnish an easy means of computing numerical values corresponding to special values of the variable for many functions where such computation would be extremely difficult or even impossible by any other means. For example, the cosine of an angle x may be defined by the series

$$\cos x = 1 - \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \dots,$$

where the unit of measure for x is the radian which is the equivalent of $57^\circ.2957795 +$. This series is convergent for all values of x since the test ratio reduces to $-\frac{x^2}{2n(2n-1)}$, and

$$\lim_{n \rightarrow \infty} \left| -\frac{x^2}{2n(2n-1)} \right| = 0$$

for all values of x .

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The value of the cosine of one radian is therefore

$$\cos 1 = 1 - \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \dots$$

Five terms of this series give

$$\cos 1 = .540302 +,$$

a result which is correct to the fifth decimal place.

Three power series are of especial importance in investment problems. They are the *binomial series*, the *exponential series*, and the *logarithmic series*. They will be considered in the following sections.

11. The binomial series. In elementary algebra we are made familiar with the expansion of a power of a binomial when the exponent is a positive integer. Such an expansion has the form

$$(a + x)^n = a^n + \frac{n}{1} a^{n-1}x + \frac{n(n-1)}{1 \cdot 2} a^{n-2}x^2 + \dots + x^n. \quad (1)$$

This expansion is a power series consisting of a finite number of terms with coefficients

$$a^n, \quad \frac{n}{1} a^{n-1}, \quad \frac{n(n-1)}{1 \cdot 2} a^{n-2}, \quad \dots, \quad 1,$$

and whose general term is

$$\frac{n(n-1)(n-2) \dots (n-r+1)}{1 \cdot 2 \cdot 3 \dots (r-1)r} \cdot a^{n-r}x^r. \quad (2)$$

The series (1) may also be looked upon as an infinite power series all of whose coefficients after the $(n+1)$ st are zero. If the exponent n is not a positive integer, it is still possible to obtain a development for $(a+x)^n$ which will be similar in form to (1) and for which the general term will be (2). There is this great difference, however: when n is not a positive integer, the series will never end but is a true infinite series.

The power series

$$a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2} a^{n-2}x^2 + \dots \quad (3)$$

is called the *binomial series*.

To determine the character of the binomial series, consider the test ratio, viz.

$$\frac{\frac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots(r-1)r}a^{n-r}x^r}{\frac{n(n-1)(n-2)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}a^{n-r+1}x^{r-1}} = \frac{n-r+1}{r} \cdot \frac{x}{a}.$$

The limit of the test ratio as r increases is

$$\begin{aligned} t &= \lim_{r \rightarrow \infty} \frac{n-r+1}{r} \cdot \frac{x}{a} \\ &= \frac{x}{a} \lim_{r \rightarrow \infty} \frac{n-r+1}{r} \\ &= -\frac{x}{a}. \end{aligned}$$

But in order that the series (3) should be convergent, t must be less in absolute value than unity, i.e.

$$\left| -\frac{x}{a} \right| < 1, \quad \text{or} \quad |x| < |a|. \quad (4)$$

If, on the other hand, $|x| > |a|$,

the series (3) will be divergent.

We write

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^2 + \dots$$

for all values of n , with the understanding that the power series represents the development of $(a+x)^n$ for those values of x which satisfy the convergence condition (4).

A case of special importance is that for which $a=1$; then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \dots \quad (5)$$

for values of x such that $|x| < 1$.

The series (5) is of great use in computing many quantities which occur in the theory of interest. As a simple example, suppose the value of the expression $(1.04)^{1\frac{1}{2}}$ is required. This value may readily be obtained by replacing x by .04 and n by $1\frac{1}{2}$ in the binomial expansion (5).

By the binomial series

$$(1.04)^{\frac{1}{12}} = 1 + \frac{1}{12} \cdot .04 + \frac{\frac{1}{12}(\frac{1}{12}-1)}{1 \cdot 2} (.04)^2 \\ + \frac{\frac{1}{12}(\frac{1}{12}-1)(\frac{1}{12}-2)}{1 \cdot 2 \cdot 3} (.04)^3 + \dots$$

Four terms of the series give the result

$$(1.04)^{\frac{1}{12}} = 1.003273 +,$$

which is accurate to the sixth decimal place.

This sort of an expression occurs in almost every financial problem that involves monthly payments.

12. The exponential series. The expression

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n \quad (1)$$

occurs so frequently and is of such importance in mathematics that it may well be called a fundamental limit.

To find an expression for it that will be useful in practical work, suppose that n approaches infinity through the series of natural numbers, i.e. through the positive integers taken in order of increasing magnitude. For any positive integral value of n

$$\left(1 + \frac{x}{n}\right)^n = 1 + n \frac{x}{n} + \frac{n(n-1)}{1 \cdot 2} \frac{x^2}{n^2} \\ + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \frac{x^3}{n^3} + \dots + \frac{x^n}{n^n}. \quad (2)$$

To find the value of the limit (1) by roughly approximate methods, divide out the n 's from the numerators and denominators. Then (2) takes the form

$$\left(1 + \frac{x}{n}\right)^n = 1 + \frac{x}{1} + \frac{1 - \frac{1}{n}}{1 \cdot 2} x^2 \\ + \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)}{1 \cdot 2 \cdot 3} x^3 + \dots \text{to } n+1 \text{ terms.} \quad (3)$$

If we assume that the limit of the sum on the right in (3) can be found by taking the sum of the limits of the separate terms, even though the number of terms be indefinitely increased, we shall have

$$\lim_{n=\infty} \left(1 + \frac{x}{n}\right)^n = 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \dots \quad (4)$$

The power series on the right is called the *exponential series* and is ordinarily denoted by e^x . By definition of e^x ,

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \dots \quad (5)$$

The series (5) is convergent for all finite values of the variable x , for the general term is

$$\frac{x^n}{1 \cdot 2 \cdot 3 \dots n}$$

and the test ratio is

$$\frac{\frac{x^{n+1}}{1 \cdot 2 \cdot 3 \dots n \cdot (n+1)}}{\frac{x^n}{1 \cdot 2 \cdot 3 \dots n}} = \frac{x}{n+1}.$$

Consequently, $t = \lim_{n=\infty} \frac{x}{n+1} = x \cdot \lim_{n=\infty} \frac{1}{n+1} = x \cdot 0$.

The number t is therefore zero for every finite value of x , and the statement is proved.

It is not difficult to prove that the expression e^x behaves exactly like an ordinary power when combined with other expressions of the same sort. In particular,

$$e^x \cdot e^y = e^{x+y}, \quad e^x \div e^y = e^{x-y}, \quad \text{and} \quad (e^x)^y = e^{xy}.$$

To find the value of e^1 or e , substitute 1 for x in the series (5). Then

$$e = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \dots$$

Ten terms of the series will give

$$e = 2.71828,$$

accurate to the fifth decimal place. The student is advised to carry out the computation.

13. The logarithmic series. The series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (1)$$

is called the *logarithmic series*. It is convergent for values of x which are numerically less than unity, for the general term is

$$\frac{x^n}{n}$$

and the test ratio is $\frac{x^{n+1}}{n+1} \div \frac{x^n}{n} = \frac{n}{n+1} \cdot x$.

The limiting value of the test ratio is then

$$\begin{aligned} t &= \lim_{n \rightarrow \infty} \frac{n}{n+1} x \\ &= x \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = x. \end{aligned}$$

The series (1) is therefore convergent for

$$|t| = |x| < 1,$$

which was to be proved.

The series (1) is called the logarithmic series because for values of x numerically less than 1, i.e. for values for which the series is convergent, it represents the value of the expression $\log(1+x)$. By definition,*

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (2)$$

The logarithmic series is not well adapted to computation, since it converges very slowly and a large number of terms are required to obtain a good approximation. The actual work of computing logarithms is best accomplished by the use of some such series as that of Example 9 below, which is derived from the logarithmic series.

*The logarithm that is here defined is the hyperbolic, or Napierian, logarithm. For a definition of different systems of logarithms, see Chapter IV.

EXAMPLES

1. Find the value of $(1.05)^8$ to five decimal places.
2. Find the fifth root of 1.05, i.e. the value of $(1 + .05)^{\frac{1}{5}}$, to five decimal places.
3. Find the value of e^{-1} to five decimal places by means of the power series

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \cdots.$$

4. If $\sin x$ is defined by the series

$$\sin x = x - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \cdots,$$

find the value of $\sin(.1)$ to the fourth decimal place.

5. For what values of x is the series

$$\frac{1}{1 \cdot 2} + \frac{x}{2 \cdot 3} + \frac{x^2}{3 \cdot 4} + \frac{x^3}{4 \cdot 5} + \cdots + \frac{x^n}{(n+1)(n+2)} + \cdots$$

convergent?

6. For what values of x is the series

$$\frac{1}{1} + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots + \frac{1}{x^n} + \cdots$$

convergent?

7. Prove by actual multiplication that the product of

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \cdots$$

and

$$e^y = 1 + \frac{y}{1} + \frac{y^2}{1 \cdot 2} + \frac{y^3}{1 \cdot 2 \cdot 3} + \cdots$$

is e^{x+y} .

8. Compute the value of e^3 to the fifth decimal place.
9. Compute the value of the logarithm of 2 from the series

$$\log \frac{M}{N} = 2 \left[\frac{M-N}{M+N} + \frac{1}{3} \left(\frac{M-N}{M+N} \right)^3 + \frac{1}{5} \left(\frac{M-N}{M+N} \right)^5 + \cdots \right]$$

by making $M = 2$ and $N = 1$.

CHAPTER III

INTERPOLATION

14. Functions. When two quantities x and y are so related that y may be found when x is known, y is said to depend on x or to be a *function* of x . If a functional relation of the sort just described exists between x and y , the fact is indicated by the notation

$$y = f(x). \quad (1)$$

We call x the *argument* of the function y . When the exact form of the function is known we frequently replace the general functional symbol f by an abbreviation for the name of the function. Thus for logarithm of x we write $\log x$; for cosine of x , $\cos x$; with similar abbreviations for other well-known functions. There are many different kinds of functions which will play an important part in this book.

15. The tabulation of a function. Many of the more important functions occur so frequently that their values for a large number of values of the argument are computed in advance and are arranged in tables for ready reference, such as the following:

Table of logarithms of integers from 34500 to 34509 inclusive.

x	$\log x$	$p.p.$	
			126
34500	4.537 8191		
34501	4.537 8317	1	12.6
34502	4.537 8443	2	25.2
34503	4.537 8569	3	37.8
34504	4.537 8694	4	50.4
34505	4.537 8820	5	63.0
34506	4.537 8946	6	75.6
34507	4.537 9072	7	88.2
34508	4.537 9198	8	100.8
34509	4.537 9324	9	113.4

When the function depends upon a single variable, the table may consist of two columns, one for values of the argument and the other for values of the function. Such a table is called a *table of single entry*. The preceding brief table gives the logarithms, to seven places of decimals, of the integral numbers from 34500 to 34509 inclusive. This table is a table of single entry.

The significance of the column headed p.p. will be explained in § 19. Other tables will be found at the back of this book.

16. The graphic representation of a function. The properties of a function are much more easily comprehended if the function can be represented graphically. The graphic representation of a function of one argument is accomplished if we lay off on a given straight line segments representing the values of the argument, taking a fixed point as origin, and then at the extremities of these segments erect perpendiculars whose lengths and directions correspond to the values of the functions. A continuous line drawn

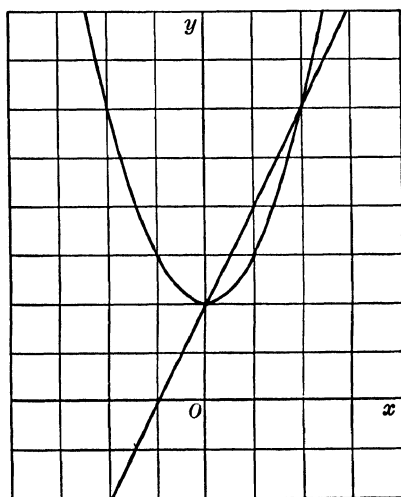


FIG. 1

through the extremities of the perpendicular lines is called the *graph of the function*. The graphs of the functions $2 + 2x$ and $2 + x^2$ are shown in Fig. 1. The tables of the functions for values of the argument from 0 to 5 inclusive, are as follows:

x	$2 + 2x$	x	$2 + x^2$
0	2	0	2
1	4	1	3
2	6	2	6
3	8	3	11
4	10	4	18
5	12	5	27

Both the graphs and the tables of the functions $2 + 2x$ and $2 + x^2$ are strikingly different. The one property most important for us to note at the present time is that the tabular differences for the function $2 + 2x$ are constant, while they are not constant for the function $2 + x^2$.

A function whose tabular differences are constant is called a *linear function*. It is not difficult to prove that the graph of a function whose first differences are constant is a straight line.

17. Interpolation by first differences. Clearly it would be impossible to construct a table giving the logarithms of all numbers, or, for that matter, a table giving the values of any function for an infinite number of arguments. It may be necessary, however, to find from the table of § 15 the logarithm of such a number as 34502.25 which lies between 34502 and 34503. The process of finding the value of a function for a value of the argument intermediate to two arguments for which the function is given is called *interpolation*.

The difference between two consecutive values of a function given by a table is called the *first difference*, or *difference of first order*. It is also called the *tabular difference*.

It may be noted that in the table of § 15 the differences, with a single exception, are equal to .0000126. It must not be thought, however, that the tabular differences are constant for a table of logarithms. They are approximately constant for small ranges of the argument only.

The fact that for small ranges of the argument the first differences are approximately constant may be stated as follows:

The difference between the logarithms of two numbers which differ by a small amount is approximately proportional to the difference between the numbers themselves.

If, then, the logarithm of 34502.25 is to be found, we may denote by y the *correction* that must be added to the logarithm of 34502, and we have $y : .0000126 = .25 : 1$

or $y = .0000126 \times .25 = .00000315$.

Consequently, $\log 34502.25 = 4.5378443 + y = 4.5378474$.

18. General theory of first-difference interpolation. In first-difference interpolation it is assumed that for small differences of the argument the tabular differences are proportional to the differences between the arguments, and consequently the graph is assumed to be a straight line for the values of the argument

under consideration. A figure will enable us to see the relations more clearly.

Let x_1 and x_2 (Fig. 2) be two consecutive values of the argument and y_1 and y_2 the known values of the function corresponding to x_1 and x_2 . The problem is to find the value y of the function corresponding to value x of the argument, which lies between x_1 and x_2 .

Since the differences of the function are supposed to be proportional to the differences of the argument, the graph of the function between x_1 and x_2 is assumed to be a straight line.

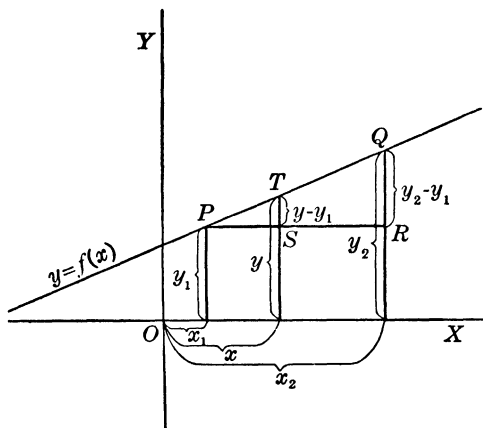


FIG. 2

The problem is then to find the length of the line ST , whose measure is the correction that must be added to the value y_1 .

From the triangle PRQ ,

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1},$$

or

$$y - y_1 = \frac{x - x_1}{x_2 - x_1} (y_2 - y_1). \quad (1)$$

The number $x_2 - x_1$ is the interval or difference between two consecutive values of the argument measured in terms of the unit interval under consideration. If we were using the table of § 15, $x_2 - x_1$ would be 1. If we were using a table of logarithmic sines which gave the values of the function for values of the angle differing by 10 seconds, $x_2 - x_1$ would be 10 seconds. The number $x - x_1$ has a similar meaning. The fraction

$$p = \frac{x - x_1}{x_2 - x_1}, \quad (2)$$

which is always less than 1, is called a *proportionality factor*.

Furthermore, $y_2 - y_1$ is the first, or tabular, difference of the function and $y - y_1$ is the correction which must be added to y_1 to obtain the required value for y .

Denoting the first difference $y_2 - y_1$ by D_1 , we may write equation (1) in the form

$$y - y_1 = pD_1; \quad (3)$$

or, finally,
$$y = y_1 + pD_1. \quad (4)$$

EXAMPLES

1. From the two logarithms $\log 81895 = 4.9132574$ and $\log 81896 = 4.9132627$ find $\log 81895.32$.

Solution. The interval between the two values of the argument 81895 and 81896 is 1, so that $p = .32$. The tabular difference is 53. The correction is therefore $.32 \times 53 = 16.96$.

Consequently, $\log 81895.32 = 4.9132574 + .0000017 = 4.9132591$.

2. The present values of \$1 for 20 years at $1\frac{1}{4}\%$ and $1\frac{1}{2}\%$ are 0.7800085 and 0.7424704 respectively. Find the present value of \$1 for the same time at $1\frac{3}{8}\%$.

Solution. The tabular difference is $-.0375381$. The interval $x_2 - x_1$ for the two arguments is .0025. The difference between the smaller argument and the argument for which the function is required is .00125. Consequently $p = \frac{1}{2}$. The correction is therefore $\frac{1}{2} \times (-.0375381) = -.0187691$, and the required value for the present value of \$1 for 50 years at $1\frac{3}{8}\%$ is approximately

$$0.7800085 - 0.0187691 = 0.7612394.$$

3. From the table in § 15 find $\log 34503.17$.

4. From the same table find $\log 34503.171$.

5. The function $f(i) = \frac{i}{1 - (1 + i)^{-n}}$, when i is the rate of interest and n the time, has the values

for rate $3\frac{1}{2}\%$ and time 50 years, $f(i) = 0.0426337$;

for rate 4% and time 50 years, $f(i) = 0.0465502$.

Find the approximate value of the function for rate $3\frac{3}{4}\%$ and time 50 years.

HINT. In this case the interval for the argument is .005.

6. The function in Example 5 has the values

$f(i) = 0.0426337$ for rate $3\frac{1}{2}\%$ and time 50 years;

$f(i) = 0.0423216$ for rate $3\frac{1}{2}\%$ and time 51 years.

Find the value of the function for 50 years and 6 months.

19. Proportional parts. By means of formula (3), § 18, the determination of the correction is reduced to multiplication of the first difference by the proportionality factor p . This multiplication is in turn reduced to the multiplication of the difference by a series of integers. These latter products are frequently tabulated as an auxiliary table called a table, or column, of *proportional parts*. To illustrate, the table of § 15 gives for D_1 the value .0000126. We may drop the zeros and the decimal point and remember also that p is a pure decimal. The table of proportional parts for the difference 126 then takes the form of the column headed " $p.p.$ " in the table of § 15.

By means of this auxiliary table the work of finding $\log 34502.25$ is done without actually performing the multiplication. The work may be arranged as follows:

$$\begin{array}{r} \log 34502 = 4.5378443 \\ \text{correction for } .2 = \quad \quad 25 \\ \text{correction for } .05 = \quad \quad 6 \\ \hline \log 34502.25 = 4.5378474 \end{array}$$

Of course, a column of proportional parts can be constructed for every tabular difference.

EXAMPLES

From the table of § 15, find

- | | | |
|------------------------|------------------------|------------------------|
| 1. $\log 34503.7$. | 4. $\log 34503.75$. | 7. $\log 34503.753$. |
| 2. $\log 34505.82$. | 5. $\log 34506.937$. | 8. $\log 34503.923$. |
| 3. $\log 34505.3734$. | 6. $\log 34508.2373$. | 9. $\log 34507.3213$. |

10. Construct columns of proportional parts for differences 247, 359, and 1837.

20. Interpolation by means of second differences. For most functions that are tabulated the first differences are not constant. Indeed, in the table given in § 15 the first differences are constant only because the values of the functions are carried out to a comparatively small number of decimal places. The situation would be wholly different if more decimal places were used.

As a simple illustration of a table where the first differences are not constant let us consider a table of compound amounts of \$1 for

10 years at rates .03 to .06 at intervals of one half of 1%. In the following table the second, third, and fourth differences have been computed. For this table the fifth differences are approximately constant, so that the sixth differences may be taken to be zero.

Table of compound amounts on \$1 for 10 years at rates 3 to 6% at intervals of one half of 1%, together with first, second, third, and fourth differences.

Rate	Amount on \$1	First differences	Second differences	Third differences	Fourth differences
.03	1.3439164	.0666824			
.035	1.4105988	.0690455	.0029631		
.04	1.4802443	.0727251	.0030796	.0001165	.0000040
.045	1.5529694	.0759252	.0032001	.0001205	.0000041
.05	1.6288946	.0792499	.0033247	.0001246	.0000040
.055	1.7081445	.0827032	.0034533	.0001286	
.06	1.7908477				

21. Higher differences in general terms. Let $y_1, y_2, y_3, \dots$ be the values of the function corresponding to the values $x_1, x_1 + d, x_1 + 2d, \dots$. The table showing all differences up to and including the fourth may be arranged as follows:

Argument	Function	First differences	Second differences	Third differences	Fourth differences
x_1	y_1				
$x_1 + d$	y_2	$y_2 - y_1$			
$x_1 + 2d$	y_3	$y_3 - y_2$	$y_3 - 2y_2 + y_1$	$y_4 - 3y_3 + 3y_2 - y_1$	
$x_1 + 3d$	y_4	$y_4 - y_3$	$y_4 - 2y_3 + y_2$	$y_5 - 3y_4 + 3y_3 - y_2$	$y_5 - 4y_4 + 6y_3 - 4y_2 + y_1$
$x_1 + 4d$	y_5	$y_5 - y_4$	$y_5 - 2y_4 + y_3$		

If now $D_1, D_2, D_3, \dots$ denote the differences at the tops of the respective difference columns,

$$\left. \begin{aligned} D_1 &= -y_1 + y_2 \\ D_2 &= y_1 - 2y_2 + y_3 \\ D_3 &= -y_1 + 3y_2 - 3y_3 + y_4 \\ D_4 &= y_1 - 4y_2 + 6y_3 - 4y_4 + y_5 \end{aligned} \right\} \quad (1)$$

From the equations (1) we obtain, by eliminating y_2 from the second, y_2 and y_3 from the third, and so on,

$$\left. \begin{aligned} y_1 &= y_1 &= y_1 \\ y_2 &= y_1 + D_1 &= y_1 + D_1 \\ y_3 &= -y_1 + 2y_2 + D_2 = y_1 + 2D_1 + D_2 \\ y_4 &= \dots &= y_1 + 3D_1 + 3D_2 + D_3 \\ y_5 &= \dots &= y_1 + 4D_1 + 6D_2 + 4D_3 + D_4 \end{aligned} \right\} \quad (2)$$

Careful inspection of the final results shows that the coefficients of the D 's are exactly the binomial coefficients for a power one less than the subscript of the y on the left. It may be proved that this statement is true for all integral values of n . Consequently,

$$\begin{aligned} y_n &= y_1 + (n-1)D_1 + \frac{(n-1)(n-2)}{1 \cdot 2} D_2 \\ &\quad + \frac{(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3} D_3 + \dots \end{aligned} \quad (3)$$

The value y_n is distant from y_1 by $n-1$ intervals. It follows therefore that formula (3) furnishes a means of computing the value y_n of the function corresponding to the value x_n of the argument, which is distant from x_1 by $n-1$ intervals. In other words, we may find y_n corresponding to $x_n = x_1 + (n-1)$ intervals.

Suppose we wish to find the value y corresponding to $x = x_1 + p$ intervals; or, more briefly, $x = x_1 + p$, where p is the length along the x -axis measured in intervals. Assuming formula (3) to hold for any value of $n-1$, either integral or fractional,

we may replace $n-1$ by p . The formula thus obtained from formula (3) is

$$y = y_1 + pD_1 + \frac{p(p-1)}{1 \cdot 2} D_2 + \frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3} D_3 + \dots \quad (4)$$

Formula (4) is one of the most important of the general interpolation formulas for finding a value y of the function corresponding to a value $x = x_1 + p$, where p indicates a fractional part of an interval.

If we assume that all differences of order greater than two are zero, we obtain

$$y = y_1 + pD_1 + \frac{p(p-1)}{1 \cdot 2} D_2. \quad (5)$$

Fig. 3 will help to make clear the relation between formulas (4) of § 18 and (4) and (5)

of § 21. If $y=f(x)$ be represented graphically by the curve PTQ , the line ST will be the correction that would be made if one were able to find the absolutely correct value of the function corresponding to $x = x_1 + p$. The line pD_1 represents the correction obtained by first-difference interpolation.

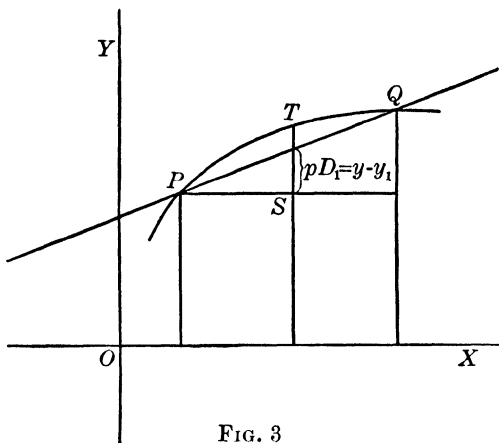


FIG. 3

The correction obtained by second differences would be some line intermediate between the line pD_1 and ST .

EXAMPLES

1. Find the compound amount on \$1 for 10 years at 4.25% by means of the table of § 20.

Solution. The difference between two consecutive values of the argument is .005, while the value of y corresponding to

$$x = .04 + .0025$$

is required. When the interval is .005, .0025 is one half an interval; i.e. $p = \frac{1}{2}$ and

$$x = .04 + (.5 \text{ interval}).$$

From the table, $D_1 = .0727251$ and $D_2 = .0032001$. Consequently, formula (5) of § 21 becomes

$$\begin{aligned} y &= 1.4802443 + .5 \times .0727251 + \frac{.5(.5 - 1)}{1 \cdot 2} \times .0032001 \\ &= 1.5162068. \end{aligned}$$

The result is accurate to the fifth decimal place.

2. Find the compound amount on \$1 for 10 years at 5.25% by means of the table of § 20, using second differences.

3. Find the compound amount on \$1 for 10 years at 4.10% by means of the table of § 20, using second differences.

4. By means of the same table find the compound amount on \$1 for 10 years at 4.36%.

5. By means of Table III at the back of this book find the compound amount on \$1 for 10 years at 5.25%, using third differences.

6. By means of Table IV find the present value of \$1 due in 10 years without interest when money is worth 4.25%, using second differences.

7. Solve Example 5, using third differences.

8. By means of Table IV find the present value of \$1 due in 10 years when money is worth 4.36%, using second differences.

9. By means of the following table of cube roots of the numbers from 550 to 561 inclusive, find the cube root of 552.25:

Number	Cube root	Number	Cube root
550	8.19321	556	8.22290
551	8.19818	557	8.22783
552	8.20313	558	8.23275
553	8.20808	559	8.23766
554	8.21303	560	8.24257
555	8.21797	561	8.24747

10. From the table in Example 9 find $\sqrt[3]{556.32}$.

11. Construct a table of squares of numbers from 1 to 10 inclusive and find the differences of first and second orders.

12. Construct a table of cubes of numbers from 1 to 10 inclusive and find the differences of various orders.

13. Find the differences for a table of fourth powers.

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14. What is your guess as to the n th order differences for a table of x^n ? Prove your statement for $n = 2$.

15. Construct a table and find the differences for the function $x^3 - 2x^2$ for the range $x = 3$ to $x = 8$.

16. Construct a table and find all non-zero differences for the function $x^4 - 2x^3 + 4x^2$ for the range $x = 1$ to $x = 6$.

22. Interpolation of values of the argument; first differences. If a value of the function not listed in the table is given, and we wish to find the corresponding value of the argument, the problems of the previous sections are reversed; in other words, the fraction p is the unknown number.

Suppose the value $y = y_1 + w$ of the function is given where w is less than the first difference, and it is desired to find the corresponding value $x = x_1 + p$ of the argument. If we have regard for first differences only, we may find p from the equation

$$y = y_1 + p(y_2 - y_1). \quad (1) \quad (\text{See (3), § 18.})$$

The required value of p is

$$p = \frac{y - y_1}{y_2 - y_1} = \frac{y - y_1}{D_1}, \quad (2)$$

where p is expressed as a fraction of an interval for the argument.

EXAMPLES

1. If $\log x = 4.5378875$, find x from the table of § 15.

Solution. By inspection it may be seen that the required value of x lies between $x_1 = 34505$ and $x_2 = 34506$, for which the corresponding logarithms are $y_1 = 4.5378820$ and $y_2 = 4.5378946$. Clearly

$$D_1 = .0000126 \quad \text{and} \quad p = \frac{.5}{8} = .44.$$

In this case the interval is 1. Therefore

$$x = x_1 + p = 34505.44.$$

2. The logarithms of the tangents of $49^\circ 10'$ and $49^\circ 20'$ are .0633895 and .0659441 respectively. Find the logarithm of the tangent of $49^\circ 15'$.

3. Find the logarithm of the tangent of $49^\circ 13'$ from the data of Example 2.

4. The logarithm of an unknown number x is 4.5378900. Find x by means of the table of § 15.

23. Interpolation of values of the argument ; second differences.

When second differences are taken into account we must make use of the formula

$$y = y_1 + pD_1 + \frac{p(p-1)}{1 \cdot 2}D_2. \quad (1) \text{ (See (5), § 21.)}$$

This formula may be put into the form

$$p = \frac{y - y_1}{D_1 + \frac{(p-1)}{1 \cdot 2}D_2}. \quad (2)$$

If now we replace p on the right by the value obtained from equation (1) or (2) of § 22, we shall find a second, and usually a better, approximation for p .

EXAMPLES

1. Suppose the compound amount of \$1 for 10 years at an unknown rate is 1.5932641. What is the rate?

Solution. From the table of § 20 it is seen that the required rate lies between $x_1 = .045$ and $x_2 = .05$. The corresponding values of the function are $y_1 = 1.5529694$ and $y_2 = 1.6288946$. The table gives

$$D_1 = .0759252 \text{ and } D_2 = .0033247.$$

Moreover,

$$y - y_1 = .0402947,$$

so that the value of p given by first-difference interpolation is

$$p_1 = \frac{.0402947}{.0759252} = .531.$$

If this value of p be substituted in formula (2), § 23, the result is

$$\begin{aligned} p &= \frac{.0402947}{.0759252 + \frac{(.531-1)}{2} \times .0033247} \\ &= \frac{.0402947}{.0751456} = .536. \end{aligned}$$

It must be remembered that p is a fractional part of an interval which in this case is not 1 but .005. Consequently,

$$\begin{aligned} x &= .045 + .536 \times .005 \\ &= .047680. \end{aligned}$$

The result is correct to the sixth decimal place. Interpolation by first differences would have given the result $x = x_1 + p_1 = .047655$.

2. Given $\log 33510 = 4.5251744$ and $\log 33511 = 4.5251874$. Find $\log 33510.25$.

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3. By direct calculation we find $25^2 = 625$, $26^2 = 676$, $27^2 = 729$. Find $(26.5)^2$ by interpolation, using first differences only.

4. From the data in Example 3 find $(26.5)^4$, using first and second differences.

5. From the table of § 20 find the compound amount on \$1 for 10 years at 3.75%, using first and second differences.

6. From $\sqrt{51} = 7.141$, $\sqrt{52} = 7.211$, and $\sqrt{53} = 7.280$, find $\sqrt{51.3}$.

7. Using the data of Example 6, find the number of which 7.245 is the approximate square root. Solve the problem, using first differences alone, then using first and second differences.

8. The following are approximate values: $\frac{1}{6.2} = .161$, $\frac{1}{6.3} = .159$, $\frac{1}{6.4} = .156$, $\frac{1}{6.5} = .154$. Find the approximate values of $\frac{1}{6.32}$, $\frac{1}{6.33}$, and $\frac{1}{6.34}$.

9. The population of the United States by decades has been as follows: in 1890, 62,622,250; in 1900, 75,994,575; in 1910, 91,972,266; in 1920, 105,709,000. What was the approximate population in 1917?

10. The function $f(i) = \frac{1 - (1 + i)^{-20}}{i}$ has the values given by the following table:

i	$f(i)$
.025	15.58916229
.02625	15.40665946
.0275	15.22725213
.02875	15.05087768
.03	14.87747486
.03125	14.70698385

Find the approximate value of $f(.028125) = \frac{1 - (1.028125)^{-20}}{.028125}$.

11. If $f(1) = 1.06$, $f(2) = 1.1236$, $f(3) = 1.191016$, $f(4) = 1.26247696$, $f(5) = 1.33822558$, what is $f(3.5)$?

12. The compound amounts on \$1 for 10 years at various rates are as follows:

for 5%, 1.62889463; for $5\frac{1}{2}\%$, 1.70814446;

for 6%, 1.79084770; for $6\frac{1}{2}\%$, 1.87713747.

For what approximate rate is 1.73854962 the compound amount?

24. Interpolation for functions of two variables; tables of double entry. So far it has been assumed that the functions under consideration were functions of one variable and that the tables were tables of single entry. As a matter of fact, most of the functions whose tables are given in this

book are functions of two variables and the tables are tables of double entry. For example, the compound amount of one dollar is given by

$$S_n = (1 + i)^n,$$

which is a function of the rate i and the time n .

Problems in interpolation for functions of two variables may be solved easily by two successive single interpolations. Suppose it is required to find the compound amount of \$1 for 10.25 years at $2\frac{1}{4}\%$ from the following table:

n	2%	$2\frac{1}{2}\%$	3%
10	1.218 9944	1.280 0845	1.343 9164
11	1.243 3743	1.312 0867	1.384 2339
12	1.268 2418	1.344 8888	1.425 7609

The first line of the table may be considered as a table of single entry. Using first differences only,

$$(1.0225)^{10} = 1.2189944 + \frac{1}{2}(.0610901) = 1.2495394. \quad (1)$$

In the same way

$$(1.0225)^{11} = 1.2433743 + \frac{1}{2}(.0687124) = 1.2777305. \quad (2)$$

The results given by (1) and (2) may be looked upon as a table of single entry according to the time, since the rate .0225 is fixed. We have therefore

$$D_1 = .0281911$$

$$\text{and} \quad (1.0225)^{10.25} = 1.2495394 + \frac{1}{4}(.0281911) = 1.2565872. \quad (3)$$

By a longer, though not more difficult, computation, the values of $(1.0225)^{10}$, $(1.0225)^{11}$, and $(1.0225)^{12}$ might have been found by means of first and second differences. The work is as follows:

$$(1.0225)^{10} = 1.2189944 + \frac{1}{2}(.0610901) - \frac{1}{8}(.0027418) = 1.2491967. \quad (4)$$

$$(1.0225)^{11} = 1.2433743 + \frac{1}{2}(.0687124) - \frac{1}{8}(.0034348) = 1.2773011. \quad (5)$$

$$(1.0225)^{12} = 1.2682418 + \frac{1}{2}(.0766470) - \frac{1}{8}(.0042251) = 1.3060372. \quad (6)$$

Using (4), (5), and (6) we find

$$(1.0225)^{10.25} = 1.2491967 + \frac{1}{4}(.0281044) - \frac{3}{8}(.0006317) = 1.2561637. \quad (7)$$

The final result as given by (7) is much more accurate than that given by (3).

EXAMPLES

1. By means of Table IV find $(1.0425)^{-45.5}$, using first differences alone.
2. Solve Example 1, using second differences.
3. From Table VII find the value of $\frac{.0425}{1 - (1.0425)^{-44.25}}$. Solve by using first differences alone and also by using second differences.

CHAPTER IV

LOGARITHMS

25. Definitions and preliminary notions. In elementary algebra the power a^x is defined for positive, negative, zero, and fractional values of x . If

$$a^x = y, \quad (A)$$

the number a is called a *base*, y is a number, and x is called the *logarithm* of y to the base a . The logarithm of a number to a given base is defined to be *the exponent which indicates the power to which the base must be raised to produce the given number*. We write

$$x = \log_a y, \quad (B)$$

so that equations (A) and (B) are only different ways of saying the same thing.

It can be shown that when $a > 0$ and $a \neq 1$, a single real positive value of y exists for every real value of x , even though x be an incommensurable number, like $\sqrt{2}$; and, conversely, for every real positive value of y a single real value, which may be positive or negative, exists for x . In other words, given a positive base a , different from unity, corresponding to every real logarithm there exists a single positive number; and, conversely, for every real positive number there exists a single real logarithm.

Negative numbers have no real logarithms, for the base a is always taken as a positive number, and the negative numbers that might otherwise exist, as in an equation like $10^{\frac{1}{2}} = y$, have been deliberately excluded from the definition of the expression a^x .

A set of logarithms x of all numbers y , corresponding to a given base a , is called a *system of logarithms*. Two systems are in use: the common, or Briggsian, system, which is used in practical

computations, and the hyperbolic, or Napierian, system, which is always used in analytical work. The base of the common system is 10, while the base of the Napierian system is the number $e = 2.71828+$, which was found in § 12.

The following theorems hold for all systems of logarithms:

I. *The logarithm of 1 is zero.*

For, suppose $\log_a 1 = x$. This fact may be written $a^x = 1$. But when a is different from unity, the only power of a that can be equal to 1 is a^0 .

$$\therefore \log_a 1 = 0.$$

II. *The logarithm of the base is 1.*

For, let $\log_a a = x$. This statement written in the form of equation (A) is

$$a^x = a,$$

and $x=1$ is the only real value of x which will satisfy this equation.

III. *The logarithm of a product is equal to the sum of the logarithms of the factors.*

For, let x and y be the factors of a product and m and n their respective logarithms. Then

$$\log_a x = m \quad \text{and} \quad \log_a y = n,$$

$$\text{or} \quad x = a^m \quad \text{and} \quad y = a^n.$$

$$\text{Consequently,} \quad xy = a^{m+n}$$

$$\text{and} \quad \log_a xy = m + n = \log_a x + \log_a y.$$

IV. *The logarithm of a power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

Let x be the number, n the exponent of the power, and l the logarithm of x . Then we have

$$l = \log_a x, \quad \text{or} \quad a^l = x;$$

$$\text{whence} \quad x^n = a^{nl},$$

$$\text{and, consequently,} \quad \log_a x^n = nl = n \log_a x.$$

COROLLARY. *The logarithm of a quotient is equal to the logarithm of the dividend diminished by the logarithm of the divisor.*

For the quotient $\frac{x}{y}$ may be written xy^{-1} . We have then

$$\log_a \frac{x}{y} = \log_a xy^{-1} = \log_a x + \log_a y^{-1} = \log_a x - \log_a y.$$

26. Characteristic and mantissa. A logarithm which is not an integer, like every other fractional number, is the sum of two parts, one a whole number and the other a proper fraction which may be expressed exactly or approximately as a decimal. Moreover, any negative number may be written in such form that its decimal part is positive. For example,

$$-.6989700 = -1 + .3010300,$$

which may be written in the form $\bar{1}.3010300$. Similarly,

$$-1.2351423 = \bar{2}.7648577.$$

In every case the change is accomplished by adding 1 to the decimal part and subtracting it from the integral part.

When a logarithm is written in such form that the decimal part is positive, the decimal part is called the *mantissa* and the integral part the *characteristic*.

THEOREM. *In the system of common, or Briggsian, logarithms the mantissa is the same for all numbers having the same sequence of figures.*

PROOF. Moving the decimal point to the right or to the left is equivalent to multiplying or dividing the number by an integral power of 10. Since 10 is the base, the logarithm of any power of 10 is an integer. By Theorems III and IV, multiplying a number by an integral power of 10 will change the logarithm by an integer and will not therefore affect the mantissa.

From this point the word logarithm shall be understood to mean common logarithm, i.e. a logarithm with base 10.

The mantissas of logarithms are obtained from a table of logarithms prepared in advance, while the characteristic, which depends upon the decimal point, is determined by inspection.

The characteristic is easily obtained from a table of powers of 10, such as

$$10^{-4} = .0001$$

$$10^{-3} = .001$$

$$10^{-2} = .01$$

$$10^{-1} = .1$$

$$10^0 = 1$$

$$10^1 = 10$$

$$10^2 = 100$$

$$10^3 = 1000$$

$$10^4 = 10000.$$

From this table we may construct a brief table of logarithms, which may be written in the following form:

NUMBER	.0001	.001	.01	.1	1	10	100	1000	10000
LOGARITHM	- 4	- 3	- 2	- 1	0	1	2	3	4

From the last table it is easy to see that the characteristic of any number which lies between 100 and 1000, i.e. which has three figures to the left of the decimal point, is 2. Similarly, if the number has four figures to the left of the decimal point the characteristic is 3.

An examination of that part of the number scale which lies to the left of 1 makes it clear, for example, that a number which lies between .001 and .01 must have a logarithm greater than - 3 and less than - 2. In other words, the logarithm is - 3 *plus* a mantissa. Any number that lies between .001 and .01 must have two zeros and no more between the decimal point and the first significant figure.

From these considerations we obtain the

RULE FOR FINDING THE CHARACTERISTIC: *If there are figures to the left of the decimal point, the characteristic is positive and is one less than the number of such figures; if there are no significant figures to the left of the decimal point, the characteristic is negative and numerically one more than the number of zeros between the decimal point and the first significant figure to the right.*

Turning this rule about, we obtain the

RULE FOR POINTING A NUMBER WHEN THE CHARACTERISTIC OF ITS LOGARITHM IS GIVEN: *If the characteristic is positive or zero, the number of figures to the left of the decimal point is one more than the number of units in the characteristic; if the characteristic is negative, the number is a pure decimal and the number of zeros between the point and the first significant figure is one less than the numerical value of the characteristic.*

Negative characteristics are usually written with the negative sign above the characteristic instead of in front of it. Thus we may write $\bar{2}.3010300$. A still better method is to express the characteristic as the difference between a positive integer and 10, writing the -10 at the end. Thus,

$$\bar{2}.3010300 = 8.3010300 - 10.$$

To facilitate division of logarithms we may increase both the positive and the negative parts of the characteristic by such a number that the quotient of the negative part by the divisor is -10 . For example, if division of $\bar{2}.3010300$ by 7 is required, we proceed as follows:

$$(8.3010300 - 10) \div 7 = (68.3010300 - 70) \div 7 = 9.7572900 - 10.$$

27. Computation by means of logarithms. One of the chief values of logarithms lies in the fact that by means of them many computations which otherwise would be difficult, or even impossible, are easily and quickly performed. For any given computation that table of logarithms should be chosen which will yield the desired degree of accuracy. For financial problems seven-place or eight-place tables are required. For the logarithmic computations required in this book it will be assumed that a seven-place table is at hand. From a seven-place table *the logarithm of any number expressed by a sequence of five significant figures can be read off directly from the table.*

The methods used in computation are best illustrated by an example.

Let it be required to find the value of

$$M = \frac{\sqrt{349.236} \times \sqrt[3]{.03452318}}{2378.41}.$$

Solution. By the theorems on logarithms (§ 25),

$$\log M = \frac{1}{2} \log 349.236 + \frac{1}{3} \log .03452318 - \log 2378.41.$$

To find $\log 349.236$ we may look for the mantissa of $\log 34923.6$, since the position of the decimal point does not affect the mantissa, and the logarithm of any number expressed by a sequence of five digits can be read directly from the table. From the table,

$$\text{Mant. } \log 34923. = .5431115.$$

To find the correction for .6 we use the interpolation method of § 17 and note that .6 is six tenths of the interval between the consecutive numbers 34923 and 34924. The tabular difference is .0000125. Consequently the correction is .0000075. Omitting decimals, we may say the tabular difference is 125 and the correction 75. Adding this correction and writing in the characteristic determined by the rule, we have

$$\log 349.236 = 2.5431190.$$

Similarly,

$$\log .034523 = 8.5381085 - 10.$$

The tabular difference is 126. The correction must be made for .18 of one interval. From the table of proportional parts the correction for .1 is 13 and the correction for .08 is 10. Adding the total correction and writing the characteristic in a form adapted for convenient division by 7, we have

$$\log .03452318 = 68.5381108 - 70.$$

In a similar way we find $\log 2378.41 = 3.3762867$.

The work may then be arranged as follows:

$$\begin{array}{rcl} \frac{1}{2} \log 349.236 & = & 1.2715595 \\ \frac{1}{3} \log .03452318 & = & \frac{9.7911587 - 10}{11.0627182 - 10} \\ \log 2378.41 & = & 3.3762867 \\ \log M & = & \frac{ 7.6864315 - 10}{} \end{array}$$

To find M we must use the table in a reverse way, using the method of § 18 for first differences. The tabular mantissa nearest .6864315, but smaller, is .6864307, which differs from the given mantissa by .0000008. The tabular difference is .0000089. The number corresponding to the mantissa .6864307 is 48577, and to find the number corresponding to the mantissa .6864315 we must add $\frac{8}{89}$ or .09, giving us the sequence of figures 4857709. By the rule for pointing,

$$\log M = .004857709.$$

EXAMPLES

Find by logarithms the values of the expressions in Examples 1-3 and state the degree of accuracy in each case.

$$1. \frac{\sqrt{293} \cdot (.024)^5}{2345} \quad 2. \frac{\sqrt[5]{2356} \cdot \sqrt[3]{.02426}}{(.02345)^8} \quad 3. \left(\frac{(35.62^2) \cdot (.004993)^5}{(29342)^{\frac{1}{2}}} \right).$$

Find by logarithms the values of the expressions in Examples 4-6.

$$4. \frac{\sqrt{235.8926} \times \sqrt[3]{.043768}}{293.8426} \quad 5. \sqrt{\frac{.0832714 \times 938.42}{.0763824 \times 1000.89}} \quad 6. \sqrt[23]{.00069382}.$$

7. Find an approximate value for the twentieth term of the geometric progression

$$1 + \frac{2}{3} + \frac{4}{9} + \cdots.$$

8. Find the simple interest on \$389427.31 for 287 days at 5.7%.

HINT. 287 days = $\frac{2}{3}\frac{2}{3}\frac{7}{10}$ yr.

9. Find the value of $(1.05)^{\frac{1}{12}}$.

10. Find the value of $(1.05)^{-17}$.

11. Find the volume of a right circular cone whose base is 6.37 feet in diameter and whose altitude is 16 ft. 8 in., assuming the formula $\frac{1}{3}\pi r^2 h$ for volume.

12. Find the value of a solid copper sphere 15 inches in diameter at 10.75 cents per pound, knowing that the specific gravity of copper is 8.838 and that a cubic foot of water weighs 62.425 pounds.

28. Exponential equations. In most of the equations that occur in elementary algebra the exponents are known and the number affected by the exponent is unknown; for example,

$$x^2 + 5x + 6 = 0.$$

It was noted in § 4 that when the number of terms of a geometric progression is unknown, it is ordinarily impossible to find it by elementary means, since it occurs as an exponent. For example, if the first term, the last term, and the ratio of a geometric progression are 5, 98415, and 3 respectively, equation (C) of § 3 gives

$$98415 = 5(3)^{n-1},$$

where n , the unknown, is a part of the exponent. This equation is of the same type as the equation

$$a^x = y$$

and may be treated in the same way.

An equation in which the unknown number occurs as an exponent is called an *exponential equation*.

The simplest form of an exponential equation is given by the equation

$$a^x = b. \quad (1)$$

This equation can be solved readily by taking logarithms of both sides. By the theorem for the logarithm of a power;

$$x \log a = \log b;$$

whence
$$x = \frac{\log b}{\log a}. \quad (2)$$

It must be carefully noted that the expression on the right side of equation (2) is the quotient of two logarithms and not the logarithm of a quotient, so that it cannot be written as $\log b - \log a$.

No general rule for the solution of exponential equations can be given, but the solutions of two of the following examples will serve to illustrate the method to be used in the simpler cases.

EXAMPLES

1. Solve the equation $2^x = 3$.

Solution. Taking logarithms of both sides, we find

$$x \log 2 = \log 3,$$

and solving this linear equation for x , $x = \frac{\log 3}{\log 2}$;

or, using a four-place table,
$$x = \frac{.4771}{.3010}.$$

By actual division (which may of course be performed by logarithms)

$$x = 1.585 +.$$

2. Solve $37^{x+2} \times 15^{x-8} = 14^{2x+1}$ for x .

Solution. Taking logarithms of both sides, we have

$$(x+2)\log 37 + (x-3)\log 15 = (2x+1)\log 14,$$

or $x \log 37 + x \log 15 - 2x \log 14 = -2 \log 37 + 3 \log 15 + \log 14.$

$$\begin{aligned} \text{Solving for } x, \text{ we find } x &= \frac{-2 \log 37 + 3 \log 15 + \log 14}{\log 37 + \log 15 - 2 \log 14} \\ &= \frac{-3.1364 + 3.5283 + 1.1461}{1.5682 + 1.1761 - 2.2922} \\ &= \frac{1.5380}{.4521} \\ &= 3.402 +. \end{aligned}$$

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3. Find x from the equation $30^x = 3000$.
4. Given $\frac{30^x \cdot 42^x}{17^x} = 29$; find x .
5. Given $2^x \cdot 3^y = 2000$ and $3y = 5x$; find x and y .
6. If $(1.04)^n = 1.80094$, what is n ?
7. If $\frac{1}{(1.05)^n} = 0.783526$, what is n ?
8. Find n from the equation

$$\frac{(1.04)^n - 1}{.04} = 12.06107.$$

9. Find n from the equation

$$\frac{1 - (1.04)^{-n}}{.04} = 8.110896.$$

10. Given $l = ar^{n-1}$; find n .

11. Given the two equations $l = ar^{n-1}$,

$$s = \frac{rl - a}{r - 1};$$

find n in terms of a , l , and s .

12. Find n from the equation $s = P(1 + i)^n$.

13. Find n from the equation $\frac{(1 + i)^n - 1}{i} = S$.

29. The transformation of logarithms of one system into logarithms of another system. In theoretical work the so-called Napierian logarithms, whose base is the number $e = 2.71828 +$, are almost invariably used, while for numerical computation the common logarithms, whose base is 10, are used. It is therefore frequently necessary to transform a Napierian logarithm into a common logarithm, and vice versa. This transformation is a special case of a more general problem, viz. *to express a logarithm with a given base a in terms of the logarithm with another base b .*

Let N be the number whose Napierian logarithm is sought and l its logarithm to the base 10. Then, by hypothesis,

$$\log_{10} N = l. \tag{1}$$

Changing equation (1) to the exponential form,

$$N = 10^l, \tag{2}$$

Taking logarithms to the base e of both sides of equation (2),

$$\log_e N = l \log_e 10. \quad (3)$$

If l be replaced by its value as given by equation (1), the result is

$$\log_e N = \log_e 10 \cdot \log_{10} N. \quad (4)$$

To find the common logarithm when the Napierian logarithm is given, solve (4) for $\log_{10} N$ and obtain

$$\log_{10} N = \frac{1}{\log_e 10} \cdot \log_e N. \quad (5)$$

The factor $\frac{1}{\log_e 10}$, by means of which Napierian logarithms are converted into common logarithms, is called the *modulus of the common system* and is usually denoted by the symbol M , so that by definition

$$M = \frac{1}{\log_e 10}. \quad (6)$$

The number M and its reciprocal,

$$\frac{1}{M} = \log_e 10, \quad (7)$$

have been carefully computed to a large number of decimal places. Their values to the sixth decimal place are

$$M = .434294$$

and
$$\frac{1}{M} = 2.302585.$$

If the values of M and $\frac{1}{M}$ be introduced into equations (4) and (5), these equations become

$$\log_e N = 2.302585 \log_{10} N \quad (4')$$

and
$$\log_{10} N = .434294 \log_e N. \quad (5')$$

From equations (4') and (5') the following rules for the transformation of logarithms of the one system into logarithms of the other system are easily obtained.

To find the Napierian logarithm when the common logarithm is given, multiply the common logarithm by 2.302585.

To find the common logarithm when the Napierian logarithm is given, multiply the Napierian logarithm by .434294.

EXAMPLES

1. The common logarithm of 2 is approximately .3010300. What is its Napierian logarithm?

2. By means of a table of common logarithms and one of the formulas of the present section find the Napierian logarithm of 258.

3. Prove that $\log_{10} e = \frac{1}{\log_e 10}$ and that, in general, $\log_b a = \frac{1}{\log_a b}$. Why do we write $M = \frac{1}{\log_e 10}$ rather than $M = \log_{10} e$?

4. Compute the common logarithm of 1.1 to five places by starting from the power series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

5. Compute the common logarithm of 3 to five decimal places by making $M=3$ and $N=1$ in the series

$$\log \frac{M}{N} = 2 \left[\frac{M-N}{M+N} + \frac{1}{3} \left(\frac{M-N}{M+N} \right)^3 + \frac{1}{5} \left(\frac{M-N}{M+N} \right)^5 + \cdots \right].$$

PART II. INTEREST AND ANNUITIES

CHAPTER V

INTEREST AND DISCOUNT

30. Simple interest. Broadly speaking, *interest* is defined as the income on capital profitably invested. In a more restricted sense it is the sum paid for the use of money that is loaned. The sum invested, whether it be in the form of money loaned or capital invested in a business enterprise, is called the *principal*. Interest is computed as so many hundredths of the principal earned in a given unit of time. The unit of time is almost invariably one year. The *rate of interest** is that fraction which expresses the ratio of the interest earned in the unit of time to the principal. It may also be defined as the fraction which expresses the amount paid for the use of a unit of principal for a unit of time. *Simple interest* is interest which is proportional to the time. The sum of the principal and the interest is called the *amount*.

If we use the symbols P , I , i , n , and A , to denote the principal, the interest, the rate, the time, and the amount, respectively, the fundamental formulas for simple interest are

$$I = Pni \tag{1}$$

and

$$A = P + I,$$

or

$$A = P(1 + ni). \tag{2}$$

All problems that can arise in simple interest may be solved by means of formulas (1) and (2).

* Care must be taken to note that the rate of interest is always a fraction. We use the expression 6% meaning that the rate is .06.

ILLUSTRATIVE EXAMPLE. Let it be required to find the principal which will yield \$500 in interest at 5% in 2 years and 6 months.

By formula (1)

$$500 = P \times .05 \times 2.5;$$

whence

$$\begin{aligned} P &= \frac{500}{.05 \times 2.5} \\ &= \$4000. \end{aligned}$$

31. Ordinary and exact interest. In practice the greater number of problems that occur in simple interest are problems involving short periods of time. For reasons of convenience it is customary in ordinary transactions to compute the interest on the basis of 360 instead of 365 days in a year. Interest computed on the basis of 360 days in a year is called *ordinary interest*. Interest computed on the basis of 365 days in a year is called *exact interest*.

To find the relation between the two kinds of interest, let I denote ordinary and I' exact interest. If d denote the time expressed in days,

$$I = Pi \frac{d}{360}, \quad (1)$$

and

$$I' = Pi \frac{d}{365}. \quad (2)$$

To find the relation between I and I' , divide equation (1) by equation (2) and obtain

$$\frac{I}{I'} = \frac{365}{360} = 1 + \frac{1}{72};$$

whence

$$I = I' + \frac{I'}{72}. \quad (3)$$

Translating this formula into words, we obtain as the rule for deriving ordinary from exact interest: *Ordinary interest is equal to the exact interest increased by one seventy-second of itself.*

Solving formula (3) for I' ,

$$I' = \frac{72}{73} I,$$

or

$$I' = I - \frac{I}{73}. \quad (4)$$

From formula (4) we have the rule for deriving exact interest from ordinary interest: *Exact interest is equal to ordinary interest diminished by one seventy-third of itself.*

ILLUSTRATIVE EXAMPLE. Find the exact interest on \$5278.17 from May 11, 1911, to June 25, 1911, at 6%.

The time is 45 days, which is $\frac{1}{8}$ of 360 days. Hence the ordinary interest is

$$\$5278.17 \times .06 \times \frac{1}{8} = \$39.586.$$

The exact interest is then

$$\$39.586 - \frac{\$39.586}{73} = \$39.044.$$

32. Computation of simple interest. The computation of simple interest, however the work may be arranged, depends upon one of the formulas

$$I = Pni,$$

$$I = Pi \frac{d}{360},$$

and

$$I' = Pi \frac{d}{365}.$$

If a large amount of such work is to be done, an interest table is almost a necessity. Elaborate tables giving the interest on all sums up to \$1000, for times less than a year and at rates from 3 to 8 or 9%, are published for the use of bankers and others who wish to obtain results quickly. For ordinary purposes a much simpler table is sufficient.

From Table II we may find the exact interest at 5%, on all amounts from \$1 to \$100,000, for times up to 365 days, with a small amount of labor. A table like Table I, which gives the number of the day of the year, counting forward from January 1, is a useful aid in finding the time between two dates.

Since simple interest is proportional to the rate, we may find the interest at a rate differing from 5% by increasing or diminishing the interest at 5% by the proper number of fifths of itself. Thus, to find the interest at 6% we add to the interest at 5% one fifth of itself.

EXAMPLES

1. Compute the exact interest on \$43,729.29 from April 1 to May 18 at 5%.

The time is 47 days. From the table we find

Interest on	\$10000	=	\$257.534
Interest on	3000	=	19.315
Interest on	700	=	4.507
Interest on	20	=	.129
Interest on	9	=	.058
Interest on	.29	=	.001
Interest on	\$43729.29	=	\$281.544

The exact interest on the same principal and for the same time at 6% would be

$$\$281.54 + \$56.31 = \$337.85.$$

2. Find the simple interest at 6% on \$23,738.42 from February 1, 1908, to March 30, 1908.

3. Find the simple interest at 6% on \$38,472.27 from March 21 to April 11 of the same year.

4. Find the simple interest on \$17,927.34 from January 3, 1915, to February 17, 1916, at 7%.

5. Find by multiplication the ordinary simple interest on

- (a) \$17,927.34 for 13 days at 7%.
- (b) \$3271.25 for 90 days at 7%.
- (c) \$934.29 for 30 days at 7%.
- (d) \$839.27 for 6 months and 13 days at 6%.
- (e) \$932.19 for 1 year, 2 months, and 16 days at 5%.

6. Find the simple interest on

- (a) \$379.27 from January 19, 1920, to September 21, 1920, at 6%.
- (b) \$289.14 from May 3, 1919, to July 2, 1920, at 7%.
- (c) \$329.17 from March 13, 1920, to June 7, 1922, at 7%.
- (d) \$842.14 from May 7, 1919, to July 3, 1922, at 7%.
- (e) \$3427.18 from August 13, 1916, to June 3, 1918, at 6%.

7. A man made an emergency loan of \$75 for 30 days, agreeing to pay 7%. At the end of 10 days he paid the loan together with the bank's minimum charge of 25 cents. What rate of interest did he pay?

8. Construct the graph of the function $100 \times .05 t$ where t is expressed in years and show how it may be used as an interest table for full-year periods.

9. Construct the graph of the simple-interest function $100 \times .05 t$ where t is expressed in months and demonstrate its use.

10. Find the exact interest for Examples (a)–(e) of 6 by using Table I to find the time and use the results to check answers already obtained for these examples.

11. Find the exact interest for Examples (a)–(e) of 6 by means of Tables I and II and use results to check answers already obtained.

33. Compound interest. Interest, whether it be in the form of money paid for the use of money that has been loaned, or whether it be in the form of dividends on capital profitably invested, should be paid promptly when due. When paid, it may itself be put at interest, so that it is, in effect, added to the principal to form a new principal. This process may go on to any length. In case the interest is not paid by the borrower, it may still be considered as added to the principal at stated intervals.

When the interest is added to the principal at stated intervals throughout a given time, the difference between the original principal and the sum due at the end of the time is called *compound interest*. The total amount due is called the *compound amount*. The interest is usually added to the principal at the end of each successive year or half year or quarter year. We say the interest is "*compounded*, or *converted into principal*, annually, semiannually, or quarterly, or m times a year," or that interest is "*payable* annually, or semiannually, or quarterly, or m times a year." The time elapsing between two successive conversions of interest into principal is called the *conversion interval*.

In all theoretical work in compound interest the principal is assumed to be a unit of money, without regard to the particular currency to which the unit belongs. Thus, we speak of the "compound amount on 1" for a given time, and the formulas obtained will apply equally well whether the unit be a dollar, a pound, a mark, or a franc. The compound amount on any principal is then found by multiplying the given principal by the compound amount on a unit principal, since the amount is proportional to the principal employed.

PROBLEM. *To find the compound amount on 1 for n years with interest at rate i payable annually.*

At the end of the first year the principal becomes $1+i$, and at the end of the second it is

$$1+i+i(1+i)=(1+i)^2;$$

at the end of the third year it is

$$(1+i)^2+i(1+i)^2=(1+i)^3;$$

and so on. At the end of the n th year it will be the n th term of the geometric progression

$$1+i, (1+i)^2, (1+i)^3, \dots,$$

or, denoting the amount by s ,

$$s=(1+i)^n. \quad (1)$$

The compound amount of any principal, P , is

$$S=P(1+i)^n. \quad (2)$$

It should be noted that, while the compound amount is proportional to the principal, it is *not* proportional to the rate of interest or to the time, as is simple interest.

ILLUSTRATIVE EXAMPLE. Find the compound amount on \$175.50 for 3 years at 5%.

Solution. The compound amount of 1 for 3 years is $(1.05)^3$, or 1.157625. Multiplying this amount by 175.50, we find that the compound amount of \$175.50 is \$203.16.

PROBLEM. *To find the compound amount on 1 compounded m times a year for n years.*

For reasons that will appear later, we use the letter j for the rate instead of i . We assume the rate for the m th part of a year to be the m th part of the rate for a year. The rate for one interval of time will then be $\frac{j}{m}$. The problem is exactly like the first problem, except that the rate per interval of time is $\frac{j}{m}$ and the number of intervals is mn .

The required expression is, then,

$$s = \left(1 + \frac{j}{m}\right)^{mn}. \quad (3)$$

Similarly, the compound amount of any principal, P , with interest convertible m times a year at rate j is

$$S = P \left(1 + \frac{j}{m}\right)^{mn}. \quad (4)$$

The foregoing definitions and formulas apply only when the time contains an exact number of conversion intervals. Consequently, compound interest can have no meaning when the time does not contain an exact number of conversion intervals. To obviate this difficulty *we assume that formulas (1) and (3), viz.*

$$s = (1+i)^n \text{ and } s = \left(1 + \frac{j}{m}\right)^{mn},$$

hold for all values of n when n is expressed in years.

This assumption is of course equivalent to a definition. According to the definition the compound amount of 1 for 1 year and 6 months at 5%, convertible annually, would be

$$(1.05)^{\frac{3}{2}} = \sqrt{(1.05)^3} = 1.07593.$$

Similarly, the compound amount of 1 for 6 months, at rate .05, convertible annually, would be $(1.05)^{\frac{1}{2}}$, which is *less* than the amount, at simple interest, of the same principal at the same rate and for the same time.

However, in practice the compound amount is frequently computed for the integral number of years, and the amount of simple interest is then figured on this amount for the fractional part of the year remaining. For example, the compound amount on 1 for 5 years and 6 months at 5%, payable annually, would be given as

$$(1.05)^5 (1.025) = 1.30189.$$

The expression $(1+i)^n$, or its equivalent $\left(1 + \frac{j}{m}\right)^{mn}$, is sometimes called an *accumulation factor* (see § 38).

It is important to note that when interest is payable oftener than once a year, the year is replaced by the interval and the rate is $\frac{j}{m}$ per interval. The rate j is then called the nominal rate. (See § 35.) For example, if \$500 is put at interest at 6% nominal, convertible semiannually, for 3 years, the problem is identical with that of finding the compound amount of \$500 for 6 years at 3%, convertible annually.

EXAMPLES

1. Find by actual multiplication the compound amounts on

- (a) \$250 for 3 years at 4%.
- (b) \$300 for 3 years at 5%.
- (c) \$289.17 for 3 years at $5\frac{1}{2}\%$.
- (d) \$389.27 for 4 years at 4%.
- (e) \$247.21 for 4 years at 5%.

2. Find by actual multiplication the compound amounts on

- (a) \$250 for 2 years compounded semiannually at 2% per conversion interval.
- (b) \$250 for 1 year compounded quarterly at 1% per conversion interval.
- (c) \$250 for 2 years compounded semiannually at 3% per conversion interval.
- (d) \$300 for 2 years and 6 months compounded semiannually at 2% per conversion interval.
- (e) \$500 for 1 year and 3 months compounded quarterly at 4% per annum.

3. A boy deposits \$10 January 1, April 1, July 1, and October 1 for 2 years in a savings bank that pays 4% compounded quarterly. How much money will he have at the end of the time?

34. The rate of interest and the time. If all but one of the elements occurring in formulas (2) and (4) of the preceding section are given, that one may be found. The problems occurring most frequently in practice are those in which the rate of interest or the time is the unknown element. The problem in which the principal is the unknown element is treated in a later section.

PROBLEM. *To find the rate of interest when the principal, the compound amount, and the time are given.*

(a) If interest is payable annually, the problem is to find i from the formula (2), § 33, viz.

$$S = P(1 + i)^n.$$

Dividing through by P and extracting the n th root of both sides of the resulting equation, we find

$$1 + i = \sqrt[n]{\frac{S}{P}},$$

and, finally,
$$i = \sqrt[n]{\frac{S}{P}} - 1. \quad (1)$$

(b) If interest is convertible m times a year, the relation connecting P , S , n , j , and m is, by (4), § 33,

$$S = P \left(1 + \frac{j}{m} \right)^{mn}.$$

Solving this equation for j , we find

$$j = m \left(\sqrt[mn]{\frac{S}{P}} - 1 \right). \quad (2)$$

The two expressions $\sqrt[n]{\frac{S}{P}}$ and $\sqrt[mn]{\frac{S}{P}}$ are easily found by logarithms when the values of S , P , n , and m are known.

ILLUSTRATIVE EXAMPLE. In 5 years \$1000, placed at interest convertible quarterly, amounts to \$1346.85. What is the rate?

Solution. By formula (2)
$$j = 4 \left(\sqrt[20]{\frac{1346.85}{1000}} - 1 \right).$$

Since
$$\log \sqrt[20]{\frac{1346.85}{1000}} = \frac{1}{20} \times .1293192$$

$$= .0064660,$$

we must have
$$\sqrt[20]{\frac{1346.85}{1000}} - 1 = 1.0150 - 1$$

$$= .0150,$$

and, finally,
$$j = .06.$$

PROBLEM. *To find the time, when the principal, the amount, and the rate of interest are given.*

Again, it is convenient to divide the solution into two parts, according as interest is convertible one or m times a year.

(a) When interest is payable annually, we have, as before,

$$S = P(1+i)^n,$$

and the problem is reduced to the solution of this exponential equation for n .

Taking logarithms of both sides, we obtain

$$\log S = \log P + n \log (1+i),$$

so that n is easily found to be given by the formula

$$n = \frac{\log S - \log P}{\log (1+i)}. \quad (3)$$

(b) If the interest is convertible m times a year, we find in a similar way, from equation (4) of § 33,

$$n = \frac{\log S - \log P}{m \log \left(1 + \frac{i}{m}\right)}. \quad (4)$$

The problem of finding the conversion interval when the other elements are given can be solved only by the method of interpolation, since m occurs both as an exponent and as a base. It may usually be found by trial, since it is nearly always one of the three integers 2, 4, or 12.

NOTE. The student should in every case start with formula (2) or formula (4) of § 33 and not attempt to memorize formulas. Or, better still, he should substitute values in (2) or (4), as the case may be, and solve the resulting numerical equation.

EXAMPLES

1. In what time would \$236.41 amount to \$421.32, with interest at 6%, convertible quarterly?

$$\begin{aligned} \text{By formula (4), § 34, } n &= \frac{\log 421.32 - \log 236.41}{4 \log \left(1 + \frac{.06}{4}\right)} \\ &= \frac{0.2509463}{0.0258640} \\ &= 9.70. \end{aligned}$$

2. In what time will \$2394.62 amount to \$10,000 when placed at compound interest at 5%, convertible semiannually?

3. In 15 years a merchant's capital of \$20,000 increased to \$37,423. What was the annual rate of increase for the period?

4. A philanthropist gave \$1,000,000 to a university with instructions that half the income should be converted into principal each year until the total amount should reach \$20,000,000, when the whole income should be available for use. How long will it be before the whole income will be available if the bequest can be made to yield 4%?

5. Solve Problem 4 on the assumption that the principal can be made to yield 3%.

6. In 1890 the population of Madison, Wisconsin, was 13,426 and in 1920 it was 38,378. What was the average annual rate of increase during the 30 years? What was the average rate of increase per decade?

7. If the rate of increase of population in Example 6 should continue, how long will it take the city to reach a population of 100,000?

35. Nominal and effective rates. The two formulas

$$s = (1 + i)^n \quad \text{and} \quad s = \left(1 + \frac{j}{m}\right)^{mn}$$

both give the compound amount on one dollar for a given number of conversion intervals at a given rate per interval. In the first, the number of intervals is n and the rate per interval is i . In the second, the number of intervals is mn and the rate per interval is $\frac{j}{m}$. If the time is one year the formulas become

$$s = 1 + i \quad \text{and} \quad s = \left(1 + \frac{j}{m}\right)^m.$$

Wherever it occurs, the integer m denotes the number of conversion intervals in a year. The number j , which is *the product of the rate per interval by the number of intervals in one year*, is called the *nominal rate*. It is the rate named in the contract and indicates what the return on one dollar would be if no interest received during the year were to be reinvested until the end of the year.

To illustrate, suppose \$100 is invested with interest at 6%, payable semiannually. At the end of the first half year \$3 will

be due. If this be collected and laid away until the end of the second half year, when another \$3 will be due, the investor will have \$6 as interest for the year.

But no prudent business man would conduct his affairs in the way indicated in the illustration. Rather he would reinvest the \$3 received at the end of the first period as soon as he received it. In this way he would have as a total return on his \$100 invested for 1 year the amount of \$6.09. The nominal rate is 6%, but the total return is greater than \$6. When interest is convertible *oftener* than once a year the *total return on one dollar for one year* is called the *effective rate*. The effective rate is *greater* than the nominal rate. In the example just given the effective rate is .0609, while the nominal rate is .06.

PROBLEM. *To express the effective rate in terms of the nominal rate, and vice versa.*

Let i be the effective rate and j the nominal rate convertible m times a year. By formula (3) of § 33 the compound amount of 1 for one year, when the interest is convertible m times a year at rate $\frac{j}{m}$ per conversion interval, is

$$\left(1 + \frac{j}{m}\right)^m.$$

The total return on 1 for one year, i.e. the effective rate of interest, is therefore

$$i = \left(1 + \frac{j}{m}\right)^m - 1. \quad (1)$$

We obtain a very useful form of the relation between effective and nominal rates by simply transposing the 1; thus,

$$1 + i = \left(1 + \frac{j}{m}\right)^m. \quad (2)$$

To obtain j in terms of i we have only to solve (1) or (2) for j . Extracting the m th root of both sides of (2), transposing 1, and, finally, multiplying through by m , we find

$$j = m \left\{ (1 + i)^{\frac{1}{m}} - 1 \right\}. \quad (3)$$

For reasons that will appear later m is frequently replaced by p , and we further emphasize the fact that the nominal rate j depends upon p as well as i by writing $j_{(p)}$ instead of j . We have then by definition

$$j_{(p)} = p \{ (1 + i)^{\frac{1}{p}} - 1 \}. \quad (3')$$

The quantity $j_{(p)}$ is an important factor in many computations that will occur later on. Its value for $p = 2, 4$, and 12 and for various rates of interest is given in Table IX. Clearly

$$j_{(1)} = i. \quad (4)$$

The computation required in finding the effective rate when the nominal rate is given, or vice versa, is easily accomplished by means of the binomial expansion, since according to § 11 the series will be convergent; or the value of $\left(1 + \frac{j}{m}\right)^m$ may be found by logarithms.

ILLUSTRATIVE EXAMPLE. If the nominal rate is .05, convertible quarterly, the effective rate is

$$\begin{aligned} i &= \left(1 + \frac{.05}{4}\right)^4 - 1 \\ &= (1.0125)^4 - 1 \\ &= 4 \times .0125 + 6 \times (.0125)^2 \\ &\quad + 4 \times (.0125)^3 + (.0125)^4 \\ &= .0509 +. \end{aligned}$$

The result, true to the fourth decimal place, is obtained by neglecting the last two terms in the binomial expansion.

If the effective rate were .05, and we wished to find the nominal rate when interest is convertible quarterly, we would have

$$\begin{aligned} j &= 4 \{ (1 + .05)^{\frac{1}{4}} - 1 \} \\ &= 4 \left\{ 1 + \frac{1}{4} \times .05 + \frac{\frac{1}{4}(\frac{1}{4} - 1)}{1 \cdot 2} (.05)^2 \right. \\ &\quad \left. + \frac{\frac{1}{4}(\frac{1}{4} - 1)(\frac{1}{4} - 2)}{1 \cdot 2 \cdot 3} (.05)^3 + \dots - 1 \right\} \\ &= .0491 -. \end{aligned}$$

Here again the result obtained by taking four terms of the binomial expansion is accurate to the fourth decimal place.

A briefer solution is obtained by finding the value of $(1.05)^{\frac{1}{4}}$ from Table VIII.

EXAMPLES

1. If the rate is 4% nominal, compounded quarterly, what is the effective rate?
2. If the rate is 5% nominal, compounded quarterly, what is the effective rate?
3. If the rate is 6% nominal, compounded monthly, what is the effective rate?
4. What nominal rate compounded semiannually corresponds to 6% effective?
5. What nominal rate convertible quarterly corresponds to 6% effective?
6. Find the effective rate corresponding to the nominal rate 6% compounded quarterly.
7. What is the effective rate on liberty bonds yielding $4\frac{1}{4}\%$, payable semiannually, if the bonds are purchased at par?
8. What is the effective rate corresponding to 6% payable monthly?
9. A money lender is willing to take 6% effective, but requires that the interest be paid quarterly. What rate should be named in the note?
10. What nominal rate convertible quarterly is equivalent to 7% effective?
11. A pawnbroker charges 1% per month on loans. What effective rate does he receive?

36. Computation of compound interest. The formulas

$$\begin{aligned}
 S &= P(1+i)^n \\
 &= P\left(1+\frac{j}{m}\right)^{mn}
 \end{aligned}$$

are admirably adapted to computation by logarithms, but in order to secure accurate results for compound amounts on principals up to \$10,000, one would need at least six-place logarithms. Compound amounts play such an important part in practical affairs that compound-interest tables have been constructed for all the rates in common use and for times up to two hundred years. Such a table gives the amount for unit principal, but the amount for any principal is easily found by multiplication.

Table III gives the amount of 1 at compound interest at rates $1\frac{1}{2}\%$, $1\frac{3}{4}\%$, 2%, $2\frac{1}{2}\%$, 3%, $3\frac{1}{2}\%$, 4%, $4\frac{1}{2}\%$, 5%, and 6% for times up to one hundred years. The solutions of the examples that follow will show

how to use the table most effectively. A little care in noting the number of decimal places that are required will often save a considerable amount of work. For example, if the principal does not exceed \$100, four-place tables will give results accurate to the cent, but if the principal is as large as \$100,000, accurate results cannot be obtained without tables carried out to at least seven places. Eight-place tables would be still better.

If logarithms are not used in performing the multiplication, it is well to begin with the left-hand figure of the multiplier, so that unnecessary figures can be dropped out as the multiplication proceeds.

ILLUSTRATIVE EXAMPLE. Required the compound amount of \$236.41 for 10 years with interest payable annually at 5%.

Solution. Using four decimal places we find from Table III that the compound amount on \$1 for 10 years at 5% is 1.6289. The multiplication may then be arranged as follows:

$$\begin{array}{r}
 \$1.6289 \\
 236.41 \\
 \hline
 325.78 \\
 48.87 \\
 9.77 \\
 .65 \\
 .02 \\
 \hline
 \$385.09
 \end{array}$$

In some cases it may be advisable to use enough decimal places to carry the result to the third decimal place. Practice will enable the student to obtain accurate results with the least amount of work.

By utilizing the properties of the power of a base the range of the table may be considerably extended.

ILLUSTRATIVE EXAMPLE. Let it be required to find the compound amount of \$231.41 for 75 years at 5% with only a 50-year table at hand.

$$\begin{array}{ll}
 \text{Solution. Since } (1.05)^{75} = (1.05)^{50} \times (1.05)^{25} \text{ we have} & \\
 \text{compound amount} & = 236.41 \times (1.05)^{50} \times (1.05)^{25} \\
 & = 236.41 \times 11.46740 \times 3.38635.
 \end{array}$$

The result may be found by abbreviated multiplication or by logarithms.

Where division by large numbers is required, the work may be abbreviated by methods similar to those used in multiplication.

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ILLUSTRATIVE EXAMPLE. What principal put at compound interest for 30 years at $4\frac{1}{2}\%$ effective will amount to \$34,275.27?

Solution. From Table III the compound amount of \$1 for 30 years at $4\frac{1}{2}\%$ effective is 3.7453181. The equation by means of which P is determined is therefore

$$34275.27 = P \times 3.7453181.$$

If a table of logarithms is not at hand, the division required to find P may be arranged as follows:

$$\begin{array}{r} 3.7453181 \overline{) 34275.270(9151.498} \\ \underline{33707\ 863} \\ 567\ 407 \\ \underline{374\ 532} \\ 192\ 875 \\ \underline{187\ 266} \\ 5\ 609 \\ \underline{3\ 745} \\ 1\ 864 \\ \underline{1\ 498} \\ 366 \\ \underline{337} \\ 29 \end{array}$$

The result to the nearest cent is \$9151.50. It is obtained by discarding one figure of the divisor each time a new partial division is made. A little practice will enable the student to determine where to begin to discard figures of the divisor.

If interest is convertible oftener than once a year, we can use the table for the rate $\frac{j}{m}$ and for mn years. For example, to find the amount of 1 for 10 years at 5%, convertible semiannually, we have to look up the amount of 1 for 20 years at $2\frac{1}{2}\%$, convertible annually, since, by the formula, we have

$$s = \left(1 + \frac{.05}{2}\right)^{10 \times 2} = (1.025)^{20}.$$

If the time does not contain the conversion interval an exact number of times, it will ordinarily be necessary to use logarithms or straight-out multiplication to reach the result. If, for example, we wish to find the amount of \$250 for 5 years and 6 months at 5% when interest is payable annually, we must find the value of the product

$$250 \times (1.05)^{\frac{11}{2}}.$$

If a table of logarithms is not at hand, the value of $(1.05)^{\frac{11}{2}}$ may be separated into the two factors $(1.05)^5$ and $(1.05)^{\frac{1}{2}}$, and the value of both factors may be found by the binomial theorem.

MISCELLANEOUS EXAMPLES IN COMPOUND INTEREST

1. Find the compound amounts in the following examples :

- (a) \$2342.75 for 15 years at 5% effective.
- (b) \$2458.23 for 10 years at 6% nominal, convertible semiannually.
- (c) \$25942.63 for 10 years at 5% nominal, convertible quarterly.
- (d) \$347.28 for 25 years at $4\frac{1}{4}\%$ nominal, convertible semiannually.

2. How long will it take a sum of money to double itself at 5% effective?

3. How long will it take a sum of money to double itself at 5% nominal, convertible semiannually?

4. How long will it take \$5000 to amount to \$8000 at 6% nominal, compounded semiannually?

5. The attendance of long-course and summer-session students at the University of Wisconsin in 1909 was 4386. Assuming an average rate of increase of $5\frac{1}{2}\%$, what should the attendance have been in 1919? What will it be in 1929?

6. How long will it take a university to double its attendance if the average annual rate of increase is 6%?

7. In 1921-1922 the attendance of high-school pupils in a certain city was 1953, and during a period of 10 years preceding the average rate of increase had been 8%. At that rate what will be the attendance in 1932?

8. The attendance at the University of Wisconsin, exclusive of extension students, was 11,367 in 1921-1922. Assuming an average annual increase of 5%, what will be the attendance in the same departments in 1931-1932?

9. Assuming the attendance and the rate of increase given in 1921-1922 for Example 8, when would the attendance reach 15,000?

10. The school population of a certain city has been increasing at the rate of 7% each year for 10 years. If it was 9327 in 1920, what will it be in 1925? in 1942?

11. How long will it take the school population in Example 10 to treble itself?

12. A sum of money amounting to \$10,000 is to accumulate for 30 years. The interest rate at present is 5%, but it is assumed that it will fall to $4\frac{1}{2}\%$ in 10 years and to 4% in 20 years. What will be the amount at the end of 30 years?

13. The assessed valuation of a certain city was \$85,437,629. If it be assumed that the valuation will increase at an average rate of $3\frac{1}{2}\%$, what will be the valuation at the end of 30 years?

14. A timber plantation is estimated to contain 200,000 feet of merchantable lumber in 1922. If the annual growth amounts to 4% each year, what will be the amount of lumber contained in 1937?

15. An apple orchard produced one hundred dollars' worth of fruit the first year it came into bearing, and it is assumed that the production will increase 10% every 2 years until the production is a maximum in 20 years. What will be the value of the maximum production?

16. If the production in Example 15 should increase at the rate of 5% a year to the period of maximum production in 20 years, would the result be the same? Explain.

17. In 1907 the coal consumption of the United States was 480,363,000 tons, which was 7.36% in excess of the consumption in 1906. Assuming the rate of increase to remain constant, what will be the consumption in 1930?

18. If \$5000 put into business increases to \$15,000 in 15 years, what is the average annual rate of increase?

19. A city's population increased 50% in 10 years. What was the average annual rate of increase?

20. If the rate of increase in the coal consumption as given in Example 17 were to continue, when would the consumption reach 1,000,000,000 tons?

21. In 1910 the population of Madison, Wisconsin, was 25,531 and in 1920 it was 38,378. At that rate when will it reach 100,000? What will it be in 1950?

22. The census of 1910 gave 93,402,151 as the population of the United States and the census of 1920 gave 105,709,000. What will the 1930 census give, on the assumption that the rate of increase will continue?

23. Construct the graph of the function

$$s_n = (1.05)^n,$$

where n is assumed to be the independent variable.

24. A rule in common use for finding the time in which a sum of money will double itself at compound interest is "Divide .69 by the rate of interest and add one third of a year." Prove that this rule is approximately correct. (The Napierian logarithm and not the common logarithm of 2 must be used.)

25. Find a rough rule for the time in which a sum of money will quadruple itself at compound interest.

26. A man leaves to a university an estate valued at \$2,000,000, on the condition that half the income shall be added to the principal each year until the amount reaches \$20,000,000, after which one fourth of the income shall be added until the amount reaches \$30,000,000, when the whole income shall be available. When will the university come into the possession of the full income if the funds can be made to realize 4%?

27. A banker charges 7%, payable in advance, on loans made for 90 days. What is the effective rate?

37. Instantaneous compound interest. Force of interest. From the fundamental formula

$$i = \left(1 + \frac{j}{m}\right)^m - 1 \quad (1)$$

it follows that the nominal rate is identical with the effective rate when the interest is converted into principal once a year. Moreover, the effective rate increases as the number of conversion intervals increases, since the positive part of i , viz. $\left(1 + \frac{j}{m}\right)^m$, increases. However, it does not increase indefinitely as the number of conversion intervals in a year increases, but approaches a definite and well-known limit. This limit is found by taking the limit of both sides of equation (1). We have, then,

$$\begin{aligned} \lim_{m \rightarrow \infty} i &= \lim_{m \rightarrow \infty} \left[\left(1 + \frac{j}{m}\right)^m - 1 \right] \\ &= \lim_{m \rightarrow \infty} \left(1 + \frac{j}{m}\right)^m - 1. \end{aligned}$$

But, by equations (4) and (5), § 12,

$$\lim_{m \rightarrow \infty} \left(1 + \frac{j}{m}\right)^m = e^j$$

where

$$e = 2.71828 +.$$

Denoting $\lim i$ by k for the moment, we have

$$k = e^j - 1. \quad (2)$$

The expression $e^j - 1$, which we have denoted by k , is the *effective rate corresponding to the nominal rate j convertible instantaneously*.

On the other hand, if the effective rate is fixed, and the number of intervals is increased, the corresponding nominal rate as given by the formula

$$j = m\{(1 + i)^{\frac{1}{m}} - 1\}$$

is decreased. It is not diminished indefinitely, however, but approaches a limit which is again well known. To find this limit by rough methods, we

expand $(1+i)^{\frac{1}{m}}$ by means of the binomial theorem. This method is legitimate, because the resulting series is convergent (see § 11). We have, then,

$$\begin{aligned} j &= m\{(1+i)^{\frac{1}{m}} - 1\} \\ &= m\left\{1 + \frac{1}{m}i + \frac{\frac{1}{m}(\frac{1}{m}-1)}{1 \cdot 2}i^2 + \frac{\frac{1}{m}(\frac{1}{m}-1)(\frac{1}{m}-2)}{1 \cdot 2 \cdot 3}i^3 + \dots - 1\right\} \\ &= i + \frac{(\frac{1}{m}-1)}{1 \cdot 2}i^2 + \frac{(\frac{1}{m}-1)(\frac{1}{m}-2)}{1 \cdot 2 \cdot 3}i^3 + \dots \end{aligned}$$

Assuming that we may take the limit of the right member by taking the sum of the limits of the separate terms, we find

$$\begin{aligned} \lim j &= \lim_{m=\infty} \left[i + \frac{(\frac{1}{m}-1)}{1 \cdot 2}i^2 + \frac{(\frac{1}{m}-1)(\frac{1}{m}-2)}{1 \cdot 2 \cdot 3}i^3 + \dots \right] \\ &= i - \frac{i^2}{2} + \frac{i^3}{3} - \frac{i^4}{4} + \dots \end{aligned}$$

The series on the right is exactly the logarithmic series which was discussed in § 13, so that

$$\lim j = \log_e(1+i). \quad (3)$$

It is customary to denote the limit of j by the Greek letter δ (delta), so that

$$\delta = \log_e(1+i). \quad (4)$$

The number δ or its equivalent is called the *force of interest*. The force of interest is the limit of the nominal rate as the number of conversion intervals in a year is indefinitely increased, corresponding to the fixed effective rate i .

To express the effective rate in terms of the force of interest, it is only necessary to write equation (4) in the exponential form. The result is

$$1+i = e^{\delta}, \quad (5)$$

or

$$i = e^{\delta} - 1. \quad (6)$$

Formula (6) is identical in form with (2), as indeed it should be, for δ is precisely the limit toward which j approaches as the number of conversion intervals in a year is indefinitely increased, and i is the corresponding effective rate.

The value of i in terms of δ is easily found by means of equation (5) of § 12, which gives

$$i = 1 + \frac{\delta}{1} + \frac{\delta^2}{1 \cdot 2} + \frac{\delta^3}{1 \cdot 2 \cdot 3} + \dots - 1,$$

or

$$i = \frac{\delta}{1} + \frac{\delta^2}{1 \cdot 2} + \frac{\delta^3}{1 \cdot 2 \cdot 3} + \dots \quad (7)$$

For example, if

$$\begin{aligned}\delta &= .05, \\ i &= \frac{.05}{1} + \frac{(.05)^2}{1 \cdot 2} + \frac{(.05)^3}{1 \cdot 2 \cdot 3} + \dots \\ &= .05127 + .\end{aligned}$$

To find δ when i is given, the logarithmic series

$$\log_e(1+i) = i - \frac{i^2}{2} + \frac{i^3}{3} - \frac{i^4}{4} + \dots$$

may be used. For example, if $i = .05$,

$$\begin{aligned}\delta &= \log_e(1.05) \\ &= .05 - \frac{(.05)^2}{2} + \frac{(.05)^3}{3} - \frac{(.05)^4}{4} + \dots \\ &= .048790 + .\end{aligned}$$

It is easy to find the value of δ by common logarithms, for, by (5') of § 29,

$$\delta = \log_e(1.05) = \frac{1}{M} \cdot \log_{10}(1.05),$$

where

$$\frac{1}{M} = 2.302585 + \dots$$

Consequently,

$$\begin{aligned}\delta &= 2.302585 \times .021189 + \dots \\ &= .048790 + \dots,\end{aligned}$$

which agrees with the result already obtained.

Instantaneous compound interest and force of interest have no real existence in practical life, though they are approximated in the business of large concerns which are receiving interest and making loans every day, indeed many times every day. Their value consists chiefly in the fact that by their use many computations are greatly simplified.

If we remember that we have, by formula (6),

$$1+i = e^\delta,$$

and that the fundamental relation between nominal and effective rates of interest may take the form

$$1+i = \left(1 + \frac{j}{m}\right)^m,$$

the formulas for compound interest, viz.

$$s = (1+i)^n = \left(1 + \frac{j}{m}\right)^{mn}$$

and

$$S = P(1+i)^n = P\left(1 + \frac{j}{m}\right)^{mn},$$

are expressible in the simple forms

$$s = e^{n\delta} \tag{8}$$

and

$$S = Pe^{n\delta}. \tag{9}$$

These forms justify the name *compound interest law*, which has been applied to the relation having the form

$$y = A_0 a^x. \quad (10)$$

Formulas (8) and (9) are only two of many cases in which a simplification is effected by means of the introduction of the force of interest.

EXAMPLES

1. By means of the exponential series, find the effective rate when the nominal rate is .06 and the interest is compounded momentarily.
2. Find the force of interest corresponding to the effective rate .06.
3. If the force of interest is .06, find the compound amount of \$1259 for 3 years and 6 months.

38. Discount and present value. The word *discount* has a variety of meanings in the world of business. To the merchant buying a stock of goods it means a reduction in his bill for the payment of cash; to the same merchant selling his goods it means a reduction from the marked price of an article to secure prompt sale; to the banker it usually means simple interest payable in advance. The problems that arise in such cases are usually problems in percentage or in simple interest, and for that reason they do not require discussion here.

Discount, as we shall use the term, is a consideration for the payment of a sum of money before it is due. It is the difference between the value of a sum of money payable at some future time, with or without interest, at the time when it is due, and its value at some earlier time, usually the present. We sometimes say of a sum of money that it is *accumulated* to a certain date, meaning thereby that it is put at interest until the date in question; on the other hand, we say that a sum of money is *discounted* to a certain date, meaning that we seek to find the value of the sum at the date in question as compared with its value given at some assigned later date.

The value *now* of a sum due at some future date, with or without interest, is called the *present worth*, or *present value*. The present value of a sum due at some future date may also be defined as that sum which, put at compound interest now, would amount to the sum in question on the given future date. *The*

processes of accumulation and discounting are then exactly the reverse of each other. A clear recognition of this fact will be of the greatest aid in understanding future sections.

PROBLEM. *To find the present value of 1 due in n years without interest.*

Let x be the present value of 1 and let i be the rate of interest that could be realized if the money were in hand to be put at interest. By definition of present value

$$x(1+i)^n = 1;$$

so that

$$x = \frac{1}{(1+i)^n}.$$

The function $\frac{1}{(1+i)^n}$ is one of the important functions occurring in the mathematics of finance and is everywhere denoted by v . Clearly, if

$$v = \frac{1}{1+i}, \quad (1)$$

$$v^n = \frac{1}{(1+i)^n} = (1+i)^{-n}. \quad (2)$$

If the principal be not 1 but P , and V_n be used to denote the present value of P due in n years without interest,

$$V_n = P \cdot \frac{1}{(1+i)^n} = P(1+i)^{-n} = P v^n. \quad (3)$$

The values of v^n are tabulated in Table IV.

If interest is convertible oftener than once a year and the nominal rate is given, the expression $1+i$ must be replaced as in previous problems by its value $\left(1 + \frac{j}{m}\right)^m$.

ILLUSTRATIVE EXAMPLE. Find the present value of \$2365.29 due 3 years hence, if money could be invested at 5%.

Solution. By formula (3)

$$V_3 = \$2365.29(1.05)^{-3}.$$

The value of the product on the right may be obtained by means of a table of six-place logarithms, or the value of $(1.05)^{-3}$ may be taken directly from Table IV. If neither the table of logarithms nor the table of present values is

at hand, the value of $(1.05)^{-3}$ may be obtained by the binomial expansion. Using the last-named method, we have

$$(1.05)^{-3} = 1 + (-3)(.05) + \frac{(-3)(-3-1)}{1 \cdot 2} (.05)^2 + \frac{(-3)(-3-1)(-3-2)}{1 \cdot 2 \cdot 3} (.05)^3 + \dots$$

Six terms of this series give the result

$$(1.05)^{-3} = .863837 + \dots$$

Consequently,

$$\begin{aligned} V_3 &= \$2365.29 \times .863837 \\ &= \$2043.23. \end{aligned}$$

PROBLEM. *To find the present value of 1 due n years hence with interest.*

Let k be the rate of interest borne by the principal. If no interest payments are made, the principal will amount to $(1+k)^n$ in n years. Let v'^n be the present value; then

$$v'^n (1+i)^n = (1+k)^n.$$

And, consequently,
$$v'^n = \frac{(1+k)^n}{(1+i)^n}. \quad (4)$$

If the principal be P and the corresponding present value be denoted by V'_n ,

$$V'_n = P \frac{(1+k)^n}{(1+i)^n} = P(1+k)^n v'^n. \quad (5)$$

Formulas (1)–(5) may be expressed in terms of the nominal rates j and k' by means of the fundamental relations

$$(1+i) = \left(1 + \frac{j}{p}\right)^p \quad \text{and} \quad 1+k = \left(1 + \frac{k'}{p}\right)^p.$$

The expression $(1+i)^{-n}$, or its equivalent $\left(1 + \frac{j}{p}\right)^{-np}$, is frequently called the *discount factor*.

ILLUSTRATIVE EXAMPLE. Find the present value of \$379.25, with interest payable annually at 4%, and due 3 years hence, when money is worth 5% nominal, payable semiannually.

Solution. By formula (5),

$$V'_3 = \$379.25 \frac{(1.04)^3}{(1.025)^6},$$

where the nominal rate has been introduced by means of the relation

$$\left(1 + \frac{.05}{2}\right)^2 = (1+i).$$

The result is found by logarithms to be \$367.86.

EXAMPLES

1. Find the present values on :

- (a) \$1250 due in 2 years without interest, money worth 5%.
- (b) \$2500 due in 5 years without interest, money worth 6%.
- (c) \$5000 due in 6 years without interest, money worth 4%.
- (d) \$3000 due in 4 years without interest, money worth 5%.
- (e) \$3275.29 due in 5 years without interest, money worth 6%.
- (f) \$378.24 due in 3 years without interest, money worth 6%.
- (g) \$384.29 due in 4 years without interest, money worth $4\frac{1}{2}\%$.
- (h) \$2324.75 due in 3 years without interest, money worth $3\frac{1}{2}\%$.

2. Find the present values on :

- (a) \$1250 due in 2 years with interest at 4%, money worth 5%.
- (b) \$1500 due in 3 years with interest at 3%, money worth 6%.
- (c) \$379.24 due in 4 years with interest at 5%, money worth 5%.
- (d) \$289.27 due in 5 years with interest at 7%, money worth 5%.
- (e) \$325.42 due in 3 years with interest at 6%, money worth 4%.

3. Find the present values on :

- (a) \$1250 due in 3 years without interest, money worth 6% nominal, convertible semiannually.
- (b) \$1500 due in 4 years without interest, money worth 6% nominal, convertible quarterly.
- (c) \$1250 due in 3 years with interest at 4%, money worth 6%.
- (d) \$1250 due in 4 years with interest at 4% nominal, convertible semiannually, money worth 6% nominal, convertible semiannually.
- (e) \$378.42 due in 5 years with interest at 6% nominal, convertible semiannually, money worth 5%.

4. Find the present value of \$3000 due in $2\frac{1}{2}$ years without interest, if money is worth 6%.

5. Find the present value of \$2000 due in 3 years without interest, if money is worth 6%, payable quarterly.

6. Find the present value of \$687.14 with interest payable quarterly at 5%, if money is worth 4%.

7. A father wishes to set aside, at the birth of his son, an amount sufficient to accumulate to \$10,000 by the time the boy is 21. What is the sum required, if the money can be made to earn 5%?

8. A wholesale merchant sells goods on 90 days' time or 3% off for cash. What is the highest rate a buyer could afford to pay to borrow money and pay cash?

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9. One man buys a farm for \$5000, agreeing to pay \$1000 cash and \$1000, with interest at 6% on all sums remaining due, at the end of each year, until the whole amount is paid. How much does he pay each year?

Another man buys a farm, agreeing to pay \$1000 cash, \$1240 at the end of the first year, \$1180 at the end of the second, \$1120 at the end of the third, and \$1060 at the end of the fourth. If money is worth 6%, what is the cash price of the second man's farm?

Explain the significance of these two problems in the light of the note at the end of § 39.

10. A restaurant keeper sells a meal ticket good for four dollars' worth of meals to be served in a month for \$3.75, payable in advance. Assuming that the \$4 could be paid equitably at the middle of a month of 30 days, what rate of interest is equivalent to the discount?

11. Construct the graph of the function $(1.04)^{-n}$ with the time as the variable. What is the significance of that part of the graph that lies to the left of the axis of ordinates?

12. The assessed valuation of a certain city was \$54,375,000 in 1900 and \$81,398,000 in 1920. Assuming a constant rate of growth, what should it have been in 1890?

39. The rate of discount. The measure of the discount on unit principal for unit time is called the *rate of discount*. Since present value has been defined as the principal diminished by the discount, we have, as a defining equation,

$$v = 1 - d, \quad (1)$$

where d denotes the rate of discount.

From equation (1) we may easily find the relation between the rate of discount and the rate of interest, for, replacing v by its value as given by (1) (§ 38),

$$\frac{1}{1+i} = 1 - d.$$

Solving this equation for d ,

$$d = \frac{i}{1+i} = iv. \quad (2)$$

Solving the same equation for i ,

$$i = \frac{d}{1-d}. \quad (3)$$

Formulas (2) and (3) furnish the means for the direct computation of i when d is known, and vice versa. For example, if $i = .06$, then, by simple division,

$$d = \frac{.06}{1.06} = .05660 + \dots$$

In many cases, however, it is even easier to express the one quantity in terms of the other by means of a power series, and to use the power series as a means of computation. By division we obtain from (2) the formula

$$d = i - i^2 + i^3 - i^4 + \dots \quad (4)$$

Similarly, from (3), $i = d + d^2 + d^3 + \dots \quad (5)$

ILLUSTRATIVE EXAMPLE. What is the rate of discount when the rate of interest is .06?

Solution. By (4),
$$d = .06 - (.06)^2 + (.06)^3 - (.06)^4 + \dots = .05660 + \dots$$

Additional terms would not change this result unless more decimal places were used.

The computation of i by means of (5) is even simpler, since all the terms of the series are positive.

NOTE. The so-called *true discount* is defined as the difference between the principal and the sum which, put at *simple* interest, would amount to the principal when the latter becomes due. What compound discount must be used in all problems where interest is paid when it is due is seen from problem 9 of § 38.

EXAMPLES

1. When the interest rate is 5%, what is the corresponding rate of discount?
2. What is the rate of interest corresponding to a discount rate of 5% for a year?
3. What annual rate of interest corresponds to 5% "off for cash," when goods are sold on 30-day time?
4. What 90-day rate of interest corresponds to 5% "off for cash"?
5. When interest is compounded momentarily the corresponding rate of discount would be expressed by the equation

$$\lim_{p \rightarrow \infty} \frac{1}{\left(1 + \frac{j}{p}\right)^p} = 1 - \delta.$$

The number δ is called the *force of discount*. Find the force of discount when the nominal rate of interest is .06.

CHAPTER VI

THE EQUATION OF VALUE AND EQUATION OF PAYMENTS

40. The equation of value. It is frequently necessary to compare the value of one set of sums due at various times with another set due at other times. To make such comparison possible it is necessary to observe the following

FUNDAMENTAL PRINCIPLE: *To compare the values of several sums due at various times, all the sums must be accumulated or discounted to the same date.*

The relation expressing the fact that one set of obligations is equivalent to another set is called the *equation of value*.

The formation of the equation of value is best illustrated by an example. Suppose a man owing \$1000 due in 18 months and \$500 due in 2 years has an opportunity of discharging the two obligations by two equal payments, one made in 6 months and one in 1 year. The problem is to find the amount of the two equal payments.

Let the value of the two equal payments be denoted by X . If all sums be discounted to the present, the present values will be as follows:

$$Xv^{\frac{1}{2}}, \quad Xv, \quad 1000v^{\frac{3}{2}}, \quad 500v^2.$$

If the first two obligations are equivalent to the last two the numbers must be connected by the equation

$$Xv^{\frac{1}{2}} + Xv = 1000v^{\frac{3}{2}} + 500v^2. \quad (1)$$

This equation suffices to determine the unknown number X .

The time to which the various sums are accumulated or discounted is wholly immaterial. For example, all the sums may be accumulated to the last date. The accumulation periods are then $1\frac{1}{2}$, 1, $\frac{1}{2}$, and 0 years respectively and the equation becomes

$$X(1+i)^{\frac{3}{2}} + X(1+i) = 1000(1+i)^{\frac{1}{2}} + 500(1+i)^0.$$

If this equation be divided through by $(1+i)^2$, the resulting equation is

$$X(1+i)^{-\frac{1}{2}} + X(1+i)^{-1} = 1000(1+i)^{-\frac{3}{2}} + 500(1+i)^{-2},$$

which is identical with equation (1). Again, if we multiply equation (1) by v^{100} we obtain an equation of value in which all sums have been discounted to a date 100 years ago. However, the equation differs from (1) in form only.

To find the equation in its most general form suppose the sums $X_1, X_2, \dots, X_r$ due in $n_1, n_2, \dots, n_r$ years are together equivalent to the sums $y_1, y_2, \dots, y_s$ due in $m_1, m_2, \dots, m_s$ years. When all sums are discounted to the present we obtain the equation

$$X_1 v^{n_1} + X_2 v^{n_2} + \dots + X_r v^{n_r} = y_1 v^{m_1} + y_2 v^{m_2} + \dots + y_s v^{m_s}. \quad (2)$$

If all the numbers $X_1, X_2, \dots, X_r, y_1, y_2, \dots, y_s$ are known except one, that one can be found from equation (2).

The principle involved in the equation of value is of the utmost importance in formulating many problems in finance, and should be thoroughly mastered.

EXAMPLES

1. What sum due in 1 year will discharge equitably two debts each of \$500, one due now and the other in 2 years, when money is worth 4%?
2. What would be the answer to Example 1 if the rate were 6%?
3. Three sums each of \$300 will fall due in 1, 2, and 3 years respectively. What is the cash equivalent, money worth 5%?
4. Three sums of \$300 each will fall due in 1, 2, and 3 years respectively. What will be the cash equivalent at the end of the third year, money worth 5%?
5. A man owing \$10,000 is able to arrange with his creditor for the payment of the debt in five equal annual installments, the first payable 1 year hence. If money is worth 6%, what will the annual installment be?
6. The sum of \$10,000 without interest will be due 5 years hence. The creditor wishes to have the money in five equal annual installments, the first payable immediately. If money is worth 6%, what will the annual installment be?

7. Money is worth 6%. One sum of \$500 is due today, another of \$600 is 1 year overdue, and one of \$700 will be due in 2 years. What annual payment, the first in cash, will discharge the debt by the end of the third year?

8. A owes \$200 due now, \$300 due in 1 year without interest, \$500 due in 2 years without interest. What sum will discharge the three obligations at the end of a year and a half if money is worth 6% nominal, convertible semiannually?

41. Equation of payments. The term *equation of payments*, or *equation of accounts*, is used to denote a rule for determining the time at which several debts, due at different times, can be equitably discharged by the payment of a single sum equal in amount to the sum of all the debts. The rule is found by solving an equation of value. The time thus found is called the *equated time*.

PROBLEM. *To determine the equated time for the payment of several debts due at various times.*

Let $s_1, s_2, s_3, \dots, s_r$, denote the sums due at later dates, and $n_1, n_2, n_3, \dots, n_r$, the times to elapse before they fall due. If n denote the unknown equated time, and all sums be discounted to the present, the equation of value will be

$$(s_1 + s_2 + \dots + s_r)v^n = s_1v^{n_1} + s_2v^{n_2} + s_3v^{n_3} + \dots + s_rv^{n_r}. \quad (1)$$

Solving this equation for v^n , we find

$$v^n = \frac{s_1v^{n_1} + s_2v^{n_2} + \dots + s_rv^{n_r}}{s_1 + s_2 + \dots + s_r}. \quad (2)$$

Solving this equation for n , we find

$$n = \frac{\log[s_1v^{n_1} + s_2v^{n_2} + \dots + s_rv^{n_r}] - \log[s_1 + s_2 + \dots + s_r]}{\log v}. \quad (3)$$

If v be replaced by its value $(1+i)^{-1}$ and the sign of the numerator be changed at the same time, equation (3) may be written in the convenient form

$$n = \frac{\log[s_1 + s_2 + \dots + s_r] - \log[s_1v^{n_1} + s_2v^{n_2} + \dots + s_rv^{n_r}]}{\log(1+i)}. \quad (4)$$

The equated time may be computed by means of formula (3) or formula (4).

42. The ordinary rule. The ordinary rule for the equation of payments is easily obtained by substituting approximate values for v^n , v^{n_1} , v^{n_2} , $\dots$, v^{n_r} , in equation (1), § 41. This rule is as follows:

Multiply each sum by the time to elapse before it becomes due, add the products, and divide the sum by the sum of the amounts to fall due.

The formula which gives rise to this rule may be obtained by the use of so-called *bank discount*, for which the discount factor is $1 - ni$, or it may be obtained directly from equation (1), § 41.

If we recall that $v = (1+i)^{-n}$ and expand each of the expressions $(1+i)^{-n}$, $(1+i)^{-n_1}$, $\dots$, $(1+i)^{-n_s}$, by the binomial theorem, we find

$$v^n = (1+i)^{-n} = 1 - ni + \frac{(-n)(-n-1)}{1 \cdot 2} i^2 + \dots$$

$$v^{n_1} = (1+i)^{-n_1} = 1 - n_1 i + \frac{(-n_1)(-n_1-1)}{1 \cdot 2} i^2 + \dots$$

$$v^{n_r} = (1+i)^{-n_r} = 1 - n_r i + \frac{(-n_r)(-n_r-1)}{1 \cdot 2} i^2 + \dots$$

If all powers of i higher than the first be dropped and the approximate values

$$1 - ni, \quad 1 - n_i i, \quad \dots, \quad (1 - n_r),$$

be substituted for $v^n, v^{n_1}, \dots, v^{n_r}$, respectively, the formula (1), § 41, takes the form

$$(s_1 + s_2 + \cdots + s_r)(1 - ni) = s_1(1 - n_1i) + \cdots + s_r(1 - n_ri).$$

When this equation is solved for n the result is

$$n = \frac{n_1 s_1 + n_2 s_2 + \dots + n_r s_r}{s_1 + s_2 + \dots + s_r}. \quad (1)$$

This formula, which is independent of the rate of interest, gives the ordinary rule when translated into words. The ordinary rule favors the debtor by increasing the time slightly, though it is reasonably accurate when the times are short. (See Todhunter, "Institute of Actuaries' Text-Book," chap. ii, Art. 8.)

EXAMPLES

1. Find by the ordinary rule the equated time for the payment, in one sum of \$1800, of the following sums due without interest, money worth 5%: \$600 due in 18 months, \$700 due in 2 years, \$400 due in 3 years, and \$100 due now.

2. Find the difference by the exact rule, and by the ordinary rule the time for payment in a single sum of \$8000, for the following sums due without interest in the times indicated, money worth 4%: \$1000 due in 1 year, \$2000 due in 2 years, \$3000 due in 18 months, \$2000 due now.

3. Solve problem 1 by discounting all sums by bank discount; i.e. by subtracting simple interest for the time.

4. A loan and investment concern loans \$100 for 1 year and takes out \$8 in advance for interest. It requires the borrower to take out an investment certificate upon which he pays \$8.33 at the end of each month for 11 months and \$8.37 at the end of the twelfth month. At the end of 12 months the borrower is paid \$2.20 interest, and the debt is canceled by the surrender of the investment certificate. Assuming that interest charged and interest paid is simple interest and that interest may be paid at any time during the year, what is the rate of interest charged when the investment certificate is looked upon as a mode of repayment of the loan? — *Problem submitted to the Securities Division of the Wisconsin Railroad Commission*

HINT. Assume face of debt to be \$100. The debtor has the use of \$8.37 for 12 months, of \$8.33 for 11 months, of \$8.33 for 10 months, and so on. The problem will be solved when we know to what sum due in 1 year these various sums will be equivalent. The net interest charged is \$5.80.

CHAPTER VII

ANNUITIES

43. Definitions and notation. An *annuity* is a series of payments, usually equal in amount, made at equal intervals of time, called *intervals of payment*. Unless the contrary is specifically stated, the first payment is made at the *end* of the first interval. If the first payment is made at the beginning instead of at the end of the first interval, the annuity is called an *annuity due*.

An *annuity certain* is one for which the payments begin and end at fixed dates. The time to elapse between the beginning of the first interval of payment and the end of the last is called the *term* of the annuity certain.

If the date either of the first or of the last payment depends upon some event the time of whose occurrence cannot be foretold, the annuity is called a *contingent annuity*. An annuity whose payments begin or end with the death of an individual would be a contingent annuity.

A *deferred annuity* is one for which a certain specified number of intervals must elapse before the first payment is made. A deferred annuity becomes an ordinary annuity after the lapse of the specified number of intervals. To be exact, if the annuity is deferred m intervals, the first payment is made at the end of the $(m + 1)$ st interval.

A *perpetuity* is an annuity whose payments are supposed to continue forever.

When the payments are equal in amount, the sum of the payments made in one year is called the *annual rent*. If the annual rent be R , we speak of an *annuity of R per annum*.

If the payments of an annuity are allowed to accumulate to the end of the term during which they are made, the annuity is said to be *forborne*, and the total sum, payments and interest, due at the end of this term is called the *amount* of the annuity.

The *present value*, or the cash equivalent, of an annuity is the sum of the present values of all the payments of the annuity.

One has only to cite a few examples to see the wide range of application of the theory of annuities. The rental of a house, the interest payments on a mortgage note, the annual premiums on a life-insurance policy, the stated return from an interest-bearing bond, a pension, are all examples of annuities.

The following notation is standard among English-speaking writers on the subject of annuities:

$s_{\overline{n}|}$ denotes the amount of an annuity of 1 per annum, payable annually for n years.

$s_{\overline{n}|}^{(p)}$ denotes the amount of an annuity of 1 per annum, payable in p installments at equal intervals throughout the year for n years.

$a_{\overline{n}|}$ denotes the present value of 1 per annum, payable annually for n years.

$a_{\overline{n}|}^{(p)}$ denotes the present value of 1 per annum, payable p times a year for n years.

${}_m|a_{\overline{n}|}^{(p)}$ denotes the present value of a deferred annuity of 1 per annum, payable p times a year for n years, the first payment to be made after the lapse of $m + \frac{1}{p}$ years.

If the annuity is due, roman letters in black type are used. Thus, $\mathbf{a}_{\overline{n}|}$ and $\mathbf{a}_{\overline{n}|}^{(p)}$ are used to denote the present values of the annuities due, the first of 1 per annum, payable annually for n years, and the second of 1 per annum, payable p times a year for n years. Similarly, $\mathbf{s}_{\overline{n}|}$ and $\mathbf{s}_{\overline{n}|}^{(p)}$ denote the amounts of annuities due when payable annually and p times a year respectively.

When interest is convertible oftener than once a year, the rate of interest may be given as either effective or nominal. All formulas involving the rate of interest may be expressed in the one or the other of these two rates at will by means of the fundamental relation

$$(1+i) = \left(1 + \frac{j}{m}\right)^m.$$

All theoretical computations are made for an annuity of 1 per annum.

44. The amount of an annuity.

PROBLEM. *To find the amount of a forborne annuity of 1 per annum, payable annually for n years.*

Let i be the effective rate.

The first payment made one year from the beginning of the term of the annuity will be accumulated for $n-1$ years and will amount to $(1+i)^{n-1}$. Likewise, the second will amount to $(1+i)^{n-2}$, the third to $(1+i)^{n-3}$, and so on, while the last term will be cash. The series of amounts will be

$$(1+i)^{n-1}, (1+i)^{n-2}, \dots, (1+i), 1$$

Denoting the total amount by $s_{\overline{n}|}$, according to the notation established in § 43, and writing the terms in reverse order, we have

$$s_{\overline{n}|} = 1 + (1+i) + (1+i)^2 + \dots + (1+i)^{n-2} + (1+i)^{n-1}.$$

The expression on the right is a geometrical progression of n terms with ratio $1+i$, and the sum is found by the formula

$$s = \frac{rl - a}{r - 1}. \quad (\text{See formula } (D'), \text{ § 3.})$$

Substituting for r , l , and a their values $1+i$, $(1+i)^n$, and 1, we find, as the required formula,

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}. \quad (1)$$

If interest is convertible m times a year at nominal rate j , we may replace i and $(1+i)^n$ by their values as obtained in the fundamental relations (1) and (2) of § 35. The result will then take the form

$$s_{\overline{n}|} = \frac{\left(1 + \frac{j}{m}\right)^{mn} - 1}{\left(1 + \frac{j}{m}\right)^m - 1}. \quad (2)$$

Formula (2), which gives the amount of the annuity in terms of the nominal rate j , may be written in the form

$$s_{\overline{n}|} = \frac{s_{\overline{mn}|}}{s_{\overline{m}|}} \left(\text{at rate } \frac{j}{m} \right), \quad (3)$$

as is easily seen when numerator and denominator are divided by $\frac{j}{m}$.

The amount of an annuity of R per annum for n years is

$$K = Rs_{\overline{n}|}. \quad (4)$$

EXAMPLES

1. A man puts \$100 into a savings bank at the end of every year for 5 years. If the bank pays 4%, payable annually, what will his savings amount to at the end of the 5 years?

Solution. The amount will be found by substituting the values $n = 5$ and $i = .04$ in formula (1), and then multiplying by 100. The result is

$$\$100 \cdot \frac{(1.04)^5 - 1}{.04} = \$541.63.$$

2. Find the amount of an annuity of \$600 a year for 10 years, accumulated at 4%.

3. A man saves \$400 a year for 10 years; what will his savings amount to in 10 years if they are accumulated at 4%?

4. A father begins to put by a fund for the education of his son when the son is born. He saves \$100 each year until the boy is ready to enter college at the age of 17. How much will there be in the fund if each year's savings are put into an institution that will accumulate the amounts at 5%?

5. If in Example 4 interest on the annual payments is convertible twice a year, what will be the amount?

6. If in Example 4 the interest rate were 4.3, what would the amount be?

7. If \$100 set aside at the end of each year for 12 years amounts to \$1546.50, what is the approximate rate of interest?

8. If \$500 set aside at the end of each year for 20 years amounts to \$15,685, what is the approximate rate of interest?

9. A father wishing to provide a college education for his son begins in the year of the son's birth to lay aside a certain sum to be accumulated at 5%. It is estimated that \$3000 should be available on the son's seventeenth birthday. How much should the father put aside at the end of each year up to and including the seventeenth?

PROBLEM. *To find the amount of an annuity of 1 per annum, payable p times a year for n years.*

Suppose the rate of interest is the effective rate i . The time to elapse between the making of the first payment and the end of the whole transaction will be $n - \frac{1}{p}$ years; and, similarly, the times for the second and succeeding payments will be $n - \frac{2}{p}$ years, $n - \frac{3}{p}$ years, and so on. The last payment will be cash. The amounts of the various payments, beginning with the first, will be

$$\frac{1}{p}(1+i)^{n-\frac{1}{p}}, \quad \frac{1}{p}(1+i)^{n-\frac{2}{p}}, \quad \dots, \quad \frac{1}{p}(1+i)^{\frac{1}{p}}, \quad \frac{1}{p}.$$

Writing these amounts in reverse order and denoting their sum by $s_{\overline{n}|}^{(p)}$, according to § 43, we have

$$s_{\overline{n}|}^{(p)} = \frac{1}{p} + \frac{1}{p}(1+i)^{\frac{1}{p}} + \frac{1}{p}(1+i)^{\frac{2}{p}} + \dots + \frac{1}{p}(1+i)^{n-\frac{1}{p}}.$$

This is again a geometric progression with first term $\frac{1}{p}$, ratio $(1+i)^{\frac{1}{p}}$, and last term $\frac{1}{p}(1+i)^{n-\frac{1}{p}}$. By formula (D') of § 3 the sum will be

$$s_{\overline{n}|}^{(p)} = \frac{(1+i)^n - 1}{p[(1+i)^{\frac{1}{p}} - 1]}. \quad (5)$$

Formula (5), which expresses the amount of an annuity of 1 per annum payable p times a year with interest at effective rate i , may be put in a form better adapted for computation by noting that the denominator is $j_{(p)}$ and then multiplying numerator and denominator by i . The result is

$$s_{\overline{n}|}^{(p)} = \frac{(1+i)^n - 1}{i} \cdot \frac{i}{j_{(p)}} = s_{\overline{n}|} \cdot \frac{i}{j_{(p)}}. \quad (6)$$

Both $s_{\overline{n}|}$ and $\frac{i}{j_{(p)}}$ may be taken directly from the tables, or, better still, if a table of logarithms of these functions is available, the work of actual multiplication may be avoided. (See Glover's "Tables of Compound Interest Functions.")

If we desire the formula for the amount in terms of a nominal rate j , convertible m times a year, we have only to replace $1+i$ by its value in terms of j , viz. by $\left(1+\frac{j}{m}\right)^m$, and we obtain the formula

$$s_{\overline{n}|}^{(p)} = \frac{1}{p} \frac{\left(1+\frac{j}{m}\right)^{mn} - 1}{\left(1+\frac{j}{m}\right)^{\frac{m}{p}} - 1}. \quad (7)$$

Finally, if the payment interval and the conversion interval for interest are the same, $m=p$ and formula (7) takes the important form

$$s_{\overline{n}|}^{(p)} = \frac{1}{p} \cdot \frac{\left(1+\frac{j}{p}\right)^{np} - 1}{\frac{j}{p}} = \frac{1}{p} s_{\overline{np}|} \left(\text{at rate } \frac{j}{p}\right). \quad (8)$$

If the annual rent is not 1, but R , it is only necessary to multiply the right member of each of the foregoing formulas by R . For example, if the annual rent is R and the corresponding amount be denoted by K , formula (8) becomes

$$K = \frac{R}{p} s_{\overline{np}|} \left(\text{at rate } \frac{j}{p}\right). \quad (9)$$

EXAMPLES

1. Find the amount of an annuity of \$1000 per year, payable quarterly for 10 years at 5% effective.

Solution. The easiest solution is by formula (6). By the tables

$$s_{\overline{10}|} \text{ at } 5\% = 12.5778925,$$

$$\frac{i}{j^{(4)}} \text{ at } 5\% = 1.0185594.$$

The result is therefore

$$\$1000 \times 12.577892 \times 1.0185594 = \$12,811.33.$$

2. Find the amount of an annuity of \$1000 per year, payable quarterly for 10 years, if interest is 5% nominal, convertible quarterly.

Solution. The problem is equivalent to that of finding the amount of one fourth of \$1000 for four times the number of years at one fourth the rate.

$$\begin{aligned} \$250 \times s_{\overline{40}|} \text{ at rate } 1\frac{1}{4}\% &= \$250 \times 51.489557 \\ &= \$12,872.39. \end{aligned}$$

3. A man owns \$10,000 in Liberty Bonds yielding $4\frac{1}{2}\%$ nominal, payable semiannually. If he places the income in a bank paying $4\frac{1}{2}\%$ nominal, convertible semiannually, what will be the amount of his accumulations at the end of 10 years?

4. What would be the amount of the income from the bonds in Example 3 if the interest payments were accumulated at $4\frac{1}{2}\%$ effective?

5. A man buys \$10,000 of preferred stock yielding 7% nominal, payable quarterly. If the income be accumulated at 5% nominal, convertible semiannually, what will be the amount of the accumulations at the end of 10 years?

6. A man sets aside \$50 at the end of each month for 20 years. If the savings can be invested without delay at 5% effective, how much will he have at the end of the time?

7. If the accumulations in Example 6 are made at 5% nominal, payable semiannually, what will be the total amount in the savings account in 20 years?

8. How much will a man have to save each year in order to have \$20,000 in 20 years, if the savings are invested at 5% effective?

9. How much will a man need to save each month in order to have \$20,000 at the end of 20 years, if the savings can be invested at 6% effective?

10. Find the amount of 1 per annum payable p times a year for 1 year at effective rate i .

45. The present value of an annuity.

PROBLEM. *To find the present value of an annuity of 1 per annum, payable annually for n years.*

Let it be required to find the present value in terms of the effective rate i . The several annual payments will be, in effect, a series of sums due in 1, 2, 3, $\dots$, n years. Their present values will then be, by (2), § 38,

$$(1+i)^{-1}=v, \quad (1+i)^{-2}=v^2, \quad \dots, \quad (1+i)^{-n}=v^n.$$

Denoting the sum of the present values by $a_{\overline{n}|}$, we have

$$a_{\overline{n}|}=v+v^2+v^3+\dots+v^n.$$

The right member is a geometric progression with first term v , last term v^n , and ratio v . The sum is therefore

$$a_{\overline{n}|}=\frac{v^{n+1}-v}{v-1}=\frac{v-v^{n+1}}{1-v}.$$

If we remember that v is defined by the equation

$$v = (1 + i)^{-1},$$

the expression on the right is easily reduced, by dividing numerator and denominator by v , to the form

$$a_{\overline{n}|} = \frac{1 - v^n}{i} = \frac{1 - (1 + i)^{-n}}{i}. \quad (1)$$

If the nominal rate j , convertible m times a year, is given, the formula (1) may be written

$$a_{\overline{n}|} = \frac{1 - v^n}{\left(1 + \frac{j}{m}\right)^m - 1} = \frac{1 - \left(1 + \frac{j}{m}\right)^{-mn}}{\left(1 + \frac{j}{m}\right)^m - 1}. \quad (2)$$

EXERCISE

Show that formula (2) may be written in the form

$$a_{\overline{n}|} = \frac{a_{\overline{mn}|}}{s_{\overline{m}|}} \left(\text{at rate } \frac{j}{p} \right). \quad (2')$$

EXAMPLES

1. A man buys a house, agreeing to pay \$1000 cash and \$1000 at the end of each year for 5 years. What would be the cash price of the house (a) if money is worth 6% effective? (b) if money is worth 6% nominal, payable semiannually?

Solution. The first \$1000 constitutes a cash payment whose present value is \$1000, and the other five payments constitute an annuity of \$1000 per annum. For the first case the present value of the annuity, as given by formula (1), is

$$\$1000 \frac{1 - (1.06)^{-5}}{.06} = \$4212.36.$$

The cash price would then be \$5212.36.

For the second case the present value of the annuity, as given by formula (2), is

$$\$1000 \frac{1 - (1.03)^{-10}}{(1.03)^2 - 1} = \$4202.07,$$

and the cash price would be \$5202.07.

2. Find the present value of an annuity of \$395.47 per annum, payable twice a year, for 10 years, with interest at 6% nominal, convertible twice a year.

3. The annual charge for a certain preparatory school is \$900 per year, payable in advance. What would be the charge for the four years if payment is made at the beginning of the first year, money worth 6%?

4. What would be the charge for the four years in Example 3 if money is worth 6% nominal, payable twice a year, and the amount is paid in cash at the beginning of the first year?

5. A property owner holding 60 feet of frontage pays an assessment of \$15 per year for 10 years for street paving. What is the equivalent cash price paid at the beginning of the term?

6. A house is sold for \$1000 cash and \$1000 at the end of each year for 5 years. If money is worth 6%, what is the selling price of the house?

7. A man is offered \$10,000 cash for his house, or \$2000 in cash and \$2000 at the end of each year for 5 years. If money is worth 6% effective, which is the better offer and how much better is it?

8. If money is worth 8% effective, which of the two offers in Example 7 is the better and how much?

9. A father begins when his son is 7 years of age to lay aside \$400 at the end of each year for the son's college education. If all sums can be accumulated at 5% until the fund is exhausted, how much will be available at the beginning of each of the four years of the boy's college course, if he starts to college at the age of 17?

10. How much money should be set aside at the end of each year from a boy's 8th to his 17th year inclusive in order to provide him with \$800 per year during each year of his college course, if he enters college at the age of 17 and money can be made to earn 5%?

11. A man purchases a farm for \$20,000, agreeing to pay \$5000 in cash and the balance, principal and interest at 6%, in ten equal annual payments. How much is the annual payment?

PROBLEM. *To find the present value of an annuity of 1 per annum, payable p times a year for n years.*

The stated payment will be $\frac{1}{p}$, and the times to elapse before the several payments are to be made will be, in years and fractions of a year, $\frac{1}{p}, \frac{2}{p}, \frac{3}{p}, \dots, n - \frac{1}{p}, n$.

The present values in terms of the effective rate i will be

$$\frac{1}{p}(1+i)^{-\frac{1}{p}} = \frac{1}{p}v^{\frac{1}{p}}, \quad \frac{1}{p}(1+i)^{-\frac{2}{p}} = \frac{1}{p}v^{\frac{2}{p}}, \quad \dots, \quad \frac{1}{p}(1+i)^{-n} = \frac{1}{p}v^n.$$

Denoting the sum of these present values by $a_{\overline{n}|}^{(p)}$, we have

$$a_{\overline{n}|}^{(p)} = \frac{1}{p} (v^{\frac{1}{p}} + v^{\frac{2}{p}} + \dots + v^{n-\frac{1}{p}} + v^n).$$

The right member is a geometric progression of np terms, with first term $\frac{1}{p} v^{\frac{1}{p}}$, last term $\frac{1}{p} v^n$, and ratio $v^{\frac{1}{p}}$. Consequently, the sum is

$$a_{\overline{n}|}^{(p)} = \frac{1}{p} \frac{v^{n+\frac{1}{p}} - v^{\frac{1}{p}}}{v^{\frac{1}{p}} - 1}.$$

Dividing numerator and denominator by $v^{\frac{1}{p}}$, and noting that

$$v = (1+i)^{-1},$$

we find

$$a_{\overline{n}|}^{(p)} = \frac{1-v^n}{p[(1+i)^{\frac{1}{p}}-1]} = \frac{1-v^n}{j_{(p)}}. \quad (3)$$

Formula (3) may be put in a form better adapted for computation if the right member be multiplied and divided by i . We have then

$$a_{\overline{n}|}^{(p)} = \frac{1-v^n}{i} \cdot \frac{i}{j_{(p)}} = a_{\overline{n}|} \cdot \frac{i}{j_{(p)}}. \quad (3')$$

If the nominal rate j , convertible m times a year, is given, $1+i$ must be replaced by $(1+\frac{j}{m})^m$; so that

$$a_{\overline{n}|}^{(p)} = \frac{1}{p} \frac{1 - \left(1 + \frac{j}{m}\right)^{-mn}}{\left(1 + \frac{j}{m}\right)^{\frac{m}{p}} - 1}. \quad (4)$$

Finally, if the conversion interval for interest coincides with the payment interval for the annuity, $m=p$, and (4) becomes

$$a_{\overline{n}|}^{(p)} = \frac{1}{p} \frac{1-v^n}{\frac{j}{p}} = \frac{1}{p} \frac{1 - \left(1 + \frac{j}{p}\right)^{-np}}{\frac{j}{p}} = \frac{1}{p} a_{\overline{np}|} \left(\text{at rate } \frac{j}{p}\right). \quad (5)$$

The case covered by formula (5) occurs frequently in practice.

EXAMPLES

1. Find the present value of \$1000 per annum, payable quarterly for 10 years, when money is worth 6%.

Solution. The problem is solved most easily by means of formula (3'). We have therefore for the present value of 1

$$\begin{aligned} a_{\overline{10}|}^{(4)} \text{ at } 6\% &= a_{\overline{10}|} \text{ at } 6\% \times \frac{i}{j_{(4)}} \text{ at } 6\% \\ &= 7.3600871 \times 1.0222269 \\ &= 7.5236349. \end{aligned}$$

The present value of \$1000 per annum is therefore \$7523.63.

2. Find the present value of \$1000 per annum, payable quarterly for 10 years, if money is worth 6% nominal, convertible quarterly.

Solution. The problem is equivalent to that of finding the present value of \$250 per quarter for 40 quarters at $1\frac{1}{2}\%$. The result is therefore

$$\begin{aligned} \$250 \times a_{\overline{40}|} \text{ at } 1\frac{1}{2}\% &= \$250 \times 29.915845 \\ &= \$7478.96. \end{aligned}$$

3. Find the present value of \$500 per annum, payable semiannually for 20 years, if money is worth 5% effective.

4. Find the present value of \$500 per annum, payable semiannually for 20 years, if money is worth 5% nominal, convertible semiannually, on the supposition that the payment interval and the conversion interval coincide.

5. A man pays \$10 down and \$10 per month for 3 years for a piano. If money is worth 6%, what should be the cash price of the instrument?

6. If the quarterly premium on a life-insurance policy is \$20.24 payable in advance, what would the annual premium payable in advance be, if money is worth 6%?

7. What payment made at the end of each month is equivalent to an annual payment of \$100 made at the beginning of the year, when money is worth 6%?

8. A man may purchase a piano for \$400 cash or \$10 down and fixed monthly payments for 3 years. If interest is charged at the rate of 7%, what is the amount of the monthly payment?

46. Annuities due. An annuity due has been defined (§ 43) as one for which the first payment is made at the beginning instead of at the end of the first interval.

PROBLEM. *To find the present value of an annuity due.*

The solution is almost self-evident, for the present value consists of (1) a cash payment and (2) the present value of an ordinary annuity whose term is less by one payment interval

than the term of the annuity due. Using the notation of § 43 we have, for an annuity due payable annually,

$$a_{\overline{n}|} = a_{\overline{n-1}|} + 1. \quad (1)$$

For an annuity due payable p times a year,

$$a_{\overline{n}|}^{(p)} = a_{\overline{n-\frac{1}{p}}|}^{(p)} + \frac{1}{p}, \quad (2)$$

when the symbol $a_{\overline{n-\frac{1}{p}}|}^{(p)}$ is defined by the equation

$$a_{\overline{n-\frac{1}{p}}|}^{(p)} = \frac{1 - (1+i)^{-(n-\frac{1}{p})}}{p[(1+i)^{\frac{1}{p}} - 1]}. \quad (3)$$

The expression for $a_{\overline{n-\frac{1}{p}}|}^{(p)}$ may be obtained by summing the series for the present value of an ordinary annuity payable p times a year for $n - \frac{1}{p}$ years.

Clearly, formula (1) is a special case of (2).

A useful form for $a_{\overline{n}|}^{(p)}$ is obtained if $a_{\overline{n-\frac{1}{p}}|}^{(p)}$ in (2) be replaced by its value as given by (3) and the resulting expression be reduced to a common denominator. The result is

$$a_{\overline{n}|}^{(p)} = \frac{-(1+i)^{-(n-\frac{1}{p})} + (1+i)^{\frac{1}{p}}}{j_{(p)}}, \quad (4)$$

which contains the factor $(1+i)^{\frac{1}{p}}$. Consequently,

$$a_{\overline{n}|}^{(p)} = (1+i)^{\frac{1}{p}} \cdot \frac{1 - (1+i)^{-n}}{j_{(p)}} = (1+i)^{\frac{1}{p}} \cdot a_{\overline{n}|} \cdot \frac{i}{j_{(p)}}. \quad (5)$$

Formula (5) is well adapted to computation, since the three factors on the right, $(1+i)^{\frac{1}{p}}$, $a_{\overline{n}|}$, and $\frac{i}{j_{(p)}}$, and their logarithms, have all been tabulated.

PROBLEM. *To find the amount of an annuity due.*

If the annuity is payable annually, the first payment is on interest for n years, the second for $n-1$ years, and so on to the last which is on interest for 1 year. The amount of the annuity due is therefore

$$s_{\overline{n}|} = (1+i)^n + (1+i)^{n-1} + \dots + (1+i).$$

The amount of an ordinary annuity for $n+1$ years is

$$s_{\overline{n+1}|} = (1+i)^n + (1+i)^{n-1} + \dots + (1+i) + 1.$$

A comparison of these two sums shows that $s_{\overline{n}|}$ differs from $s_{\overline{n+1}|}$ by the omission of the last term of $s_{\overline{n+1}|}$, which is 1. Consequently,

$$s_{\overline{n}|} = s_{\overline{n+1}|} - 1. \quad (6)$$

Similarly,

$$s_{\overline{n}|}^{(p)} = s_{\overline{n+\frac{1}{p}}|}^{(p)} - \frac{1}{p}, \quad (7)$$

where

$$s_{\overline{n+\frac{1}{p}}|}^{(p)} = \frac{(1+i)^{n+\frac{1}{p}} - 1}{j_{(p)}}. \quad (8)$$

Formula (7) is not well adapted for computation. However, if $s_{\overline{n+\frac{1}{p}}|}^{(p)}$ in (7) be replaced by its value as given by (8), the result be reduced to a common denominator, and the factor $(1+i)^{\frac{1}{p}}$ be set aside, (7) can be written in the form

$$s_{\overline{n}|}^{(p)} = (1+i)^{\frac{1}{p}} \cdot s_{\overline{n}|} \cdot \frac{i}{j_{(p)}}. \quad (9)$$

The similarity between the formulas for $a_{\overline{n}|}$ and $\ddot{a}_{\overline{n}|}^{(p)}$ and those for $s_{\overline{n}|}$ and $s_{\overline{n}|}^{(p)}$ is noteworthy.

EXAMPLES

1. Find the present value of \$237.50 payable at the beginning of each year for 15 years, when money is worth 5%.
2. Find the present value of \$1000 per year payable in quarterly installments at the beginning of each year for 10 years, when money is worth 6% effective.
3. Find the amount of an annuity of \$1500 per annum payable at the beginning of each year for 20 years, when money is worth 6% effective.
4. Find the amount of \$1000 per year payable quarterly for 15 years, if the payments are made at the beginning of each quarter.
5. The annual premium on a life-insurance policy is \$97.50, payable in advance. What should be the quarterly premium to be paid in advance?
6. A man deposits \$450 at the beginning of each quarter in a savings bank which pays 4% nominal, convertible semiannually. What will his savings amount to by the end of 5 years?

47. Deferred annuities. A deferred annuity is one whose payments begin at some future time, say m years hence. The amount of an annuity deferred m years is denoted by ${}_m|s_{\overline{n}|}^{(p)}$, and the present value by ${}_m|a_{\overline{n}|}^{(p)}$.

Clearly
$${}_m|s_{\overline{n}|}^{(p)} = s_{\overline{n}|}^{(p)} = s_{\overline{n}|} \cdot \frac{i}{j_{(p)}} \quad (1)$$
 for all values of m .

The present value of an annuity deferred m years may be looked upon as the present value of an annuity for $n + m$ years from which the present value of a similar annuity for m years has been subtracted. Consequently,

$${}_m|a_{\overline{n}|}^{(p)} = a_{\overline{n+m}|}^{(p)} - a_{\overline{m}|}^{(p)} = (a_{\overline{n+m}|} - a_{\overline{m}|}) \cdot \frac{i}{j_{(p)}} = v^m a_{\overline{n}|} \cdot \frac{i}{j_{(p)}}. \quad (2)$$

EXAMPLES

1. A mine is estimated to yield \$10,000 net per year for 20 years, but production cannot begin for 5 years. What is the present value of the total estimated output on a 6% basis?

2. Find the present value of an annuity of \$250 per quarter for 12 years on a 5% basis, if the first payment is to be made $8\frac{1}{4}$ years from date.

3. Find the present value of the annuity of Example 2 if the money rate is 5% nominal, convertible semiannually.

48. Summary of annuity formulas; tables. A close examination of the annuity formulas will reveal the fact that not more than ten distinct formulas have been developed. Of these ten, four are by far the most important; and of the four, two are special cases of the other two. The formulas are as follows:

For the amount of 1 per annum, payable annually,

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}. \quad (1)$$

For the present value of 1 per annum, payable annually,

$$a_{\overline{n}|} = \frac{1 - (1+i)^{-n}}{i}. \quad (2)$$

For the amount of 1 per annum, payable p times a year,

$$s_{\overline{n}|}^{(p)} = \frac{1 - (1+i)^{-n}}{\frac{i}{p} \left((1+i)^{\frac{1}{p}} - 1 \right)} = s_{\overline{n}|} \cdot \frac{i}{j_{(p)}}. \quad (3)$$

For the present value of 1 per annum, payable p times a year,

$$a_{\overline{n}|}^{(p)} = \frac{1}{p} \frac{1 - (1+i)^{-n}}{(1+i)^{\frac{1}{p}} - 1} = a_{\overline{n}|} \cdot \frac{i}{j_{(p)}}. \quad (4)$$

Clearly formulas (1) and (2) are special cases of (3) and (4), for which $p = 1$.

Of the following six formulas (5) and (6) are special cases of (3) and (4). When the rate is nominal and the conversion interval for interest corresponds with the payment interval,

$$s_{\overline{n}|}^{(p)} \text{ (at nominal rate } j) = \frac{1}{p} \cdot s_{\overline{np}|} \left(\text{at rate } \frac{j}{p} \right), \quad (5)$$

$$a_{\overline{n}|}^{(p)} \text{ (at nominal rate } j) = \frac{1}{p} \cdot a_{\overline{np}|} \left(\text{at rate } \frac{j}{p} \right). \quad (6)$$

If the annuity is due,

$$s_{\overline{n}|}^{(p)} = s_{\overline{n+\frac{1}{p}}|}^{(p)} - \frac{1}{p} = (1+i)^{\frac{1}{p}} \cdot s_{\overline{n}|} \cdot \frac{i}{j_{(p)}}, \quad (7)$$

$$a_{\overline{n}|}^{(p)} = a_{\overline{n-\frac{1}{p}}|}^{(p)} + \frac{1}{p} = (1+i)^{\frac{1}{p}} \cdot a_{\overline{n}|} \cdot \frac{i}{j_{(p)}}. \quad (8)$$

If the annuity is deferred for m years,

$${}_m|s_{\overline{n}|}^{(p)} = s_{\overline{n}|}^{(p)}, \quad (9)$$

$${}_m|a_{\overline{n}|}^{(p)} = (a_{\overline{m+n}|} - a_{\overline{m}|}) \cdot \frac{i}{j_{(p)}} = v^m \cdot a_{\overline{n}|} \cdot \frac{i}{j_{(p)}}. \quad (10)$$

If the annuity is an annuity of R per annum, each of the foregoing expressions will be multiplied by R .

The most superficial examination of the formulas given in the foregoing summary will convince us that computations will be greatly simplified by means of tables of values of the four standard functions

$$s_{\overline{n}|}, \quad a_{\overline{n}|}, \quad (1+i)^{\frac{1}{p}}, \quad \text{and} \quad \frac{i}{j_{(p)}},$$

where n and p are integers. Indeed, values of these four functions are all that are absolutely essential in the numerical solution of all except the most special problems involving annuities. Seven-place tables for these functions for the more common rates

of interest are given at the end of this book. (See Tables V, VI, VIII, and X.) Eight-place tables of the functions together with seven-place logarithms of the functions are given in Glover's invaluable "Tables of Compound Interest Functions," George Wahr, Ann Arbor, 1923.

49. The annuity that will amount to 1.

PROBLEM. *To find the annuity that will amount to 1 in n years.*

The problem requires the determination of the annual rent of the annuity that will amount to 1 in n years.

Let R be the annual rent. The amount of an annuity with annual rent R is, by § 44, $Rs_{\overline{n}|}$, or $Rs_{\overline{n}|}^{(p)}$, as the case may be. For an annuity payable annually and amounting to 1 after n years,

$$Rs_{\overline{n}|} = 1;$$

$$\text{whence} \quad R = \frac{1}{s_{\overline{n}|}} = \frac{i}{(1+i)^n - 1}. \quad (1)$$

If the annuity is payable p times a year, clearly

$$R = \frac{1}{s_{\overline{n}|}^{(p)}} = \frac{p[(1+i)^{\frac{1}{p}} - 1]}{(1+i)^n - 1} = \frac{j_{(p)}}{i} \cdot \frac{1}{s_{\overline{n}|}}. \quad (2)$$

The formula for the annual rent of an annuity that will amount to K is found by multiplying equation (1), or equation (2), as the case may be, by K . If the annual rent of an annuity that will amount to K be denoted by R' , where

$$R' = KR, \quad (3)$$

$$\text{then} \quad R' = K \cdot \frac{1}{s_{\overline{n}|}} = \frac{Ki}{(1+i)^n - 1}. \quad (4)$$

A similar formula is derived from (2) when the annuity is payable oftener than once a year.

When interest is convertible m times a year, the appropriate formula is found by replacing i and $1+i$, in formulas (1), (2), or (4), by their values as given by the fundamental relation

$$i = \left(1 + \frac{j}{m}\right)^m - 1.$$

For reasons that will appear later, the problem of the present section may be called the *sinking-fund problem*, and equation (1), or its equivalent, (4), is called the *sinking-fund equation*. The importance of the problem consists in the fact that it is equivalent to the problem of determining the amount that must be set aside annually to meet a given obligation at the end of a given time. For example, if a city bonds itself for \$200,000 for twenty years, in order to erect a new high school, some provision must be made for the payment of the bonds when they fall due. A convenient method is to set aside the same amount each year until the debt becomes due. The amount to be set aside each year is found by the formula (1). Substituting the values $K = 200,000$, $i = .04$, $n = 20$, we have, as the required amount,

$$\begin{aligned} \$200,000 \times \frac{.04}{(1.04)^{20} - 1} &= \$200,000 \times .03358175 \\ &= \$6716.35. \end{aligned}$$

The subject will be taken up later, in the sections on sinking funds.

The symbol $\frac{1}{s_{\overline{n}|i}}$ should be looked upon as the symbol for *the annuity that will amount to 1* rather than as the reciprocal of $s_{\overline{n}|i}$.

EXAMPLES

1. A man gives a mortgage for \$10,000, to be repaid in 5 years, with interest payable annually at 6%. If the interest is paid promptly, what sum must be set aside each year to repay the principal when it falls due, provided the money set aside can be invested at 4%?

2. What sum must be set aside annually to provide for the rebuilding, after 25 years, of a bridge costing \$25,000, provided the money set aside can be invested at 4%?

3. What sum must a father set aside at the end of each year to provide \$1000 for his son at the beginning of each of 4 college years, if he begins on the boy's fifth birthday and the boy enters college at 17, money worth 5%?

4. A mine is bought for \$400,000 and will produce for 30 years. How much must be set aside annually out of the income to provide for the repayment of the original capital when the mine is exhausted, if money can be made to earn 5%?

50. The annuity that 1 will purchase. The annuity whose present value is 1 is usually spoken of as *the annuity that 1 will purchase*. It plays a large part in the solution of many important problems.

PROBLEM. *To determine the annual rent of an annuity that 1 will purchase.*

Let R be the required annual rent. Since the present value of an annuity whose annual rent is R , is $Ra_{\overline{n}|i}$, the value of R will be determined from the equation

$$Ra_{\overline{n}|i} = 1.$$

Consequently,
$$R = \frac{1}{a_{\overline{n}|i}} = \frac{i}{1 - v^n} \quad (1)$$
 is the required formula.

If the annuity is payable p times a year, clearly

$$R = \frac{1}{a_{\overline{n}|i^{(p)}}} = \frac{1}{a_{\overline{n}|i}} \cdot \frac{j_{(p)}}{i}. \quad (2)$$

To find the annuity that a given sum A will purchase, it is only necessary to multiply both sides of (1) or (2) by A . For the general case,

$$R' = AR = A \cdot \frac{1}{a_{\overline{n}|i^{(p)}}} = A \cdot \frac{1}{a_{\overline{n}|i}} \cdot \frac{j_{(p)}}{i}. \quad (2')$$

If the interval for annuity payments does not correspond with the conversion interval, the appropriate formula may be found by replacing $1+i$ by its value $\left(1 + \frac{j}{m}\right)^m$.

Formula (1) is called the *amortization equation of Euler*. It will be considered more fully in the chapter on amortization.

Suppose, for example, a man wishes to pay a debt of \$5000, bearing interest at 6% effective, in five equal annual installments, the first installment to be paid one year hence. The annual payment is required. The payments constitute an annuity whose annual rent R' is required. By formula (1),

$$\begin{aligned} R &= \$5000 \cdot \frac{.06}{1 - (1.06)^{-5}} \\ &= \$5000 \times .237396 \\ &= \$1186.98. \end{aligned}$$

EXAMPLES

1. A man makes a cash payment of \$2000 on a farm purchased for \$10,000, and wishes to pay the balance, with interest at 5%, in five equal annual installments. What will the annual installment be?

2. A farmer took out a loan of \$5000 under the long-time amortization plan, agreeing to pay the principal and interest in equal annual installments over a period of 40 years. What was the amount of the annual installment?

3. A man bought a piano for \$500, agreeing to pay \$50 down and the balance with interest in monthly payments equal in amount during a period of 3 years. What was the amount of the monthly payment?

4. Find the formula for the annuity, payable p times a year, for n years, that can be purchased for 1, when interest is at nominal rate j , convertible m times a year.

51. Fundamental relation between $\frac{1}{s_{\overline{n}|}^{(p)}}$ and $\frac{1}{a_{\overline{n}|}^{(p)}}$.

THEOREM. *The annual rent of the annuity that 1 will purchase, diminished by the annual rent of the annuity that will amount to 1, is equal to $j_{(p)}$, where $j_{(p)}$ is defined by the equation*

$$j_{(p)} = p[(1+i)^{\frac{1}{p}} - 1]. \quad (\text{See (3')}, \S 35.)$$

In symbols the theorem to be proved is

$$\frac{1}{a_{\overline{n}|}^{(p)}} - \frac{1}{s_{\overline{n}|}^{(p)}} = j_{(p)}. \quad (1)$$

By (2) of § 49 and (2) of § 50,

$$\frac{1}{a_{\overline{n}|}^{(p)}} = \frac{1}{a_{\overline{n}|}} \cdot \frac{j_{(p)}}{i} \quad \text{and} \quad \frac{1}{s_{\overline{n}|}^{(p)}} = \frac{1}{s_{\overline{n}|}} \cdot \frac{j_{(p)}}{i}. \quad (2)$$

Consequently,

$$\begin{aligned} \frac{1}{a_{\overline{n}|}^{(p)}} - \frac{1}{s_{\overline{n}|}^{(p)}} &= \left(\frac{1}{a_{\overline{n}|}} - \frac{1}{s_{\overline{n}|}} \right) \frac{j_{(p)}}{i} \\ &= \left[\frac{1}{1 - (1+i)^{-n}} - \frac{1}{(1+i)^n - 1} \right] \cdot j_{(p)} \\ &= \left[\frac{(1+i)^n}{(1+i)^n - 1} - \frac{1}{(1+i)^n - 1} \right] \cdot j_{(p)} \\ &= j_{(p)}. \end{aligned}$$

The theorem is therefore proved.

The most important case is that for which $p=1$. When $p=1$, $s_{\overline{n}|}^{(1)} = s_{\overline{n}|}$, $a_{\overline{n}|}^{(1)} = a_{\overline{n}|}$, and $j_{(1)} = i$. Consequently,

$$\frac{1}{a_{\overline{n}|}} - \frac{1}{s_{\overline{n}|}} = i. \quad (3)$$

By means of formula (3) the value of either $\frac{1}{a_{\overline{n}|}}$ or of $\frac{1}{s_{\overline{n}|}}$ can be found when the other is known. In practice this fact saves the printing of one table. In this book the table for $\frac{1}{a_{\overline{n}|}}$ is given (Table VII). The values of $\frac{1}{s_{\overline{n}|}}$ are found by subtracting i from the corresponding values of $\frac{1}{a_{\overline{n}|}}$.

EXERCISE

Prove the relation (3) directly by means of the expressions for $s_{\overline{n}|}$ and $a_{\overline{n}|}$.

52. The term of an annuity.

PROBLEM. *To find the term of an annuity when the amount, the annual rent, and the rate of interest are given.*

By (6) of § 44 the amount K of an annuity whose annual rent is R is given by

$$Rs_{\overline{n}|} \cdot \frac{i}{j_{(p)}} = K. \quad (1)$$

From this equation

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i} = \frac{K}{R} \cdot \frac{j_{(p)}}{i}. \quad (2)$$

From equation (2)

$$(1+i)^n = 1 + \frac{K}{R} \cdot j_{(p)}. \quad (3)$$

The exponential equation (3) solved for n gives

$$n = \frac{\log \left(1 + \frac{K}{R} \cdot j_{(p)} \right)^*}{\log(1+i)}. \quad (4)$$

If $p=1$, $j_{(p)} = i$ and equation (4) takes the very useful form

$$n = \frac{\log \left(1 + \frac{K}{R} i \right)}{\log(1+i)}. \quad (5)$$

* For the meaning of an annuity with a fractional term see § 58.

PROBLEM. *To find the term of an annuity when the present value, the annual rent, and the rate of interest are given.*

If A denote the present value and R the annual rent, the present value is, by (3') of § 45,

$$Ra_{\overline{n}|} \cdot \frac{i}{j_{(p)}} = A. \quad (6)$$

The unknown number n occurs in $a_{\overline{n}|}$ only and is found in a manner exactly like that employed in the previous problem. The result is

$$n = - \frac{\log \left(1 - \frac{A}{R} j_{(p)} \right)}{\log(1+i)}. \quad (7)$$

If $p=1$, equation (7) takes the form

$$n = - \frac{\log \left(1 - \frac{A}{R} i \right)}{\log(1+i)}. \quad (8)$$

Approximate values for n may also be found by means of the tables.

EXAMPLES

1. A man wishes to accumulate the purchase price, \$1200, of an automobile by setting aside \$100 a month. If he can realize 4% effective on his money, how long will it be before he can pay cash for the car?

2. A man buys an automobile priced at \$1200, agreeing to pay \$100 down and the balance in monthly installments of \$100 each, with interest at 6%. How long will it take him to complete payment?

3. How long will it take to pay for a piano valued at \$500 by monthly payments of \$10 each, if the first payment is cash and interest at 6% is charged?

4. Under the Federal Land Loan Act of 1916 a farmer may borrow money for 5%, with 1% for amortization, i.e. he makes during each year of the life of the contract a payment equal to the first year's interest plus 1% of the principal. How long will it take him to pay his loan?

5. A real-estate firm sells houses on the monthly payment plan, taking a "land contract" until the property is half paid for, when it gives a deed and takes a mortgage. On June 1, 1923, a man purchased a house and lot for \$8000, agreeing to pay \$50 per month. If interest is charged at 6%, when should the land contract be changed to a mortgage?

53. The rate of interest borne by an annuity.

PROBLEM. *To find the rate of interest borne by an annuity which is payable annually, when the amount (or the present value), the annual rent, and the term are given.*

Solution by interpolation. (See Chapter III.) This method, which is the most useful method, can be best understood by a concrete problem. Suppose, for example, it is required to find the rate of interest at which a series of payments, each of \$500, made at the end of each year for 10 years, will amount to \$6250.

$$\text{From the data given,} \quad 6250 = 500 \cdot s_{\overline{10}|}; \quad (1)$$

$$\text{so that} \quad s_{\overline{10}|} = \frac{6250}{500} = 12.50 = y.$$

From the table for $s_{\overline{n}|}$ we find

$$s_{\overline{10}|} \text{ at } 4\frac{1}{2}\% = 12.2882094 = y_1,$$

$$s_{\overline{10}|} \text{ at } 5\% = 12.5778925 = y_2,$$

$$s_{\overline{10}|} \text{ at } 5\frac{1}{2}\% = 12.8753538 = y_3.$$

From these figures it is clear that i lies between .045 and .05, since $s_{\overline{10}|}$ at the unknown rate is equal to 12.50.

The two tabular differences are .2896831 and .2974613. The first and second differences are therefore

$$D_1 = .2896831 \text{ and } D_2 = .0077782.$$

Using first differences only, we have to find p by means of the equation

$$p = \frac{y - y_1}{D_1} = \frac{y - y_1}{y_2 - y_1}. \quad (\text{Formula (2), § 22.})$$

$$\begin{aligned} \text{We have} \quad p &= \frac{12.50 - 12.2882094}{.2896831} = \frac{.2117906}{.2896831} \\ &= .73111, \end{aligned}$$

where p is measured in terms of the interval from .045 to .05. This interval is .005, so that the correction to be added to .045 is

$$.005 \times .73111 = .003656.$$

The approximate value of i is therefore given by

$$i = .048656.$$

This result is correct to the fifth decimal place.

The student should become so familiar with this interpolation process that he can carry it out without referring to the formulas.

If a closer approximation is desired we may use the second differences, and insert the value of p found above in the right member of the formula.

$$p = \frac{y - y_1}{D_1 + \frac{p_1 - 1}{1 \cdot 2} D_2} \quad (\text{Formula (2), § 23.})$$

The result is

$$p = \frac{.2117906}{.2896831 - (.13444) \times (.0077782)} \\ = .73376.$$

The correction to be added to .045 is

$$.005 \times .73376 = .0036688.$$

The second approximation is therefore given by

$$i = .0486688.$$

This result is accurate to the sixth decimal place.

Solution by algebraic approximation. Formula (1) is an equation of degree ten in the unknown number i . When cleared of fractions it may be written in the form

$$(1 + i)^{10} - 12.5i - 1 = 0. \quad (2)$$

From the tables it is clear that i lies between .045 and .05. We may then write

$$i = .05 + h, \quad (3)$$

where h is a negative number as yet unknown.

$$\text{Then} \quad (1.05 + h)^{10} - 12.5(.05 + h) - 1 = 0. \quad (4)$$

Expanding $(1.05 + h)^{10}$ by the binomial theorem and collecting powers of h , we find

$$(1.05)^{10} - 1.625 + 10(1.05)^9 h - 12.5h + \text{higher powers of } h = 0. \quad (5)$$

The number h is known to be a very small number. Consequently, the higher powers of h will be nearly negligible, and if *first powers only* be used, the equation

$$[10(1.05)^9 - 12.5]h + (1.05)^{10} - 1.625 = 0 \quad (6)$$

will serve to determine h approximately.

The values of $(1.05)^9$ and $(1.05)^{10}$ may be taken directly from Table III. Equation (6) is then reduced to the form

$$3.01328h + .003895 = 0; \quad (7)$$

or

$$h = -.0013.$$

The first approximation for i is then

$$i = .05 - .0013 = .0487.$$

The approximation process may be repeated by making

$$i = .0487 + h',$$

where h' is smaller than h .

Equation (2) now becomes

$$(1.0487 + h)^{10} - 12.5(.0487 + h) - 1 = 0. \quad (8)$$

Treating this equation exactly as equation (4) was treated, we find for h' the approximate value

$$h' = \frac{12.5 \times .0487 + 1 - (1.0487)^{10}}{10 \times (1.0487)^9 - 12.5}. \quad (9)$$

The values of $(1.0487)^{10}$ and $(1.0487)^9$ may be found by logarithms. The final result is

$$h' = -.000032.$$

The second approximation for i is therefore

$$\begin{aligned} i &= .0487 - .000032 \\ &= .048668. \end{aligned}$$

This result is correct to within less than .000002.

The solution when the present value is given is entirely similar to the solution when the amount is given.

The method may be formulated in the following rule:

To find the rate of interest for an annuity when the annual rent, the time, and the amount (or the present value) are given, find from the tables the two values between which i lies; select as an approximate value i' for i one of these values or some convenient value lying between them, and in the appropriate formula replace i by $i' + h$. Integralize the resulting equation, expand all powers of $1 + i' + h$ in powers of h , drop powers of h above the first, and solve the resulting equation for h . The value of

$$i' + h$$

thus determined will then be a first approximation to i . If a closer approximation is desired, repeat the process, using the approximate value $i' + h$ in place of i' .

EXERCISES

1. A man invested \$500 a year in a savings account, and found, at the end of 10 years, that the amount to his credit was \$6317.29. Assuming that the amounts were paid in at the end of each year, what rate of interest did he receive? Solve by first-difference interpolation, by second-difference interpolation, and by algebraic approximation.

2. Solve Example 1 by one method on the supposition that the annual payments are made at the beginning instead of at the end of the year.

3. A farmer assumes a mortgage for \$5000, agreeing to pay principal and interest in twenty equal annual installments of \$425 per year. Assuming the payments to be made at the end of the year, what rate of interest is he paying?

4. A man buys a piano for \$500, paying \$25 down and agreeing to pay the balance in thirty-six equal monthly installments of \$15 each. What rate of interest does he pay? Solve by interpolation.

5. A man paid \$10 per month into a building and loan association. His stock matured to \$2000 in 11 years and 10 months. On the assumption that the payments were made at the end of the month, what rate of interest did he receive for the period?

MISCELLANEOUS EXAMPLES AND PROBLEMS

1. A mine yields \$5000 net payable to the owner in two equal semi-annual installments at the middle and the end of the year. If the payments are accumulated at 5% nominal, convertible semiannually, what will the amount be at the end of 20 years?

2. Find the amount in the previous problem if the payments are accumulated at 6% effective.

3. A mine yielding \$2500 net at the end of each half year will be "worked out" in 20 years, and the equipment will be worth \$1000 at that time. What is the value of the mine if money is worth 6%?

4. A man gives a mortgage to secure a loan on his property. How much must he invest at the end of each year, in a savings bank that pays 4% effective, in order to discharge the mortgage at the end of 10 years, provided interest is paid as it falls due?

5. If in the previous problem the payments are made to the savings bank at the end of each half year, and interest is convertible half yearly at nominal rate .04, what would the half-yearly payment be?

6. A man pays \$20 per month for 12 years into a building and loan association which pays $5\frac{1}{2}\%$ nominal, convertible semiannually. What will his stock be worth at the end of the time?

7. A life-insurance policy matures with a value of \$5000, and the insurance company gives the option between the whole amount in cash and ten equal annual installments, the first to be paid 1 year after the maturity of the policy. What will be the annual installment if the interest rate is .04?

8. A father, wishing to provide for the education of his son, sets aside, on the day of the son's birth, a sum which, invested at 5%, will yield \$500 a year for 4 years, the first installment to be paid on the boy's seventeenth birthday. What is the sum to be set aside?

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9. What sum must be paid each month to a building and loan association which pays 5.75% nominal, convertible semiannually, to enable a man to discharge a debt of \$4000 in 12 years?

10. A man buys a piano for \$500, agreeing to pay \$20 a month with interest at 6% effective on all sums remaining due. If the first payment is made on the date of purchase, how long will it take to pay for the instrument?

11. A bridge costing \$25,000 new must be replaced after 25 years. If money can be invested at 5% effective, what sum should be set aside annually to provide for the rebuilding of the bridge, provided it can be rebuilt for the same amount as the cost of the original structure?

12. How long will it take a man to accumulate \$100,000 by saving \$1000 a year and investing it at 6% effective?

13. What is the rate of interest on an annuity of \$100 per year, payable annually, which amounts to \$1300 in 10 years?

14. The present value of an annuity of \$100 per year, payable annually for 10 years, is \$780. What is the rate?

15. The amount of an annuity of \$100 per annum, payable annually, is \$1300 and the present value is \$750. What is the rate of interest?

SUGGESTION. Use the fundamental relation between $\frac{1}{a_{\overline{n}|}}$ and $\frac{1}{s_{\overline{n}|}}$, or solve directly by eliminating n between the equations giving the values of $a_{\overline{n}|}$ and $s_{\overline{n}|}$.

16. Prove that $s_{\overline{n}|}^{(p)} = (1+i)^n a_{\overline{n}|}^{(p)}$, and interpret the result in words.

17. Find the relation $\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$

by eliminating n from the two equations

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i} \quad \text{and} \quad a_{\overline{n}|} = \frac{1 - (1+i)^{-n}}{i}.$$

18. Prove that when the conversion interval for interest coincides with the interval of payment for the annuity,

$$a_{\overline{n}|}^{(p)} = \frac{1}{p} a_{\overline{np}|} \text{ at rate } \frac{j}{p}.$$

19. Prove that $s_{\overline{n}|}^{(m)} = s_{\overline{n}|} \cdot \frac{i}{j_{(m)}}$,

where $s_{\overline{n}|}$ is computed at the effective rate i .

20. Prove that

$$a_{\overline{n}|}^{(m)} = a_{\overline{n}|} \cdot \frac{i}{j_{(m)}},$$

where $a_{\overline{n}|}$ is computed at the effective rate i .

21. Prove that when the conversion interval coincides with the interval of payment,

$$\frac{1}{s_{\overline{n}|}^{(p)}} = \frac{p}{s_{\overline{np}|}} \text{ at rate } \frac{j}{p},$$

and

$$\frac{1}{a_{\overline{n}|}^{(p)}} = \frac{p}{a_{\overline{np}|}} \text{ at rate } \frac{j}{p}.$$

22. Find the expression for an annuity of 1 per annum payable every 5 years for n years (*a*) when n is an exact multiple of 5; (*b*) when n contains 5 q times with a remainder r .

23. If $s_{\overline{n}|}$ denote the amount of an "annuity due" at the end of n years, prove that

$$s_{\overline{n}|} = s_{\overline{n-1}|} + 1.$$

CHAPTER VIII

SOME SPECIAL TYPES OF ANNUITIES

54. Perpetuities and capitalization. A *perpetuity* is an annuity whose payments are to be continued forever. Obviously the amount of a perpetuity would be an expression without meaning, since, as time went on, the amount of an annuity would increase beyond all bounds. On the contrary, the present value of a perpetuity is a definite sum, viz. the limit of $a_{\overline{n}|}$ as n increases indefinitely. The notion of a perpetuity is one of great importance in practical business affairs, as we shall soon see.

PROBLEM. *To find the present value of a perpetuity whose payments are made annually.*

Denoting the present value of the perpetuity of 1 per annum by a_{∞} , we have, by definition,

$$a_{\infty} = \lim_{n \rightarrow \infty} a_{\overline{n}|} = \lim_{n \rightarrow \infty} \frac{1 - v^n}{i}.$$

But
$$v = \frac{1}{1+i}$$

is a fraction whose value is less than unity; hence

$$\lim_{n \rightarrow \infty} v^n = \lim_{n \rightarrow \infty} \frac{1}{(1+i)^n} = 0,$$

and, consequently,
$$\lim_{n \rightarrow \infty} \frac{1 - v^n}{i} = \frac{1}{i}.$$

We have therefore
$$a_{\infty} = \frac{1}{i}, \tag{1}$$

which represents the amount of money that must be invested permanently to yield an income of 1 per annum payable at the *end* of each year forever.

The problem may also be solved directly by noting that the perpetuity is the limit of the sum of the geometric progression

$$v + v^2 + v^3 + \dots + v^n,$$

as n approaches infinity.

If the annual rent of the perpetuity is R , the present value will be

$$Ra_{\infty} = \frac{R}{i}. \quad (1')$$

Equation (1') shows that an annual income of R per year indefinitely continued is equivalent to a cash sum of $\frac{R}{i}$. The result is trivial, but the idea is important. For example, if money is worth 5%, an income of \$2000 per year would be equivalent to \$40,000 in cash. Again, an acre of land that can be made to yield a net income of \$5 per annum would be worth \$100 with money at 5%. The cash equivalent of an annual expenditure may be found in the same way. For example, if it costs \$5 a year to dig the weeds from a block of cobblestone gutter, it would pay to spend \$100, in addition to the cost of the cobblestone gutter, to substitute a cement gutter needing no cleaning, when money is worth 5%.

The process of determining the cash equivalent of an annual income or an annual outgo that is supposed to continue indefinitely is called *capitalization*. It is customary to use the expression "to capitalize an income, or an outgo," at such and such a rate. In the example relating to the net income of the acre of land, we say that "the annual income of \$5 capitalized at 5% is \$100."

PROBLEM. *To find the present value of a perpetuity whose payments are made at intervals of k years.*

If the first payment be made at the end of k years, the second at the end of $2k$ years, and so on, the sum of the present values of the payments will be $v^k + v^{2k} + v^{3k} + \dots$,

and if the payments be continued for nk years, the sum may be denoted by $a_{nk, k}$. We have

$$a_{nk, k} = v^k \frac{1 - v^{nk}}{1 - v^k}$$

as the present value of the first n payments. Dividing the numerator and denominator of the expression on the right by v^k , and remembering that

$$v^k = \frac{1}{(1+i)^k},$$

we have, finally,

$$a_{nk, k} = \frac{1 - v^{nk}}{(1+i)^k - 1}. \quad (2)$$

If now the number of payments be increased indefinitely, nk will be increased indefinitely, and so

$$\lim_{n \rightarrow \infty} v^{nk} = 0.$$

Denoting $\lim_{n \rightarrow \infty} a_{nk, k}$ by $a_{\infty, k}$, we have at once

$$a_{\infty, k} = \frac{1}{(1+i)^k - 1}. \quad (3)$$

If the right members of formulas (2) and (3) be multiplied and divided by i they may be written in the forms

$$a_{nk, k} = a_{nk} \cdot \frac{1}{s_{\overline{k}|}} \quad (2')$$

and

$$a_{\infty, k} = \frac{1}{i} \cdot \frac{1}{s_{\overline{k}|}}, \quad (3')$$

respectively.

If the periodic payment be R , both sides of these equations should be multiplied by R .

EXAMPLES

1. A bridge costing \$25,000 must be renewed every 30 years. What sum should be set aside when the bridge is first built to provide for an indefinite number of renewals, on the assumption that the cost of renewal will remain constant and that money will remain at 4%?

Solution. By formula (3') the required sum is

$$\frac{\$25,000}{.04} \cdot \frac{1}{s_{\overline{30}|} \text{ at } 4\%} = \$11,143.75.$$

2. It takes 30 men at a cost of \$800 a year each to guard the grade crossings of a certain city. How much could the railroads afford to expend on track elevation, if money is worth 4%?

3. What is the value per acre of a farm that will produce \$10 per acre per year net on a 5% basis? on a 4% basis?

4. What is the value per acre of the farm in Example 3 if the farm must lie fallow every other year?

5. A building costing \$100,000 must be renewed every 40 years. If the cost of the renewals should be constant and money should remain at 4%, what sum would be required to keep the building permanently on the site?

The importance of problems like Example 1 becomes evident when we note that the \$11,143.75 is *the amount that might be expended over and above the \$25,000, to make the structure permanent.*

55. Capitalized cost. The *capitalized cost* of an article is defined as the first cost plus the present value of the cost of indefinite renewals. If C denote the first cost, C_∞ may be used to denote the capitalized cost. On the supposition that the cost of each successive renewal is the same as the first cost, by (3') of § 54 the cost of renewals is

$$\frac{C}{i} \cdot \frac{1}{s_{\overline{k}|}}$$

so that

$$C_\infty = C + \frac{C}{i} \frac{1}{s_{\overline{k}|}} = C \left(1 + \frac{1}{i} \cdot \frac{1}{s_{\overline{k}|}} \right). \quad (1)$$

When $s_{\overline{k}|}$ is replaced by its value and the expression in the parenthesis is reduced to a common denominator, it is easily seen that the expression for C_∞ reduces to the form

$$C_\infty = C \frac{(1+i)^k}{(1+i)^k - 1}. \quad (2)$$

Dividing numerator and denominator by $(1+i)^k$, multiplying both by i , and noting that

$$\frac{1}{(1+i)^k} = v^k,$$

we find for C_∞ the very convenient form

$$C_\infty = \frac{1}{i} \cdot \frac{1}{a_{\overline{k}|}}. \quad (3)$$

EXAMPLES

1. What is the capitalized cost for a railroad tie costing \$2 if it has to be renewed every 10 years and money is worth $4\frac{1}{2}\%$?
2. If a telephone pole costs \$50 placed in position, what is the capitalized cost if renewals have to be made every 10 years and money is worth 4%?
3. Under the old Jewish law the land lay fallow every seventh year. What should be the value of land capable of yielding \$10 per acre per year net under this system of agriculture?

56. Applications of capitalized cost. The notion of capitalized cost is helpful in the solution of many engineering problems. For example, a railroad company wishes to find out how much may be expended for treating ties costing \$2 laid, to extend the life of the tie from 7 to 15 years. The problem, stated in general terms, is as follows:

PROBLEM. *To determine what amount may be expended on an article costing C dollars in order to extend its life from k years to k' years.*

If X be the amount to be added to the original cost, the increased cost will be $C + X$. By (3) of § 55, the capitalized costs of the two articles will be

$$\frac{C}{i} \cdot \frac{1}{a_{\overline{k}|}} \quad \text{and} \quad \frac{C+X}{i} \cdot \frac{1}{a_{\overline{k'}|}}, \text{ respectively.}$$

If in the long run the use of one article is to be just as economical as the other, the capitalized costs will be equal. Consequently, we have for the determination of X the equation

$$\frac{C+X}{i} \cdot \frac{1}{a_{\overline{k'}|}} = \frac{C}{i} \cdot \frac{1}{a_{\overline{k}|}}. \quad (1)$$

From equation (1) it is easy to find

$$X = C(a_{\overline{k'}|} - a_{\overline{k}|}) \cdot \frac{1}{a_{\overline{k}|}}. \quad (2)$$

The right member of (2) may be put in still more compact form by noting that

$$a_{\overline{k'}|} - a_{\overline{k}|} = \frac{1 - v^{k'}}{i} - \frac{1 - v^k}{i} = \frac{v^k - v^{k'}}{i} = v^k \frac{1 - v^{k'-k}}{i} = v^k a_{\overline{k'-k}|}$$

and that

$$v^k \cdot \frac{1}{a_{\overline{k}|}} = \frac{1}{v^k a_{\overline{k}|}} = \frac{1}{s_{\overline{k}|}}$$

Equation (2) may then be written in the form

$$X = C \cdot a_{\overline{k'-k}|} \cdot \frac{1}{s_{\overline{k}|}} \quad (2')$$

The crosstie problem may be solved easily by means of formula (2) or (2'). If the tie costs originally \$2 and treatment will extend its life from 7 to 15 years,

$$X = 2 \cdot a_{\overline{15-7}|} \cdot \frac{1}{s_{\overline{7}|}}$$

Assuming money to be worth 5%,

$$\begin{aligned} X &= \$2 \times 6.463 \times 0.1047 \\ &= \$1.35. \end{aligned}$$

Formulas (2) and (2') can easily be modified to take into consideration the cost of maintenance and other items connected with the use of the article. (See Hickerson, "Formulas for Investment Calculations," reprinted from the *Elisha Mitchell Journal*, Chapel Hill, North Carolina.)

EXAMPLES

1. A farm can be made to yield a net annual income of \$1000, after improvements are kept up, fertility maintained, and current expenses met. If money is worth 5%, what is the farm worth?

2. A man is offered \$10,000 a year for a 99-year lease, or \$200,000 cash for a business site when money is worth 5%. Assuming a constant interest rate, which is the better offer? (Solve by inspection.)

3. How much can a railroad company afford to pay to abolish a grade crossing which is guarded by two watchmen, each receiving \$600 per year, when money can be invested at 4%?

4. A philanthropist wishes to give an endowment to be used for erecting and keeping in place a building costing \$100,000 and requiring to be rebuilt every 50 years. How much money will be required if money is worth 4%?

5. On a certain railroad there is a grade which necessitates the use of a "helper" engine and two crews. If the cost of wages, fuel, oil, and repairs is \$10,000 per annum, and a new engine costing \$12,000 must be purchased every 20 years, how much could the company afford to expend in reducing the grade to the point where the helper could be dispensed with, money worth 4%?

6. A building costs \$100,000 to erect. Annual repairs cost \$500. Every 10 years it must be thoroughly overhauled at a cost of \$5000, and every 75 years it must be rebuilt at the original cost. What amount of endowment would be necessary in order to build and maintain the structure indefinitely, if money is worth 4%?

7. What amount can be expended in treating a telegraph pole costing \$8, to extend its life from 8 to 15 years, if the cost of setting the pole is \$5, on the supposition that money is worth 5%?

8. Prove that the amount to be expended in doubling the life of an article is

$$X = Cv^k, \quad (3)$$

where k years is the life of the article without the added expenditure. (Obtain proof by means of formula (2').)

9. If an article of cost C has at the end of its period of service a scrap value S , prove that the amount that may be expended in extending its life from k years to k' years is given by the formula

$$X = a_{\overline{k'-k}|} \cdot \frac{1}{s_{\overline{k}|}} (C - S). \quad (4)$$

10. A man buys a team of horses for \$500 and a wagon and harness for \$150. The team will have to be replaced in 10 years and the wagon and harness in 20 years. Feed and blacksmith bills cost \$200 a year, taxes and insurance \$20, and wages of the driver \$600. How much must he make in a year in order to realize 8% on his investment, if the investment is capitalized at 5%?

57. Continuous annuities. We may think of an annuity for which the total amount paid in a year is a fixed sum, but the payments are infinitesimally small and are made momentarily. Such a hypothetical annuity is called a *continuous annuity*. Such annuities do not exist in the actual business world, but they are approximated by the business of large concerns which are receiving many small sums every day.

PROBLEM. To find the amount of a continuous annuity of 1 per annum for n years.

We will denote the amount of a continuous annuity by $\bar{s}_{\overline{n}|}$, and the present value by $\bar{a}_{\overline{n}|}$. Since p becomes infinite,

$$\bar{s}_{\overline{n}|} = \lim_{p=\infty} s_{\overline{n}|}^{(p)} = \lim_{p=\infty} \frac{(1+i)^n - 1}{p \left[(1+i)^{\frac{1}{p}} - 1 \right]}.$$

As the variable p does not occur in the numerator, we have only to consider the limit of the denominator. But we have already found that

$$\begin{aligned} \lim_{p=\infty} p \left[(1+i)^{\frac{1}{p}} - 1 \right] &= \log_e (1+i) \\ &= \delta, \quad (\text{See (3) and (4), § 37.}) \end{aligned}$$

where δ is the force of interest. Using these values for the limit in the denominator, we have

$$\bar{s}_{n|} = \lim_{p=\infty} s_{n|}^{(p)} = \frac{(1+i)^n - 1}{\delta} = s_{n|} \cdot \frac{i}{\delta}. \quad (1)$$

From the expression for $a_{n|}^{(p)}$, we obtain, in exactly the same manner,

$$\bar{a}_{n|} = \lim_{p=\infty} a_{n|}^{(p)} = \frac{1 - v^n}{\log_e(1+i)} = \frac{1 - v^n}{\delta} = a_{n|} \cdot \frac{i}{\delta}. \quad (2)$$

ILLUSTRATIVE EXAMPLE. Find the amount of a continuous annuity of \$1000 per annum for 5 years at 4%.

Solution.

$$\begin{aligned} \bar{s}_{5|} &= s_{5|} \times \frac{.04}{.04 - \frac{(.04)^2}{2} + \frac{(.04)^3}{3} - \dots} \\ &= \frac{.2166529}{.0392207} \\ &= 5.52394. \end{aligned}$$

The amount of the continuous annuity of \$1000 per annum is \$5523.94.

One advantage of the continuous annuity is that it may be used to find approximate results for annuities for which the payments are made frequently. For example, some fraternal societies require a weekly payment. The amount of such weekly payments made over a series of years would be given by

$$52p \times s_{n|}^{(52)} = 52p \times \frac{(1+i)^n - 1}{52[(1+i)^{\frac{1}{52}} - 1]},$$

where p is the amount of the weekly payment. The approximate value of the right member is

$$\frac{(1+i)^n - 1}{\log_e(1+i)} = \frac{(1+i)^n - 1}{\delta}.$$

EXAMPLES

1. Find the approximate amount of an annuity of \$52 per year, paid at the rate of \$1 per week for 10 years, accumulated at 4% effective.

2. A man spends 25 cents a day for cigars. If the sums had been accumulated at 5% effective, what would be the approximate amount at the end of 20 years?

58. Annuities with fractional terms. In certain methods of payment the term of an annuity frequently comes out a fractional number. The expressions for $s_{n|}$ and $a_{n|}$ have been found on the assumption that n is an integer, and have no meaning on any other basis.

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In order to extend the meanings for these symbols we will assume that they are defined by

$$\frac{(1+i)^n-1}{i} \quad \text{and} \quad \frac{1-v^n}{i}, \text{ respectively,}$$

for positive fractional as well as for positive integral values of n . When this is done the expressions no longer represent sums of geometric progressions as in sections 44 and 45. In order to interpret the meanings of these algebraic expressions let

$$n = n_1 + \frac{p}{q} \tag{1}$$

where $\frac{p}{q}$ is a proper fraction. Then

$$s_{\overline{n_1 + \frac{p}{q}}|} = \frac{(1+i)^{n_1 + \frac{p}{q}} - 1}{i};$$

or

$$\begin{aligned} s_{\overline{n_1 + \frac{p}{q}}|} &= \frac{(1+i)^{n_1} \cdot (1+i)^{\frac{p}{q}} - 1}{i} \\ &= (1+i)^{\frac{p}{q}} \frac{[(1+i)^{n_1} - (1+i)^{-\frac{p}{q}}]}{i} \\ &= (1+i)^{\frac{p}{q}} \frac{[(1+i)^{n_1} - 1 + 1 - (1+i)^{-\frac{p}{q}}]}{i} \\ &= (1+i)^{\frac{p}{q}} \frac{(1+i)^{n_1} - 1}{i} + (1+i)^{\frac{p}{q}} \frac{1 - (1+i)^{-\frac{p}{q}}}{i} \\ &= (1+i)^{\frac{p}{q}} s_{\overline{n_1}|} + \frac{(1+i)^{\frac{p}{q}} - 1}{i}; \end{aligned}$$

$$\text{or, finally,} \quad s_{\overline{n_1 + \frac{p}{q}}|} = (1+i)^{\frac{p}{q}} s_{\overline{n_1}|} + s_{\overline{\frac{p}{q}}|.} \tag{2}$$

By methods entirely similar we find

$$\begin{aligned} a_{\overline{n_1 + \frac{p}{q}}|} &= \frac{1 - v^{n_1 + \frac{p}{q}}}{i} = v^{\frac{p}{q}} \frac{(v^{-\frac{p}{q}} - v^{n_1})}{i} \\ &= v^{\frac{p}{q}} \left[\frac{1 - v^{n_1}}{i} + \frac{v^{-\frac{p}{q}} - 1}{i} \right], \end{aligned}$$

or

$$a_{\overline{n_1 + \frac{p}{q}}|} = v^{\frac{p}{q}} a_{\overline{n_1}|} + a_{\overline{\frac{p}{q}}|.} \tag{3}$$

The meanings of equations (2) and (3) may be seen easily. In the case of $s_{\overline{n_1 + \frac{p}{q}}|i}$ the amount of the annuity for the fractional term is the amount for the integral part of the term accumulated to the end of the term and the accumulation increased by the amount of 1 per annum for the fractional part of the term.

EXAMPLES

1. Find the amount of an annuity of \$500 per annum for $10\frac{1}{2}$ years at 6% effective.
2. Find the amount of an annuity of \$500 per annum for 10 years and 9 months at effective rate 6%.
3. Find the present value of \$500 per year for 10 years and 6 months, when money is worth 6%.
4. Find the present value of an annuity of \$1000 per annum for 12 years and 6 months, if money is worth 6%.
5. Find the present value of an annuity of \$1000 per annum, which is to begin 10 years hence, and to continue for 12 years and 6 months.

CHAPTER IX

THE EXTINCTION OF INTEREST-BEARING DEBTS BY PERIODICAL PAYMENTS—AMORTIZATION

59. The amortization of an interest-bearing debt by equal annual installments. In finance the term *amortization* means the extinction of a debt, usually regardless of the method employed; in the present section it is used in connection with a particular method, viz. the repayment of a debt, principal and interest, in equal annual installments. A bond is issued or a loan is made, and it is agreed that a fixed amount, which must be in excess of a year's interest, shall be paid each year. Of this fixed amount a portion goes to the payment of interest due, and the balance is applied toward the reduction of the principal. For example, a savings bank makes a loan of \$1000, on which 6% is to be paid, and it is arranged that \$100 shall be paid each year. At the end of the first year an interest payment of \$60 will be due, leaving \$40 to apply toward the reduction of the principal. The principal for the second year will be \$960, and the interest payment will be \$57.60, so that \$42.40 will be available for the reduction of the principal. The process continues in this way (in this case through some sixteen years) until the debt is entirely repaid. The method has obvious advantages for the debtor, since by means of it payments are uniform to the end of the transaction.

In the example just given, the amount of the annual payment was given, and the time required to pay the debt was unknown. Frequently the time is given and the amount of the annual payment is the unknown. There are, then, two fundamental problems in connection with this sort of amortization: viz. to find the annual installment when the principal and the time are known, and to find the time when the principal and the annual payment are known.

PROBLEM. *To determine the periodical payment required to extinguish a given interest-bearing debt in a given time.*

Let A denote the face of the debt, n the time in years, i the rate of interest, and p the number of payments made in a year. The several payments constitute an annuity whose present value is A . If R denote the sum of all payments made in a year, then by (3'), § 45,

$$A = R \cdot \frac{1 - (1+i)^{-n}}{p[(1+i)^{\frac{1}{p}} - 1]} = R \cdot a_{\overline{n}|} \cdot \frac{i}{j_{(p)}}. \quad (1)$$

From (1) we obtain directly

$$R = A \cdot \frac{1}{a_{\overline{n}|}} \cdot \frac{j_{(p)}}{i}, \quad (2)$$

where R is the sum of all payments made in one year. If p payments be made in a year, the periodic payment is given by

$$\frac{R}{p} = \frac{1}{p} \cdot A \cdot \frac{1}{a_{\overline{n}|}} \cdot \frac{j_{(p)}}{i}. \quad (3)$$

From formula (2) R is easily found by logarithms, using Table VII for the determination of $\frac{1}{a_{\overline{n}|}}$ and Table IX for the determination of $\frac{j_{(p)}}{i}$. Much time and labor may be saved by the use of a set of tables like that of Glover and Carver, which gives the logarithms of $\frac{1}{a_{\overline{n}|}}$ and $\frac{j_{(p)}}{i}$.

The most important case of formula (2) is that in which payments are made once a year and consequently $p=1$. When $p=1$, $j_{(1)}=i$, so that formula (2) takes the simple form

$$R = A \cdot \frac{1}{a_{\overline{n}|}}. \quad (4)$$

PROBLEM. *To find the time required to extinguish a given interest-bearing debt by means of periodic payments equal in amount.*

The time is found by solving equation (1) for n . The result as given by (7), § 52, is

$$n = - \frac{\log\left(1 - \frac{A}{R} j_{(p)}\right)}{\log(1+i)}. \quad (5)$$

Again the special case where $p=1$ and $j_{(1)}=i$ is important. The special case gives

$$n = -\frac{\log\left(1 - \frac{A}{R}i\right)}{\log(1+i)}. \quad (6)$$

EXAMPLES

1. What annual installment will extinguish a debt of \$1000, bearing 6% interest, in 15 years?

Solution. In this problem $A = \$1000$, $n = 15$, and $i = .06$.

Consequently, $\$1000 = R \cdot a_{\overline{15}|}$,

and $R = \$1000 \frac{1}{a_{\overline{15}|}}$.

By the tables, $\frac{1}{a_{\overline{15}|}}$ at 6% = .10296,

so that $R = \$1000 \frac{1}{a_{\overline{15}|}} = \$102.96$.

2. A man buys a piano for \$500, agreeing to pay \$100 down and the balance in equal monthly installments of \$20, with interest at 6%. How long will it take him to complete the payment?

Solution. By Formula (5), we have

$$n = -\frac{\log\left(1 - \frac{400}{240}(.06)_{(12)}\right)}{\log(1.06)}.$$

From the table of values for $j_{(p)}$,

$$.06_{(12)} = .05841,$$

so that

$$\begin{aligned} n &= -\frac{\log .90265}{\log 1.06} \\ &= -\frac{9.955519 - 10}{.025306} \\ &= \frac{.044481}{.025306} \\ &= 1.758 \text{ years, approximately.} \end{aligned}$$

3. What is the monthly payment required to pay for a piano costing \$500, with interest at 6%, in 5 years?

Solution. In this problem we have

$$A = \$500, n = 5, i = .06, \text{ and } p = 12.$$

By formula (3), monthly payment $= \frac{R}{12} = \frac{500}{12 \times .06} \cdot (.06)_{(12)} \cdot \frac{1}{a_5}$.

The tables give $(.06)_{(12)} = .05841$

and $\frac{1}{a_5} \text{ at } 6\% = .23740.$

By logarithms we find $\frac{R}{12} = \$9.63.$

4. A man buys a farm, making a cash payment and leaving unpaid the sum of \$5000, which he agrees to pay, principal and interest at 6%, in five equal annual installments, the first to be paid 1 year hence. What will be the amount of the annual installment?

5. Find the annual payment required to extinguish a debt of \$5000 with interest at 6%, if the first of the equal annual payments is cash.

6. Prove formula (1) by first accumulating the annuity to the end of the term and then finding the annuity that will amount to the accumulation.

7. What monthly payment, continued for 10 years, will suffice to repay a loan of \$4000, principal and interest at 6%, the first payment to be made one month from the date on which the debt was incurred?

8. A city having an assessed valuation of \$30,000,000 votes bonds to the amount of \$200,000 for the erection of a high-school building, and arranges to pay principal and interest at 6%, payable annually, in twenty equal annual installments. How much will the rate of taxation be increased, and how much will the tax of a man whose property is assessed at \$40,000 be increased?

60. The amount remaining due after payments have been made for r years. It is of great importance to know the exact relation of debtor to creditor on any given date during the time that the process of extinguishing the debt is in progress. The city clerk, for example, must be able, at any time, to know the amount of the city's indebtedness, just as a business man should know his liabilities and his assets. The following problem is therefore an important one.

PROBLEM. *To find the amount remaining due after payments have been made for r years.*

First solution. If no accounting were to be made until the end of r years, the creditor would be entitled to interest on the face value of the debt to that time, and the debtor would be entitled to interest on all amounts paid in from the dates when the payments were made. Clearly the difference would be the

amount remaining due. If A_r denote the amount remaining due, the equation of values would be

$$A_r = A(1+i)^r - Rs_{\overline{r}|i}^{(p)}. \quad (1)$$

Both quantities on the right are easily computed by means of the tables.

If the payments are made once a year, $p = 1$ and formula (1) becomes

$$A_r = A(1+i)^r - Rs_{\overline{r}|i}. \quad (2)$$

Second solution. After payments have been made for r years, it will be necessary to make further payments for $n - r$ years. These payments will constitute an annuity whose present value is the amount sought. With the notation used above,

$$A_r = Ra_{\overline{n-r}|i}^{(p)}; \quad (3)$$

$$\text{or, if } p = 1, \quad A_r = Ra_{\overline{n-r}|i}. \quad (4)$$

It is easy to show that the values of A_r as given by (1) and (2) are identical with those given by (3) and (4). All that is necessary is to replace R in (1) or (2) by its value as given by (2) or (3) of § 59, and make the proper reductions.

Formula (1) leads to an interesting solution of the problem for the determination of the sum R of the annual payments; for it is clear that (1) holds for all values of r up to and including $r = n$. Moreover, it is clear that by definition $A_n = 0$. Consequently,

$$A(1+i)^n - Rs_n^{(p)} = 0; \quad (5)$$

$$\text{or} \quad A(1+i)^n - R \frac{(1+i)^n - 1}{i} \cdot \frac{i}{j_{(p)}} = 0. \quad (6)$$

Either equation (5) or its equivalent (6) will serve to determine R when A and n are known.

When equation (6) is divided through by $(1+i)^n$, it takes the form

$$A - R \frac{1 - (1+i)^{-n}}{i} \cdot \frac{i}{j_{(p)}} = 0;$$

$$\text{or} \quad A - R \cdot a_{\overline{n}|i}^{(p)} \cdot \frac{i}{j_{(p)}} = 0.$$

From the last equation

$$R = A \cdot \frac{1}{a_{\overline{n}|i}^{(p)}} \cdot \frac{j_{(p)}}{i}, \quad (7)$$

which is identical with equation (2), § 59.

NOTE. The special case of equation (6), for which $p = 1$, is called Euler's *amortization equation*, after Leonhard Euler, who published it in 1748. See Cantor, "Politische Arithmetik," 2d ed., p. 50.

EXAMPLES

1. A man arranges to pay a debt of \$10,000, with interest at 5%, payable annually, in ten equal annual installments; but after paying six installments, he obtains permission to pay the balance remaining due on the date when he would have paid the seventh installment. How much does he have to pay at this time?

First solution. By (2) of § 59, the annual installment is

$$\begin{aligned} R &= \$10,000 \left(\frac{1}{a_{10}} \text{ at } 5\% \right) \\ &= \$10,000 \times .129505 \\ &= \$1295.05. \end{aligned}$$

Consequently, by (1), the amount remaining due after the seventh installment has been paid would be

$$\begin{aligned} A_7 &= \$10,000(1.05)^7 - \$1295.05(s_{71} \text{ at } 5\%) \\ &= \$10,000 \times 1.407100 - \$1295.05 \times 8.142008 \\ &= \$14,071.00 - \$10,544.31 \\ &= \$3526.69. \end{aligned}$$

To this amount must be added the seventh payment, which is to be paid at the same time. The total amount to be paid is therefore \$4821.74.

Second solution. The annual payment was found by the first solution to be \$1295.05. By formula (2) the amount remaining due after the seventh installment is paid will be

$$\begin{aligned} A_7 &= R(a_{31} \text{ at } 5\%) \\ &= \$1295.05 \times 2.723248 \\ &= \$3526.74. \end{aligned}$$

The amount due *just before* the seventh installment would have been paid is therefore

$$\$3526.74 + \$1295.05 = \$4821.79.$$

The discrepancy of 5 cents in the two results is due to the fact that sufficiently extended tables have not been used.

2. A debt of \$100,000 is to be paid, principal and interest at 5%, in twenty equal annual installments, the first installment due in 1 year. What is the annual payment, and what remains due after the tenth payment is made?

3. On January 1, 1905, a city borrowed, for the construction of a system of waterworks, \$400,000, to be repaid, principal and interest at 5%, payable annually, in twenty equal annual installments, the first installment to be paid January 1, 1906. What amount will remain due immediately after the payment is made on January 1, 1915?

4. Suppose a railway company were to borrow \$10,000,000, to be repaid, principal and interest at 5%, in twenty equal annual installments, the first installment to be paid 5 years after the date of making the loan. What would the amount of the annual installment be?

5. Prove that the two values for A_r , viz.

$$A_r = A(1+i)^n - Rs_{\overline{n}|i}$$

and

$$A_r = Ra_{\overline{n-r}|i},$$

as given by formulas (1) and (4), § 60, are identical.

SUGGESTION. Replace A by $Ra_{\overline{n}|i}$ or R by $A \cdot \frac{1}{a_{\overline{n}|i}}$.

6. Suppose the face value of the debt A , the annual payment R , and the rate of interest are given. Find the expression for the time from the amortization equation (3), § 60.

7. Find the amortization equation when p payments, whose aggregate amount is R , are made in a year.

8. Find the formula for the amount remaining due after the r th payment is made on a debt bearing interest at effective rate i , if a payment R is made regularly every p th part of a year.

9. A man agrees to pay M dollars per month toward a debt of P dollars, principal and interest. If the rate is 6% nominal, compounded semi-annually, prove that the amount due in n years is

$$M \left[202.4857 + \left(\frac{P}{M} - 202.4857 \right) (1.03)^{2n} \right].$$

10. A real-estate dealer sold a house on the long-payment plan at a stipulated price of \$10,000. The purchaser gave a land contract and agreed to pay \$100 per month, beginning 1 month from date of contract, with interest at 6%. It was agreed that when the amount due should be reduced to \$5000 the seller would give a deed for the property and exchange the land contract for a mortgage. How long a time will elapse before the deed is given?

11. Solve the foregoing problem in general terms, assuming the purchase price to be A , the monthly payment to be M , the rate i , and the time r .

12. A real-estate dealer sold a house to a customer for \$4500, agreeing to take \$500 cash and \$50 per month until the remainder of the debt, with interest at 6%, should have been fully paid. How long will it take to complete the payment, if the interest rate is 6% effective? if it is 6% nominal, payable semiannually? if it is 6% nominal, payable monthly?

13. Solve the preceding problem if P denote the purchase price, p the monthly payment, and j the nominal rate.

14. A man buys a house at a stipulated price of \$6000, agreeing to pay \$50 per month. If the rate of interest is 6% nominal, convertible twelve times a year, show that the amount remaining due just after the r th payment has been made is given by the formula

$$R = 4000[2.5 - (1.005)^r].$$

61. The amount remaining due when both the payments and the time intervals are unequal.

PROBLEM. *To find the amount remaining due on a debt bearing interest convertible annually, after several unequal payments have been made at unequal intervals of time.*

Let $R_1, R_2, \dots, R_r$ be the several payments made at times $t_1, t_2, \dots, t_r$ after the debt was incurred, and let A_r be the amount remaining due after the payment R_r is made at the end of t_r years.

If all sums be accumulated to the date t_r after the debt was incurred, the equation of value for that date will be

$$A_r = A(1+i)^{t_r} - R_1(1+i)^{t_r-t_1} - R_2(1+i)^{t_r-t_2} - \dots - R_r, \quad (1)$$

since R_1 will be on interest for $t_r - t_1$ years, R_2 for $t_r - t_2$ years, and so on.

Formula (1) gives the rule for "partial payments" when the interest is convertible annually. If interest is convertible m times a year, $1+i$ must be replaced by $\left(1+\frac{j}{m}\right)^m$, in accordance with the fundamental relation

$$1+i = \left(1+\frac{j}{m}\right)^m.$$

EXAMPLE

A man bought a farm for \$10,000, paying \$3000 in cash, giving a mortgage note for the balance with interest at 6%, payable annually. He paid \$2800 at the end of the first year, \$1500 at the end of the second, \$2000 at the end of the third, and the balance at the end of the fourth year. What was the amount of the last payment?

62. The amortization schedule. For the purpose of keeping a record of such transactions as have been described in §§ 59, 60, and 61, it is advisable to construct a table which will show the following things: the principal remaining due, or *outstanding*, immediately after the annual payment, the interest for the interval, and the amount of principal repaid. Such a table is called an *amortization schedule*. Such a schedule, or its equivalent, is almost a necessity in the bookkeeping of any institution which has to do with the amortization of debts by equal annual installments. The construction will be most easily understood by examining a concrete case.

Let it be required to construct a schedule showing the amortization of a loan of \$10,000, with interest at 5%, in five equal annual installments.

By solving the amortization equation (equation (6) of § 60) for the annual rent R , or by equation (4) of § 59, we find the expression for the annual installment, viz.

$$R = A \cdot \left(\frac{1}{a_{\overline{5}|}} \text{ at } 5\% \right).$$

From the table for the annuity that 1 will purchase,

$$\frac{1}{a_{\overline{5}|}} \text{ at } 5\% = 0.230975.$$

Consequently, the annual payment is

$$R = \$2309.75.$$

Of this amount \$500 must go to pay interest due, leaving \$1809.75 to apply to the reduction of the principal. For the second year the principal outstanding will therefore be \$8190.25, and the interest on this sum for the year at 5% will be \$409.51, so that, after interest is paid, there will be available from the second installment \$1900.24 for the reduction of the principal. The remaining parts of the schedule are computed in a similar manner. The computation is of the simplest character, though it is rather long when the amortization is carried out, as is frequently the case, through a long period of time.

Schedule for the amortization of a debt of \$10,000, with interest at 5%, payable annually, in five equal annual installments :

Year	Principal outstanding at beginning of year	Interest at 5%	Annuity	Principal repaid
1	\$10,000.00	\$500.00	\$2,309.75	\$1,809.75
2	8,190.25	409.51	2,309.75	1,900.24
3	6,290.01	314.50	2,309.75	1,995.25
4	4,294.76	214.74	2,309.75	2,035.01
5	2,199.75	109.99	2,309.75	2,199.76
	\$30,974.77	\$1,548.74	\$11,548.75	\$10,000.01

Several checks upon the accuracy of the work are possible. The first and most important is that the sum of the amounts in the column for "principal repaid" should be equal to the original principal. In the above schedule the slight discrepancy is caused by the fact that the tables employed are not sufficiently extended. Seven-place tables would give, for the annuity that 1 would purchase for 5 years at 5%, \$0.2309748, and if this value were used, the results would check exactly.

In the second place, the total interest paid should equal the interest for 1 year on the amount found by adding together the amounts in the column of "principal outstanding." Finally, the total interest plus the original principal should be equal to the total sum paid, i.e. five times the annual installment.

EXAMPLES

1. Construct the schedule for the amortization of a loan of \$5000, at 5%, to be repaid in five equal annual installments.

2. Construct the amortization schedule for the repayment of a loan of \$5000, principal and interest at 6% nominal, payable semiannually, in ten semiannual payments.

3. Construct a schedule for the amortization of a loan of \$10,000, principal and interest at 6%, in ten equal annual payments.

63. **The amortization schedule in general terms.** If we look upon the debt as the present value of an annuity of R per annum, we may easily construct the schedule in general terms. To do this, let R denote the

annual payment. The principal outstanding at the beginning of the first year is then

$$Ra_{\overline{n}|} = R \frac{1 - v^n}{i}; \quad (1)$$

the interest due for the first year is

$$Ra_{\overline{n}|} i = R(1 - v^n); \quad (2)$$

the principal repaid is the annual payment diminished by the interest, or

$$R - R(1 - v^n) = Rv^n; \quad (3)$$

and, finally, the principal remaining unpaid, or the principal outstanding for the second year, is

$$\begin{aligned} Ra_{\overline{n}|} - Rv^n &= R \left[\frac{1 - v^n}{i} - v^n \right] \\ &= R \left[\frac{1 - (1 + i)v^n}{i} \right]. \end{aligned}$$

But

$$(1 + i)v^n = v^{n-1},$$

since

$$(1 + i) = \frac{1}{v}.$$

We have, then,

$$Ra_{\overline{n}|} - Rv^n = R \frac{1 - v^{n-1}}{i} = Ra_{\overline{n-1}|}. \quad (4)$$

Starting with $Ra_{\overline{n}|}$ as the amount of the debt, we have for the first line of the table the numbers

$$Ra_{\overline{n}|}, \quad R(1 - v^n), \quad R, \quad Rv^n.$$

If we take $Ra_{\overline{n-1}|}$ as the principal outstanding at the beginning of the second year, the other quantities in the line may be computed in the same way as the quantities in the first line, or, better yet, they may be written down by analogy with the first line. The remaining lines present no difficulty. The schedule is, then, as follows:

Schedule for the amortization of a debt of $Ra_{\overline{n}|}$ in n equal annual installments:

Number of interval	Principal outstanding at beginning of interval	Interest for interval	Annuity	Principal repaid at end of interval
1	$Ra_{\overline{n} }$	$R(1 - v^n)$	R	Rv^n
2	$Ra_{\overline{n-1} }$	$R(1 - v^{n-1})$	R	Rv^{n-1}
3	$Ra_{\overline{n-2} }$	$R(1 - v^{n-2})$	R	Rv^{n-2}
...	...	...	..	...
n	$Ra_{\overline{1} }$	$R(1 - v)$	R	Rv

It is important to note, from the schedule in general terms, that, for the particular case, the greater part of the schedule could be derived almost directly from the annuity tables. The amounts in the column for principal

outstanding would be obtained from a table for the present value of an annuity, while the v^n occurring in two of the other columns is nothing but the present value of 1 for n years. It is of interest to note also that the amounts of principal repaid in successive years form a geometric progression with ratio $\frac{1}{v}$.

64. The amortization schedule when the debt is expressed in bonds of a given denomination. When a debt is repaid by the retirement of bonds of a given denomination, the amount of principal repaid at a given time must be a multiple of the denomination of the bonds. For example, if the denomination of the bonds is \$1000, the principal repaid must be \$1000 or \$2000 or some other multiple of \$1000. One could not pay \$2375 on the principal. In such cases the annual payments are made (as nearly as possible) equal to the amount required to pay the debt in equal annual installments. An example will make the matter clear.

Suppose it is required to construct a schedule for the amortization, in 10 years, of a loan of \$100,000 at 5%, consisting of 100 bonds, each of denomination \$1000, with annual payments as nearly equal as possible.

The amount that would be required to repay such a loan in equal annual installments is found by the amortization equation to be \$12,950.46, and of this amount \$5000 must go to the payment of interest, leaving \$7950.46 available for the reduction of the principal on the basis of equal annual payments. But since only whole bonds can be retired, \$8000 is the amount nearest to \$7950.46 that it is possible to pay.

The annual payment would then be \$13,000, and the principal unpaid would be \$92,000. The second year's interest will be \$4600, and the amount available for repayment of principal about \$8300, so that the payment on the principal should be again \$8000. The remainder of the schedule explains itself.*

* When a loan expressed in terms of bonds of a given denomination is to be repaid gradually over a series of years, the determination of the precise bonds that are to be repaid each year is a matter that must be considered. This determination is usually effected by what are called drawings. The bonds are numbered consecutively, and, at some convenient time before a payment is to be made, the numbers of the bonds remaining unpaid are put into a box and thoroughly shaken; numbers are then drawn out one by one, until a sufficient number have been drawn to exhaust the funds available after the interest has been paid. The numbers thus drawn are

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Schedule for the repayment of a loan of \$100,000, expressed in bonds of \$1000, in ten annual installments as nearly equal as possible :

Year	Principal outstanding	Interest at 5%	Annual payment	Principal repaid	Number of bonds retired
1	\$100,000	\$5,000	\$13,000	\$8,000	8
2	92,000	4,600	12,600	8,000	8
3	84,000	4,200	13,200	9,000	9
4	75,000	3,750	12,750	9,000	9
5	66,000	3,300	13,300	10,000	10
6	56,000	2,800	12,800	10,000	10
7	46,000	2,300	13,300	11,000	11
8	35,000	1,750	12,750	11,000	11
9	24,000	1,200	13,200	12,000	12
10	12,000	600	12,600	12,000	12
	\$590,000	\$29,500	\$129,500	\$100,000	100

EXAMPLES

1. A city makes a loan of \$400,000, to be repaid, with interest at 5%, in twenty equal annual installments. Construct a schedule.

2. Construct a schedule for the repayment of a loan of \$1,000,000, with interest at 4%, to be repaid in twenty annual installments as nearly equal as possible if the bonds are of denomination \$1000.

3. A man borrows \$2000 at 6% interest, and agrees to repay it at the rate of \$240 a year. Construct the schedule.

NOTE. When loans are repayable in this manner, it is customary to say that the loan bears interest at 6% *with 6% for amortization*.

4. Construct the schedule for the repayment of a loan of \$4000 bearing 6% interest with 6% for amortization.

5. Construct the schedule for the repayment of a loan of \$4000 bearing 5% interest with 6% for amortization.

6. Construct the general amortization schedule when the principal is payable in np equal installments to be paid p times a year.

published, together with the statement that, after a certain date, interest will cease and the principal named in the bond will be repaid. This method is frequently followed in the case of European loans, though it has not as yet found much favor in this country, possibly for the reason that the time to elapse before the bond is to be redeemed, which is an element in its valuation, cannot be determined in advance (see Chapter X).

65. The amortization schedule when the payment is fixed. Under the plan adopted by the Federal Land Bank for making farm loans the term of the amortization is not given, but the borrower pays a fixed sum each year, usually a given percentage of the original principal. It is sometimes said in such cases that the obligation bears a certain rate of interest with such and such per cent for amortization.

To illustrate, a man borrows \$5000 on his farm, agreeing to pay \$300 per year until the debt is amortized. In this case it would be said that he pays "5% interest with 1% for amortization." The 1% for amortization ceases to be true as soon as the first payment is made.

The construction of the amortization schedule in such cases differs in no way from that in which the annuity term is fixed first and the annual payment is determined afterwards. It will turn out, however, that the last payment is smaller than the others.

The student who is interested in amortization plans should consult Schlimbach's "Politische Arithmetik," Frankfurt am Main, 1902. In this book no less than twenty-seven different plans are given.

EXAMPLES

1. A man makes a loan of \$5000 at 6%, to be repaid, principal and interest, at the rate of \$600 per year. How long will it take to complete payment? Construct the schedule.

2. The Federal Land Bank charges 5% interest and allows a farmer to repay his loan, principal and interest, in equal annual installments, at such a rate that the first payment is 5% for interest and 1% for amortization. How long will it take to repay a loan? Construct a schedule for the repayment of a loan of \$5000.

CHAPTER X

BONDS

66. Definitions. For present purposes a *bond* may be defined as a certificate of ownership in a definite portion of a debt due from a government, a city, a business corporation, or an individual. In its simplest form it is a promise to pay a stipulated sum on or after a given date, and to pay interest or dividends at a specified *dividend rate* and at definite intervals. The interval between interest payments is either a year, a half year, or a quarter year. The amount named in the bond is called the *par value*, or sometimes the *nominal par value*. The bond is said to be *redeemed* when the face value has been repaid and the bond surrendered to the debtor.

Bonds are usually redeemable *at par*; i.e. the sum named on the face of the bond is paid by the debtor, though sometimes, in order to secure an apparent reduction in the interest rate, they are made redeemable at a higher figure, or *above par*. Government bonds are usually redeemable at the option of the debtor after a certain date. For example, the law of 1870, under which the debt of the United States was refunded, provided for the issue of three kinds of bonds: one redeemable after 10 years and bearing 5% interest; one redeemable after 15 years and bearing $4\frac{1}{2}\%$ interest; and one redeemable after 15 years and bearing 4% interest.

A bond is described briefly by giving the dividend rate and the due date. For example, "Consumer's Power Company 5s, 1952," means a bond bearing a 5% dividend rate, and redeemable at par in 1952. This description is not complete, since it does not indicate how often dividends are payable.

Bonds are usually sold on the open market for what they will bring. If the selling price is the same as the face value, the bonds are said to be sold *at par*; if the selling price is greater than the par value, they are said to be sold *at a premium*; if it is less than the par value, they are said to be sold *at a discount*.

In determining the value of a bond, five things are taken into consideration: they are (1) the character of the security that the debtor offers for the prompt payment of interest and principal as they fall due, (2) the price at which the bond is to be redeemed, (3) the rate of interest or dividend to be paid, (4) the rate of interest that is to be realized by the buyer, (5) the length of time to elapse before the bond will be redeemed.

The first of these elements cannot be measured mathematically, and so cannot be taken into consideration in any theoretical treatment of the problem of valuation. We have to consider only bonds concerning which, theoretically at least, there is no doubt as to the prompt and full payment of both interest and principal, and which bear a definite rate of interest, payable at definite dates.

It should be carefully noted that the rate of interest, or the dividend rate, named in the bond, has little or no relation to the current rate of interest, i. e. the rate to be realized by the investor. The dividend rate named in the bond is given merely for the sake of determining the periodical payment that is to be made to the investor. It is, however, an important element in the determination of the value of the bond. Neither the borrower nor the investor would be seriously affected by naming a higher or a lower dividend rate, for the price to be paid for the bond would be adjusted to meet the changed conditions.

67. The price of a bond. The purchaser of a bond is interested in the income rate. When that is known, the selling price may be computed without difficulty.

PROBLEM. *To find the price of a bond purchased to yield a given rate of interest.*

Let C be the par value,

C_1 the price to be paid on redemption, usually par value,

n the number of years to elapse before redemption,

r the ratio of the annual dividend to C , i. e. the *dividend rate*, or *cash rate*,

i the rate to be realized — the *income rate*, or *investment rate*,

$1 + k$ the ratio of C_1 to C ,

A_n the purchase price n years before redemption.

The annual dividend will be rC , and if it is payable p times a year, the periodical dividend will be $\frac{rC}{p}$. The dividends constitute an annuity with annual rent rC for n years. Consequently, the value of the bond consists of two elements: viz. the present value of the redemption price and the present value of the annuity constituted by the periodical dividend payments. The amount A_n which the investor can afford to pay in order to secure a specified income rate is therefore the sum of the present values of these two elements. The present value of the redemption price is $C_1 v^n$, and the present value of the annuity constituted by the periodical dividend payments is $rC a_n^{(p)}$ where both v^n and $a_n^{(p)}$ are to be computed at the investment rate i . The purchase price is, therefore, given by the formula

$$A_n = C_1 v^n + rC \cdot a_n^{(p)}; \quad (1)$$

or, if the values of v^n and $a_n^{(p)}$ be written out in full,

$$A_n = C_1 (1+i)^{-n} + \frac{rC}{p} \cdot \frac{1 - (1+i)^{-n}}{(1+i)^{\frac{1}{p}} - 1}. \quad (2)$$

If the investment rate is nominal and convertible m times a year,

$$1+i = \left(1 + \frac{j}{m}\right)^m$$

by the formula for the relation between nominal and effective rates, and equation (2) takes the form

$$A_n = C_1 \left(1 + \frac{j}{m}\right)^{-mn} + \frac{rC}{p} \cdot \frac{1 - \left(1 + \frac{j}{m}\right)^{-mn}}{\left(1 + \frac{j}{m}\right)^{\frac{m}{p}} - 1}. \quad (3)$$

Finally, if the conversion interval for the interest coincides with the dividend period, $m = p$, and

$$A_n = C_1 \left(1 + \frac{j}{p}\right)^{-np} + \frac{rC}{p} \cdot \frac{1 - \left(1 + \frac{j}{p}\right)^{-np}}{\frac{j}{p}}, \quad (4)$$

or

$$A_n = C \left[\frac{C_1}{C} v^{np} + \frac{r}{p} \cdot a_{np} \right] \left(\text{at rate } \frac{j}{p} \right). \quad (5)$$

If C_1 be expressed in terms of C by means of the relation $C_1 = (1+k)C$, formula (5) may be written in the form

$$A_n = C \left[(1+k)v^{np} + \frac{r}{p} \cdot a_{\overline{np}|} \right] \left(\text{at rate } \frac{j}{p} \right). \quad (6)$$

The purchase price, A_n , includes all expenses, such as brokerage and stamp taxes, incidental to the transfer of the bond from seller to buyer, for the bond is bought to yield a certain rate of income to the buyer. If such expenses have been incurred, let the total amount be denoted by E . Then the amount realized by the seller is

$$A_n - E = C_1 v^n + r C a_{\overline{n}|}^{(p)} - E. \quad (7)$$

Formula (7) may be modified to correspond with any one of the formulas (2), (3), (4) or (6), as occasion may require.

EXAMPLES

1. How much must be paid for a bond of \$100, with dividend rate at 6% nominal, payable January 1 and July 1, and redeemable after 20 years at par, to yield 5% nominal, convertible semiannually?

Solution. Since the conversion interval for the interest coincides with the dividend period, the solution will be given by formula (4). The present value of the redemption price will be

$$\$100 \times (1.025)^{-40} = \$37.243.$$

The total dividend paid in a year is \$6, and the semiannual dividend is \$3. The present value of the annuity constituted by the dividend payments is therefore

$$\begin{aligned} \$3 \times \frac{1 - (1.025)^{-40}}{.025} &= \$3 \times 25.103 \\ &= \$75.309. \end{aligned}$$

It follows that the purchase price of the bond will be

$$\$37.243 + \$75.309 = \$112.55.$$

The premium to be paid is \$12.55.

2. How much must be paid for a bond of \$100 with dividend rate at 5% nominal, payable January 1 and July 1 and redeemable at par in 20 years, to yield 6% nominal, convertible semiannually?

3. Find the price of a bond for \$100 with dividend rate at 6% nominal, payable semiannually, for 20 years, to yield the purchaser 5%.

4. A bond for \$100 redeemable after 10 years at 108 and paying a dividend rate of 6% nominal, payable semiannually, is bought to yield $5\frac{1}{2}\%$ nominal, payable semiannually. What is the price?

5. If the bond in Example 4 were bought to yield 6% nominal, what would be the price?

68. The computation of the premium. Of course the premium is known as soon as the purchase price is known. It turns out, however, that it is easier in most cases to compute the premium directly than to compute the purchase price, so that the best way to compute the purchase price is to compute the premium.

PROBLEM. To find the premium that must be paid on a bond purchased to yield a given rate of interest.

Let P denote the premium. Then, by the notation of § 67,

$$P = A_n - C. \quad (1)$$

If A_n is replaced by its value as given by (1) of the previous section, equation (1) may be written in the form

$$P = C_1 v^n + r C a_{\overline{n}|}^{(p)} - C; \quad (2)$$

or, when C_1 is replaced by $(1+k)C$, and the value of $a_{\overline{n}|}^{(p)}$ written out in full,

$$P = C \left[r \cdot \frac{1-v^n}{j_{(p)}} - (1-v^n) + kv^n \right]. \quad (3)$$

When the first two terms in the brackets are reduced to a common denominator it is easily seen that $a_{\overline{n}|}^{(p)}$ is a common factor, so that (3) may be written

$$P = C[(r - j_{(p)}) a_{\overline{n}|}^{(p)} + kv^n]. \quad (4)$$

The most important case is that for which the investment interval and the dividend interval correspond. In this case $m = p$, $j_{(p)}$ reduces to j , and $a_{\overline{n}|}^{(p)}$ reduces to $\frac{1}{p} a_{\overline{np}|}$ at rate $\frac{j}{p}$. (See § 45.)

Formula (4) then takes the simple form

$$P = C \left[\left(\frac{r-j}{p} \right) a_{\overline{np}|} + kv^{np} \right] \left(\text{at rate } \frac{j}{p} \right). \quad (5)$$

When the bonds are redeemed at par k is of course zero, so that the formula for the premium takes the still simpler form

$$P = C \frac{r - j}{p} a_{\overline{np}|} \left(\text{at rate } \frac{j}{p} \right). \quad (6)$$

For the convenience of persons who buy and sell bonds, tables giving the prices to yield a given income rate have been published. Sprague's tables, which give the price to the nearest cent of a bond for \$1,000,000, are among the best. The following table * gives the yield of a 6% bond for \$1000 to the nearest cent, to yield rates from 4.50 to 5.00% with intervals of .05% and for times from 3 to 5 years with half-year intervals.

Values to the nearest cent of a 6% bond for \$1000, with interest payable semiannually:

Investment rate	3 years	3½ years	4 years	4½ years	5 years
4.50	1 041.66	1 048.08	1 054.35	1 060.49	1 066.50
4.55	1 040.24	1 046.43	1 052.49	1 058.41	1 064.20
4.60	1 038.82	1 044.70	1 050.62	1 056.33	1 061.90
4.65	1 037.40	1 043.15	1 048.76	1 054.25	1 059.61
4.70	1 035.98	1 041.51	1 046.91	1 052.18	1 057.33
4.75	1 034.57	1 039.87	1 045.05	1 050.11	1 055.05
4.80	1 033.16	1 038.24	1 043.20	1 048.05	1 052.78
4.85	1 031.75	1 036.61	1 041.36	1 045.99	1 050.52
4.90	1 030.35	1 034.99	1 039.52	1 043.94	1 048.26
4.95	1 028.94	1 033.37	1 037.68	1 041.90	1 046.01
5.00	1 027.54	1 031.75	1 035.85	1 039.85	1 043.76

EXAMPLES

1. Find the purchase price of a bond for \$100 with dividend rate at 6% nominal, payable January 1 and July 1, and redeemable after 20 years at par, to yield 5% nominal, convertible semiannually.

Solution. We have $C = \$100$, $r = .06$, $j = .05$, $p = m = 2$, and $n = 20$. Hence, by formula (6),

$$\begin{aligned}
 P &= \frac{100(.06 - .05)}{2} (a_{\overline{40}|} \text{ at rate } .025) \\
 &= .50 \times 25.103 \\
 &= \$12.55.
 \end{aligned}$$

* Adapted with permission from Sprague's "Extended Bond Tables." Copyrighted, The Ronald Press Company, 1915.

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2. A bond for \$100 pays 6% nominal, payable January 1 and July 1, and is to be redeemed at par after 5 years. At what price must it be bought to yield 4% nominal, convertible January 1 and July 1?

3. Prove that after the k th interest payment has been made, the value of a bond redeemable at par after n years is given by the formula

$$A_{n-\frac{k}{p}} = C + C \frac{r-j}{p} \left(a_{\overline{n-p-k}|} \text{ at rate } \frac{j}{p} \right),$$

where C denotes the par value.

4. When bonds, redeemable on a certain date at par but due at a later date, are sold at a premium, their value is found on the supposition that they will be redeemed at the earlier date. Why?

HINT. The debtor can redeem the bond at par at any time after the earlier date. The question to be considered is whether it is worth more or less than par.

5. The following is clipped from an advertisement of the bond department of the National City Bank of New York for July, 1923. Assuming that where two investment rates are given, the second refers to the later due date, and that the due date will be July 1 of the year given, find the price in each case as of date July 1, 1923.

Issue	Due	Approximate yield %
(a) State of Kansas $4\frac{1}{2}$ s	1929-48	4.30-4.25
(b) Chicago Illinois Sanitary Dist. 4s	1925-43	4.35-4.25
(c) Jersey City $4\frac{3}{4}$ s	1929-55	4.35-4.25
(d) Shelby County Tenn. $4\frac{3}{4}$ s	1933-57	4.65-4.60
(e) Standard Oil Co. (Calif.) Serial 5s	1925-32	5.00-5.17
(f) Delaware & Hudson Co. 4s	1943	5.10
(g) Chicago & Northwestern Equip. 5s	1926-38	5.25
(h) Fruit Growers Express Equip. 5s	1937-38	5.40
(i) Republic of Cuba 30 yr. $5\frac{1}{2}$ s	1953	5.55
(j) Toledo Edison Co. 1st Mtge. 5s	1947	5.70
(k) Consumers Power Co. 5s	1952	5.75
(l) Interborough Equip. 6s	1927-28	6.00
(m) Chili Copper Co. Conv. 6s	1932	6.05
(n) Detroit City Gas 1st Mtge. 6s	1947	6.05
(o) Tennessee Electric Power 6s	1947	6.45
(p) Republic of Haiti 30 yr. 6s	1952	6.50
(q) Chicago & Eastern Illinois 5s	1951	6.57
(r) Camaguey Sugar Co. 1st S. F. 7s	1942	7.20
(s) Czechoslovak Republic Secured 8s	1951	8.60

69. The amortization of the premium on a bond bought above par. When a bond which is to be redeemed at par is bought at a premium, some provision must be made for restoring to the original capital the amount of the premium; otherwise the capital will be impaired when the bond is redeemed. To illustrate, suppose the guardian of a minor child buys a bond with par value \$1000 at a premium of \$500, intending to use the income for the maintenance of his ward. He has invested \$1500 belonging to his ward, and if the annual income from the bond has all been used for living expenses, *there will be only \$1000 to reinvest, instead of the original \$1500, when the bond is redeemed at par.* In other words, \$500 of the capital has disappeared as effectually as if it had been stolen. Indeed, the guardian should be held responsible for the disappearance of the \$500, just as though it had been stolen.

To remedy this difficulty, the premium should be considered as a debt to be repaid out of the surplus interest received. The gradual extinction of the premium through the application of the surplus interest is called the *amortization of the premium*. The value of the bond redeemable at par will diminish with each successive reduction of the premium until the date of redemption, when it should stand exactly at par. The value of a bond at any given date, when purchased to yield a given rate of interest, is termed the *book value*. An amortization schedule should be prepared exactly as in § 62. Such a schedule should show the book value, the net income, and the amount of amortization at the end of every dividend period until the bond is redeemed.

ILLUSTRATIVE EXAMPLE. Consider the amortization of the premium on a bond for \$100, with interest at 6%, payable January 1 and July 1, and redeemable at par January 1, 1928, bought January 1, 1923, to yield 5% nominal, convertible semiannually.

Solution. By (6) of § 68 the premium is found to be \$4.376, and consequently the purchase price is \$104.376. At the end of each half year \$3 will be received, while 5% nominal on \$104.376 for a half year is only \$2.609; so that there is a surplus of \$0.391 available for amortization. Applying this surplus to the reduction of the premium, we see that the value of the bond is reduced to \$103.985. For the second half year the net income will be $2\frac{1}{2}\%$ of \$103.985, or \$2.600, which will leave \$0.400 for amortization. The process is continued in this manner until, on January 1, 1928, the book value stands at \$100. The schedule may be arranged as follows:

Schedule of amortization of premium on 6% bond of the ———, redeemable January 1, 1928, January and July, bought January 1, 1923, to yield 5% :

Date	Total interest at 6%	Net income at 5%	For amorti- zation	Book value
1923 Jan. 1				\$104.376
July 1	\$3.00	\$2.609	\$0.391	103.985
1924 Jan. 1	3.00	2.600	.400	103.585
July 1	3.00	2.590	.410	103.175
1925 Jan. 1	3.00	2.579	.421	102.754
July 1	3.00	2.569	.431	102.323
1926 Jan. 1	3.00	2.558	.442	101.881
July 1	3.00	2.547	.453	101.428
1927 Jan. 1	3.00	2.536	.464	100.964
July 1	3.00	2.524	.476	100.488
1928 Jan. 1	3.00	2.512	.488	100.000

If the bond is redeemable at a price different from par the arithmetic process is identical with that used in the above schedule. In this case, however, the book value will be written down to the redemption price. For example, if the 6% bond in the schedule had been redeemable at 110 the purchase price would have been \$112.188 and this book value would have been written down to \$110.

EXAMPLES

1. Construct the schedule of amortization for a 6% bond with par value \$1000, payable in 5 years, and with dividend dates January 1 and July 1, bought January 1, 1923, to yield $4\frac{1}{2}\%$ nominal, convertible January 1 and July 1.

2. Find the price of a 6% bond redeemable January 1, 1928, at 110, bought January 1, 1923, to yield 5%, payable January 1 and July 1, and construct the schedule.

70. The accumulation of the discount on a bond bought below par. When the dividend rate is less than the specified income rate, the bond must be bought at a discount, and the value of the bond increases as the date of redemption approaches. At no time will the interest received on the bond be equal to the net

income that should be received on the investment. The difference must be found by "writing up" the value of the bond, i.e. by adding the difference to the book value. The process of increasing the value of the bond at each dividend period by the difference between the required net income and the periodic installment of interest is called the *accumulation of the discount*. A schedule of accumulation similar to the schedule of amortization of § 69 should be constructed when a bond is bought at a discount. An example will make clear the process by which the schedule may be constructed.

ILLUSTRATIVE EXAMPLE. Consider the accumulation of the discount on a bond for \$100, with interest at 4% nominal, payable January 1 and July 1, and redeemable January 1, 1928, bought January 1, 1923, to yield 5% nominal, convertible semiannually.

Solution. By (6) of § 68 the discount is found to be \$4.376, so that the price of the bond is \$95.624. At the end of every half year an interest payment of \$2 is made, but for the first half year the net income at $2\frac{1}{2}\%$ per half year must be \$2.391. In this case the interest received lacks \$0.391 of meeting the required income, so that the value of the bond must be "written up" by this amount. The value upon which income at $2\frac{1}{2}\%$ per half year must be computed is, then, \$96.015, and the required income at $2\frac{1}{2}\%$ per half year is \$2.40, of which \$0.40 must be written on to the value of the bond. With this explanation the construction of the schedule is easily understood.

Schedule of accumulation of discount on 4% bond of the ———, payable January 1, 1928, January and July, bought January 1, 1923, to yield 5% :

Date	Total interest at 4%	Income at 5%	Accumulation	Book value
1923 Jan. 1 . . .				\$95.624
July 1 . . .	\$2.00	\$2.391	\$0.391	96.015
1924 Jan. 1 . . .	2.00	2.400	.400	96.415
July 1 . . .	2.00	2.410	.410	96.825
1925 Jan. 1 . . .	2.00	2.421	.421	97.246
July 1 . . .	2.00	2.431	.431	97.677
1926 Jan. 1 . . .	2.00	2.442	.442	98.119
July 1 . . .	2.00	2.453	.453	98.572
1927 Jan. 1 . . .	2.00	2.464	.464	99.036
July 1 . . .	2.00	2.476	.476	99.512
1928 Jan. 1 . . .	2.00	2.488	.488	100.000

EXAMPLES

1. Construct the schedule for the accumulation of the discount on a bond of \$100, with interest at 5%, payable January 1 and July 1, redeemable January 1, 1929, purchased January 1, 1923, to yield 6% nominal, payable semiannually.

2. A 6% bond redeemable after 5 years at 110 is bought to yield 5% nominal, payable semiannually. Find the price and construct the schedule for the amortization of the premium.

71. Analysis of the schedule for the amortization of the premium. The book value of a bond redeemable at par and purchased s years before maturity is

$$A_s = C + C \frac{r-j}{p} a_{\overline{sp}|} \quad (\text{equation 6, § 68}), \quad (1)$$

where C is par value and the nominal rate is j , payable p times a year.

The income at rate j on the purchase price is, for the period,

$$\begin{aligned} A_s \cdot \frac{j}{p} &= C \frac{j}{p} + C \frac{r-j}{p} a_{\overline{sp}|} \cdot \frac{j}{p} \\ &= C \frac{j}{p} + C \frac{r-j}{p} (1 - v^p) \\ &= \frac{rC}{p} - C \frac{r-j}{p} v^{sp}. \end{aligned} \quad (2)$$

The amortization for the period is the dividend less the income on the purchase price at rate j . If a_s denote the amortization and equation (2) be taken into account,

$$a_s = \frac{rC}{p} - A_s \cdot \frac{j}{p} = C \frac{r-j}{p} v^{sp}. \quad (3)$$

The book value for the next period is

$$\begin{aligned} A_{s-\frac{1}{p}} &= A_s - a_s \\ &= C + C \frac{r-j}{p} a_{\overline{sp}|} - C \frac{r-j}{p} v^{sp} \\ &= C + C \frac{r-j}{p} (a_{\overline{sp}|} - v^{sp}) \\ &= C + C \frac{r-j}{p} a_{\overline{sp-1}|}, \end{aligned} \quad (4)$$

which agrees with the result obtained by writing down directly the value of a bond having $sp - 1$ periods to run. The process may be continued at will.

If the bond is bought at a discount the dividend rate r will be less than the investment rate j , so that every number containing the factor $\frac{r-j}{p}$ will be negative. With this exception the analysis of the schedule for the accumulation of the discount is exactly like that of the amortization of the premium.

If the bond is redeemable at a price different from par, say at $C(1+k)$, the expression for A_s will contain the additional term Ckv^{sp} in accordance with formula (4), § 68, and the book value will be "written up" or "written down," as the case may be, to the value $C(1+k)$ instead of to the value C . When the bond has s years to run the amortization of the premium or the accumulation of the discount will be given by

$$a_s = \frac{C(r-j-jk)}{p} v^{sp}. \quad (5)$$

EXERCISES

1. Prove that a bond redeemable in n years at $(1+k)C$ and bought just after the m th interest payment has been made, to yield rate j must be bought for

$$A_{n-\frac{m}{p}} = C + C \frac{r-j}{p} a_{\frac{np-m}{p}} + Ckv^{np-m}.$$

2. Prove that the amortization on the bond described in Example 1 must be, for the first period after purchase,

$$a_{n-\frac{m}{p}} = C \frac{(r-j-jk)}{p} v^{np-m}.$$

3. Prove that if $\sum_1^{\frac{np-m}{p}} a_s$ represent the sum of all amortizations to the date of redemption,

$$\begin{aligned} A_{n-\frac{m}{p}} - \sum_1^{\frac{np-m}{p}} a_s &= A_{n-\frac{m}{p}} - C \frac{r-j-jk}{p} \sum_1^{\frac{np-m}{p}} v^s \\ &= C(1+k). \end{aligned}$$

State the result in words.

72. Bonds bought between two interest dates. When a bond is bought between two interest dates the seller is entitled to, and the buyer can afford to pay for, a portion of the interest at the dividend rate that has accrued on the bond since the last interest date. In such cases the bond is usually sold at a quoted price "with accrued interest." The "quoted price"

may be a so-called "flat rate" or the bond may be "priced to yield a certain investment rate."

In the first case the price is the flat rate plus the accrued interest on the par value at the dividend rate. For example, if a 6% bond which is priced at "102 flat with accrued interest" is bought two months after an interest date, the actual price paid will be 103.

The *theoretical value* of a bond bought between two interest dates would be the book value at the earlier date accumulated at compound interest for the time that has elapsed since this date. The *value in practice* is found by increasing this book value by simple interest for the elapsed time. The following considerations show the basis upon which the value in practice rests.

When the yield is specified the "quoted price" is the price at the beginning of the period and, if the bond is not bought at par, must be adjusted to take into account the amortization of the premium or the accumulation of the discount as the case may be. To make this point clear note that the book value of the bond for which the amortization schedule is given in § 69 is 104.376 on January 1, 1923, and 103.985 on July 1, 1923. If this bond had been bought on March 1, 1923, the value *without* the accrued interest would have been some value between these two.

PROBLEM. *To find the value of a bond bought between two interest dates.*

First solution. In practice it is assumed that for a part of a dividend period the interest and the amortization are both proportional to the time. Suppose the bond is bought one k th part of a period after an interest date. The accrued interest will then be $\frac{rC}{kp}$, and the amortization for the portion of the period

will be $\frac{C(r-j)}{kp} v^{np}$. If the value of the bond be denoted by

$A_{n-\frac{1}{kp}}$, we shall have

$$A_{n-\frac{1}{kp}} = A_n + \frac{rC}{kp} - \frac{C(r-j)}{kp} v^{np}. \quad (1)$$

If $\frac{C(r-j)}{kp}$ be added to and subtracted from the right member of (1), we have

$$\begin{aligned} A_{n-\frac{1}{kp}} &= A_n + \frac{rC}{kp} + \frac{C(r-j)}{kp} (1 - v^{np}) - \frac{C(r-j)}{kp} \\ &= A_n + \frac{Cj}{kp} + \frac{C(r-j)}{kp} (1 - v^{np}) \\ &= A_n + \frac{j}{kp} \left[C + \frac{C(r-j)}{p} \frac{1 - v^{np}}{\frac{j}{p}} \right] \\ &= A_n + \frac{j}{kp} A_n, \end{aligned}$$

or
$$A_{n-\frac{1}{kp}} = A_n \left(1 + \frac{j}{kp} \right). \quad (2)$$

Translated into words, formula (2) says that *the value in practice of a bond bought one kth part of a dividend period after an interest date is the value at the last interest date increased by simple interest at the investment rate for that part of the dividend period which has elapsed.*

ILLUSTRATIVE EXAMPLE. What is the price of a 6% bond, interest payable semiannually, bought 4 years and 10 months before maturity, to yield 5% nominal, payable semiannually?

Solution. By (6) of § 68 the price of the bond bought 5 years before maturity would be 104.376. One third of the dividend period of 6 months has elapsed. The simple interest on \$104.376 for 2 months at 5% is .870. The price is, therefore

$$104.376 + .870 = 105.246.$$

Second solution. One of the best methods for the determination of the price of a bond bought between two interest date is the method of interpolation (§§ 17 and 18). In the example just given, we have

$$\text{Book value Jan. 1, 1923} = 104.376^*,$$

$$\text{Book value July 1, 1923} = 103.985.$$

The difference is 0.391, and the difference for 2 months, or one third of the dividend period after Jan. 1, 1923, will be one third of 0.391, or 0.130. Since the book value is decreasing

*The book values may be computed or may be found from a bond table if it available.

this difference must be subtracted from the book value at the earlier date. The price on March 1, 1923, would therefore be

$$104.376 - .130 + 1.00 = 105.246.$$

The result is identical with that obtained by the first method. To show that the two methods are in fact identical, we may suppose that a bond is bought at a date which is one k th part of the dividend period after an interest date. The time to run will then be $n - \frac{1}{kp}$ years, where n was the time to run at the last interest date. The difference between book values at the two consecutive interest dates is, therefore,

$$A_n - A_{n - \frac{1}{p}}.$$

The book value one k th of a period after the interest date has been denoted by $A_{n - \frac{1}{kp}}$. Remembering that the book value must be increased by a proportionate amount of interest, we have

$$A_{n - \frac{1}{kp}} = A_n - \frac{1}{k}(A_n - A_{n - \frac{1}{p}}) + \frac{rC}{kp}. \quad (3)$$

That this formula is identical with formula (1) is clear when we note that $A_n - A_{n - \frac{1}{p}}$ is the amortization and $\frac{rC}{p}$ the interest for the dividend period.

If the bond is bought at a discount, the difference $A_n - A_{n - \frac{1}{p}}$ will change its sign and must be added instead of subtracted.

EXAMPLES

1. Find the value of a bond for \$100, with interest at 6% nominal, payable January 1 and July 1, and redeemable January 1, 1916, bought March 1, 1911, to yield 5% nominal, convertible semiannually.

Solution. By the formula (6) of § 68 the value on January 1, 1911, was \$104.376. The time since the last interest payment may be taken as $\frac{1}{3}$ of a period, or $\frac{1}{6}$ of a year. The theoretical value is, therefore,

$$\begin{aligned} A_{5 - \frac{1}{6}} &= \$104.376(1.025)^{\frac{1}{3}} \\ &= \$105.239. \end{aligned}$$

To find the value in practice, we have, by formula (2),

$$\begin{aligned} A_{5-\frac{1}{2}} &= \$104.376 \times \left(1 + \frac{.05}{6}\right) \\ &= \$105.246. \end{aligned}$$

The result shows, as we might know from theoretical considerations, that the method in practice is slightly to the buyer's disadvantage.

2. Find the prices for the bonds listed in Example 5 of § 68, on the supposition that purchase is made November 1, 1923.

3. Construct the schedule for the amortization of the premium on a 6% bond for \$100, interest payable January 1 and July 1, redeemable January 1, 1921, purchased October 1, 1911, to yield 5%, convertible semiannually.

4. Prove the statement made above, that the value of a bond purchased one k th part of a period after an interest date, defined to be the value at the last interest date, plus one k th part of the interest on the bond for the period, minus one k th part of the amortization for the particular period within which the purchase is made, is equivalent to the value at the last interest date accumulated at simple interest from the last interest date to the date of purchase.

5. Show that the theoretical value of a bond bought between two interest dates will be the same whether the value at the last interest date be accumulated to the date of purchase or the value at the next interest date be discounted to the date of purchase. Will this be true of the value in practice?

73. The investment rate for a bond sold at a given price. Bonds are usually advertised by means of a prospectus containing a description of the bonds, some account of the property values supporting the bond issue, and a statement of the price at which the bonds are sold to the investor. The statement of the price usually contains the words "priced to yield about ——." For example, on the first of July, 1923, a prospectus listed "1st and Refunding Mortgage 5% Gold Bonds" of the Illinois Bell Telephone Company "priced at 95.25 to yield about 5.30%."

When the description of the bond and the price at which it is sold are given, the investment rate cannot be found exactly, but it is of great importance that it be determined approximately.

PROBLEM. *To determine the investment rate on a bond sold at a given price.*

The various methods of solving this important problem can be best understood through a concrete illustration.

EXAMPLE

A $4\frac{1}{2}\%$ bond for \$100, with interest payable semiannually, and redeemable at par in 10 years, is bought for \$97.50. What rate of interest will the investor receive?

1. *Solution by means of bond tables; interpolation.* By Sprague's tables a $4\frac{1}{2}\%$ investment with 10 years to run will yield 4.80% when bought at 97.6393 and 4.85% when bought at 97.2525. The required yield must then lie between 4.80% and 4.85%. The investment-rate interval is .05.

The tabular difference is $-.3868$ and the difference between the first price and 97.50 is $-.1393$. The correction is then found by

$$p = \frac{y - y_1}{y_2 - y_1} = \frac{-.1393}{-.3868} = .360, \quad (\text{See (2), § 22.})$$

where p is a fraction of the investment-rate interval .05. The correction is therefore .0180 and the approximate investment rate is

$$4.80 + .0180 = 4.8180,$$

or $j = .048180$, approximately.

If a closer approximation is required, second differences may be found and the formula

$$p = \frac{y - y_1}{D_1 + \frac{p_1 - 1}{2} D_2}$$

may be used.

To find the second differences three entries from the bond table are needed. The computation may be arranged as follows:

Income rate	Price	First differences	Second differences
4.80	97.6393		
4.85	97.2525	$-.3868$	
4.90	96.8674	$-.3851$	$+.0017$

$$\text{Then } p = \frac{-.1393}{-.3868 + \left(\frac{.360 - 1}{2}\right)(+.0017)} = .361.$$

Writing the income rate as a decimal, the interval is .0005, so that

$$j = .0480 + .361 \times .0005 = .0481805.$$

2. *Solution by means of annuity tables.* The preceding solution is based on the interpolation of the argument j by means of known functions of j ; namely, the entries found in a bond table for known values of j . The interpolation could be carried out just as well for any other function of j which determines the price of the bond. Such a function is found in the expression of the premium in equation (6), § 68.

The expression for the premium is given by

$$P = C \frac{(r-j)}{p} \left(a_{\overline{np}|} \text{ at rate } \frac{j}{p} \right). \quad (1)$$

If r , C , p , and n are known, the right member is a function of j . We obtain a somewhat simpler function of j by dividing both sides of (1) by $\frac{C}{p}$. Then

$$\frac{pP}{C} = (r-j) \left(a_{\overline{np}|} \text{ at rate } \frac{j}{p} \right). \quad (2)$$

If we write $f(j) = (r-j) \left(a_{\overline{np}|} \text{ at rate } \frac{j}{p} \right), \quad (3)$

the value of $f(j)$ for any *assumed* value of j is easily computed by means of a table for $a_{\overline{n}|}$, and $\frac{pP}{C}$ is, under the conditions of the problem, known. The problem is then to interpolate the value j of the argument corresponding to the known value $\frac{pP}{C}$.

In the given example $r = .045$, $p = 2$, $n = 10$, $C = 100$, $P = -2.50$. The known value of $f(j)$ is therefore

$$f(j) = \frac{2(-2.50)}{100} = -.05.$$

We know that $j > .045$, since the bond with dividend rate $r = .045$ is bought at a discount. Moreover, for $j = .05$, the premium, computed by formula (1), would be -3.90 . It is certain, therefore, that j lies between $.045$ and $.05$. Computing $f(.045)$ and $f(.05)$ by formula (3), we find

$$\begin{aligned} f(.045) &= 0 \\ f(.05) &= (.045 - .05) 15.5816 \\ &= -.0779458. \end{aligned}$$

The difference between $f(.045)$ and $f(.05)$ is therefore $-.0779458$, while the difference between $f(.045)$ and $f(j) = -.05$ is $-.05$. Consequently j is given approximately by

$$j = .045 + \frac{.05}{.0779458} \times .005 = .04821.$$

3. *Second solution by means of annuity tables.* The value of j is usually known to within 1% or less. A fair approximation for j may be obtained by finding from the tables the value of $a_{\overline{np}|}$ for a rate differing from the true value by not more than 1% and then solving equation (1) for the j that remains.

4. *Solution by algebraic approximation.* When the investment rate j is given, the premium on a bond redeemable at par, and with dividends payable p times a year, is computed by the formula

$$P = C \frac{(r-j)}{p} \left(a_{\overline{np}|} \text{ at rate } \frac{j}{p} \right). \quad (4)$$

When the purchase price is known and the investment rate is desired, (4) becomes an equation of degree $np+1$ in j .

The equation for the premium may be divided through by P and $a_{\overline{np}|}$ written out in full. It then takes the form

$$\frac{j}{p} = \frac{C}{P} \left(\frac{r}{p} - \frac{j}{p} \right) \left[1 - \left(1 + \frac{j}{p} \right)^{-np} \right]. \quad (5)$$

In the present problem, $C=100$, $p=2$, $P=-2.50$, $n=10$, $r=.045$. Inserting these values in equation (5), we obtain

$$\frac{j}{2} = -40 \left(.0225 - \frac{j}{2} \right) \left[1 - \left(1 + \frac{j}{2} \right)^{-20} \right].$$

It is not difficult to show that j lies between .045 and .05. We may then write

$$\frac{j}{2} = .0225 + h,$$

where h is a very small number, and the equation will take the form, after division by -1 ,

$$-(.0225 + h) = 40 [.0225 - (.0225 + h)] [1 - (1.0225 + h)^{-20}],$$

or, after simplification,

$$.0225 + h = 40 h [1 - (1.0225)^{-20} + \text{terms containing } h].$$

Dropping all terms containing powers of h above the first, we obtain

$$\begin{aligned}.0225 + h &= 40 [1 - .64081647]h \\ &= 14.36734 h.\end{aligned}$$

Solving for h ,
$$h = \frac{.0225}{13.36734} = .001683.$$

Consequently,
$$\begin{aligned}\frac{j}{2} &= .0225 + .001683 \\ &= .024183;\end{aligned}$$

and, finally, we have for the first approximation

$$j = .04837.$$

If we write
$$\frac{j}{2} = .02418 + h'$$

and repeat the process we will obtain a very close approximation.

EXAMPLES

1. A bond for \$100 bearing 6% interest, payable semiannually, is bought 10 years before maturity for 105. What rate of interest will the investor receive?

2. Find the yield in the following examples on the assumption that interest is payable semiannually, January 1 and June 1, and the bonds are bought on January 1, 1924:

- (a) State of Kansas $4\frac{1}{2}$ s, due 1929, bought at 97.50.
- (b) Chicago Illinois Territory District 4s, due 1925, bought at 98.
- (c) Chicago and Eastern Illinois 5s, due 1951, bought at 70.
- (d) Toledo Edison Co. 1st Mortgage 50, due 1947, bought at 90.
- (e) Republic of Haiti 30 years 6s, due 1952, bought at 93.50.
- (f) Czechoslovak Republic Secured 8s, due 1951, bought at 90.

3. A Liberty bond bearing $4\frac{1}{4}$ % payable April 15 and October 15, and due April 15, 1933, is bought April 15, 1924, for 96.25. What rate of interest will the investor receive?

4. A bond for \$100, bearing 6% interest payable January 1 and July 1, payable January 1, 1934, is bought April 15, 1924, for 105 and accrued interest. What rate of interest will the investor receive?

5. If the bond in Example 4 were bought April 1, 1924, for 97.50, what would the investment rate be?

74. The rate of interest paid by the borrower. If bonds could be placed on the market without expense, the rate of interest paid by the borrower would be exactly the same as that realized by the investor, since the elements to be considered, viz. the payment of a stipulated sum $C(1+k)$ after the lapse of a given time, and the payment of a definite annuity of rC per annum, are identical for both borrower and investor. If the expense of placing a series of bonds having the same maturity on the market be denoted by E , and the price paid to the debtor by A_n , the rate paid by the borrower (not the dividend rate) is the unknown number j determined by the equation

$$A_n - E = C(1+k)v^{np} + C \frac{r-j}{p} a_{\overline{np}|j}, \quad (1)$$

where j is the rate for both v^{np} and $a_{\overline{np}|j}$.

Equation (1) is precisely like that used in finding the investment rate except that the known term is $A_n - E$ instead of A_n . The methods of solution would be exactly like those of § 73 and for that reason need not be repeated.

EXAMPLES

1. An issue of 6% bonds, with interest payable July 1 and January 1, and redeemable after 10 years at par, are sold to investors at 94.50. The bond house floating the issue receives 2% on par value for its service and the bond salesman receives $1\frac{1}{2}\%$. What rate of interest does the borrower pay?

2. Six-per-cent bonds, with interest payable January 1 and July 1, and redeemable in 20 years at par, are sold on the date of issue to yield 7%. If the expense of issuing and selling the bonds is 3% of the par value, what rate does the borrower pay?

MISCELLANEOUS EXAMPLES AND PROBLEMS

1. Find the purchase price of 5% bonds, interest payable quarterly, and redeemable at par in 10 years, to yield 6% effective.

2. Find the purchase price of 6% bonds, interest payable quarterly, redeemable at par in 10 years, to yield the purchaser 5% effective on the investment.

3. Find the purchase price of 4% bonds, interest payable half yearly, and redeemable in 20 years at 120, to yield 5% effective to the purchaser.

4. A man owns some bonds paying 6% nominal, payable quarterly, and having 8 years to run, when they will be redeemed at 110. He pays a commission man $\frac{3}{4}\%$, computed on the face value, for selling them, and the government imposes a stamp tax of 2 cents on every hundred dollars of par value. What is the selling price if the purchaser is to realize 6% on his investment?

5. If 5% bonds are redeemable after 20 years at 120, at the option of the debtor, and money can be borrowed at $4\frac{1}{2}\%$ at the end of the period, will the debtor be likely to redeem the bonds or not? Explain why.

6. Find the expression for the purchase price of a bond by accumulating the interest to the date of maturity of the bond and finding the present value of the sum that would be due at maturity. Show that the result agrees with the result obtained in formula (2) of § 67.

7. One thousand bonds bearing interest at 5% nominal, payable semi-annually, are issued in denomination of \$1000 each. At the end of 3 years, and at the end of each year thereafter up to and including the twentieth, such a number of bonds is to be redeemed at 105 that the whole issue shall be paid in 20 years. How many bonds must be redeemed each year in order that the sum of the interest charge on the unpaid bonds and the amount paid to redeem bonds shall be the same for each of the 18 years? (See *Engineering News*, October 30, 1902.)

75. Installment bonds. An *installment bond* is an interest-bearing bond payable, principal and interest, in equal annual installments. The problem of determining the annual installment to be paid on such a bond is therefore precisely the problem of § 59, where the annual payment required to extinguish a debt A by a series of equal annual installments is found.

Let C be the face value of the bond, n the time to elapse before the bond is paid, r the rate of interest named in the bond, and R the annual installment. By (4), § 59, the annual installment is

$$R = C \left(\frac{1}{a_{\overline{n}|r}} \text{ at rate } r \right). \quad (1)$$

PROBLEM. *To find the price at which an installment bond must be purchased to yield a given rate on the investment.*

Let A be the purchase price and i the rate of interest to the investor. The purchaser is buying an annuity of R per annum, which must yield interest at rate i . The purchase price A is

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therefore the present value of an annuity of R per annum with interest at rate i , and the value of A is given by the formula

$$A = R(a_{\overline{n}|} \text{ at rate } i). \quad (2)$$

If R be replaced in (2) by its value as given by (1), the result is

$$A = C \cdot a_{\overline{n}|} \text{ (at rate } i) \cdot \frac{1}{a_{\overline{n}|} \text{ at rate } r}, \quad (3)$$

or, written out in full,

$$A = C \cdot \frac{r}{i} \frac{1 - (1 + i)^{-n}}{1 - (1 + r)^{-n}}. \quad (4)$$

EXAMPLES

1. What is the value of a 4% installment bond with face value \$1000, to be paid in ten annual installments, in order that the purchaser may receive 3% on his investment?

Solution. By formula (3) the solution would be, using five-place tables,

$$\begin{aligned} A &= \$1000 \frac{a_{\overline{10}|} \text{ at rate } .03}{a_{\overline{10}|} \text{ at rate } .04} \\ &= \$1000 \frac{8.53020}{8.11090} \\ &= \$1051.70. \end{aligned}$$

The solution could have been obtained by finding $\frac{1}{a_{\overline{10}|}}$ at rate .04 from the tables, and thus avoiding the division. In this way

$$\begin{aligned} A &= \$1000 \times 8.53020 \times .12329 \\ &= \$1051.69. \end{aligned}$$

If six-place tables were used, the results would agree exactly.

2. For what price must a 5% installment bond for \$100, to be paid in 10 years, be purchased to yield 6% to the purchaser?

3. Find the formula for the purchase price of an installment bond, payable quarterly for n years, with interest at rate r , to yield interest at rate i to the purchaser.

4. Prove formulas (3) and (4) by accumulating the purchase price and setting the result equal to the amount of the annuity.

EXAMPLES

1. A city issued school bonds to the amount of \$200,000, paying $4\frac{1}{2}\%$, payable semiannually, and one twentieth of the issue to be redeemed at the end of each year for 20 years. They were sold to yield 4%. What amount of cash was received for the issue?

2. Suppose the bonds in Example 1 are to be redeemed in twenty equal annual payments, the first redemption to take place at the end of 5 years. What will be received for the issue?

76. Serial bonds. A bond issue is frequently so arranged that a certain number of the bonds shall be redeemed each year. For example, in 1921 the city of Madison, Wisconsin, issued bonds to the amount of \$200,000 for the construction of a high-school building, payable at the rate of \$10,000 per year for 20 years, beginning 1 year from the date of issue. Neither the amounts to be paid nor the intervals of time between payments need be uniform.

A bond of any given series in a serial issue does not differ from an ordinary bond and therefore presents no new problem to the purchaser. It is worth while, however, to exhibit the formula for the amount that must be paid for the whole issue.

PROBLEM. *To determine the price which must be paid for an issue of serial bonds.*

Let A be the price to be paid for the whole issue, $A_1, A_2, A_3, \dots, A_r$ the prices to be paid for the bonds of the several series, $n_1, n_2, \dots, n_r$ the times, j the investment rate, $C_1, C_2, \dots, C_r$ the par values, $1 + k_1, 1 + k_2, \dots, 1 + k_r$ the ratios of the redemption values to the par values.

The price of a single issue will then be, by (5), § 67,

$$A_s = C_s + C_s \frac{r-j}{p} a_{\overline{n_s}|p} + C_s v^{n_s} k_s. \quad (1)$$

The price to be paid for the whole issue will be the sum of the prices paid for the separate issues, i.e.

$$A = \sum_1^r A_s = \sum_1^r \left[C_s + C_s \frac{r-j}{p} a_{\overline{n_s}|p} + C_s v^{n_s} k_s \right]. \quad (2)$$

If the investment rate j be required, it must be determined from equation (2) by interpolation methods.

CHAPTER XI

SINKING FUNDS

77. Sinking funds defined. A *sinking fund* is a fund created for the purpose of paying a debt when the debt falls due. The sums set aside for sinking funds are supposed to be productively invested. If the same amount is set aside at regular intervals, the amounts form an annuity for which either the amount, the present value, or the annual rent may be computed when the requisite data are given.*

PROBLEM. *To find the amount that must be set aside annually to extinguish a given debt K due in n years.*

Suppose the debt K falls due in n years, and that sums paid to the sinking fund are accumulated at rate i . The problem is, then, to determine the annual rent R when the amount K , the time n , and the rate i are known. This is exactly the problem of § 49. We shall have

$$Rs_{\overline{n}|i} = K;$$

whence
$$R = K \cdot \frac{1}{s_{\overline{n}|i}}. \quad (1)$$

This equation is identical with equation (4) of § 49, which was called the *sinking-fund equation*. The value of $\frac{1}{s_{\overline{n}|i}}$ may be found from a table of values of "the annuity that will amount to 1," or, if such a table is not at hand, it may be found from a table

* Under the terms of many bond issues the debtor has been required to set aside each year one n th part of the principal for a sinking fund. That this method is scarcely fair to the debtor is easily shown by an example. Suppose a bond issue is made for \$100,000, payable in 20 years, and the debtor is required to set aside \$5000 annually for a sinking fund. If the annuity of \$5000 per annum be accumulated at 4% (a savings-bank rate), the amount in the fund when the debt falls due will be \$148,890.39, which exceeds the required amount by nearly 50%.

of values of "the annuity that 1 will purchase" by means of the fundamental relation

$$\frac{1}{s_{\overline{n}|i}} = \frac{1}{a_{\overline{n}|i}} - i. \quad (\text{See (3), § 51.})$$

ILLUSTRATIVE EXAMPLE. A city incurs a debt of \$100,000, agreeing to pay interest promptly and to establish a sinking fund to repay the principal at the end of 20 years. If the payments to the sinking fund can be invested at 4% effective, what will be the annual payment to the fund?

Solution. By formula (1),

$$\begin{aligned} R &= \$100,000 \left(\frac{1}{s_{\overline{20}|.04}} \right) \\ &= \$100,000 \left(\frac{1}{a_{\overline{20}|.04}} - .04 \right) \\ &= \$3358.175. \end{aligned}$$

PROBLEM. To find the amount that must be set aside annually to pay interest at rate i on a debt K and create a sinking fund that will extinguish the debt in n years, when the rate of accumulation differs from i .

Let i' be the rate at which the sinking fund can be accumulated, and let R' be the amount that must be set aside. It will require Ki to pay the interest as it falls due each year. The payment into the sinking fund will be, by (1),

$$R = K \cdot \left(\frac{1}{s_{\overline{n}|i'}} \right).$$

The required amount is, therefore,

$$R' = Ki + K \cdot \left(\frac{1}{s_{\overline{n}|i'}} \right). \quad (2)$$

If $\frac{1}{s_{\overline{n}|i'}}$ be replaced by its value $\frac{1}{a_{\overline{n}|i'}} - i'$, according to § 51,

$$R' = Ki + K \left(\frac{1}{a_{\overline{n}|i'}} - i' \right),$$

or
$$R' = K(i - i') + K \left(\frac{1}{a_{\overline{n}|i'}} \right). \quad (3)$$

COROLLARY. If $i' = i$, the first term of (3) disappears and the formula is simply

$$R' = K \left(\frac{1}{a_{\overline{n}|i}} \right). \quad (4)$$

But formula (4) is precisely the formula (4) of § 59 for the annual payment that will extinguish an interest-bearing debt in equal annual installments. This important fact may be stated as follows:

When the sinking fund can be accumulated at the rate of interest borne by the debt, the amount required each year for interest and for the sinking-fund payment is exactly equal to the amount that would be required to pay the debt, principal and interest, in the same number of equal annual installments.

PROBLEM. *To find the amount in the sinking fund at any given time.*

Let n_1 be the time that has elapsed since the sinking fund was established, and let S_{n_1} be the amount in the sinking fund at the expiration of n_1 years. The annual payment into the sinking fund is, by (1), $K \cdot \frac{1}{s_{\overline{n_1}|i}}$. This sum is the annual rent of the annuity whose amount, S_{n_1} , we wish to find. By the formula for the amount of an annuity whose annual rent is known (see (4), § 44), the amount is

$$S_{n_1} = K \cdot \frac{1}{s_{\overline{n_1}|i}} \cdot s_{\overline{n_1}|i}, \quad (5)$$

or

$$S_{n_1} = K \frac{(1+i)^{n_1} - 1}{(1+i)^n - 1}. \quad (5')$$

Either (5) or (5') is well adapted to logarithmic computation, since $s_{\overline{n_1}|i}$, $s_{\overline{n_1}|i}$, $(1+i)^n$, and $(1+i)^{n_1}$ may be found directly from the tables.

78. The repurchase by the borrower of his own bonds. Sometimes when a bond issue has been sold, sinking-fund payments are used to purchase a part of the identical bonds for the redemption of which the sinking fund was established.* When sinking funds are so invested, the bonds thus purchased may be "kept alive" for interest purposes or they may be canceled. In case the bonds are kept alive, the company simply transfers to its

* H. R. Hatfield, *Modern Accounting*. D. Appleton and Company, 1909.

sinking-fund account on each interest date the interest that would be due on the bonds in hand. From the mathematical point of view this method of investing the sinking fund does not differ from the ordinary method, except that the rate of interest realized will probably be somewhat higher than the savings-bank rate. It is of course immaterial whether a company receives interest on its own bonds or on the bonds of another corporation.

If, however, the bonds are retired, that portion of the debt which the retired bonds represent is paid and ceases to bear interest. There is in such cases an interesting problem which is formulated as follows:

PROBLEM. *To determine the annual payment that will be required to pay interest on a bonded debt, and to purchase for retirement a sufficient number of bonds so that the bonds will all be retired at the end of n years.*

Let K denote the amount of the debt, i the effective rate of interest paid by the bonds, $1+k$ the price the bonds will bring in the open market, and x the required annual payment.

At the end of the first year the accrued interest will be Ki and after this is paid, the amount available for the purchase of bonds will be $x - Ki$; the face value of the bonds purchased at the price $1+k$ will be $\frac{x - Ki}{1+k}$; and the amount of the debt remaining unpaid will be $K - \frac{x - Ki}{1+k}$. Similarly, at the end of the second year we shall have

$$\text{interest due} = Ki - \frac{x - Ki}{1+k} i;$$

$$\text{amount available for purchasing bonds} = x - Ki + \frac{x - Ki}{1+k} i;$$

$$\text{par value of bonds purchased} = \frac{x - Ki}{1+k} + \frac{x - Ki}{(1+k)^2} i;$$

$$\text{debt outstanding} = K - 2 \frac{x - Ki}{1+k} - \frac{x - Ki}{(1+k)^2} i.$$

If the process be carried a step farther, the law by which the various quantities may be computed will begin to show itself. At the end of the third year the situation will be as follows:

$$\text{interest payment} = Ki - 2 \frac{x - Ki}{1 + k} i - \frac{x - Ki}{(1 + k)^2} i^2; \quad (1)$$

$$\text{amount paid for bonds} = x - Ki + 2 \frac{x - Ki}{1 + k} i + \frac{x - Ki}{(1 + k)^2} i^2; \quad (2)$$

$$\text{par value of bonds purchased} = \frac{x - Ki}{1 + k} + 2 \frac{x - Ki}{(1 + k)^2} i + \frac{x - Ki}{(1 + k)^3} i^2; \quad (3)$$

$$\text{debt outstanding} = K - 3 \frac{x - Ki}{1 + k} - 3 \frac{x - Ki}{(1 + k)^2} i - \frac{x - Ki}{(1 + k)^3} i^2. \quad (4)$$

If we take out of the first and second of these expressions, as far as is possible, the factor $x - Ki$, out of the third the factor $\frac{x - Ki}{1 + k}$, and out of the fourth the factor $\frac{x - Ki}{i}$, and compare the remaining factors with the binomial expansions for powers of $1 + \frac{i}{1 + k}$, the expressions (1), (2), (3), and (4) are seen to have the following forms for the r th year:

$$\text{interest payment} = Ki - (x - Ki) \left[\left(1 + \frac{i}{1 + k} \right)^{r-1} - 1 \right]; \quad (5)$$

$$\text{amount paid for bonds} = (x - Ki) \left(1 + \frac{i}{1 + k} \right)^{r-1}; \quad (6)$$

$$\text{par value of bonds purchased} = \frac{x - Ki}{1 + k} \left(1 + \frac{i}{1 + k} \right)^{r-1}; \quad (7)$$

$$\text{debt outstanding} = \frac{x - Ki}{i} \left[\left(1 + \frac{i}{1 + k} \right)^r - 1 \right]. \quad (8)$$

That these results are correct may be shown by mathematical induction; for if they are assumed to be true for $n = r$, it can be shown that they are true for $n = r + 1$, and consequently for all values of n .

But if the debt is to be paid in n years, the debt outstanding at the end of n years must be zero; i.e.

$$K - \frac{x - Ki}{i} \left[\left(1 + \frac{i}{1+k} \right)^n - 1 \right] = 0. \quad (9)$$

Transposing K , dividing by the coefficient of $x - Ki$, and finally transposing Ki , we find

$$x = Ki \left[1 + \frac{1}{\left(1 + \frac{i}{1+k} \right)^n - 1} \right]. \quad (10)$$

When the complex fraction inside the bracket in (10) is reduced to a simple fraction, x may be expressed in the form

$$x = Ki \frac{(1+i+k)^n}{(1+i+k)^n - (1+k)^n}. \quad (11)$$

Of course no actual business transaction could be carried out in exact accordance with formula (11), since in the first place an integral number of bonds must be retired each year, and in the second place the assumed market value of the bonds $1+k$ would probably vary from year to year. Nevertheless (and this is the important point) *the value of x given by formula (11) will furnish a starting point by means of which the process of retirement may be approximately shown.*

ILLUSTRATIVE EXAMPLE. A city issues bonds for \$200,000, to pay for a high-school building. The bonds are of denomination \$1000 and bear interest at 5% effective. If they can be repurchased at 98, what amount must be set aside annually to pay interest and buy back bonds so that the debt will all be paid in 20 years?

Solution. For this problem $K = \$200,000$, $i = .05$, $k = -.02$, $n = 20$. We have, then,

$$\begin{aligned} x &= \$200,000 \times .05 \frac{(1.03)^{20}}{(1.03)^{20} - (.98)^{20}} \\ &= \frac{\$18,061.112}{1.13851} \\ &= \$15,864, \text{ nearly.} \end{aligned}$$

For reasons noted above it would be useless to carry the division farther. After the interest, \$10,000, is paid, there would be approximately \$5864 available for the purchase of bonds at 98. The par value of the bonds purchased

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would be a little less than \$6000, so that 6 bonds of face value \$1000 or 60 bonds of face value \$100 could be purchased. The following year the interest charge would be \$4700, and the same number of bonds could be retired. The process furnishes little more than the means of making a rough estimate as to what would happen.

EXAMPLES

1. A debt of \$200,000 is to be paid in 20 years by means of a sinking fund established at the time the debt is incurred. If the sinking fund can be accumulated at 4% effective, how large will the fund be at the end of 10 years?

Solution. By formula (5), the amount in the fund will be

$$\begin{aligned} S_{10} &= \$200,000 \cdot \frac{1}{s_{\overline{20}|}} \cdot s_{\overline{10}|} \\ &= \$200,000 \times .03358175 \times 12.006107 \\ &= \$80,637.216. \end{aligned}$$

Or, by formula (5'), $S_{10} = \$200,000 \cdot \frac{(1.04)^{10} - 1}{(1.04)^{20} - 1}$

$$\begin{aligned} &= \$200,000 \cdot \frac{.48024428}{1.19112314} \\ &= \$80,637.216. \end{aligned}$$

2. What will be the annual payment into a sinking fund to be accumulated at 4% effective, established for the purpose of extinguishing a debt of \$225,000, due in 20 years?

3. If a sinking fund can be accumulated at 4% nominal, convertible semiannually, what must be the semiannual payment to a sinking fund which will provide for the extinction of the debt in Example 1 at the end of 20 years?

4. A man gives a mortgage for \$4000 on his home, paying interest at the rate of 6%, payable annually, and depositing a certain sum each year in a savings bank which pays 4% nominal, convertible semiannually. What sum must be set aside annually to pay interest and to pay off the mortgage in 10 years?

5. A city incurs a debt of \$200,000. Which would be better, to pay the debt, principal and interest at $6\frac{1}{4}\%$, in twenty equal annual installments, or to pay 6% interest each year on the debt and pay a fixed amount annually into a sinking fund to be accumulated for 20 years at 4%?

6. A debt of \$10,000, due in 10 years and bearing interest at the rate of 6%, payable annually, can be paid by means of a sinking fund which can be accumulated at 4% effective. What is the rate if the debtor can arrange to pay principal and interest in ten equal annual installments, in order to make the terms equivalent to the sinking-fund plan?

7. A city issues twenty-year bonds of denomination \$100, bearing interest at 5% nominal, payable semiannually, to the amount of \$200,000. A sinking fund can be accumulated at 4% effective. Which would be better, to accumulate a sinking fund, to which equal annual payments shall be made, or to buy back bonds in such manner that the annual payments for interest and for bonds repurchased will be as nearly equal as possible throughout the twenty-year period?

8. Show that formula (11) may be put into the form

$$x = K(1 + k) \left(\frac{1}{a_{\overline{n}|}} \text{ at rate } \frac{i}{1 + k} \right). \quad (12)$$

CHAPTER XII

DEPRECIATION

79. Definitions. *Depreciation* has been defined as "loss in the value of physical property due to use, which cannot be made good by current repairs." No matter how much care is taken to keep machinery or buildings in good repair, there will come a time when a new machine or new buildings must be put in place of the old, and this is true of practically every kind of physical property that is used in the conduct of business. To neglect to make definite provision for the replacement of property that will be destroyed through use is to invite certain financial disaster in the not far-distant future.

To illustrate, suppose a man pays \$1000 cash for an auto truck, and engages in the business of hauling freight. He earns, say, \$20 a day, out of which he pays costs of operation and maintenance. The remainder he spends for living expenses.

At the end of a period varying from 5 to 10 years his machine is worn out beyond all possibility of repair, and, having no money with which to purchase a new machine, he must go out of business. The case would not have been essentially different had the machine been bought on time.

The trouble might have been averted if this man had set aside each year a definite sum which would have accumulated to an amount sufficient to enable him to purchase a new machine when the old one was worn out.

A fund set aside to replace a piece of depreciable property when it is worn out is sometimes called a *depreciation fund*. A better term would be *replacement fund*.

A replacement fund is essentially a sinking fund, as may be seen if we think of the cost of the depreciable property as an obligation to be met when the property is cast aside. For the sake of brevity the depreciable property will be called an *asset*.

It is usually composed of several distinct units, as buildings, steam engines, locomotives, and the like.

A piece of depreciable property usually possesses a residual or salvage value when it must be thrown aside as worn out. This residual value is called the *scrap value*. The difference between the cost new and the scrap value is called the *wearing value* or the *depreciable value*. If C denote cost new, S the scrap value, and W the wearing value, clearly,

$$W = C - S. \quad (1)$$

The length of time to elapse before the value of an asset is reduced to scrap value by reason of wear is called the *probable life*, or simply the *life*. The determination of the life of an asset is a matter of observation and must be made by the engineer or the architect, or by some person familiar with the construction and use of the asset. For example, the engineers of the Interstate Commerce Commission have fixed the probable life of a steam locomotive at 30 years.

80. The problem. Two aspects of the problem of depreciation present themselves. When a business unit, whether it be a building or a machine, is put into service, its cost is placed upon the books of the owner. This value is called the *book value*, or, strictly speaking, the *initial book value*. From year to year the book value is decreased as the unit depreciates, until at the end of the life of the unit its book value has been reduced to the scrap value. One important problem is to determine for the accountant how much should be "charged off" or "written off" each year so that at the end of the life the value on the books shall stand at scrap value. The amount "written off" each year may be called a *depreciation charge*. The depreciation charge may be defined in several ways. (See §§ 81, 82, 83.)

This depreciation charge must be deducted from the earnings of the business before profits are distributed. It is just as much a legitimate charge as the charge for labor, or materials, or insurance. The amount may be invested in an outside enterprise or, as is usually the case, it may be put back into the business.

The other aspect of the problem is the determination of the amount that must be set aside each year as a reserve, so that the

accumulated reserve will enable the owner to replace the unit when it is worn out. Such an annual charge may properly be called a *replacement charge* and the reserve itself a *replacement fund*. The replacement charge is not necessarily equal to the depreciation charge, as the following pages will show.

81. The straight-line method. The straight-line method of determining the depreciation charge consists in setting aside or "writing off" one n th part of the wearing value at the end of each year during the n years of the life of the asset. If C be the cost new, S the scrap value, and D the annual depreciation charge, the formula for D is

$$D = \frac{C - S}{n} = \frac{W}{n}. \quad (1)$$

The total depreciation D_r at the end of r years will be

$$D_r = \frac{C - S}{n} r = \frac{W}{n} r; \quad (2)$$

and the book value at the same date will be

$$V_r = C - \frac{W}{n} r. \quad (3)$$

This method is called the straight-line method because the total depreciation charge is proportional to the time, so that when plotted against the time r the graph will be a straight line. The graph of the book values is also a straight line.

EXAMPLES

1. What is the annual depreciation charge, by the straight-line method, of an electric locomotive costing new \$150,000, and having a scrap value of \$1000 at the end of its life of 30 years?

2. What will be the book value of the locomotive in Example 1 at the end of 20 years?

3. Construct the curve showing the depreciation of the locomotive in Example 1.

4. Construct the curve of book values of the locomotive in Example 1.

This method has the advantage of extreme simplicity and ease of application. However, it takes no account of the earning power of the sums written off and for that reason is a fair method only in

case the funds are not supposed to be productively invested. It is clear that if the depreciation charge $\frac{W}{n}$ be written off each year, at the end of n years the book value will coincide with the scrap value.

82. The sinking-fund method. Under the sinking-fund method the owner sets aside annually a fixed amount, sufficiently large so that when the amounts set aside are accumulated at a specified rate the amount of the accumulations will suffice, together with the scrap value, to replace the worn-out unit at the end of its life. The amount may be set aside as a book account or as an actual reserve. In either case it is understood that the funds will be productively invested.

PROBLEM. *To find the amount which must be set aside annually to provide a fund for the replacement of an asset when the end of its life is reached.**

The problem is precisely that of §§ 49 and 77. If R denote the fixed annual charge to provide for replacement, W the wearing value, and n the life, R is given by the formula

$$R = W \cdot \frac{1}{s_{\overline{n}|}} = (C - S) \frac{1}{s_{\overline{n}|}}, \quad (1)$$

where $s_{\overline{n}|}$ is computed at the specified rate of interest.

This charge R is, properly speaking, a *replacement* charge rather than a *depreciation* charge. The sum of the annual charges will not equal the wearing value, as should be the case with a true depreciation charge.

PROBLEM. *To find the amount in the replacement fund at the end of r years.*

The annual charge is, by (1), $W \cdot \frac{1}{s_{\overline{n}|}}$. The amount in the replacement fund at the end of r years is then the amount of the annuity of $W \cdot \frac{1}{s_{\overline{n}|}}$ per annum, payable annually for r years. If the amount in the fund at the end of r years be denoted by R_r , we shall have

$$R_r = W \cdot \frac{1}{s_{\overline{n}|}} \cdot s_{\overline{r}|} = (C - S) \frac{1}{s_{\overline{n}|}} \cdot s_{\overline{r}|}. \quad (2)$$

* An asset is not necessarily discarded at the end of its estimated life, but from that time forward its book value should be the scrap value.

If the amount of invested capital is to remain intact, the book value at the end of r years should be exactly the original cost less the amount in the replacement fund. That is to say, if V_r denote the value remaining, or the book value at the end of r years,

$$V_r = C - (C - S) \frac{1}{s_{\overline{n}|}} \cdot s_{\overline{r}|}. \quad (3)$$

It will be profitable to work out in detail an example showing the progress of the replacement fund and the book values from year to year.

ILLUSTRATIVE EXAMPLE. A man buys an automobile for \$1000. If the life is 5 years, the scrap value is \$100, and money is worth 6%, what is the annual replacement charge, and what are the book values at the end of each succeeding year?

Solution. By formula (1) the annual replacement charge will be

$$R = W \cdot \frac{1}{s_{\overline{5}|}} = 900 \cdot \frac{1}{s_{\overline{5}|}} = \$159.66.$$

The book value at the beginning of the first year will be \$1000. At the end of the first year or at the beginning of the second year this book value must be diminished by the amount placed in the reserve fund, i.e. by \$159.66. It will then stand at \$840.34. The amounts in the replacement, or reserve, fund will be the amounts of the annuity of \$159.66 per annum accumulated to the date in question, while the corresponding book values will be found by subtracting the amounts in the reserve fund from \$1000. In this way the invested capital will remain intact, since the amount originally invested will be found in the book value and the reserve fund combined.

The whole process may be made clear by the following schedule:

Schedule showing depreciation on a 6% basis of an automobile which costs new \$1000, and is reduced to a scrap value of \$100 at the end of 5 years:

Year end	Replacement charge	Amount in replacement fund	Book value	Increase in replacement fund
.			\$1000.00	
1	\$159.66	\$159.66	840.34	\$159.66
2	159.66	328.90	671.10	169.24
3	159.66	508.29	491.71	179.39
4	159.66	698.44	301.56	190.15
5	159.66	900.00	100.00	201.56
				\$900.00

Since the purpose of a theory of depreciation is to enable the business man to keep his invested capital intact, it is clear that the book value for any year end may now be defined as the difference between the initial book value and the amount in the replacement fund. The *annual* charge for depreciation will then be the *decrease in book value for the year* or, what comes to the same thing, the increase in the amount of the reserve fund for the year. The depreciation charges for the five years are shown in the last column of the schedule.

The schedule makes clear the difference under the present method between a replacement charge, which is constant, and a depreciation charge, which increases from year to year. The two are the same for the first year.

EXAMPLES

1. A building costing \$100,000 has an estimated life of 40 years, with no scrap value. Determine, on a 6% basis, the annual replacement charge. What will be the total amount in the replacement fund and the book value at the end of 30 years?

2. Construct the schedule showing the progress of the replacement fund and the book values for the first 5 years of the life of the building in Example 1.

3. Construct the schedule showing the progress of the replacement fund and the book values for the last 5 years in Example 1.

4. A locomotive costing new \$100,000 has a life of 30 years and a scrap value of \$1000. On a 5% basis, what is the annual replacement charge? What will be the book value at the end of the tenth year? What are the depreciation charges for the first 5 years?

5. Construct a curve showing the amounts in the depreciation fund for the locomotive in Example 4.

6. Construct a curve showing the book values for the locomotive in Example 4.

7. Prove that the increase in the replacement fund for the r th year over that for the $(r - 1)$ st year is

$$(C - S) \frac{1}{s^n} (1 + i)^{r-1}. \quad (4)$$

PROBLEM. *To find the annual payment to the replacement fund when the depreciable property consists of several kinds of assets having probable lives of various lengths.*

Let $W_1, W_2, W_3, \dots$, be the wearing values of the various parts of a "plant," and $n_1, n_2, n_3, \dots$, their probable lives. The total charge for replacement will be the sum of the charges for the separate parts. The separate charges will be, by formula (1),

$$W_1 \cdot \frac{1}{s_{\overline{n_1}|}}, \quad W_2 \cdot \frac{1}{s_{\overline{n_2}|}}, \quad W_3 \cdot \frac{1}{s_{\overline{n_3}|}}, \quad \dots,$$

and consequently the total annual charge will be

$$R = W_1 \cdot \frac{1}{s_{\overline{n_1}|}} + W_2 \cdot \frac{1}{s_{\overline{n_2}|}} + W_3 \cdot \frac{1}{s_{\overline{n_3}|}} + \dots \quad (5)$$

Suppose, for example, we wish to find the annual charge for depreciation for an electric-power plant which consists of four parts: viz. a building with wearing value \$20,000 and life 50 years, an engine with wearing value \$5000 and life 25 years, a boiler with wearing value \$2000 and life 15 years, a dynamo with wearing value \$6000 and life 18 years, when money can be accumulated at 4%.

By formula (5), the total annual charge to provide for replacement will be

$$\begin{aligned} R &= \$20,000 \cdot \frac{1}{s_{\overline{50}|}} + \$5000 \frac{1}{s_{\overline{25}|}} + \$2000 \frac{1}{s_{\overline{15}|}} + \$6000 \frac{1}{s_{\overline{18}|}} \\ &= \$131.00 + \$120.06 + \$99.88 + \$233.96 \\ &= \$584.90. \end{aligned}$$

These results may be exhibited in the form of a table, as follows:

Part	Life of part	Wearing value	Annual replacement charge
Building	50 years	\$20,000	\$131.00
Engine	25 years	5,000	120.06
Boiler	15 years	2,000	99.88
Dynamo	18 years	6,000	233.96
		\$33,000	\$584.90

If the probable life has been correctly given, it is clear that the renewal of each part of the plant will be provided for when the part must be discarded as worn out, since the replacement charge on each part has been figured separately.

The amount in the replacement fund at the end of r years will be

$$R_r = \left(W_1 \cdot \frac{1}{s_{\overline{n_1}|}} + W_2 \cdot \frac{1}{s_{\overline{n_2}|}} + W_3 \cdot \frac{1}{s_{\overline{n_3}|}} + \cdots \right) s_{\overline{n}|}, \quad (6)$$

provided r is not greater than the smallest n . For example, if we were using the data of the problem given above, r could not be greater than 15.

EXAMPLES

1. On a 4% basis, find the annual charge for replacement for a plant consisting of four parts, with costs, scrap values, and probable lives as given in the following table:

Part	Cost new	Scrap value	Life
A	\$100,000	\$5000	50 years
B	25,000	3000	25 years
C	20,000	1000	15 years
D	30,000	2500	20 years

2. A building costs new \$200,000. Small repairs cost \$1000 per annum, and every 10 years the building must be thoroughly renovated at a cost of \$20,000. If the building must be torn down in 50 years, with a salvage value of \$5000, what sum must be set aside annually to cover the cost of repairs, renovation, and replacement?

3. What should be the book value of the plant in Example 1 at the end of 5 years?

4. Using the definition of depreciation charge given on page 167, what would be the annual depreciation charges for the first 5 years in Example 1?

5. The rules of the Interstate Commerce Commission (1909) require that a monthly charge for depreciation shall be allowed toward operating expenses for all steam and electric locomotives. What would be the monthly replacement charge for a steam locomotive costing new \$50,000, having a probable life of 30 years and a scrap value of \$1000, provided the monthly charge is taken to be one twelfth of the annual replacement charge and the rate of interest is 3% effective? What would be the monthly replacement charge computed by the formula for an annuity payable twelve times a year? What would be the annual depreciation charge for the first 5 years of the life of the locomotive? Find the depreciation charge for the eleventh year by formula (4).

6. In a recent investigation of a public utility the values of the various parts were found to be as follows:

Part	Life	Wearing value
A	10 years	\$3,157,930
B	25 years	3,076,239
C	40 years	2,677,425
D	15 years	674,867
E	12 years	666,176
F	12 years	1,036,062
G	30 years	146,059
H	10 years	2,862,091
I	10 years	679,206
J	50 years	1,648,378
K	5 years	134,828

What should be the annual charge for replacement on a 3% basis?—
W. J. Hagenah, *Chicago Telephone Company Investigation*

83. The interest-on-investment method. By this method a constant charge is made each year to cover depreciation for the year and interest on the value remaining at the end of the previous year after the depreciation charge for that year has been written off. Since the sum of interest and depreciation charges is constant, the interest charge, which is based on book value, will decrease and the depreciation charge will increase from year to year.

PROBLEM. *To determine the constant charge for interest on value remaining and for depreciation.* •

Let C be the original cost, K the total annual charge for interest and depreciation, $V_0, V_1, V_2, V_3, \dots, V_r$ the book values remaining at the *beginning* of the first, second, third, $\dots$, and $(r+1)$ st years, respectively. Then

$$V_0 = C,$$

$$V_1 = C(1+i) - K,$$

$$V_2 = [C(1+i) - K](1+i) - K = C(1+i)^2 - [(1+i) + 1]K,$$

$$V_3 = \{C(1+i)^2 - [(1+i) + 1]K\}(1+i) - K$$

$$= C(1+i)^3 - [(1+i)^2 + (1+i) + 1]K,$$

$$\vdots$$

$$V_r = C(1+i)^r - [(1+i)^{r-1} + (1+i)^{r-2} + \dots + (1+i) + 1]K.$$

The number in the last bracket is $s_{\overline{r}|}$, so that

$$V_r = C(1+i)^r - s_{\overline{r}|}K. \quad (1)$$

By hypothesis, the value remaining at the *end* of n years, that is to say, the value remaining at the *beginning* of the $(n+1)$ st year, is the scrap value S . Consequently, the equation

$$V_n = C(1+i)^n - s_{\overline{n}|}K = S \quad (2)$$

will serve to determine the required number K . We find from (2) that

$$K = [C(1+i)^n - S] \cdot \frac{1}{s_{\overline{n}|}}. \quad (3)$$

The quantity K may be put into more convenient form by noting that from the equation defining $s_{\overline{n}|}$ we have identically

$$(1+i)^n = 1 + i s_{\overline{n}|}. \quad (4)$$

When $(1+i)^n$ in (3) is replaced by its value as given by (4) and proper simplifications are made, the value of K takes the convenient form

$$K = Ci + (C - S) \frac{1}{s_{\overline{n}|}}. \quad (5)$$

The first term in the expression for K is the interest charge for the first year, while the second is the replacement charge as given by (1) of § 82.

Since the sum of the interest charge and the depreciation charge is to be constant, the quantity $(C - S) \frac{1}{s_{\overline{n}|}}$ is the depreciation charge *for the first year*.

If the problem be merely to determine the value of K it is completely solved. However, there is nothing in formula (5) to indicate how the accountant is to determine the depreciation charge after the first year. To show how this may be done it is necessary to solve two further problems. These problems are to find the book value at the end of r years and to find the depreciation charge for the r th year.

EXERCISES

1. What sum will cover interest and depreciation on a machine costing \$10,000 and having a scrap value of \$500 at the end of its life of 15 years, money worth 6%?

2. A building costs \$100,000 and must be abandoned at the end of 40 years. The labor cost of razing the building offsets the value of the material that is salvaged. How much should be charged annually for interest at 6% on values remaining and for depreciation?

PROBLEM. *To find the value remaining after r years in terms of C , S , n , and r .*

If K in equation (1) is replaced by its value as given by (5), equation (1) gives for V_r the value

$$\begin{aligned} V_r &= C(1+i)^r - s_{\overline{r}|} \cdot \left[Ci + (C-S) \cdot \frac{1}{s_{\overline{n}|}} \right] \\ &= C[(1+i)^r - i s_{\overline{r}|}] - (C-S) \cdot \frac{1}{s_{\overline{n}|}} \cdot s_{\overline{r}|}. \end{aligned}$$

From the equation defining $s_{\overline{r}|}$, we note that

$$(1+i)^r - i s_{\overline{r}|} = 1;$$

so that
$$V_r = C - (C-S) \cdot \frac{1}{s_{\overline{n}|}} \cdot s_{\overline{r}|}. \quad (6)$$

Alternative solution. The annual replacement charge is, by (1) of § 82,

$$(C-S) \frac{1}{s_{\overline{n}|}},$$

and at the end of r years the amount in the replacement fund should be

$$(C-S) \cdot \frac{1}{s_{\overline{n}|}} \cdot s_{\overline{r}|}.$$

If capital invested is to remain intact, the value remaining at the end of r years should be the difference between the original cost and the amount in the replacement fund, or

$$V_r = C - (C-S) \cdot \frac{1}{s_{\overline{n}|}} \cdot s_{\overline{r}|}. \quad (6)$$

Formula (6), which is the same as formula (3) of § 82, shows the substantial identity of the sinking-fund method and the interest-on-investment method, since both methods lead to the same book values and consequently to the same depreciation charges.

EXERCISES

1. A machine costing \$10,000 and having a scrap value of \$500 has a life of 15 years. On a 6% basis, what should be the book value at the end of r years?

2. If 6% is to be realized on values remaining, what should be the book value of a building costing \$100,000 and having no scrap value, at the end of a life of 40 years?

PROBLEM. *To break up the constant charge K into two parts, one of which shall be the interest on the book value for the previous year and the other the depreciation charge.*

If D_r denote the depreciation charge for the r th year, D_r will be the difference between the constant charge K and the interest charge on the book value at the beginning of the year. Then

$$K = V_{r-1}i + D_r, \quad (7)$$

or
$$D_r = K - V_{r-1} \cdot i \quad (8)$$

$$\begin{aligned} &= Ci + (C - S) \frac{1}{s_{\overline{n}|}} - \left[C - (C - S) \frac{1}{s_{\overline{n}|}} \frac{s_{r-1}}{s_{\overline{n}|}} \right] i \\ &= (C - S) \frac{1}{s_{\overline{n}|}} [1 + i s_{r-1}]. \end{aligned}$$

But $1 + i s_{r-1} = (1 + i)^{r-1}$, from the equation defining s_{r-1} . Consequently,

$$D_r = (C - S) \frac{1}{s_{\overline{n}|}} (1 + i)^{r-1}. \quad (9)$$

Finally, replacing $V_{r-1}i$ and D_r in (7) by their values as given by (6) and (9),

$$K = \left[C - (C - S) \cdot \frac{1}{s_{\overline{n}|}} \cdot \frac{s_{r-1}}{s_{\overline{n}|}} \right] i + (C - S) \frac{1}{s_{\overline{n}|}} (1 + i)^{r-1}. \quad (10)$$

Formula (10) shows K broken up in the required manner. The actual computations will be made by means of formulas (5), (6), and (9).

ILLUSTRATIVE EXAMPLE. A man buys an automobile having a life of 5 years and a scrap value of \$100, for \$1000. Find (1) the total charge for interest and depreciation and (2) the depreciation charge for each year,

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and construct a schedule showing the book values and the depreciation charges during the life of the automobile. Assume that money is worth 6%.

Solution. The replacement charge which is the same as the depreciation charge for the first year is $(C - S)\frac{1}{s_n}$, or, for the present problem,

$$(\$1000 - \$100) \times 0.1773964 = \$159.66.$$

The interest charge for the first year will be \$60. The total annual charge will therefore be

$$\$60 + \$159.66 = \$219.66.$$

The book value at the beginning of the second year will be

$$\$1000 - \$159.66 = \$840.34.$$

The computation may be carried on year by year, either by computing successive interest charges or by computing successive depreciation charges. The following schedule shows in detail the steps by which the original value is reduced to scrap value by charging off depreciation.

Schedule showing the depreciation on a 6% basis of an automobile which costs new \$1000, and which is reduced to a scrap value of \$100 in 5 years:

Year end	Interest	Depreciation	Total charge	Book value
.				\$1000.00
1	\$60.00	\$159.66	\$219.66	840.34
2	50.42	169.24	219.66	671.10
3	40.27	179.39	219.66	491.71
4	25.91	190.15	219.66	301.66
5	18.10	201.56	219.66	100.00
		\$900.00		

The schedule shows that the sum of the depreciation charges is exactly the wearing value, \$900. That this is true in all cases may be shown by taking the sum of the quantities obtained by giving to r the values 0, 1, 2, 3, . . . , $r - 1$ in formula (9).

The student may compare the amounts in the column for depreciation in the above schedule with the amounts in the difference column of the schedule in § 82.

EXAMPLES

1. A building costing \$100,000 has an estimated life of 50 years and no scrap value. On a 6% basis what would be the total charge for interest and depreciation? Give the depreciation charges for the first 5 years of the life; for the last 5 years.

2. Give the book values of the building in Example 1 for the first 5 years; for the last 5 years.

3. A building costing \$100,000 has an estimated life of 40 years, and no scrap value. Find the annual charge required for interest and depreciation on a 6% basis. What is the annual interest charge and the annual depreciation charge for each of the first 5 years? for each of the last 5 years? Compare the results with the results in Example 1, p. 167.

4. Show that formula 10, § 83, may be proved by multiplying the last term of

$$K = Ci + (C - s) \frac{1}{s_{\overline{n}|i}}$$

by the expression $(1 + i)^{r-1} - i s_{\overline{r-1}|i}$, which, by the formula defining $s_{\overline{r-1}|i}$, is identically equal to 1.

84. Modification of the interest-on-investment method. A modification of the method of the preceding section is obtained by assuming that the interest on the value remaining is computed at a rate different from the going rate. In this case the interest charge will differ from that of § 83, but the depreciation charge will remain the same. The total charge will no longer be constant.

Formula (10) of § 83 for the combined interest and depreciation charges becomes for the r th year

$$K_r = \left[C - (C - S) \cdot \frac{1}{s_{\overline{n}|i}} \cdot s_{\overline{r-1}|i} \right] i' + (C - S) \cdot \frac{1}{s_{\overline{n}|i}} \cdot (1 + i)^{r-1}, \quad (1)$$

where $s_{\overline{n}|i}$ and $s_{\overline{r-1}|i}$ are computed at rate i .

The following schedule shows the progress of the various charges in the illustrative example of § 83, when two interest rates are involved.

Schedule showing the depreciation on a truck costing \$1000 and having a scrap value of \$100 at the end of 5 years, when interest on values remaining is computed at 8% and the current rate is 6% :

Year end	Book value	Interest charge	Depreciation charge	Total interest and depreciation
.	\$1000.00			
1	840.34	\$80.00	\$159.66	\$239.66
2	671.10	67.23	169.24	236.66
3	491.71	53.69	179.39	233.08
4	301.66	39.34	190.15	229.49
5	100.00	24.12	201.56	225.68

EXAMPLES

1. A building costs new \$50,000. It is estimated that it can be used for 40 years, at the end of which time there will be no salvage value. Determine the charges for interest and depreciation for the first 5 years on a basis of 6% for current rate and 7% for interest on book values.

2. Determine interest and depreciation charges and book values for the last 5 years of the life of the building in Example 1.

85. The unit-cost method. The unit-cost method is based on a comparison of an old machine with a new machine designed to perform the same service. It is better adapted to the process of setting a valuation upon the old machine than to the needs of the bookkeeper, whose business it is to determine the amount to be charged off for depreciation against the old machine.

Let w_r be the value of the old machine after r years of service, and s its scrap value at the end of its estimated life. Further, let

n = the total life of the old machine,

q = the average annual cost of operation for the remainder of the life,

p = the average annual cost of repairs for the remainder of the life,

u = the average cost of a unit of output,

y = the average number of units of annual output.

Also let V be the cost of a new machine to replace the old machine, S the scrap value of the new machine, N , P , Q , U , and Y the numbers for the new machine corresponding to n , p , q , u , and y for the old.

The cost of the annual output of the old machine will then be made up of a replacement charge, an interest charge, and the charges for operation and repairs. Consequently,

$$uy = (w_r - S) \frac{1}{s_{n-r}|} + w_r i + q + p. \quad (1)$$

If account be taken of the relation

$$\frac{1}{s_{n-r}|} + i = \frac{1}{a_n|}$$

and both sides of equation (1) be divided by y , this equation may be written in the form

$$u = \frac{(w_r - s) \frac{1}{a_{n-r}} + si + q + p}{y}. \quad (2)$$

Similarly, the unit cost for the new machine would be

$$U = \frac{(V - S) \frac{1}{a_N} + Si + Q + P}{Y}. \quad (3)$$

Theoretically the unit costs (2) and (3) should be equal, so that

$$\frac{(w_r - s) \frac{1}{a_{n-r}} + si + q + p}{y} = \frac{(V - S) \frac{1}{a_N} + Si + Q + P}{Y}. \quad (4)$$

When (4) is solved for $w_r - s$ the result is

$$w_r - s = \left[\frac{y}{Y} (V - S) \frac{1}{a_N} + (Si + Q + P) \frac{y}{Y} - si - q - p \right] a_{n-r}; \quad (5)$$

$$\text{or } w_r - s = \frac{y}{Y} (V - S) \cdot \frac{1}{a_N} \cdot a_{n-r} + \frac{(yS - Ys)i + y(Q + P) - Y(q + p)}{Y} \cdot a_{n-r}. \quad (6)$$

By means of the identical relation

$$a_k = v^k s_k = \frac{s_k}{(1+i)^k},$$

the product $\frac{1}{a_N} \cdot a_{n-r}$ may be reduced to the form

$$s_N \cdot s_{n-r} \cdot (1+i)^{N-n-r};$$

so that finally

$$w_r = s + \frac{y}{Y} (V - S) \frac{1}{s_N} \cdot s_{n-r} (1+i)^{N-n-r} + \frac{(yS - Ys)i + y(Q + P) - Y(q + p)}{Y} a_{n-r}. \quad (7)$$

Either formula (6) or formula (7) may be used for computing w_r .*

* It must be noted that these formulas furnish the basis for a comparison of values of the new and the old machine rather than the basis for determining depreciation. This can be seen clearly from the fact that the value w_0 , obtained by making $r = 0$ in (6) or (7), is not the initial value of the old machine unless $y = Y$, $s = S$, $n = N$, $p = P$, and $q = Q$; in other words, unless the two machines are identical except as to age.

EXAMPLES

1. A machine having an estimated life of 20 years has been in use for 10 years and turns out 100 units of work each year. The operating cost is \$500 per year and repairs are \$300 per year. A new machine costing \$2000 has a probable life of 20 years and will turn out 100 units of work per year. For the new machine operation costs \$400 and repairs cost \$300 per year. Neither machine is supposed to have any salvage value. What is the value of the old machine on the basis of a 6% interest rate?

Solution. From the data given, $y = 100$, $Y = 100$, $V = \$2000$, $n = 20$, $N = 20$, $r = 10$, $p = \$300$, $P = \$300$, $q = \$500$, $Q = \$400$, $s = S = 0$, $i = .06$. Formula (7) then gives

$$\begin{aligned} w_{10} &= \frac{100}{100} 2000 \cdot \frac{1}{s_{\overline{20}|}} \cdot s_{\overline{10}|} \cdot (1.06)^{10} + \frac{100 \times 700 - 100 \times 800}{100} a_{\overline{10}|} \\ &= 2000 \times .0271846 \times 13.1807949 \times 1.7908475 + (-100) \times 7.3600870 \\ &= 1283.37 - 736.01 \\ &= \$547.36. \end{aligned}$$

2. What would be the value of the old machine in Example 1 if the new machine would produce 125 units in a year? Interpret the result.

3. What output for the new machine in Example 1 would render the value of the old machine zero?

4. Solve Example 1 in general terms.

5. Suppose that all data for both machines in Example 1 are the same except the items of operation and repairs. What total amount for these two items will render the old machine valueless at the end of 10 years?

6. A steam locomotive costing \$100,000 has an estimated life of 30 years. It requires for a year's operation 3000 tons of coal costing \$4 per ton; and other material costing \$500. The men who operate it receive \$5500, and after the tenth year it is estimated that repairs will cost \$1500 per year. An electric locomotive having one and one-half times the capacity costs \$200,000 and has an estimated life of 35 years. It consumes current costing \$15,000 and requires for operation other materials costing \$1000. Repairs are estimated at \$2000 per year, and the men who operate it receive \$6000 per year. What is the value of the steam locomotive after 10 years of service, if money is worth 5%?

86. **The relation between the unit-cost method and the sinking-fund method.** It is easy to show that the sinking-fund method is in a certain sense a special case of the unit-cost method, though each method contains elements not found in the other. To this end, let us suppose that the new and the old machines have the same scrap values, will produce the same outputs, and require the same amounts for operation and repairs. In other

words, the new machine is practically the same machine that the old one was at the beginning of its life.

If $y = Y$, $s = S$, $p = P$, and $q = Q$, the second term of the right member of equation (7) vanishes and

$$W_r - s = (V - s) \cdot \frac{1}{s_{\overline{N}|}} \cdot s_{\overline{n-r}|} (1+i)^{N-\overline{n-r}}. \quad (1)$$

If in addition $n=N$ and $r=0$, equation (1) is reduced to

$$W_0 = V, \text{ or } W_0 - s = (V - s) \cdot \frac{1}{s_{\overline{N}|}} \cdot s_{\overline{N}|}. \quad (2)$$

Now the total depreciation on the machine having an initial value W_0 is

$$D_r = W_0 - W_r. \quad (3)$$

We may therefore obtain D_r by subtracting equation (2) from equation (1). The resulting equation is

$$D_r = W_0 - W_r = (V - s) \cdot \frac{1}{s_{\overline{n}|}} [s_{\overline{n}|} - (1+i)^r s_{\overline{n-r}|}]. \quad (4)$$

But

$$s_{\overline{n}|} - (1+i)^r s_{\overline{n-r}|} = s_{\overline{r}|}, \text{ and } V = W_0,$$

so that

$$D_r = W_0 - W_r = (W_0 - s) \cdot \frac{1}{s_{\overline{n}|}} \cdot s_{\overline{r}|}. \quad (5)$$

Equation (5) is identical with equation (2), § 82.

87. The composite life of a plant. The notion of the life of a plant as a whole is of considerable importance in several connections. The *composite life* of a plant is defined to be the time in which the total annual replacement charge for the plant, found by the sinking-fund method, would amount to the total wearing value when accumulated at an assumed rate of interest.

PROBLEM. *To determine the composite life of a plant composed of several parts having different probable lives.*

According to the definition the composite life would be the value of n determined from the equation

Total wearing value = $s_n \times$ total annual replacement charge.

Let W denote the total wearing value, i.e. the sum of the wearing values of the several parts, and let D be the total annual replacement charge. Then,

$$W = W_1 + W_2 + \dots + W_r, \quad (1)$$

$$D = D_1 + D_2 + \dots + D_r, \quad (2)$$

where $W_1, W_2, \dots, W_r$ and $D_1, D_2, \dots, D_r$ are the wearing values and the annual replacement charges for the various parts of the plant. By definition,

$$W = s_{\overline{n}|i} D. \quad (3)$$

When $s_{\overline{n}|i}$ is replaced by its value, viz. $\frac{(1+i)^n - 1}{i}$, equation (1) becomes

$$W = \frac{(1+i)^n - 1}{i} D, \quad (4)$$

which in turn reduces easily to the form

$$(1+i)^n = 1 + \frac{W}{D} i.$$

Solving this exponential equation for n , we find

$$n = \frac{\log\left(1 + \frac{W}{D} i\right)}{\log(1+i)}. \quad (5)$$

The ratio of the total annual replacement charge D to the total wearing value W may be called the *rate of depreciation*. Denoting the rate of depreciation by d , we have, by definition,

$$d = \frac{D}{W} = \frac{1}{s_{\overline{n}|i}}. \quad (6)$$

If the value of the ratio $\frac{D}{W}$ be replaced by d in equation (5), that formula reduces to

$$n = \frac{\log\left(1 + \frac{i}{d}\right)}{\log(1+i)} = \frac{\log(1 + i s_{\overline{n}|i})}{\log(1+i)}. \quad (7)$$

ILLUSTRATIVE EXAMPLE. Suppose we wish to determine the composite life of the electric-power plant discussed in § 82.

Solution. On a 4% basis the annual replacement charge for the building with life 50 years and wearing value \$20,000 was found to be \$131.00; for the engine with life 25 years and wearing value \$5000, \$120.06; for the boiler with life 15 years and wearing value \$2000, \$99.88; for the dynamo with life 18 years and wearing value \$6000, \$233.96. The total wearing value W is \$33,000, and

the total annual charge for depreciation is \$584.90. Consequently, the rate of depreciation is

$$d = \frac{584.90}{33,000} = .017724.$$

By formula (5),

$$\begin{aligned} n &= \frac{\log \left(1 + \frac{.04}{.017724} \right)}{\log(1.04)} \\ &= \frac{\log 3.25682}{\log 1.04} \\ &= \frac{0.51279}{0.01703} \\ &= 30.111 \text{ years.} \end{aligned}$$

It is easy to get an approximate solution directly from the tables, for from formula (6) we have

$$\frac{1}{s_{\overline{n}|}} = \frac{D}{W} = d. \quad (8)$$

But

$$\frac{1}{s_{\overline{n}|}} = \frac{1}{a_{\overline{n}|}} - i,$$

so that

$$\frac{1}{a_{\overline{n}|}} = d + i. \quad (9)$$

In the present problem

$$\begin{aligned} \frac{1}{a_{\overline{n}|}} &= .017724 + .04 \\ &= .057724. \end{aligned}$$

From the table for "the annuity that 1 will purchase," i.e. $\frac{1}{a_{\overline{n}|}}$, we see that .057724 is very nearly the value corresponding to $n = 30$.

The notion of composite life is important when considered in connection with bonds secured by the property of a plant. For example, in the case just considered, bonds secured by the property of the plant should not be issued for a time exceeding the composite life of the plant, viz. twenty-six years. Indeed, to provide a proper margin of safety, bonds should not be issued for a time exceeding twenty years.

EXAMPLES

1. Find the composite life, on a 3% basis, of a plant consisting of three parts, A, B, and C, as follows: A cost new \$10,000, scrap value \$600, probable life 30 years; B cost new \$5000, scrap value \$100, probable life 25 years; C cost new \$8000, scrap value \$300, probable life 20 years.

2. Find the composite life, on a 4% basis, of the plant in Example 1 of § 82.

3. On a 3% basis find the composite life of the plant in Example 6 of § 82 (p. 170). What is the rate of depreciation?

4. The *condition per cent* is defined as the ratio of depreciable value existing to the depreciable value new, i.e. $\frac{W_t}{W}$, where W and W_t are the depreciable values new and at the end of t years respectively. Prove that if r_t denote the condition per cent at age t , theoretically, its value would be given by the formula

$$r_t = 1 - \frac{s_{\bar{t}|}}{s_{\bar{n}|}} = \frac{(1+i)^n - (1+i)^t}{(1+i)^n - 1}.$$

5. The *composite age* of an asset is the length of time in which the asset would theoretically reach a given per-cent condition, or, what amounts to the same thing, a certain existing wearing value. If t denote the composite age, prove by means of the formula of example (4) that, theoretically, the composite age of an asset having a probable life of n years and having condition per cent r is given by the formula

$$t = \frac{\log[(1+i)^n(1-r) + r]}{\log(1+i)}.*$$

6. If W_1, W_2, W_3 , be the wearing values of the parts of a plant consisting of three parts, and n_1, n_2, n_3 , their probable lives, prove that an approximate value for the composite life n is given by the formula

$$n = \frac{n_3(W_1 + W_2 + W_3)}{W_1 \frac{n_3}{n_1} + W_2 \frac{n_3}{n_2} + W_3},$$

where n_3 is the life of the most durable part.

Suggestion. Formula (5) of § 82 is equivalent to

$$(W_1 + W_2 + W_3) \frac{1}{s_{\bar{n}|}} = W_1 \frac{1}{s_{\bar{n}_1|}} + W_2 \frac{1}{s_{\bar{n}_2|}} + W_3 \frac{1}{s_{\bar{n}_3|}}.$$

Replace $\frac{1}{s_{\bar{n}|}}, \frac{1}{s_{\bar{n}_1|}}, \frac{1}{s_{\bar{n}_2|}},$ and $\frac{1}{s_{\bar{n}_3|}}$ by their values, expand powers of $1+i$, and drop powers of i above the first. What is the significance of the formula thus obtained?

7. In § 55 it was shown that

$$C_\infty = C + \frac{C}{i} \cdot \frac{1}{s_{\bar{k}|}}$$

is the capitalized cost of an asset with depreciable value C and probable life of k years. Prove that computing an annual return at rate i on C_∞ would be exactly equivalent to allowing interest on the original investment at rate i and allowing for depreciation by formula (5) of § 83.

*For the formula in Example 5 I am indebted to Mr. Edwin Gruhl, formerly with the Wisconsin Railroad Commission.

88. Note on depreciation. An adequate theory of depreciation should satisfy the following requirements:

1. It should furnish the means by which the accountant may determine in a simple manner the amount to be charged off annually until the book value of the asset is reduced to scrap value.
2. It should enable the accountant to determine easily and quickly the amount that should be in the replacement fund at any given time, so that at the end of the life of the asset this reserve will be sufficient in amount to replace the asset.
3. The depreciation charges should be fairly uniform throughout the life of the asset.

Viewed in the light of these requirements, either the straight-line method, the sinking-fund method, or the interest-on-investment method is adequate.

It should be noted, however, that if depreciation be defined as decrease in book value (§ 82), the sinking-fund method and the interest-on-investment method lead to identical results.

Any theory of depreciation is of necessity based largely on conjecture and arbitrary assumptions. Nevertheless some theory is a necessity. Nowhere does this necessity appear more clearly than in the wide fields of taxation and regulation of public utilities.

During recent years the subject has received much attention both from engineers and from accountants. One of the most valuable as well as one of the most authoritative discussions is to be found in Chapter VI of the "Final Report of the Special Committee to formulate Principles and Methods for the Valuation of Railroad Property and Other Public Utilities," published in Vol. LXXXI of the *Transactions of the American Society of Civil Engineers*, 1917. The authors of this excellent report apparently fail to note that the sinking-fund method and the interest-on-investment method (which they call the compound-interest method) are practically identical, since they lead to the same book values.

One valuable feature of the above-mentioned report is the very careful discussion of the accounting methods to be used in connection with the various methods of determining depreciation.

It should, perhaps, be said that the present tendency of the larger corporations is in the direction of placing in the maintenance column many charges that could be carried under the head of depreciation. This is particularly true of items whose cost is relatively small. It would be impossible for a concern like the American Telephone and Telegraph Company to keep a depreciation account for each telephone pole. For similar reasons a railway corporation operating a thousand locomotives might find it wise to charge the replacement of a single locomotive to maintenance. On the other hand, it would be folly for a company operating a single electrical generator not to make definite provision for its replacement by means of an annual depreciation charge.

CHAPTER XIII

THE THEORY OF MINE VALUATION

89. The valuation of mining properties. When a sum of money is loaned, the person making the loan not only receives interest at a stipulated rate at the end of each year, but at the end of a stated period he receives his original capital back again in full. There are, however, some forms of investment where the original capital is not returned to the investor at the end of the period. In such cases provision must be made for the *redemption* of the capital by setting aside some portion of the annual income as a *redemption fund*. A mine is a typical example of this sort of investment, for from it a definite amount of mineral can be removed, and when this has been done, the mine will be valueless.

It is the business of the mining engineer to determine the amount of mineral that can be taken from a mine and the cost of getting the mineral to market. When these elements are determined, together with the rate at which the mineral is to be removed, the problem of determining the *value of the mine* becomes fairly definite. The net annual income is the rent of an annuity, and the value of the mine is nothing but the present value of the annuity.

In what follows, a mine is assumed to be a body of ore, definite in extent, with machinery installed ready to bring the ore to the mine mouth, where it may be sold for a definite price. Departures from this definition are supposed to be taken care of by making allowances for engineering and financial contingencies.

It will be seen at once that the contingencies are sufficiently numerous and sufficiently important to constitute a considerable "risk" to the investor. For this reason investors usually demand a higher rate from investments in mining ventures than is accepted from more stable investments.

PROBLEM. *To find the value of a mine when the net annual income is known.*

Let N denote the net annual income and V_n the value of a mine supposed to have a life of n years. Let S be the salvage value of the land and the equipment. The value of the mine is the present value of the annuity of N per annum together with the discounted salvage value. Then

$$V_n = Na_{\overline{n}|i} + Sv^n = N \frac{1-v^n}{i} + Sv^n. \quad (1)$$

That this valuation is adequate to restore fully the capital invested may be shown as follows:

The annual interest on the investment will be

$$V_n i = N(1-v^n) + Sv^n i,$$

and the remainder left each year to apply to the redemption fund will be

$$N - V_n i = Nv^n - Sv^n i.$$

This amount will be exactly sufficient to redeem the original capital, *provided the annual payments can be accumulated at the investment rate, i* . For the amount in the redemption fund at the end of n years will be

$$\begin{aligned} (N - V_n i) s_{\overline{n}|i} &= Nv^n s_{\overline{n}|i} - Sv^n i s_{\overline{n}|i} \\ &= Na_{\overline{n}|i} + Sv^n - S. \end{aligned} \quad (2)$$

Clearly the original capital is fully restored when the salvage value S is added to the accumulations in the redemption fund.

The foregoing discussion would be trivial except for the fact that it furnishes the simplest example of a redemption fund.

It has already been stated that the investment rate for mining enterprises is a high rate. The investment rate has been called the *stipulated rate*, while the rate at which the redemption fund can be accumulated is called the *practicable rate*.*

Let i be the practicable rate and i' the stipulated rate; then the interest paid annually to the investor will be $V_n i'$, and the amount available annually for the redemption fund will be

$$N - V_n i'.$$

*Hoskold, The Engineer's Valuing Assistant. London, 1905.

If these annual payments be accumulated for n years, the amount will be

$$(N - V_n i') s_{\overline{n}|i'}$$

which, by hypothesis, must equal the original value. Consequently,

$$(N - V_n i') s_{\overline{n}|i'} = V_n. \quad (3)$$

From equation (3) the value V_n of the mine is found to be

$$V_n = \frac{N}{\frac{1}{s_{\overline{n}|i'}} + i'} = \frac{N}{\frac{1}{a_{\overline{n}|i}} + i' - i}. \quad (4)$$

Extensive tables giving the values of V_n directly for practically every case that might occur have been given by Hoskold in the work cited above. If such tables are not at hand, computation by logarithms, or even by direct division, is not difficult.

90. The net annual income. The net annual income has been assumed to be known. It must not be thought, however, that it is merely the difference in the value of the output and the cost of labor and materials for a year. The land may have been acquired by lease or purchase, or the privilege of extracting the mineral may have been given in return for a *royalty* to be paid to the owner of the land. Frequently costly shafts or tunnels must be constructed before the face of the ore is reached. Spur tracts must be laid and surface buildings erected. Such costs should be distributed over the life of the mine and added each year to the cost of operation and maintenance.

PROBLEM. *To find the net annual income from a mine.*

Let L be the cost of the land or of the leasehold, D the cost of development, m the number of years elapsing between the date when the land was purchased and the date when production began, n the life of the mine in years, C the cost of labor, materials, and other current expense, i the money rate.

The cost of the leasehold on the date when production begins will be

$$L(1+i)^m, \quad (1)$$

and the total cost of the mine when production begins is

$$L(1+i)^m + D. \quad (2)$$

This cost is equivalent to the present value of an annuity of

$$[L(1+i)^m + D] \frac{1}{a_n} \quad (3)$$

per annum payable annually during the lifetime of the mine. The total annual expenditure will then be

$$K = [L(1+i)^m + D] \cdot \frac{1}{a_n} + C. \quad (4)$$

If payment for the mining privilege be made in the form of a royalty to be paid as the ore is removed, $L = 0$, and in place of $L(1+i)^m$ we have a different term.

To determine this new term, let A be the total tonnage, s a factor introduced to provide for faulty estimates, and r the royalty per ton. The total royalty paid in a year will be

$$\frac{Asr}{n},$$

and the total annual expenditure will then be

$$K' = D \cdot \frac{1}{a_n} + \frac{Asr}{n} + C. \quad (5)$$

To find the gross annual return, let p denote the price per ton realized for the ore and S the salvage value to be obtained from the sale of the land and equipment. Then the salvage value S is the amount of an annuity of $S \cdot \frac{1}{s_n}$ per annum, distributed over the lifetime of the mine. The gross annual return is therefore

$$G = \frac{As}{n} \cdot p + S \cdot \frac{1}{s_n}. \quad (6)$$

The net annual income is then

$$N = G - K = \frac{As}{n} p + S \cdot \frac{1}{s_n} - [L(1+i)^m + D] \frac{1}{a_n} - C, \quad (7)$$

when the mining rights have been acquired by purchase; and

$$N = G - K' = \frac{As}{n} (p - r) + S \cdot \frac{1}{s_n} - D \cdot \frac{1}{a_n} - C, \quad (8)$$

when the mining rights have been acquired by payment of royalty, to be paid as the ore is mined.

Lumbering operations would give rise to a similar theory.

EXAMPLES

1. A coal mine can be made to yield \$10,000 net annually for 25 years. What is its value if the stipulated rate is 7% and the redemption fund can be accumulated at 3%?

Solution. From the table we find

$$\begin{aligned} \frac{1}{s_{25}|} &= .0274279. \\ \text{Then } V_{25} &= \frac{10,000}{.0274279 + .07} \\ &= \frac{10,000}{.0974279} \\ \log V_{25} &= 5.0113167 \\ V_{25} &= \$102,640. \end{aligned}$$

2. Verify the result in Example 1 by showing that the surplus income, after 7% has been deducted for the investor, will accumulate at 3% to the value V_{25} , as found.

3. It is estimated that from a certain mine the mineral will be exhausted in 20 years, and that during that time it can be made to yield \$5000 per year net. What should be the purchase price to yield 8% to the investor, if the redemption fund can be accumulated at $3\frac{1}{2}\%$?

4. A coal company purchases 1000 acres of land underlaid by a vein of coal 4 feet thick, paying \$100 per acre for the land. After 10 years they develop the mine at a cost of \$50,000, and plan to work it at a uniform rate such that it will be worked out in 40 years. The miners receive \$500,000 annually, and taxes are \$2000 per year. It is expected that the land will be worth \$40 per acre after the coal has been removed. If a cubic foot of coal weighs 90 pounds, what is the mine worth on a 6% basis and an assumed rate of \$4.25 per ton for the coal at the mine mouth?

CHAPTER XIV

BUILDING AND LOAN ASSOCIATIONS

91. Definitions and first principles. A *building and loan association* is an association of persons formed for the purpose of enabling its members to accumulate money by periodical payments into its treasury, to be invested from time to time in loans to those of its members who wish to build homes. The membership of such associations usually consists of two classes: shareholders who are investors only, and shareholders who are at the same time investors and borrowers.

There are many plans for building and loan associations, but most of them require a small monthly or weekly payment, which, in the case of borrowers, provides for the payment of interest and for the establishment of a sinking fund which will extinguish the indebtedness when the stock *matures*; and in the case of investors constitutes a savings account which is augmented from time to time by the dividends which may be declared out of profits earned by the association. The sources of profits are interest earned on loans made to borrowing members; gains due to the difference between the *book value* and the *withdrawal value* of stock belonging to members who retire before their stock has matured; fines assessed upon members who may have been delinquent in the payment either of dues or of interest; fees charged on new business; and, finally, any undivided surplus that may have remained over at the last distribution.*

The *net profits* are the profits remaining after all the expenses of conducting the business have been paid. The net profits are sometimes distributed on the basis of the book values immediately after the last distribution of profits, and sometimes

* One other source of profit frequently utilized by loan associations is the difference between the interest received on money borrowed by the association at a lower rate than that charged to borrowing shareholders, and the interest paid on loans thus made. The amount of such loans is usually limited by law. For example, in Wisconsin the limit is 20% of the assets of the association.

the basis of distribution is found by adding to the book values immediately after the last distribution a portion of the amount paid in dues since the last distribution. The latter plan is more equitable, since each shareholder's profits are more nearly proportional to the amount of money he has invested. The former plan will give a slightly higher *rate* of profit, though not a higher actual profit.

PROBLEM. *To determine the rate of profit for a period.*

The rate of profit for a period cannot be determined in advance, since it depends upon elements that cannot be known until the end of the period. For example, loans in force at the beginning of the period may be repaid within the period, or a considerable amount of stock may be withdrawn, or part of the funds of the association may remain uninvested. At the end of the period, when the elements of the problem are all known, it reduces to a simple problem in percentage.

Let I denote the total interest received during the period, f the amount received from fees and fines, g all other gains for the period, and e the total expenses and losses. Further, let B be the book value immediately after the last distribution of all stock remaining in force to the end of the period, and b the amount added to the book values to enable the amount paid in as dues during the period to participate equitably in the profits. The net profits N will then be

$$N = I + f + g - e, \quad (1)$$

and the rate of profit r will be

$$r = \frac{N}{B} = \frac{I + f + g - e}{B} \quad (2)$$

if the distribution is made on the basis of book values at the beginning of the period; or

$$r = \frac{N}{B + b} = \frac{I + f + g - e}{B + b} \quad (3)$$

if dues paid in *during* the period are allowed to participate in the profits.

To find b , one has, in effect, a problem in the equation of payments for every shareholder. If, for example, a shareholder pays \$1 a month per share, and pays promptly on the first of the month for six months, each share of his stock will contribute \$3.50 toward b . The stock of a shareholder delinquent in all dues for the period would contribute nothing to b . The stock of a member who paid \$1 the first month, \$1 the second month, nothing the third and fourth months, \$3 the fifth month, and \$1 the sixth month, would contribute \$3 to b , since, by the approximate rule for equation of accounts, \$1 for six months, \$1 for five months, \$3 for two months, and \$1 for one month would be equivalent to \$3 for six months.

It would be impossible to give even a concrete example of the determination of the rate of profit without taking into account the receipts and disbursements, and the status of every shareholder's stock for an entire period.

The rate of profit once determined, the apportionment of the net profits to each shareholder's stock is a simple matter of multiplying the book value of each shareholder's stock by the rate. This book value is either the book value at the beginning of the period or at some later date, according as the rate is computed by formula (2) or formula (3).

EXAMPLES

1. B holds 10 shares in the Provident Building and Loan Association, which, immediately after the distribution of profits on January 1, 1910, had a book value of \$72.98 per share. During the following half-year period he paid as follows: January 1, \$10; February 1, \$10; May 1, \$30; June 1, \$10. On July 1 the rate of profits was .0254. What was his share of the profits?

Solution. To find the amount of the dues participating in the profits, we note that B had invested \$10 for 6 months, \$10 for 5 months, \$30 for 2 months, and \$10 for 1 month. These sums would be equivalent to a sum x for 6 months, where x is determined from the equation

$$\begin{aligned} 6x &= 10 \times 6 + 10 \times 5 + 30 \times 2 + 10 \times 1 \\ &= 180. \end{aligned}$$

Therefore $x = 30$.

The profits on B's stock will then be computed on \$729.80 + \$30, or \$759.80. Consequently, B's share of the profits will be

$$\$759.80 \times .0254 = \$19.30.$$

2. A has 10 shares in a building association, and on January 1, 1911, these shares were worth \$63.92 per share. During the next six months he paid dues as follows: January 15, \$10; February 15, \$10; May 15, \$30; June 15, \$10. The association earned net profits amounting to \$3479.72 on total book values of \$126,426.20. What was his share of the profits for the period from January 1 to July 1?

92. The effective rate realized by the investing shareholder.

PROBLEM. *To find the effective rate realized by the investing shareholder.*

The problem is similar to that of § 53 where the rate of interest paid by an annuity was determined. We will suppose that the stock matures immediately after the last monthly payment is made.*

Let C be the value at maturity, M the monthly payment, and n the time from the date of the first payment to maturity. The first payment will be on interest for n years and the remaining payments constitute an annuity of $12M$ per annum, payable monthly for n years. Consequently,

$$C = M(1+i)^n + 12M s_{\overline{n}|i}^{(12)}; \quad (1)$$

$$\text{or} \quad \frac{C}{M} = (1+i)^n + 12s_{\overline{n}|i} \cdot \frac{i}{j_{(12)}} = f(i). \quad (2)$$

For any assumed value of i the right member of (2) is easily computed. The value of i can then be found with sufficient accuracy by interpolation. To illustrate, suppose the stock of an investing shareholder paying \$1 per month matures in exactly 12 years, and it is required to find the rate. Suppose also that the association loans its funds at 6%, payable monthly.

The required rate is almost certainly less than 6%, since all expenses must be paid out of earnings. Let the right member be denoted by $f(i)$. Then

$$f(.05) = 1.80 + 12 \times 15.92 \times 1.023 = 197.23,$$

$$f(.06) = 2.01 + 12 \times 16.87 \times 1.027 = 209.92,$$

while for the unknown rate i

$$f(i) = \frac{C}{M} = 200.$$

* The last interval is usually fractional. When it is seen that the next payment and interest would bring the book value above the maturing value, the maturity date may be found by finding the time in which the book value would amount to the maturing value, taking for the rate the average rate of profits for the preceding year.

The tabular difference is 12.69, and the difference

$$f(i) - f(.05) = 2.77.$$

The correction is, therefore,

$$p(.01) = \frac{2.77}{12.69} \times (.01) = .0022,$$

and the required rate is approximately

$$i = .0522.$$

If a closer approximation is desired, the value of $f(.055)$ may be computed, and the interpolation may be made by means of second differences. Or, the problem may be solved by the more laborious method of making the substitution

$$i = .05 + h$$

in equation (1) and then finding an approximate algebraic solution of the resulting equation in h , as was done in the last solution in § 53. There is, however, no practical necessity for finding a close approximation.

If the number of years is fractional the computation for the right member of (2) cannot be made by means of the table. This computation is not difficult when done by means of logarithms.

EXAMPLES

1. An investing shareholder paid \$10 per month for 10 shares in the Provident Building and Loan Association, and the stock matured in exactly 12 years. What rate of interest did he receive?

2. If the time of maturity in Example 1 had been 11 years and 6 months, what would the rate have been?

3. An investing shareholder pays \$20 per month for 10 shares of stock in a building and loan association, and the stock matures in $7\frac{1}{2}$ years. What is the rate of interest received?

93. The rate of interest paid by the borrowing shareholder. We may look upon the repayment of a loan made to a borrowing shareholder in two ways:

In the first place, we may consider the loan as an entirely distinct transaction which has nothing whatever to do with the ownership of stock in the association. From this point of view

the rate of interest is simply the effective rate corresponding to the nominal rate charged by the association and payable twelve times a year.

To find the rate when the loan is considered as a separate transaction, we may assume that the interest is payable at the end of each month instead of at the beginning. The effective rate is, then,

$$i = \left(1 + \frac{j}{12}\right)^{12} - 1. \quad (1)$$

In other words, the rate paid by the borrower is exactly the rate stipulated in the mortgage rate, while the payment of the dues is looked upon as a separate transaction for the purpose of creating a sinking fund that will meet the payment of the note when it falls due.

On the other hand, we may consider the transaction as the repayment of a loan, principal and interest, in equal monthly installments. For example, a shareholder borrows \$4000 from an association which issues stock in shares of \$200 each, matured by the payment of \$1 per month per share, and which charges interest at 6% nominal, payable monthly in advance. He must take out 20 shares of stock, for which he pays \$20 a month in dues. Besides, he must pay \$20 each month for interest, making \$40 a month in all. The \$40 per month may be considered as a series of payments that will extinguish the debt when the stock matures. Looked at from this point of view the determination of the rate of interest is a more difficult matter.

PROBLEM. *To determine the rate of interest paid by a borrowing shareholder when the monthly interest payment and the monthly dues are together considered as a single sum for the repayment of the principal and interest by a series of monthly installments.*

The problem differs from that of § 53 only in that in the present case the annuity is "due"; i.e. the first payment is made at the beginning of the first period instead of at the end. The problem can be considered from the standpoint of a single share of stock, since, if it requires the value at maturity of m shares of stock, m will be a common factor which may be divided out from both sides of the equations involved.

Let C be the value to which the stock matures, which is also the face value of the debt, M the monthly dues, n the number of years required for the stock to mature, k the nominal rate charged on loans, i the unknown rate paid by the borrower, and M_1 the total monthly payment.

The monthly interest payment will be $\frac{Ck}{12}$, so that

$$M_1 = \frac{Ck}{12} + M. \quad (2)$$

The present value of the annuity is the face of the debt and is the same as the value to which the stock matures. It is given by

$$C = M_1 + 12 M_1 \cdot a_{\overline{n}|}^{(12)}. \quad (3)$$

Equation (3) may be written in the form

$$\frac{C - M_1}{12 M_1} = a_{\overline{n}|}^{(12)} = a_{\overline{n}|} \cdot \frac{i}{j_{(12)}}, \quad (4)$$

where C and M_1 are both known and the right member contains the unknown rate i . The problem of determining i is then similar to the problem of finding the rate received by the investing shareholder which was solved in § 92. The same difficulty that appeared in that section is encountered when the time required for the stock to mature is fractional.

EXAMPLES

1. A man borrows \$1200 from a building and loan association, agreeing to pay \$6 per month interest and \$1 per month on each of six shares to mature to \$200. When the transaction was completed it was found that the time of maturity was exactly 12 years. What rate did the borrower pay?

Solution. The monthly payment M_1 is 12, and equation (4) becomes

$$\frac{1200 - 12}{144} = 8.25 = a_{\overline{12}|} \cdot \frac{i}{j_{(12)}} = f(i).$$

By means of the table we find

$$f(.06) = a_{\overline{12}|} \cdot \frac{i}{j_{(12)}} \text{ at } 6\% = 8.612,$$

$$f(.07) = a_{\overline{12}|} \cdot \frac{i}{j_{(12)}} \text{ at } 7\% = 8.194.$$

The tabular difference is $-.418$ and the difference between $f(i)$ and $f(.06)$ is $-.362$, so that the correction is $\frac{3}{4} \frac{6}{1} \frac{2}{8} \times .01$, or .00866. The approximate value of i is therefore given by

$$i = .06866.$$

2. A borrowing shareholder paid \$25 per month interest and \$25 per month in dues on a loan of \$5000. The stock matured in 11 years and 9 months. What rate of interest did he pay if the total monthly payment is looked upon as a payment for principal and interest?

3. A man borrows \$200 from a loan association charging 6% nominal, payable monthly in advance, and finds that his one share matures to \$200 in exactly 11 years. What rate of interest did he pay if the monthly payment of \$2, consisting of due and interest, be considered as one of the equal monthly installments for repaying the debt, principal and interest?

4. A building and loan association has \$142,951.74 loaned to its members. The book value of its stock is \$119,953.19. The expenses for the period from June 30, 1909, to December 31, 1909, were \$607.69. Fees, fines, and other profits amounted to \$108.32. The association charges 6% on its loans. Assuming that all interest due was paid, and that no stock was withdrawn during the period, what was the rate of profit for the half year?

5. A man paid \$5 per month to a building and loan association for 11 years and 9 months, when his stock matured to \$1000. What rate of interest did he receive?

6. An association is able to pay a semiannual dividend of .0275 on the dollar. How long will it take a share of stock on which \$1 per month is paid to mature to \$200?

7. If a man is able to pay off a mortgage of \$2000 by paying \$20 to a building and loan association in 145 monthly payments, what effective rate of interest is he paying on his loan?

8. Two men each borrow \$2000 from a building and loan association which charges 6% nominal, payable monthly in advance. One man pays \$20 and the other \$30 per month. If the association pays a semiannual dividend of $2\frac{1}{2}\%$ on book values, how long will it take for each man to pay his loan, and which pays the greater rate of interest when the transactions are looked upon from the standpoint of the payment of the loans, principal and interest, by equal monthly installments?

94. The time required for stock to mature.

PROBLEM. To find the approximate time in which stock will mature when the maturing value, the monthly payment, and the approximate rate of interest are known.

Equation (1) of § 92 is easily reduced to the form

$$\frac{C}{12M} = \frac{(1+i)^{n+\frac{1}{12}} - 1}{j_{(12)}}. \quad (1)$$

This equation cleared of fractions takes the form

$$(1+i)^{n+\frac{1}{12}} = 1 + \frac{C}{12M} \cdot j_{(12)}. \quad (3)$$

From (3) we find, by taking logarithms of both sides, dividing by $\log(1+i)$, and transposing the term $\frac{1}{12}$,

$$n = -\frac{1}{12} + \frac{\log\left(1 + \frac{C}{12M} \cdot j_{(12)}\right)}{\log(1+i)}. \quad (4)$$

If the nominal rate payable semiannually is given, $1+i$ in (4) must be replaced by $\left(1+\frac{j}{2}\right)^2$. The value of n is then given by

$$n = -\frac{1}{12} + \frac{\log\left[1 + \frac{C}{6M} \cdot \left(\frac{j}{2}\right)\right]}{2 \log\left(1 + \frac{j}{2}\right)}. \quad (4')$$

For ordinary rates $\left(\frac{j}{2}\right)_6$, which is written for $6\left[\left(1+\frac{j}{2}\right)^{\frac{1}{6}} - 1\right]$, can be taken directly from the tables.

EXAMPLES

1. A building and loan association issues stock maturing to \$200 per share, with a monthly payment of \$1 per share. Four successive semi-annual distributions of profits were at rates .0287, .0283, .0254, and .0307. About how long will it take the stock to mature?

Solution. The arithmetic mean of the rates for the two years is .0283, so that we may put $j = .0566$. Moreover, $C = 200$ and $M = 1$, so that formula (4') becomes

$$n = -\frac{1}{12} + \frac{\log\{1 + 200[(1.0283)^{\frac{1}{6}} - 1]\}}{2 \log(1.0283)}.$$

Using logarithms, we find $\log(1.0283) = .0121198$,
 $(1.0283)^{\frac{1}{6}} = 1.00466$.

$$\begin{aligned} \text{Consequently, } n &= -\frac{1}{12} + \frac{\log 1.9324}{2 \log(1.0283)} \\ &= -\frac{1}{12} + \frac{.2860970}{.0242396} \\ &= -\frac{1}{12} + 11.80 \\ &= 11.72 +. \end{aligned}$$

The required time is therefore 11 years, 8 months, and 19 days.

2. Suppose the association in Example 1 decides to issue stock on which payments of \$2 per month per share are made. How long will it take for such stock to mature to \$200 per share?

3. Four successive semiannual distributions of profits of an association were at rates .0287, .0290, .030, and .0295. How long will it take a share costing \$1 per month to mature to \$200?

4. If in Example 3 the monthly payment had been \$2, how long would it have taken for the stock to mature?

95. Note on the rate of interest and the time. The questions of rate of interest and time of maturity are not fundamental problems in the practical management of a building and loan association, since everything is, so to speak, worked from the other end. That is to say, the rate of profit is computed for each half year separately and the profits thus determined are added to the book value of each shareholder's stock.

This rate is ordinarily not the same for any two consecutive half years. The time of maturity is not fixed in advance but depends on the rapidity with which profits can be accumulated.

It is important, however, to know what can be done with building and loan association stock on the basis of certain arbitrary assumptions made in advance. This is all the more true in view of the exaggerated statements that are frequently made both for and against such stock.

PART III. PROBABILITY AND ITS APPLICATIONS TO FINANCIAL PROBLEMS

CHAPTER XV

THE THEORY OF PROBABILITY

96. Definitions and first principles. If an ordinary die, with faces numbered from one to six points, be thrown, there are six possible positions in which it may come to rest; i.e. it may fall with any one of its six faces uppermost. Moreover, if the die is a perfect cube made of homogeneous material, it is equally likely that any one of its six faces will fall uppermost. There are six *cases*, or six *events*, in question. If we are interested in having the die fall with a given face uppermost, we say that one case is favorable to us and five unfavorable. In everyday language, we say that there is one chance in six in our favor.

Again, if a bag contains four white balls and two black ones, and we wish to draw out a white ball, there are six possible cases, of which four are favorable and two unfavorable.

We may say that it is probable that in a single trial a white ball would be drawn, but it would be impossible to draw conclusions without a carefully framed definition to start with. The following definition is made the basis of the theory of probability.

The probability that, among several equally likely events, a given event will happen is the ratio of the number of favorable cases to the whole number of possible cases.

Thus, in the first example, the probability that the die will fall with one point uppermost is $\frac{1}{6}$, since one case among six possible ones is favorable. In the second example the probability that in a single trial a white ball will be drawn is $\frac{4}{6}$, or $\frac{2}{3}$.

The foregoing definition is purely arbitrary and must be taken to mean that if the same situation were to come into existence a very great number of times, the ratio of the number of times it will actually happen to the number of times the situation recurs will be *very near* the ratio of the number of the favorable cases to the whole number of possible cases. For example, if a single die be thrown 6000 times, we expect it to fall with one point uppermost 1000 times. If, however, the trial were to be made, the result might be different. The probability that a coin will fall with "heads" uppermost is $\frac{1}{2}$, but in an actual trial in which 100 throws were made, the coin fell with heads uppermost 42 times, and not 50 times, as we should have expected. Nevertheless, we believe that *in the long run* the frequency with which a given face of the die would fall uppermost would be one sixth of the total number of throws, and the frequency with which the coin would fall "heads" would be one half the total number of trials.

The restriction that all the cases must be equally probable is a fundamental one, as a single example will show. We might say with absolute certainty that a given child ten years of age will either die or not die within a year. There are, then, two events in question — living one year and dying within the year; and we might say that the probability that the child will live one year is $\frac{1}{2}$. This would be very far from the truth, however, for from a very large number of records it has been found that out of 100,000 children who under certain conditions have reached the age of ten years, only 749 die within the year. The probability that a healthy child ten years of age will die before reaching his eleventh birthday is .00749, and not .5. The error consisted in assuming that living one year and dying within the year are equally likely for a child aged ten years.

If the number of favorable cases is a and the number of unfavorable cases is b , the total number of cases is $a + b$, and the probability that an event will happen takes the mathematical form

$$p = \frac{a}{a + b}. \quad (1)$$

If there are no unfavorable cases, $b = 0$ and

$$p = \frac{a}{a + 0} = 1.$$

Moreover, the event is certain to happen. *Certainty* is therefore expressed by 1.

If there are no favorable cases, the event is impossible, so that the expression for *impossibility* is

$$p = \frac{0}{0 + b} = 0.$$

Two events are said to be *complementary* if the happening of one excludes the possibility of the other, and the sum of their probabilities is 1. For example, if we draw a ball from a bag containing four white and two black balls, the probability of drawing a white ball is $\frac{2}{3}$ and the probability of drawing a black ball is $\frac{1}{3}$. Moreover, the two events cannot happen at the same time. They are therefore complementary.

THEOREM. *If the probability that an event will happen is p , the probability that it will fail is $1 - p$.*

Proof. Let q be the probability of failure. Then

$$p = \frac{a}{a + b} \quad \text{and} \quad q = \frac{b}{a + b}.$$

The happening and the failure of the event are then complementary, since they are mutually exclusive, and

$$p + q = \frac{a}{a + b} + \frac{b}{a + b} = 1;$$

consequently,

$$q = 1 - p, \tag{2}$$

as was to be proved.

97. Simple problems in probability. Many of the simpler problems in the theory of probability may be solved by means of the definition of probability, and the fundamental theorems and formulas from the theory of permutations and combinations. The truth of the following propositions concerning permutations and combinations will be assumed.

1. If one act can be performed in p ways, and if, after this act is completed, a second unrelated act can be performed in q ways, the number of ways in which the two acts can be performed in succession is pq .

2. The number of permutations of n things taken r at a time is

$$A\binom{n}{r} = \frac{\lfloor n \rfloor}{\lfloor n-r \rfloor}, \quad (1)$$

and for the special case where $n = r$

$$A_n = \lfloor n. \quad (2)$$

3. The number of combinations of n things taken r at a time is

$$C\binom{n}{r} = \frac{\lfloor n \rfloor}{\lfloor r \rfloor \lfloor n-r \rfloor}. \quad (3)$$

EXAMPLES

1. A die is thrown once. What is the probability that the number of points is less than 5?

Suggestion. There are four favorable cases.

2. Ten balls, exactly alike except that they are numbered from 1 to 10, are put into a bag, and a single ball is drawn at random. What is the probability that the ball is numbered 1? What is the probability that the ball is numbered either 1 or 2?

3. If two of the balls described in Example 2 are drawn simultaneously, what is the probability of drawing the pair numbered 3 and 5?

Suggestion. There are $C\binom{10}{2} = 45$ ways of selecting a pair from 10 numbered balls.

4. A bag contains n balls numbered consecutively from 1 to n , and from it three are drawn simultaneously at random. What is the probability that the numbers are 1, 2, and 3?

5. Two coins are tossed into the air at the same time (or in succession). What is the probability that both will fall heads?

Suggestion. By proposition 1, two coins may fall in any one of four ways.

6. What is the probability that 10 coins tossed into the air at the same time will all fall heads? What is the probability that a single coin tossed into the air 10 times will fall heads every time?

7. Two balls are drawn at the same time from a bag containing 3 white and 5 black balls. What is the probability that both will be white? that both will be black? that one will be black and one white?

Suggestion. There are C_2^8 possible pairs of balls and C_2^3 possible pairs of white balls.

8. What is the chance of throwing one, and only one, 5 with one throw of two dice?

98. Total and partial probability. The probability that an event belonging to a given class in a series of n mutually exclusive events consisting of several classes will happen, is called *partial*, or *relative*, *probability*. The probability that *any event whatever* of the series will happen is called *total probability*.

To illustrate, suppose a bag contains 11 white balls, of which 5 are marked with crosses and 6 not; 7 black balls, 4 with and 3 without crosses; 7 yellow balls, 2 with and 5 without crosses. The drawing of a ball with a cross occurs with any one of a series of three independent events: viz., (1) drawing a white ball with a cross, (2) drawing a black ball with a cross, (3) drawing a yellow ball with a cross. The probability that a given one of these three events—for example, drawing a white ball with a cross—will happen is partial probability, while the probability of drawing a ball with a cross regardless of color is total probability.

In the foregoing examples there are four things that interest us:

1. The probability of drawing a white ball with a cross is $\frac{5}{25}$.
2. The probability of drawing a black ball with a cross is $\frac{4}{25}$.
3. The probability of drawing a yellow ball with a cross is $\frac{2}{25}$.
4. The probability of drawing a ball with a cross is $\frac{11}{25}$.

The striking fact is that the sum of the partial probabilities, $\frac{5}{25}$, $\frac{4}{25}$, and $\frac{2}{25}$, is equal to the total probability. This fact is true in the general case and forms one of the two fundamental rules for the computation of probabilities.

THEOREM. *The total probability of an event is equal to the sum of its partial probabilities.*

Proof. Suppose that there are m possible cases and n events in the series. Let $a_1, a_2, \dots, a_n$ be the number of cases favorable

to the happening of the events belonging to the respective classes of the series, and let $p_1, p_2, \dots, p_n$, be the probabilities of these events. By definition,

$$p_1 = \frac{a_1}{m}, \quad p_2 = \frac{a_2}{m}, \quad \dots, \quad p_n = \frac{a_n}{m}. \quad (1)$$

But the total probability that an event of the series will happen is

$$p = \frac{a_1 + a_2 + a_3 + \dots + a_n}{m} = \frac{a_1}{m} + \frac{a_2}{m} + \dots + \frac{a_n}{m}, \quad (2)$$

since the number of favorable cases is $a_1 + a_2 + \dots + a_n$. Replacing the fractions $\frac{a_1}{m}, \frac{a_2}{m}, \dots, \frac{a_n}{m}$, in (2) by their values, we have the final result,

$$p = p_1 + p_2 + \dots + p_n. \quad (3)$$

It is necessary that the events in question be mutually exclusive. To illustrate, suppose that the probability that A can solve a given problem is $\frac{1}{4}$, and the probability that B can solve it is $\frac{1}{8}$. The probability that the problem will be solved if both work at it is not $\frac{1}{8} + \frac{1}{4} = \frac{3}{8}$, as we might infer, because the two events are not mutually exclusive. The mutually exclusive events of the series are (1) A succeeds and B fails; (2) A fails and B succeeds; (3) both succeed. It will be subsequently shown that the probabilities of these mutually exclusive events are $\frac{1}{8}, \frac{1}{4}$, and $\frac{1}{8}$, so that the total probability that the problem will be solved is

$$\frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2}.$$

On the other hand, if A and B, with others, are competitors in a race, and the probability that A will win is $\frac{1}{4}$, while the probability that B will win is $\frac{1}{8}$, the probability that A or B will win is

$$\frac{1}{4} + \frac{1}{8} = \frac{3}{8},$$

since the two events, A wins and B wins, are mutually exclusive.

99. Compound probability. Two events are said to be *independent* if the happening of one has no influence upon the other, and vice versa. *Compound probability* is the probability that two

independent events will happen simultaneously or in succession. For example, the probability that A and B, working independently, will solve a given problem is compound and, as we shall soon see, depends upon the probabilities that each will solve it. The statement of this relation is the second fundamental proposition in the theory of probability.

THEOREM. *The compound probability of two independent events is the product of the probabilities of the two events taken singly.*

Proof. Let p_1 be the probability of the first event, and p_2 the probability of the second. Suppose that the number of cases possible in connection with the first event is m_1 , and the number in connection with the second is m_2 . Suppose also that of the m_1 cases a_1 are favorable to the happening of the first event, and of the m_2 cases a_2 are favorable to the happening of the second event. By definition,

$$p_1 = \frac{a_1}{m_1} \text{ and } p_2 = \frac{a_2}{m_2}.$$

Moreover, we may combine every one of the possible cases for the first event with every one for the second, so that, by the first proposition on combinations and permutations (§ 97), the number of possible cases in connection with the simultaneous happening of the two events is $m_1 m_2$. Similarly, we may show that the number of cases favorable to the happening of the two events simultaneously is $a_1 a_2$. If, therefore, p represent the compound probability that the two events will happen simultaneously,

$$p = \frac{a_1 a_2}{m_1 m_2}.$$

But

$$\frac{a_1 a_2}{m_1 m_2} = \frac{a_1}{m_1} \cdot \frac{a_2}{m_2} = p_1 \cdot p_2.$$

Therefore

$$p = p_1 \cdot p_2. \quad (1)$$

COROLLARY I. If p be the compound probability that n independent events whose separate probabilities are $p_1, p_2, \dots, p_n$, will happen simultaneously (or in succession),

$$p = p_1 p_2 \cdots p_n. \quad (2)$$

COROLLARY II. If $p_1, p_2, \dots, p_n$, are the separate probabilities of n independent events, the probability that they will all fail is

$$(1 - p_1)(1 - p_2) \cdots (1 - p_n), \quad (3)$$

and the probability that the first r will happen and the remainder fail is

$$p_1 p_2 \cdots p_r (1 - p_{r+1}) \cdots (1 - p_n). \quad (4)$$

EXAMPLES

1. The probability that A working alone can solve a given problem is $\frac{1}{4}$, and the probability that B working alone can solve it is $\frac{1}{3}$. What is the probability that the problem will be solved if both work at it, each alone.

Solution. The mutually exclusive events, any one of which would bring about the solution of the problem, are (1) A succeeds, B fails; (2) A fails, B succeeds; (3) both succeed. Moreover, the success or failure of one has nothing to do with the success or failure of the other. The compound probability of the simultaneous success of A and failure of B is, then,

$$\frac{1}{4}(1 - \frac{1}{3}) = \frac{1}{6}.$$

Likewise, the compound probability that A will fail and B succeed is

$$(1 - \frac{1}{4})\frac{1}{3} = \frac{1}{6};$$

and, finally, the probability that both will succeed is

$$\frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}.$$

The *total* probability that the problem will be solved is the sum of the partial probabilities, or

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{12} = \frac{1}{2}.$$

2. A, B, and C run in a race with other competitors. The probability that A will win is $\frac{1}{8}$, that B will win is $\frac{1}{8}$, and that C will win is $\frac{1}{4}$. What is the probability that one of the three will win?

3. Two dice are thrown simultaneously. What is the probability that the throw will be greater than 9?

4. A single die is thrown twice. What is the probability that the first throw will be less than 3 and the second less than 5?

5. What is the probability that a single throw of two dice will yield either 5 or 6 points?

100. Probability of an event when several trials are made.

THEOREM I. *The probability that an event will happen exactly r times in n trials is*

$$\frac{n(n-1) \dots (n-r+1)}{1 \cdot 2 \cdot 3 \dots r} p^r q^{n-r},$$

where p is the probability that it will happen and q the probability that it will fail in a single trial.

Proof. By (4) of § 99 the compound probability that the event will happen in a given trial and fail in the other $n-1$ is

$$pq^{n-1}.$$

The total probability that it will happen in some one of the n trials is the sum of the probabilities that it will happen in the separate trials, viz.

$$pq^{n-1} + pq^{n-1} + \dots \text{to } n \text{ terms} = npq^{n-1}.$$

Again, the compound probability that an event will happen in two assigned trials and fail in the other $n-2$, say the fifth and the eleventh, is

$$p^2q^{n-2},$$

and the total probability that it will happen in any two trials whatever is

$$p^2q^{n-2} + p^2q^{n-2} + \dots,$$

where the number of terms is equal to the number of ways of specifying two trials out of n , i.e. $C(\frac{n}{2})$. But, by proposition (3) of § 97,

$$C(\frac{n}{2}) = \frac{n(n-1)}{1 \cdot 2}.$$

Therefore the number of terms is $\frac{n(n-1)}{1 \cdot 2}$, and consequently the total probability that an event will happen twice and fail $n-2$ times in n trials is

$$\frac{n(n-1)}{1 \cdot 2} p^2 q^{n-2}.$$

In general, the probability that the event will happen in r assigned trials and fail in the other $n-r$ trials is

$$p^r q^{n-r}.$$

But there are

$$C(r) = \frac{n(n-1)(n-2) \cdots (n-r+1)}{1 \cdot 2 \cdot 3 \cdots r}$$

ways of specifying r trials out of n trials. The total probability that the event will happen exactly r times in n trials is

$$p^r q^{n-r} + p^r q^{n-r} + \cdots, \text{ to } C(r) \text{ terms.}$$

It is therefore

$$C(r)p^r q^{n-r} = \frac{n(n-1)(n-2) \cdots (n-r+1)}{1 \cdot 2 \cdot 3 \cdots r} p^r q^{n-r}.$$

THEOREM II. *The probability that an event will happen at least r times in n trials is*

$$p^n + np^{n-1}q + \frac{n(n-1)}{1 \cdot 2} p^{n-2}q^2 + \cdots + \frac{n(n-1) \cdots (r+1)}{1 \cdot 2 \cdot 3 \cdots (n-r)} p^r q^{n-r},$$

where p is the probability that the event will happen, and q the probability that it will fail in a single trial.

The event will happen r times if it happens n times, or if it happens $n-1$ times, or if it happens $n-2$ times, and so on to $n-(n-r)$ times. Consequently, the required probability is the total probability made up of the partial probabilities that it will happen n times, $n-1$ times, and so on. But, by Theorem I, the partial probability that the event will happen n times in n trials is p^n ; that it will happen $n-1$ times is $np^{n-1}q$; that it will happen $n-2$ times is $\frac{n(n-1)}{1 \cdot 2} p^{n-2}q^2$; and, finally, the probability that it will happen r times is $\frac{n(n-1) \cdots (r+1)}{1 \cdot 2 \cdot 3 \cdots (n-r)} p^r q^{n-r}$. Consequently, the total probability that it will happen at least r times is

$$p^n + np^{n-1}q + \frac{n(n-1)}{1 \cdot 2} p^{n-2}q^2 + \cdots + \frac{n(n-1) \cdots (r+1)}{1 \cdot 2 \cdot 3 \cdots (n-r)} p^r q^{n-r}.$$

EXAMPLES

1. Find the probability of throwing 1 and only 1 point in two trials with 1 die.
2. Find the probability of throwing at least 1 point in 2 throws of 1 die.
3. If, on an average, 99 out of 100 ships reach port safely, find the probability that at least 2 out of 10 will arrive.

101. Mathematical expectation. The name *mathematical expectation* has been given to the product of a sum whose payment depends upon the happening of some contingent event, multiplied by the probability that the event will happen. For example, if a stake of \$6000 were offered, to be paid if the throw of a die is 1, the mathematical expectation of a player is \$1000. This does not mean, of course, that a man having \$1000 could afford to risk it on the throw of a single die, but rather that, by paying \$1000 for each throw, in a very large number of throws the player would come out approximately even in the long run, in a perfectly fair game.

A second example may serve to make the matter clearer. Suppose that 100,000 men, all aged forty years, agree to make up a fund of such an amount that at the end of a year each survivor may receive one dollar. What is the mathematical expectation of each participant?

It has been found by the American life-insurance companies that, of every 100,000 men aged forty years, approximately 979 will die within a year. The probability that any one of these men will live one year is therefore $\frac{99,021}{100,000} = .99021$. If we neglect interest, the mathematical expectation of any one of the number is, then, \$0.99021.

The matter will be still clearer if we examine it from another point of view. If 979 die within a year, there will be 99,021 survivors, each to receive a dollar, so that \$99,021 must be contributed. The amount that each man must contribute is therefore

$$\frac{\$99,021}{100,000} = \$0.99021.$$

While the notion of mathematical expectation had its origin at the gaming table, it has been made the basis of one of the great economic developments of modern times, viz. the development of the business of life insurance. The example just given shows that if a man forty years old wishes to provide \$10,000 for his estate in case his death should occur within one year, he would have to pay at least \$97.90 to the insurance company

agreeing to pay the amount to his estate, since the mathematical expectation would be the probability of the man's dying, viz. .00979 times the amount to be received. The cost would be greater than this figure, of course, since interest, cost of administering the business, and a fair profit to the company would have to be allowed.

Here, again, it should be carefully observed that it would be gambling, pure and simple, for an insurance company to promise, for a consideration of \$125, to pay \$10,000 to the estate of a single individual forty years of age in the event of the death of the individual within a year, though it would find the business not only perfectly safe but quite profitable if it could make 100,000 such contracts.

EXAMPLES

1. At a gaming table a stake of \$100 is made contingent upon the event of a throw of a die being less than 4. What is the mathematical expectation of the player?

2. If 999 out of 1000 ships of a given class, and in a given condition, reach port safely, what would be the cost of insuring a ship belonging to the class in question, and its cargo, for \$500,000 for a single voyage, if interest, expenses, and profits are neglected?

3. Experience has shown that, of 100,000 children aged 10 years, 749 die within 1 year. What should be the minimum cost of insuring the life of a 10-year-old child for \$1000 for 1 year?

4. If it were shown that 2 out of every 1000 dwelling houses worth \$5000 burn annually, what would be the amount of the "risk" assumed in insuring such a house for one year?

102. The mortality table. One of the most important applications of the theory of probability to human affairs is the application to problems having to do with the duration of human life. The greater number of such problems can be solved if we can answer the question, Given a certain number of persons all of the same age, how many will be alive at the end of each succeeding year until all are dead? The answer to this question, so far as it can be answered, must be obtained as a result of observation upon a large number of lives, and not by

any process of mathematics. The results of such observation carried on through a series of years are embodied in a table called a *mortality table*.

In its simplest form a mortality table is a table showing the number of persons, out of a large number all born on the same day, that survive at the end of each successive year until every individual of the group is dead. It would consist, therefore, of two columns: one a column of ages beginning with the age at which every member of the group is supposed to be alive; the other the number of persons living at each succeeding age. In practice at least three more columns, deduced from these two, are put into the mortality table: viz. one for the number of persons dying within each year; one for the probability that an individual will die between two consecutive ages; and one for the probability that an individual will live one year beyond any given age.

It is of course impossible to trace the careers of a very large number of children, say 100,000, all born on the same day, until all are dead, so that a mortality table cannot be constructed in this way. However, it is not at all necessary to do this. We are concerned with the probability that a person of a given age will live or die within a year, and, this being the case, the probabilities for two different ages may be determined wholly independently of each other. For example, the number dying within a year out of a group aged 40 and the number dying within a year out of a group aged 41 may be found from two wholly different groups. Indeed, it is not even necessary that the two groups be of the same size, provided both are large enough to furnish a safe basis upon which to work.

A mortality table is not only the foundation of all safe life-insurance business, but it is the basis of all computations having to do with life annuities and old-age pensions, and with many problems having to do with the administration of inheritance-tax laws. It will differ for different countries, for different periods in the same country, for different races living side by side in the same country, for persons of the same race engaged in different occupations. There are marked differences in tables constructed for males and for females.

The actual construction of a mortality table from observed data is a matter of considerable difficulty, and it would be out of place to attempt to follow out the work in detail here. The reader is referred for such information to standard textbooks on insurance. Many excellent tables have been constructed, some from the records of insurance companies, and others from vital statistics collected in various other ways.

Table XI, known as the American Experience Table, is based upon the records of the Mutual Life Insurance Company of New York, and was first published in 1868. It is the table most used in this country, and will be used in this book in all subsequent computations having to do with mortality statistics.

In what follows, the age of a person will be denoted by x , the number living at age x , out of the number with which the table starts, by l_x , and the number dying between ages x and $x+1$, by d_x . The probability that a person of age x will live at least one year is denoted by p_x , and the probability that a person of age x will die within the year, by q_x . The symbol (x) is frequently used to denote a person or a life aged x .

With the notation indicated above, and by means of the definition of probability, we have the following formulas:

$$d_x = l_x - l_{x+1}, \quad (1)$$

$$p_x = \frac{l_{x+1}}{l_x}, \quad (2)$$

$$q_x = \frac{d_x}{l_x} = \frac{l_x - l_{x+1}}{l_x}. \quad (3)$$

The probability that a person will live at least n years is the probability that he will be alive at the end of n years. It is denoted by ${}_np_x$. Therefore

$${}_np_x = \frac{l_{x+n}}{l_x}. \quad (4)$$

The probability that (x) will not live n years is denoted by ${}_nq_x$. We have

$${}_nq_x = 1 - {}_np_x = \frac{l_x - l_{x+n}}{l_x}. \quad (5)$$

A few typical examples will suffice to show how to attack problems that may arise.

PROBLEM. *To find the probability that (x) and (y) will both survive n years.*

The survivals of the two lives are independent events, and the survival of both is the compound probability made up of the product of the two simple probabilities. Therefore, if ${}_np_{xy}$ denote the probability required,

$${}_np_{xy} = {}_np_x \cdot {}_np_y. \quad (6)$$

PROBLEM. *To find the probability that the life (x) will survive n years and the life (y) will fail within n years.*

Again we have to do with independent events, and consequently the required probability is

$${}_np_x \cdot {}_nq_y = {}_np_x(1 - {}_np_y).$$

PROBLEM. *To find the probability that at least one of two lives, (x) and (y), will survive n years.*

There are three mutually exclusive events, for each of which the *partial* probability is known. These partial probabilities are (1) the probability that both will survive, or ${}_np_{xy}$; (2) the probability that (x) will survive and (y) die, or ${}_np_x(1 - {}_np_y)$; (3) the probability that (x) will die and (y) survive, or $(1 - {}_np_x){}_np_y$.

The *total* probability that at least one will survive n years is the sum of the three partial probabilities, or

$${}_np_x \cdot {}_np_y + {}_np_x(1 - {}_np_y) + (1 - {}_np_x){}_np_y = {}_np_x + {}_np_y - {}_np_x \cdot {}_np_y.$$

EXAMPLES

1. Find the probability that a man 35 years of age will live to be 45.

Solution. By formula (4),

$${}_{10}p_{35} = \frac{l_{45}}{l_{35}},$$

and the table gives $l_{35} = 81,822$ and $l_{45} = 74,173$.

The required probability is therefore

$${}_{10}p_{35} = \frac{74,173}{81,822} = .90652.$$

2. A father is 40 years of age and his son is 10. What is the probability that both will be alive 11 years hence?

Solution. The probability that the father will live at least 11 years is

$${}_{11}P_{40} = \frac{68,842}{78,106} = .88139,$$

and the probability that the son will live 11 years is

$${}_{11}P_{10} = \frac{91,914}{100,000} = .91914.$$

The probability that both will live 11 years is the compound probability

$$.88139 \times .91914 = .81012.$$

3. Find the probability that at least one of the two persons in Example 2 will survive 11 years.

Solution. The mutually exclusive events are as follows:

1. The survival of both.
2. The survival of the father and the death of the son.
3. The death of the father and the survival of the son.

The probability of the survival of both has been found in Example 2 to be .81021. The probability of the survival of the father and the death of the son is

$$\begin{aligned} {}_{11}P_{40}(1 - {}_{11}P_{10}) &= .88139 \times (1 - .91914) \\ &= .07127. \end{aligned}$$

The probability of the survival of the son and the death of the father is

$$\begin{aligned} {}_{11}P_{10}(1 - {}_{11}P_{40}) &= .91914 \times (1 - .88139) \\ &= .10902. \end{aligned}$$

The total probability that one of the two will survive is the sum of these partial probabilities, viz.

$$.81012 + .07127 + .10902 = .99041.$$

This solution is an excellent illustration of the fact that in all cases it is better to work out the simplest possible formula before beginning the numerical computation. The formula of the last problem of the present section gives directly

$$\begin{aligned} {}_{11}P_{40} + {}_{11}P_{10} - {}_{11}P_{40} \cdot {}_{11}P_{10} &= .88139 + .91914 - .81012 \\ &= .99041. \end{aligned}$$

4. A man aged 50 starts an enterprise requiring 10 years for its completion. What is the probability that he will live to see the end of the work?

5. A man and his wife are 28 and 26 years old, respectively, when their first child is born. What is the probability that both will live until the twenty-first anniversary of the child's birth?

6. Suppose that the man and wife and child in Example 5 are all living when the child is 10 years old. What is the probability that they will all be alive when the child is 21?

7. What is the probability that, of the two parents in Example 5, at least one will be alive on the twenty-first anniversary of the child's birth? that one and only one will be alive when the child is 21?

8. Under a certain pension system a man who had served 25 years in a given capacity was eligible to a retiring allowance. What would be the probability that under such a system a man beginning his service at the age of 30 would live to take advantage of a retiring allowance?

9. Find the probability that at least one of the two lives (x) and (y) will fail within n years?

10. Find the probability that (x) will live exactly n years, i.e. the probability that he will die between the ages $x + n$ and $x + n + 1$.

11. Approximately 2,800,000 men were mustered out of the two armies at the close of the Civil War in 1865. If we suppose that they were all 27 years of age, how many will probably be alive in 1915?

12. From the American Experience Table, plot the curve showing the number of deaths per 100,000 occurring annually from age 10 to the limit of the table.

13. Plot the curve showing the probability of dying for each year from the age of 10 according to the American Experience Table.

CHAPTER XVI

LIFE ANNUITIES

103. Endowments and life annuities. A *life annuity* is an annuity the end of whose term is determined by the duration of one or more lives. If no confusion can arise, it is called simply an annuity. If the end of the term depends upon the duration of more than one life, the annuity is called a *joint life annuity*. The annual payments of a pension constitute a life annuity. Joint life annuities occur frequently in connection with inheritance-tax computations. For example, a father leaves an estate of \$100,000, the income of which is to go to his two sons as long as they shall both live. If the bequest is subject to an inheritance tax, the problem of determining its value to the two sons is at once in question. In most states the annual income would be estimated at 5% of the estate, so that the problem resolves itself into that of finding the present value of a joint life annuity of \$5000 per annum on a 5% basis.

A life annuity may be *due*, in which case the first payment is made immediately instead of at the end of one year; it may be *deferred*, in which case the first payment is made at the expiration of $n + 1$ years, where n is the number of years the annuity is deferred; or it may be *temporary*, in which case the payments cease at the expiration of a given time, even though the annuitant be still alive. The payments of a deferred annuity will never begin should the annuitant die before the expiration of $n + 1$ years. Nevertheless, the annuity has a definite present value.

The important thing to be determined in the case of life annuities is the present value of the annuity, and the present value can be obtained most easily through the notion of an *endowment*, or, as it is sometimes called, a *pure endowment*. An endowment is a sum payable to an individual, the *nominee*, at a

given future date, provided he survive to that date. The *present value of an endowment* of 1, payable at the end of n years to a person aged x , if he should live to the age $n + x$, is denoted by ${}_nE_x$.

PROBLEM. *To find the present value of an endowment of 1, payable after n years to a person aged x .*

First proof. If the sum of 1 were certain to be paid at the end of n years, the present value would be v^n ; but since the payment is contingent upon the probability that the nominee will live to receive the sum of 1, the present value will be the mathematical expectation of receiving v^n contingent upon a known probability. The probability that a person aged x years will live at least n years is, by § 102,

$${}_np_x = \frac{l_{x+n}}{l_x}.$$

Therefore the present value of the endowment is

$${}_nE_x = v^n \cdot {}_np_x = v^n \cdot \frac{l_{x+n}}{l_x}. \quad (1)$$

In connection with this proof it is important to note that no equitable arrangement would be possible by which a single person aged x could pay the sum ${}_nE_x$ with the expectation of receiving 1 at the end of n years, for if the sum ${}_nE_x$ were put at interest for n years, the amount would be less than 1, and if the person were living, the amount would not be sufficient to redeem the promise. On the other hand, if the individual should die before n years, there would be at the end of n years the sum $(1+i)^n {}_nE_x$, with no one to claim it. The fact that such an arrangement could be made if participated in by a considerable number of persons is made clear in the alternative proof.

Second proof. Suppose that l_x persons, all of age x , enter into an agreement to contribute a sum sufficient to secure the payment of one dollar to each of the survivors at the end of n years. According to the mortality table the number of survivors would

be l_{x+n} , so that l_{x+n} dollars would be required at the end of the time. It follows that the sum required now would be $v^n l_{x+n}$, and the amount to be contributed to the fund by each one, if all share equally, is exactly

$$v^n \cdot \frac{l_{x+n}}{l_x},$$

which agrees with (1).

The present value of an endowment of a sum different from 1 is, of course, proportional to the amount to be received; for example, the endowment of \$100 payable in n years to a person aged x is

$$\$100 {}_nE_x.$$

EXAMPLES

1. A father bequeathes to his son the sum of \$25,000, to be paid when the son is 21. If the father dies when the son is 15, what is the value, on a 5% basis, of the son's inheritance at the time of the father's death?

Solution. The probability that the son will live to receive the bequest is

$$\frac{l_{21}}{l_{15}} = \frac{91,914}{96,285} = .95460.$$

Furthermore, v^6 at 5% = .746215.

The present value of the bequest to the son is therefore

$$\$25,000 \times .746215 \times .95460 = \$17,808.42.$$

2. A father, dying when his son is 25, bequeathes to the son the sum of \$10,000, to be paid when the son is 30. What is the present value of the bequest on a 5% basis?

PROBLEM. *To find the present value of a life annuity of 1 per annum payable to a person aged x .*

Let a_x denote the present value of an annuity of 1 per annum payable to a person aged x . The several annual payments may be looked upon as so many endowments whose present values are

$${}_1E_x, {}_2E_x, {}_3E_x, \dots$$

Since each of these present values contains the probability of survival as a factor, the series will continue until the probability of survival becomes zero. It will continue, therefore, until $x+n$ in

the expression l_{x+n} reaches the highest age given in the mortality table, or, as we shall say for short, to table limit. We have, then,

$$a_x = {}_1E_x + {}_2E_x + {}_3E_x + \cdots \text{ to table limit.} \quad (2)$$

If we replace each term of the series in (2) by its value in terms of $l_x, l_{x+1}, \dots, l_{x+n}$, and powers of v , and note that l_x is a common denominator, we have

$$a_x = \frac{vl_{x+1} + v^2l_{x+2} + \cdots \text{ to table limit.}}{l_x}. \quad (3)$$

EXAMPLES

1. A pension of \$500 per annum, payable at the end of each year, is granted to a man 92 years old. What is the present value of his estate, according to the American Experience Table, at $3\frac{1}{2}\%$?

Solution. By formula (3),

$$a_{92} = \frac{(1.035)^{-1} \cdot l_{93} + (1.035)^{-2} \cdot l_{94} + (1.035)^{-3} \cdot l_{95}}{l_{92}}.$$

The American Experience Table gives $l_{92} = 216$, $l_{93} = 79$, $l_{94} = 21$, and $l_{95} = 3$. Consequently,

$$\begin{aligned} a_{92} &= \frac{(1.035)^{-1} \cdot 79 + (1.035)^{-2} \cdot 21 + (1.035)^{-3} \cdot 3}{216} \\ &= .45666. \end{aligned}$$

The value of the annuity of \$500 per annum is, therefore,

$$\$500 a_{92} = \$228.329.$$

2. Find the present value of the annuity in Example 1, on a 5% basis.

104. The computation of life annuities ; commutation columns. Example 1 of the previous section, which was purposely chosen with the age of the annuitant near the limit of the table, shows us that the labor of computation for the younger ages would be very great. For example, if we wish to compute a_{21} by means of the American Experience Table, the numerator of (3), § 103, will be a series consisting of no less than 74 terms, each one of which must be computed separately. Fortunately, it is possible to abridge the computation greatly by means of what are known as *commutation columns*.

To get an idea of what is meant by commutation columns, let numerator and denominator of the right member of (3), § 103, be multiplied by v^x . The result is

$$a_x = \frac{v^{x+1}l_{x+1} + v^{x+2}l_{x+2} + \dots \text{ to table limit}}{v^x l_x}. \quad (1)$$

Every term of both numerator and denominator of (1) is now of the same form, viz. $v^{x+r}l_{x+r}$.

If we define D_{x+r} by the equation

$$D_{x+r} = v^{x+r}l_{x+r}, \quad (2)$$

equation (1) will take the form

$$a_x = \frac{D_{x+1} + D_{x+2} + \dots \text{ to table limit}}{D_x}. \quad (3)$$

Again, if we define N_{x+1} by the equation

$$N_{x+1} = D_{x+1} + D_{x+2} + \dots \text{ to table limit}, \quad (4)$$

we have for a_x the very simple form

$$a_x = \frac{N_{x+1}}{D_x}. \quad (5)$$

We have also

$$1 + a_x = 1 + \frac{N_{x+1}}{D_x} = \frac{D_x + D_{x+1} + \dots}{D_x} = \frac{N_x}{D_x}. \quad (6)$$

Tables giving the values of N and D for various ages, and at rates of interest in common use, have been computed and are available for use. Such tables are called *commutation tables* or *commutation columns** (see Table XII). Other commutation columns will be described later on.

* The name *commutation columns* was given in 1840 by Professor De Morgan, who noticed that they formed a means of commuting, or exchanging, one benefit for another. To illustrate, the definitions of D_x and D_{x+n} enable us to write

$$\frac{D_{x+n}}{D_x} = \frac{v^{x+n}l_{x+n}}{v^x l_x};$$

whence

$$D_{x+n} = v^n D_x \cdot \frac{l_{x+n}}{l_x}$$

= present value of D_x due n years hence into the probability that a person aged x will survive n years.

In words, one could give up D_{x+n} now for D_x payable n years hence. The other quantities, N_x , C_x , and M_x , have similar properties.

By means of the proper commutation columns the computation of life annuities becomes a matter of simple division. Great care must be taken, however, to use columns based upon the mortality table and the rate of interest that should be used in the problem in question. For example, at the present time, in Wisconsin, the American Experience Table is the standard and 5% interest is specified by law for all inheritance-tax computations in which annuities are involved; on the other hand, the law of Massachusetts specifies that for similar computations a table known as the Actuaries', or Combined Experience, Table and 4% must be used. Still other tables are used in other states. Table No. XII is the American Experience Table with interest assumed at $3\frac{1}{2}\%$. This table and interest at $3\frac{1}{2}\%$ is a combination widely used for insurance work in this country. It will be used exclusively in this book.

EXAMPLES

1. What is the present value of a life annuity of \$500 per annum for a person aged 21 years?

$$\begin{aligned}\text{Solution. By formula (5), } a_{21} &= \frac{N_{22}}{D_{21}} \\ &= \frac{893,212.5}{44,630.8} \\ &= 20.013365.\end{aligned}$$

The annuity of \$500 has a present value of

$$\$500 \times 20.013365 = \$10,006.68.$$

2. A man left an estate worth approximately \$2,000,000, and in his will directed that the income of the estate should go to his widow, aged 60 years at the time of his death, for the remainder of her life. Suppose the law specifies that the annual income shall be considered as 5% of the estate, and that an inheritance tax of $1\frac{1}{2}\%$ must be paid. What will be the amount of the inheritance tax upon the wife's interest?

3. Express ${}_nE_x$ in terms of D 's.

105. Deferred and temporary life annuities.

PROBLEM. *To find the present value of a life annuity payable to a person aged x after the lapse of n years.*

By the definition of a deferred annuity the first payment is to be made at the end of $n+1$ years, and the payments will

continue until the life fails. The present value will be the sum of the present values of a series of endowments due $n+1$, $n+2$, $\dots$, years, to the table limit. Denoting the present value of the annuity by ${}_n|a_x$, we have

$${}_n|a_x = {}_{n+1}E_x + {}_{n+2}E_x + {}_{n+3}E_x + \dots \text{ to table limit,} \quad (1)$$

or, replacing each one of the E 's by their values as given by (1), § 103, and taking out the common denominator l_x ,

$${}_n|a_x = \frac{v^{n+1}l_{x+n+1} + v^{n+2}l_{x+n+2} + \dots \text{ to table limit}}{l_x}. \quad (2)$$

To express ${}_n|a_x$ in terms of the commutation symbols D and N , we multiply numerator and denominator by v^x . The result is, as can easily be seen,

$${}_n|a_x = \frac{N_{x+n+1}}{D_x}, \quad (3)$$

where $n+1$ is the number of years to elapse before the first payment is made.

PROBLEM. *To find the present value of a temporary annuity.*

A temporary annuity has been defined as one whose first payment begins one year hence and continues n years, provided the annuitant should live so long. It consists of n payments contingent upon the continued life of the annuitant. Its present value is denoted by the symbol ${}_n|a_x$, where x is the age of the annuitant.

To solve the problem, note that a life annuity may be considered as a temporary annuity for n years, followed by a life annuity deferred n years. The equation

$$a_x = {}_n|a_x + {}_n|a_x \quad (4)$$

is, therefore, identically true, so that the value of the temporary annuity is given by the formula

$${}_n|a_x = a_x - {}_n|a_x. \quad (5)$$

By means of (5) of § 104 and (3) of the present section, (5) is reduced to the form

$${}_n|a_x = \frac{N_{x+1} - N_{x+n+1}}{D_x}. \quad (6)$$

EXAMPLES

1. A father bequeathes to his son an annuity of \$500, to begin when the son is 40. What is the value of the son's interest at the time of the father's death, which occurs when the son is 30?

Solution. Formula (3) gives

$$\begin{aligned} {}_{10}|a_{30} &= \frac{N_{41}}{D_{30}} \\ &= \frac{324,440.0}{30,440.8} = 10.65803. \end{aligned}$$

The deferred annuity of \$500 per annum has a present value of

$$\$500 \times 10.65803 = \$5829.02.$$

2. What is the present value of a temporary annuity of \$500 per annum for 10 years, payable to a person whose age is 21?

Solution. By formula (6),

$$\begin{aligned} {}_{10}a_{21} &= \frac{N_{22} - N_{32}}{D_{21}} \\ &= \frac{893,212.5 - 537,199.3}{44,630.8} \\ &= 7.976850 \end{aligned}$$

The present value of the annuity of \$500 per annum under similar circumstances is, therefore,

$$\$500 \times 7.976850 = \$3988.42.$$

3. The Carnegie Foundation for the Advancement of Teaching grants, as a retiring allowance to a professor who has reached age 70, an annual stipend equal in amount to one half the professor's active pay. What is the privilege worth to a professor aged 50 years whose active pay is \$4000?

4. Find the value of ${}_n|a_x$ by using the notion of the endowment.

5. Find the value of ${}_n a_x$ by using the notion of the endowment.

CHAPTER XVII

SOME PROBLEMS IN LIFE INSURANCE

106. Definitions. An *insurance*, in its broadest sense, is an indemnity against loss. The business of furnishing insurance is conducted by a corporation or an association. It consists, primarily, in collecting sums of money from those who wish themselves or their heirs to be protected against loss incurred through the happening of a stipulated event, and distributing the funds thus collected to those to whom the stipulated event has happened, or to their heirs. In life insurance the stipulated event is the death of the insured or the attainment of a certain age.

The written agreement, or contract, between the insured person and the corporation or association furnishing the insurance is called a *policy*, and the insured person is called a *policyholder*. The amount received under the terms of the policy is called a *benefit*, and the person to whom it is paid is called the *beneficiary*.

Among the many forms of life insurance in existence we shall consider only the simpler forms furnished by the so-called *legal-reserve*, or *old-line*, insurance companies. A legal-reserve life insurance company offers insurance for a specified sum, called the *gross*, or *office premium*, or simply the *premium*. The gross premium may be defined as "the amount paid, or agreed to be paid, in one sum, or periodically, for a contract of insurance." Whether the premium is paid in a lump sum or in annual installments, it is invariably paid in advance. The period of twelve calendar months, reckoned from the date on which the policy goes into effect, or from any anniversary of this date, is called the *policy year*.

The first problem that confronts the life insurance company is the determination of the amount to be paid by the policyholder. The two elements having most to do with this determination are,

first, the *death rate*, and second, the *rate of interest that can be realized on investments*.

The death rate is defined as the number of deaths that will occur in a definite group of individuals of given age and class in one year. It is determined by means of a mortality table, such as has been considered in § 102, and, for insurance purposes, is expressed as the probability that a person of given age will die within one year.

The kinds of life insurance offered by life insurance companies may be divided, for convenience, into three classes, viz. *whole life insurance*, *term insurance*, and *endowment insurance*.

A whole life insurance is an insurance in which the company agrees to pay the benefit on satisfactory proofs of the policyholder's death. A term insurance is one in which the benefit is payable on the death of the insured, provided death should occur within a specified term of years.

The first problem in the mathematics of life insurance, after one has constructed a mortality table, is the determination of what is called the *net premium*. The net premium is the sum that would be paid by the insured as the equivalent of the benefit guaranteed by the company under the following conditions:

- (1) That the death rate is exactly the rate given by the mortality table adopted as the standard.
- (2) That the benefit is payable at the end of the policy year in which death occurs.
- (3) That the interest rate is exactly the assumed rate.
- (4) That the business is conducted without expense.

The net premium, when expressed as a single sum, is called the *net single premium*. The net single premium is, under the assumptions made, the mathematical equivalent of the benefit; i.e. it is the present value of the benefit.

The solutions of the problems in life insurance are very similar to the solutions of the problems of life annuity, the difference being that the life annuity has to do with the probability of living, while the life insurance has to do, for the most part, with the probability of dying. For the sake of simplicity the benefit is assumed to be 1 in all computations.

107. Whole life insurance.

PROBLEM. *To find the net single premium for a whole life insurance of 1 on the life of a person aged x years.*

Suppose l_x persons, all of age x , agree to pay to a company a sum sufficient to secure to the estate of each one the payment of \$1, payable at the end of the policy year in which death occurs. The number of persons dying during the successive years will be, according to the mortality table,

$$d_x, \quad d_{x+1}, \quad d_{x+2}, \quad \text{to table limit.}$$

The sum d_x required at the end of the first year would have a present value of vd_x . Likewise, the sums required at the end of each of the succeeding years would have for their present values

$$v^2d_{x+1}, \quad v^3d_{x+2}, \quad \dots, \quad \text{to table limit.}$$

The total amount required will be, therefore,

$$vd_x + v^2d_{x+1} + v^3d_{x+2} + \dots \text{ to table limit.}$$

If this amount be shared equally among the l_x persons, the amount paid by each one will be the net single premium, or A_x .

Therefore

$$A_x = \frac{vd_x + v^2d_{x+1} + \dots \text{ to table limit.}}{l_x}. \quad (1)$$

The computation required to find A_x from equation (1) is quite simple, though it would be very long for the younger ages. To adapt (1) for computation, we make use of the same device that was used in computing a_x in § 104. Multiplying numerator and denominator by v^x , we note that the denominator is changed to D_x , defined by equation (2) in § 104. If, moreover, we define C_x by the equation

$$C_x = v^{x+1}d_x, \quad (2)$$

equation (1) will take the form

$$A_x = \frac{C_x + C_{x+1} + C_{x+2} + \dots \text{ to table limit.}}{D_x}. \quad (3)$$

It is customary to denote the sum of the terms in the numerator by M_x , so that, by definition,

$$M_x = C_x + C_{x+1} + C_{x+2} + \dots \text{ to table limit.} \quad (4)$$

With this notation A_x takes the simple form

$$A_x = \frac{M_x}{D_x}. \quad (5)$$

The expressions C_x and M_x , which possess the same peculiar property as the expressions D_x and N_x of § 104, are usually tabulated along with the commutation columns for D_x and N_x (see Table XII).

EXAMPLES

1. What is the net single premium on a whole life policy for \$10,000 on the life of a person aged 35?

$$\begin{aligned} \text{Solution. By formula (5), } A_{35} &= \frac{M_{35}}{D_{35}} \\ &= \frac{9094.955}{24,544.7} \\ &= .370547. \end{aligned}$$

The net single premium for \$10,000 is, therefore,

$$\$10,000 \times A_{35} = \$3705.47$$

2. Find the net single premium for a whole life policy for \$10,000 on a life aged 30 years.

When the premium is paid in annual installments, the annual payments may continue throughout the life of the policyholder, or they may continue for a definite term of years. A whole life policy on which the payments continue through life is called an *ordinary life policy*, while a whole life policy on which the payments continue for a stated number of years is called a *limited payment policy*, or an *n-payment life policy*, where n denotes the number of annual payments that are to be made unless death should occur earlier.

The sum which, if paid at the beginning of each policy year for a term of years, or for life, would make the equivalent of the net single premium is called the *net annual premium*.

PROBLEM. To find the net annual premium for an ordinary life policy on the life of a person aged x .

By the definitions the net annual premiums for an ordinary life policy constitute a life annuity which, since the premiums

are paid in advance, is *due*. Let P_x denote the net annual premium. The present value of the annuity is, therefore,

$$P_x + P_x \cdot a_x,$$

which must equal the net single premium. Consequently, we have the equation

$$P_x + P_x \cdot a_x = A_x,$$

from which we find
$$P_x = \frac{A_x}{1 + a_x}. \quad (6)$$

If we express A_x and a_x in terms of commutation symbols according to (6) of § 104 and (5) of the present section, we have, after reduction,

$$P_x = \frac{M_x}{N_x}. \quad (7)$$

PROBLEM. *To find the net annual premium for an n -payment life policy on the life of a person aged x .*

The net annual premiums for an n -payment life policy constitute a cash payment and a temporary annuity for $n-1$ years, unless death should occur earlier. If ${}_nP_x$ denote the net annual premium, we have

$${}_nP_x + {}_nP_x \cdot |_{n-1}a_x = A_x,$$

from which we find
$${}_nP_x = \frac{A_x}{1 + |_{n-1}a_x}. \quad (8)$$

In commutation symbols, (8) has the form

$${}_nP_x = \frac{M_x}{D_x + N_{x+1} - N_{x+n}} = \frac{M_x}{N_x - N_{x+n}}. \quad (9)$$

EXAMPLES

1. Find the net annual premium for an ordinary life policy of \$10,000 on a life aged 27 years?

Solution. The net annual premium is given by formula (7), so that we have

$$\begin{aligned} P_{27} &= \frac{M_{27}}{N_{27}} \\ &= \frac{11,053.97}{696,333.8} = .0158745. \end{aligned}$$

The net annual premium on an ordinary life policy for \$10,000 would be

$$\$10,000 \times .0158745 = \$158.745.$$

2. Find the net annual premium for a twenty-payment life policy of \$10,000 on the life of a person aged 25 years.

3. Find the expression for A_x by using the probability of death for each age in the mortality table, instead of the number of persons dying out of the group l_x .

4. Prove that $M_x = vN_x - N_{x+1}$.

108. Term insurance.

PROBLEM. *To find the net single premium for a term insurance of 1 for n years on the life of a person aged x .*

The probability that a person aged x will die within the first year is $\frac{d_x}{l_x}$, and the present value of the mathematical expectation that the benefit will have to be paid is $v \frac{d_x}{l_x}$. Similarly, the probability that the person will die within the second year is $\frac{d_{x+1}}{l_x}$, and the present value of the mathematical expectation that the benefit will have to be paid at the end of the second year is $v^2 \frac{d_{x+1}}{l_x}$, and so on for n years. The present value of the mathematical expectation for the n th year will be $v^n \frac{d_{x+n-1}}{l_x}$. Denoting the net single premium by ${}_nA_x$, we have

$${}_nA_x = \frac{vd_x + v^2d_{x+1} + \dots + v^nd_{x+n-1}}{l_x}. \quad (1)$$

If numerator and denominator of (1) be multiplied by v^x , and note be taken of the fact that, by (2) of § 107,

$$v^{x+r+1}d_{x+r} = C_{x+r},$$

equation (1) becomes

$${}_nA_x = \frac{C_x + C_{x+1} + \dots + C_{x+n-1}}{D_x}. \quad (2)$$

The numerator of (2) is the numerator of (3) of § 107, or, what amounts to the same thing, of (5) of § 107, less the terms of the sum

$$C_{x+n} + C_{x+n+1} + \dots \text{to table limit.} \quad (3)$$

The numerator of (5), § 107, is M_x , while the sum of the terms of (3) is $M_{x+n} = C_{x+n} + C_{x+n+1} + \dots$ to table limit.

Therefore equation (2) is reduced to the final form,

$${}_nA_x = \frac{M_x - M_{x+n}}{D_x}. \quad (4)$$

EXAMPLE

Find the net single premium for a term insurance of \$10,000 for 10 years on the life of a person aged 46.

Solution. By formula (4),

$$\begin{aligned} {}_{10}A_{46} &= \frac{M_{46} - M_{56}}{D_{46}} \\ &= \frac{7022.682 - 5335.898}{15,070.0} = .111930. \end{aligned}$$

The net single premium is, therefore,

$$\$10,000 \times .111930 = \$1119.30.$$

PROBLEM. *To find the net annual premium for a term insurance of 1 for n years on the life of a person aged x years.*

The net annual premiums will constitute a temporary life annuity which is due, since the premiums are payable in advance. Let the net annual premium be denoted by ${}_nP_x$. The present value of the annuity will therefore be

$${}_nP_x + {}_nP_x \cdot {}_{n-1}a_x,$$

which must be equal to the net single premium ${}_nA_x$. The unknown quantity ${}_nP_x$ is therefore determined by the equation

$${}_nP_x + {}_nP_x \cdot {}_{n-1}a_x = {}_nA_x.$$

Consequently,

$${}_nP_x = \frac{{}_nA_x}{1 + {}_{n-1}a_x}. \quad (5)$$

If we substitute for ${}_{n-1}a_x$ and ${}_nA_x$ their values in terms of D 's, N 's, and M 's as given by (6) of § 105 and (4) of the present section, we find

$${}_nP_x = \frac{M_x - M_{x+n}}{D_x + N_{x+1} - N_{x+n}} = \frac{M_x - M_{x+n}}{N_x - N_{x+n}}. \quad (6)$$

EXAMPLES

1. Find the net annual premium for a term insurance of \$10,000 for 10 years on the life of a person aged 46 years.

Solution. Formula (6) gives

$$\begin{aligned} {}_{10}P_{46} &= \frac{M_{46} - M_{56}}{N_{46} - N_{56}} \\ &= \frac{7022.682 - 5335.898}{237,971.9 - 115,142.4} \\ &= \frac{1686.784}{122,829.5} \\ &= .0137327. \end{aligned}$$

The net annual premium on a policy for \$10,000 under the same conditions is, therefore,

$$\$0.0137327 \times 10,000 = \$137.327.$$

2. Find the net annual premium on a policy of \$10,000 for 10 years on the life of a person aged 42 years.

3. Find the expression for ${}_n P_x$ in terms of N 's alone.

109. Endowment insurance. An *endowment insurance* is a form of insurance in which the company agrees to pay a given sum in the event either of the death of the policyholder or of his survival to the end of a specified term of years.

PROBLEM. *To find the net single premium for an endowment insurance of 1 on the life of a person aged x years.*

An endowment insurance may be looked upon as a term insurance for n years, together with a pure endowment payable at the end of n years in case the life survives. American writers use the symbol ${}_e\Pi_{x:n}$ to denote the net single premium of an endowment assurance. We have, therefore,

$${}_e\Pi_{x:n} = {}_nA_x + {}_nE_x. \quad (1)$$

But, by (4), §108,

$${}_nA_x = \frac{M_x - M_{x+n}}{D_x}.$$

By (1), §103, and (2), §104,

$${}_nE_x = \frac{v^n l_{x+n}}{l_x} = \frac{v^{x+n} l_{x+n}}{v^x l_x} = \frac{D_{x+n}}{D_x}.$$

Substituting these values in (1), we obtain

$${}_e\Pi_{xn} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}. \quad (2)$$

PROBLEM. *To find the net annual premium for an endowment insurance of 1 for n years on the life of a person aged x years.*

Let ${}_eP_{xn}$ denote the net annual premium. The first installment is a cash payment of ${}_eP_{xn}$, and the remaining payments constitute a temporary annuity of ${}_eP_{xn}$ for $n-1$ years. To determine ${}_eP_{xn}$, we have the equation

$${}_eP_{xn} + {}_eP_{xn} \cdot |_{n-1}a_x = {}_e\Pi_{xn}, \quad (3)$$

and, consequently,

$${}_eP_{xn} = \frac{{}_e\Pi_{xn}}{1 + |_{n-1}a_x}. \quad (4)$$

By (2) of the present section and (6) of §105, equation (4) is reduced to the final form,

$${}_eP_{xn} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}}. \quad (5)$$

EXAMPLES

1. Compute the net annual premium on a thirty-year endowment insurance of \$1000 on the life of a person aged 24 years.

Solution. By formula (5)

$$\begin{aligned} {}_eP_{24\ 30} &= \frac{D_{54} + M_{24} - M_{54}}{N_{24} - N_{54}} \\ &= \frac{10,252.4 + 11,935.38 - 5682.861}{809,420.6 - 135,128.2} \\ &= \frac{16,504.92}{674,292.4} \\ &= .024477. \end{aligned}$$

The premium on a policy for \$1000, under the same conditions, would be \$24.48.

2. Compute the net single premium on a thirty-year endowment policy for \$1000 on a life aged 24 years.

3. Express ${}_eP_{xn}$ in terms of N 's, D 's, and v .

110. Reserves; valuation of policies. Complete mortality tables show that from the age of ten or twelve years onward the death rate increases. If, therefore, the annual premium is a *level premium*, i.e. a premium which remains constant throughout the whole time that the policy remains in force, it will be higher than the net annual premium for a one-year insurance during the earlier years and lower than for a one-year insurance during the later years. For example, the probability that a person aged twenty-five years will die within one year is, by the American Experience Table, .008065. Consequently, on a $3\frac{1}{2}\%$ basis the net single premium on a one-year policy for \$1000 would be

$$\frac{\$0.008065}{1.035} \times 1000 = \$7.79.$$

Likewise, the premium for a one-year insurance on a policy for \$1000 at age sixty would be \$25.80. The net level premium on an ordinary life policy for \$1000 purchased at age twenty-five is \$15.10. The excess paid during the earlier years is an overpayment on the part of the policyholder and constitutes a reserve to meet the deficiency that will occur in later years.

The total excess accumulated to the end of a given policy year at the assumed rate of interest, and on the assumption that the death rate is exactly the tabular rate, is called the *terminal reserve* for that year and is known as the *value of the policy* or the *net value of the policy* at the end of the year in question.

To view the matter from another standpoint, let us examine an ordinary life policy for \$1000 on a life of age thirty. According to the American Experience Table the net annual premium at $3\frac{1}{2}\%$ is \$17.19. The number living at age thirty, out of 100,000 who were alive at age ten, is 85,441. Assuming that 85,441 policies of \$1000 are issued at the same time on 85,441 lives, all aged thirty years, the amount of the premiums received will be \$1,468,730.79, which, at $3\frac{1}{2}\%$, will amount in one year to \$1,520,136.37. According to the table 720 persons will die within the year, so that the death losses will be \$720,000. This will leave with the company a total of \$800,136.37, which would enable it to place \$9.45 to the credit of each of the 84,721

survivors. At the beginning of the second year there will be in the hands of the company the reserve fund of \$800,136.37, together with the premiums paid by the 84,721 survivors, each paying \$17.19, or a total of \$2,256,490.36. Beginning with this amount the computation for the reserve for the second year will be exactly like that for the first year. The following table will explain itself.

Table showing terminal reserves on an ordinary life policy for \$1000 on the life of a person aged 30 years.

Age of policy in years	Funds on hand at beginning of year,	Fund accumulated at 3½%	Death losses	Fund at end of year	Amount to credit of each survivor
1	\$1,468,730.79	\$1,520,136.37	\$720,000	\$800,136.37	\$9.45
2	2,256,490.36	2,335,467.52	721,000	1,614,467.52	19.22
3	3,058,427.52	3,165,472.48	723,000	2,442,472.48	29.33
4	3,874,004.11	4,009,594.25	726,000	3,283,594.25	39.78
...	...	...	...	...	...

The amounts in the last column are the terminal reserves required. If the table were continued to the limit of the mortality table, the terminal reserve for the sixty-fifth year of the age of the policy, i.e. the ninety-fifth year of the age of the policyholder, would be exactly \$1000. Such reserves exist for all level premium policies covering a period longer than one year.

PROBLEM. *To find the terminal reserve for an ordinary life policy in general terms.*

We note that, at the end of any given year after the policy has been issued, the sum of the terminal reserve and the present value of the premiums yet unpaid will be equal to the net single premium for a new policy on the life of the insured person at the age attained. Let n denote the number of years that have elapsed since the policy was issued, and ${}_nV_x$ the reserve at the end of n years on a policy issued on a life aged x years. The age of the policyholder on the date in question will be $x + n$ years, and by §107 the present value of the unpaid premiums would be

$$P_x(1 + a_{x+n}),$$

while the net single premium for a new policy at the attained age, $x+n$, would be

$$A_{x+n}.$$

Consequently, the equation for the determination of ${}_nV_x$ is

$$A_{x+n} = {}_nV_x + P_x(1 + a_{x+n}). \quad (1)$$

From equation (1),

$${}_nV_x = A_{x+n} - P_x(1 + a_{x+n}). \quad (2)$$

The known quantities on the right may be easily expressed in terms of commutation symbols, for, by (6) of § 104 and (5) and (7) of § 107, we have

$$1 + a_{x+n} = \frac{N_{x+n}}{D_{x+n}},$$

$$A_{x+n} = \frac{M_{x+n}}{D_{x+n}},$$

$$P_x = \frac{M_x}{N_x}.$$

Substituting these values in (2), we have

$${}_nV_x = \frac{M_{x+n}}{D_{x+n}} - \frac{M_x}{N_x} \cdot \frac{N_{x+n}}{D_{x+n}}, \quad (3)$$

or

$${}_nV_x = \frac{N_x M_{x+n} - M_x N_{x+n}}{N_x \cdot D_{x+n}}. \quad (4)$$

Formulas similar to (2), (3), and (4) may be found for other forms of policies.

The question of reserves is one of very great importance in determining the condition of an insurance company. Indeed, the maintenance of a fund sufficient to cover the terminal reserves on all policies in force is the test of solvency and should be enforced by law.

The reserve is also the basis upon which the so-called *loan value*, or the *cash surrender value*, is determined. The cash surrender value is the amount returned to the policyholder in case the policy is allowed to lapse. It is usually equal to the loan value, or the amount which the company will loan with the policy as collateral security.

For various reasons it would not be fair to the company to give to the policyholder who withdraws the full amount of the reserve on his policy, at least during the earlier years of the life of the policy. The most important reason is the fact that it costs much more to get "new business" than to carry business already on the company's books. The insuring company is therefore allowed a *surrender charge* if the policyholder asks for the value of his policy either in cash or in extended insurance. The amount of the surrender charge is in a measure arbitrary and in most states is determined by law. It varies from $2\frac{1}{2}\%$ to 3% of the amount insured.

EXAMPLES

1. A man takes out an ordinary life policy for \$1000 at age 30. When he is 50 years of age, the company decides to go out of business. What sum is due him?

Solution. By formula (4),

$$\begin{aligned} {}_{20}V_{30} &= \frac{M_{50}N_{30} - M_{30}N_{50}}{N_{30} \cdot D_{50}} \\ &= \frac{6,335.436 \times 596,803.6 - 10,259.02 \times 181,663.4}{596,803.6 \times 12,498.6} \\ &= .25704. \end{aligned}$$

The terminal reserve on the policy of \$1000 would then be \$257.04.

2. Find the terminal reserve on the above policy 60 years after it is issued, provided the holder is still alive.

3. Verify the result obtained in Example 2 by carrying out a table similar to that on page 210 and using \$17.19 as the net annual premium on \$1000 at age 30.

Suggestion. The reserve should be large enough to meet the demand that, according to the mortality table, will surely be made upon it at the end of the sixty-fifth year.

111. Loading; gross premiums. An insurance company could not do business on the basis of net premiums alone, for the net premium is found on the supposition that the business is conducted without expense. The most important items of expense in connection with the business are

1. The expenses incurred in getting new business, such as the agent's commission, which takes a large part of the first annual premium, cost of medical examination, and so on.

2. The cost of administering the business, such as salaries of administrative officers, cost of collecting premiums, and so on.

3. A small amount to cover unforeseen contingencies.

To meet these expenses the net premium is increased, sometimes by a percentage of itself, sometimes by a fixed charge, and sometimes by both. The increase is called *loading*. The sum of the net premium and the loading is called the *gross* or *office premium*. The gross premium is the amount actually paid by the policyholder.

It is customary to denote the gross premium by the same symbol that is used to denote the net premium, but with an accent. Thus, for an ordinary life policy P_x denotes the net annual premium and P'_x the gross annual premium. If k denote the rate at which the percentage charge is made, and c the fixed charge, the formula for P'_x in terms of P_x would be

$$P'_x = P_x(1 + k) + c. \quad (1)$$

If the whole of the loading is considered as a fixed charge, k will be zero, while if it is looked upon as a percentage charge, c will be zero.

It would seem to be more in accord with scientific method if one could determine the sum total of the loading and then distribute it as an annual charge throughout the life of the policy. For example, suppose that the initial expenses incurred in connection with an ordinary life policy amount to K , and that all other expenses are met by an annual charge c . To distribute the amount K throughout the life of the policy, it must be looked upon as the present value of a life annuity. The annual rent R will then be given by the formula

$$K = R(1 + a_x), \quad (2)$$

and we have
$$R = \frac{K}{1 + a_x}. \quad (3)$$

The total loading would then be $R + c$, and the gross premium would be

$$P'_x = P_x + R + c, \quad (4)$$

or
$$P'_x = P_x + \frac{K}{1 + a_x} + c.$$

Similar expressions may easily be obtained for the gross annual premiums on other forms of policies. It is scarcely worth while to spend time in deriving formulas which are not in agreement with current practice, however faulty such practice may be, except in so far as they may enable the student to discuss intelligently propositions looking toward better practice.

EXAMPLES

1. Derive the formula for the gross annual premium on a twenty-payment life policy on the supposition that the initial expense is to be distributed as an annual charge to be made so long as the payments continue.

2. Express formula (4) above in terms of commutation symbols, so far as is possible.

3. Find the gross premium in terms of commutation symbols on an n -year endowment policy when the initial expense is distributed as a constant annual charge during the life of the policy.

112. Concluding note on life insurance. In the foregoing sections an attempt has been made to present in the barest outline a few of the more important problems that present themselves in the theory of life insurance. It may not be amiss to call attention to some of the subjects that the student of life insurance should take up.

1. In the first place, a more extended study of reserves will be indispensable.

2. No mention has been made of the "*surplus*." The net premium has been computed on the assumptions (1) that the death rate is exactly that of the mortality table in use; (2) that the interest rate is exactly the assumed rate; (3) that no policies will be allowed to lapse before the death of the policyholder or the expiration of the term for which the insurance was written. Moreover, the loading required to pay all expenses was supposed to be known in advance. Sound business policy requires that every one of these assumptions shall be made conservatively. The careful actuary will therefore adopt a mortality table that will surely cover the mortality. The loading should be made large enough to provide for every possible contingency that may arise.

It is clear, then, that at the end of each year a surplus may remain after all expenses have been met and sufficient provision has been made for the maintenance of adequate reserves for all policies in force. If the company is a mutual company, this surplus belongs to the policyholders, and an equitable method for distributing it must be found.

3. Many policies are written, under the terms of which the benefit is payable on the failure of one or both of two lives. Such a policy is called a joint life policy. The theory of joint life insurance is not essentially different from insurances on single lives, though many modifications must be made and new commutation columns must be constructed.

PART IV. TABLES

TABLE I.—THE NUMBER OF EACH DAY OF THE
YEAR COUNTING FROM JANUARY 1

DAY OF MONTH	JAN.	FEB.	MAR.	APRIL	MAY	JUNE	JULY	AUG.	SEPT.	OCT.	NOV.	DEC.	DAY OF MONTH
1	1	32	60	91	121	152	182	213	244	274	305	335	1
2	2	33	61	92	122	153	183	214	245	275	306	336	2
3	3	34	62	93	123	154	184	215	246	276	307	337	3
4	4	35	63	94	124	155	185	216	247	277	308	338	4
5	5	36	64	95	125	156	186	217	248	278	309	339	5
6	6	37	65	96	126	157	187	218	249	279	310	340	6
7	7	38	66	97	127	158	188	219	250	280	311	341	7
8	8	39	67	98	128	159	189	220	251	281	312	342	8
9	9	40	68	99	129	160	190	221	252	282	313	343	9
10	10	41	69	100	130	161	191	222	253	283	314	344	10
11	11	42	70	101	131	162	192	223	254	284	315	345	11
12	12	43	71	102	132	163	193	224	255	285	316	346	12
13	13	44	72	103	133	164	194	225	256	286	317	347	13
14	14	45	73	104	134	165	195	226	257	287	318	348	14
15	15	46	74	105	135	166	196	227	258	288	319	349	15
16	16	47	75	106	136	167	197	228	259	289	320	350	16
17	17	48	76	107	137	168	198	229	260	290	321	351	17
18	18	49	77	108	138	169	199	230	261	291	322	352	18
19	19	50	78	109	139	170	200	231	262	292	323	353	19
20	20	51	79	110	140	171	201	232	263	293	324	354	20
21	21	52	80	111	141	172	202	233	264	294	325	355	21
22	22	53	81	112	142	173	203	234	265	295	326	356	22
23	23	54	82	113	143	174	204	235	266	296	327	357	23
24	24	55	83	114	144	175	205	236	267	297	328	358	24
25	25	56	84	115	145	176	206	237	268	298	329	359	25
26	26	57	85	116	146	177	207	238	269	299	330	360	26
27	27	58	86	117	147	178	208	239	270	300	331	361	27
28	28	59	87	118	148	179	209	240	271	301	332	362	28
29	29		88	119	149	180	210	241	272	302	333	363	29
30	30		89	120	150	181	211	242	273	303	334	364	30
31	31		90		151		212	243		304		365	31

NOTE.—For leap years the number of the day is one greater than the tabular number after February 28.

TABLE II. — EXACT INTEREST AT 5% FOR TIMES
FROM 1 TO 365 DAYS

$$I = Pi \frac{d}{365}$$

DAYS	\$ 1,000	\$ 2,000	\$ 3,000	\$ 4,000	\$ 5,000	\$ 6,000	\$ 7,000	\$ 8,000	\$ 9,000	DAYS
1	0.1370	0.2740	0.4110	0.5749	0.6849	0.8219	0.9589	1.0959	1.2329	1
2	0.2740	0.5479	0.8219	1.0959	1.3699	1.6438	1.9178	2.1918	2.4658	2
3	0.4110	0.8219	1.2329	1.6438	2.0548	2.4657	2.8777	3.2877	3.6986	3
4	0.5479	1.0959	1.6438	2.1918	2.7397	3.2877	3.8356	4.3836	4.9315	4
5	0.6849	1.3699	2.0548	2.7397	3.4247	4.1096	4.7945	5.4795	6.1644	5
6	0.8219	1.6438	2.4658	3.2877	4.1096	4.9315	5.7534	6.5753	7.3973	6
7	0.9589	1.9178	2.8767	3.8356	4.7945	5.7534	6.7123	7.6712	8.6301	7
8	1.0959	2.1918	3.2877	4.3836	5.4795	6.5753	7.6712	8.7671	9.8630	8
9	1.2329	2.4658	3.6986	4.9315	6.1644	7.3973	8.6301	9.8630	11.0959	9
10	1.3699	2.7397	4.1096	5.4795	6.8493	8.2192	9.5890	10.9589	12.3288	10
20	2.7397	5.4795	8.2192	10.9589	13.6986	16.4384	19.1781	21.9178	24.6575	20
30	4.1096	8.2192	12.3288	16.4384	20.5479	24.6575	28.7671	32.8767	36.9863	30
40	5.4795	10.9589	16.4384	21.9178	27.3973	32.8767	38.3562	43.8356	49.3151	40
50	6.8493	13.6986	20.5479	27.3973	34.2466	41.0959	47.9452	54.7945	61.6438	50
60	8.2192	16.4384	24.6575	32.8767	41.0959	49.3151	57.5342	65.7534	73.9726	60
70	9.5890	19.1781	28.7671	38.3562	47.9452	57.5342	67.1233	76.7123	86.3014	70
80	10.9589	21.9178	32.8767	43.8356	54.7945	65.7334	76.7123	87.6712	98.6301	80
90	12.3288	24.6575	36.9863	49.3151	61.6438	73.9726	86.3014	98.6301	110.9589	90
100	13.6986	27.3973	41.0959	54.7945	68.4931	82.1918	95.8904	109.5890	123.2877	100
110	15.0685	30.1370	45.2055	60.2740	75.3425	90.4110	105.4795	120.5479	135.6164	110
120	16.4384	32.8767	49.3151	65.7534	82.1918	98.6301	115.0685	131.5068	147.9452	120
130	17.8082	35.6164	53.4247	71.2329	89.0411	106.8493	124.6575	142.4658	160.2740	130
140	19.1781	38.3562	57.5342	76.7123	95.8904	115.0685	134.2466	153.4247	172.6027	140
150	20.5479	41.0959	61.6438	82.1918	102.7397	123.2877	143.8356	164.3836	184.9315	150
160	21.9178	43.8356	65.7534	87.6712	109.5890	131.5068	153.4247	175.3425	197.2603	160
170	23.2877	46.5753	69.8630	93.1507	116.4384	139.7260	163.0137	186.3014	209.5890	170
180	24.6575	49.3151	73.9726	98.6301	123.2877	147.9452	172.6027	197.2603	221.9178	180
190	26.0274	52.0548	78.0822	104.1096	130.1370	156.1644	182.1918	208.2192	234.2466	190
200	27.3973	54.7945	82.1918	109.5890	136.9863	164.3836	191.7808	219.1781	246.5753	200
210	28.7671	57.5342	86.3014	115.0685	143.8356	172.6027	201.3699	230.1370	258.9041	210
220	30.1370	60.2740	90.4110	120.5479	150.6849	180.8219	210.9589	241.0959	271.2329	220
230	31.5068	63.0137	94.5205	126.0274	157.5342	189.0411	220.5479	252.0548	283.5616	230
240	32.8767	65.7534	98.6301	131.5068	164.3836	197.2603	230.1370	263.0137	295.8904	240
250	34.2466	68.4932	102.7397	136.9863	171.2329	205.4795	239.7260	273.9726	308.2192	250
260	35.6164	71.2329	106.8493	142.4658	178.0822	213.6986	249.3151	284.9315	320.5479	260
270	36.9863	73.9726	110.9589	147.9452	184.9315	221.9178	258.9041	295.8904	332.8767	270
280	38.3562	76.7123	115.0685	153.4247	191.7808	230.1370	268.4932	306.8493	345.2055	280
290	39.7260	79.4521	119.1781	158.9041	198.6301	238.3562	278.0822	317.8082	357.5342	290
300	41.0959	82.1918	123.2877	164.3836	205.4795	246.5753	287.6712	328.7671	369.8630	300
310	42.4658	84.9315	127.3973	169.8630	212.3288	254.7945	297.2603	339.7260	382.1918	310
320	43.8356	87.6712	131.5068	175.3425	219.1781	263.0137	306.8493	350.6849	394.5205	320
330	45.2055	90.4110	135.6164	180.8219	226.0274	271.2329	316.4384	361.6438	406.8493	330
340	46.5753	93.1507	139.7260	186.3014	232.8767	279.4521	326.0274	372.6027	419.1781	340
350	47.9452	95.8904	143.8356	191.7808	239.7260	287.6712	335.6164	383.5616	431.5068	350
360	49.3151	98.6301	147.9452	197.2603	246.5753	295.8904	345.2055	394.5205	443.8356	360

TABLE III.—COMPOUND AMOUNT ON 1

$$s = (1 + i)^n$$

<i>n</i>	1 $\frac{1}{4}$ %	1 $\frac{1}{2}$ %	2 %	2 $\frac{1}{2}$ %	3 %	<i>n</i>
1	1.012 5000	1.015 0000	1.020 0000	1.025 0000	1.030 0000	1
2	1.025 1563	1.030 2250	1.040 4000	1.050 6250	1.060 9000	2
3	1.037 9707	1.045 6784	1.061 2080	1.076 8906	1.092 7270	3
4	1.050 9453	1.061 3636	1.082 4322	1.103 8129	1.125 5088	4
5	1.064 0822	1.077 2840	1.104 0808	1.131 4082	1.159 2741	5
6	1.077 3832	1.093 4433	1.126 1624	1.159 6934	1.194 0523	6
7	1.090 8505	1.109 8449	1.148 6857	1.188 6858	1.229 8739	7
8	1.104 4861	1.126 4926	1.171 6594	1.218 4029	1.266 7701	8
9	1.118 2922	1.143 3900	1.195 0926	1.248 8630	1.304 7732	9
10	1.132 2708	1.160 5408	1.218 9944	1.280 0845	1.343 9164	10
11	1.146 4242	1.177 9489	1.243 3743	1.312 0867	1.384 2339	11
12	1.160 7545	1.195 6182	1.268 2418	1.344 8888	1.425 7609	12
13	1.175 2639	1.213 5524	1.293 6066	1.378 5110	1.468 5337	13
14	1.189 9547	1.231 7557	1.319 4788	1.412 9738	1.512 5897	14
15	1.204 8292	1.250 2321	1.345 8683	1.448 2982	1.557 9674	15
16	1.219 8895	1.268 9855	1.372 7857	1.484 5056	1.604 7064	16
17	1.235 1382	1.288 0203	1.400 2414	1.521 6183	1.652 8476	17
18	1.250 5774	1.307 3406	1.428 2463	1.559 6587	1.702 4331	18
19	1.266 2096	1.326 9507	1.456 8112	1.598 6502	1.753 5061	19
20	1.282 0372	1.346 8550	1.485 9474	1.638 6164	1.806 1112	20
21	1.298 0627	1.367 0578	1.515 6663	1.679 5819	1.860 2946	21
22	1.314 2885	1.387 5637	1.545 9797	1.721 5714	1.916 1034	22
23	1.330 7171	1.408 3771	1.576 8993	1.764 6107	1.973 5865	23
24	1.347 3511	1.429 5028	1.608 4373	1.808 7260	2.032 7941	24
25	1.364 1929	1.450 9454	1.640 6060	1.853 9441	2.093 7780	25
26	1.381 2454	1.472 7095	1.673 4181	1.900 2927	2.156 5913	26
27	1.398 5109	1.494 8002	1.706 8865	1.947 8000	2.221 2890	27
28	1.415 9923	1.517 2222	1.741 0242	1.996 4950	2.287 9277	28
29	1.433 6922	1.539 9805	1.775 8447	2.046 4074	2.356 5655	29
30	1.451 6134	1.563 0802	1.811 3616	2.097 5676	2.427 2625	30
31	1.469 7585	1.586 5264	1.847 5888	2.150 0068	2.500 0803	31
32	1.488 1305	1.610 3243	1.884 5406	2.203 7569	2.575 0828	32
33	1.506 7321	1.634 4792	1.922 2314	2.258 8509	2.652 3352	33
34	1.525 5663	1.658 9964	1.960 6760	2.315 3221	2.731 9053	34
35	1.544 6359	1.683 8813	1.999 8896	2.373 2052	2.813 8625	35
36	1.563 9438	1.709 1395	2.039 8873	2.432 5353	2.898 2783	36
37	1.583 4931	1.734 7766	2.080 6851	2.493 3487	2.985 2267	37
38	1.603 2868	1.760 7983	2.122 2988	2.555 6824	3.074 7835	38
39	1.623 3279	1.787 2102	2.164 7448	2.619 5745	3.167 0270	39
40	1.643 6195	1.814 0184	2.208 0397	2.685 0638	3.262 0378	40
41	1.664 1647	1.841 2287	2.252 2005	2.752 1904	3.359 8989	41
42	1.684 9668	1.868 8471	2.297 2445	2.820 9952	3.460 6959	42
43	1.706 0289	1.896 8798	2.343 1894	2.891 5201	3.564 5168	43
44	1.727 3542	1.925 3330	2.390 0531	2.963 8081	3.671 4523	44
45	1.748 9461	1.954 2130	2.437 8542	3.037 9033	3.781 5958	45
46	1.770 8080	1.983 5262	2.486 6113	3.113 8509	3.895 0437	46
47	1.792 9431	2.013 2791	2.536 3435	3.191 6971	4.011 8950	47
48	1.815 3549	2.043 4783	2.587 0704	3.271 4896	4.132 2519	48
49	1.838 0468	2.074 1305	2.638 8118	3.353 2768	4.256 2194	49
50	1.861 0224	2.105 2424	2.691 5880	3.437 1087	4.383 9060	50

TABLE III.—COMPOUND AMOUNT ON 1

$$s = (1 + i)^n$$

<i>n</i>	1 $\frac{1}{4}$ %	1 $\frac{1}{2}$ %	2%	2 $\frac{1}{2}$ %	3%	<i>n</i>
51	1.884 2851	2.136 8211	2.745 4198	3.523 0364	4.515 4232	51
52	1.907 8387	2.168 8734	2.800 3282	3.611 1123	4.650 8859	52
53	1.931 6867	2.201 4065	2.856 3347	3.701 3902	4.790 4125	53
54	1.955 8328	2.234 4276	2.913 4614	3.793 9249	4.934 1248	54
55	1.980 2807	2.267 9440	2.971 7307	3.888 7730	5.082 1486	55
56	2.005 0342	2.301 9631	3.031 1653	3.985 9924	5.234 6130	56
57	2.030 0971	2.336 4926	3.091 7886	4.085 6422	5.391 6514	57
58	2.055 4733	2.371 5400	3.156 6244	4.187 7832	5.553 4010	58
59	2.081 1668	2.407 1131	3.216 6969	4.292 4778	5.720 0030	59
60	2.107 1884	2.443 2198	3.281 0308	4.399 7897	5.891 6031	60
61	2.133 5211	2.479 8681	3.346 6514	4.509 7845	6.068 3512	61
62	2.160 1901	2.517 0661	3.413 5844	4.622 5291	6.250 4017	62
63	2.187 1925	2.554 8221	3.481 8561	4.738 0923	6.437 9138	63
64	2.214 5324	2.593 1444	3.551 4932	4.856 5446	6.631 0512	64
65	2.242 2141	2.632 0416	3.622 5231	4.977 9583	6.829 9827	65
66	2.270 2417	2.671 5222	3.694 9736	5.102 4072	7.034 8822	66
67	2.298 6198	2.711 5950	3.768 8730	5.229 9674	7.245 9287	67
68	2.327 3525	2.752 2690	3.844 2505	5.360 7166	7.463 3065	68
69	2.356 4444	2.793 5530	3.921 1355	5.494 7345	7.687 2057	69
70	2.385 9000	2.835 4563	3.999 5582	5.632 1029	7.917 8219	70
71	2.415 7237	2.877 9881	4.079 5494	5.772 9054	8.155 3566	71
72	2.445 9203	2.921 1580	4.161 1404	5.917 2281	8.400 0173	72
73	2.476 4943	2.964 9753	4.244 3632	6.065 1588	8.652 0178	73
74	2.507 4505	3.009 4500	4.329 2504	6.216 7877	8.911 5783	74
75	2.538 7936	3.054 5917	4.415 8355	6.372 2074	9.178 9257	75
76	2.570 5285	3.100 4106	4.504 1522	6.531 5126	9.454 2934	76
77	2.602 6601	3.146 9167	4.594 2352	6.694 8004	9.737 9222	77
78	2.635 1934	3.194 1205	4.686 1199	6.862 1704	10.030 0599	78
79	2.668 1333	3.242 0323	4.779 8423	7.033 7247	10.330 9617	79
80	2.701 4849	3.290 6628	4.875 4392	7.209 5678	10.640 8906	80
81	2.735 2535	3.340 0227	4.972 9479	7.389 8070	10.960 1173	81
82	2.769 4442	3.390 1231	5.072 4069	7.574 5522	11.288 9208	82
83	2.804 0622	3.440 9749	5.173 8550	7.763 9160	11.627 5884	83
84	2.839 1130	3.492 5895	5.277 3321	7.958 0139	11.976 4161	84
85	2.874 6019	3.544 9784	5.382 8788	8.156 9642	12.335 7085	85
86	2.910 5344	3.598 1531	5.490 5364	8.360 8883	12.705 7798	86
87	2.946 9161	3.652 1253	5.600 3471	8.569 9106	13.086 9532	87
88	2.983 7526	3.706 9072	5.712 3540	8.784 1583	13.479 5618	88
89	3.021 0495	3.762 5108	5.826 6011	9.003 7623	13.883 9487	89
90	3.058 8126	3.818 9485	5.943 1331	9.228 8563	14.300 4671	90
91	3.097 0477	3.876 2327	6.061 9958	9.459 5777	14.729 4811	91
92	3.135 7609	3.934 3762	6.183 2357	9.696 0672	15.171 3656	92
93	3.174 9579	3.993 3919	6.306 9004	9.938 4689	15.626 5065	93
94	3.214 6448	4.053 2927	6.433 0384	10.186 9306	16.095 3017	94
95	3.254 8279	4.114 0921	6.561 6992	10.441 6038	16.578 1608	95
96	3.295 5132	4.175 8035	6.692 9332	10.702 6439	17.075 5056	96
97	3.336 7072	4.238 4406	6.826 7918	10.970 2100	17.587 7708	97
98	3.378 4160	4.302 0172	6.963 3277	11.244 4653	18.115 4039	98
99	3.420 6462	4.366 5474	7.102 5942	11.525 5769	18.658 8660	99
100	3.463 4043	4.432 0457	7.244 6461	11.813 7164	19.218 6320	100

TABLE III.—COMPOUND AMOUNT ON 1

$$s = (1 + i)^n$$

<i>n</i>	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	<i>n</i>
1	1.035 0000	1.040 0000	1.045 0000	1.050 0000	1.060 0000	1
2	1.071 2250	1.081 6000	1.092 0250	1.102 5000	1.123 6000	2
3	1.108 7179	1.124 8640	1.141 1661	1.157 6250	1.191 0160	3
4	1.147 5230	1.169 8586	1.192 5186	1.215 5063	1.262 4770	4
5	1.187 6863	1.216 6529	1.246 1819	1.276 2816	1.338 2256	5
6	1.229 2553	1.265 3190	1.302 2601	1.340 0956	1.418 5191	6
7	1.272 2793	1.315 9318	1.360 8618	1.407 1004	1.503 6303	7
8	1.316 8090	1.368 5691	1.422 1006	1.477 4554	1.593 8481	8
9	1.362 8974	1.423 3118	1.486 0951	1.551 3282	1.689 4790	9
10	1.410 5988	1.480 2443	1.552 9694	1.628 8946	1.790 8477	10
11	1.459 9697	1.539 4541	1.622 8530	1.710 3394	1.898 2986	11
12	1.511 0687	1.601 0322	1.695 8814	1.795 8563	2.012 1965	12
13	1.563 9561	1.665 0735	1.772 1961	1.885 6491	2.132 9283	13
14	1.618 6945	1.731 6764	1.851 9449	1.979 9316	2.260 9040	14
15	1.675 3488	1.800 9435	1.935 2824	2.078 9282	2.396 5582	15
16	1.733 9860	1.872 9812	2.022 3702	2.182 8746	2.540 3517	16
17	1.794 6756	1.947 9005	2.113 3768	2.292 0183	2.692 7728	17
18	1.857 4892	2.025 8165	2.208 4788	2.406 6192	2.854 3392	18
19	1.922 5013	2.106 8492	2.307 8603	2.526 9502	3.025 5995	19
20	1.989 7889	2.191 1231	2.411 7140	2.653 2977	3.207 1355	20
21	2.059 4315	2.278 7681	2.520 2412	2.785 9626	3.399 5636	21
22	2.131 5116	2.369 9188	2.633 6520	2.925 2607	3.603 5374	22
23	2.206 1145	2.464 7155	2.752 1663	3.071 5238	3.819 7497	23
24	2.283 3285	2.563 3042	2.876 0138	3.225 0999	4.048 9346	24
25	2.363 2450	2.665 8363	3.005 4345	3.386 3549	4.291 8707	25
26	2.445 9586	2.772 4698	3.140 6790	3.555 6727	4.549 3830	26
27	2.531 5671	2.883 3686	3.282 0096	3.733 4563	4.822 3459	27
28	2.620 1720	2.998 7033	3.429 7000	3.920 1291	5.111 6867	28
29	2.711 8780	3.118 6515	3.584 0365	4.116 1356	5.418 3879	29
30	2.806 7937	3.243 3975	3.745 3181	4.321 9424	5.743 4912	30
31	2.905 0315	3.373 1334	3.913 8575	4.538 0395	6.088 1006	31
32	3.006 7076	3.508 0587	4.089 9810	4.764 9415	6.453 3867	32
33	3.111 9424	3.648 3811	4.274 0302	5.003 1885	6.840 5899	33
34	3.220 8603	3.794 3163	4.466 3615	5.253 3480	7.251 0253	34
35	3.333 5904	3.946 0890	4.667 3478	5.516 0154	7.686 0868	35
36	3.450 2661	4.103 9326	4.877 3785	5.791 8161	8.147 2520	36
37	3.571 0254	4.268 0899	5.096 8605	6.081 4069	8.636 0871	37
38	3.696 0113	4.438 8135	5.326 2192	6.385 4773	9.154 2523	38
39	3.825 3717	4.616 3660	5.565 8991	6.704 7512	9.703 5075	39
40	3.959 2597	4.801 0206	5.816 3645	7.039 9887	10.285 7179	40
41	4.097 8338	4.993 0615	6.078 1009	7.391 9881	10.902 8610	41
42	4.241 2580	5.192 7839	6.351 6155	7.761 5876	11.557 0327	42
43	4.389 7020	5.400 4953	6.637 4382	8.149 6669	12.250 4546	43
44	4.543 3416	5.616 5151	6.936 1229	8.557 1503	12.985 4819	44
45	4.702 3586	5.841 1757	7.248 2484	8.985 0078	13.764 6108	45
46	4.866 9411	6.074 8227	7.574 4196	9.434 2582	14.590 4875	46
47	5.037 2840	6.317 8156	7.915 2685	9.905 9711	15.465 9167	47
48	5.213 5890	6.570 5282	8.271 4556	10.401 2696	16.393 8717	48
49	5.396 0646	6.833 3494	8.643 6711	10.921 3331	17.377 5040	49
50	5.584 9269	7.106 6833	9.032 6363	11.467 3998	18.420 1543	50

TABLE III.—COMPOUND AMOUNT ON 1

$$s = (1 + i)^n$$

<i>n</i>	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	<i>n</i>
51	5.780 3993	7.390 9507	9.439 1049	12.040 7698	19.525 3635	51
52	5.982 7133	7.686 5887	9.863 8646	12.642 8083	20.696 8853	52
53	6.192 1082	7.994 0523	10.307 7385	13.274 9487	21.938 6985	53
54	6.408 8320	8.313 8143	10.771 5868	13.938 6961	23.255 0204	54
55	6.633 1411	8.646 3669	11.256 3082	14.635 6309	24.650 3216	55
56	6.865 3011	8.992 2216	11.762 8420	15.367 4125	26.129 3409	56
57	7.105 5866	9.351 9105	12.292 1699	16.135 7831	27.697 1013	57
58	7.354 2822	9.725 9869	12.845 3176	16.942 5722	29.358 9274	58
59	7.611 6820	10.115 0264	13.423 3569	17.789 7009	31.120 4631	59
60	7.878 0909	10.519 6274	14.027 4079	18.679 1859	32.987 6909	60
61	8.153 8241	10.940 4125	14.658 6413	19.613 1452	34.966 9523	61
62	8.439 2079	11.378 0290	15.318 2801	20.593 8025	37.064 9694	62
63	8.734 5802	11.833 1502	16.007 6028	21.623 4926	39.288 8676	63
64	9.040 2905	12.306 4762	16.727 9449	22.704 6672	41.646 1997	64
65	9.356 7007	12.798 7352	17.480 7024	23.839 9006	44.144 9717	65
66	9.684 1852	13.310 6846	18.267 3340	25.031 8956	46.793 6699	66
67	10.023 1317	13.843 1120	19.089 3640	26.283 4904	49.601 2901	67
68	10.373 9413	14.396 8365	19.948 3854	27.597 6649	52.577 3676	68
69	10.737 0292	14.972 7100	20.846 0628	28.977 5481	55.732 0096	69
70	11.112 8253	15.571 6184	21.784 1356	30.426 4255	59.075 9302	70
71	11.501 7741	16.194 4831	22.764 4217	31.947 7468	62.620 4860	71
72	11.904 3362	16.842 2624	23.788 8207	33.545 1342	66.377 7152	72
73	12.320 9880	17.515 9529	24.859 3176	35.222 3909	70.360 3781	73
74	12.752 2226	18.216 5910	25.977 9869	36.983 5104	74.582 0007	74
75	13.198 5504	18.945 2547	27.146 9963	38.832 6859	79.056 9208	75
76	13.660 4996	19.703 0648	28.368 6111	40.774 3202	83.800 3360	76
77	14.138 6171	20.491 1874	29.645 1986	42.813 0362	88.828 3562	77
78	14.633 4687	21.310 8349	30.979 2326	44.953 6880	94.158 0576	78
79	15.145 6401	22.163 2683	32.373 2980	47.201 3724	99.807 5410	79
80	15.675 7375	23.049 7991	33.830 0964	49.561 4411	105.795 9935	80
81	16.224 3884	23.971 7910	35.352 4508	52.039 5131	112.143 7531	81
82	16.792 2419	24.930 6627	36.943 3111	54.641 4888	118.872 3782	82
83	17.379 9704	25.927 8892	38.605 7601	57.373 5632	126.004 7210	83
84	17.988 2694	26.965 0047	40.343 0193	60.242 2414	133.565 0042	84
85	18.617 8588	28.043 6049	42.158 4551	63.254 3534	141.578 9045	85
86	19.269 4839	29.165 3491	44.055 5856	66.417 0711	150.073 6388	86
87	19.943 9158	30.331 9631	46.038 0870	69.737 9247	159.078 0571	87
88	20.641 9529	31.545 2416	48.109 8009	73.224 8209	168.622 7405	88
89	21.364 4212	32.807 0513	50.274 7419	76.886 0620	178.740 1049	89
90	22.112 1759	34.119 3333	52.537 1053	80.730 3651	189.464 5112	90
91	22.886 1021	35.484 1067	54.901 2750	84.766 8833	200.832 3819	91
92	23.687 1157	36.903 4709	57.371 8324	89.005 2275	212.882 3248	92
93	24.516 1647	38.379 6098	59.953 5649	93.455 4888	225.655 2643	93
94	25.374 2305	39.914 7942	62.651 4753	98.128 2633	239.194 5801	94
95	26.262 3286	41.511 3859	65.470 7917	103.034 6765	253.546 2550	95
96	27.181 5101	43.171 8414	68.416 9773	108.186 4103	268.759 0303	96
97	28.132 8629	44.898 7150	71.495 7413	113.595 7308	284.884 5721	97
98	29.117 5131	46.694 6636	74.713 0496	119.275 5173	301.977 6464	98
99	30.136 6261	48.562 4502	78.075 1369	125.239 2932	320.096 3052	99
100	31.191 4080	50.504 9482	81.588 5180	131.501 2579	339.302 0835	100

TABLE IV.—PRESENT VALUE OF 1

$$v^n = (1 + i)^{-n}$$

<i>n</i>	1 $\frac{1}{4}$ %	1 $\frac{1}{2}$ %	2%	2 $\frac{1}{2}$ %	3%	<i>n</i>
1	0.987 6543	0.985 2217	0.980 3922	0.975 6098	0.970 8738	1
2	0.975 4611	0.970 6617	0.961 1688	0.951 8144	0.942 5959	2
3	0.963 4183	0.956 3170	0.942 3223	0.928 5994	0.915 1417	3
4	0.951 5243	0.942 1842	0.923 8454	0.905 9506	0.888 4870	4
5	0.939 7771	0.928 2603	0.905 7308	0.883 8543	0.862 6088	5
6	0.928 1749	0.914 5422	0.887 9714	0.862 2969	0.837 4843	6
7	0.916 7159	0.901 0268	0.870 5602	0.841 2652	0.813 0915	7
8	0.905 3985	0.887 7111	0.853 4904	0.820 7466	0.789 4092	8
9	0.894 2207	0.874 5922	0.836 7553	0.800 7284	0.766 4167	9
10	0.883 1809	0.861 6672	0.820 3483	0.781 1984	0.744 0939	10
11	0.872 2775	0.848 9332	0.804 2630	0.762 1448	0.722 4213	11
12	0.861 5086	0.836 3874	0.788 4932	0.743 5559	0.701 3799	12
13	0.850 8727	0.824 0270	0.773 0325	0.725 4204	0.680 9513	13
14	0.840 3681	0.811 8493	0.757 8750	0.707 7272	0.661 1178	14
15	0.829 9932	0.799 8515	0.743 0147	0.690 4656	0.641 8619	15
16	0.819 7463	0.788 0310	0.728 4458	0.673 6249	0.623 1669	16
17	0.809 6260	0.776 3853	0.714 1626	0.657 1951	0.605 0164	17
18	0.799 6306	0.764 9116	0.700 1594	0.641 1659	0.587 3946	18
19	0.789 7587	0.756 6075	0.686 4308	0.625 5277	0.570 2860	19
20	0.780 0085	0.742 4704	0.672 9713	0.610 2709	0.553 6758	20
21	0.770 3788	0.731 4980	0.659 7758	0.595 3863	0.537 5493	21
22	0.760 8680	0.720 6876	0.646 8390	0.580 8647	0.521 8925	22
23	0.751 4745	0.710 0371	0.634 1559	0.566 6972	0.506 6917	23
24	0.742 1971	0.699 5439	0.621 7215	0.552 8754	0.491 9337	24
25	0.733 0341	0.689 2058	0.609 5309	0.539 3906	0.477 6056	25
26	0.723 9843	0.679 0205	0.597 5793	0.526 2347	0.463 6947	26
27	0.715 0463	0.668 9857	0.585 8620	0.513 3997	0.450 1891	27
28	0.706 2185	0.659 0993	0.574 3746	0.500 8778	0.437 0768	28
29	0.697 4998	0.649 3589	0.563 1123	0.488 6613	0.424 3464	29
30	0.688 8887	0.639 7624	0.552 0709	0.476 7427	0.411 9868	30
31	0.680 3839	0.630 3078	0.541 2460	0.465 1148	0.399 9871	31
32	0.671 9841	0.620 9929	0.530 6333	0.453 7705	0.388 3370	32
33	0.663 6880	0.611 8157	0.520 2287	0.442 7030	0.377 0262	33
34	0.655 4943	0.602 7741	0.510 0282	0.431 9053	0.366 0449	34
35	0.647 4018	0.593 8661	0.500 0276	0.421 3711	0.355 3834	35
36	0.639 4092	0.585 0897	0.490 2232	0.411 0937	0.345 0324	36
37	0.631 5152	0.576 4431	0.480 6109	0.401 0670	0.334 9829	37
38	0.623 7187	0.567 9242	0.471 1872	0.391 2849	0.325 2262	38
39	0.616 0185	0.559 5313	0.461 9482	0.381 7414	0.315 7535	39
40	0.608 4133	0.551 2623	0.452 8904	0.372 4306	0.306 5568	40
41	0.600 9021	0.543 1156	0.444 0102	0.363 3469	0.297 6280	41
42	0.593 4835	0.535 0893	0.435 3041	0.354 4848	0.288 9592	42
43	0.586 1566	0.527 1815	0.426 7688	0.345 8389	0.280 5429	43
44	0.578 9201	0.519 3907	0.418 4007	0.337 4038	0.272 3718	44
45	0.571 7729	0.511 7149	0.410 1968	0.329 1744	0.264 4386	45
46	0.564 7140	0.504 1527	0.402 1537	0.321 1458	0.256 7365	46
47	0.557 7422	0.496 7021	0.394 2684	0.313 3129	0.249 2588	47
48	0.550 8565	0.489 3617	0.386 5376	0.305 6712	0.241 9988	48
49	0.544 0558	0.482 1298	0.378 9584	0.298 2158	0.234 9503	49
50	0.537 3391	0.475 0047	0.371 5279	0.290 9422	0.228 1071	50

TABLE IV.—PRESENT VALUE OF 1

$$v^n = (1 + i)^{-n}$$

<i>n</i>	1 $\frac{1}{4}$ %	1 $\frac{1}{2}$ %	2%	2 $\frac{1}{2}$ %	3%	<i>n</i>
51	0.530 7052	0.467 9849	0.364 2430	0.283 8461	0.221 4632	51
52	0.524 1533	0.461 0689	0.357 1010	0.276 9230	0.215 0128	52
53	0.517 6823	0.454 2550	0.350 0990	0.270 1688	0.208 7503	53
54	0.511 2912	0.447 5419	0.343 2343	0.263 5793	0.202 6702	54
55	0.504 9789	0.440 9280	0.336 5042	0.257 1505	0.196 7672	55
56	0.498 7446	0.434 4118	0.329 9061	0.250 8785	0.191 0361	56
57	0.492 5873	0.427 9919	0.323 4374	0.244 7596	0.185 4719	57
58	0.486 5059	0.421 6669	0.317 0955	0.238 7898	0.180 0698	58
59	0.480 4997	0.415 4354	0.310 8779	0.232 9657	0.174 8251	59
60	0.474 5676	0.409 2960	0.304 7823	0.227 2836	0.169 7331	60
61	0.468 7087	0.403 2473	0.298 8061	0.221 7401	0.164 7894	61
62	0.462 9222	0.397 2879	0.292 9472	0.216 3318	0.159 9897	62
63	0.457 2071	0.391 4167	0.287 2031	0.211 0554	0.155 3298	63
64	0.451 5626	0.385 6322	0.281 5717	0.205 9077	0.150 8057	64
65	0.445 9878	0.379 9332	0.276 0507	0.200 8856	0.146 4133	65
66	0.440 4817	0.374 3184	0.270 6379	0.195 9859	0.142 1488	66
67	0.435 0437	0.368 7866	0.265 3313	0.191 2058	0.138 0085	67
68	0.429 6728	0.363 3366	0.260 1287	0.186 5422	0.133 9889	68
69	0.424 3682	0.357 9671	0.255 0282	0.181 9924	0.130 0863	69
70	0.419 1291	0.352 6769	0.250 0276	0.177 5536	0.126 2974	70
71	0.413 9546	0.347 4650	0.245 1251	0.173 2230	0.122 6188	71
72	0.408 8441	0.342 3300	0.240 3187	0.168 9980	0.119 0474	72
73	0.403 7966	0.337 2709	0.235 6066	0.164 8761	0.115 5800	73
74	0.398 8115	0.332 2866	0.230 9869	0.160 8548	0.112 2136	74
75	0.393 8879	0.327 3760	0.226 4577	0.156 9315	0.108 9452	75
76	0.389 0251	0.322 5379	0.222 0174	0.153 1039	0.105 7721	76
77	0.384 2223	0.317 7714	0.217 6641	0.149 3697	0.102 6913	77
78	0.379 4788	0.313 0752	0.213 3962	0.145 7265	0.099 7003	78
79	0.374 7939	0.308 4485	0.209 2119	0.142 1722	0.096 7964	79
80	0.370 1668	0.303 8901	0.205 1097	0.138 7046	0.093 9771	80
81	0.365 5968	0.299 3992	0.201 0880	0.135 3215	0.091 2399	81
82	0.361 0833	0.294 9745	0.197 1451	0.132 0210	0.088 5824	82
83	0.356 6255	0.290 6153	0.193 2795	0.128 8010	0.086 0024	83
84	0.352 2227	0.286 3205	0.189 4897	0.125 6595	0.083 4974	84
85	0.347 8743	0.282 0892	0.185 7742	0.122 5946	0.081 0655	85
86	0.343 5795	0.277 9204	0.182 1316	0.119 6045	0.078 7043	86
87	0.339 3378	0.273 8132	0.178 5604	0.116 6873	0.076 4120	87
88	0.335 1484	0.269 7667	0.175 0592	0.113 8413	0.074 1864	88
89	0.331 0108	0.265 7800	0.171 6266	0.111 0647	0.072 0256	89
90	0.326 9242	0.261 8522	0.168 2614	0.108 3558	0.069 9278	90
91	0.322 8881	0.257 9824	0.164 9622	0.105 7130	0.067 8911	91
92	0.318 9019	0.254 1699	0.161 7276	0.103 1346	0.065 9136	92
93	0.314 9648	0.250 4137	0.158 5565	0.100 6191	0.063 9938	93
94	0.311 0764	0.246 7130	0.155 4475	0.098 1650	0.062 1299	94
95	0.307 2359	0.243 0670	0.152 3995	0.095 7707	0.060 3203	95
96	0.303 4429	0.239 4749	0.149 4113	0.093 4349	0.058 5634	96
97	0.299 6967	0.235 9358	0.146 4817	0.091 1560	0.056 8577	97
98	0.295 9967	0.232 4491	0.143 6095	0.088 9326	0.055 2016	98
99	0.292 3424	0.229 0139	0.140 7936	0.086 7635	0.053 5938	99
100	0.288 7333	0.225 6294	0.138 0330	0.084 6474	0.052 0328	100

TABLE IV.—PRESENT VALUE OF 1

$$v^n = (1 + i)^{-n}$$

<i>n</i>	3½%	4%	4½%	5%	6%	<i>n</i>
1	0.966 1836	0.961 5385	0.956 9378	0.952 3810	0.943 3962	1
2	0.933 5107	0.924 5562	0.915 7300	0.907 0295	0.889 9964	2
3	0.901 9427	0.888 9964	0.876 2966	0.863 8376	0.839 6193	3
4	0.871 4422	0.854 8042	0.838 5613	0.822 7025	0.792 0937	4
5	0.841 9732	0.821 9271	0.802 4510	0.783 5262	0.747 2582	5
6	0.813 5006	0.790 3145	0.767 8957	0.746 2154	0.704 9605	6
7	0.785 9910	0.759 9178	0.734 8285	0.710 6813	0.665 0571	7
8	0.759 4116	0.730 6902	0.703 1851	0.676 8394	0.627 4124	8
9	0.733 7310	0.702 5867	0.672 9044	0.644 6089	0.591 8985	9
10	0.708 9188	0.675 5642	0.643 9277	0.613 9133	0.558 3948	10
11	0.684 9457	0.649 5809	0.616 1987	0.584 6793	0.526 7875	11
12	0.661 7833	0.624 5970	0.589 6639	0.556 8374	0.496 9694	12
13	0.639 4042	0.600 5741	0.564 2716	0.530 3214	0.468 8390	13
14	0.617 7818	0.577 4751	0.539 9729	0.505 0680	0.442 3010	14
15	0.596 8906	0.555 2645	0.516 7204	0.481 0171	0.417 2651	15
16	0.576 7059	0.533 9082	0.494 4693	0.458 1115	0.393 6463	16
17	0.557 2038	0.513 3732	0.473 1764	0.436 2967	0.371 3644	17
18	0.538 3611	0.493 6281	0.452 8004	0.415 5207	0.350 3438	18
19	0.520 1557	0.474 6424	0.433 3018	0.395 7340	0.330 5130	19
20	0.502 5659	0.456 3869	0.414 6429	0.376 8895	0.311 8047	20
21	0.485 5709	0.438 8336	0.396 7874	0.358 9424	0.294 1554	21
22	0.469 1506	0.421 9554	0.379 7009	0.341 8499	0.277 5051	22
23	0.453 2856	0.405 7263	0.363 3501	0.325 5713	0.261 7973	23
24	0.437 9571	0.390 1215	0.347 7035	0.310 0679	0.246 9785	24
25	0.423 1470	0.375 1168	0.332 7306	0.295 3028	0.232 9986	25
26	0.408 8377	0.360 6892	0.318 4025	0.281 2407	0.219 8100	26
27	0.395 0122	0.346 8166	0.304 6914	0.267 8483	0.207 3679	27
28	0.381 6543	0.333 4775	0.291 5707	0.255 0936	0.195 6301	28
29	0.368 7482	0.320 6514	0.279 0150	0.242 9463	0.184 5567	29
30	0.356 2784	0.308 3187	0.267 0000	0.231 3774	0.174 1101	30
31	0.344 2303	0.296 4603	0.255 5024	0.220 3595	0.164 2548	31
32	0.332 5897	0.285 0579	0.244 4999	0.209 8662	0.154 9574	32
33	0.321 3427	0.274 0942	0.233 9712	0.199 8725	0.146 1862	33
34	0.310 4761	0.263 5521	0.223 8959	0.190 3548	0.137 9115	34
35	0.299 9769	0.253 4155	0.214 2544	0.181 2903	0.130 1052	35
36	0.289 8327	0.243 6687	0.205 0282	0.172 6574	0.122 7408	36
37	0.280 0316	0.234 2968	0.196 1992	0.164 4356	0.115 7932	37
38	0.270 5619	0.225 2854	0.187 7504	0.156 6054	0.109 2389	38
39	0.261 4125	0.216 6206	0.179 6655	0.149 1480	0.103 0555	39
40	0.252 5725	0.208 2890	0.171 9287	0.142 0457	0.097 2222	40
41	0.244 0314	0.200 2779	0.164 5251	0.135 2816	0.091 7190	41
42	0.235 7791	0.192 5749	0.157 4403	0.128 8396	0.086 5274	42
43	0.227 8059	0.185 1682	0.150 6605	0.122 7044	0.081 6296	43
44	0.220 1023	0.178 0463	0.144 1728	0.116 8613	0.077 0091	44
45	0.212 6592	0.171 1984	0.137 9644	0.111 2965	0.072 6501	45
46	0.205 4679	0.164 6139	0.132 0233	0.105 9967	0.068 5378	46
47	0.198 5197	0.158 2826	0.126 3381	0.100 9492	0.064 6583	47
48	0.191 8065	0.152 1948	0.120 8977	0.096 1421	0.060 9984	48
49	0.185 3202	0.146 3411	0.115 6916	0.091 5639	0.057 5457	49
50	0.179 0534	0.140 7126	0.110 7096	0.087 2027	0.054 2884	50

TABLE IV.—PRESENT VALUE OF 1

$$v^n = (1 + i)^{-n}$$

<i>n</i>	3½%	4%	4½%	5%	6%	<i>n</i>
51	0.172 9984	0.135 3006	0.105 9422	0.083 0512	0.051 2154	51
52	0.167 1482	0.130 0967	0.101 3801	0.079 0964	0.048 3164	52
53	0.161 4959	0.125 0930	0.097 0145	0.075 3299	0.045 5816	53
54	0.156 0347	0.120 2817	0.092 8368	0.071 7427	0.043 0015	54
55	0.150 7581	0.115 6555	0.088 8391	0.068 3264	0.040 5674	55
56	0.145 6600	0.111 2072	0.085 0135	0.065 0728	0.038 2712	56
57	0.140 7343	0.106 9300	0.081 3526	0.061 9741	0.036 1049	57
58	0.135 9752	0.102 8173	0.077 8494	0.059 0229	0.034 0612	58
59	0.131 3770	0.098 8628	0.074 4970	0.056 2123	0.032 1332	59
60	0.126 9343	0.095 0604	0.071 2890	0.053 5355	0.030 3143	60
61	0.122 6418	0.091 4042	0.068 2191	0.050 9862	0.028 5984	61
62	0.118 4945	0.087 8887	0.065 2815	0.048 5583	0.026 9797	62
63	0.114 4875	0.084 5084	0.062 4703	0.046 2460	0.025 4525	63
64	0.110 6159	0.081 2580	0.059 7802	0.044 0438	0.024 0118	64
65	0.106 8753	0.078 1327	0.057 2059	0.041 9465	0.022 6526	65
66	0.103 2611	0.075 1276	0.054 7425	0.039 9490	0.021 3704	66
67	0.099 7692	0.072 2381	0.052 3852	0.038 0467	0.020 1608	67
68	0.096 3954	0.069 4597	0.050 1294	0.036 2349	0.019 0196	68
69	0.093 1356	0.066 7882	0.047 9707	0.034 5095	0.017 9430	69
70	0.089 9861	0.064 2194	0.045 9050	0.032 8662	0.016 9274	70
71	0.086 9431	0.061 7494	0.043 9282	0.031 3011	0.015 9692	71
72	0.084 0030	0.059 3744	0.042 0366	0.029 8106	0.015 0653	72
73	0.081 1623	0.057 0908	0.040 2264	0.028 3910	0.014 2125	73
74	0.078 4177	0.054 8950	0.038 4941	0.027 0391	0.013 4081	74
75	0.075 7659	0.052 7837	0.036 8365	0.025 7515	0.012 6491	75
76	0.073 2038	0.050 7535	0.035 2502	0.024 5252	0.011 9331	76
77	0.070 7283	0.048 8015	0.033 7323	0.023 3574	0.011 2577	77
78	0.068 3365	0.046 9245	0.032 2797	0.022 2451	0.010 6204	78
79	0.066 0256	0.045 1197	0.030 8897	0.021 1858	0.010 0193	79
80	0.063 7929	0.043 3843	0.029 5595	0.020 1770	0.009 4522	80
81	0.061 6356	0.041 7157	0.028 2866	0.019 2162	0.008 9171	81
82	0.059 5513	0.040 1112	0.027 0685	0.018 3011	0.008 4124	82
83	0.057 5375	0.038 5685	0.025 9029	0.017 4296	0.007 9362	83
84	0.055 5918	0.037 0851	0.024 7874	0.016 5996	0.007 4870	84
85	0.053 7119	0.035 6588	0.023 7200	0.015 8092	0.007 0632	85
86	0.051 8955	0.034 2873	0.022 6986	0.015 0564	0.006 6634	86
87	0.050 1406	0.032 9685	0.021 7211	0.014 3394	0.006 2862	87
88	0.048 4450	0.031 7005	0.020 7858	0.013 6566	0.005 9304	88
89	0.046 8068	0.030 4813	0.019 8907	0.013 0063	0.005 5947	89
90	0.045 2240	0.029 3089	0.019 0342	0.012 3869	0.005 2780	90
91	0.043 6946	0.028 1816	0.018 2145	0.011 7971	0.004 9793	91
92	0.042 2170	0.027 0977	0.017 4302	0.011 2353	0.004 6974	92
93	0.040 7894	0.026 0555	0.016 6796	0.010 7003	0.004 4315	93
94	0.039 4101	0.025 0534	0.015 9613	0.010 1907	0.004 1807	94
95	0.038 0774	0.024 0898	0.015 2740	0.009 7055	0.003 9441	95
96	0.036 7897	0.023 1632	0.014 6163	0.009 2433	0.003 7208	96
97	0.035 5456	0.022 2724	0.013 9868	0.008 8031	0.003 5102	97
98	0.034 3436	0.021 4157	0.013 3845	0.008 3840	0.003 3115	98
99	0.033 1822	0.020 5920	0.012 8082	0.007 9847	0.003 1241	99
100	0.032 0601	0.019 8000	0.012 2566	0.007 6045	0.002 9473	100

TABLE V.—THE AMOUNT OF 1 PER ANNUM

$$s_n = \frac{(1+i)^n - 1}{i}$$

<i>n</i>	1 $\frac{1}{4}$ %	1 $\frac{1}{2}$ %	2%	2 $\frac{1}{2}$ %	3%	<i>n</i>
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1
2	2.012 5000	2.015 0000	2.020 0000	2.025 0000	2.030 0000	2
3	3.037 6562	3.045 2250	3.060 4000	3.075 6250	3.090 9000	3
4	4.075 6269	4.090 9034	4.121 6080	4.152 5156	4.183 6270	4
5	5.126 5723	5.152 2669	5.204 0402	5.256 3285	5.309 1358	5
6	6.190 6544	6.229 5509	6.308 1210	6.387 7367	6.468 4099	6
7	7.268 0376	7.322 9942	7.434 2834	7.547 4301	7.662 4622	7
8	8.358 8881	8.432 8391	8.582 9691	8.736 1159	8.892 3360	8
9	9.463 3742	9.559 3317	9.754 6284	9.954 5188	10.159 1061	9
10	10.581 6664	10.702 7217	10.949 7210	11.203 3818	11.463 8793	10
11	11.713 9372	11.863 2625	12.168 7154	12.483 4663	12.807 7957	11
12	12.860 3614	13.041 2114	13.412 0897	13.795 5530	14.192 0296	12
13	14.021 1159	14.236 8296	14.680 3315	15.140 4418	15.617 7904	13
14	15.196 3799	15.450 3820	15.973 9382	16.518 9528	17.086 3242	14
15	16.386 3346	16.682 1378	17.293 4169	17.931 9267	18.598 9139	15
16	17.591 1638	17.932 3698	18.639 2853	19.380 2248	20.156 8813	16
17	18.811 0534	19.201 3554	20.012 0710	20.864 7304	21.761 5877	17
18	20.046 1915	20.489 3757	21.412 3124	22.386 3487	23.414 4354	18
19	21.296 7689	21.796 7164	22.840 5586	23.946 0074	25.116 8684	19
20	22.562 9785	23.123 6671	24.297 3698	25.544 6576	26.870 3745	20
21	23.845 0158	24.470 5221	25.783 3172	27.183 2741	28.676 4857	21
22	25.143 0785	25.837 5799	27.298 9835	28.862 8559	30.536 7803	22
23	26.457 3669	27.225 1436	28.844 9632	30.584 4273	32.452 8837	23
24	27.788 0840	28.633 5208	30.421 8625	32.349 0380	34.426 4702	24
25	29.135 4351	30.063 0236	32.030 2997	34.157 7639	36.459 2643	25
26	30.499 6280	31.513 9690	33.670 9057	36.011 7080	38.553 0423	26
27	31.880 8734	32.986 6785	35.344 3238	37.912 0007	40.709 6335	27
28	33.279 3843	34.481 4787	37.051 2103	39.859 8007	42.930 9225	28
29	34.695 3766	35.998 7009	38.792 2345	41.856 2958	45.218 8502	29
30	36.129 0688	37.538 6814	40.568 0792	43.902 7032	47.575 4157	30
31	37.580 6822	39.101 7616	42.379 4408	46.000 2707	50.002 6782	31
32	39.050 4407	40.688 2880	44.227 0296	48.150 2775	52.502 7585	32
33	40.538 5712	42.298 6123	46.111 5702	50.354 0344	55.077 8413	33
34	42.045 3033	43.933 0915	48.033 8016	52.612 8853	57.730 1765	34
35	43.570 8696	45.592 0879	49.994 4776	54.928 2074	60.462 0818	35
36	45.115 5055	47.275 9692	51.994 3672	57.301 4126	63.275 9443	36
37	46.679 4493	48.985 1087	54.034 2545	59.733 9479	66.174 2226	37
38	48.262 9424	50.719 8854	56.114 9396	62.227 2966	69.159 4493	38
39	49.866 2292	52.480 6837	58.237 2384	64.782 9791	72.234 2328	39
40	51.489 5571	54.267 8939	60.401 9832	67.402 5535	75.401 2597	40
41	53.133 1765	56.081 9123	62.610 0228	70.087 6174	78.663 2975	41
42	54.797 3412	57.923 1410	64.862 2233	72.839 8078	82.023 1965	42
43	56.482 3080	59.791 9881	67.159 4678	75.660 8030	85.483 8923	43
44	58.188 3369	61.688 8679	69.502 6571	78.552 3231	89.048 4091	44
45	59.915 6911	63.614 2010	71.892 7103	81.516 1312	92.719 8614	45
46	61.664 6372	65.568 4140	74.330 5645	84.554 0344	96.501 4572	46
47	63.435 4452	67.551 9402	76.817 1758	87.667 8853	100.396 5009	47
48	65.228 3884	69.565 2193	79.353 5193	90.859 5824	104.408 3960	48
49	67.043 7431	71.608 6976	81.940 5897	94.131 0720	108.540 6479	49
50	68.881 7899	73.682 8280	84.579 4015	97.484 3288	112.796 8673	50

TABLE V.—THE AMOUNT OF 1 PER ANNUM

$$s_n = \frac{(1+i)^n - 1}{i}$$

<i>n</i>	1 $\frac{1}{4}$ %	1 $\frac{1}{2}$ %	2%	2 $\frac{1}{2}$ %	3%	<i>n</i>
51	70.742 8123	75.788 0705	87.270 9895	100.921 4575	117.180 7733	51
52	72.627 0974	77.924 8915	90.016 4093	104.444 4939	121.696 1965	52
53	74.534 9361	80.093 7649	92.816 7375	108.055 6063	126.347 0824	53
54	76.466 6228	82.295 1714	95.673 0722	111.756 9965	131.137 4949	54
55	78.422 4556	84.529 5989	98.586 5337	115.550 9214	136.071 6197	55
56	80.402 7363	86.797 5429	101.558 2643	119.439 6944	141.153 7683	56
57	82.407 7705	89.099 5061	104.589 4296	123.425 6868	146.388 3814	57
58	84.437 8676	91.435 9987	107.681 2182	127.511 3289	151.780 0328	58
59	86.493 3410	93.807 5386	110.834 8426	131.699 1121	157.333 4338	59
60	88.574 5078	96.214 6517	114.051 5394	135.991 5900	163.053 4368	60
61	90.681 6891	98.657 8715	117.332 5702	140.391 3797	168.945 0399	61
62	92.815 2102	101.137 7396	120.679 2216	144.901 1642	175.013 3911	62
63	94.975 4003	103.654 8057	124.092 8060	149.523 6933	181.263 7928	63
64	97.162 5928	106.209 6277	127.574 6622	154.261 7856	187.701 7066	64
65	99.377 1253	108.802 7722	131.126 1554	159.118 3303	194.332 7578	65
66	101.619 3393	111.434 8137	134.748 6785	164.096 2885	201.162 7406	66
67	103.889 5811	114.106 3359	138.443 6521	169.198 6957	208.197 6228	67
68	106.188 2008	116.817 9310	142.212 5251	174.428 6631	215.443 5515	68
69	108.515 5533	119.570 1999	146.056 7756	179.789 3797	222.906 8580	69
70	110.871 9978	122.363 7529	149.977 9111	185.284 1142	230.594 0637	70
71	113.257 8977	125.199 2092	153.977 4694	190.916 2171	238.511 8856	71
72	115.673 6215	128.077 1974	158.057 0188	196.689 1225	246.667 2422	72
73	118.119 5417	130.998 3553	162.218 1591	202.606 3506	255.067 2595	73
74	120.596 0360	133.963 3307	166.462 5223	208.671 5093	263.719 2773	74
75	123.103 4864	136.972 7806	170.791 7728	214.888 2970	272.630 8556	75
76	125.642 2800	140.027 3723	175.207 6082	221.260 5045	281.809 7813	76
77	128.212 8085	143.127 7829	179.711 7604	227.792 0171	291.264 0747	77
78	130.815 4686	146.274 6997	184.305 9956	234.486 8175	301.001 9969	78
79	133.450 6620	149.468 8202	188.992 1155	241.348 9879	311.032 0568	79
80	136.118 7953	152.710 8525	193.771 9578	248.382 7126	321.363 0185	80
81	138.820 2802	156.001 5153	198.647 3970	255.592 2805	332.003 9091	81
82	141.555 5337	159.341 5380	203.620 3449	262.982 0875	342.964 0264	82
83	144.324 9779	162.731 6611	208.692 7518	270.556 6397	354.252 9472	83
84	147.129 0401	166.172 6360	213.866 6068	278.320 5557	365.880 5356	84
85	149.968 1531	169.665 2255	219.143 9390	286.278 5695	377.856 9517	85
86	152.842 7550	173.210 2039	224.526 8177	294.435 5338	390.192 6602	86
87	155.753 2895	176.808 3569	230.017 3541	302.796 4221	402.898 4400	87
88	158.700 2056	180.460 4823	235.617 7012	311.366 3327	415.985 3932	88
89	161.683 9581	184.167 3895	241.330 0552	320.150 4910	429.464 9550	89
90	164.705 0076	187.929 9004	247.156 6563	329.154 2533	443.348 9037	90
91	167.763 8202	191.748 8489	253.099 7894	338.383 1096	457.649 3708	91
92	170.860 8680	195.625 0816	259.161 7852	347.842 6873	472.378 8519	92
93	173.996 6288	199.559 4578	265.345 0209	357.538 7545	487.550 2174	93
94	177.171 5867	203.552 8497	271.651 9214	367.477 2234	503.176 7240	94
95	180.386 2315	207.606 1425	278.084 9598	377.664 1540	519.272 0257	95
96	183.641 0594	211.720 2346	284.646 6590	388.105 7578	535.850 1865	96
97	186.936 5726	215.896 0381	291.339 5922	398.808 4018	552.925 6920	97
98	190.273 2798	220.134 4787	298.166 3840	409.778 6118	570.513 4628	98
99	193.651 6958	224.436 4959	305.129 7117	421.023 0771	588.628 8667	99
100	197.072 3420	228.803 0433	312.232 3059	432.548 6540	607.287 7327	100

TABLE V.—THE AMOUNT OF 1 PER ANNUM

$$s_n = \frac{(1+i)^n - 1}{i}$$

<i>n</i>	3½ %	4 %	4½ %	5 %	6 %	<i>n</i>
1	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1.000 0000	1
2	2.035 0000	2.040 0000	2.045 0000	2.050 0000	2.060 0000	2
3	3.106 2250	3.121 6000	3.137 0250	3.152 5000	3.183 6000	3
4	4.214 9429	4.246 4640	4.278 1911	4.310 1250	4.374 6160	4
5	5.362 4659	5.416 3226	5.470 7097	5.525 6312	5.637 0930	5
6	6.550 1522	6.632 9755	6.716 8917	6.801 9128	6.975 3185	6
7	7.779 4075	7.898 2945	8.019 1518	8.142 0085	8.393 8376	7
8	9.051 6868	9.214 2263	9.380 0136	9.549 1089	9.897 4679	8
9	10.368 4958	10.582 7953	10.802 1142	11.026 5643	11.491 3160	9
10	11.731 3932	12.006 1071	12.288 2094	12.577 8925	13.180 7949	10
11	13.141 9919	13.486 3514	13.841 1788	14.206 7872	14.971 6426	11
12	14.601 9616	15.025 8055	15.464 0318	15.917 1265	16.869 9412	12
13	16.113 0303	16.626 8377	17.159 9132	17.712 9828	18.882 1377	13
14	17.676 9864	18.291 9112	18.932 0194	19.598 6320	21.015 0659	14
15	19.295 6809	20.023 5876	20.784 0543	21.578 5636	23.275 9699	15
16	20.971 0297	21.824 5311	22.719 3367	23.657 4918	25.672 5281	16
17	22.705 0157	23.697 5124	24.741 7069	25.840 3664	28.212 8798	17
18	24.499 6913	25.645 4129	26.855 0837	28.132 3847	30.905 6525	18
19	26.357 1805	27.671 2294	29.063 5625	30.539 0039	33.759 9917	19
20	28.279 6818	29.778 0786	31.371 4228	33.065 9541	36.785 5912	20
21	30.269 4707	31.969 2017	33.783 1368	35.719 2518	39.992 7267	21
22	32.328 9021	34.247 9698	36.303 3780	38.505 2144	43.392 2903	22
23	34.460 4137	36.617 8886	38.937 0300	41.430 4751	46.995 8277	23
24	36.666 5282	39.082 6041	41.689 1963	44.501 9989	50.815 5774	24
25	38.949 8567	41.645 9083	44.565 2101	47.727 0988	54.864 5120	25
26	41.313 1017	44.311 7446	47.570 6446	51.113 4538	59.156 3827	26
27	43.759 0602	47.084 2144	50.711 3236	54.669 1264	63.705 7657	27
28	46.290 6273	49.967 5830	53.993 3332	58.402 5828	68.528 1116	28
29	48.910 7993	52.966 2863	57.423 0332	62.322 7119	73.639 7983	29
30	51.622 6773	56.084 9377	61.007 0697	66.438 8475	79.058 1862	30
31	54.429 4710	59.328 3353	64.752 3878	70.760 7899	84.801 6774	31
32	57.334 5025	62.701 4687	68.666 2452	75.298 8294	90.889 7780	32
33	60.341 2101	66.209 5274	72.756 2263	80.063 7708	97.343 1647	33
34	63.453 1524	69.857 9085	77.030 2565	85.066 9594	104.183 7546	34
35	66.674 0127	73.652 2249	81.496 6180	90.320 3074	111.434 7799	35
36	70.007 6032	77.598 3138	86.163 9658	95.836 3227	119.120 8667	36
37	73.467 8693	81.702 2464	91.041 3443	101.628 1389	127.268 1187	37
38	77.028 8947	85.970 3363	96.138 2048	107.709 5458	135.904 2058	38
39	80.724 9060	90.409 1497	101.464 4240	114.095 0231	145.058 4581	39
40	84.550 2777	95.025 5157	107.030 3231	120.799 7742	154.761 9656	40
41	88.509 5375	99.826 5363	112.846 6876	127.839 7630	165.047 6836	41
42	92.607 3713	104.819 5978	118.924 7885	135.231 7511	175.950 5446	42
43	96.848 6293	110.012 3817	125.276 4040	142.993 3387	187.507 5772	43
44	101.238 3313	115.412 8770	131.913 8422	151.143 0056	199.758 0319	44
45	105.781 6729	121.029 3920	138.849 9651	159.700 1559	212.743 5138	45
46	110.484 0314	126.870 5677	146.098 2135	168.685 1637	226.508 1246	46
47	115.350 9725	132.945 3904	153.672 6331	178.119 4218	241.098 6121	47
48	120.388 2566	139.263 2060	161.587 9016	188.025 3929	256.564 5288	48
49	125.601 8456	145.833 7343	169.859 3572	198.426 6626	272.958 4005	49
50	130.997 9102	152.667 0837	178.503 0283	209.347 9957	290.335 9046	50

TABLE V.—THE AMOUNT OF 1 PER ANNUM

$$s_n = \frac{(1+i)^n - 1}{i}$$

<i>n</i>	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	<i>n</i>
51	136.582 8370	159.773 7670	187.535 6646	220.815 3955	308.756 0589	51
52	142.363 2363	167.164 7177	196.974 7695	232.856 1653	328.281 4224	52
53	148.345 9496	174.851 3064	206.838 6341	245.498 9735	348.978 3077	53
54	154.538 0578	182.845 3586	217.146 3726	258.773 9222	370.917 0062	54
55	160.946 8898	191.159 1730	227.917 9594	272.712 6183	394.172 0266	55
56	166.580 0310	199.805 5399	239.174 2676	287.348 2492	418.882 3482	56
57	174.445 3321	208.797 7615	250.937 1096	302.715 6617	444.951 6890	57
58	181.550 9187	218.149 6720	263.229 2795	318.851 4448	472.648 7904	58
59	188.905 2008	227.875 6588	276.074 5971	335.794 0170	502.007 7178	59
60	196.516 8829	237.990 6852	289.497 9540	353.583 7179	533.128 1809	60
61	204.394 9738	248.510 3126	303.525 3619	372.262 9038	566.115 8717	61
62	212.548 7979	259.450 7251	318.184 0032	391.876 0490	601.082 8240	62
63	220.988 0058	270.828 7541	333.502 2833	412.469 8514	638.147 7935	63
64	229.722 5860	282.661 9043	349.509 8861	434.093 3440	677.436 6611	64
65	238.762 8765	294.968 3805	366.237 8310	456.798 0112	719.082 8608	65
66	248.119 5772	307.767 1157	383.718 5333	480.637 9117	763.227 8324	66
67	257.803 7624	321.077 8003	401.985 8673	505.669 8073	810.021 5024	67
68	267.826 8941	334.920 9123	421.075 2314	531.953 2977	859.622 7925	68
69	278.200 8354	349.317 7488	441.023 6168	559.550 9626	912.200 1600	69
70	288.937 8646	364.290 4588	461.869 6795	588.528 5107	967.932 1696	70
71	300.050 6899	379.862 0771	483.653 8151	618.954 9362	1027.008 0998	71
72	311.552 4640	396.056 5602	506.418 2368	650.902 6831	1089.628 5858	72
73	323.456 8002	412.898 8226	530.207 0575	684.447 8172	1156.006 3010	73
74	335.777 7882	430.414 7755	555.066 3751	719.670 2081	1226.366 6790	74
75	348.530 0108	448.631 3665	581.044 3619	756.653 7185	1300.948 6798	75
76	361.728 5612	467.576 6212	608.191 3582	795.486 4044	1380.005 6006	76
77	375.389 0609	487.279 6860	636.559 9694	836.260 7246	1463.805 9366	77
78	389.527 6780	507.770 8735	666.205 1680	879.073 7608	1552.634 2928	78
79	404.161 1467	529.081 7084	697.184 4005	924.027 4489	1646.792 3503	79
80	419.306 7868	551.244 9767	729.557 6985	971.228 8213	1746.599 8914	80
81	434.982 5244	574.294 7758	763.387 7950	1020.790 2624	1852.395 8849	81
82	451.206 9127	598.266 5668	798.740 2457	1072.829 7755	1964.539 6379	82
83	467.999 1547	623.197 2295	835.683 5568	1127.471 2643	2083.412 0162	83
84	485.379 1251	649.125 1187	874.289 3169	1184.844 8275	2209.416 7372	84
85	503.367 3945	676.090 1235	914.632 3361	1245.087 0689	2342.981 7414	85
86	521.985 2533	704.133 7284	956.790 7912	1308.341 4223	2484.560 6459	86
87	541.254 7372	733.299 0775	1000.846 3769	1374.758 4935	2634.634 2847	87
88	561.198 6530	763.631 0406	1046.884 4638	1444.496 4181	2793.712 3417	88
89	581.840 6058	795.176 2823	1094.994 2647	1517.721 2390	2962.335 0822	89
90	603.205 0270	827.983 3335	1145.269 0066	1594.607 3010	3141.075 1872	90
91	625.317 2030	862.102 6669	1197.806 1119	1675.337 6660	3330.539 6984	91
92	648.203 3051	897.586 7736	1252.707 3869	1760.104 5493	3531.372 0803	92
93	671.890 4207	934.490 2445	1310.079 2193	1849.109 7768	3744.254 4051	93
94	696.406 5855	972.869 8543	1370.032 7842	1942.565 2656	3969.909 6694	94
95	721.780 8160	1012.784 6485	1432.684 2595	2040.693 5289	4209.104 2496	95
96	748.043 1445	1054.296 0344	1498.155 0512	2143.728 2054	4462.650 5046	96
97	775.224 6546	1097.467 8758	1566.572 0285	2251.914 6156	4731.409 5348	97
98	803.357 5175	1142.366 5908	1638.067 7698	2365.510 3464	5016.294 1070	98
99	832.475 0306	1189.061 2544	1712.780 8194	2484.785 8637	5318.271 7534	99
100	862.611 6567	1237.623 7046	1790.855 9563	2610.025 1569	5638.368 0586	100

TABLE VI.—THE PRESENT VALUE OF 1 PER ANNUM

$$a_n = \frac{1 - v^n}{i}$$

<i>n</i>	1¼ %	1½ %	2 %	2½ %	3 %	<i>n</i>
1	0.987 6543	0.985 2217	0.980 3922	0.975 6098	0.970 8738	1
2	1.963 1154	1.955 8834	1.941 5609	1.927 4242	1.913 4697	2
3	2.926 5337	2.912 2004	2.883 8833	2.856 0236	2.828 6114	3
4	3.878 0580	3.854 3846	3.807 7287	3.761 9742	3.717 0984	4
5	4.817 8350	4.782 6450	4.713 4595	4.645 8285	4.579 7072	5
6	5.746 0099	5.697 1872	5.601 4309	5.508 1254	5.417 1914	6
7	6.662 7258	6.598 2140	6.471 9911	6.349 3906	6.230 2830	7
8	7.568 1243	7.485 9251	7.325 4814	7.170 1372	7.019 6922	8
9	8.462 3450	8.360 5173	8.162 2367	7.970 8655	7.786 1089	9
10	9.345 5259	9.222 1846	8.982 5850	8.752 0639	8.530 2028	10
11	10.217 8034	10.071 1178	9.786 8480	9.514 2087	9.252 6241	11
12	11.079 3120	10.907 5052	10.575 3412	10.257 7646	9.954 0040	12
13	11.930 1847	11.731 5322	11.348 3737	10.983 1850	10.634 9553	13
14	12.770 5527	12.543 3815	12.106 2488	11.690 9122	11.296 0731	14
15	13.600 5459	13.343 2330	12.849 2635	12.381 3777	11.937 9351	15
16	14.420 2923	14.131 2641	13.577 7093	13.055 0027	12.561 1020	16
17	15.229 9183	14.907 6493	14.291 8719	13.712 1977	13.166 1185	17
18	16.029 5489	15.672 5609	14.992 0312	14.353 3636	13.753 5131	18
19	16.819 3076	16.426 1684	15.678 4620	14.978 8913	14.323 7991	19
20	17.599 3161	17.168 6388	16.351 4333	15.589 1623	14.877 4749	20
21	18.369 6949	17.900 1367	17.011 2092	16.184 5486	15.415 0241	21
22	19.130 5629	18.620 8244	17.658 0482	16.765 4132	15.936 9166	22
23	19.882 0374	19.330 8614	18.292 2041	17.332 1105	16.443 6084	23
24	20.624 2345	20.030 4054	18.913 9256	17.884 9858	16.935 5421	24
25	21.357 2686	20.719 6112	19.523 4565	18.424 3764	17.413 1477	25
26	22.081 2530	21.398 6317	20.121 0358	18.950 6111	17.876 8424	26
27	22.796 2993	22.067 6175	20.706 8978	19.464 0109	18.327 0315	27
28	23.502 5178	22.726 7167	21.281 2724	19.964 8887	18.764 1082	28
29	24.200 0176	23.376 0756	21.844 3847	20.453 5499	19.188 4546	29
30	24.888 9062	24.015 8380	22.396 4556	20.930 2926	19.600 4413	30
31	25.569 2901	24.646 1458	22.937 7015	21.395 4074	20.000 4285	31
32	26.241 2742	25.267 1387	23.468 3348	21.849 1780	20.388 7655	32
33	26.904 9621	25.878 9544	23.988 5636	22.291 8809	20.765 7918	33
34	27.560 4564	26.481 7285	24.498 5917	22.723 7863	21.131 8367	34
35	28.207 8582	27.075 5946	24.998 6193	23.145 1573	21.487 2201	35
36	28.847 2674	27.660 6843	25.488 8425	23.556 2511	21.832 2525	36
37	29.478 7826	28.237 1274	25.969 4534	23.957 3181	22.167 2354	37
38	30.102 5013	28.805 0516	26.440 6406	24.348 6030	22.492 4616	38
39	30.718 5198	29.364 5829	26.902 5888	24.730 3444	22.808 2151	39
40	31.326 9332	29.915 8452	27.355 4792	25.102 7750	23.114 7720	40
41	31.927 8352	30.458 9608	27.799 4895	25.466 1220	23.412 4000	41
42	32.521 3187	30.994 0500	28.234 7936	25.820 6068	23.701 3592	42
43	33.107 4753	31.521 2316	28.661 5623	26.166 4457	23.981 9021	43
44	33.686 3954	32.040 6222	29.079 9631	26.503 8495	24.254 2739	44
45	34.258 1682	32.552 3372	29.490 1599	26.833 0239	24.518 7125	45
46	34.822 8822	33.056 4898	29.892 3136	27.154 1696	24.775 4491	46
47	35.380 6244	33.553 1919	30.286 5820	27.467 4826	25.024 7078	47
48	35.931 4809	34.042 5536	30.673 1196	27.773 1537	25.266 7066	48
49	36.475 5367	34.524 6834	31.052 0780	28.071 3695	25.501 6569	49
50	37.012 8757	34.999 6881	31.423 6059	28.362 3117	25.729 7640	50

TABLE VI.—THE PRESENT VALUE OF 1 PER ANNUM

$$a_n = \frac{1 - v^n}{i}$$

<i>n</i>	1¼%		1½%		2%		2½%		3%		<i>n</i>
51	37.543	5810	35.467	6730	31.787	8489	28.646	1577	25.951	2272	51
52	38.067	7343	35.928	7418	32.144	9499	28.923	0807	26.166	2400	52
53	38.585	4166	36.382	9969	32.495	0489	29.193	2495	26.374	9903	53
54	39.096	7078	36.830	5388	32.838	2833	29.456	8288	26.577	6605	54
55	39.601	6867	37.271	4668	33.174	7875	29.713	9793	26.774	4276	55
56	40.100	4313	37.705	8786	33.504	6936	29.964	8578	26.965	4637	56
57	40.593	0185	38.133	8706	33.828	1310	30.209	6174	27.150	9357	57
58	41.079	5245	38.555	5375	34.145	2265	30.448	4072	27.331	0055	58
59	41.560	0242	38.970	9729	34.456	1044	30.681	3729	27.505	8306	59
60	42.034	5918	39.380	2689	34.760	8867	30.908	6565	27.675	5637	60
61	42.503	3005	39.783	5161	35.059	6928	31.130	3966	27.840	3531	61
62	42.966	2227	40.180	8041	35.352	6400	31.346	7284	28.000	3428	62
63	43.423	4299	40.572	2208	35.639	8432	31.557	7838	28.155	6726	63
64	43.874	9925	40.957	8530	35.921	4149	31.763	6915	28.306	4783	64
65	44.320	9802	41.337	7862	36.197	4655	31.964	5771	28.452	8915	65
66	44.761	4619	41.712	1046	36.468	1035	32.160	5630	28.595	0403	66
67	45.196	5056	42.080	8912	36.733	4348	32.351	7688	28.733	0488	67
68	45.626	1784	42.444	2278	36.993	5635	32.538	3110	28.867	0377	68
69	46.050	5466	42.802	1949	37.248	5917	32.720	3034	28.997	1240	69
70	46.469	6756	43.154	8718	37.498	6193	32.897	8570	29.123	4214	70
71	46.883	6302	43.502	3368	37.743	7444	33.071	0800	29.246	0401	71
72	47.292	4743	43.844	6668	37.984	0631	33.240	0780	29.365	0875	72
73	47.696	2709	44.181	9377	38.219	6697	33.404	9542	29.480	6675	73
74	48.095	0824	44.514	2243	38.450	6566	33.565	8089	29.592	8811	74
75	48.488	9703	44.841	6003	38.677	1143	33.722	7404	29.701	8263	75
76	48.877	9953	45.164	1383	38.899	1317	33.875	8443	29.807	5983	76
77	49.262	2176	45.481	9096	39.116	7958	34.025	2140	29.910	2896	77
78	49.641	6964	45.794	9848	39.330	1919	34.170	9405	30.009	9899	78
79	50.016	4903	46.103	4333	39.539	4039	34.313	1127	30.106	7863	79
80	50.386	6571	46.407	3235	39.744	5136	34.451	8172	30.200	7634	80
81	50.752	2539	46.706	7226	39.945	6016	34.587	1388	30.292	0033	81
82	51.113	3372	47.001	6972	40.142	7466	34.719	1598	30.380	5858	82
83	51.469	9626	47.292	3125	40.336	0261	34.847	9607	30.466	5881	83
84	51.822	1853	47.578	6330	40.525	5158	34.973	6202	30.550	0856	84
85	52.170	0596	47.860	7222	40.711	2900	35.096	2149	30.631	1510	85
86	52.513	6391	48.138	6425	40.893	4216	35.215	8194	30.709	8554	86
87	52.852	9769	48.412	4557	41.071	9819	35.332	5067	30.786	2673	87
88	53.188	1253	48.682	2224	41.247	0411	35.446	3480	30.860	4537	88
89	53.519	1361	48.948	0023	41.418	6677	35.557	4127	30.932	4794	89
90	53.846	0603	49.209	8545	41.586	9292	35.665	7685	31.002	4071	90
91	54.168	9485	49.467	8370	41.751	8913	35.771	4814	31.070	2982	91
92	54.487	8504	49.722	0069	41.913	6190	35.874	6160	31.136	2118	92
93	54.802	8152	49.972	4205	42.072	1754	35.975	2352	31.200	2057	93
94	55.113	8915	50.219	1335	42.227	6230	36.073	4002	31.262	3356	94
95	55.421	1274	50.462	2005	42.380	0225	36.169	1709	31.322	6559	95
96	55.724	5703	50.701	6754	42.529	4339	36.262	6057	31.381	2193	96
97	56.024	2670	50.937	6112	42.675	9155	36.353	7617	31.438	0770	97
98	56.320	2637	51.170	0603	42.819	5250	36.442	6943	31.493	2787	98
99	56.612	6061	51.399	0742	42.960	3187	36.529	4579	31.546	8725	99
100	56.901	3394	51.624	7037	43.098	3516	36.614	1053	31.598	9053	100

TABLE VI.—THE PRESENT VALUE OF 1 PER ANNUM

$$a_n = \frac{1 - v^n}{i}$$

<i>n</i>	$3\frac{1}{2}\%$	4%	$4\frac{1}{2}\%$	5%	6%	<i>n</i>
1	0.966 1836	0.961 5385	0.956 9378	0.952 3810	0.943 3962	1
2	1.899 6943	1.886 0947	1.872 6678	1.859 4104	1.833 3927	2
3	2.801 6370	2.775 0910	2.748 9644	2.723 2480	2.673 0119	3
4	3.673 0792	3.629 8952	3.587 5257	3.545 9505	3.465 1056	4
5	4.515 0524	4.451 8223	4.389 9767	4.329 4767	4.212 3638	5
6	5.328 5530	5.242 1369	5.157 8725	5.075 6921	4.917 3243	6
7	6.114 5440	6.002 0547	5.892 7009	5.786 3734	5.582 3814	7
8	6.873 9555	6.732 7449	6.595 8861	6.463 2128	6.209 7938	8
9	7.607 6865	7.435 3316	7.268 7905	7.107 8217	6.801 6923	9
10	8.316 6053	8.110 8958	7.912 7182	7.721 7349	7.360 0871	10
11	9.001 5510	8.760 4767	8.528 9169	8.306 4142	7.886 8746	11
12	9.663 3343	9.385 0738	9.118 5808	8.863 2516	8.383 8439	12
13	10.302 7385	9.985 6478	9.682 8524	9.393 5730	8.852 6830	13
14	10.920 5203	10.563 1229	10.222 8253	9.898 6409	9.294 9839	14
15	11.517 4109	11.118 3874	10.739 5457	10.379 6580	9.712 2490	15
16	12.094 1168	11.652 2956	11.234 0150	10.837 7696	10.105 8953	16
17	12.651 3206	12.165 6689	11.707 1914	11.274 0662	10.477 2597	17
18	13.189 6817	12.659 2970	12.159 9918	11.689 5869	10.827 6035	18
19	13.709 8374	13.133 9394	12.593 2936	12.085 3209	11.158 1165	19
20	14.212 4033	13.590 3263	13.007 9365	12.462 2103	11.469 9212	20
21	14.697 9742	14.029 1599	13.404 7239	12.821 1527	11.764 0766	21
22	15.167 1248	14.451 1153	13.784 4248	13.163 0026	12.041 5817	22
23	15.620 4105	14.856 8417	14.147 7749	13.488 5739	12.303 3790	23
24	16.058 3676	15.246 9631	14.495 4784	13.798 6418	12.550 3575	24
25	16.481 5146	15.622 0799	14.828 2090	14.093 9446	12.783 3562	25
26	16.890 3523	15.982 7692	15.146 6114	14.375 1853	13.003 1662	26
27	17.285 3645	16.329 5857	15.451 3028	14.643 0336	13.210 5341	27
28	17.667 0188	16.663 0632	15.742 8735	14.898 1273	13.406 1643	28
29	18.035 7670	16.983 7146	16.021 8885	15.141 0736	13.590 7210	29
30	18.392 0454	17.292 0333	16.288 8885	15.372 4510	13.764 8312	30
31	18.736 2758	17.588 4936	16.544 3910	15.592 8105	13.929 0860	31
32	19.068 8655	17.873 5515	16.788 8909	15.802 6767	14.084 0434	32
33	19.390 2082	18.147 6457	17.022 8621	16.002 5492	14.230 2296	33
34	19.700 6842	18.411 1978	17.246 7580	16.192 9040	14.368 1411	34
35	20.000 6611	18.664 6132	17.461 0124	16.374 1943	14.498 2464	35
36	20.290 4938	18.908 2820	17.666 0406	16.546 8517	14.620 9871	36
37	20.570 5254	19.142 5788	17.862 2398	16.711 2873	14.736 7803	37
38	20.841 0874	19.367 8642	18.049 9902	16.867 8927	14.846 0192	38
39	21.102 4999	19.584 4848	18.229 6557	17.017 0407	14.949 0747	39
40	21.355 0723	19.792 7739	18.401 5844	17.159 0864	15.046 2969	40
41	21.599 1037	19.993 0518	18.566 1095	17.294 3680	15.138 0159	41
42	21.834 8828	20.185 6267	18.723 5498	17.423 2076	15.224 5433	42
43	22.062 6887	20.370 7949	18.874 2103	17.545 9120	15.306 1729	43
44	22.282 7910	20.548 8413	19.018 3831	17.662 7733	15.383 1820	44
45	22.495 4503	20.720 0397	19.156 3474	17.774 0698	15.455 8321	45
46	22.700 9181	20.884 6536	19.288 3707	17.880 0665	15.524 3699	46
47	22.899 4378	21.042 9361	19.414 7088	17.981 0157	15.589 0282	47
48	23.091 2443	21.195 1309	19.535 6065	18.077 1578	15.650 0266	48
49	23.276 5645	21.341 4720	19.651 2981	18.168 7217	15.707 5723	49
50	23.455 6179	21.482 1846	19.762 0078	18.255 9255	15.761 8606	50

TABLE VI.—THE PRESENT VALUE OF 1 PER ANNUM

$$a_n = \frac{1 - v^n}{i}$$

<i>n</i>	3½%	4%	4½%	5%	6%	<i>n</i>
51	23.628 6163	21.617 4852	19.867 9500	18.338 9766	15.813 0761	51
52	23.795 7645	21.747 5819	19.969 3302	18.418 0730	15.861 3925	52
53	23.957 2604	21.872 6749	20.066 3447	18.493 4028	15.906 9741	53
54	24.113 2951	21.992 9567	20.159 1815	18.565 1456	15.949 9755	54
55	24.264 0532	22.108 6122	20.248 0206	18.633 4720	15.990 5430	55
56	24.409 7133	22.219 8194	20.333 0340	18.698 5447	16.028 8141	56
57	24.550 4476	22.326 7494	20.414 3866	18.760 5188	16.064 9190	57
58	24.686 4228	22.429 5668	20.492 2360	18.819 5417	16.098 9802	58
59	24.817 7998	22.528 4296	20.566 7330	18.875 7540	16.131 1134	59
60	24.944 7341	22.623 4900	20.638 0220	18.929 2895	16.161 4277	60
61	25.067 3760	22.714 8942	20.706 2412	18.980 2757	16.190 0261	61
62	25.185 8705	22.802 7829	20.771 5227	19.028 8340	16.217 0058	62
63	25.300 3580	22.887 2912	20.833 9930	19.075 0800	16.242 4583	63
64	25.410 9739	22.968 5493	20.893 7732	19.119 1238	16.266 4701	64
65	25.517 8492	23.046 6820	20.950 9791	19.161 0703	16.289 1227	65
66	25.621 1103	23.121 8096	21.005 7217	19.201 0194	16.310 4931	66
67	25.720 8795	23.194 0477	21.058 1068	19.239 0661	16.330 6539	67
68	25.817 2749	23.263 5074	21.108 2362	19.275 3010	16.349 6735	68
69	25.910 4105	23.330 2956	21.156 2069	19.309 8105	16.367 6165	69
70	26.000 3966	23.394 5150	21.202 1119	19.342 6766	16.384 5439	70
71	26.087 3398	23.456 2644	21.246 0401	19.373 9778	16.400 5131	71
72	26.171 3428	23.515 6388	21.288 0766	19.403 7883	16.415 5784	72
73	26.252 5051	23.572 7297	21.328 3030	19.432 1794	16.429 7909	73
74	26.330 9228	23.627 6247	21.366 7971	19.459 2185	16.443 1990	74
75	26.406 6887	23.680 4083	21.403 6336	19.484 9700	16.455 8481	75
76	26.479 8924	23.731 1619	21.438 8838	19.509 4952	16.467 7812	76
77	26.550 6207	23.779 9633	21.472 6161	19.532 8526	16.479 0389	77
78	26.618 9572	23.826 8878	21.504 8958	19.555 0977	16.489 6593	78
79	26.684 9828	23.872 0075	21.535 7854	19.576 2835	16.499 6786	79
80	26.748 7757	23.915 3918	21.565 3449	19.596 4605	16.509 1308	80
81	26.810 4113	23.957 1075	21.593 6315	19.615 6767	16.518 0479	81
82	26.869 9626	23.997 2188	21.620 7000	19.633 9778	16.526 4603	82
83	26.927 5001	24.035 7873	21.646 6029	19.651 4074	16.534 3965	83
84	26.983 0919	24.072 8724	21.671 3903	19.668 0070	16.541 8835	84
85	27.036 8037	24.108 5312	21.695 1103	19.683 8162	16.548 9467	85
86	27.088 6993	24.142 8184	21.717 8089	19.698 8726	16.555 6101	86
87	27.138 8399	24.175 7869	21.739 5301	19.713 2120	16.561 8963	87
88	27.187 2849	24.207 4874	21.760 3159	19.726 8686	16.567 8267	88
89	27.234 0917	24.237 9687	21.780 2066	19.739 8748	16.573 4214	89
90	27.279 3156	24.267 2776	21.799 2407	19.752 2617	16.578 6994	90
91	27.323 0103	24.295 4592	21.817 4553	19.764 0588	16.583 6787	91
92	27.365 2273	24.322 5569	21.834 8854	19.775 2941	16.588 3762	92
93	27.406 0167	24.348 6124	21.851 5650	19.785 9944	16.592 8077	93
94	27.445 4268	24.373 6658	21.867 5263	19.796 1851	16.596 9884	94
95	27.483 5042	24.397 7556	21.882 8003	19.805 8906	16.600 9324	95
96	27.520 2939	24.420 9188	21.897 4166	19.815 1339	16.604 6532	96
97	27.555 8395	24.443 1912	21.911 4034	19.823 9370	16.608 1634	97
98	27.590 1831	24.464 6069	21.924 7379	19.832 3210	16.611 4749	98
99	27.623 3653	24.485 1990	21.937 5961	19.840 3057	16.614 5990	99
100	27.655 4254	24.504 9990	21.949 8527	19.847 9102	16.617 5462	100

TABLE VII.—THE ANNUITY THAT 1 WILL PURCHASE

$$\frac{1}{a_n} = \frac{i}{1-v^n} \cdot \left(\frac{1}{s_n} = \frac{1}{a_n} - i \right)$$

<i>n</i>	1 $\frac{1}{4}$ %	1 $\frac{1}{2}$ %	2 %	2 $\frac{1}{2}$ %	3 %	<i>n</i>
1	1.012 5000	1.015 0000	1.020 0000	1.025 0000	1.030 0000	1
2	0.509 3944	0.511 2779	0.515 0495	0.518 8272	0.522 6108	2
3	0.341 7012	0.343 3830	0.346 7547	0.350 1372	0.353 5304	3
4	0.257 8610	0.259 4448	0.262 6238	0.265 8179	0.269 0270	4
5	0.207 5621	0.209 0893	0.212 1584	0.215 2469	0.218 3546	5
6	0.174 0338	0.175 5252	0.178 5258	0.181 5500	0.184 5975	6
7	0.150 0887	0.151 5562	0.154 5120	0.157 4954	0.160 5064	7
8	0.132 1331	0.133 5840	0.136 5098	0.139 4673	0.142 4564	8
9	0.118 1705	0.119 6098	0.122 5154	0.125 4569	0.128 4339	9
10	0.107 0031	0.108 4342	0.111 3265	0.114 2588	0.117 2305	10
11	0.097 8684	0.099 2938	0.102 1779	0.105 1060	0.108 0774	11
12	0.090 2583	0.091 6800	0.094 5596	0.097 4871	0.100 4621	12
13	0.083 8210	0.085 2404	0.088 1184	0.091 0483	0.094 0295	13
14	0.078 3051	0.079 7233	0.082 6020	0.085 5365	0.088 5263	14
15	0.073 5265	0.074 9444	0.077 8255	0.080 7665	0.083 7666	15
16	0.069 3467	0.070 7651	0.073 6501	0.076 5990	0.079 6108	16
17	0.065 6602	0.067 0797	0.069 9698	0.072 9278	0.075 9525	17
18	0.062 3848	0.063 8058	0.066 7021	0.069 6701	0.072 7087	18
19	0.059 4555	0.060 8785	0.063 7818	0.066 7606	0.069 8139	19
20	0.056 8204	0.058 2457	0.061 1567	0.064 1471	0.067 2157	20
21	0.054 4375	0.055 8655	0.058 7847	0.061 7873	0.064 8718	21
22	0.052 2724	0.053 7033	0.056 6314	0.059 6466	0.062 7474	22
23	0.050 2967	0.051 7308	0.054 6681	0.057 6964	0.060 8139	23
24	0.048 4866	0.049 9241	0.052 8711	0.055 9128	0.059 0474	24
25	0.046 8225	0.048 2635	0.051 2204	0.054 2759	0.057 4279	25
26	0.045 2873	0.046 7320	0.049 6992	0.052 7687	0.055 9383	26
27	0.043 8668	0.045 3153	0.048 2931	0.051 3769	0.054 5642	27
28	0.042 5486	0.044 0011	0.046 9897	0.050 0879	0.053 2932	28
29	0.041 3223	0.042 7788	0.045 7784	0.048 8913	0.052 1147	29
30	0.040 1785	0.041 6392	0.044 6499	0.047 7776	0.051 0193	30
31	0.039 1094	0.040 5743	0.043 5963	0.046 7390	0.049 9989	31
32	0.038 1079	0.039 5771	0.042 6106	0.045 7683	0.049 0466	32
33	0.037 1679	0.038 6414	0.041 6865	0.044 8594	0.048 1561	33
34	0.036 2839	0.037 7619	0.040 8187	0.044 0067	0.047 3220	34
35	0.035 4511	0.036 9336	0.040 0022	0.043 2056	0.046 5393	35
36	0.034 6653	0.036 1524	0.039 2329	0.042 4516	0.045 8038	36
37	0.033 9227	0.035 4144	0.038 5068	0.041 7409	0.045 1116	37
38	0.033 2198	0.034 7161	0.037 8206	0.041 0701	0.044 4593	38
39	0.032 5537	0.034 0546	0.037 1711	0.040 4362	0.043 8439	39
40	0.031 9214	0.033 4271	0.036 5557	0.039 8362	0.043 2624	40
41	0.031 3206	0.032 8311	0.035 9719	0.039 2679	0.042 7124	41
42	0.030 7491	0.032 2643	0.035 4173	0.038 7288	0.042 1917	42
43	0.030 2047	0.031 7246	0.034 8899	0.038 2169	0.041 6981	43
44	0.029 6856	0.031 2104	0.034 3879	0.037 7304	0.041 2298	44
45	0.029 1901	0.030 7198	0.033 9096	0.037 2675	0.040 7852	45
46	0.028 7167	0.030 2512	0.033 4534	0.036 8268	0.040 3625	46
47	0.028 2641	0.029 8034	0.033 0179	0.036 4067	0.039 9605	47
48	0.027 8307	0.029 3750	0.032 6018	0.036 0060	0.039 5778	48
49	0.027 4156	0.028 9648	0.032 2040	0.035 6235	0.039 2131	49
50	0.027 0176	0.028 5717	0.031 8232	0.035 2581	0.038 8655	50

TABLE VII.—THE ANNUITY THAT 1 WILL PURCHASE

$$\frac{1}{a_n} = \frac{i}{1 - v^n} \quad \left(\frac{1}{s_n} = \frac{1}{a_n} - i \right)$$

<i>n</i>	1 $\frac{1}{4}$ %	1 $\frac{1}{2}$ %	2 %	2 $\frac{1}{2}$ %	3 %	<i>n</i>
51	0.026 6357	0.028 1947	0.031 4586	0.034 9087	0.038 5338	51
52	0.026 2690	0.027 8329	0.031 1091	0.034 5745	0.038 2172	52
53	0.025 9165	0.027 4854	0.030 7739	0.034 2545	0.037 9147	53
54	0.025 5776	0.027 1514	0.030 4523	0.033 9480	0.037 6256	54
55	0.025 2514	0.026 8302	0.030 1434	0.033 6542	0.037 3491	55
56	0.024 9374	0.026 5211	0.029 8466	0.033 3724	0.037 0845	56
57	0.024 6348	0.026 2234	0.029 5612	0.033 1020	0.036 8311	57
58	0.024 3430	0.025 9366	0.029 2867	0.032 8424	0.036 5885	58
59	0.024 0616	0.025 6601	0.029 0224	0.032 5931	0.036 3559	59
60	0.023 7899	0.025 3934	0.028 7680	0.032 3534	0.036 1330	60
61	0.023 5276	0.025 1360	0.028 5228	0.032 1229	0.035 9191	61
62	0.023 2741	0.024 8875	0.028 2864	0.031 9013	0.035 7139	62
63	0.023 0290	0.024 6474	0.028 0585	0.031 6879	0.035 5168	63
64	0.022 7920	0.024 4153	0.027 8385	0.031 4825	0.035 3276	64
65	0.022 5627	0.024 1909	0.027 6262	0.031 2846	0.035 1458	65
66	0.022 3406	0.023 9739	0.027 4212	0.031 0940	0.034 9711	66
67	0.022 1256	0.023 7638	0.027 2232	0.030 9102	0.034 8031	67
68	0.021 9172	0.023 5603	0.027 0317	0.030 7330	0.034 6416	68
69	0.021 7153	0.023 3633	0.026 8467	0.030 5621	0.034 4862	69
70	0.021 5194	0.023 1724	0.026 6676	0.030 3971	0.034 3366	70
71	0.021 3294	0.022 9873	0.026 4945	0.030 2379	0.034 1927	71
72	0.021 1450	0.022 8078	0.026 3268	0.030 0842	0.034 0540	72
73	0.020 9660	0.022 6337	0.026 1645	0.029 9357	0.033 9205	73
74	0.020 7921	0.022 4647	0.026 0074	0.029 7922	0.033 7919	74
75	0.020 6232	0.022 3007	0.025 8551	0.029 6536	0.033 6680	75
76	0.020 4591	0.022 1415	0.025 7075	0.029 5196	0.033 5485	76
77	0.020 2995	0.021 9868	0.025 5645	0.029 3900	0.033 4333	77
78	0.020 1444	0.021 8365	0.025 4258	0.029 2646	0.033 3222	78
79	0.019 9934	0.021 6904	0.025 2912	0.029 1434	0.033 2151	79
80	0.019 8465	0.021 5483	0.025 1607	0.029 0260	0.033 1117	80
81	0.019 7036	0.021 4102	0.025 0340	0.028 9125	0.033 0120	81
82	0.019 5644	0.021 2758	0.024 9111	0.028 8025	0.032 9158	82
83	0.019 4288	0.021 1451	0.024 7917	0.028 6961	0.032 8228	83
84	0.019 2968	0.021 0178	0.024 6758	0.028 5930	0.032 7331	84
85	0.019 1681	0.020 8940	0.024 5632	0.028 4931	0.032 6465	85
86	0.019 0427	0.020 7733	0.024 4538	0.028 3963	0.032 5628	86
87	0.018 9204	0.020 6558	0.024 3475	0.028 3025	0.032 4820	87
88	0.018 8012	0.020 5414	0.024 2442	0.028 2116	0.032 4039	88
89	0.018 6849	0.020 4298	0.024 1437	0.028 1235	0.032 3285	89
90	0.018 5715	0.020 3211	0.024 0460	0.028 0381	0.032 2556	90
91	0.018 4608	0.020 2152	0.023 9510	0.027 9552	0.032 1851	91
92	0.018 3527	0.020 1118	0.023 8586	0.027 8749	0.032 1169	92
93	0.018 2472	0.020 0110	0.023 7687	0.027 7969	0.032 0511	93
94	0.018 1442	0.019 9127	0.023 6812	0.027 7213	0.031 9874	94
95	0.018 0437	0.019 8168	0.023 5960	0.027 6479	0.031 9258	95
96	0.017 9454	0.019 7232	0.023 5131	0.027 5766	0.031 8662	96
97	0.017 8494	0.019 6319	0.023 4324	0.027 5075	0.031 8086	97
98	0.017 7556	0.019 5427	0.023 3538	0.027 4403	0.031 7528	98
99	0.017 6639	0.019 4556	0.023 2773	0.027 3752	0.031 6989	99
100	0.017 5743	0.019 3706	0.023 2027	0.027 3119	0.031 6467	100

TABLE VII.—THE ANNUITY THAT 1 WILL PURCHASE

$$\frac{1}{a_n} = \frac{i}{1-v} \quad \left(\frac{1}{a_n} = \frac{1}{a_{\overline{n}|i}} - i \right)$$

<i>n</i>	3½ %	4 %	4½ %	5 %	6 %	<i>n</i>
1	1.035 0000	1.040 0000	1.045 0000	1.050 0000	1.060 0000	1
2	0.526 4005	0.530 1961	0.533 9976	0.537 8049	0.545 4369	2
3	0.356 9342	0.360 3485	0.363 7734	0.367 2086	0.374 1098	3
4	0.272 2511	0.275 4900	0.278 7436	0.282 0118	0.288 5915	4
5	0.221 4814	0.224 6271	0.227 7916	0.230 9748	0.237 3964	5
6	0.187 6682	0.190 7619	0.193 8784	0.197 0175	0.203 3626	6
7	0.163 5445	0.166 6096	0.169 7015	0.172 8198	0.179 1350	7
8	0.145 4766	0.148 5278	0.151 6097	0.154 7218	0.161 0359	8
9	0.131 4460	0.134 4930	0.137 5745	0.140 6901	0.147 0222	9
10	0.120 2414	0.123 2909	0.126 3788	0.129 5046	0.135 8680	10
11	0.111 0920	0.114 1490	0.117 2482	0.120 3889	0.126 7929	11
12	0.103 4839	0.106 5522	0.109 6662	0.112 8254	0.119 2770	12
13	0.097 0616	0.100 1437	0.103 2754	0.106 4558	0.112 9601	13
14	0.091 5707	0.094 6690	0.097 8203	0.101 0240	0.107 5849	14
15	0.086 8251	0.089 9411	0.093 1138	0.096 3423	0.102 9628	15
16	0.082 6848	0.085 8200	0.089 0154	0.092 2699	0.098 9521	16
17	0.079 0431	0.082 1985	0.085 4176	0.088 6991	0.095 4448	17
18	0.075 8168	0.078 9933	0.082 2369	0.085 5462	0.092 3565	18
19	0.072 9403	0.076 1386	0.079 4073	0.082 7450	0.089 6209	19
20	0.070 3611	0.073 5817	0.076 8761	0.080 2426	0.087 1846	20
21	0.068 0366	0.071 2801	0.074 6006	0.077 9961	0.085 0046	21
22	0.065 9321	0.069 1988	0.072 5456	0.075 9705	0.083 0456	22
23	0.064 0188	0.067 3091	0.070 6825	0.074 1368	0.081 2785	23
24	0.062 2728	0.065 5868	0.068 9870	0.072 4709	0.079 6790	24
25	0.060 6740	0.064 0120	0.067 4390	0.070 9525	0.078 2267	25
26	0.059 2054	0.062 5674	0.066 0214	0.069 5643	0.076 9043	26
27	0.057 8524	0.061 2385	0.064 7195	0.068 2919	0.075 6972	27
28	0.056 6026	0.060 0130	0.063 5208	0.067 1225	0.074 5926	28
29	0.055 4454	0.058 8799	0.062 4146	0.066 0455	0.073 5796	29
30	0.054 3713	0.057 8301	0.061 3915	0.065 0514	0.072 6489	30
31	0.053 3724	0.056 8554	0.060 4434	0.064 1321	0.071 7922	31
32	0.052 4415	0.055 9486	0.059 5632	0.063 2804	0.071 0023	32
33	0.051 5724	0.055 1036	0.058 7445	0.062 4900	0.070 2729	33
34	0.050 7597	0.054 3148	0.057 9819	0.061 7554	0.069 5984	34
35	0.049 9983	0.053 5773	0.057 2704	0.061 0717	0.068 9739	35
36	0.049 2842	0.052 8869	0.056 6058	0.060 4345	0.068 3948	36
37	0.048 6132	0.052 2396	0.055 9840	0.059 8398	0.067 8574	37
38	0.047 9821	0.051 6319	0.055 4017	0.059 2842	0.067 3581	38
39	0.047 3878	0.051 0608	0.054 8557	0.058 7646	0.066 8938	39
40	0.046 8273	0.050 5235	0.054 3431	0.058 2782	0.066 4615	40
41	0.046 2982	0.050 0174	0.053 8616	0.057 8223	0.066 0589	41
42	0.045 7983	0.049 5402	0.053 4087	0.057 3947	0.065 6834	42
43	0.045 3254	0.049 0899	0.052 9823	0.056 9933	0.065 3331	43
44	0.044 8777	0.048 6645	0.052 5807	0.056 6163	0.065 0061	44
45	0.044 4534	0.048 2625	0.052 2020	0.056 2617	0.064 7005	45
46	0.044 0511	0.047 8820	0.051 8447	0.055 9282	0.064 4149	46
47	0.043 6692	0.047 5219	0.051 5073	0.055 6142	0.064 1477	47
48	0.043 3065	0.047 1806	0.051 1886	0.055 3184	0.063 8977	48
49	0.042 9617	0.046 8571	0.050 8872	0.055 0396	0.063 6636	49
50	0.042 6337	0.046 5502	0.050 6021	0.054 7767	0.063 4443	50

TABLE VII.—THE ANNUITY THAT 1 WILL PURCHASE

$$\frac{1}{a_n} = \frac{i}{1 - v^n} \cdot \left(\frac{1}{s_n} = \frac{1}{a_n} - i \right)$$

<i>n</i>	3½%	4%	4½%	5%	6%	<i>n</i>
51	0.042 3216	0.046 2588	0.050 3323	0.054 5287	0.063 2388	51
52	0.042 0243	0.045 9821	0.050 0768	0.054 2945	0.063 0462	52
53	0.041 7410	0.045 7191	0.049 8347	0.054 0733	0.062 8655	53
54	0.041 4709	0.045 4691	0.049 6052	0.053 8644	0.062 6960	54
55	0.041 2132	0.045 2312	0.049 3875	0.053 6669	0.062 5370	55
56	0.040 9673	0.045 0049	0.049 1811	0.053 4801	0.062 3876	56
57	0.040 7325	0.044 7893	0.048 9851	0.053 3034	0.062 2474	57
58	0.040 5081	0.044 5840	0.048 7990	0.053 1363	0.062 1157	58
59	0.040 2937	0.044 3884	0.048 6222	0.052 9780	0.061 9920	59
60	0.040 0886	0.044 2018	0.048 4543	0.052 8282	0.061 8757	60
61	0.039 8925	0.044 0240	0.048 2946	0.052 6823	0.061 7664	61
62	0.039 7048	0.043 8543	0.048 1428	0.052 5518	0.061 6637	62
63	0.039 5251	0.043 6924	0.047 9985	0.052 4244	0.061 5670	63
64	0.039 3531	0.043 5378	0.047 8611	0.052 3037	0.061 4762	64
65	0.039 1883	0.043 3902	0.047 7305	0.052 1892	0.061 3907	65
66	0.039 0303	0.043 2492	0.047 6061	0.052 0806	0.061 3102	66
67	0.038 8789	0.043 1145	0.047 4876	0.051 9776	0.061 2345	67
68	0.038 7337	0.042 9858	0.047 3749	0.051 8799	0.061 1633	68
69	0.038 5945	0.042 8627	0.047 2675	0.051 7872	0.061 0963	69
70	0.038 4609	0.042 7451	0.047 1651	0.051 6992	0.061 0331	70
71	0.038 3328	0.042 6325	0.047 0676	0.051 6156	0.060 9737	71
72	0.038 2097	0.042 5249	0.046 9747	0.051 5363	0.060 9177	72
73	0.038 0916	0.042 4219	0.046 8861	0.051 4610	0.060 8651	73
74	0.037 9782	0.042 3233	0.046 8016	0.051 3895	0.060 8154	74
75	0.037 8692	0.042 2290	0.046 7210	0.051 3216	0.060 7687	75
76	0.037 7645	0.042 1387	0.046 6442	0.051 2571	0.060 7246	76
77	0.037 6639	0.042 0522	0.046 5709	0.051 1958	0.060 6831	77
78	0.037 5672	0.041 9694	0.046 5010	0.051 1376	0.060 6441	78
79	0.037 4743	0.041 8901	0.046 4343	0.051 0822	0.060 6072	79
80	0.037 3849	0.041 8141	0.046 3707	0.051 0296	0.060 5725	80
81	0.037 2989	0.041 7413	0.046 3099	0.050 9796	0.060 5398	81
82	0.037 2163	0.041 6715	0.046 2520	0.050 9321	0.060 5090	82
83	0.037 1368	0.041 6046	0.046 1966	0.050 8869	0.060 4800	83
84	0.037 0602	0.041 5405	0.046 1438	0.050 8440	0.060 4526	84
85	0.036 9866	0.041 4791	0.046 0933	0.050 8032	0.060 4268	85
86	0.036 9158	0.041 4202	0.046 0452	0.050 7643	0.060 4024	86
87	0.036 8476	0.041 3637	0.045 9992	0.050 7274	0.060 3796	87
88	0.036 7819	0.041 3095	0.045 9552	0.050 6923	0.060 3580	88
89	0.036 7187	0.041 2576	0.045 9132	0.050 6589	0.060 3376	89
90	0.036 6578	0.041 2078	0.045 8732	0.050 6271	0.060 3184	90
91	0.036 5992	0.041 1600	0.045 8349	0.050 5969	0.060 3003	91
92	0.036 5427	0.041 1141	0.045 7983	0.050 5681	0.060 2832	92
93	0.036 4883	0.041 0701	0.045 7633	0.050 5408	0.060 2671	93
94	0.036 4359	0.041 0279	0.045 7299	0.050 5148	0.060 2519	94
95	0.036 3855	0.040 9874	0.045 6980	0.050 4900	0.060 2376	95
96	0.036 3368	0.040 9485	0.045 6675	0.050 4665	0.060 2241	96
97	0.036 2899	0.040 9112	0.045 6383	0.050 4441	0.060 2114	97
98	0.036 2448	0.040 8754	0.045 6105	0.050 4227	0.060 1994	98
99	0.036 2012	0.040 8410	0.045 5838	0.050 4024	0.060 1880	99
100	0.036 1593	0.040 8080	0.045 5584	0.050 3831	0.060 1774	100

TABLE VIII.—COMPOUND AMOUNT FOR TIMES LESS THAN A YEAR

$$s = (1 + i)^n$$

<i>n</i>	1 $\frac{1}{4}$ %	1 $\frac{1}{2}$ %	2 %	2 $\frac{1}{2}$ %	3 %	<i>n</i>
$\frac{1}{12}$	1.001 0357	1.001 2415	1.001 6516	1.002 0598	1.002 4663	$\frac{1}{12}$
$\frac{1}{4}$	1.003 1105	1.003 7291	1.004 9629	1.006 1922	1.007 4171	$\frac{1}{4}$
$\frac{1}{2}$	1.006 2306	1.007 4721	1.009 9505	1.012 4228	1.014 8892	$\frac{1}{2}$
<i>n</i>	3 $\frac{1}{2}$ %	4 %	5 %	6 %	7 %	<i>n</i>
$\frac{1}{12}$	1.002 8709	1.003 2737	1.004 0741	1.004 8676	1.005 6541	$\frac{1}{12}$
$\frac{1}{4}$	1.008 6374	1.009 8534	1.012 2722	1.014 6738	1.017 0585	$\frac{1}{4}$
$\frac{1}{2}$	1.017 3495	1.019 8039	1.024 6951	1.029 5630	1.034 4080	$\frac{1}{2}$

TABLE IX.—THE VALUE OF

$$j_{(p)} = p [(1 + i)^{\frac{1}{p}} - 1]$$

<i>p</i>	1 $\frac{1}{4}$ %	1 $\frac{1}{2}$ %	2 %	2 $\frac{1}{2}$ %	3 %	<i>p</i>
12	.012 4290	.014 8979	.019 8190	.024 7180	.029 5952	12
4	.012 4418	.014 9164	.019 8517	.024 7690	.029 6683	4
2	.012 4612	.014 9442	.019 9010	.024 8457	.029 7783	2
<i>p</i>	3 $\frac{1}{2}$ %	4 %	5 %	6 %	7 %	<i>p</i>
12	.034 4508	.039 2849	.048 8895	.058 4106	.067 8497	12
4	.034 5498	.039 4136	.049 0889	.058 6954	.068 2341	4
2	.034 6990	.039 6078	.049 3902	.059 1260	.068 8161	2

TABLE X.—THE VALUE OF

$$s_{1|}^{(p)} = \frac{i}{j_{(p)}} = \frac{i}{p [(1 + i)^{\frac{1}{p}} - 1]}$$

<i>p</i>	1 $\frac{1}{4}$ %	1 $\frac{1}{2}$ %	2 %	2 $\frac{1}{2}$ %	3 %	<i>p</i>
12	1.005 7163	1.006 8565	1.009 1339	1.011 4072	1.013 6766	12
4	1.004 6754	1.005 6075	1.007 4691	1.009 3268	1.011 1807	4
2	1.003 1153	1.003 7360	1.004 9752	1.006 2114	1.007 4446	2
<i>p</i>	3 $\frac{1}{2}$ %	4 %	5 %	6 %	7 %	<i>p</i>
12	1.015 9420	1.018 2035	1.022 7148	1.027 2107	1.031 6914	12
4	1.013 0309	1.014 8774	1.018 5594	1.022 2269	1.025 8800	4
2	1.008 6748	1.009 9020	1.012 3475	1.014 7815	1.017 2040	2

**TABLE XI.—AMERICAN EXPERIENCE TABLE OF
MORTALITY**

AGE	NUMBER LIVING l_x	NUMBER DYING d_x	YEARLY PROBABILI- TY OF DYING q_x	YEARLY PROBABILI- TY OF LIVING p_x	AGE	NUMBER LIVING l_x	NUMBER DYING d_x	YEARLY PROBABILI- TY OF DYING q_x	YEARLY PROBABILI- TY OF LIVING p_x
10	100 000	749	.007 490	.992 510	53	66 797	1 091	.016 333	.983 667
11	99 251	746	.007 516	.992 484	54	65 706	1 143	.017 396	.982 604
12	98 505	743	.007 543	.992 457	55	64 563	1 199	.018 571	.981 429
13	97 762	740	.007 569	.992 421	56	63 364	1 260	.019 885	.980 115
14	97 022	737	.007 596	.992 404	57	62 104	1 325	.021 335	.978 665
15	96 285	735	.007 634	.992 366	58	60 779	1 394	.022 936	.977 064
16	95 550	732	.007 661	.992 339	59	59 385	1 468	.024 720	.975 280
17	94 818	729	.007 688	.992 312	60	57 917	1 546	.026 693	.973 307
18	94 089	727	.007 727	.992 273	61	56 371	1 628	.028 880	.971 120
19	93 362	725	.007 765	.992 235	62	54 743	1 713	.031 292	.968 708
20	92 637	723	.007 805	.992 195	63	53 030	1 800	.033 943	.966 057
21	91 914	722	.007 855	.992 145	64	51 230	1 889	.036 873	.963 127
22	91 192	721	.007 906	.992 094	65	49 341	1 980	.040 129	.959 871
23	90 471	720	.007 958	.992 042	66	47 361	2 070	.043 707	.956 293
24	89 751	719	.008 011	.991 989	67	45 291	2 158	.047 647	.952 353
25	89 032	718	.008 065	.991 935	68	43 133	2 243	.052 002	.947 998
26	88 314	718	.008 130	.991 870	69	40 890	2 321	.056 762	.943 238
27	87 596	718	.008 197	.991 803	70	38 569	2 391	.061 993	.938 007
28	86 878	718	.008 264	.991 736	71	36 178	2 448	.067 665	.932 335
29	86 160	719	.008 345	.991 655	72	33 730	2 487	.073 733	.926 267
30	85 441	720	.008 427	.991 573	73	31 243	2 505	.080 178	.919 822
31	84 721	721	.008 510	.991 490	74	28 738	2 501	.087 028	.912 972
32	84 000	723	.008 607	.991 393	75	26 237	2 476	.094 371	.905 629
33	83 277	726	.008 718	.991 282	76	23 761	2 431	.102 311	.897 689
34	82 551	729	.008 831	.991 169	77	21 330	2 369	.111 064	.888 936
35	81 822	732	.008 946	.991 054	78	18 961	2 291	.120 827	.879 173
36	81 090	737	.009 089	.990 911	79	16 670	2 196	.131 734	.868 266
37	80 353	742	.009 234	.990 766	80	14 474	2 091	.144 466	.855 534
38	79 611	749	.009 408	.990 592	81	12 383	1 964	.158 605	.841 395
39	78 862	756	.009 586	.990 414	82	10 419	1 816	.174 297	.825 703
40	78 106	765	.009 794	.990 206	83	8 603	1 648	.191 561	.808 439
41	77 341	774	.010 008	.989 992	84	6 955	1 470	.211 359	.788 641
42	76 567	785	.010 252	.989 748	85	5 485	1 292	.235 552	.764 448
43	75 782	797	.010 517	.989 483	86	4 193	1 114	.265 681	.734 319
44	74 985	812	.010 829	.989 171	87	3 079	933	.303 020	.696 980
45	74 173	828	.011 163	.988 837	88	2 146	744	.346 692	.653 308
46	73 345	848	.011 562	.988 438	89	1 402	555	.395 863	.604 137
47	72 497	870	.012 000	.988 000	90	847	385	.454 545	.545 455
48	71 627	896	.012 509	.987 491	91	462	246	.532 466	.467 534
49	70 731	927	.013 106	.986 894	92	216	137	.634 259	.365 741
50	69 804	962	.013 781	.986 219	93	79	58	.734 177	.265 823
51	68 842	1 001	.014 541	.985 459	94	21	18	.857 143	.142 857
52	67 841	1 044	.015 389	.984 611	95	3	3	1.000 000	.000 000

TABLE XII.—COMMUTATION COLUMNS

AMERICAN EXPERIENCE, THREE AND ONE-HALF PER CENT

AGE	D_x	$N_x = D_x + D_{x+1} + \dots$	M_x	AGE	D_x	$N_x = D_x + D_{x+1} + \dots$	M_x
10	70 891.9	1 575 535.3	17 612.91	53	10 787.4	145 915.7	5 853.095
11	67 981.5	1 504 643.4	17 099.89	54	10 252.4	135 128.2	5 682.861
12	65 189.0	1 436 661.9	16 606.20	55	9 733.40	124 875.8	5 510.544
13	62 509.4	1 371 472.9	16 131.12	56	9 229.60	115 142.4	5 335.898
14	59 938.4	1 308 963.5	15 673.96	57	8 740.17	105 912.8	5 158.573
15	54 471.6	1 249 025.0	15 234.05	58	8 264.44	97 172.64	4 978.405
16	55 104.2	1 191 553.4	14 810.17	59	7 801.83	88 908.20	4 795.266
17	52 832.9	1 136 449.2	14 402.30	60	7 351.65	81 106.38	4 608.926
18	50 653.9	1 083 616.2	14 009.83	61	6 913.44	73 754.73	4 419.322
19	48 562.8	1 032 962.4	13 631.68	62	6 486.75	66 841.28	4 226.413
20	46 556.2	984 399.6	13 267.32	63	6 071.27	60 354.54	4 030.296
21	44 630.8	937 843.3	12 916.25	64	5 666.85	54 283.27	3 831.187
22	42 782.8	893 212.5	12 577.53	65	5 273.33	48 616.41	3 629.300
23	41 009.2	850 429.7	12 250.71	66	4 890.55	43 343.08	3 424.843
24	39 307.1	809 420.5	11 935.38	67	4 518.65	38 452.53	3 218.321
25	37 673.6	770 113.4	11 631.14	68	4 157.82	33 933.88	3 010.299
26	36 106.1	732 439.8	11 337.59	69	3 808.32	29 776.06	2 801.396
27	34 601.5	696 333.7	11 053.97	70	3 470.67	25 967.74	2 592.538
28	33 157.4	661 733.2	10 779.94	71	3 145.43	22 497.07	2 384.657
29	31 771.3	628 574.8	10 515.18	72	2 833.42	19 351.64	2 179.018
30	30 440.8	596 803.6	10 259.02	73	2 535.75	16 518.22	1 977.167
31	29 163.5	566 362.9	10 011.17	74	2 253.57	13 982.47	1 780.731
32	27 937.5	537 199.3	9 771.375	75	1 987.87	11 728.90	1 591.240
33	26 760.5	509 261.8	9 539.044	76	1 739.39	9 741.028	1 409.988
34	25 630.1	482 501.3	9 313.638	77	1 508.63	8 001.633	1 238.047
35	24 544.7	456 871.2	9 094.955	78	1 295.73	6 492.999	1 076.158
36	23 502.5	432 326.5	8 882.798	79	1 100.65	5 197.271	924.893 7
37	22 501.4	408 824.0	8 676.415	80	923.338	4 096.624	784.804 6
38	21 539.7	386 322.6	8 475.658	81	763.234	3 173.286	655.924 5
39	20 615.5	364 782.9	8 279.860	82	620.465	2 410.052	538.965 7
40	19 727.4	344 167.4	8 088.915	83	494.995	1 789.587	434.477 6
41	18 873.6	324 440.0	7 902.231	84	386.641	1 294.592	342.862 4
42	18 052.9	305 566.3	7 719.738	85	294.610	907.951 3	263.905 9
43	17 263.6	287 513.4	7 540.910	86	217.598	613.341 7	196.856 9
44	16 504.4	270 249.8	7 365.489	87	154.383	395.743 8	141.000 3
45	15 773.6	253 745.5	7 192.809	88	103.963	241.360 9	95.801 07
46	15 070.0	237 971.9	7 022.682	89	65.623 1	137.397 8	60.976 81
47	14 392.1	222 901.9	6 854.337	90	38.304 7	71.774 70	35.877 55
48	13 738.5	208 509.8	6 687.466	91	20.186 9	33.470 01	19.055 09
49	13 107.9	194 771.3	6 521.419	92	9.118 89	13.283 09	8.669 695
50	12 498.6	181 663.4	6 355.436	93	3.222 36	4.164 21	3.081 545
51	11 909.6	169 164.7	6 189.012	94	0.827 611	0.941 84	.795 762
52	11 339.5	157 255.2	6 021.696	95	0.114 232	0.114 23	.110 369

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