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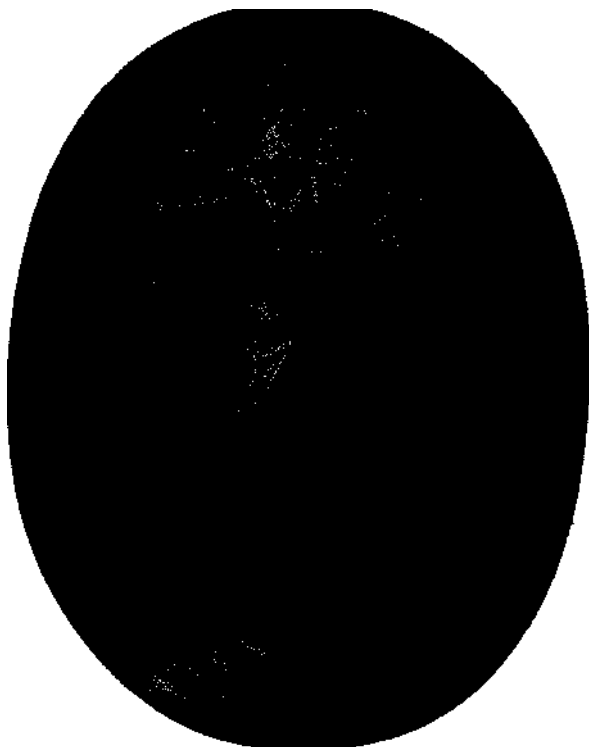
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NEWTON'S PRINCIPIA

MOTTE'S TRANSLATION REVISED



S IR ISA AC NEWTON

(See Appendix, Note, 1, page 627)

SIR ISAAC NEWTON'S
MATHEMATICAL PRINCIPLES
OF
NATURAL PHILOSOPHY AND HIS
SYSTEM OF THE WORLD

Translated into English by Andrew Motte
in 1729

The translations revised, and supplied with an historical
and explanatory appendix

by

FLORIAN CAJORI

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1934

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Editor's Note to the Present Revision

DROFESSOR FLORIAN CAJORI died August 15, 1930. In May of the following year I was invited by the University of California Press to edit this work. After much delay, due in part to unavoidable circumstances and in part to the time consumed in the extraordinary care taken in reading, checking, and rereading the proofs, this edition of Newton's *Principia* is now ready to be run off the press.

The manuscript as presented to the Press contained no Preface. Much of the material that would be included in the usual Preface is contained in the first few notes of the Appendix, pages 627 ff. Professor Cajori probably intended to prepare a Preface while the book was in the process of manufacture. There being none, the customary acknowledgment of thanks to various persons who assisted him in one way or another is lacking. Lest I unknowingly omit some to whom thanks are due, I refrain from attempting any such acknowledgment on behalf of the author.

As the title page states, this is a revision of Motte's translation of the *Principia*. From many conversations with Professor Cajori, I know that he had long cherished the idea of revising Newton's immortal work by rendering certain parts into modern phraseology (e.g., to change the reading of "reciprocally in the subduplicate ratio of" to "inversely as the square root of") and to append historical and critical notes which would provide instruction to some readers and interest to all. This is his last work; one most fitting to crown a life devoted to investigation and to writing the history of the sciences in his chosen field.

R. T. CRAWFORD

Berkeley, California,
March 31, 1934

PHILOSOPHIC I
N A T U R A L I S
P R I N C I P I A
M A T H E M A T I C A

Autore *J S. NEWT ON*, *Trin. Coll, Cantab, Soc. Mathefeos*
Profeffore Lucajhno, & Societatis Regalis Sodali.

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plures Bibliopoks. Anno MDCLXXXVII.

The Ode Dedicated to Newton by Edmund Halley

THIS ODE PREFIXED TO THE PRINCIPIA OF NEWTON
IS HERE TRANSLATED BY LEON J. RICHARDSON
PROFESSOR OF LATIN
IN THE UNIVERSITY OF CALIFORNIA
FROM THE VERSION AS GIVEN IN THE FIRST EDITION

TO THE ILLUSTRIOUS MAN

ISAAC NEWTON

AND THIS HIS WORK

DONE IN FIELDS OF THE MATHEMATICS AND PHYSICS
A SIGNAL DISTINCTION OF OUR TIME AND RACE

*Lo, for your gaze, the pattern of the skies!
What balance of the mass, what reckonings
Divine I Here ponder too the Laws which God,
Framing the universe, set not aside
But made the fixed foundations of his work-*

*The inmost places of the heavens, now gained,
Breaks into view, nor longer hidden is
The force that turns the farthest orb. The sun
Exalted on his throne bids all things tend
Toward him by inclination and descent,
Nor suffers that the courses of the stars
Be straight, as through the boundless void they move,*

ODE TO NEWTON

*But with himself as centre speeds them on
In motionless ellipses. Now we know
The sharply veering ways of comets, once
A source of dread, nor longer do we quail
Beneath appearances of bearded stars.*

*At last we learn wherefore the silver moon
Once seemed to travel with unequal steps,
As if she scorned to suit her pace to numbers—
Till now made clear to no astronomer,
Why, though the Seasons go and then return,
The Hours move ever forward on their way,
Explained too are the forces of the deep,
How roaming Cynthia bestirs the tides,
Whereby the surf, deserting now the kelp
Along the shore, exposes shoals of sand
Suspected by the sailors, now in turn
Driving its billows high upon the beach*

*Matters that vexed the mtnds of ancient seers,
And for our learned doctors often led
To loud and vain contention, now are seen
In reason 's light, the clouds of ignorance
Dispelled at last by science. Those on whom
Delusion cast its gloomy pall of doubt,
Upborne now on the wings that genius lends,
May penetrate the mansions of the gods
And scale the heights of heaven. O mortal men,
Ansel And, casting off your earthly cares,*

ODE TO NEWTON

*Learn ye the potency of heaven-born mind,
Its thought and life far from the herd withdrawn'*

*The man who through the tables of the laws
Once banished theft and murder, who suppressed
Adultery and crimes of broken faith,
And put the roving peoples into cities
Girt round with walls, was founder of the state,
While he who blessed the race with Ceres' gift,
Who pressed from grapes an anodyne to care,
Or showed how on the tissue made from reeds
Growing beside the Nile one may inscribe
Symbols of sound and so present the voice
For sight to grasp, did lighten human lot,
Offsetting thus the miseries of life
With some felicity But now, behold,
Admitted to the banquets of the gods,
We contemplate the politics of heaven,
And spelling out the secrets of the earth,
Discern the changeless order of the world
And all the aeons of its history.*

*Then ye who now on heavenly nectar fare,
Come celebrate with me in song the name
Of Newton, to the Muses dear; for he
Unlocked the hidden treasures of Truth
So richly through his mind had Phoebus cast
The radiance of his own divinity.
Nearer the gods no mortal may approach.*

Newton's Preface to the First Edition

SINCE THE ANCIENTS (as we are told by *Pappus*) esteemed the science of mechanics of greatest importance in the investigation of natural things, and the moderns, rejecting substantial forms and occult qualities, have endeavored to subject the phenomena of nature to the laws of mathematics, I have in this treatise cultivated mathematics as far as it relates to philosophy. The ancients considered mechanics in a twofold respect; as rational, which proceeds accurately by demonstration, and practical. To practical mechanics all the manual arts belong, from which mechanics took its name. But as artificers do not work with perfect accuracy, it comes to pass that mechanics is so distinguished from geometry that what is perfectly accurate is called geometrical; what is less so, is called mechanical. However, the errors are not in the art, but in the artificers. He that works with less accuracy is an imperfect mechanic; and if any could work with perfect accuracy, he would be the most perfect mechanic of all, for the description of right lines and circles, upon which geometry is founded, belongs to mechanics. Geometry does not teach us to draw these lines, but requires them to be drawn, for it requires that the learner should first be taught to describe these accurately before he enters upon geometry, then it shows how by these operations problems may be solved. To describe right lines and circles are problems, but not geometrical problems. The solution of these problems is required from mechanics, and by geometry the use of them, when so solved, is shown; and it is the glory of geometry that from those few principles, brought from without, it is able to produce so many things. Therefore geometry is founded in mechanical practice, and is nothing but that part of universal mechanics which accurately proposes and demonstrates the art of measuring. But since the manual arts are chiefly employed in the moving of bodies, it happens that geometry is commonly referred to their magnitude, and mechanics to their motion. In this sense rational mechanics will be the science of motions resulting from any forces whatsoever, and of the forces required to produce any motions, accurately proposed and demonstrated. This part of mechanics, as far as it extended to the five powers which relate to manual arts, was cultivated by the ancients, who considered gravity (it not being a manual power) no otherwise than in moving weights by those powers. But I consider philosophy rather than arts and write not concerning manual but natural powers, and consider chiefly those things which relate to gravity, levity, elastic force, the resistance of fluids, and the like forces, whether attractive or impulsive; and therefore I offer this work as the mathematical principles of philosophy, for the whole burden of philosophy seems to consist in this—from the phenomena of motions to investigate the forces of nature, and then from these forces to dem-

onstrate the other phenomena; and to this end the general propositions in the first and second Books are directed. In the third Book I give an example of this in the explication of the System of the World; for by the propositions mathematically demonstrated in the former Books, in the third I derive from the celestial phenomena the forces of gravity with which bodies tend to the sun and the several planets. Then from these forces, by other propositions which are also mathematical, I deduce the motions of the planets, the comets, the moon, and the sea. I wish we could derive the rest of the phenomena of Nature by the same kind of reasoning from mechanical principles, for I am induced by many reasons to suspect that they may all depend upon certain forces by which the particles of bodies, by some causes hitherto unknown, are either mutually impelled towards one another, and cohere in regular figures, or are repelled and recede from one another. These forces being unknown, philosophers have hitherto attempted the search of Nature in vain, but I hope the principles here laid down will afford some light either to this or some truer method of philosophy.

In the publication of this work the most acute and universally learned Mr. *Edmund Halley* not only assisted me in correcting the errors of the press and preparing the geometrical figures, but it was through his solicitations that it came to be published; for when he had obtained of me my demonstrations of the figure of the celestial orbits, he continually pressed me to communicate the same to the *Royal Society*, who afterwards, by their kind encouragement and entreaties, engaged me to think of publishing them. But after I had begun to consider the inequalities of the lunar motions, and had entered upon some other things relating to the laws and measures of gravity and other forces; and the figures that would be described by bodies attracted according to given laws; and the motion of several bodies moving among themselves; the motion of bodies in resisting mediums; the forces, densities, and motions, of mediums; the orbits of the comets, and such like, I deferred that publication till I had made a search into those matters, and could put forth the whole together. What relates to the lunar motions (being imperfect), I have put all together in the corollaries of Prop. LXVI, to avoid being obliged to propose and distinctly demonstrate the several things there contained in a method more prolix than the subject deserved and interrupt the series of the other propositions. Some things, found out after the rest, I chose to insert in places less suitable, rather than change the number of the propositions and the citations I heartily beg that what I have here done may be read with forbearance; and that my labors in a subject so difficult may be examined, not so much with the view to censure, as to remedy their defects

IS. NEWTON

Cambridge, Trinity College, May 8, 1686.¹

[¹ Appendix, Note 3]

Newton's Preface to the Second Edition

IN THIS SECOND EDITION of the *Principia* there are many emendations and some additions.¹ In the second section of the first Book, the determination of forces, by which bodies may be made to revolve in given orbits, is illustrated and enlarged. In the seventh section of the second Book the theory of the resistances of fluids was more accurately investigated, and confirmed by new experiments. In the third Book the lunar theory and the precession of the equinoxes were more fully deduced from their principles; and the theory of the comets was confirmed by more examples of the calculation of their orbits, done also with greater accuracy.

IS NEWTON

London, March 28, 1713.

¹ Appendix, Note 4]

Cotes's Preface to the Second Edition¹

WE HFREBY PRESENT to the benevolent reader the long-awaited new edition **W** at Newton's Philosophy, now greatly amended and increased The principal contents of this celebrated work may be gathered from the adjoining Table. What has been added or modified is indicated in the author's Preface. There remains for us to add something relating to the method of this philosophy.

Those who have treated of natural philosophy may be reduced to about three classes Of these some have attributed to the several species of things, specific and occult qualities, according to which the phenomena of particular bodies are supposed to proceed in some unknown manner The sum of the doctrine of the Schools derived from *Aristotle* and the *Peripatetics* is founded on this principle They affirm that the several effects of bodies arise from the particular natures of those bodies But whence it is that bodies derive those natures they don't tell us, and therefore they tell us nothing And being entirely employed in giving names to things, and not in searching into things themselves, they have invented, we may say, a philosophical way of speaking, but they have not made known to us true philosophy.

Others have endeavored to apply their labors to greater advantage by rejecting that useless medley of words. They assume that all matter is homogeneous, and that the variety of forms which is seen in bodies arises from some very plain and simple relations of the component particles And by going on from simple things to those which are more compounded they certainly proceed right, if they attribute to those primary relations no other relations than those which Nature has given But when they take a liberty of imagining at pleasure unknown figures and magnitudes, and uncertain situations and motions of the parts, and moreover of supposing occult fluids, freely pervading the pores of bodies, endued with an all-performing subtilty, and agitated with occult motions, they run out into dreams and chimeras, and neglect the true constitution of things, which certainly is not to be derived from fallacious conjectures, when we can scarce reach it by the most certain observations Those who assume hypotheses as first principles of their speculations, although they afterwards proceed with the greatest accuracy from those principles, may indeed form an ingenious romance, but a romance it will still be

I here is left then the third class, which possess experimental philosophy. These indeed derive the causes of all things from the most simple principles possible; but then they assume nothing as a principle, that is not proved by phenomena. They frame no hypotheses, nor receive them into philosophy otherwise than as questions whose truth may be disputed They proceed therefore in a twofold

[¹ Appendix, Note 5]

method, synthetical and analytical From some select phenomena they deduce by analysis the forces of Nature and the more simple laws of forces; and from thence by synthesis show the constitution of the rest. This is that incomparably best way of philosophizing, which our renowned author most justly embraced in preference to the rest, and thought alone worthy to be cultivated and adorned by his excellent labors. Of this he has given us a most illustrious example, by the explication of the System of the World, most happily deduced from the Theory of Gravity. That the attribute of gravity was found in all bodies,¹ others suspected, or imagined before him, but he was the only and the first philosopher that could demonstrate it from appearances, and make it a solid foundation to the most noble speculations.

I know indeed that some persons, and those of great name, too much prepossessed with certain prejudices, are unwilling to assent to this new principle, and are ready to prefer uncertain notions to certain. It is not my intention to detract from the reputation of these eminent men; I shall only lay before the reader such considerations as will enable him to pass an equitable judgment in this dispute.

Therefore, that we may begin our reasoning from what is most simple and nearest to us, let us consider a little what is the nature of gravity in earthly bodies, that we may proceed the more safely when we come to consider it in the heavenly bodies that he at the remotest distance from us. It is now agreed by all philosophers that all circumterrestrial bodies gravitate towards the earth. That no bodies having no weight are to be found, is now confirmed by manifold experience. That which is relative levity is not true levity, but apparent only, and arises from the preponderating gravity of the contiguous bodies.

Moreover, as all bodies gravitate towards the earth, so does the earth gravitate again towards all bodies. That the action of gravity is mutual and equal on both sides, is thus proved. Let the mass of the earth be divided into any two parts whatever, either equal or unequal; now if the weights of the parts towards each other were not mutually equal, the lesser weight would give way to the greater, and the two parts would move on together indefinitely in a right line towards that point to which the greater weight tends, which is altogether contrary to experience. Therefore we must say that the weights with which the parts tend to each other are equal; that is, that the action of gravity is mutual and equal in contrary directions.

The weights of bodies at equal distances from the centre of the earth are as the quantities of matter in the bodies. This is inferred from the equal acceleration of all bodies that fall from a state of rest by their weights, for the forces by which unequal bodies are equally accelerated must be proportional to the quantities of the matter to be moved. Now, that all falling bodies are equally accelerated, appears from this, that when the resistance of the air is taken away, as it is under

¹ Appendix, Note 6.]

an exhausted receiver of Mr. *Boyle*, they describe equal spaces in equal times; but this is yet more accurately proved by the experiments with pendulums.

The attractive forces of bodies at equal distances are as the quantities of matter in the bodies. For since bodies gravitate towards the earth, and the earth again towards bodies with equal moments, the weight of the earth towards each body, or the force with which the body attracts the earth, will be equal to the weight of the same body towards the earth. But this weight was shown to be as the quantity of matter in the body; and therefore the force with which each body attracts the earth, or the absolute force of the body, will be as the same quantity of matter.

Therefore the attractive force of the entire bodies arises from and is composed of the attractive forces of the parts, because, as was just shown, if the bulk of the matter be augmented or diminished, its power is proportionately augmented or diminished. We must therefore conclude that the action of the earth is composed of the united actions of its parts, and therefore that all terrestrial bodies must attract one another mutually, with absolute forces that are as the matter attracting. This is the nature of gravity upon earth; let us now see what it is in the heavens.

That every body continues in its state either of rest or of moving uniformly in a right line, unless so far as it is compelled to change that state by external force, is a law of Nature universally received by all philosophers. But it follows from this that bodies which move in curved lines, and are therefore continually bent from the right lines that are tangents to their orbits, are retained in their curvilinear paths by some force continually acting. Since, then, the planets move in curvilinear orbits, there must be some force operating, by the incessant actions of which they are continually made to deflect from the tangents.

Now it is evident from mathematical reasoning, and rigorously demonstrated, that all bodies that move in any curved line described in a plane, and which, by a radius drawn to any point, whether at rest or moved in any manner, describe areas about that point proportional to the times, arc urged by forces directed towards that point. This must therefore be granted. Since, then, all astronomers agree that the primary planets describe about the sun, and the secondary about the primary, areas proportional to the times, it follows that the forces by which they are continually turned aside from the rectilinear tangents, and made to revolve in curvilinear orbits, are directed towards the bodies that are placed in the centres of the orbits. This force may therefore not improperly be called centripetal in respect of the revolving body, and in respect of the central body attractive, from whatever cause it may be imagined to arise.

Moreover, it must be granted, as being mathematically demonstrated, that, if several bodies revolve with an equable motion in concentric circles, and the squares of the periodic times are as the cubes of the distances from the common centre, the centripetal forces will be inversely as the squares of the distances. Or,

if bodies revolve in orbits that are very nearly circular and the apsides of the orbits are at rest, the centripetal *forces* of the revolving bodies will be inversely as the squares of the distances. That both these facts hold for all the planets, all astronomers agree. Therefore the centripetal forces of all the planets are inversely as the squares of the distances from the centres of their orbits. If any should object, that the apsides of the planets, and especially of the moon, are not perfectly at rest, but are carried progressively with a slow kind of motion, one may give this answer, that, though we should grant that this very slow motion arises from a slight deviation of the centripetal force from the law of the square of the distance, yet we are able to compute mathematically the quantity of that aberration, and find it perfectly insensible. For even the ratio of the lunar centripetal force itself, which is the most irregular of them all, will vary inversely as a power a little greater than the square of the distance, but will be well-nigh sixty times nearer to the square than to the cube of the distance. But we may give a truer answer, by saying that this progression of the apsides arises not from a deviation from the law of inverse squares of the distance, but from a quite different cause, as is most admirably shown in this work. It is certain then that the centripetal forces with which the primary planers tend to the sun, and the secondary planets to their primary, are accurately as the inverse squares of the distances.

From what has been hitherto said, it is plain that the planets are retained in their orbits by some force continually acting upon them; it is plain that this force is always directed towards the centres of their orbits; it is plain that its intensity is increased in its approach and is decreased in its recession from the centre, and that it is increased in the same ratio in which the square of the distance is diminished, and decreased in the same ratio in which the square of the distance is augmented. Let us now see whether, by making a comparison between the centripetal forces of the planets and the force of gravity, we may not by chance find them to be of the same kind. Now, they will be of the same kind if we find on both sides the same laws and the same attributes. Let us then first consider the centripetal force of the moon, which is nearest to us.

The rectilinear spaces which bodies let fall from rest describe in a given time at the very beginning of the motion, when the bodies are urged by any forces whatsoever, are proportional to the forces. This appears from mathematical reasoning. Therefore the centripetal force of the moon revolving in its orbit is to the force of gravity at the surface of the earth, as the space which in a very small interval of time the moon, deprived of all its circular force and descending by its centripetal force towards the earth, would describe, is to the space which a heavy body would describe, when falling by the force of its gravity near to the earth, in the same small interval of time. The first of these spaces is equal to the versed sine of the arc described by the moon in the same time, because that versed sine

measures the translation of the moon from the tangent, produced by the centripetal force, and therefore may be computed, if the periodic time of the moon and its distance from the centre of the earth are given. The last space is found by experiments with pendulums, as Mr. *Wren* has shown. Therefore by making a calculation we shall find that the first space is to the latter, or the centripetal force of the moon revolving in its orbit will be to the force of gravity at the surface of the earth, as the square of the semidiameter of the earth to the square of the semidiameter of the orbit. But by what was shown before, the very same ratio holds between the centripetal force of the moon revolving in its orbit, and the centripetal force of the moon near the surface of the earth. Therefore the centripetal force near the surface of the earth is equal to the force of gravity. Therefore these are not two different forces, but one and the same; for if they were different, these forces united would cause bodies to descend to the earth with twice the velocity they would fall with by the force of gravity alone. Therefore it is plain that the centripetal force, by which the moon is continually either impelled or attracted out of the tangent and retained in its orbit, is the very force of terrestrial gravity reaching up to the moon. And it is very reasonable to believe that this force should extend itself to vast distances, since upon the tops of the highest mountains we find no sensible diminution of it. Therefore the moon gravitates towards the earth; but on the other hand, the earth by a mutual action equally gravitates towards the moon, which is also abundantly confirmed in this philosophy, where the tides in the sea and the precession of the equinoxes are treated of, which arise from the action both of the moon and of the sun upon the earth. Hence lastly, we discover by what law the force of gravity decreases at great distances from the earth. For since gravity is noways different from the moon's centripetal force, and this is inversely proportional to the square of the distance, it follows that it is in that very ratio that the force of gravity decreases.

Let us now go on to the other planets. Because the revolutions of the primary planets about the sun and of the secondary about Jupiter and Saturn are phenomena of the same kind with the revolution of the moon about the earth, and because it has been moreover demonstrated that the centripetal forces of the primary planets are directed towards the centre of the sun and those of the secondary towards the centres of Jupiter and Saturn, in the same manner as the centripetal force of the moon is directed towards the centre of the earth, and since, besides, all these forces are inversely as the squares of the distances from the centres, in the same manner as the centripetal force of the moon is as the square of the distance from the earth, we must of course conclude that the nature of all is the same. Therefore as the moon gravitates towards the earth and the earth again towards the moon, so also all the secondary planets will gravitate towards their

primary, and the primary planets again towards their secondary, and so all the primary towards the sun, and the sun again towards the primary.

Therefore the sun gravitates towards all the planets, and all the planets towards the sun. For the secondary planets, while they accompany the primary, revolve the meanwhile with the primary about the sun. Therefore, by the same argument, the planets of both kinds gravitate towards the sun and the sun towards them. That the secondary planets gravitate towards the sun is moreover abundantly clear from the inequalities of the moon, a most accurate theory of which, laid open with a most admirable sagacity, we find explained in the third Book of this work.

That the attractive force of the sun is propagated on all sides to prodigious distances and is diffused to every part of the wide space that surrounds it, is most evidently shown by the morion of the comets, which, coming from places immensely distant from the sun, approach very near to it, and sometimes so near that in their perihelia they almost touch its body. The theory of these bodies was altogether unknown to astronomers till in our own times our excellent author most happily discovered it and demonstrated the truth of it by most certain observations. So that it is now apparent that the comets move in conic sections having their foci in the sun's centre, and by radii drawn to the sun describe areas proportional to the times. But from these phenomena it is manifest and mathematically demonstrated, that those forces by which the comets are retained in their orbits are directed towards the sun and are inversely proportional to the squares of the distances from its centre. Therefore the comets gravitate towards the sun, and therefore the attractive force of the sun not only acts on the bodies of the planets, placed at given distances and very nearly in the same plane, but reaches also the comets in the most different parts of the heavens, and at the most different distances. This therefore is the nature of gravitating bodies, to exert their force at all distances to all other gravitating bodies. But from thence it follows that all the planets and comets attract one another mutually, and gravitate towards one another, which is also confirmed by the perturbation of Jupiter and Saturn, observed by astronomers, and arising from the mutual actions of these two planets upon each other, as also from that very slow motion of the apsides, above taken notice of, which arises from a like cause.

We have now proceeded so far, that it must be acknowledged that the sun, and the earth, and all the heavenly bodies attending the sun, attract one another mutually. Therefore all the least particles of matter in every one must have their several attractive forces proportional to their quantities of matter, as was shown above of the terrestrial bodies. At different distances these forces will be also inversely as the squares of their distances; for it is mathematically demonstrated, that globes attracting according to this law are composed of particles attracting according to the same law.

The foregoing conclusions are grounded on this axiom which is received by all philosophers, namely, that effects of the same kind, whose known properties are the same, take their rise from the same causes and have the same unknown properties also. For if gravity be the cause of the descent of a stone in *Europe*, who doubts that it is also the cause of the same descent in *America*? If there is a mutual gravitation between a stone and the earth in *Europe*, who will deny the same to be mutual in *America*? If in *Europe* the attractive force of a stone and the earth is composed of the attractive forces of the parts, who will deny the like composition in *America*? If in *Europe* the attraction of the earth be propagated to all kinds of bodies and to all distances, why may we not say that it is propagated in like manner in *America*? All philosophy is founded on this rule; for if that be taken away, we can affirm nothing as a general truth. The constitution of particular things is known by observations and experiments; and when that is done, no general conclusion of the nature of things can thence be drawn, except by this rule.

Since, then, all bodies, whether upon earth or in the heavens, are heavy, so far as we can make any experiments or observations concerning them, we must certainly allow that gravity is found in all bodies universally. And in like manner as we ought not to suppose that any bodies can be otherwise than extended, movable, or impenetrable, so we ought not to conceive that any bodies can be otherwise than heavy. The extension, mobility, and impenetrability of bodies become known to us only by experiments; and in the very same manner their gravity becomes known to us. All bodies upon which we can make any observations, are extended, movable, and impenetrable; and thence we conclude all bodies, and those concerning which we have no observations, are extended and movable and impenetrable. So all bodies on which we can make observations, we find to be heavy; and thence we conclude all bodies, and those we have no observations of, to be heavy also. If anyone should say that the bodies of the fixed stars are not heavy because their gravity is not yet observed, they may say for the same reason that they are neither extended nor movable nor impenetrable, because these properties of the fixed stars are not yet observed. In short, either gravity must have a place among the primary qualities of all bodies, or extension, mobility, and impenetrability must not. And if the nature of things is not rightly explained by the gravity of bodies, it will not be rightly explained by their extension, mobility, and impenetrability.

Some I know disapprove this conclusion, and mutter something about occult qualities. They continually are cavilling with us, that gravity is an occult property, and occult causes are to be quite banished from philosophy. But to this the answer is easy: that those are indeed occult causes whose existence is occult, and imagined but not proved; but not those whose real existence is clearly demonstrated by

observations. Therefore gravity can by no means be called an occult cause of the celestial motions, because it is plain from the phenomena that such a power does really exist. Those rather have recourse to occult causes, who set imaginary vortices of a matter entirely fictitious and imperceptible by our senses, to direct those motions.

But shall gravity be therefore called an occult cause, and thrown out of philosophy, because the cause of gravity is occult and not yet discovered? Those who affirm this, should be careful not to fall into an absurdity that may overturn the foundations of all philosophy. For causes usually proceed in a continued chain from those that are more compounded to those that are more simple; when we are arrived at the most simple cause we can go no farther. Therefore no mechanical account or explanation of the most simple cause is to be expected or given; for if it could be given, the cause were not the most simple. These most simple causes will you then call occult, and reject them? Then you must reject those that immediately depend upon them, and those which depend upon these last, till philosophy is quite cleared and disencumbered of all causes.

Some there are who say that gravity is preternatural, and call it a perpetual miracle. Therefore they would have it rejected, because preternatural causes have no place in physics. It is hardly worth while to spend time in answering this ridiculous objection which overturns all philosophy. For either they will deny gravity to be in bodies, which cannot be said, or else, they will therefore call it preternatural because it is not produced by the other properties of bodies, and therefore not by mechanical causes. But certainly there are primary properties of bodies; and these, because they are primary, have no dependence on the others. Let them consider whether all these are not in like manner preternatural, and in like manner to be rejected; and then what kind of philosophy we are like to have.

Some there are who dislike this celestial physics because it contradicts the opinions of *Descartes*, and seems hardly to be reconciled with them. Let these enjoy their own opinion, but let them act fairly, and not deny the same liberty to us which they demand for themselves. Since the *Newtonian* Philosophy appears true to us, let us have the liberty to embrace and retain it, and to follow causes proved by phenomena, rather than causes only imagined and not yet proved. The business of true philosophy is to derive the natures of things from causes truly existent, and to inquire after those laws on which the Great Creator actually chose to found this most beautiful Frame of the World, not those by which he might have done the same, had he so pleased. It is reasonable enough to suppose that from several causes, somewhat differing from one another, the same effect may arise; but the true cause will be that from which it truly and actually does arise; the others have no place in true philosophy. The same motion of the hour-

hand in a clock may be occasioned either by a weight hung, or a spring shut up within. But if a certain clock should be really moved with a weight, we should laugh at a man that would suppose it moved by a spring, and from that principle, suddenly taken up without further examination, should go about to explain the motion of the index; for certainly the way he ought to have taken would have been actually to look into the inward parts of the machine, that he might find the true principle of the proposed motion. The like judgment ought to be made of those philosophers who will have the heavens to be filled with a most subtile matter which is continually carried round in vortices. For if they could explain the phenomena ever so accurately by their hypotheses, we could not yet say that they have discovered true philosophy and the true causes of the celestial motions, unless they could either demonstrate that those causes do actually exist, or at least that no others do exist. Therefore if it be made clear that the attraction of all bodies is a property actually existing *in rerum natura*, and if it be also shown how the motions of the celestial bodies may be solved by that property, it would be very impertinent for anyone to object that these motions ought to be accounted for by vortices; even though we should allow such an explication of those motions to be possible. But we allow no such thing; for the phenomena can by no means be accounted for by vortices, as our author has abundantly proved from the clearest reasons. So that men must be strangely fond of chimeras, who can spend their time so idly as in patching up a ridiculous figment and setting it off with new comments of their own.

If the bodies of the planets and comets are carried round the sun in vortices, the bodies so carried, and the parts of the vortices next surrounding them, must be carried with the same velocity and the same direction, and have the same density, and the same inertia, answering to the bulk of the matter. But it is certain, the planets and comets, when in the very same parts of the heavens, are carried with various velocities and various directions. Therefore it necessarily follows that those parts of the celestial fluid, which are at the same distances from the sun, must revolve at the same time with different velocities in different directions; for one kind of velocity and direction is required for the motion of the planets, and another for that of the comets. But since this cannot be accounted for, we must either say that all celestial bodies are not carried about by vortices, or else that their motions are derived, not from one and the same vortex, but from several distinct ones, which fill and pervade the spaces round about the sun.

But if several vortices are contained in the same space, and are supposed to penetrate one another, and to revolve with different motions, then because these motions must agree with those of the bodies carried about by them, which are perfectly regular, and performed in conic sections which are sometimes very eccentric, and sometimes nearly circles, one may very reasonably ask how it comes

to pass that these vortices remain entire, and have suffered no manner of perturbation in so many ages from the actions of the conflicting matter. Certainly if these fictitious motions are more compounded and harder to be accounted for than the true motions of the planets and comets, it seems to no purpose to admit them into philosophy, since every cause ought to be more simple than its effect. Allowing men to indulge their own fancies, suppose any man should affirm that the planets and comets are surrounded with atmospheres like our earth, which hypothesis seems more reasonable than that of vortices; let him then affirm that these atmospheres by their own nature move about the sun and describe conic sections, which motion is much more easily conceived than that of the vortices penetrating one another; lastly, that the planets and comets are carried about the sun by these atmospheres of theirs: and then applaud his own sagacity in discovering the causes of the celestial motions. He that rejects this fable must also reject the other; for two drops of water are not more like than this hypothesis of atmospheres, and that of vortices.

Galileo has shown that when a stone projected moves in a parabola, its deflection into that curve from its rectilinear path is occasioned by the gravity of the stone towards the earth, that is, by an occult quality. But now somebody, more cunning than he, may come to explain the cause after this manner. He will suppose a certain subtle matter, not discernible by our sight, our touch, or any other of our senses, which fills the spaces which are near and contiguous to the surface of the earth, and that this matter is carried with different directions, and various, and often contrary, motions, describing parabolic curves. Then see how easily he may account for the deflection of the stone above spoken of. The stone, says he, floats in this subtle fluid, and following its motion, can't choose but describe the same figure. But the fluid moves in parabolic curves, and therefore the stone must move in a parabola, of course. Would not the acuteness of this philosopher be thought very extraordinary, who could deduce the appearances of Nature from mechanical causes, matter and motion, so clearly that the meanest man may understand it? Or indeed should not we smile to see this new *Galileo* taking so much mathematical pains to introduce occult qualities into philosophy, from whence they have been so happily excluded? But I am ashamed to dwell so long upon trifles.

The sum of the matter is this: the number of the comets is certainly very great; their motions are perfectly regular and observe the same laws with those of the planets. The orbits in which they move are conic sections, and those very eccentric. They move every way towards all parts of the heavens, and pass through the planetary regions with all possible freedom, and their motion is often contrary to the order of the signs. These phenomena are most evidently confirmed by astronomical observations, and cannot be accounted for by vortices. Nay, indeed,

they are utterly irreconcilable with the vortices of the planets. There can be no room for the motions of the comets, unless the celestial spaces be entirely cleared of that fictitious matter.

For if the planets are carried about the sun in vortices, the parts of the vortices which immediately surround every planet must be of the same density with the planet, as was shown above. Therefore all the matter contiguous to the perimeter of the earth's orbit¹ must be of the same density as the earth. But this great orb and the orb of Saturn must have either an equal or a greater density. For to make the constitution of the vortex permanent, the parts of less density must be near the centre, and those of greater density must go farther from it. For since the periodic times of the planets vary as the $\frac{3}{2}$ th powers of their distances from the sun, the periods of the parts of the vortices must also preserve the same ratio. Thence it will follow that the centrifugal forces of the parts of the vortex must be inversely as the squares of their distances. Those parts therefore which are more remote from the centre endeavor to recede from it with less force; whence, if their density be deficient, they must yield to the greater force with which the parts that be nearer the centre endeavor to ascend. Therefore the denser parts will ascend, and those of less density will descend, and there will be a mutual change of places, till all the fluid matter in the whole vortex be so adjoined and disposed, that being reduced to an equilibrium its parts become quiescent. If two fluids of different density be contained in the same vessel, it will certainly come to pass that the fluid of greater density will sink the lower; and by a like reasoning it follows that the denser parts of the vortex by their greater centrifugal force will ascend to the higher places. Therefore all that far greater part of the vortex which lies without the earth's orb, will have a density, and by consequence an inertia, answering to the bulk of the matter, which cannot be less than the density and inertia of the earth. But from hence will arise a mighty resistance to the passage of the comets, such as must be very sensible, not to say enough to put a stop to and absorb their motions entirely. But it appears from the perfectly regular motion of the comets, that they suffer no resistance that is in the least sensible, and therefore that they do not meet with matter of any kind that has any resisting force or, by consequence, any density or inertia. For the resistance of mediums arises either from the inertia of the matter of the fluid, or from its want of lubricity. That which arises from the want of lubricity is very small, and is scarcely observable in the fluids commonly known, unless they be very tenacious like oil and honey. The resistance we find in air, water, quicksilver, and the like fluids that are not tenacious, is almost all of the first kind, and cannot be diminished by a greater degree of subtilty, if the density and inertia, to which this resistance is proportional, remains, as is most evidently demonstrated by our author in his noble theory of resistances in the second Book.

Bodies in going on through a fluid communicate their motion to the ambient fluid by little and little, and by that communication lose their own motion, and by losing it are retarded. Therefore the retardation is proportional to the motion communicated, and the communicated motion, when the velocity of the moving body is given, is as the density of the fluid; and therefore the retardation or resistance will be as the same density of the fluid; nor can it be taken away, unless the fluid, coming about to the hinder parts of the body, restore the motion lost. Now this cannot be done unless the impression of the fluid on the hinder parts of the body be equal to the impression of the fore parts of the body on the fluid; that is, unless the relative velocity with which the fluid pushes the body behind is equal to the velocity with which the body pushes the fluid; that is, unless the absolute velocity of the recurring fluid be twice as great as the absolute velocity with which the fluid is driven forwards by the body, which is impossible. Therefore the resistance of fluids arising from their inertia can by no means be taken away. So that we must conclude that the celestial fluid has no inertia, because it has no resisting force; that it has no force to communicate motion with, because it has no inertia; that it has no force to produce any change in one or more bodies, because it has no force wherewith to communicate motion; that it has no manner of efficacy, because it has no faculty wherewith to produce any change of any kind. Therefore certainly this hypothesis may be justly called ridiculous and unworthy a philosopher, since it is altogether without foundation and does not in the least serve to explain the nature of things.¹ Those who would have the heavens filled with a fluid matter, but suppose it void of any inertia, do indeed in words deny a vacuum, but allow it in fact. For since a fluid matter of that kind can noways be distinguished from empty space, the dispute is now about the names and not the natures of things. If any are so fond of matter that they will by no means admit of a space void of body, let us consider where they must come at last.

For either they will say that this constitution of a world everywhere full was made so by the will of God to this end, that the operations of Nature might be assisted everywhere by a subtile ether pervading and filling all things; which cannot be said, however, since we have shown from the phenomena of the comets, that this ether is of no efficacy at all; or they will say, that it became so by the same will of God for some unknown end, which ought not be said, because for the same reason a different constitution may be as well supposed; or lastly, they will not say that it was caused by the will of God, but by some necessity of its nature. Therefore they will at last sink into the mire of that infamous herd who dream that all things are governed by fate and not by providence, and that matter exists by the necessity of its nature always and everywhere, being infinite and eternal. But supposing these things, it must be also everywhere uniform; for variety of forms is entirely inconsistent with necessity. It must be also unmoved;

for if it be necessarily moved in any determinate direction, with any determinate velocity, it will by a like necessity be moved in a different direction with a different velocity; but it can never move in different directions with different velocities; therefore it must be unmoved. Without all doubt this world, so diversified with that variety of forms and motions we find in it, could arise from nothing but the perfectly free will of God directing and presiding over all.

From this fountain it is that those laws, which we call the laws of Nature, have flowed, in which there appear many traces indeed of the most wise contrivance, but not the least shadow of necessity. These therefore we must not seek from uncertain conjectures, but learn them from observations and experiments. He who is presumptuous enough to think that he can find the true principles of physics and the laws of natural things by the force alone of his own mind, and the internal light of his reason, must either suppose that the world exists by necessity, and by the same necessity follows the laws proposed; or if the order of Nature was established by the will of God, that himself, a miserable reptile, can tell what was fittest to be done. All sound and true philosophy is founded on the appearances of things; and if these phenomena inevitably draw us, against our wills, to such principles as most clearly manifest to us the most excellent counsel and supreme dominion of the All-wise and Almighty Being, they are not therefore to be laid aside because some men may perhaps dislike them. These men may call them miracles or occult qualities, but names maliciously given ought not to be a disadvantage to the things themselves, unless these men will say at last that all philosophy ought to be founded in atheism. Philosophy must not be corrupted in compliance with these men, for the order of things will not be changed.

Fair and equal judges will therefore give sentence in favor of this most excellent method of philosophy, which is founded on experiments and observations. And it can hardly be said or imagined, what light, what splendor, hath accrued to that method from this admirable work of our illustrious author, whose happy and sublime genius, resolving the most difficult problems, and reaching to discoveries of which the mind of man was thought incapable before, is deservedly admired by all those who are somewhat more than superficially versed in these matters. The gates are now set open, and by the passage he has revealed we may freely enter into the knowledge of the hidden secrets and wonders of natural things. He has so clearly laid open and set before our eyes the most beautiful frame of the System of the World, that if King *Alphonso* were now alive, he would not complain for want of the graces either of simplicity or of harmony in it. Therefore we may now more nearly behold the beauties of Nature, and entertain ourselves with the delightful contemplation; and, which is the best and most valuable fruit of philosophy, be thence incited the more profoundly to reverence and adore the great Maker and Lord of all. He must be blind who from

the most wise and excellent contrivances of things cannot see the infinite Wisdom and Goodness of their Almighty Creator, and he must be mad and senseless who refuses to acknowledge them.

Newton's distinguished work will be the safest protection against the attacks of atheists, and nowhere more surely than from this quiver can one draw forth missiles against the band of godless men. This was felt long ago and first surprisingly demonstrated in learned English and Latin discourses by *Richard Bentley*, who, excelling in learning and distinguished as a patron of the highest arts, is a great ornament of his century and of our academy, the most worthy and upright Master of our *Trinity College*. To him in many ways I must express my indebtedness. And you too, benevolent reader, will not withhold the esteem due him. For many years an intimate friend of the celebrated author (since he aimed not only that the author should be esteemed by those who come after, but also that these uncommon writings should enjoy distinction among the literati of the world), he cared both for the reputation of his friend and for the advancement of the sciences. Since copies of the previous edition were very scarce and held at high prices, he persuaded by frequent entreaties and almost by chidings, the splendid man, distinguished alike for modesty and for erudition, to grant him permission for the appearance of this new edition, perfected throughout and enriched by new parts, at his expense and under his supervision. He assigned to me, as he had a right, the not unwelcome task of looking after the corrections as best I could.

ROGER COTER

Fellow of Trinity College.,
Plumian Professor of Astronomy
And [Experimental](#) Philosophy

Cambridge, May 12, 1713.

Newton's Preface to the Third Edition

IN THIS THIRD EDITION, prepared with much care by *Henry Pemberton*, M D , a man of the greatest skill in these matters, some things in the second Book on the resistance of mediums are somewhat more comprehensively handled than before, and new experiments on the resistance of heavy bodies falling in air are added. In the third Book, the argument to prove that the moon is retained in its orbit by the force of gravity is more fully stated; and there are added new observations made by Mr. *Pound*, concerning the ratio of the diameters of Jupiter to one another. Some observations are also added on the comet which appeared in the year 1680, made in *Germany* in the month of *November* by Mr. *Kirk*, which have lately come to my hands. By the help of these it becomes apparent how nearly parabolic orbits represent the motions of comets. The orbit of that comet is determined somewhat more accurately than before, by the computation of Dr. *Halley*, in an ellipse. And it is shown that, in this elliptic orbit, the comet took its course through the nine signs of the heavens, with as much accuracy as the planets move in the elliptic orbits given in astronomy. The orbit of the comet which appeared in the year 1723 is also added, computed by Mr. *Bradley*, Professor of Astronomy at *Oxford*

Is. NEWTON

London, Jan. 12, 1725-6.

[' Appendix, Note 9]

MATHEMATICAL PRINCIPLES OF NATURAL PHILOSOPHY¹

D *efinitions*

DEFINITION I

*The quantity of matter is the measure of the same, arising from its density and bulk conjointly.*²

THUS AIR of a double density, in a double space, is quadruple in quantity; in a triple space, sextuple in quantity. The same thing is to be understood of snow, and fine dust or powders, that are condensed by compression or liquefaction, and of all bodies that are by any causes whatever differently condensed. I have no regard in this place to a medium, if any such there is, that freely pervades the interstices between the parts of bodies. It is this quantity that I mean hereafter everywhere under the name of body or mass. And the same is known by the weight of each body, for it is proportional to the weight, as I have found by experiments on pendulums, very accurately made, which shall be shown hereafter.

DEFINITION II'

The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjointly.

The motion of the whole is the sum of the motions of all the parts; and therefore in a body double in quantity, with equal velocity, the motion is double; with twice the velocity, it is quadruple.

[¹ Appendix, Note 10] [² Appendix, Note 11.] [³ Appendix, Note 12]

DEFINITION III

The vis insita, or innate force of matter, is a power of resisting, by which every body, as much as in it lies, continues in its present state, whether it be of rest, or of moving uniformly forwards in a right line.

This force is always proportional to the body whose force it is and differs nothing from the inactivity of the mass, but in our manner of conceiving it. A body, from the inert nature of matter, is not without difficulty put out of its state of rest or motion. Upon which account, this *vis insita* may, by a most significant name, be called inertia (*vis inertia*) or force of inactivity. But a body only exerts this force when another force, impressed upon it, endeavors to change its condition; and the exercise of this force may be considered as both resistance and impulse; it is resistance so far as the body, for maintaining its present state, opposes the force impressed; it is impulse so far as the body, by not easily giving way to the impressed force of another, endeavors to change the state of that other. Resistance is usually ascribed to bodies at rest, and impulse to those in motion; but motion and rest, as commonly conceived, are only relatively distinguished; nor are those bodies always truly at rest, which commonly are taken to be so.

DEFINITION IV

An impressed force is an action exerted upon a body, in order to change its state, either of rest, or of uniform motion in a right line.

This force consists in the action only, and remains no longer in the body when the action is over. For a body maintains every new state it acquires, by its inertia only. But impressed forces are of different origins, as from percussion, from pressure, from centripetal force.

DEFINITION V

A centripetal force is that by which bodies are drawn or impelled, or any way tend, towards a point as to a centre.

Of this sort is gravity, by which bodies tend to the centre of the earth; magnetism, by which iron tends to the loadstone; and that force, whatever it is, by which the planets are continually drawn aside from the rectilinear motions, which otherwise they would pursue, and made to revolve in curvilinear orbits. A stone, whirled about in a sling, endeavors to recede from the

hand that turns it; and by that endeavor, distends the sling, and that with so much the greater force, as it is revolved with the greater velocity, and as soon as it is let go, flies away. That force which opposes itself to this endeavor, and by which the sling continually draws back the stone towards the hand, and retains it in its orbit, because it is directed to the hand as the centre of the orbit, I call the centripetal force. And the same thing is to be understood of all bodies, revolved in any orbits. They all endeavor to recede from the centres of their orbits; and were it not for the opposition of a contrary force which restrains them to, and detains them in their orbits, which I therefore call centripetal, would fly off in right lines, with an uniform motion. A projectile, if it was not for the force of gravity, would not deviate towards the earth, but would go off from it in a right line, and that with an uniform motion, if the resistance of the air was taken away. It is by its gravity that it is drawn aside continually from its rectilinear course, and made to deviate towards the earth, more or less, according to the force of its gravity, and the velocity of its motion. The less its gravity is, or the quantity of its matter, or the greater the velocity with which it is projected, the less will it deviate from a rectilinear course, and the farther it will go. If a leaden ball, projected from the top of a mountain by the force of gunpowder, with a given velocity, and in a direction parallel to the horizon, is carried in a curved line to the distance of two miles before it falls to the ground; the same, if the resistance of the air were taken away, with a double or decuple velocity, would fly twice or ten times as far. And by increasing the velocity, we may at pleasure increase the distance to which it might be projected, and diminish the curvature of the line which it might describe, till at last it should fall at the distance of 10, 30, or 90 degrees, or even might go quite round the whole earth before it falls; or lastly, so that it might never fall to the earth, but go forwards into the celestial spaces, and proceed in its motion *in infinitum*. And after the same manner that a projectile, by the force of gravity, may be made to revolve in an orbit, and go round the whole earth, the moon also, either by the force of gravity, if it is endued with gravity, or by any other force, that impels it towards the earth, may be continually drawn aside towards the earth, out of the rectilinear way which by its innate force it would pursue; and would be made to revolve in the orbit which it now describes; nor could the moon without some such force be retained in its

orbit. If this force was too small, it would not sufficiently turn the moon out of a rectilinear course; if it was too great, it would turn it too much, and draw down the moon from its orbit towards the earth. It is necessary that the force be of a just quantity, and it belongs to the mathematicians to find the force that may serve exactly to retain a body in a given orbit with a given velocity; and *vice versa*, to determine the curvilinear way into which a body projected from a given place, with a given velocity, may be made to deviate from its natural rectilinear way, by means of a given force.

The quantity of any centripetal force may be considered as of three kinds: absolute, accelerative, and motive.

DEFINITION VI

The absolute quantity of a centripetal force is the measure of the same, proportional to the efficacy of the cause that propagates it from the centre, through the spaces round about.

Thus the magnetic force is greater in one loadstone and less in another, according to their sizes and strength of intensity.

DEFINITION VII

The accelerative quantity of a centripetal force is the measure of the same, proportional to the velocity which it generates in a given time.

Thus the force of the same loadstone is greater at a less distance, and less at a greater: also the force of gravity is greater in valleys, less on tops of exceeding high mountains; and yet less (as shall hereafter be shown), at greater distances from the body of the earth; but at equal distances, it is the same everywhere; because (taking away, or allowing for, the resistance of the air), it equally accelerates all falling bodies, whether heavy or light, great or small.

DEFINITION VIII

The motive quantity of a centripetal force is the measure of the same, proportional to the motion which it generates in a given time.

Thus the weight is greater in a greater body, less in a less body; and, in the same body, it is greater near to the earth, and less at remoter distances. This sort of quantity is the centripetency, or propension of the whole body towards the centre, or, as I may say, its weight; and it is always known by

the quantity of an equal and contrary force just sufficient to hinder the descent of the body.

These quantities of forces, we may, for the sake of brevity, call by the names of motive, accelerative, and absolute forces; and, for the sake of distinction, consider them with respect to the bodies that tend to the centre, to the places of those bodies, and to the centre of force towards which they tend; that is to say, I refer the motive force to the body as an endeavor and propensity of the whole towards a centre, arising from the propensities of the several parts taken together; the accelerative force to the place of the body, as a certain power diffused from the centre to all places around to move the bodies that are in them; and the absolute force to the centre, as endued with some cause, without which those motive forces would not be propagated through the spaces round about; whether that cause be some central body (such as is the magnet in the centre of the magnetic force, or the earth in the centre of the gravitating force), or anything else that does not yet appear. For I here design only to give a mathematical notion of those forces, without considering their physical causes and seats.

Wherefore the accelerative force will stand in the same relation to the motive, as celerity does to motion. For the quantity of motion arises from the celerity multiplied by the quantity of matter; and the motive force arises from the accelerative force multiplied by the same quantity of matter. For the sum of the actions of the accelerative force, upon the several particles of the body, is the motive force of the whole. Hence it is, that near the surface of the earth, where the accelerative gravity, or force productive of gravity, in all bodies is the same, the motive gravity or the weight is as the body; but if we should ascend to higher regions, where the accelerative gravity is less, the weight would be equally diminished, and would always be as the product of the body, by the accelerative gravity. So in those regions, where the accelerative gravity is diminished into one-half, the weight of a body two or three times less, will be four or six times less.

I likewise call attractions and impulses, in the same sense, accelerative, and motive; and use the words attraction, impulse, or propensity of any sort towards a centre, promiscuously, and indifferently, one for another; considering those forces not physically, but mathematically: wherefore the reader is not to imagine that by those words I anywhere take upon me to

define the kind, or the manner of any action, the causes or the physical reason thereof, or that I attribute forces, in a true and physical sense, to certain centres (which are only mathematical points); when at any time I happen to speak of centres as attracting, or as endued with attractive powers.

SCHOLIUM¹

Hitherto I have laid down the definitions of such words as are less known, and explained the sense in which I would have them to be understood in the following discourse. I do not define time, space, place, and motion, as being well known to all. Only I must observe, that the common people conceive those quantities under no other notions but from the relation they bear to sensible objects. And thence arise certain prejudices, for the removing of which it will be convenient to distinguish them into absolute and relative, true and apparent, mathematical and common.

I. Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external, and by another name is called duration: relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, which is commonly used instead of true time; such as an hour, a day, a month, a year.

II. Absolute space, in its own nature, without relation to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position to bodies; and which is commonly taken for immovable space; such is the dimension of a subterraneous, an aerial, or celestial space, determined by its position in respect of the earth. Absolute and relative space are the same in figure and magnitude; but they do not remain always numerically the same. For if the earth, for instance, moves, a space of our air, which relatively and in respect of the earth remains always the same, will at one time be one part of the absolute space into which the air passes; at another time it will be another part of the same, and so, absolutely understood, it will be continually changed.

III. Place is a part of space which a body takes up, and is according to the space, either absolute or relative. I say, a part of space; not the situation, nor the external surface of the body. For the places of equal solids are always

[¹ Appendix, Note 13.]

equal; but their surfaces, by reason of their dissimilar figures, are often unequal. Positions properly have no quantity, nor are they so much the places themselves, as the properties of places. The motion of the whole is the same with the sum of the motions of the parts; that is, the translation of the whole, out of its place, is the same thing with the sum of the translations of the parts out of their places; and therefore the place of the whole is the same as the sum of the places of the parts, and for that reason, it is internal, and in the whole body.

IV. Absolute motion is the translation of a body from one absolute place into another; and relative motion, the translation from one relative place into another. Thus in a ship under sail, the relative place of a body is that part of the ship which the body possesses; or that part of the cavity which the body fills, and which therefore moves together with the ship: and relative rest is the continuance of the body in the same part of the ship, or of its cavity. But real, absolute rest, is the continuance of the body in the same part of that immovable space, in which the ship itself, its cavity, and all that it contains, is moved. Wherefore, if the earth is really at rest, the body, which relatively rests in the ship, will really and absolutely move with the same velocity which the ship has on the earth. But if the earth also moves, the true and absolute motion of the body will arise, partly from the true motion of the earth, in immovable space, partly from the relative motion of the ship on the earth; and if the body moves also relatively in the ship, its true motion will arise, partly from the true motion of the earth, in immovable space, and partly from the relative motions as well of the ship on the earth, as of the body in the ship; and from these relative motions will arise the relative motion of the body on the earth. As if that part of the earth, where the ship is, was truly moved towards the east, with a velocity of 10010 parts; while the ship itself, with a fresh gale, and full sails, is carried towards the west, with a velocity expressed by 10 of those parts; but a sailor walks in the ship towards the east, with 1 part of the said velocity; then the sailor will be moved truly in immovable space towards the east, with a velocity of 10001 parts, and relatively on the earth towards the west, with a velocity of 9 of those parts.

Absolute time, in astronomy, is distinguished from relative, by the equation or correction of the apparent time. For the natural days are truly un-

equal, though they are commonly considered as equal, and used for a measure of time; astronomers correct this inequality that they may measure the celestial motions by a more accurate time. It may be, that there is no such thing as an equable motion, whereby time may be accurately measured. All motions may be accelerated and retarded, but the flowing of absolute time is not liable to any change. The duration or perseverance of the existence of things remains the same, whether the motions are swift or slow, or none at all: and therefore this duration ought to be distinguished from what are only sensible measures thereof; and from which we deduce it, by means of the astronomical equation. The necessity of this equation, for determining the times of a phenomenon, is evinced as well from the experiments of the pendulum clock, as by eclipses of the satellites of Jupiter.

As the order of the parts of time is immutable, so also is the order of the parts of space. Suppose those parts to be moved out of their places, and they will be moved (if the expression may be allowed) out of themselves. For times and spaces are, as it were, the places as well of themselves as of all other things. All things are placed in time as to order of succession; and in space as to order of situation. It is from their essence or nature that they are places; and that the primary places of things should be movable, is absurd. These are therefore the absolute places; and translations out of those places, are the only absolute motions.

But because the parts of space cannot be seen, or distinguished from one another by our senses, therefore in their stead we use sensible measures of them. For from the positions and distances of things from any body considered as immovable, we define all places; and then with respect to such places, we estimate all motions, considering bodies as transferred from some of those places into others. And so, instead of absolute places and motions, we use relative ones; and that without any inconvenience in common affairs; but in philosophical disquisitions, we ought to abstract from our senses, and consider things themselves, distinct from what are only sensible measures of them. For it may be that there is no body really at rest, to which the places and motions of others may be referred.

But we may distinguish rest and motion, absolute and relative, one from the other by their properties, causes, and effects. It is a property of rest, that bodies really at rest do rest in respect to one another. And therefore as it is

possible, that in the remote regions of the fixed stars, or perhaps far beyond them, there may be some body absolutely at rest; but impossible to know, from the position of bodies to one another in our regions, whether any of these do keep the same position to that remote body, it follows that absolute rest cannot be determined from the position of bodies in our regions.

It is a property of motion, that the parts, which retain given positions to their wholes, do partake of the motions of those wholes. For all the parts of revolving bodies endeavor to recede from the axis of motion; and the impetus of bodies moving forwards arises from the joint impetus of all the parts. Therefore, if surrounding bodies are moved, those that are relatively at rest within them will partake of their motion. Upon which account, the true and absolute motion of a body cannot be determined by the translation of it from those which only seem to rest; for the external bodies ought not only to appear at rest, but to be really at rest. For otherwise, all included bodies, besides their translation from near the surrounding ones, partake likewise of their true motions; and though that translation were not made, they would not be really at rest, but only seem to be so. For the surrounding bodies stand in the like relation to the surrounded as the exterior part of a whole does to the interior, or as the shell does to the kernel; but if the shell moves, the kernel will also move, as being part of the whole, without any removal from near the shell.

A property, near akin to the preceding, is this, that if a place is moved, whatever is placed therein moves along with it; and therefore a body, which is moved from a place in motion, partakes also of the motion of its place. Upon which account, all motions, from places in motion, are no other than parts of entire and absolute motions; and every entire motion is composed of the motion of the body out of its first place, and the motion of this place out of its place; and so on, until we come to some immovable place, as in the before-mentioned example of the sailor. Wherefore, entire and absolute motions can be no otherwise determined than by immovable places; and for that reason I did before refer those absolute motions to immovable places, but relative ones to movable places. Now no other places are immovable but those that, from infinity to infinity, do all retain the same given position one to another; and upon this account must ever remain unmoved; and do thereby constitute immovable space.

The causes by which true and relative motions are distinguished, one from the other, are the forces impressed upon bodies to generate motion. True motion is neither generated nor altered, but by some force impressed upon the body moved; but relative motion may be generated or altered without any force impressed upon the body. For it is sufficient only to impress some force on other bodies with which the former is compared, that by their giving way, that relation may be changed, in which the relative rest or motion of this other body did consist. Again, true motion suffers always some change from any force impressed upon the moving body; but relative motion does not necessarily undergo any change by such forces. For if the same forces are likewise impressed on those other bodies, with which the comparison is made, that the relative position may be preserved, then that condition will be preserved in which the relative motion consists. And therefore any relative motion may be changed when the true motion remains unaltered, and the relative may be preserved when the true suffers some change. Thus, true motion by no means consists in such relations.

The effects which distinguish absolute from relative motion are, the forces of receding from the axis of circular motion. For there are no such forces in a circular motion purely relative, but in a true and absolute circular motion, they are greater or less, according to the quantity of the motion. If a vessel, hung by a long cord, is so often turned about that the cord is strongly twisted, then filled with water, and held at rest together with the water; thereupon, by the sudden action of another force, it is whirled about the contrary way, and while the cord is untwisting itself, the vessel continues for some time in this motion; the surface of the water will at first be plain, as before the vessel began to move; but after that, the vessel, by gradually communicating its motion to the water, will make it begin sensibly to revolve, and recede by little and little from the middle, and ascend to the sides of the vessel, forming itself into a concave figure (as I have experienced), and the swifter the motion becomes, the higher will the water rise, till at last, performing its revolutions in the same times with the vessel, it becomes relatively at rest in it. This ascent of the water shows its endeavor to recede from the axis of its motion; and the true and absolute circular motion of the water, which is here directly contrary to the relative, becomes known, and may be measured by this endeavor. At first, when the relative

motion of the water in the vessel was greatest, it produced no endeavor to recede from the axis; the water showed no tendency to the circumference, nor any ascent towards the sides of the vessel, but remained of a plain surface, and therefore its true circular motion had not yet begun. But afterwards, when the relative motion of the water had decreased, the ascent thereof towards the sides of the vessel proved its endeavor to recede from the axis; and this endeavor showed the real circular motion of the water continually increasing, till it had acquired its greatest quantity, when the water rested relatively in the vessel. And therefore this endeavor docs not depend upon any translation of the water in respect of the ambient bodies, nor can true circular motion be defined by such translation. There is only one real circular motion of any one revolving body, corresponding to only one power of endeavoring to recede from its axis of motion, as its proper and adequate effect; but relative motions, in one and the same body, are innumerable, according to the various relations it bears to external bodies, and, like other relations, are altogether destitute of any real effect, any other-wise than they may perhaps partake of that one only true motion. And therefore in their system who suppose that our heavens, revolving below the sphere of the fixed stars, carry the planets along with them; the several parts of those heavens, and the planets, which are indeed relatively at rest in their heavens, do yet really move. For they change their position one to another (which never happens to bodies truly at rest), and being carried together with their heavens, partake of their motions, and as parts of revolving wholes, endeavor to recede from the axis of their motions.

Wherefore relative quantities are not the quantities themselves, whose names they bear, but those sensible measures of them (either accurate or inaccurate), which are commonly used instead of the measured quantities themselves. And if the meaning of words is to be determined by their use, then by the names time, space, place, and motion, their [sensible] measures are properly to be understood; and the expression will be unusual, and purely mathematical, if the measured quantities themselves are meant. On this account, those violate the accuracy of language, which ought to be kept precise, who interpret these words for the measured quantities. Nor do those less defile the purity of mathematical and philosophical truths, who confound real quantities with their relations and sensible measures.

It is indeed a matter of great difficulty to discover, and effectually to distinguish, the true motions of particular bodies from the apparent; because the parts of that immovable space, in which those motions are performed, do by no means come under the observation of our senses. Yet the thing is not altogether desperate; for we have some arguments to guide us, partly from the apparent motions, which are the differences of the true motions; partly from the forces, which are the causes and effects of the true motions. For instance, if two globes, kept at a given distance one from the other by means of a cord that connects them, were revolved about their common centre of gravity, we might, from the tension of the cord, discover the endeavor of the globes to recede from the axis of their motion, and from thence we might compute the quantity of their circular motions. And then if any equal forces should be impressed at once on the alternate faces of the globes to augment or diminish their circular motions, from the increase or decrease of the tension of the cord, we might infer the increment or decrement of their motions; and thence would be found on what faces those forces ought to be impressed, that the motions of the globes might be most augmented; that is, we might discover their hindmost faces, or those which, in the circular motion, do follow. But the faces which follow being known, and consequently the opposite ones that precede, we should likewise know the determination of their motions. And thus we might find both the quantity and the determination of this circular motion, even in an immense vacuum, where there was nothing external or sensible with which the globes could be compared. But now, if in that space some remote bodies were placed that kept always a given position one to another, as the fixed stars do in our regions, we could not indeed determine from the relative translation of the globes among those bodies, whether the motion did belong to the globes or to the bodies. But if we observed the cord, and found that its tension was that very tension which the motions of the globes required, we might conclude the motion to be in the globes, and the bodies to be at rest; and then, lastly, from the translation of the globes among the bodies, we should find the determination of their motions. But how we are to obtain the true motions from their causes, effects, and apparent differences, and the converse, shall be explained more at large in the following treatise. For to this end it was that I composed it.

AXIOMS, OR LAWS OF MOTION¹

LAW I

Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

PROJECTILES continue in their motions, so far as they are not retarded by the resistance of the air, or impelled downwards by the force of gravity. A top, whose parts by their cohesion are continually drawn aside from rectilinear motions, does not cease its rotation, otherwise than as it is retarded by the air. The greater bodies of the planets and comets, meeting with less resistance in freer spaces, preserve their motions both progressive and circular for a much longer time.

LAW II²

The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

If any ibrc generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force be impressed altogether and at once, or gradually and successively. And this motion (being always directed the same way with the generating force), if the body moved before, is added to or subtracted from the former motion, according as they directly conspire with or are directly contrary to each other; or obliquely joined, when they are oblique, so as to produce a new motion compounded from the determination of both.

LAW III

To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

Whatever draws or presses another is as much drawn or pressed by that other. If you press a stone with your finger, the finger is also pressed by the

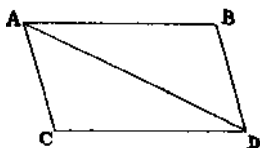
[¹ Appendix, Note 14] [² Appendix, Note 15]

stone. If a horse draws a stone tied to a rope, the horse (if I may so say) will be equally drawn back towards the stone; for the distended rope, by the same endeavor to relax or unbend itself, will draw the horse as much towards the stone as it does the stone towards the horse, and will obstruct the progress of the one as much as it advances that of the other. If a body impinge upon another, and by its force change the motion of the other, that body also (because of the equality of the mutual pressure) will undergo an equal change, in its own motion, towards the contrary part. The changes made by these actions are equal, not in the velocities but in the motions of bodies; that is to say, if the bodies are not hindered by any other impediments. For, because the motions are equally changed, the changes of the velocities made towards contrary parts are inversely proportional to the bodies. This law takes place also in attractions, as will be proved in the next Scholium.

COROLLARY I

A body, acted on by two forces simultaneously, wdl describe the diagonal of a parallelogram in the same time as it would describe the sides by those forces separately.

If a body in a given time, by the force M impressed apart in the place A, should with an uniform motion be carried from A to B, and by the force N impressed apart in the same place, should be carried from A to C, let the parallelogram ABCD be completed, and, by both forces acting together, it will in the same time be carried in the diagonal from A to D. For since the force N acts in the direction of the line AC, parallel to BD, this force (by the second Law) will not at all alter the velocity generated by the other force M, by which the body is carried towards the line BD. The body therefore will arrive at the line BD in the same time, whether the force N be impressed or not; and therefore at the end of that time it will be found somewhere in the line BD. By the same argument, at the end of the same time it will be found somewhere in the line CD. Therefore it will be found in the point D, where both lines meet. But it will move in a right line from A to D, by Law 1.



COROLLARY II

And hence is explained the composition of any one direct force AD, out of any two oblique forces AC and CD; and, on the contrary, the resolution of any one direct force AD into two oblique forces AC and CD: which composition and resolution are abundantly confirmed from mechanics.

As if the unequal radii OM and ON drawn from the centre O of any wheel, should sustain the weights A and P by the cords MA and NP; and the forces of those weights to move the wheel were required. Through the centre O draw the right line KOL, meeting the cords perpendicularly in K and L; and from the centre O, with OL the greater of the distances OK and OL, describe a circle, meeting the cord MA in D; and drawing OD, make AC parallel and DC perpendicular thereto. Now, it being in-different whether the points K, L, D, of the cords be fixed to the plane of the wheel or not, the weights will have the same effect whether they are suspended from the points K and L, or from D and L. Let the whole force of the weight A be represented by the line AD, and let it be resolved into the forces AC and CD, of which the force AC, drawing the radius OD directly from the centre, will have no effect to move the wheel; but the other force DC, drawing the radius DO perpendicularly, will have the same effect as if it drew perpendicularly the radius OL equal to OD; that is, it will have the same effect as the weight P, if

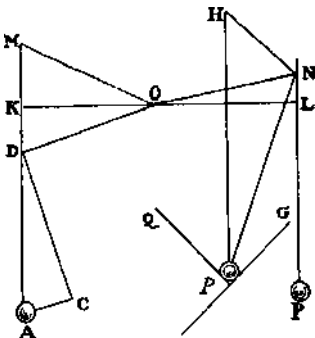
$$P : A - DC : DA,$$

but because the triangles ADC and DOK are similar,

$$DC:DA = OK:OD = OK:OL.$$

Therefore,

$P : A = \text{radius OK} : \text{radius OL}.$



As these radii lie in the same right line they will be equipollent, and so remain in equilibrium; which is the well-known property of the balance, the lever, and the wheel. If either weight is greater than in this ratio, its force to move the wheel will be so much greater.

If the weight p - P , is partly suspended by the cord Np , partly sustained by the oblique plane pG ; draw pH , NH , the former perpendicular to the horizon, the latter to the plane pG ; and if the force of the weight p tending downwards is represented by the line pH , it may be resolved into the forces pN , HN . If there was any plane pQ , perpendicular to the cord pN , cutting the other plane pG in a line parallel to the horizon, and the weight p was supported only by those planes pQ , pG , it would press those planes perpendicularly with the forces pN , HN ; to wit, the plane pQ with the force pN , and the plane pG with the force HN . And therefore if the plane pQ was taken away, so that the weight might stretch the cord, because the cord, now sustaining the weight, supplied the place of the plane that was removed, it would be strained by the same force pN which pressed upon the plane before. Therefore, the

$$\text{tension of } pN : \text{tension of } PN - \text{line } pN : \text{line } pH.$$

Therefore, if

$$p : A = OK : OL - \text{line } pH : \text{line } pN,$$

then the weights p and A will have the same effect towards moving the wheel, and will therefore sustain each other; as anyone may find by experiment.

But the weight p pressing upon those two oblique planes, may be considered as a wedge between the two internal surfaces of a body split by it; and hence the forces of the wedge and the mallet may be determined: because the force with which the weight p presses the plane pQ is to the force with which the same, whether by its own gravity, or by the blow of a mallet, is impelled in the direction of the line pH towards both the planes, as

$$pN : pH;$$

and to the force with which it presses the other plane pG , as

$$pN : NH.$$

And thus the force of the screw may be deduced from a like resolution of forces; it being no other than a wedge impelled with the force of a lever. Therefore the use of this Corollary spreads far and wide, and by that diffusive extent the truth thereof is further confirmed. For on what has been said depends the whole doctrine of mechanics variously demonstrated by different authors. For from hence are easily deduced the forces of machines, which are compounded of wheels, pullics, levers, cords, and weights, ascending directly or obliquely, and other mechanical powers; as also the force of the tendons to move the bones of animals.

COROLLARY III

The quantity of motion, which is obtained by taking the sum of the motions directed towards the same parts, and the difference of those that are directed to contrary parts, suffers no change from the action of bodies among themselves.

For action and its opposite reaction are equal, by Law 10, and therefore, by Law 11, they produce in the motions equal changes towards opposite parts. Therefore if the motions are directed towards the same parts, whatever is added to the motion of the preceding body will be subtracted from the motion of that which follows; so that the sum will be the same as before. If the bodies meet, with contrary motions, there will be an equal deduction from the motions of both; and therefore the difference of the motions directed towards opposite parts will remain the same.

Thus, if a spherical body A is 3 times greater than the spherical body B, and has a velocity = 2, and B follows in the same direction with a velocity = 10, then the

$$\text{motion of A : motion of B} = 6 : 10.$$

Suppose, then, their motions to be of 6 parts and of 10 parts, and the sum will be 16 parts. Therefore, upon the meeting of the bodies, if A acquire 3, 4, or 5 parts of motion, B will lose as many; and therefore after reflection A will proceed with 9, 10, or 11 parts, and B with 7, 6, or 5 parts; the sum remaining always of 16 parts as before. If the body A acquire 9, 10, 11, or 12 parts of motion, and therefore after meeting proceed with 15, 16, 17, or

18 parts, the body B, losing so many parts as A has got, will either proceed with i part, having lost 9, or stop and remain at rest, as having lost its whole progressive motion of 10 parts; or it will go back with 1 part, having not only lost its whole motion, but (if I may so say) one part more; or it will go back with 2 parts, because a progressive motion of 12 parts is taken off. And so the sums of the conspiring motions,

$$15 + 1 \text{ or } 16 + 0,$$

and the differences of the contrary motions,

$$17-1 \text{ and } 18-2,$$

will always be equal to 16 parts, as they were before the meeting and reflection of the bodies. But the motions being known with which the bodies proceed after reflection, the velocity of either will be also known, by taking the velocity after to the velocity before reflection, as the motion after is to the motion before. As in the last case, where the

$$\begin{aligned} &\text{motion of A before reflection (6) : motion of A after (18)} \\ &= \text{velocity of A before (2) : velocity of A after (x);} \end{aligned}$$

that is,

$$6 : 18 = 2 : x, x=6.$$

But if the bodies are either not spherical, or, moving in different right lines, impinge obliquely one upon the other, and their motions after reflection are required, in those cases we are first to determine the position of the plane that touches the bodies in the point of impact, then the motion of each body (by Cor. 11) is to be resolved into two, one perpendicular to that plane, and the other parallel to it. This done, because the bodies act upon each other in the direction of a line perpendicular to this plane, the parallel motions are to be retained the same after reflection as before; and to the perpendicular motions we are to assign equal changes towards the contrary parts; in such manner that the sum of the conspiring and the difference of the contrary motions may remain the same as before. From such kind of reflections sometimes arise also the circular motions of bodies about their own centres. But these are cases which I do not consider in what follows; and it would be too tedious to demonstrate every particular case that relates to this subject.

COROLLARY IV

The common centre of gravity of two or more bodies does not alter its state of motion or rest by the actions of the bodies among themselves; and therefore the common centre of gravity of all bodies acting upon each other (excluding external actions and impediments) is either at rest, or moves uniformly in a right line.

For if two points proceed with an uniform motion in right lines, and their distance be divided in a given ratio, the dividing point will be either at rest, or proceed uniformly in a right line. This is demonstrated hereafter in Lem. XXIII and Corollary, when the points are moved in the same plane; and by a like way of arguing, it may be demonstrated when the points are not moved in the same plane. Therefore if any number of bodies move uniformly in right lines, the common centre of gravity of any two of them is either at rest, or proceeds uniformly in a right line; because the line which connects the centres of those two bodies so moving is divided at that common centre in a given ratio. In like manner the common centre of those two and that of a third body will be either at rest or moving uniformly in a right line; because at that centre the distance between the common centre of the two bodies, and the centre of this last, is divided in a given ratio. In like manner the common centre of these three, and of a fourth body, is either at rest, or moves uniformly in a right line; because the distance between the common centre of the three bodies, and the centre of the fourth, is there also divided in a given ratio, and so on *in infinitum*. Therefore, in a system of bodies where there is neither any mutual action among themselves, nor any foreign force impressed upon them from without, and which consequently move uniformly in right lines, the common centre of gravity of them all is either at rest or moves uniformly forwards in a right line.

Moreover, in a system of two bodies acting upon each other, since the distances between their centres and the common centre of gravity of both are reciprocally as the bodies, the relative motions of those bodies, whether of approaching to or of receding from that centre, will be equal among themselves. Therefore since the changes which happen to motions are equal and directed to contrary parts, the common centre of those bodies, by their mutual action between themselves, is neither accelerated nor retarded, nor

suffers any change as to its state of motion or rest. But in a system of several bodies, because the common centre of gravity of any two acting upon each other suffers no change in its state by that action; and much less the common centre of gravity of the others with which that action does not intervene; but the distance between those two centres is divided by the common centre of gravity of all the bodies into parts inversely proportional to the total sums of those bodies whose centres they are; and therefore while those two centres retain their state of motion or rest, the common centre of all does also retain its state: it is manifest that the common centre of all never suffers any change in the state of its motion or rest from the actions of any two bodies between themselves. But in such a system all the actions of the bodies among themselves either happen between two bodies, or are composed of actions interchanged between some two bodies; and therefore they do never produce any alteration in the common centre of all as to its state of motion or rest. Wherefore since that centre, when the bodies do not act one upon another, either is at rest or moves uniformly forwards in some right line, it will, notwithstanding the mutual actions of the bodies among themselves, always continue in its state, either of rest, or of proceeding uniformly in a right line, unless it is forced out of this state by the action of some power impressed from without upon the whole system. And therefore the same law takes place in a system consisting of many bodies as in one single body, with regard to their persevering in their state of motion or of rest. For the progressive motion, whether of one single body, or of a whole system of bodies, is always to be estimated from the motion of the centre of gravity.

C O R O L L A R Y V

The motions of bodies included in a given space are the same among themselves, whether that space is at rest, or moves uniformly forwards in a right line without any circular motion.

For the differences of the motions tending towards the same parts, and the sums of those that tend towards contrary parts, are, at first (by supposition), in both cases the same; and it is from those sums and differences that the collisions and impulses do arise with which the bodies impinge one upon another. Wherefore (by Law II), the effects of those collisions will be

equal in both cases; and therefore the mutual motions of the bodies among themselves in the one case will remain equal to the motions of the bodies among themselves in the other. A clear proof of this we have from the experiment of a ship; where all motions happen after the same manner, whether the ship is at rest, or is carried uniformly forwards in a right line.

COROLLARY VI

If bodies, moved in any manner among themselves, are urged in the direction of parallel lines by equal accelerative forces, they will all continue to move among themselves, after the same manner as if they had not been urged by those forces.

For these forces acting equally (with respect to the quantities of the bodies to be moved), and in the direction of parallel lines, will (by Law u) move all the bodies equally (as to velocity), and therefore will never produce any change in the positions or motions of the bodies among themselves.

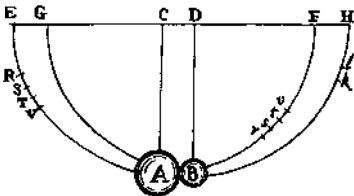
SCHOLIUM¹

Hitherto I have laid down such principles as have been received by mathematicians, and are confirmed by abundance of experiments. By the first two Laws and the first two Corollaries, *Galileo* discovered that the descent of bodies varied as the square of the time (in *duplicata ratione temporis*) and that the motion of projectiles was in the curve of a parabola; experience agreeing with both, unless so far as these motions are a little retarded by the resistance of the air. When a body is falling, the uniform force of its gravity acting equally, impresses, in equal intervals of time, equal forces upon that body, and therefore generates equal velocities; and in the whole time impresses a whole force, and generates a whole velocity proportional to the time. And the spaces described in proportional times are as the product of the velocities and the times; that is, as the squares of the times. And when a body is thrown upwards, its uniform gravity impresses forces and reduces velocities proportional to the times; and the times of ascending to the greatest heights are as the velocities to be taken away, and those heights are as the product of the velocities and the times, or as the squares of the velocities. And if a body be projected in any direction, the motion arising from its projection is compounded with the motion arising from its gravity. Thus,

^[1] Appendix, Note 16]

if the body A by its motion of projection alone could describe in a given time the right line AB, and with its motion of falling alone could describe in the same time the altitude AC; complete the parallelogram ABCD, and the body by that compounded motion will at the end of the time be found in the place D; and the curved line AED, which that body describes, will be a parabola, to which the right line AB will be a tangent at A; and whose ordinate BD will be as the square of the line AB. On the same Laws and Corollaries depend those things which have been demonstrated concerning the times of the vibration of pendulums, and are confirmed by the daily experiments of pendulum clocks. By the same, together with Law in, Sir *Christopher Wren*, Dr. *Walks*, and Mr. *Huygens*, the greatest geometers of our times, did severally determine the rules of the impact and reflection of hard bodies, and about the same time communicated their discoveries to the *Royal Society*, exactly agreeing among themselves as to those rules. Dr. *Wallis*, indeed, was somewhat earlier in the publication; then followed Sir *Christopher Wren*, and, lastly, Mr. *Huygens*. But Sir *Christopher Wren* confirmed the truth of the thing before the *Royal Society* by the experiments on pendulums, which M. *Martotte* soon after thought fit to explain in a treatise entirely upon that subject.¹ But to bring this experiment to an accurate agreement with the theory, we are to have due regard as well to the resistance of the air as to the elastic force of the concurring bodies.

Let the spherical bodies A, B be suspended by the parallel and equal strings AC, BD, from the centres C, D. About these centres, with those lengths as radii, describe the semicircles EAF, GBH, bisected respectively by the radii CA, DB. Bring the



body A to any point R of the arc EAF, and (withdrawing the body B) let it go from thence, and after one oscillation suppose it to return to the point

¹ Appendix, Note 17]

V: then RV will be the retardation arising from the resistance of the air. Of this RV let ST be a fourth part, situated in the middle, namely, so that

$$RS = TV,$$

and

$$RS:ST = 3:2,$$

then will ST represent very nearly the retardation during the descent from S to A. Restore the body B to its place: and, supposing the body A to be let fall from the point S, the velocity thereof in the place of reflection A, without sensible error, will be the same as if it had descended *in vacuo* from the point T. Upon which account this velocity may be represented by the chord of the arc TA. For it is a proposition well known to geometers, that the velocity of a pendulous body in the lowest point is as the chord of the arc which it has described in its descent. After reflection, suppose the body A comes to the place s, and the body B to the place k. Withdraw the body B, and find the place v, from which if the body A, being let go, should after one oscillation return to the place r, st may be a fourth part of rv, so placed in the middle thereof as to leave rs equal to tv, and let the chord of the arc /A represent the velocity which the body A had in the place A immediately after reflection. For t will be the true and correct place to which the body A should have ascended, if the resistance of the air had been taken off. In the same way we are to correct the place k to which the body B ascends, by finding the place l to which it should have ascended *in vacuo*. And thus everything may be subjected to experiment, in the same manner as if we were really placed *in vacuo*. These things being done, we are to take the product (if I may so say) of the body A, by the chord of the arc TA (which represents its velocity), that we may have its motion in the place A immediately before reflection; and then by the chord of the arc /A, that we may have its motion in the place A immediately after reflection. And so we are to take the product of the body B by the chord of the arc B/, that we may have the motion of the same immediately after reflection. And in like manner, when two bodies are let go together from different places, we are to find the motion of each, as well before as after reflection; and then we may compare the motions between themselves, and collect the effects of the reflection. Thus trying the thing with pendulums of 10 feet, in unequal as well as equal bodies, and making the bodies to concur after a descent through large

spaces, as of 8, 12, or 16 feet, I found always, without an error of 3 inches, that when the bodies concurred together directly, equal changes towards the contrary parts were produced in their motions, and, of consequence, that the action and reaction were always equal. As if the body A impinged upon the body B at rest with 9 parts of motion, and losing 7, proceeded after reflection with 2, the body B was carried backwards with those 7 parts. If the bodies concurred with contrary motions, A with 12 parts of motion, and B with 6, then if A receded with 2, B receded with 8; namely, with a deduction of 14 parts of motion on each side. For from the motion of A subtracting 12 parts, nothing will remain; but subtracting 2 parts more, a motion will be generated of 2 parts towards the contrary way; and so, from the motion of the body B of 6 parts, subtracting 14 parts, a motion is generated of 8 parts towards the contrary way. But if the bodies were made both to move towards the same way, A, the swifter, with 14 parts of motion, B, the slower, with 5, and after reflection A went on with 5, B likewise went on with 14 parts; 9 parts being transferred from A to B. And so in other cases. By the meeting and collision of bodies, the quantity of motion, obtained from the sum of the motions directed towards the same way, or from the difference of those that were directed towards contrary ways, was never changed. For the error of an inch or two in measures may be easily ascribed to the difficulty of executing everything with accuracy. It was not easy to let go the two pendulums so exactly together that the bodies should impinge one upon the other in the lowermost place AB; nor to mark the places s, and k, to which the bodies ascended after impact. Nay, and some errors, too, might have happened from the unequal density of the parts of the pendulous bodies themselves, and from the irregularity of the texture proceeding from other causes.

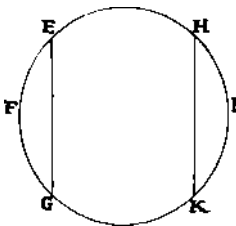
But to prevent an objection that may perhaps be alleged against the rule, for the proof of which this experiment was made, as if this rule did suppose that the bodies were either absolutely hard, or at least perfectly elastic (whereas no such bodies are to be found in Nature), I must add, that the experiments we have been describing, by no means depending upon that quality of hardness, do succeed as well in soft as in hard bodies. For if the rule is to be tried in bodies not perfectly hard, we are only to diminish the reflection in such a certain proportion as the quantity of the elastic force

requires. By the theory of *Wren* and *Huygens*, bodies absolutely hard return one from another with the same velocity with which they meet. But this may be affirmed with more certainty of bodies perfectly elastic. In bodies imperfectly elastic the velocity of the return is to be diminished together with the elastic force; because that force (except when the parts of bodies are bruised by their impact, or suffer some such extension as happens under the strokes of a hammer) is (as far as I can perceive) certain and determined, and makes the bodies to return one from the other with a relative velocity, which is in a given ratio to that relative velocity with which they met. This I tried in balls of wool, made up tightly, and strongly compressed. For, first, by letting go the pendulous bodies, and measuring their reflection, I determined the quantity of their elastic force; and then, according to this force, estimated the reflections that ought to happen in other cases of impact. And with this computation other experiments made afterwards did accordingly agree; the balls always receding one from the other with a relative velocity, which was to the relative velocity with which they met as about 5 to 9. Balls of steel returned with almost the same velocity; those of cork with a velocity something less; but in balls of glass the proportion was as about 15 to 16. And thus the third Law, so far as it regards percussions and reflections, is proved by a theory exactly agreeing with experience.

In attractions, I briefly demonstrate the thing after this manner. Suppose an obstacle is interposed to hinder the meeting of any two bodies A, B, attracting one the other: then if either body, as A, is more attracted towards the other body B, than that other body B is towards the first body A, the obstacle will be more strongly urged by the pressure of the body A than by the pressure of the body B, and therefore will not remain in equilibrium: but the stronger pressure will prevail, and will make the system of the two bodies, together with the obstacle, to move directly towards the parts on which B lies; and in free spaces, to go forwards *in infinitum* with a motion continually accelerated; which is absurd and contrary to the first Law. For, by the first Law, the system ought to continue in its state of rest, or of moving uniformly forwards in a right line; and therefore the bodies must equally press the obstacle, and be equally attracted one by the other. I made the experiment on the loadstone and iron. If these, placed apart in proper

vessels, are made to float by one another in standing water, neither of them will propel the other; but, by being equally attracted, they will sustain each other's pressure, and rest at last in an equilibrium.

So the gravitation between the earth and its parts is mutual. Let the earth FI be cut by any plane EG into two parts EGF and EGI, and their weights one towards the other will be mutually equal. For if by another plane HK, parallel to the former EG, the greater part EGI is cut into two parts EGKH



and HKI, whereof HKI is equal to the part EFG, first cut off, it is evident that the middle part EGKH will have no propension by its proper weight towards either side, but will hang as it were, and rest in an equilibrium between both. But the one extreme part HKI will with its whole weight bear upon and press the middle part towards the other extreme part EGF; and therefore the force with

which EGI, the sum of the parts HKI and EGKH, tends towards the third part EGF, is equal to the weight of the part HKI, that is, to the weight of the third part EGF. And therefore the weights of the two parts EGI and EGF, one towards the other, are equal, as I was to prove. And indeed if those weights were not equal, the whole earth floating in the nonresisting ether would give way to the greater weight, and, retiring from it, would be carried off *in infinitum*.

And as those bodies are equipollent in the impact and reflection, whose velocities are inversely as their innate forces, so in the use of mechanic instruments those agents are equipollent, and mutually sustain each the contrary pressure of the other, whose velocities, estimated according to the determination of the forces, are inversely as the forces.

So those weights are of equal force to move the arms of a balance, which during the play of the balance are inversely as their velocities upwards and downwards; that is, if the ascent or descent is direct, those weights are of equal force, which are inversely as the distances of the points at which they are suspended from the axis of the balance; but if they are turned aside by the interposition of oblique planes, or other obstacles, and made to ascend or descend obliquely, those bodies will be equipollent, which are inversely

as the heights of their ascent and descent taken according to the perpendicular; and that on account of the determination of gravity downwards.

And in like manner in the pulley, or in a combination of pulleys, the force of a hand drawing the rope directly, which is to the weight, whether ascending directly or obliquely, as the velocity of the perpendicular ascent of the weight to the velocity of the hand that draws the rope, will sustain the weight.

In clocks and such like instruments, made up from a combination of wheels, the contrary forces that promote and impede the motion of the wheels, if they are inversely as the velocities of the parts of the wheel on which they are impressed, will mutually sustain each other.

The force of the screw to press a body is to the force of the hand that turns the handles by which it is moved as the circular velocity of the handle in that part where it is impelled by the hand is to the progressive velocity of the screw towards the pressed body.

The forces by which the wedge presses or drives the two parts of the wood it cleaves are to the force of the mallet upon the wedge as the progress of the wedge in the direction of the force impressed upon it by the mallet is to the velocity with which the parts of the wood yield to the wedge, in the direction of lines perpendicular to the sides of the wedge. And the like account is to be given of all machines.

The power and use of machines consist only in this, that by diminishing the velocity we may augment the force, and the contrary; from whence, in all sorts of proper machines, we have the solution of this problem: *To move a given weight with a given power*, or with a given force to overcome any other given resistance. For if machines are so contrived that the velocities of the agent and resistant are inversely as their forces, the agent will just sustain the resistant, but with a greater disparity of velocity will overcome it. So that if the disparity of velocities is so great as to overcome all that resistance which commonly arises either from the friction of contiguous bodies as they slide by one another, or from the cohesion of continuous bodies that are to be separated, or from the weights of bodies to be raised, the excess of the force remaining, after all those resistances are overcome, will produce an acceleration of motion proportional thereto, as well in the parts of the machine as in the resisting body. But to treat of mechanics is

not my present business. I was aiming only to show by those examples the great extent and certainty of the third Law *of* Motion. For if we estimate the action of the agent from the product of its force and velocity, and likewise the reaction of the impediment from the product of the velocities of its several parts, and the forces of resistance arising from the friction, cohesion, weight, and acceleration of those parts, the action and reaction in the use of all sorts of machines will be found always equal to one another. And so far as the action is propagated by the intervening instruments, and at last impressed upon the resisting body, the ultimate action will be always contrary to the reaction.

Book One

THE MOTION OF BODIES

SECTION I

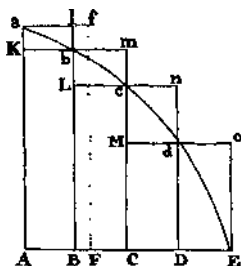
The method of first and last ratios of quantities, by the help of which we demonstrate the propositions that follow.

LEMMA I

Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer to each other than by any given difference, become ultimately equal.

If you deny it, suppose them to be ultimately unequal, and let D be their ultimate difference. Therefore they cannot approach nearer to equality than by that given difference D; which is contrary to the supposition.

LEMMA II



/f in any figure AacE, terminated by the right lines Aa, AE, and the curve acE, there be inscribed any number of parallelograms Ab, Be, Cd, &c, comprehended under equal bases AB, BC, CD, &c, and the sides, Bb, Cc, Dd, &c, parallel to one side Aa of the figure; and the parallelograms aKbl, bLcm, cMdn, &c, are completed: then if the breadth of those parallelograms be supposed to be diminished, and their number to be augmented in infinitum, I say, that the ultimate ratios which the

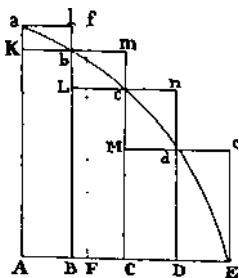
inscribed figure $AKbLcMdD$, the circumscribed figure $AalbmcndoE$, and curvilinear figure $AabcdE$, will have to one another, are ratios of equality.

For the difference of the inscribed and circumscribed figures is the sum of the parallelograms Kl , Lm , Mn , Do , that is (from the equality of all their bases), the rectangle under one of their bases Kb and the sum of their altitudes ha , that is, the rectangle $ABla$. But this rectangle, because its breadth AB is supposed diminished in *infinitum*, becomes less than any given space. And therefore (by Lem. i) the figures inscribed and circumscribed become ultimately equal one to the other; and much more will the intermediate curvilinear figure be ultimately equal to either. Q.E.D.

LEMMA III

The same ultimate ratios are also ratios of equality, when the breadths AB , BC , DC , &c, of the parallelograms are unequal, and are all diminished in infinitum.

For suppose AF equal to the greatest breadth, and complete the parallelogram $FAaf$. This parallelogram will be greater than the difference of the inscribed and circumscribed figures; but, because its breadth AF is diminished in *infinitum*, it will become less than any given rectangle. Q.E.D.



COR. i Hence the ultimate sum of those evanescent parallelograms will in all parts coincide with the curvilinear figure.

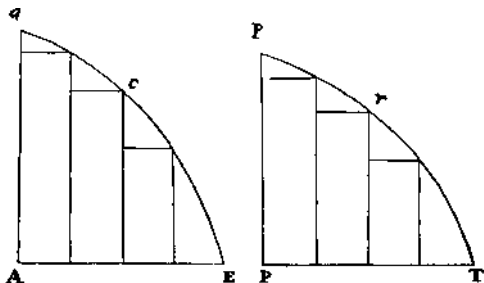
COR. II. Much more will the rectilinear figure comprehended under the chords of the evanescent arcs ab , be , cd , &c, ultimately coincide with the curvilinear figure.

COR. in. And also the circumscribed rectilinear figure comprehended under the tangents of the same arcs.

COR. iv. And therefore these ultimate figures (as to their perimeters acE) are not rectilinear, but curvilinear limits of rectilinear figures.

LEMMA IV

If in two figures AacE, PprT, there are inscribed (as before) two series of parallelograms, an equal number in each series, and, their breadths being diminished in infinitum, if the ultimate ratios of the parallelograms in one figure to those in the other, each to each respectively, are the same: I say, that those two figures, AacE, PprT, are to each other in that same ratio.



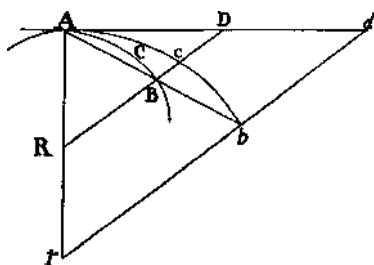
For as the parallelograms in the one are severally to the parallelograms in the other, so (by composition) is the sum of all in the one to the sum of all in the other; and so is the one figure to the other; because (by Lem. 111) the former figure to the former sum, and the latter figure to the latter sum, are both in the ratio of equality. Q.E.D.

COR. Hence if two quantities of any kind are divided in any manner into an equal number of parts, and those parts, when their number is augmented, and their magnitude diminished *in infinitum*, have a given ratio to each other, the first to the first, the second to the second, and so on in order, all of them taken together will be to each other in that same given ratio. For if, in the figures of this Lemma, the parallelograms are taken to each other in the ratio of the parts, the sum of the parts will always be as the sum of the parallelograms; and therefore supposing the number of the parallelograms and parts to be augmented, and their magnitudes diminished *in infinitum*, those sums will be in the ultimate ratio of the parallelogram in the one figure to the correspondent parallelogram in the other; that is (by the supposition), in the ultimate ratio of any part of the one quantity to the correspondent part of the other.

L E M M A V

All homologous sides of similar figures, whether curvilinear or rectilinear, are proportional; and the areas are as the squares of the homologous sides.

L E M M A VI



/f any arc ACB, given in position, is subtended by its chord AB, and in any point A, in the middle of the continued curvature, is touched by a right line AD, produced both ways; then if the points A and B approach one another and meet, I say, the angle BAD, contained between the chord and the tan-

gent, will be diminished in infinitum, and ultimately will vanish.

For if that angle does not vanish, the arc ACB will contain with the tangent AD an angle equal to a rectilinear angle; and therefore the curvature at the point A will not be continued, which is against the supposition.

L E M M A VII

The same things being supposed, I say that the ultimate ratio of the arc, chord, and tangent, any one to any other, is the ratio of equality.

For while the point B approaches towards the point A, consider always AB and AD as produced to the remote points *b* and *d*; and parallel to the secant BD draw *bd*; and let the arc *Acb* be always similar to the arc ACB. Then, supposing the points A and B to coincide, the angle *dAb* will vanish, by the preceding Lemma; and therefore the right lines *Ab*, *Ad* (which are always finite), and the intermediate arc *Acb*, will coincide, and become equal among themselves. Wherefore, the right lines AB, AD, and the intermediate arc ACB (which are always proportional to the former), will vanish, and ultimately acquire the ratio of equality. Q.E.D.

COR. i. Whence if through B we draw BF parallel to the tangent, always cutting any right line AF passing through A in F, this line BF will be

ultimately in the ratio of equality with the evanescent arc ACB; because, completing the parallelogram AFBD, it is always in a ratio of equality with AD.

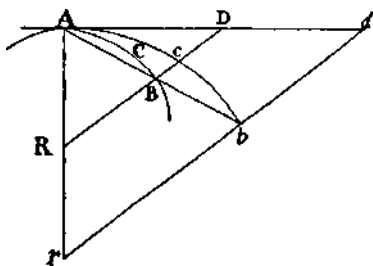
COR. II. And if through B and A more right lines are drawn, as BE, BD, AF, AG, cutting the tangent AD and its parallel BF; the ultimate ratio of all the abscissas AD, AE, BF, BG, and of the chord and arc AB, any one to any other, will be the ratio of equality.

COR. HI. And therefore in all our reasoning about ultimate ratios, we may freely use any one of those lines for any other.

LEMMA VIII

If the right lines AR, BR, with the arc ACB, the chord AB, and the tangent AD, constitute three triangles RAB, RACB, RAD, and the points A and B approach and meet: I say, that the ultimate form of these evanescent triangles is that of similitude, and their ultimate ratio that of equality.

For while the point B approaches towards the point A, consider always AB, AD, AR, as produced to the remote points *b*, *d*, and *r*, and *rbd* as drawn



parallel to RD, and let the arc *Acb* be always similar to the arc ACB. Then supposing the points A and B to coincide, the angle *bkd* will vanish; and therefore the three triangles *rAb*, *rAc*, *rAd* (which are always finite), will coincide, and on that account become both similar and equal. And

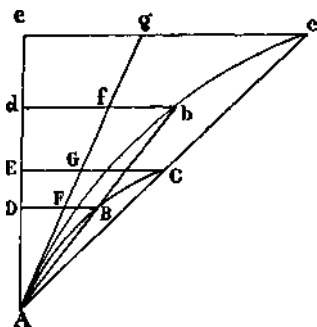
therefore the triangles RAB, RACB, RAD, which are always similar and proportional to these, will ultimately become both similar and equal among themselves. Q.E.D.

COR. And hence in all reasonings about ultimate ratios, we may use any one of those triangles for any other.

L E M M A I X

If a right line AE, and a curved line ABC, both given by position, cut each other in a given angle, A; and to that right line, in another given angle, BD, CE are ordinately applied, meeting the curve in B, C; and the points B and C together approach towards and meet in the point A: I say, that the areas of the triangles ABD, ACE, will ultimately be to each other as the squares of homologous sides.

For while the points B, C, approach towards the point A, suppose always AD to be produced to the remote points *d* and *e*, so as *Ad*, *Ae* may be proportional to AD, AE; and the ordinates *db*, *ec*, to be drawn parallel to the



ordinates DB and EC, and meeting AB and AC produced in *b* and *c*. Let the curve *Abe* be similar to the curve ABC, and draw the right line *Ag* so as to touch both curves in A, and cut the ordinates DB, EC, *db*, *ec*, in F, G, *f* g. Then, supposing the length *Ae* to remain the same, let the points B and C meet in the point A; and the angle *cAg* vanishing, the curvilinear areas *Abd*, *Ace* will coincide with the rectilinear areas *Afd*, *Age*; and

therefore (by Lem. v) will be one to the other in the duplicate ratio of the sides *Ad*, *Ac*. But the areas ABD, ACE are always proportional to these areas; and so the sides AD, AE are to these sides. And therefore the areas ABD, ACE are ultimately to each other as the squares of the sides AD, AE. Q.E.D.

L E M M A X

The spaces which a body describes by any finite force urging it, whether that force is determined and immutable, or is continually augmented or continually diminished, are in the very beginning of the motion to each other as the squares of the times.

Let the times be represented by the lines AD, AE, and the velocities generated in those times by the ordinates DB, EC. The spaces described with

these velocities will be as the areas ABD, ACE, described by those ordinates, that is, at the very beginning of the motion (by Lem. ix), in the duplicate ratio of the times AD, AE. Q.E.D.

COR. I. And hence one may easily infer, that the errors of bodies describing similar parts of similar figures in proportional times, the errors being generated by any equal forces similarly applied to the bodies, and measured by the distances of the bodies from those places of the similar figures, at which, without the action of those forces, the bodies would have arrived in those proportional times—are nearly as the squares of the times in which they are generated.

COR. II. But the errors that are generated by proportional forces, similarly applied to the bodies at similar parts of the similar figures, are as the product of the forces and the squares of the times.

COR. HI. The same thing is to be understood of any spaces whatsoever described by bodies urged with different forces; all which, in the very beginning of the motion, are as the product of the forces and the squares of the times.

COR. iv. And therefore the forces are directly as the spaces described in the very beginning of the motion, and inversely as the squares of the times.

COR. V. And the squares of the times are directly as the spaces described, and inversely as the forces.

SCHOLIUM

If in comparing with each other indeterminate quantities of different sorts, any one is said to be directly or inversely as any other, the meaning is, that the former is augmented or diminished in the same ratio as the latter, or as its reciprocal. And if any one is said to be as any other two or more, directly or inversely, the meaning is, that the first is augmented or diminished in the ratio compounded of the ratios in which the others, or the reciprocals of the others, are augmented or diminished. Thus, if A is said to be as B directly, and C directly, and D inversely, the meaning is, that A is augmented or diminished in the same ratio as $B \cdot C^{1/D}$, that is to say, that A and B^{BC}/D are to each other in a given ratio.

COR. I. Therefore since the tangents AD , Ad , the arcs AB , Ab , and their sines, BC , be , become ultimately equal to the chords AB , Ab , their squares will ultimately become as the subtenses BD , bd .

COR. II. Their squares are also ultimately as the versed sines of the arcs, bisecting the chords, and converging to a given point. For those versed sines are as the subtenses BD , bd .

COR. HI. And therefore the versed sine is as the square of the time in which a body will describe the arc with a given velocity.

COR. IV. The ultimate proportion,

$$A ADB : AAdb = AD^8 : Ad'' - DB^{9/2} : db^{3/2}$$

is derived from

$$A ADB : A Adb = AD DB : Ad db$$

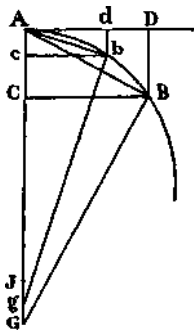
and from the ultimate proportion

$$AD^2 : Ad^2 = DK : db.$$

So also is obtained ultimately

$$A A B C : A A \wedge = B C^3 : \wedge^8.$$

COR. v. And because DB , db are ultimately parallel and as the squares of the lines AD , Ad , the ultimate curvilinear areas ADB , Adb will be (by the nature of the parabola) two-thirds of the rectilinear triangles ADB , Adb , and the segments AB , Ab will be one-third of the same triangles. And thence those areas and those segments will be as the squares of the tangents AD , Ad , and also of the chords and arcs AB , Ab .



SCHOLIUM

But we have all along supposed the angle of contact to be neither infinitely greater nor infinitely less than the angles of contact made by circles and their tangents; that is, that the curvature at the point A is neither infinitely small nor infinitely great, and that the interval AJ is of a finite magnitude. For DB may be taken as AD^1 : in which case no circle can be drawn through the point A , between the tangent AD and the curve AB , and therefore the angle of contact will be infinitely less than those of circles. And by a like reasoning, if DB be made successively as AD^4 , AD' , AD^{fl} , AD^7 , &c, we shall have a series of angles of contact, proceeding in *infinitum*, wherein every succeeding term is infinitely less than the preceding.

And if DB be made successively as AD^2 , $AD^{3/2}$, $AD^{3/4}$, $AD^{3/4}$, $AD^{7/9}$, $AD^{7/6}$ &c, we shall have another infinite series of angles of contact, the first of which is of the same sort with those of circles, the second infinitely greater, and every succeeding one infinitely greater than the preceding. But between any two of these angles another series of intermediate angles of contact may be interposed, proceeding both ways *in infinitum*, wherein every succeeding angle shall be infinitely greater or infinitely less than the preceding. As if between the terms AD^2 and AD^3 there were interposed the series $AD^{13/9}$, $AD^{11/5}$, $AD^{11/5}$, $AD^{7/3}$, $AD^{5/2}$, $AD^{3/3}$, $AD^{11/4}$, $AD^{14/5}$, $AD^{17/6}$, &c. And again, between any two angles of this series, a new series of intermediate angles may be interposed, differing from one another by infinite intervals. Nor is Nature confined to any bounds.

Those things which have been demonstrated of curved lines, and the surfaces which they comprehend, may be easily applied to the curved surfaces and contents of solids. These Lemmas are premised to avoid the tediousness of deducing involved demonstrations *ad absurdum*, according to the method of the ancient geometers. For demonstrations arc shorter by the method of indivisibles; but because the hypothesis of indivisibles seems somewhat harsh, and therefore that method is reckoned less geometrical, I chose rather to reduce the demonstrations of the following Propositions to the first and last sums and ratios of nascent and evanescent quantities, that is, to the limits of those sums and ratios, and so to premise, as short as I could, the demonstrations of those limits. For hereby the same thing is performed as by the method of indivisibles; and now those principles being demonstrated, we may use them with greater safety. Therefore if hereafter I should happen to consider quantities as made up of particles, or should use little curved lines for right ones, I would not be understood to mean indivisibles, but evanescent divisible quantities; not the sums and ratios of determinate parts, but always the limits of sums and ratios; and that the force of such demonstrations always depends on the method laid down in the foregoing Lemmas.

Perhaps it may be objected, that there is no ultimate proportion of evanescent quantities; because the proportion, before the quantities have vanished, is not the ultimate, and when they are vanished, is none. But by the same argument it may be alleged that a body arriving at a certain place, and

there stopping, has no ultimate velocity; because the velocity, before the body comes to the place, is not its ultimate velocity; when it has arrived, there is none. But the answer is easy; for by the ultimate velocity is meant that with which the body is moved, neither before it arrives at its last place and the motion ceases, nor after, but at the very instant it arrives; that is, that velocity with which the body arrives at its last place, and with which the motion ceases. And in like manner, by the ultimate ratio of evanescent quantities is to be understood the ratio of the quantities not before they vanish, nor afterwards, but with which they vanish. In like manner the first ratio of nascent quantities is that with which they begin to be. And the first or last sum is that with which they begin and cease to be (or to be augmented or diminished). There is a limit which the velocity at the end of the motion may attain, but not exceed. This is the ultimate velocity. And there is the like limit in all quantities and proportions that begin and cease to be. And since such limits are certain and definite, to determine the same is a problem strictly geometrical. But whatever is geometrical we may use in determining and demonstrating any other thing that is also geometrical.

It may also be objected, that if the ultimate ratios of evanescent quantities are given, their ultimate magnitudes will be also given: and so all quantities will consist of indivisibles, which is contrary to what *Euclid* has demonstrated concerning incommensurables, in the tenth Book of his *Elements*. But this objection is founded on a false supposition. For those ultimate ratios with which quantities vanish are not truly the ratios of ultimate quantities, but limits towards which the ratios of quantities decreasing without limit do always converge; and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished *in infinitum*. This thing will appear more evident in quantities infinitely great. If two quantities, whose difference is given, be augmented *in infinitum*, the ultimate ratio of these quantities will be given, namely, the ratio of equality; but it does not from thence follow, that the ultimate or greatest quantities themselves, whose ratio that is, will be given. Therefore if in what follows, for the sake of being more easily understood, I should happen to mention quantities as least, or evanescent, or ultimate, you are not to suppose that quantities of any determinate magnitude are meant, but such as are conceived to be always diminished without end.

Draw cC parallel to BS , meeting BC in C ; and at the end of the second part of the time, the body (by Cor. 1 of the Laws) will be found in C , in the same plane with the triangle ASB . Join SC , and, because SB and O are parallel, the triangle SBC will be equal to the triangle SBr , and therefore also to the triangle SAB . By the like argument, if the centripetal force acts successively in C , D , E , &c, and makes the body, in each single particle of time, to describe the right lines CD , DE , EF , &c., they will all lie in the same plane; and the triangle SCD will be equal to the triangle SBC , and SDE to SCD , and SEF to SDE . And therefore, in equal times, equal areas are described in one immovable plane: and, by composition, any sums $SADS$, $SAFS$, of those areas, are to each other as the times in which they are described. Now let the number of those triangles be augmented, and their breadth diminished *in infinitum*; and (by Cor. iv, Lem. in) their ultimate perimeter ADF will be a curved line: and therefore the centripetal force, by which the body is continually drawn back from the tangent of this curve, will act continually; and any described areas $SADS$, $SAFS$, which are always proportional to the times of description, will, in this case also, be proportional to those times. Q.E.D.

COR. 1. The velocity of a body attracted towards an immovable centre, in spaces void of resistance, is inversely as the perpendicular let fall from that centre on the right line that touches the orbit. For the velocities in those places A , B , C , D , E , are as the bases AB , BC , CD , DE , EF , of equal triangles; and these bases are inversely as the perpendiculars let fall upon them.

COR. 11. If the chords AB , BC of two arcs, successively described in equal times by the same body, in spaces void of resistance, are completed into a parallelogram $ABCV$, and the diagonal BV of this parallelogram, in the position which it ultimately acquires when those arcs are diminished *in infinitum*, is produced both ways, it will pass through the centre of force.

COR. in. If the chords AB , BC , and DE , EF , of arcs described in equal times, in spaces void of resistance, are completed into the parallelograms $ABCV$, $DEFZ$, the forces in B and E are one to the other in the ultimate ratio of the diagonals BV , EZ , when those arcs are diminished *in infinitum*. For the motions BC and EF of the body (by Cor. 1 of the Laws) are compounded of the motions Bc , BV , and Ef , EZ ; but BV and EZ , which are equal to Cc and Ff , in the demonstration of this Proposition, were generated

by the impulses of the centripetal force in B and E, and are therefore proportional to those impulses.

COR. IV. The forces by which bodies, in spaces void of resistance, are drawn back from rectilinear motions, and turned into curvilinear orbits, are to each other as the versed sines of arcs described in equal times; which versed sines tend to the centre of force, and bisect the chords when those arcs are diminished to infinity. For such versed sines are the halves of the diagonals mentioned in Cor. in.

COR. V. And therefore those forces are to the force of gravity as the said versed sines to the versed sines perpendicular to the horizon of those parabolic arcs which projectiles describe in the same time.

COR. vi. And the same things do all hold good (by Cor. v of the Laws) when the planes in which the bodies are moved, together with the centres of force which are placed in those planes, are not at rest, but move uniformly forwards in right lines.

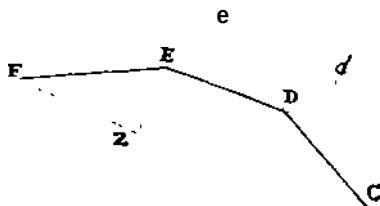
PROPOSITION II. THEOREM II

Every body that moves in any curved line described in a plane, and by a radius drawn to a point either immovable, or moving forwards with an uniform rectilinear motion, describes about that point areas proportional to the times, is urged by a centripetal force directed to that point.

CASE I. For every body that moves in a curved line is (by Law i) turned aside from its rectilinear course by the action of some force that impels it. And that force by which the body is turned off from its rectilinear course, and is made to describe, in equal times, the equal least triangles SAB, SBC, SCD, &c, about the immovable point S (by Prop, XL, Book i, *Elem. of Euclid*, and Law n), acts in the place B, according to the direction of a line parallel to cC , that is, in the direction of the line BS; and in the place C, according to the direction of a line parallel to dD , that is, in the direction of the line CS, &c; and therefore acts always in the direction of lines tending to the immovable point S. Q.E.D.

CASE 2. And (by Cor. v of the Laws) it is indifferent whether the surface in which a body describes a curvilinear figure be at rest, or moves together with the body, the figure described, and its point S, uniformly forwards in a right line.

COR. i. In nonresisting spaces or mediums, if the areas are not proportional to the times, the forces are not directed to the point in which the radii meet, but deviate therefrom towards the part to which the motion is directed, if the description of the areas is accelerated, and away from that part, if retarded.



A

COR. II. And even in resisting mediums, if the description of the areas is accelerated, the directions of the forces deviate from the point in which the radii meet, towards the part to which the motion tends.

SCHOLIUM

A body may be urged by a centripetal force compounded of several forces; in which case the meaning of the Proposition is, that the force which results out of all tends to the point S. But if any force acts continually in the direction of lines perpendicular to the described surface, this force will make the body to deviate from the plane of its motion; but will neither augment nor diminish the area of the described surface, and is therefore to be neglected in the composition of forces.

PROPOSITION III. THEOREM III

Every body, that by a radius drawn to the centre of another body, howsoever moved, describes areas about that centre proportional to the times, is urged by a force compounded of the centripetal force tending to that other body, and of all the acccleratwe force by which that other body is impelled.

Let L represent the one, and T the other body; and (by Cor. vi of the Laws) if both bodies are urged in the direction of parallel lines, by a new force equal and contrary to that by which the second body T is urged, the first body L will go on to describe about the other body T the same areas as before: but the force by which that other body T was urged will be now destroyed by an equal and contrary force; and therefore (by Law i) that other body T, now left to itself, will either rest, or move uniformly forwards in a right line: and the first body L, impelled by the difference of the forces, that is, by the force remaining, will go on to describe about the other body T areas proportional to the times. And therefore (by Theor. n) the difference of the forces is directed to the other body T as its centre. Q.E.D.

COR. I. Hence if the one body L, by a radius drawn to the other body T, describes areas proportional to the times; and from the whole force, by which the first body L is urged (whether that force is simple, or, according to Cor. II of the Laws, compounded out of several forces), we subtract (by the same Cor.) that whole accelerative force by which the other body is urged; the whole remaining force by which the first body is urged will tend to the other body T, as its centre.

COR. II. And, if these areas are proportional to the times nearly, the remaining force will tend to the other body T nearly.

COR. HI. And *vice versa*, if the remaining force tends nearly to the other body T, those areas will be nearly proportional to the times.

COR. IV. If the body L, by a radius drawn to the other body T, describes areas, which, compared with the times, are very unequal; and that other body T be either at rest, or moves uniformly forwards in a right line: the action of the centripetal force tending to that other body T is either none at all, or it is mixed and compounded with very powerful actions of other forces: and the whole force compounded of them all, if they are many, is directed to another (immovable or movable) centre. The same thing

obtains, when the other body is moved by any motion whatsoever; provided that centripetal force is taken, which remains after subtracting that whole force acting upon that other body T.

SCHOLIUM

Since the equable description of areas indicates that there is a centre to which tends that force by which the body is most affected, and by which it is drawn back from its rectilinear motion, and retained in its orbit, why may we not be allowed, in the following discourse, to use the equable description of areas as an indication of a centre, about which all circular motion is performed in free spaces?

PROPOSITION IV. THEOREM IV¹

The centripetal forces of bodies, which by equable motions describe different circles, tend to the centres of the same circles; and are to each other as the squares of the arcs described in equal times divided respectively by the radii of the circles.

These forces tend to the centres of the circles (by Prop. n, and Cor. n, Prop. i), and are to one another as the versed sines of the least arcs described in equal times (by Cor. iv, Prop. i); that is, as the squares of the same arcs divided by the diameters of the circles (by Lem. VII); and therefore since those arcs are as arcs described in any equal times, and the diameters are as the radii, the forces will be as the squares of any arcs described in the same time divided by the radii of the circles. Q.E.D.

COR. i. Therefore, since those arcs are as the velocities of the bodies, the centripetal forces are as the squares of the velocities divided by the radii.

COR. n. And since the periodic times are as the radii divided by the velocities, the centripetal forces are as the radii divided by the square of the periodic times.

COR. HI. Whence if the periodic times are equal, and the velocities therefore as the radii, the centripetal forces will be also as the radii; and conversely.

COR. IV. If the periodic times and the velocities are both as the square roots of the radii, the centripetal forces will be equal among themselves; and conversely.

¹ [Appendix, Note 15]

COR. V. If the periodic times are as the radii, and therefore the velocities equal, the centripetal forces will be inversely as the radii; and conversely.

COR. vi. If the periodic times are as the $3/2$ th powers of the radii, and therefore the velocities inversely as the square roots of the radii, the centripetal forces will be inversely as the squares of the radii; and conversely.

COR. VII. And universally, if the periodic time is as any power R^n of the radius R , and therefore the velocity inversely as the power R^{n-1} of the radius, the centripetal force will be inversely as the power R^{2n-1} of the radius; and conversely.

COR. VIII. The same things hold concerning the times, the velocities, and the forces by which bodies describe the similar parts of any similar figures that have their centres in a similar position with those figures; as appears by applying the demonstration of the preceding cases to those. And the application is easy, by only substituting the equable description of areas in the place of equable motion, and using the distances of the bodies from the centres instead of the radii.

COR. ix. From the same demonstration it likewise follows, that the arc which a body, uniformly revolving in a circle with a given centripetal force, describes in any time, is a mean proportional between the diameter of the circle, and the space which the same body falling by the same given force would describe in the same given time.

SCHOLIUM

The case of the sixth Corollary obtains in the celestial bodies (as Sir *Christopher Wren*, Dr. *Hooke*, and Dr. *Halley* have severally observed); and therefore in what follows, I intend to treat more at large of those things which relate to centripetal force decreasing as the squares of the distances from the centres.

Moreover, by means of the preceding Proposition and its Corollaries, we may discover the proportion of a centripetal force to any other known force, such as that of gravity. For if a body by means of its gravity revolves in a circle concentric to the earth, this gravity is the centripetal force of that body. But from the descent of heavy bodies, the time of one entire revolution, as well as the arc described in any given time, is given (by Cor. ix of this Prop.). And by such propositions, Mr. *Huygens*, in his excellent book

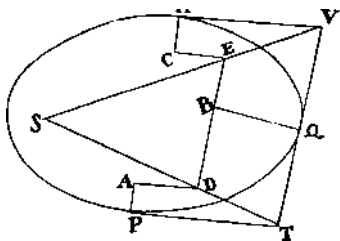
De horologio oscillatorio, has compared the force of gravity with the centrifugal forces of revolving bodies.

The preceding Proposition may be likewise demonstrated after this manner. In any circle suppose a polygon to be inscribed of any number of sides. And if a body, moved with a given velocity along the sides of the polygon, is reflected from the circle at the several angular points, the force, with which at every reflection it strikes the circle, will be as its velocity: and therefore the sum of the forces, in a given time, will be as the product of that velocity and the number of reflections; that is (if the species of the polygon be given), as the length described in that given time, and increased or diminished in the ratio of the same length to the radius of the circle; that is, as the square of that length divided by the radius; and therefore the polygon, by having its sides diminished *in infinitum*, coincides with the circle, as the square of the arc described in a given time divided by the radius. This is the centrifugal force, with which the body impels the circle; and to which the contrary force, wherewith the circle continually repels the body towards the centre, is equal.

PROPOSITION V. PROBLEM I

There being given, in any places, the velocity with which a body describes a given figure, by means of forces directed to some common centre: to find that centre.

Let the three right lines PT, TQV, VR touch the figure described in as many points, P, Q, R, and meet in T and V. On the tangents erect the perpendiculars PA, QB, RC, inversely proportional to the velocities of the body in the points P, Q, R, from which the perpendiculars were raised; that is, so that PA may be to QB



as the velocity in Q to the velocity in P, and QB to RC as the velocity in R to the velocity in Q. Through the ends A, B, C of the perpendiculars draw

AD, DBE, EC, at right angles, meeting in D and E: and the right lines TD, VE produced, will meet in S, the centre required.

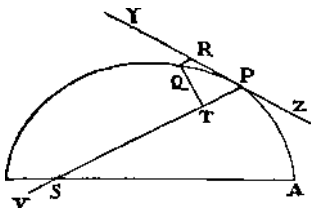
For the perpendiculars let fall from the centre S on the tangents PT, QT, are inversely as the velocities of the bodies in the points P and Q (by Cor. i, Prop. i), and therefore, by construction, directly as the perpendiculars AP, BQ; that is, as the perpendiculars let fall from the point D on the tangents. Whence it is easy to infer that the points S, D, T are in one right line. And by the like argument the points S, E, V are also in one right line; and therefore the centre S is in the point where the right lines TD, VE meet. Q.E.D.

PROPOSITION VI. THEOREM V

In a space void of resistance, if a body revolves in any orbit about an immovable centre, and in the least time describes any arc just then nascent; and the versed sine of that arc is supposed to be drawn bisecting the chord, and produced passing through the centre of force: the centripetal force in the middle of the arc will be directly as the versed sine and inversely as the square of the time.

For the versed sine in a given time is as the force (by Cor. iv, Prop. i); and augmenting the time in any ratio, because the arc will be augmented in the same ratio, the versed sine will be augmented in the square of that ratio (by Cor. n and in, Lem. xi), and therefore is as the force and the square of the time. Divide both sides by the square of the time, and the force will be directly as the versed sine, and inversely as the square of the time. Q.E.D.

And the same thing may also be easily demonstrated by Cor. iv, Lem. x.



COR. i. If a body P revolving about the centre S describes a curved line APQ, which a right line ZPR touches in any point P; and from any other point Q of the curve, QR is drawn parallel to the distance SP, meeting the tangent in R; and QT is drawn perpendicular to the distance SP; the

centripetal force will be inversely as the solid $\frac{SP^2 QT^2}{QR}$, if the solid be

taken of that magnitude which it ultimately acquires when the points P and Q coincide. For QR is equal to the versed sine of double the arc QP, whose middle is P: and double the triangle SQP, or $SP \cdot QT$ is proportional to the time in which that double arc is described; and therefore may be used to represent the time.

COR. II. By a like reasoning, the centripetal force is inversely as the solid $SY^2 \cdot QP^2$ $\frac{SY^2 \cdot QP^2}{QR}$; if SY is a perpendicular from the centre of force on PR, the

tangent of the orbit. For the rectangles S Y QP and SP QT are equal.

COR. in. If the orbit is either a circle, or touches or cuts a circle concentrically, that is, contains with a circle the least angle of contact or section, having the same curvature and the same radius of curvature at the point P; and if PV be a chord of this circle, drawn from the body through the centre of force; the centripetal force will be inversely as the solid $SY^{*1} \cdot PV$. For PV is $\frac{OP^2}{SY}$.

COR. IV. The same things being supposed, the centripetal force is as the square of the velocity directly, and that chord inversely. For the velocity is reciprocally as the perpendicular SY, by Cor. i, Prop. i.

COR. V. Hence if any curvilinear figure APQ is given, and therein a point S is also given, to which a centripetal force is continually directed, that law of centripetal force may be found, by which the body P will be continually drawn back from a rectilinear course, and, being detained in the perimeter of that figure, will describe the same by a continual revolution. That is, we

are to find, by computation, either the solid $\frac{SP^2 \cdot QT^2}{QK}$ or the solid $SY'' \cdot PV$,

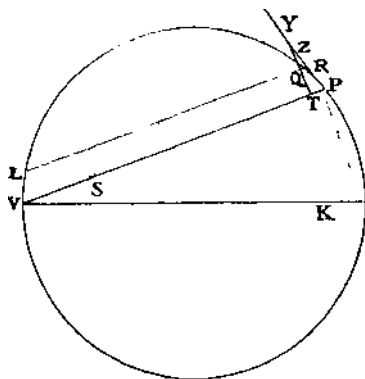
inversely proportional to this force. Examples of this we shall give in the following Problems.

PROPOSITION VII. PROBLEM II

If a body revolves in the circumference of a circle, it is proposed to find the law of centripetal force directed to any given point.

Let VQPA be the circumference of the circle; S the given point to which as to a centre the force tends; P the body moving in the circumference; Q the next place into which it is to move; and PRZ the tangent of the circle

at the preceding place. Through the point S draw the chord PV, and the diameter VA of the circle; join AP, and draw QT perpendicular to SP, which produced, may meet the tangent PR in Z; and lastly, through the



point Q, draw LR parallel to SP, meeting the circle in L, and the tangent PZ in R. And, because of the similar triangles ZQR, ZTP, VPA, we shall have

$$RP^2 : QT^2 = AV^2 : PV^2$$

Since $RP^2 = RL \cdot QR$,

$$QT^2 = \frac{RL \cdot QR \cdot PV^2}{AV^2}$$

Multiply those equals by $\frac{SF}{QR}$

and the points P and Q coinciding, for RL write PV; then we shall have

$$\frac{SP^2}{AV^2} \cdot \frac{PV^3}{QR} = SP^2 \cdot \frac{QT^3}{QR}$$

And therefore (by Cor. 1 and v, Prop. vi) the centripetal force is inversely as $\frac{SP^2 \cdot PV^3}{AV^2}$; that is (because AV^2 is given), inversely as the product of SP^2 and PV^3 . Q.E.I.

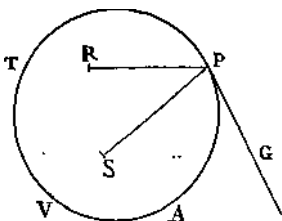
The same otherwise.

On the tangent PR produced let fall the perpendicular SY; and (because of the similar triangles SYP, VPA) we shall have AV to PV as SP to SY, and therefore $\frac{SP \cdot PV}{AV} = SY$, and $\frac{SP^2 \cdot PV^3}{AV^2} : SY^2 \cdot PV$. And therefore (by

Cor. in and v, Prop. vi) the centripetal force is inversely as $\frac{SP^2 \cdot PV^3}{AV^2}$ that is (because AV is given), inversely as $SP^2 \cdot PV^3$. Q.E.I.

COR. I. Hence if the given point S, to which the centripetal force always tends, is placed in the circumference of the circle, as at V, the centripetal force will be inversely as the fifth power of the altitude SP.

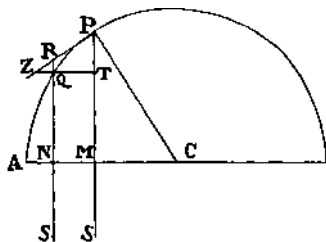
COR. II. The force by which the body P in the circle APTV revolves about the centre of force S is to the force by which the same body P may revolve in the same circle, and in the same periodic time, about any other centre of force R, as $RP^2 \cdot SP$ to the cube of the right line SG, which from the first centre of force S is drawn parallel to the distance PR of the body from the second centre of force R, meeting the tangent PG of the orbit in G. For by the construction of this Proposition, the former force is



to the latter as $RP^2 \cdot PT^3$ to $SP^2 \cdot PV$; that is, as $SP \cdot RP^2$ to $SP^3 \cdot PY^3$ or (because of the similar triangles PSG, TPV) to SG^3 .

COR. in. The force by which the body P in any orbit revolves about the centre of force S, is to the force by which the same body may revolve in the same orbit, and the same periodic time, about any other centre of force R, as the solid $SP \cdot RP^2$, contained under the distance of the body from the first centre of force S, and the square of its distance from the second centre of force R, to the cube of the right line SG, drawn from the first centre of the force S, parallel to the distance RP of the body from the second centre of force R, meeting the tangent PG of the orbit in G. For the force in this orbit at any point P is the same as in a circle of the same curvature.

PROPOSITION VIII. PROBLEM III



If a body moves in the semicircumference PQA; it is proposed to find the law of the centripetal force tending to a point S, so remote, that all the lines PS, RS drawn thereto, may be taken for parallels.

From C, the centre of the semicircle, let the semidiameter CA be drawn, cutting the parallels at right angles in M and N, and join CP. Because of the similar triangles CPM,

be changed in the ratio of PR^2 or QT^2 . Therefore the ratio $\frac{QT^2}{QR}$ remains the same as before, that is, as SP . And $\frac{QT^2 \cdot SP^2}{QR}$ is as SP^3 , and therefore (by Cor. 1 and v, Prop. vi) the centripetal force is inversely as the cube of the distance SP . Q.E.I.

The same otherwise.

The perpendicular SY let fall upon the tangent, and the chord PV of the circle concentrically cutting the spiral, are in given ratios to the height SP ; and therefore SP^3 is as $SY^2 \cdot PV$, that is (by Cor. in and v, Prop. vi) inversely as the centripetal force.

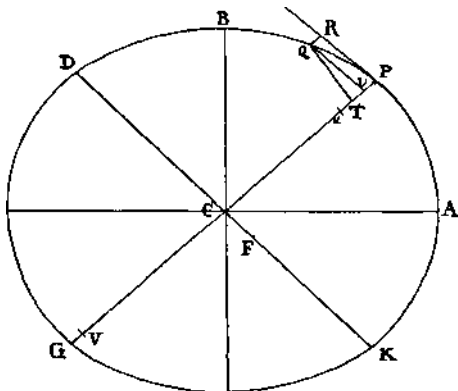
LEMMA XII

All parallelograms circumscribed about any conjugate diameters of a given ellipse or hyperbola are equal among themselves.

This is demonstrated by the writers on the conic sections.

PROPOSITION X. PROBLEM V

If a body revolves in an ellipse, it is proposed to find the law of the centripetal force tending to the centre of the ellipse.



Suppose CA , CB to be semiaxes of the ellipse; GP , DK , conjugate diameters; PF , QT , perpendiculars to those diameters; Qv , an ordinate to the

diameter GP; and if the parallelogram QvPR be completed, then (by the properties of the conic sections) $Pv \cdot vG : Qv^2 = PC^2 : CD^2$, and, because of the similar triangles QvT, PCF, $Qv^2 : QT^2 = PC^2 : PF^2$; and by eliminating Qv^2 , $vG : \frac{QT^2}{Pv} = PC^2 : \frac{CD^2 \cdot PF^2}{PC^2}$. Since $QR = Pv$, and (by Lem. XII) $BC \cdot CA = CD \cdot PF$, and, when the points P and Q coincide, $2PC = vG$, we shall have, multiplying the extremes and means together,

$$\frac{QT^2 \cdot PC^2}{QR} = \frac{2BC^2 \cdot CA^2}{PC}.$$

Therefore (by Cor. v, Prop. vi), the centripetal force is inversely as $\frac{2BC^2 \cdot CA^2}{PC}$; that is (because $2BC^2 \cdot CA^2$ is given), inversely as $\frac{1}{PC}$; that is, directly as the distance PC. Q.E.I.

The same otherwise.

In the right line PG on the other side of the point T, take the point u so that Tu may be equal to Tv ; then take uV , such that $uV : vG = DC^2 : PC^2$. Since, by the conic sections, $Qv^2 : Pv \cdot vG = DC^2 : PC^2$, we have $Qv^2 = Pv \cdot uV$. Add $Pu \cdot Pv$ to both sides, and the square of the chord of the arc PQ will be equal to the rectangle $PV \cdot Pv$; and therefore a circle which touches the conic section in P, and passes through the point Q, will pass also through the point V. Now let the points P and Q meet, and the ratio of uV to vG , which is the same with the ratio of DC^2 to PC^2 , will become the ratio of PV to PG , or PV to $2PC$; and therefore PV will be equal to $\frac{2DC^2}{PC}$. And therefore the force by which the body P revolves in the ellipse will be inversely as $\frac{2DC^2}{PC} \cdot PF^2$ (by Cor. III, Prop. vi); that is (because $2DC^2 \cdot PF^2$ is given), directly as PC. Q.E.I.

COR. I. And therefore the force is as the distance of the body from the centre of the ellipse; and, *vice versa*, if the force is as the distance, the body will move in an ellipse whose centre coincides with the centre of force, or perhaps in a circle into which the ellipse may degenerate.

COR. II. And the periodic times of the revolutions made in all ellipses whatsoever about the same centre will be equal. For those times in similar

ellipses will be equal (by Cor. in and vin, Prop, iv); but in ellipses that have their greater axis common, they are to each other as the whole areas of the ellipses directly, and the parts of the areas described in the same time inversely; that is, as the lesser axes directly, and the velocities of the bodies in their principal vertices inversely; that is, as those lesser axes directly, and the ordinates to the same point of the common axes inversely; and therefore (because of the equality of the direct and inverse ratios) in the ratio of equality, $i : i$.

SCHOLIUM

If the ellipse, by having its centre removed to an infinite distance, degenerates into a parabola, the body will move in this parabola; and the force, now tending to a centre infinitely remote, will become constant. This is *Galileo's* theorem. And if the parabolic section of the cone (by changing the inclination of the cutting plane to the cone) degenerates into an hyperbola, the body will move in the perimeter of this hyperbola, having its centripetal force changed into a centrifugal force. And in like manner as in the circle, or in the ellipse, if the forces are directed to the centre of the figure placed in the abscissa, those forces by increasing or diminishing the ordinates in any given ratio, or even by changing the angle of the inclination of the ordinates to the abscissa, are always augmented or diminished in the ratio of the distances from the centre; provided the periodic times remain equal; so also in all figures whatsoever, if the ordinates are augmented or diminished in any given ratio, or their inclination is any way changed, the periodic time remaining the same, the forces directed to any centre placed in the abscissa are in the several ordinates augmented or diminished in the ratio of the distances from the centre.

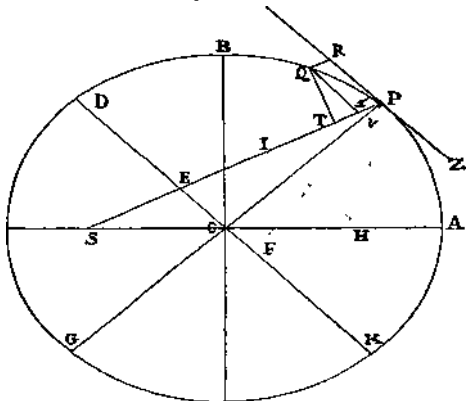
SECTION III

The motion of bodies in eccentric conic sections.

PROPOSITION XL PROBLEM VI

If a body revolves in an ellipse, it is required to find the law of the centripetal force tending to the focus of the ellipse.

Let S be the focus of the ellipse. Draw SP cutting the diameter DK of the ellipse in E, and the ordinate Qv in x; and complete the parallelogram Q*PR. It is evident that EP is equal to the greater semiaxis AC: for drawing HI from the other focus H of the ellipse parallel to EC, because CS, CH are equal, ES, EI will be also equal; so that EP is the half-sum of PS, PI,



that is (because of the parallels HI, PR, and the equal angles IPR, HPZ), of PS, PH, which taken together are equal to the whole axis 2AC. Draw QT perpendicular to SP, and putting L for the principal latus rectum of the ellipse (or for $\frac{2BC^2}{AC}$), we shall have

$$L \cdot QR : L \cdot P\nu = QR : P\nu = PE : PC = AC : PC,$$

also, $L \cdot Pv : Gv \cdot Pv = L : Gv$, and, $Gv \cdot Pv : Qv^2 = PC^2 : CD^2$.

By Cor. II, Lem. VII, when the points P and Q coincide, $Qv^2 = Qx^2$, and

Qx^2 or Qv^2 : $QT^2 = EP^2$: $PF^2 = CA^2$: PF^2 , and (by Lem. XII) $= CD^2$: CB^2 . Multiplying together corresponding terms of the four proportions, and simplifying, we shall have

$L \cdot QR : QT^2 = AC \cdot L \cdot PC^2 \cdot CD^4 : PC \cdot Gv \cdot CD^2 \cdot CB^4 = 2PC : Gv$, since $AC \cdot L = 2BC^4$. But the points Q and P coinciding, $2PC$ and Gv are equal. And therefore the quantities $L \cdot QR$ and QT^2 , proportional to these, will be also equal. Let those equals be multiplied by $\frac{SP^2}{QR}$, and $L \cdot SP^2$ will become equal to $\frac{SP^2 \cdot QT^2}{QR}$. And therefore (by Cor. 1 and v, Prop. VI) the centripetal force is inversely as $L \cdot SP^2$, that is, inversely as the square of the distance SP. Q.E.I.

The same otherwise.

Since the force tending to the centre of the ellipse, by which the body P may revolve in that ellipse, is (by Cor. i, Prop. x) as the distance CP of the body from the centre C of the ellipse, let CE be drawn parallel to the tangent PR of the ellipse; and the force by which the same body P may revolve about any other point S of the ellipse, if CE and PS intersect in E, will be as PE^3

(by Cor. in, Prop. vn); that is, if the point S is the focus of the ellipse, and therefore PE be given as SP^{-2} reciprocally. Q.E.I.

With the same brevity with which we reduced the fifth Problem to the parabola, and hyperbola, we might do the like here; but because of the dignity of the Problem and its use in what follows, I shall confirm the other cases by particular demonstrations.

PROPOSITION XII. PROBLEM VII

Suppose a body to move in an hyperbola, it is required to find the law of the centripetal force tending to the focus of that figure.

Let CA, CB be the semiaxes of the hyperbola; PG, KD other conjugate diameters; PF a perpendicular to the diameter KD; and Qv an ordinate to the diameter GP. Draw SP cutting the diameter DK. in E, and the ordinate Qv in x, and complete the parallelogram QRPx. It is evident that EP is equal to the semitransverse axis AC; for drawing HI, from the other focus H of the hyperbola, parallel to EC, because CS, CH are equal, ES, EI will

Multiplying together corresponding terms of the four proportions, and simplifying,

$L \cdot QR : QT^2 = AC \cdot L \cdot PC^2 \cdot CD^2 : PC \cdot Gv \cdot CD^2 \cdot CB^2 = 2PC : Gv$,
 since $AC \cdot L = 2BC^2$. But the points P and Q coinciding, $2PC$ and Gv are equal. And therefore the quantities $L \cdot QR$ and QT^2 , proportional to them, will also be equal. Let those equals be drawn into $\frac{SP^2}{QR}$, and we shall have $L \cdot SP^2$ equal to $\frac{SP^2 \cdot QT^2}{QR}$. And therefore (by Cor. 1 and v, Prop. vi) the centripetal force is inversely as $L \cdot SP^2$, that is, inversely as the square of the distance SP. Q.E.I.

The same otherwise.

Find out the force tending from the centre C of the hyperbola. This will be proportional to the distance CP. But from thence (by Cor. in, Prop. vn) the force tending to the focus S will be as $\frac{PE^3}{SP^2}$, that is, because PE is given reciprocally as SP^2 Q.E.I.

And the same way may it be demonstrated, that the body having its centripetal changed into a centrifugal force, will move in the conjugate hyperbola.

LEMMA XIII

The latus rectum of a parabola belonging to any vertex is four times the distance of that vertex from the focus of the figure.

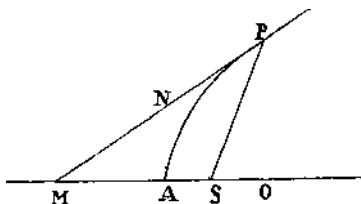
This is demonstrated by the writers on the conic sections.

LEMMA XIV

The perpendicular, let fall from the focus of a parabola on its tangent, is a mean proportional between the distances of the focus from the point of contact, and from the principal vertex of the figure.

For, let AP be the parabola, S its focus, A its principal vertex, P the point of contact, PO an ordinate to the principal diameter, PM the tangent meet-

ing the principal diameter in M, and SN the perpendicular from the focus on the tangent: join AN, and because of the equal lines MS and SP, MN



and NP, MA and AO, the right lines AN, OP will be parallel; and thence the triangle SAN will be right-angled at A, and similar to the equal triangles SNM, SNP; therefore PS is to SN as SN is to SA. Q.E.D.

COR. i. PS^2 is to SN^2 as PS is to SA.

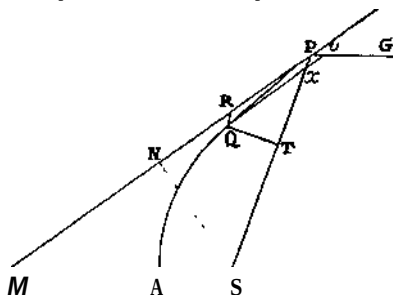
COR. II. And because SA is given, SN^2 will vary as PS.

COR. in. And the intersection of any tangent PM, with the right line SN, drawn from the focus perpendicular on the tangent, falls in the right line AN that touches the parabola in the principal vertex.

PROPOSITION XIII. PROBLEM VIII

If a body moves in the perimeter of a parabola; it is required to find the law of the centripetal force tending to the focus of that figure.

Retaining the construction of the preceding Lemma, let P be the body in the perimeter of the parabola; and from the place Q, into which it is next



to succeed, draw QR parallel and QT perpendicular to SP, as also Qv parallel to the tangent, and meeting the diameter PG in v, and the distance SP in x. Now, because of the similar triangles Pvx, SPM, and of the equal

sides SP, SM of the one, the sides Px or QR and Pv of the other will be also equal. But (by the conic sections) the square of the ordinate Qv is equal to the rectangle under the latus rectum and the segment Pv of the diameter; that is (by Lem. xii), to the rectangle 4PS · Pv, or 4PS · QR; and the points P and Q coinciding, (by Cor. II, Lem. vii), Qx = Qv. And therefore Qx², in this case, becomes equal to the rectangle 4PS · QR. But (because of the similar triangles QxT, SPN),

$$\begin{aligned} Qx^2 : QT^2 &= PS^2 : SN^2 = PS : SA \text{ (by Cor. I, Lem. xiv),} \\ &= 4PS \cdot QR : 4SA \cdot QR. \end{aligned}$$

Therefore (by Prop. ix, Book v, *Elem. of Euclid*), $QT^2 = 4SA \cdot QR$. Multiply these equals by $\frac{SP^2}{QR}$, and $\frac{SP^2 \cdot QT^2}{QR}$ will become equal to $SP^2 \cdot 4SA$; and therefore (by Cor. I and v, Prop. vi), the centripetal force is inversely as $SP^2 \cdot 4SA$; that is, because 4SA is given, inversely as the square of the distance SP. Q.E.I.

COR. I. From the three last Propositions it follows, that if any body P goes from the place P with any velocity in the direction of any right line PR, and at the same time is urged by the action of a centripetal force that is inversely proportional to the square of the distance of the places from the centre, the body will move in one of the conic sections, having its focus in the centre of force; and conversely. For the focus, the point of contact, and the position of the tangent, being given, a conic section may be described, which at that point shall have a given curvature. But the curvature is given from the centripetal force and velocity of the body being given; and two orbits, touching one the other, cannot be described by the same centripetal force and the same velocity.

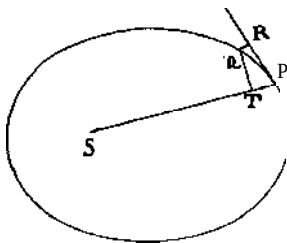
COR. II. If the velocity with which the body goes from its place P is such, that in any infinitely small moment of time the small line PR may be thereby described; and the centripetal force such as in the same time to move the same body through the space QR; the body will move in one of the conic

QT"

sections, whose principal latus rectum is the quantity $\frac{QT^2}{SP^2}$ in its ultimate state, when the small lines PR, QR are diminished *in infinitum*. In these Corollaries I consider the circle as an ellipse; and I except the case where the body descends to the centre in a right line.

PROPOSITION XIV. THEOREM VI

If several bodies revolve about one common centre, and the centripetal force is inversely as the square of the distance of places from the centre. I say, that the principal later a recta of their orbits are as the squares of the areas, which the bodies by radii drawn to the centre describe in the same time.



For (by Cor. 11, Prop, XIII) the latus rectum L is equal to the quantity $\frac{QT^2}{QR}$ in its ultimate state when the points P and Q coincide. But the small line QR in a given time is as the generating centripetal force; that is (by supposition), QT .

And therefore $QT \propto SP^2$; that is, the latus rectum L is as the square of the area $QT \propto SP$. Q.E.D.

COR. Hence the whole area of the ellipse, and the rectangle under the axes, which is proportional to it, is as the product of the square root of the latus rectum, and the periodic time. For the whole area is as the area $QT \propto SP$, described in a given time, multiplied by the periodic time.

PROPOSITION XV. THEOREM VII

The same things being supposed, I say, that the periodic times in ellipses are as the $3/2$ th power (in ratione sesquiphcata) of their greater axes.

For the lesser axis is a mean proportional between the greater axis and the latus rectum; and, therefore, the product of the axes is equal to the product of the square root of the latus rectum and the $3/2$ th power of the greater axis. But the product of the axes (by Cor., Prop, xiv) varies as the product of the square root of the latus rectum, and the periodic time. Divide both sides by the square root of the latus rectum and it follows that the $3/2$ th power of the greater axis varies as the periodic time. Q.E.D.

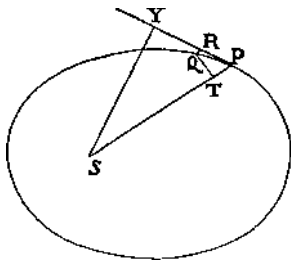
COR. Therefore the periodic times in ellipses are the same as in circles whose diameters are equal to the greater axes of the ellipses.

PROPOSITION XVI. THEOREM VIII

The same things being supposed, and right lines being drawn to the bodies that shall touch the orbits, and perpendiculars being let fall on those tangents from the common focus: I say, that the velocities of the bodies vary inversely as the perpendiculars and directly as the square roots of the principal latera recta.

From the focus S draw SY perpendicular to the tangent PR, and the velocity of the body P varies inversely as the square root of the quantity SY^2

For that velocity is as the infinitely small arc PQ described in a given moment of time, that is (by Lem. vn), as the tangent PR; that is (because of the proportion, $PR : QT = SP : SY$), as $SP \cdot QT$; or inversely as SY, and directly as $SP \cdot QT$; but $SP \cdot QT$ is as the area described in the given time, that is (by Prop. xiv), as the square root of the latus rectum. Q.E.D.



COR. 1. The principal latera recta vary as the squares of the perpendiculars and the squares of the velocities.

COR. 11. The velocities of bodies, in their greatest and least distances from the common focus, are inversely as the distances and directly as the square root of the principal latera recta. For those perpendiculars are now the distances.

COR. in. And therefore the velocity in a conic section, at its greatest or least distance from the focus, is to the velocity in a circle, at the same distance from the centre, as the square root of the principal latus rectum is to the double of that distance.

COR. iv. The velocities of the bodies revolving in ellipses, at their mean distances¹ from the common focus, are the same as those of bodies revolving in circles, at the same distances; that is (by Cor. vi, Prop. iv), inversely as the

¹ [Appendix, Note 18]

square root of the distances. For the perpendiculars are now the lesser semi-axes, and these are as mean proportionals between the distances and the latera recta. Let the inverse of this ratio [of the minor semiaxes] be multiplied by the square root of the direct ratio of the latera recta, and we shall have the square root of the inverse ratio of the distances.

COR. v. In the same figure, or even in different figures, whose principal latera recta are equal, the velocity of a body is inversely as the perpendicular let fall from the focus on the tangent.

COR. vi. In a parabola, the velocity is inversely as the square root of the ratio of the distance of the body from the focus of the figure; it is more variable in the ellipse, and less in the hyperbola, than according to this ratio. For (by Cor. 11, Lem. xiv) the perpendicular let fall from the focus on the tangent of a parabola is as the square root of the ratio of the distance. In the hyperbola the perpendicular is less variable; in the ellipse, more.

COR. vii. In a parabola, the velocity of a body at any distance from the focus is to the velocity of a body revolving in a circle, at the same distance from the centre, as the square root of the ratio of the number 2 to 1; in the ellipse it is less, and in the hyperbola greater, than according to this ratio. For (by Cor. 11 of this Prop.) the velocity at the vertex of a parabola is in this ratio, and (by Cor. vi of this Prop, and Prop, iv) the same proportion holds in all distances. And hence, also, in a parabola, the velocity is everywhere equal to the velocity of a body revolving in a circle at half the distance; in the ellipse it is less, and in the hyperbola greater.

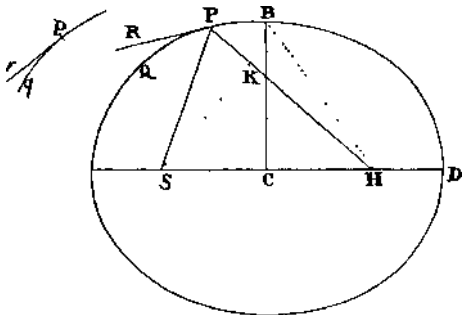
COR. viii. The velocity of a body revolving in any conic section is to the velocity of a body revolving in a circle, at the distance of half the principal latus rectum of the section, as that distance to the perpendicular let fall from the focus on the tangent of the section. This appears from Cor. v.

COR. IX. Wherefore, since (by Cor. vi, Prop, iv) the velocity of a body revolving in this circle is to the velocity of another body revolving in any other circle, inversely as the square root of the ratio of the distances; therefore, likewise, the velocity of a body revolving in a conic section will be to the velocity of a body revolving in a circle at the same distance as a mean proportional between that common distance, and half the principal latus rectum of the section, to the perpendicular let fall from the common focus upon the tangent of the section.

PROPOSITION XVII. PROBLEM IX

Supposing the centripetal force to be inversely proportional to the squares of the distances of places from the centre, and that the absolute value of that force is known; it is required to determine the line which a body will describe that is let go from a given place with a given velocity in the direction of a given right line.

Let the centripetal force tending to the point S be such as will make the body p revolve in any given orbit pq ; and suppose the velocity of this body in the place p is known. Then from the place P suppose the body P to be let go with a given velocity in the direction of the line PR ; but by virtue of a centripetal force to be immediately turned aside from that right line into the conic section PQ . This, the right line PR will therefore touch in P . Suppose likewise that the right line pr touches the orbit pq in p ; and if from



So you suppose perpendiculars let fall on those tangents, the principal latus rectum of the conic section (by Cor. 1, Prop. xvi) will be to the principal latus rectum of that orbit in a ratio compounded of the squared ratio of the perpendiculars, and the squared ratio of the velocities; and is therefore given. Let this latus rectum be L ; the focus S of the conic section is also given. Let the angle RPH be the supplement of the angle RPS , and the line PH , in which the other focus H is placed, is given by position. Let fall SK

perpendicular on PH, and erect the conjugate semiaxis BC; this done, we shall have

$$\begin{aligned} SP^2 - 2PH \cdot PK + PH^2 &= SH^2 - 4CH^2 = 4(BH^2 - BC^2) = \\ (SP + PH)^2 - L(SP + PH) &= SP^2 + 2PS \cdot PH + PH^2 - L(SP + PH). \end{aligned}$$

Add on both sides

$$2PK \cdot PH - SP^2 - PH^2 + L(SP + PH),$$

and we shall have

$$\begin{aligned} L(SP + PH) - 2PS \cdot PH + 2PK \cdot PH, \text{ or} \\ (SP + PH) : PH &:: 2(SP + KP) : L. \end{aligned}$$

Hence PH is given both in length and position. That is, if the velocity of the body in P is such that the latus rectum L is less than $2SP + 2KP$, PH will be on the same side of the tangent PR with the line SP; and therefore the figure will be an ellipse, which from the given foci S, H, and the principal axis $SP + PH$, is given also. But if the velocity of the body is so great, that the latus rectum L becomes equal to $2SP + 2KP$, the length PH will be infinite; and therefore, the figure will be a parabola, which has its axis SH parallel to the line PK, and is thence given. But if the body goes from its place P with a yet greater velocity, the length PH is to be taken on the other side the tangent; and so the tangent passing between the foci, the figure will be an hyperbola having its principal axis equal to the difference of the lines SP and PH, *And* thence is given, for if the body, in these cases, revolves in a conic section so found, it is demonstrated in Prop, xi, xn, and xm, that the centripetal force will be inversely as the square of the distance of the body from the centre of force S, and therefore we have rightly determined the line PQ, which a body let go from a given place P with a given velocity, and in the direction of the right line PR given by position, would describe with such a force. Q.E.F.

COR 1. Hence in every conic section, from the principal vertex D, the latus rectum L, and the focus S given, the other focus H is given, by taking L/H to DS as the latus rectum to the difference between the latus rectum and $4DS$. For the proportion

$$SP + PH : PH :: 2SP + 2KP : L$$

becomes, in the case of this Corollary,

$$DS + DH : DH = 4DS : L,$$

$$\text{and } DS : DH = 4DS - L : L.$$

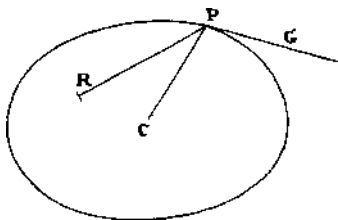
COR. II. Whence if the velocity of a body in the principal vertex D is given, the orbit may be readily found; namely, by taking its latus rectum to twice the distance DS, in the squared ratio of this given velocity to the velocity of a body revolving in a circle at the distance DS (by Cor. m, Prop, xvi), and then taking DH to DS as the latus rectum to the difference between the latus rectum and 4DS.

COR. III. Hence also if a body move in any conic section, and is forced out of its orbit by any impulse, you may discover the orbit in which it will afterwards pursue its course. For by compounding the proper motion of the body with that motion, which the impulse alone would generate, you will have the motion with which the body will go off from a given place of impulse in the direction of a right line given in position.

COR. IV. And if that body is continually disturbed by the action of some foreign force, we may nearly know its course, by collecting the changes which that force introduces in some points, and estimating the continual changes it will undergo in the intermediate places, from the analogy that appears in the progress of the series.

SCHOLIUM

If a body P, by means of a centripetal force tending to any given point R, move in the perimeter of any given conic section whose centre is C; and the law of the centripetal force is required: draw CG parallel to the radius RP, and meeting the tangent PG of the orbit in G; and the force required (by Cor. 1 and Schol., Prop, x, and Cor. 111, Prop, VII) will be as

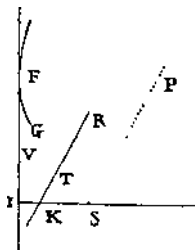


or a point P and a tangent TR , we are to describe two circles. Let H be their common intersection, and from the foci S, H , with the given axis describe the conic: I say, the thing is done. For (because $PH + SP$ in the ellipse, and $PH - SP$ in the hyperbola, is equal to the axis) the described conic will pass through the point P , and (by the preceding Lemma) will touch the right line TR . And by the same argument it will either pass through the two points P, p , or touch the two right lines TR, tr . Q.E.F.

PROPOSITION XIX. PROBLEM XI

About a given focus, to describe a parabola which shall pass through given points and touch right lines given by position.

Let S be the focus, P a point, and TR a tangent of the curve to be described. About P as a centre, with the radius PS , describe the circle FG . From the focus let fall ST perpendicular on the tangent, and produce the same to V , so as TV may be equal to ST . After the same manner another circle fg is to be described, if another point p is given; or another point v is to be found, if another tangent tr is given; then draw the right line IF , which shall touch the two circles FG, fg , if two points P, p are given; or pass through the two points V, v , if two tangents TR, tr , are given; or touch the circle FG , and pass through the point V , if the point P and the tangent TR are given. On FI let fall the perpendicular SI , and bisect the same in K ; and with the axis SK and principal vertex K describe a parabola: I say, the thing is done. For this parabola (because SK is equal to IK , and SP to FP) will pass through the point P ; and (by Cor. in, Lem. xiv) because ST is equal to TV , and STR a right angle, it will touch the right line TR . Q.E.F.

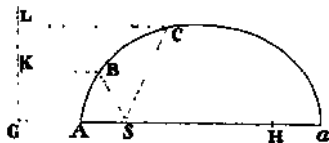


PROPOSITION XX. PROBLEM XII

About a given focus, to describe any given conic which shall pass through given points and touch right lines given by position.

CASE I. About the focus S it is required to describe a conic ABC , passing through two points B, C . Because the conic is given in kind, the ratio of the

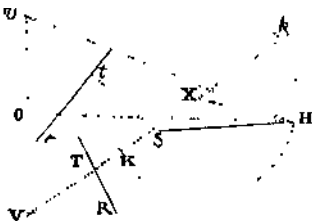
principal axis to the distance of the foci will be given. In that ratio take KB to BS, and LC to CS. About the centres B, C, with the intervals BK, CL, describe two circles; and on the right line KL, that touches the same in K and L, let fall the perpendicular SG; which cut in A and *a*, so that GA may



be to AS, and *Ga* to *aS*, as KB to BS; and with the axis A*a*, and vertices A, *a*, describe a conic: I say, the thing is done. For let H be the other focus of the described figure, and seeing that $GA : AS = Ga : aS$, we shall have

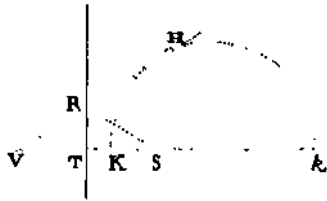
$Ga = GA : aS \cdot AS = GA : AS$, or $Aa : SH = GA : AS$, and therefore GA and AS are in the ratio which the principal axis of the figure to be described has to the distance of its foci; and therefore the described figure is of the same kind with the figure which was to be described. And since KB to BS, and LC to CS, are in the same ratio, this figure will pass through the points B, C, as is manifest from the conic sections.

CASE 2. About the focus S it is required to describe a conic which shall somewhere touch two right lines TR, *tr*. From the focus on those tangents let fall the perpendiculars ST, *Si*, which produce to V, *v*, so that TV, *tv* may be equal to TS *tS*. Bisect Vy in O and erect the indefinite perpendicular OH, and cut the right line VS infinitely produced in K and *k*, so that VK be to KS, and *Vk* to *kS*, as the principal axis of the conic to be described is to the distance of its foci. On the diameter K*k* describe a circle cutting OH in H; and with the foci S, H, and principal axis equal to VH, describe a conic: I say, the thing is done. For bisecting K*s* in X, and joining HX, HS, HV, W*v*, because VK is to KS as *Vk* to *kS*; and by composition, as $VK + Vk$ to $KS + kS$; and by subtraction, as $Vk - VK$ to $4S - KS$, that is, as $2VX$ to $2KX$, and $2KX$ to $2SX$, and therefore as VX to HX and HX to SX, the triangles VXH, HXS will be similar; therefore VH



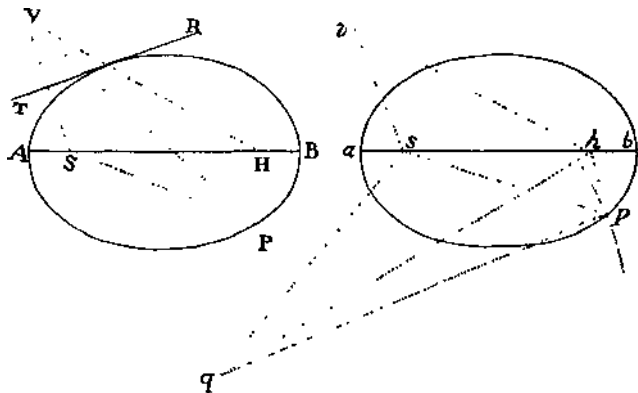
will be to SH as VX to XH; and therefore as VK to KS. Wherefore VH, the principal axis of the described conic, has the same ratio to SH, the distance of the foci, as the principal axis of the conic which was to be described has to the distance of its foci; and is therefore of the same kind. And seeing VH, eH are equal to the principal axis, and VS, vS are perpendicularly bisected by the right lines TR, *tr*, it is evident (by Lem. xv) that those right lines touch the described conic. Q.E.F.

CASE 3. About the focus S it is required to describe a conic which shall touch a right line TR in a given point R. On the right line TR let fall the perpendicular ST, which produce to V, so that TV may be equal to ST; join VR, and cut the right line VS indefinitely produced in K and *k*, so that VK may be to SK, and *Vk* to *Sk*, as the principal axis of the ellipse to be described to the distance of its foci; and on the diameter K*k* describing a circle, cut the right line VR produced in H; then with the foci S, H, and principal axis equal to VH, describe a conic: I say, the thing is done. For $VH:SH = VK:SK$, and therefore as the principal axis of the conic which was to be described to the distance of its foci (as appears from what we have demonstrated in Case 2); and therefore the described conic is of the same kind with that which was to be described; but that the right line TR, by which the angle VRS is bisected, touches the conic in the point R, is certain from the properties of the conic sections. Q.E.F.



CASE 4. About the focus S it is required to describe a conic APB that shall touch a right line TR, and pass through any given point P without the tangent, and shall be similar to the figure *apb*, described with the principal axis *ab*, and foci *s*, *h*. On the tangent TR let fall the perpendicular ST, which produce to V, so that TV may be equal to ST; and making the angles *hsq*, *shq*, equal to the angles VSP, SVP, about *q* as a centre, and with a radius which shall be to *ab* as SP to VS, describe a circle cutting the figure *apb* in *p*. Join *sp*, and draw SH such that it may be to *sh* as SP is to *sp*, and may make the angle PSH equal to the angle *psh*, and the angle VSH equal to the

angle psq . Then with the foci S , H , and principal axis AB , equal to the distance VH , describe a conic section: I say, the thing is done; for vis is drawn so that it shall be to sp as sh is to sq , and shall make the angle vsp equal to the angle hsq , and the angle vsh equal to the angle psq , the triangles svh , spq , will be similar, and therefore vh will be to pq as sh is to sq ; that is (because of the similar triangles VSP , hsq), as VS is to SP , or as ah to pq . Wherefore vh and ab are equal. But, because of the similar triangles VSH ,



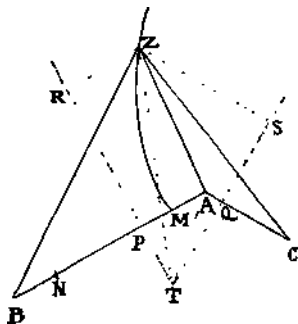
vsh , VH is to SH as vh to sh ; that is, the axis of the conic section now described is to the distance of its foci as the axis ab to the distance of the foci sh ; and therefore the figure now described is similar to the figure aph . But, because the triangle PSH is similar to the triangle psh , this figure passes through the point P ; and because VH is equal to its axis, and VS is perpendicularly bisected by the right line TR , the said figure touches the right line TR . Q.E.F.

LEMMA XVI

From three given points to draw to a fourth point that is not given three right lines whose differences either shall be given or are zero.

CASE I. Let the given points be A , B , C , and Z the fourth point which we are to find; because of the given difference of the lines AZ , BZ , the locus of the point Z will be an hyperbola whose foci are A and B , and whose principal

axis is the given difference. Let that axis be MN . Taking PM to MA as MN to AB , erect PR perpendicular to AB , and let fall ZR perpendicular to PR ; then from the nature of the hyperbola, $ZR : AZ = MN : AB$. And by the like argument, the locus of the point Z will be another hyperbola, whose foci are A, C , and whose principal axis is the difference between AZ and CZ ; and QS a perpendicular on AC may be drawn, to which (QS) if from any point Z of this hyperbola a perpendicular ZS is let fall, (this ZS) shall be to AZ as the difference between AZ and CZ is to AC . Wherefore the ratios of ZR and ZS to AZ are given, and consequently the ratio of ZR to ZS one to the other; and therefore if the right lines RP, SQ , meet in T , and TZ and TA are drawn, the figure $TRZS$ will be given in kind, and the right line TZ , in which the point Z is somewhere placed, will be given in position. There will be given also the right line TA , and the angle ATZ ; and because the ratios of AZ and TZ to ZS are given, their ratio to each other is given also; and thence will be given likewise the triangle ATZ , whose vertex is the point Z . Q.E.I.



CASE 2. If two of the three lines, for example AZ and BZ , are equal, draw the right line TZ so as to bisect the right line AB ; then find the triangle ATZ as above. Q.E.I.

CASE 3. If all the three are equal, the point Z will be placed in the centre of a circle that passes through the points A, B, C . Q.E.I.

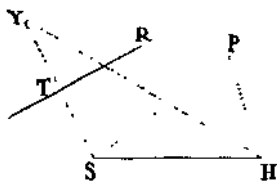
This problematic Lemma is likewise solved in the *Book of Tactions of Apollonius* restored by *Vieta*.

PROPOSITION XXL PROBLEM XIII

About a given focus, to describe a conic that shall pass through given points and touch right lines given by position.

Let the focus S , the point P , and the tangent TR be given, and suppose that the other focus H is to be found. On the tangent let fall the perpen-

dicular ST , which produce to Y , so that TY may be equal to ST , and YH will be equal to the principal axis. Join SP , HP , and SP will be the difference between HP and the principal axis. After this manner, if more tangents

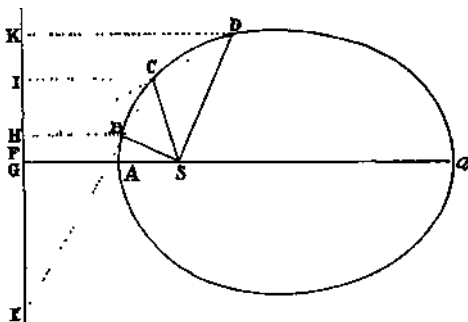


TR are given, or more points P , we shall always determine as many lines YH , or PH , drawn from the said points Y or P , to the focus H , which either shall be equal to the axes, or differ from the axes by given lengths SP ; and therefore which shall either be equal among themselves, or shall have given differences;

from whence (by the preceding Lemma), that other focus H is given. But having the foci and the length of the axis (which is either YH , or, if the conic be an ellipse, $PH + SP$; or $PH - SP$, if it be an hyperbola), the conic is given. Q.E.I.

SCHOLIUM

When the conic is an hyperbola, I do not include its conjugate hyperbola under the name of this conic. For a body going on with a continued motion can never pass out of one hyperbola into its conjugate hyperbola.



The case when three points are given is more readily solved thus. Let B , C , D be the given points. Join BC , CD , and produce them to E , F , so as EB may be to EC as SB to SC ; and FC to FD as SC to SD . On EF drawn and

produced let fall the perpendiculars SG, BH, and in GS produced indefinitely take GA to AS, and Ga to aS , as HB is to BS: then A will be the vertex, and Aa the principal axis of the conic; which, according as GA is greater than, equal to, or less than AS, will be either an ellipse, a parabola, or an hyperbola; the point a in the first case falling on the same side of the line GF as the point A; in the second, going off to an infinite distance; in the third, falling on the other side of the line GF. For if on GF the perpendiculars CI, DK are let fall, IC will be to HB as EC to EB; that is, as SC to SB; and by permutation, IC to SC as HB to SB, or as GA to SA. And, by the like argument, we may prove that KD is to SD in the same ratio. Wherefore the points B, C, D be in a conic section described about the focus S, in such manner that all the right lines drawn from the focus S to the several points of the section, and the perpendiculars let fall from the same points on the right line GF, are in that given ratio.

That excellent geometer M. *de la Hire* has solved this Problem much after the same way, in his *Comes*, Prop, xxv, Book VIII.

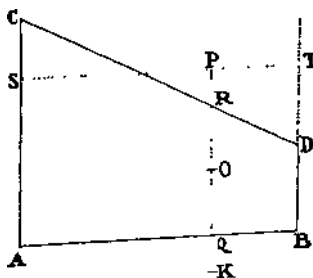
SECTION V

How the orbits are to be found when neither focus is given.

LEMMA XVII

If from any point P of a given conic section, to the four produced sides AB, CD, AC, DB of any trapezium¹ ABDC inscribed in that section, as many right lines PQ, PR, PS, PT are drawn in given angles, each line to each side; the rectangle PQ · PR of those on the opposite sides AB, CD, will be to the rectangle PS · PT of those on the other two opposite sides AC, BD, in a given ratio.

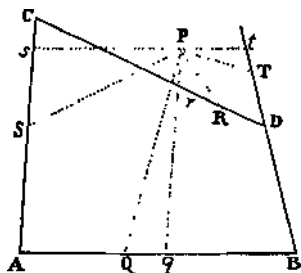
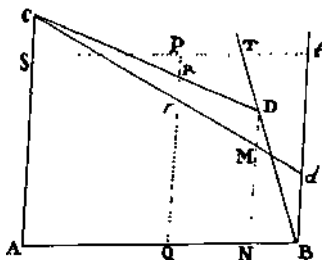
CASE I. Let us suppose, first, that the lines drawn to one pair of opposite sides are parallel to either of the other sides; as PQ and PR to the side AC, and PS and PT to the side AB. And further, that one pair of the opposite



sides, as AC and BD, are parallel between themselves; then the right line which bisects those parallel sides will be one of the diameters of the conic section, and will likewise bisect RQ. Let O be the point in which RQ is bisected, and PO will be an ordinate to that diameter. Produce PO to K, so that OK may be equal to PO, and OK will be an ordinate on the other side of that diameter.

Since, therefore, the points A, B, P, and K are placed in the conic section, and PK cuts AB in a given angle, the rectangle PQ · QK (by Prop. xvii, xix, xxi, and xxiii, Book in, *Comes of Apollonius*) will be to the rectangle AQ · QB in a given ratio. But QK and PR are equal, as being the differences of the equal lines OK, OP, and OQ, OR; whence the rectangles PQ · QK and PQ · PR are equal; and therefore the rectangle PQ · PR is to the rectangle AQ · QB, that is, to the rectangle PS · PT, in a given ratio. Q.E.D.

CASE 2. Let us next suppose that the opposite sides AC and BD of the trapezium are not parallel. Draw Vd parallel to AC, and meeting as well the right line ST in t , as the conic section in d . Join Cd cutting PQ in r , and draw DM parallel to PQ, cutting Cd in M, and AB in N. Then (because of the similar triangles BT/, DBN) Bt or $PQ : Tt = DN : NB$. And so $Rr : AQ$ or $PS = DM : AN$. Wherefore, by multiplying the antecedents by the antecedents, and the consequents by the consequents, as the rectangle PQ Rr is to the rectangle PS Tt , so will the rectangle DN DM be to the rectangle NA NB; and (by Case 1) so is the rectangle PQ $\cdot$ Pr to the rectangle PS $\cdot$ Pt, and, by division, so is the rectangle PQ $\cdot$ PR to the rectangle PS $\cdot$ PT. Q.E.D.



CASE 3. Let us suppose, lastly, the four lines PQ, PR, PS, PT not to be parallel to the sides AC, AB, but any way inclined to them. In their place draw Vq , Vr , parallel to AC; and Vs , Vt parallel to AB; and because the angles of the triangles VQq , PRr , VSs , PTt are given, the ratios of PQ to Pq , PR to Pr , PS to Ps , PT to Pt will be also given; and therefore the compounded ratios PQ $\cdot$ PR to Pq Pr , and PS $\cdot$ PT to Ps Pt are given. But from

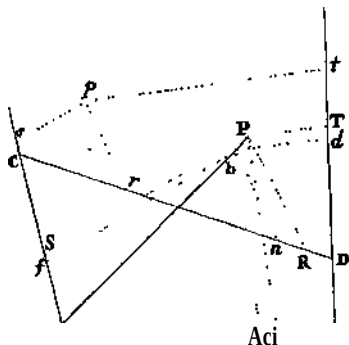
what we have demonstrated before, the ratio of Pq Pr to Ps Pt is given; and therefore also the ratio of PQ $\cdot$ PR to PS $\cdot$ PT. Q.E.D.

LEMMA XVIII

The same things supposed, if the rectangle PQ $\cdot$ PR of the lines drawn to the two opposite sides of the trapezium is to the rectangle PS $\cdot$ PT of those

drawn to the other two sides in a given ratio, the point *P*, from whence those lines are drawn, will be placed in a conic section described about the trapezium.

Conceive a conic section to be described passing through the points *A*, *B*, *C*, *D*, and any one of the infinite number of points *P*, as for example *p*: I say, the point *P* will be always placed in this section. If you deny the thing,



join *AP* cutting this conic section somewhere else, if possible, than in *P*, as in *b*. Therefore if from those points *p* and *b*, in the given angles to the sides of the trapezium, we draw the right lines *pq*, *pr*, *ps*, *pt*, and *bk*, *bn*, *bf*, *bd*, we shall have, as $bk \cdot bn$ to $bf \cdot bd$, so (by Lem. xvii) $pq \cdot pr$ to $ps \cdot pt$; and so (by supposition) $PQ \cdot PR$ to $PS \cdot PT$. And because of the similar trapezia *bkAf* *PQAS*, as *bk* to *bf*, so *PQ* to *PS*. Wherefore by dividing

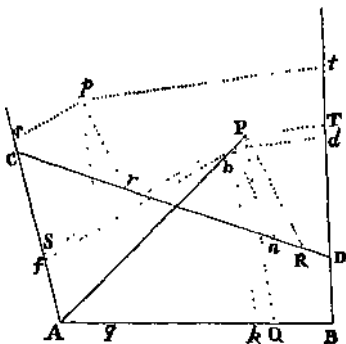
the terms of the preceding proportion by the correspondent terms of this, we shall have *bn* to *bd* as *PR* to *PT*. And therefore the equiangular trapezia *Dnbd*, *DRPT*, are similar, and consequently their diagonals *Db*, *DP* do coincide. Wherefore *b* falls in the intersection of the right lines *AP*, *DP*, and consequently coincides with the point *P*. And therefore the point *P*, wherever it is taken, falls within the assigned conic section. Q.E.D.

COR. Hence if three right lines *PQ*, *PR*, *PS* are drawn from a common point *P*, to as many other right lines given in position, *AB*, *CD*, *AC*, each to each, in as many angles respectively given, and the rectangle $PQ \cdot PR$ under any two of the lines drawn be to the square of the third *PS* in a given ratio; the point *P*, from which the right lines are drawn, will be placed in a conic section that touches the lines *AB*, *CD* in *A* and *C*; and the contrary. For the position of the three right lines *AB*, *CD*, *AC* remaining the same, let the line *BD* approach to and coincide with the line *AC*; then let the line

PT come likewise to coincide with the line PS; and the rectangle $PS \cdot PT$ will become PS^2 , and the right lines AB, CD, which before did cut the curve in the points A and B, C and D, can no longer cut, but only touch, the curve in those coinciding points.

S C H O L I U M

In this Lemma, the name of conic section is to be understood in a large sense, comprehending as well the rectilinear section through the vertex of the cone, as the circular one parallel to the base. For if the point p happens to be in a right line, by which the points A and D, or C and B are joined, the conic section will be changed into two right lines, one of which is that right line upon which the point p falls, and the other is a right line that joins the other two of the four points. If the two opposite angles of the trapezium taken together are equal to two right angles, and if the four lines PQ, PR, PS, PT are drawn to the sides thereof at right angles, or any other equal angles, and the rectangle $PQ \cdot PR$ under two of the lines drawn PQ and PR, is equal to the rectangle $PS \cdot PT$ under the other two PS and PT, the conic section will become a circle. And the same thing will happen if the four lines are drawn in any angles, and the rectangle PQ PR, under one pair of the lines drawn, is to the rectangle PS PT under the other pair as the rectangle under the sines of the angles S, T, in which the two last lines PS, PT are drawn, to the rectangle under the sines of the angles Q, R, in which the first two PQ, PR are drawn. In all other cases the locus of the point P will be one of the three figures which pass commonly by the name of the conic sections. But in place of the trapezium ABCD, we may substitute a quadrilateral figure whose two opposite sides cross one another like diagonals. And one or two of the four points A, B, C, D may be sup-

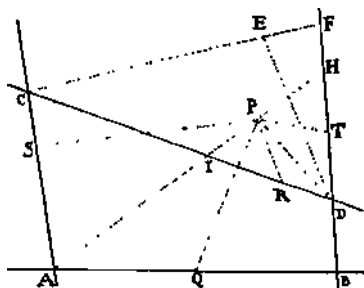


posed to be removed to an infinite distance, by which means the sides of the figure which converge to those points, will become parallel; and in this case the conic section will pass through the other points, and will go the same way as the parallels *in infinitum*.

LEMMA XIX

To find a point P from which if four right lines PQ, PR, PS, PT are drawn to as many other right lines AB, CD, AC, BD, given by position, each to each, at given angles, the rectangle PQ PR, under any two of the lines drawn, shall be to the rectangle PS PT, under the other two, in a given ratio.

Suppose the lines AB, CD, to which the two right lines PQ, PR, containing one of the rectangles, are drawn to meet two other lines, given by position, in the points A, B, C, D. From one of those, as A, draw any right

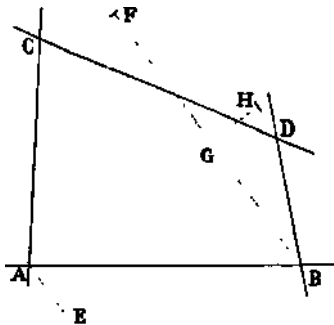


line AH, in which you would find the point P. Let this cut the opposite lines BD, CD, in H and I; and, because all the angles of the figure are given, the ratio of PQ to PA, and PA to PS, and therefore of PQ to PS, will be also given. This ratio taken as a divisor of the given ratio of PQ PR to PS PT, gives the ratio of PR to PT; and multiplying

the given ratios of PI to PR, and PT to PH, the ratio of PI to PH, and therefore the point P, will be given. Q.E.I.

COR. I. Hence also a tangent may be drawn to any point D of the locus of all the points P. For the chord PD, where the points P and D meet, that is, where AH is drawn through the point D, becomes a tangent. In which case the ultimate ratio of the evanescent lines IP and PH will be found as above. Therefore draw CF parallel to AD, meeting BD in F, and cut it in E in the same ultimate ratio, then DE will be the tangent; because CF and the evanescent IH are parallel, and similarly cut in E and P.

COR. II. Hence also the locus of all the points P may be determined. Through any of the points A, B, C, D, as A, draw AE touching the locus, and through any other point B, parallel to the tangent, draw BF meeting the locus in F; and find the point F by this Lemma. Bisect BF in G, and, drawing the indefinite line AG, this will be the position of the diameter to which BG and FG are ordinates. Let this AG meet the locus in H, and AH will be its diameter or latus transversum, to which the latus rectum will be as BG^2 to AG GH. If AG nowhere meets the locus, the line AH being infinite, the locus will be a parabola; and its latus rectum corresponding to the diameter AG



will be AG^2 . But if it does meet it

anywhere, the locus will be an hyperbola, when the points A and H are placed on the same side of the point G; and an ellipse, if the point G falls between the points A and H; unless, perhaps, the angle AGB is a right angle, and at the same time BG^2 equal to the rectangle GA GH, in which case the locus will be a circle.

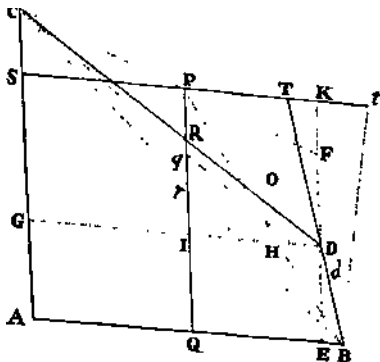
And so we have given in this Corollary a solution of that famous Problem of the ancients concerning four lines, begun by *Euclid*, and carried on by *Apollonius*; and this not an analytical calculus but a geometrical composition, such as the ancients required.

LEMMA XX

If the two opposite angular points A and P of any parallelogram ASPQ touch any conic section in the points A and P; and the sides AQ, AS of one of those angles, indefinitely produced, meet the same conic section in B and C; and from the points of meeting B and C to any fifth point D of the conic section, two right lines BD, CD are drawn meeting the two other sides PS, PQ of the parallelogram, indefinitely produced in T and R; the parts PR

and PT , cut off from the sides, will always be one to the other in a given ratio. And conversely, if those parts cut off are one to the other in a given ratio, the locus of the point D will be a conic section passing through the four points A, B, C, P .

CASE 1. Join BP, CP , and from the point D draw the two right lines DG, DE , of which the first DG shall be parallel to AB , and meet PB, PQ, CA , in H, I, G ; and the other DE shall be parallel to AC , and meet PC, PS, AB , in



F, K, E ; and (by Lem. xvn) the rectangle $DE \cdot DF$ will be to the rectangle $DG \cdot DH$ in a given ratio. But PQ is to DE (or IQ) as PB to HB , and consequently as PT to DH ; and by permutation PQ is to PT as DE to DH . Likewise PR is to DF as RC to DC , and therefore as $(IG$ or) PS to DG ; and by permutation PR is to PS as DF to DG ; and, by compounding those ratios, the rectangle $PQ \cdot PR$ will be to the

rectangle $PS \cdot PT$ as the rectangle $DE \cdot DF$ is to the rectangle $DG \cdot DH$, and consequently in a given ratio. But PQ and PS are given, and therefore the ratio of PR to PT is given. Q.E.D.

CASE 2. But if PR and PT are supposed to be in a given ratio one to the other, then by going back again, by a like reasoning, it will follow that the rectangle $DE \cdot DF$ is to the rectangle $DG \cdot DH$ in a given ratio; and so the point D (by Lem. xvni) will be in a conic section passing through the points A, B, C, P , as its locus. Q.E.D.

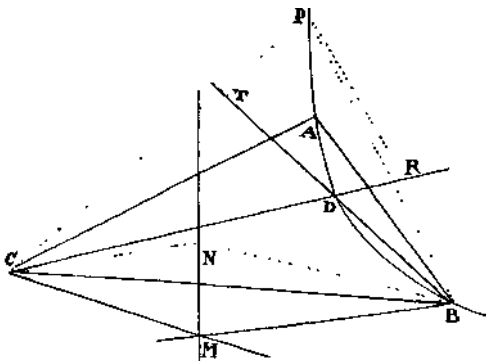
COR. 1. Hence if we draw BC cutting PQ in r and in PT take Yt to Pr in the same ratio which PT has to PR ; then Bt will touch the conic section in the point B . For suppose the point D to coalesce with the point B , so that the chord BD vanishing, BT shall become a tangent; and CD and BT will coincide with CB and Bt .

COR. 11. And, *vice versa*, if Bt is a tangent, and the lines BD, CD meet in any point D of a conic section, PR will be to PT as Vr to Ft. And, on the contrary, if PR is to PT as Pr to P/, then BD and CD will meet in some point D of a conic section.

COR. in. One conic section cannot cut another conic section in more than four points. For, if it is possible, let two conic sections pass through the five points A, B, C, P, O; and let the right line BD cut them in the points D, *d*, and the right line Cd cut the right line PQ in *q*. Therefore PR is to PT as Vq to PT: whence PR and Vq are equal one to the other, against the supposition.

LEMMA XXI

If two movable and indefinite right lines BM, CM drawn through given points B, C, as poles, do by their point of meeting M describe a third right line MN given by position; and other two indefinite right lines BD, CD are drawn, making with the former two at those given points B, C, given angles, MBD, MCD: I say, that those two right lines BD, CD will by their point of meeting D describe a conic section passing through the points B, C. And conversely, if the right lines BD, CD do by their point of meeting D describe a conic section passing through the given points B, C, A, and the angle DBM is always equal to the given angle ABC, as well as the angle DCM always equal to the given angle ACB, the point M will lie in a right line given by position, as its locus.



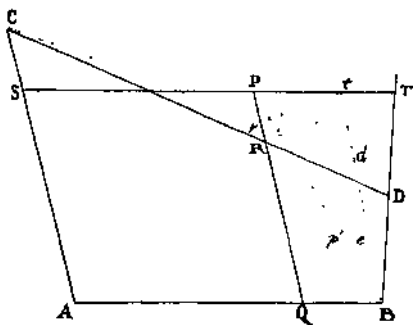
this line nN will be the continual locus of that movable point M . For, if possible, let the point M be placed in any curved line. Therefore the point D will be placed in a conic section passing through the five points B, C, A, p, P , when the point M is continually placed in a curved line. But from what was demonstrated before, the point D will be also placed in a conic section passing through the same five points B, C, A, p, P , when the point M is continually placed in a right line. Wherefore the two conic sections will both pass through the same five points, against Cor. in, Lem. xx. It is therefore absurd to suppose that the point M is placed in a curved line. Q.E.D.

PROPOSITION XXII. PROBLEM XIV

To describe a conic that shall pass through five given points.

Let the five given points be A, B, C, P, D . From any one of them, as A , to any other two as B, C , which may be called the poles, draw the right lines AB, AC , and parallel to those the lines TPS, PRQ through the fourth point P . Then from the two

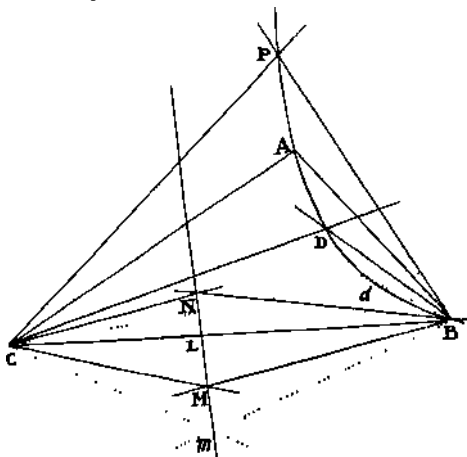
poles B, C , draw through the fifth point D two indefinite lines BDT, CRD , meeting with the last drawn lines TPS, PRQ (the former with the former, and the latter with the latter) in T and R . And then draw the right line tr parallel to TR , cutting off from the right lines PT, PR , any segments Pt, Vr , pro-



portional to PT, PR ; and if through their extremities t, r , and the poles B, C , the right lines Bt, Cr are drawn, meeting in d , that point d will be placed in the conic required. For (by Lem. xx) that point d is placed in a conic section passing through the four points A, B, C, P ; and the lines Rr, Tt vanishing, the point d comes to coincide with the point D . Wherefore the conic section passes through the five points A, B, C, P, D . Q.E.D.

The same otherwise.

Of the given points join any three, as A, B, C; and about two of them B, C, as poles, making the angles ABC, ACB of a given magnitude to revolve, apply the legs BA, CA, first to the point D, then to the point P, and mark the points M, N, in which the other legs BL, CL intersect each other in both cases. Draw the indefinite right line MN, and let those movable angles revolve about their poles B, C, in such manner that the intersection, which



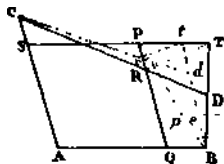
is now supposed to be m , of the legs BL, CL, or BM, CM, may always fall in that indefinite right line MN; and the intersection, which is now supposed to be d , of the legs BA, CA, or BD, CD, will describe the conic required, PADdB. For (by Lem. xxi) the point d will be placed in a conic section passing through the points B, C; and when the point m comes to coincide with the points L, M, N, the point d will (by construction) come to coincide with the points A, D, P. Wherefore a conic section will be described that shall pass through the five points A, B, C, P, D. Q.E.F.

COR. I. Hence a right line may be readily drawn which shall be a tangent to the conic in any given point B. Let the point d come to coincide with the point B, and the right line Bd will become the tangent required.

COR. 11. Hence also may be found the centres, diameters, and latera recta of the conies, as in Cor. II, Lem. xix.

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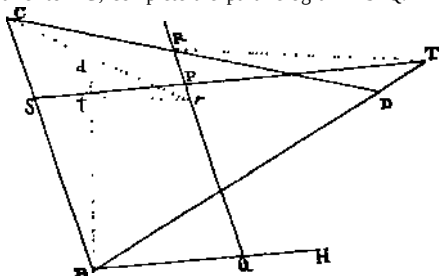
The former of these constructions will become something more simple by joining B, P , and in that line, produced, if need be, taking Bp to BP as PR is to PT ; and through p draw the indefinite right line pe parallel to SPT , and in that line pe taking always pe equal to Pr ; and draw the right lines Be, Cr to meet in d . For since Pr to Pt , PR to PT , pB to PB , pe to Pt , are all in the same ratio, pe and Pr will be always equal. After this manner the points of the conic are most readily found, unless you would rather describe the curve mechanically, as in the second construction.



PROPOSITION XXIII. PROBLEM XV

To describe a conic that shall pass through four given points, and touch a given right line.

CASE I. Suppose that HB is the given tangent, B the point of contact, and C, D, P , the three other given points. Join BC , and draw PS parallel to BH , and PQ parallel to BC ; complete the parallelogram $BSPQ$. Draw BD cut-



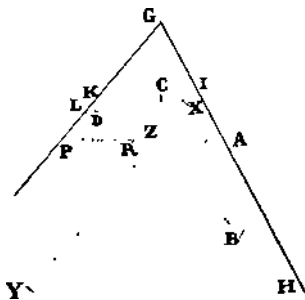
ting SP in T , and CD cutting PQ in R . Lastly, draw any line tr parallel to TR , cutting off from PQ, PS , the segments Pr, Pt proportional to PR, PT respectively, and draw Cr, Bt ; their point of intersection d will (by Lem. xx) always fall on the conic to be described.

product of the mean proportional between GD and GB, and the mean proportional between PI and IC, and A will be the point of contact. For if HX, a parallel to the right line PI, cuts the conic in any points X and Y, the point A (by the properties of the conic sections) will come to be so placed, that HA^2 will become to AI^2 in a ratio that is compounded out of the ratio of the rectangle HX HY to the rectangle BH HD, or of the rectangle CG GP to the rectangle DG GB; and the ratio of the rectangle BH HD to the rectangle PI IC. But after the point of contact A is found, the conic will be described as in the first Case. Q.E.F. But the point A may be taken either between or without the points H and I, upon which account a two-fold conic may be described.

PROPOSITION XXIV. PROBLEM XVI

To describe a conic that shall pass through three given points, and touch two given right lines.

Suppose HI, KL to be the given tangents and B, C, D the given points. Through any two of those points, as B, D, draw the indefinite right line BD meeting the tangents in the points H, K. Then likewise through any other two of these points, as C, D, draw the indefinite right line CD meeting the tangents in the points I, L. Cut the lines drawn in R and S, so that HR may be to KR as the mean proportional between BII and HD is to the mean proportional between BK and KD, and IS to LS as the mean proportional between CI and ID is to the mean proportional between CL and LD. But you may cut, at pleasure, either within or between the points K and H, I and L, or without them. Then draw RS cutting the tangents in A and P, and A and P will be the points of contact. For if A and P are supposed to be the points of contact, situated anywhere else in the tangents, and through any of the points H, I, K, L, as I, situated in either tangent HI,



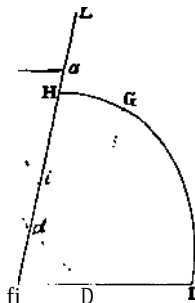
a right line IY is drawn parallel to the other tangent KL, and meeting the curve in X and Y, and in that right line there be taken IZ equal to a mean proportional between IX and IY, the rectangle XI IY or IZ² will (by the properties of the conic sections) be to LP² as the rectangle CI ID is to the rectangle CL LD; that is (by the construction), as SI is to SL², and therefore IZ : LP² :: SI : SL. Wherefore the points S, P, Z are in one right line. Moreover, since the tangents meet in G, the rectangle XI IY or IZ² will (by the properties of the conic sections) be to IA² as GP² is to GA², and consequently IZ : IA :: GP : GA. Wherefore the points P, Z, A lie in one right line, and therefore the points S, P, and A are in one right line. And the same argument will prove that the points R, P, and A are in one right line. Wherefore the points of contact A and P lie in the right line RS. But after these points are found, the conic may be described, as in the first Case of the preceding Problem. Q.E.F.

In this Proposition, and Case 2 of the foregoing, the constructions are the same, whether the right line XY cuts the conic in X and Y, or not; neither do they depend upon that section. But the constructions being demonstrated where that right line does cut the conic, the constructions where it does not are also known; and therefore, for brevity's sake, I omit any further demonstration of them.

LEMMA XXII¹

To transform figures into other figures of the same kind.

Suppose that any figure HGI is to be transformed. Draw, at pleasure, two parallel lines AO, BL, cutting any given third line AB in A and B, and from any point G of the figure, draw out any right line GD, parallel to OA, till it meets the right line AB. Then from any given point O in the line OA, draw to the point



[¹ Appendix, Note 20]

D the right line OD, meeting BL in d ; and from the point of intersection raise the right line dg containing any given angle with the right line BL, and having such ratio to Od as DG has to OD; and g will be the point in the new figure hgi , corresponding to the point G. And in like manner the several points of the first figure will give as many correspondent points of the new figure. If we therefore conceive the point G to be carried along by a continual motion through all the points of the first figure, the point g will be likewise carried along by a continual motion through all the points of the new figure, and describe the same. For distinction's sake, let us call DG the first ordinate, dg the new ordinate, AD the first abscissa, ad the new abscissa, O the pole, OD the abscinding radius, OA the first ordinate radius, and Oa (by which the parallelogram $OABa$ is completed) the new ordinate radius.

I say, then, that if the point G is placed in a given right line, the point g will be also placed in a given right line. If the point G is placed in a conic section, the point g will be likewise placed in a conic section. And here I understand the circle as one of the conic sections. But further, if the point G is placed in a line of the third analytical order, the point g will also be placed in a line of the third order, and so on in curved lines of higher orders. The two lines in which the points G, g are placed, will be always of the same analytical order. For as $ad : OA = Od : OD = dg : DG = AB : AD$; and therefore AD is equal to $\frac{OA \cdot AB}{ad}$, and DG equal to $\frac{OA \cdot dg}{ad}$. Now if the point G is placed in a right line, and therefore, in any equation by which the relation between the abscissa AD and the ordinate GD is expressed, those indetermincd lines AD and DG rise no higher than to one dimension, by writing this equation $\frac{OA \cdot AB}{ad}$ in place of AD, and $\frac{OA \cdot dg}{ad}$ in place of DG, a new equation will be produced, in which the new abscissa ad and new ordinate dg rise only to one dimension; and which therefore must denote a right line. But if AD and DG (or either of them) had risen to two dimensions in the first equation, ad and dg would likewise have risen to two dimensions in the second equation. And so on in three or more dimensions. The indetermincd lines, ad , dg in the second equation, and AD, DG in the

first, will always rise to the same number of dimensions; and therefore the lines in which the points G, g are placed are of the same analytical order.

I say, further, that if any right line touches the curved line in the first figure, the same right line transferred the same way with the curve into the new figure will touch that curved line in the new figure, and conversely. For if any two points of the curve in the first figure are supposed to approach one the other till they come to coincide, the same points transferred will approach one the other till they come to coincide in the new figure; and therefore the right lines with which those points are joined will become together tangents of the curves in both figures. I might have given demonstrations of these assertions in a more geometrical form; but I study to be brief.

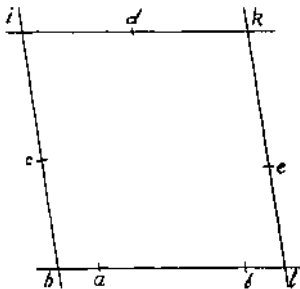
Wherefore if one rectilinear figure is to be transformed into another, we need only transfer the intersections of the right lines of which the first figure consists, and through the transferred intersections to draw right lines in the new figure. But if a curvilinear figure is to be transformed, we must transfer the points, the tangents, and other right lines, by means of which the curved line is defined. This Lemma is of use in the solution of the more difficult Problems; for thereby we may transform the proposed figures, if they are intricate, into others that are more simple. Thus any right lines converging to a point are transformed into parallels, by taking for the first ordinate radius any right line that passes through the point of intersection of the converging lines, and that because their point of intersection is by this means made to go oft in *infinitum*; and parallel lines are such as tend to a point infinitely remote. And after the problem is solved in the new figure, if by the inverse operations we transform the new into the first figure, we shall have the solution required.

This Lemma is also of use in the solution of solid problems. For as often as two conic sections occur, by the intersection of which a problem may be solved, any one of them may be transformed, if it is an hyperbola or a parabola, into an ellipse, and then this ellipse may be easily changed into a circle. So also a right line and a conic section, in the construction of plane problems, may be transformed into a right line and a circle.

PROPOSITION XXV. PROBLEM XVII

To describe a come that shall pass through two given points, and touch three given right lines.

Through the intersection of any two of the tangents one with the other, and the intersection of the third tangent with the right line which passes through the two given points, draw an indefinite right line; and, taking this line for the first ordinate radius, transform the figure by the preceding Lemma into a new figure. In this figure those two tangents will become parallel to each other, and the third tangent will be parallel to the right line that passes through the two given points. Suppose hi , kl to be those two parallel tangents, ik the third tangent, and hl a right line parallel thereto, passing through those points a , b , through which the conic section ought to pass in this new figure; and completing the parallelogram $hikl$, let the right lines hi , ik , kl be so cut in c , d , c , that he may be to the square root of the rectangle ahb , ic to id , and ke to kd , as the sum of the right lines hi and kl is to the sum of the three lines, the first whereof is the right line ik , and the other two are the square roots of the rectangles ahb and alb ; and c , d , e will be the points of contact. For by the properties of the conic sections,



$$hc^2 : ah \cdot hb :: ic^2 : id^2 :: ke^2 : kd^2 :: el^2 : al \cdot lb.$$

Therefore,

$$\begin{aligned} hc : \sqrt{(ah \cdot hb)} &= ic : id = ke : kd = el : \sqrt{(al \cdot lb)} \\ &= hc : ic + ke + el : \sqrt{(ah \cdot hb)} + id + kd : \sqrt{(al \cdot lb)} \\ &= hi : kl : \sqrt{(ah \cdot hb)} + ik : \sqrt{(al \cdot lb)}. \end{aligned}$$

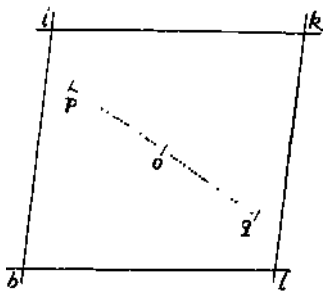
Wherefore from that given ratio we have the points of contact c , d , e , in the new figure. By the inverted operations of the last Lemma, let those points be transferred into the first figure, and the conic will be there described by Prob. xiv. Q.E.F. But according as the points a , b , fall between the points h , l , or without them, the points c , d , e must be taken either between the

points h, i, k, l , or without them. If one of the points a, b falls between the points h, i , and the other without the points h, l , the Problem is impossible.

PROPOSITION XXVI. PROBLEM XVIII

To describe a conic that shall pass through a given point, and touch four given right lines.

From the common intersections of any two of the tangents to the common intersection of the other two, draw an indefinite right line; and taking this line for the first ordinate radius, transform the figure (by Lem. xxn)



into a new figure, and the two pairs of tangents, each of which before concurred in the first ordinate radius, will now become parallel. Let hi and kl , ik and hl , be those pairs of parallels completing the parallelogram $hikl$. And let p be the point in this new figure corresponding to the given point in the first figure. Through O the centre of the figure draw pq : and Oq being equal to Op , q will be

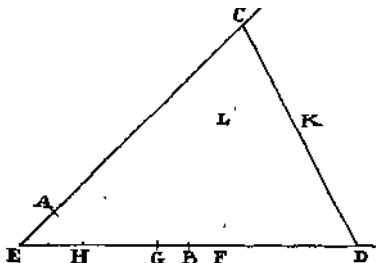
the other point through which the conic section must pass in this new figure. Let this point be transferred, by the inverse operation of Lem. xxn, into the first figure, and there we shall have the two points through which the conic is to be described. But through those points that conic may be described by Prop. XVII.

LEMMA XXIII

If two given right lines, as AC, BD , terminating in given points A, B , are in a given ratio one to the other, and the right line CD , by which the indetermined points C, D are joined is cut in K in a given ratio: I say, that the point K will be placed in a given right line.

For let the right lines AC, BD meet in E , and in BE take BG to AE as BD is to AC , and let FD be always equal to the given line EG ; and, by construction, EC will be to GD , that is, to EF , as AC to BD , and therefore in

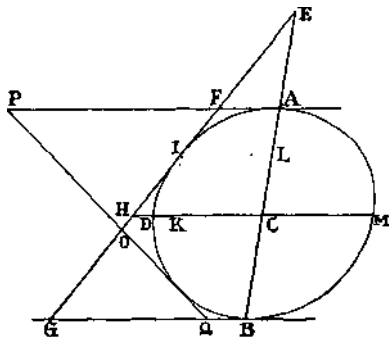
a given ratio; and therefore the triangle EFC will be given in kind. Let CF be cut in L so as CL may be to CF in the ratio of CK to CD; and because that is a given ratio, the triangle EFL will be given in kind, and therefore the point L will be placed in the given right line EL. Join LK, and the triangles CLK, CFD will be similar; and because FD is a given line, and LK is to FD in a given ratio, LK will be also given. To this let EH be taken equal, and ELKH will be always a parallelogram. And therefore the point K is always placed in the given side HK of that parallelogram. Q.E.D.



COR. Because the figure EFLC is given in kind, the three right lines EF, EL, and EC, that is, GD, HK, and EC, will have given ratios to each other.

LEMMA XXIV

If three right lines, two whereof are parallel, and given in position, touch any conic section: I say, that the semidiameter of the section which is parallel to those two is a mean proportional between the segments of those two that are intercepted between the points of contact and the third tangent.



Let AF, GB be the two parallels touching the conic section ADB in A and B; EF the third right line touching the conic section in I, and meeting the two former tangents in F and G, and let CD be the semidiameter of the figure parallel to those tangents: I say, that AF, CD, BG are continually proportional.

For if the conjugate diameters AB, DM meet the tangent FG in E and H, and cut one the other in C, and the parallelogram IKCL be completed; from the nature of the conic sections,

$$EC : CA = CA : CL;$$

thence, $EC - CA : CA - CL = EC : CA$

or $EA : AL = EC : CA;$

thence, $EA : EA + AL - EC : EC + CA$

or $EA : EL = EC : EB.$

Therefore, because of the similitude of the triangles EAF, ELI, ECH, EBG,

$$AF : LI = CH : BG.$$

Likewise, from the nature of the conic sections,

$$LI \text{ or } CK : CD = CD : CH.$$

Taking the products of corresponding terms in the last two proportions and simplifying,

$$AF : CD = CD : BG.$$

Q.E.D.

COR. I. Hence if two tangents FG, PQ meet two parallel tangents AF, BG in F and G, P and Q, and cut one the other in O; then by the Lemma applied to EG and PQ,

$$AF : CD = CD : BG,$$

$$BQ : CD = CD : AP.$$

Therefore, $AF : AP = BQ : BG$

and $AP - AF : AP = BG - BQ : BG$

or $PF : AP = GQ : BG,$

and $AP : BG = PF : GQ = FO : GO = AF : BQ.$

COR. II. Whence also the two right lines PG, FQ drawn through the points P and G, F and Q, will meet in the right line ACB passing through the centre of the figure and the points of contact A, B.

LEMMA XXV

If four sides of a parallelogram indefinitely produced touch any conic section, and are cut by a fifth tangent: I say, that, taking those segments of any two conterminous sides that terminate in opposite angles of the parallelo-

gram, either segment is to the side from which it is cut off as that part of the other conterminous side which is intercepted between the point of contact and the third side is to the other segment.

Let the four sides ML, IK, KL, MI of the parallelogram MLIK touch the conic section in A, B, C, D; and let the fifth tangent FQ cut those sides in F, Q, H, and E; and taking the segments ME, KQ of the sides MI, KI, or the segments KH, MF of the sides KL, ML: I say, that

$$ME : MI = BK : KQ,$$

and $KH : KL = AM : MF.$

For, by Cor. 1 of the preceding Lemma,

$$ME : EI = AM \text{ or } BK : BQ,$$

and by addition,

$$ME : MI = BK : KQ.$$

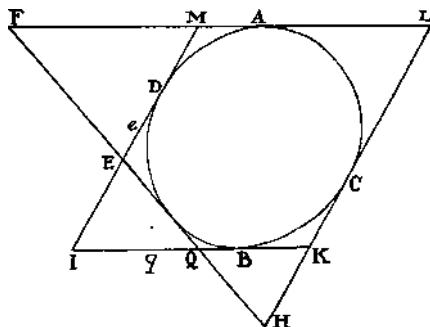
Q.E.D.

Also, $KH : HL = BK \text{ or } AM : AF,$

and by subtraction,

$$KH : KL = AM : MF.$$

Q.E.D.



COR. I. Hence if a parallelogram I K L M described about a given conic section is given, the rectangle KQ ME, as also the rectangle KH MF equal thereto, will be given. For, by reason of the similar triangles KQH, MFE, those rectangles are equal.

COR. II. And if a sixth tangent eq is drawn meeting the tangents KI, MI in q and e , the rectangle KQ ME will be equal to the rectangle $Kq Me$,

and $KQ : Me - Kq : ME$,
and by subtraction

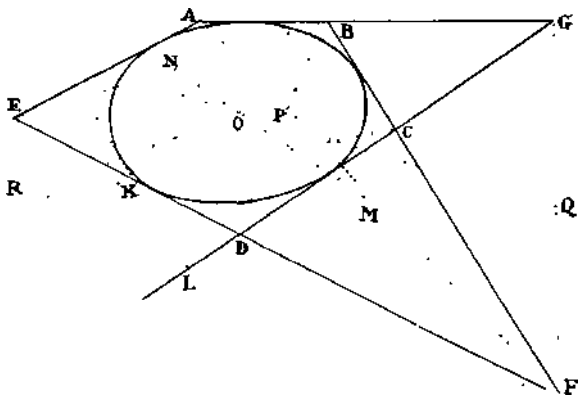
$$KQ : Me = Qq : Ee.$$

COR. HI. Hence, also, if Eq , eQ are joined and bisected, and a right line is drawn through the points of bisection, this right line will pass through the centre of the conic section. For since $Qq : Ee = KQ : Me$, the same right line will pass through the middle of all the lines Eq , eQ , MK (by Lem. xxIII), and the middle point of the right line MK is the centre of the section.

PROPOSITION XXVII. PROBLEM XIX

To describe a come that may touch five right lines given in position.

Supposing ABG , BCF , GCD , FDE , EA to be the tangents given in position. Bisect in M and N , AF , BE , the diagonals of the quadrilateral figure $ABFE$ contained under any four of them; and (by Cor. III, Lem. xxv) the



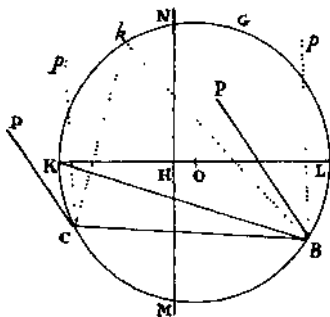
right line MN drawn through the points of bisection will pass through the centre of the conic. Again, bisect in P and Q the diagonals (if I may so call them) BD , GF of the quadrilateral figure $BGDF$ contained under any other four tangents, and the right line PQ drawn through the points of bisection will pass through the centre of the conic; and therefore the centre

will be given in the intersection of the bisecting lines. Suppose it to be O. Parallel to any tangent BC draw KL at such distance that the centre O may be placed in the middle between the parallels; this KL will touch the conic to be described. Let this cut any other two tangents GCD, FDE, in L and K. Through the points C and K, F and L, where the tangents not parallel, CL, FK, meet the parallel tangents CF, KL, draw CK, FL meeting in R; and the right line OR, drawn and produced, will cut the parallel tangents CF, KL, in the points of contact. This appears from Cor. II, Lem. xxiv. And by the same method the other points of contact may be found, and then the conic may be described by Prob. xiv. Q.E.F.

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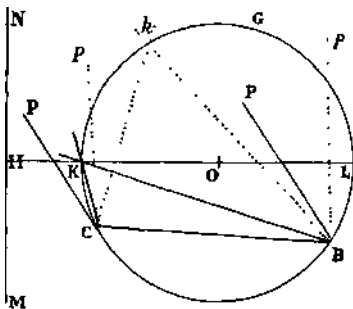
Under the preceding Propositions are comprehended those Problems wherein either the centres or asymptotes of the conies are given. For when points and tangents and the centre are given, as many other points and as many other tangents are given at an equal distance on the other side of the centre. And an asymptote is to be considered as a tangent, and its infinitely remote extremity (if we may say so) is a point of contact. Conceive the point of contact of any tangent removed *in infinitum*, and the tangent will degenerate into an asymptote, and the constructions of the preceding Problems will be changed into the constructions of those Problems wherein the asymptote is given.

After the conic is described, we may find its axes and loci in this manner. In the construction and figure of Lem. xxi, let those legs BP, CP, of the movable angles PBN, PCN, by the intersection of which the conic was described, be made parallel one to the other; and retaining that position, let them revolve about their poles B, C, in that figure. In the meanwhile let the other legs CN, BN, of those angles, by their intersection K or



k, describe the circle BKGK. Let O be the centre of this circle; and from this centre upon the ruler MN, wherein those legs CN, BN did concur while the conic was described, let fall the perpendicular OH meeting the circle in K and L. And when those other legs CK, BK meet in the point K that is nearest to the ruler, the first legs CP, BP will be parallel to the greater axis, and perpendicular on the lesser; and the contrary will happen if those legs meet in the remotest point L. Whence if the centre of the conic is given, the axes will be given; and those being given, the foci will be readily found.

But the squares of the axes are one to the other as KH to LH , and thence it is easy to describe a conic given in kind through four given points. For if two of the given points are made the poles C, B , the third will give the



movable angles PCK, PBK; but those being given, the circle BGKC may be described. Then, because the conic is given in kind, the ratio of OH to OK, and therefore OH itself, will be given. About the centre O, with the interval OH, describe another circle, and the right line that touches this circle, and passes through the intersection of the legs CK, BK, when the first legs CP, BP

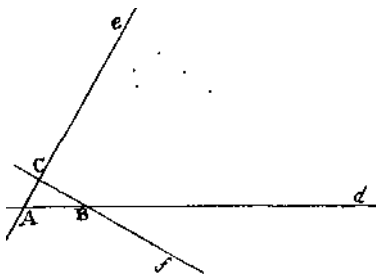
meet in the fourth given point, will be the ruler MN, by means of which the conic may be described. Whence also on the other hand a trapezium given in kind (excepting a few cases that are impossible) may be inscribed in a given conic section.

There are also other Lemmas, by the help of which conies given in kind may be described through given points, and touching given lines. Of such a sort is this, that if a right line is drawn through any point given in position, that may cut a given conic section in two points, and the distance of the intersections is bisected, the point of bisection will touch another conic section of the same kind with the former, and having its axes parallel to the axes of the former. But I hasten to things of greater use.

LEMMA XXVI

To place the three angles of a triangle, given both in kind and in magnitude, in respect to as many right lines given in position, provided they are not all parallel among themselves, in such manner that the several angles may touch the several lines.

Three indefinite right lines AB, AC, BC arc given in position, and it is required so to place the triangle DEF that its angle D may touch the line AB, its angle E the line AC, and its angle F the line BC. Upon DE, DF, and EF describe three segments of circles DRE, DGF, EMF, capable of angles equal to the angles BAC, ABC, ACB respectively. But those segments are to be described towards such sides of the lines DE, DF, EF, that the letters DRED may turn round about in the same order with the letters BACB; the letters DGFD in the same order with the letters ABCA; and the letters EMFE in the same order with the letters ACBA; then, completing those segments into entire circles, let the two former circles cut each other in G, and suppose P and Q to be their centres. Then joining GP, PQ, take



Ga: AB-GP:PQ;

and about the centre G, with the interval Ga, describe a circle that may cut the first circle DGE in a. Join aD cutting the second circle DFG in b, as well as aE cutting the third circle EMF in c. Complete the figure ABCdef similar and equal to the figure abcDEF: I say, the thing is done.

For drawing $\angle c$ meeting aD in n, and joining aG, bG, QG, QD, PD, by construction the angle EaD is equal to the angle CAB, and the angle acF equal to the angle ACB; and therefore the triangle anc equiangular to the triangle ABC. Wherefore the angle anc or FnD is equal to the angle ABC, and consequently to the angle FbD; and therefore the point n falls on the point b. Moreover the angle GPQ, which is half the angle GPD at the cen-

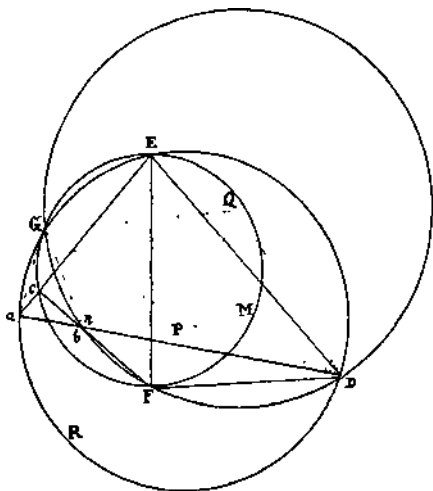
tre, is equal to the angle GaD at the circumference; and the angle GQP , which is half the angle GQD at the centre, is equal to the supplement of the angle GbD at the circumference, and therefore equal to the angle Gba . Upon which account the triangles GPQ , Gab are similar, and

$$Ga : ab :: GP : PQ$$

and, by construction,

$$GP : PQ = Ga : AB.$$

Wherefore ab and AB are equal; and consequently the triangles abc , ABC , which we have now proved to be similar, are also equal. And therefore since



the angles D , E , F of the triangle DEF do respectively touch the sides ab , ac , bc of the triangle abc , the figure $ABCdef$ may be completed similar and equal to the figure $abcDEF$, and by completing it the Problem will be solved. Q.E.F.

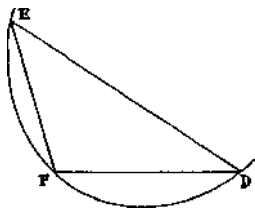
COR. Hence a right line may be drawn whose parts given in length may be intercepted between three right lines given in position. Suppose the triangle DEF , by the approach of its point D to the side EF , and by having

the sides DE, DF placed into the same straight line, to be itself changed into a right line whose given part DE is to be placed between the right lines AB, AC given in position; and its given part DF is to be placed between the right lines AB, BC given in position; then, by applying the preceding construction to this case, the Problem will be solved.

PROPOSITION XXVIII. PROBLEM XX

To describe a conic given both in kind and in magnitude, given parts of which shall be placed between three right lines given in position.

Suppose a conic is to be described that may be similar and equal to the curved line DEF, and may be cut by three right lines AB, AC, BC, given in position, into parts DE and EF, similar and equal to the given parts of this curved line.



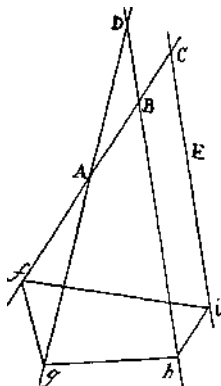
Draw the right lines DE, EF, DF; and place the angles D, E, F, of this triangle DEF, so as to touch those right lines given in position (by Lem. xxvi). Then about the triangle describe the conic, similar and equal to the curve DEF. Q.E.F.

LEMMA XXVII

To describe a trapezium given in kind, the angles whereof may respectively touch four right lines given in position, that are neither all parallel among themselves, nor converge to one common point.

Let the four right lines ABC, AD, BD, CE be given in position; the first cutting the second in A, the third in B, and the fourth in C; and suppose a trapezium *fghi* is to be described that may be similar to the trapezium FGHI, and whose angle *f*, equal to the given angle F, may touch the right

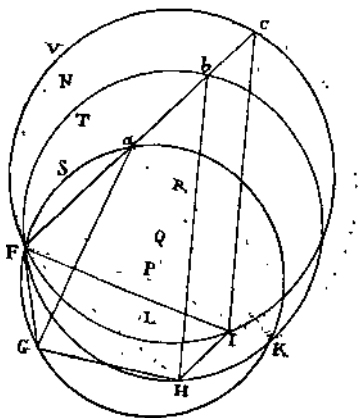
line ABC; and the other angles g, h, i , equal to the other given angles G, H, I , may touch the other lines AD, BD, CE respectively. Join FH, and



upon FG, FH, FI describe as many segments of circles FSG, FTH, FVI, the first of which FSG may be capable of an angle equal to the angle BAD; the second FTH capable of an angle equal to the angle CBD; and the third FVI of an angle equal to the angle ACE. But the segments are to be described towards those sides of the lines FG, FH, FI, that the circular order of the letters FSGF may be the same as of the letters BADB, and that the letters FTHF may turn about in the same order as the letters CBDC, and the letters FVIF in the same order as the letters ACEA. Complete the segments into entire circles, and let P be the centre of the first circle FSG, Q the centre of the second FTH. Join and

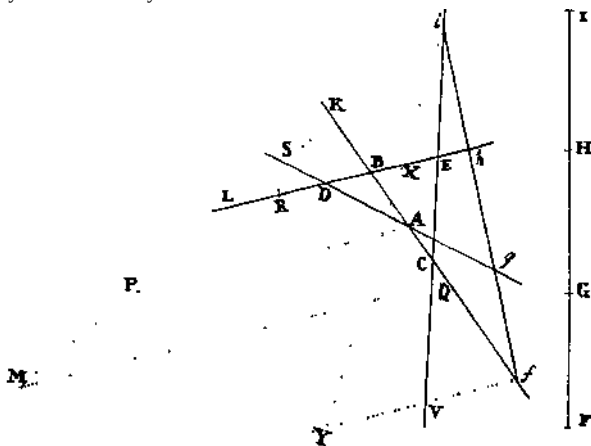
produce both ways the line PQ, and in it take QR so that $QR : PQ = BC : AB$. But QR is to be taken towards that side of the point Q, that the order of the letters P, Q, R may be the same as of the letters A, B, C; and about the centre R with the radius RF describe a fourth circle FNr cutting the third circle FVI in c. Join Fc cutting the first circle in a, and the second in b. Draw aG, bH, cI, and let the figure ABCfghi be made similar to the figure abcFGHI; and the trapezium fghi will be that which was required to be described.

For let the two first circles FSG, FTH cut one the other in K; join PK, QK, RK, aK,



bK , cK , and produce QP to L . The angles FaK , FbK , FcK at the circumferences are the halves of the angles FPK , FQK , FRK at the centres, and therefore equal to LPK , LQK , LRK , the halves of those angles. Therefore the figure $PQRK$ is equiangular and similar to the figure $abcK$, and consequently ab is to bc as PQ to QR , that is, as AB to BC . But by construction the angles $fk g$, fbh , $fc i$ are equal to the angles FaG , FbH , Fcl . And therefore the figure $ABCfghi$ may be completed similar to the figure $abcFGHI$. This done, a trapezium $fghi$ will be constructed similar to the trapezium $FGHI$, and by its angles f , g, h, i will touch the right lines ABC , AD , BD , CE . Q.E.F.

COR. Hence a right line may be drawn whose parts intercepted in a given order, between four right lines given by position, shall have a given proportion among themselves. Let the angles FGH , GHI be so far increased that the right lines FG , GH , HI may lie in the same line; and by constructing the Problem in this case, a right line $fghi$ will be drawn, whose parts fg , gh , hi , intercepted between the four right lines given in position, AB and AD , AD and BD , BD and CE , will be to each other as the lines FG , GH , HI , and will observe the same order among themselves. But the same thing may be more readily done in this manner.



Produce AB to K and BD to L, so as BK may be to AB as HI to GH; and DL to BD as GI to FG; and join KL meeting the right line CE in *i*. Produce *iL* to M, so as LM may be to *iL* as GH to HI; then draw MQ parallel to LB, and meeting the right line AD in *g*, and join *gi* cutting AB, BD in *f*, *h* 'I say, the thing is done.

For let *Mg* cut the right line AB in Q, and AD the right line KL in S, and draw AP parallel to BD and meeting *iL* in P, and *gM* to *Lh* (*gi* to *hi*, *Mi* to *Li*, GI to HI, AK to BK) and AP to BL will be in the same ratio. Cut DL in R, so as DL to RL may be in that same ratio; and because *gS* to *gM*, AS to AP, and DS to DL are proportional; therefore, as *gS* to *Lh*, so will AS be to BL, and DS to RL; and mixtly, BL - RL to *Lh* - BL, as AS - DS to *gS* - AS. That is, BR is to *Bh* as AD is to *Ag*, and therefore as BD to *gQ*. And alternately BR is to BD as *Bh* to *gQ*, or as *fh* to *fg*. But by construction the line BL was cut in D and R in the same ratio as the line FI in G and H; and therefore BR is to BD as FH to FG. Therefore *fh* is to *fg* as FH to FG. Since, therefore, *gt* to *hi* likewise is as *Mi* to *hi*, that is, as GI to HI, it is manifest that the lines FI, *fi* are similarly cut in G and H, *g* and *h*. Q.E.F.

In the construction of this Corollary, after the line LK is drawn cutting CE in *i*, we may produce *iE* to V, so as EV may be to *Ei* as FH to HI, and then draw Vf parallel to BD. It will come to the same, if about the centre *i* with an interval IH, we describe a circle cutting BD in X, and produce *iX* to Y so as *iY* may be equal to IF, and then draw Yi parallel to BD.

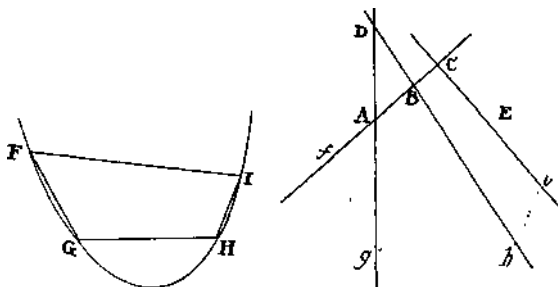
Sir Christopher Wren and Dr. Wallis have long ago given other solutions of this Problem.

PROPOSITION XXIX. PROBLEM XXI

To describe a conic given in kind, that may be cut by four right lines given in position, into parts given in order, kind, and proportion.

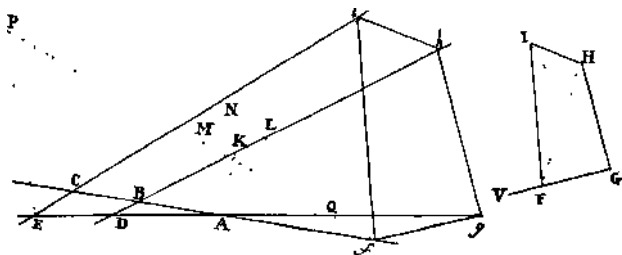
Suppose a conic is to be described that may be similar to the curved line FGHI, and whose parts, similar and proportional to the parts FG, GH, HI of the other, may be intercepted between the right lines AB and AD, AD and BD, BD and CE given in position, viz., the first between the first pair of those lines, the second between the second, and the third between the third. Draw the right lines FG, GH, HI, FI; and (by Lem. xxvii) describe

a trapezium $fghi$ that may be similar to the trapezium $FGHI$, and whose angles f, g, h, i may touch the right lines given in position AB, AD, BD, CE , severally according to their order. And then about this trapezium describe a conic, that conic will be similar to the curved line $FGHI$.



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This problem may be likewise constructed in the following manner. Joining FG, GH, HI, FI , produce GF to V , and join FH, IG , and make the angles CAK, DAL equal to the angles FGH, VFH . Let AK, AL meet the right line BD in K and L , and thence draw KM, LN , of which let KM make the angle AKM equal to the angle GHI , and be itself to AK as HI is to GH ; and let LN make the angle ALN equal to the angle FHI , and be



itself to AL as HI to FH . But AK, KM, AL, LN are to be drawn towards those sides of the lines AD, AK, AL , that the letters $CAKMC, ALKA, DALND$ may be carried round in the same order as the letters $FGHIF$;

and draw MN meeting the right line CE in *i*. Make the angle /EP equal to the angle IGF, and let PE be to Ei as FG to GI; and through P draw *PQf* that may with the right line ADE contain an angle PQE equal to the angle FIG, and may meet the right line AB in *f*, and join *fi*. But PE and PQ are to be drawn towards those sides of the lines CE, PE that the circular order of the letters PEiP and PEQP may be the same as of the letters FGHIF; and if upon the line *fi*, in the same order of letters, and similar to the trapezium FGHI, a trapezium *fghi* is constructed, and a conic given in kind is circumscribed about it, the Problem will be solved.

So far concerning the finding of the orbits. It remains that we determine the motions of bodies in the orbits so found.

SECTION VI

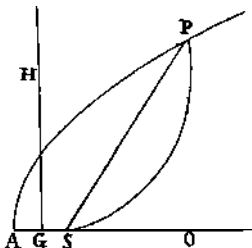
How the motions are to be found in given orbits.

PROPOSITION XXX. PROBLEM XXII

To find at any assigned time the place of a body moving in a given parabola.

Let S be the focus, and A the principal vertex of the parabola; and suppose $4AS \cdot M$ equal to the parabolic area to be cut off APS, which either was described by the radius SP, since the body's departure from the vertex, or is to be described thereby before its

arrival there. Now the quantity of that area to be cut off is known from the time which is proportional to it. Bisect AS in G, and erect the perpendicular GH equal to $3M$, and a circle described about the centre H, with the radius HS, will cut the parabola in the place P required. For letting fall PO perpendicular on the axis, and drawing PH, there will be $AG^2 + GH^2 (= HP^2 =$



$(AO - AG)^2 + (PO - GH)^2 = AO^2 + PO^2$

$- 2AO \cdot AG - 2GH \cdot PO + AG^2 + GH^2$. Whence $2GH \cdot PO (= AO^2 + PO^2 -$

$2AO \cdot AG) = AO^2 + \frac{3}{4} PO^2$. For AO^2 write $AO \cdot \frac{PO^2}{4AS}$; then dividing all

the terms by $3PO$, and multiplying them by $2AS$, we shall have $\frac{1}{3} GH \cdot AS$

$(= \frac{1}{6} AO \cdot PO + \frac{1}{2} AS \cdot PO = \frac{AO + 3AS}{6} \cdot PO = \frac{4AO - 3SO}{6}$ PO = to the

area, $APQ - SPQ$) = to the area APS. But GH was $3M$, and therefore $\frac{1}{3} GH \cdot AS$ is $4AS \cdot M$. Therefore the area cut off APS is equal to the area that was to be cut off $4AS \cdot M$. Q.E.D.

COR. 1. Hence GH is to AS as the time in which the body described the arc AP to the time in which the body described the arc between the vertex A and the perpendicular erected from the focus S upon the axis.

COR. II. And supposing a circle ASP continually to pass through the moving body P, the velocity of the point H is to the velocity which the body had in the vertex A as 3 to 8; and therefore in the same ratio is the line GH to the right line which the body, in the time of its moving from A to P, would describe with that velocity which it had in the vertex A.

COR. in. Hence, also, on the other hand, the time may be found in which the body has described any assigned arc AP. Join AP, and on its middle point erect a perpendicular meeting the right line GH in H.

LEMMA XXVIII¹

There is no oval figure whose area, cut off by right lines at pleasure, can be universally found by means of equations of any number of finite terms and dimensions.

Suppose that within the oval any point is given, about which as a pole a right line is continually revolving with an uniform motion, while in that right line a movable point going out from the pole moves always forwards with a velocity proportional to the square of that right line within the oval. By this motion that point will describe a spiral with infinite circumgyrations. Now if a portion of the area of the oval cut off by that right line could be found by a finite equation, the distance of the point from the pole, which is proportional to this area, might be found by the same equation, and therefore all the points of the spiral might be found by a finite equation also; and therefore the intersection of a right line given in position with the spiral might also be found by a finite equation. But every right line infinitely produced cuts a spiral in an infinite number of points; and the equation by which any one intersection of two lines is found at the same time exhibits all their intersections by as many roots, and therefore rises to as many dimensions as there are intersections. Because two circles cut one another in two points, one of those intersections is not to be found but by an equation of two dimensions, by which the other intersection may be also found. Because there may be four intersections of two conic sections, any one of them is not to be found universally, but by an equation of four dimensions, by which they may be all found together. For if those intersections are severally sought, because the law and condition of all is the same, the calculus will be the same in every case, and therefore the condu-

¹ Appendix, Note 21.]

sion always the same, which must therefore comprehend all those intersections at once within itself, and exhibit them all indifferently. Hence it is that the intersections of the conic sections with the curves of the third order, because they may amount to six, come out together by equations of six dimensions; and the intersections of two curves of the third order, because they may amount to nine, come out together by equations of nine dimensions. If this did not necessarily happen, we might reduce all solid to plane Problems, and those higher than solid to solid Problems. But here I speak of curves irreducible in power. For if the equation by which the curve is defined may be reduced to a lower power, the curve will not be one single curve, but composed of two, or more, whose intersections may be severally found by different calculi. After the same manner the two intersections of right lines with the conic sections come out always by equations of two dimensions; the three intersections of right lines with the irreducible curves of the third order, by equations of three dimensions; the four intersections of right lines with the irreducible curves of the fourth order, by equations of four dimensions; and so on *in infinitum*. Wherefore the innumerable intersections of a right line with a spiral, since this is but one simple curve, and not reducible to more curves, require equations infinite in number of dimensions and roots, by which they may be all exhibited together. For the law and calculus of all is the same. For if a perpendicular is let fall from the pole upon that intersecting right line, and that perpendicular together with the intersecting line revolves about the pole, the intersections of the spiral will mutually pass the one into the other, and that which was first or nearest, after one revolution, will be the second, after two, the third; and so on: nor will the equation in the meantime be changed but as the magnitudes of those quantities are changed, by which the position of the intersecting line is determined. Therefore since those quantities after every revolution return to their first magnitudes, the equation will return to its first form; and consequently one and the same equation will exhibit all the intersections, and will therefore have an infinite number of roots, by which they may be all exhibited. Therefore the intersection of a right line with a spiral cannot be universally found by any finite equation; and hence there is no oval figure whose area, cut off by right lines at pleasure, can be universally exhibited by any such equation.

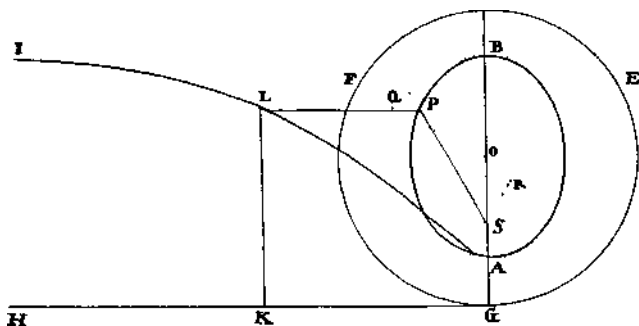
By the same argument, if the interval of the pole and point by which the spiral is described is taken proportional to that part of the perimeter of the oval which is cut off, it may be proved that the length of the perimeter cannot be universally exhibited by any finite equation. But here I speak of ovals that are not touched by conjugate figures running out *in infinitum*.

COR. Hence the area of an ellipse, described by a radius drawn from the focus to the moving body, is not to be found from the time given by a finite equation; and therefore cannot be determined by the description of curves geometrically rational. Those curves I call geometrically rational, all the points whereof may be determined by lengths that are definable by equations; that is, by the complicated ratios of lengths. Other curves (such as spirals, quadratrixes, and cycloids) I call geometrically irrational. For the lengths which are or are not as number to number (according to Book x, *Elem. of Euclid*) are arithmetically rational or irrational. And therefore I cut off an area of an ellipse proportional to the time in which it is described by a curve geometrically irrational, in the following manner:

PROPOSITION XXXI. PROBLEM XXIII

To find the place of a body moving in a given ellipse at any assigned time.

Suppose A to be the principal vertex, S the focus, and O the centre of the ellipse APB; and let P be the place of the body to be found. Produce OA to G so that $OG : OA :: OA : OS$. Erect the perpendicular GH; and about the centre O, with the radius OG, describe the circle GEF; and on the ruler

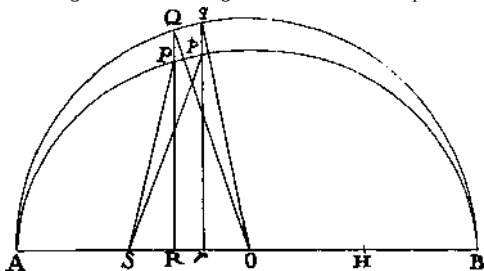


GH, as a base, suppose the wheel GEF to move forwards, revolving about its axis, and in the meantime by its point A describing the cycloid ALL. This done, take GK to the perimeter GEFG of the wheel, in the ratio of the time in which the body proceeding from A described the arc AP, to the time of a whole revolution in the ellipse. Erect the perpendicular KL meeting the cycloid in L; then LP drawn parallel to KG will meet the ellipse in P, the required place of the body.

For about the centre O with the radius OA describe the semicircle AQB, and let LP, produced, if need be, meet the arc AQ in Q, and join SQ, OQ. Let OQ meet the arc EFG in F, and upon OQ let fall the perpendicular SR. The area APS varies as the area AQS, that is, as the difference between the sector OQA and the triangle OQS, or as the difference of the rectangles $\frac{1}{2}$ OQ AQ, and $\frac{1}{2}$ OQ SR, that is, because $\frac{1}{2}$ OQ is given, as the difference between the arc AQ and the right line SR; and therefore (because of the equality of the given ratios SR to the sine of the arc AQ, OS to OA, OA to OG, AQ to GF; and by division, AQ-SR to GF-sine of the arc AQ) as GK, the difference between the arc GF and the sine of the arc AQ. O.E.D.

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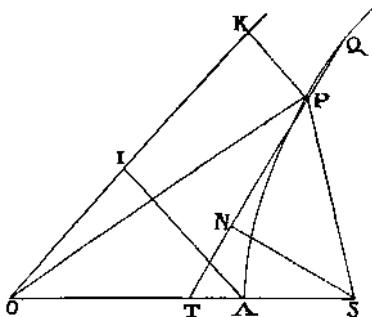
But since the description of this curve is difficult, a solution by approximation will be preferable.¹ First, then, let there be found a certain angle B which may be to an angle of 57.29578 degrees, which an arc equal to the radius



subtends, as SH, the distance of the foci, to AB, the diameter of the ellipse. Secondly, a certain length L, which may be to the radius in the same ratio inversely. And these being found, the Problem may be solved by the follow-

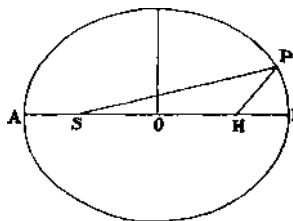
ing analysis. By any construction (or even by conjecture), suppose we know P the place of the body near its true place p . Then letting fall on the axis of the ellipse the ordinate PR from the proportion of the diameters of the ellipse, the ordinate RQ of the circumscribed circle AQB will be given; which ordinate is the sine of the angle AOQ , supposing AO to be the radius, and also cuts the ellipse in P . It will be sufficient if that angle is found by a rude calculus in numbers near the truth. Suppose we also know the angle proportional to the time, that is, which is to four right angles as the time in which the body described the arc hp to the time of one revolution in the ellipse. Let this angle be N . Then take an angle D , which may be to the angle B as the sine of the angle AOQ to the radius; and an angle E which may be to the angle $N - AOQ$ as the length L to the same length L diminished by the cosine of the angle AOQ , when that angle is less than a right angle, or increased thereby when greater. In the next place, take an angle F that may be to the angle B as the sine of the angle $AOQ + E$ to the radius, and an angle G , that may be to the angle $N - AOQ - E + F$ as the length L to the same length L diminished by the cosine of the angle $AOQ + E$, when that angle is less than a right angle, or increased thereby when greater. For the third time take an angle H , that may be to the angle B as the sine of the angle $AOQ + E + G$ to the radius; and an angle I to the angle $N - AOQ - E - G + H$, as the length L is to the same length L diminished by the cosine of the angle $AOQ + E + G$, when that angle is less than a right angle, or increased thereby when greater. And so we may proceed *in infinitum*. Lastly, take the angle AOQ equal to the angle $AOQ + E + G + I + \&c$, and from its cosine Or and the ordinate pr , which is to its sine qr as the lesser axis of the ellipse to the greater, we shall have p the correct place of the body. When the angle $N - AOQ + D$ happens to be negative, the sign $!$ of the angle E must be everywhere changed into $-$, and the sign $-$ into $!$. And the same thing is to be understood of the signs of the angles G and I , when the angles $N - AOQ - E + F$, and $N - AOQ - E - G + H$ come out negative. But the infinite series $AOQ + E + G + I + \&c$, converges so very fast, that it will be scarcely ever needful to proceed beyond the second term E . And the calculus is founded upon this Theorem, that the area APS varies as the difference between the arc AQ and the right line let fall from the focus S perpendicularly upon the radius OQ .

And by a calculus not unlike, the Problem is solved in the hyperbola. Let its centre be O , its vertex A , its focus S , and asymptote OK ; and suppose the amount of the area to be cut off is known, as being proportional to the time. Let that be A , and by conjecture suppose we know the position of a right line SP , that cuts off an area APS near the truth. Join OP , and from A and P to the asymptote draw AI , PK , parallel to the other asymptote; and by the table of logarithms the area $AIKP$ will be given, and equal thereto the area OPA , which, subtracted from the triangle OPS , will leave the area cut off APS . And by applying $2APS - 2A$, or $2A - 2APS$, the double difference of the area A that was to be cut off, and the area APS that is cut off, to the line SN that is let fall from the focus S , perpendicular upon the tangent TP , we shall have the length of the chord PQ . Winch chord PQ is to be inscribed between A and P , if the area APS that is cut off be greater than the area A that was to be cut off, but towards the contrary side of the point P , if otherwise: and the point Q will be the place of the body more



accurately. And by repeating the computation the place may be found continually to greater and greater accuracy.

And by such computations we have a general analytical resolution of the Problem. But the particular calculus that follows is better fitted for astronomical purposes. Supposing AO , OB , OD to be the semiaxes of the



ellipse, and L its latus rectum, and D the difference between the lesser semiaxis OD , and $\frac{1}{2}L$ the half of the latus rectum: let an angle Y be found,

whose sine may be to the radius as the rectangle under that difference D , and $AO + OD$ the half sum of the axes, to the square of the greater axis AB . Find also an angle Z , whose sine may be to the radius as the double rectangle under the distance of the foci SH and that difference D , to triple the square of half the greater semiaxis AO . Those angles being once found, the place of the body may be thus determined. Take the angle T proportional to the time in which the arc BP was described, or equal to what is called the mean motion; and take an angle V , the first equation of the mean motion, to the angle Y , the greatest first equation, as the sine of double the angle T is to the radius; and take an angle X , the second equation, to the angle Z , the second greatest equation, as the cube of the sine of the angle T is to the cube of the radius. Then take the angle BHP , the mean equated motion either equal to $T + X + V$, the sum of the angles T , V , X , if the angle T is less than a right angle, or equal to $T + X - V$, the difference of the same, if that angle T is greater than one and less than two right angles; and if HP meets the ellipse in P , draw SP , and it will cut off the area BSP , nearly proportional to the time.

This practice seems to be expeditious enough, because the angles V and X , taken in fractions of seconds, if you please, being very small, it will be sufficient to find two or three of their first figures. But it is likewise sufficiently accurate to answer to the theory of the planets' motions. For even in the orbit of Mars, where the greatest equation of the centre amounts to ten degrees, the error will scarcely exceed one second. But when the angle of the mean motion equated BHP is found, the angle of the true motion BSP , and the distance SP , are readily had by the known methods.

And so far concerning the motion of bodies in curved lines. But it may also come to pass that a moving body shall ascend or descend in a right line; and I shall now go on to explain what belongs to such kind of motions.

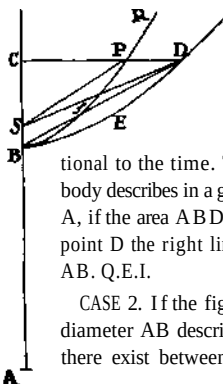
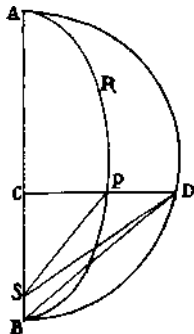
SECTION VII

The rectilinear ascent and descent of bodies.

PROPOSITION XXXII. PROBLEM XXIV

Supposing that the centripetal force is inversely proportional to the square of the distance of the places from the centre; it is required to define the spaces which a body, falling directly, describes in given times.

CASE I. If the body does not fall perpendicularly, it will (by Cor. 1, Prop. xm) describe some conic section whose focus is placed in the centre of force. Suppose that conic section to be $ARPB$ and its focus S . And, first, if the figure be an ellipse, upon the greater axis thereof AB describe the semicircle ADB , and let the right line DPC pass through the falling body, making right angles with the axis; and drawing DS , PS , the area ASD will be proportional to the area ASP , and therefore also to the time. The axis AB still remaining the same, let the breadth of the ellipse be continually diminished, and the area ASD will always remain pro-

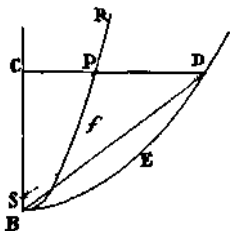


portional to the time. Suppose that breadth to be diminished *in infinitum*; and the orbit APB in that case coinciding with the axis AB , and the focus S with the extreme point of the axis B , the body will descend in the right line AC , and the area ABD will become propor-

tional to the time. Therefore the space AC will be given which the body describes in a given time by its perpendicular fall from the place A , if the area ABD is taken proportional to the time, and from the point D the right line DC is let fall perpendicularly on the right line AB . Q.E.I.

CASE 2. If the figure RPB is an hyperbola, on the same principal diameter AB describe the rectangular hyperbola BED ; and because there exist between the several areas and the heights CP and CD

relations, $CSP : CSD - CBfP : CBED = SPfB : SDEB = CP : CD$, and since the area $SPfB$ varies as the time in which the body P will move through the arc PfB , the area $SDEB$ will also vary as that time. Let the latus rectum of the hyperbola RPB be diminished *in infinitum*, the transverse axis remaining the same; and the arc PB will come to coincide with the right line CB , and the focus S with the vertex B , and the right line SD with the right line



BD . And therefore the area $BDEB$ will vary as the time in which the body C , by its perpendicular descent, describes the line CB . Q.E.I.

CASE 3. And by the like argument, if the figure RPB is a parabola, and to the same principal vertex B another parabola BED is described, that may always remain given while the former parabola in whose perimeter the body P moves, by having its latus rectum diminished and reduced to nothing,

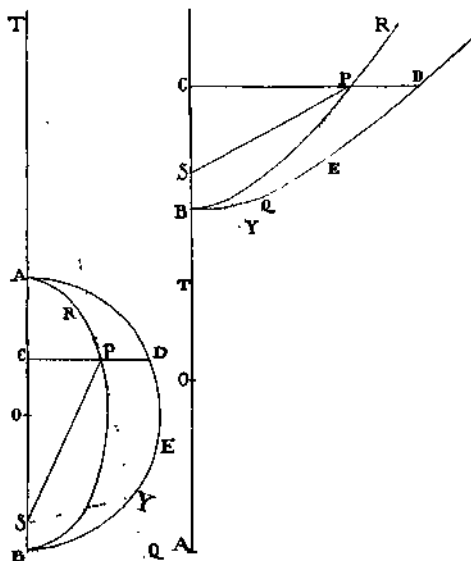
comes to coincide with the line CB , the parabolic segment $BDEB$ will vary as the time in which that body P or C will descend to the centre S or B . Q.E.I.

PROPOSITION XXXIII. THEOREM IX

The things above found being supposed, I say, that the velocity of a falling body in any place C is to the velocity of a body, describing a circle about the centre B at the distance BC , as the square root of the ratio of AC , the distance of the body from the remoter vertex A of the circle or rectangular hypetbola, to $MJAB$, the principal semidiameter of the figure.

Let AB , the common diameter of both figures RPB , DEB , be bisected in O ; and draw the right line PT that may touch the figure RPB in P , and likewise cut that common diameter AB (produced, if need be) in T ; and let SY be perpendicular to this line, and BQ perpendicular to this diameter, and suppose the latus rectum of the figure RPB to be L . From Cor. ix, Prop. xvi, it is manifest that the velocity of a body, moving in the line RPB about the centre S , in any place P , is to the velocity of a body describing a circle about the same centre, at the distance SP , as the square root of the

ratio of the rectangle $\frac{1}{2}L \cdot SP$ to SY^2 . For by the properties of the conic sections $AC \cdot CB$ is to CP^2 as $2AO$ to L , and therefore $\frac{2CP^2 \cdot AO}{AC \cdot CB}$ is equal to L . Therefore those velocities are to each other as the square root of the ratio of $\frac{CP^2 \cdot AO \cdot SP}{AC \cdot CB}$ to SY^2 . Moreover, by the properties of the conic sec-



tions,
thence,
and
From this,
and
and, since

$$\begin{aligned} CO : BO &= BO : TO, \\ CO + BO : BO &= BO + TO : TO, \\ CO : BO &= CB : BT. \\ BO - CO : BO &= BT - CB : BT \\ AC : AO &= TC : BT = CP : BQ; \\ CP &= \frac{BQ \cdot AC}{AO}, \end{aligned}$$

one obtains $\frac{CP^2 \cdot AO \cdot SP}{AC \cdot CB}$ equal to $\frac{BQ^2 \cdot AC \cdot SP}{AO \cdot BC}$.

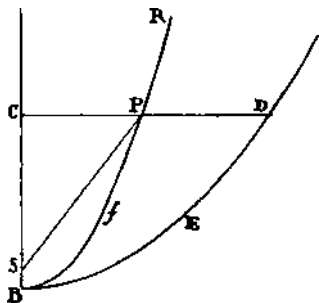
Now suppose CP, the breadth of the figure RPB, to be diminished *in infinitum*, so that the point P may come to coincide with the point C, and the point S with the point B, and the line SP with the line BC, and the line SY with the line BQ; and the velocity of the body now descending perpendicularly in the line CB will be to the velocity of a body describing a circle about the centre B, at the distance BC, as the square root of the ratio of $\frac{BQ^2 \cdot AC \cdot SP}{AO \cdot BC}$ to SY, that is (neglecting the ratios of equality of SP to BC, and BQ^2 to SY^2), as the square root of the ratio of AC to AO, or VsAB. Q.E.D.

COR. I. When the points B and S come to coincide, TC will become to TS as AC to AO.

COR. II. A body revolving in any circle at a given distance from the centre, by its motion converted upwards, will ascend to double its distance from the centre.

PROPOSITION XXXIV. THEOREM X

If the figure BED is a parabola, I say, that the velocity of a falling body in any place C is equal to the velocity by which a body may uniformly describe a circle about the centre B at half the interval BC.



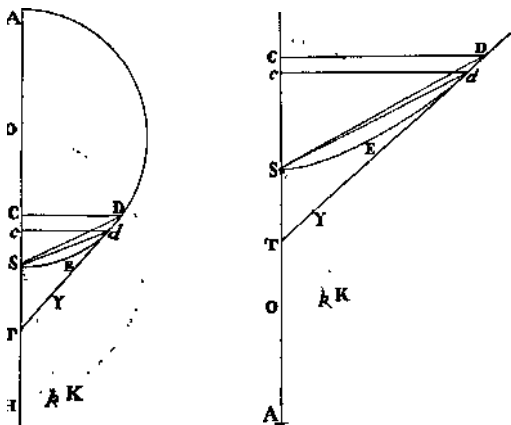
For (by Cor. vn, Prop, xvi) the velocity of a body describing a parabola RPB about the centre S, in any place P, is equal to the velocity of a body uniformly describing a circle about the same centre S at half the interval SP. Let the breadth CP of the parabola be diminished *in infinitum*, so that the parabolic arc Pfb may come to coincide with the right line CB, the centre S with the vertex B, and the

interval SP with the interval BC, and the Proposition will be manifest. Q.E.D.

PROPOSITION XXXV. THEOREM XI

The same things supposed, I say, that the area of the figure DES, described by the indefinite radius SD, is equal to the area which a body with a radius equal to half the latus rectum of the figure DES describes in the same time, by uniformly revolving about the centre S.

For suppose a body C in the smallest moment of time describes in falling the infinitely little line Cc, while another body K, uniformly revolving about the centre S in the circle OKk, describes the arc Kk. Erect the perpendiculars CD, cd, meeting the figure DES in D, d. Join SD, Sd, SK, Sk, and draw Dd meeting the axis AS in T, and thereon let fall the perpendicular SY.



CASE I. If the figure DES is a circle, or a rectangular hyperbola, bisect its transverse diameter AS in O, and SO will be half the latus rectum. And

because $TC : TD = Cc : Dd$

and $TD : TS = CD : Sy$,

there follows $TC : TS = CD : Cc : SY : Dd$.

But (by Cor. I, Prop. xxxiii)

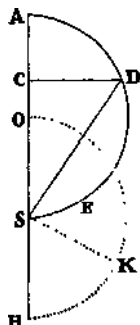
$TC : TS = AC : AO$,

of SK to $\frac{1}{2} CD$. Therefore $\frac{1}{2} SK \cdot Kk$ is equal to $\frac{1}{4} CD \cdot Cc$, and therefore equal to $\frac{1}{2} SY \cdot Dd$; that is, the area $KS k$ is equal to the area SDd , as above. Q.E.D.

PROPOSITION XXXVI. PROBLEM XXV

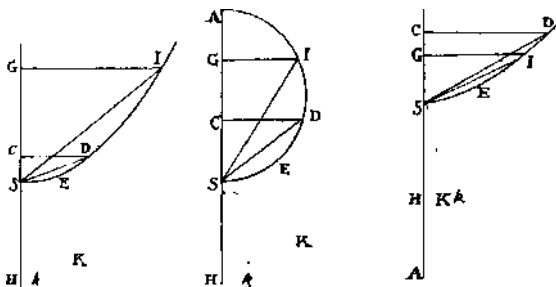
To determine the times of the descent of a body falling from a given place A.

Upon the diameter AS , the distance of the body from the centre at the beginning, describe the semicircle ADS , as likewise the semicircle OKH equal thereto, about the centre S . From any place C of the body erect the ordinate CD . Join SD , and make the sector OSK equal to the area ASD . It is evident (by Prop. xxxv) that the body in falling will describe the space AC in the same time in which another body, uniformly revolving about the centre S , may describe the arc OK . Q.E.F.



PROPOSITION XXXVII. PROBLEM XXVI

To define the times of the ascent or descent of a body projected upwards or downwards from a given place.

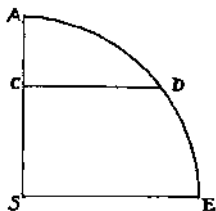


Suppose the body to go off from the given place G , in the direction of the line GS , with any velocity. Take GA to $Vz AS$ as the square of the ratio of this velocity to the uniform velocity in a circle, with which the body may

revolve about the centre S at the given interval SG . If that ratio is the same as of the number 2 to 1, the point A is infinitely remote; in which case a parabola is to be described with any latus rectum to the vertex S , and axis SG ; as appears by Prop, xxxiv. But if that ratio is less or greater than the ratio of 2 to 1, in the former case a circle, in the latter a rectangular hyperbola, is to be described on the diameter SA ; as appears by Prop, XXXIII. Then about the centre S , with a radius equal to half the latus rectum, describe the circle HkK ; and at the place G of the ascending or descending body, and at any other place C , erect the perpendiculars GI , CD , meeting the conic section or circle in I and D . Then joining SI , SD , let the sectors HSK , HSk be made equal to the segments $SEIS$, $SEDS$, and (by Prop, xxxv) the body G will describe the space GC in the same time in which the body K may describe the arc Kk . Q.E.F.

PROPOSITION XXXVIII. THEOREM XII

Supposing that the centripetal force is proportional to the altitude or distance of places from the centre, I say, that the times and velocities of falling bodies, and the spaces which they describe, are respectively proportional to the arcs, and the sines and versed sines of the arcs.



Suppose the body to fall from any place A in the right line AS ; and about the centre of force S , with the radius AS , describe the quadrant of a circle AE ; and let CD be the sine of any arc AD ; and the body A will in the time AD in falling describe the space AC , and in the place C will acquire the velocity CD .

This is demonstrated the same way from Prop. x, as Prop, XXXII was demonstrated from Prop. xi.

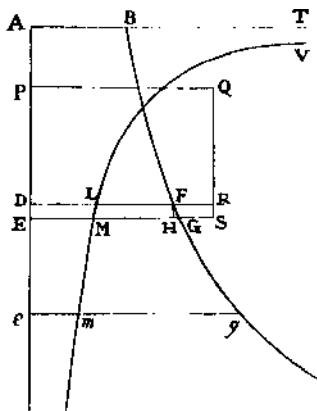
COR. 1. Hence the times are equal in which one body falling from the place A arrives at the centre S , and another body revolving describes the quadrantal arc ADE .

COR. 11. Therefore all the times are equal in which bodies falling from whatsoever places arrive at the centre. For all the periodic times of revolving bodies are equal (by Cor. 111, Prop. iv).

PROPOSITION XXXIX. PROBLEM XXVII

Supposing a centripetal force of any $\propto md$, and granting the quadratures of curvilinear figures; it is required to find the velocity of a body, ascending or descending in a right line, in the several places through which it passes, as also the time in which it will arrive at any place; and conversely.

Suppose the body E to fall from any place A in the right line ADEC; and from its place E imagine a perpendicular EG always erected proportional



to the centripetal force in that place tending to the centre C; and let BFG be a curved line, the locus of the point G. And in the beginning of the motion suppose EG to coincide with the perpendicular AB; and the velocity of the body in any place E will be as a right line whose square is equal to the curvilinear area ABGE. Q.E.I.

In EG take EM inversely proportional to a right line whose square is equal to the area ABGE, and let VLM be a curved line wherein the point M is always placed, and to which the right line AB produced is an asymptote; and the time in which the body in falling describes the line AE, will be

as the curvilinear area ABTVME. Q.E.I.

For in the right line AE let there be taken the very small line DE of a given length, and let DLF be the place of the line EMG, when the body was in D; and if the centripetal force be such, that a right line, whose square is equal to the area ABGE, is as the velocity of the descending body, the area itself will be as the square of that velocity; that is, if for the velocities in D and E we write V and V + I, the area ABFD will be as VV , and the area ABGE as $VV + 2VI + II$; and by subtraction, the area DFGE as $2VI + II$, and therefore $\frac{DFGE}{DE}$ will be as $\frac{2VI + II}{DE}$; that

is, if we take the first ratios of those quantities when just nascent, the length DF is as the quantity $\frac{2VI}{DE}$, and therefore also as half that quantity

$\frac{I \cdot V}{DE}$.

r. But the time in which the body in falling describes the very small line DE, is directly as that line and inversely as the velocity V; and the force will be directly as the increment I of the velocity and inversely as the time; and therefore if we take the first ratios when those quantities are just nascent, as $\frac{I \cdot V}{DE}$, that is, as the length DF. Therefore a force proportional to DF or EG will cause the body to descend with a velocity that is as the right line whose square is equal to the area ABGE. Q.E.D.

Moreover, since the time in which a very small line DE of a given length may be described is inversely as the velocity and therefore also inversely as a right line whose square is equal to the area ABFD; and since the line DL, and by consequence the nascent area DLME, will be inversely as the same right line, the time will be as the area DLME, and the sum of all the times will be as the sum of all the areas; that is (by Cor., Lem. iv), the whole time in which the line AE is described will be as the whole area ATVME. Q.E.D.

COR. I. Let P be the place from whence a body ought to fall, so as that, when urged by any known uniform centripetal force (such as gravity is commonly supposed to be), it may acquire in the place D a velocity equal to the velocity which another body, falling by any force whatever, hath acquired in that place D. In the perpendicular DF let there be taken DR, which may be to DF as that uniform force to the other force in the place D. Complete the rectangle PDRQ, and cut off the area ABFD equal to that rectangle. Then A will be the place from whence the other body fell. For completing the rectangle DRSE, since the area ABFD is to the area DFG E as VV to $2VI$, and therefore as $\frac{1}{2} V$ to I , that is, as half the whole velocity to the increment of the velocity of the body falling by the variable force; and in like manner the area PQRD to the area DRSE as half the whole velocity to the increment of the velocity of the body falling by the uniform force; and since those increments (by reason of the equality of the nascent times) are as the generating forces, that is, as the ordinates DF, DR, and consequently as the nascent areas DFG E, DRSE; therefore, the whole areas

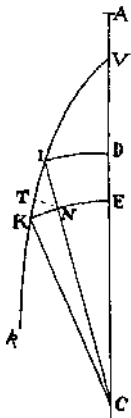
SECTION VIII

The determination of orbits in which bodies will revolve, being acted upon by any sort of centripetal force.

PROPOSITION XL. THEOREM XIII

If a body, acted upon by any centripetal force, it moved in any manner, and another body ascends or descends in a right line, and their velocities be equal in any one case of equal altitudes, their velocities will be also equal at all equal altitudes.

Let a body descend from A through D and E, to the centre C; and let another body move from V in the curved line VIKk. From the centre C, with any distances, describe the concentric circles DI, EK, meeting the right line AC in D and E, and the curve VIK in I and K. Draw IC meeting KE in N, and on IK let fall the perpendicular NT; and let the interval DE or IN between the circumferences of the circles be very small; and imagine the bodies in D and I to have equal velocities. Then because the distances CD and CI arc equal, the centripetal forces in D and I will be also equal. Let those forces be expressed by the equal short lines DE and IN; and let the force IN (by Cor. n of the Laws of Motion) be resolved into two others, NT and IT. Then the force NT acting in the direction of the line NT perpendicular to the path ITK of the body will not at all affect or change the velocity of the body in that path, but only draw it aside from a rectilinear course, and make it deflect continually from the tangent of the orbit, and proceed in the curvilinear path ITKk. That whole force, therefore, will be spent in producing this effect; but the other force IT, acting in the direction of the course of the body, will be all employed in accelerating it, and in the least given time will produce an acceleration proportional to itself. Therefore the accelerations of



the bodies in D and I, produced in equal times, are as the lines DE, IT (if we take the first ratios of the nascent lines DE, IN, IK, IT, NT); and in unequal times as the product of those lines and the times. But the times in which DE and IK are described, are, by reason of the equal velocities (in D and I), as the spaces described DE and IK, and therefore the accelerations in the course of the bodies through the lines DE and IK arc as DE and IT, and DE and IK conjointly; that is, as the square of DE to the rectangle IT IK. But the rectangle IT IK is equal to the square of IN, that is, equal to the square of DE; and therefore the accelerations generated in the passage of the bodies from D and I to E and K are equal. Therefore the velocities of the bodies in E and K arc also equal: and by the same reasoning they will always be found equal in any subsequent equal distances. Q.E.D.

By the same reasoning, bodies of equal velocities and equal distances from the centre will be equally retarded in their ascent to equal distances. Q.E.D.

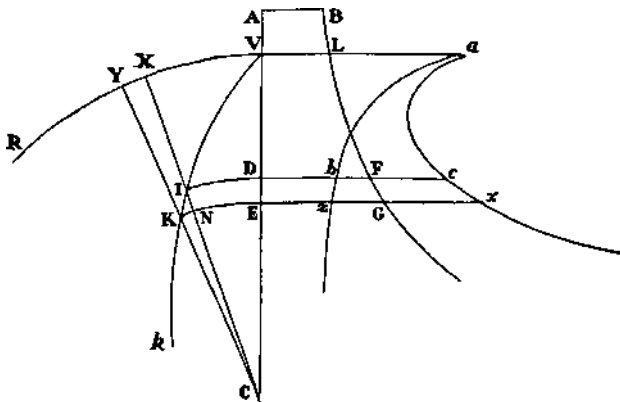
COR. 1. Therefore if a body either oscillates by hanging to a string, or by any polished and perfectly smooth impediment is forced to move in a curved line; and another body ascends or descends in a right line, and their velocities be equal at any one equal altitude, their velocities will be also equal at all other equal altitudes. For by the string of the pendulous body, or by the impediment of a vessel perfectly smooth, the same thing will be effected as by the transverse force NT. The body is neither accelerated nor retarded by it, but only is obliged to leave its rectilinear course.

COR. II. Suppose the quantity P to be the greatest distance from the centre to which a body can ascend, whether it be oscillating, or revolving in a curve, and so the same projected upwards from any point of a curve with the velocity it has in that point. Let the quantity A be the distance of the body from the centre in any other point of the orbit; and let the centripetal force be always as the power A^{n-1} , of the quantity A, the index of which power $n-1$ is any number n diminished by unity. Then the velocity in every altitude A will be as $\sqrt{P-A^n}$, and therefore will be given. For by Prop. xxxix, the velocity of a body ascending and descending in a right line is in that very ratio.

PROPOSITION XLI. PROBLEM XXVIII

Supposing a centripetal force of any kind, and granting the quadratures of curvilinear figures; it is required to find as well the curves in which bodies will move, as the times of their motions in the curves found.

Let any centripetal force tend to the centre C, and let it be required to find the curve VIKk. Let there be given the circle VR, described from the centre C with any radius CV; and from the same centre describe any other circles ID, KE, cutting the curve in I and K, and the right line CV



in D and E. Then draw the right line CNIX cutting the circles KE, VR in N and X, and the right line CKY meeting the circle VR in Y. Let the points I and K be indefinitely near; and let the body go on from V through I and K to k; and let the point A be the place from which another body is to fall, so as in the place D to acquire a velocity equal to the velocity of the first body in I. And things remaining as in Prop, xxxix, the short line IK, described in the least given time, will be as the velocity, and therefore as the right line whose square is equal to the area ABFD, and the triangle ICK proportional to the time will be given, and therefore KN will be inversely

as the altitude IC; that is (if there be given any quantity Q, and the altitude IC be called A), as $\frac{Q}{A}$. This quantity $\frac{Q}{A}$ call Z, and suppose the magnitude of Q to be such that in some one case

$$\sqrt{ABFD} : Z = IK : KN,$$

and then in all cases

$$\sqrt{ABFD} : Z = IK : KN,$$

and

$$ABFD : ZZ = IK^2 : KN^2,$$

and by subtraction,

$$ABFD - ZZ : ZZ = IN^2 : KN^2,$$

and therefore

$$\sqrt{(ABFD - ZZ)} : Z \text{ or } \frac{Q}{A} = IN : KN,$$

and

$$A \cdot KN = \frac{Q \cdot IN}{\sqrt{(ABFD - ZZ)}}.$$

Since

$$YX \cdot XC : A \cdot KN = CX^2 : AA,$$

it follows that

$$YX \cdot XC = \frac{Q \cdot IN \cdot CX^2}{AA \sqrt{(ABFD - ZZ)}}.$$

Therefore in the perpendicular DF let there be taken continually Db, Dc

equal to $\frac{Q}{\sqrt{(ABFD - ZZ)}}, \frac{Q \cdot CX^2}{AA \sqrt{(ABFD - ZZ)}}$ respectively, and let the

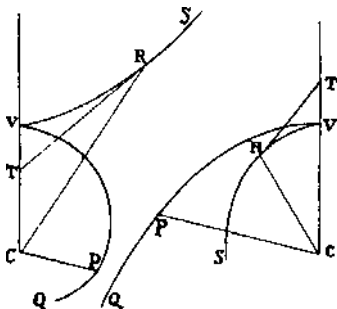
curved lines *ab*, *ac*, the foci of the points *b* and *c*, be described; and from the point V let the perpendicular Va be erected to the line AC, cutting off the curvilinear areas VDb*a*, VD*c*a, and let the ordinates Ez, Ex, be erected also. Then because the rectangle Db • IN or DbzE is equal to half the rectangle A • KN, or to the triangle ICK; and the rectangle Dc • IN or Dcx*E* is equal to half the rectangle YX • XC, or to the triangle XCY; that is, because the nascent particles Dbz*E*, ICK of the areas VDb*a*, VIC are always equal; and the nascent particles Dcx*E*, XCY of the areas VD*c*a, VCX are always equal: therefore the generated area VDb*a* will be equal to the generated area VIC, and therefore proportional to the time; and the generated area VD*c*a is equal to the generated sector VCX. If, therefore, any time be given during which the body has been moving from V, there will be also given the area proportional to it VDb*a*; and thence will be given the alti-

tude of the body CD or CI ; and the area $VDca$, and the sector VCX equal thereto, together with its angle VCI . But the angle VCI , and the altitude CI being given, there is also given the place I , in which the body will be found at the end of that time. Q.E.I.

COR. 1. Hence the greatest and least altitudes of the bodies, that is, the apsides of the curves, may be found very readily. For the apsides are those points in which a right line IC drawn through the centre falls perpendicularly upon the curves VIK ; which comes to pass when the right lines IK and NK become equal; that is, when the area $ABFD$ is equal to ZZ .

COR. 11. So also the angle KIN , in which the curve at any place cuts the line IC , may be readily found by the given altitude IC of the body; namely, by making the sine of that angle to the radius as KN to IK , that is, as Z to the square root of the area $ABFD$.

COR. 111. If to the centre C , and the principal vertex V , there be described a conic section VRS ; and from any point thereof, as R , there be drawn the tangent RT meeting the axis CV indefinitely produced in the point T ; and



then joining CR there be drawn the right line CP , equal to the abscissa CT , making an angle VCP proportional to the sector VCR ; and if a centripetal force inversely proportional to the cubes of the distances of the places from the centre, tends to the centre C ; and from the place V there sets out a body with a just velocity in the direction of a line perpendicular to the right line CV ; that body will proceed in a curve VPQ , which the point P will

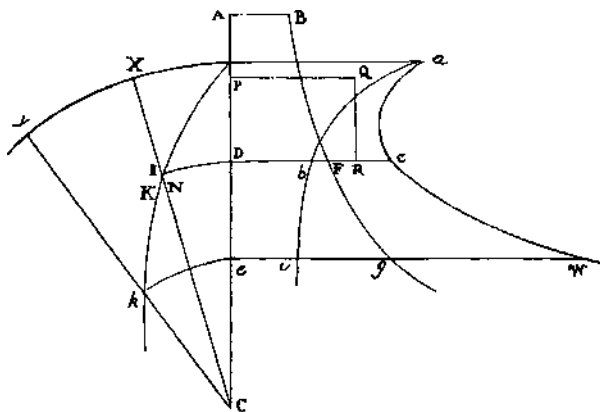
always touch; and therefore if the conic section VRS be an hyperbola, the body will descend to the centre; but if it be an ellipse, it will ascend continually, and go farther and farther off in infinitum. And, on the contrary, if a body endued with any velocity goes off from the place V , and according as it begins either to descend obliquely to the centre, or to ascend obliquely

from it, the figure VRS be either an hyperbola or an ellipse, the curve may be found by increasing or diminishing the angle VCP in a given ratio. And the centripetal force becoming centrifugal, the body will ascend obliquely in the curve VPQ, which is found by taking the angle VCP proportional to the elliptic sector VRC, and the length CP equal to the length CT, as before. All these things follow from the foregoing Proposition, by the quadrature of a certain curve, the invention of which, as being easy enough, for brevity's sake I omit.

PROPOSITION XLII. PROBLEM XXIX

The law of centripetal force being given, it is required to find the motion of a body setting out from a given place, with a given velocity, in the direction of a given right line.

Suppose the same things as in the three preceding Propositions; and let the body go off from the place I in the direction of the little line IK, with



the same velocity as another body, by falling with an uniform centripetal force from the place P, may acquire in D; and let this uniform force be to the force with which the body is at first urged in I, as DR to DF. Let the

body go on towards k ; and about the centre C , with the radius Ck , describe the circle ke , meeting the right line PD in e , and let there be erected the lines eg , ev , ew , ordinately applied to the curves BFg , abv , acw . From the given rectangle $PDRQ$ and the given law of centripetal force, by which the first body is acted on, the curved line BFg is also given, by the construction of Prop. XXVII, and its Cor. i. Then from the given angle CIK is given the proportion of the nascent lines IK , KN ; and thence, by the construction of Prob. xxviii, there is given the quantity Q , with the curved lines abv , acw and therefore, at the end of any time $Dbve$, there is given both the altitude of the body Ce or Ck , and the area $Dcwe$, with the sector equal to it XCy , the angle ICk , and the place k , in which the body will then be found. Q.E.I.

We suppose in these Propositions the centripetal force to vary in its recess from the centre according to some law, which anyone may imagine at pleasure, but at equal distances from the centre to be everywhere the same.

I have hitherto considered the motions of bodies in immovable orbits. It remains now to add something concerning their motions in orbits which revolve round the centres of force.

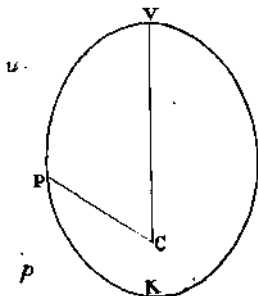
SECTION IX

The motion of bodies in movable orbits; and the motion of the apsides.

PROPOSITION XLIII. PROBLEM XXX

If is required to make a body move in a curve that revolves about the centre of force in the same manner as another body in the same curve at rest.

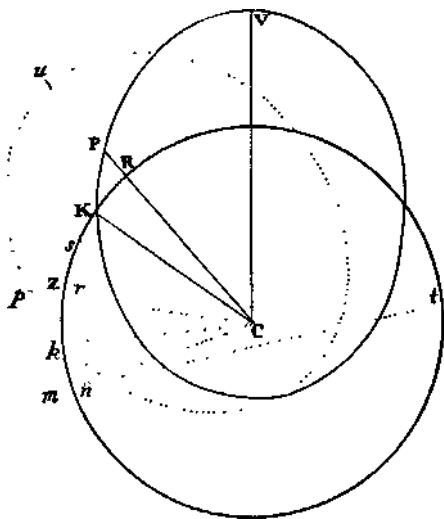
In the fixed orbit VPK , let the body P revolve, proceeding from V towards K . From the centre C let there be continually drawn Cp , equal to CP , making the angle VCp proportional to the angle VCP ; and the area which the line Cp describes will be to the area VCP , which the line CP describes at the same time, as the velocity of the describing line Cp to the velocity of the describing line CP ; that is, as the angle VCp to the angle VCP , therefore in a given ratio, and therefore proportional to the time. Since, then, the area described by the line Cp in a fixed plane is proportional to the time, it is manifest that a body, being acted upon by a suitable centripetal force, may revolve with the point p in the curved line which the same point p , by the method just now explained, may be made to describe in a fixed plane. Make the angle VCu equal to the angle VCp , and the line Cu equal to CV , and the figure uCp equal to the figure VCP , and the body being always in the point p , will move in the perimeter of the revolving figure uCp , and will describe its (revolving) arc up in the same time that the other body P describes the similar and equal arc VP in the fixed figure VPK . Find, then, by Cor. v, Prop. vi, the centripetal force by which the body may be made to revolve in the curved line which the point p describes in a fixed plane, and the Problem will be solved. Q.E.F.



PROPOSITION XLIV. THEOREM XIV

The difference of the forces, by which two bodies may be made to move equally, one in a fixed, the other in the same orbit revolving, varies inversely as the cube of their common altitudes.

Let the parts of the fixed orbit VP , PK be similar and equal to the parts of the revolving orbit up , pk ; and let the distance of the points P and K be supposed of the utmost smallness. Let fall a perpendicular $k'r$ from the point k , to the right line pC , and produce it to m , so that mr may be to kr



as the angle YCp to the angle VCP . Because the altitudes of the bodies PC and pC , KC and kC , are always equal, it is manifest that the increments or decrements of the lines PC and pC are always equal; and therefore if each of the several motions of the bodies in the places P and p be resolved into two (by Cor. 11 of the Laws of Motion), one of which is directed towards the centre, or according to the lines PC , pC , and the other, transverse

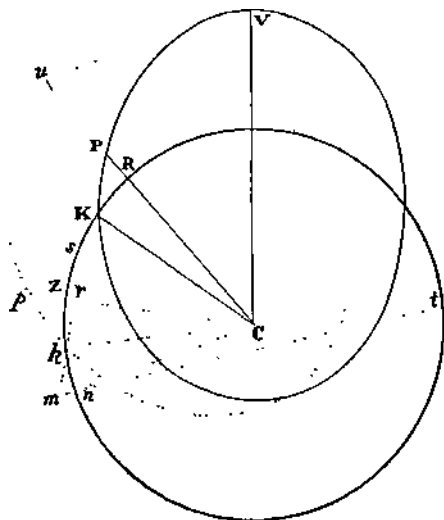
to the former, hath a direction perpendicular to the lines PC and pC; the motions towards the centre will be equal, and the transverse motion of the body p will be to the transverse motion of the body P as the angular motion of the line pC to the angular motion of the line PC; that is, as the angle VCp to the angle VCP. Therefore, at the same time that the body P, by both its motions, comes to the point K, the body p, having an equal motion towards the centre, will be equally moved from p towards C; and therefore that time being expired, it will be found somewhere in the line mkr, which, passing through the point k, is perpendicular to the line pC; and by its transverse motion will acquire a distance from the line pC, that will be to the distance which the other body P acquires from the line PC as the transverse motion of the body p to the transverse motion of the other body P. Therefore since kr is equal to the distance which the body P acquires from the line PC, and mr is to kr as the angle VCp to the angle VCP, that is, as the transverse motion of the body p to the transverse motion of the body P, it is manifest that the body p, at the expiration of that time, will be found in the place m. These things will be so, if the bodies p and P are equally moved in the directions of the lines pC and PC, and are therefore urged with equal forces in those directions. But if we take an angle pC« that is to the angle pCk as the angle VCp to the angle VCP, and nC be equal to kC, in that case the body p at the expiration of the time will really be in n; and is therefore urged with a greater force than the body P, if the angle nCp is greater than the angle kCp, that is, if the orbit upk moves either progressively, or in a retrograde direction, with a velocity greater than the double of that with which the line CP is carried forwards; and with a less force if the retrograde motion of the orbit is slower. And the difference of the forces will be as the interval mn of the places through which the body would be carried by the action of that difference in that given space of time. About the centre C with the interval Cn or Ck suppose a circle described cutting the lines mr, mn produced in s and t, and the rectangle mn • mt will be equal to the rectangle mk • ms, and therefore mn will be equal to $\frac{mk \cdot ms}{mt}$. But since the triangles pCk, pCn, in a given time, are of a given magnitude, kr and mr, and their difference mk, and their sum ms, are inversely as the altitude pC, and therefore the rectangle mk ms

is inversely as the square of the altitude pC . Moreover, mt is directly as $\frac{1}{2}mt$, that is, as the altitude pC . These are the first ratios of the nascent lines; and hence $\frac{mk \cdot ms}{mt}$, that is, the nascent short line mn , and the difference of the forces proportional thereto, are inversely as the cube of the altitude pC . Q.E.D.

COR. 1. Hence the difference of the forces in the places P and p , or K and k , is to the force with which a body may revolve with a circular motion from R to K , in the same time that the body P in a fixed orbit describes the arc PK , as the nascent line mn to the versed sine of the nascent arc RK , that is, as $\frac{mk \cdot ms}{mt}$ to $\frac{rk^2}{2kC}$, or as $mk \cdot ms$ to the square of rk ; that is, if we take given quantities F and G in the same ratio to each other as the angle VCP bears to the angle VCp , as $GG - FF$ to FF . And, therefore, if from the centre C , with any distance CP or Cp , there be described a circular sector equal to the whole area VPC , which the body revolving in a fixed orbit hath by a radius drawn to the centre described in any certain time, the difference of the forces, with which the body P revolves in a fixed orbit, and the body p in a movable orbit, will be to the centripetal force, with which another body by a radius drawn to the centre can uniformly describe that sector in the same time as the area VPC is described, as $GG - FF$ to FF . For that sector and the area pCk are to each other as the times in which they are described.

COR. II. If the orbit VPK be an ellipse, having its focus C , and its highest apse V , and we suppose the ellipse upk similar and equal to it, so that pC may be always equal to PC , and the angle VCp be to the angle VCP in the given ratio of G to F ; and for the altitude PC or pC we put A , and $2R$ for the latus rectum of the ellipse, the force with which a body may be made to revolve in a movable ellipse will be as $\frac{FF}{AA} + \frac{RGG - RFF}{A^3}$, and conversely. Let the force with which a body may revolve in a fixed ellipse be expressed by the quantity $\frac{FF}{AA}$, and the force in V will be $\frac{FF}{CV^2}$. But the force with which a body may revolve in a circle at the distance CV , with the same velocity as a body revolving in an ellipse has in V , is to the force with which a body

revolving in an ellipse is acted upon in the apse V, as half the latus rectum of the ellipse to the semidiameter CV of the circle, and therefore is as $\frac{RFF}{CV^3}$; and the force which is to this as GG - FF to FF, is as $\frac{RGG - RFF}{CV^3}$; and this force (by Cor. 1 of this Prop.) is the difference of the forces in V, with which the body P revolves in the fixed ellipse VPK, and the body p

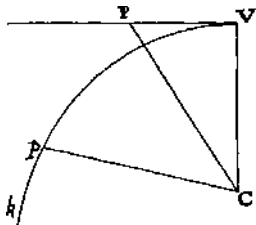


in the movable ellipse upk . Then since by this Proposition that difference at any other altitude A is to itself at the altitude CV as $\frac{1}{A^3}$ to $\frac{1}{CV^3}$, the same difference in every altitude A will be as $\frac{RGG - RFF}{A^3}$. Therefore to the force $\frac{FF}{AA}$, by which the body may revolve in a fixed ellipse VPK, add the excess $\frac{RGG - RFF}{A^3}$, and the sum will be the whole force $\frac{FF}{AA} + \frac{RGG - RFF}{A^3}$ by which a body may revolve in the same time in the movable ellipse upk .

COR. in. In the same manner it will be found, that, if the fixed orbit VPK be an ellipse having its centre in the centre of the forces C, and there be supposed a movable ellipse upl, similar, equal, and concentric to it; and 2R be the principal latus rectum of that ellipse, and 2T the latus transversum, or greater axis; and the angle VCp be continually to the angle VCP as G to F; the forces with which bodies may revolve in the fixed and movable ellipse, in equal times, will be as $\frac{FFA}{T^2}$ and $\frac{FFA}{T^2} + \frac{RGG - RFF}{A^3}$ respectively.

COR. IV. And universally, if the greatest altitude CV of the body be called T, and the radius of the curvature which the orbit VPK has in V, that is, the radius of a circle equally curved, be called R, and the centripetal force with which a body may revolve in any fixed curve VPK at the place V be called $\frac{VFF}{TT}$, and in other places P be indefinitely styled X; and the altitude CP be called A, and G be taken to F in the given ratio of the angle VCp to the angle VCP; the centripetal force with which the same body will perform the same motions in the same time, in the same curve upl revolving with a circular motion, will be as the sum of the forces $X + \frac{VRGG - VRFF}{A^3}$.

COR. v. Therefore the motion of a body in a fixed orbit being given, its angular motion round the centre of the forces may be increased or diminished in a given ratio; and thence new fixed orbits may be found in which bodies may revolve with new centripetal forces.



COR. vi. Therefore if there be erected the line VP of an indeterminate length, perpendicular to the line CV given by position, and CP be drawn, and Cp equal to it, making the angle VCp having a given ratio to the angle VCP, the force with which a body may revolve in the curved line Vpk, which the point p is continually describing, will be inversely as the cube of the altitude Cp. For the body P, by its inertia alone, no other force impelling it, will proceed uniformly in the right line VP. Add,

then, a force tending to the centre C inversely as the cube of the altitude CP or Cp , and (by what was just demonstrated) the body will deflect from the rectilinear motion into the curved line Vpk . But this curve Vpk is the same with the curve VPQ found in Cor. in, Prop, XLI, in which, I said, bodies attracted with such forces would ascend obliquely.

PROPOSITION XLV. PROBLEM XXXI

To find the motion of the apsides in orbits approaching very near to circles}

This problem is solved arithmetically by reducing the orbit, which a body revolving in a movable ellipse (as in Cor. 11 and 111 of the above Prop.) describes in a fixed plane, to the figure of the orbit whose apsides arc required; and then seeking the apsides of the orbit which that body describes in a fixed plane. But orbits acquire the same figure, if the centripetal forces with which they are described, compared between themselves, are made proportional at equal altitudes. Let the point V be the highest apse, and write T for the greatest altitude CV, A for any other altitude CP or Cp , and X for the difference of the altitudes CV - CP; and the force with which a body moves in an ellipse revolving about its focus C (as in Cor. 11), and which in Cor. II was as $\frac{FF}{AA} + \frac{RGG - RFF}{A^3}$, that is, as $\frac{FFA + RGG - RFF}{A^3}$, by substituting $T - X$ for A, will become as $\frac{RGG - RFF + TFF - FFX}{A^3}$. In like manner any other centripetal force is to be reduced to a fraction whose denominator is A^3 , and the numerators are to be made analogous by collating together the homologous terms. This will be made plainer by Examples.

EXAM. I. Let us suppose the centripetal force to be uniform, and therefore as $\frac{A^3}{A^3}$, or, writing $T - X$ for A in the numerator, as

$$\frac{T^3 - 3TTX + 3TXX - X^3}{A^3}.$$

Then collating together the correspondent terms of the numerators, that is, those that consist of given quantities with those of given quantities, and

those of quantities not given with those of quantities not given, it will become

$$\begin{aligned} \text{RGG} - \text{RFF} + \text{TFF} : \text{T}^3 &= -\text{FFX} : 3\text{T}^2\text{TX} + 3\text{TXX} - \text{X}^3 \\ &= -\text{FF} : 3\text{T}^2 + 3\text{TX} - \text{XX}. \end{aligned}$$

Now since the orbit is supposed extremely near to a circle, let it coincide with a circle; and because in that case R and T become equal, and X is infinitely diminished, the last ratios will be

$$\text{GG} : \text{T}^2 = -\text{FF} : -3\text{T}^2$$

and again, $\text{GG} : \text{FF} = \text{TT} : 3\text{T}^2 :: 1 : 3$;

and therefore G is to F, that is, the angle VCP to the angle VCP, as 1 to $\sqrt{3}$. Therefore since the body, in a fixed ellipse, in descending from the upper to the lower apse, describes an angle, if I may so speak, of 180° , the other body in a movable ellipse, and therefore in the fixed plane we are treating of, will in its descent from the upper to the lower apse, describe an angle VCP of $\frac{180^\circ}{\sqrt{3}}$. And this comes to pass by reason of the likeness of this orbit

which a body acted upon by an uniform centripetal force describes, and of that orbit which a body performing its circuits in a revolving ellipse will describe in a fixed plane. By this collation of the terms, these orbits are made similar; not universally, indeed, but then only when they approach very near to a circular figure. A body, therefore, revolving with an uniform centripetal force in an orbit nearly circular, will always describe an angle of $\frac{180^\circ}{\sqrt{3}}$, or $103^\circ 55' 23''$ at the centre; moving from the upper apse to the lower apse when it has once described that angle, and thence returning to the upper apse when it has described that angle again; and so on *in infinitum*.

EXAM. 2. Suppose the centripetal force to be as any power of the altitude A, as, for example, A^{n-3} , or $\frac{A^n}{A^3}$; where $n-3$ and n signify any indices of powers whatever, whether integers or fractions, rational or surd, affirmative or negative. That numerator A^n or $(T-X)^n$ being reduced to an indeterminate series by my method of converging series, will become

$$T^n - nXT^{n-1} + \frac{n(n-1)}{2} X^2T^{n-2}, \&c.$$

And comparing these terms with the terms of the other numerator

$$RGG - RFF + TFF - FFX,$$

it becomes $RGG - RFF + TFF : T^n = -FF : -nT^{n-1} + \frac{nn-n}{2} \cdot XT^{n-2}$, &c.

And taking the last ratios where the orbits approach to circles, it becomes

$$RGG : T^n = -FF : -nT^{n-1},$$

or, $GG : T^{n-1} = FF : nT^{n-1},$

and again, $GG : FF = T^{n-1} : nT^{n-1} = 1 : n;$

and therefore G is to F, that is, the angle VCp to the angle VCP , as 1 to $\sqrt{n}$.

Therefore since the angle VCP , described in the descent of the body from the upper apse to the lower apse in an ellipse, is of 180° , the angle VCp , described in the descent of the body from the upper apse to the lower apse in an orbit nearly circular which a body describes with a centripetal force proportional to the power A^{n-2} , will be equal to an angle of $\frac{180^\circ}{\sqrt{n}}$, and this

angle being repeated, the body will return from the lower to the upper apse, and so on *in infinitum*. As if the centripetal force be as the distance of the body from the centre, that is, as A , or $\frac{A^*}{A}$, n will be equal to 4 , and

$\sqrt{n}$ equal to 2 ; and therefore the angle between the upper and the lower apse will be equal to $\frac{180^\circ}{2}$, or 90° . Therefore the body having performed a

fourth part of one revolution, will arrive at the lower apse, and having performed another fourth part, will arrive at the upper apse, and so on *in infinitum*. This appears also from Prop. x. For a body acted on by this centripetal force will revolve in a fixed ellipse, whose centre is the centre of force. If the centripetal force is inversely as the distance, that is, directly as

$\frac{1}{A}$ or $\frac{A'}{A^2}$, n will be equal to 2 ; and therefore the angle between the upper and the lower apse will be $\frac{180^\circ}{\sqrt{2}}$, or $127^\circ 16' 45''$; and hence a body revolving

with such a force will, by a continual repetition of this angle, move alternately from the upper to the lower and from the lower to the upper apse forever. So, also, if the centripetal force be inversely as the fourth root of the eleventh power of the altitude, that is, inversely as $A^{11/4}$, and therefore

directly as $\frac{1}{A^{\frac{1}{14}}}$, or as $\frac{A^m}{A^{\frac{1}{14}}}$, n will be equal to $\frac{1}{4}$, and $\frac{180^\circ}{\sqrt{n}}$ will be equal to 360° ; and therefore the body parting from the upper apse, and from thence continually descending, will arrive at the lower apse when it has completed one entire revolution; and thence ascending continually, when it has completed another entire revolution, it will arrive again at the upper apse; and so alternately forever.

EXAM. 3. Taking m and n for any indices of the powers of the altitude, and b and c for any given numbers, suppose the centripetal force to be as $(bA^m + cA^n) - A^3$, that is, as $[b(T-X)^m + c(T-X)^n] \div A^3$, or (by the method of converging series above mentioned) as

$$[bT^m + cT^n - mbXT^{m-1} - ncXT^{n-1} + \frac{mm-m}{2} bXXT^{m-2} + \frac{nn-n}{2} \\ - cXXT^{n-2}, \&c.] \div A^3;$$

and comparing the terms of the numerators, there will arise,

$$RGG - RFF + TFF : bT^m + cT^n = -FF : -mbT^{m-1} - ncT^{n-1} + \\ \frac{mm-m}{2} bXT^{m-2} + \frac{nn-n}{2} cXT^{n-2}, \&c.$$

And taking the last ratios that arise when the orbits come to a circular form, there will come forth

$$GG : bT^{m-1} + cT^{n-1} = FF : mbT^{m-1} + ncT^{n-1};$$

and again, $GG : FF = bT^{m-1} + cT^{n-1} : mbT^{m-1} + ncT^{n-1}$.

This proportion, by expressing the greatest altitude CV or T arithmetically

by unity, becomes, $GG : FF = b + c : mb + nc = 1 : \frac{mb + nc}{b + c}$. Whence G be-

comes to F, that is, the angle VCP to the angle VCP, as 1 to $\sqrt{\frac{mb + nc}{b + c}}$.

And therefore, since the angle VCP between the upper and the lower apse, in a fixed ellipse, is of 180° , the angle VCP between the same apsides in an orbit which a body describes with a centripetal force, that is, as $\frac{bA^m + cA^n}{A^3}$, will be equal to an angle of $180^\circ \sqrt{\frac{b + c}{mb + nc}}$. And by the same

reasoning, if the centripetal force be as $\frac{bA^m - cA^n}{A^3}$, the angle between the

apsides will be found equal to $180^\circ \sqrt{\frac{b-c}{mb-nc}}$. After the same manner the

Problem is solved in more difficult cases. The quantity to which the centripetal force is proportional must always be resolved into a converging series whose denominator is A^3 . Then the given part of the numerator arising from that operation is to be supposed in the same ratio to that part of it which is not given, as the given part of this numerator RGG - RFF + TFF - FFX is to that part of the same numerator which is not given. And taking away the superfluous quantities, and writing unity for T, the proportion of G to F is obtained.

COK. 1. Hence if the centripetal force be as any power of the altitude, that power may be found from the motion of the apsides; and conversely. That is, if the whole angular motion, with which the body returns to the same apse, be to the angular motion of one revolution, or 360° , as any number as m to another as n , and the altitude be called A; the force will be as the

power $A^{\frac{3}{\frac{nn}{mm}-3}}$ of the altitude A; the index of which power is $\frac{nn}{mm}-3$. This

appears by the second Example. Hence it is plain that the force in its recess from the centre cannot decrease in a greater than a cubed ratio of the altitude. A body revolving with such a force, and parting from the apse, if it once begins to descend, can never arrive at the lower apse or least altitude, but will descend to the centre, describing the curved line treated of in Cor. in, Prop. XLI. But if it should, at its parting from the lower apse, begin to ascend ever so little, it will ascend *in infinitum*, and never come to the upper apse; but will describe the curved line spoken of in the same Cor., and Cor. vi, Prop. XLIV. SO that where the force in its recess from the centre decreases in a greater than a cubed ratio of the altitude, the body at its parting from the apse, will either descend to the centre, or ascend *in infinitum*, according as it descends or ascends at the beginning of its motion. But if the force in its recess from the centre either decreases in a less than a cubed ratio of the altitude, or increases in any ratio of the altitude whatsoever, the body will never descend to the centre, but will at some time arrive at the lower apse; and, on the contrary, if the body alternately ascending and descending from one apse to another never comes to the centre, then either the force increases in the recess from the centre, or it decreases in a less than

a cubed ratio of the altitude; and the sooner the body returns from one apse to another, the farther is the ratio of the forces from the cubed ratio. As if the body should return to and from the upper apse by an alternate descent and ascent in 8 revolutions, or in 4, or 2, or $1\frac{1}{2}$; that is, if m should be to n as 8, or 4, or 2, or $1\frac{1}{2}$ to 1, and therefore $\frac{nn}{mm} - 3$, be $\frac{3}{8} - 3$, or $\frac{1}{4} - 3$, or $\frac{1}{2} - 3$, or $\frac{3}{4} - 3$, or $\frac{1}{9} - 3$; then the force will be as $A^{\frac{3}{8}-3}$, or $A^{\frac{1}{4}-3}$, or $A^{\frac{1}{2}-3}$, or $A^{\frac{3}{4}-3}$; that is, it will be inversely as $A^{2-\frac{3}{8}}$, or $A^{2-\frac{1}{4}}$, or $A^{2-\frac{1}{2}}$, or $A^{2-\frac{3}{4}}$. If the body after each revolution returns to the same apse, and the apse remains unmoved, then m will be to n as 1 to 1, and therefore $A^{\frac{nn}{mm}-3}$ will be equal to A^{-2} , or $\frac{1}{AA}$; and therefore the decrease of the forces will be in a squared ratio of the altitude; as was demonstrated above. If the body in three fourth parts, or two thirds, or one third, or one fourth part of an entire revolution, return to the same apse; m will be to n as $\frac{3}{4}$ or $\frac{2}{3}$ or $\frac{1}{3}$ or $\frac{1}{4}$ to 1, and therefore $A^{\frac{nn}{mm}-3}$ is equal to $A^{\frac{3}{4}-3}$, or $A^{\frac{2}{3}-3}$, or $A^{\frac{1}{3}-3}$, or $A^{\frac{1}{4}-3}$; and therefore the force is either inversely as $A^{\frac{9}{4}}$ or $A^{\frac{7}{3}}$, or directly as $A^{\frac{1}{3}}$ or $A^{\frac{1}{4}}$. Lastly if the body in its progress from the upper apse to the same upper apse again, goes over one entire revolution and three degrees more, and therefore that apse in each revolution of the body moves forward three degrees, then m will be to n as $363''$ to $360''$, or as 121 to 120, and therefore $A^{\frac{nn}{mm}-3}$ will be equal to $A^{-\frac{29521}{14641}}$, and therefore the centripetal force will be inversely as $A^{\frac{29521}{14641}}$, or inversely as $A^{2\frac{4}{343}}$ very nearly. Therefore the centripetal force decreases in a ratio something greater than the squared ratio; but approaching $59\frac{3}{4}$ times nearer to the squared than the cubed.

COR. II. Hence also if a body, urged by a centripetal force which is inversely as the square of the altitude, revolves in an ellipse whose focus is in the centre of the forces; and a new and foreign force should be added to or subtracted from this centripetal force, the motion of the apsides arising from that foreign force may (by the third Example) be known; and conversely: If the force with which the body revolves in the ellipse be as $\frac{1}{AA}$; and the foreign force as cA , and therefore the remaining force as $\frac{A - cA^4}{A^5}$;

then (by the third Example) b will be equal to 1, m equal to 1, and n equal to 4; and therefore the angle of revolution between the apsides is equal to $180^\circ \sqrt{\frac{1-c}{1-4c}}$. Suppose that foreign force to be 357.45 times less than the other force with which the body revolves in the ellipse; that is, c to be $\frac{100}{85745}$, A or T being equal to 1; and then $180^\circ \sqrt{\frac{1-c}{1-4c}}$ will be $180^\circ \sqrt{\frac{35845}{85145}}$ or $180^\circ 7623$, that is, $180^\circ 45' 44''$. Therefore the body, parting from the upper apse, will arrive at the lower apse with an angular motion of $180^\circ 45' 44''$, and this angular motion being repeated, will return to the upper apse; and therefore the upper apse in each revolution will go forward $1^\circ 31' 28''$. The apse of the moon is about twice as swift.

So much for the motion of bodies in orbits whose planes pass through the centre of force. It now remains to determine those motions in eccentric planes. For those authors who treat of the motion of heavy bodies used to consider the ascent and descent of such bodies, not only in a perpendicular direction, but at all degrees of obliquity upon any given planes; and for the same reason we are to consider in this place the motions of bodies tending to centres by means of any forces whatsoever, when those bodies move in eccentric planes. These planes are supposed to be perfectly smooth and polished, so as not to retard the motion of the bodies in the least. Moreover, in these demonstrations, instead of the planes upon which those bodies roll or slide, and which are therefore tangent planes to the bodies, I shall use planes parallel to them, in which the centres of the bodies move, and by that motion describe orbits. And by the same method I afterwards determine the motions of bodies performed in curved surfaces.

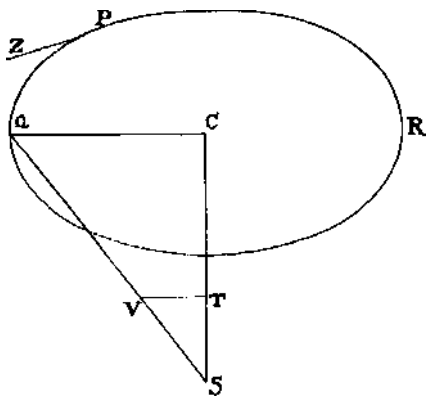
SECTION X

The motion of bodies in given surfaces; and the oscillating pendulous motion of bodies.

PROPOSITION XLVI. PROBLEM XXXII

Any kind of centripetal force being supposed, and the centre of force, and any plane whatsoever in which the body revolves, being given, and the quadratures of curvilinear figures being allowed; it is required to determine the motion of a body going off from a given place with a given velocity, in the direction of a given right line in that plane.

Let S be the centre of force, SC the least distance of that centre from the given plane, P a body issuing from the place P in the direction of the right line PZ , Q the same body revolving in its curve, and PQR the curve itself



which is required to be found, described in that given plane. Join CQ , QS , and if in QS we take SV proportional to the centripetal force with which the body is attracted towards the centre S , and draw VT parallel to CQ , and meeting SC in T ; then will the force SV be resolved into two (by Cor. 11 of the Laws of Motion), the force ST , and

the force TV ; of which ST attracting the body in the direction of a line perpendicular to that plane, does not at all change its motion in that plane. But the action of the other force TV , coinciding with the position of the plane itself, attracts the body directly towards the given point C in that plane; and therefore causes the body to move in the plane in the same

manner as if the force ST were taken away, and the body were to revolve in free space about the centre C by means of the force TV alone. But there being given the centripetal force TV with which the body Q revolves in free space about the given centre C, there is given (by Prop, XLII) the curve PQR which the body describes; the place Q, in which the body will be found at any given time; and, lastly, the velocity of the body in that place Q. And conversely. Q.E.I.

PROPOSITION XLVII. THEOREM XV

Supposing the centripetal force to be proportional to the distance of the body from the centre; all bodies revolving in any planes whatsoever will describe ellipses, and complete their revolutions in equal times, and those which move in right lines, running backwards and forwards alternately, will complete their several periods of going and returning in the same times.

For letting all things stand as in the foregoing Proposition, the force SV, with which the body Q revolving in any plane PQR is attracted towards the centre S, is as the distance SQ; and therefore because SV and SQ, TV and CQ are proportional, the force TV with which the body is attracted towards the given point C in the plane of the orbit, is as the distance CQ. Therefore the forces with which bodies found in the plane PQR are attracted towards the point C, are in proportion to the distances equal to the forces with which the same bodies are attracted every way towards the centre S; and therefore the bodies will move in the same times, and in the same figures, in any plane PQR about the point C, as they would do in free spaces about the centre S; and therefore (by Cor. n, Prop, x, and Cor. 11, Prop, XXVIII) they will in equal times either describe ellipses in that plane about the centre C, or move to and fro in right lines passing through the centre C in that plane; completing the same periods of time in all cases. Q.E.D.

SCHOLIUM

The ascent and descent of bodies in curved surfaces has a near relation to these motions we have been speaking of. Imagine curved lines to be described on any plane, and to revolve about any given axes passing through the centre of force, and by that revolution to describe curved surfaces; and

that the bodies move in such sort that their centres may be always found in those surfaces. If those bodies oscillate to and fro with an oblique ascent and descent, their motions will be performed in planes passing through the axis, and therefore in the curved lines, by whose revolution those curved surfaces were generated. In those cases, therefore, it will be sufficient to consider the motion in those curved lines.

PROPOSITION XLVIII. THEOREM XVI

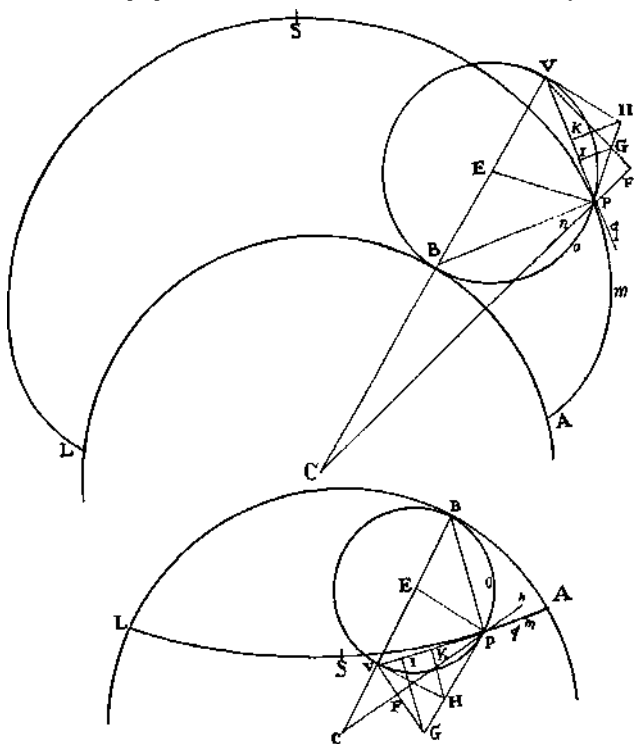
If a wheel stands upon the outside of a globe at right angles thereto, and revolving about its own axis goes forwards in a great circle, the length of the curvilinear path which any point, given in the perimeter of the wheel, hath described since the time that it touched the globe (which curvilinear path we may call the cycloid or epicycloid), will be to double the versed sine of half the arc which since that time hath touched the globe in passing over it, as the sum of the diameters of the globe and the wheel to the semidiameter of the globe.

PROPOSITION XLIX. THEOREM XVII

If a wheel stands upon the inside of a concave globe at right angles thereto, and revolving about its own axis goes forwards in one of the great circles of the globe, the length of the curvilinear path which any point, given in the perimeter of the wheel, hath described since it touched the globe, will be to the double of the versed sine of half the arc which in all that time hath touched the globe in passing over it, as the difference of the diameters of the globe and the wheel to the semidiameter of the globe.

Let ABL be the globe, C its centre, BPV the wheel resting on it, E the centre of the wheel, B the point of contact, and P the given point in the perimeter of the wheel. Imagine this wheel to proceed in the great circle ABL from A through B towards L, and in its progress to revolve in such a manner that the arcs AB, PB may be always equal one to the other, and the given point P in the perimeter of the wheel may describe in the meantime the curvilinear path AP. Let AP be the whole curvilinear path described since the wheel touched the globe in A, and the length of this path AP will be to twice the versed sine of the arc $\frac{1}{2}PB$ as $2CE$ to CB . For let the right

line CE (produced if need be) meet the wheel in V, and join CP, BP, EP, VP; produce CP, and let fall thereon the perpendicular VF. Let PH, VH, meeting in H, touch the circle in P and V, and let PH cut VF in G, and to VP let fall the perpendiculars GI, HK. From the centre C with any radius



let there be described the circle *nom*, cutting the right line CP in *n*, the perimeter of the wheel BP in *o*, and the curvilinear path AP in *m*; and from the centre V with the radius Vo let there be described a circle cutting VP produced in *q*.

Because the wheel in its progress always revolves about the point of contact B, it is manifest that the right line BP is perpendicular to that curved line AP which the point P of the wheel describes, and therefore that the right line VP will touch this curve in the point P. Let the radius of the circle *tiorn* be gradually increased or diminished so that at last it becomes equal to the distance CP; and by reason of the similitude of the evanescent figure *Vnomq*, and the figure PFGVI, the ultimate ratio of the evanescent short lines *Vm*, *Vn*, *Vo*, *Vq*, that is, the ratio of the momentary increments of the curve AP, the right line CP, the circular arc BP, and the right line VP, will be the same as of the lines PV, PF, PG, PI, respectively. But since VF is perpendicular to CF, and VH to CV, and therefore the angles HVG, VCF equal; and the angle VHG (because the angles of the quadrilateral HVEP arc right in V and P) is equal to the angle CEP, the triangles VHG, CEP will be similar; and thence it will come to pass that

$$EP : CE = HG : HV \text{ or } HP - KI : PK,$$

and by addition or subtraction,

$$CB : CE = PI : PK,$$

and $CM : 2CE - PI : PV = Pq : Pm.$

Therefore the decrement of the line VP, that is, the increment of the line BV - VP to the increment of the curved line AP is in a given ratio of CB to 2CE, and therefore (by Cor., Lem. iv) the lengths BV - VP and AP, generated by those increments, are in the same ratio. But if BV be radius, VP is the cosine of the angle BVP or $\frac{1}{2}BEP$, and therefore BV - VP is the versed sine of the same angle, and therefore in this wheel, whose radius is $\frac{1}{2}BV$, BV - VP will be double the versed sine of the arc VJBP. Therefore AP is to double the versed sine of the arc $\frac{1}{2}BP$ as 2CE to CB. Q.E.D.

The line AP in the former of these Propositions we shall name the cycloid without the globe, the other in the latter Proposition the cycloid within the globe, for distinction's sake.

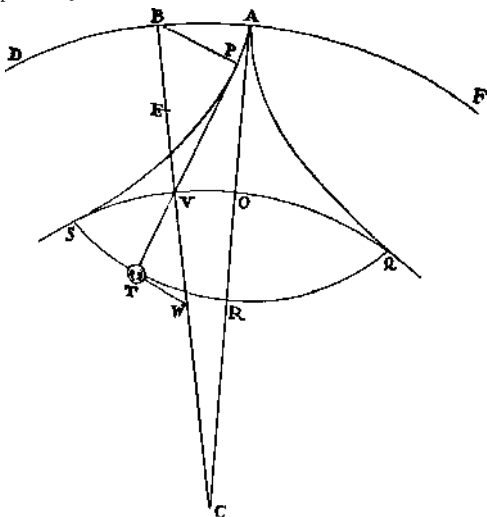
COR. 1. Hence if there be described the entire cycloid ASL, and the same be bisected in S, the length of the part PS will be to the length PV (which is the double of the sine of the angle VBP, when EB is radius) as 2CE to CB, and therefore in a given ratio.

COR. 11. And the length of the semidiameter of the cycloid AS will be equal to a right line which is to the diameter of the wheel BV as 2CE to CB.

PROPOSITION L. PROBLEM XXXIII

To cause a pendulous body to oscillate in a given cycloid.

Let there be given within the globe QVS described with the centre C, the cycloid QRS, bisected in R, and meeting the surface of the globe with its extreme points Q and S on either hand. Let there be drawn CR bisecting



the arc QS in O, and let it be produced to A in such sort that CA may be to CO as CO to CR. About the centre C, with the radius CA, let there be described an exterior globe DAF; and within this globe, by a wheel whose diameter is AO, let there be described two semicycloids AQ, AS, touching the interior globe in Q and S, and meeting the exterior globe in A. From that point A, with a thread APT in length equal to the line AR, let the body T be suspended and oscillated in such manner between the two semicycloids AQ, AS, that, as often as the pendulum parts from the perpendicular AR, the upper part of the thread AP may be applied to that semicycloid APS towards which the motion tends, and fold itself round that curved line, as if it were some solid obstacle, the remaining part of the same thread PT

which has not yet touched the semicycloid continuing straight. Then will the weight T oscillate in the given cycloid QRS. Q.E.F.

For let the thread PT meet the cycloid QRS in T, and the circle QOS in V, and let CV be drawn; and to the rectilinear part of the thread PT from the extreme points P and T let there be erected the perpendiculars BP, TW, meeting the right line CV in B and W. It is evident, from the construction and generation of the similar figures AS, SR, that those perpendiculars PB, TW, cut off from CV the lengths VB, VW, equal the diameters of the wheels OA, OR. Therefore TP is to VP (which is double the sine of the angle VBP when $\frac{1}{2}BV$ is radius) as BW to BV, or AO OR to AO, that is (since CA and CO, CO and CR, and by division AO and OR are proportional), as CA CO to CA, or, if BV be bisected in E, as 2CE to CB. Therefore (by Cor. 1, Prop, XLIX), the length of the rectilinear part of the thread PT is always equal to the arc of the cycloid PS, and the whole thread APT is always equal to half the cycloid APS, that is (by Cor. 11, Prop, XLIX), to the length AR. And conversely, if the string is always equal to the length AR, the point T will always move in the given cycloid QRS. Q.E.D.

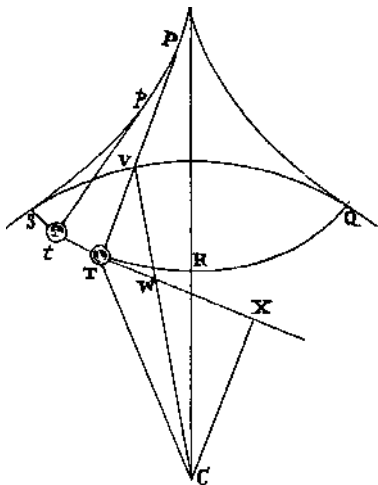
COR. The string AR is equal to the semicycloid AS, and therefore has the same ratio to AC, the semidiameter of the exterior globe, as the like semicycloid SR has to CO, the semidiameter of the interior globe.

PROPOSITION LI. THEOREM XVIII

If a centripetal force tending on all sides to the centre C of a globe, be in all places as the distance of the place from the centre; and, by this force alone acting upon it, the body T oscillate (in the manner above described) in the per meter of the cycloid QRS: I say, that all the oscillations, howsoever unequal in themselves, will be performed in equal times.

For upon the tangent TW indefinitely produced let fall the perpendicular CX, and join CT. Because the centripetal force with which the body T is impelled towards C is as the distance CT, let this (by Cor. 11 of the Laws) be resolved into the parts CX, TX, of which CX impelling the body directly from P stretches the thread PT, and by the resistance the thread makes to it is totally employed, producing no other effect; but the other part TX, impelling the body transversely or towards X, directly accelerates the motion in the cycloid. Then it is plain that the acceleration of the body, propor-

tional to this accelerating force, will be every moment as the length TX , that is (because CV , WV , and TX , TW proportional to them are given), as the length TW , that is (by Cor. 1, Prop. XLIX), as the length of the arc of the cycloid TR . If therefore two pendulums APT , Apt , be unequally drawn aside from the perpendicular AR , and let fall together, their accelerations will be always as the arcs to be described TR , tR . But the parts described at the beginning of the motion arc as the accelerations, that is, as the whole spaces that arc to be described at the beginning, and therefore the parts which remain to be described, and the subse-



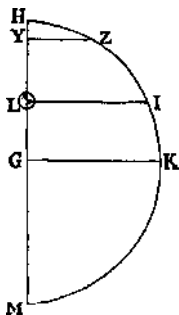
quent accelerations proportional to those parts, are also as the whole, and so on. Therefore the accelerations, and consequently the velocities generated, and the parts described with those velocities, and the parts to be described, are always as the whole; and therefore the parts to be described preserving a given ratio to each other will vanish together, that is, the two bodies oscillating will arrive together at the perpendicular AR . And since on the other hand the ascent of the pendulums from the lowest place R through the same cycloidal arcs with a retrograde motion, is retarded in the several places they pass through by the same forces by which their descent was accelerated, it is plain that the velocities of their ascent and descent through the same arcs are equal, and consequently performed in equal times; and, therefore, since the two parts of the cycloid RS and RQ lying on either side of the perpendicular are similar and equal, the two pendulums will perform as well the whole as the half of their oscillations in the same times. Q.E.D.

COR. The force with which the body T is accelerated or retarded in any place T of the cycloid, is to the whole weight of the same body in the highest place S or Q as the arc of the cycloid TR is to the arc SR or QR.

PROPOSITION LII. PROBLEM XXXIV

To define the velocities of pendulums in the several places, and the times in which both the entire oscillations and their several parts are performed.

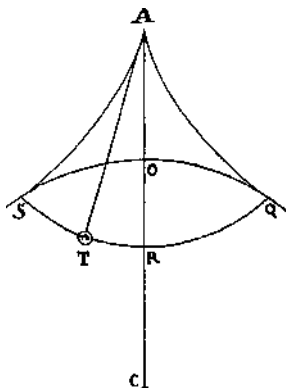
About any centre G, with the radius GH equal to the arc of the cycloid RS, describe a semicircle HKM bisected by the semidiameter GK. And if a centripetal force proportional to the distance of the places from the



centre tend to the centre G, and it be in the perimeter HIK equal to the centripetal force in the perimeter of the globe QOS tending towards its centre, and at the same time that the pendulum T is let fall from the highest place S, a body, as L, is let fall from H to G; then because the forces which act upon the bodies arc equal at the beginning, and always proportional to the spaces to be described TR, LG, and therefore if TR and LG are equal, are also equal in the places T and L, it is plain that those bodies describe at the beginning equal spaces ST, HL, and therefore are still acted upon equally, and continue to describe equal spaces. Therefore by

Prop. XXXVIII, the time in which the body describes the arc ST is to the time of one oscillation, as the arc HI the time in which the body H arrives at L, to the semiperiphery HKM, the time in which the body H will come to M. And the velocity of the pendulous body in the place T is to its velocity in the lowest place R, that is, the velocity of the body H in the place L to its velocity in the place G, or the momentary increment of the line HL to the momentary increment of the line HG (the arcs HI, HK increasing with an uniform velocity) as the ordinate LI to the radius GK, or as $V(SR - TR')$ to SR. Hence, since in unequal oscillations there are described in equal times arcs proportional to the entire arcs of the oscillations, there are obtained, from the times given, both the velocities and the arcs described in all the oscillations universally. Which was first required.

Let now any pendulous bodies oscillate in different cycloids described within different globes, whose absolute forces are also different; and if the absolute force of any globe QOS be called V , the accelerative force with which the pendulum is acted on in the circumference of this globe, when it begins to move directly towards its centre, will be as the distance of the pendulous body from that centre and the absolute force of the globe conjointly, that is, as $CO \cdot V$. Therefore the short line HY , which is as this accelerated force $CO \cdot V$, will be described in a given time; and if there be erected the perpendicular YZ meeting the circumference in Z , the nascent arc HZ will denote that given time. But that nascent arc HZ varies as the square root of the rectangle $GH \cdot HY$, and therefore as $\sqrt{(GH \cdot CO \cdot V)}$. Whence the time of an entire oscillation in the cycloid QRS (it being as the semipenphery HKM , which denotes that entire oscillation, directly; and as the arc HZ , which in like manner denotes a given time, inversely) will be as GH directly and $\sqrt{(GH \cdot CO \cdot V)}$



inversely; that is, because GH and SR are equal, as $\sqrt{\frac{SR}{CO \cdot V}}$ or (by Cor.,

Prop. L), as $\sqrt{\frac{AR}{AC \cdot v}}$. Therefore the oscillations in all globes and cycloids, performed with any absolute forces whatever, vary directly as the square root of the length of the string, and inversely as the square root of the distance between the point of suspension and the centre of the globe, and also inversely as the square root of the absolute force of the globe. Q.E.I.

COR. 1. Hence also the times of oscillating, falling, and revolving bodies may be compared among themselves. For if the diameter of the wheel with which the cycloid is described within the globe is supposed equal to the semidiameter of the globe, the cycloid will become a right line passing through the centre of the globe, and the oscillation will be changed into a

descent and subsequent ascent in that right line. Hence there is given both the time of the descent from any place to the centre, and the time equal to it in which the body revolving uniformly about the centre of the globe at any distance describes an arc of a quadrant. For this time (by Case 2) is to the time of half the oscillation in any cycloid QRS as 1 to $\sqrt{\frac{AR}{AC}}$.

COR. 11. Hence also follow what Sir *Christopher Wren* and Mr. *Huygens* have discovered concerning the common cycloid. For if the diameter of the globe be infinitely increased, its spherical surface will be changed into a plane, and the centripetal force will act uniformly in the direction of lines perpendicular to that plane, and our cycloid will become the same with the common cycloid. But in that case the length of the arc of the cycloid between that plane and the describing point will become equal to four times the versed sine of half the arc of the wheel between the same plane and the describing point, as was discovered by Sir *Christopher Wren*. And a pendulum between two such cycloids will oscillate in a similar and equal cycloid in equal times, as Mr. *Huygens* demonstrated. The descent of heavy bodies also in the time of one oscillation will be the same as Mr. *Huygens* exhibited.

The Propositions here demonstrated are adapted to the true constitution of the earth, so far as wheels moving in any of its great circles will describe, by the motions of nails fixed in their perimeters, cycloids without the globe; and pendulums, in mines and deep caverns of the earth, must oscillate in cycloids within the globe, that those oscillations may be performed in equal times. For gravity (as will be shown in the third Book) decreases in its progress from the surface of the earth; upwards as the square root of the distances from the centre of the earth; downwards as these distances.

PROPOSITION LIU. PROBLEM XXXV

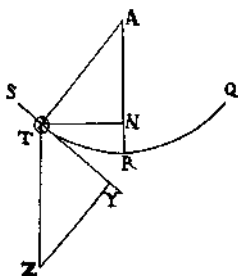
Granting the quadratures of curvilinear figures, it is required to find the forces with which bodies moving in given curved lines may always perform their oscillations in equal times.

Let the body T oscillate in any curved line STRQ, whose axis is AR passing through the centre of force C. Draw TX touching that curve in any place of the body T, and in that tangent TX take TY equal to the arc TR.

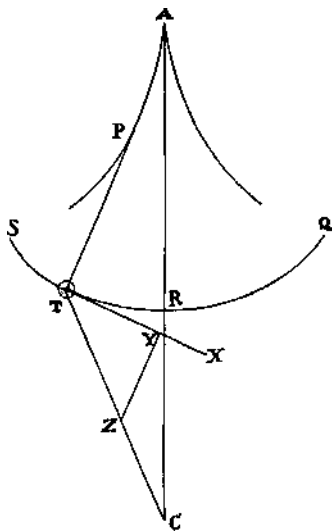
The length of that arc is known from the common methods used for the quadratures of figures. From the point Y draw the right line YZ perpendicular to the tangent. Draw CT meeting YZ in Z, and the centripetal force will be proportional to the right line TZ. Q.E.I.

For if the force with which the body is attracted from T towards C be expressed by the right line TZ taken proportional to it, that force will be resolved into two forces TY, YZ, of which YZ, drawing the body in the direction of the length of the thread PT, docs not at all change its motion; whereas the other force TY directly accelerates or retards its motion in the curve STRQ. Therefore since that force is as the space to be described TR, the accelerations or retardations of the body in describing two proportional parts (a greater and a less)

of two oscillations, will be always as those parts, and therefore will cause those parts to be described together. But bodies which continually describe in the same tune parts proportional to the whole, will describe the whole in the same time. Q.E.D.



COR. i. Hence if the body T, hanging by a rectilinear thread AT from the centre A, describe the circular arc STRQ, and in the meantime be acted on by any force tending downwards with parallel directions, which is to the uniform force of gravity as the arc TR to its sine TN, the times of the several oscillations will be equal. For because TZ,



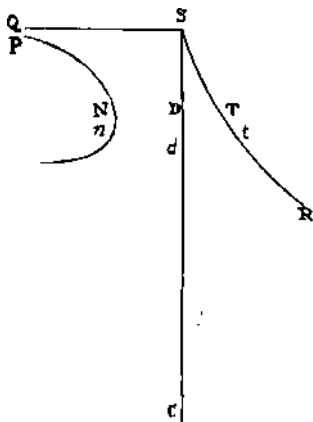
AR are parallel, the triangles ATN, ZTY are similar; and therefore TZ will be to AT as TY to TN; that is, if the uniform force of gravity be expressed by the given length AT, the force TZ, by which the oscillations become isochronous, will be to the force of gravity AT, as the arc TR equal to TY is to TN the sine of that arc.

COR. ii. And therefore in clocks, if forces are impressed by some machine upon the pendulum which continues the motion, and so compounded with the force of gravity that the whole force tending downwards will be always as a line which is obtained by dividing the product of the arc TR and the radius AR, by the sine TN, then all the oscillations will become isochronous.

PROPOSITION LIV. PROBLEM XXXVI

Granting the quadratures of curvilinear figures, it is required to find the times in which bodies by means of any centripetal force will descend or ascend in any curved lines in a plane passing through the centre of force.

Let the body descend from any place S, and move in any curve ST/R given in a plane passing through the centre of force C. Join CS, and let it be divided into innumerable equal parts, and let Yd be one of those parts.



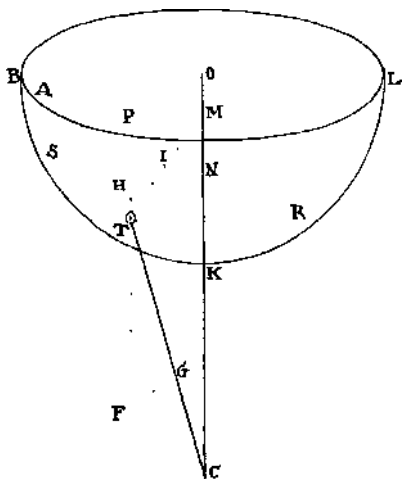
From the centre C, with the radii CD, Cd, let the circles DT, dt be described, meeting the curved line STtR in T and t. And because the law of centripetal force is given, and also the altitude CS from which the body at first fell, there will be given the velocity of the body in any other altitude CT (by Prop, xxxix). But the time in which the body describes the short line Tt is as the length of that short line, that is, directly as the secant of the angle tTC and inversely as the velocity. Let the ordinate DN, proportional to this time, be made perpendicular to the right line CS at the point D, and because Dd

is given, the rectangle Dd DN, that is, the area $DNnd$, will be proportional to the same time. Therefore if PNn be a curved line which the point N continually touches, and its asymptote be the right line SQ standing upon the line CS at right angles, the area $SQPND$ will be proportional to the time in which the body in its descent hath described the line ST ; and therefore that area being found, the time is also given. Q.E.I.

PROPOSITION LV. THEOREM XIX

If a body move in any curved surface, whose axis passes through the centre of force, and from the body a perpendicular be let fall upon the axis; and a line parallel and equal thereto be drawn from any given point of the axis: I say, that this parallel line will describe an area proportional to the time.

Let BKL be a curved surface, T a body revolving in it, STR a curve which the body describes in the same, S the beginning of the curve, OMK the axis of the curved surface, TN a right line let fall perpendicularly from the body to the axis; OP a line parallel and equal thereto drawn from the given point O in the axis; AP the path described by the point P in the plane AOP in which the revolving line OP is found; A the beginning of that path answering to the



point S ; TC a right line drawn from the body to the centre; TG a part thereof proportional to the centripetal force with which the body tends towards the centre C ; TM a right line perpendicular to the curved surface; TI a part thereof proportional to the force of pressure with which the body

urges the surface, and therefore with which it is again repelled by the surface towards M; PTF a right line parallel to the axis and passing through the body, and GF, IH right lines let fall perpendicularly from the points G and I upon that parallel PHTF. I say, now, that the area AOP, described by the radius OP from the beginning of the motion, is proportional to the time. For the force TG (by Cor. n of the Laws of Motion) is resolved into the forces TF, FG; and the force TI into the forces TH, HI; but the forces TF, TH, acting in the direction of the line PF perpendicular to the plane AOP, introduce no change in the motion of the body but in a direction perpendicular to that plane. Therefore its motion, so far as it hath the same direction with the position of the plane, that is, the motion of the point P, by which the projection AP of the curve is described in that plane, is the same as if the forces TF, TH were taken away, and the body were acted on by the forces FG, HI alone; that is, the same as if the body were to describe in the plane AOP the curve AP by means of a centripetal force tending to the centre ($\circ$), and equal to the sum of the forces FG and HI. But with such a force as that (by Prop, i) the area AOP will be described proportional to the time. Q.E.D.

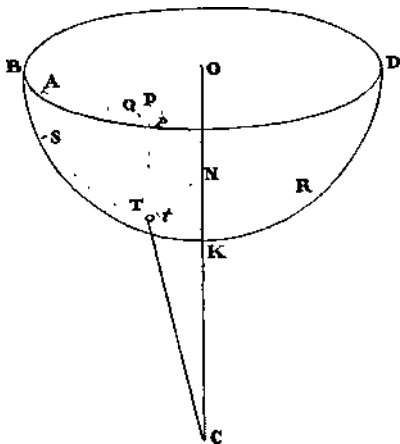
COR. By the same reasoning, if a body, acted on by forces tending to two or more centres in the same given right line CO, should describe in a free space any curved line ST, the area AOP would be always proportional to the time.

PROPOSITION LVI. PROBLEM XXXVII

Granting the quadratures of curvilinear figures, and supposing that there are given both the law of centripetal force tending to a given centre, and the curved surface whose axis passes through that centre; it is required to find the curve which a body will describe in that surface, when going off from a given place with a given velocity, and in a given direction in that surface.

The last construction remaining, let the body T go from the given place S, in the direction of a line given by position, and turn into the curve sought STR, whose orthographic projection in the plane BDO is AP. And from the given velocity of the body in the altitude SC, its velocity in any other alti-

tude TC will be also given. With that velocity, in a given moment of time, let the body describe the segment Tt of its curve and let Pp be the projection of that segment described in the plane AOP . Join Op , and a little circle being described upon the curved surface about the centre T with the radius Tt , let the projection of that little circle in the plane AOP be the ellipse pQ . And because the magnitude of that little circle Tt , and TN or PO its distance from the axis CO is also given, the ellipse pQ will be given both in kind and magnitude, as also its position to the right line PO . And since the area PQp is proportional to the time, and



therefore given because the time is given, the angle VOp will be given. And thence will be given p the common intersection of the ellipse and the right line Op , together with the angle OPp , in which the projection APp of the curve cuts the line OP . But from thence (by comparing Prop, XLI, with its Cor. 11) the manner of determining the curve AP easily appears. Then from the several points P of that projection erecting to the plane AOP , the perpendiculars PT meeting the curved surface in T , there will be given the several points T of the curve. Q.E.I.

SECTION XI

The motions of bodies tending to each other with centripetal forces.

I have hitherto been treating of the attractions of bodies towards an immovable centre; though very probably there is no such thing existent in nature. For attractions are made towards bodies, and the actions of the bodies attracted and attracting are always reciprocal and equal, by Law 111; so that if there are two bodies, neither the attracted nor the attracting body is truly at rest, but both (by Cor. iv of the Laws of Motion), being as it were mutually attracted, revolve about a common centre of gravity. And if there be more bodies, which either are attracted by one body, which is attracted by them again, or which all attract each other mutually, these bodies will be so moved among themselves, that their common centre of gravity will either be at rest, or move uniformly forwards in a right line. I shall therefore at present go on to treat of the motion of bodies attracting each other; considering the centripetal forces as attractions; though perhaps in a physical strictness they may more truly be called impulses. But these Propositions are to be considered as purely mathematical; and therefore, laying aside all physical considerations, I make use of a familiar way of speaking, to make myself the more easily understood by a mathematical reader.

PROPOSITION LVII. THEOREM XX

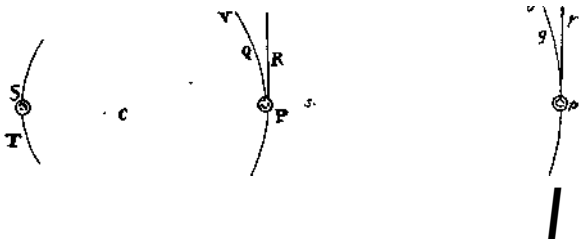
Two bodies attracting each other mutually describe similar figures about their common centre of gravity, and about each other mutually.

For the distances of the bodies from their common centre of gravity are inversely as the bodies; and therefore in a given ratio to each other; and thence, by composition of ratios, in a given ratio to the whole distance between the bodies. Now these distances are carried round their common extremity with an uniform angular motion, because lying in the same right line they never change their inclination to each other. But right lines that arc in a given ratio to each other, and arc carried round their extremities with an uniform angular motion, describe upon planes, which either rest together with them, or are moved with any motion not angular, figures entirely similar round those extremities. Therefore the figures described by the revolution of these distances are similar. Q.E.D.

PROPOSITION LVIII. THEOREM XXI

If two bodies attract each other with forces of any kind, and revolve about the common centre of gravity: I say, that, by the same forces, there may be described round either body unmoved a figure similar and equal to the figures which the bodies so moving describe round each other.

Let the bodies S and P revolve about their common centre of gravity C, proceeding from S to T, and from P to Q. From the given point *s* let there be continually drawn *sp*, *sq*, equal and parallel to SP, TQ; and the curve *pqv*, which the point *p* describes in its revolution round the fixed point *s*,



will be similar and equal to the curves which the bodies S and P describe about each other; and therefore, by Theor. xx, similar to the curves ST and PQV which the same bodies describe about their common centre of gravity C; and that because the proportions of the lines SC, CP, and SP or *sp*, to each other, are given.

CASF 1. The common centre of gravity C (by Cor. iv of the Laws of Motion) is either at rest, or moves uniformly in a right line. Let us first suppose it at rest, and in *s* and *p* let there be placed two bodies, one immovable in *s*, the other movable in *p*, similar and equal to the bodies S and P. Then let the right lines PR and *pr* touch the curves PQ and *pq* in P and *p*, and produce CQ and *sq* to R and *r*. And because the figures CPRQ, *sprq* are similar, RQ will be to *rq* as CP to *sp*, and therefore in a given ratio. Hence if the force with which the body P is attracted towards the body S, and by consequence towards the intermediate centre C, were to the force with which the body *p* is attracted towards the centre *s*, in the same given ratio, these forces would in equal times attract the bodies from the tangents PR, *pr* to the arcs PQ, *Pq*, through the intervals proportional to them RQ, *rq*; and therefore this last force (tending to *s*) would make the body *p* revolve in the curve *pqv*, which would become similar to the curve PQV, in which the first force

obliges the body P to revolve; and their revolutions would be completed in the same times. But because those forces are not to each other in the ratio of CP to sp , but (by reason of the similarity and equality of the bodies S and s , P and p , and the equality of the distances SP , sp) mutually equal, the bodies in equal times will be equally drawn from the tangents; and therefore that the body p may be attracted through the greater interval rq , there is required a greater time, which will vary as the square root of the intervals; because, by Lem. x, the spaces described at the beginning of the motion are as the square of the times. Suppose, then, the velocity of the body p to be to the velocity of the body P as the square root of the ratio of the distance ' p ' to the distance CP , so that the arcs pq , PQ , which are in a simple proportion to each other, may be described in times that are as the square root of the distances; and the bodies P , p , always attracted by equal forces, will describe round the fixed centres C and s similar figures PQV , pqv , the latter of which pqv is similar and equal to the figure which the body P describes round the movable body S . Q.E.D.

CASE 2. Suppose now that the common centre of gravity, together with the space in which the bodies are moved among themselves, proceeds uniformly in a right line; and (by Cor. vi of the Laws of Motion) all the motions in this space will be performed in the same manner as before; and therefore the bodies will describe about each other the same figures as before, which will be therefore similar and equal to the figure pqv . Q.E.D.

COR. 1. Hence two bodies attracting each other with forces proportional to their distance, describe (by Prop, x), both round their common centre of gravity, and round each other, concentric ellipses; and, conversely, if such figures are described, the forces are proportional to the distances.

COR. 11. And two bodies, whose forces are inversely proportional to the square of their distance, describe (by Prop, xi, xii, xiii), both round their common centre of gravity, and round each other, conic sections having their focus in the centre about which the figures are described. And, conversely, if such figures are described, the centripetal forces are inversely proportional to the square of the distance.

COR. II. Any two bodies revolving round their common centre of gravity describe areas proportional to the times, by radii drawn both to that centre and to each other.

PROPOSITION LIX. THEOREM XXII

The periodic time of two bodies S and P revolving round their common centre of gravity C, is to the periodic time of one of the bodies P revolving round the other S remaining fixed, and describing a figure similar and equal to those which the bodies describe about each other, as $\sqrt{S}$ it to $\sqrt{S+P}$.

For, by the demonstration of the last Proposition, the times in which any similar arcs PQ and pq are described are as $\sqrt{CP}$ is to $\sqrt{SP}$, or $\sqrt{sp}$, that is, as $\sqrt{S}$ is to $\sqrt{S+P}$. And by composition of ratios, the sums of the times in which all the similar arcs PQ and pq are described, that is, the whole times in which the whole similar figures are described, are in the same ratio, $\sqrt{S}$ to $\sqrt{S+P}$. Q.E.D.

PROPOSITION LX. THEOREM XXIII

If two bodies S and P, attracting each other with forces inversely proportional to the square of their distance, revolve about their common centre of gravity: I say, that the principal axis of the ellipse which either of the bodies, as P, describes by this motion about the other S, will be to the principal axis of the ellipse, which the same body P may describe in the same periodic time about the other body S fixed, as the sum of the two bodies S + P to the first of two mean proportionals¹ between that sum and the other body S.

For if the ellipses described were equal to each other, their periodic times by the last Theorem would be as the square root of the ratio of the body S to the sum of the bodies S + P. Let the periodic time in the latter ellipse be diminished in that ratio, and the periodic times will become equal; but, by Prop, xv, the principal axis of the ellipse will be diminished in a ratio which is the $3/2$ th power of the former ratio; that is, in a ratio to which the ratio of S to S + P is the cube, and therefore that axis will be to the principal axis of the other ellipse as the first of two mean proportionals between S + P and S to S + P. And inversely the principal axis of the ellipse described about the movable body will be to the principal axis of that described round the immovable as S + P to the first of two mean proportionals between S + P and S. Q.E.D.

[¹ Appendix, Note 24]

PROPOSITION LXI. THEOREM XXIV

If two bodies attracting each other with any kind of forces, and not otherwise agitated or obstructed, are moved in any manner whatsoever, those motions will be the same as if they did not at all attract each other, but were both attracted with the same forces by a third body placed in their common centre of gravity; and the law of the attracting forces will be the same in respect of the distance of the bodies from the common centre, as in respect of the distance between the two bodies.

For those forces with which the bodies attract each other, by tending to the bodies, tend also to the common centre of gravity lying directly between them; and therefore are the same as if they proceeded from an intermediate body. Q.E.D.

And because there is given the ratio of the distance of either body from that common centre to the distance between the two bodies, there is given, of course, the ratio of any power of one distance to the same power of the other distance; and also the ratio of any quantity derived in any manner from one of the distances compounded in any manner with given quantities, to another quantity derived in like manner from the other distance, and as many given quantities having that given ratio of the distances to the first. Therefore if the force with which one body is attracted by another be directly or inversely as the distance of the bodies from each other, or as any power of that distance; or, lastly, as any quantity derived after any manner from that distance compounded with given quantities; then will the same force with which the same body is attracted to the common centre of gravity be in like manner directly or inversely as the distance of the attracted body from the common centre, or as any power of that distance; or, lastly, as a quantity derived in like sort from that distance compounded with analogous given quantities. That is, the law of attracting force will be the same with respect to both distances. Q.E.D.

PROPOSITION LXII. PROBLEM XXXVIII

To determine the motions of two bodies which attract each other with forces inversely proportional to the squares of the distance between them, and are let fall from given places.

The bodies, by the last Theorem, will be moved in the same manner as if they were attracted by a third placed in the common centre of their

gravity; and by the hypothesis that centre will be fixed at the beginning of their motion, and therefore (by Cor. iv of the Laws of Motion) will be always fixed. The motions of the bodies are therefore to be determined (by Prob. xxv) in the same manner as if they were impelled by forces tending to that centre; and then we shall have the motions of the bodies attracting each other. Q.E.I.

PROPOSITION LXIII. PROBLEM XXXIX

To determine the motions of two bodies attracting each other with forces inversely proportional to the squares of their distance, and going off from given places in given directions with given velocities.

The motions of the bodies at the beginning being given, there is given also the uniform motion of the common centre of gravity, and the motion of the space which moves along with this centre uniformly in a right line, and also the very first, or beginning motions of the bodies in respect of this space. Then (by Cor. v of the Laws, and the last Theorem) the subsequent motions will be performed in the same manner in that space, as if that space together with the common centre of gravity were at rest, and as if the bodies did not attract each other, but were attracted by a third body placed in that centre. The motion therefore in this movable space of each body going off from a given place, in a given direction, with a given velocity, and acted upon by a centripetal force tending to that centre, is to be determined by Prob. ix and xxvi, and at the same time will be obtained the motion of the other round the same centre. With this motion compound the uniform progressive motion of the entire system of the space and the bodies revolving in it, and there will be obtained the absolute motion of the bodies in immovable space. Q.E.I.

PROPOSITION LXIV. PROBLEM XL

Supposing forces with which bodies attract each other to increase in a simple ratio of their distances from the centres; it is required to find the motions of several bodies among themselves.

Suppose the first two bodies T and L to have their common centre of gravity in D. These, by Cor. i, Theor. xxi, will describe ellipses having their centres in D, the magnitudes of which ellipses are known by Prob. v.

Let now a third body S attract the two former T and L with the accelerative forces ST, SL, and let it be attracted again by them. The force ST (by Cor. 11 of the Laws of Motion) is resolved into the forces SD, DT; and the force SL into the forces SD and DL. Now the forces DT, DL, which are as

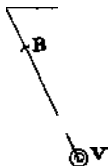
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their sum TL, and therefore as the accelerative forces with which the bodies T and L attract each other, added to the forces of the bodies T and L, the first to the first, and the last to the last, compose forces proportional to the distances DT and DL as before, but only greater than those former forces; and therefore (by Cor. 1, Prop. x, and Cor. 1 and vm, Prop. iv) they will cause those bodies to describe ellipses as before, but with a swifter motion. The remaining accelerative forces SD and DL, by the motive forces SD T and SD L, which are as the bodies attracting those bodies equally and in the direction of the lines TI, LK parallel to DS, do not at all change their situations with respect to one another, but cause them equally to approach to the line IK; which must be imagined drawn through the middle of the body S, and perpendicular to the line DS. But that approach to the line IK will be hindered by causing the system of the bodies T and L on one side, and the body S on the other, with proper velocities, to revolve round the common centre of gravity C. With such a motion the body S, because the sum of the motive forces SD T and SD L is proportional to the distance CS, tends to the centre C, and will describe an ellipse round that centre; and the point D, because the lines CS and CD are proportional, will describe a like ellipse over against it. Rut the bodies T and L, attracted by the motive forces SD • T and SD L, the first by the first, and the last by the last, equally and in the direction of the parallel lines TI and LK, as was said before, will (by Cor. v and vi of the Laws of Motion) continue to describe their ellipses round the movable centre D, as before. Q.E.I.

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Let there be added a fourth body V, and, by the like reasoning, it will be demonstrated that this body and the point C will describe ellipses about the

common centre of gravity B; the motions of the bodies T, L, and S round the centres D and C remaining the same as before, but accelerated. And by the same method one may add yet more bodies at pleasure. Q.E.I.

This would be the case, though the bodies T and L should attract each other with accelerative forces greater or less than those with which they attract the other bodies in proportion to their distance. Let all the accelerative attractions be to each other as the distances multiplied into the attracting bodies; and from what has gone before it will easily be concluded that all the bodies will describe different ellipses with equal periodic times about their common centre of gravity B, in an immovable plane. Q.L.I.

PROPOSITION LXV. THEOREM XXV

Bodies, whose forces decrease as the square of their distances from their centres, may move among themselves in ellipses; and by radii drawn to the foci may describe areas very nearly proportional to the times.

In the last Proposition we demonstrated that case in which the motions will be performed exactly in ellipses. The more distant the law of the forces is from the law in that case, the more will the bodies disturb each other's motions; neither is it possible that bodies attracting each other according to the law supposed in this Proposition should move exactly in ellipses, unless by keeping a certain proportion of distances from each other. However, in the following cases the orbits will not much differ from ellipses.

CASE I. Imagine several lesser bodies to revolve about some very great one at different distances from it, and suppose absolute forces tending to every one of the bodies proportional to each. And because (by Cor. w of the Laws) the common centre of gravity of them all is either at rest, or moves uniformly forwards in a right line, suppose the lesser bodies so small that the great body may be never at a sensible distance from that centre; and then the great body will, without any sensible error, be either at rest, or move uniformly forwards in a right line; and the lesser will revolve about that great one in ellipses, and by radii drawn thereto will describe areas proportional to the times; if we except the errors that may be introduced by the receding of the great body from the common centre of gravity, or by the actions of the lesser bodies upon each other. But the lesser bodies may be so far diminished, as that this recess and the actions of the bodies on

each other may become less than any assignable; and therefore so as that the orbits may become ellipses, and the areas answer to the times, without any error that is not less than any assignable. Q.E.O.

CASE 2. Let us imagine a system of lesser bodies revolving about a very great one in the manner just described, or any other system of two bodies revolving about each other, to be moving uniformly forwards in a right line, and in the meantime to be impelled sideways by the force of another vastly greater body situate at a great distance. And because the equal accelerative forces with which the bodies are impelled in parallel directions do not change the situation of the bodies with respect to each other, but only oblige the whole system to change its place while the parts still retain their motions among themselves, it is manifest that no change in those motions of the attracted bodies can arise from their attractions towards the greater, unless by the inequality of the accelerative attractions, or by the inclinations of the lines towards each other, in whose directions the attractions are made. Suppose, therefore, all the accelerative attractions made towards the great body to be among themselves inversely as the squares of the distances; and then, by increasing the distance of the great body till the differences of the right lines drawn from that to the others in respect of their length, and the inclinations of those lines to each other, be less than any given, the motions of the parts of the system will continue without errors that are not less than any given. And because, by the small distance of those parts from each other, the whole system is attracted as if it were but one body, it will therefore be moved by this attraction as if it were one body; that is, its centre of gravity will describe about the great body one of the conic sections (that is, a parabola or hyperbola when the attraction is but languid and an ellipse when it is more vigorous); and by radii drawn thereto, it will describe areas proportional to the times, without any errors but those which arise from the distances of the parts, and these are by the supposition exceedingly small, and may be diminished at pleasure. Q.E.O.

By a like reasoning one may proceed to more complicated cases *in infinitum*.

COR 1. In the second Case, the nearer the very great body approaches to the system of two or more revolving bodies, the greater will the perturbation be of the motions of the parts of the system among themselves; because

the inclinations of the lines drawn from that great body to those parts become greater; and the inequality of the proportion is also greater.

COR. 11. But the perturbation will be greatest of all, if we suppose the accelerative attractions of the parts of the system towards the greatest body of all are not to each other inversely as the squares of the distances from that great body; especially if the inequality of this proportion be greater than the inequality of the proportion of the distances from the great body. For if the accelerative force, acting in parallel directions and equally, causes no perturbation in the motions of the parts of the system, it must of course, when it acts unequally, cause a perturbation somewhere, which will be greater or less as the inequality is greater or less. The excess of the greater impulses acting upon some bodies, and not acting upon others, must necessarily change their situation among themselves. And this perturbation, added to the perturbation arising from the inequality and inclination of the lines, makes the whole perturbation greater.

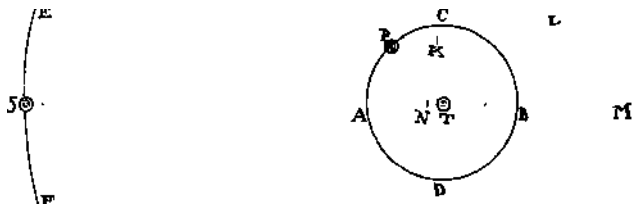
COR. in. Hence if the parts of this system move in ellipses or circles without any remarkable perturbation, it is manifest that, if they are at all impelled by accelerative forces tending to any other bodies, the impulse is very weak, or else is impressed very near equally and in parallel directions upon all of them.

PROPOSITION LXVI. THEOREM XXVI

If three bodies, whose forces decrease as the square of the distances, attract each other; and the accelerative attractions of any two towards the third be between themselves inversely as the squares of the distances; and the two least revolve about the greatest: I say, that the interior of the two revolving bodies will, by radii drawn to the innermost and greatest, describe round that body areas more proportional to the times, and a figure more approuching to that of an ellipse having its focus in the point of intersection of the radii, if that great body be agitated by those attractions, than it would do if that great body were not attracted at all by the lesser, but remained at rest; or than it would do if that great body were very much more or very much less attracted, or very much more or very much less agitated, by the attractions.

This appears plainly enough from the demonstration of the second Corollary of the foregoing Proposition; but it may be made out after this manner by a way of reasoning more distinct and more universally convincing.

CASE I. Let the lesser bodies P and S revolve in the same plane about the greatest body T, the body P describing the interior orbit PAB, and S the exterior orbit ESE. Let SK be the mean distance of the bodies P and S; and let the accelerative attraction of the body P towards S, at that mean distance, be expressed by that line SK. Make SL to SK as the square of SK to the square of SP, and SL will be the accelerative attraction of the body P towards S at any distance SP. Join PT, and draw LM parallel to it meet-



ing ST in M; and the attraction SL will be resolved (by Cor. n of the Laws of Motion) into the attractions SM, LM. And so the body P will be urged with a threefold accelerative force. One of these forces tends towards T, and arises from the mutual attraction of the bodies T and P. By this force alone the body P would describe round the body T, by the radius PT, areas proportional to the times, and an ellipse whose focus is in the centre of the body T; and this it would do whether the body T remained unmoved, or whether it were agitated by that attraction. This appears from Prop. xi, and Cor. n and in of Thcor. xxi. The other force is that of the attraction LM, which, because it tends from P to T, will be superadded to and coincide with the former force; and cause the areas to be still proportional to the times, by Cor. in, Thcor. xxi. But because it is not inversely proportional to the square of the distance PT, it will compose, when added to the former, a force varying from that proportion; this variation will be the greater by as much as the proportion of this force to the former is greater, other things remaining the same. Therefore, since by Prop. xi, and by Cor. II, Theor. xxi, the force with which the ellipse is described about the focus T ought to be directed to that

focus, and to be inversely proportional to the square of the distance PT , that compounded force varying from that proportion will make the orbit PAB vary from the figure of an ellipse that has its focus in the point T ; and so much the more by as much as the variation from that proportion is greater; and in consequence by as much as the proportion of the second force LM to the first force is greater, other things remaining the same. But now the third force SM , attracting the body P in a direction parallel to ST , composes with the other forces a new force which is no longer directed from P to T ; and this varies so much more from this direction by as much as the proportion of the third force to the other forces is greater, other things remaining the same; and therefore causes the body P to describe, by the radius TP , areas no longer proportional to the times; and therefore makes the variation from that proportionality so much greater by as much as the proportion of this force to the others is greater. But this third force will increase the variation of the orbit PAB from the elliptical figure before mentioned upon two accounts: first, because that force is not directed from P to T ; and, secondly, because it is not inversely proportional to the square of the distance PT . These things being premised, it is manifest that the areas are then most nearly proportional to the times, when that third force is the least possible, the rest preserving their *former* quantity; and that the orbit PAB does then approach nearest to the elliptical figure above mentioned, when both the second and third, but especially the third force, is the least possible; the first force remaining in its former quantity.

Let the accelerative attraction of the body T towards S be expressed by the line SN ; then if the accelerative attractions SM and SN were equal, these, attracting the bodies T and P equally and in parallel directions, would not at all change their situation with respect to each other. The motions *at* the bodies between themselves would be the same in that case as if those attractions did not act at all, by Cor. vi of the Laws of Motion. And, by a like reasoning, if the attraction SN is less than the attraction SM , it will take away out of the attraction SM the part SN , so that there will remain only the part (of the attraction) MN to disturb the proportionality of the areas and times, and the elliptical figure of the orbit. And in like manner if the attraction SN be greater than the attraction SM , the perturbation of the orbit and proportion will be produced by the difference MN alone.

After this manner the attraction SN reduces always the attraction SM to the attraction MN, the first and second attractions remaining perfectly unchanged; and therefore the areas and times come then nearest to proportionality, and the orbit PAB to the above-mentioned elliptical figure, when the attraction MN is either none, or the least that is possible; that is, when the accelerative attractions of the bodies P and T approach as near as possible to equality; that is, when the attraction SN is neither none at all, nor less than the least of all the attractions SM, but is, as it were, a mean between the greatest and least of all those attractions SM, that is, not much greater nor much less than the attraction SK. Q.E.D.

CASE 2. Let now the lesser bodies P, S revolve about a greater T in different planes; and the force LM, acting in the direction of the line PT situated in the plane of the orbit PAB, will have the same effect as before; neither will it draw the body P from the plane of its orbit. But the other force NM, acting in the direction of a line parallel to ST (and therefore, when the body S is without the line of the nodes, inclined to the plane of the orbit PAB), besides the perturbation of the motion just now spoken of as to longitude, introduces another perturbation also as to latitude, attracting the body P out of the plane of its orbit. And this perturbation, in any given situation of the bodies P and T to each other, will be as the generating force MN; and therefore becomes least when the force MN is least, that is (as was just now shown), where the attraction SN is not much greater nor much less than the attraction SK. Q.E.D.

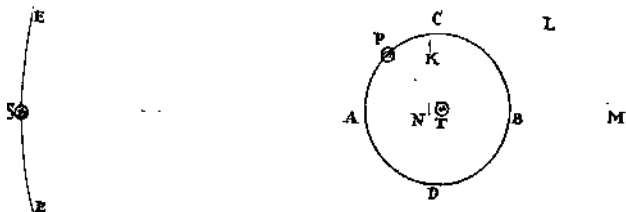
COR. 1. Hence it may be easily inferred, that if several less bodies P, S, R, &c, revolve about a very great body T, the motion of the innermost revolving body P will be least disturbed by the attractions of the others, when the great body is as well attracted and agitated by the rest (according to the ratio of the accelerative forces) as the rest are by each other.

COR. 11. In a system of three bodies T, P, S, if the accelerative attractions of any two of them towards a third be to each other inversely as the squares of the distances, the body P, by the radius PT, will describe its area about the body T swifter near the conjunction A and the opposition B than it will near the quadratures C and D. For every force with which the body P is acted on and the body T is not, and which does not act in the direction of the line PT, does either accelerate or retard the description of the area,

according as its direction is the same as, or contrary to that of the motion of the body. Such is the force NM. This force in the passage of the body P from C to A tends in the direction in which the body is moving, and therefore accelerates it; then as far as D, it tends in the opposite direction, and retards the motion; then in the direction of the body, as far as B; and lastly in a contrary direction, as it moves from B to C.

COR. in. And from the same reasoning it appears that the body P, other things remaining the same, moves more swiftly in the conjunction and opposition than in the quadratures.

COR. iv. The orbit of the body P, other things remaining the same, is more curved at the quadratures than at the conjunction and opposition. For the swifter bodies move, the less they deflect from a rectilinear path. And besides, the force KL, or NM, at the conjunction and opposition, is contrary to the force with which the body T attracts the body P, and therefore diminishes that force; but the body P will deflect the less from a rectilinear path the less it is impelled towards the body T.

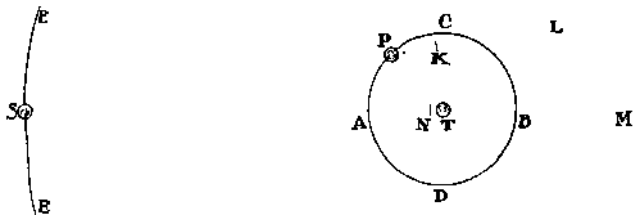


COR. v. Hence the body P, other things remaining the same, goes farther from the body T at the quadratures than at the conjunction and opposition. This is said, however, when no account is taken of the variable eccentricity. For if the orbit of the body P be eccentric, its eccentricity (as will be shown presently by Cor. ix) will be greatest when the apsides are in the syzygies; and thence it may sometimes come to pass that the body P, in its near approach to the farther apse, may go farther from the body T at the syzygies than at the quadratures.

COR. vi. Because the centripetal force of the central body T, by which the body P is retained in its orbit, is increased at the quadratures by the addition caused by the force LM, and diminished at the syzygies by the subtraction

of the force KL , and, because the force KL is greater than LM , it is more diminished than increased; and, moreover, since that centripetal force (by Cor. 11, Prop. iv) varies directly as the radius TP , and inversely as the square of the periodical time, it is plain that the resulting ratio is diminished by the action of the force KL ; and therefore that the periodical time, supposing the radius of the orbit PT to remain the same, will be increased, and that as the square root of that ratio in which the centripetal force is diminished; and, therefore, supposing this radius increased or diminished, the periodical time will be increased more or diminished less than in the $3/2$ th power of this radius, by Cor. vi, Prop. iv. If that force of the central body should gradually decay, the body P being less and less attracted would go farther and farther from the centre T ; and, on the contrary, if it were increased, it would draw nearer to it. Therefore if the action of the distant body S , by which that force is diminished, were to increase and decrease by turns, the radius TP would be also increased and diminished by turns; and the periodical time would be increased and diminished in a ratio compounded of the $3/2$ th power of the ratio of the radius, and of the square root of that ratio in which the centripetal force of the central body T was diminished or increased, by the increase or decrease of the action of the distant body S .

COR vii. It also follows, from what was before laid down, that the axis of the ellipse described by the body P , or the line of the apsides, does as to its angular motion go forwards and backwards by turns, but more forwards than backwards, and by the excess of its direct motion is on the whole



carried forwards. For the force with which the body P is urged to the body T at the quadratures, where the force MN vanishes, is compounded of the force LM and the centripetal force with which the body T attracts the body

P. The first force LM , if the distance PT be increased, is increased in nearly the same proportion with that distance, and the other force decreases as the square of the ratio of the distance; and therefore the sum of these two forces decreases in less than the square of the ratio of the distance PT ; and therefore, by Cor. I, Prop. XLV, will make the line of the apsides, or, which is the same thing, the upper apse, to go backwards. But at the conjunction and opposition the force with which the body P is urged towards the body T is the difference of the force KL , and of the force with which the body T attracts the body P ; and that difference, because the force KL is very nearly increased in the ratio of the distance PT , decreases in more than the square of the ratio of the distance PT ; and therefore, by Cor. i, Prop. xi/v, causes the line of the apsides to go forwards. In the places between the syzygies and the quadratures, the motion of the line of the apsides depends upon both of these causes conjointly, so that it either goes forwards or backwards in proportion to the excess of one of these causes above the other. Therefore since the force KL in the syzygies is almost twice as great as the force LM in the quadratures, the excess will be on the side of the force KL , and by consequence the line of the apsides will be carried forwards. The truth of this and the foregoing Corollary will be more easily understood by conceiving the system of the two bodies T and P to be surrounded on every side by several bodies S, S, S , &c, disposed about the orbit $EwSE$. For by the actions of these bodies the action of the body T will be diminished on every side, and decrease in more than the square of the ratio of the distance.

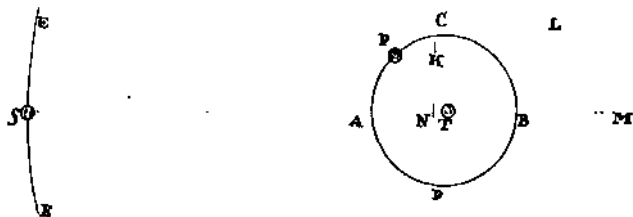
COR. VIII. But since the direct or retrograde motion of the apsides depends upon the decrease of the centripetal force, that is, upon its being in a greater or less ratio than the square of the ratio of the distance TP , in the passage of the body from the lower apse to the upper; and upon a like increase in its return to the lower apse again; and therefore becomes greatest where the proportion of the force at the upper apse to the force at the lower apse recedes farthest from the inverse square of the ratio of the distances; it is plain that, when the apsides are in the syzygies, they will, by reason of the subtracted force KL or $NM - LM$, go forwards more swiftly; and in the quadratures by the additional force LM go backwards more slowly. Because the velocity of the progression or the slowness of the retrogression is continued for a long time, this inequality becomes exceedingly great.

COR. IX. If a body is obliged, by a force inversely proportional to the square of its distance from any centre, to revolve in an ellipse round that centre; and afterwards in its descent from the upper apse to the lower apse, that force by a continual accession of new force is increased in more than the square of the ratio of the diminished distance; it is manifest that the body, being impelled always towards the centre by the continual accession of this new force, will incline more towards that centre than if it were urged by that force alone which decreases as the square of the diminished distance, and therefore will describe an orbit interior to that elliptical orbit, and at the lower apse approaching nearer to the centre than before. Therefore the orbit by the accession of this new force will become more eccentric. If now, while the body is returning from the lower to the upper apse, it should decrease by the same degrees by which it increased before, the body would return to its first distance; and therefore if the force decreases in a yet greater ratio, the body, being now less attracted than before, will ascend to a still greater distance, and so, the eccentricity of the orbit will be increased still more. Therefore if the ratio of the increase and decrease of the centripetal force be augmented with each revolution, the eccentricity will be augmented also; and, on the contrary, if that ratio decrease, it will be diminished.

Now, therefore, in the system of the bodies T, P, S, when the apsides of the orbit PAR are in the quadratures, the ratio of that increase and decrease is least of all, and becomes greatest when the apsides are in the syzygies. If the apsides are placed in the quadratures, the ratio near the apsides is less, and near the syzygies greater, than the square of the ratio of the distances; and from that greater ratio arises a direct motion of the line of the apsides, as was just now said. But if we consider the ratio of the whole increase or decrease in the progress between the apsides, this is less than the square of the ratio of the distances. The force in the lower is to that in the upper apse in less than the square of the ratio of the distance of the upper apse from the focus of the ellipse to the distance of the lower apse from the same focus; and conversely, when the apsides are placed in the syzygies, the force in the lower apse is to the force in the upper apse in a greater than the square of the ratio of the distances. For the forces LM in the quadratures added to the forces of the body T, compose forces in a less ratio; and the

forces KL in the syzygies subtracted from the forces of the body T, leave the forces in a greater ratio. Therefore the ratio of the whole increase and decrease in the passage between the apsides is least at the quadratures and greatest at the syzygies; and therefore in the passage of the apsides from the quadratures to the syzygies it is continually augmented, and increases the eccentricity of the ellipse; and in the passage from the syzygies to the quadratures it is continually decreasing, and diminishes the eccentricity.

COR. x. That we may give an account of the errors of latitude, let us suppose the plane of the orbit EST to remain immovable; and from the cause of the errors above explained, it is manifest that, of the two forces NM, ML, which are the only and entire cause of them, the force ML acting always in the plane of the orbit PAB never disturbs the motions as to latitude; and that the force NM, when the nodes are in the syzygies, acting also in the same plane of the orbit, does not at that time affect those motions. But when the nodes are in the quadratures, it disturbs them very much,



and, attracting the body P continually out of the plane of its orbit, it diminishes the inclination of the plane in the passage of the body from the quadratures to the syzygies, and again increases the same in the passage from the syzygies to the quadratures. Hence it comes to pass that when the body is in the syzygies, the inclination is then least of all, and returns to the first magnitude nearly, when the body arrives at the next node. But if the nodes are situated at the octants after the quadratures, that is, between C and A, D and B, it will appear, from what was just now shown, that in the passage of the body P from either node to the ninetieth degree from thence, the inclination of the plane is continually diminished; then, in the passage through the next 45 degrees to the next quadrature, the inclination is in-

creased; and afterwards, again, in its passage through another 45 degrees to the next node, it is diminished. Therefore the inclination is more diminished than increased, and is therefore always less in the subsequent node than in the preceding one. And, by a like reasoning, the inclination is more increased than diminished when the nodes are in the other octants between A and D, B and C. The inclination, therefore, is the greatest of all when the nodes are in the syzygies. In their passage from the syzygies to the quadratures the inclination is diminished at each appuls of the body to the nodes; and becomes least of all when the nodes are in the quadratures, and the body in the syzygies; then it increases by the same degrees by which it decreased before; and, when the nodes come to the next syzygies, returns to its former magnitude.

COR. XI. Because when the nodes are in the quadratures the body P is continually attracted from the plane of its orbit; and because this attraction is made towards S in its passage from the node C through the conjunction A to the node I); and in the opposite direction in its passage from the node D through the opposition B to the node C; it is manifest that, in its motion from the node C, the body recedes continually from the former plane CD of its orbit till it comes to the next node; and therefore at that node, being now at its greatest distance from the first plane CD, it will pass through the plane of the orbit FST not in D, the other node of that plane, but in a point that lies nearer to the body S, which therefore becomes a new place of the node behind its former place. And, by a like reasoning, the nodes will continue to recede in their passage from this node to the next. The nodes, therefore, when situated in the quadratures, recede continually; and at the syzygies, where no perturbation can be produced in the motion as to latitude, are quiescent; in the intermediate places they partake of both conditions, and recede more slowly; and, therefore, being always either retrograde or stationary, they will be carried backwards, or made to recede in each revolution.

Cor. xii. All the errors described in these Corollaries are a little greater at the conjunction of the bodies P, S than at their opposition; because the generating forces NM and ML are greater.

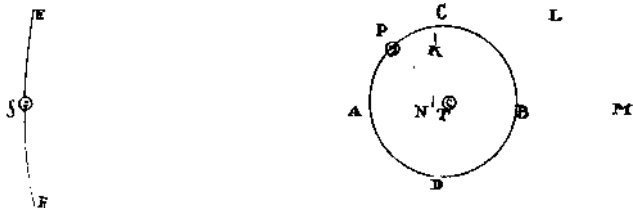
COR. xiii. And since the causes and proportions of the errors and variations mentioned in these Corollaries do not depend upon the magnitude of

the body S, it follows that all things before demonstrated will happen, if the magnitude of the body S be imagined so great that the system of the two bodies P and T may revolve about it. And from this increase of the body S, and the consequent increase of its centripetal force, from which the errors of the body P arise, it will follow that all these errors, at equal distances, will be greater in this case, than in the other where the body S revolves about the system of the bodies P and T.

COR. XIV. But since the forces NM, ML, when the body S is exceedingly distant, are very nearly as the force SK and the ratio PT to ST conjointly; that is, if both the distance PT and the absolute force of the body S be given, inversely as ST'' ; and since those forces NM, ML are the causes of all the errors and effects treated of in the foregoing Corollaries; it is manifest that all those effects, if the system of bodies T and P continue as before, and only the distance ST and the absolute force of the body S be changed, will be very nearly in a ratio compounded of the direct ratio of the absolute force of the body S, and the cubed inverse ratio of the distance ST. Hence if the system of bodies T and P revolve about a distant body S, those forces NM, ML, and their effects, will be (by Cor. 11 and vi, Prop. IV) inversely as the square of the periodical time. And thence, also, if the magnitude of the body S be proportional to its absolute force, those forces NM, ML, and their effects, will be directly as the cube of the apparent diameter of the distant body S viewed from T; and conversely. For these ratios are the same as the compounded ratio above mentioned.

COR. xv. If the orbits ESE and PAB, retaining their figure, proportions, and inclination to each other, should alter their magnitude, and if the forces of the bodies S and T should either remain unaltered or be changed in any given ratio, then these forces (that is, the force of the body T, which obliges the body P to deflect from a rectilinear course into the orbit PAB, and the force of the body S, which causes the body P to deviate from that orbit) will act always in the same manner, and in the same proportion. Consequently it follows, that all the effects will be similar and proportional, and the times of those effects will be proportional also; that is, that all the linear errors will be as the diameters of the orbits, the angular errors the same as before; and the times of similar linear errors, or equal angular errors, are as the periodical times of the orbits.

COR. XVI. Therefore if the figures of the orbits and their inclination to each other be given, and the magnitudes, forces, and distances of the bodies be changed in any manner, we may, from the errors and times of those errors in one case, obtain very nearly the errors and times of the errors in any other case. Rut this may be done more expeditiously by the following method. The forces NM , ML , other things remaining unaltered, are as the radius TP ; and their periodical effects (by Cor. 11, Lem. x) are as the forces and the square of the periodical time of the body P jointly. These are the linear errors of the body P ; and hence the angular errors as they appear from the centre T (that is, the motion of the apsides and of the nodes, and all the apparent errors of longitude and latitude) are in each revolution of the body P as the square of the time of the revolution, very nearly. Let these ratios be compounded with the ratios in Cor. xiv, and in any system of bodies T , P , S , where P revolves about T very near to it, and T revolves about S at a great distance, the angular errors of the body P , observed from



the centre T , will be in each revolution of the body P directly as the square of the periodical time of the body P , and inversely as the square of the periodical time of the body T . And therefore the mean motion of the line of the apsides will be in a given ratio to the mean motion of the nodes; and both those motions will be ductly as the periodical time of the body P , and inversely as the square of the periodical time of the body T . The increase or diminution of the eccentricity and inclination of the orbit PAB makes no sensible variation in the motions of the apsides and nodes, unless that increase or diminution be very great indeed.

COR. XVII. Since the line LM becomes sometimes greater and sometimes less than the radius PT , let the mean quantity of the force LM be expressed by that radius PT ; and then diat mean force will be to the mean force SK

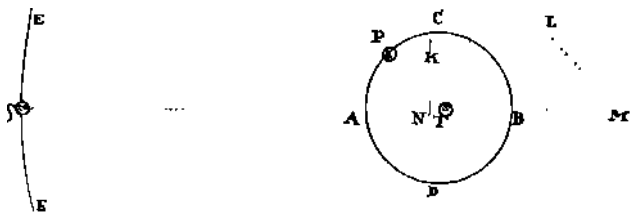
or SN (which may be also expressed by ST) as the length PT to the length ST. But the mean force SN or ST, by which the body T is retained in the orbit it describes about S, is to the force with which the body P is retained in its orbit about T in a ratio compounded of the ratio of the radius ST to the radius PT, and the squared ratio of the periodical time of the body P about T to the periodical time of the body T about S. And, consequently, the mean force LM is to the force by which the body P is retained in its orbit about T (or by which the same body P might revolve at the distance PT in the same periodical time about any immovable point T) in the same squared ratio of the periodical times. The periodical times therefore being given, together with the distance PT, the mean force LM is also given; and that force being given, there is given also the force MN, very nearly, by the analogy of the lines PT and MN.

COR. XVIII. By the same laws by which the body P revolves about the body T, let us suppose many fluid bodies to move round T at equal distances from it; and to be so numerous, that they may all become contiguous to each other, so as to form a fluid annulus, or ring, of a round figure, and concentric to the body T; and the several parts of this ring, performing their motions by the same law as the body P, will draw nearer to the body T, and move swifter in the conjunction and opposition of themselves and the body S, than in the quadratures. And the nodes of this ring or its intersections with the plane of the orbit of the body S or T, will rest at the syzygies; but out of the syzygies they will be carried backwards, or in a retrograde direction, with the greatest swiftness in the quadratures, and more slowly in other places. The inclination of this ring also will vary, and its axis will oscillate in each revolution, and when the revolution is completed will return to its former situation, except only that it will be carried round a little by the precession of the nodes.

COR. XIX. Suppose now the spherical body T, consisting of some matter not fluid, to be enlarged, and to extend itself on every side as far as that ring, and that a channel were cut all round its circumference containing water; and that this sphere revolves uniformly about its own axis in the same periodical time. This water being accelerated and retarded by turns (as in the last Corollary), will be swifter at the syzygies, and slower at the quadratures, than the surface of the globe, and so will ebb and flow in its channel

after the manner of the sea. If the attraction of the body S were taken away, the water would acquire no motion of flux and reflux by revolving round the quiescent centre of the globe. The case is the same of a globe moving uniformly forwards in a right line, and in the meantime revolving about its centre (by Cor. v of the Laws of Motion), and of a globe uniformly attracted from its rectilinear course (by Cor. vi of the same Laws). But let the body S come to act upon it, and by its varying attraction the water will receive this new motion; for there will be a stronger attraction upon that part of the water that is nearest to the body, and a weaker upon that part which is more remote. And the force LM will attract the water downwards at the quadratures, and depress it as far as the syzygies; and the force KL will attract it upwards in the syzygies, and withhold its descent, and make it rise as far as the quadratures; except only so far as the motion of flux and reflux may be directed by the channel, and be a little retarded by friction.

COR. xx. If now, the ring becomes hard, and the globe is diminished, the motion of flux and reflux will cease; but the oscillating motion of the inclination and the precession of the nodes will remain. Let the globe have the same axis with the ring, and perform its revolutions in the same times, and at its surface touch the ring within, and adhere to it; then the globe partaking of the motion of the ring, this whole body will oscillate, and the nodes will go backwards, for the globe, as we shall show presently, is perfectly



indifferent to the receiving of all impressions. The greatest angle of the inclination of the ring alone is when the nodes are in the syzygies. Thence in the progress of the nodes to the quadratures, it endeavors to diminish its inclination, and by that endeavor impresses a motion upon the whole globe. The globe retains this motion impressed, till the ring by a contrary endeavor destroys that motion, and impresses a new motion in a contrary direction.

And by this means the greatest motion of the decreasing inclination happens when the nodes are in the quadratures, and the least angle of inclination in the octants after the quadratures; and, again, the greatest motion of the reclination happens when the nodes are in the syzygies; and the greatest angle of inclination in the octants following. And the case is the same of a globe without this ring, if it be a little higher or a little denser in the equatorial than in the polar regions; for the excess of that matter in the regions near the equator supplies the place of the ring. And although we should suppose the centripetal force of this globe to be increased in any manner, so that all its parts tend downwards, as the parts of our earth gravitate to the centre, yet the phenomena of this and the preceding Corollary would scarce be altered; except that the places of the greatest and least height of the water will be different; for the water is now no longer sustained and kept in its orbit by its centrifugal force, but by the channel in which it flows. And, besides, the force LM attracts the water downwards most in the quadratures, and the force KL or N M - L M attracts it upwards most in the syzygies. And these forces conjoined cease to attract the water downwards, and begin to attract it upwards in the octants before the syzygies; and cease to attract the water upwards, and begin to attract the water downwards in the octants after the syzygies. And thence the greatest height of the water may happen about the octants after the syzygies; and the least height about the octants after the quadratures; excepting only so far as the motion of ascent or descent impressed by these forces may by the inertia of the water continue a little longer, or be stopped a little sooner by impediments in its channel.

COR. xxi. For the same reason that redundant matter in the equatorial regions of a globe causes the nodes to go backwards, and therefore by the increase of that matter that retrograde motion is increased, by the diminution is diminished, and by the removal quite ceases; it follows, that, if more than that redundant matter be taken away, that is, if the globe be either more depressed, or of a rarer consistence near the equator than near the poles, there will arise a direct motion of the nodes.

COR. xxii. And thence from the motion of the nodes is known the constitution of the globe. That is, if the globe retains unalterably the same poles, and the motion (of the nodes) is retrograde, there is a redundancy of the

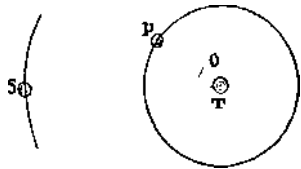
matter near the equator; but if that motion is direct, a deficiency. Suppose a uniform and exactly spherical globe to be first at rest in a free space; then by some impulse made obliquely upon its surface to be driven from its place, and to receive a motion partly circular and partly straight forward. Since this globe is perfectly indifferent to all the axes that pass through its centre, nor has a greater propensity to one axis or to one situation of the axis than to any other, it is manifest that by its own force it will never change its axis, or the inclination of its axis. Let now this globe be impelled obliquely by a new impulse in the same part of its surface as before; and since the effect of an impulse is not at all changed by its coming sooner or later, it is manifest that these two impulses, successively impressed, will produce the same motion, as if they had been impressed at the same time; that is, the same motion, as if the globe had been impelled by a simple force compounded of them both (by Cor. II of the Laws), that is, a simple motion about an axis of a given inclination. And the case is the same if the second impulse were made upon any other place of the equator of the first motion; and also if the first impulse were made upon any place in the equator of the motion which would be generated by the second impulse alone; and therefore, also, when both impulses are made in any places whatsoever; for these impulses will generate the same circular motion as if they were impressed together, and at once, in the place of the intersections of the equators of those motions, which would be generated by each of them separately. Therefore, a homogeneous and perfect globe will not retain several motions distinct, but will unite all those that are impressed on it, and reduce them into one; revolving, as far as in it lies, always with a simple and uniform motion about one single given axis, with an inclination always invariable. And the inclination of the axis, or the velocity of the rotation, will not be changed by centripetal force. For if the globe be supposed to be divided into two hemispheres, by any plane whatsoever passing through its own centre, and the centre to which the force is directed, that force will always urge each hemisphere equally; and therefore will not incline the globe to any side with respect to its motion round its own axis. But let there be added anywhere between the pole and the equator a heap of new matter like a mountain, and this, by its continual endeavor to recede from the centre of its motion, will disturb the motion of the globe, and

cause its poles to wander about its surface describing circles about themselves and the points opposite to them. Neither can this enormous deviation of the poles be corrected otherwise than by placing that mountain either in one of the poles, in which case, by Cor. xxi, the nodes of the equator will go forwards; or in the equatorial regions, in which case, by Cor. xx, the nodes will go backwards; or, lastly, by adding on the other side of the axis a new quantity of matter, by which the mountain may be balanced in its motion; and then the nodes will either go forwards or backwards, as the mountain and this newly added matter happen to be nearer to the pole or to the equator.

PROPOSITION LXVII. THEOREM XXVII

The same laws of attraction being supposed, I say, that the exterior body S does, by radii drawn to the point O, the common centre of gravity of the interior bodies P and T, describe round that centre areas more proportional to the times, and an orbit more approaching to the form of an ellipse having its focus in that centre, than it can describe round the innermost and greatest body T by radii drawn to that body.

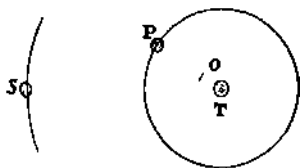
For the attractions of the body S towards T and P compose its absolute attraction, which is more directed towards O, the common centre of gravity of the bodies T and P, than it is to the greatest body T; and which approaches nearer to the inverse proportion of the square of the distance SO, than of the square of the distance ST; as will easily appear by a little consideration.



PROPOSITION LXVIII. THEOREM XXVIII

The same laws of attraction supposed, I say, that the exterior body S will, by radii drawn to O, the common centre of gravity of the interior bodies P and T, describe round that centre areas more proportional to the times, and an orbit more approaching to the form of an ellipse having its focus in that centre, if the innermost and greatest body be agitated by these attractions as well as the rest, than it would do if that body either were at rest and not attracted at all, or were much more or much less attracted, or were much more or much less agitated.

This may be demonstrated after the same manner as Prop, LXVI, but by a more prolix reasoning, which I therefore pass over. It will be sufficient to consider it after this manner. From the demonstration of the last Proposition it is plain, that the centre, towards which the body S is urged by the two forces conjointly, is very near to the common centre of gravity of those two other bodies. If this centre were to coincide with that common centre, and moreover the common centre of gravity of all the three bodies were at rest, the body S on one side, and the common centre of gravity of the other two bodies on the other side, would describe true ellipses about that quiescent common centre. This appears from Cor. 11, Prop, LVIII, compared



with what was demonstrated in Prop. LXIV, and LXV. Now this accurate elliptical motion will be disturbed a little by the distance of the centre of the two bodies from the centre towards which the third body S is attracted. Let there be added, moreover, a motion to the

common centre of the three, and the perturbation will be increased yet more. Therefore the perturbation is least when the common centre of the three bodies is at rest; that is, when the innermost and greatest body T is attracted according to the same law as the rest are; and is always greatest when the common centre of the three, by the diminution of the motion of the body T, begins to be moved, and is more and more agitated.

COR. And hence if several smaller bodies revolve about the great one, it may easily be inferred that the orbits described will approach nearer to ellipses; and the descriptions of areas will be more nearly uniform, if all the bodies attract and agitate each other with accelerative forces that are directly as their absolute forces, and inversely as the squares of the distances, and if the focus of each orbit be placed in the common centre of gravity of all the interior bodies (that is, if the focus of the first and innermost orbit be placed in the centre of gravity of the greatest and innermost body; the focus of the second orbit in the common centre of gravity of the two innermost bodies; the focus of the third orbit in the common centre of gravity of the three innermost; and so on), than if the innermost body were at rest, and was made the common focus of all the orbits.

PROPOSITION LXIX. THEOREM XXIX

In a system of several bodies A, B, C, D, &c, if any one of those bodies, as A, attract all the rest, B, C, D, &c, with accelerative forces that are inversely as the squares of the distances from the attracting body; and another body, as B, attracts also the rest, A, C, D, &c, with forces that are inversely as the squares of the distances from the attracting body; the absolute forces of the attracting bodies A and B will be to each other as those very bodies A and B to which those forces belong.

For the accelerative attractions of all the bodies B, C, I), towards A, are by the supposition equal to each other at equal distances; and in like manner the accelerative attractions of all the bodies towards B are also equal to each other at equal distances. But the absolute attractive force of the body A is to the absolute attractive force of the body B as the accelerative attraction of all the bodies towards A is to the accelerative attraction of all the bodies towards B at equal distances; and so is also the accelerative attraction of the body B towards A to the accelerative attraction of the body A towards B. But the accelerative attraction of the body B towards A is to the accelerative attraction of the body A towards B as the mass of the body A is to the mass of the body B; because the motive forces which (by the second, seventh and eighth Definitions) are as the accelerative forces and the bodies attracted conjointly are here equal to one another by the third Law. Therefore the absolute attractive force of the body A is to the absolute attractive force of the body B as the mass of the body A is to the mass of the body B. Q.E.D.

COR. i. Therefore if each of the bodies of the system A, B, C, D, &c, does singly attract all the rest with accelerative forces that are inversely as the squares of the distances from the attracting body, the absolute forces of all those bodies will be to each other as the bodies themselves.

COR. 11. By a like reasoning, if each of the bodies of the system A, B, C, D, &c, does singly attract all the rest with accelerative forces, which are either inversely or directly in the ratio of any power whatever of the distances from the attracting body; or which are defined by the distances from each of the attracting bodies according to any common law; it is plain that the absolute forces of those bodies are as the bodies themselves.

COR. HI. In a system of bodies whose forces decrease as the square of the distances, if the lesser revolve about one very great one in ellipses, having their common focus in the centre of that great body, and of a figure exceedingly accurate; and moreover by radii drawn to that great body describe areas proportional to the times exactly; the absolute forces of those bodies to each other will be either accurately or very nearly in the ratio of the bodies. And so conversely. This appears from Cor. of Prop, XLVIII, compared with the first Corollary of this Proposition.

SCHOLIUM

These Propositions naturally lead us to the analogy there is between centripetal forces and the central bodies to which those forces are usually directed; for it is reasonable to suppose that forces which are directed to bodies should depend upon the nature and quantity of those bodies, as we see they do in magnectical experiments. And when such cases occur, we are to compute the attractions of the bodies by assigning to each of their particles its proper force, and then finding the sum of them all. I here use the word *attraction* in general for any endeavor whatever, made by bodies to approach to each other, whether that endeavor arise from the action of the bodies themselves, as tending to each other or agitating each other by spirits emitted; or whether it arises from the action of the ether or of the air, or of any medium whatever, whether corporeal or incorporeal, in any manner impelling bodies placed therein towards each other. In the same general sense I use the word *impulse*, not defining in this treatise the species or physical qualities of forces, but investigating the quantities and mathematical proportions of them; as I observed before in the Definitions. In mathematics we are to investigate the quantities of forces with their proportions consequent upon any conditions supposed; then, when we enter upon physics, we compare those proportions with the phenomena of Nature, that we may know what conditions of those forces answer to the several kinds of attractive bodies. And this preparation being made, we argue more safely concerning the physical species, causes, and proportions of the forces. Let us see, then, with what forces spherical bodies consisting of particles endued with attractive powers in the manner above spoken of must act upon one another; and what kind of motions will follow from them.

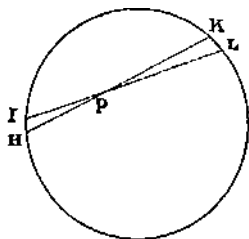
SECTION XII

The attractive forces of spherical bodies.

PROPOSITION LXX. THEOREM XXX

If to every point of a spherical surface there tend equal centripetal forces decreasing as the square of the distances from those points, I say, that a corpuscle placed within that surface will not be attracted by those forces any way.

Let $HIKL$ be that spherical surface, and P a corpuscle placed within. Through P let there be drawn to this surface two lines HK , IL , intercepting very small arcs HI , KL ; and because (by Cor. in, Lem. vii) the triangles HPI , LPK are alike, those arcs will be proportional to the distances HP , LP ; and any particles at HI and KL of the spherical surface, terminated by right lines passing through P , will be as the square of those distances. Therefore the forces of these particles exerted upon the body P are equal between themselves. For the forces are directly as the particles, and inversely as the square of the distances. And these two ratios compose the ratio of equality, 1:1. The attractions therefore, being equal, but exerted in opposite directions, destroy each other. And by a like reasoning all the attractions through the whole spherical surface are destroyed by contrary attractions. Therefore the body P will not be any way impelled by those attractions. Q.E.D.

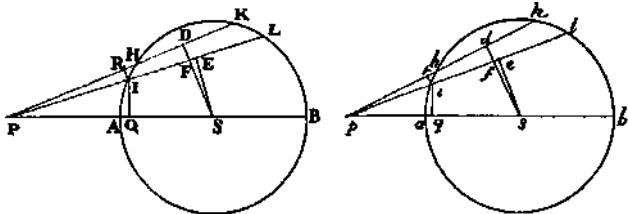


PROPOSITION LXXI. THEOREM XXXI

The same things supposed as above, I say, that a corpuscle placed without the spherical surface is attracted towards the centre of the sphere with a force inversely proportional to the square of its distance from that centre.

Let $AHKB$, $ahkb$ be two equal spherical surfaces described about the centres S , s ; their diameters AB , ab ; and let P and p be two corpuscles situate

without the spheres in those diameters produced. Let there be drawn from the corpuscles the lines PHK , PL , phk , pil , cutting off from the great circles AHB , ahb , the equal arcs HK , hk , IL , i ; and to those lines let fall the perpendiculars SD , sd , SE , sc , IR , ir ; of which let SD , sd , cut PL , pl , in F and f .



Let fall also to the diameters the perpendiculars IQ , iq . Let now the angles DPE , dpc vanish; and because DS and ds , ES and es are equal, the lines PE , PF , and pe , pf , and the short lines DF , df may be taken for equal; because their last ratio, when the angles DPE , dpc vanish together, is the ratio of equality. These things being thus determined, it follows that

$$PI : PF - RI : DF$$

and $pf : pi = df$ or $DF : rt$.

Multiplying corresponding terms,

$$PI \cdot pf : PF \cdot pi - RI : ri - \text{arc } IH : \text{arc } ih \text{ (by Cor. III, Lem. vn).}$$

Again, $PI : PS = IQ : SE$

and $ps : pi - se$ or $SE : iq$.

Hence, $PI \cdot ps : VS \cdot pi = IQ : iq$.

Multiplying together corresponding terms of this and the similarly derived preceding proportion,

$$PI^2 \cdot pf \cdot ps : pr \cdot 2PFPS \cdot HI \quad IQ : th - iq,$$

that is, as the circular surface which is described by the arc IH , as the semicircle AKB revolves about the diameter AB , is to the circular surface described by the arc th as the semicircle akb revolves about the diameter ab . And the forces with which these surfaces attract the corpuscles P and p in the direction of lines tending to those surfaces are directly, by the hypothesis, as the surfaces themselves, and inversely as the squares of the distances of the surfaces from those corpuscles; that is, as $pf \cdot ps$ to $PF \cdot PS$. And these

forces again are to the oblique parts of them which (by the resolution of forces as in Cor. 11 of the Laws) tend to the centres in the directions of the lines PS, *ps*, as PI to PQ, and *pi* to *pq*; that is (because of the like triangles PIQ and PSF, *piq* and *psf*), as PS to PF and *ps* to *pf*. Thence, the attraction of the corpuscle P towards S is to the attraction of the corpuscle

p towardssas $\frac{PF-pf-ps}{PS}$. $\frac{PfPF}{ps}$ is to —————, that is, as *ps* to PS". And, by

a like reasoning, the forces with which the surf ices described by the revolution of the arcs KL, *kl* attract those corpuscles, will be as *pi* to PS². And in the same ratio will be the forces of all the circular surfaces into which each of the spherical surfaces may be divided by taking *sd* always equal to SD, and *se* equal to SE. And therefore, by composition, the forces of the entire spherical surfaces exerted upon those corpuscles will be in the same ratio. Q.E.D.

PROPOSITION LXXII. THEOREM XXXII

If to the several points of a sphere there tend equal centripetal forces decreasing as the square of the distances from those points; and there be given both the density of the sphere and the ratio of the diameter of the sphere to the distance of the corpuscle from its centre: I say, that the force with which the corpuscle is attracted is proportional to the semidiameter of the sphere.

For conceive two corpuscles to be severally attracted by two spheres, one by one, the other by the other, and their distances from the centres of the spheres to be proportional to the diameters of the spheres respectively; and the spheres to be resolved into like particles, disposed in a like situation to the corpuscles. Then the attractions of one corpuscle towards the several particles of one sphere will be to the attractions of the other towaidas as many analogous particles of the other sphere in a ratio compounded of the ratio of the particles directly, and the square of the distances inversely. But the particles are as the spheres, that is, as the cubes of the diameters, and the distances are as the diameters; and the first ratio directly with the last ratio taken twice inversely, becomes the ratio of diameter to diameter. Q.E.D.

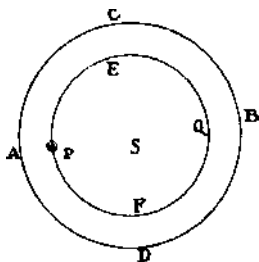
COR. 1. Hence if corpuscles revolve in circles about spheres composed of matter equally attracting, and the distances from the centres of the spheres be proportional to their diameters, the periodic times will be equal.

COR. 11. And, *vice versa*, if the periodic times are equal, the distances will be proportional to the diameters. These two Corollaries appear from Cor. in, Prop. iv.

COR. in. If to the several points of any two solids whatever, of like figure and equal density, there tend equal centripetal forces decreasing as the square of the distances from those points, the forces, with which corpuscles placed in a like situation to those two solids will be attracted by them, will be to each other as the diameters of the solids.

PROPOSITION LXXIII. THEOREM XXXIII

If to the several points of a given sphere there tend equal centripetal forces decreasing as the square of the distances from the points, I say, that a corpuscle placed within the sphere is attracted by a force proportional to its distance from the centre.



In the sphere ACBD, described about the centre S, let there be placed the corpuscle P; and about the same centre S, with the interval SP, conceive described an interior sphere PEQF. It is plain (by Prop, LXX) that the concentric spherical surfaces of which the difference AEBF of the spheres is composed, have no effect at all upon the body P, their attractions being destroyed by contrary attractions. There remains, therefore, only the attraction of the interior sphere PEQF. And (by Prop, LXXII) this is as the distance PS. Q.E.D.

SCHOLIUM

By the surfaces of which I here imagine the solids composed, I do not mean surfaces purely mathematical, but orbs so extremely thin, that their thickness is as nothing; that is, the evanescent orbs of which the sphere will at last consist, when the number of the orbs is increased, and their thickness diminished without end. In like manner, by the points of which lines, surfaces, and solids are said to be composed, are to be understood equal particles, whose magnitude is perfectly inconsiderable.

PROPOSITION LXXIV. THEOREM XXXIV

The same things supposed, I say, that a corpuscle situated without the sphere is attracted with a force inversely proportional to the square of its distance from the centre.

For suppose the sphere to be divided into innumerable concentric spherical surfaces, and the attractions of the corpuscle arising from the several surfaces will be inversely proportional to the square of the distance of the corpuscle from the centre of the sphere (by Prop. LXXI). And, by composition, the sum of those attractions, that is, the attraction of the corpuscle towards the entire sphere, will be in the same ratio. Q.E.D.

COR. 1. Hence the attractions of homogeneous spheres at equal distances from the centres will be as the spheres themselves. For (by Prop. LXXII) if the distances be proportional to the diameters of the spheres, the forces will be as the diameters. Let the greater distance be diminished in that ratio; and the distances now being equal, the attraction will be increased as the square of that ratio; and therefore will be to the other attraction as the cube of that ratio; that is, in the ratio of the spheres.

COR. II. At any distances whatever the attractions are as the spheres applied to the squares of the distances.

COR. III. If a corpuscle placed without an homogeneous sphere is attracted by a force inversely proportional to the square of its distance from the centre, and the sphere consists of attractive particles, the force of every particle will decrease as the square of the distance from each particle.

PROPOSITION LXXV. THEOREM XXXV

If to the several points of a given sphere there tend equal centripetal forces decreasing as the square of the distances from the point, I say, that another similar sphere will be attracted by it with a force inversely proportional to the square of the distance of the centres.¹

For the attraction of every particle is inversely as the square of its distance from the centre of the attracting sphere (by Prop. LXXIV), and is therefore the same as if that whole attracting force issued from one single corpuscle placed in the centre of this sphere. But this attraction is as great as

^[1] Appendix, Note 25]

on the other hand the attraction of the same corpuscle would be, if that were itself attracted by the several particles of the attracted sphere with the same force with which they are attracted by it. But that attraction of the corpuscle would be (by Prop, LXXIV) inversely proportional to the square of its distance from the centre of the sphere; therefore the attraction of the sphere, equal thereto, is also in the same ratio. Q.E.D.

COR. 1. The attractions of spheres towards other homogeneous spheres are as the attracting spheres applied to the squares of the distances of their centres from the centres of those which they attract.

COR. 11. The case is the same when the attracted sphere does also attract. For the several points of the one attract the several points of the other with the same force with which they themselves are attracted by the others again; and therefore since in all attractions (by Law 111) the attracted and attracting point are both equally acted on, the force will be doubled by their mutual attractions, the proportions remaining.

COR. in. Those several truths demonstrated above concerning the motion of bodies about the focus of the conic sections will take place when an attracting sphere is placed in the focus, and the bodies move without the sphere.

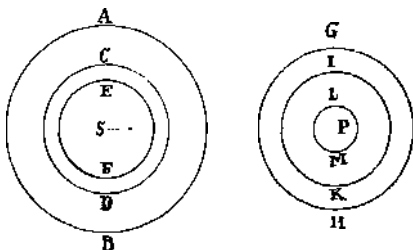
COR. IV. Those things which were demonstrated before of the motion of bodies about the centre of the conic sections take place when the motions are performed within the sphere.

PROPOSITION LXXVI. THEOREM XXXVI

If spheres be however dissimilar (as to density of matter and attractive force) in the same ratio on wards from the centre to the circumference; but everywhere similar, at every given distance from the centre, on all sides round about; and the attractive force of every point decreases as the square of the distance of the body attracted: I say, that the whole force with which one of these spheres attracts the other will be inversely proportional to the square of the distance of the centres.

Imagine several concentric similar spheres AB, CD, EF, &c., the innermost of which added to the outermost may compose a matter more dense towards the centre, or subtracted from them may leave the same more lax and rare. Then, by Prop, LXXV, these spheres will attract other similar con-

centric spheres GH, IK, LM, &c, each the other, with forces inversely proportional to the square of the distance SP. And, by addition or subtraction, the sum of all those forces, or the excess of any of them above the others; that is, the entire force with which the whole sphere AB (composed of any concentric spheres or of their differences) will attract the whole sphere GH (composed of any concentric spheres or of their differences) in the same ratio. Let the number of the concentric spheres be increased in *infinitum*, so that the



density of the matter together with the attractive force may, in the progress from the circumference to the centre, increase or decrease according to any given law; and by the addition of matter not attractive, let the deficient density be supplied, that so the spheres may acquire any form desired; and the force with which one of these attracts the other will be still, by the former reasoning, in the same inverse ratio of the square of the distance. Q.E.D.

COR. 1. Hence if many spheres of this kind, similar in all respects, attract each other, the accelerative attractions of each to each, at any equal distances of the centres, will be as the attracting spheres.

COR. 11. And at any unequal distances, as the attracting spheres divided by the squares of the distances between the centres.

COR. in. The motive attractions, or the weights of the spheres towards one another, will be at equal distances of the centres conjointly as the attracting and attracted spheres; that is, as the products arising from multiplying the spheres into each other.

COR. IV. And at unequal distances directly as those products and inversely as the squares of the distances between the centres.

COR. V. These proportions hold true also when the attraction arises from the attractive power of both spheres exerted upon each other. For the attraction is only doubled by the conjunction of the forces, the proportions remaining as before.

COR. VI. If spheres of this kind revolve about others at rest, each about each, and the distances between the centres of the quiescent and revolving bodies are proportional to the diameters of the quiescent bodies, the periodic times will be equal.

COR. VII. And, again, if the periodic times are equal, the distances will be proportional to the diameters.

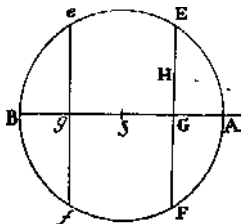
COR. VIII. All those truths above demonstrated, relating to the motions of bodies about the foci of conic sections, will take place when an attracting sphere, of any form and condition like that above described, is placed in the focus.

COR. IX. And also when the revolving bodies are also attracting spheres of any condition like that above described.

PROPOSITION LXXVII. THEOREM XXXVII

If to the several points of spheres there tend centripetal forces proportional to the distances of the points from the attracted bodies, I say, that the compounded force with which two spheres attract each other is as the distance between the centres of the spheres.

CASE I. Let AEBF be a sphere; S its centre; P a corpuscle attracted; PASB the axis of the sphere passing through the centre of the corpuscle; EF, *ef* two planes cutting the sphere, and perpendicular to the axis, and equidistant, one on one side, the other



on the other, from the centre of the sphere; G and *g* the intersections of the planes and the axis; and H any point in the plane EF. The centripetal force of the point H upon the corpuscle P, exerted in the direction of the line PH, is as the distance PH; and (by Cor. II of the

Laws) the same exerted in the direction of the line PG, or towards the centre S, is as the length PG. Therefore the force of all the points in the plane EF (that is, of that whole plane) by which the corpuscle P is attracted

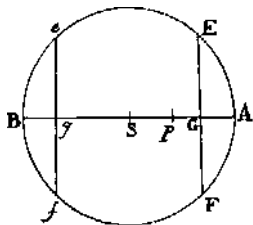
towards the centre S is as the distance PG multiplied by the number of those points, that is, as the solid contained under that plane EF and the distance PG . And in like manner the force of the plane ef , by which the corpuscle P is attracted towards the centre S , is as that plane multiplied by its distance Pg , or as the equal plane EF multiplied by that distance Pg ; and the sum of the forces of both planes as the plane EF multiplied by the sum of the distances $PG + Pg$, that is, as that plane multiplied by twice the distance PS of the centre and the corpuscle; that is, as twice the plane EF multiplied by the distance PS , or as the sum of the equal planes $EF + ef$ multiplied by the same distance. And, by a like reasoning, the forces of all the planes in the whole sphere, equidistant on each side from the centre of the sphere, are as the sum of those planes multiplied by the distance PS , that is, as the whole sphere and the distance PS conjointly. Q.E.D.

CASE 2. Let now the corpuscle P attract the sphere $AERF$. And, by the same reasoning, it will appear that the force with which the sphere is attracted is as the distance PS . Q.E.D.

CASE 3. Imagine another sphere composed of innumerable corpuscles P ; and because the force with which every corpuscle is attracted is as the distance of the corpuscle from the centre of the first sphere, and as the same sphere conjointly, and is therefore the same as if it all proceeded from a single corpuscle situated in the centre of the sphere, the entire force with which all the corpuscles in the second sphere are attracted, that is, with which that whole sphere is attracted, will be the same as if that sphere were attracted by a force issuing from a single corpuscle in the centre of the first sphere; and is therefore proportional to the distance between the centres of the spheres. Q.E.D.

CASE 4. Let the spheres attract each other, and the force will be doubled, but the proportion will remain. Q.E.D.

CASE 5. Let the corpuscle p be placed within the sphere $AhBF$; and because the force of the plane ef upon the corpuscle is as the solid contained under that plane and the distance pg ; and the contrary force of the plane



EF as the solid contained under that plane and the distance pG ; the force compounded of both will be as the difference of the solids, that is, as the sum of the equal planes multiplied by half the difference of the distances; that is, as that sum multiplied by pS , the distance of the corpuscle from the centre of the sphere. And, by a like reasoning, the attraction of all the planes EF, ef throughout the whole sphere, that is, the attraction of the whole sphere, is conjointly as the sum of all the planes, or as the whole sphere, and as pS , the distance of the corpuscle from the centre of the sphere. Q.E.D.

CASE 6. And if there be composed a new sphere out of innumerable corpuscles such as p , situated within the first sphere AEBF, it may be proved, as before, that the attraction, whether single of one sphere towards the other, or mutual of both towards each other, will be as the distance pS of the centres. Q.E.D.

PROPOSITION LXXVIII. THEOREM XXXVIII

If spheres in the progress from the centre to the circumference be however dissimilar and unequale, but similar on every side round about at all given distances from the centre; and the attractive force of every point be as the distance of the attracted body: I say, that the entire force with which two spheres of this kind attract each other mutually is proportional to the distance between the centres of the spheres.

This is demonstrated from the foregoing Proposition, in the same manner as Prop. LXXVI was demonstrated from Prop. LXXV.

COR. Those things that were above demonstrated in Prop. x and LXIV, of the motion of bodies round the centres of conic sections, take place when all the attractions are made by the force of spherical bodies of the condition above described, and the attracted bodies are spheres of the same kind.

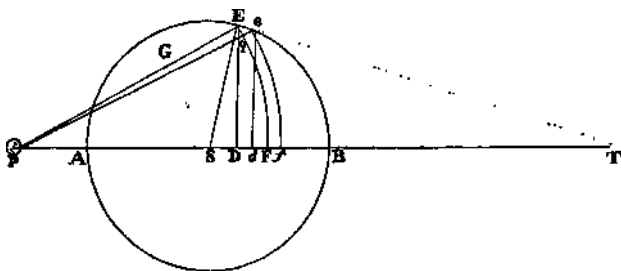
SCHOLIUM

I have now explained the two principal cases of attractions; to wit, when the centripetal forces decrease as the square of the ratio of the distances, or increase in a simple ratio of the distances, causing the bodies in both cases to revolve in conic sections, and composing spherical bodies whose centripetal forces observe the same law of increase or decrease in the recess from the

centre as the forces of the particles themselves do; which is very remarkable. It would be tedious to run over the other cases, whose conclusions are less elegant and important, so particularly as I have done these. I choose rather to comprehend and determine them all by one general method as follows.

L E M M A X X I X

If about the centre S there be described any circle as AEB, and about the centre P there be also described two circles EF, ef, cutting the first in E and e, and the line PS in F and f; and there be let fall to PS the perpendiculars ED, cd: I say, that if the distance of the arcs EF, ef be supposed to be infinitely diminished, the last ratio of the evanescent line Dd to the evanescent line Ff is the same as that of the line PE to the line PS.



For if the line Ve cut the arc EF in q ; and the right line Ee , which coincides with the evanescent arc Ee , be produced, and meet the right line PS in T ; and there be let fall from S to PE the perpendicular SG ; then, because of the like triangles DTE , dTe , DES ,

$$Dd : Ee = DT : TE = DE : ES;$$

and because the triangles, Eeq , ESG (by Lem. vin, and Cor. in, Lem. VII) are similar,

$$Ee : eq \text{ or } Ff = ES : SG.$$

Multiplying together corresponding terms of the two proportions,

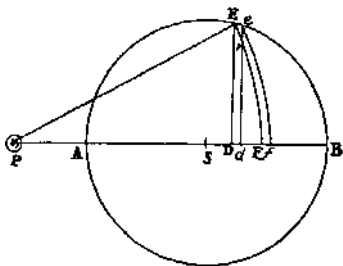
$$Dd : Ff - DE : SG - PE : PS$$

(because of the similar triangles PDE , PGS). Q.E.D.

PROPOSITION LXXIX. THEOREM XXXIX

Suppose a surface as EFfe to have its breadth infinitely diminished, and to be just vanishing; and that the same surface by its revolution round the axis PS describes a spherical concavoconvex solid, to the several equal particles of which there tend equal centripetal forces: I say, that the force with which that solid attracts a corpuscle situated in P is in a ratio compounded of the ratio of the solid DE² Ff and the ratio of the force with which the given particle in the place Ff would attract the same corpuscle.

For if we consider, first, the force of the spherical surface FE which is generated by the revolution of the arc FE, and is cut anywhere, as in r , by the line de , the annular part of the surface generated by the revolu-



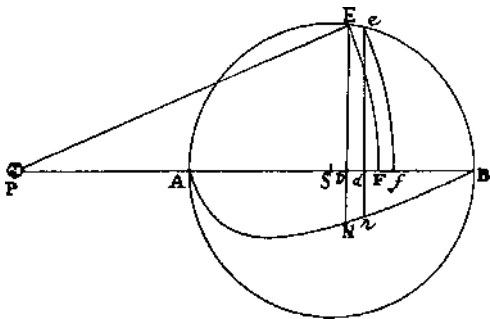
tion of the arc rE will be as the short line Dd , the radius of the sphere PE remaining the same; as *Archimedes* has demonstrated in his *Book of the Sphere and Cylinder*. And the force of this surface exerted in the direction of the lines PE or Fr situated all round in the conical surface, will be as this annular surface itself; that is, as the short line Dd , or,

which is the same, as the rectangle under the given radius PE of the sphere and the short line Dd ; but that force, exerted in the direction of the line PS tending to the centre S, will be less in the ratio PD to PE, and therefore will be as $PD \cdot Dd$. Suppose now the line DF to be divided into innumerable little equal particles, each of which call Dd , and then the surface FE will be divided into so many equal annuli, whose forces will be as the sum of all the rectangles $PD \cdot Dd$, that is, as $\frac{1}{2} PF^2 - \frac{1}{2} PD^2$, and therefore as DE' . Let now the surface FE be multiplied by the altitude Ff; and the force of the solid EFfe exerted upon the corpuscle P will be as $DE'^2 Ff$; that is, if the force be given which any given particle as Ff exerts upon the corpuscle P at the distance PF. But if that force be not given, the force of the solid EF/c- will be conjointly as the solid $DE'^2 Ff$ and that force not given. Q.E.D.

PROPOSITION LXXX. THEOREM XL

If to the several equal parts of a sphere ABE described about the centre S there tend equal centripetal forces; and from the several points D in the axis of the sphere AB in which a corpuscle, as P, is placed, there be erected the perpendiculars DE meeting the sphere in E, ^{and if in those perpen-} ^{$\frac{DE^2 \cdot PS}{PE}$} diculars the lengths DN be taken as the quantity $\frac{DE^2 \cdot PS}{PE}$, and as the force which a particle of the sphere situated in the axis exerts at the distance PE upon the corpuscle P conjointly: I say, that the whole force with which the corpuscle P is attracted towards the sphere is as the area ANB, comprehended under the axis of the sphere AB, and the curved line ANB, the locus of the point N.

For supposing the construction in the last Lemma and Theorem to stand, conceive the axis of the sphere AB to be divided into innumerable equal particles Dd, and the whole sphere to be divided into so many sphetical



concavoconvex laminae EFfe; and erect the perpendicular dn. By the last Theorem, the force with which the laminae EFfe attract the corpuscle P is as $DE^2 \cdot Ff$ and the force of one particle exerted at the distance PE or PF, conjointly. But (by the last Lemma) Dd is to Ff as PE to PS, and therefore

Ff is equal to $\frac{PS \cdot Dd}{PE}$; and $DE^2 \cdot Ff$ is equal to $Dd \cdot \frac{DE^2 \cdot PS}{PE}$, and therefore the force of the lamina EFfe is as $Dd \cdot \frac{DE^2 \cdot PS}{PE}$ and the force of a par-

tide exerted at the distance PF conjointly; that is, by the supposition, as $DN \cdot Dd$, or as the evanescent area $DNnd$. Therefore the forces of all the laminae exerted upon the corpuscle P are as all the areas $DNnd$, that is, the whole force of the sphere will be as the whole area ANB. Q.E.D.

COR. I. Hence if the centripetal force tending to the several particles remain always the same at all distances, and DN be made as $\frac{DE' \cdot PS}{PE}$, the whole force with which the corpuscle is attracted by the sphere is as the area ANB.

COR. II. If the centripetal force of the particles be inversely as the distance of the corpuscle attracted by it, and DN be made as $-\frac{DE' \cdot PS}{PE^2}$, the force with which the corpuscle P is attracted by the whole sphere will be as the area ANB.

COR. in. If the centripetal force of the particles be inversely as the cube of the distance of the corpuscle attracted by it, and DN be made as $\frac{DE'' \cdot PS}{PE^3}$, the force with which the corpuscle is attracted by the whole sphere will be as the area ANB.

COR. IV. And universally if the centripetal force tending to the several particles of the sphere be supposed to be inversely as the quantity V; and DN be made as $\frac{DE'' \cdot PS}{V}$; the force with which a corpuscle is attracted by the whole sphere will be as the area ANB.

PROPOSITION LXXXI. PROBLEM XLI

The things remaining as above, it is required to measure the area ANB.

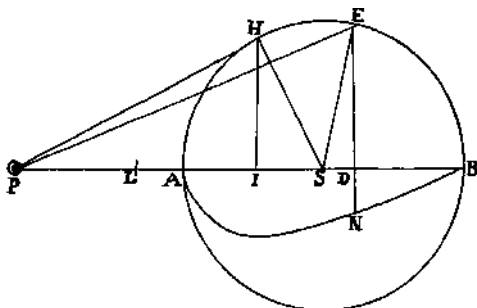
From the point P let there be drawn the right line PH touching the sphere in H; and to the axis PAB, letting fall the perpendicular HI, bisect PI in L; and (by Prop, XII, Book II, *Elem. of Euclid*) PE'' is equal to $PS' + SE^2 \div 2 PS \cdot SD$. But because the triangles SPH, SHI are alike, SE^2 or SH'' is equal to the rectangle PS · IS. Therefore PE^2 is equal to the rectangle contained under PS and PS-I-SI I 2SD; that is, under PS and 2LS I 2SD; that is, under PS and 2LD. Moreover DE^2 is equal to $SE^2 - SD^2$, or

$$SE^2 - LS^2 + 2LS \cdot LD - LD^2,$$

that is, $2LS \cdot LD - LD^2 - LA \cdot LB$.

For $LS^2 - SE^2$ or $LS''' - SA^2$ (by Prop, vi, Book 11, *Elem. of Euclid*) is equal to the rectangle $LA \cdot LB$. Therefore if instead of DE^2 we write

$$2LS \cdot LD - LD^2 - LA \cdot LB,$$



the quantity $\frac{DE^2 \cdot PS}{PE \cdot V}$, which (by Cor. iv of the foregoing Prop.) is as the length of the ordinate DN , will now resolve itself into three parts

$$\frac{2SLD \cdot PS}{PE \cdot V} - \frac{LD^2 \cdot PS}{PE \cdot V} - \frac{ALB \cdot PS}{PE \cdot V};$$

where if instead of V we write the inverse ratio of the centripetal ibcc, and instead of PE the mean proportional between PS and $2LD$, those three parts will become ordinates to so many curved lines, whose areas are discovered by the common methods. Q.E.D.

EXAM. I. If the centripetal force tending to the several particles of the sphere be inversely as the distance; instead of V write PE the distance, then

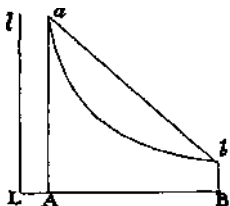
$2PS \cdot LD$ for PE^2 ; and DN will become as $SL - \frac{1}{2}LD - \frac{LA \cdot LB}{2LD}$. Suppose

DN equal to its double $2SL - LD - \frac{LA \cdot LB}{LD}$; and $2SL$ the given part of

the ordinate drawn into the length AB will describe the rectangular area $2SL \cdot AB$; and the indefinite part LD , drawn perpendicularly into the same length with a continued motion, in such sort as in its motion one way or another it may either by increasing or decreasing remain always equal to

the length LD , will describe the area $\frac{LB^2 - LA^2}{2}$, that is, the area $SL \cdot AB$;

which taken from the former area $2SL \cdot AB$, leaves the area $SL \cdot AB$. But the third part $\frac{LA \cdot LB}{LD}$, drawn after the same manner with a continued motion perpendicularly into the same length, will describe the area of an hyperbola, which subtracted from the area $SL \cdot AB$ will leave ANB the area sought. Whence arises this construction of the Problem. At the points L, A, B , erect the perpendiculars Ll, Aa, Bb ; making Aa equal to LB , and Bb equal to LA . Making Ll and LB asymptotes, describe through the points a, b the hyperbolic curve ab . And the chord ba being drawn, will inclose the area aba equal to the area sought ANB .



EXAM. 2. If the centripetal force tending to the several particles of the sphere be inversely as the cube of the distance, or (which is the

same thing) as that cube applied to any given plane; write $\frac{PE^2}{2AS^2}$ for V , and $2PS \cdot LD$ for PE^2 ; and DN will become as

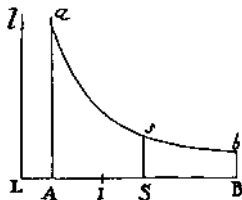
$$\frac{SL \cdot AS^2}{PS \cdot LD} - \frac{AS^2}{2PS} - \frac{LA \cdot LB \cdot AS^2}{2PS \cdot LD^2},$$

that is (because PS, AS, SI are continually proportional), as

$$\frac{LSI}{LD} - \frac{1}{2}SI - \frac{LA \cdot LB \cdot SI}{2LD^2}.$$

If we draw then these three parts into the length AB , the first $\frac{SL \cdot SI}{LD}$ will generate the area of an hyperbola; the second $\frac{1}{2}SI$ the area $\frac{1}{2}AB \cdot SI$; the third $\frac{LA \cdot LB \cdot SI}{2LD^2}$ the area $\frac{LA \cdot LB \cdot SI}{2LA \cdot B} - \frac{LA \cdot LB \cdot SI}{2LB \cdot A}$, that is, $\frac{1}{2}AB \cdot SI$.

From the first subtract the sum of the second and third, and there will remain ANB the area sought. Whence arises this construction of the Problem. At the points L, A, S, B , erect the perpendiculars U, Aa, Ss, Bb , of which suppose Ss equal to SI ; and through the point s , to the asymptotes LI, LB , describe the hyperbola asb meeting the perpen-

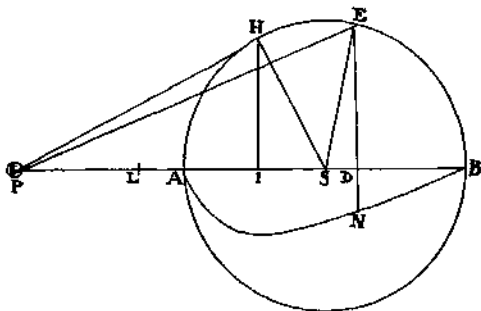


diculars Aa , Bb in a and b ; and the rectangle $2SA \cdot SI$ subtracted from the hyperbolic area $AasbB$, will leave ANB the area sought.

EXAM. 3. If the centripetal force tending to the several particles of the spheres decrease as the fourth power of the distance from the particles; write $\frac{PE^4}{2AS^2}$ for V , then $\sqrt{(2PS+LD)}$ for PE , and DN will become as

$$\frac{SI^2 \cdot SL}{\sqrt{2}SI} \cdot \frac{1}{\sqrt{LD}} - \frac{SI^2}{2\sqrt{2}SI} \cdot \frac{1}{\sqrt{LD}} - \frac{SI^2 \cdot LA \cdot LB}{2\sqrt{2}SI} \cdot \frac{1}{\sqrt{LD}}.$$

These three parts drawn into the length AB , produce so many arcs, viz.,



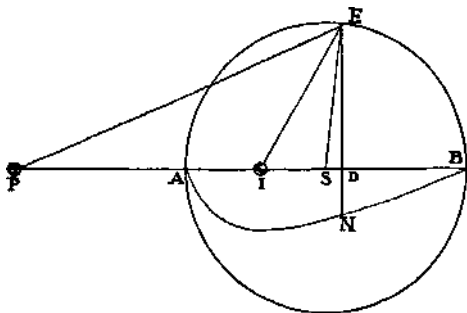
$\frac{2SI^2 \cdot SL}{\sqrt{2}SI}$ into $\left(\frac{1}{\sqrt{LA}} - \frac{1}{\sqrt{LB}}\right)$; $\frac{SI^2}{2\sqrt{2}SI}$ into $\sqrt{(LB - \sqrt{LA})}$; and $\frac{SI^2 \cdot LA \cdot LB}{3\sqrt{2}SI}$ into $\left(\frac{1}{\sqrt{LA}} - \frac{1}{\sqrt{LB}}\right)$. And these after due reduction come forth $\frac{2SI^2 \cdot SL}{LI}$, SI^2 , and $SI^2 + \frac{2SI^2}{3LI}$. And these by subtracting the last from the first, become $\frac{4SI^2}{3LI}$. Therefore the entire force with which the corpuscle P is attracted towards the centre of the sphere is as $\frac{SI^2}{PI^3}$, that is, inversely as $PS^3 \cdot PI$. Q.E.I.

By the same method one may determine the attraction of a corpuscle situated within the sphere, but more expeditiously by the following Theorem.

PROPOSITION LXXXII. THEOREM XLI

In a sphere described about the centre S with the radius SA, if there be taken SI, SA, SP continually proportional: I say, that the attraction of a corpuscle within the sphere in any place I is to its attraction without the sphere in the place P in a ratio compounded of the square root of the ratio of IS, PS, the distances from the centre, and the square root of the ratio of the centripetal forces tending to the centre in those places P and I.

As, if the centripetal forces of the particles of the sphere be inversely as the distances of the corpuscle attracted by them, the force with which the corpuscle situated in I is attracted by the entire sphere will be to the force with which it is attracted in P in a ratio compounded of the square root of the ratio of the distance SI to the distance SP, and the square root of the ratio



of the centripetal force in the place I arising from any particle in the centre to the centripetal force in the place P arising from the same particle in the centre; that is, inversely as the square root of the ratio of the distances SI, SP to each other. These two square roots of ratios compose the ratio of equality, and therefore the attractions in I and P produced by the whole sphere are equal. By the like calculation, if the forces of the particles of the sphere are inversely as the square of the ratio of the distances, it will be found that the attraction in I is to the attraction in P as the distance SP to the semidiameter SA of the sphere. If those forces are inversely as the cube of the ratio of the distances, the attractions in I and P will be to each other as SP^2 to SA^2 ; if

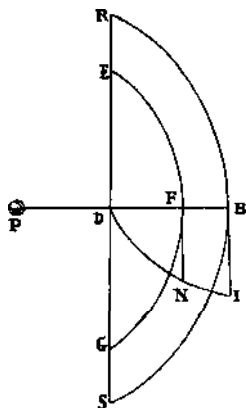
as the fourth power of the ratio, as SP^3 to SA^3 . Therefore since the attraction in P was found in this last case to be inversely as $PS^a \cdot PI$, the attraction in I will be inversely as $SA^3 \cdot PI$, that is, because SA^3 is given, inversely as PI . And the progression is the same *in infinitum*. The demonstration of this Theorem is as follows:

The things remaining as above constructed, and a corpuscle being in any place P, the ordinate DN was found to be as $\frac{DE^2 \cdot PS}{PE \cdot V}$. Therefore if IE be drawn, that ordinate for any other place of the corpuscle, as I, will become (other things being equal) as $\frac{DE^2 \cdot IS}{IE \cdot V}$. Suppose the centripetal forces flowing from any point of the sphere, as E, to be to each other at the distances IE and PE as PE^n to IE^n (where the number n denotes the index of the powers of PE and IE), and those ordinates will become as $\frac{DE^2 \cdot PS}{PE \cdot PE^n}$ and $\frac{DE^2 \cdot IS}{IE \cdot IE^n}$, whose ratio to each other is as $PS \cdot IE \cdot IE^n$ to $IS \cdot PE \cdot PE^n$. Because SI, SE, SP are in continued proportion, the triangles SPE, SEI are alike; and thence IE is to PE as IS to SE or SA. For the ratio of IE to PE write the ratio of IS to SA; and the ratio of the ordinates becomes that of $PS \cdot IE^n$ to $SA \cdot PE^n$. But the ratio of PS to SA is the square root of that of the distances PS, SI; and the ratio of IE^n to PE^n (because IE is to PE as IS to SA) is the square root of that of the forces at the distances PS, IS. Therefore the ordinates, and consequently the areas which the ordinates describe, and the attractions proportional to them, are in a ratio compounded of the square root of those ratios. Q.E.D.

PROPOSITION LXXXIII. PROBLEM XLII

To find the force with which a corpuscle placed in the centre of a sphere is attracted towards any segment of that sphere whatsoever.

Let P be a body in the centre of that sphere, and R BSD a segment thereof contained under the plane RDS and the spherical surface RBS. Let DB be cut in F by a spherical surface EFG described from the centre P, and let the segment be divided into the parts BREFGS, FEDG. Let us suppose that segment to be not a purely mathematical but a physical surface, having



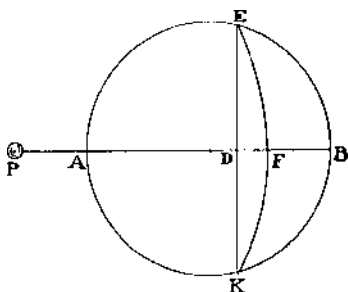
some, but a perfectly inconsiderable thickness. Let that thickness be called O , and (by what *Archimedes* hath demonstrated) that surface will be as $PF \cdot DF \cdot O$. Let us suppose, besides, the attractive forces of the particles of the sphere to be inversely as that power of the distances, of which n is index; and the force with which the surface EFG attracts the body P will be (by Prop, LXXIX) as $\frac{DE \cdot O}{PF^n}$, that is, as $\frac{2DF \cdot O}{PF^{n-1}} - \frac{DF \cdot O}{PF^n}$. Let the perpendicular FN multiplied by O be proportional to this quantity; and the curvilinear area BDI , which the ordinate FN , drawn through the length DB with a continued motion will describe, will be as the

whole force with which the whole segment $RBSD$ attracts the body P . Q.E.I.

PROPOSITION LXXXIV. PROBLEM XLIII

To find the force with which a corpuscle, placed without the centre of a sphere in the axis of any segment, is attracted by that segment.

Let the body P placed in the axis ADB of the segment EBK be attracted by that segment. About the centre P , with the radius PE , let the spherical surface EFK be described; and let it divide the segment into two parts $EBKFE$ and $EFKDE$. Find the force of the first of those parts by Prop, LXXXI, and the force of the latter part by Prop, LXXXIII, and the sum of the forces will be the force of the whole segment $EBKDE$. Q.E.I.



SCHOLIUM

The attractions of spherical bodies being now explained, it comes next in order to treat of the laws of attraction in other bodies consisting in like manner of attractive particles; but to treat of them particularly is not necessary to my design. It will be sufficient to add some general Propositions relating to the forces of such bodies, and the motions thence arising, because the knowledge of these will be of some little use in philosophical inquiries.

SECTION XIII

The attractive forces of bodies which are not spherical.

PROPOSITION LXXXV. THEOREM XLII

If a body be attracted by another, and its attraction be vastly stronger when it is contiguous to the attracting body than when they are separated from each other by a very small interval, the forces of the particles of the attracting body decrease, in the recess of the body attracted, in more than the squar ed ratio of the distance of the particles.

For if the forces decrease as the square of the distances from the particles, the attraction towards a spherical body being (by Prop, LXXIV) inversely as the square of the distance of the attracted body from the centre of the sphere, will not be sensibly increased by the contact, and it will be still less increased by it, if the attraction, in the recess of the body attracted, decreases in a still less proportion. The Proposition, therefore, is evident concerning attractive spheres. And the case is the same of concave spherical orbs attracting external bodies. And much more does it appear in orbs that attract bodies placed within them, because there the attractions diflused through the cavities of those orbs are (by Prop, LXX) destroyed by contrary attractions, and therefore have no effect even in the place of contact. Now if from these spheres and spherical orbs we take away any parts remote from the place of contact, and add new parts anywhere at pleasure, we may change the figures of the attractive bodies at pleasure; but the parts added or taken away, being remote from the place of contact, will cause no remarkable excess of the attraction arising from the contact of the two bodies. Therefore the Proposition holds good in bodies of all figures. Q. E. D.

PROPOSITION LXXXVI. THEOREM XLIII

If the forces of the particles of which an attractive body is composed decrease, in the recession of the attractive body, as the third or more than the third power of the distance from the particles, the attraction will be vastly stronger in the point of contact than when the attracting and attracted bodies are separated from each other, though by ever so small an interval.

For that the attraction is infinitely increased when the attracted corpuscle comes to touch an attracting sphere of this kind, appears, by the solution

of Problem xu, exhibited in the second and third Examples. The same will also appear (by comparing those Examples and Theor xu together) of attractions of bodies made towards concavoconvx orbs, whether the attracted bodies be placed without the orbs, or in the cavities within them. And by adding to or taking from those spheres and orbs any attractive matter anywhere without the place of contact, so that the attractive bodies may receive any assigned figure, the Proposition will hold good of all bodies universally. Q.E.D.

PROPOSITION LXXXVII. THEOREM XLIV

If two bodies similar to each other, and consisting of matter equally attractive, attract separately two corpuscles proportional to those bodies, and in a like situation to them, the accelerative attractions of the corpuscles towards the entire bodies will be as the accelerative attractions of the corpuscles towards particles of the bodies proportional to the wholes, and similarly situated in them.

For if the bodies are divided into particles proportional to the wholes, and alike situated in them, it will be, as the attraction towards any particle of one of the bodies to the attraction towards the correspondent particle in the other body, so are the attractions towards the several particles of the first body, to the attractions towards the several correspondent particles of the other body; and, by composition, so is the attraction towards the first whole body to the attraction towards the second whole body. Q.E.D.

COR. 1. Therefore if, as the distances of the corpuscles attracted increase, the attractive forces of the particles decrease in the ratio of any power of the distances, the accelerative attractions towards the whole bodies will be directly as the bodies, and inversely as those powers of the distances. As if the forces of the particles decrease as the square of the distances from the corpuscles attracted, and the bodies are as A' and B^3 , and therefore both the cubic sides of the bodies, and the distance of the attracted corpuscles from the bodies, are as A and B ; the accelerative attractions towards the bodies will be as $\frac{A^3}{A^2}$ and $\frac{B^3}{B^2}$ that is, as A and B the cubic sides of those bodies. If the forces of the panicles decieasc as the cube of the distances from the attracted corpuscles, the accelerative attractions towards the whole bodies

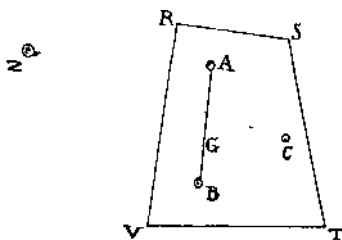
will be as $\frac{A^3}{A^4}$ and $\frac{B^3}{B^4}$, that is, equal. If the forces decrease as the fourth power, the attractions towards the bodies will be as $\frac{A^3}{A^4}$ and $\frac{B^3}{B^4}$, that is, inversely as the cubic sides A and B. And so in other cases.

COR. ii. Hence, on the other hand, from the forces with which like bodies attract corpuscles similarly situated, may be obtained the ratio of the decrease of the attractive forces of the particles as the attracted corpuscle recedes from them; if only that decrease is directly or inversely in any ratio of the distances.

PROPOSITION LXXXVIII. THEOREM XLV

If the attractive forces of the equal pay tides of any body be as the distance of the places from the particles, the force of the whole body will tend to its centre of gravity, and will be the same with the force of a globe, consisting of similar and equal matter, and having its centre in the centre of gravity.

Let the particles A, B of the body RSTV attract any corpuscle Z with forces which, supposing the particles to be equal between themselves, are as the distances AZ, BZ; but, if they are supposed unequal, are as those



particles and their distances AZ, BZ conjointly, or (if I may so speak) as those particles multiplied by their distances AZ, BZ respectively. And let those forces be expressed by the contents under A AZ, and B BZ. Join AB, and let it be cut in G, so that AG may be to BG as the particle B to the particle A; and G will

be the common centre of gravity of the particles A and B. The force A AZ will (by Cor. II of the Laws) be resolved into the forces A GZ and A AG; and the force B BZ into the forces B GZ and B BG. Now the forces A AG and B BG, because A is proportional to B, and BG to AG, are equal, and therefore having contrary directions destroy one another. There remain

then the forces $A\ GZ$ and $B\ GZ$. These tend from Z towards the centre G , and compose the force $(A + B)\ GZ$; that is, the same force as if the attractive particles A and B were placed in their common centre of gravity G , composing there a little globe.

By the same reasoning, if there be added a third particle C , and the force of it be compounded with the force $(A + B)\ GZ$ tending to the centre G , the force thence arising will tend to the common centre of gravity of that globe in G and of the particle C ; that is, to the common centre of gravity of the three particles A , B , C ; and will be the same as if that globe and the particle C were placed in that common centre composing a greater globe there; and so we may go on *in infinitum*. Therefore the whole force of all the particles of any body whatever $RSTV$ is the same as if that body, without removing its centre of gravity, were to put on the form of a globe. Q.E.D.

COR. Hence the motion of the attracted body Z will be the same as if the attracting body $RSTV$ were spherical; and therefore if that attracting body be either at rest, or proceed uniformly in a right line, the body attracted will move in an ellipse having its centre in the centre of gravity of the attracting body.

PROPOSITION LXXXIX. THEOREM XLVI

If there be several bodies consisting of equal particles whose forces are as the distances of the places from each, the force compounded of all the forces by which any corpuscle is attracted will tend to the common centre of gravity of the attracting bodies; and will be the same as if those attracting bodies, preserving their common centre of gravity, should unite there, and be formed into a globe.

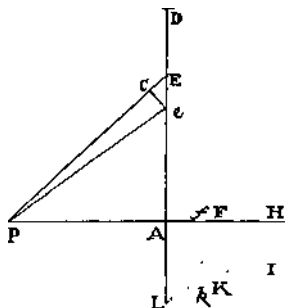
This is demonstrated after the same manner as the foregoing Proposition.

COR. Therefore the motion of the attracted body will be the same as if the attracting bodies, preserving their common centre of gravity, should unite there, and be formed into a globe. And, therefore, if the common centre of gravity of the attracting bodies be either at rest, or proceed uniformly in a right line, the attracted body will move in an ellipse having its centre in the common centre of gravity of the attracting bodies.

PROPOSITION XC. PROBLEM XLIV

If to the several points of any circle there tend equal centripetal forces, increasing or decreasing in any ratio of the distances; it is required to find the force with which a corpuscle is attracted, that is, situated anywhere in a right line which stands at right angles to the plane of the circle at its centre.

Suppose a circle to be described about the centre A with any radius AD in a plane to which the right line AP is perpendicular; and let it be required to find the force with which a corpuscle P is attracted towards the same. From any point E of the circle, to the attracted corpuscle P, let there be drawn the right line PE. In the right line PA take PF equal to PE, and



make a perpendicular FK, erected at F, to be as the force with which the point E attracts the corpuscle P. And let the curved line IKL be the locus of the point K. Let that curve meet the plane of the circle in L. In PA take PII equal to PD, and erect the perpendicular HI meeting that curve in I; and the attraction of the corpuscle P towards the circle will be as the area AHIL multiplied by the altitude AP. Q.E.I.

For let there be taken in AE a very small line Ee. Join Ve, and in PE, PA take PC, Pf both equal to Pe. And because the force, with which any point E of the ring described about the centre A with the radius AE in the afore-said plane attracts to itself the body P, is supposed to be as FK; and, therefore, the force with which that point attracts the body P towards A is as AP FK

; and the force with which the whole ring attracts the body P towards A is as the ring and $\frac{AP \cdot FK}{PE}$ conjointly; and that ring also is as the rectangle under the radius AE and the breadth Ee, and this rectangle (because PE and AE, Ee and CE are proportional) is equal to the rectangle PE CE or PE Ff; the force with which that ring attracts the body P to-

wards A will be as PE Ff and $\frac{AP \cdot FK}{PE}$ conjointly; that is, as the content under Ff FK AP, or as the area FKkf multiplied by AP. And therefore the sum of the forces with which all the rings, in the circle described about the centre A with the radius AD, attract the body P towards A, is as the whole area AHIKL multiplied by AP. Q.E.D.

COR. 1. Hence if the forces of the points decrease as the square of the distances, that is, if FK be as $\frac{1}{PF^2}$, and therefore the area AHIKL as $\frac{1}{PA} - \frac{1}{PH}$; the attraction of the corpuscle P towards the circle will be as

$$1 - \frac{PA}{PH}; \text{ that is, as } \frac{AH}{PH}.$$

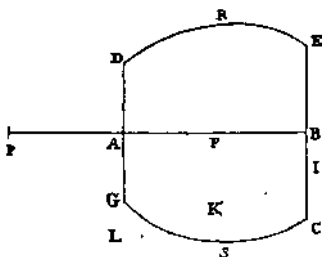
COR. II. And universally if the forces of the points at the distances D be inversely as any power D^n of the distances; that is, if FK be as $\frac{1}{D^n}$, and therefore the area AHIKL as $\frac{1}{PA^{n-1}} - \frac{1}{PH^{n-1}}$; the attraction of the corpuscle P towards the circle will be as $\frac{1}{PA^{n-2}} - \frac{PA}{PH^{n-2}}$.

COR. HI. And if the diameter of the circle be increased *in infinitum*, and the number n be greater than unity; the attraction of the corpuscle P towards the whole infinite plane will be inversely as PA^{n-1} , because the other term $\frac{PA}{PH^{n-2}}$ vanishes.

PROPOSITION XCI. PROBLEM XLV

To find the attraction of a corpuscle situated in the axis of a round solid, to whose several points there tend equal centripetal forces decreasing in any ratio of the distances whatsoever.

Let the corpuscle P, situated in the axis AB of the solid DECG, be attracted towards that solid. Let the solid be cut by any circle as RFS, perpendicular to the axis; and in its semidiameter FS, in any plane PALKB



passing through the axis, let there be taken (by Prop. xc) the length FK proportional to the force with which the corpuscle P is attracted towards that circle. Let the locus of the point K be the curved line LKI, meeting the planes of the outermost circles AL and BI in L and I; and the attraction of the corpuscle P towards the solid will be as the area LABI. Q.E.I.

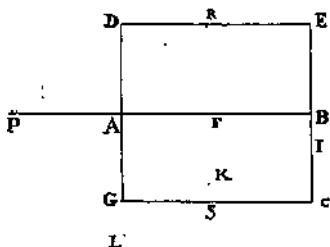
COR. i. Hence if the solid be a cylinder described by the parallelogram ADEB revolved about the axis AB, and the centripetal forces tending to the several points be inversely as the squares of the distances from the points; the attraction of the corpuscle P towards this cylinder will be as $AB - PE + PD$. For the ordinate FK (by Cor. i, Prop. xc) will

be as $1 - \frac{PF}{PR}$. The part 1 of this quantity, multiplied by the length AB, describes the area $I \cdot AB$; and

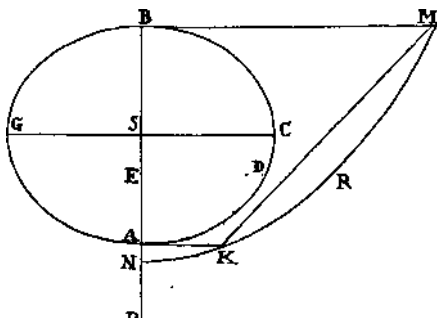
the other part $\frac{PF}{PR}$, multiplied by the length PB, describes the area

$1 \cdot (PE - AD)$ (as may be easily shown from the quadrature of the curve LKI); and, in like manner, the same part multiplied by the length PA describes the area $1 \cdot (PD - AD)$, and multiplied by AB, the difference of PB and PA, describes $1 \cdot (PE - PD)$, the difference of the areas. From the first content $1 \cdot AB$ take away the last content $1 \cdot (PE - PD)$, and there will remain the area LABI equal to $1 \cdot (AB - PE + PD)$. Therefore the force, being proportional to this area, is as $AB - PE + PD$.

COR. II. Hence also is known the force by which a spheroid AGBC attracts any body P situate externally in its axis AB. Let NKR be a conic section whose ordinate ER perpendicular to PE may be always equal to the



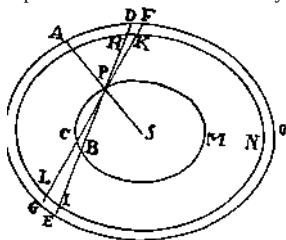
length of the line PD, continually drawn to the point D in which that ordinate cuts the spheroid. From the vertices A, B of the spheroid, let there be erected to its axis AB the perpendiculars AK, BM, respectively equal to AP, BP, and therefore meeting the conic section in K and M; and join KM



cutting off from it the segment KMRK. Let S be the centre of the spheroid, and SC its greatest semidiameter; and the force with which the spheroid attracts the body P will be to the force with which a sphere described with

the diameter AB attracts the same body as $\frac{AS \cdot CS' - PS \cdot KMRK}{PS^2 + CS'^2 - AS^2}$ is to $\frac{AS^3}{3PS^2}$. And by a calculation founded on the same principles may be found the forces of the segments of the spheroid.

COR. in. If the corpuscle be placed within the spheroid and in its axis, the attraction will be as its distance from the centre. This may be easily inferred from the following reasoning, whether the particle be in the axis or in any other given diameter. Let AGOF be an attracting spheroid, S its centre, and P the body attracted. Through the body P let there be drawn the semidiameter SPA, and two right lines DE, FG meeting the spheroid in D and E, F and G; and let PCM, HLN be the surfaces of two interior spheroids similar



and concentric to the exterior, the first of which passes through the body P, and cuts the right lines DE, FG in B and C; the latter cuts the same right lines in H and I, K and L. Let the spheroids have all one common axis, and the parts of the right lines intercepted on both sides DP and BE, FP and CG, DH and IE, FK and LG, will be mutually equal; because the right lines DE, PB, and HI are bisected in the same point, as are also the right lines FG, PC, and KL. Conceive now DPF, EPG to represent opposite cones described with the infinitely small vertical angles DPF, EPG, and the lines DH, EI to be infinitely small also. Then the particles of the cones DHKF, GLIE, cut off by the spheroidal surfaces, by reason of the equality of the lines DH and EI, will be to one another as the squares of the distances from the body P, and will therefore attract that corpuscle equally. And by a like reasoning if the spaces DPF, EGCB be divided into particles by the surfaces of innumerable similar spheroids concentric to the former and having one common axis, all these particles will equally attract on both sides the body P towards contrary parts. Therefore the forces of the cone DPF, and of the conic segment EGCB, arc equal, and by their opposed actions destroy each other. And the case is the same of the forces of all the matter that lies without the interior spheroid PCBM. Therefore the body P is attracted by the interior spheroid PCBM alone, and therefore (by Cor in, Prop, i xxu) its attraction is to the force with which the body A is attracted by the whole spheroid AGOD as the distance PS is to the distance AS. Q.E.D.

PROPOSITION XCII. PROBLEM XLVI

An attracting body being given, it is required to find the ratio of the decrease of the centripetal forces tending to its several points.

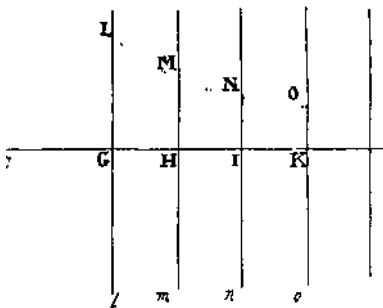
The body given must be formed into a sphere, a cylinder, or some regular figure, whose law of attraction answering to any ratio of decrease may be found by Prop, LXXX, LXXXI, and xci. Then, by experiments, the force of the attractions must be found at several distances, and the law of attraction towards the whole, made known by that means, will give the ratio of the decrease of the forces of the several parts; which was to be found.

PROPOSITION XCIII. THEOREM XLVII

If a solid be plane on one side, and infinitely extended on all other sides, and consist of equal particles equally attractive, whose forces decrease, in receding from the solid, in the ratio of any power greater than the square of the distances; and a corpuscle placed towards either part of the plane is attracted by the force of the whole solid. I say, that the attractive force of the whole solid, in receding from its plane surface will decrease in the ratio of a power whose side is the distance of the corpuscle from the plane, and its index less by 3 than the index of the power of the distances.

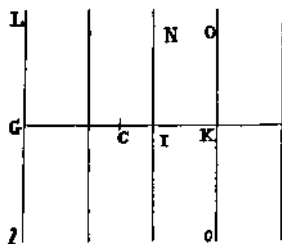
CASE I. Let LGL be the plane by which the solid is terminated. Let the solid lie on that side of the plane that is towards I, and let it be resolved into innumerable planes mHM, nIN, oKO, &c., parallel to GL. And first let the attracted body C be placed without the solid. Let there be drawn CGHI perpendicular to those innumerable planes, and let the attractive forces of the points of the solid decrease in the ratio of a power of the distances whose index is the number n not less than 3. Therefore (by Cor. III, Prop, xc) the force with which any plane w H M attracts the point C is inversely as CH^{n-2} . In the plane mHM take the length HM inversely proportional to CH^{n-2} , and that force

will be as HM. In like manner in the several planes /GL, nIN, oKO, &c., take the lengths GL, IN, KO, &c, inversely proportional to CG^{n-2} , CP^{n-2} , CK^{n-2} , &c, and the forces of those planes will be as the lengths so taken, and therefore the sum of the forces as the sum of the lengths, that is, the force of the whole solid as



the area GLOK produced infinitely towards OK. But that area (by the known methods of quadratures) is inversely as CG^{n-2} , and therefore the force of the whole solid is inversely as CG^{n-3} . Q.K.D.

CASE 2. Let the corpuscle C be now placed on that side of the plane /GL that is within the solid, and take the distance CK equal to the distance CG.



And the part of the solid LGloKO terminated by the parallel planes /GL, oKO, will attract the corpuscle C, situated in the middle, neither one way nor another, the contrary actions of the opposite points destroying one another by reason of their equality. Therefore the corpuscle C is attracted by the force only of the solid situated beyond the plane OK. But this force (by Case 1)

is inversely as CK^{n-3} , that is (because CG, CK are equal), inversely as CG^{n-3} . Q.E.D.

COR. i. Hence if the solid LGIN be terminated on each side by two infinite parallel planes LG, IN, its attractive force is known, subtracting from the attractive force of the whole infinite solid LGKO the attractive force of the more distant part NIKO infinitely produced towards KO.

COR. II. If the more distant part of this solid be rejected, because its attraction compared with the attraction of the nearer part is inconsiderable, the attraction of that nearer part will, as the distance increases, decrease nearly in the ratio of the power CG^{n-3} .

COR. III. And hence if any finite body, plane on one side, attract a corpuscle situated over against the middle of that plane, and the distance between the corpuscle and the plane compared with the dimensions of the attracting body be extremely small; and the attracting body consist of homogeneous particles, whose attractive forces decrease in the ratio of any power of the distances greater than the fourth; the attractive force of the whole body will decrease very nearly in the ratio of a power whose side is that very small distance, and the index less by 3 than the index of the former power. This assertion does not hold good, however, of a body consisting of particles whose attractive forces decrease in the ratio of the third power of the distances; because, in that case, the attraction of the remoter part of the infinite body in the second Corollary is always infinitely greater than the attraction of the nearer part.

S C H O L I U M

If a body is attracted perpendicularly towards a given plane, and from the law of attraction given, the motion of the body be required; the Problem will be solved by seeking (by Prop, xxxix) the motion of the body descending in a right line towards that plane, and (by Cor. n of the Laws) compounding that motion with an uniform motion performed in the direction of lines parallel to that plane. And, on the contrary, if there be required the law of the attraction tending towards the plane in perpendicular directions, by which the body may be caused to move in any given curved line, the Problem will be solved by working after the manner of the third Problem.

But the operations may be contracted by resolving the ordinates into converging series. As if to a base A the length B be ordinately applied in any given angle, and that length be as any power of the base A^n ; and there be sought the force with which a body, either attracted towards the base or driven from it in the direction of that ordinate, may be caused to move in the curved line which that ordinate always describes with its superior extremity; I suppose the base to be increased by a very small part O, and I resolve the ordinate $(A + O)^m$ into an infinite series

$$A^m + \frac{m}{n} OA^{m-n} + \frac{mm-mn}{2nn} OOA^{m-2n} \&c.,$$

and I suppose the force proportional to the term of this series in which O is of two dimensions, that is, to the term $\frac{mm-mn}{2nn} OOA^{m-2n}$. Therefore

the force sought is as $\frac{mm-mn}{nn} A^{m-2n}$, or, which is the same thing, as

$\frac{mm-mn}{nn} B^{m-2n}$. As if the ordinate describe a parabola, m being = 2, and

$n=1$. the force will be as the given quantity $2B^0$. and therefore is given. Therefore with a given force the body will move in a parabola, as *Galileo* hath demonstrated. If the ordinate describe an hyperbola, m being = 0-1, and $n=1$, the force will be as $2A^{-3}$ or $2B^3$; and therefore a force which is as the cube of the ordinate will cause the body to move in an hyperbola. But leaving Propositions of this kind, I shall go on to some others relating to motion which I have not yet touched upon.

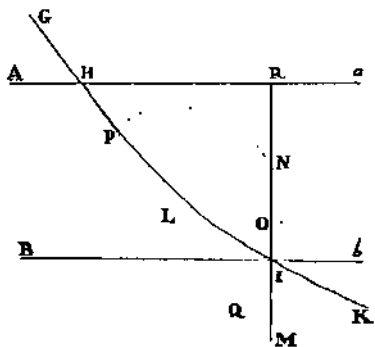
SECTION XIV

The motion of very small bodies when agitated by centripetal forces tending to the several parts of any very great body.

PROPOSITION XCIV. THEOREM XLVIII

If two similar mediums be separated from each other by a space terminated on both sides by parallel planes, and a body in its passage through that space be attracted or impelled perpendicularly towards either of those mediums, and not agitated or hindered by any other force; and the attraction be everywhere the same at equal distances from either plane, taken towards the same side of the plane' I say, that the sine of incidence upon either plane will be to the sine of emergence from the other plane in a given ratio.

CASE I. Let Aa and Bb be two parallel planes, and let the body light upon the first plane Aa in the direction of the line GH , and in its whole passage through the intermediate space let it be attracted or impelled towards the medium of incidence, and by that action let it be made to describe a curved



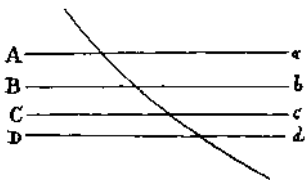
line HI , and let it emerge in the direction of the line IK . Let there be erected IM perpendicular to Kb the plane of emergence, and meeting the line of incidence GH prolonged in M , and the plane of incidence Aa in R ; and let the line of emergence KI be produced and meet HM in L . About the centre L , with the radius LI , let a circle be described cutting both HM in P and Q , and MI produced in

N ; and, first, if the attraction or impulse be supposed uniform, the curve HI (by what *Galileo* hath demonstrated) will be a parabola, whose property

is that of a rectangle under its given latus rectum, and the line IM equal to the square of HM; and moreover the line HM will be bisected in L. Hence if to MI there be let fall the perpendicular LO, then MO, OR will be equal; and adding the equal lines ON, OI, the wholes MN, IR will be equal also. Therefore since IR is given, MN is also given, and the rectangle MI MN is to the rectangle under the latus rectum and IM, that is, to HM^2 in a given ratio. But the rectangle MI MN is equal to the rectangle MP MQ, that is, to the difference of the squares ML^2 , and PL^2 or LP; and HM^2 hath a given ratio to its fourth part ML^2 ; therefore the ratio of $ML^2 - LP$ to ML^2 is given, and by conversion the ratio of LP to ML^2 , and its square root, the ratio of LI to ML. But in every triangle, as LMI, the sines of the angles arc proportional to the opposite sides. Therefore the ratio of the sine of the angle of incidence LMR to the sine of the angle of emergence LIR is given. Q.E.D.

CASE 2. Let now the body pass successively through several spaces terminated with parallel planes AabB, BbcC, &c, and let it be acted on by a force which is uniform in each of them separately, but different in the different spaces; and by what was just

demonstrated, the sine of the angle of incidence on the first plane Aa is to the sine of emergence from the second plane Bb in a given ratio; and this sine of incidence upon the second plane Bb will be to the sine of emergence from the third plane

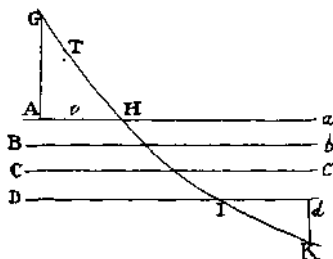


O in a given ratio; and this sine to the sine of emergence from the fourth plane Del in a given ratio; and so on *in infinitum*, and, by multiplication of equals, the sine of incidence on the first plane is to the sine of emergence from the last plane in a given ratio. Let now the intervals of the planes be diminished, and their number be infinitely increased, so diat the action of attraction or impulse, exerted according to any assigned law, may become continual, and the ratio of *The* sine of incidence on the first plane to the sine of emergence from the last plane being all along given, will be given then also. Q.E.D.

PROPOSITION XCV. THEOREM XLIX

The same things being supposed, I say, that the velocity of the body before its incidence is to its velocity after emergence as the sine of emergence to the sine of incidence.

Make AH and Id equal, and erect the perpendiculars AG , dK meeting the lines of incidence and emergence GH , IK in G and K . In GH take TH equal to IK , and to the plane ha let fall a perpendicular Tv . And (by



Cor. II of the Laws of Motion) let the motion of the body be resolved into two, one perpendicular to the planes Aa , Bb , Cc , &c, and another parallel to them. The force of attraction or impulse, acting in directions perpendicular to those planes, does not at all alter the motion in parallel directions; and therefore the body proceeding with this motion will in equal

times go through those equal parallel intervals that he between the line AG and the point H , and between the point I and the line dK ; that is, they will describe the lines GH , IK in equal times. Therefore the velocity before incidence is to the velocity after emergence as GH to IK or TH , that is, as AH or Id to vH , that is (supposing TH or IK radius), as the sine of emergence to the sine of incidence.¹ Q.E.D.

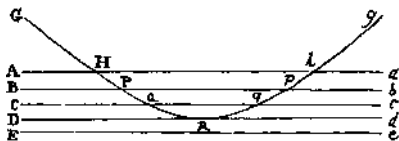
PROPOSITION XCVI. THEOREM L

The same things being supposed, and that the motion before incidence is swifter than afterwards. I say, that if the line of incidence be inclined continually, the body will be at last reflected, and the angle of refraction will be equal to the angle of incidence.

For conceive the body passing between the parallel planes ha , Bb , Cc , &c, to describe parabolic arcs as above; and let those arcs be HP , PQ , QR , &c. And let the obliquity of the line of incidence GH to the first plane Aa

[¹ Appendix, Note 26]

be such that the sine of incidence may be to the radius of the circle whose sine it is, in the same ratio which the same sine of incidence hath to the sine of emergence from the plane *Dd* into the space *DdeE*; and because the sine of emergence is now become equal to the radius, the angle of emergence will be a right one, and therefore the line of emergence will coincide with the plane *Dd*. Let the body

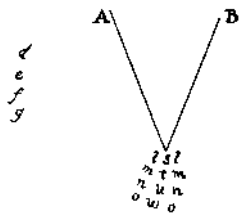


come to this plane in the point *R*; and because the line of emergence coincides with that plane, it is manifest that the body can proceed no farther towards the plane *Ee*. But neither can it proceed in the line of emergence *Kd*; because it is perpetually attracted or impelled towards the medium of incidence. It will return, therefore, between the planes *Cc*, *Dd*, describing an arc of a parabola *QRq*, whose principal vertex (by what *Galileo* hath demonstrated) is in *R*, cutting the plane *Cc* in the same angle at *q*, that it did before at *Q*; then going on in the parabolic arcs *qp*, *ph*, &c, similar and equal to the former arcs *QP*, *PH*, &c, it will cut the rest of the planes in the same angles at *p*, *h*, &c, as it did before in *P*, *H*, &c, and will emerge at last with the same obliquity at *h* with which it first impinged on that plane at *H*. Conceive now the intervals of the planes *ha*, *Bb*, *Cc*, *Dd*, *he*, &c, to be infinitely diminished, and the number infinitely increased, so that the action of attraction or impulse, exerted according to any assigned law, may become continual; and, the angle of emergence remaining all along equal to the angle of incidence, will be equal to the same also at last. Q.E.D.

SCHOLIUM

These attractions bear a great resemblance to the reflections and refractions of light made in a given ratio of the secants, as was discovered by *Snell*; and consequently in a given ratio of the sines, as was exhibited by *Descartes*. For it is now certain from the phenomena of Jupiter's satellites, confirmed by the observations of different astronomers, that light is propagated in succession, and requires about seven or eight minutes to travel from the sun to the earth. Moreover, the rays of light that are in our

air (as lately was discovered by *Grimaldi*, by the admission of light into a dark room through a small hole, which I have also tried) in their passage near the angles of bodies, whether transparent or opaque (such as the circular and rectangular edges of gold, silver, and brass coins, or of knives,



or broken pieces of stone or glass), are bent or inflected¹ round those bodies as if they were attracted to them; and those rays which in their passage come nearest to the bodies are the most inflected, as if they were most attracted; which thing I myself have also carefully observed. And those which pass at

greater distances are less inflected; and those at still greater distances are a little inflected the contrary way, and form three fringes of colors. In the figure *s* represents the edge of a knife, or any kind of wedge *Ash*; and *gowog*, *fnunf*, *emtme*, *dlsld* are rays inflected towards the knife in the arcs *owo*, *nvu*, *mtm*, *Isl*; which inflection is greater or less according to their distance from the knife. Now since this inflection of the rays is performed in the air without the knife, it follows that the rays which fall upon the knife are first inflected in the air before they touch the knife. And the case is the same of the rays falling upon glass. The refraction, therefore, is made not in the point of incidence, but gradually, by a continual inflection of the rays; which is done partly in the air before they touch the glass, partly (if I mistake not) within the glass, after they have entered it; as is represented in the rays *ckjzc*, *biyb*, *ahxa*, falling upon *r*, *q*, *p*, and inflected between *k* and *z*, *i* and *j*, *l*; and *x*. Therefore because of the analogy there is between the propagation of the rays of light and the motion of bodies, I thought it not amiss to add the following Propositions for optical uses; not at all considering the nature of the rays of light, or inquiring

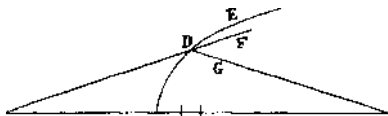
[¹ Appendix, Note 27]

whether they are bodies or not; but only determining the curves of bodies which are extremely like the curves of the rays.

PROPOSITION XCVII. PROBLEM XLVII

Supposing the sine of incidence upon any surface to be in a given ratio to the sine of emergence; and that the inflection of the paths of those bodies near that surface is performed in a very short space, which may be considered as a point, it is required to determine such a surface as may cause all the corpuscles issuing from any one given place to converge to another given place.

Let A be the place from whence the corpuscles diverge; B the place to which they should converge; CDE the curved line which by its revolution round the axis AB describes the surface sought; D, E any two points of that curve; and EF, EG perpendiculars let fall on the paths of the bodies AD, DB. Let the point D approach to and coalesce with the point E; and the ultimate ratio of the line DF by which AD is increased, to the line DG by which DB is diminished, will be the same as that of the sine of incidence to the sine of emergence. Therefore the ratio of the increment of the line AD to the decrement of the line DB is given; and therefore if in the axis AB there be taken anywhere the point C through which the curve CDE must pass, and CM the increment of AC be taken in that given ratio to CN the decrement of BC, and from the centres A, B, with the radii AM, BN, there be described two circles cutting each other in D; that point D will touch the curve sought CDE, and, by touching it anywhere at pleasure, will determine that curve. Q.E.I.



COR. 1. By causing the point A or B to go off sometimes *in infinitum*, and sometimes to move towards other parts of the point C, will be obtained all those figures which *Descartes* has exhibited in his *Optus* and *Geometry* relating to refractions. The invention of which *Descartes* having thought fit to conceal, is here laid open in this Proposition.

point F will touch the line EF, which, being turned round the axis AB, will describe the surface sought. Q.E.F.

For conceive the lines CP, CQ to be everywhere perpendicular to AD, DF, and the lines ER, ES to FB, FD respectively, and therefore QS to be always equal to CE; and (by Cor. II, Prop, XCVII) PD will be to QD as M to N, and therefore as DL to DK, or FB to FK; and by subtraction, as DL - FB or PH - PD - FB to FD or FQ - QD; and by addition as PH - FB to FQ, that is (because PH and CG, QS and CE, are equal), as CE + BG - FR to CE - FS. But (because BG is to CE as M - N to N) it comes to pass also that CE + BG is to CE as M to N; and therefore, by subtraction, FR is to FS as M to N; and therefore (by Cor. n, Prop, XCVII) the surface EF compels a body, falling upon it in the direction DF, to go on in the line FR to the place B. Q.E.D.

SCHOLIUM

In the same manner one may go on to three or more surfaces. But of all figures the spherical is the most proper for optical uses. If the object glasses of telescopes were made of two glasses of a spherical figure, containing water between them, it is not unlikely that the errors of the refractions made in the extreme parts of the surfaces of the glasses may be accurately enough corrected by the refractions of the water. Such object glasses are to be preferred before elliptic and hyperbolic glasses, not only because they may be formed with more ease and accuracy, but because the pencils of rays situated without the axis of the glass would be more accurately refracted by them. But the different refrangibility of different rays is the real obstacle that hinders optics from being made perfect by spherical or any other figures. Unless the errors thence arising can be corrected, all the labor spent in correcting the others is quite thrown away.

Book Two

THE MOTION OF BODIES

(IN RESISTING MEDIUMS)

SECTION I

The motion of bodies that are resisted in the ratio of the velocity.

PROPOSITION I. THEOREM I

If a body is resisted in the ratio of its velocity, the motion lost by resistance is as the space gone over in its motion.

For since the motion lost in each equal interval of time is as the velocity, that is, as the small increment of space gone over, then, by composition, the motion lost in the whole time will be as the whole space gone over. Q.E.D.

COR. Therefore if the body, destitute of all gravity, move by its innate force only in free spaces, and there be given both its whole motion at the beginning, and also the motion remaining after some part of the way is gone over, there will be given also the whole space which the body can describe in an infinite time. For that space will be to the space now described as the whole motion at the beginning is to the part lost of that motion.

L E M M A I

Quantities proportional to their differences are continually proportional.

Let $A : A - B = B : B - C = C : C - D = \&c;$
then, by subtraction,

$$A : B - B : C - C : D - \&c.$$

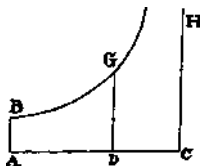
Q.E.D.

PROPOSITION II. THEOREM II

If a body is resisted in the ratio of its velocity, and moves, by its inertia only, through an homogeneous medium, and the times be taken equal, the velocities in the beginning of each of the times are in a geometrical progression, and the spaces described in each of the times are as the velocities.

CASE I. Let the time be divided into equal intervals; and if at the very beginning of each interval we suppose the resistance to act with one single impulse which is as the velocity, the decrement of the velocity in each of the intervals of time will be as the same velocity. Therefore the velocities are proportional to their differences, and therefore (by Lem. 1, Book 11) continually proportional. Therefore if out of an equal number of intervals there be compounded any equal portions of time, the velocities at the beginning of those times will be as terms in a continued progression, which are taken by jumps, omitting everywhere an equal number of intermediate terms. But the ratios of these terms are compounded of the equal ratios of the intermediate terms equally repeated, and therefore are equal. Therefore the velocities, being proportional to those terms, are in geometrical progression. Let those equal intervals of time be diminished, and their number increased *in infinitum*, so that the impulse of resistance may become continual; and the velocities at the beginnings of equal times, always continually proportional, will be also in this case continually proportional. Q.E.D.¹

CASE 2. And, by division, the differences of the velocities, that is, the parts of the velocities lost in each of the times, are as the wholes; but the spaces described in each of the times are as the lost parts of the velocities (by Prop. 1, Book 1), and therefore are also as the wholes. Q.E.D.



COR. Hence if to the rectangular asymptotes AC, CH, the hyperbola BG is described, and AB, DG be drawn perpendicular to the asymptote AC, and both the velocity of the body, and the

resistance of the medium, at the very beginning of the motion, be expressed by any given line AC, and, after some time is elapsed, by the indefinite line DC; the time may be expressed by the area ABGD, and the space described

[¹ Appendix. Note 28 1

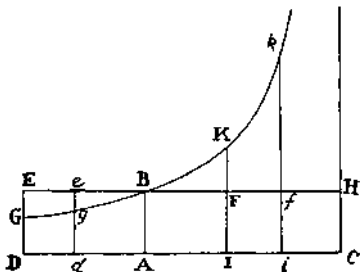
in that time by the line AD. For if that area, by the motion of the point D, be uniformly increased in the same manner as the time, the right line DC will decrease in a geometrical ratio in the same manner as the velocity; and the parts of the right line AC, described in equal times, will decrease in the same ratio.

PROPOSITION III. PROBLEM I

To define the motion of a body which, in an homogeneous medium, ascends or descends in a right line, and is resisted in the ratio of its velocity, and acted upon by an uniform force of gravity.

The body ascending, let the gravity be represented by any given rectangle BACH; and the resistance of the medium, at the beginning of the ascent,

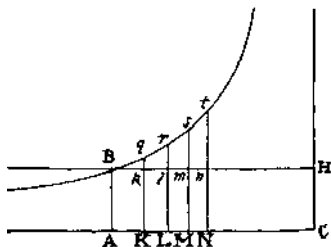
by the rectangle BADE, taken on the contrary side of the right line AB. Through the point B, with the rectangular asymptotes AC, CH, describe an hyperbola, cutting the perpendiculars DE, de in G, g; and the body ascending will in the time DGgd describe the space EGge; in the time DGBA, the space of the whole ascent EGB; in the time ABKI, the space of descent BFK; and in



the time IKki the space of descent KFfk; and the velocities of the bodies (proportional to the resistance of the medium) in these periods of time will be ABED, ABed, o, ABFI, ABfi respectively; and the greatest velocity which the body can acquire by descending will be BACH.

For let the rectangle BACH be resolved into innumerable rectangles $A^{\wedge}$, Yd, Lm, Mn, &c, which shall be as the increments of the velocities produced in so many equal times; then will o, Ak, Al, Am, An, &c, be as the whole velocities, and therefore (by supposition) as the resistances of the medium in the beginning of each of the equal times. Make AC to AK, or ABHC to ABkK, as the force of gravity to the resistance in the beginning of the second time; then from the force of gravity subtract the resistances,

and ABHC, KkHC, LlHC, MmHC, &c, will be as the absolute forces with which the body is acted upon in the beginning of each of the times, and therefore (by Law 1) as the increments of the velocities, that is, as the rec-



tangles Ak, Kl, Lm, Mn, &c, and therefore (by Lem. 1, Book 11) in a geometrical progression. Therefore, if the right lines Kk, Ll, Mm, Nn, &c, are produced so as to meet the hyperbola in q, r, s, t, &c, the areas ABqK, KqrL, LrsM, MstN, &c, will be equal, and therefore analogous to the equal times and equal gravitating forces. But the area ABqK (by Cor. iii, Lem. vii

and viii, Book 1) is to the area Bqk as Kq to $\frac{1}{2}kq$, or AC to $\frac{1}{2}AK$, that is, as the force of gravity to the resistance in the middle of the first time. And by the like reasoning, the areas qKLr, rLMs, sMNt, &c, are to the areas qklr, rims, smnt, &c, as the gravitating forces to the resistances in the middle of the second, third, fourth time, and so on. Therefore since the equal areas BKKq, qKLr, rLMs, sMNt, &c, are analogous to the gravitating forces, the areas Bqk, qklr, rims, smnt, &c, will be analogous to the resistances in the middle of each of the times, that is (by supposition), to the velocities, and so to the spaces described. Take the sums of the analogous quantities, and the areas Blq, Blr, Bms, Bnt, &c, will be analogous to the whole spaces described; and also the areas ABqK, ABrL, AB/M, AB/N, &c, to the times. Therefore the body, in descending, will in any time ABrL describe the space Blr, and in the time Lr/N the space rnt. Q.E.D. And the like demonstration holds in ascending motion.

COR. 1. Therefore the greatest velocity that the body can acquire by falling is to the velocity acquired in any given time as the given force of gravity which continually acts upon it to the resisting force which opposes it at the end of that time.

COR. 11. But the time being augmented in an arithmetical progression, the sum of that greatest velocity and the velocity in the ascent, and also their difference in the descent, decreases in a geometrical progression.

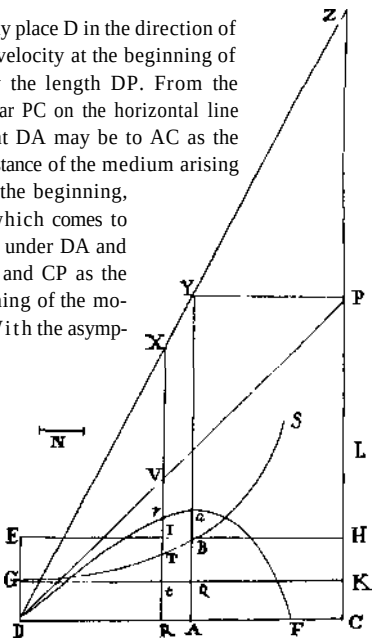
COR. HI. Also the differences of the spaces, which are described in equal differences of the times, decrease in the same geometrical progression.

COR. IV. The space described by the body is the difference of two spaces, whereof one is as the time taken from the beginning of the descent, and the other as the velocity; which [spaces] also at the beginning of the descent are equal among themselves.

PROPOSITION IV. PROBLEM II

Supposing the force of gravity in any homogeneous medium to be uniform, and to tend perpendicularly to the plane of the horizon: to define the motion of a projectile therein, which suffers resistance proportional to its velocity.

Let the projectile go from any place D in the direction of any right line DP, and let its velocity at the beginning of the motion be represented by the length DP. From the point P let fall the perpendicular PC on the horizontal line DC, and cut DC in A, so that DA may be to AC as the vertical component of the resistance of the medium arising from the motion upwards at the beginning, to the force of gravity; or (which comes to the same) so that the rectangle under DA and DP may be to that under AC and CP as the whole resistance at the beginning of the motion, to the force of gravity. With the asymptotes DC, CP describe any hyperbola GTBS cutting the perpendiculars DG, AB in G and B; complete the parallelogram DGKC, and let its side GK cut AB in Q. Take a line N in the same ratio to QB as DC is in to CP; and from any point R of the right line DC erect RT perpendicular to it, meeting the hyperbola in T,



and the right lines EH, GK, DP in I, t , and V; in that perpendicular take Vr equal to $\frac{rGT}{N}$, or, which is the same thing, take Rr equal to $t\frac{GTIE}{N}$; and the projectile in the time DRTG will arrive at the point r, describing the curved line DraF, the locus of the point r; thence it will come to its greatest height a in the perpendicular AB; and afterwards ever approach to the asymptote PC. And its velocity in any point r will be as the tangent rL to the curve. Q.E.I.

For $N : QB = DC : CP = DR : RV$,

and therefore RV is equal to $\frac{DR \cdot QB}{N}$, and Rr (that is, $RV - Vr$, or $\frac{DR \cdot QB - rGT}{N}$) is equal to $\frac{DR \cdot AB - RDGT}{N}$. Now let the time be rep-

resented by the area RDGT, and (by Laws, Cor. 11) distinguish the motion of the body into two others, one of ascent, the other lateral. And since the resistance is as the motion, let that also be distinguished into two parts proportional and contrary to the parts of the motion: and therefore the length described by the lateral motion will be (by Prop. 11, Book 11) as the line DR, and the height (by Prop. in, Book 11) as the area DR · AB - RDGT, that is, as the line Rr. But in the very beginning of the motion the area RDGT is equal to the rectangle DR · AQ, and therefore that line Rr (or $\frac{DR \cdot AB - DR \cdot AQ}{N}$) will then be to DR as AB - AQ or QB to N,

that is, as CP to DC; and therefore as the motion upwards to the motion lengthwise at the beginning. Since, therefore, Rr is always as the height, and DR always as the length, and Rr is to DR at the beginning as the height to the length, it follows, that Rr is always to DR as the height to the length; and therefore that the body will move in the line DraF, which is the locus of the point r. Q.E.D.

COR. 1. Therefore Rr is equal to $\frac{DR \cdot AB}{N} - \frac{RDGT}{N}$; and therefore if RT

be produced to X so that RX may be equal to $\frac{DR \cdot AB}{N}$, that is, if the parallelogram ACPY be completed, and DY cutting CP in Z be drawn, and

RT be produced till it meets DY in X; Xr will be equal to $\frac{RDGT}{N}$, and therefore proportional to the time.

COR. II. Whence if innumerable lines CR, or, which is the same, innumerable lines ZX, be taken in a geometrical progression, there will be as many lines Xr in an arithmetical progression. And hence the curve *DraF* is easily delineated by the table of logarithms.

COR. in. If a parabola be constructed to the vertex D, and the diameter DG produced downwards, and its latus rectum is to 2 DP as the whole resistance at the beginning of the motion to the gravitating force, the velocity with which the body ought to go from the place D, in the direction of the right line DP, so as in an uniform resisting medium to describe the curve DraF, will be the same as that with which it ought to go from the same place D in the direction of the same right line DP, so as to describe a parabola in a nonresisting medium. For the latus rectum of this parabola, at the very beginning of the motion, is $\frac{DV^2}{Vr}$; and Vr is $\frac{GT}{N}$ or $\frac{DR \cdot Tt}{2N}$. But a right line which, if drawn, would touch the hyperbola GTS in G, is parallel to DK, and therefore Tt is $\frac{CK \cdot DK}{DC}$, and N is $\frac{QB \cdot DC}{CP}$. And therefore Vr is equal to $\frac{DR^2 \cdot CK \cdot CP}{2DC^2 \cdot QB}$, that is (because DR and DC, DV and DP are proportionals), to $\frac{DV^2 \cdot CK \cdot CP}{2DP^2 \cdot QB}$; and the latus rectum $\frac{DV^2}{Vr}$ comes out $\frac{2DP^2 \cdot QB}{CK \cdot CP}$, that is (because QB and CK, DA and AC are proportional), $\frac{2DP^2 \cdot DA}{AC \cdot CP}$, and therefore is to 2DP as DP · DA to CP · AC; that is, as the resistance to the gravity. Q.E.D.

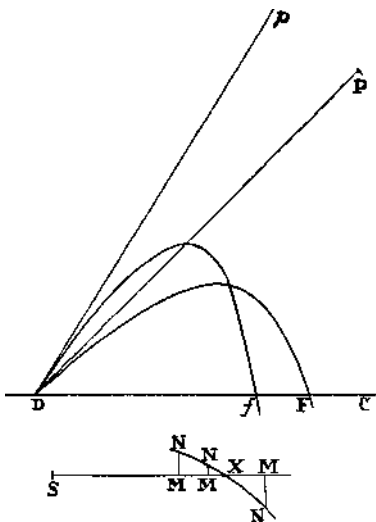
COR VII. Hence appears the method of determining the curve $DraF$ nearly from the phenomena, and thence finding the resistance and velocity with which the body is projected. Let two similar and equal bodies be projected with the same velocity,

from the place D , in different angles CDP , CDp ; and let the places F , f , where they fall upon the horizontal plane DC , be known. Then taking any length for DP or Dp suppose the resistance in D to be to the gravity in any ratio whatsoever, and let that ratio be represented by any length SM . Then, by computation, from that assumed length DP , find the lengths DF , Df ;

and from the ratio $\frac{Ff}{DF}$, found

by calculation, subtract the same ratio as found by experiment; and let the difference be represented by the perpendicular MN . Repeat the same

a second and a third time, by assuming always a new ratio SM of the resistance to the gravity, and collecting a new difference MN . Draw the positive differences on one side of the right line SM , and the negative on the other side; and through the points N , N , N , draw a regular curve NNN , cutting the right line $SMMM$ in X , and SX will be the true ratio of the resistance to the gravity, which was to be found. From this ratio the length DF is to be found by calculation; and a length, which is to the assumed length DP as the length DF known by experiment to the length DF just now found, will be the true length DP . This being known, you will have both the curved line $DraF$ which the body describes, and also the velocity and resistance of the body in each place.



S C H O L I U M

However, that the resistance of bodies is in the ratio of the velocity, is more a mathematical hypothesis than a physical one. In mediums void of all tenacity, the resistances made to bodies are as the square of the velocities. For by the action of a swifter body, a greater motion in proportion to a greater velocity is communicated to the same quantity of the medium in a less time; and in an equal time, by reason of a greater quantity of the disturbed medium, a motion is communicated as the square of the ratio greater; and the resistance (by Law *n* and *in*) is as the motion communicated. Let us, therefore, see what motions arise from this law of resistance.

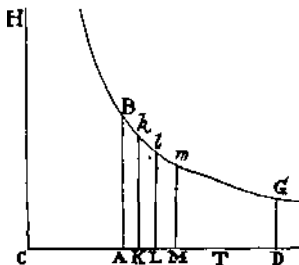
SECTION II

The motion of bodies that are resisted as the square of their velocities.

PROPOSITION V. THEOREM III

If a body is resisted as the square of its velocity, and moves by its innate force only through an homogeneous medium; and the times be taken in a geometrical progression, proceeding from less to greater terms: I say, that the velocities at the beginning of each of the times are in the same geometrical progression inversely; and that the spaces are equal, which are described in each of the times.

For since the resistance of the medium is proportional to the square of the velocity, and the decrement of the velocity is proportional to the resistance: if the time be divided into innumerable equal intervals, the squares of the velocities at the beginning of each of the times will be proportional to the differences of the same velocities. Let those intervals of time be AK , KL , LM , &c, taken in the right line CD ; and erect the perpendiculars AB , Kk , Ll , Mm , &c, meeting the hyperbola $BklmG$, described with the centre C , and the rectangular asymptotes CD , CH , in B , k , l , m , &c; then AB will be to Kk as CK to CA , and, by division, $AB - Kk$ to Kk as AK to CA , and alternately, $AB - Kk$ to AK as Kk to CA ; and therefore as $AB - Kk$ to $AB - CA$. Therefore since AK and



$AB - CA$ are given, $AB - Kk$ will be as $AB - Kk$; and, lastly, when AB and Kk coincide, as AB^2 . And, by the like reasoning, $Kk - Ll$, $Ll - Mm$, &c, will be as Kk^2 , Ll^2 , &c. Therefore the squares of the lines AB , Kk , Ll , Mm , &c, are as their differences; and, therefore, since the squares of the velocities were shown above to be as their differences, the progression of both will be alike. This being demonstrated it follows also that the areas described by these lines are in a like progression with the spaces described by these veloci-

ties. Therefore if the velocity at the beginning of the first time AK be represented by the line AB, and the velocity at the beginning of the second time KL by the line Kk, and the length described in the first time by the area AKkB, all the following velocities will be represented by the following lines *LI*, *Mm*, &c, and the lengths described by the areas *Yd*, *Lm*, &c. And, by composition, if the whole time be represented by AM, the sum of its parts, the whole length described will be represented by AMwB, the sum of its parts. Now conceive the time AM to be divided into the parts AK, KL, LM, &c, so that CA, CK, CL, CM, &c, may be in a geometrical progression; and those parts will be in the same progression, and the velocities AB, K[^], LI, Mm, &c, will be in the same progression inversely, and the spaces described Ak, Kl, Lm, &c, will be equal. Q.E.D.

COR. 1. Hence it appears, that if the time be represented by any part AD of the asymptote, and the velocity in the beginning of the time by the ordinate AB, the velocity at the end of the time will be represented by the ordinate DG; and the whole space described by the adjacent hyperbolic area ABGD; and the space which any body can describe in the same time AD, with the first velocity AB, in a nonresisting medium, by the rectangle ABAD.

COR. 11. Hence the space described in a resisting medium is given, by taking it to the space described with the uniform velocity AB in a non-resisting medium, as the hyperbolic area ABGD to the rectangle AB • AD.

COR. in. The resistance of the medium is also given, by making it equal, in the very beginning of the motion, to an uniform centripetal force, which could generate, in a body falling through a nonresisting medium, the velocity AB in the time AC. For if BT be drawn touching the hyperbola in B, and meeting the asymptote in T, the right line AT will be equal to AC, and will express the time in which the first resistance, uniformly continued, may take away the whole velocity AB.

COR. iv. And thence is also given the proportion of this resistance to the force of gravity, or any other given centripetal force.

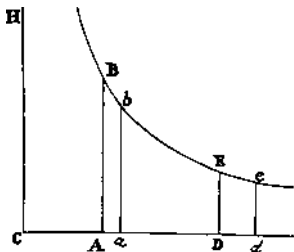
COR. v. And, conversely, if there is given the proportion of the resistance to any given centripetal force, the time AC is also given, in which a centripetal force equal to the resistance may generate any velocity as AB; and thence is given the point B, through which the hyperbola, having CH,

CD for its asymptotes, is to be described; as also the space ABGD, which a body, by beginning its motion with that velocity AB, can describe in any time AD, in an homogeneous resisting medium.

PROPOSITION VI. THEOREM IV

Homogeneous and equal spherical bodies, opposed by resistances that are as the square of the velocities, and moving on by their innate force only, will, in times which are inversely as the velocities at the beginning, describe equal spaces, and lose parts of their velocities proportional to the wholes.

To the rectangular asymptotes CD, CH describe any hyperbola BEd*, cutting the perpendiculars AB, *ab*, DE, *de* in B, *b*, E, *e*; let the initial velocities be represented by the perpendiculars AB, DE, and the times by the lines Aa, Dd. Therefore as Aa is to Dd, so (by the hypothesis) is DE to AB, and so (from the nature of the hyperbola) is CA to CD; and, by composition, so is Ca to Cd. Therefore the areas ABba, DEed, that is, the spaces described, are equal among themselves, and the first velocities AB, DE are proportional to the last *ab*, *de*; and therefore, by subtraction, proportional to the parts of the velocities lost, AB - *ab*, DE - *de*. Q.E.D.



PROPOSITION VII. THEOREM V

*If spherical bodies are resisted as the squares of their velocities, in times which are directly as the first motions, and inversely as the first resistances, they will lose parts of their motions proportional to the wholes, and will describe spaces proportional to the product of those times and the first velocities.*¹

For the parts of the motions lost are as the product of the resistances and times. Therefore, that those parts may be proportional to the wholes, the product of the resistance and time ought to be as the motion. Therefore the

[¹ Appendix, Note 29]

time will be as the motion directly and the resistance inversely. Therefore the intervals of the times being taken in that ratio, the bodies will always lose parts of their motions proportional to the wholes, and therefore will retain velocities always proportional to their first velocities. And because of the given ratio of the velocities, they will always describe spaces which are as the product of the first velocities and the times. Q.E.D.

COR. 1. Therefore if bodies equally swift are resisted as the square of their diameters, homogeneous globes moving with any velocities whatsoever, by describing spaces proportional to their diameters, will lose parts of their motions proportional to the wholes. For the motion of each globe will be as the product of its velocity and mass, that is, as the product of the velocity and the cube of its diameter; the resistance (by supposition) will be as the product of the square of the diameter and the square of the velocity; and the time (by this Proposition) is in the former ratio directly, and in the latter inversely, that is, as the diameter directly and the velocity inversely; and therefore the space, which is proportional to the time and velocity, is as the diameter.

COR. 11. If bodies equally swift are resisted as the $1/4$ th power of their diameters, homogeneous globes, moving with any velocities whatsoever, by describing spaces that are as the $3/2$ th power of the diameters, will lose parts of their motions proportional to the wholes.

COR. in. And universally, if equally swift bodies are resisted in the ratio of any power of the diameters, the spaces, in which homogeneous globes, moving with any velocity whatsoever, will lose parts of their motions proportional to the wholes, will be as the cubes of the diameters applied to that power. Let those diameters be D and E ; and if the resistances, where the velocities are supposed equal, are as D^n and E^n ; the spaces in which the globes, moving with any velocities whatsoever, will lose parts of their motions proportional to the wholes, will be as D^{3-n} and E^{3-n} . And therefore homogeneous globes, in describing spaces proportional to D^{8-n} and E^{3-n} , will retain their velocities in the same ratio to one another as at the beginning.

COR. iv. Now if the globes are not homogeneous, the space described by the denser globe must be augmented in the ratio of the density. For the motion, with an equal velocity, is greater in the ratio of the density, and the

time (by this Proposition) is augmented in the ratio of motion directly, and the space described in the ratio of the time.

COR. V. And if the globes move in different mediums, the space, in a medium which, other things being equal, resists the most, must be diminished in the ratio of the greater resistance. For the time (by this Proposition) will be diminished in the ratio of the augmented resistance, and the space in the ratio of the time.

L E M M A I I

The moment of any genitum is equal to the moments of each of the generating sides multiplied by the indices of the powers of those sides, and by their coefficients continually.

I call any quantity a *genitum* which is not made by addition or subtraction of divers parts, but is generated or produced in arithmetic by the multiplication, division, or extraction of the root of any terms whatsoever; in geometry by the finding of contents and sides, or of the extremes and means of proportionals. Quantities of this kind are products, quotients, roots, rectangles, squares, cubes, square and cubic sides, and the like. These quantities I here consider as variable and indetermined, and increasing or decreasing, as it were, by a continual motion or flux; and I understand their momentary increments or decrements by the name of moments; so that the increments may be esteemed as added or affirmative moments; and the decrements as subtracted or negative ones. But take care not to look upon finite particles as such.¹ Finite particles are not moments, but the very quantities generated by the moments. We are to conceive them as the just nascent principles of finite magnitudes. Nor do we in this Lemma regard the magnitude of the moments, but their first proportion, as nascent. It will be the same thing, if, instead of moments, we use either the velocities of the increments and decrements (which may also be called the motions, mutations, and fluxions of quantities), or any finite quantities proportional to those velocities. The coefficient of any generating side is the quantity which arises by applying the genitum to that side.

Wherefore the sense of the Lemma is, that if the moments of any quantities $A, B, C, \&c$, increasing or decreasing by a continual flux, or the velocities of the mutations which are proportional to them, be called $a, b, c, \&c$,

[¹ Appendix, Note 30]

the moment or mutation of the generated rectangle AB will be $aB + bA$; the moment of the generated content ABC will be $aBC + bAC + cAB$; and the moments of the generated powers $A^2, A^3, A^4, A^{1/2}, A^{3/2}, A^{5/2}, A^{7/2}, A^{-1}, A^{-2}, A^{-3/2}$ will be $2aA, 3aA^2, 4aA^3, \frac{1}{2}aA^{-1/2}, \frac{3}{2}aA^{1/2}, \frac{5}{2}aA^{3/2}, \frac{7}{2}aA^{5/2}, -aA^{-2}, -2aA^{-3}, -\frac{1}{2}aA^{-5/2}$ respectively; and, in general, that the moment of any power A^m will be $\frac{n}{m} aA^{m-1}$. Also, that the moment of the generated quantity A^2B will be $2aAB + bA^2$; the moment of the generated quantity $A^3B^2C^2$ will be $3aA^2B^2C^2 + 4bA^3B^2C^2 + 2cA^3B^2C$; and the moment of the generated quantity $\frac{A^3}{B^2}$ or A^3B^{-2} will be $3aA^2B^{-2} - 2bA^3B^{-3}$; and so on. The Lemma is thus demonstrated.¹

CASE 1. Any rectangle, as AB , augmented by a continual flux, when, as yet, there wanted of the sides A and B half their moments $\frac{1}{2}a$ and $\frac{1}{2}b$, was $A - \frac{1}{2}a$ into $B - \frac{1}{2}b$, or $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$; but as soon as the sides A and B are augmented by the other half-moments, the rectangle becomes $A + \frac{1}{2}a$ into $B + \frac{1}{2}b$, or $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$. From this rectangle subtract the former rectangle, and there will remain the excess $aB + bA$. Therefore with the whole increments a and b of the sides, the increment $aB + bA$ of the rectangle is generated. Q.E.D.

CASE 2. Suppose AB always equal to G , and then the moment of the content ABC or GC (by Case 1) will be $gC + cG$, that is (putting AB and $aB + bA$ for G and g), $aBC + bAC + cAB$. And the reasoning is the same for contents under ever so many sides. Q.E.D.

CASE 3. Suppose the sides A, B , and C , to be always equal among themselves; and the moment $aB + bA$, of A^2 , that is, of the rectangle AB , will be $2aA$; and the moment $aBC + bAC + cAB$ of A^3 , that is, of the content ABC , will be $3aA^2$. And by the same reasoning the moment of any power A^n is naA^{n-1} . Q.E.D.

CASE 4. Therefore since $\frac{1}{A}$ into A is 1, the moment of $\frac{1}{A}$ multiplied by A , together with $\frac{1}{A}$ multiplied by a , will be the moment of 1, that is, nothing. Therefore the moment of $\frac{1}{A}$, or of A^{-1} , is $-\frac{a}{A^2}$. And generally since $\frac{1}{A^n}$ into A^n is 1, the moment of $\frac{1}{A^n}$ multiplied by A^n together with $\frac{1}{A^n}$ into

[¹ Appendix, Note 31.]

naA^{n-1} will be nothing. And, therefore, the moment of $\frac{1}{A^n}$ or A^{-n} will be $-\frac{na}{A^{n+1}}$. Q.E.D.

CASE 5. And since $A^{\frac{1}{2}}$ into $A^{\frac{1}{2}}$ is A , the moment of $A^{\frac{1}{2}}$ multiplied by $2A^{\frac{1}{2}}$ will be a (by Case 3); and, therefore, the moment of $A^{\frac{1}{2}}$ will be $\frac{a}{2A^{\frac{1}{2}}}$ or $\frac{1}{2}aA^{-\frac{1}{2}}$. And generally, putting A^m equal to B , then A^m will be equal to B^n , and therefore maA^{m-1} equal to nbB^{n-1} , and maA^{-1} equal to nbB^{-1} , or $nbA^{-\frac{m}{n}}$; and therefore $\frac{m}{n}aA^{\frac{n-m}{n}}$ is equal to b , that is, equal to the moment of $A^{\frac{m}{n}}$. Q.E.D.

CASE 6. Therefore the moment of any generated quantity A^mB^n is the moment of A^m multiplied by B^n , together with the moment of B^n multiplied by A^m , that is, $maA^{m-1}B^n + nbB^{n-1}A^m$; and that whether the indices m and n of the powers be whole numbers or fractions, affirmative or negative. And the reasoning is the same for higher powers. Q.E.D.

COR. I. Hence in quantities continually proportional,¹ if one term is given, the moments of the rest of the terms will be as the same terms multiplied by the number of intervals between them and the given term. Let A, B, C, D, E, F be continually proportional; then if the term C is given, the moments of the rest of the terms will be among themselves as $—2A, —B, D, 2E, 3F$.

COR. II. And if in four proportionals the two means are given, the moments of the extremes will be as those extremes. The same is to be understood of the sides of any given rectangle.

Cor. in. And if the sum or difference of two squares is given, the moments of the sides will be inversely as the sides.

SCHOLIUM²

In a letter of mine to Mr. J. Collins, dated *December 10, 1672*, having described a method of tangents, which I suspected to be the same with Sluse's method, which at that time was not made public, I added these words: *This is one particular, or rather a Corollary, of a general method, which extends itself, without any troublesome calculation, not only to the drawing of tangents to any curved lines, whether geometrical or median-*

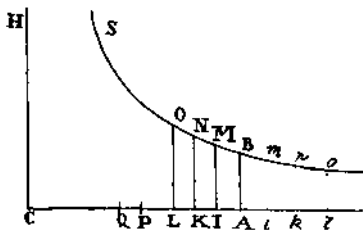
¹ Appendix, Note 32] ² Appendix, Note 33]

ical or in any manner respecting right lines or other curves, but also to the resolving other abstruser kinds of problems about the crookedness, areas, lengths, centres of gravity of curves, &c; nor is it (as Hudden's method de maximis et minimis) limited to equations which are free from surd quantities. This method I have interwoven with that other of wording in equations, by reducing them to infinite series. So far that letter. And these last words relate to a treatise I composed on that subject in the year 1671. The foundation of that general method is contained in the preceding Lemma.

PROPOSITION VIII. THEOREM VI

If a body in an uniform medium, being uniformly acted upon by the force of gravity, ascends or descends in a right line; and the whole space described be divided into equal parts, and in the beginning of each of the parts (by adding or subtracting the resisting force of the medium to or from the force of gravity, when the body ascends or descends) you derive the absolute forces: I say, that those absolute forces are in a geometrical progression.

Let the force of gravity be represented by the given line AC; the force of resistance by the indefinite line AK; the absolute force in the descent of the body by the difference KC; the velocity of the body by a line AP, which



shall be a mean proportional between AK and AC, and therefore as the square root of the resistance; the increment of the resistance made in a given interval of time by the short line KL, and the contemporaneous increment of the velocity by the short line PQ; and with the centre

C, and rectangular asymptotes CA, CH, describe any hyperbola BNS meeting the erected perpendiculars AB, KN, LO in B, N, and O. Because AK is as AP^2 , the moment KL of the one will be as the moment $2AP \cdot PQ$ of the other, that is, as $AP \cdot KC$; for the increment PQ of the velocity is (by Law n) proportional to the generating force KC. Let the ratio of KL be

multiplied by the ratio KN , and the rectangle $KL KN$ will become as $AP KC - KN$; that is (because the rectangle $KC KN$ is given), as AP . But the ultimate ratio of the hyperbolic area $KNOL$ to the rectangle $KL KN$ becomes, when the points K and L coincide, the ratio of equality. Therefore that hyperbolic evanescent area is as AP . Therefore the whole hyperbolic area $ABOL$ is composed of intervals $KNOL$ which are always proportional to the velocity AP ; and therefore is itself proportional to the space described with that velocity. Let that area be now divided into equal parts, as $ABMI$, $IMNK$, $KNOL$, &c., and the absolute forces AC , IC , KC , LC , &c., will be in a geometrical progression. Q.E.D. And by a like reasoning, in the ascent of the body, taking, on the contrary side of the point A , the equal areas $ABmi$, $imnk$, $hnol$, &c., it will appear that the absolute forces AC , iC , kC , IC , &c., are continually proportional. Therefore if all the spaces in the ascent and descent arc taken equal, all the absolute forces $/C$, ^{C} , iC , AC , IC , KC , LC , &c., will be continually proportional. Q.E.D.

COR. 1. Hence if the space described be represented by the hyperbolic area $ABNK$, the force of gravity, the velocity of the body, and the resistance of the medium, may be represented by the lines AC , AP , and AK respectively; and conversely.

COR. 11. And the greatest velocity which the body can ever acquire in an infinite descent will be represented by the line AC .

COR. in. Therefore if the resistance of the medium answering to any given velocity be known, the greatest velocity will be found, by taking it to that given velocity, as the square root of the ratio which the force of gravity bears to that known resistance of the medium.

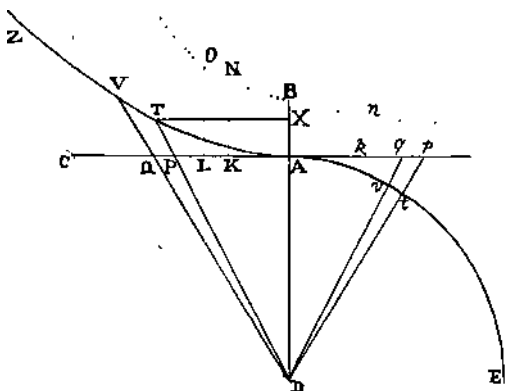
PROPOSITION IX. THEOREM VII

Supposing what is above demonstrated, I say, that if the tangents of the angles of the sector of a circle, and of an hyperbola, be taken proportional to the velocities, the radius being of a fit magnitude, all the time of the ascent to the highest place will be as the sector of the circle, and all the time of descending from the highest place as the sector of the hyperbola.

To the right line AC , which expresses the force of gravity, let AD be drawn perpendicular and equal. From the centre D , with the semidiameter AD describe as well the quadrant AtE of a circle, as the rectangular hyper-

bola AVZ, whose axis is AK, principal vertex A, and asymptote DC. Let Dp , DP be drawn; and the circular sector AtD will be as all the time of the ascent to the highest place; and the hyperbolic sector ATD as all the time of descent from the highest place; if so be that the tangents hp , AP of those sectors be as the velocities.

CASE I. Draw Dvq cutting off the moments or least intervals tDv and qDp , described in the same time, of the sector ADt and of the triangle



ADp . Since those intervals (because of the common angle D) are as the square of the sides, the interval tDv will be as $\frac{qDp \cdot tD^2}{pD^2}$, that is (because tD is given), as $\frac{qDp}{pD^2}$. But pD^2 is $AD^2 + Ap^2$, that is, $AD^2 + AD \cdot Ak$, or $AD \cdot Ck$; and qDp is $\frac{1}{2}AD \cdot pq$. Therefore tDv , the interval of the sector, is as $\frac{pq}{Ck}$; that is, directly as the least decrement pq of the velocity, and inversely as the force Ck which diminishes the velocity; and therefore as the interval of time answering to the decrement of the velocity. And, by composition, the sum of all the intervals tDv in the sector ADr will be as the sum of the intervals of time answering to each of the lost intervals pq of

COR. 1. Hence if AB be equal to a fourth part of AC, the space which a body will describe by falling in any time will be to the space which the body could describe, by moving uniformly on in the same time with its greatest velocity AC, as the area ABNK, which expresses the space described in falling to the area ATD, which expresses the time. For since

$$AC:AP = AP:AK,$$

and by Cor. 1, Lem. 11, of this Book,

$$LK:PQ = 2AK:AP - 2AP:AC,$$

therefore $LK:4PQ = AP:\frac{1}{4}AC$ or AB,

and since $KN:AC$ or $AD=AD:CK$,

multiplying together corresponding terms,

$$LKNO:DPQ = AP:CK.$$

As shown above,

$$DPQ:DTV = CK:AC.$$

Hence, $LKNO:DTV = AP:AC$;

that is, as the velocity of the falling body to the greatest velocity which the body by falling can acquire. Since, therefore, the moments LKNO and DTV of the areas ABNK and ATD are as the velocities, all the parts of those areas generated in the same time will be as the spaces described in the same time; and therefore the whole areas ABNK and ADT, generated from the beginning, will be as the whole spaces described from the beginning of the descent. Q.E.D.

COR. 11. The same is true also of the space described in the ascent. That is to say, that all that space is to the space described in the same time, with the uniform velocity AC, as the area ABnk is to the sector ADt.

COR. in. The velocity of the body, falling in the time ATD, is to the velocity which it would acquire in the same time in a nonresisting space, as the triangle APD to the hyperbolic sector ATD. For the velocity in a nonresisting medium would be as the time ATD, and in a resisting medium is as AP, that is, as the triangle APD. And those velocities, at the beginning of the descent, are equal among themselves, as well as those areas ATD, APD.

COR. iv. By the same argument, the velocity in the ascent is to the velocity with which the body in the same time, in a nonresisting space, would lose all its motion of ascent, as the triangle ApD to the circular sector A/D; or as the right line Ap to the arc At.

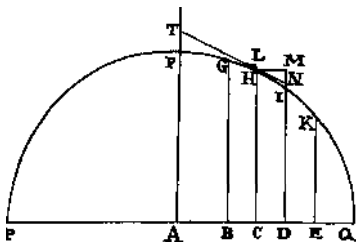
COR. v. Therefore the time in which a body, by falling in a resisting medium, would acquire the velocity AP , is to the time in which it would acquire its greatest velocity AC , by falling in a nonresisting space, as the sector ADT to the triangle ADC ; and the time in which it would lose its velocity Ap , by ascending in a resisting medium, is to the time in which it would lose the same velocity by ascending in a nonresisting space, as the arc At to its tangent Ap .

COR. VI. Hence from the given time there is given the space described in the ascent or descent. For the greatest velocity of a body descending *in infinitum* is given (by Cor. 11 and 111, Theor. vi, of this Book); and thence the time is given in which a body would acquire that velocity by falling in nonresisting space. Taking the sector ADT or ADt to the triangle ADC in the ratio of the given time to the time just found, there will be given both the velocity AP or Ap , and the area $ABNK$ or $ABnk$, which is to the sector ADT , or ADt , as the space sought to that which would, in the given time, be uniformly described with that greatest velocity found just before.

COR. VII. And by going backwards, from the given space of ascent or descent $ABnk$ or $ABNK$, there will be given the time ADt or ADT .

PROPOSITION X. PROBLEM III¹

Suppose the uniform force of gravity to tend directly to the plane of the horizon, and the resistance to be as the product of the density of the medium and the square of the velocity: it is proposed to find the density of the medium in each place, which shall make the body move in any given curved line, the velocity of the body, and the resistance of the medium in each place.



Let PQ be a plane perpendicular to the plane of the scheme itself; $PFHQ$ a curved line meeting that plane in the points P and Q ; G, H, I, K four places of the body going on in this curve from F to Q ; and GB, HC, ID, KE four parallel ordinates let fall from these points to the horizon, and

[¹ Appendix, Note 34.]

standing on the horizontal line PQ at the points B, C, D, E; and let the distances BC, CD, DE of the ordinates be equal among themselves. From the points G and H let the right lines GL, HN be drawn touching the curve in G and H, and meeting the ordinates CH, DI, produced upwards, in L and N; and complete the parallelogram HCDM. And the times in which the body describes the arcs GH, HI, will be as the square root of the altitudes LH, NI, which the bodies would describe in those times, by falling from the tangents; and the velocities will be directly as the lengths described GH, HI, and inversely as the times. Let the times be represented by T and t,

and the velocities by $\frac{GH}{T}$ and $\frac{HI}{t}$; and the decrement of the velocity produced in the time t will be represented by $\frac{GH}{T} - \frac{HI}{t}$. This decrement arises

from the resistance which retards the body, and from the gravity which accelerates it. Gravity, in a falling body, which in its fall describes the space NI, produces a velocity with which it would be able to describe twice that space in the same time, as *Galileo* hath demonstrated; that is, the velocity $\frac{2NI}{t}$; but if the body describes the arc HI, it augments that arc only by

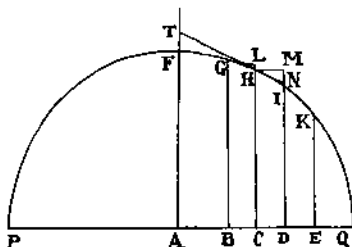
the length HI - HN or $\frac{MI \cdot NI}{HI}$; and therefore generates only the velocity $\frac{2MI \cdot NI}{t \cdot HI}$. Let this velocity be added to the before-mentioned decrement,

and we shall have the decrement of the velocity arising from the resistance alone, that is, $\frac{GH}{T} - \frac{HI}{t} + \frac{2MI \cdot NI}{t \cdot HI}$. Therefore since, in the same time, the action of gravity generates, in a falling body, the velocity $\frac{2NI}{t}$, the resistance will be to the gravity as

$$\frac{GH}{T} - \frac{HI}{t} + \frac{2MI \cdot NI}{t \cdot HI} \text{ to } \frac{2NI}{t} \text{ or as } \frac{t}{T} \frac{GH}{HI} - 1 + \frac{2MI}{HI} \text{ to } 2NI.$$

Now, for the abscissas CB, CD, CE, put -o, o, 2o. For the ordinate CH put P; and for MI put any series $Qo + Ro^2 + So^3 + \&c.$ And all the terms of the series after the first, that is, $Ro^2 + So^3 + \&c.$, will be NI; and the ordinates DI, EK, and BG will be $P - Qo - Ro^2 - So^3 - \&c.$, $P - 2Qo - 4Ro^2 - 8So^3 - \&c.$, and $P + Qo - Ro^2 + So^3 - \&c.$, respectively. And by squaring the differences of the ordinates BG - CH and CH - DI, and to the

squares thence produced adding the squares of BC and CD themselves, you will have $oo + QQoo - 2QRoo^3 +$, &c., and $oo + QQoo + 2QRoo^3 +$, &c., the squares of the arcs GH, HI; whose roots $o\sqrt{(1+QQ)} - \frac{QRoo}{\sqrt{(1+QQ)}}$ and $o\sqrt{(1+QQ)} + \frac{QRoo}{\sqrt{(1+QQ)}}$ are the arcs GH and HI. Moreover, if from the ordinate CH there be subtracted half the sum of the ordinates BG and DI, and from the ordinate DI there be subtracted half the sum of the ordinates CH and EK, there will remain Roo and $Roo + 3So^3$, the versed sines of the arcs GI and HK. And these are proportional to the short lines LH and NI, and therefore are as the squares of the infinitely small times T and t:



and thence the ratio $\frac{t}{T}$ varies as the square root of $\frac{R + 3So}{R}$ or $\frac{R + \frac{3}{2}So}{R}$; and $\frac{t \times GH}{T} - HI + \frac{2MI \times NI}{HI}$ by substituting the values of $\frac{t}{T}$, GH, HI, MI, and NI just found, becomes $\frac{3Soo}{2R} \cdot \sqrt{(1+QQ)}$. And since $2NI$ is $2Roo$, the resistance will be now to the gravity as $\frac{3Soo}{2R} \sqrt{(1+QQ)}$ to $2Roo$, that is, as $3S\sqrt{(1+QQ)}$ to $4RR$.

And the velocity will be such, that a body going off therewith from any place H, in the direction of the tangent HN, would describe, in a vacuum, a parabola, whose diameter is HC, and its latus rectum $\frac{HN^3}{NI}$ or $\frac{1+QQ}{R}$.

And the resistance is as the product of the density of the medium and the square of the velocity; and therefore the density of the medium is directly as the resistance, and inversely as the square of the velocity; that is, directly as $\frac{3S\sqrt{(1+QQ)}}{4RR}$ and inversely as $\frac{1+QQ}{R}$; that is, as $\frac{S}{R\sqrt{(1+QQ)}}$. Q.E.I.

COR. I. If the tangent HN be produced both ways, so as to meet any ordinate AF in T, $\frac{HT}{AC}$ will be equal to $\sqrt{(1+QQ)}$, and therefore in what has gone before may be put for $\sqrt{(1+QQ)}$. By this means the resistance will be to the gravity as $3S \cdot HT$ to $4RR \cdot AC$; the velocity will be as $\frac{HT}{AC\sqrt{R}}$, and the density of the medium will be as $\frac{S \cdot AC}{R \cdot HT}$.

COR. II. And hence, if the curved line PFHQ be defined by the relation between the base or abscissa AC and the ordinate CH, as is usual, and the value of the ordinate be resolved into a converging series, the Problem will be expeditiously solved by the first terms of the series; as in the following Examples.

EXAM. I. Let the line PFHQ be a semicircle described upon the diameter PQ; to find the density of the medium that shall make a projectile move in that line.

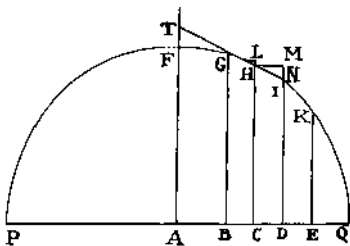
Bisect the diameter PQ in A; and call AQ, n ; AC, a ; CH, e ; and CD, o ; then DI^2 or $AQ^2 - AD^2 = nn - aa - 2ao - oo$, or $ee - 2ao - oo$; and the root being extracted by our method, will give

$$DI = e - \frac{ao}{e} - \frac{oo}{2e} - \frac{aao}{2e^3} - \frac{ao^3}{2e^5} - \frac{a^3o^5}{2e^7} \dots \&c.$$

Here put nn for $ee + aa$, and DI will become $= ee - \frac{ao}{e} - \frac{nnoo}{2e^3} - \frac{a^3noo^3}{2e^7} \dots \&c.$

In such a series I distinguish the successive terms after this manner: I call that the first term in which the infinitely small quantity o is not found; the second, in which that quantity is of one dimension only; the third, in which it arises to two dimensions; the fourth, in which it is of three; and so *ad infinitum*. And the first term, which here is e , will always denote the length of the ordinate CH, erected at the starting point of the indefinite quantity o . The second term, which here is $—$, will denote the difference between CH and DN; that is, the short line MN which is cut off by completing the parallelogram IICDM; and therefore always determines the position of the tangent HN; as, in this case, by taking MN; $IIM - \therefore o = a:c$. The

third term, which here is $\frac{3nco}{2c^3}$, will represent the short line IN, which lies between the tangent and the curve; and therefore determines the angle of contact IHN, or the curvature which the curved line has in H. If that short line IN is of a finite magnitude, it will be expressed by the third term, together with those that follow *in infinitum*. But if that short line be diminished *in infinitum*, the terms following become infinitely less than the third term, and therefore may be neglected. The fourth term determines the variation of the curvature; the fifth, the variation of the variation; and so on.



From this, by the way, appears the use, not to be disdained, which may be made of these series in the solution of problems that depend upon tangents, and the curvature of curves.

Now compare the series

$$e = \frac{ao}{e} - \frac{nnoo}{2e^b} - \frac{annoo}{2e^a} - \&c.,$$

with the series

P-Qo-Roo-So'-&c.,

and for P, Q, R and S, put $e, \frac{a}{e}, \frac{nn}{2e}$, and $\frac{ann}{2e}$, and for $\sqrt{(1+QQ)}$ put $\sqrt{(1+\frac{aa}{ee})}$ or $\frac{n}{e}$; and the density of the medium will come out as $\frac{a}{nc}$; that is (because n is given), as $\frac{a}{e}$ or $\frac{AC}{CH}$, that is, as that length of the tangent HT, which is

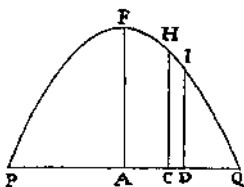
terminated at the semidiameter AF standing perpendicularly on PQ: and the resistance will be to the gravity as $3a$ to m , that is, as $3AC$ to the diameter PQ of the circle; and the velocity will be as $\sqrt{CR}$. Therefore if the body goes from the place F, with a due velocity, in the direction of a line parallel to PQ, and the density of the medium in each of the places H is as the length of the tangent HT, and the resistance also in any place H is to the force of gravity as $3AC$ to PQ, that body will describe the quadrant FHQ of a circle. Q.E.I.

But if the same body should go from the place P, in the direction of a line perpendicular to PQ, and should begin to move in an arc of the semicircle PFQ, we must take AC or a on the contrary side of the centre A; and therefore its sign must be changed, and we must put $-a$ for $+a$. Then the density of the medium would come out as $-$. But Nature does not admit of a negative density, that is, a density which accelerates the motion of bodies; and therefore it cannot naturally come to pass that a body by ascending from P should describe the quadrant PF of a circle. To produce such an effect, a body ought to be accelerated by an impelling medium, and not impeded by a resisting one.

EXAM. 2. Let the line PFQ be a parabola, having its axis AF perpendicular to the horizon PQ; to find the density of the medium, which will make a projectile move in that line.

From the nature of the parabola, the rectangle -PD DQ is equal to the rectangle under the ordinate DI and some given right line; that is, if that right line be called b ; PC, a ; PQ, c ; CH, e ; and CD, o ; the rectangle

$$(a + o)(c - a - o) = ac - aa - 2ao + co - oo - b DI;$$



$$\text{therefore } DI = \frac{ac - aa}{b} + \frac{c - 2a}{b} o - \frac{oo}{b}.$$

Now the second term $\frac{c - 2a}{b} o$ of this series is to be put for Qo , and the third term $\frac{oo}{b}$ for Ro . But since there are no more terms, the coefficient S of the fourth term will vanish;

and therefore the quantity $\frac{S}{R\sqrt{(1+Oo)}}$, to which the density of the medium is proportional, will be nothing. Therefore, where the medium is of no density, the projectile will move in a parabola; as *Galileo* hath heretofore demonstrated. Q.E.I.

EXAM. 3. Let the line AGK be an hyperbola, having its asymptote NX perpendicular to the horizontal plane AK; to find the density of the medium that will make a projectile move in that line.

Let MX be the other asymptote, meeting the ordinate DG produced in V; and from the nature of the hyperbola, the rectangle of XV into VG will

be given. There is also given the ratio of DN to VX, and therefore the rectangle of DN into VG is given. Let that be bb ; and, completing the parallelogram DNZX, let BN be called a ; BD, o ; NX, c ; and let the given ratio of VZ to ZX or DN be $\frac{m}{n}$. Then DN will be equal to $a-o$, VG equal to $\frac{bb}{a-o}$, VZ equal to $\frac{m}{n} (a-o)$ and GD or NX-VZ-VG equal to

$$c - \frac{m}{n} a + \frac{m}{n} o - \frac{bb}{a-o}.$$

Let the term $\frac{bb}{a-o}$ be resolved into the converging series

$$\frac{bb}{a} + \frac{bb}{aa} o + \frac{bb}{a^3} oo + \frac{bb}{a^4} o^3, \&c.,$$

and GD will become equal to

$$c - \frac{m}{n} a + \frac{bb}{a} + \frac{m}{n} o - \frac{bb}{aa} o - \frac{bb}{a^3} o^2 - \frac{bb}{a^4} o^3, \&c.$$

The second term $\frac{m}{n} o - \frac{bb}{aa} o$ of this series is to be used for Qo; the third $\frac{bb}{a^3} o^2$, with its sign changed for Ro²; and the fourth $\frac{bb}{a^4} o^3$, with its sign

changed also for So³, and their coefficients $\frac{m}{n} - \frac{bb}{aa}$, $\frac{bb}{a^3}$, and $\frac{bb}{a^4}$ are to be put for Q, R, and S in the former rule. Which being done, the density of the medium will come out as

$$\frac{bb}{a^3} \sqrt{\left(1 + \frac{mm}{nn} - \frac{2mbb}{naa} + \frac{b^4}{a^4}\right)}$$

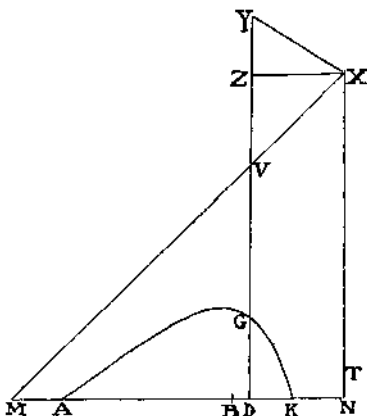
or

$$\sqrt{\left(aa + \frac{mm}{nn} aa - \frac{2mbb}{n} + \frac{b^4}{aa}\right)},$$

that is, if in VZ you take VY

equal to VG, as $\frac{1}{XY}$. For

$$aa \text{ and } \frac{m^2}{n^2} a^2 - \frac{2mbb}{n} + \frac{b^4}{aa}$$



are the squares of XZ and ZY. But the ratio of the resistance to gravity is found to be that of 3XY to 2YG; and the velocity is that with which the body would describe a parabola, whose vertex is G, diameter DG, latus rectum $\frac{XY^2}{VG}$. Suppose, therefore, that the densities of the medium in each of the places G are inversely as the distances XY, and that the resistance in any place G is to the gravity as 3XY to 2YG; and a body let go from the place A, with a due velocity, will describe that hyperbola AGK. Q.E.I.

EXAM. 4. Suppose, indefinitely, the line AGK to be an hyperbola described with the centre X, and the asymptotes MX, NX, so that, having constructed the rectangle XZDN, whose side ZD cuts the hyperbola in G and its asymptote in V, VG may be inversely as any power DNⁿ of the line

ZX or DN, whose index is the number *n*, to find the density of the medium in which a projected body will describe this curve.

For BN, BD, NX, put A, O, C, respectively, and let VZ be to XZ or DN as *d* to *e*, and VG be equal to $\frac{bb}{DN^n}$; then DN will be equal to A - O,

$$VG = \frac{bb}{(A - O)^n}$$

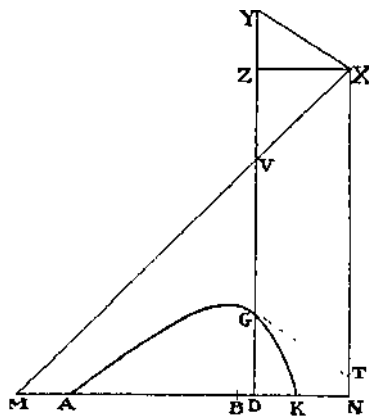
$$VZ = \frac{d}{e}(A - O)$$

and GD or NX - VZ - VG equal to

$$C - \frac{d}{e}A + \frac{d}{e}O - \frac{bb}{(A - O)^n}$$

Let the term $\frac{bb}{(A - O)^n}$ be resolved into an infinite series

$$\frac{bb}{A^n} + \frac{nbb}{A^{n+1}} O + \frac{nn + n}{2A^{n+2}} bb O^2 + \frac{n^3 + 3nn + 2n}{6A^{n+3}} bb O^3, \&c.,$$



and GD will be equal to

$$C - \frac{d}{c} A - \frac{bb}{A^n} + \frac{d}{c} O - \frac{nb b}{A^{n+1}} O - \frac{+nn+n}{2A^{n+2}} bbO^2 - \frac{+n^3+3nn+2n}{6A^{n+3}} bbO^3, \&c.$$

The second term $\frac{d}{c} O - \frac{nb b}{A^{n+1}} O$ of this series is to be used for Qo , the third $\frac{nn+n}{2A^{n+2}} bbO^2$ for Ro , the fourth $\frac{n^3+3nn+2n}{6A^{n+3}} bbO^3$ for So . And thence the

density of the medium $\frac{S}{R\sqrt{(1+QQ)}}$, in any place G, will be

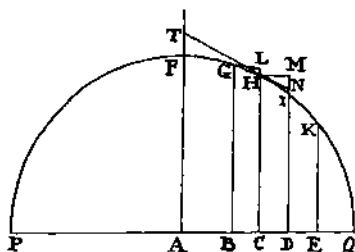
$$\frac{n+2}{3\sqrt{\left(A^2 + \frac{dd}{cc} A^2 - \frac{2dnbb}{cA^n} A + \frac{nnb^2}{A^{2n}}\right)'}}$$

and therefore if in VZ you take VY equal to $n \cdot VG$, that density is reciprocally as XY. For A^2 and $\frac{dd}{cc} A^2 - \frac{2dnbb}{cA^n} A + \frac{nnb^2}{A^{2n}}$ are the squares of XZ and ZY. But the resistance in the same place G is to the force of gravity as $3S \cdot \frac{XY}{A}$ to $4RR$, that is, as XY to $\frac{2nn+2n}{n+2} VG$. And the velocity there is the same wherewith the projected body would move in a parabola, whose vertex is G, diameter GD, and latus rectum $\frac{1+QQ}{R}$ or $\frac{2XY^2}{(nn+n) VG}$. Q.E.I.

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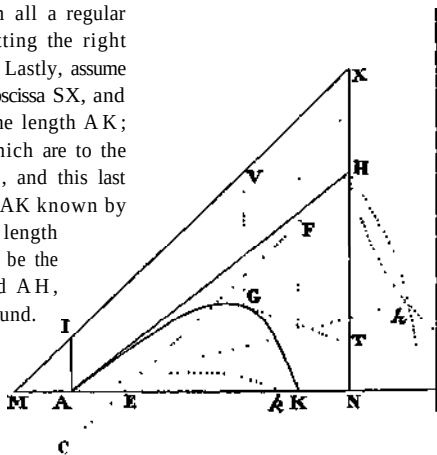
In the same manner that the density of the medium comes out to be as $\frac{S \cdot AC}{R \cdot HT}$, in Cor. 1, if the resistance is put as any power V^n of the velocity V, the density of the medium will come out to be as

$$\frac{S}{R^{\frac{1}{1-n}}} \cdot \left(\frac{AC}{HT}\right)^{n-1}.$$



And therefore, if a curve can be found, such that the ratio of $\frac{S}{R^{\frac{1}{1-n}}}$ to $\left(\frac{HT}{AC}\right)^{n-1}$, or of $\frac{S^2}{R^{\frac{2}{1-n}}}$ to $(1+QQ)^{n-1}$ may be given; the body, in an uni-

projected with the same velocity, in different angles HAK , hAl , and let them fall upon the plane of the horizon in K and k ; and note the proportion of AK to Ak . Let it be as d to e . Then erecting a perpendicular AI of any length, assume any length AH or Ah , and thence graphically, or by scale and compass, collect the lengths AK , Ak (by Rule 6). If the ratio of AK to Ak be the same with that of d to e , the length of AH was rightly assumed. If not, take on the indefinite right line SM , the length SM equal to the assumed AH ; and erect a perpendicular MN equal to the difference $\frac{AK}{Ak} - \frac{d}{e}$ of the ratios multiplied by any given right line. By the like method, from several assumed lengths AH , you may find several points N ; and draw through them all a regular curve $NNXN$, cutting the right line SMM in X . Lastly, assume AH equal to the abscissa SX , and thence find again the length AK ; and the lengths, which are to the assumed length AI , and this last AH , as the length AK known by experiment, to the length AK last found, will be the true lengths AI and AH , which were to be found.

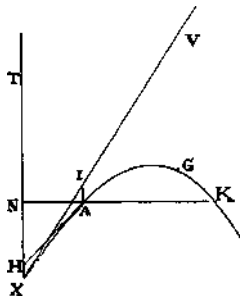


But these being given, there will be given also the resisting force of the medium in the place A, it being to the force of gravity as AH to ${}^4/3AI$. Let the density of the medium be increased by Rule 4, and if the resisting force just found be increased in the same ratio, it will become still more accurate.

RULE 8. The lengths AH, HX being found; let there be now required the position of the line AH, according to which a projectile thrown with that given velocity shall fall upon any point K. At the points A and K, erect the

lines AC, KF perpendicular to the horizon; whereof let AC be drawn downwards, and be equal to AI or $\frac{1}{2}HX$. With the asymptotes AK, KF, describe an hyperbola, whose conjugate shall pass through the point C; and from the centre A, with the interval AH, describe a circle cutting that hyperbola in the point H; then the projectile thrown in the direction of the right line AH will fall upon the point K. Q.E.I. For the point H, because of the given length AH, must be somewhere in the circumference of the described circle. Draw CH meeting AK and KF in E and F; and because CH, MX are parallel, and AC, AI equal, AE will be equal to AM, and therefore also equal to KN. But CE is to AE as FH to KN, and therefore CE and FH are equal. Therefore the point H falls upon the hyperbolic curve described with the asymptotes AK, KF whose conjugate passes through the point C; and is therefore found in the common intersection of this hyperbolic curve and the circumference of the described circle. Q.E.D. It is to be observed that this operation is the same, whether the right line AKN be parallel to the horizon, or inclined thereto in any angle; and that from two intersections H, *h*, there arise two angles NAH, NAA; and that in mechanical practice it is sufficient once to describe a circle, then to apply a ruler CH, of an indeterminate length, so to the point C, that its part FH, intercepted between the circle and the right line FK, may be equal to its part CE placed between the point C and the right line AK.

What has been said of hyperbolas may be easily applied to parabolas. For if a parabola be represented by XAGK, touched by a right line XV in the vertex X, and the ordinates IA, VG be as any powers XIⁿ, XVⁿ, of the abscissas XI, XV; draw XT, GT, AH, whereof let XT be parallel to VG, and let GT, AH touch the parabola in G and A: and a body projected from any place A, in the direction of the right line AH, with a due velocity, will describe this parabola, if the density of



the medium in each of the places G be inversely as the tangent GT. In that case the velocity in G will be the same as would cause a body, moving in a nonresisting space, to describe a conic

parabola, having G for its vertex, VG produced downwards for its diameter, and $-\frac{2GT^2}{(nn-n) \cdot VG}$ for its latus rectum. And the resisting force in G will be to the force of gravity as GT to $\frac{2nn-2r}{n-2} \cdot VG$. Therefore if NAK represent an horizontal line, and both the density of the medium at A, and the velocity with which the body is projected, remaining the same, the angle NAH be anyhow altered, the lengths AH, AI, HX will remain; and thence will be given the vertex X of the parabola, and the position of the right line XI; and by taking VG to IA as XV" to XI", there will be given all the points G of the parabola, through which the projectile will pass.

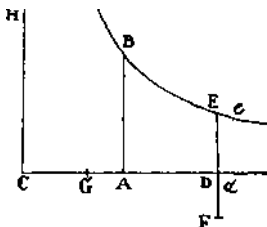
SECTION III

The motion of bodies that are resisted partly in the ratio of the velocities, and partly as the square of the same ratio.

PROPOSITION XL THEOREM VIII

If a body be resisted partly in the ratio and partly as the square of the ratio of its velocity, and moves in a similar medium by its innate force only; and the times be taken in arithmetical progression: then quantities inversely proportional to the velocities, increased by a certain given quantity, will be in geometrical progression.

With the centre C, and the rectangular asymptotes CADd and CH, describe an hyperbola BEe, and let AB, DE, de be parallel to the asymptote CH. In the asymptote CD let A, G be given points; and if the time be represented by the hyperbolic area ABED uniformly increasing, I say, that the velocity may be expressed by the length DF, whose reciprocal GD, together with the given line CG, compose the length CD increasing in a geometrical progression.



For let the small area DEed be the least given increment of the time, and Dd will be inversely as DE, and therefore directly

as CD. Therefore the decrement of $\frac{1}{GD}$, which (by Lem. II, Book II) is

$\frac{Dd}{GD^2}$, will be also as $\frac{CD}{GD^2}$ or $\frac{CG + GD}{GD^2}$, that is, as $\frac{1}{GD} + \frac{CG}{GD^2}$. Therefore, the time ABED uniformly increasing by the addition of the given intervals

EDde, it follows that $\frac{1}{GD}$ decreases in the same ratio with the velocity. For

the decrement of the velocity is as the resistance, that is (by the supposition), as the sum of two quantities, whereof one is as the velocity, and the

other as the square of the velocity; and the decrement of $\frac{1}{GD}$ is as the sum of the quantities $\frac{1}{GD}$ and $\frac{CG}{GD^2}$, whereof the first is $\frac{1}{GD}$ itself, and the last

$\frac{CG}{GD}$ is as $\frac{1}{GD^2}$: therefore $\frac{1}{GD}$ is as the velocity, the decrements of both being analogous. And if the quantity GD inversely proportional to $\frac{1}{GD}$, be augmented by the given quantity CG ; the sum CD , the time $ABED$ uniformly increasing, will increase in a geometrical progression. Q.E.D.

COR. i. Therefore, if, having the points A and G given, the time be represented by the hyperbolic area $ABED$, the velocity may be represented by

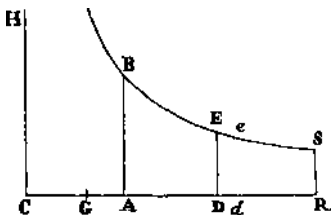
$\frac{1}{GD}$ the reciprocal of GD .

COR. II. And by taking GA to GD as the reciprocal of the velocity at the beginning to the reciprocal of the velocity at the end of any time $ABED$, the point G will be found. And that point being found, the velocity may be found from any other time given.

PROPOSITION XII. THEOREM IX

The same things being supposed, I say, that if the spaces described are taken in arithmetical progression, the velocities augmented by a certain given quantity will be in geometrical progression.

In the asymptote CD let there be given the point R , and, erecting the perpendicular RS meeting the hyperbola in S , let the space described be represented by the hyperbolic area $RSED$; and the velocity will be as the length GD , which, together with the given line CG , composes a length CD decreasing in a geometrical progression, while the space $RSED$ increases in an arithmetical progression.



For, because the increment $EDde$ of the space is given, the short line Dd , which is the decrement of GD , will be reciprocally as ED , and therefore directly as CD ; that is, as the sum of the same GD and the given length CG . But the decrement of the velocity, in a time inversely proportional thereto, in which the given interval of space $DdeE$ is described, is as the resistance and the time conjointly, that is, directly as the sum of two quantities, whereof one

IS as the velocity, the other as the square of the velocity, and inversely as the velocity; and therefore directly as the sum of two quantities, one of which is given, the other is as the velocity. Therefore the decrement both of the velocity and of the line GD is as a given quantity and a decreasing quantity conjointly; and, because the decrements are analogous, the decreasing quantities will always be analogous; viz., the velocity, and the line GD. Q.E.D.

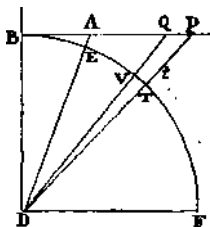
COR. I. If the velocity be represented by the length GD, the space described will be as the hyperbolic area DESR.

COR. ii. And if the point R be assumed anywhere, the point G will be found, by taking GR to GD as the velocity at the beginning to the velocity after any space RSED is described. The point G being given, the space is given from the given velocity; and conversely.

COR. III. Whence since (by Prop, xi) the velocity is given from the given time, and (by this Proposition) the space is given from the given velocity, the space will be given from the given time; and conversely.

PROPOSITION XIII. THEOREM X

Supposing that a body attracted downwards by an uniform gravity ascends or descends in a right line; and that the same is resisted partly in the ratio of its velocity, and partly as the square of the ratio thereof: I say, that, if right lines parallel to the diameters of a circle and an hyperbola be drawn through the ends of the conjugate diameters, and the velocities be as some segments of those parallels drawn from a given point, the times will be as the sectors of the areas cut off by right lines drawn from the centre to the ends of the segments; and conversely.

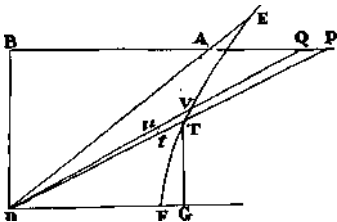


CASE I. Suppose first that the body is ascending, and from the centre D, with any semidiameter DB, describe a quadrant BETF of a circle, and through the end B of the semidiameter DB draw the indefinite line BAP, parallel to the semidiameter DF. In that line let there be given the point A, and take the segment AP proportional to the velocity. And since one part of the resistance is as the velocity, and another part as the square of the velocity, let the whole resistance be as $AP^2 + 2BA \cdot AP$. Join DA, DP, cutting the

circle in E and T, and let the gravity be represented by DA^2 , so that the gravity shall be to the resistance in P as DA^2 to $AP^2 + 2BA \cdot AP$; and the time of the whole ascent will be as the sector EDT of the circle.

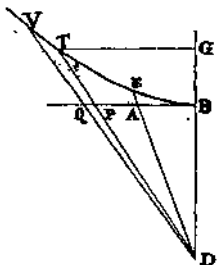
For draw DVQ, cutting off the moment PQ of the velocity AP, and the moment DTV of the sector DET answering to a given moment of time; and that decrement PQ of the velocity will be as the sum of the forces of gravity DA^2 and of resistance $AP^2 + 2BA \cdot AP$; that is (by Prop. xn, Book 11, *Elem. of Euclid*), as DP^2 . Then the area DPQ, proportional to PQ, is as DP^2 , and the area DTV, which is to the area DPQ as DT^2 to DP^2 , is as the given quantity DT^2 . Therefore the area EDT decreases uniformly according to the rate of the future time, by subtraction of given intervals DTV, and is therefore proportional to the time of the whole ascent. Q.E.D.

CASE 2. If the velocity in the ascent of the body be represented by the length AP as before, and the resistance be made as $AP^2 + 2BA \cdot AP$; and if the force of gravity be less than can be expressed by DA^2 ; take BD of such a length, that $AB^2 - BD^2$ may be proportional to the gravity, and let DF be perpendicular and equal to DB, and through the vertex F describe the hyperbola FTVE, whose conjugate semidiameters are DB and DF, and which cuts DA in E, and DP, DQ in T and V; and the time of the whole ascent will be as the hyperbolic sector TDE.



For the decrement PQ of the velocity, produced in a given interval of time, is as the sum of the resistance $AP^2 + 2BA \cdot AP$ and of the gravity $AB^2 - BD^2$, that is, as $BP^2 - BD^2$. But the area DTV is to the area DPQ as DT^2 to DP^2 ; and, therefore, if GT be drawn perpendicular to DF, as GT' or $GD^2 - DF^2$ to BD^2 , and as GD^2 to BP^2 , and, by *Euclid*, as DF^2 to $BP^2 - BD^2$. Therefore since the area DPQ is as PQ, that is, as $BP^2 - BD^2$, the area DTV will be as the given quantity DF^2 . Therefore the area EDT decreases uniformly in each of the equal intervals of time, by the subtraction of so many given intervals DTV, and therefore is proportional to the time. Q.E.D.

CASE 3. Let AP be the velocity in the descent of the body, and $AP^2 + 2BA \cdot AP$ the force of resistance, and $BD^2 - AB^2$ the force of gravity, the angle DBA being a right one. And if with the centre D , and the principal vertex B , there be described a rectangular hyperbola $BETV$ cutting DA , DP , and DQ produced in E , T , and V ; the sector DET of this hyperbola will be as the whole time of descent.



For the increment PQ of the velocity, and the area DPQ proportional to it, is as the excess of the gravity above the resistance, that is, as

$$BD^2 - AB^2 - 2BA \cdot AP - AP^2$$

or $BD^2 - BP^2$. And the area DTV is to the area DPQ as DT to DP^2 ; and therefore as GT or $GD^2 - BD^2$ to BP^2 ; and as GD^2 to BD^2 , and, by subtraction, as BD^2 to $BD^2 - BP^2$. Therefore since

the area DPQ is as $BD^2 - BP^2$, the area DTV will be as the given quantity BD^2 . Therefore the area EDT increases uniformly in the several equal intervals of time by the addition of as many given intervals DTV , and therefore is proportional to the time of the descent. Q.E.D.

COR. If with the centre D and the semidiameter DA there be drawn through the vertex A an arc At similar to the arc ET , and similarly subtending the angle ADT , the velocity AP will be to the velocity which the body in the time EDT , in a nonresisting space, can lose in its ascent, or acquire in its descent, as the area of the triangle DAP to the area of the sector $DA/$; and therefore is given from the time given. For the velocity in a nonresisting medium is proportional to the time, and therefore to this sector; in a resisting medium, it is as the triangle; and in both mediums, where it is least, it approaches to the ratio of equality, as the sector and triangle do.

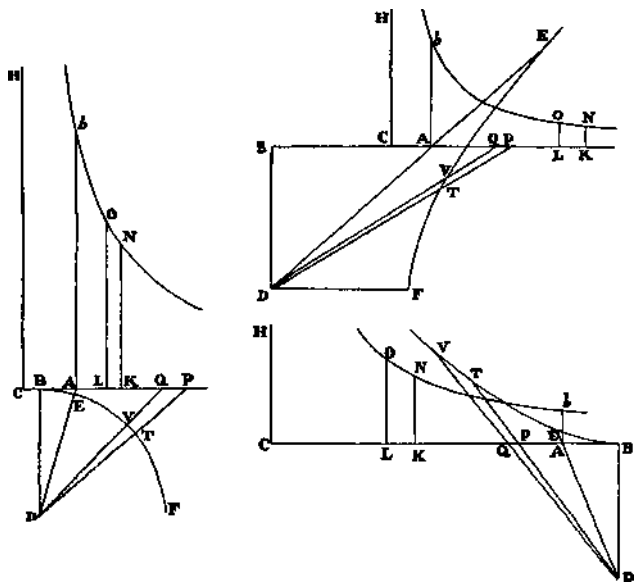
SCHOLIUM

One may demonstrate also that case in the ascent of the body, where the force of gravity is less than can be expressed by DA^2 or AB^2 or BD^2 , and greater than can be expressed by $AB^2 - DB^2$, and must be expressed by AB^2 . But I hasten to other things.

PROPOSITION XIV. THEOREM XI

The same things being supposed, I say, that the space described in the ascent or descent is as the difference of the area by which the time is expressed, and of some other area which is augmented or diminished in an arithmetical progression; if the forces compounded of the resistance and the gravity be taken in a geometrical progression.

Take AC (in these three figures) proportional to the gravity, and AK to the resistance; but take them on the same side of the point A, if the body is descending, otherwise on the contrary. Erect Kb, which make to DB as



DB^2 to $4 BA CA$; and to the rectangular asymptotes CK, CH, describe the hyperbola bN ; and, erecting KN perpendicular to CK , the area $AbNK$ will be augmented or diminished in an arithmetical progression, while the

forces CK are taken in a geometrical progression. I say, therefore, that the distance of the body from its greatest altitude is as the excess of the area AbNK above the area DET.

For since AK is as the resistance, that is, as $AP^2 + 2BA \cdot AP$; assume any given quantity Z, and put AK equal to $\frac{AP^2 + 2BA \cdot AP}{Z}$; then (by Lem. II of this Book) the moment KL of AK will be equal to $\frac{2PQ \cdot AP + 2BA \cdot PQ}{Z}$ or $\frac{2PQ \cdot BP}{Z}$, and the moment KLON of the area AbNK will be equal to $\frac{2PQ \cdot BP \cdot LO}{Z}$ or $\frac{PQ \cdot BP \cdot BD^2}{2Z \cdot CK \cdot AB}$.

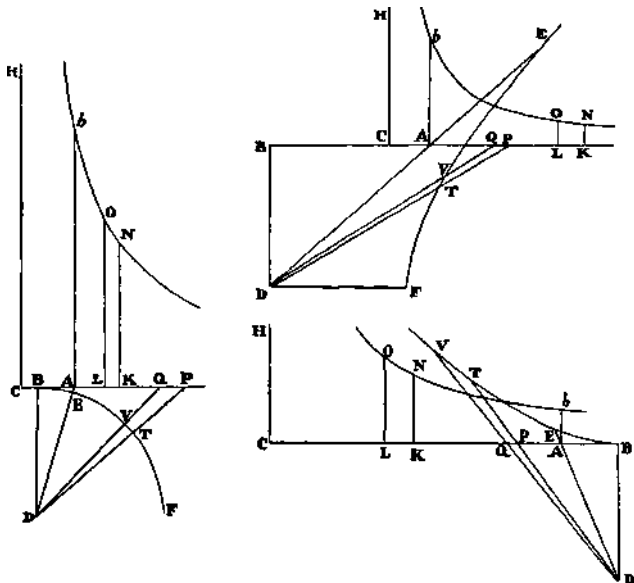
CASE 1. Now if the body ascends, and the gravity be as $AB^2 + BD^2$, BET being a circle, the line AC, which is proportional to the gravity, will be $\frac{AB^2 + BD^2}{Z}$, and DP^2 or $AP^2 + 2BA \cdot AP + AB^2 + BD^2$ will be $AK \cdot Z + AC \cdot Z$ or $CK \cdot Z$; and therefore the area DTV will be to the area DPQ as DT^2 or DB^2 to $CK \cdot Z$.

CASE 2. If the body ascends, and the gravity be as $AB^2 - BD^2$, the line AC will be $\frac{AB^2 - BD^2}{Z}$, and DT^2 will be to DP^2 as DF^2 or DB^2 to $BP^2 - BD^2$ or $AP^2 + 2BA \cdot AP + AB^2 - BD^2$, that is, to $AK \cdot Z + AC \cdot Z$ or $CK \cdot Z$. And therefore the area DTV will be to the area DPQ as DB^2 to $CK \cdot Z$.

CASE 3. And by the same reasoning, if the body descends, and therefore the gravity is as $BD^2 - AB^2$, and the line AC becomes equal to $\frac{BD^2 - AB^2}{Z}$; the area DTV will be to the area DPQ as DB^2 to $CK \cdot Z$: as above.

Since, therefore, these areas are always in this ratio, if for the area DTV, by which the moment of the time, always equal to itself, is expressed, there be put any determinate rectangle, as $BD \cdot m$, the area DPQ, that is, $\frac{1}{2}BD \cdot PQ$, will be to $BD \cdot m$ as $CK \cdot Z$ to BD^2 . And thence $PQ \cdot BD^2$ becomes equal to $2BD \cdot m \cdot CK \cdot Z$, and the moment KLON of the area AbNK, found before, becomes $\frac{BP \cdot BD \cdot m}{AB}$. From the area DET subtract its moment DTV or $BD \cdot m$, and there will remain $\frac{AP \cdot BD \cdot m}{AB}$. Therefore the difference of

the moments, that is, the moment of the difference of the areas, is equal to $\frac{AP \cdot BD \cdot m}{AB}$; and therefore (because of the given quantity $\frac{3D \cdot m}{AB}$) is the velocity AP ; that is, as the moment of the space which the body describes in its ascent or descent. And therefore the difference of the areas, and that space, increasing or decreasing by proportional moments, and beginning together or vanishing together, are proportional. Q.E.D.



COR. If the length, which arises by applying the area DET to the line BD , be called M ; and another length V be taken in that ratio to the length M , which the line DA has to the line DE ; the space which a body, in a resisting medium, describes in its whole ascent or descent, will be to the space which a body, in a nonresisting medium, falling from rest, can describe in the same time, as the difference of the aforesaid areas to $\frac{BD \cdot V^2}{AB}$; and there-

fore is given from the time given. For the space in a nonresisting medium is as the square of the time, or as V^2 ; and, because BD and AB are given, as $\frac{BD \cdot V^2}{AB}$. This area is equal to the area $\frac{DA^2 \cdot BD \cdot M^2}{DE^2 \cdot AB}$ and the moment of M is m ; and therefore the moment of this area is $\frac{DA^2 \cdot BD \cdot 2M \cdot m}{DE^2 \cdot AB}$. But this moment is to the moment of the difference of the aforesaid areas DET and $A\delta NK$, viz., to $\frac{AP \cdot BD \cdot m}{AB}$, as $\frac{DA^2 \cdot BD \cdot M}{DE^2}$ to $\frac{1}{2}BD \cdot AP$, or as $\frac{DA^2}{DE^2}$ into DET to DAP ; and, therefore, when the areas DET and DAP are least, in the ratio of equality. Therefore the area $\frac{BD \cdot V^2}{AB}$ and the difference of the areas DET and $A\delta NK$, when all these areas are least, have equal moments; and are therefore equal. Therefore since the velocities, and therefore also the spaces in both mediums described together, in the beginning of the descent, or the end of the ascent, approach to equality, and therefore are then one to another as the area $\frac{BD \cdot V^2}{AB}$, and the difference of the areas DET and $A\delta NK$; and moreover since the space, in a nonresisting medium, is continually as $\frac{BD \cdot V^2}{AB}$, and the space, in a resisting medium, is continually as the difference of the areas DET and $A\delta NK$; it necessarily follows, that the spaces, in both mediums, described in any equal times, are one to another as that area $\frac{BD \cdot V^2}{AB}$, and the difference of the areas DET and $A\delta NK$. Q.E.D.

SCHOLIUM

The resistance of spherical bodies in fluids arises partly from the tenacity, partly from the attrition, and partly from the density of the medium. And that part of the resistance which arises from the density of the fluid is, as I said, as the square of the velocity; the other part, which arises from the tenacity of the fluid, is uniform, or as the moment of the time; and, therefore, we might now proceed to the motion of bodies, which are resisted partly by an uniform force, or in the ratio of the moments of the time, and

partly as the square of the velocity. But it is sufficient to have cleared the way to this speculation in Prop, viii and ix foregoing, and their Corollaries. For in those Propositions, instead of the uniform resistance made to an ascending body arising from its gravity, one may substitute the uniform resistance which arises from the tenacity of the medium, when the body moves by its inertia alone; and when the body ascends in a right line, add this uniform resistance to the force of gravity, and subtract it when the body descends in a right line. One might also go on to the motion of bodies which are resisted in part uniformly, in part in the ratio of the velocity, and in part as the square of the same velocity. And I have opened a way to this in Prop, xii and xiv foregoing, in which the uniform resistance arising from the tenacity of the medium may be substituted for the force of gravity, or be compounded with it as before. But I hasten to other things.

SECTION IV

The circular motion of bodies in resisting mediums.

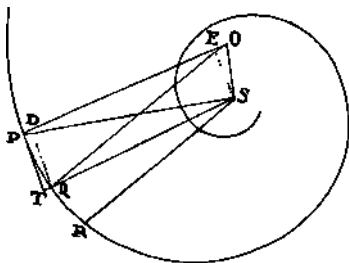
LEMMA III

Let PQR be a spiral cutting all the radii SP, SQ, SR, &c, in equal angles. Draw the right line PT touching the spiral in any point P, and cutting the radius SQ w T; draw PO, QO perpendicular to the spiral, and meeting in O, and join SO: / say, that if the points P and Q approach and coincide, the angle PSO will become a right angle, and the ultimate ratio of the rectangle TQ · 2PS to PQ^2 will be the ratio of equality.

For, from the right angles OPQ, OQR, subtract the equal angles SPQ, SQR, and there will remain the equal angles OPS, OQS. Therefore a circle which passes through the points OSP will pass also through the point Q.

Let the points P and Q coincide, and this circle will touch the spiral in the place of coincidence PQ, and will therefore cut the right line OP perpendicularly. Therefore OP will become a diameter of this circle, and the angle OSP, being in a semicircle, becomes a right one. Q.E.D.

Draw QD, SE perpendicular to OP, and the ultimate ratios of



the lines will be as follows:

$$TQ : PD = TS \text{ or } PS : PE - 2PO : 2PS;$$

and $PD : PQ = PQ : 2PO;$

multiplying together corresponding terms of equal ratios,

$$TQ : PQ = PQ : 2PS.$$

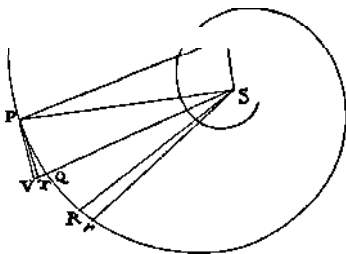
Whence PQ^2 becomes equal to $TQ \cdot 2PS$. Q.E.D.

PROPOSITION XV. THEOREM XII

If the density of a medium in each place thereof be inversely as the distance of the places from an immovable centre, and the centripetal force be as the square of the density • / say, that a body may revolve in a spiral which cuts all the radii drawn from that centre in a given angle.

Suppose everything to be as in the foregoing Lemma, and produce SQ to V so that SV may be equal to SP. In any time let a body, in a resisting medium, describe the least arc PQ, and in double the time the least arc PR; and the decrements of those arcs arising from the resistance, or their differences from the arcs which would be described in a nonresisting medium in the same times, will be to each other as the squares of the times in which they are generated; therefore the decrement of the arc PQ is the fourth part of the decrement of the arc PR. Whence also if the area QSr be taken equal to the area PSQ, the decrement

of the arc PQ will be equal to half the short line Rr; and therefore the force of resistance and the centripetal force are to each other as the short line ViRr and TQ which they generate in the same time. Because the centripetal force with which the body is urged in P is inversely as SP^2 , and (by Lem. x, Book i) the



short line TQ, which is generated by that force, is in a ratio compounded of the ratio of this force and the squared ratio of the time in which the arc PQ is described (for in this case I neglect the resistance, as being infinitely less than the centripetal force), it follows that $TQ \propto SP^2$, that is (by the last Lemma), $\frac{1}{2}PQ^2SP$, will be as the square of the time, and therefore the time is as $PQ \sqrt{SP}$; and the velocity of the body, with which the arc PQ

is described in that time, as $\frac{PQ}{PO \cdot \sqrt{SP}}$ or $\frac{1}{\sqrt{SP}}$, that is, inversely as the

square root of SP. And, by a like reasoning, the velocity with which the arc QR is described, is inversely as the square root of SQ. Now those arcs PQ and QR are as the describing velocities to each other; that is, as the square root of the ratio of SQ to SP, or as SQ to $V(SP - SQ)$; and, because of the equal angles SPQ, SQr, and the equal areas PSQ, QSr, the arc PQ is to the arc Qr as SQ to SP. Take the differences of the proportional consequents, and the arc PQ will be to the arc Rr as SQ to $SP - V(SP - SQ)$, or $VaVQ$. For, the points P and Q coinciding, the ultimate ratio of $SP - \sqrt{SP - SQ}$ to

V/VQ is the ratio of equality. Since the decrement of the arc PQ arising from the resistance, or its double Rr , is as the resistance and the square of the time conjointly, the resistance will be as $\frac{Rr}{PQ^2 \cdot SP}$. But PQ was to Rr as SQ to $\frac{1}{2}VQ$, and thence $\frac{Rr}{PQ^2 \cdot SP}$ becomes as $\frac{\frac{1}{2}VQ}{PQ \cdot SP \cdot SQ}$, or as $\frac{\frac{1}{2}OS}{OP \cdot SP^2}$. For, the points P and Q coinciding, SP and SQ coincide also, and the angle PVQ becomes a right one; and, because of the similar triangles PVQ , PSO , PQ becomes to $\frac{1}{2}VQ$ as OP to $\frac{1}{2}OS$. Therefore $\frac{OS}{OP \cdot SP^2}$ is as the resistance, that is, in the ratio of the density of the medium in P and the squared ratio of the velocity conjointly. Subtract the squared ratio of the velocity, namely, the ratio $\frac{1}{SP^2}$, and there will remain the density of the medium in P , as $\frac{OS}{OP \cdot SP}$. Let the spiral be given, and, because of the given ratio of OS to OP , the density of the medium in P will be as $\frac{1}{SP}$. Therefore in a medium whose density is inversely as SP the distance from the centre, a body will revolve in this spiral. Q.E.D.

COR. I. The velocity in any place P , is always the same wherewith a body in a nonresisting medium with the same centripetal force would revolve in a circle, at the same distance SP from the centre.

COR. II. The density of the medium, if the distance SP be given, is as $\frac{OS}{OP}$, but if that distance is not given, as $\frac{OS}{OP \cdot SP}$. And thence a spiral may be fitted to any density of the medium.

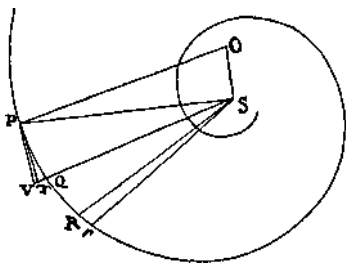
COR. III. The force of the resistance in any place P is to the centripetal force in the same place as $\frac{1}{2}OS$ to OP . For those forces are to each other as $\frac{1}{2}Rr$ and TQ , or as $\frac{\frac{1}{4}VQ \cdot PQ}{SQ}$ and $\frac{\frac{1}{2}PQ^2}{SP}$, that is, as $\frac{1}{2}VQ$ and PQ , or $\frac{1}{2}OS$ and OP . The spiral therefore being given, there is given the proportion of the resistance to the centripetal force; and, conversely, from that proportion given the spiral is given.

COR. IV. Therefore the body cannot revolve in this spiral, except where the force of resistance is less than half the centripetal force. Let the resistance

be made equal to half the centripetal force, and the spiral will coincide with the right line PS, and in that right line the body will descend to the centre with a velocity that is to the velocity with which it was proved before in the case of the parabola (Thcor. x, Book i) that the descent would be made in a nonresisting medium, as the square root of the ratio of unity to the number 2. And the times of the descent will be here inversely as the velocities, and therefore given.

COR. V. And because at equal distances from the centre the velocity is the same in the spiral PQR as it is in the right line SP, and the length of the spiral is to the length of the right line PS in a given ratio, namely, in the ratio of OP to OS; the time of the descent in the spiral will be to the time of the descent in the right line SP in the same given ratio, and therefore given.

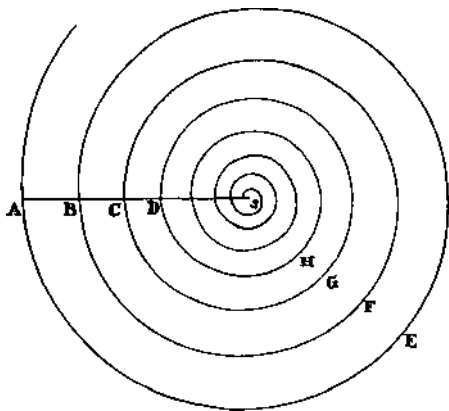
COR. VI. If from the centre S, with any two given radii, two circles are described; and these circles remaining, the angle



which the spiral makes with the radius PS be changed in any manner; the number of revolutions which the body can complete in the space between the circumferences of those circles going round in the spiral from one circumference to another, will be as $\frac{PS}{OS}$, or as the tangent of the angle which the spiral makes with the radius PS; and the time of the same revolutions will be as $\frac{OP}{OS}$, that is, as the secant of the same angle, or inversely as the density of the medium.

COR. VII. If a body, in a medium whose density is inversely as the distances of places from the centre, revolves in any curve AEB about that centre, and cuts the first radius AS in the same angle in B as it did before in A, and that with a velocity that shall be to its velocity in A inversely as the square root of the distances from the centre (that is, as AS to a mean proportional between AS and BS), that body will continue to describe innumerable simi-

lar revolutions BFC, CGD, &c, and by its intersections will divide the radius AS into parts AS, BS, CS, DS, &c, that are continually proportional. But the times of the revolutions will be directly as the perimeters of the orbits AEB, BFC, CGD, &c, and inversely as the velocities at the beginnings A, B, C of those orbits; that is, as $AS^{3/2}$, $BS^{3/2}$, $CS^{3/2}$. And the whole time in which the body will arrive at the centre, will be to the time of the first revolution as the sum of all the continued proportionals $AS^{3/2}$, $BS^{3/2}$, $CS^{3/2}$, going on *ad infinitum*, is to the first term $AS^{3/2}$; that is, as the first term $AS^{3/2}$ is to the difference of the two first $AS^{3/2} - BS^{3/2}$, or as $\frac{3}{2}AS$ is to AB, very nearly. Whence the whole time may be easily found.



COR. vm. From hence also may be deduced, near enough, the motions of bodies in mediums whose density is either uniform, or observes any other assigned law. From the centre S, with radii SA, SB, SC, &c, continually proportional, describe as many circles; and suppose the time of the revolutions between the perimeters of any two of those circles, in the medium whereof we treated, to be to the time of the revolutions between the same in the medium proposed as the mean density of the proposed medium between those circles is to the mean density of the medium whereof we treated, between the same circles, nearly; and that the secant of the angle in which the spiral above determined, in the medium whereof we treated,

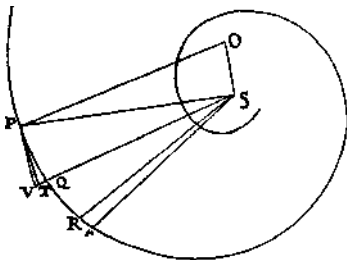
cuts the radius AS, is in the same ratio to the secant of the angle in which the new spiral, in the proposed medium, cuts the same radius; and also that the number of all the revolutions between the same two circles is nearly as the tangents of those angles. If this be done everywhere between every two circles, the motion will be continued through all the circles. And by this means one may without difficulty ascertain at what rate and in what time bodies ought to revolve in any regular medium.

COR. IX. And although these motions becoming eccentric should be performed in spirals approaching to an oval figure, yet, assuming the several revolutions of those spirals to be at the same distances from each other, and to approach to the centre by the same degrees as the spiral above described, we may also understand how the motions of bodies may be performed in spirals of that kind.

PROPOSITION XVI. THEOREM XIII

If the density of the medium in each of the places be inversely as the distance of the places from the immovable centre, and the centripetal force be inversely as any power of the same distance. I say, that the body may revolve in a spiral intersecting all the radii drawn from that centre in a given

This is demonstrated in the same manner as the foregoing Proposition. For if the centripetal force in P be inversely as any power SP^{n+1} of the distance SP whose index is $w+1$; it will be concluded, as above, that the time in which the body describes any arc PQ, will be as $PQ \cdot PS^{1/2}$;



and the resistance in P as $\frac{Rr}{PQ \cdot SP^n}$, or as $\frac{(1-\frac{1}{2}n) \cdot VQ}{PQ \cdot SP^n \cdot SQ}$, and therefore as $\frac{(1-\frac{1}{2}n) \cdot OS}{OP \cdot SP^{n+1}}$, that is (because $\frac{(1-\frac{1}{2}n) \cdot OS}{OP}$ is a given quantity), inversely as SP^{n+1} . And therefore, since the velocity is inversely as $SP^{1/2}$, the density in P will be reciprocally as SP.

PROPOSITION XVIII. PROBLEM V

The law of centripetal force being given, to find the density of the medium in each of the places thereof, by which a body may describe a given spiral.

From the centripetal force the velocity in each place must be found; then from the retardation of the velocity the density of the medium is found, as in the foregoing Proposition.

But I have explained the method of managing these Problems in the tenth Proposition and second Lemma of this Book; and will no longer detain the reader in these complicated investigations. I shall now add some things relating to the forces of progressive bodies, and to the density and resistance of those mediums in which the motions hitherto discussed, and those akin to them, are performed.

SECTION V

The density and compression of fluids; hydrostatics.

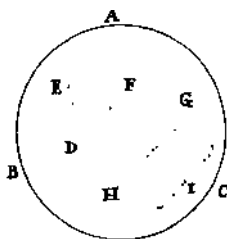
THE DEFINITION OF A FLUID

A fluid is any body whose parts yield to any force impressed on it, and, by yielding, are easily moved among themselves.

PROPOSITION XIX. THEOREM XIV

All the parts of an homogeneous and unmoved fluid included in any unmoved vessel, and compressed on every side (setting aside the consideration of condensation, gravity, and all centripetal forces), will be equally pressed on every side, and remain in their places without any motion arising from that pressure.

CASE I. Let a fluid be included in the spherical vessel ABC, and uniformly compressed on every side: I say, that no part of it will be moved by that pressure. For if any part, as D, be moved, all such parts at the same distance from the centre on every side must necessarily be moved at the same time



by a like motion; because the pressure of them all is similar and equal; and all other motion is excluded that does not arise from that pressure. But if these parts come all of them nearer to the centre, the fluid must be condensed towards the centre, contrary to the supposition. If they recede from it, the fluid must be condensed towards the circumference; which is also contrary to the supposition. Neither can they move in any one direction retaining their distance

from the centre, because, for the same reason, they may move in a contrary direction; but the same part cannot be moved contrary ways at the same time. Therefore no part of the fluid will be moved from its place. Q.E.D.

CASE 2. I say now, that all the spherical parts of this fluid are equally pressed on every side. For let EF be a spherical part of the fluid; if this be not pressed equally on every side, augment the lesser pressure till it be pressed equally on every side; and its parts (by Case 1) will remain in their

places. But before the increase of the pressure, they would remain in their places (by Case 1); and by the addition of a new pressure they will be moved, by the definition of a fluid, from those places. Now these two conclusions contradict each other. Therefore it was false to say that the sphere EF was not pressed equally on every side. Q.E.D.

CASE 3.1 say besides, that different spherical parts have equal pressures. For the contiguous spherical parts press each other mutually and equally in the point of contact (by Law in). But (by Case 2) they are pressed on every side with the same force. Therefore any two spherical parts not contiguous, since an intermediate spherical part can touch both, will be pressed with the same force. Q.E.D.

CASE 4. I say now, that all the parts of the fluid are everywhere pressed equally. For any two parts may be touched by spherical parts in any points whatever; and there they will equally press those spherical parts (by Case 3), and are in reaction equally pressed by them (by Law 111). Q.E.D.

CASE 5. Since, therefore, any part GHI of the fluid is inclosed by the rest of the fluid as in a vessel, and is equally pressed on every side; and also its parts equally press one another, and are at rest among themselves; it is manifest that all the parts of any fluid as GHI, which is pressed equally on every side, do press each other mutually and equally, and are at rest among themselves. Q.E.D.

CASE 6. Therefore if that fluid be included in a vessel of a yielding substance, or that is not rigid, and be not equally pressed on every side, the same will give way to a stronger pressure, by the definition of fluidity.

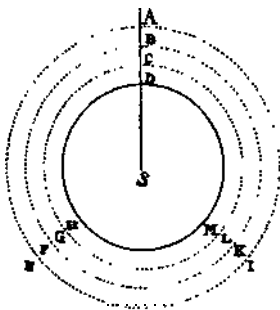
CASE 7. And therefore, in an inflexible or rigid vessel, a fluid will not sustain a stronger pressure on one side than on the other, but will give way to it, and that in a moment of time; because the rigid side of the vessel does not follow the yielding liquor. But the fluid, by thus yielding, will press against the opposite side, and so the pressure will tend on every side to equality. And because the fluid, as soon as it endeavors to recede from the part that is most pressed, is withstood by the resistance of the vessel on the opposite side, the pressure will on every side be reduced to equality, in a moment of time, without any local motion; and from thence the parts of the fluid (by Case 5) will press each other mutually and equally, and be at rest among themselves. Q.E.D.

COR. Hence neither will a motion of the parts of the fluid among themselves be changed by a pressure communicated to the external surface, except so far as either the figure of the surface may be somewhere altered, or that all the parts of the fluid, by pressing one another more intensely or remissly, may slide with more or less difficulty among themselves.

PROPOSITION XX. THEOREM XV

If all the parts of a spherical fluid, homogeneous at equal distances from the centre, lying on a spherical concentric bottom, gravitate towards the centre of the whole, the bottom will sustain the weight of a cylinder, whose base is equal to the surface of the bottom, and whose altitude is the same with that of the incumbent fluid.

Let DHM be the surface of the bottom, and AEI the upper surface of the fluid. Let the fluid be divided into concentric orbs of equal thickness, by the innumerable spherical surfaces BFK, CGL; and conceive the force of gravity to act only in the upper surface of every orb, and the actions to be



equal on the equal parts of the surfaces. Therefore the upper surface AE is pressed by the single force of its own gravity, by which all the parts of the upper orb, and the second surface BFK, will (by Prop, xix), according to its measure, be equally pressed. The second surface BFK is pressed likewise by the force of its own gravity, which, added to the former force, makes the pressure double. The third surface CGL is, according to its measure, acted on by this pressure and the force of its own gravity

besides, which makes its pressure triple. And in like manner the fourth surface receives a quadruple pressure, the fifth surface a quintuple, and so on. Therefore the pressure acting on every surface is not as the solid quantity of the incumbent fluid, but as the number of the orbs reaching to the upper surface of the fluid; and is equal to the gravity of the lowest orb

multiplied by the number of orbs; that is, to the gravity of a solid whose ultimate ratio to the cylinder above mentioned (when the number of the orbs is increased and their thickness diminished, *ad infinitum*, so that the action of gravity from the lowest surface to the uppermost may become continued) is the ratio of equality. Therefore the lowest surface sustains the weight of the cylinder above determined. Q.E.D. And by a like reasoning the Proposition will be evident, where the gravity of the fluid decreases in any assigned ratio of the distance from the centre, and also where the fluid is more rare above and denser below. Q.E.D.

COR. 1. Therefore the bottom is not pressed by the whole weight of the incumbent fluid, but only sustains that part of it which is described in the Proposition; the rest of the weight being sustained archwise by the spherical figure of the fluid.

COR. 11. The quantity of the pressure is the same always at equal distances from the centre, whether the surface pressed be parallel to the horizon, or perpendicular, or oblique; or whether the fluid, continued upwards from the compressed surface, rises perpendicularly in a rectilinear direction, or creeps obliquely through crooked cavities and canals, whether those passages be regular or irregular, wide or narrow. That the pressure is not altered by any of these circumstances, may be inferred by applying the demonstration of this Theorem to the several cases of fluids.

COR. in. From the same demonstration it may also be concluded (by Prop, xix), that the parts of a heavy fluid acquire no motion among themselves by the pressure of the incumbent weight, except that motion which arises from condensation.

COR. iv. And therefore if another body of the same specific gravity, incapable of condensation, be immersed in this fluid, it will acquire no motion by the pressure of the incumbent weight: it will neither descend nor ascend, nor change its figure. If it be spherical, it will remain so, notwithstanding the pressure; if it be square, it will remain square; and that, whether it be soft or fluid; whether it swims freely in the fluid, or lies at the bottom. For any internal part of a fluid is in the same state with the submersed body; and the case of all submersed bodies that have the same magnitude, figure, and specific gravity, is alike. If a submersed body, retaining its weight, should dissolve and put on the form of a fluid, this body, if before it should

have ascended, descended, or from any pressure assumed a new figure, would now likewise ascend, descend, or put on a new figure; and that, because its gravity and the other causes of its motion remain. But (by Case 5, Prop, xix) it would now be at rest, and retain its figure. Therefore also in the former case.

COR. V. Therefore a body that is specifically heavier than a fluid contiguous to it will sink; and that which is specifically lighter will ascend, and attain so much motion and change of figure as that excess or defect of gravity is able to produce. For that excess or defect is the same thing as an impulse, by which a body, otherwise in equilibrium with the parts of the fluid, is acted on; and may be compared with the excess or defect of a weight in one of the scales of a balance.

COR. VI. Therefore bodies placed in fluids have a twofold gravity: the one true and absolute, the other apparent, common, and comparative. Absolute gravity is the whole force with which the body tends downwards; relative and common gravity is the excess of gravity with which the body tends downwards more than the ambient fluid. By the first kind of gravity the parts of all fluids and bodies gravitate in their proper places; and therefore their weights taken together compose the weight of the whole. For the whole taken together is heavy, as may be experienced in vessels full of liquor; and the weight of the whole is equal to the weights of all the parts, and is therefore composed of them. By the other kind of gravity bodies do not gravitate in their places; that is, compared with one another, they do not preponderate, but, hindering one another's endeavor to descend, remain in their proper places as if they were not heavy. Those things which are in the air, and do not preponderate, are commonly looked on as not heavy. Those which do preponderate are commonly reckoned heavy, inasmuch as they are not sustained by the weight of the air. The common weights are nothing else but the excess of the true weights above the weight of the air. Hence also, commonly, those things are called light which are less heavy, and, by yielding to the preponderating air, mount upwards. But these are only comparatively light, and not truly so, because they descend in a vacuum. Thus, in water, bodies which, by their greater or less gravity, descend or ascend, are comparatively and apparently heavy or light; and their comparative and apparent gravity or levity is the excess or defect by

which their true gravity either exceeds the gravity of the water or is exceeded by it. But those things which neither by preponderating descend, nor, by yielding to the preponderating fluid, ascend, although by their true weight they do increase the weight of the whole, yet comparatively, and as commonly understood, they do not gravitate in the water. For these cases are alike demonstrated.

COR. VII. These things which have been demonstrated concerning gravity take place in any other centripetal forces.

COR. VIII. Therefore if the medium in which any body moves be acted on either by its own gravity, or by any other centripetal force, and the body be urged more powerfully by the same force; the difference of the forces is that very motive force, which, in the foregoing Proposition, I have considered as a centripetal force. But if the body be more lightly urged by that force, the difference of the forces becomes a centrifugal force, and is to be considered as such.

COR. IX. But since fluids by pressing the included bodies do not change their external figures, it appears also (by Cor., Prop, xix) that they will not change the situation of their internal parts in relation to one another; and therefore if animals were immersed therein, and if all sensation did arise from the motion of their parts, the fluid would neither hurt the immersed bodies, nor excite any sensation, unless so far as those bodies might be condensed by the compression. And the case is the same of any system of bodies encompassed with a compressing fluid. All the parts of the system will be agitated with the same motions as if they were placed in a vacuum, and would only retain their comparative gravity; unless so far as the fluid may somewhat resist their motions, or be requisite to unite them by compression.

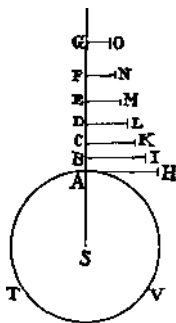
PROPOSITION XXI. THEOREM XVI

Let the density of any fluid be proportional to the compression, and its parts be attracted downwards by a centripetal force inversely proportional to the distances from the centre: I say, that, if those distances be taken continually proportional, the densities of the fluid at the same distances will be also continually proportional.

Let ATV denote the spherical bottom of the fluid, S the centre, SA, SB, SC, SD, SE, SF, &c, distances continually proportional. Erect the perpen-

diculars AH, BI, CK, DL, EM, FN, &c, which shall be as the densities of the medium in the places A, B, C, D, E, F; and the specific gravities in those places will be as $\frac{AH}{AS}, \frac{BI}{BS}, \frac{CK}{CS}, \&c.$, or, which is all one, as $\frac{AH}{AB}, \frac{BI}{BC}, \frac{CK}{CD}, \&c.$

Suppose, first, these gravities to be uniformly continued from A to B, from B to C, from C to D, &c, the decrements in the points B, C, D, &c, being taken by steps. And these gravities multiplied by the altitudes AB, BC, CD, &c, will give the pressures AH, BI, CK, &c, by which the bottom ATV is acted on (by Theor. xv). Therefore the particle A sustains all the pressures

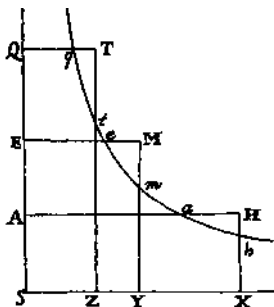


AH, BI, CK, DL, &c, proceeding *in infinitum*; and the particle B sustains the pressures of all but the first AH; and the particle C all but the two first AH, BI; and so on: and therefore the density AH of the first particle A is to the density BI of the second particle B as the sum of all AH + BI + CK + DL, *in infinitum*, to the sum of all BI + CK + DL, &c. And BI the density of the second particle B is to CK the density of the third C, as the sum of all BI + CK + DL, &c, to the sum of all CK + DL, &c. Therefore these sums are proportional to their differences AH, BI, CK, &c, and therefore continually proportional (by Lem. i of this Book); and therefore the differences AH, BI, CK, &c.,

proportional to the sums, are also continually proportional. Therefore since the densities in the places A, B, C, &c, are as AH, BI, CK, &c, they will also be continually proportional. Proceed intermissively, and, at the distances SA, SC, SE, continually proportional, the densities AH, CK, EM will be continually proportional. And by the same reasoning, at any distances SA, SD, SG, continually proportional, the densities AH, DL, GO will be continually proportional. Let now the points A, B, C, D, E, &c, coincide, so that the progression of the specific gravities from the bottom A to the top of the fluid may be made continual; and at any distances SA, SD, SG, continually proportional, the densities AH, DL, GO, being all along continually proportional, will still remain continually proportional. Q.E.D.

COR. Hence if the density of the fluid in two places, as A and E, be given, its density in any other place Q may be obtained. With the centre S, and the

rectangular asymptotes SQ, SX, describe an hyperbola cutting the perpendiculars AH, EM, QT in a , e , and q , as also the perpendiculars HX, MY, TZ, let fall upon the asymptote SX, in h , m , and t . Make the area YmtZ to the given area YmhX as the given area EeqQ to the given area EeaA; and the line Z/ produced will cut off the line QT proportional to the density. For if the lines SA, SE, SQ are continually proportional, the areas EeqQ, EeaA will be equal, and thence the areas YmtZ, XhmY, proportional to them, will be also equal; and the lines SX, SY, SZ, that is, AH, EM, QT continually proportional, as they ought to be. And if the lines SA, SE, SQ obtain any other order in the series of continued proportionals, the lines AH, EM, QT, because of the proportional hyperbolic areas, will obtain the same order in another series of quantities continually proportional.

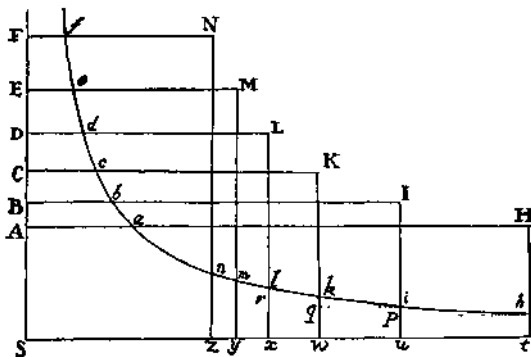


PROPOSITION XXII. THEOREM XVII

Let the density of any fluid be proportioned to the compression, and its parts be attracted downwards by a gravitation inversely proportional to the squares of the distances from the centre: I say, that if the distances be taken in harmonic progression, the densities of the fluid at those distances will be in a geometrical progression.

Let S denote the centre, and SA, SB, SC, SD, SE the distances in geometrical progression. Erect the perpendiculars AH, BI, CK, &c, which shall be as the densities of the fluid in the places A, B, C, D, E, &c, and the specific gravities thereof in those places will be as $\frac{AH}{SA^2}, \frac{BI}{SB^2}, \frac{CK}{SC^2}$, &c. Suppose these gravities to be uniformly continued, the first from A to B, the second from B to C, the third from C to D, &c. And these multiplied by the altitudes AB, BC, CD, DE, &c, or, which is the same thing, by the distances SA, SB, SC, &c., proportional to those altitudes, will give $\frac{AH}{SA^2}, \frac{BI}{SB^2}, \frac{CK}{SC^2}$, &c., repre-

sending the pressures. Therefore since the densities are as the sums of those pressures, the differences $AH - BI$, $BI - CK$, &c, of the densities will be as the differences of those sums $\frac{AH}{SA}$, $\frac{BI}{SB}$, $\frac{CK}{SC}$, &c. With the centre S , and the asymptotes SA , Sx , describe any hyperbola, cutting the perpendiculars AH , BI , CK , &c., in a , b , c , &c, and the perpendiculars Ht , lu , Kw , let fall upon



the asymptote Sx , in h , i , k ; and the differences of the densities, tu , uw , &c., will be as $\frac{AH}{SA}$, $\frac{BI}{SB}$, &c. And the rectangles tu th , uw ui , &c., or tp , uq , &c., as $\frac{AH \cdot th}{SA}$, $\frac{BI \cdot ui}{SB}$, &c., that is, as Aa , Bb , &c. For, by the nature of the hyperbola, SA is to AH or St as th to Aa , and therefore $\frac{AH \cdot th}{SA}$ is equal to Aa . And, by a like reasoning, $\frac{BI \cdot ui}{SB}$ is equal to Bb , &c. But Aa , Bb , Cc , &c., are continually proportional, and therefore proportional to their differences $Aa - Bb$, $Bb - Cc$, &c, therefore the rectangles tp , uq , &c, are proportional to those differences; as also the sums of the rectangles $tp + uq$ or $tp + uq$ $I - wr$ to the sums of the differences $Aa - Cc$ or $Aa - Dd$. Suppose several of these terms, and the sum of all the differences, as $Aa - Ff$, will be proportional to the sum of all the rectangles, as $zthn$. Increase the number of terms, and diminish the distances of the points A , B , C , &c, in *infinitum*, and those

rectangles will become equal to the hyperbolic area $zthn$, and therefore the difference $Aa - Ff$ is proportional to this area. Take now any distances, as SA, SD, SF , in harmonic progression, and the differences $Aa - Dd, Dd - Ff$ will be equal; and therefore the areas $thlx, xlnz$, proportional to those differences, will be equal among themselves, and the densities St, Sx, Sz , that is, AH, DL, FN , continually proportional. Q.E.D.

COR. Hence if any two densities of the fluid, as AH and BI , be given, the area $thiu$, answering to their difference tu , will be given; and thence the density FN will be found at any height SF , by taking the area $thnz$ to that given area $thiu$ as the difference $Aa - Ff$ to the difference $Aa - Bb$.

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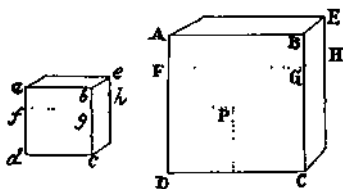
By a like reasoning it may be proved, that if the gravity of the particles of a fluid diminishes as the cube of the distances from the centre, and the reciprocals of the squares of the distances SA, SB, SC , &c, (namely, $\frac{SA^3}{SA^2}, \frac{SA^3}{SB^2}, \frac{SA^3}{SC^2}$) be taken in an arithmetical progression, the densities AH, BI, CK , &c, will be in a geometrical progression. And if the gravity be diminished as the fourth power of the distances, and the reciprocals of the cubes of the distances [$\text{as } \frac{SA^4}{SA^3}, \frac{SA^4}{SB^3}, \frac{SA^4}{SC^3}$, &c.] be taken in arithmetical progression, the densities AH, BI, CK , &c, will be in geometrical progression. And so *in infinitum*. Again; if the gravity of the particles of the fluid be the same at all distances, and the distances be in arithmetical progression, the densities will be in a geometrical progression, as Dr. *Halley* hath found. If the gravity be as the distance, and the squares of the distances be in arithmetical progression, the densities will be in geometrical progression. And so *in infinitum*. These things will be so, when the density of the fluid condensed by compression is as the force of compression; or, which is the same thing, when the space possessed by the fluid is inversely as this force. Other laws of condensation may be supposed, as that the cube of the compressing force may be as the fourth power of the density, or the cube of the ratio of the force the same with the fourth power of the ratio of the density: in which case, if the gravity be inversely as the square of the distance from

the centre, the density will be inversely as the cube of the distance. Suppose that the cube of the compressing force be as the fifth power of the density; and if the gravity be inversely as the square of the distance, the density will be inversely as the $3/2$ th power of the distance. Suppose the compressing force to be as the square of the density, and the gravity inversely as the square of the distance, then the density will be inversely as the distance. To run over all the cases that might be offered would be tedious. But as to our own air, this is certain from experiment, that its density is either accurately, or very nearly at least, as the compressing force; and therefore the density of the air in the atmosphere of the earth is as the weight of the whole incumbent air, that is, as the height of the mercury in the barometer.

PROPOSITION XXIII. THEOREM XVIII

If a fluid be composed of particles fleeing from each other, and the density be as the compression, the centrifugal forces of the particles will be inversely proportional to the distances of their centres. And, conversely, particles fleeing from each other, with forces that are inversely proportional to the distances of their centres, compose an elastic fluid, whose density is as the compression.

Let the fluid be supposed to be included in a cubic space ACE, and then to be reduced by compression into a lesser cubic space ace; and the distances of the particles retaining a like situation with respect to each other in both the spaces, will be as the sides AB, ab of the cubes; and the densities of the



mediums will be inversely as the containing spaces AB^3 , ab^3 . In the plane side of the greater cube ABCD take the square DP equal to the plane side db of the lesser cube; and, by the supposition, the pressure with which the square DP urges the inclosed

fluid will be to the pressure with which that square db urges the inclosed fluid as the densities of the mediums are to each other, that is, as ab^3 to AB^3 . But the pressure with which the square DB urges the included fluid is to

the pressure with which the square DP urges the same fluid as the square DB to the square DP, that is, as AB^2 to ab^2 . Therefore, multiplying together corresponding terms of the proportions, the pressure with which the square DB urges the fluid is to the pressure with which the square db urges the fluid as ab to AB . Let the planes FGH , fgh be drawn through the interior of the two cubes, and divide the fluid into two parts. These parts will press each other with the same forces with which they are themselves pressed by the planes AC , ac , that is, in the proportion of ab to AB : and therefore the centrifugal forces by which these pressures are sustained are in the same ratio. The number of the particles being equal, and the situation alike, in both cubes, the forces which all the particles exert, according to the planes FGH , fgh , upon all, are as the forces which each exerts on each. Therefore the forces which each exerts on each, according to the plane FGH in the greater cube, are to the forces which each exerts on each, according to the plane fgh in the lesser cube, as ab to AB , that is, inversely as the distances of the particles from each other. Q.E.D.

And, conversely, if the forces of the single particles are inversely as the distances, that is, inversely as the sides of the cubes AB , ab ; the sums of the forces will be in the same ratio, and the pressures of the sides DB , db as the sums of the forces; and the pressure of the square DP to the pressure of the side DB as ab^1 to AB^2 . And, multiplying together corresponding terms of the proportions, one obtains the pressure of the square DP to the pressure of the side db as aP to AB^3 ; that is, the force of compression in the one is to the force of compression in the other as the density in the former to the density in the latter. Q.E.D.

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By a like reasoning, if the centrifugal forces of the particles are inversely as the square of the distances between the centres, the cubes of the compressing forces will be as the fourth power of the densities. If the centrifugal forces be inversely as the third or fourth power of the distances, the cubes of the compressing forces will be as the fifth or sixth power of the densities. And universally, if D be put for the distance, and E for the density of the compressed fluid, and the centrifugal forces be inversely as

any power D^n of the distance, whose index is the number n , the compressing forces will be as the cube roots of the power E^{n+2} , whose index is the number $n + 2$; and conversely. All these things are to be understood of particles whose centrifugal forces terminate in those particles that are next them, or are diffused not much farther. We have an example of this in magnetic bodies. Their attractive force is terminated nearly in bodies of their own kind that are next them. The force of the magnet is reduced by the interposition of an iron plate, and is almost terminated at it: for bodies farther off are not attracted by the magnet so much as by the iron plate. If in this manner particles repel others of their own kind that lie next them, but do not exert their force on the more remote, particles of this kind will compose such fluids as are treated of in this Proposition. If the force of any particle diffuse itself every way *in infinitum*, there will be required a greater force to produce an equal condensation of a greater quantity of the fluid. But whether elastic fluids do really consist of particles so repelling each other, is a physical question. We have here demonstrated mathematically the property of fluids consisting of particles of this kind, that hence philosophers may take occasion to discuss that question.

SECTION VI

The motion and resistance of pendulous bodies.

PROPOSITION XXIV. THEOREM XIX

The quantities of matter in pendulous bodies, whose centres of oscillation are equally distant from the centre of suspension, are in a ratio compounded of the ratio of the weights and the squared ratio of the times of the oscillations in a vacuum.

For the velocity which a given force can generate in a given matter in a given time is directly as the force and the time, and inversely as the matter. The greater the force or the time is, or the less the matter, the greater the velocity generated. This is manifest from the second Law of Motion. Now if pendulums are of the same length, the motive forces in places equally distant from the perpendicular are as the weights: and therefore if two bodies by oscillating describe equal arcs, and those arcs are divided into equal parts; since the times in which the bodies describe each of the correspondent parts of the arcs are as the times of the whole oscillations, the velocities in the correspondent parts of the oscillations will be to each other directly as the motive forces and the whole times of the oscillations, and inversely as the quantities of matter: and therefore the quantities of matter are directly as the forces and the times of the oscillations, and inversely as the velocities. But the velocities are inversely as the times, and therefore the times are directly and the velocities inversely as the squares of the times; and therefore the quantities of matter are as the motive forces and the squares of the times, that is, as the weights and the squares of the times. Q.E.D.

COR. 1. Therefore if the times are equal, the quantities of matter in each of the bodies are as the weights.

COR. 11. If the weights are equal, the quantities of matter will be as the squares of the times.

COR. HI. If the quantities of matter are equal, the weights will be inversely as the squares of the times.

COR. IV. Since the squares of the times, other things being equal, are as the lengths of the pendulums, therefore if both the times and the quantities of matter are equal, the weights will be as the lengths of the pendulums.

COR. V. And, in general, the quantity of matter in the pendulous body is directly as the weight and the square of the time, and inversely as the length of the pendulum.

COR. vi. But in a nonresisting medium, the quantity of matter in the pendulous body is directly as the comparative weight and the square of the time, and inversely as the length of the pendulum. For the comparative weight is the motive force of the body in any heavy medium, as was shown above; and therefore does the same thing in such a nonresisting medium as the absolute weight does in a vacuum.

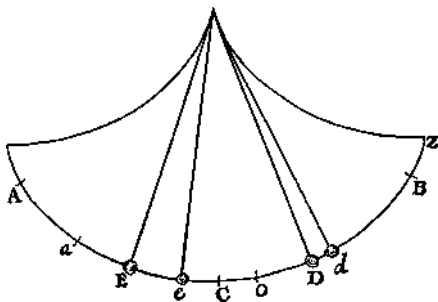
COR. vn. And hence appears a method both of comparing bodies one with another, as to the quantity of matter in each; and of comparing the weights of the same body in different places, to know the variation of its gravity. And by experiments made with the greatest accuracy, I have always found the quantity of matter in bodies to be proportional to their weight.

PROPOSITION XXV. THEOREM XX

Pendulous bodies that are, in any medium, resisted in the ratio of the moments of time, and pendulous bodies that move in a nonresisting medium of the same specific gravity, perform their oscillations in a cycloid in the same time, and describe proportional parts of arcs together.

Let AB be an arc of a cycloid, which a body D, by vibrating in a nonresisting medium, shall describe in any time. Bisect that arc in C, so that C may be the lowest point thereof; and the accelerative force with which the body is urged in any place D, or d , or E, will be as the length of the arc CD, or Cd, or CE. Let that force be expressed by that same arc; and since the resistance is as the moment of the time, and therefore given, let it be expressed by the given part CO of the cycloidal arc, and take the arc Od in the same ratio to the arc CD that the arc OB has to the arc CB: and the force with which the body in d is urged in a resisting medium, being the excess of the force Cd above the resistance CO, will be expressed by the arc Od, and will therefore be to the force with which the body D is urged in a nonresisting medium in the place D, as the arc Od to the arc CD; and therefore also in the place B, as the arc OB to the arc CB. Therefore if two bodies D, d go from the place B, and are urged by these forces; since the forces at the begin-

ning are as the arcs CB and OB, the first velocities and arcs first described will be in the same ratio. Let those arcs be BD and Bd , and the remaining arcs CD, Od will be in the same ratio. Therefore the forces, being proportional to those arcs CD, Od , will remain in the same ratio as at the beginning, and therefore the bodies will continue describing together arcs in the same ratio. Therefore the forces and velocities and the remaining arcs CD, Od , will be always as the whole arcs CB, OB, and therefore those remaining arcs will be described together. Therefore the two bodies D and d will



arrive together at the places C and O; that which moves in the nonresisting medium, at the place C, and the other, in the resisting medium, at the place O. Now since the velocities in C and O are as the arcs CB, OB, the arcs which the bodies describe when they go firther will be in the same ratio. Let those arcs be CE and Oe . The force with which the body D in a nonresisting medium is retarded in E is as CE, and the force with which the body d in the resisting medium is retarded in e , is as the sum of the force Ce and the resistance CO, that is, as Oe ; and therefore the forces with which the bodies are retarded are as the arcs CB, OB, proportional to the arcs CE, Oe ; and therefore the velocities, retarded in that given ratio, remain in the same given ratio. Therefore the velocities and the arcs described with those velocities are always to each other in that given ratio of the arcs CB and OB; and therefore if the entire arcs AB, aB are taken in the same ratio, the bodies D and d will describe those arcs together, and in the places A and a will lose all their motion together. Therefore the whole oscillations are

isochronal, or are performed in equal times; and any parts of the arcs, as BD , Bd , or BE , Be , that are described together, are proportional to the whole arcs BA , Ba . Q.E.D.

COR. Therefore the swiftest motion in a resisting medium does not fall upon the lowest point C , but is found in that point O , in which the whole arc described Ba is bisected. And the body, proceeding from thence to a , is retarded at the same rate with which it was accelerated before in its descent from B to O .

PROPOSITION XXVI. THEOREM XXI

Pendulous bodies, that are resisted in the ratio of the velocity, have their oscillations in a cycloid isochronal.

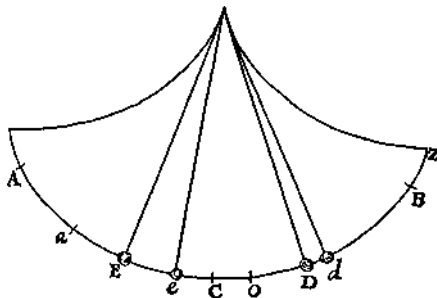
For if two bodies, equally distant from their centres of suspension, describe, in oscillating, unequal arcs, and the velocities in the correspondent parts of the arcs be to each other as the whole arcs; the resistances, proportional to the velocities, will be also to each other as the same arcs. Therefore if these resistances be subtracted from or added to the motive forces arising from gravity which are as the same arcs, the differences or sums will be to each other in the same ratio of the arcs; and since the increments and decrements of the velocities are as these differences or sums, the velocities will be always as the whole arcs; therefore if the velocities are in any one case as the whole arcs, they will remain always in the same ratio. But at the beginning of the motion, when the bodies begin to descend and describe those arcs, the forces, which at that time are proportional to the arcs, will generate velocities proportional to the arcs. Therefore the velocities will be always as the whole arcs to be described, and therefore those arcs will be described in the same time. Q.E.D.

PROPOSITION XXVII. THEOREM XXII

If pendulous bodies are resisted as the square of their velocities, the differences between the times of the oscillations in a resisting medium, and the times of the oscillations in a nonresisting medium of the same specific gravity, will be proportional to the arcs described in oscillating, nearly.

For let equal pendulums in a resisting medium describe the unequal arcs A , B ; and the resistance of the body in the arc A will be to the resistance of

the body in the correspondent part of the arc B as the square of the velocities, that is, as AA to BB, nearly. If the resistance in the arc B were to the resistance in the arc A as AB to AA, the times in the arcs A and B would be equal (by the last Proposition). Therefore the resistance A A in the arc A,



or AB in the arc B, causes the excess of the time in the arc A above the time in a nonresisting medium; and the resistance BB causes the excess of the time in the arc B above the time in a nonresisting medium. But those excesses are as the efficient forces AB and BB nearly, that is, as the arcs A and B. Q.E.D.

COR. 1. Hence from the times of the oscillations in unequal arcs in a resisting medium, may be known the times of the oscillations in a nonresisting medium of the same specific gravity. For the difference of the times will be to the excess of the time in the shorter arc above the time in a nonresisting medium as the difference of the arcs is to the shorter arc.

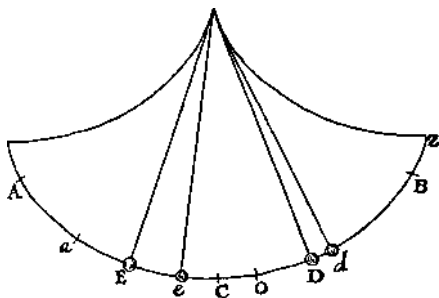
COR. 11. The shorter oscillations are more isochronal, and very short ones are performed nearly in the same times as in a nonresisting medium. But the times of those which are performed in greater arcs are a little greater, because the resistance in the descent of the body, by which the time is prolonged, is greater, in proportion to the length described in the descent than the resistance in the subsequent ascent, by which the time is contracted. But the time of the oscillations, both short and long, seems to be prolonged in some measure by the motion of the medium. For retarded bodies are resisted somewhat less in proportion to the velocity, and accelerated bodies

somewhat more than those that proceed uniformly forwards; because the medium, by the motion it has received from the bodies, going forwards the same way with them, is more agitated in the former case, and less in the latter; and so conspires more or less with the bodies moved. Therefore it resists the pendulums in their descent more, and in their ascent less, than in proportion to the velocity; and these two causes concurring prolong the time.

PROPOSITION XXVIII. THEOREM XXIII

If a pendulous body, oscillating in a cycloid, be resisted in the ratio of the moments of the time, its resistance will be to the force of gravity, as the excess of the arc described in the whole descent above the arc described in the subsequent ascent is to twice the length of the pendulum.

Let BC represent the arc described in the descent, Ca the arc described in the ascent, and Aa the difference of the arcs: and things remaining as they were constructed and demonstrated in Prop, xxv, the force with which the oscillating body is urged in any place D will be to the force of resistance as

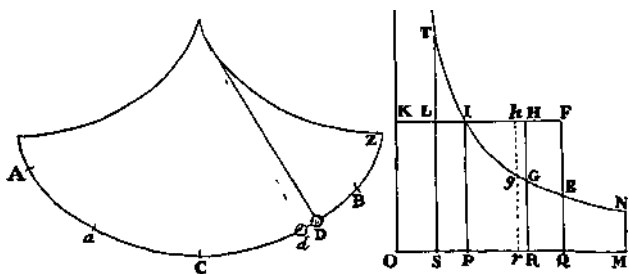


the arc CD to the arc CO, which is half of that difference Aa. Therefore the force with which the oscillating body is urged at the beginning or the highest point of the cycloid, that is, the force of gravity, will be to the resistance as the arc of the cycloid, between that highest point and the lowest point C, is to the arc CO; that is (doubling those arcs), as the whole cycloidal arc, or twice the length of the pendulum, is to the arc Aa. Q.E.D.

PROPOSITION XXIX. PROBLEM VI

Supposing that a body oscillating in a cycloid is resisted as the square of the velocity; to find the resistance in each place.

Let Ba be an arc described in one entire oscillation, C the lowest point of the cycloid, and CZ half the whole cycloidal arc, equal to the length of the pendulum; and let it be required to find the resistance of the body in any place D . Cut the indefinite right line OQ in the points O, S, P, Q , so that (erecting the perpendiculars OK, ST, PI, QE , and with the centre O , and the asymptotes OK, OQ , describing the hyperbola $TIGE$ cutting the perpendiculars ST, PI, QE in T, I, E , and through the point I drawing KF , parallel to the asymptote OQ , meeting the asymptote OK in K , and



the perpendiculars ST and QE in L and F) the hyperbolic area $PIEQ$ may be to the hyperbolic area $PITS$ as the arc BC , described in the descent of the body, is to the arc Ca described in the ascent; and that the area IEF may be to the area ILT as OQ to OS . Then with the perpendicular MN cut off the hyperbolic area $PINM$, and let that area be to the hyperbolic area $PIEQ$ as the arc CZ to the arc BC described in the descent. And if the perpendicular RG cuts off the hyperbolic area $PIGR$, which shall be to the area $PIEQ$ as any arc CD is to the arc BC described in the whole descent, the resistance in any place D will be to the force of gravity as the area $\frac{OR}{OQ}$ $IEF - IGH$ is to the area $PINM$.

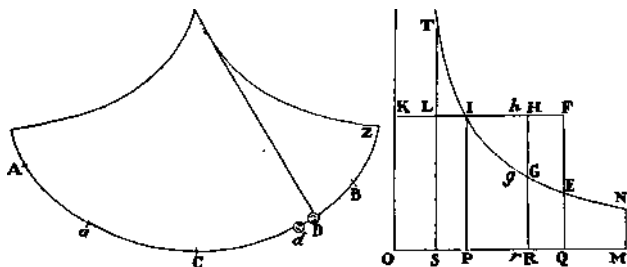
For since the forces arising from gravity with which the body is urged in the places Z, B, D, *a* are as the arcs CZ, CB, CD, *Ga*, and those arcs are as the areas PINM, PIEQ, PIGR, PITS; let those areas represent both the arcs and the forces respectively. Let *Dd* be a very small space described by the body in its descent; and let it be expressed by the very small area *RGgr*, comprehended between the parallels RG, *rg*; and produce *rg* to *h*, so that *GHhg* and *RGgr* may be the contemporaneous decrements of the areas IGH, PIGR. And the increment $GHhg - \frac{Rr}{OQ} IEF$, or $Rr \cdot HG - \frac{Rr}{OQ} IEF$, of the area $\frac{OR}{OQ} IEF - IGH$ will be to the decrement *RGgr*, or $Rr \cdot RG$, of the area PIGR, as $HG - \frac{IEF}{OQ}$ is to RG; and therefore as $OR \cdot HG - \frac{OR}{OQ} IEF$ is to $OR \cdot GR$ or $OP \cdot PI$, that is (because of the equal quantities $OR \cdot HG$, $OR \cdot HR - OR \cdot GR$, $ORHK - OPIK$, $PIHR$ and $PIGR + IGH$), as $PIGR + IGH - \frac{OR}{OQ} IEF$ is to $OPIK$. Therefore if the area $\frac{OR}{OQ} IEF - IGH$ be called Y, and *RGgr* the decrement of the area PIGR be given, the increment of the area Y will be as $PIGR - Y$.

Then if V represent the force arising from the gravity, proportional to the arc CD to be described, by which the body is acted upon in D, and R be put for the resistance, $V - R$ will be the whole force with which the body is urged in D. Therefore the increment of the velocity is as $V - R$ and the interval of time in which it is generated conjointly. But the velocity itself is directly as the contemporaneous increment of the space described and inversely as the same interval of time. Therefore, since the resistance is, by the supposition, as the square of the velocity, the increment of the resistance will (by Lem. II) be as the velocity and the increment of the velocity conjointly, that is, as the moment of the space and $V - R$ conjointly; and, therefore, if the moment of the space be given, as $V - R$; that is, if for the force V we put its expression PIGR, and the resistance R be expressed by any other area Z, as $PIGR - Z$.

Therefore the area PIGR uniformly decreasing by the subtraction of given moments, the area Y increases in proportion of $PIGR - Y$, and the area Z in proportion of $PIGR - Z$. And therefore if the areas Y and Z begin together, and at the beginning are equal, these, by the addition of equal

moments, will continue to be equal; and in like manner decreasing by equal moments, will vanish together. And, conversely, if they together begin and vanish, they will have equal moments and be always equal. For, if the resistance Z be augmented, then the velocity together with the arc Ca , described in the ascent of the body, will be diminished; and, the point in which all the motion together with the resistance ceases, coming nearer to the point C , then the resistance vanishes sooner than the area Y . And the contrary will happen when the resistance is diminished.

Now the area Z begins and ends where the resistance is nothing, that is, at the beginning of the motion where the arc CD is equal to the arc CB, and the right line RG falls upon the right line QE; and at the end of the motion where the arc CD is equal to the arc Ca, and RG falls upon the right line ST. And the area Y or $\frac{OR}{OQ}$ IEF – IGH begins and ends also where the resistance is nothing, and therefore where $\frac{OR}{OQ}$ IEF and IGH are equal; that is (by the construction), where the right line RG falls successively upon the



right lines QE and ST. Therefore those areas begin and vanish together, and are therefore always equal. Hence, the area $\frac{OR}{OQ}$ IEF - IGH is equal to the area Z, by which the resistance is expressed, and therefore is to the area PINM, by which the gravity is expressed, as the resistance is to the gravity. Q.E.D.

COR. i. Therefore the resistance in the lowest place C is to the force of gravity as the area $\frac{OP}{IEF}$ is to the area PINM.

if in DE you take DK in the same ratio to the length of the pendulum as the resistance is to the gravity, DK will be the exponent of the resistance. From the centre C with the interval CA or CB describe a semicircle *BEeA*. Let the body describe, in the least time, the space *Dd*; and, erecting the perpendiculars DE, *de*, meeting the circumference in E and *e*, they will be as the velocities which the body descending in a vacuum from the point B would acquire in the places D and *d*. This appears by Prop, LII, Book i. Let, therefore, these velocities be expressed by those perpendiculars DE, *de*; and let DF be the velocity which it acquires in D by falling from B in the resisting medium. And if from the centre C with the interval CF we describe the circle F/M meeting the right lines *de* and AB in / and M, then M will be the place to which it would thenceforward, without further resistance, ascend, and *df* the velocity it would acquire in *d*. Hence, also, if Fg represent the moment of the velocity which the body D, in describing the least space *Dd*, loses by the resistance of the medium; and CN be taken equal to Cg; then will N be the place to which the body, if it met no further resistance, would thenceforward ascend, and MN will be the decrement of the ascent arising from the loss of that velocity. Draw Fm perpendicular to *df*, and the decrement Fg of the velocity DF generated by the resistance DK will be to the increment *fm* of the same velocity, generated by the force CD, as the generating force DK to the generating force CD. But because of the similar triangles Fmf, Fhg, FDC, *fm* is to Fm or *Dd* as CD to DF; and, by multiplication of corresponding terms, Fg to *Dd* as DK to DF. Also Fh is to Fg as DF to CF; and, again by multiplication of corresponding terms, Fh or MN to *Dd* as DK to CF or CM; and therefore the sum of all the MN • CM will be equal to the sum of all the *Dd* • DK. At the movable point M suppose always a rectangular ordinate erected equal to the indeterminate CM, which by a continual motion is multiplied by the whole length Aa; and the trapezium described by that motion, or its equal, the rectangle Aa • $\frac{1}{2}$ aB, will be equal to the sum of all the MN • CM, and therefore to the sum of all the *Dd* DK, that is, to the area BKVTtf. Q.E.D.

COR. Hence from the law of resistance, and the difference Aa of the arcs Ca, CB, may be derived the proportion of the resistance to the gravity, nearly.

For if the resistance DK be uniform, the figure $BKTa$ will be a rectangle under Ba and DK ; and hence the rectangle under $ViBa$ and ha will be equal to the rectangle under Ba and DK , and DK will be equal to ^xkha . Therefore since DK represents the resistance, and the length of the pendulum represents the gravity, the resistance will be to the gravity as $V'zha$ is to the length of the pendulum; altogether as in Prop, xxviii is demonstrated.

If the resistance be as the velocity, the figure $BKTa$ will be nearly an ellipse. For if a body, in a nonresisting medium, by one entire oscillation, should describe the length BA , the velocity in any place D would be as the ordinate DE of the circle described on the diameter AB . Therefore since Ba in the resisting medium, and BA in the nonresisting one, are described nearly in the same times; and therefore the velocities in each of the points of Ba are to the velocities in the corresponding points of the length BA nearly as Ba is to BA , the velocity in the point D in the resisting medium will be as the ordinate of the circle or ellipse described upon the diameter Ba ; and therefore the figure $BKVTa$ will be nearly an ellipse. Since the resistance is supposed proportional to the velocity, let OV represent the resistance in the middle point O ; and an ellipse $BRV Sa$ described with the centre O , and the semiaxes OB , OV , will be nearly equal to the figure $BKVTtf$, and to its equal the rectangle ha BO . Therefore ha BO is to $OV \cdot BO$ as the area of this ellipse to $OV \cdot BO$; that is, ha is to OV as the area of the semicircle is to the square of the radius, or as 11 to 7 nearly; and, therefore, Vn ha is to the length of the pendulum as the resistance of the oscillating body in O is to its gravity.

Now if the resistance DK varies as the square of the velocity, the figure $BKVTtf$ will be almost a parabola having V for its vertex and OV for its axis, and therefore will be nearly equal to the rectangle under $^{2/3}Ba$ and OV . Therefore the rectangle under $^{1/2}Ba$ and ha is equal to the rectangle ^2ABa OV , and therefore OV is equal to $^{3/4}Aa$; and therefore the resistance in O made to the oscillating body is to its gravity as $^{3/4}ha$ is to the length of the pendulum.

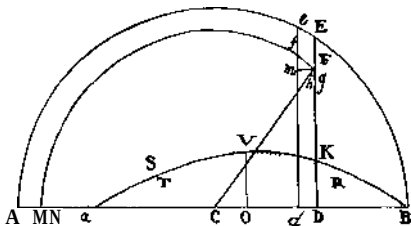
And I take these conclusions to be accurate enough for practical uses. For since an ellipse or parabola $BRV Sa$ falls in with the figure $BKVTa$ in the middle point V , that figure, if greater towards the part BRV or YSa , is less towards the contrary part, and is therefore nearly equal to it.

PROPOSITION XXXI. THEOREM XXV

/f the resistance made to an oscillating body in each of the proportional parts of the arcs described be augmented or diminished in a given ratio, the difference between the arc described in the descent and the arc described in the subsequent ascent will be augmented or diminished in the same ratio.

For that difference arises from the retardation of the pendulum by the resistance of the medium, and therefore is as the whole retardation and the retarding resistance proportional thereto. In the foregoing Proposition the rectangle under the right

line $\frac{1}{2}aB$ and the difference ha of the arcs CB , Ca , was equal to the area $BKTa$. And that area, if the length ah remains, is augmented or diminished in the ratio of the ordinates DK ; that is, in the ratio of the resistance.



and is therefore as the length aB and the resistance conjointly. And therefore the rectangle under Aa and $\frac{1}{2}aB$ is as aB and the resistance conjointly, and therefore Aa is as the resistance. Q.E.D.

COR. 1. Hence if the resistance be as the velocity, the difference of the arcs in the same medium will be as the whole arc described; and conversely.

COR. 11. If the resistance varies as the square of the velocity, that difference will vary as the square of the whole arc; and conversely.

COR. HI. And generally, if the resistance varies as the third or any other power of the velocity, the difference will vary as the same power of the whole arc; and conversely.

COR. IV. If the resistance varies partly as the first power of the velocity and partly as the square of the same, the difference will vary partly as the first power and partly as the square of the whole arc; and conversely. So that the law and ratio of the resistance will be the same for the velocity as the law and ratio of that difference for the length of the arc.

COR. V. And therefore if a pendulum describe successively unequal arcs, and we can find the ratio of the increment or decrement of this difference for the length of the arc described, there will be had also the ratio of the increment or decrement of the resistance for a greater or less velocity.

GENERAL SCHOLIUM

From these Propositions we may find the resistance of mediums by pendulums oscillating therein. I found the resistance of the air by the following experiments. I suspended a wooden globe or ball weighing y/vi ounces troy, its diameter $6\frac{7}{8}$ London inches, by a fine thread on a firm hook, so that the distance between the hook and the centre of oscillation of the globe was $10\frac{1}{2}$ feet. I marked on the thread a point 10 feet and i inch distant from the centre of suspension; and even with that point I placed a ruler divided into inches, by the help of which I observed the lengths of the arcs described by the pendulum. Then I numbered the oscillations in which the globe would lose $\frac{1}{8}$ part of its motion. If the pendulum was drawn aside from the perpendicular to the distance of 2 inches, and then let go, so that in its whole descent it described an arc of 2 inches, and in the first whole oscillation, compounded of the descent and subsequent ascent, an arc of almost 4 inches, the pendulum in 164 oscillations lost Vs part of its motion, so as in its last ascent to describe an arc of $1\frac{3}{4}$ inches. If in the first descent it described an arc of 4 inches, it lost $\frac{1}{8}$ part of its motion in 121 oscillations, so as in its last ascent to describe an arc of $3\frac{1}{2}$ inches. If in the first descent it described an arc of 8, 16, 32, or 64 inches, it lost $\frac{1}{8}$ part of its motion in 69, $35\frac{1}{2}$, $18\frac{1}{2}$, $9\frac{2}{3}$ oscillations, respectively. Therefore the difference between the arcs described in the first descent and the last ascent was in the 1st, 2d, 3d, 4th, 5th, 6th cases, $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 4, 8 inches, respectively. Divide those differences by the number of oscillations in each case, and in one mean oscillation, in which an arc of $3\frac{3}{4}$, $7\frac{1}{2}$, 15, 30, 60, 120 inches was described, the difference of the arcs described in the descent and subsequent ascent will be $\frac{1}{656}$, $\frac{1}{242}$, $\frac{1}{69}$, $\frac{4}{71}$, $\frac{8}{37}$, $\frac{24}{29}$ parts of an inch, respectively. But these differences in the greater oscillations are as the square of the arcs described, nearly, but in lesser oscillations somewhat greater than in that ratio; and therefore (by Cor. 11, Prop. xxxi of this Book) the resistance of the globe,

when it moves very swiftly, varies as the square of the velocity, nearly; and when it moves slowly, in a somewhat greater ratio.

Now let V represent the greatest velocity in any oscillation, and let A , B , and C be given quantities, and let us suppose the difference of the arcs to be $AV - BV^{3/2} + CV^2$. Since the greatest velocities are in the cycloid as V the arcs described in oscillating, and in the circle as $V/2$ the chords of those arcs; and therefore in equal arcs are greater in the cycloid than in the circle in the ratio of $1/2$ the arcs to their chords; but the times in the circle are greater than in the cycloid, in a ratio inversely as the velocity; it is plain that the differences of the arcs (which are as the resistance and the square of the time conjointly) are nearly the same in both curves: for in the cycloid those differences must be on the one hand augmented, with the resistance, in about the squared ratio of the arc to the chord, because of the velocity augmented in the simple ratio of the same; and on the other hand diminished, with the square of the time, in the same squared ratio. Therefore to reduce these observations to the cycloid, we must take the same differences of the arcs as were observed in the circle, and suppose the greatest velocities analogous to the half, or the whole arcs, that is, to the numbers $1/2$, 1 , 2 , 4 , 8 , 16 . Therefore in the 2d, 4th, and 6th cases put 1 , 4 , and 16 for V ; and the difference of the arcs in the 2d case will become $1/121 = A + B + C$; in the 4th case, $2/35\frac{1}{2} = 4A + 8B + 16C$; in the 6th case, $8/9\frac{2}{3} = 16A + 64B + 256C$. These equations reduced give $A = 0.0000916$, $B = 0.0010847$, and $C = 0.0029558$. Therefore the difference of the arcs is as $0.0000916V + 0.0010847V^{1/2} + 0.0029558V^2$; and therefore since (by Cor., Prop. xxx, applied to this case) the resistance of the globe in the middle of the arc described in oscillating, where the velocity is V , is to its weight as $V \sqrt{1 - AV + \frac{7}{10}BV^{3/2} + \frac{3}{4}CV^2}$ is to the length of the pendulum, if for A , B , and C you put the numbers found, the resistance of the globe will be to its weight as $0.0000583V + 0.0007593V^{3/2} + 0.0022169V^2$ is to the length of the pendulum between the centre of suspension and the ruler, that is, to 121 inches. Therefore since V in the second case represents 1 , in the 4th case 4 , and in the 6th case 16 , the resistance will be to the weight of the globe, in the 2d case, as 0.0030345 is to 121 ; in the 4th, as 0.041748 is to 121 ; in the 6th, as 0.61705 is to 121 .

The arc, which the point marked in the thread described in the 6th case, was $120 - \frac{8}{92\frac{2}{3}}$, or $119\frac{1}{2}$ inches. And therefore since the radius was 121 inches,

and the length of the pendulum between the point of suspension and the centre of the globe was 126 inches, the arc which the centre of the globe described was $124\frac{3}{31}$ inches. Because the greatest velocity of the oscillating body, by reason of the resistance of the air, does not fall on the lowest point of the arc described, but near the middle place of the whole arc, this velocity will be nearly the same as if the globe in its whole descent in a nonresisting medium should describe $62\frac{3}{62}$ inches, the half of that arc, and that in a cycloid, to which we have above reduced the motion of the pendulum; and therefore that velocity will be equal to that which the globe would acquire by falling perpendicularly from a height equal to the versed sine of that arc. But that versed sine in the cycloid is to that arc $62\frac{3}{62}$ as the same arc to twice the length of the pendulum 252, and therefore equal to 15.278 inches. Therefore the velocity of the pendulum is the same which a body would acquire by falling, and in its fall describing a space of 15.278 inches. Therefore with such a velocity the globe meets with a resistance which is to its weight as 0.61705 is to 121, or (if we take that part only of the resistance which is in the squared ratio of the velocity) as 0.56752 to 121.

I found, by an hydrostatical experiment, that the weight of this wooden globe was to the weight of a globe of water of the same magnitude as 55 to 97; and therefore since 121 is to 213.4 in the same ratio, the resistance made to this globe of water, moving forwards with the above-mentioned velocity, will be to its weight as 0.56752 to 213.4, that is, as 1 to $376\frac{1}{50}$. Since the weight of a globe of water, in the time in which the globe with a velocity uniformly continued describes a length of 30.556 inches, will generate all that velocity in the falling globe, it is manifest that the force of resistance uniformly continued in the same time will take away a velocity, which will be less than the other in the ratio of 1 to $376\frac{1}{50}$, that is, the $\frac{1}{376\frac{1}{50}}$ part of the whole velocity. And therefore in the time that the globe, with the same velocity uniformly continued, would describe the length of its semidiameter, or $3\frac{7}{16}$ inches, it would lose the $\frac{1}{3312}$ part of its motion.

I also counted the oscillations in which the pendulum lost $\frac{1}{4}$ part of its motion. In the following table the upper numbers denote the length of the arc described in the first descent, expressed in inches and parts of an inch; the middle numbers denote the length of the arc described in the last ascent; and in the lowest place are the numbers of the oscillations. I give

an account of this experiment, as being more accurate than that in which only $\frac{1}{8}$ part of the motion was lost. I leave the calculation to such as are disposed to make it.

<i>First descent</i>		2	4	8	16	32	64
<i>Last ascent</i>		$1\frac{1}{2}$	3	6	12	24	48
<i>No. of oscillations</i>	. .	374	272	$162\frac{1}{2}$	$83\frac{1}{3}$	$41\frac{2}{3}$	$22\frac{2}{3}$

I afterwards suspended a leaden globe of 2 inches in diameter, weighing $26\frac{1}{4}$ ounces troy by the same thread, so that between the centre of the globe and the point of suspension there was an interval of $10\frac{1}{2}$ feet, and I counted the oscillations in which a given part of the motion was lost. The first of the following tables exhibits the number of oscillations in which $\frac{1}{8}$ part of the whole motion was lost; the second the number of oscillations in which there was lost $\frac{1}{4}$ part of the same.

<i>First descent</i>	1	2	4	8	16	32	64
<i>Last ascent</i>	. . . ,	$\frac{7}{8}$	$\frac{7}{4}$	$3\frac{1}{2}$	7	14	28
<i>No. of oscillations</i>	226	228	193	140	$90\frac{1}{2}$	53	30
<i>First descent</i>	1	2	4	8	16	32	64
<i>Last ascent</i>	. .	$\frac{3}{4}$	$1\frac{1}{2}$	3	6	12	24
<i>No. of oscillations</i>	510	518	420	318	204	121	70

Selecting in the first table the 3d, 5th, and 7th observations, and expressing the greatest velocities in these observations particularly by the numbers 1, 4, 16, respectively, and generally by the quantity V as above, there will come out in the 3d observation $\frac{1}{2}/_{193} = A + B + C$, in the 5th observation $\frac{2}{90\frac{1}{2}} = 4A + 8B + 16C$, in the 7th observation $\frac{8}{36} = 16A + 64B + 256C$, These equations reduced give $A = 0.001414$, $B = 0.000297$, $C = 0.000879$. And thence the resistance of the globe moving with the velocity V will be to its weight $26\frac{1}{4}$ ounces in the same ratio as $0.0009V + 0.000208V^{\frac{3}{2}} + 0.000659V^2$ to 121 inches, the length of the pendulum. And if we regard that part only of the resistance which is as the square of the velocity, it will be to the weight of the globe as $0.000659V^2$ to 121 inches. But this part of the resistance in the first experiment was to the weight of the wooden globe of $57\frac{7}{22}$ ounces as $0.002217V^2$ to 121; hence the resistance of the wooden globe is to the resistance of the leaden one (their velocities being equal) as $57\frac{7}{22}$ into 0.002217

to 26V4 into 0.000659, that is as $7^{1/3}$ to 1. The diameters of the two globes were 6% and 2 inches, and the squares of these are to each other as 47% and 4, or $11^{13/16}$ and 1, nearly. Therefore the resistances of these equally swift globes were in less than a squared ratio of the diameters. But we have not yet considered the resistance of the thread, which was certainly very considerable, and ought to be subtracted from the resistance of the pendulums here found. I could not determine this accurately, but I found it greater than $\frac{1}{10}$ part of the whole resistance of the lesser pendulum; hence I gathered that the resistances of the globes, when the resistance of the thread is subtracted, are nearly in the squared ratio of their diameters. For the ratio of $7^{1/2} - \frac{1}{3}$ to $1 - \frac{1}{3}$, or $10^{1/2}$ to 1 is not very different from the squared ratio of the diameters $11^{13/16}$ to 1.

Since the resistance of the thread is of less moment in greater globes, I tried the experiment also with a globe whose diameter was $18\frac{3}{4}$ inches. The length of the pendulum between the point of suspension and the centre of oscillation was $122\frac{1}{2}$ inches, and between the point of suspension and the knot in the thread $109\frac{1}{2}$ inches. The arc described by the knot at the first descent of the pendulum was 32 inches. The arc described by the same knot in the last ascent after five oscillations was 28 inches. The sum of the arcs, or the whole arc described in one mean oscillation, was 60 inches; the difference of the arcs, 4 inches. The $\frac{1}{10}$ part of this, or the difference between the descent and ascent in one mean oscillation, is $\frac{2}{5}$ of an inch. Then as the radius $109\frac{1}{2}$ is to the radius $122\frac{1}{2}$, so is the whole arc of 60 inches described by the knot in one mean oscillation to the whole arc of $67\frac{1}{8}$ inches described by the centre of the globe in one mean oscillation; and so is the difference 75 to a new difference 0.4475. If the length of the arc described were to remain, and the length of the pendulum should be augmented in the ratio of 126 to $122\frac{1}{4}$, the time of the oscillation would be augmented, and the velocity of the pendulum would be diminished as the square root of that ratio; so that the difference 0.4475 of the arcs described in the descent and subsequent ascent would remain. Then if the arc described be augmented in the ratio of $124\frac{3}{31}$ to $67\frac{1}{8}$, that difference 0.4475 would be augmented as the square of that ratio, and so would become 1.5295. These things would be so upon the supposition that the resistance of the pendulum were as the square of the velocity. Therefore if the pendulum describe the whole arc

of $124\frac{3}{31}$ inches, and its length between the point of suspension and the centre of oscillation be 126 inches, the difference of the arcs described in the descent and subsequent ascent would be 1.5295 inches. And this difference multiplied by the weight of the pendulous globe, which was 208 ounces, produces 318.136. Again, in the pendulum above mentioned, made of a wooden globe, when its centre of oscillation, being 126 inches from the point of suspension, described the whole arc of $124\frac{3}{31}$ inches, the difference of the arcs described in the descent and ascent was $\frac{126}{121}$ into $\frac{8}{9\frac{3}{4}}$. This multiplied by the weight of the globe, which was $57\frac{7}{22}$ ounces, produces 49.396. But I multiply these differences by the weights of the globes, in order to find their resistances. For the differences arise from the resistances, and are as the resistances directly and the weights inversely. Therefore the resistances are as the numbers 318.136 and 49.396. But that part of the resistance of the lesser globe, which is as the square of the velocity, was to the whole resistance as 0.56752 to 0.61675, that is, as 45.453 to 49.396, whereas that part of the resistance of the greater globe is almost equal to its whole resistance, and so those parts are nearly as 318.136 and 45.453, that is, as 7 and 1. But the diameters of the globes are 1874 and $6\frac{7}{8}$; and their squares $351\frac{9}{16}$ and $47\frac{17}{64}$ are as 7.438 and 1, that is, nearly as the resistances of the globes 7 and 1. The difference of these ratios is barely greater than may arise from the resistance of the thread. Therefore those parts of the resistances which are, when the globes are equal, as the squares of the velocities, are also, when the velocities are equal, as the squares of the diameters of the globes.

But the greatest of the globes I used in these experiments was not perfectly spherical, and therefore in this calculation I have, for brevity's sake, neglected some little niceties; being not very solicitous for an accurate calculus in an experiment that was not very accurate. So that I could wish that these experiments were tried again with other globes, of a larger size, more in number, and more accurately formed; since the demonstration of a vacuum depends thereon. If the globes be taken in a geometrical proportion, whose diameters, let us suppose, are 4, 8, 16, 32 inches; one may infer from the progression observed in the experiments what would happen if the globes were still larger.

In order to compare the resistances of different fluids with each other, I made the following trials. I procured a wooden vessel 4 feet long, 1 foot

broad, and i foot high. This vessel, being uncovered, I filled with spring water, and, having immersed pendulums therein, I made them oscillate in the water. And I found that a leaden globe weighing $166\frac{1}{6}$ ounces, and in diameter $3\frac{3}{4}$ inches, moved therein as it is set down in the following table; the length of the pendulum from the point of suspension to a certain point marked in the thread being 126 inches, and to the centre of oscillation $134\frac{3}{8}$ inches.

The arc described in the first descent, by a point marked in the thread was inches

64 . 32 . 16 . 8 . 4 . 2 . 1 . $\frac{1}{2}$. $\frac{1}{4}$

The arc described in the last ascent was inches

48 . 24 . 12 . 6 . 3 . $1\frac{1}{2}$. $\frac{3}{4}$. $\frac{3}{8}$. $\frac{3}{16}$

The difference of the arcs, proportional to the motion lost, was inches

16 . 8 . 4 . 2 . 1 . $\frac{1}{2}$. $\frac{1}{4}$. $\frac{1}{8}$. $\frac{1}{16}$

The number of the oscillations in water

$29\frac{9}{10}$. $1\frac{1}{8}$. 3 . 7 . $11\frac{1}{4}$. $12\frac{2}{3}$. $13\frac{1}{2}$

The number of the oscillations in air

$85\frac{1}{2}$. 287 . 535

In the experiments of the 4th column there were equal motions lost in 535 oscillations made in the air, and iVr , in water. The oscillations in the air were indeed a little swifter than those in the water. But if the oscillations in the water were accelerated in such a ratio that the motions of the pendulums might be equally swift in both mediums, there would be still the same number of $1\frac{1}{5}$ oscillations in the water, and by these the same quantity of motion would be lost as before; because the resistance is increased, and the square of the time diminished in the same squared ratio. The pendulums, therefore, being of equal velocities, there were equal motions lost in 535 oscillations in the air, and $1\frac{1}{5}$ in the water; and therefore the resistance of the pendulum in the water is to its resistance in the air as 535 to $1\frac{1}{5}$. This is the proportion of the whole resistances in the case of the 4th column.

Now let AV-I CV" represent the difference of the arcs described in the descent and subsequent ascent by the globe moving in air with the greatest

velocity V ; and since the greatest velocity is in the case of the 4th column to the greatest velocity in the case of the 1st column as 1 is to 8; and that difference of the arcs in the case of the 4th column to the difference in the case of the 1st column as $\frac{7}{595}$ to $\frac{16}{85\frac{1}{2}}$, or as $85\frac{1}{2}$ to 4280; put in these cases 1 and 8 for the velocities, and $85\frac{1}{2}$ and 4280 for the differences of the arcs, and $A + C$ will be $= 85\frac{1}{2}$, and $8A + 64C = 4280$ or $A + 8C = 535$; and then, by reducing these equations, there will come out $7C = 449\frac{1}{2}$ and $C = 64\frac{3}{4}$; and $A = 21\frac{2}{7}$; and therefore the resistance, which is as $\frac{1}{11}AV + \frac{3}{4}CV^2$, will become as $13\frac{9}{11}V + 48\frac{9}{66}V^2$. Therefore in the case of the 4th column, where the velocity was 1, the whole resistance is to its part proportional to the square of the velocity as $13\frac{9}{11} + 48\frac{9}{66}$ or $61\frac{12}{17}$ to $48\frac{9}{66}$; and therefore the resistance of the pendulum in water is to that part of the resistance in air, which is proportional to the square of the velocity, and which in swift motions is the only part that deserves consideration, as $61\frac{12}{17}$ to $48\frac{9}{66}$ and 535 to $1\frac{1}{2}$ conjointly, that is, as 571 to 1. If the whole thread of the pendulum oscillating in the water had been immersed, its resistance would have been still greater; so that the resistance of the pendulum oscillating in the water, that is, that part which is proportional to the square of the velocity, and which only needs to be considered in swift bodies, is to the resistance of the same whole pendulum, oscillating in air with the same velocity, as about 850 to 1, that is, as the density of water is to the density of air, nearly.

In this calculation we ought also to have taken in that part of the resistance of the pendulum in the water which was as the square of the velocity; but I found (which will perhaps seem strange) that the resistance in the water was augmented in more than a squared ratio of the velocity. In searching after the cause, I thought upon this, that the vessel was too narrow for the magnitude of the pendulous globe, and by its narrowness obstructed the motion of the water as it yielded to the oscillating globe. For when I immersed a pendulous globe, whose diameter was one inch only, the resistance was augmented nearly as the square of the velocity. I tried this by making a pendulum of two globes, of which the lesser and lower oscillated in the water, and the greater and higher was fastened to the thread just above the water, and, by oscillating in the air, assisted the motion of the pendulum, and continued it longer. The experiments made by this contrivance resulted as shown in the following table (p. 324).

<i>Arc described in first descent</i>	16	.	8	.	4	.	2	.	1	.	$\frac{1}{2}$	.	$\frac{1}{4}$
<i>Arc described in last ascent</i>	12	.	6	.	3	.	$1\frac{1}{4}$	.	$\frac{3}{4}$	.	$\frac{3}{8}$	.	$\frac{3}{16}$
<i>Difference of arcs, proportional to motion lost</i>	4	.	2	.	1	.	$\frac{1}{2}$	.	$\frac{1}{4}$	.	$\frac{1}{8}$	.	$\frac{1}{16}$
<i>Number of oscillations</i>	$3\frac{3}{8}$	.	$6\frac{1}{2}$	.	$12\frac{1}{2}$	.	$21\frac{1}{8}$	.	34	.	53	.	$62\frac{1}{8}$

In comparing the resistances of the mediums with each other, I also caused iron pendulums to oscillate in quicksilver. The length of the iron wire was about 3 feet, and the diameter of the pendulous globe about $\frac{1}{3}$ of an inch. To the wire, just above the quicksilver, there was fixed another leaden globe of a bigness sufficient to continue the motion of the pendulum for some time. Then a vessel, that would hold about 3 pounds of quicksilver, was filled by turns with quicksilver and common water, so that, by making the pendulum oscillate successively in these two different fluids, I might find the proportion of their resistances; and the resistance of the quicksilver proved to be to the resistance of water as about 13 or 14 to 1; that is, as the density of quicksilver to the density of water. When I made use of a pendulous globe something bigger, as of one whose diameter was about $\frac{1}{2}$ or $\frac{2}{3}$ of an inch, the resistance of the quicksilver proved to be to the resistance of the water as about 12 or 10 to 1. But the former experiment is more to be relied on, because in the latter the vessel was too narrow in proportion to the magnitude of the immersed globe; for the vessel ought to have been enlarged together with the globe. I intended to repeat these experiments with larger vessels, and in melted metals, and other liquors both cold and hot; but I had not leisure to try all; and besides, from what is already described, it appears sufficiently that the resistance of bodies moving swiftly is nearly proportional to the densities of the fluids in which they move. I do not say accurately; for more tenacious fluids, of equal density, will undoubtedly resist more than those that are more liquid; as cold oil more than warm, warm oil more than rain water, and water more than spirit of wine. But in liquors, which are sensibly fluid enough, as in air, in salt and fresh water, in spirit of wine, of turpentine, and salts, in oil cleared of its feces by distillation and warmed, in oil of vitriol, and in mercury, and melted metals, and any other such like, that are fluid enough to retain for some time the motion impressed upon them by the agitation of the vessel, and which being poured out are easily resolved into drops, I doubt not that

the rule already laid down may be accurate enough, especially if the experiments be made with larger pendulous bodies and more swiftly moved.

Lastly, since it is the opinion of some that there is a certain ethereal medium extremely rare and subtile, which freely pervades the pores of all bodies; and from such a medium, so pervading the pores of bodies, some resistance must needs arise; in order to try whether the resistance, which we experience in bodies in motion, be made upon their outward surfaces only, or whether their internal parts meet with any considerable resistance upon their surfaces, I thought of the following experiment. I suspended a round deal box by a thread 11 feet long, on a steel hook, by means of a ring of the same metal, so as to make a pendulum of the aforesaid length. The hook had a sharp hollow edge on its upper part, so that the upper arc of the ring pressing on the edge might move the more freely; and the thread was fastened to the lower arc of the ring. The pendulum being thus prepared, I drew it aside from the perpendicular to the distance of about 6 feet, and that in a plane perpendicular to the edge of the hook, lest the ring, while the pendulum oscillated, should slide to and fro on the edge of the hook; for the point of suspension, in which the ring touches the hook, ought to remain immovable. I therefore accurately noted the place to which the pendulum was brought, and letting it go, I marked three other places, to which it returned at the end of the 1st, 2d, and 3d oscillation. This I often repeated, that I might find those places as accurately as possible. Then I filled the box with lead and other heavy metals that were near at hand. But, first, I weighed the box when empty, and that part of the thread that went round it, and half the remaining part, extended between the hook and the suspended box; for the thread so extended always acts upon the pendulum, when drawn aside from the perpendicular, with half its weight. To this weight I added the weight of the air contained in the box. And this whole weight was about *Vis* of the weight of the box when filled with the metals. Then because the box when full of the metals, by extending the thread with its weight, increased the length of the pendulum, I shortened the thread so as to make the length of the pendulum, when oscillating, the same as before. Then drawing aside the pendulum to the place first marked, and letting it go, I reckoned about 77 oscillations before the box returned to the second mark, and as many afterwards before it came to the third

mark, and as many after that before it came to the fourth mark. From this I conclude that the whole resistance of the box, when full, had not a greater proportion to the resistance of the box, when empty, than 78 to 77. For if their resistances were equal, the box, when full, by reason of its inertia, which was 78 times greater than the inertia of the same when empty, ought to have continued its oscillating motion so much the longer, and therefore to have returned to those marks at the end of 78 oscillations. But it returned to them at the end of 77 oscillations.

Let, therefore, A represent the resistance of the box upon its external surface, and B the resistance of the empty box on its internal surface, and if the resistances to the internal parts of bodies equally swift be as the matter, or the number of particles that are resisted, then $78B$ will be the resistance made to the internal parts of the box, when full; and therefore the whole resistance $A + B$ of the empty box will be to the whole resistance $A + 78B$ of the full box as 77 to 78, and, by subtraction, $A + B$ to $77B$ as 77 to 1; and thence $A + B$ to B as 77 to 1, and, by subtraction, again, A to B as 5928 to 1. Therefore the resistance of the empty box in its internal parts will be above 5000 times less than the resistance on its external surface. This reasoning depends upon the supposition that the greater resistance of the full box arises not from any other latent cause, but only from the action of some subtile fluid upon the included metal.

This experiment is related by memory, the paper being lost in which I had described it; so that I have been obliged to omit some fractional parts, which are slipped out of my memory; and I have no leisure to try it again. The first time I made it, the hook being weak, the full box was retarded sooner. The cause I found to be, that the hook was not strong enough to bear the weight of the box; so that, as it oscillated to and fro, the hook was bent sometimes this and sometimes that way. I therefore procured a hook of sufficient strength, so that the point of suspension might remain unmoved, and then all things happened as is above described.

SECTION VII

The motion of fluids, and the resistance made to projected bodies.

PROPOSITION XXXII. THEOREM XXVI

Suppose two similar systems of bodies consisting of an equal number of particles, and let the correspondent particles be similar and proportional, each in one system to each in the other, and have a like situation among themselves, and the same given ratio of density to each other; and let them begin to move among themselves in proportional times, and with like motions (that is, those in one system among one another, and those in the other among one another). And if the particles that are in the same system do not touch one another, except in the moments of reflection; nor attract, nor repel each other, except with acceleratwe forces that are inversely as the diameters of the correspondent particles, and directly as the squares of the velocities: I say, that the particles of those systems will continue to move among themselves with like motions and in proportional times.

Like bodies in like situations are said to be moved among themselves with like motions and in proportional times, when their situations at the end of those times are always found alike in respect of each other; as suppose we compare the particles in one system with the correspondent particles in the other. Hence the times will be proportional, in which similar and proportional parts of similar figures will be described by correspondent particles. Therefore if we suppose two systems of this kind, the correspondent particles, by reason of the similitude of the motions at their beginning, will continue to be moved with like motions, so long as they move without meeting one another; for if they are acted on by no forces, they will go on uniformly in right lines, by the first Law. But if they agitate one another with some certain forces, and those forces are inversely as the diameters of the correspondent particles and directly as the squares of the velocities, then, because the particles are in like situations, and their forces are proportional, the whole forces with which correspondent particles are agitated, and which are compounded of each of the agitating forces (by Cor. 11 of the Laws), will have like directions, and have the same effect as if they respected centres placed alike among the particles; and those whole forces will be to each other as the several forces which compose them, that is, in-

versely as the diameters of the correspondent particles and directly as the squares of the velocities: and therefore will cause correspondent particles to continue to describe like figures. These things will be so (by Cor. 1 and vm, Prop, iv, Book 1), if those centres are at rest; but if they are moved, yet, by reason of the similitude of the translations, their situations among the particles of the system will remain similar, so that the changes introduced into the figures described by the particles will still be similar. So that the motions of correspondent and similar particles will continue similar till their first meeting with each other; and thence will arise similar collisions, and similar reflections; which will again beget similar motions of the particles among themselves (by what was just now shown), till they mutually fall upon one another again, and so on *ad infinitum*. Q.E.D.

COR. 1. Hence if any two bodies, which are similar and in like situations to the correspondent particles of the systems, begin to move amongst them in like manner and in proportional times, and their magnitudes and densities be to each other as the magnitudes and densities of the corresponding particles, these bodies will continue to be moved in like manner and in proportional times; for the case of the greater parts of both systems and of the particles is the very same.

COR. 11. And if all the similar and similarly situated parts of both systems be at rest among themselves; and two of them, which are greater than the rest, and mutually correspondent in both systems, begin to move in lines alike posited, with any similar motion whatsoever, they will excite similar motions in the rest of the parts of the systems, and will continue to move among those parts in like manner and in proportional times; and will therefore describe spaces proportional to their diameters.

PROPOSITION XXXIII. THEOREM XXVII

The same things being supposed, I say, that the greater parts of the systems are resisted in a ratio compounded of the squared ratio of their velocities, and the squared ratio of their diameters, and the simple ratio of the density of the parts of the systems.

For the resistance arises partly from the centripetal or centrifugal forces with which the particles of the system act on each other, partly from the collisions and reflections of the particles and the greater parts. The resistances

of the first kind are to each other as the whole motive forces from which they arise, that is, as the whole accelerative forces and the quantities of matter in corresponding parts; that is (by the supposition), directly as the squares of the velocities and inversely as the distances of the corresponding particles, and directly as the quantities of matter in the correspondent parts: and therefore since the distances of the particles in one system are to the correspondent distances of the particles in the other, as the diameter of one particle or part in the former system to the diameter of the correspondent particle or part in the other, and since the quantities of matter are as the densities of the parts and the cubes of the diameters, the resistances are to each other as the squares of the velocities and the squares of the diameters and the densities of the parts of the systems. Q.E.D. The resistances of the latter sort are as the number of correspondent reflections and the forces of those reflections conjointly; but the number of the reflections are to each other directly as the velocities of the corresponding parts and inversely as the spaces between their reflections. And the forces of the reflections are as the velocities and the magnitudes and the densities of the corresponding parts conjointly; that is, as the velocities and the cubes of the diameters and the densities of the parts. And, joining all these ratios, the resistances of the corresponding parts are to each other as the squares of the velocities and the squares of the diameters and the densities of the parts conjointly. Q.E.D.

COR. 1. Therefore if those systems are two elastic fluids, like our air, and their parts are at rest among themselves; and two similar bodies proportional in magnitude and density to the parts of the fluids, and similarly situated among those parts, be in any manner projected in the direction of lines similarly posited; and the accelerative forces with which the particles of the fluids act upon each other are inversely as the diameters of the bodies projected and directly as the squares of their velocities; those bodies will excite similar motions in the fluids in proportional times, and will describe similar spaces and proportional to their diameters.

COR. 11. Therefore in the same fluid a projected body that moves swiftly meets with a resistance that is as the square of its velocity, nearly. For if the forces with which distant particles act upon one another should be augmented as the square of the velocity, the projected body would be resisted in the same squared ratio accurately; and therefore in a medium, whose

parts when at a distance do not act with any force on one another, the resistance is as the square of the velocity, accurately. Let there be, therefore, three mediums A, B, C, consisting of similar and equal parts regularly disposed at equal distances. Let the parts of the mediums A and B recede from each other with forces that are among themselves as T and V; and let the parts of the medium C be entirely destitute of any such forces. And if four equal bodies D, E, F, G move in these mediums, the two first D and E in the two first A and B, and the other two F and G in the third C; and if the velocity of the body D be to the velocity of the body E, and the velocity of the body F to the velocity of the body G, as the square root of the ratio of the force T to the force V; then the resistance of the body D to the resistance of the body E, and the resistance of the body F to the resistance of the body G, will be as the square of the velocities; and therefore the resistance of the body D will be to the resistance of the body F as the resistance of the body E to the resistance of the body G. Let the bodies D and F be equally swift, as also the bodies E and G; and, augmenting the velocities of the bodies D and F in any ratio, and diminishing the forces of the particles of the medium B as the square of the same ratio, the medium B will approach to the form and condition of the medium C at pleasure; and therefore the resistances of the equal and equally swift bodies E and G in these mediums will continually approach to equality, so that their difference will at last become less than any given. Therefore since the resistances of the bodies D and F are to each other as the resistances of the bodies E and G, those will also in like manner approach to the ratio of equality. Therefore the bodies D and F, when they move with very great swiftness, meet with resistances very nearly equal; and therefore since the resistance of the body F is in a squared ratio of the velocity, the resistance of the body D will be nearly in the same ratio.

COR. in. Hence the resistance of a body moving very swiftly in an elastic fluid is almost the same as if the parts of the fluid were destitute of their centrifugal forces, and did not fly from each other; provided only that the elasticity of the fluid arise from the centrifugal forces of the particles, and the velocity be so great as not to allow the particles time enough to act.

COR. IV. Since the resistances of similar and equally swift bodies, in a medium whose distant parts do not fly from each other, are as the squares of the diameters, therefore the resistances made to bodies moving with very

great and equal velocities in an elastic fluid will be as the squares of the diameters, nearly.

COR. V. And since similar, equal, and equally swift bodies, moving through mediums of the same density, whose particles do not fly from each other, will strike against an equal quantity of matter in equal times, whether the particles of which the medium consists be more and smaller, or fewer and greater, and therefore impress on that matter an equal quantity of motion, and in return (by the third Law of Motion) suffer an equal reaction from the same, that is, are equally resisted; it is manifest, also, that in elastic fluids of the same density, when the bodies move with extreme swiftness, their resistances are nearly equal, whether the fluids consist of gross parts, or of parts ever so subtile. For the resistance of projectiles moving with exceedingly great celerities is not much diminished by the subtilty of the medium.

COR. VI. All these things are so in fluids whose elastic force takes its rise from the centrifugal forces of the particles. But if that force arise from some other cause, as from the expansion of the particles after the manner of wool, or the boughs of trees, or any other cause, by which the particles are hindered from moving freely among themselves, the resistance, by reason of the lesser fluidity of the medium, will be greater than in the Corollaries above.

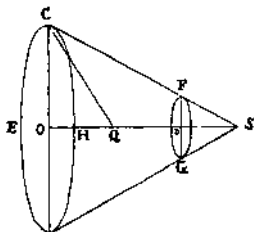
PROPOSITION XXXIV. THEOREM XXVIII

If in a rare medium, consisting of equal particles freely disposed at equal distances from each other, a globe and a cylinder described on equal diameters move with equal velocities in the direction of the axis of the cylinder, the resistance of the globe will be but half as great as that of the cylinder.

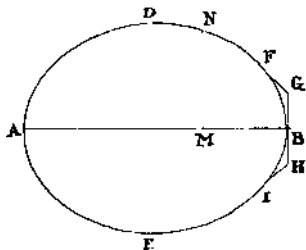
For since the action of the medium upon the body is the same (by Cor. v of the Laws) whether the body move in a quiescent medium, or whether the particles of the medium impinge with the same velocity upon the quiescent body, let us consider the body as if it were quiescent, and see with what force it would be impelled by the moving medium. Let, therefore, ABKI represent a spherical body described from the centre C with the semi-diameter CA, and let the particles of the medium impinge with a given velocity upon that spherical body in the directions of right lines parallel to

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By the same method other figures may be compared together as to their resistance; and those may be found which are most apt to continue their motions in resisting mediums. As if upon the circular base CEBH from the centre O, with the radius OC, and the altitude OD, one would construct a frustum CBGF of a cone, which should meet with less resistance than any other frustum constructed with the same base and altitude, and going forwards towards D in the direction of its axis: bisect the altitude OD in Q, and produce OQ to S, so that QS may be equal to QC, and S will be the vertex of the cone whose frustum is sought.



Incidentally, since the angle CSB is always acute, it follows from the above that, if the solid ADBE be generated by the convolution of an elliptical or oval figure ADBE about its axis AB, and the generating figure be touched by three right lines FG, GH, HI, in the points F, B, and I, so that



GH shall be perpendicular to the axis in the point of contact B, and FG, HI may be inclined to GH in the angles FGB, BHI of 135 degrees: the solid arising from the convolution of the figure ADFGHIE about the same axis AB will be less resisted than

the former solid, provided that both move forwards in the direction of their axis AB, and that the extremity B of each go foremost. This Proposition I conceive may be of use in the building of ships.

If the figure DNFG be such a curve, that if, from any point thereof, as N, the perpendicular NM be let fall on the axis AB, and from the given point G there be drawn the right line GR parallel to a right line touching the

¹ Appendix, Note 35.]

figure in N, and cutting the axis produced in R, MN becomes to GR as GR^3 to $4BR \cdot GB^2$, the solid described by the revolution of this figure about its axis AB, moving in the before-mentioned rare medium from A towards B, will be less resisted than any other circular solid whatsoever, described of the same length and breadth.

PROPOSITION XXXV. PROBLEM VII

If a rare medium consist of very small quiescent particles of equal magnitudes, and freely disposed at equal distances from one another: to find the resistance of a globe moving uniformly forwards in this medium.

CASE 1. Let a cylinder described with the same diameter and altitude be conceived to go forwards with the same velocity in the direction of its axis through the same medium; and let us suppose that the particles of the medium, on which the globe or cylinder falls, fly back with as great a force of reflection as possible. Then since the resistance of the globe (by the last Proposition) is but half the resistance of the cylinder, and since the globe is to the cylinder as 2 to 3, and since the cylinder by falling perpendicularly on the particles, and reflecting them with the utmost force, communicates to them a velocity double to its own: it follows that the cylinder in moving forwards uniformly half the length of its axis, will communicate a motion to the particles which is to the whole motion of the cylinder as the density of the medium to the density of the cylinder; and that the globe, in the time it describes one length of its diameter in moving uniformly forwards, will communicate the same motion to the particles; and, in the time that it describes two-thirds of its diameter, will communicate a motion to the particles which is to the whole motion of the globe as the density of the medium to the density of the globe. And therefore the globe meets with a resistance, which is to the force by which its whole motion may be either taken away or generated in the time in which it describes two-thirds of its diameter moving uniformly forwards, as the density of the medium is to the density of the globe.

CASE 2. Let us suppose that the particles of the medium incident on the globe or cylinder are not reflected; and then the cylinder falling perpendicularly on the particles will communicate its own simple velocity to them,

and therefore meets a resistance but half so great as in the former case, and the globe also meets with a resistance but half so great.

CASE 3. Let us suppose the particles of the medium to fly back from the globe with a force which is neither the greatest, nor yet none at all, but with a certain mean force; then the resistance of the globe will be in the same mean ratio between the resistance in the first case and the resistance in the second. Q.E.I.

COR. 1. Hence if the globe and the particles are infinitely hard, and destitute of all elastic force, and therefore of all force of reflection, the resistance of the globe will be to the force by which its whole motion may be destroyed or generated, in the time that the globe describes four third parts of its diameter, as the density of the medium is to the density of the globe.

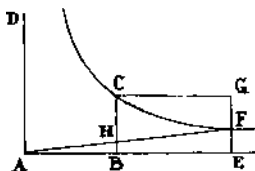
COR. 11. The resistance of the globe, other things being equal, varies as the square of the velocity.

COR. in. The resistance of the globe, other things being equal, varies as the square of the diameter.

COR. IV. The resistance of the globe, other things being equal, varies as the density of the medium.

COR. V. The resistance of the globe vanes jointly as the square of the velocity, as the square of the diameter, and as the density of the medium.

COR. VI. The motion of the globe and its resistance may be thus represented. Let AB be the time in which the globe may, by its resistance uniformly continued, lose its whole motion. Erect AD, BC perpendicular to AB. Let BC be that whole motion, and through the point C, the asymptotes being AD, AB, describe the hyperbola CF. Produce AB to any point E. Erect the perpendicular EF meeting the hyperbola in F. Complete the parallelogram CBEG, and draw AF meeting BC in H. Then if the globe in any time BE, with its first motion BC uniformly continued, describes in a nonresisting medium the space CBEG represented by the area of the parallelogram, the same in a resisting medium will describe the space CBEF, represented by the area of the hyperbola; and its motion at the end of that time will be represented by EF, the ordinate of the hyperbola, there



being lost of its motion the part FG. And its resistance at the end of the same time will be represented by the length BH, there being lost of its resistance the part CH. All these things appear by Cor. 1 and in, Prop, v, Book 11.

COR. vii. Hence if the globe in the time T by the resistance R uniformly continued lose its whole motion M , the same globe in the time t in a resisting medium, wherein the resistance R decreases as the square of the velocity, will lose out of its motion M the part $\frac{2M}{T+t}$, the part $\frac{TM}{T+t}$ remaining; and will describe a space which is to the space described in the same time t , with the uniform motion M , as the logarithm of the number $—^{\wedge}$ multiplied by the number 2.302585092994 is to the number $—^{\wedge}$ because the hyperbolic area BCFE is to the rectangle BCGE in that proportion.

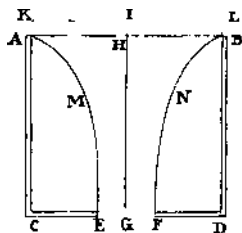
S C H O L I U M

I have exhibited in this Proposition the resistance and retardation of spherical projectiles in mediums that are not continued, and shown that this resistance is to the force by which the whole motion of the globe may be destroyed or produced in the time in which the globe can describe two-thirds of its diameter, with a velocity uniformly continued, as the density of the medium is to the density of the globe, provided the globe and the particles of the medium be perfectly elastic, and are endued with the utmost force of reflection; and that this force, where the globe and particles of the medium are infinitely hard and void of any reflecting force, is diminished one-half. But in continued mediums, as water, hot oil, and quicksilver, the globe as it passes through them does not immediately strike against all the particles of the fluid that generate the resistance made to it, but presses only the particles that lie next to it, which press the particles beyond, which press other particles, and so on; and in these mediums the resistance is diminished one other half. A globe in these extremely fluid mediums meets with a resistance that is to the force by which its whole motion may be destroyed or generated in the time wherein it can describe, with that motion uniformly continued, eight third parts of its diameter, as the density of the medium is to the density of the globe. This I shall endeavor to show in what follows.

PROPOSITION XXXVI. PROBLEM VIII

To find *the* motion of water running out of a cylindrical vessel through a hole made at the bottom}

Let ACDB be a cylindrical vessel, AB the mouth of it, CD the bottom parallel to the horizon, EF a circular hole in the middle of the bottom, G the centre of the hole, and GH the axis of the cylinder perpendicular to the horizon. And suppose a cylinder of ice APQB to be of the same breadth with the cavity of the vessel, and to have the same axis, and to descend continually with an uniform motion, and that its parts, as soon as they touch the surface AB, dissolve into water, and flow down by their weight into the vessel, and in their fall compose the cataract or column of water ABNFEM, passing through the hole EF, and filling up the same exactly. Let the uniform velocity of the descending ice and of the contiguous water in the circle AB be that which the water would acquire by falling through the space IH; and let IH and HG lie in the same right line; and through the point I let there be drawn the right line KL parallel to the horizon, and meeting the ice on both the sides thereof in K and L. Then the velocity of the water running out at the hole EF will be the same that it would acquire by falling from I through the space IG. Therefore, by *Galileo's* Theorems, IG will be to IH as the square of the velocity of the water that runs out at the hole to the velocity of the water in the circle AB, that is, as the square of the ratio of the circle AB to the circle EF; those circles being inversely as the velocities of the water which in the same time and in equal quantities passes through each of them, and completely fills them both. We are now considering the velocity with which the water tends to the plane of the horizon. But the motion parallel to the same, by which the parts of the falling water approach to each other, is not here taken notice of; since it is neither produced by gravity, nor at all changes the motion perpendicular to the horizon which the gravity produces. We suppose, indeed, that the parts of the



[¹ Appendix, Note 36]

water cohere a little, that by their cohesion they may in falling approach to each other with motions parallel to the horizon in order to form one single cataract, and to prevent their being divided into several; but the motion parallel to the horizon arising from this cohesion does not come under our present consideration.

CASE I. Conceive now the whole cavity in the vessel, which surrounds the falling water ABNFEM, to be full of ice, so that the water may pass through the ice as through a funnel. Then if the water pass very near to the ice only, without touching it; or, which is the same thing, if by reason of the perfect smoothness of the surface of the ice, the water, though touching it, glides over it with the utmost freedom, and without the least resistance; the water will run through the hole EF with the same velocity as before, and the whole weight of the column of water ABNFEM will be taken up as before in forcing out the water, and the bottom of the vessel will sustain the weight of the ice surrounding that column.

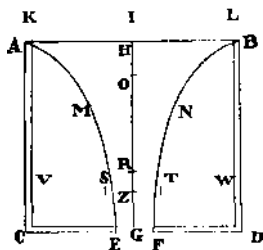
Let now the ice in the vessel dissolve into water; but the efflux of the water will remain, as to its velocity, the same as before. It will not be less, because the ice now dissolved will endeavor to descend; it will not be greater, because the ice, now become water, cannot descend without hindering the descent of other water equal to its own descent. The same force ought always to generate the same velocity in the effluent water.

But the hole at the bottom of the vessel, by reason of the oblique motions of the particles of the effluent water, must be a little greater than before. For now the particles of the water do not all of them pass through the hole perpendicularly, but, flowing down on all parts from the sides of the vessel, and converging towards the hole, pass through it with oblique motions; and in tending downwards they meet in a stream whose diameter is a little smaller below the hole than at the hole itself; its diameter being to the diameter of the hole as 5 to 6, or as $5\sqrt{2}$ to $6\frac{1}{2}$, very nearly, if I measured those diameters rightly. I procured a thin flat plate, having a hole pierced in the middle, the diameter of the circular hole being five eighth parts of an inch. And that the stream of running water might not be accelerated in falling, and by that acceleration become narrower, I fixed this plate not to the bottom, but to the side of the vessel, so as to make the water *go* out in the direction of a line parallel to the horizon. Then, when the vessel was

full of water, I opened the hole to let it run out; and the diameter of the stream, measured with great accuracy at the distance of about half an inch from the hole, was $\frac{21}{40}$ of an inch. Therefore the diameter of this circular hole was to the diameter of the stream very nearly as 25 to 21. So that the water in passing through the hole converges on all sides, and, after it has run out of the vessel, becomes smaller by converging in that manner, and by becoming smaller is accelerated till it comes to the distance of half an inch from the hole, and at that distance flows in a smaller stream and with greater celerity than in the hole itself, and this in the ratio of 25 to 21, or 17 to 12, very nearly; that is, in about the ratio of A^2 to 1. Now it is certain from experiments, that the quantity of water running out in a given time through a circular hole made in the bottom of a vessel is equal to the quantity, which, flowing freely with the aforesaid velocity, would run out in the same time through another circular hole, whose diameter is to the diameter of the former as 21 to 25. And therefore this running water in passing through the hole itself has a velocity downwards nearly equal to that which a heavy body would acquire in falling through half the height of the stagnant water in the vessel. But then, after it has run out, it is still accelerated by converging, till it arrives at a distance from the hole that is nearly equal to its diameter, and acquires a velocity greater than the other in about the ratio of $\sqrt{2}$ to 1; this velocity a heavy body would nearly acquire by falling freely through the whole height of the stagnant water in the vessel.

Therefore in what follows let the diameter of the stream be represented by that lesser hole which we shall call EF. And imagine another plane VW above the hole EF, and parallel to the plane thereof, to be placed at a distance equal to the diameter of the same hole, and to be pierced through with a greater hole ST, of such a magnitude that a stream which will exactly fill the lower hole EF may pass through it; the

diameter of this hole will therefore be to the diameter of the lower hole nearly as 25 to 21. By this means the water will run perpendicularly out at



the lower hole; and the quantity of the water running out will be, according to the magnitude of this last hole, very nearly the same as that which the solution of the Problem requires. The space included between the two planes and the falling stream may be considered as the bottom of the vessel. But to make the solution more simple and mathematical, it is better to take the lower plane alone for the bottom of the vessel, and to suppose that the water which flowed through the ice as through a funnel, and ran out of the vessel through the hole EF made in the lower plane, preserves its motion continually, and that the ice continues at rest. Therefore in what follows let ST be the diameter of a circular hole described from the centre Z, and let the stream run out of the vessel through that hole, when the water in the vessel is all fluid. And let EF be the diameter of the hole, which the stream, in falling through, exactly fills up, whether the water runs out of the vessel by that upper hole ST, or flows through the middle of the ice in the vessel, as through a funnel. And let the diameter of the upper hole ST be to the diameter of the lower EF as about 25 to 21, and let the perpendicular distance between the planes of the holes be equal to the diameter of the lesser hole EF. Then the velocity of the water downwards, in running out of the vessel through the hole ST, will be in that hole the same that a body may acquire by falling freely from half the height IZ; and the velocity of both the falling streams will be in the hole EF, the same which a body would acquire by falling freely from the whole height IG.

CASE 2. If the hole EF be not in the middle of the bottom of the vessel, but in some other part thereof, the water will still run out with the same velocity as before, if the magnitude of the hole be the same. For though a heavy body takes a longer time in descending to the same depth, by an oblique line, than by a perpendicular line, yet in both cases it acquires in its descent the same velocity; as *Galileo* hath demonstrated.

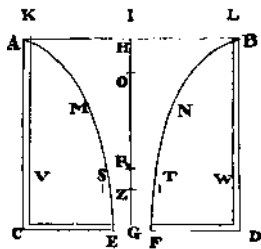
CASE 3. The velocity of the water is the same when it runs out through a hole in the side of the vessel. For if the hole be small, so that the interval between the surfaces AB and KL may vanish as to sense, and the stream of water horizontally issuing out may form a parabolic figure; from the latus rectum of this parabola one may see, that the velocity of the effluent water is that which a body may acquire by falling the height IG or HG of the stagnant water in the vessel. For, by making an experiment, I found that if

the height of the stagnant water above the hole were 20 inches, and the height of the hole above a plane parallel to the horizon were also 20 inches, a stream of water springing out from thence would fall upon the plane, at the distance of very nearly 37 inches, from a perpendicular let fall upon that plane from the hole. For without resistance the stream would have fallen upon the plane at the distance of 40 inches, the latus rectum of the parabolic stream being 80 inches.

CASE 4. If the effluent water tend upwards, it will still issue forth with the same velocity. For the small stream of water springing upwards, ascends with a perpendicular motion to GH or GI, the height of the stagnant water in the vessel; except so far as its ascent is hindered a little by the resistance of the air; and therefore it springs out with the same velocity that it would acquire in falling from that height. Every particle of the stagnant water is equally pressed on all sides (by Prop, xix, Book 11), and, yielding to the pressure, tends always with an equal force, whether it descends through the hole in the bottom of the vessel, or gushes out in an horizontal direction through a hole in the side, or passes into a canal, and springs up from thence through a little hole made in the upper part of the canal. And it may not only be inferred from reasoning, but is manifest also from the well-known experiments just mentioned, that the velocity with which the water runs out is the very same that is assigned in this Proposition.

CASE 5. The velocity of the effluent water is the same, whether the figure of the hole be circular, or square, or triangular, or of any other figure whatever equal to the circular; for the velocity of the effluent water does not depend upon the figure of the hole, but arises from such depth of the hole as it may have below the plane KL.

CASE 6. If the lower part of the vessel ABDC be immersed into stagnant water, and the height of the stagnant water above the bottom of the vessel be GR, the velocity with which the water that is in the vessel will run out at the hole EF into the stagnant water will be the same which the water would acquire by falling from the height IR; for the weight of all the water



in the vessel that is below the surface of the stagnant water will be sustained in equilibrium by the weight of the stagnant water, and therefore does not at all accelerate the motion of the descending water in the vessel. This case will also become evident from experiments, measuring the times in which the water will run out.

COR. i. Hence if CA, the depth of the water, be produced to K, so that AK may be to CK as the square of the ratio of the area of a hole made in any part of the bottom to the area of the circle AB, the velocity of the effluent water will be equal to the velocity which the water would acquire by falling freely from the height KC.

COR. II. And the force with which the whole motion of the effluent water may be generated is equal to the weight of a cylindric column of water, whose base is the hole EF, and its altitude 2GI or 2CK. For the effluent water, in the time it becomes equal to this column, may acquire, by falling by its own weight from the height GI, a velocity equal to that with which it runs out.

COR. in. The weight of all the water in the vessel ABDC is to that part of the weight which is employed in forcing out the water as the sum of the circles AB and EF is to twice the circle EF. For let IO be a mean proportional between IH and IG, and the water running out at the hole EF will, in the time that a drop falling from I would describe the altitude IG, become equal to a cylinder whose base is the circle EF and its altitude 2IG, that is, to a cylinder whose base is the circle AB, and whose altitude is 2IO. For the circle EF is to the circle AB as the square root of the ratio of the altitude IH to the altitude IG; that is, in the simple ratio of the mean proportional IO to the altitude IG. Moreover, in the time that a drop falling from I can describe the altitude IH, the water that runs out will have become equal to a cylinder whose base is the circle AB, and its altitude 2IH; and in the time that a drop falling from I through H to G describes HG, the difference of the altitudes, the effluent water, that is, the water contained within the solid ABNFEM, will be equal to the difference of the cylinders, that is, to a cylinder whose base is AB, and its altitude 2HO. And therefore all the water contained in the vessel ABDC is to the whole falling water contained in the said solid ABNFEM as HG is to 2HO, that is, as HO + OG to 2HO, or IH f IO to 2IH. But the weight of all the water in the solid ABNFEM is

employed in forcing out the water; and therefore the weight of all the water in the vessel is to that part of the weight that is employed in forcing out the water as $IH + 10$ is to $2IH$, and therefore as the sum of the circles EF and AB is to twice the circle EF .

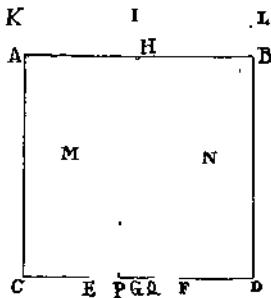
COR. IV. And hence the weight of all the water in the vessel $ABDC$ is to the other part of the weight which is sustained by the bottom of the vessel as the sum of the circles AB and EF is to the difference of the same circles.

COR. v. And that part of the weight which the bottom of the vessel sustains is to the other part of the weight employed in forcing out the water as the difference of the circles AB and EF is to twice the lesser circle EF , or as the area of the bottom to twice the hole.

COR. vi. That part of the weight which presses upon the bottom is to the whole weight of the water perpendicularly incumbent thereon as the circle AB is to the sum of the circles AB and EF , or as the circle AB is to the excess of twice the circle AB above the area of the bottom. For that part of the weight which presses upon the bottom is to the weight of the whole water in the vessel as the difference of the circles AB and EF is to the sum of the same circles (by Cor. iv); and the weight of the whole water in the vessel is to the weight of the whole water perpendicularly incumbent on the bottom as the circle AB is to the difference of the circles AB and EF . Therefore, multiplying together corresponding terms of the two proportions, that part of the weight which presses upon the bottom is to the weight of the whole water perpendicularly incumbent thereon as the circle AB to the sum of the circles AB and EF , or the excess of twice the circle AB above the bottom.

COR. VII. If in the middle of the hole EF there be placed the little circle PQ described about the centre G , and parallel to the horizon, the weight of water which that little circle sustains is greater than the weight of a third part of a cylinder of water whose base is that little circle and its height GH . For let $ABNFEM$ be the cataract or column of falling water whose axis is GH , as above, and let all the water, whose fluidity is not requisite for the ready and quick descent of the water, be supposed to be congealed, as well round about the cataract, as above the little circle. And let PHQ be the column of water congealed above the little circle, whose vertex is H , and its altitude GH . And suppose this cataract to fall with its whole

weight downwards, and not in the least to lie against or to press PHQ, but to glide freely by it without any friction, unless, perhaps, just at the very vertex of the ice, where the cataract at the beginning of its fall may tend to a concave figure. And as the congealed water AMEC, BNFD, lying



round the cataract, is convex in its internal surfaces AME, BNF towards the falling cataract, so this column PHQ will be convex towards the cataract also, and will therefore be greater than a cone whose base is that little circle PQ and its altitude GH; that is, greater than a third part of a cylinder described with the same base and altitude. Now that little circle sustains the weight of this column, that is, a weight greater than the weight of the cone, or a third part of the cylinder.

COR. vin. The weight of water which the circle PQ, when very small, sustains, seems to be less than the weight of two-thirds of a cylinder of water whose base is that little circle, and its altitude HG. For, things standing as above supposed, imagine the half of a spheroid described whose base is that little circle, and its semiaxis or altitude HG. This figure will be equal to two-thirds of that cylinder, and will comprehend within it the column of congealed water PHQ, the weight of which is sustained by that little circle. For though the motion of the water tends directly downwards, the external surfaces of that column must yet meet the base PQ in an angle somewhat acute, because the water in its fall is continually accelerated, and by reason of that acceleration becomes narrower. Therefore, since that angle is less than a right one, this column in the lower parts thereof will lie within the hemispheroid. In the upper parts also it will be acute or pointed; because to make it otherwise, the horizontal motion of the water must be at the vertex infinitely more swift than its motion towards the horizon. And the less this circle PQ is, the more acute will the vertex of this column be; and the circle being diminished *in infinitum*, the angle PHQ will be diminished *in infinitum*, and therefore the column will lie within the hemispheroid.

Therefore that column is less than that hemispheroid, or than two third parts of the cylinder whose base is that little circle, and its altitude GH . Now the little circle sustains a force of water equal to the weight of this column, the weight of the ambient water being employed in causing its efflux out at the hole.

COR. IX. The weight of water which the little circle PQ sustains, when it is very small, is very nearly equal to the weight of a cylinder of water whose base is that little circle, and its altitude $V2GH$; for this weight is an arithmetical mean between the weights of the cone and the hemispheroid above mentioned. But if that little circle be not very small, but on the contrary increased till it be equal to the hole EF , it will sustain the weight of all the water lying perpendicularly above it, that is, the weight of a cylinder of water whose base is that little circle, and its altitude GH .

COR. x. And (as far as I can judge) the weight which this little circle sustains is always to the weight of a cylinder of water whose base is that little circle, and its altitude $\frac{1}{2}GH$, as EF^2 is to $EF^2 - \frac{1}{2}PQ^2$, or as the circle EF is to the excess of this circle above half the little circle PQ , very nearly.

LEMMA IV

If a cylinder moves uniformly forwards in the direction of its length, the resistance made thereto is not at all changed by augmenting or diminishing that length; and is therefore the same with the resistance of a circle, described with the same diameter, and moving forwards with the same velocity in the direction of a right line perpendicular to its plane.

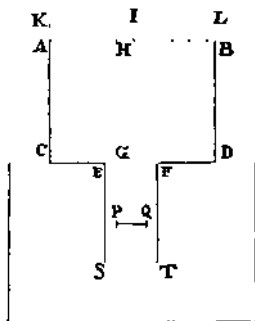
For the sides are not at all opposed to the motion; and a cylinder becomes a circle when its length is diminished in *infinitum*.

PROPOSITION XXXVII. THEOREM XXIX

If a cylinder moves uniformly forwards in a compressed, infinite, and non-elastic fluid, in the direction of its length, the resistance arising from the magnitude of its transverse section is to the force by which its whole motion may be destroyed or generated, in the time that it moves four times its length, as the density of the medium is to the density of the cylinder, nearly.

For let the vessel $ABDC$ touch the surface of stagnant water with its bottom CD , and let the water run out of this vessel into the stagnant water

through the cylindric canal EFTS perpendicular to the horizon; and let the little circle PQ be placed parallel to the horizon anywhere in the middle of the canal; and produce CA to K, so that AK may be to CK as the square of the ratio, which the excess of the orifice of the canal EF above



the little circle PQ bears to the circle AB. Then it is manifest (by Case 5, Case 6, and Cor. 1, Prop, xxxvi) that the velocity of the water passing through the annular space between the little circle and the sides of the vessel will be the very same as that which the water would acquire by falling, and in its fall describing the altitude KC or IG.

And (by Cor. x, Prop, xxxvi) if the breadth of the vessel be infinite, so that the short line HI may vanish, and the altitudes IG, HG become equal; the force of the water that flows down and presses upon the circle will be to the weight of a cylinder whose base is that little circle, and the altitude MJIG, as EF^2 is to $EF^2 - \frac{1}{2}PQ^2$, very nearly. For the force of the water flowing downwards uniformly through the whole canal will be the same upon the little circle PQ in whatsoever part of the canal it be placed.

Let now the orifices of the canal EF, ST be closed, and let the little circle ascend in the fluid compressed on every side, and by its ascent let it oblige the water that lies above it to descend through the annular space between the little circle and the sides of the canal. Then will the velocity of the ascending little circle be to the velocity of the descending water as the difference of the circles EF and PQ is to the circle PQ; and the velocity of the ascending little circle will be to the sum of the velocities, that is, to the relative velocity of the descending water with which it passes by the little circle in its ascent, as the difference of the circles EF and PQ is to the circle EF, or as $EF^2 - PQ^2$ to EF^2 . Let that relative velocity be equal to the velocity with which it was shown above that the water would pass through the annular space, if the circle were to remain unmoved, that is, to the velocity which the water would acquire by falling, and in its fall describing the altitude IG; and the

force of the water upon the ascending circle will be the same as before (by Cor. v of the Laws of Motion); that is, the resistance of the ascending little circle will be to the weight of a cylinder of water whose base is that little circle, and its altitude $V2IG$, as EF^2 is to $EF^2 - \frac{1}{2}PQ^2$, nearly. But the velocity of the little circle will be to the velocity which the water acquires by falling, and in its fall describing the altitude IG , as $EF^2 - PQ^2$ is to EF^2 .

Let the breadth of the canal be increased *in infinitum*; and the ratios between $EF' - PQ^2$ and EF , and between EF^2 and $EF^2 - \frac{1}{2}PQ^2$, will become at last ratios of equality. And therefore the velocity of the little circle will now be the same as that which the water would acquire in falling, and in its fall describing the altitude IG ; and the resistance will become equal to the weight of a cylinder whose base is that little circle, and its altitude half the altitude IG , from which the cylinder must fall to acquire the velocity of the ascending circle; and with this velocity the cylinder in the time of its fall will describe four times its length. But the resistance of the cylinder moving forwards with this velocity in the direction of its length is the same with the resistance of the little circle (by Lem. iv), and is therefore nearly equal to the force by which its motion may be generated while it describes four times its length.

If the length of the cylinder be augmented or diminished, its motion, and the time in which it describes four times its length, will be augmented or diminished in the same ratio, and therefore the force by which the motion, so increased or diminished, may be destroyed or generated, will continue the same; because the time is increased or diminished in the same proportion; and therefore that force remains still equal to the resistance of the cylinder, because (by Lem. iv) that resistance will also remain the same.

If the density of the cylinder be augmented or diminished, its motion, and the force by which its motion may be generated or destroyed in the same time, will be augmented or diminished in the same ratio. Therefore the resistance of any cylinder whatsoever will be to the force by which its whole motion may be generated or destroyed, in the time during which it moves four times its length, as the density of the medium is to the density of the cylinder, nearly. Q.E.D.

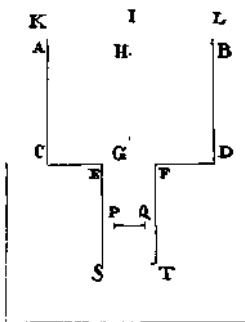
A fluid must be compressed to become continued; it must be continued and nonelastic, that all the pressure arising from its compression may be

propagated in an instant; and so, acting equally upon all parts of the body moved, may produce no change of the resistance. The pressure arising from the motion of the body is spent in generating a motion in the parts of the fluid, and this creates the resistance. But the pressure arising from the compression of the fluid, be it ever so forcible, if it be propagated in an instant, generates no motion in the parts of a continued fluid, produces no change at all of motion therein; and therefore neither augments nor lessens the resistance. This is certain, that the action of the fluid arising from the compression cannot be stronger on the hinder parts of the body moved than on its fore parts, and therefore cannot lessen the resistance described in this Proposition. And if its propagation be infinitely swifter than the motion of the body pressed, it will not be stronger on the fore parts than on the hinder parts. But that action will be infinitely swifter, and propagated in an instant, if the fluid be continued and nonelastic.

COR. 1. The resistances, made to cylinders going uniformly forwards in the direction of their lengths through continued infinite mediums, are in a ratio compounded of the square of the ratio of the velocities and the square of the ratio of the diameters, and the ratio of the density of the mediums.

COR. 11. If the breadth of the canal be not infinitely increased, but the cylinder go forwards in the direction of its length through an included quiescent medium, its axis all the while coinciding with the axis of the canal, its resistance will be to the force by which its whole motion, in the time in which it describes four times its length, may be generated or destroyed, in a ratio compounded of the ratio of EF^2 to $EF^2 - \frac{1}{2}PQ^2$, and the square of the ratio of EF^2 to $EF^2 - PQ^2$, and the ratio of the density of the medium to the density of the cylinder.

COR. in. The same thing supposed, and that a length L is to four times the length of the cylinder in a ratio compounded of the ratio $EF^2 - \frac{1}{2}PQ'$ to EF' , and the square of the ratio of $EF^2 - PQ^2$ to EF^2 :

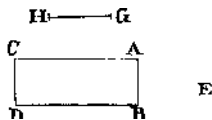


the resistance of the cylinder will be to the force by which its whole motion, in the time during which it describes the length L , may be destroyed or generated, as the density of the medium is to the density of the cylinder.

S C H O L I U M

In this Proposition we have investigated that resistance alone which arises from the magnitude of the transverse section of the cylinder, neglecting that part of the same which may arise from the obliquity of the motions. For as, in Case i of Prop, xxxvi, the obliquity of the motions with which the parts of the water in the vessel converged on every side to the hole EF hindered the efflux of the water through the hole, so, in this Proposition, the obliquity of the motions, with which the parts of the water, pressed by the antecedent extremity of the cylinder, yield to the pressure, and diverge on all sides, retards their passage through the places that lie round that antecedent extremity, towards the hinder parts of the cylinder, and causes the fluid to be moved to a greater distance; which increases the resistance, and that in the same ratio almost in which it diminished the efflux of the water out of the vessel, that is, in the squared ratio of 25 to 21, nearly. And as, in Case 1 of that Proposition, we made the parts of the water pass through the hole EF perpendicularly and in the greatest plenty, by supposing all the water in the vessel lying round the cataract to be frozen, and that part of the water whose motion was oblique and useless to remain without motion, so in this Proposition, that the obliquity of the motions may be taken away, and the parts of the water may give the freest passage to the cylinder, by yielding to it with the most direct and quick motion possible, so that only so much resistance may remain as arises from the magnitude of the transverse section, and as is incapable of diminution, unless by diminishing the diameter of the cylinder; we must conceive those parts of the fluid whose motions are oblique and useless, and produce resistance, to be at rest among themselves at both extremities of the cylinder, and there to cohere, and be joined to the cylinder.

Let $ABCD$ be a rectangle, and let AE and BE be two parabolic arcs, described with the axis AB , and with a latus rectum that is to the space HG , which must be



described by the cylinder in falling, in order to acquire the velocity with which it moves, as HG to $\frac{1}{2}AB$. Let CF and DF be two other parabolic arcs described with the axis CD , and a latus rectum four times the former; and by the revolution of the figure about the axis EF let there be generated a solid, whose middle part $ABDC$ is the cylinder we are here speaking of, and whose extreme parts ABE and CDF contain the parts of the fluid at rest among themselves, and concreted into two hard bodies, adhering to the cylinder at each end like a head and tail. Then if this solid $EACFDB$ move in the direction of the length of its axis FE towards the parts beyond E , the resistance will be nearly the same as that which we have here determined in this Proposition; that is, it will have the same ratio to the force with which the whole motion of the cylinder may be destroyed or generated, in the time that it is describing the length $4AC$ with that motion uniformly continued, as the density of the fluid has to the density of the cylinder, nearly. And (by Cor. vn, Prop, xxxvi) the resistance must be to this force in the ratio of 2 to 3, at the least.

L E M M A V

If a cylinder, a sphere, and a spheroid, of equal breadths be placed successively in the middle of a cylindric canal, so that their axes may coincide with the axis of the canal, these bodies will equally hinder the passage of the water through the canal.

For the spaces lying between the sides of the canal, and the cylinder, sphere, and spheroid, through which the water passes, are equal; and the water will pass equally through equal spaces.

This is true, upon the supposition that all the water above the cylinder, sphere, or spheroid, whose fluidity is not necessary to make the passage of the water the quickest possible, is congealed, as was explained above in Cor. vn, Prop, xxxvi.

L E M M A V I

The same supposition remaining, the fore-mentioned bodies are equally acted on by the water flowing through the canal.

This appears by Lem. v and the third Law. For the water and the bodies act upon each other mutually and equally.

LEMMA VII

If the water be at rest in the canal, and these bodies move with equal velocity and in opposite directions through the canal, their resistances will be equal among themselves.

This appears from the last Lemma, for the relative motions remain the same among themselves.

SCHOLIUM

The case is the same for all convex and round bodies, whose axes coincide with the axis of the canal. Some difference may arise from a greater or less friction; but in these Lemmas we suppose the bodies to be perfectly smooth, and the medium to be void of all tenacity and friction; and that those parts of the fluid which by their oblique and superfluous motions may disturb, hinder, and retard the flux of the water through the canal, are at rest among themselves; being fixed like water by frost, and adhering to the fore and hinder parts of the bodies in the manner explained in the Scholium of the last Proposition; for in what follows we consider the very least resistance that round bodies described with the greatest given transverse sections can possibly meet with.

Bodies swimming upon fluids, when they move straight forwards, cause the fluid to ascend at their fore parts and subside at their hinder parts, especially if they are of an obtuse figure; and hence they meet with a little more resistance than if they were acute at the head and tail. And bodies moving in elastic fluids, if they are obtuse behind and before, condense the fluid a little more at their fore parts, and relax the same at their hinder parts; and therefore meet also with a little more resistance than if they were acute at the head and tail. But in these Lemmas and Propositions we are not treating of elastic but nonelastic fluids; not of bodies floating on the surface of the fluid, but deeply immersed therein. And when the resistance of bodies in nonelastic fluids is once known, we may then augment this resistance a little in elastic fluids, as our air; and in the surfaces of stagnating fluids, as lakes and seas.

PROPOSITION XXXVIII. THEOREM XXX

If a globe move uniformly forwards in a compressed, infinite, and non-elastic fluid, its resistance is to the force by which its whole motion may be destroyed or generated, in the time that it describes eight third parts of its diameter, as the density of the fluid is to the density of the globe, very nearly.

For the globe is to its circumscribed cylinder as 2 to 3; and therefore the force which can destroy all the motion of the cylinder, while the same cylinder is describing the length of four of its diameters, will destroy all the motion of the globe, while the globe is describing two-thirds of this length, that is, eight third parts of its own diameter. Now the resistance of the cylinder is to this force very nearly as the density of the fluid is to the density of the cylinder or globe (by Prop. xxxvn), and the resistance of the globe is equal to the resistance of the cylinder (by Lem. v, vi, vii). Q.E.D.

COR. 1. The resistances of globes in infinite compressed mediums are in a ratio compounded of the squared ratio of the velocity, and the squared ratio of the diameter, and the ratio of the density of the mediums.

COR. 11. The greatest velocity, with which a globe can descend by its comparative weight through a resisting fluid, is the same as that which it may acquire by falling with the same weight, and without any resistance, and in its fall describing a space that is to four third parts of its diameter as the density of the globe is to the density of the fluid. For the globe in the time of its fall, moving with the velocity acquired in falling, will describe a space that will be to eight third parts of its diameter as the density of the globe is to the density of the fluid; and the force of its weight which generates this motion will be to the force that can generate the same motion, in the time that the globe describes eight third parts of its diameter, with the same velocity as the density of the fluid is to the density of the globe; and therefore (by this Proposition) the force of weight will be equal to the force of resistance, and therefore cannot accelerate the globe.

COR. in. If there be given both the density of the globe and its velocity at the beginning of the motion, and the density of the compressed quiescent fluid in which the globe moves, there is given at any time both the velocity of the globe and its resistance, and the space described by it (by Cor. vii, Prop. xxxv).

COR. IV. A globe moving in a compressed quiescent fluid of the same density with itself will lose half its motion before it can describe the length of two of its diameters (by the same Cor. vn).

PROPOSITION XXXIX. THEOREM XXXI

If a globe move uniformly forwards through a fluid inclosed and compressed in a cylindric canal, its resistance is to the force by which its whole motion may be generated or destroyed, in the time in which it describes eight third parts of its diameter, in a ratio compounded of the ratio of the orifice of the canal to the excess of that orifice above half the greatest circle of the globe; and the squared ratio of the orifice of the canal to the excess of that orifice above the greatest circle of the globe; and the ratio of the density of the fluid to the density of the globe, nearly.

This appears by Cor. II, Prop, xxxvii, and the demonstration proceeds in the same manner as in the foregoing Proposition.

SCHOLIUM

In the last two Propositions we suppose (as was done before in Lem. v) that all the water which precedes the globe, and whose fluidity increases the resistance of the same, is congealed. Now if that water becomes fluid, it will somewhat increase the resistance. But in these Propositions that increase is so small, that it may be neglected, because the convex surface of the globe produces the very same effect almost as the congelation of the water.

PROPOSITION XL. PROBLEM IX

To find by experiment the resistance of a globe moving through a perfectly fluid compressed medium.

Let A be the weight of the globe in a vacuum, B its weight in the resisting medium, D the diameter of the globe, F a space which is to ^4tD as the density of the globe is to the density of the medium, that is, as A is to A - B, G the time in which the globe falling with the weight B without resistance describes the space F, and H the velocity which the body acquires by that fall. Then H will be the greatest velocity with which the globe can possibly descend with the weight B in the resisting medium, by Cor. II, Prop, xxxviii; and the resistance which the globe meets with, when descending with that

velocity, will be equal to its weight B; and the resistance it meets with in any other velocity will be to the weight B as the square of the ratio of that velocity to the greatest velocity H, by Cor. i, Prop. XXXVIII.

This is the resistance that arises from the inactivity of the matter of the fluid. That resistance which arises from the elasticity, tenacity, and friction of its parts, may be thus investigated.

Let the globe be let fall so that it may descend in the fluid by the weight B; and let P be the time of falling, and let that time be expressed in seconds, if the time G be given in seconds. Find the absolute number N agreeing to the logarithm 0.4342944819 $\frac{2P}{G}$, and let L be the logarithm of the number $\frac{N+1}{N}$; and the velocity acquired in falling will be $\frac{N-1}{N+1}H$, and the height described will be $\frac{2PF}{G} - 1.3862943611F + 4.605170186LF$. If the fluid be of a sufficient depth, we may neglect the term 4.605170186LF; and $\frac{2PF}{G} - 1.3862943611F$ will be the altitude described, nearly. These things appear by Prop. ix, Book 11, and its Corollaries, and are true upon this supposition, that the globe meets with no other resistance but that which arises from the inactivity of matter. Now if it really meet with any resistance of another kind, the descent will be slower, and from the amount of that retardation will be known the amount of this new resistance.

That the velocity and descent of a body falling in a fluid might more easily be known, I have composed the following table (p. 355), the first column of which denotes the times of descent; the second shows the velocities acquired in falling, the greatest velocity being 100000000; the third exhibits the spaces described by falling in those times, 2F being the space which the body describes in the time G with the greatest velocity; and the fourth gives the spaces described with the greatest velocity in the same times. The numbers in the fourth column are $\frac{2P}{G}$, and by subtracting the number 1.3862944-4.6051702L, are found the numbers in the third column; and these numbers must be multiplied by the space F to obtain the spaces described in falling. A fifth column is added to all these, containing the spaces described in the same times by a body falling in a vacuum with the force of B its comparative weight.

The Times F	Velocities of the body falling in the fluid	The spaces described in falling in the fluid	The spaces described with the greatest motion	The spaces described by falling in a vacuum
0 001G	99999 ²⁰ / ₃₀	0 000001F	0 002F	0 000001F
0 01G	999967	0 0001F	0 02F	0 0001F
0 1G	9966799	0 0099834F	0 2F	0 01F
0 2G	19737532	0 0397361F	0 4F	0 04F
0 3G	29131261	0 0886815F	0 6F	0 09F
0 4G	37994896	0 1559070F	0 8F	0 16F
0 5G	46211716	0 2402290F	1 0F	0 25F
0 6G	53704957	0 3402706F	1 2F	0 36F
0 7G	60436778	0 4545405F	1 4F	0 49F
0 8G	66403677	0 5815071F	1 6F	0 64F
0 9G	71629787	0 7196609F	1 8F	0 81F
1G	76159416	0 8675617F	2F	1F
2G	96402758	2 6500055F	4F	4F
3G	99505475	4 6186570F	6F	9F
4G	99932930	6 6143765F	8F	16F
5G	99990920	8 6137964F	10F	25F
6G	99998771	10 6137179F	12F	36F
7G	99999834	12 6137073F	14F	49F
8G	99999980	14 6137059F	16F	64F
9G	99999997	16 6137057F	18F	81F
10G	99999999 ² / ₅	18 6137056F	20F	100F

S C H O L I U M

In order to investigate the resistances of fluids from experiments, I procured a square wooden vessel, whose length and breadth on the inside was 9 inches *English* measure, and its depth $9\frac{1}{2}$ feet; this I filled with rain water; and having provided globes made up of wax, and lead included therein, I noted the times of the descents of these globes, the height through which they descended being 112 inches. A solid cubic foot of *English* measure contains 76 pounds troy weight of rain water; and a solid inch contains $\frac{19}{36}$ ounces troy weight, or $253\frac{1}{3}$ grains; and a globe of water of one inch in diameter contains 132.645 grains in air, or 132.8 grains in a vacuum; and any other globe will be as the excess of its weight in a vacuum above its weight in water.

EXPER. 1. A globe whose weight was $156\frac{1}{4}$ grains in air, and 77 grains in water, described the whole height of 112 inches in 4 seconds. And, upon repeating the experiment, the globe spent again the very same time of 4 seconds in falling.

The weight of this globe in a vacuum is $156\frac{13}{38}$ grains; and the excess of this weight above the weight of the globe in water is $79\frac{13}{38}$ grains. Hence the diameter of the globe appears to be 0.84224 parts of an inch. Then it will be, as that excess to the weight of the globe in a vacuum, so is the density of the water to the density of the globe; and so is $\frac{8}{3}$ parts of the diameter of the globe (viz., 2.24597 inches) to the space 2F, which will be therefore 4.4256 inches. Now a globe falling in a vacuum with its whole weight of $156\frac{13}{38}$ grains in one second of time will describe $193\frac{1}{3}$ inches; and falling in water in the same time with the weight of 77 grains without resistance, will describe 95.219 inches; and in the time G, which is to one second of time as the square root of the ratio of the space F, or of 2.2128 inches to 95.219 inches, will describe 2.2128 inches, and will acquire the greatest velocity H with which it is capable of descending in water. Therefore the time G is 0.15244 sec. And in this time G, with that greatest velocity H, the globe will describe the space 2F, which is 4.4256 inches; and therefore in 4 seconds will describe a space of 116.1245 inches. Subtract the space 1.3862944F, or 3.0676 inches, and there will remain a space of 113.0569 inches, which the globe falling through water in a very wide vessel will describe in 4 seconds. But this space, by reason of the narrowness of the wooden vessel before mentioned, ought to be diminished in a ratio compounded of the square root of the ratio of the orifice of the vessel to the excess of this orifice above half a great circle of the globe, and of the simple ratio of the same orifice to its excess above a great circle of the globe, that is, in a ratio of 1 to 0.9914. This done, we have a space of 112.08 inches, which a globe falling through the water in this wooden vessel in 4 seconds of time ought nearly to describe by this theory; but it described 112 inches by the experiment.

EXPER. 2. Three equal globes, whose weights were severally $y6Vs$ grains in air, and $5\frac{1}{16}$ grains in water, were let fall successively; and every one fell through the water in 15 seconds of time, describing in its fall a height of 112 inches.

By computation, the weight of each globe in a vacuum is $76\frac{5}{12}$ 2 grains; the excess of this weight above the weight in water is $71\frac{17}{48}$ grains; the diameter of the globe 0.81296 of an inch; % parts of this diameter 2.16789 inches; the space 2F is 2.3217 inches; the space which a globe of 5Mr. grains in weight would describe in one second without resistance, 12.808 inches, and the time G 0.301056 sec. Therefore the globe, with the greatest velocity it is capable of receiving from a weight of $5\frac{1}{16}$ grains in its descent through water, will describe in the time 0.301056 seconds the space 2.3217 inches; and in 15 seconds the space 115.678 inches. Subtract the space 1.3862944F, or 1.609 inches, and there remains the space 114.069 inches; which therefore the falling globe ought to describe in the same time, if the vessel were very wide. But because our vessel was narrow, the space ought to be diminished by about 0.895 of an inch. And so the space will remain 113.174 inches, which a globe falling in this vessel ought nearly to describe in 15 seconds. But by the experiment it described 112 inches. The difference is not sensible.

EXPER. 3. Three equal globes, whose weights were severally 121 grains in air, and 1 grain in water, were successively let fall; and they fell through the water in the times 46 seconds, 47 seconds, and 50 seconds, describing a height of 112 inches.

By the theory, these globes ought to have fallen in about 40 sec. Now whether their falling more slowly were occasioned from the consideration that in slow motions the resistance arising from the force of inactivity does really bear a less proportion to the resistance arising from other causes; or whether it is to be attributed to little bubbles that might chance to stick to the globes, or to the rarefaction of the wax by the warmth of the weather, or of the hand that let them fall; or, lastly, whether it proceeded from some insensible errors in weighing the globes in the water, I am not certain. Therefore the weight of the globe in water should be of several grains, that the experiment may be certain, and to be depended on.

EXPER. 4.1 began the foregoing Experiments to investigate the resistances of fluids, before I was acquainted with the theory laid down in the Propositions immediately preceding. Afterwards, in order to examine the theory after it was discovered, I procured a wooden vessel, whose breadth on the inside was 8% inches, and its depth $15\frac{1}{3}$ feet. Then I made four globes of wax, with lead included, each of which weighed 139V1 grains in air, and

$7\frac{1}{8}$ grains in water. These I let fall, measuring the times of their falling in the water with a pendulum oscillating to half-seconds. The globes were cold, and had remained so some time, both when they were weighed and when they were let fall; because warmth rarefies the wax, and by rarefying it diminishes the weight of the globe in the water; and wax, when rarefied, is not instantly reduced by cold to its former density. Before they were let fall, they were totally immersed under water, lest, by the weight of any part of them that might chance to be above the water, their descent should be accelerated in its beginning. Then, when after their immersion they were perfectly at rest, they were let go with the greatest care, that they might not receive any impulse from the hand that let them down. And they fell successively in the times of $47\frac{1}{2}$, $48\frac{1}{2}$, 50, and 51 oscillations, describing a height of 15 feet and 2 inches. But the weather was now a little colder than when the globes were weighed, and therefore I repeated the experiment another day; and then the globes fell in the times of 49, $49\frac{1}{2}$, 50, and 53; and at a third trial in the times of $49\frac{1}{2}$, 50, 51, and 53 oscillations. And by making the experiment several times over, I found that the globes fell mostly in the times of $49\frac{1}{2}$ and 50 oscillations. When they fell slower, I suspect them to have been retarded by striking against the sides of the vessel.

Now, computing from the theory, the weight of the globe in a vacuum is $139\frac{2}{5}$ grains; the excess of this weight above the weight of the globe in water $132\frac{11}{40}$ grains; the diameter of the globe 0.99868 of an inch; $\frac{8}{3}$ parts of the diameter 2.66315 inches; the space 2F 2.8066 inches; the space which a globe weighing $7\frac{1}{8}$ grains falling without resistance describes in a second of time 9.88164 inches; and the time G 0.376843 sec. Therefore the globe with the greatest velocity with which it is capable of descending through the water by the force of a weight of $7\frac{1}{8}$ grains, will in the time 0.376843 sec. describe a space of 2.8066 inches, and in one second of time a space of 7.44766 inches, and in the time 25 sec, or in 50 oscillations, the space 186.1915 inches. Subtract the space 1.386294F, or 1.9454 inches, and there will remain the space 184.2461 inches which the globe will describe in that time in a very wide vessel. Because our vessel was narrow, let this space be diminished in a ratio compounded of the square root of the ratio of the orifice of the vessel to the excess of this orifice above half a great circle of the globe, and of the simple ratio of the same orifice to its excess above a

great circle of the globe; and we shall have the space of 181.86 inches, which the globe ought by the theory to describe in this vessel in the time of 50 oscillations, nearly. But it described the space of 182 inches, by experiment, in $49\frac{1}{2}$ or 50 oscillations.

EXPER. 5. Four globes weighing $154\frac{3}{8}$ grains in air, and $21\frac{1}{2}$ grains in water, being let fall several times, fell in the times of $28\frac{1}{2}$, 29, $29\frac{1}{2}$, and 30, and sometimes of 31, 32, and 33 oscillations, describing a height of 15 feet and 2 inches.

They ought by the theory to have fallen in the time of 29 oscillations, nearly.

EXPER. 6. Five globes, weighing $212\frac{3}{8}$ grains in air, and $79\frac{1}{2}$ in water, being several times let fall, fell in the times of 15, $15\frac{1}{2}$, 16, 17, and 18 oscillations, describing a height of 15 feet and 2 inches.

By the theory they ought to have fallen in the time of 15 oscillations, nearly.

EXPER. 7. Four globes, weighing $293\frac{3}{8}$ grains in air, and $35\frac{7}{8}$ grains in water, being let fall several times, fell in the times of $29\frac{1}{2}$, 30, $30\frac{1}{2}$, 31, 32, and 33 oscillations, describing a height of 15 feet and $1\frac{1}{2}$ inches.

By the theory they ought to have fallen in the time of 28 oscillations, nearly.

In searching for the cause that occasioned these globes of the same weight and magnitude to fall, some swifter and some slower, I hit upon this: that the globes, when they were first let go and began to fall, oscillated about their centres; that side which chanced to be the heavier descending first, and producing an oscillating motion. Now by oscillating thus, the globe communicates a greater motion to the water than if it descended without any oscillations; and by this communication loses part of its own motion with which it should descend; and therefore as this oscillation is greater or less, it will be more or less retarded. Besides, the globe always recedes from that side of itself which is descending in the oscillation, and by so receding comes nearer to the sides of the vessel, so as even to strike against them sometimes. And the heavier the globes are, the stronger this oscillation is; and the greater they are, the more is the water agitated by it. Therefore to diminish this oscillation of the globes, I made new ones of lead and wax, sticking the lead in one side of the globe very near its surface; and I let fall

he globe in such a manner, that, as near as possible, the heavier side might be lowest at the beginning of the descent. By this means the oscillations Decame much less than before, and the times in which the globes fell were lot so unequal: as in the following Experiments.

EXPER. 8. Four globes weighing 139 grains in air, and 6V2 in water, were et fall several times, and fell mostly in the time of 51 oscillations, never in more than 52, or in fewer than 50, describing a height of 182 inches.

By the theory they ought to fall in about the time of 52 oscillations.

EXPER. 9. Four globes weighing $273\frac{1}{4}$ grains in air, and $140\frac{3}{4}$ in water, being several times let fall, fell in never fewer than 12, and never more than 13 oscillations, describing a height of 182 inches.

These globes by the theory ought to have fallen in the time of $11\frac{1}{3}$ oscillations, nearly.

EXPER. 10. Four globes, weighing 384 grains in air, and $119\frac{1}{2}$ in water, being let fall several times, fell in the times of $17\frac{3}{4}$, 18, $18\frac{1}{2}$, and 19 oscillations, describing a height of $181\frac{1}{2}$ inches. And when they fell in the time of 19 oscillations, I sometimes heard them hit against the sides of the vessel before they reached the bottom.

By the theory they ought to have fallen in the time of $15\frac{5}{8}$ oscillations, nearly.

EXPER. 11. Three equal globes, weighing 48 grains in air, and $3\frac{29}{32}$ in water, being several times let fall, fell in the times of $43\frac{1}{2}$, 44, $44\frac{1}{2}$, 45, and 46 oscillations, and mostly in 44 and 45, describing a height of $182\frac{1}{2}$ inches, nearly.

By the theory they ought to have fallen in the time of $46\frac{5}{8}$ oscillations, nearly.

EXPER. 12. Three equal globes, weighing 141 grains in air, and $4\frac{3}{8}$ in water, being let fall several times, fell in the times of 61, 62, 63, 64, and 65 oscillations. describing a space of 182 inches.

And by the theory they ought to have fallen in $64\frac{1}{2}$ oscillations, nearly.

From these Experiments it is manifest, that when the globes fell slowly, as in the second, fourth, fifth, eighth, eleventh, and twelfth Experiments, the times of falling are rightly exhibited by the theory; but when the globes fell more swiftly, as in the sixth, ninth, and tenth Experiments, the resistance was somewhat greater than the square of the velocity. For the globes

in falling oscillate a little; and this oscillation, in those globes that are light and fall slowly, soon ceases by the weakness of the motion; but in greater and heavier globes, the motion being strong, it continues longer, and is not to be checked by the ambient water till after several oscillations. Besides, the more swiftly the globes move, the less are they pressed by the fluid at their hinder parts; and if the velocity be continually increased, they will at last leave an empty space behind them, unless the compression of the fluid be increased at the same time. For the compression of the fluid ought to be increased (by Prop. xxxn and xxxin) as the square of the velocity, in order to maintain the resistance in the same squared ratio. But because this is not done, the globes that move swiftly are not so much pressed at their hinder parts as the others; and by the defect of this pressure it comes to pass that their resistance is a little greater than the square of their velocity.

So that the theory agrees with the experiments on bodies falling in water. It remains that we examine the observations of bodies falling in air.

EXPER. 13. From the top of *St. Paul's Church in London*, in *June*, 1710, there were let fall together two glass globes, one full of quicksilver, the other of air; and in their fall they described a height of 220 *English* feet. A wooden table was suspended upon iron hinges on one side, and the other side of the table was supported by a wooden pin. The two globes lying upon this table were let fall together by pulling out the pin by means of an iron wire reaching thence down to the ground; so that, the pin being removed, the table, which had then no support but the iron hinges, fell downwards, and turning round upon the hinges, gave leave to the globes to drop off from it. At the same instant, with the same pull of the iron wire that took out the pin, a pendulum oscillating to seconds was let go, and began to oscillate. The diameters and weights of the globes, and their times of falling, are exhibited in the accompanying table (p. 362).

But the times observed must be corrected; for the globes of mercury (by *Galileo's* theory), in 4 seconds of time, will describe 257 *English* feet, and 220 feet in only 3 sec. 42 thirds.¹ So that the wooden table, when the pin was taken out, did not turn upon its hinges so quickly as it ought to have done; and the slowness of that revolution hindered the descent of the globes at the beginning. For the globes lay about the middle of the table, and indeed were rather nearer to the axis upon which it turned than to the pin. And

^[1] Appendix, Note 37]

hence the times of falling were prolonged about 18 thirds;¹ and therefore ought to be corrected by subtracting that excess, especially in the larger globes, which, by reason of the largeness of their diameters, lay longer upon the revolving table than the others. This being done, the times in which the six larger globes fell will come forth 8 sec. 12 thirds, 7 sec. 42 thirds, 7 sec. 42 thirds, 7 sec. 57 thirds, 8 sec. 12 thirds, and 7 sec. 42 thirds.

The globes filled with mercury			The globes full of air		
Weights.	Diameters	falling"	Weights	Diameters	falling
<i>grains</i>		<i>seconds</i>			<i>seconds</i>
908	0 8	4	510	51	
983	0 8	4 -	642	5 2	8
866	0 8	4	599	5 1	8
747	0 75	4+	515	5 0	1¼
808	0 75	4	483	5 0	
784	0 75	4+	641	5 2	8

Therefore the fifth in order among the globes that were full of air being 5 inches in diameter, and 483 grains in weight, fell in 8 sec. 12 thirds, describing a space of 220 feet. The weight of a bulk of water equal to this globe is 16600 grains; and the weight of an equal bulk of air is $\frac{1}{10}$ grains, or $19\frac{3}{10}$ grains; and therefore the weight of the globe in a vacuum is $502\frac{3}{10}$ grains; and this weight is to the weight of a bulk of air equal to the globe as $502\frac{3}{10}$ is to $19\frac{3}{10}$; and so is 2F to $\frac{8}{3}$ of the diameter of the globe, that is, to $13\frac{1}{3}$ inches. Hence 2F becomes 28 feet 11 inches. A globe, falling in a vacuum with its whole weight of $502\frac{3}{10}$ grains, will in one second of time describe $193\frac{1}{3}$ inches as above; and with the weight 483 grains will describe 185,905 inches; and with that weight 483 grains in a vacuum will describe the space F, or 14 feet $5\frac{1}{2}$ inches, in the time of 57 thirds and 58 fourths, and acquire the greatest velocity it is capable of descending with in the air. With this velocity the globe in 8 sec. 12 thirds of time will describe 245 feet and $5\frac{1}{3}$ inches. Subtract 1.3863 F, or 20 feet and $\frac{1}{2}$ an inch, and there remain 225 feet 5 inches. This space, therefore, the falling globe ought by the theory to describe in 8 sec. 12 thirds. But by the experiment it described a space of 220 feet. The difference is inappreciable.

[¹ Appendix, Note 37.]

By like calculations applied to the other globes full of air, I composed the following table.

The weights of the globes	The diameters	The times of falling from a height of 220 feet		The spaces discrbr by the		The excesses	
<i>grams</i>	<i>inches</i>	<i>seconds</i>	<i>thirds</i>	<i>feet</i>	<i>inches</i>	<i>feet</i>	<i>inches</i>
510	5 1	8	12	226	11	6	11
642	5 2	7	42	230	9	10	9
599	5 1	7	42	227	10	7	0
515	5	7	57	224	5	4	5
483	5	8	12	225	5	5	5
641	5 2	7	42	230	7	10	7

EXPER. 14. In the year 1719, in the month of *July*, Dr. *Desaguliers* made some experiments of this kind again, by forming hogs' bladders into spherical orbs; which was *done* by means of a concave wooden sphere, which the bladders, being wetted well first, were put into. After that, being blown full of air, they were obliged to fill up the spherical cavity that contained them; and then, when dry, were taken out. These were let fall from the lantern on the top of the cupola of the same church, namely, from a height of 272 feet; and at the same moment of time there was let fall a leaden globe, whose weight was about 2 pounds troy weight. And in the meantime some persons standing in the upper part of the church where the globes were let fall observed the whole times of falling; and others standing on the ground observed the differences of the times between the fall of the leaden weight and the fall of the bladder. The times were measured by pendulums oscillating to half-seconds. And one of those that stood upon the ground had a machine vibrating four times in one second; and another had another machine accurately made with a pendulum vibrating four times in a second also. One of those also who stood at the top of the church had a like machine; and these instruments were so contrived, that their motions could be stopped or renewed at pleasure. Now the leaden globe fell in about $4\frac{1}{4}$ seconds of time; and from the addition of this time to the difference of time above spoken of, was obtained the whole time in which the bladder was falling. The times which the five bladders spent in falling, after the leaden globe had reached the ground, were, the first time, $14\frac{3}{4}$ sec, $12\frac{3}{4}$

hence the times of falling were prolonged about 18 thirds;¹ and therefore ought to be corrected by subtracting that excess, especially in the larger globes, which, by reason of the largeness of their diameters, lay longer upon the revolving table than the others. This being done, the times in which the six larger globes fell will come forth 8 sec. 12 thirds, 7 sec. 42 thirds, 7 sec. 57 thirds, 8 sec. 12 thirds, and 7 sec. 42 thirds.

The globci filled with mercury			The globes full of air		
We.ghti	Diameters	l lms in falling	Wrights	Diameters	Times in falling
	<i>Inches</i>	<i>seconds</i>			
508	0 8	4	510	51	
! 983	0 8	4 -	642	5 2	8
866	0 8	4	599	51	8
747	0 75	44-	515	5 0	
808	0 75	4	483	5 0	
784	0 75	4+	641	5 2	8

Therefore the fifth in order among the globes that were full of air being 5 inches in diameter, and 483 grains in weight, fell in 8 sec. 12 thirds, describing a space of 220 feet. The weight of a bulk of water equal to this globe is 16600 grains; and the weight of an equal bulk of air is $\frac{16600}{860}$ grains, or $19\frac{3}{10}$ grains; and therefore the weight of the globe in a vacuum is $502\frac{3}{10}$ grains; and this weight is to the weight of a bulk of air equal to the globe as $502\frac{3}{10}$ to $19\frac{3}{10}$; and so is 2F to % of the diameter of the globe, that is, to $13\frac{3}{8}$ inches. Hence 2F becomes 28 feet 11 inches. A globe, falling in a vacuum with its whole weight of $502\frac{3}{10}$ grains, will in one second of time describe $193\frac{3}{8}$ inches as above; and with the weight 483 grains will describe 185.905 inches; and with that weight 483 grains in a vacuum will describe the space F, or 14 feet $5\frac{1}{2}$ inches, in the time of 57 thirds and 58 fourths, and acquire the greatest velocity it is capable of descending with in the air. With this velocity the globe in 8 sec. 12 thirds of time will describe 245 feet and $5\frac{1}{3}$ inches. Subtract 1.3863 -F, or 20 feet and $\frac{1}{2}$ an inch, and there remain 225 feet 5 inches. This space, therefore, the falling globe ought by the theory to describe in 8 sec. 12 thirds. But by the experiment it described a space of 220 feet. The difference is inappreciable.

[¹ Appendix, Note 37.]

By like calculations applied to the other globes full of air, I composed the following table.

The weights of the globes	The diameters	The times of falling from a height of 220 feet		The spaces which they would describe by the		The cesses	
	<i>inches</i>	<i>seconds</i>	<i>thirds</i>	<i>feet</i>	<i>inches</i>	<i>feet</i>	<i>inches</i>
510	5 1	8	12	226	11	6	11
642	5 2	7	42	230	9	10	9
599	5 1	7	42	227	10	7	0
515	5	7	57	224	5	4	5
483	5	8	12	225	5	5	5
641	5 2	7	42	230	7	10	7

EXPER. 14. In the year 1719, in the month of *July*, Dr. *Desaguher*s made some experiments of this kind again, by forming hogs' bladders into spherical orbs; which was done by means of a concave wooden sphere, which the bladders, being wetted well first, were put into. After that, being blown full of air, they were obliged to fill up the spherical cavity that contained them; and then, when dry, were taken out. These were let fall from the lantern on the top of the cupola of the same church, namely, from a height of 272 feet; and at the same moment of time there was let fall a leaden globe, whose weight was about 2 pounds troy weight. And in the meantime some persons scanding in the upper part of the church where the globes were let fall observed the whole times of falling; and others standing on the ground observed the differences of the times between the fall of the leaden weight and the fall of the bladder. The times were measured by pendulums oscillating to half-seconds. And one of those that stood upon the ground had a machine vibrating four times in *one* second; and another had another machine accurately made with a pendulum vibrating four times in a second also. One of those also who stood at the top of the church had a like machine; and these instruments were so contrived, that their motions could be stopped or renewed at pleasure. Now the leaden globe fell in about $4\frac{1}{4}$ seconds of time; and from the addition of this time to the difference of time above spoken of, was obtained the whole time in which the bladder was falling. The times which the five bladders spent in falling, after the leaden globe had reached the ground, were, the first time, $14\frac{3}{4}$ sec, $12\frac{3}{4}$

sec., $14\frac{5}{8}$ sec., $17\frac{3}{4}$ sec., and $16\frac{7}{8}$ sec.; and the second time, $14\frac{1}{2}$ sec., $14\frac{1}{4}$ sec., 14 sec., 19 sec., and $16\frac{3}{4}$ sec. Add to these $4\frac{1}{4}$ sec., the time in which the leaden globe was falling, and the whole times in which the five bladders fell were, the first time, 19 sec., 17 sec., $18\frac{7}{8}$ sec., 22 sec., and $21\frac{1}{8}$ sec.; and the second time, $18\frac{3}{4}$ sec., $18\frac{1}{2}$ sec., $18\frac{1}{4}$ sec., $23\frac{1}{4}$ sec., and 21 sec. The times observed at the top of the church were, the first time, $19\frac{7}{8}$ sec., $17\frac{1}{4}$ sec., $18\frac{3}{4}$ sec., $22\frac{3}{8}$ sec., and $21\frac{5}{8}$ sec.; and the second time, 19 sec., $18\frac{5}{8}$ sec., $18\frac{3}{8}$ sec., 24 sec., and $21\frac{1}{4}$ sec. But the bladders did not always fall directly down, but sometimes fluttered a little in the air, and waved to and fro, as they were descending. And by these motions the times of their falling were prolonged, and increased by half a second sometimes, and sometimes by a whole second. The second and fourth bladders fell most directly the first time, and the first and third the second time. The fifth bladder was wrinkled, and by its wrinkles was a little retarded. I found their diameters by their circumferences measured with a very fine thread wound about them twice. In the following table I have compared the experiments with the theory; making the density of air to be to the density of rain water as 1 to 860, and computing the spaces which by the theory the globes ought to describe in falling.

The weights of the bladders	The diameters	The times of falling from a height of 272 feet	The spaces which by the theory ought to have been described in those times		The difference between the theory and the experiments	
			feet	inches	feet	inches
128 grams	5 28 inches	19 seconds	271	11	- 0	1
156	5 19	17	272	$0\frac{1}{2}$	+ 0	$0\frac{1}{2}$
$137\frac{1}{2}$	5 3	18	272	7	+ 0	7
$97\frac{1}{2}$	5 26	22	277	4	+ 5	4
$99\frac{1}{8}$	5	$21\frac{1}{8}$	282	0	+ 10	0

Our theory, therefore, exhibits rightly, within a very little, all the resistance that globes moving either in air or in water meet with; which appears to be proportional to the densities of the fluids in globes of equal velocities and magnitudes.

In the Scholium subjoined to the sixth Section, we showed, by experiments of pendulums, that the resistances of equal and equally swift globes

moving in air, water, and quicksilver, are as the densities of the fluids. We here prove the same more accurately by experiments of bodies falling in air and water. For pendulums at each oscillation excite a motion in the fluid always contrary to the motion of the pendulum in its return; and the resistance arising from this motion, as also the resistance of the thread by which the pendulum is suspended, makes the whole resistance of a pendulum greater than the resistance deduced from the experiments of falling bodies. For by the experiments of pendulums described in that Scholium, a globe of the same density as water in describing the length of its semidiameter in air would lose the $\frac{1}{3412}$ part of its motion. But by the theory delivered in this seventh Section, and confirmed by experiments of falling bodies, the same globe in describing the same length would lose only a part of its motion equal to $\frac{1}{4580}$, supposing the density of water to be to the density of air as 860 to 1. Therefore the resistances were found greater by the experiments of pendulums (for the reasons just mentioned) than by the experiments of falling globes; and that in the ratio of about 4 to 3. But yet since the resistances of pendulums oscillating in air, water, and quicksilver, are alike increased by like causes, the proportion of the resistances in these mediums will be rightly enough exhibited by the experiments of pendulums, as well as by the experiments of falling bodies. And from all this it may be concluded, that the resistances of bodies, moving in any fluids whatsoever, though of the most extreme fluidity, are, other things being equal, as the densities of the fluids.

These things being thus established, we may now determine what part of its motion any globe projected in any fluid whatsoever would nearly lose in a given time. Let D be the diameter of the globe, and V its velocity at the beginning of its motion, and T the time in which a globe with the velocity V can describe in a vacuum a space that is to the space $\frac{8}{3}D$ as the density of the globe to the density of the fluid; and the globe projected in that fluid will, in any other time t lose the part $\frac{tV}{T+t}$, the part $\frac{TV}{T+t}$ remaining; and will describe a space, which will be to that described in the same time in a vacuum with the uniform velocity V , as the logarithm of the number $\frac{T+t}{T}$ multiplied by the number 2.302585093 is to the number $\frac{t}{T}$ by Cor. vii,

Prop. xxxv. In slow motions the resistance may be a little less, because the figure of a globe is more adapted to motion than the figure of a cylinder described with the same diameter. In swift motions the resistance may be a little greater, because the elasticity and compression of the fluid do not increase as the square of the velocity. But these little niceties I take no notice of.

And though air, water, quicksilver, and the like fluids, by the division of their parts *in infinitum*, should be subtilized, and become mediums infinitely fluid, nevertheless, the resistance they would make to projected globes would be the same. For the resistance considered in the preceding Propositions arises from the inactivity of the matter; and the inactivity of matter is essential to bodies, and always proportional to the quantity of matter. By the division of the parts of the fluid the resistance arising from the tenacity and friction of the parts may be indeed diminished; but the quantity of matter will not be at all diminished by this division; and if the quantity of matter be the same, its force of inactivity will be the same; and therefore the resistance here spoken of will be the same, as being always proportional to that force. To diminish this resistance, the quantity of matter in the spaces through which the bodies move must be diminished; and therefore the celestial spaces, through which the globes of the planets and comets are continually passing towards all parts, with the utmost freedom, and without the least sensible diminution of their motion, must be utterly void of any corporeal fluid, excepting, perhaps, some extremely rare vapors and the rays of light.

Projectiles excite a motion in fluids as they pass through them, and this motion arises from the excess of the pressure of the fluid at the fore parts of the projectile above the pressure of the same at the hinder parts; and cannot be less in mediums infinitely fluid than it is in air, water, and quicksilver, in proportion to the density of matter in each. Now this excess of pressure does, in proportion to its quantity, not only excite a motion in the fluid, but also acts upon the projectile so as to retard its motion; and therefore the resistance in every fluid is as the motion excited by the projectile in the fluid; and cannot be less in the most subtile ether in proportion to the density of that ether, than it is in air, water, and quicksilver, in proportion to the densities of those fluids.

SECTION VIII

The motion propagated through fluids.

PROPOSITION XLI. THEOREM XXXII

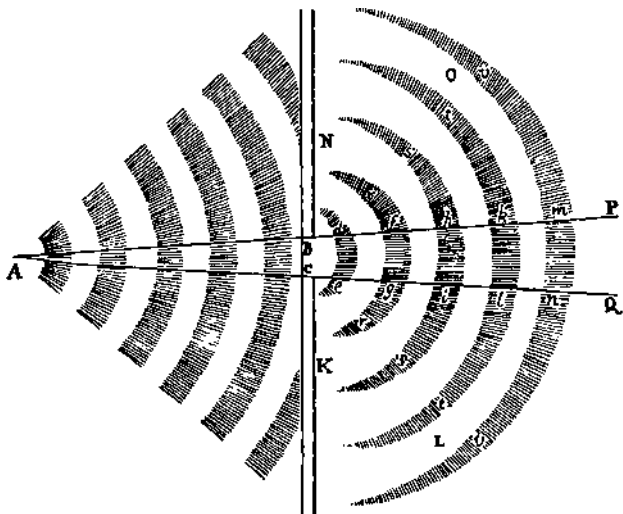
A pressure is not propagated through a fluid in rectilinear directions except where the particles of the fluid lie in a right line.

If the particles *a, b, c, d, e* lie in a right line, the pressure may be indeed directly propagated from *a* to *e*; but then the particle *e* will urge the obliquely posited particles *f* and *g* obliquely, and those particles *f* and *g* will not sustain this pressure, unless they be supported by the particles *h* and *k* lying beyond them; but the particles that support them are also pressed by them; and those particles cannot sustain that pressure, without being supported by, and pressing upon, those particles that lie still farther, as *l* and *m*, and so on *in infinitum*. Therefore the pressure, as soon as it is propagated to particles that lie out of right lines, begins to deflect towards one hand and the other, and will be propagated obliquely *in infinitum*; and after it has begun to be propagated obliquely, if it reaches more distant particles lying out of the right line, it will deflect again on each hand; and this it will do as often as it lights on particles that do not lie exactly in a right line. Q.E.D.



COR. If any part of a pressure, propagated through a fluid from a given point, be intercepted by any obstacle, the remaining part, which is not intercepted, will deflect into the spaces behind the obstacle. This may be demonstrated also after the following manner. Let a pressure be propagated from the point *A* towards any part, and, if it be possible, in rectilinear directions; and the obstacle *NBCK* being perforated in *BC*, let all the pressure be intercepted but the coniform part *APQ* passing through the circular hole *BC*. Let the cone *APQ* be divided into frustums by the transverse planes, *de, fg, hi*. Then while the cone *ABC*, propagating the pressure, urges the conic frustum *degf* beyond it on the surface *de*, and this frustum urges the next frustum *jgih* on the surface *fg*, and that frustum urges a third frustum,

and so *in infinitum*; it is manifest (by the third Law) that the first frustum *defg* is, by the reaction of the second frustum *fghi*, as much urged and pressed on the surface *fg*, as it urges and presses that second frustum. Therefore the frustum *defg* is compressed on both sides, that is, between the cone *Ade* and the frustum *fhi*g; and therefore (by Case 6, Prop, xix) cannot preserve its figure, unless it be compressed with the same force on all sides. Therefore with the same force with which it is pressed on the surfaces *de*,

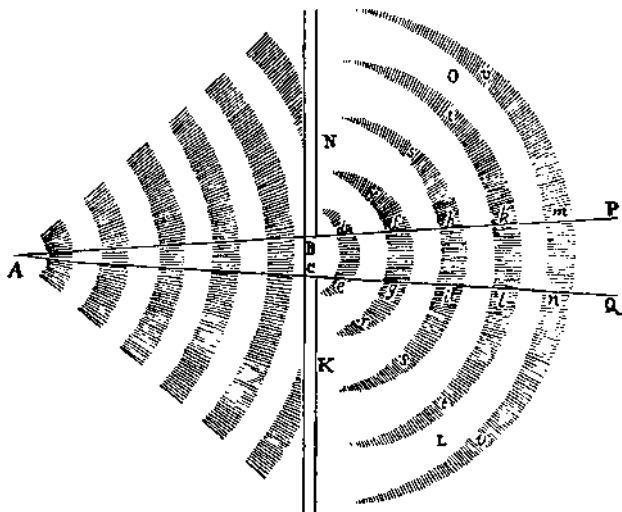


fg, it will endeavor to break forth at the sides *df*, *eg*; and there (being not in the least tenacious or hard, but perfectly fluid) it will run out, expanding itself, unless there be an ambient fluid opposing that endeavor. Therefore, by the effort it makes to run out, it will press the ambient fluid, at its sides *df*, *eg*, with the same force that it does the frustum *fghi*; and therefore, the pressure will be propagated as much from the sides *df*, *eg*, into the spaces *NO*, *KL* this way and that way, as it is propagated from the surface *fg* towards *PQ*. Q.E.D.

PROPOSITION XLII. THEOREM XXXIII

All motion propagated through a fluid diverges from a rectilinear progress into the unmoved spaces.

CASE 1. Let a motion be propagated from the point A through the hole BC, and, if it be possible, let it proceed in the conic space BCQP according to right lines diverging from the point A. And let us first suppose this motion to be that of waves in the surface of standing water; and let *de*, *fg*, *hi*, *kl*, &c, be the tops of the several waves, divided from each other by as



many intermediate valleys or hollows. Then, because the water in the ridges of the waves is higher than in the unmoved parts of the fluid *KL*, *NO*, it **will** run down from off the tops of those ridges, *e*, *g*, *i*, *l*, &c, *d*, *f*, *h*, *k*, &c., this way and that way towards *KL* and *NO*; and because the water is more depressed in the hollows of the waves than in the unmoved parts of the fluid *KL*, *NO*, it will run down into those hollows out of those unmoved parts. By the first deflux the ridges of the waves will dilate themselves this

way and that way, and be propagated towards KL and NO. And because the motion of the waves from A towards PQ is carried on by a continual deflux from the ridges of the waves into the hollows next to them, and therefore cannot be swifter than in proportion to the celerity of the descent; and the descent of the water on each side towards KL and NO must be performed with the same velocity: it follows that the dilatation of the waves on each side towards KL and NO will be propagated with the same velocity as the waves themselves go forwards with directly from A to PQ. And therefore the whole space this way and that way towards KL and NO will be filled by the dilated waves *rfgr*, *shis*, *tklt*, *vmnv*, &c. Q.E.D. That these things are so, anyone may find by making the experiment in still water.

CASE 2. Let us suppose that *de*, *fg*, *hi*, *kl*, *mn* represent pulses successively propagated from the point A through an elastic medium. Conceive the pulses to be propagated by successive condensations and rarefactions of the medium, so that the densest part of every pulse may occupy a spherical surface described about the centre A, and that equal intervals intervene between the successive pulses. Let the lines *de*, *fg*, *hi*, *kl*, &c, represent the densest parts of the pulses, propagated through the hole BC; and because the medium is denser there than in the spaces on either side towards KL and NO, it will dilate itself as well towards those spaces KL, NO, on each hand, as towards the rare intervals between the pulses; and hence the medium, becoming always more rare next the intervals, and more dense next the pulses, will partake of their motion. And because the progressive motion of the pulses arises from the continual relaxation of the denser parts towards the antecedent rare intervals; and since the pulses will relax themselves on each hand towards the quiescent parts of the medium KL, NO with very near the same celerity; therefore the pulses will dilate themselves on all sides into the unmoved parts KL, NO with almost the same celerity with which they are propagated directly from the centre A; and therefore will fill up the whole space KLON. Q.E.D. And we find the same by experience also in sounds which are heard through a mountain interposed; and, if they come into a chamber through the window, dilate themselves into all the parts of the room, and are heard in every corner; and not as reflected from the opposite walls, but directly propagated from the window, as far as our sense can judge.

CASE 3. Let us suppose, lastly, that a motion of any kind is propagated from A through the hole BC. Then since the cause of this propagation is that the parts of the medium that are near the centre A disturb and agitate those which lie farther from it; and since the parts which are urged are fluid, and therefore recede every way towards those spaces where they are less pressed: they will by consequence recede towards all the parts of the quiescent medium, as well to the parts on each hand, as KL and NO, as to those right before, as PQ; and by this means all the motion, as soon as it has passed through the hole BC, will begin to dilate itself, and from thence, as from its principle and centre, will be propagated directly every way. Q.E.D.

PROPOSITION XLIII. THEOREM XXXIV

Every tremulous body in an elastic medium propagates the motion of the pulses on every side straight forwards, but in a nonelastic medium excites a circular motion.

CASE I. The parts of the tremulous body, alternately going and returning, do in going urge and drive before them those parts of the medium that be nearest, and by that impulse compress and condense them; and in returning suffer those compressed parts to recede again, and expand themselves. Therefore the parts of the medium that lie nearest to the tremulous body move to and fro by turns, in like manner as the parts of the tremulous body itself do; and for the same cause that the parts of this body agitate these parts of the medium, these parts, being agitated by like tremors, will in their turn agitate others next to themselves; and these others, agitated in like manner, will agitate those that lie beyond them, and so on in *infinitum*. And in the same manner as the first parts of the medium were condensed in going, and relaxed in returning, so will the other parts be condensed every time they go, and expand themselves every time they return. And therefore they will not be all going and all returning at the same instant (for in that case they would always maintain determined distances from each other, and there could be no alternate condensation and rarefaction); but since, in the places where they are condensed, they approach to, and, in the places where they are rarefied, recede from each other, therefore some of them will be going while others are returning; and so on in *infinitum*. The parts so going, and in their going condensed, are pulses, by reason of

the progressive motion with which they strike obstacles in their **way**; and therefore the successive pulses produced by a tremulous body will be propagated in rectilinear directions; and that at nearly equal distances from each other, because of the equal intervals of time in which the body, by its several tremors, produces the several pulses. And though the parts of the tremulous body go and return in some certain and determinate direction, yet the pulses propagated from thence through the medium will dilate themselves towards the sides, by the foregoing Proposition; and will be propagated on all sides from that tremulous body, as from a common centre, in surfaces nearly spherical and concentric, as in waves excited by shaking a finger in water, which proceed not only forwards and backwards agreeably to the motion of the finger, but spread themselves in the manner of concentric circles all round the finger, and are propagated on every side. For the gravity of the water supplies the place of elastic force.

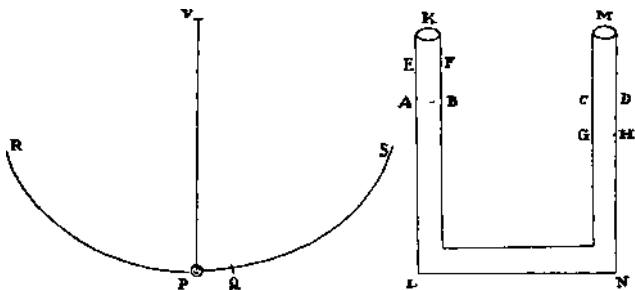
CASE 2. If the medium be not elastic, then, because its parts cannot be condensed by the pressure arising from the vibrating parts of the tremulous body, the motion will be propagated in an instant towards the parts where the medium yields most easily, that is, to the parts which the tremulous body would otherwise leave vacuous behind it. The case is the same with that of a body projected in any medium whatever. A medium yielding to projectiles does not recede *in infinitum*, but with a circular motion comes round to the spaces which the body leaves behind it. Therefore as often as a tremulous body tends to any part, the medium yielding to it comes round in a circle to the parts which the body leaves; and as often as the body returns to the first place, the medium will be driven from the place it came round to, and return to its original place. And though the tremulous body be not firm and hard, but every way flexible, yet if it continue of a given magnitude, since it cannot impel the medium by its tremors anywhere without yielding to it somewhere else, the medium receding from the parts of the body where it is pressed will always come round in a circle to the parts that yield to it. Q.E.D.

COR. Hence it is a mistake to think that the agitation of the parts of flame conduces to the propagation of a pressure in rectilinear directions through an ambient medium. Such a pressure must be derived not from the agitation only of the parts of flame, but from the dilatation of the whole.

PROPOSITION XLIV. THEOREM XXXV

If water ascend and descend alternately in the erected legs KL, MN of a canal or pipe; and a pendulum be constructed whose length between the point of suspension and the centre of oscillation is equal to half the length of the water in the canal: I say, that the water will ascend and descend in the same times in which the pendulum oscillates.

I measure the length of the water along the axes of the canal and its legs, and make it equal to the sum of those axes; and take no notice of the resistance of the water arising from its attrition by the sides of the canal. Let, therefore, AB, CD represent the mean height of the water in both legs; and when the water in the leg KL ascends to the height EF, the water will descend in the leg MN to the height GH. Let P be a pendulous body, VP



the thread, V the point of suspension, RPQS the cycloid which the pendulum describes, P its lowest point, PQ an arc equal to the height AE. The force with which the motion of the water is accelerated and retarded alternately is the excess of the weight of the water in one leg above the weight in the other; and, therefore, when the water in the leg KL ascends to EF, and in the other leg descends to GH, that force is double the weight of the water EABF, and therefore is to the weight of the whole water as AE or PQ to VP or PR. The force also with which the body P is accelerated or retarded in any place, as Q, of a cycloid, is (by Cor., Prop, LI, Book i) to its whole weight as its distance PQ from the lowest place P to the length PR of the cycloid. Therefore the motive forces of the water and pendulum, descnb-

ing the equal spaces AE, PQ, are as the weights to be moved; and therefore if the water and pendulum are quiescent at first, those forces will move them in equal times, and will cause them to go and return together with a reciprocal motion. Q.E.D.

COR. I. Therefore the reciprocations of the water in ascending and descending are all performed in equal times, whether the motion be more or less intense or remiss.

COR. II. If the length of the whole water in the canal be of 6 $\frac{1}{2}$ feet of *French* measure, the water will descend in one second of time, and will ascend in another second, and so on by turns *in infinitum*; for a pendulum of $3\frac{1}{18}$ such feet in length will oscillate in one second of time.

COR. III. But if the length of the water be increased or diminished, the time of the reciprocation will be increased or diminished as the square root of the length.

PROPOSITION XLV. THEOREM XXXVI

The velocity of waves varies as the square root of the breadths.

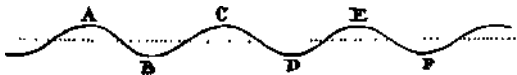
This follows from the construction of the following Proposition.

PROPOSITION XLVI. PROBLEM X

To find the velocity of waves.

Let a pendulum be constructed, whose length between the point of suspension and the centre of oscillation is equal to the breadth of the waves, and in the time that the pendulum will perform one single oscillation the waves will advance forwards nearly a space equal to their breadth.

That which I call the breadth of waves is the transverse measure lying between the deepest part of the hollows, or the tops of the ridges. Let ABCDEF represent the surface of stagnant water ascending and descend-



ing in successive waves; and let A, C, E, &c, be the tops of the waves; and let B, D, F, &c, be the intermediate hollows. Because the motion of the waves is carried on by the successive ascent and descent of the water, so that

the parts thereof, as A, C, E, &c, which are highest at one time, become lowest immediately after; and because the motive force, by which the highest parts descend and the lowest ascend, is the weight of the elevated water, that alternate ascent and descent will be analogous to the reciprocal motion of the water in the canal, and will observe the same laws as to the times of ascent and descent; and therefore (by Prop, XLIV) if the distances between the highest places of the waves A, C, E and the lowest B, D, F be equal to twice the length of any pendulum, the highest parts A, C, E will become the lowest in the time of one oscillation, and in the time of another oscillation will ascend again. Therefore with the passage of each wave, the time of two oscillations will occur; that is, the wave will describe its breadth in the time that pendulum will oscillate twice; but a pendulum of four times that length, and therefore equal to the breadth of the waves, will just oscillate once in that time. Q.E.I.

COR. i. Therefore, waves, whose breadth is equal to *VIH French* feet, will advance through a space equal to their breadth in one second of time; and therefore in one minute will go over a space of 183[^] feet; and in an hour a space of 11000 feet, nearly.

COR. II. And the velocity of greater or less waves will be augmented or diminished as the square root of their breadth.

These things are true upon the supposition that the parts of water ascend or descend in a straight line; but, in truth, that ascent and descent is rather performed in a circle; and therefore I give the time defined by this Proposition as only approximate.

PROPOSITION XLVII. THEOREM XXXVII

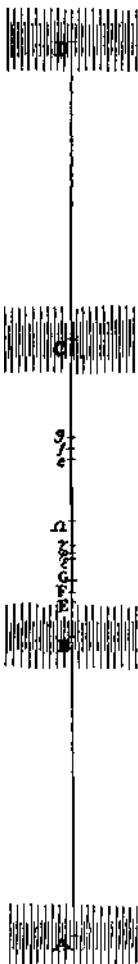
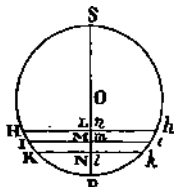
If pulses are propagated through a fluid, the several particles of the fluid, going and returning with the shortest reciprocal motion, are always accelerated or retarded according to the law of the oscillating pendulum.

Let AB, BC, CD, &c, represent equal distances of successive pulses; ABC the line of direction of the motion of the successive pulses propagated from A to B; E, F, G three physical points of the quiescent medium situate in the right line AC at equal distances from each other; Ee, Ff, Gg equal spaces of extreme shortness, through which those points go and return with

a reciprocal motion in each vibration; c, Y any intermediate places of the same points; EF, FG physical short lines, or linear parts of the medium lying between those points, and successively transferred into the places eS ,

and ef, fg . Let there be drawn the right line PS equal to the right line Ee . Bisect the same in O , and from the centre O , with the radius OP , describe the circle $SIPi$. Let the whole time of one vibration, with its proportional parts, be represented by the whole circumference of this circle and its parts, in such sort, that, when any time PH or $PHSh$ is completed, if there be let fall to PS the perpendicular HL or hi , and there be taken Ee equal to PL or PI , the physical point E may be found in e . A point, as E , moving according to this law with a reciprocal motion, in its going from E through e to e , and returning again through E to E , will perform its several vibrations with the same degrees of acceleration and retardation with those of an oscillating pendulum. We are now to prove that the several physical points of the medium will be agitated with such a kind of motion. Let us suppose, then, that a medium hath such a motion excited in it from any cause whatsoever, and consider what will follow from thence.

In the circumference $PHSh$ let there be taken the equal arcs, HI, IK , or hi, ik , having the same ratio to the whole circumference as the equal right lines EF, FG have to BC , the whole interval of the pulses. Let fall the perpendiculars IM, KN , or tm, kn ; then because the points E, F, G are successively agitated with like motions, and perform their entire vibrations composed of their going and return, while the pulse is transferred from B to C ; if PH or $PHSh$ be the time elapsed since



the beginning of the motion of the point E, then will PI or PHSi be the time elapsed since the beginning of the motion of the point F, and PK or PHSk the time elapsed since the beginning of the motion of the point G; and therefore Ec, F, GY will be respectively equal to PL, PM, PN, while the points are going, and to PI, Pm, Pn, when the points are returning. Therefore fY or $\text{EG} + \text{GY} - \text{Ee}$ will, when the points are going, be equal to $\text{EG} - \text{LN}$, and in their return equal to $\text{EG} + \text{ln}$. But CY is the breadth or expansion of the part EG of the medium in the place ey; and therefore the expansion of that part in its going is to its mean expansion as $\text{EG} - \text{LN}$ to EG; and in its return, as $\text{EG} + \text{ln}$ or $\text{EG} + \text{LN}$ is to EG. Therefore since LN is to KH as IM to the radius OP, and KH to EG as the circumference PHShP to BC; that is, if we put V for the radius of a circle whose circumference is equal to BC the interval of the pulses, as OP is to V; and, multiplying together corresponding terms of the proportions, we obtain LN to EG as IM to V; the expansion of the part EG, or of the physical point F in the place fY , is to the mean expansion of the same part in its first place EG, as $\text{V} - \text{IM}$ is to V in going, and as $\text{V} - \text{im}$ is to V in its return. Hence the elastic force of the point F in the place CY IS to its mean elastic force in the place EG as $\frac{1}{\text{V} - \text{IM}}$ is to $\frac{1}{\text{V}}$ in its going, and as $\frac{1}{\text{V} + \text{im}}$ is to $\frac{1}{\text{V}}$ in its return. And by the same reasoning the elastic forces of the physical points E and G in going are as $\frac{1}{\text{V} - \text{HL}}$ and $\frac{1}{\text{V} - \text{KN}}$ is to $\frac{1}{\text{V}}$; and the difference of the forces is to the mean elastic force of the medium as $\frac{\text{HL} - \text{KN}}{\text{V} \cdot \text{V} - \text{V} \cdot \text{HL} - \text{V} \cdot \text{KN} + \text{HL} \cdot \text{KN}}$ is to $\frac{1}{\text{V}}$; that is, as $\frac{\text{HL} - \text{KN}}{\text{V} \cdot \text{V}}$ is to $\frac{1}{\text{V}}$, or as $\text{HL} - \text{KN}$ is to V; if we suppose (by reason of the very short extent of the vibrations) HL and KN to be indefinitely less than the quantity V. Therefore since the quantity V is given, the difference of the forces is as $\text{HL} - \text{KN}$; that is (because $\text{HL} - \text{KN}$ is proportional to HK, and OM to OI or OP; and because HK and OP are given), as OM; that is, if F/ be bisected in . And for the same reason the difference of the elastic forces of the physical points t and Y, in the return of the physical short line SY, is as . But that difference (that is, the excess of the elastic force of the point c above the elastic force of the point Y) IS the very force by which

the intervening physical short line *ey* of the medium is accelerated in going, and retarded in returning; and therefore the accelerative force of the physical short line *cy* is as its distance from *ft*, the middle place of the vibration. Therefore (by Prop, xxxvm, Book 1) the time is rightly represented by the arc *PI*; and the linear part of the medium *ey* is moved according to the law above mentioned, that is, according to the law of a pendulum oscillating; and the case is the same of all the linear parts of which the whole medium is compounded. Q.E.D.

COR. Hence it appears that the number of the pulses propagated is the same with the number of the vibrations of the tremulous body, and is not multiplied in their progress. For the physical short line *cy* as soon as it returns to its first place is at rest; neither will it move again, unless it receives a new motion either from the impulse of the tremulous body, or of the pulses propagated from that body. As soon, therefore, as the pulses cease to be propagated from the tremulous body, it will return to a state of rest, and move no more.

PROPOSITION XLVIII.¹ THEOREM XXXVIII

The velocities of pulses propagated in an elastic fluid are in a ratio compounded of the square root of the ratio of the elastic force directly, and the square root of the ratio of the density inversely; supposing the elastic force of the fluid to be proportional to its condensation.

CASE I. If the mediums be homogeneous, and the distances of the pulses in those mediums be equal amongst themselves, but the motion in one medium is more intense than in the other, the contractions and dilatations of the corresponding parts will be as those motions; not that this proportion is perfectly accurate. However, if the contractions and dilatations are not exceedingly intense, the error will not be sensible; and therefore this proportion may be considered as physically exact. Now the motive elastic forces are as the contractions and dilatations; and the velocities generated in the same time in equal parts are as the forces. Therefore equal and corresponding parts of corresponding pulses will go and return together, through spaces proportional to their contractions and dilatations, with velocities that are as those spaces; and therefore the pulses, which in the time of one going and returning advance forwards a space equal to their breadth, and

¹ [Appendix, Note 38]

are always succeeding into the places of the pulses that immediately go before them, will, by reason of the equality of the distances, go forwards in both mediums with equal velocity.

CASE 2. If the distances of the pulses or their lengths are greater in one medium than in another, let us suppose that the correspondent parts describe spaces, in going and returning, each time proportional to the breadths of the pulses; then will their contractions and dilatations be equal; and therefore if the mediums are homogeneous, the motive elastic forces, which agitate them with a reciprocal motion, will be equal also. Now the matter to be moved by these forces is as the breadth of the pulses; and the space through which they move every time they go and return is in the same ratio. And, moreover, the time of one going and returning is in a ratio compounded of the square root of the matter and the square root of the space; and therefore is as the space. But the pulses advance a space equal to their breadths in the times of going once and returning once; that is, they go over spaces proportional to the times, and therefore are equally swift.

CASE 3. And therefore in mediums of equal density and elastic force, all the pulses are equally swift. Now if the density or the elastic force of the medium were augmented, then, because the motive force is increased in the ratio of the elastic force, and the matter to be moved is increased in the ratio of the density, the time which is necessary for producing the same motion as before will be increased as the square root of the ratio of the density, and will be diminished as the square root of the ratio of the elastic force. And therefore the velocity of the pulses will be in a ratio compounded of the square root of the inverse ratio of the density of the medium, and the square root of the direct ratio of the elastic force. Q.E.D.

This Proposition will be made clearer from the construction of the following Problem.

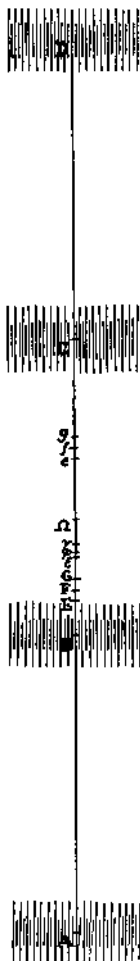
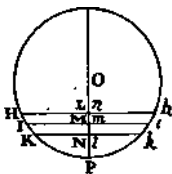
PROPOSITION XLIX. PROBLEM XI

The density and elastic force of a medium being given, to find the velocity of the pulses.

Suppose the medium to be pressed by an incumbent weight after the manner of our air; and let A be the height of an homogeneous medium, whose weight is equal to the incumbent weight, and whose density is the

same with the density of the compressed medium in which the pulses are propagated. Suppose a pendulum to be constructed whose length between the point of suspension and the centre of oscillation is A: and in the time in which that pendulum will perform one entire oscillation composed of its going and returning, the pulse will be propagated right onwards through a space equal to the circumference of a circle described with the radius A.

For, letting those things stand which were constructed in Prop, XLVII, if any physical line, as EF, describing the space PS in each vibration, be acted on in the extremities P and S of every going and return that it makes by an elastic force that is equal to its weight, it will perform its several vibrations in the time in which the same might oscillate in a cycloid whose whole perimeter is equal to the length PS; and that because equal forces will impel equal corpuscles through equal spaces in the same or equal times. Therefore since the times of the oscillations are as the square root of the lengths of the pendulums, and the length of the pendulum is equal to half the arc of the whole cycloid, the time of one vibration would be to the time of the oscillation of a pendulum whose length is A as the square root of the length Vi:PS or PO to the length A. But the elastic force with which the physical short line EG is urged, when it is found in its extreme places P, S, was (in the demonstration of Prop, XLVII) to its whole elastic force as H L - KN is to V, that is (since the point K now falls upon P), as HK to V; and all that force, or, which is the same thing, the incumbent weight by which the short line EG is compressed, is to the weight of the short line



as the altitude of the incumbent weight is to EG the length of the short line; and therefore, taking the product of corresponding terms, the force with which the short line EG is urged in the places P and S is to the weight of that short line as HK A is to V EG; or as PO A is to V V; because HK was to EG as PO to V. Therefore, since the times in which equal bodies are impelled through equal spaces are inversely as the square root of the forces, the time of one vibration, produced by the action of that elastic force, will be to the time of a vibration, produced by the impulse of the weight, as the square root of the ratio of VV to PO A, and therefore to the time of the oscillation of a pendulum whose length is A as the square root of the ratio of VV to PO A, and as the square root of the ratio of PO to A conjointly; that is, in the entire ratio of V to A. But in the time of one vibration composed of the going and returning of the pendulum, the pulse will be propagated right onwards through a space equal to its breadth BC. Therefore the time in which a pulse runs over the space BC is to the time of one oscillation composed of the going and returning of the pendulum as V is to A, that is, as BC is to the circumference of a circle whose radius is A. But the time in which the pulse will run over the space BC is to the time in which it will run over a length equal to that circumference in the same ratio; and therefore in the time of such an oscillation the pulse will run over a length equal to that circumference. Q.E.D.

COR. 1. The velocity of the pulses is equal to that which heavy bodies acquire by falling with an equally accelerated motion, and in their fall describing half the altitude A. For the pulse will, in the time of this fall, supposing it to move with the velocity acquired by that fall, run over a space that will be equal to the whole altitude A; and therefore in the time of one oscillation composed of one going and return, will go over a space equal to the circumference of a circle described with the radius A; for the time of the fall is to the time of oscillation as the radius of a circle to its circumference.

COR. 11. Therefore since that altitude A is directly as the elastic force of the fluid, and inversely as the density of the same, the velocity of the pulses will be in a ratio compounded of the square root of the ratio of the density inversely, and the square root of the ratio of the elastic force directly.

PROPOSITION L. PROBLEM XII

To find the distances of the pulses.

Let the number of the vibrations of the body, by whose tremor the pulses are produced, be found to any given time. By that number divide the space which a pulse can go over in the same time, and the part found will be the breadth of one pulse. Q.E.I.

S C H O L I U M

The last Propositions respect the motions of light and sounds; for since light is propagated in right lines, it is certain that it cannot consist in action alone (by Prop, XLI and XLII). As to sounds, since they arise from tremulous bodies, they can be nothing else but pulses of the air propagated through it (by Prop, XLIII); and this is confirmed by the tremors which sounds, if they be loud and deep, excite in the bodies near them, as we experience in the sound of drums; for quick and short tremors are less easily excited. But it is well known that any sounds, falling upon strings in unison with the sonorous bodies, excite tremors in those strings. This is also confirmed from the velocity of sounds; for since the specific gravities of rain water and quicksilver are to one another as about 1 to $13\frac{3}{4}$, and when the mercury in the barometer is at the height of 30 inches of our measure, the specific gravities of the air and of rain water are to one another as about 1 to 870, therefore the specific gravities of air and quicksilver are to each other as 1 to 11890. Therefore when the height of the quicksilver is at 30 inches, a height of uniform air, whose weight would be sufficient to compress our air to the density we find it to be of, must be equal to 356700 inches, or 29725 feet of our measure; and this is that very height of the medium, which I have called A in the construction of the foregoing Proposition. A circle whose radius is 29725 feet is 186768 feet in circumference. And since a pendulum $39\frac{1}{8}$ inches in length completes one oscillation, composed of its going and return, in two seconds of time, as is commonly known, it follows that a pendulum 29725 feet, or 356700 inches in length will perform a like oscillation in $190\frac{3}{4}$ seconds. Therefore in that time a sound will go right onwards 186768 feet, and therefore in one second 979 feet.

But in this computation we have made no allowance for the crassitude of the solid particles of the air, by which the sound is propagated instantaneously. Because the weight of air is to the weight of water as 1 to 870, and because salts are almost twice as dense as water; if the particles of air are supposed to be of about the same density as those of water or salt, and the rarity of the air arises from the intervals of the particles; the diameter of one particle of air will be to the interval between the centres of the particles as 1 to about 9 or 10, and to the interval between the particles themselves as 1 to 8 or 9. Therefore to 979 feet, which, according to the above calculation, a sound will advance forwards in one second of time, we may add $\frac{9}{10}$, or about 109 feet, to compensate for the crassitude of the particles of the air: and then a sound will go forwards about 1088 feet in one second of time.

Moreover, the vapors floating in the air being of another spring, and a different tone, will hardly, if at all, partake of the motion of the true air in which the sounds are propagated. Now if these vapors remain unmoved, that motion will be propagated the swifter through the true air alone, and that as the square root of the defect of the matter. So if the atmosphere consist often parts of true air and one part of vapors, the motion of sounds will be swifter as the square root of the ratio of 11 to 10, or very nearly in the entire ratio of 21 to 20, than if it were propagated through eleven parts of true air: and therefore the motion of sounds above discovered must be increased in that ratio. By this means the sound will pass through 1142 feet in one second of time.

These things will be found true in spring and autumn, when the air is rarefied by the gentle warmth of those seasons, and by that means its elastic force becomes somewhat more intense. But in winter, when the air is condensed by the cold, and its elastic force is somewhat remitted, the motion of sounds will be slower as the square root of the density; and, on the other hand, swifter in the summer.

Now by experiments it actually appears that sounds do really advance in one second of time about 1142 feet of *English* measure, or 1070 feet of *French* measure.

The velocity of sounds being known, the intervals of the pulses are known also. For M. *Sauveur*, by some experiments that he made, found

that an open pipe about five *Paris* feet in length gives a sound of the same tone with a viol string that vibrates a hundred times in one second. Therefore there are near 100 pulses in a space of 1070 *Pans* feet, which a sound runs over in a second of time; and therefore one pulse fills up a space of about ~~1070~~ $10\frac{7}{10}$ *Pans* feet, that is, about twice the length of the pipe. From this it is probable that the breadths of the pulses, in all sounds made in open pipes, are equal to twice the length of the pipes.

Moreover, from the Corollary of Prop, XLVII appears the reason why the sounds immediately cease with the motion of the sonorous body, and why they are heard no longer when we are at a great distance from the sonorous bodies than when we are very near them. And besides, from the foregoing principles, it plainly appears how it comes to pass that sounds are so mightily increased in speaking-trumpets; for all reciprocal motion tends to be increased by the generating cause at each return. And in tubes hindering the dilatation of the sounds, the motion decays more slowly, and recurs more forcibly; and therefore is the more increased by the new motion impressed at each return. And these are the principal phenomena of sounds.

SECTION IX

The circular motion of fluids.

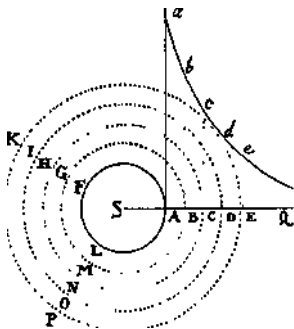
HYPOTHESIS

The resistance arising from the want of lubricity in the parts of a fluid, is, other things being equal, proportional to the velocity with which the parts of the fluid are separated from one another.

PROPOSITION LI. THEOREM XXXIX

If a solid cylinder infinitely long, in an uniform and infinite fluid, revolves with an uniform motion about an axis given in position, and the fluid be forced round by only this impulse of the cylinder, and every part of the fluid continues uniformly in its motion: I say, that the periodic times of the parts of the fluid are as their distances from the axis of the cylinder.

Let AFL be a cylinder turning uniformly about the axis S, and let the concentric circles BGM, CHN, DIO, EKP, &c, divide the fluid into innumerable concentric cyhndric solid orbs of the same thickness. Then, because the fluid is homogeneous, the impressions which the contiguous orbs make upon each other will be (by the Hypothesis) as their translations from each other, and as the contiguous surfaces upon which the impressions are made. If the impression made upon any orb be greater or less on its concave than on its convex side, the stronger impression will prevail, and will either accelerate or retard the motion of the orb, according as it agrees with, or is contrary to, the motion of the same. Therefore, that every orb may continue uniformly in its motion, the impressions made on both sides must be equal and their directions contrary. Therefore since the impressions are as the contiguous surfaces, and as their translations from one another, the translations will be inversely as the surfaces, that is, inversely as



the distances of the surfaces from the axis. But the differences of the angular motions about the axis are as those translations applied to the distances, or directly as the translations and inversely as the distances; that is, joining these ratios together, inversely as the squares of the distances. Therefore if there be erected the lines ha , Bb , Cc , Dd , Ee , &c, perpendicular to the several parts of the infinite right line $SABCDEQ$, and inversely proportional to the squares of SA , SB , SC , SD , SR , &c, and through the extremities of those perpendiculars there be supposed to pass an hyperbolic curve, the sums of the differences, that is, the whole angular motions, will be as the correspondent sums of the lines Aa , Bb , Cc , Dd , Ee , that is (if to constitute a medium uniformly fluid the number of the orbs be increased and their breadth diminished *in infinitum*), as the hyperbolic areas AaQ , BbQ , OQ , DdQ , EeQ , &c, analogous to the sums; and the times, inversely proportional to the angular motions, will be also inversely proportional to those areas. Therefore the periodic time of any particle, as D , is inversely as the area DdQ , that is (as appears from the known methods of quadratures of curves), directly as the distance SD . Q.E.D.

COR. 1. Hence the angular motions of the particles of the fluid are inversely as their distances from the axis of the cylinder, and the absolute velocities are equal.

COR. 11. If a fluid be contained in a cylindric vessel of an infinite length, and contain another cylinder within, and both the cylinders revolve about one common axis, and the times of their revolutions be as their semidiameters, and every part of the fluid continues in its motion, the periodic times of the several parts will be as the distances from the axis of the cylinders.

COR. HI. If there be added or taken away any common quantity of angular motion from the cylinder and fluid moving in this manner, yet because this new motion will not alter the mutual attrition of the parts of the fluid, the motion of the parts among themselves will not be changed; for the translations of the parts from one another depend upon the attrition. Any part will continue in that motion, which, by the attrition made on both sides with contrary directions, is no more accelerated than it is retarded.

COR. IV. Therefore if there be taken away from this whole system of the cylinders and the fluid all the angular motion of the outward cylinder, we shall have the motion of the fluid in a quiescent cylinder.

COR. V. Therefore if the fluid and outward cylinder are at rest, and the inward cylinder revolve uniformly, there will be communicated a circular motion to the fluid, which will be propagated by degrees through the whole fluid; and will go on continually increasing, till such time as the several parts of the fluid acquire the motion determined in Cor. iv.

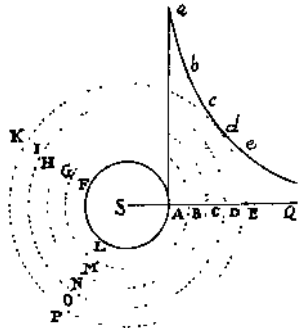
COR. vi. And because the fluid endeavors to propagate its motion still farther, its impulse will carry the outmost cylinder also about with it, unless the cylinder be forcibly held back; and accelerate its motion till the periodic times of both cylinders become equal with each other. But if the outward cylinder be forcibly held fast, it will make an effort to retard the motion of the fluid; and unless the inward cylinder preserve that motion by means of some external force impressed thereon, it will make it cease by degrees.

All these things will be found true by making the experiment in deep standing water.

PROPOSITION LII. THEOREM XL

If a solid sphere, in an uniform and infinite fluid, revolves about an axis given in position with an uniform motion, and the fluid be forced round by only this impulse of the sphere; and every part of the fluid continues uniformly in its motion: I say, that the periodic times of the parts of the fluid are as the squares of their distances from the centre of the sphere.

CASE I. Let AFL be a sphere turning uniformly about the axis S, and let the concentric circles BGM, CHN, DIO, EKP, &c, divide the fluid into innumerable concentric orbs of the same thickness. Suppose those orbs to be solid; and, because the fluid is homogeneous, the impressions which the contiguous orbs make one upon another will be (by the supposition) as their translations from one another, and the contiguous surfaces upon which the impressions are made. If



the impression upon any orb be greater or less upon its concave than upon its convex side, the more forcible impression will prevail, and will either accelerate or retard the velocity of the orb, according as it is directed with a conspiring or contrary motion to that of the orb. Therefore that every orb may continue uniformly in its motion, it is necessary that the impressions made upon both sides of the orb should be equal, and have contrary directions. Therefore since the impressions are as the contiguous surfaces, and as their translations from one another, the translations will be inversely as the surfaces, that is, inversely as the squares of the distances of the surfaces from the centre. But the differences of the angular motions about the axis are as those translations applied to the distances, or directly as the translations and inversely as the distances; that is, by compounding those ratios, inversely as the cubes of the distances. Therefore, if upon the several parts of the infinite right line *SABCDEQ* there be erected the perpendiculars *Aa, Bb, Cc, Dd, Ee, &c.*, inversely proportional to the cubes of *SA, SB, SC, SD, SE, &c.*, the sums of the differences, that is, the whole angular motions, will be as the corresponding sums of the lines *Aa, Bb, Cc, Dd, Ee, &c.*, that is (if to constitute an uniformly fluid medium the number of the orbs be increased and their thickness diminished *in infinitum*), as the hyperbolic areas *AaQ, BbQ, CcQ, DdQ, EeQ, &c.*, analogous to the sums; and the periodic times being inversely proportional to the angular motions, will be also inversely proportional to those areas. Therefore the periodic time of any orb *DIO* is inversely as the area *DdQ*, that is (by the known methods of quadratures), directly as the square of the distance *SD*. Which was first to be demonstrated.

CASE 2. From the centre of the sphere let there be drawn a great number of indefinite right lines, making given angles with the axis, exceeding one another by equal differences; and, by these lines revolving about the axis, conceive the orbs to be cut into innumerable annuli; then will every annulus have four annuli contiguous to it, that is, one on its inside, one on its outside, and two on each hand. Now each of these annuli cannot be impelled equally and with contrary directions by the attrition of the interior and exterior annuli, unless the motion be communicated according to the law which we demonstrated in Case 1. This appears from that demonstration. And therefore any series of annuli, taken in any right line extending

itself *in infinitum* from the globe, will move according to the law of Case 1, except we should imagine it hindered by the attrition of the annuli on each side of it. But now in a motion, according to this law, no such is, and therefore cannot be, any obstacle to the motions continuing according to that law. If annuli at equal distances from the centre revolve either more swiftly or more slowly near the poles than near the ecliptic, they will be accelerated if slow, and retarded if swift, by their mutual attrition; and so the periodic times will continually approach to equality, according to the law of Case i. Therefore this attrition will not at all hinder the motion from going on according to the law of Case i, and therefore that law will take place; that is, the periodic times of the several annuli will be as the squares of their distances from the centre of the globe. Which was to be demonstrated in the second place.

CASE 3. Let now every annulus be divided by transverse sections into innumerable particles constituting a substance absolutely and uniformly fluid; and because these sections do not at all respect the law of circular motion, but only serve to produce a fluid substance, the law of circular motion will continue the same as before. All the very small annuli will either not at all change their asperity and force of mutual attrition upon account of these sections, or else they will change the same equally. Therefore the proportion of the causes remaining the same, the proportion of the effects will remain the same also; that is, the proportion of the motions and the periodic times. Q.E.D. But now as the circular motion, and the centrifugal force thence arising, is greater at the ecliptic than at the poles, there must be some cause operating to retain the several particles in their circles; otherwise the matter that is at the ecliptic will always recede from the centre, and come round about to the poles by the outside of the vortex, and from thence return by the axis to the ecliptic with a continual circulation.

COR. 1. Hence the angular motions of the parts of the fluid about the axis of the globe are inversely as the squares of the distances from the centre of the globe, and the absolute velocities are inversely as the same squares applied to the distances from the axis.

COR. 11. If a globe revolve with an uniform motion about an axis of a given position in a similar and infinite quiescent fluid with an uniform motion, it will communicate a whirling motion to the fluid like that of a vortex, and

that motion will by degrees be propagated onwards *in infinitum*; and this motion will be increased continually in every part of the fluid, till the periodical times of the several parts become as the squares of the distances from the centre of the globe.

COR. HI. Because the inward parts of the vortex are by reason of their greater velocity continually pressing upon and driving forwards the external parts, and by that action are continually communicating motion to them, and at the same time those exterior parts communicate the same quantity of motion to those that lie still beyond them, and by this action preserve the quantity of their motion continually unchanged, it is plain that the motion is continually transferred from the centre to the circumference of the vortex, till it is quite swallowed up and lost in the boundless extent of that circumference. The matter between any two spherical surfaces concentric to the vortex will never be accelerated; because that matter will be always transferring the motion it receives from the matter nearer the centre to that matter which lies nearer the circumference.

COR. iv. Therefore, in order to continue a vortex in the same state of motion, some active principle is required from which the globe may receive continually the same quantity of motion which it is always communicating to the matter of the vortex. Without such a principle it will undoubtedly come to pass that the globe and the inward parts of the vortex, being always propagating their motion to the outward parts, and not receiving any new motion, will gradually move slower and slower, and at last be carried round no longer.

COR. v. If another globe should be swimming in the same vortex at a certain distance from its centre, and in the meantime by some force revolve constantly about an axis of a given inclination, the motion of this globe will drive the fluid round after the manner of a vortex; and at first this new and small vortex will revolve with its globe about the centre of the other; and in the meantime its motion will creep on farther and farther, and by degrees be propagated *in infinitum*, after the manner of the first vortex. And for the same reason that the globe of the new vortex was carried about before by the motion of the other vortex, the globe of this other will be carried about by the motion of this new vortex, so that the two globes will revolve about some intermediate point, and by reason of

that circular motion mutually fly from each other, unless some force restrains them. Afterwards, if the constantly impressed forces, by which the globes continue in their motions, should cease, and everything be left to act according to the laws of mechanics, the motion of the globes will languish by degrees (for the reason assigned in Cor. in and iv), and the vortices at last will quite stand still.

COR. VI. If several globes in given places should constantly revolve with determined velocities about axes given in position, there would arise from them as many vortices going on *in infinitum*. For upon the same account that any one globe propagates its motion *in infinitum*, each globe apart will propagate its motion *in infinitum* also; so that every part of the infinite fluid will be agitated with a motion resulting from the actions of all the globes. Therefore the vortices will not be confined by any certain limits, but by degrees run into each other; and by the actions of the vortices on each other, the globes will be continually moved from their places, as was shown in the last Corollary; neither can they possibly keep any certain position among themselves, unless some force restrains them. But if those forces, which are constantly impressed upon the globes to continue these motions, should cease, the matter (for the reason assigned in Cor. in and iv) will gradually stop, and cease to move in vortices.

COR. VII. If a similar fluid be inclosed in a spherical vessel, and, by the uniform rotation of a globe in its centre, be driven round in a vortex; and the globe and vessel revolve the same way about the same axis, and their periodic times be as the squares of the semidiameters: the parts of the fluid will not go on in their motions without acceleration or retardation, till their periodical times are as the squares of their distances from the centre of the vortex. No constitution of a vortex can be permanent but this.

COR. VIII. If the vessel, the inclosed fluid, and the globe, retain this motion, and revolve besides with a common angular motion about any given axis, because the mutual attrition of the parts of the fluid is not changed by this motion, the motions of the parts among themselves will not be changed; for the translations of the parts among themselves depend upon this attrition. Any part will continue in that motion in which its attrition on one side retards it just as much as its attrition on the other side accelerates it.

COR. IX. Therefore if the vessel be quiescent, and the motion of the globe be given, the motion of the fluid will be given. For conceive a plane to pass through the axis of the globe, and to revolve with a contrary motion; and suppose the sum of the time of this revolution and of the revolution of the globe to be to the time of the revolution of the globe as the square of the semidiameter of the vessel to the square of the semidiameter of the globe; and the periodic times of the parts of the fluid in respect of this plane will be as the squares of their distances from the centre of the globe.

COR. x. Therefore if the vessel move about the same axis with the globe, or with a given velocity about a different one, the motion of the fluid will be given. For if from the whole system we take away the angular motion of the vessel, all the motions will remain the same among themselves as before, by Cor. vm, and those motions will be given by Cor. ix.

COR. xi. If the vessel and the fluid are quiescent, and the globe revolves with an uniform motion, that motion will be propagated by degrees through the whole fluid to the vessel, and the vessel will be carried round by it, unless forcibly held back; and the fluid and the vessel will be continually accelerated till their periodic times become equal to the periodic times of the globe. If the vessel be either restrained by some force, or revolve with any constant and uniform motion, the medium will come little by little to the state of motion defined in Cor. vm, ix, x, nor will it ever continue in any other state. But if then the forces, by which the globe and vessel revolve with certain motions, should cease, and the whole system be left to act according to the mechanical laws, the vessel and globe, by means of the intervening fluid, will act upon each other, and will continue to propagate their motions through the fluid to each other, till their periodic times become equal among themselves, and the whole system revolves together like one solid body.

SCHOLIUM

In all these reasonings I suppose the fluid to consist of matter of uniform density and fluidity; I mean, that the fluid is such, that a globe placed anywhere therein may propagate with the same motion of its own, at distances from itself continually equal, similar and equal motions in the fluid in the same interval of time. The matter by its circular motion endeavors to recede from the axis of the vortex, and therefore presses all the matter that lies

beyond. This pressure makes the attrition greater, and the separation of the parts more difficult; and by consequence diminishes the fluidity of the matter. Again; if the parts of the fluid are in any one place denser or larger than in the others, the fluidity will be less in that place, because there are fewer surfaces where the parts can be separated from each other. In these cases I suppose the defect of the fluidity to be supplied by the smoothness or softness of the parts, or some other condition; otherwise the matter where it is less fluid will cohere more, and be more sluggish, and therefore will receive the motion more slowly, and propagate it farther than agrees with the ratio above assigned. If the vessel be not spherical, the particles will move in lines not circular, but answering to the figure of the vessel; and the periodic times will be nearly as the squares of the mean distances from the centre. In the parts between the centre and the circumference the motions will be slower where the spaces are wide, and swifter where narrow; nevertheless, the particles will not tend to the circumference at all the more because of their greater swiftness; for they then describe arcs of less curvity, and the tendency to recede from the centre is as much diminished by the lessening of this curvature as it is augmented by the increase of the velocity. As they go out of narrow into wide spaces, they recede a little farther from the centre, but in doing so are retarded; and when they come out of wide into narrow spaces, they are again accelerated; and so each particle is retarded and accelerated by turns forever. These things will come to pass in a rigid vessel; for the state of vortices in an infinite fluid is known by Cor. vi of this Proposition.

I have endeavored in this Proposition to investigate the properties of vortices, that I might find whether the celestial phenomena can be explained by them; for the phenomenon is this, that the periodic times of the planets revolving about Jupiter are as the $3/2$ th power of their distances from Jupiter's centre; and the same rule obtains also among the planets that revolve about the sun. And these rules obtain also with the greatest accuracy, as far as has been yet discovered by astronomical observation. Therefore if those planets are carried round in vortices revolving about Jupiter and the sun, the vortices must revolve according to that law. But here we found the periodic times of the parts of the vortex to be as the square of the distances from the centre of motion; and this ratio cannot be diminished

and reduced to the $3/2$ th power, unless either the matter of the vortex be more fluid the farther it is from the centre, or the resistance arising from the want of lubricity in the parts of the fluid should, as the velocity with which the parts of the fluid are separated goes on increasing, be augmented with it in a greater ratio than that in which the velocity increases. But neither of these suppositions seems reasonable. The more gross and less fluid parts will tend to the circumference, unless they are heavy towards the centre. And though, for the sake of demonstration, I proposed, at the beginning of this Section, an Hypothesis that the resistance is proportional to the velocity, nevertheless, it is in truth probable that the resistance is in a less ratio than that of the velocity; which granted, the periodic times of the parts of the vortex will be in a greater ratio than the square of the distances from its centre. If, as some think, the vortices move more swiftly near the centre, then slower to a certain limit, then again swifter near the circumference, certainly neither the $3/2$ th power, nor any other certain and determinate power, can obtain in them. Let philosophers then see how that phenomenon of the $3/2$ th power can be accounted for by vortices.

PROPOSITION LI11. THEOREM XLI

Bodies carried about in a vortex, and returning in the same orbit, are of the same density with the vortex, and are moved according to the same law with the parts of the vortex, as to velocity and direction of motion.

For if any small part of the vortex, whose particles or physical points continue a given situation among themselves, be supposed to be congealed, this particle will move according to the same law as before, since no change is made either in its density, inertia, or figure. And again; if a congealed or solid part of the vortex be of the same density with the rest of the vortex, and be resolved into a fluid, this will move according to the same law as before, except so far as its particles, now become fluid, may be moved among themselves. Neglect, therefore, the motion of the particles among themselves as not at all concerning the progressive motion of the whole, and the motion of the whole will be the same as before. But this motion will be the same with the motion of other parts of the vortex at equal distances from the centre; because the solid, now resolved into a fluid, is become exactly like the other parts of the vortex. Therefore a solid, if it be of

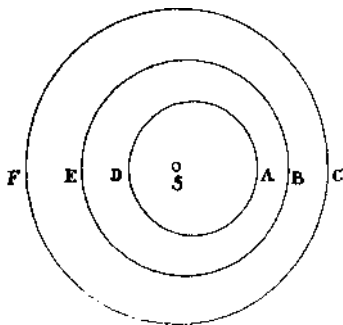
the same density with the matter of the vortex, will move with the same motion as the parts thereof, being relatively at rest in the matter that surrounds it. If it be more dense, it will endeavor more than before to recede from the centre; and therefore overcoming that force of the vortex, by which, being, as it were, kept in equilibrium, it was retained in its orbit, it will recede from the centre, and in its revolution describe a spiral, returning no longer into the same orbit. And, by the same argument, if it be more rare, it will approach to the centre. Therefore it can never continually go round in the same orbit, unless it be of the same density with the fluid. But we have shown in that case that it would revolve according to the same law with those parts of the fluid that are at the same or equal distances from the centre of the vortex.

COR. i. Therefore a solid revolving in a vortex, and continually going round in the same orbit, is relatively quiescent in the fluid that carries it.

COR. II. And if the vortex be of an uniform density, the same body may revolve at any distance from the centre of the vortex.

SCHOLIUM

Hence it is manifest that the planets are not carried round in corporeal vortices; for, according to the *Copemican* hypothesis, the planets going round the sun revolve in ellipses, having the sun in their common focus; and by radii drawn to the sun describe areas proportional to the times. But the parts of a vortex can never revolve with such a motion. For, let AD, BE, CF represent three orbits described about the sun S, of which let the outmost circle CF be concentric to the sun; let the aphe- lions of the two innermost be A, B; and their perihelions D, E. Hence a body revolving in the orb CF, describing, by a radius drawn to the sun, areas proportional to the times, will move with an uniform motion. And, according to the laws



of astronomy, the body revolving in the orbit BE will move slower in its aphelion B, and swifter in its perihelion E; whereas, according to the laws of mechanics, the matter of the vortex ought to move more swiftly in the narrow space between A and C than in the wide space between D and F; that is, more swiftly in the aphelion than in the perihelion. Now these two conclusions contradict each other. So at the beginning of the sign of Virgo, where the aphelion of Mars is at present, the distance between the orbits of Mars and Venus is to the distance between the same orbits, at the beginning of the sign of Pisces, as about 3 to 2; and therefore the matter of the vortex between those orbits ought to be swifter at the beginning of Pisces than at the beginning of Virgo in the ratio of 3 to 2; for the narrower the space is through which the same quantity of matter passes in the same time of one revolution, the greater will be the velocity with which it passes through it. Therefore if the earth being relatively at rest in this celestial matter should be carried round by it, and revolve together with it about the sun, the velocity of the earth at the beginning of Pisces would be to its velocity at the beginning of Virgo in the ratio of 3 to 2. Therefore the sun's apparent diurnal motion at the beginning of Virgo ought to be above 70 minutes, and at the beginning of Pisces less than 48 minutes; whereas, on the contrary, that apparent motion of the sun is really greater at the beginning of Pisces than at the beginning of Virgo, as experience testifies; and therefore the earth is swifter at the beginning of Virgo than at the beginning of Pisces; so that the hypothesis of vortices is utterly irreconcilable with astronomical phenomena, and rather serves to perplex than explain the heavenly motions. How these motions are performed in free spaces without vortices, may be understood by the first Book; and I shall now more fully treat of it in the following Book.

Book Three
SYSTEM OF THE WORLD
(IN MATHEMATICAL TREATMENT)

IN THE PRECEDING BOOKS I have laid down the principles of philosophy; principles not philosophical but mathematical: such, namely, as we may build our reasonings upon in philosophical inquiries. These principles are the laws and conditions of certain motions, and powers or forces, which chiefly have respect to philosophy; but, lest they should have appeared of themselves dry and barren, I have illustrated them here and there with some philosophical scholiums, giving an account of such things as are of more general nature, and which philosophy seems chiefly to be founded on; such as the density and the resistance of bodies, spaces void of all bodies, and the motion of light and sounds. It remains that, from the same principles, I now demonstrate the frame of the System of the World. Upon this subject I had, indeed, composed the third Book in a popular method, that it might be read by many; but afterwards, considering that such as had not sufficiently entered into the principles could not easily discern the strength of the consequences, nor lay aside the prejudices to which they had been many years accustomed, therefore, to prevent the disputes which might be raised upon such accounts, I chose to reduce the substance of this Book into the form of Propositions (in the mathematical way), which should be read by those only who had first made themselves masters of the principles established in the preceding Books: not that I would advise anyone to the previous study of every Proposition of those Books; for they abound with such as might cost too much time, even to readers of good mathematical learning. It is enough if one carefully reads the Definitions, the Laws of Motion, and the first three sections of the first Book. He may then pass on to this Book, and consult such of the remaining Propositions of the first two Books, as the references in this, and his occasions, shall require.

RULES OF REASONING IN PHILOSOPHY

RULE I

We are to admit no more causes of natural things than such as are both true and sufficient to explain their appearances.

To this purpose the philosophers say that Nature does nothing in vain, and more is in vain when less will serve; for Nature is pleased with simplicity, and affects not the pomp of superfluous causes.

RULE II

Therefore to the same natural effects we must, as far as possible, assign the same causes.

As to respiration in a man and in a beast; the descent of stones in *Europe* and in *America*; the light of our *cuhnai* fire and of the sun; the reflection of light in the earth, and in the planets.

RULE III

The qualities of bodies, which admit neither intensification nor remission of degrees, and which are found to belong to all bodies within the reach of our experiments, are to be esteemed the universal qualities of all bodies whatsoever.

For since the qualities of bodies are only known to us by experiments, we are to hold for universal all such as universally agree with experiments; and such as are not liable to diminution can never be quite taken away. We are certainly not to relinquish the evidence of experiments for the sake of dreams and vain fictions of our own devising; nor are we to recede from the analogy of Nature, which is wont to be simple, and always consonant to

itself. We no other way know the extension of bodies than by our senses, nor do these reach it in all bodies; but because we perceive extension in all that are sensible, therefore we ascribe it universally to all others also. That abundance of bodies are hard, we learn by experience; and because the hardness of the whole arises from the hardness of the parts, we therefore justly infer the hardness of the undivided particles not only of the bodies we feel but of all others. That all bodies are impenetrable, we gather not from reason, but from sensation. The bodies which we handle we find impenetrable, and thence conclude impenetrability to be an universal property of all bodies whatsoever. That all bodies are movable, and endowed with certain powers (which we call the inertia) of persevering in their motion, or in their rest, we only infer from the like properties observed in the bodies which we have seen. The extension, hardness, impenetrability, mobility, and inertia of the whole, result from the extension, hardness, impenetrability, mobility, and inertia of the parts; and hence we conclude the least particles of all bodies to be also all extended, and hard and impenetrable, and movable, and endowed with their proper inertia. And this is the foundation of all philosophy. Moreover, that the divided but contiguous particles of bodies may be separated from one another, is matter of observation; and, in the particles that remain undivided, our minds are able to distinguish yet lesser parts, as is mathematically demonstrated. But whether the parts so distinguished, and not yet divided, may, by the powers of Nature, be actually divided and separated from one another, we cannot certainly determine. Yet, had we the proof of but one experiment that any undivided particle, in breaking a hard and solid body, suffered a division, we might by virtue of this rule conclude that the undivided as well as the divided particles may be divided and actually separated to infinity.

Lastly, if it universally appears, by experiments and astronomical observations, that all bodies about the earth gravitate towards the earth, and that in proportion to the quantity of matter which they severally contain; that the moon likewise, according to the quantity of its matter, gravitates towards the earth; that, on the other hand, our sea gravitates towards the moon; and all the planets one towards another; and the comets in like manner towards the sun; we must, in consequence of this rule, universally allow that all bodies whatsoever are endowed with a principle of mutual gravitation.

For the argument from the appearances concludes with more force for the universal gravitation of all bodies than for their impenetrability; of which, among those in the celestial regions, we have no experiments, nor any manner of observation. Not that I affirm gravity to be essential to bodies: by their *vis insita* I mean nothing but their inertia, This is immutable. Their gravity is diminished as they recede from the earth.

R U L E I V

In experimental philosophy we are to look upon propositions inferred by general induction from phenomena as accurately or very nearly true, notwithstanding any contrary hypotheses that may be imagined, till such time as other phenomena occur, by which they may either be made more accurate, or liable to exceptions.

This rule we must follow, that the argument of induction may not be evaded by hypotheses.

[Noif In the following parts of Book III, scattered words and phrases in italics (except in Latin expression), and in names of places, months, persons, and writings) arc, in Mottc's translation, interpolations of words and phrases not in the Latin text of the *Principia*, and a few arc departures from a literal translation of the Latin]

PHENOMENA

PHENOMENONI

That the circumjoi/ial planets, by radii drawn to Jupiter's centre, describe areas proportional to the times of description; and that their periodic times, the fixed stars being at rest, are as the $^{3/2}$ th power of their distances from its centre.

This we know from astronomical observations. For the orbits of these planets differ but insensibly from circles concentric to Jupiter; and their motions in those circles are found to be uniform. And all astronomers agree that their periodic times are as the $^{3/2}$ th power of the semidiameters of their orbits; and so it manifestly appears from the following table.

The periodic times of the satellites of Jupiter.

1^d. 18^h. 27^m. 34^s., 3^d. 13^h. 13^m. 42^s., 7^d. 3^h. 42^m. 36^s., 16^d. 16^h. 32^m. 9^s.

The distances of the satellites from Jupiter's centre.

	1	2	3	4	
<i>From the observations of;</i>					
Borelli	$5^{2/3}$	$8^{2/3}$	14	$24^{2/3}$	<i>Semi-diameter of Jupiter</i>
Townly by the micrometer	5 52	8 78	13 47	24 72	
Cassini by the telescope	5	8	13	23	
Cassini by the eclipse of the satellites	$5^{2/3}$	9	$14^{23/60}$	$25^{2/10}$	
<i>From the periodic times</i>	5 667	9 017	14 384	25 299	

Mr. Pound hath determined, by the help of excellent micrometers, the diameters of Jupiter and the elongation of its satellites after the following manner. The greatest heliocentric elongation of the fourth satellite from

Jupiter's centre was taken with a micrometer in a 15-foot telescope, and at the mean distance of Jupiter from the earth was found about $8' 16''$. The elongation of the third satellite was taken with a micrometer in a telescope of 123 feet, and at the same distance of Jupiter from the earth was found $4' 42''$. The greatest elongations of the other satellites, at the same distance of Jupiter from the earth, are found from the periodic times to be $2' 56'' 47'''$, and $1' 51'' 6'''$.

The diameter of Jupiter taken with the micrometer in a 123 foot telescope¹ several times, and reduced to Jupiter's mean distance from the earth, proved always less than $40''$, never less than $38''$, generally $39''$. This diameter in shorter telescopes is $40''$, or $41''$; for Jupiter's light is a little dilated by the unequal refrangibility of the rays, and this dilatation bears a less ratio to the diameter of Jupiter in the longer and more perfect telescopes than in those which are shorter and less perfect. The times in which two satellites, the first and the third, passed over Jupiter's body, were observed, from the beginning of the ingress to the beginning of the egress, and from the complete ingress to the complete egress, with the long telescope. And from the transit of the first satellite, the diameter of Jupiter at its mean distance from the earth came forth $37\frac{1}{4}''$, and from the transit of the third $37\frac{3}{4}''$. There was observed also the time in which the shadow of the first satellite passed over Jupiter's body, and thence the diameter of Jupiter at its mean distance from the earth came out about $37''$. Let us suppose its diameter to be $37\frac{1}{4}''$, very nearly, and then the greatest elongations of the first, second, third, and fourth satellite will be respectively equal to 5.965, 9.494, 15.141, and 26.63 semidiameters of Jupiter.

P H E N O M E N O N I I

That the arcsaturnal planets, by radii drawn to Saturn's centre, describe areas proportional to the times of description; and that their periodic times, the fixed stars being at rest, are as the $\frac{3}{2}$ th power of their distances from its centre.

For, as *Cassint* from his own observations hath determined, their distances from Saturn's centre and their periodic times are as follows:

¹ Appendix, Note 39]

The periodic times of the satellites of Saturn.

1^d. 21^h. 18^m. 27^s., 2^d. 17^h. 41^m. 22^s., 4^d. 12^h. 25^m. 12^s., 15^d. 22^h. 41^m. 14^s.,
79^d. 7^h. 48^m. 00^s.

The distances of the satellites from Saturn's centre, in semidiameters of its ring.

<i>From observations</i>		1 ¹⁸ / ₂₀	2 ¹ / ₂	3 ¹ / ₂	8	24
<i>From the periodic times</i>	. .	1.93	2.47	3.45	8	23.35

The greatest elongation of the fourth satellite from Saturn's centre is commonly determined from the observations to be eight of those semidiameters, very nearly. But the greatest elongation of this satellite from Saturn's centre, when taken with an excellent micrometer in Mr. *Huygens'* telescope of 123 feet, appeared to be eight semidiameters and ⁷/₁₀ of a semidiameter. And from this observation and the periodic times the distances of the satellites from Saturn's centre in semidiameters of the ring are 2.1, 2.69, 3.75, 8.7, and 25.35. The diameter of Saturn observed in the same telescope was found to be to the diameter of the ring as 3 to 7; and the diameter of the ring, May 28-29, 1719, was found to be 43"; and hence the diameter of the ring when Saturn is at its mean distance from the earth is 42", and the diameter of Saturn 18". These things appear so in very long and excellent telescopes, because in such telescopes the apparent magnitudes of the heavenly bodies bear a greater proportion to the dilatation of light in the extremities of those bodies than in shorter telescopes. If, then, we reject all the spurious light, the diameter of Saturn will not amount to more than 16".

P H E N O M E N O N I I I

That the five primary planets, Mercury, Venus, Mars, Jupiter, and Saturn, with their several orbits, encompass the sun.

That Mercury and Venus revolve about the sun, is evident from their moon-like appearances. When they shine out with a full face, they are, in respect of us, beyond or above the sun; when they appear half full, they are about the same height on one side or other of the sun; when horned, they are below or between us and the sun; and they are sometimes, *when*

directly under, seen like spots traversing the sun's disk. That Mars surrounds the sun, is as plain from its full face when near its conjunction with the sun, and from the gibbous figure which it shows in its quadratures. And the same thing is demonstrable of Jupiter and Saturn, from their appearing full in all situations; for the shadows of their satellites that appear sometimes upon their disks make it plain that the light they shine with is not their own, but borrowed from the sun.

P H E N O M E N O N I V

That the fixed stars being at rest, the periodic times of the five primary planets, and (whether of the sun about the earth, or) of the earth about the sun, are as the $3/2$ th power of their mean distances from the sun.

This proportion, first observed by *Kepler*, is now received by all astronomers; for the periodic times are the same, and the dimensions of the orbits are the same, whether the sun revolves about the earth, or the earth about the sun. And as to the measures of the periodic times, all astronomers are agreed about them. But for the dimensions of the orbits, *Kepler* and *Boulliau*, above all others, have determined them from observations with the greatest accuracy; and the mean distances corresponding to the periodic times differ but insensibly from those which they have assigned, and for the most part fall in between them; as we may see from the following table.

The periodic times with respect to the fixed stars, of the planets and earth revolving about the sun, in days and decimal parts of a day.

♄	♅	♆	♁	♂	♀	♃
10759.275	4332.514	686.9785	365.2565	224.6176	87.9692	

The mean distances of the planets and of the earth from the sun.

	♄	♅	♆
According to <i>Kepler</i>	951000	519650	152350
“ “ <i>Boulliau</i>	954198	522520	152350
“ “ the periodic times	954006	520096	152369
	♁	♂	♀
According to <i>Kepler</i>	100000	72400	38806
“ “ <i>Boulliau</i>	100000	72398	38585
“ “ the periodic times	100000	72333	38710

As to Mercury and Venus, there can be no doubt about their distances from the sun; for they are determined by the elongations of those planets from the sun; and for the distances of the superior planets, all dispute is cut off by the eclipses of the satellites of Jupiter. For by those eclipses the position of the shadow which Jupiter projects is determined; from this we have the heliocentric longitude of Jupiter. And from its heliocentric and geocentric longitudes compared together, we determine its distance.

P H E N O M E N O N V

Then the primary planets, by radii drawn to the earth, describe areas in no wise proportional to the times; but the areas which they describe by radii drawn to the sun are proportional to the times of description.

For to the earth they appear sometimes direct, sometimes stationary, nay, and sometimes retrograde. But from the sun they are always seen direct, and to proceed with a motion nearly uniform, that is to say, a little swifter in the perihelion and a little slower in the aphelion distances, so as to maintain an equality in the description of the areas. This is a noted proposition among astronomers, and particularly demonstrable in Jupiter, from the eclipses of his satellites; by the help of these eclipses, as we have said, the heliocentric longitudes of that planet, and its distances from the sun, are determined.

P H E N O M E N O N V I

That the moon, by a radius drawn to the earth's centre, describes an area proportional to the time of description.

This we gather from the apparent motion of the moon, compared with its apparent diameter. It is true that the motion of the moon is a little disturbed by the action of the sun: but in laying down these Phenomena, I neglect those small and inconsiderable errors.

PROPOSITIONS

PROPOSITION I. THEOREM I

That the forces by which the circumjovial planets are continually drawn off from rectilinear motions, and retained in their proper orbits, tend to Jupiter's centre; and are inversely as the squares of the distances of the places of those planets from that centre.

The former part of this Proposition appears from Phen. i, and Prop. n or in, Book i; the latter from Phen. i, and Cor. vi, Prop. iv, of the same Book.

The same thing we are to understand of the planets which encompass Saturn, by Phen. n.

PROPOSITION II. THEOREM II

That the forces by which the primary planets are continually drawn off from rectilinear motions, and retained in their proper orbits, tend to the sun; and are inversely as the squares of the distances of the places of those planets from the sun's centre.

The former part of the Proposition is manifest from Phen. v, and Prop. ii, Book i; the latter from Phen. iv, and Cor. vi, Prop. iv, of the same Book. But this part of the Proposition is, with great accuracy, demonstrable from the quiescence of the aphelion points; for a very small aberration from the proportion according to the inverse square of the distances would (by Cor. i, Prop. XLV, Book i) produce a motion of the apsides sensible enough in every single revolution, and in many of them enormously great.

PROPOSITION III. THEOREM III

That the force by which the moon is retained in its orbit tends to the earth; and is inversely as the square of the distance of its place from the earth's centre.

The former part of the Proposition is evident from Phen. vi, and Prop. II or in, Book r, the latter from the very slow motion of the moon's apogee; which in every single revolution amounting but to $3^0\ 3'$ forwards, may be

neglected. For (by Cor. 1, Prop, XLV, Book 1) it appears, that, if the distance of the moon from the earth's centre is to the semidiameter of the earth as D to 1, the force, from which such a motion will result, is inversely as $D^{\frac{2543}{43}}$, i.e., inversely as the power of D , whose exponent is $2 \gg J_3$; that is to say, in the proportion of the distance somewhat greater than the inverse square, but which comes $59\frac{3}{4}$ times nearer to the proportion according to the square than to the cube. But since this increase is due to the action of the sun (as we shall afterwards show), it is here to be neglected. The action of the sun, attracting the moon from the earth, is nearly as the moon's distance from the earth; and therefore (by what we have shown in Cor. 11, Prop, XLV, Book 1) is to the centripetal force of the moon as 2 to 357.45, or nearly so; that is, as 1 to $178^{\frac{29}{40}}$. And if we neglect so inconsiderable a force of the sun, the remaining force, by which the moon is retained in its orb, will be inversely as D^2 . This will yet more fully appear from comparing this force with the force of gravity, as is done in the next Proposition.

COR. If we augment the mean centripetal force by which the moon is retained in its orb, first in the proportion of $177^{\frac{29}{40}}$ to $178^{\frac{29}{40}}$, and then in the proportion of the square of the semidiameter of the earth to the mean distance of the centres of the moon and earth, we shall have the centripetal force of the moon at the surface of the earth; supposing this force, in descending to the earth's surface, continually to increase inversely as the square of the height.

PROPOSITION IV.¹ THEOREM IV

That the moon gravitates towards the earth, and by the force of gravity is continually drawn off from a rectilinear motion, and retained in Us orbit.

The mean distance of the moon from the earth in the syzygies in semidiameters of the earth, is, according to *Ptolemy* and most astronomers, 59; according to *Vcndehn* and *Huygens*, 60; to *Copernicus*, $60\frac{1}{4}$; to *Street*, $56\frac{1}{2}$; and to *Tycho*, $56\frac{1}{2}$. But *Tycho*, and all that follow his tables of refraction, making the refractions of the sun and moon (altogether against the nature of light) to exceed the refractions of the fixed stars, and that by four or five minutes *near the horizon*, did thereby increase the moon's *horizontal* parallax by a like number of minutes, that is, by a twelfth or fifteenth part of the whole parallax. Correct this error, and the distance will become

¹ Appendix, Note 40]

about $60\frac{1}{2}$ semidiameters of the earth, near to what others have assigned. Let us assume the mean distance of 60 diameters in the syzygies; and suppose one revolution of the moon, in respect of the fixed stars, to be completed in $27^{\text{d}} 7^{\text{h}}. 43^{\text{m}}$, as astronomers have determined; and the circumference of the earth to amount to 123249600 *Paris* feet, as the *French* have found by mensuration. And now if we imagine the moon, deprived of all motion, to be let go, so as to descend towards the earth with the impulse of all that force by which (by Cor., Prop, in) it is retained in its orb, it will in the space of one minute of time, describe in its fall 15K2 *Parts* feet. This we gather by a calculus, founded either upon Prop, xxxvi, Book 1, or (which comes to the same thing) upon Cor. ix, Prop, iv, of the same Book. For the versed sine of that arc, which the moon, in the space of one minute of time, would by its mean motion describe at the distance of 60 semidiameters of the earth, is nearly $15\frac{1}{12}$ *Pans* feet, or more accurately 15 feet, 1 inch, and 1 line $\frac{4}{9}$. Wherefore, since that force, in approaching to the earth, increases in the proportion of the inverse square of the distance, and, upon that account, at the surface of the earth, is 60 60 times greater than at the moon, a body in our regions, falling with that force, ought in the space of one minute of time, to describe 60 60 $15\frac{1}{12}$ *Paris* feet; and, in the space of one second of time, to describe $15\frac{1}{12}$ of those feet; or more accurately 15 feet, 1 inch, and 1 line $\frac{4}{9}$. And with this very force we actually find that bodies here upon earth do really descend; for a pendulum oscillating seconds in the latitude of *Pans* will be 3 *Pans* feet, and 8 lines $\sqrt{2}$ in length, as Mr. *Huygens* has observed. And the space which a heavy body describes by falling in one second of time is to half the length of this pendulum as the square of the ratio of the circumference of a circle to its diameter (as Mr. *Huygens* has also shown), and is therefore 15 *Pans* feet, 1 inch, 1 line $\frac{7}{9}$. And therefore the force by which the moon is retained in its orbit becomes, at the very surface of the earth, equal to the force of gravity which we observe in heavy bodies there. And therefore (by Rule 1 and 2) the force by which the moon is retained in its orbit is that very same force which we commonly call gravity; for, were gravity another force different from that, then bodies descending to the earth with the joint impulse of both forces would fall with a double velocity, and in the space of one second of time would describe $30\frac{1}{6}$ *Pans* feet; altogether against experience.

This calculus is founded on the hypothesis of the earth's standing still; for if both earth and moon move about the sun, and at the same time about their common centre of gravity, the distance of the centres of the moon and earth from one another will be $60\frac{1}{2}$ semidiameters of the earth; as may be found by a computation from Prop, LX, Book 1.

SCHOLIUM

The demonstration of this Proposition may be more diffusely explained after the following manner. Suppose several moons to revolve about the earth, as in the system of Jupiter or Saturn; the periodic times of these moons (by the argument of induction) would observe the same law which *Kepler* found to obtain among the planets; and therefore their centripetal forces would be inversely as the squares of the distances from the centre of the earth, by Prop. 1, of this Book. Now if the lowest of these were very small, and were so near the earth as almost to touch the tops of the highest mountains, the centripetal force thereof, retaining it in its orbit, would be nearly equal to the weights of any terrestrial bodies that should be found upon the tops of those mountains, as may be known by the foregoing computation. Therefore if the same little moon should be deserted by its centrifugal force that carries it through its orbit, and be disabled from going onward therein, it would descend to the earth; and that with the same velocity, with which heavy bodies actually fall upon the tops of those very mountains, because of the equality of the forces that oblige them both to descend. And if the force by which that lowest moon would descend were different from gravity, and if that moon were to gravitate towards the earth, as we find terrestrial bodies do upon the tops of mountains, it would then descend with twice the velocity, as being impelled by both these forces conspiring together. Therefore since both these forces, that is, the gravity of heavy bodies, and the centripetal forces of the moons, are directed to the centre of the earth, and are similar and equal between themselves, they will (by Rule 1 and 2) have one and the same cause. And therefore the force which retains the moon in its orbit is that very force which we commonly call gravity; because otherwise this little moon at the top of a mountain must either be without gravity, or fall twice as swiftly as heavy bodies are wont to do.

PROPOSITION V. THEOREM V

That the circumjovial planets gravitate towards Jupiter; the circumsaturnal towards Saturn; the circumsolar towards the sun; and by the forces of their gravity are drawn off from rectilinear motions, and retained in curvilinear orbits.

For the revolutions of the circumjovial planets about Jupiter, of the circumsaturnal about Saturn, and of Mercury and Venus, and the other circumsolar planets, about the sun, are appearances of the same sort with the revolution of the moon about the earth; and therefore, by Rule 2, must be owing to the same sort of causes; especially since it has been demonstrated, that the forces upon which those revolutions depend tend to the centres of Jupiter, of Saturn, and of the sun; and that those forces, in receding from Jupiter, from Saturn, and from the sun, decrease in the same proportion, and according to the same law, as the force of gravity does in receding from the earth.

COR. 1. There is, therefore, a power of gravity tending to all the planets; for, doubtless, Venus, Mercury, and the rest, are bodies of the same sort with Jupiter and Saturn. And since all attraction (by Law in) is mutual, Jupiter will therefore gravitate towards all his own satellites, Saturn towards his, the earth towards the moon, and the sun towards all the primary planets.

COR. 11. The force of gravity which tends to any one planet is inversely as the square of the distance of places from that planet's centre.

COR. HI. All the planets do gravitate towards one another, by Cor. 1 and 11. And hence it is that Jupiter and Saturn, when near their conjunction, by their mutual attractions sensibly disturb each other's motions. So the sun disturbs the motions of the moon; and both sun and moon disturb our sea, as we shall hereafter explain.

SCHOLIUM

The force which retains the celestial bodies in their orbits has been hitherto called centripetal force; but it being now made plain that it can be no other than a gravitating force, we shall hereafter call it gravity. For the cause of that centripetal force which retains the moon in its orbit will extend itself to all the planets, by Rule 1, 2, and 4.

PROPOSITION VI. THEOREM VI

That all bodies gravitate towards every planet; and that the weights of bodies towards any one planet, at equal distances from the centre of the planet, are proportional to the quantities of matter which they severally contain.

It has been, now for a long time, observed by others, that all sorts of heavy bodies (allowance being made for the inequality of retardation which they suffer from a small power of resistance in the air) descend to the earth *from equal heights* in equal times; and that equality of times we may distinguish to a great accuracy, by the help of pendulums. I tried experiments with gold, silver, lead, glass, sand, common salt, wood, water, and wheat. I provided two wooden boxes, round and equal: I filled the one with wood, and suspended an equal weight of gold (as exactly as I could) in the centre of oscillation of the other. The boxes, hanging by equal threads of 11 feet, made a couple of pendulums perfectly equal in weight and figure, and equally receiving the resistance of the air. And, placing the one by the other, I observed them to play together forwards and backwards, for a long time, with equal vibrations. And therefore the quantity of matter in the gold (by Cor. 1 and vi, Prop, xxiv, Book 11) was to the quantity of matter in the wood as the action of the motive force (or *vis motrix*) upon all the gold to the action of the same upon all the wood; that is, as the weight of the one to the weight of the other: and the like happened in the other bodies. By these experiments, in bodies of the same weight, I could manifestly have discovered a difference of matter less than the thousandth part of the whole, had any such been. But, without all doubt, the nature of gravity towards the planets is the same as towards the earth. For, should we imagine our terrestrial bodies taken to the orbit of the moon, and there, together with the moon, deprived of all motion, to be let go, so as to fall together towards the earth, it is certain, from what we have demonstrated before, that, in equal times, they would describe equal spaces with the moon, and of consequence are to the moon, in quantity of matter, as their weights to its weight. Moreover, since the satellites of Jupiter perform their revolutions in times which observe the $\frac{3}{2}$ th power of the proportion of their distances from Jupiter's centre, their accelerative gravities towards Jupiter will be inversely as the squares of their distances from Jupiter's

centre; that is, equal, at equal distances. And, therefore, these satellites, if supposed to fall *towards Jupiter* from equal heights, would describe equal spaces in equal times, in like manner as heavy bodies do on our earth. And, by the same argument, if the circumsolar planets were supposed to be let fall at equal distances from the sun, they would, in their descent towards the sun, describe equal spaces in equal times. But forces which equally accelerate unequal bodies must be as those bodies: that is to say, the weights of the planets *towards the sun* must be as their quantities of matter. Further, that the weights of Jupiter and of his satellites towards the sun are proportional to the several quantities of their matter, appears from the exceedingly regular motions of the satellites (by Cor. in, Prop, LXV, Book i). For if some of those bodies were more strongly attracted to the sun in proportion to their quantity of matter than others, the motions of the satellites would be disturbed by that inequality of attraction (by Cor. n, Prop, LXV, Book i). If, at equal distances from the sun, any satellite, in proportion to the quantity of its matter, did gravitate towards the sun with a force greater than Jupiter in proportion to his, according to any given proportion, suppose of d to e ; then the distance between the centres of the sun and of the satellite's orbit would be always greater than the distance between the centres of the sun and of Jupiter, nearly as the square root of that proportion: as by some computations I have found. And if the satellite did gravitate towards the sun with a force, less in the proportion of e to d , the distance of the centre of the satellite's orbit from the sun would be less than the distance of the centre of Jupiter from the sun as the square root of the same proportion. Therefore if, at equal distances from the sun, the accelerative gravity of any satellite towards the sun were greater or less than the accelerative gravity of Jupiter towards the sun but by one $\frac{1}{1000}$ part of the whole gravity, the distance of the centre of the satellite's orbit from the sun would be greater or less than the distance of Jupiter from the sun by one $\frac{1}{2000}$ part of the whole distance; that is, by a fifth part of the distance of the utmost satellite from the centre of Jupiter; an eccentricity of the orbit which would be very sensible. But the orbits of the satellites are concentric to Jupiter, and therefore the accelerative gravities of Jupiter, and of all its satellites towards the sun, are equal among themselves. And by the same argument, the weights of Saturn and of his satellites towards the sun, at equal distances

from the sun, are as their several quantities of matter; and the weights of the moon and of the earth towards the sun are either none, or accurately proportional to the masses of matter which they contain. But some weight they have, by Cor. 1 and iii, Prop. v.

But further; the weights of all the parts of every planet towards any other planet are one to another as the matter in the several parts; for if some parts did gravitate more, others less, than for the quantity of their matter, then the whole planet, according to the sort of parts with which it most abounds, would gravitate more or less than in proportion to the quantity of matter in the whole. Nor is it of any moment whether these parts are external or internal; for if, for example, we should imagine the terrestrial bodies with us to be raised to the orbit of the moon, to be there compared with its body; if the weights of such bodies were to the weights of the external parts of the moon as the quantities of matter in the one and in the other respectively, but to the weights of the internal parts in a greater or less proportion, then likewise the weights of those bodies would be to the weight of the whole moon in a greater or less proportion; against what we have shown above.

COR. 1. Hence the weights of bodies do not depend upon their forms and textures; for if the weights could be altered with the forms, they would be greater or less, according to the variety of forms, in equal matter; altogether against experience.

COR. 11. Universally, all bodies about the earth gravitate towards the earth; and the weights of all, at equal distances from the earth's centre, are as the quantities of matter which they severally contain. This is the quality of all bodies within the reach of our experiments; and therefore (by Rule 3) to be affirmed of all bodies whatsoever. If the ether, or any other body, were either altogether void of gravity, or were to gravitate less in proportion to its quantity of matter, then, because (according to *Aristotle*, *Descartes*, and others) there is no difference between that and other bodies but in *mere* form of matter, by a successive change from form to form, it might be changed at last into a body of the same condition with those which gravitate most in proportion to their quantity of matter; and, on the other hand, the heaviest bodies, acquiring the first form of that body, might by degrees quite lose their gravity. And therefore the weights would depend

upon the forms of bodies, and with those forms, might be changed: contrary to what was proved in the preceding Corollary.

COR. III. All spaces are not equally full; for if all spaces were equally full, then the specific gravity of the fluid which fills the region of the air, on account of the extreme density of the matter, would fall nothing short of the specific gravity of quicksilver, or gold, or any other the most dense body; and, therefore, neither gold, nor any other body, could descend in air; for bodies do not descend in fluids, unless they are specifically heavier than the fluids. And if the quantity of matter in a given space can, by any rarefaction, be diminished, what should hinder a diminution to infinity?

COR. IV. If all the solid particles of all bodies are of the same density, and cannot be rarefied without pores, then a void, space, or vacuum must be granted. By bodies of the same density, I mean those whose inertias are in the proportion of their bulks.

COR. v. The power of gravity is of a different nature from the power of magnetism; for the magnetic attraction is not as the matter attracted. Some bodies are attracted more by the magnet; others less; most bodies not at all. The power of magnetism in one and the same body may be increased and diminished; and is sometimes far stronger, for the quantity of matter, than the power of gravity; and in receding from the magnet decreases not as the square but almost as the cube of the distance, as nearly as I could judge from some rude observations.

PROPOSITION VII. THEOREM VII

That there is a power of gravity pertaining to all bodies, proportional to the several quantities of matter which they contain.

That all the planets gravitate one towards another, we have proved before; as well as that the force of gravity towards every one of them, considered apart, is inversely as the square of the distance of places from the centre of the planet. And thence (by Prop, LXIX, Book 1, and its Corollaries) it follows, that the gravity tending towards all the planets is proportional to the matter which they contain.

Moreover, since all the parts of any planet A gravitate towards any other planet B; and the gravity of every part is to the gravity of the whole as the matter of the part to the matter of the whole; and (by Law in) to every

action corresponds an equal reaction; therefore the planet B will, on the other hand, gravitate towards all the parts of the planet A; and its gravity towards any one part will be to the gravity towards the whole as the matter of the part to the matter of the whole. Q.E.D.

COR. 1. Therefore the force of gravity towards any whole planet arises from, and is compounded of, the forces of gravity towards all its parts. Magnetic and electric attractions afford us examples of this; for all attraction towards the whole arises from the attractions towards the several parts. The thing may be easily understood in gravity, if we consider a greater planet, as formed of a number of lesser planets, meeting together in one globe; for *hence it would appear* that the force of the whole must arise from the forces of the component parts. If it is objected, that, according to this law, all bodies with us must gravitate one towards another, whereas no such gravitation anywhere appears, I answer, that since the gravitation towards these bodies is to the gravitation towards the whole earth as these bodies are to the whole earth, the gravitation towards them must be far less than to fall under the observation of our senses.

COR. II. The force of gravity towards the several equal particles of any body is inversely as the square of the distance of places from the particles; as appears from Cor. in, Prop, LXXIV, Book 1.

PROPOSITION VIII. THEOREM VIII

In two spheres gravitating each towards the other; if the matter in places on all sides round about and equidistant from the centres is similar, the weight of either sphere towards the other will be inversely as the square of the distance between their centres.

After I had found that the force of gravity towards a whole planet did arise from and was compounded of the forces of gravity towards all its parts, and towards every one part was in the inverse proportion of the squares of the distances from the part, I was yet in doubt whether that proportion inversely as the square of the distance did accurately hold, or but nearly so, in the total force compounded of so many partial ones; for it might be that the proportion which accurately enough took place in greater distances should be wide of the truth near the surface of the planet, where

the distances of the particles are unequal, and their situation dissimilar. But by the help of Prop, LXXV and LXXVI, Book 1, and their Corollaries, I was at last satisfied of the truth of the Proposition, as it now lies before us.

COR. I. Hence we may find and compare together the weights of bodies towards different planets; for the weights of bodies revolving in circles about planets are (by Cor. II, Prop, iv, Book i) directly as the diameters of the circles and inversely as the squares of their periodic times; and their weights at the surfaces of the planets, or at any other distances from their centres, are (by this Proposition) greater or less inversely as the square of the distances. Thus from the periodic times of Venus, revolving about the sun, in $224^{\text{d}}.16^{\frac{3}{4}}\text{h}.$; of the utmost circumjovial satellite revolving about Jupiter, in $16^{\text{d}}.16^{\frac{8}{15}}\text{h}.$; of the *Huygeman* satellite about Saturn, in $15^{\text{d}}.22^{\frac{2}{3}}\text{h}.$; and of the moon about the earth, in $27^{\text{d}}.7^{\text{h}}.43^{\text{m}}.$; compared with the mean distance of Venus from the sun, and with the greatest heliocentric elongations of the outmost circumjovial satellite from Jupiter's centre, $8' 16''$; of the *Huygeman* satellite from the centre of Saturn, $3' 4''$; and of the moon from the earth, $10' 33''$; by computation I found that the weight of equal bodies, at equal distances from the centres of the sun, of Jupiter, of Saturn, and of the earth, towards the sun, Jupiter, Saturn, and the earth, were one to another, as $1, 106\frac{1}{7}, 30\frac{1}{2},$ and $109\frac{1}{2}$ respectively. Then because as the distances are increased or diminished, the weights are diminished or increased in a squared ratio, the weights of equal bodies towards the sun, Jupiter, Saturn, and the earth, at the distances 10000, 997, 791, and 109 from their centres, that is, at their very surfaces, will be as 10000, 943, 529, and 435 respectively. How much the weights of bodies are at the surface of the moon, will be shown hereafter.

COR. II. Hence likewise we discover the quantity of matter in the several planets; for their quantities of matter are as the forces of gravity at equal distances from their centres; that is, in the sun, Jupiter, Saturn, and the earth, as $1, 106\frac{1}{7}, 30\frac{1}{2},$ and $109\frac{1}{2}$ respectively. If the parallax of the sun be taken greater or less than $10'' 30'''$, the quantity of matter in the earth must be augmented or diminished as the cube of that proportion.

COR. in. Hence also we find the densities of the planets; for (by Prop. LXXII, Book 1) the weights of equal and similar bodies towards similar spheres are, at the surfaces of those spheres, as the diameters of the spheres;

and therefore the densities of dissimilar spheres are as those weights applied to the diameters of the spheres. But the true diameters of the sun, Jupiter, Saturn, and the earth, were one to another as 10000, 997, 791, and 109; and the weights towards the same as 10000, 943, 529, and 435 respectively; and therefore their densities are as 100, $94\frac{1}{2}$, 67, and 400. The density of the earth, which comes out by this computation, does not depend upon the parallax of the sun, but is determined by the parallax of the moon, and therefore is here truly defined. The sun, therefore, is a little denser than Jupiter, and Jupiter than Saturn, and the earth four times denser than the sun; for the sun, by its great heat, is kept in a sort of rarefied state. The moon is denser than the earth, as shall appear afterwards.

COR. iv. The smaller the planets are, they are, other things being equal, of so much the greater density; for so the powers of gravity on their several surfaces come nearer to equality. They are likewise, other things being equal, of the greater density, as they are nearer to the sun. So Jupiter is more dense than Saturn, and the earth than Jupiter; for the planets were to be placed at different distances from the sun, that, according to their degrees of density, they might enjoy a greater or less proportion of the sun's heat. Our water, if it were removed as far as the orbit of Saturn, would be turned into ice, and in that of Mercury would quickly fly away in vapor; for the light of the sun, to which its heat is proportional, is seven times denser in the orb of Mercury than with us: and by the thermometer I have found that a sevenfold heat of our summer sun will make water boil. Nor are we to doubt that the matter of Mercury is adapted to its heat, and is therefore more dense than the matter of our earth; since, in a denser matter, the operations of Nature require a stronger heat.

PROPOSITION IX. THEOREM IX

That the force of gravity, considered downwards from the surface of the planets, decreases nearly in the proportion of the distances from the centre of the planets.

If the matter of the planet were of an uniform density, this Proposition would be accurately true (by Prop, LXXIII, Book 1). The error, therefore, can be no greater than what may arise from the inequality of the density.

PROPOSITION X. THEOREM X

That the motions of the planets in the heavens may subsist an exceedingly long time.

In the Scholium of Prop, XL, Book II, I have shown that a globe of water frozen into ice, and moving freely in our air, in the time that it would describe the length of its semidiameter, would lose by the resistance of the air part of its motion; and the same proportion holds nearly in all globes, however great, and moved with whatever velocity. But that our globe of earth is of greater density than it would be if the whole consisted of water only, I thus make out. If the whole consisted of water only, whatever was of less density than water, because of its less specific gravity, would emerge and float above. And upon this account, if a globe of terrestrial matter, covered on all sides with water, was less dense than water, it would emerge somewhere; and, the subsiding water falling back, would be gathered to the opposite side. And such is the condition of our earth, which in a great measure is covered with seas. The earth, if it was not for its greater density, would emerge from the seas, and, according to its degree of levity, would be raised more or less above their surface, the water of the seas flowing backwards to the opposite side. By the same argument, the spots of the sun, which float upon the lucid matter thereof, are lighter than that matter; and, however the planets have been formed while they were yet in fluid masses, all the heavier matter subsided to the centre. Since, therefore, the common matter of our earth on the surface thereof is about twice as heavy as water, and a little lower, in mines, is found about three, or four, or even five times heavier, it is probable that the quantity of the whole matter of the earth may be five or six times greater than if it consisted all of water; especially since I have before shown that the earth is about four times more dense than Jupiter. If, therefore, Jupiter is a little more dense than water, in the space of thirty days, in which that planet describes the length of 459 of its semidiameters, it would, in a medium of the same density with our air, lose almost a tenth part of its motion. But since the resistance of mediums decreases in proportion to their weight or density, so that water, which is 13% times lighter than quicksilver, resists less in that proportion; and air, which is 860 times lighter than water, resists less in the same proportion;

therefore in the heavens, where the weight of the medium in which the planets move is immensely diminished, the resistance will almost vanish.

It is shown in the Scholium of Prop, XXII, Book 11, that at the height of 200 miles above the earth the air is more rare than it is at the surface of the earth in the ratio of 30 to 0.0000000000003998, or as 7500000000000 to 1, nearly. And hence the planet Jupiter, revolving in a medium of the same density with that superior air, would not lose by the resistance of the medium the 1000000th part of its motion in 1000000 years. In the spaces near the earth the resistance is produced only by the air, exhalations, and vapors. When these are carefully exhausted by the air pump from under the receiver, heavy bodies fall within the receiver with perfect freedom, and without the least sensible resistance: gold itself, and the lightest down, let fall together, will descend with equal velocity; and though they fall through a space of four, six, and eight feet, they will come to the bottom at the same time; as appears from experiments. And therefore, the celestial regions being perfectly void of air and exhalations, the planets and comets meeting no sensible resistance in those spaces will continue their motions through them for an immense tract of time.

HYPOTHESIS I

That the centre of the system of the world is immovable.

This is acknowledged by all, while some contend that the earth, others that the sun, is fixed in that centre. Let us see what may from hence follow.

PROPOSITION XI. THEOREM XI

That the common centre of gravity of the earth, the sun, and all the planets, is immovable.

For (by Cor. iv of the Laws) that centre either is at rest, or moves uniformly forwards in a right line; but if that centre moved, the centre of the world would move also, against the Hypothesis.

PROPOSITION XII. THEOREM XII

That the sun is agitated by a continual motion, but never recedes far from the common centre of gravity of all the planets.

For since (by Cor. 11, Prop, VIII) the quantity of matter in the sun is to the quantity of matter in Jupiter as 1067 to 1; and the distance of Jupiter

from the sun is to the semidiameter of the sun in a proportion but a small matter greater, the common centre of gravity of Jupiter and the sun will fall upon a point a little without the surface of the sun. By the same argument, since the quantity of matter in the sun is to the quantity of matter in Saturn as 3021 to 1, and the distance of Saturn from the sun is to the semidiameter of the sun in a proportion but a small matter less, the common centre of gravity of Saturn and the sun will fall upon a point a little within the surface of the sun. And, pursuing the principles of this computation, we should find that though the earth and all the planets were placed on one side of the sun, the distance of the common centre of gravity of all from the centre of the sun would scarcely amount to one diameter of the sun. In other cases, the distances of those centres are always less; and therefore, since that centre of gravity is continually at rest, the sun, according to the various positions of the planets, must continually be moved every way, but will never recede far from that centre.

COR. Hence the common centre of gravity of the earth, the sun, and all the planets, is to be esteemed the centre of the world; for since the earth, the sun, and all the planets gravitate one towards another, and are, therefore, according to their powers of gravity, in continual agitation, as the Laws of Motion require, it is plain that their movable centres cannot be taken for the immovable centre of the world. If that body were to be placed in the centre, towards which other bodies gravitate most (according to common opinion), that privilege ought to be allowed to the sun; but since the sun itself is moved, a fixed point is to be chosen from which the centre of the sun recedes least, and from which it would recede yet less if the body of the sun were denser and greater, and therefore less apt to be moved.

PROPOSITION XIII. THEOREM XIII

The planets move in ellipses which have their common focus in the centre of the sun; and, by radii drawn to that centre, they describe areas proportional to the times of description.

We have discoursed above on these motions from the Phenomena. Now that we know the principles on which they depend, from those principles we deduce the motions of the heavens *a priori*. Because the weights of the planets towards the sun are inversely as the squares of their distances from

the sun's centre, if the sun were at rest, and the other planets did not act one upon another, their orbits would be ellipses, having the sun in their common focus; and they would describe areas proportional to the times of *description*, by Prop. 1 and xi, and Cor. 1, Prop. XIII, Book 1. But the actions of the planets one upon another are so very small, that they may be neglected; and by Prop. LXVI, Book 1, they disturb the motions of the planets around the sun in motion, less than if those motions were performed about the sun at rest.

It is true, that the action of Jupiter upon Saturn is not to be neglected; for the force of gravity towards Jupiter is to the force of gravity towards the sun (at equal distances, Cor. 11, Prop. VIII) as 1 to 1067; and therefore in the conjunction of Jupiter and Saturn, because the distance of Saturn from Jupiter is to the distance of Saturn from the sun almost as 4 to 9, the gravity of Saturn towards Jupiter will be to the gravity of Saturn towards the sun as 81 to 16 1067; or, as 1 to about 211. And hence arises a perturbation of the orbit of Saturn in every conjunction of this planet with Jupiter, so sensible, that astronomers are puzzled with it. As the planet is differently situated in these conjunctions, its eccentricity is sometimes augmented, sometimes diminished; its aphelion is sometimes carried forwards, sometimes backwards, and its mean motion is by turns accelerated and retarded; yet the whole error in its motion about the sun, though arising from so great a force, may be almost avoided (except in the mean motion) by placing the lower focus of its orbit in the common centre of gravity of Jupiter and the sun (according to Prop. LXVII, Book 1), and therefore that error, when it is greatest, scarcely exceeds two minutes; and the greatest error in the mean motion scarcely exceeds two minutes yearly. But in the conjunction of Jupiter and Saturn, the accelerative forces of gravity of the sun towards Saturn, of Jupiter towards Saturn, and of Jupiter towards the sun, arc almost as 16, 81, and———, or 156600; and therefore the difference

25

of the forces of gravity of the sun towards Saturn, and of Jupiter towards Saturn, is to the force of gravity of Jupiter towards the sun as 65 to 156609, or as 1 to 2409. But the greatest power of Saturn to disturb the motion of Jupiter is proportional to this difference; and therefore the perturbation of the orbit of Jupiter is much less than that of Saturn's. The perturbations

of the other orbits are yet far less, except that the orbit of the earth is sensibly disturbed by the moon. The common centre of gravity of the earth and moon moves in an ellipse about the sun in the focus thereof, and, by a radius drawn to the sun, describes areas proportional to the times of description. But the earth in the meantime by a menstrual motion is revolved about this common centre.

PROPOSITION XIV. THEOREM XIV

The aphelions and nodes of the orbits of the planets are fixed.

The aphelions are immovable by Prop, xi, Book i; and so are the planes of the orbits, by Prop, i of the same Book. And if the planes are fixed, the nodes must be so too. It is true, that some inequalities may arise from the mutual actions of the planets and comets in their revolutions; but these will be so small, that they may be here passed by.

COR. i. The fixed stars are immovable, seeing they keep the same position to the aphelions and nodes of the planets.

COR. II. And since these stars are liable to no sensible parallax from the annual motion of the earth, they can have no force, because of their immense distance, to produce any sensible effect in our system. Not to mention that the fixed stars, everywhere promiscuously dispersed in the heavens, by their contrary attractions destroy their mutual actions, by Prop, LXX, Book i.

SCHOLIUM

Since the planets near the sun (viz., Mercury, Venus, the earth, and Mars) are so small that they can act with but little force upon one another, therefore their aphelions and nodes must be fixed, except so far as they are disturbed by the actions of Jupiter and Saturn, and other higher bodies. And hence we may find, by the theory of gravity, that their aphelions move forwards a little, in respect of the fixed stars, and that as the $\frac{3}{2}$ th power of their several distances from the sun. So that if the aphelion of Mars, in the space of a hundred years, is carried forwards 33' 20", in respect of the fixed stars, the aphelions of the earth, of Venus, and of Mercury, will in a hundred years be carried forwards 17' 40", 10' 53", and 4' 16", respectively. But these motions are so inconsiderable, that we have neglected them in this Proposition.

PROPOSITION XV. PROBLEM I

To find the principal diameters of the orbits of the planets.

They are to be taken as the $2/3$ th power of the periodic times, by Prop, xv, Book i, and then to be severally augmented in the proportion of the sum of the masses of matter in the sun and each planet to the first of two mean proportionals between that sum and the quantity of matter in the sun, by Prop, LX, Book i.

PROPOSITION XVI. PROBLEM II

To find the eccentricities and aphelions of the planets.

This Problem is resolved by Prop, XVIII, Book i.

PROPOSITION XVII. THEOREM XV

That the diurnal motions of the planets are uniform, and that the hbration of the moon arises from its diurnal motion.

The Proposition is proved from the first Law of Motion, and Cor. xxn, Prop, LXVI, Book i. Jupiter, with respect to the fixed stars, revolves in $9^h. 56^m.$; Mars in $24^h. 39m.$; Venus in about $23^h.$; the earth in $23^h. 56^m.$; the sun in $25\frac{1}{2}^d.$, and the moon in $27^d. 7^h. 43^m.$ These things appear by the Phenomena. The spots in the sun's body return to the same situation on the sun's disk, with respect to the earth, in $27\frac{1}{2}$ days; and therefore with respect to the fixed stars the sun revolves in about $25\frac{1}{2}$ days. But because the lunar day, arising from its uniform revolution about its axis, is menstrual, *that is, equal to the time of its periodic revolution in its orbit*, hence the same face of the moon will be always nearly turned to the upper focus of its orbit; but, as the situation of that focus requires, will deviate a little to one side and to the other from the earth in the lower focus; and this is the hbration in longitude; for the hbration in latitude arises from the moon's latitude, and the inclination of its axis to the plane of the ecliptic. This theory of the hbration of the moon, Mr. *N. Mercator*, in his *Astronomy*, published at the beginning of the year 1676, explained more fully out of the letters I sent him. The utmost satellite of Saturn seems to revolve about its axis with a motion like this of the moon, respecting Saturn continually with the same face; for in its revolution round Saturn, as often as it comes to the eastern part of its orbit, it is scarcely visible, and generally quite disappears; this is probably

occasioned by some spots in that part of its body, which is then turned towards the earth, as *M. Cassini* has observed. So also the utmost satellite of Jupiter seems to revolve about its axis with a like motion, because in that part of its body which is turned from Jupiter it has a spot, which always appears as if it were in Jupiter's own body, whenever the satellite passes between Jupiter and our eye.

PROPOSITION XVIII. THEOREM XVI

That the axes of the planets are less than the diameters drawn perpendicular to the axes.

The equal gravitation of the parts on all sides would give a spherical figure to the planets, if it was not for their diurnal revolution in a circle. By that circular motion it comes to pass that the parts receding from the axis endeavor to ascend about the equator; and therefore if the matter is in a fluid state, by its ascent towards the equator it will enlarge the diameters there, and by its descent towards the poles it will shorten the axis. So the diameter of Jupiter (by the concurring observations of astronomers) is found shorter between pole and pole than from east to west. And, by the same argument, if our earth was not higher about the equator than at the poles, the seas would subside about the poles, and, rising towards the equator, would lay all things there under water.

PROPOSITION XIX. PROBLEM III

To find the proportion of the axis of a planet to the diameters perpendicular thereto.

Our countryman, Mr. *Norwood*, measuring a distance of 905751 feet of *London* measure between *London* and *York*, in 1635, and observing the difference of latitudes to be $2^0\ 28'$, determined the measure of one degree to be 367196 feet of *London* measure, that is, 57300 *Paris* toises. *M. Picard*, measuring an arc of one degree, and $22'\ 55''$ of the meridian between *Amiens* and *Malvoisine*, found an arc of one degree to be 57060 *Paris* toises. *M. Cassini*, the father, measured the distance upon the meridian from the town of *Colhoure* in *Rousstillon* to the Observatory of *Paris*; and his son added the distance from the Observatory to the Citadel of *Dunkirk*. The whole distance was $486156\frac{1}{2}$ toises and the difference of the latitudes of *Colhoure* and *Dunkirk* was 8 degrees, and $31'\ 11\frac{5}{6}''$. Hence an arc of one

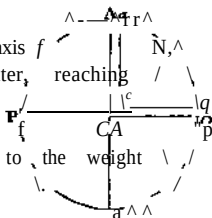
degree appears to be 57061 *Pans* toises. And from these measures we conclude that the circumference of the earth is 123249600, and its semidiameter 19615800 *Pans* feet, upon the supposition that the earth is of a spherical figure.

In the latitude of *Pans* a heavy body falling in a second of time describes 15 *Paris* feet, 1 inch, $1\frac{7}{9}$ lines, as above, that is, $2173\frac{7}{9}$ lines. The weight of the body is diminished by the weight of the ambient air. Let us suppose the weight lost thereby to be $\frac{1}{11000}$ part of the whole weight; then that heavy body falling in a vacuum will describe a height of 2174 lines in one second of time.

A body in every sidereal day of $23^{\text{H}}. 56^{\text{m}}. 4^{\text{s}}$. uniformly revolving in a circle at the distance of 19615800 feet from the centre, in one second of time describes an arc of 1433.46 feet; the versed sine of which is 0.05236561 feet, or 7.54064 lines. And therefore the force with which bodies descend in the latitude of *Paris* is to the centrifugal force of bodies in the equator arising from the diurnal motion of the earth as 2174 to 7.54064.

The centrifugal force of bodies in the equator is to the centrifugal force with which bodies recede directly from the earth in the latitude of *Pans*, $48^{\circ} 50' 10''$, as the square of the ratio of the radius to the cosine of the latitude, that is, as 7.54064 to 3.267. Add this force to the force with which bodies descend by their weight in the latitude of *Pant*, and a body, in the latitude of *Paris*, falling by its whole undiminished force of gravity, in the time of one second, will describe 2177.267 lines, or 15 *Pans* feet, 1 inch, and 5.267 lines. And the total force of gravity in that latitude will be to the centrifugal force of bodies in the equator of the earth as 2177.267 to 7.54064, or as 289 to 1.

Therefore if APBQ represent the figure of the earth, now no longer spherical, but generated by the rotation of an ellipse about its lesser PQ; and ACQ a canal full of water from the pole Qq to the centre Cc, and thence rising to the equator Aa; the weight of the water in the leg of the canal ACca will be of water in the other leg QCcq as 289 to 288, because the centrifugal force arising from the



circular motion sustains and takes off one of the 289 parts of the weight (in the one leg), and the weight of 288 in the other sustains the rest. But by computation (from Cor. 11, Prop xci, Book 1) I find, that, if the matter of the earth was all uniform, and without any motion, and its axis PQ were to the diameter AB as 100 to 101, the force of gravity in the place Q towards the earth would be to the force of gravity in the same place Q towards a sphere described about the centre C with the radius PC, or QC, as 126 to 125. And, by the same argument, the force of gravity in the place A towards the spheroid generated by the rotation of the ellipse APBQ about the axis AB is to the force of gravity in the same place A, towards the sphere described about the centre C with the radius AC, as 125 to 126. But the force of gravity in the place A towards the earth is a mean proportional between the forces of gravity towards the spheroid and this sphere; because the sphere, by having its diameter PQ diminished in the proportion of 101 to 100, is transformed into the figure of the earth; and this figure, by having a third diameter perpendicular to the two diameters AB and PQ diminished in the same proportion, is converted into the said spheroid; and the force of gravity in A, in either case, is diminished nearly in the same proportion. Therefore the force of gravity in A towards the sphere described about the centre C with the radius AC, is to the force of gravity in A towards the earth as 126 is to $125\frac{1}{2}$. And the force of gravity in the place Q towards the sphere described about the centre C with the radius QC, is to the force of gravity in the place A towards the sphere described about the centre C with the radius AC, in the proportion of the diameters (by Prop, LXXII, Book 1), that is, as 100 to 101. If, therefore, we compound those three proportions 126 to 125, 126 to $125\frac{1}{2}$, and 100 to 101, into one, the force of gravity in the place Q towards the earth will be to the force of gravity in the place A towards the earth as $126 \cdot 126 \cdot 100$ to $125 \cdot 125\frac{1}{2} \cdot 101$; or as 501 to 500.

Now since (by Cor. in, Prop, xci, Book 1) the force of gravity in either leg of the canal ACca, or QCcq, is as the distance of the places from the centre of the earth, if those legs are conceived to be divided by transverse, parallel, and equidistant surfaces, into parts proportional to the wholes, the weights of any number of parts in the one leg ACca will be to the weights of the same number of parts in the other leg as their magnitudes and the accelerative forces of their gravity conjointly, that is, as 101 to 100, and 500

to 501, or as 505 to 501. And therefore if the centrifugal force of every part in the leg *ACca*, arising from the diurnal motion, was to the weight of the same part as 4 to 505, so that from the weight of every part, conceived to be divided into 505 parts, the centrifugal force might take off four of those parts, the weights would remain equal in each leg, and therefore the fluid would rest in an equilibrium. But the centrifugal force of every part is to the weight of the same part as 1 to 289; that is, the centrifugal force, which should be $\frac{4}{505}$ parts of the weight, is only $\frac{1}{289}$ part thereof. And, therefore, I say, by the rule of proportion, that if the centrifugal force $\frac{1}{505}$ make the height of the water in the leg *ACca* to exceed the height of the water in the leg *QCcq* by $\frac{1}{100}$ part of its whole height, the centrifugal force $\frac{1}{289}$ will make the excess of the height in the leg *ACca* only $\frac{1}{229}$ part of the height of the water in the other leg *QCcq*; and therefore the diameter of the earth at the equator¹ is to its diameter from pole to pole as 230 to 229. And since the mean semidiameter of the earth, according to *Picard's* mensuration, is 19615800 *Paris* feet, or 3923.16 miles (reckoning 5000 feet to a mile), the earth will be higher at the equator than at the poles by 85472 feet, or $17\frac{1}{10}$ miles. And its height at the equator will be about 19658600 feet, and at the poles 19573000 feet.

If, the density and periodic time of the diurnal revolution remaining the same, the planet was greater or less than the earth, the proportion of the centrifugal force to that of gravity, and therefore also of the diameter between the poles to the diameter at the equator, would likewise remain the same. But if the diurnal motion was accelerated or retarded in any proportion, the centrifugal force would be augmented or diminished nearly in the same proportion squared; and therefore the difference of the diameters will be increased or diminished in the same squared ratio, very nearly. And if the density of the planet was augmented or diminished in any proportion, the force of gravity tending towards it would also be augmented or diminished in the same proportion: and the difference of the diameters on the contrary would be diminished in proportion as the force of gravity is augmented, and augmented in proportion as the force of gravity is diminished. Therefore, since the earth, in respect of the fixed stars, revolves in $23^h. 56^m.$, but Jupiter in $9^h. 56^m.$, and the squares of their periodic times are as 29 to 5, and their densities as 400 to $94\frac{1}{2}$, the difference of the diameters of Jupiter

¹ Appendix, Note 41.]

will be to its lesser diameter as $\frac{29}{5} \cdot \frac{400}{94\frac{1}{2}} \cdot \frac{1}{229}$ to 1, or as 1 to $9\frac{1}{4}$, nearly.

Therefore the diameter of Jupiter from east to west is to its diameter from pole to pole nearly as $10\frac{1}{4}$ to $9\frac{1}{4}$. Therefore, since its greatest diameter is 37", its lesser diameter lying between the poles will be 33" 25"". Add thereto about 3" for the irregular refraction of light, and the apparent diameters of this planet will become 40" and 36" 25"; which are to each other as $11\frac{1}{4}$ to $10\frac{1}{4}$, very nearly. These things are so upon the supposition that the body of Jupiter is uniformly dense. But now if its body be denser towards the plane of the equator than towards the poles, its diameters may be to each other as 12 to 11, or 13 to 12, or perhaps as 14 to 13.

And *Cassim* observed, in the year 1691, that the diameter of Jupiter reaching from east to west is greater by about a fifteenth part than the other diameter. Mr. *Pound* with his 123-foot telescope, and an excellent micrometer, measured the diameters of Jupiter in the year 1719, and found them as follows:

	The times		Greatest diameter	Lesser diameter	The diameters to each other
	days	hours	Parts	Parts	
January	28	6	13 40	12 28	As 12 to 11
March	6	7	13 12	12 20	$13\frac{3}{4}$ to $12\frac{5}{4}$
March	9	7	13 12	12 08	$12\frac{2}{4}$ to $11\frac{2}{4}$
April	9	9	12 32	11 48	$14\frac{1}{2}$ to $13\frac{1}{2}$

So that the theory agrees with the phenomena; for the planets are more heated by the sun's rays towards their equators, and therefore are a little more condensed by that heat than towards their poles.

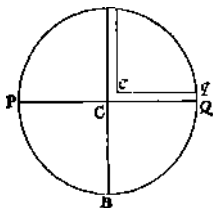
Moreover, that there is a diminution of gravity occasioned by the diurnal rotation of the earth, and therefore the earth rises higher there than it does at the poles (supposing that its matter is uniformly dense), will appear by the experiments of pendulums related under the following Proposition.

PROPOSITION XX. PROBLEM IV

To find and compare together the weights of bodies in the different regions of our earth.

Because the weights of the unequal legs of the canal of water *ACQqca* are equal; and the weights of the parts proportional to the whole legs, and

alike situated in them, are one to another as the weights of the wholes, and therefore equal between themselves; the weights of equal parts, and alike situated in the legs, will be inversely as the legs, that is, inversely as 230 to 229. And the case is the same in all homogeneous equal bodies alike situated in the legs of the canal. Their weights are inversely as the legs, that is, inversely as the distances of the bodies from the centre of the earth. Therefore if the bodies are situated in the uppermost parts of the canals, or on the surface of the earth, their weights will be one to another inversely as their distances from the centre. And, by the same argument, the weights in all other places round the whole surface of the earth are inversely as the distances of the places from the centre; and, therefore, on the hypothesis of the earth's being a spheroid, are given in proportion.



From this arises the theorem, that the increase of weight in passing from the equator to the poles is nearly as the versed sine of double the latitude; or, which comes to the same thing, as the square of the sine of the latitude; and the arcs of the degrees of latitude in the meridian increase nearly in the same proportion. And, therefore, since the latitude of *Paris* is $48^{\circ} 50'$, that of places under the equator $00^{\circ} 00'$, and that of places under the poles 90° ; and the versed sines of double those arcs are 1133400000 and 20000, the radius being 10000; and the force of gravity at the pole is to the force of gravity at the equator as 230 to 229; and the excess of the force of gravity at the pole to the force of gravity at the equator is as 1 to 229; the excess of the force of gravity in the latitude of *Paris* will be to the force of gravity at the equator as $1 \frac{11334}{20000}$ to 229, or as 5667 to 2290000. And therefore the whole forces of gravity in those places will be one to the other as 2295667 to 2290000. Therefore, since the lengths of pendulums vibrating in equal times are as the forces of gravity, and in the latitude of *Paris* the length of a pendulum vibrating seconds is 3 *Paris* feet and $8\frac{1}{2}$ lines, or rather, because of the weight of the air, $8\frac{5}{8}$ lines, the length of a pendulum vibrating in the same time under the equator will be shorter by 1.087 lines. And by a like calculus the following table is made (see p. 430).

By this table, therefore, it appears that the inequality of degrees is so small that the figure of the earth, in geographical matters, may be considered as spherical; especially if the earth be a little denser towards the plane of the equator than towards the poles.

Now several astronomers, sent into remote countries to make astronomical observations, have found that pendulum clocks do accordingly move slower near the equator than in our climates. And, first of all, in the year 1672, M. *Richer* took notice of it in the island of *Cayenne*; for when, in the month of *August*, he was observing the transits of the fixed stars over the meridian, he found his clock to go slower than it ought in respect of the mean motion of the sun at the rate of $2^m. 28^s$. a day. Therefore, fitting up a simple pendulum to vibrate in seconds, which were measured by an excellent clock, he observed the length of that simple pendulum; and this he

Latitude of the place	Length of the pendulum		Measure of one degree in the meridian		Latitude of the place	Length of the pendulum		Measure of one degree in the meridian
	feet	lines				feet	lines	
degrees			lines ¹		degrees			
0	3	7 468	56637		6	3 8 461		57022
5	3	7 482	56642		7	3 8 494		57035
10	3	7 526	56659		8	3 8 528		57048
15	3	7 596	56687		9	3 8 561		57061
20	3	7 692	56724		50	3 8 594		57074
25	3	7 812	56769		55	3 8 756		57137
30	3	7 948	56823		60	3 8 907		57196
35	3	8 099	56882		65	3 9 044		57250
40	3	8 261	56945		70	3 9 162		57295
1	3	8 294	56958		75	3 9 258		57332
2	3	8 327	56971		80	3 9 329		57360
3	3	8 361	56984		85	3 9 372		57377
4	3	8 394	56997		90	3 9 387		57382
45	3	8 428	57010					

did over and over every week for ten months together. And upon his return to *France*, comparing the length of that pendulum with the length of the pendulum at *Pans* (which was 3 *Paris* feet and 8½ lines), he found it shorter by 1¼ lines.

Afterwards, our friend Dr. *Halley*, about the year 1677, arriving at the island of *St. Helena*, found his pendulum clock to go slower there than at

¹ Appendix, Note 42]

London without marking the difference. But he shortened the rod of his clock by more than $\frac{1}{8}$ of an inch, or $1\frac{1}{2}$ lines; and, to effect this, because the length of the screw at the lower end of the rod was not sufficient, he interposed a wooden ring between the nut and the ball.

Then, in the year 1682, M. *Varin* and M. *des Hayes* found the length of a simple pendulum vibrating in seconds at the *Royal Observatory of Parts* to be 3 feet and $8\frac{5}{8}$ lines. And by the same method in the island of *Goree*, they found the length of an isochronal pendulum to be 3 feet and $6\frac{5}{8}$ lines, differing from the former by two lines. And in the same year, going to the islands of *Guadaloupe* and *Martinico*, they found that the length of an isochronal pendulum in those islands was 3 feet and $6\frac{1}{2}$ lines.

After this, M. *Couplet*, the son, in the month of *July*, 1697, ^{at the} *Royal Observatory of Parts*, so fitted his pendulum clock to the mean motion of the sun, that for a considerable time together the clock agreed with the motion of the sun. In *November* following, upon his arrival at *Lisbon*, he found his clock to go slower than before at the rate of $2^m. 13^s$. in 24 hours. And the following *March*, coming to *Paraiba*, he found his clock to go slower than at *Parts*, and at the rate $4^{111}.12^s$. in 24 hours; and he affirms that the pendulum vibrating in seconds was shorter at *Lisbon* by $2\frac{1}{2}$ lines, and at *Paraiba* by $3\frac{2}{3}$ lines, than at *Parts*. He would have done better to have reckoned those differences $1\frac{1}{8}$ and $2\frac{5}{8}$: for these differences correspond to the differences of the times $2^m. 13^s$. and $4^m. 12^s$. But this gentleman's observations are so gross, that we cannot confide in them.

In the following years, 1699 and 1700, M. *des Hayes*, making another voyage to *America*, determined that in the islands of *Cayenne* and *Granada* the length of the pendulum vibrating in seconds was a small matter less than 3 feet and $6\frac{1}{2}$ lines; that in the island of *St. Christopher* it was 3 feet and $6\frac{3}{4}$ lines; and in the island of *St. Domingo* 3 feet and 7 lines.

And, in the year 1704, *Feuille*, at *Puerto Bello* in *America*, found that the length of the pendulum vibrating in seconds was 3 *Pans* feet and only $5\frac{1}{2}$ lines, that is, almost 3 lines shorter than at *Paris*; but the observation was faulty. For afterwards, going to the island of *Martinico*, he found the length of the isochronal pendulum there 3 *Paris* feet and $5\frac{1}{4}$ lines.

Now the latitude of *Paraiba* is $6^{\circ} 38'$ south; that of *Puerto Bello*, $9^{\circ} 33'$ north; and the latitudes of the islands *Cayenne*, *Goree*, *Guadaloupe*, *Marti-*

nico, Granada, St. Christopher, and St. Domingo, are respectively $4^{\circ} 55'$, $14^{\circ} 40''$, $15^{\circ} 00'$, $14^{\circ} 44'$, $12^{\circ} 06'$, $17^{\circ} 19'$, and $19^{\circ} 48'$, north. And the excesses of the length of the pendulum at *Paris* above the lengths of the isochronal pendulums observed in those latitudes are a little greater than by the table of the lengths of the pendulum before computed. And therefore the earth is a little higher under the equator than by the preceding calculus, and a little denser at the centre than in mines near the surface, unless, perhaps, the heats of the torrid zone have a little extended the length of the pendulums.

For M. *Picard* has observed that a rod of iron, which in frosty weather in the winter season was one foot long, when heated by fire, was lengthened into one foot and $\frac{1}{4}$ line. Afterwards M. *de la Hire* found that a rod of iron, which in the like winter season was 6 feet long, when exposed to the heat of the summer sun, was extended into 6 feet and $\frac{2}{3}$ line. In the former case the heat was greater than in the latter; but in the latter it was greater than the heat of the external parts of a human body; for metals exposed to the summer sun acquire a very considerable degree of heat. But the rod of a pendulum clock is never exposed to the heat of the summer sun, nor ever acquires a heat equal to that of the external parts of a human body; and, therefore, though the 3-foot rod of a pendulum clock will indeed be a little longer in the summer than in the winter season, yet the difference will scarcely amount to $\frac{1}{4}$ line. Therefore the total difference of the lengths of isochronal pendulums in different climates cannot be ascribed to the difference of heat; nor indeed to the mistakes of the *French* astronomers. For although there is not a perfect agreement between their observations, yet the errors are so small that they may be neglected; and in this they all agree, that isochronal pendulums are shorter under the equator than at the *Royal Observatory of Paris*, by a difference not less than $1\frac{1}{4}$ lines, nor greater than $2\frac{3}{4}$ lines. By the observations of M. *Richer*, in the island of *Cayenne*, the difference was $1\frac{1}{4}$ lines. That difference being corrected by those of M. *des Hayes*, becomes $1\frac{1}{2}$ lines or $1\frac{3}{4}$ lines. By the less accurate observations of others, the same was made about 2 lines. And this disagreement might arise partly from the errors of the observations, partly from the dissimilitude of the internal parts of the earth, and the height of mountains; partly from the different temperatures of the air.

I take an iron rod 3 feet long to be shorter by a sixth part of one line in winter time with us here in *England* than in the summer. Because of the great heats under the equator, subtract this quantity from the difference of $1\frac{1}{4}$ lines observed by M. *Richer*, and there will remain $1\frac{1}{2}$ lines, which agrees very well with $1\frac{87}{1000}$ lines, obtained earlier by the theory. M. *Richer* repeated his observations, made in the island of *Cayenne*, every week for ten months together, and compared the lengths of the pendulum which he had there noted in the iron rods with the lengths thereof which he observed in *France*. This diligence and care seems to have been wanting to the other observers. If this gentleman's observations are to be depended on, the earth is higher under the equator than at the poles, and that by an excess of about 17 miles; as appeared above by the theory.

PROPOSITION XXI. THEOREM XVII

That the equinoctial points go backwards, and that the axis of the earth, by a nutation in every annual revolution, twice vibrates towards the ecliptic, and as often returns to its former position.

The Proposition appears from Cor. xx, Prop, LXVI, Book 1; but that motion of nutation must be very small, and, indeed, scarcely perceptible.

PROPOSITION XXII. THEOREM XVIII

That all the motions of the moon, and all the inequalities of those motions, follow from the principles which we have laid down.¹

That the greater planets, while they are carried about the sun, may in the meantime carry other lesser planets, revolving about them, and that those lesser planets must move in ellipses which have their foci in the centres of the greater, appears from Prop, LXV, Book 1. But then their motions will be in several ways disturbed by the action of the sun, and they will suffer such inequalities as are observed in our moon. Thus our moon (by Cor. 11, in, iv, and v, Prop, LXVI, Book 1) moves faster, and, by a radius drawn to the earth, describes an area greater for the time, and has its orbit less curved, and therefore approaches nearer to the earth in the syzygies than in the quadratures, excepting so far as these effects are hindered by the motion of eccentricity; for (by Cor. ix, Prop, LXVI, Book 1) the eccentricity is greatest when the apogee of the moon is in the syzygies, and least when the same is in the quadratures; and upon this account the perigean moon is swifter, and nearer

[¹ Appendix, Note 43.]

to us, but the apogean moon slower and farther from us, in the syzygies than in the quadratures. Moreover, the apogee goes forwards, and the nodes backwards; and this is done not with a regular but an unequal motion. For (by Cor. vii and viii, Prop, LXVI, Book i) the apogee goes more swiftly forwards in its syzygies, more slowly backwards in its quadratures; and, by the excess of its progress above its regress, advances yearly forwards. But the nodes, on the contrary (by Cor. xi, Prop, LXVI, Book i), arc quiescent in their syzygies, and go fastest back in their quadratures. Further, the greatest latitude of the moon (by Cor. x, Prop, LXVI, Book i) is greater in the quadratures of the moon than in its syzygies. And (by Cor. vi, Prop, LXVI, Book i) the mean motion of the moon is slower in the perihelion of the earth than in its aphelion. And these are the principal inequalities (of the moon) taken notice of by astronomers.

But there are yet other inequalities not observed by former astronomers, by which the motions of the moon are so disturbed that to this day we have not been able to bring them under any certain rule. For the velocities or hourly motions of the apogee and nodes of the moon, and their equations, as well as the difference between the greatest eccentricity in the syzygies and the least eccentricity in the quadratures, and that inequality which we call the variation, are (by Cor. xiv, Prop, LXVI, Book i) in the course of the year augmented and diminished as the cube of the sun's apparent diameter. And besides (by Cor. i and n, Lem. x, and Cor. xvi, Prop, LXVI, Book i) the variation is augmented and diminished nearly as the square of the time between the quadratures. But, in astronomical calculations, this inequality is commonly thrown into and combined with the equation of the moon's centre.

PROPOSITION XXIII. PROBLEM V

To derive the unequal motions of the satellites of Jupiter and Saturn from the motions of our moon.

From the motions of our moon we deduce the corresponding motions of the moons or satellites of Jupiter in this manner, by Cor. xvi, Prop, LXVI, Book i. The mean motion of the nodes of the outmost satellite of Jupiter is to the mean motion of the nodes of our moon in a proportion compounded of the squared ratio of the periodic times of the earth about the sun to the periodic times of Jupiter about the sun, and the simple ratio of the periodic

time of the satellite about Jupiter to the periodic time of our moon about the earth; and therefore, those nodes, in the space of an hundred years, are carried $8^{\circ}24'$ backwards or forwards. The mean motions of the nodes of the inner satellites are to the mean motion of the nodes of the outmost as their periodic times are to the periodic time of the former, by the same Corollary, and arc thence given. And the forward motion of the apse of every satellite is to the backward motion of its nodes as the motion of the apogee of our moon to the motion of its nodes (by the same Corollary), and is thence given. But the motions of the apsides thus found must be diminished in the proportion of 5 to 9, or of about 1 to 2, on account of a cause which I cannot here stop to explain. The greatest equations of the nodes, and of the apse of every satellite, are to the greatest equations of the nodes, and apogee of our moon respectively, as the motions of the nodes and apsides of the satellites, in the time of one revolution of the former equations, to the motions of the nodes and apogee of our moon, in the time of one revolution of the latter equations. The variation of a satellite seen from Jupiter is to the variation of our moon in the same proportion as the whole motions of their nodes respectively during the times in which the satellite and our moon (after parting from) are revolved (again) to the sun, by the same Corollary; and therefore in the outmost satellite¹ the variation does not exceed $5^{\circ}12^{\text{th}}$.

PROPOSITION XXIV. THEOREM XIX

That the flux and reflux of the sea arise from the actions of the sun and moon.

By Cor. xix and xx, Prop. Lxvi, Book 1, it appears that the waters of the sea ought twice to rise and twice to fall every day, as well lunar as solar; and that the greatest height of the waters in the open and deep seas ought to follow the approach of the luminaries to the meridian of the place by a less interval than six hours; as happens in all that eastern tract of the *Atlantic* and *Ethiopic* seas between *France* and the *Cape of Good Hope*, and on the coasts of *Chile* and *Peru* in the *South Sea*; in all which shores the flood falls out about the second, third, or fourth hour, unless where the motion propagated from the deep ocean is by the shallowness of the channels, through which it passes to some particular places, retarded to the fifth, sixth, or seventh hour, and even later. The hours I reckon from the ap-

^{f1} Appendix, Note 44]

proach of each luminary to the meridian of the place, as well under as above the horizon; and by the hours of the lunar day I understand the 24th parts of that time which the moon, by its apparent diurnal motion, employs to come about again to the meridian of the place which it left the day before. The force of the sun or moon in raising the sea is greatest in the approach of the luminary to the meridian of the place; but the force impressed upon the sea at that time continues a little while after the impression, and is afterwards increased by a new though less force still acting upon it. This makes the sea rise higher and higher, till, this new force becoming too weak to raise it any more, the sea rises to its greatest height. And this will come to pass, perhaps, in one or two hours, but more frequently near the shores in about three hours, or even more, where the sea is shallow.

The two luminaries excite two motions, which will not appear distinctly, but between them will arise one mixed motion compounded out of both. In the conjunction or opposition of the luminaries their forces will be conjoined, and bring on the greatest flood and ebb. In the quadratures the sun will raise the waters which the moon depresses, and depress the waters which the moon raises, and from the difference of their forces the smallest of all tides will follow. And because (as experience tells us) the force of the moon is greater than that of the sun, the greatest height of the waters will happen about the third lunar hour. Out of the syzygies and quadratures, the greatest tide, which by the single force of the moon ought to fall out at the third lunar hour, and by the single force of the sun at the third solar hour, by the compounded forces of both must fall out in an intermediate time that approaches nearer to the third hour of the moon than to that of the sun. And, therefore, while the moon is passing from the syzygies to the quadratures, during which time the third hour of the sun precedes the third hour of the moon, the greatest height of the waters will also precede the third hour of the moon, and that, by the greatest interval, a little after the octants of the moon; and, by like intervals, the greatest tide will follow the third lunar hour, while the moon is passing from the quadratures to the syzygies. Thus it happens in the open sea; for in the mouths of rivers the greater tides come later to their height.

But the effects of the luminaries depend upon their distances from the earth; for when they are less distant, their effects are greater, and when

more distant, their effects are less, and that as the cube of their apparent diameter. Therefore it is that the sun, in the winter time, being then in its perigee, has a greater effect, and makes the tides in the syzygies somewhat greater, and those in the quadratures somewhat less than in the summer season; and every month the moon, while in the perigee, raises greater tides than at the distance of fifteen days before or after, when it is in its apogee. From this it comes to pass that two highest tides do not follow one the other in two immediately succeeding syzygies.

The effect of either luminary doth likewise depend upon its declination or distance from the equator; for if the luminary was placed at the pole, it would constantly attract all the parts of the waters without any intensification or remission of its action, and could cause no reciprocation of motion. And, therefore, as the luminaries decline from the equator towards either pole they will, by degrees, lose their force, and on this account will excite lesser tides in the solstitial than in the equinoctial syzygies. But in the solstitial quadratures they will raise greater tides than in the quadratures about the equinoxes; because the force of the moon, then situated in the equator, most exceeds the force of the sun. Therefore the greatest tides occur in those syzygies, and the least in those quadratures, which happen about the time of both equinoxes; and the greatest tide in the syzygies is always succeeded by the least tide in the quadratures, as we find by experience. But, because the sun is less distant from the earth in winter than in summer, it comes to pass that the greatest and least tides more frequently appear before than after the vernal equinox, and more frequently after than before the autumnal.

Moreover, the effects of the luminaries depend upon the latitudes of places. Let $ApEP$ represent the earth covered with deep waters; C its centre; P, p its poles; AE the equator; F any place without the equator; F' the parallel of the place; Dd the correspondent parallel on the other side of the equator; L the place of the moon three hours before; H the place of the earth directly under it; h the opposite place; K, k the places at 90 degrees distance; CH, Ch , the greatest heights of the sea from the centre of the earth; and CK, Ck its least heights: and if with the axes Hh, Kk , an ellipse is described, and by the revolution of that ellipse about its longer axis Hh

in summer exceed those of the morning; at *Plymouth* by the height of one foot, but at *Bristol* by the height of fifteen inches, according to the observations of *Colepress* and *Sturmy*.

But the motions which we have been describing suffer some alteration from that force of reciprocation,¹ which the waters, being once moved, retain a little while *by their inertia*. Whence it comes to pass that the tides may continue for some time, though the actions of the luminaries should cease. This power of retaining the impressed motion lessens the difference of the alternate tides, and makes those tides which immediately succeed after the syzygies greater, and those which follow next after the quadratures less. And hence it is that the alternate tides at *Plymouth* and *Bristol* do not differ much more from each other than by the height of a foot or fifteen inches, and that the greatest tides at those ports are not the first but the third after the syzygies. And, besides, all the motions are retarded in their passage through shallow channels, so that the greatest tides of all, in some straits and mouths of rivers, are the fourth or even the fifth after the syzygies.

Further, it may happen that the tide may be propagated from the ocean through different channels towards the same port, and may pass quicker through some channels than through others; in which case the same tide, divided into two or more succeeding one another, may compound new motions of different kinds. Let us suppose two equal tides flowing towards the same port from different places, one preceding the other by six hours; and suppose the first tide to happen at the third hour of the approach of the moon to the meridian of the port. If the moon at the time of the approach to the meridian was in the equator, every six hours alternately there would arise equal floods, which, meeting with as many equal ebbs, would so balance each other that for that day the water would stagnate and be quiet. If the moon then declined from the equator, the tides in the ocean would be alternately greater and less, as was said; and from thence two greater and two less tides would be alternately propagated towards that port. But the two greater floods would make the greatest height of the waters to fall out in the middle time between both; and the greater and less floods would make the waters to rise to a mean height in the middle time between them, and in the middle time between the two less floods the waters would rise to their least height. Thus in the space of twenty-four hours the waters

[¹ Appendix, Note 45]

would come, not twice, as commonly, but once only to their greatest, and once only to their least height; and their greatest height, if the moon declined towards the elevated pole, would happen at the sixth or thirtieth hour after the approach of the moon to the meridian; and when the moon changed its declination, this flood would be changed into an ebb. An example of this Dr, *Halley* has given us, from the observations of seamen in the port of *Batshaw*, in the kingdom of *Ttmqmn*, in the latitude of $20^{\circ} 50'$ north. In that port, on the day which follows after the passage of the moon over the equator, the waters stagnate: when the moon declines to the north, they begin to flow and ebb, not twice, as in other ports, but once only every day; and the flood happens at the setting, and the greatest ebb at the rising of the moon. This tide increases with the declination of the moon till the seventh or eighth day; then for the seven or eight days following it decreases at the same rate as it had increased, and ceases when the moon changes its declination, crossing over the equator to the south. After which the flood is immediately changed into an ebb; and thenceforth the ebb happens at the setting and the flood at the rising of the moon; till the moon, again passing the equator, changes its declination. There are two inlets to this port and the neighboring channels, one from the seas of *China*, between the continent and the island of *Leucoma*; the other from the *Indian* sea, between the continent and the island of *Borneo*. But whether there be really two tides propagated through the said channels, one from the *Indian* sea in the space of twelve hours, and one from the sea of *China* in the space of six hours, which therefore happening at the third and ninth lunar hours, by being compounded together, produce those motions; or whether there be any other circumstances in the state of those seas, I leave to be determined by observations on the neighboring shores.

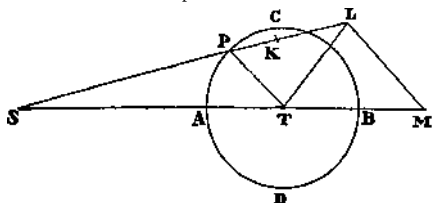
Thus I have explained the causes of the motions of the moon and of the sea. Now it is fit to subjoin something concerning the amount of those motions.

PROPOSITION XXV. PROBLEM VI

To find the forces with which the sun disturbs the motions of the moon.

Let *S* represent the sun, *T* the earth, *P* the moon, *CADB* the moon's orbit. In *SP* take *SK* equal to *ST*; and let *SL* be to *SK* as the square of *SK* to

SP: draw LM parallel to PT; and if ST or SK is supposed to represent the accelerated force of gravity of the earth towards the sun, SL will represent the accelerative force of gravity of the moon towards the sun. But that force is compounded of the parts SM and LM, of which the force LM, and that part of SM which is represented by TM, disturb the motion of the moon, as we have shown in Prop, LXVI, Book 1, and its Corollaries. Foras-



much as the earth and moon are revolved about their common centre of gravity, the motion of the earth about that centre will be also disturbed by the like forces; but we may consider the sums both of the forces and of the motions as in the moon, and represent the sum of the forces by the lines TM and ML, which are analogous to them both. The force ML (in its mean amount) is to the centripetal force by which the moon may be retained in its orbit revolving about the earth at rest, at the distance PT, as the square of the ratio of the periodic time of the moon about the earth to the periodic time of the earth about the sun (by Cor. xvn, Prop, LXVI, Book i); that is, as the square of $27^{\text{d}}. y^{\text{h}}. 43^{\text{TM}}$. to $365^{\text{d}}. 6^{\text{h}}. 9^{\text{m}}$.; or as 1000 to 178725; or as 1 to $178^{\frac{29}{40}}$. But in Prop, iv of this Book we found, that, if both earth and moon were revolved about their common centre of gravity, the mean distance of the one from the other would be nearly 60M2 mean scdiameters of the earth; and the force by which the moon may be kept revolving in its orbit about the earth at rest at the distance PT of 60Vu semidiameters of the earth, is to the force by which it may be revolved in the same time, at the distance of 60 semidiameters, as $60^{\frac{1}{2}}$ is to 60: and this force is to the force of gravity with us very nearly as 1 is to 60 60. Therefore the mean force ML is to the force of gravity on the surface of our earth as $1 \cdot 60^{\frac{1}{2}}$ to $60 \cdot 60178^{\frac{29}{40}}$, or as 1 to 638092.6; hence, by the proportion of the lines TM, ML, the force TM is also given; and these are the forces with which the sun disturbs the motions of the moon. Q.E.I.

the area in proportion as it accelerates or retards the moon. That acceleration of the moon, in its passage from the quadrature C to the conjunction A, is in every moment of time as the *generating* accelerative force EL, that

is, as $\frac{3PK \cdot TK}{TP}$. Let the time be represented by the mean motion of the

moon, or (which comes to the same thing) by the angle CTP, or even by the arc CP. At right angles upon CT erect CG equal to CT; and, supposing the quadrantal arc AC to be divided into an infinite number of equal parts Pp, &c, these *parts* may represent the like *infinite* number of the equal parts of time. Let fall pk perpendicular on CT, and draw TG meeting with KP, kj> produced in F and f; then will FK be equal to TK, and Kk be to PK as Tp is to Tp, that is, in a given proportion; and therefore FK Kk, or

the area FKkf will be as $\frac{3PK \cdot TK}{TP}$, that is, as EL; and, compounding, the

whole area GCKF will vary as the sum of all the forces EL impressed upon the moon in the whole time CP; and therefore also as the velocity generated by that sum, that is, as the acceleration of the description of the area CTP, or as the increment of the moment *thereof*. The force by which the moon may in its periodic time CADB of 27d. 7^h. 43^m. be retained revolving about the earth at rest at the distance TP, would cause a body falling in the time CT to describe the length V2CT, and at the same time to acquire a velocity equal to that with which the moon is moved in its orbit. This appears from Cor. ix, Prop, iv, Book 1. But since Kd, drawn perpendicular on TP, is *but* a third part of EL, and *equal* to the half of TP, or ML, in the octants, the force EL in the octants, where it is greatest, will exceed the force ML in the ratio of 3 to 2; and therefore will be to that force by which the moon in its periodic time may be retained revolving about the earth at rest as 100 is to $\frac{2}{3} \cdot 17872\frac{1}{2}$, or 11915; and in the time CT will generate a velocity equal to $\frac{100}{11915}$ parts of the velocity of the moon; but in the time CPA will generate a greater velocity in the proportion of CA to CT or TP. Let the greatest force EL in the octants be represented by the area FK • Kk, or by the rectangle VaTP ?p, which is equal thereto; and the velocity which that greatest force can generate in any time CP will be to the velocity which any other lesser force EL can generate in the same time as the rectangle VuTP CP to the area KCGF; but the velocities generated in the whole time CPA will

be one to the other as the rectangle $\frac{1}{2}TP \cdot CA$ is to the triangle TCG , or as the quadrantai arc CA is to the radius TP ; and therefore the latter velocity generated in the whole time will be $100/119215$ parts of the velocity of the moon. To this velocity of the moon, which is proportional to the mean moment of the area (supposing this mean moment to be represented by the number 11915), we add and subtract the half of the other velocity; the sum $11915 + 50$, or 11965, will represent the greatest moment of the area in the syzygy A ; and the difference $11915 - 50$, or 11865, the least moment thereof in the quadratures. Therefore the areas which in equal times are described in the syzygies and quadratures are one to the other as 11965 to 11865. And if to the least moment 11865 we add a moment which shall be to 100, the difference of the two former moments, as the trapezium $FKCG$ is to the triangle TCG , or, which comes to the same thing, as the square of the sine PK is to the square of the radius TP (that is, as Vd to TP), the sum will represent the moment of the area when the moon is in any intermediate place P .

But these things take place only in the hypothesis that the sun and the earth are at rest, and that the synodical revolution of the moon is finished in $27^d. 7^h. 43^m$. But since the moon's synodical period is really $29^d. 12^h. 44^{III}$, the increments of the moments must be enlarged in the same proportion as the time is, that is, in the proportion of 1080853 to 1000000. Upon which account, the whole increment, which was $100/11915$ parts of the mean moment, will now become $100 \cdot 11023$, parts thereof; and therefore the moment of the area in the quadrature of the moon will be to the moment thereof in the syzygy as $11023 - 50$ to $11023 + 50$; or as 10973 to 11073; and to the moment thereof, when the moon is in any intermediate place P , as 10973 to $10973 + Pd$; that is, supposing $TP = 100$.

The area, therefore, which the moon, by a radius drawn to the earth, describes in the several little equal parts of time, is nearly as the sum of the number 219.46, and the versed sine of the double distance of the moon from the nearest quadrature, considered in a circle which hath unity for its radius. Thus it is when the variation in the octants is in its mean quantity. But if the variation there is greater or less, that versed sine must be augmented or diminished in the same proportion.

PROPOSITION XXVII. PROBLEM VIII

From the hourly motion of the moon to find its distance from the earth.

The area which the moon, by a radius drawn to the earth, describes in every moment of time, is as the hourly motion of the moon and the square of the distance of the moon from the earth conjointly. And therefore the distance of the moon from the earth varies directly as the square root of the area and inversely as the square root of the hourly motion, taken jointly. Q.E.I.

COR. 1. Hence the apparent diameter of the moon is given; for it is inversely as the distance of the moon from the earth. Let astronomers try how accurately this rule agrees with the phenomena.

COR. 11. Hence also the orbit of the moon may be more exactly defined from the phenomena than hitherto could be done.

PROPOSITION XXVIII. PROBLEM IX

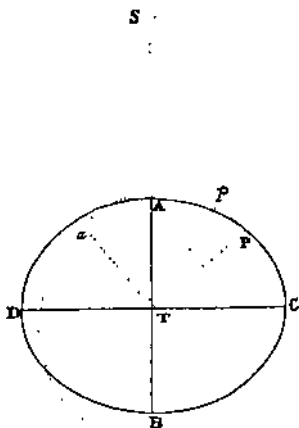
To find the diameters of the orbit, in which, without eccentricity, the moon would move.

The curvature of the orbit which a body describes, if attracted in lines perpendicular to the orbit, is directly as the force of attraction, and inversely as the square of the velocity. I estimate the curvatures of lines compared one with another according to the evanescent ratio of the sines or tangents of their angles of contact to equal radii,¹ supposing those radii to be infinitely diminished. But the attraction of the moon towards the earth in the syzygies is the excess of its gravity towards the earth above the force of the sun 2PK (see fig., Prop. xxv), by which force the accelerative gravity of the moon towards the sun exceeds the accelerative gravity of the earth towards the sun or is exceeded by it. But in the quadratures that attraction is the sum of the gravity of the moon towards the earth, and the sun's force KT, by which the moon is attracted towards the earth. And these attractions, putting N for $\frac{AT + CT}{2}$, are nearly as $\frac{178725}{AT^2} - \frac{2000}{CT N}$ and $\frac{178725}{CT^2} + \frac{1000}{AT N}$, or as $178725N CT^2 - 2000AT^2 CT$, and $178725N AT^2 + 1000CT^2 AT$. For if the accelerative gravity of the moon towards the earth

[¹ Appendix, Note 46]

be represented by the number 178725, the mean force ML, which in the quadratures is PT or TK, and draws the moon towards the earth, will be 1000, and the mean force TM in the syzygies will be 3000; from which, if we subtract the mean force ML, there will remain 2000, the force by which the moon in the syzygies is drawn from the earth, and which we above called 2PK. But the velocity of the moon in the syzygies A and B is to its velocity in the quadratures C and D as CT is to AT, and as the moment of the area, which the moon by a radius drawn to the earth describes in the syzygies, is to the moment of that area *described* in the quadratures conjointly; that is, as 11073CT is to 10973AT. Take the square of this ratio inversely, and the former ratio directly, and the curvature of the moon's orbit in the syzygies will be to the curvature thereof in the quadratures as $120406729 \cdot 178725 AT^2 \cdot CT^2 \cdot N - 120406729 \cdot 2000 AT^4 \cdot CT$ is to $122611329 \cdot 178725 AT^2 \cdot CT^2 \cdot N + 122611329 \cdot 1000 CT^4 \cdot AT$, that is, as $2151969 AT \cdot CT \cdot N - 24081 AT^4$ is to $2191371 AT \cdot CT \cdot N + 12261 CT^3$.

Because the figure of the moon's orbit is unknown, let us, in its stead, assume the ellipse DBCA, in the centre of which we suppose the earth to be situated, and the greater axis DC to lie between the quadratures as the lesser



AB between the syzygies. But since the plane of this ellipse is revolved about the earth by an angular motion, and the orbit, whose curvature we now examine, should be described in a plane void of such motion, we are to consider the figure which the moon, while it is revolved in that ellipse, describes in this plane, that is to say, the figure Cpa, the several points *p* of which are found by assuming any point P in the ellipse, which may represent the place of the moon, and drawing Tp equal to TP in such manner that the angle PTp may be equal to the apparent motion of the sun from the time of the last quadrature

in C; or (which comes to the same thing) that the angle CTp may be to the angle CTP as the time of the synodic revolution of the moon to the time of the periodic revolution thereof, or as $29^d. 12^h. 44^m.$ to $27^d. 7^h. 43^m.$ If, therefore, in this proportion we take the angle CTa to the right angle CTA, and make Ta of equal length with TA , we shall have a the lower and C the upper apse of this orbit Cpa . But, by computation, I find that the difference between the curvature of this orbit Cpa at the vertex a , and the curvature of a circle described about the centre T with the interval TA , is to the difference between the curvature of the ellipse at the vertex A, and the curvature of the same circle, as the square of the ratio of the angle CTP to the angle CTp ; and that the curvature of the ellipse in A is to the curvature of that circle as the square of the ratio of TA is to TC ; and the curvature of that circle is to the curvature of a circle described about the centre T with the radius TC as TC is to TA ; but that the curvature of this *last arch* is to the curvature of the ellipse in C as the square of the ratio of TA is to TC ; and that the difference between the curvature of the ellipse in the vertex C, and the curvature of this last circle, is to the difference between the curvature of the figure Tpa , at the vertex C, and the curvature of this same *last circle*, as the square of the ratio of the angle CTp to the angle CTP. All these relations are easily derived from the sines of the angles of contact, and of the differences of those angles. But, by comparing those ratios, we find the curvature of the figure Cpa at a to be to its curvature at C as $AT^3 - \frac{16821}{100000} CT^2 \cdot AT$ is to $CT^3 + \frac{16824}{100000} AT^2 \cdot CT$; where the number 100000 represents the difference of the squares of the angles CTP and CTp , divided by the square of the lesser angle CTP; or (which is all one) the difference of the squares of the times $27^d. 7^h. 43^m.$ and $29^d. 12^h. 44^m.$ divided by the square of the time $27^d. 7^h. 43^m.$

Since, therefore, a represents the syzygy of the moon, and C its quadrature, the ratio now found must be the same as the ratio of the curvature of the moon's orb in the syzygies to the curvature thereof in the quadratures, which we found above. Therefore, in order to find the ratio of CT to AT, let us multiply the extremes and the means of the resulting proportion, and the terms which come out, divided by $AT \cdot CT$, yield the following equation: $2062.79CT^4 - 2151969N CT^3 + 368676N \cdot AT CT^2 + 36342AT^2 CT^2 - 362047N AT^2 CT + 2191371N AT^3 + 4051.4AT^4 = 0$. Now if for the

half sum N of the terms AT and CT we put 1, and x for their half difference, then CT will be $= 1 + x$, and $AT = 1 - x$. And substituting those values in the equation, after resolving thereof, we shall find $\# = 0.00719$; and from thence the semidiameter CT $= 1.00719$, and the semidiameter AT $= 0.99281$, which numbers are nearly as $70^{1/24}$, and $69^{1/24}$. Therefore the moon's distance from the earth in the syzygies is to its distance in the quadratures (setting aside the consideration of eccentricity) as $69^{1/24}$ to $70^{1/24}$; or, in round numbers, as 69 to 70.

PROPOSITION XXIX. PROBLEM X

To find the variation of the moon.

This inequality is due partly to the elliptic figure of the moon's orbit, partly to the inequality of the moments of the area which the moon by a radius drawn to the earth describes. If the moon P revolved in the ellipse DBCA about the earth quiescent in the centre of the ellipse, and by the radius TP, drawn to the earth, described the area CTP, proportional to the time of description; and the greatest semidiameter CT of the ellipse was to the least TA as 70 to 69; the tangent of the angle CTP would be to the tangent of the angle of the mean motion, computed from the quadrature C, as the semidiameter TA of the ellipse to its semidiameter TC, or as 69 to 70. But the description of the area CTP as the moon advances from the quadrature to the syzygy, ought to be in such manner accelerated, that the moment of the area in the moon's syzygy may be to the moment thereof in its quadrature as 11073 to 10973; and that the excess of the moment in any intermediate place P above the moment in the quadrature may be as the square of the sine of the angle CTP; which we may effect with accuracy enough, if we diminish the tangent of the angle CTP in the ratio obtained from the square root of the ratio of the number 10973 to the number 11073, that is, in the ratio of the number 68.6877 to the number 69. On this account the tangent of the angle CTP will now be to the tangent of the mean motion as 68.6877 is to 70; and the angle CTP in the octants, where the mean motion is 45^0 , will be found $44^0\ 27'\ 28''$, which subtracted from 45^0 , the angle of the mean motion, leaves the greatest variation $32'\ 32''$. Thus it would be, if the moon, in passing from the quadrature to the syzygy, de-

scribed an angle CTA of 90° only. But because of the motion of the earth, by which the sun is apparently transferred forwards, the moon, before it overtakes the sun, describes an angle CTa , greater than a right angle, in the ratio of the time of the synodic revolution of the moon to the time of its periodic revolution, that is, in the ratio of $29^d. 12^h. 44^m.$ to $27^d. 7^h. 43^m.$ Whence it comes to pass that all the angles about the centre T are dilated in the same ratio; and the greatest variation, which otherwise would be *but* $32' 32''$, now augmented in the said proportion, becomes $35' 10''$.

And this is its magnitude in the mean distance of the sun from the earth, neglecting the differences which may arise from the curvature of the great orbit, and the stronger action of the sun upon the moon when horned and new, than when gibbous and full. In other distances of the sun from the earth, the greatest variation is in a ratio compounded, directly of the square of the ratio of the time of the synodic revolution of the moon (the time of the year being given), and inversely as the cube of the ratio of the distance of the sun from the earth. And, therefore, in the apogee of the sun, the greatest variation is $33' 14''$, and in its perigee $37' n''$, if the eccentricity of the sun is to the transverse semidiameter of the great orbit as $16' A0$ to 1000.

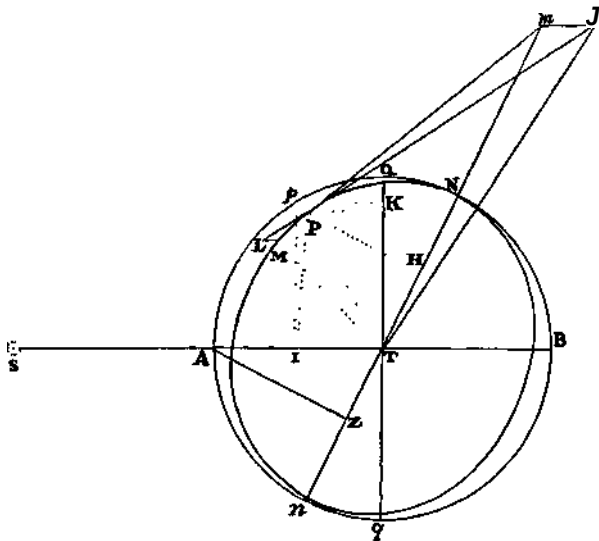
Hitherto we have investigated the variation in an orbit not eccentric, in which, to wit, the moon in its octants is always in its mean distance from the earth. If the moon, on account of its eccentricity, is more or less removed from the earth than if placed in this orbit, the variation may be something greater, or something less, than according to this rule. But I leave the excess or defect to the determination of astronomers from the phenomena.

PROPOSITION XXX. PROBLEM XI

To find the hourly motion of the nodes of the moon in a circular orbit.

Let S represent the sun, T the earth, P the moon, $NP\omega$ the orbit of the moon, Npn the orthographic projection of the orbit upon the plane of the ecliptic; N, n the nodes, $nTN\omega$ the line of the nodes produced indefinitely; PI, PK perpendiculars upon the lines ST, Qq ; Vp a perpendicular upon the plane of the ecliptic; A, B the moon's syzygies in the plane of the ecliptic; AZ a perpendicular let fall upon Nn , the line of the nodes; Q, q the quadra-

tures of the moon in the plane of the ecliptic, and pK . a perpendicular on the line Qq lying between the quadratures. The force of the sun to disturb the motion of the moon (by Prop, xxv) is twofold, one proportional to the



line LM , the other to the line MT , in the scheme of that Proposition; and the moon by the former force is drawn towards the earth, by the latter towards the sun, in a direction parallel to the right line ST joining the earth and the sun. The former force LM acts in the direction of the plane of the moon's orbit, and therefore makes no change upon the situation thereof, and is upon that account to be neglected; the latter force MT , by which the plane of the moon's orbit is disturbed, is the same with the force $3PK$ or $3IT$. And this force (by Prop, xxv) is to the force by which the moon may, in its periodic times, be uniformly revolved in a circle about the earth at rest, as $3IT$ to the radius of the circle multiplied by the number 178.725, or as IT to the radius thereof multiplied by 59.575. But in this calculus, and all that follows, I consider all the lines drawn from the moon to the sun as parallel

to the line which joins the earth and the sun; because what inclination there is almost as much diminishes all effects in some cases as it augments them in others; and we are now inquiring after the mean motions of the nodes, neglecting such niceties as are of no moment and would only serve to render the calculus more complicated.

Now suppose PM to represent an arc which the moon describes in the least moment of time, and ML a little line, the half of which the moon, by the impulse of the said force 3IT, would describe in the same time; and joining PL, MP, let them be produced to m and $/$, where they cut the plane of the ecliptic, and upon Tm let fall the perpendicular PH. Now, since the right line ML is parallel to the plane of the ecliptic, and therefore can never meet with the right line ml which lies in that plane, and yet both those right lines be in one common plane LMPw/, they will be parallel, and upon that account the triangles LMP, Imp will be similar. And seeing MPra lies in the plane of the orbit, in which the moon did move while in the place P, the point m will fall upon the line N«, which passes through the nodes N, n , of that orbit. And because the force by which the half of the little line LM is generated, if the whole had been together, and at once impressed in the point P, would have generated that whole line, and caused the moon to move in the arc whose chord is LP; that is to say, would have transferred the moon from the plane MP/wT into the plane LPIT; therefore the angular motion of the nodes generated by that force will be equal to the angle mXl . But ml is to $m?$ as ML to MP; and since MP, because of the time given, is also given, ml will be as the rectangle ML /wP, that is, as the rectangle IT $m?$. And if Tml is a right angle, the angle mTl will be as $\frac{ml}{Tm}$, and therefore as $\frac{IT \cdot Pm}{Tm}$, that is (because Tm and mP , TP and PH are proportional), as $\frac{IT \cdot PH}{Tp}$; and, therefore, because TP is given, as IT PH.

But if the angle Tml or STN is oblique, the angle mTl will be yet less, in proportion of the sine of the angle STN to the radius, or AZ to AT. And therefore the velocity of the nodes is as $IT \cdot PH \cdot AZ$, or as the product of the sines of the three angles TPI, PTN, and STN.

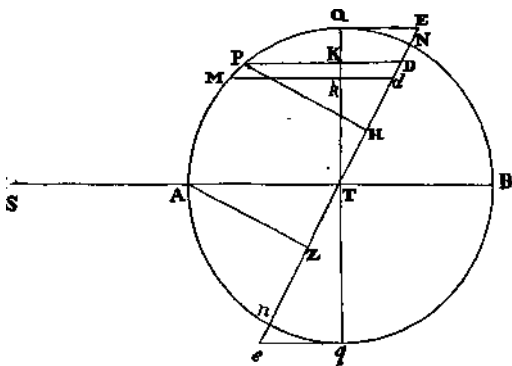
If these are right angles, as happens when the nodes are in the quadratures, and the moon in the syzygy, the little line ml will be removed to an

infinite distance, and the angle mXI will become equal to the angle $m?l$. But in this case the angle mVI is to the angle PTM , which the moon in the same time by its apparent motion describes about the earth, as i to 59.575. For the angle mVI is equal to the angle LPM , that is, to the angle of the moon's deflection from a rectilinear path; which angle, if the gravity of the moon should have then ceased, the said force of the sun $3IT$ would by itself have generated in that given time; and the angle PTM is equal to the angle of the moon's deflection from a rectilinear path; which angle, if the force of the sun $3IT$ should have then ceased, the force alone by which the moon is retained in its orbit would have generated in the same time. And these forces (as we have above shown) are the one to the other as 1 to 59.575. Since, therefore, the mean hourly motion of the moon (in respect of the fixed stars) is $32^m 56^s 27^{th} 12V2^{1V}$ the hourly motion of the node in this case will be $33^s 10^{th} 33^{IV} 12^V$. But in other cases the hourly motion will be to $33^s 10^{th} 33^{IV} 12^V$ as the product of the sines of the three angles TPI , PTN , and STN (or of the distances of the moon from the quadrature, of the moon from the node, and of the node from the sun) to the cube of the radius. And as often as the sine of any angle is changed from positive to negative, and from negative to positive, so often must the regressive be changed into a progressive, and the progressive into a regressive motion. Whence it comes to pass that the nodes are progressive as often as the moon happens to be placed between either quadrature, and the node nearest to that quadrature. In other cases they are regressive, and by the excess of the regress above the progress, they are monthly transferred backwards.

COR. 1. Hence if from P and M , the extreme points of a least arc PM , on the line Qq joining the quadratures we let fall the perpendiculars PK , $M^\wedge$, and produce the same till they cut the line of the nodes $N\#$ in D and d , the hourly motion of the nodes will be as the area $MPDd$, and the square of the line AZ , conjointly. For let PK , PH , and AZ be the three said sines, viz., PK the sine of the distance of the moon from the quadrature, PH the sine of the distance of the moon from the node, and AZ the sine of the distance of the node from the sun; and the velocity of the node will be as the product $PK \cdot PH \cdot AZ$. But PT is to PK as PM to $K\{$; and, therefore, because PT and PM are given, Kk will be as PK . Likewise AT is to PD as AZ is to PH , and therefore PH is as the rectangle $PD \cdot AZ$; and, by compounding those pro-

portions, $PK \cdot PH$ is as the solid content $Kk \cdot PD \cdot AZ$, and $PK \cdot PH \cdot AZ$ as $Kk \cdot PD \cdot AZ^2$; that is, as the area $PDdM$ and AZ^2 conjointly. Q.E.D.

COR. II. In any given position of the nodes their mean hourly motion is half their hourly motion in the moon's syzygies; and therefore is to $16^{\text{s}} 35^{\text{th}} 16^{\text{v}} 36^{\text{v}}$ as the square of the sine of the distance of the nodes from



the syzygies is to the square of the radius, or as AZ^2 to AT^2 . For if the moon, by an uniform motion, describes the semicircle QAq , the sum of all the areas $VDdM$, during the time of the moon's passage from Q to M , will make up the area $QMdE$, terminating at the tangent QE of the circle; and by the time that the moon has arrived at the point n , that sum will make up the whole area $EQAn$ described by the line PD : but when the moon proceeds from n to q , the line PD will fall without the circle, and describe the area nqe , terminating at the tangent qe of the circle, which area, because the nodes were before regressive, but are now progressive, must be subtracted from the former area, and, being itself equal to the area QEN , will leave the semicircle NQA . While, therefore, the moon describes a semicircle, the sum of all the areas $PDdM$ will be the area of that semicircle; and while the moon describes a complete circle, the sum of those areas will be the area of the whole circle. But the area $PDdM$, when the moon is in the syzygies, is the rectangle of the arc PM into the radius PT ; and the sum of all the areas, *every one* equal to this area, in the time that the moon de-

scribes a complete circle, is the rectangle of the whole circumference into the radius of the circle; and this rectangle, being double the area of the circle, will be double the former sum. If, therefore, the nodes went on with that velocity uniformly continued which they acquire in the moon's syzygies, they would describe a space double that which they describe in fact; and, therefore, the mean motion, by which, if uniformly continued, they would describe the same space with that which they do in fact describe by an unequal motion, is *but* one-half of that motion which they are possessed of in the moon's syzygies. Wherefore, since their greatest hourly motion, if the nodes are in the quadratures, is $33^s 10^h 33^{iv} 12^v$, their mean hourly motion in this case will be $16^s 35^h 16^{lv} 36^v$. And seeing the hourly motion of the nodes is everywhere as AZ^2 and the area $VDdM$ conjointly, and, therefore, in the moon's syzygies, the hourly motion of the nodes is as AZ^2 and the area $PDdM$ conjointly, that is (because the area $PDdM$ described in the syzygies is given), as AZ'' , therefore the mean motion also will be as AZ^2 ; and, therefore, when the nodes are without the quadratures, this motion will be to *ifr* $35^h 16^{lv} 36^*$ as AZ^2 to AT^2 . Q.E.D.

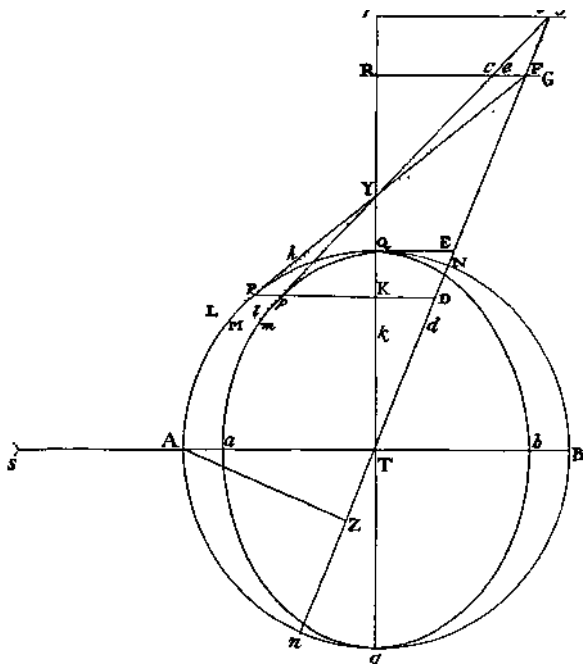
PROPOSITION XXXI. PROBLEM XII

To find the hourly motion of the nodes of the moon in an elliptic orbit.

Let $Qpmaq$ represent an ellipse described with the greater axis Qq and the less axis ab \ $QA\#B$ a circle circumscribed; T the earth in the common centre of both; S the sun; p the moon moving in this ellipse; and pm an arc which it describes in the least moment of time; N and n the nodes joined by the line $N\llcorner$; pK and m perpendiculars upon the axis Qq , produced both ways till they meet the circle in P and M , and the line of the nodes in D and d . And if the moon, by a radius drawn to the earth, describes an area proportional to the time of description, the hourly motion of the node in the ellipse will be as the area $pDdm$ and AZ^2 conjointly.

For let PF touch the circle in P , and produced meet TN in F ; and pf touch the ellipse in p , and produced meet the same TN in f , and both tangents concur in the axis TQ at Y ; and let ML represent the space which the moon, by the impulse of the above-mentioned force $3IT$ or $3PK$, would describe with a transverse motion, in the meantime while revolving in the

circle it describes the arc PM; and *ml* denote the space which the moon revolving in the ellipse would describe in the same time by the impulse of the same force 3IT or 3PK; and let LP and *lp* be produced till they meet the plane of the ecliptic in G and *g*, and FG and *fg* be joined, of which FG



produced may cut pf , pg , and TQ, in c , e , and R respectively; and fg produced may cut TQ in r . Because the force 3IT or 3PK in the circle is to the force 3IT or 3pK in the ellipse as PK to pK, or as AT to aT , the space ML generated by the former force will be to the space ml generated by the latter as PK to pK' , that is, because of the similar figures PYKp and FYRc, as FR to cR. But (because of the similar triangles PLM, PGF) ML is to FG as

PL is to PG, that is (on account of the parallels Lk, PK, GR), as *pi* is to *pe*, that is (because of the similar triangles *plm*, *cpe*), as *Im* is to *ce*; and inversely as LM is to *Im*, or as FR is to cR, so is FG to *ce*. And therefore if *fg* was to *ce* as *fy* to cY, that is, as *fr* to cR (that is, as *fr* to FR and FR to cR conjointly, that is, as *ft* to FT, and FG to *ce* conjointly), because the ratio of FG to *ce*, expunged on both sides, leaves the ratios *fg* to FG and /T to FT, *fg* would be to FG as *ft* to FT; and, therefore, the angles which FG and *fg* would subtend at the earth T would be equal to each other. But these angles (by what we have shown in the preceding Proposition) are the motions of the nodes, while the moon describes in the circle the arc PM, in the ellipse the arc *pm*; and therefore the motions of the nodes in the circle and in the ellipse would be equal to each other. Thus, I say, it would be, if *fg* was to *ce* as /Y to cY, that is, if *fg* was equal to $\frac{ce \cdot fY}{cY}$. But because of the similar triangles *fgp*, *cep*, *fg* is to *ce* as *fp* to *cp*; and therefore *fg* is equal to $\frac{ce \cdot fp}{cp}$; and therefore the angle which *fg* subtends in fact is to the former angle which FG subtends, that is to say, the motion of the nodes in the ellipse is to the motion of the same in the circle as this *fg* or $\frac{ce \cdot fp}{cp}$ to the former *fg* or $\frac{ce \cdot fY}{cY}$, that is, as *fp* cY to /Y cp, or as *fp* to /Y, and cY to *cp*; that is, if *ph* parallel to TN meet FP in *h*, as *Fh* to FY and FY to FP; that is, as FH to FP or Dp to DP, and therefore as the area *Dpmd* to the area *DPMd*. And, therefore, seeing (by Cor. 1, Prop. xxx) the latter area and AZ" conjointly are proportional to the hourly motion of the nodes in the circle, the former area and AZ² conjointly will be proportional to the hourly motion of the nodes in the ellipse. Q.E.D.

COR. Since, therefore, in any given position of the nodes, the sum of all the areas *pDdm*, in the time while the moon is carried from the quadrature to any place *m*, is the area *mpQEd* terminated at the tangent of the ellipse QE; and the sum of all those areas, in one entire revolution, is the area of the whole ellipse; the mean motion of the nodes in the ellipse will be to the mean motion of the nodes in the circle as the ellipse to the circle; that is, as *Ta* to TA, or 69 to 70. And, therefore, since (by Cor. 11, Prop. xxx)

the mean hourly motion of the nodes in the circle is to $16^s 35^{\text{th}} 16^{\text{lv}} 36^{\text{v}}$ as AZ'' to AT^2 , if we take the angle $16^s 21^{\text{th}} 3^{\text{lv}} 30^{\text{v}}$ to the angle $16^s 35^{\text{th}} 6^{\text{lv}} 36^{\text{v}}$ as 69 to 70, the mean hourly motion of the nodes in the ellipses will be to $16^s 21^{\text{th}} 3^{\text{lv}} 30^{\text{v}}$ as AZ'' to AT'' ; that is, as the square of the sine of the distance of the node from the sun to the square of the radius.

But the moon, by a radius drawn to the earth, describes the area in the syzygies with a greater velocity than it does that in the quadratures, and upon that account the time is contracted in the syzygies, and prolonged in the quadratures; and together with the time the motion of the nodes is likewise augmented or diminished. But the moment of the area in the quadratures of the moon was to the moment thereof in the syzygies as 10973¹⁰ 11073; and therefore the mean moment in the octants is to the excess in the syzygies, and to the defect in the quadratures, as 11023, the half-sum of those numbers, is to their half-difference 50. Wherefore, since the time of the moon in the several little equal parts of its orbit is inversely as its velocity, the mean time in the octants will be to the excess of the time in the quadratures, and to the defect *of the time* in the syzygies arising from this cause, nearly as 11023 to 50. But, reckoning from the quadratures to the syzygies, I find that the excess of the moments of the area, in the several places above the least moment in the quadratures, is nearly as the square of the sine of the moon's distance from the quadratures; and therefore the difference between the moment in any place, and the mean moment in the octants, is as the difference between the square of the sine of the moon's distance from the quadratures, and the square of the sine of 45 degrees, or half the square of the radius; and the increment of the time in the several places between the octants and quadratures, and the decrement thereof between the octants and syzygies, is in the same proportion. But the motion of the nodes, while the moon describes the several little equal parts of its orbit, is accelerated or retarded as the square of the time; for that motion, while the moon describes PM, is (other things being equal) as ML, and ML varies as the square of the time. Wherefore, the motion of the nodes in the syzygies, in the time while the moon describes given little parts of its orbit, is diminished as the square of the ratio of the number 11073 to the number 11023; and the decrement is to the remaining motion as 100 to 10973; but to the whole motion is as 100 to 11073, nearly. But the decre-

ment in the places between the octants and syzygies, and the increment in the places between the octants and quadratures, is to this decrement nearly as the whole motion in these places to the whole motion in the syzygies, and the difference between the square of the sine of the moon's distance from the quadrature, and the half-square of the radius, is to the half-square of the radius conjointly. Wherefore, if the nodes are in the quadratures, and we take two places, one on one side, one on the other, equally distant from the octant and other two distant by the same interval, one from the syzygy, the other from the quadrature, and from the decrements of the motions in the two places between the syzygy and octant we subtract the increments of the motions in the two other places between the octant and the quadrature, the remaining decrement will be equal to the decrement in the syzygy, as will easily appear by computation; and therefore the mean decrement, which ought to be subtracted from the mean motion of the nodes, is the fourth part of the decrement in the syzygy. The whole hourly motion of the nodes in the syzygies (when the moon by a radius drawn to the earth was supposed to describe an area proportional to the time) was $32^s 42^h 7^{IV}$. And we have shown that the decrement of the motion of the nodes, in the time while the moon, now moving with greater velocity, describes the same space, was to this motion as 100 to 11073; and therefore this decrement is $17^{th} 43^{iv} 11^v$. The fourth part of which $4^{th} 25^{iv} 48^*$ subtracted from the mean hourly motion above found, $16^s 21^{th} 3^v 30^v$, leaves $16^s 16^{th} 37^{iv} 42^v$, their correct mean hourly motion.

If the nodes are without the quadratures, and two places are considered, one on one side, one on the other, equally distant from the syzygies, the sum of the motions of the nodes, when the moon is in those places, will be to the sum of their motions, when the moon is in the same places and the nodes in the quadratures, as AZ^2 to AT^2 . And the decrements of the motions arising from the causes but now explained will be mutually as the motions themselves, and therefore the remaining motions will be mutually between themselves as AZ^1 to AT^1 ; and the mean motions will be as the remaining motions. And, therefore, in any given position of the nodes, their correct mean hourly motion is to $16^s 16^{th} 37^{iv} 42^v$ as AZ^1 to AT^1 ; that is, as the square of the sine of the distance of the nodes from the syzygies to the square of the radius.

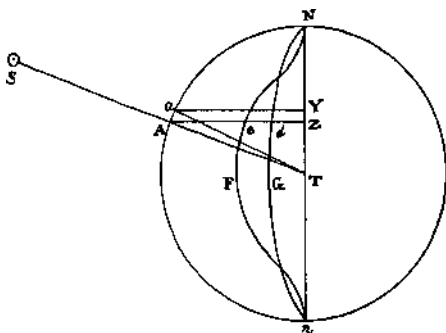
PROPOSITION XXXII. PROBLEM XIII

To find the mean motion of the nodes of the moon.

The yearly mean motion is the sum of all the mean hourly motions throughout the course of the year. Suppose that the node is in N, and that, after every hour is elapsed, it is drawn back again to its former place; so that, notwithstanding its proper motion, it may constantly remain in the same situation with respect to the fixed stars; while in the meantime the sun S, by the motion of the earth, is seen to leave the node, and to proceed till it completes its apparent annual course by an uniform motion. Let *ha* represent a given least arc, which the right line TS always drawn to the sun, by its intersection with the circle NA «, describes in the least given moment of time; and the mean hourly motion (from what we have above shown) will be as AZ^2 , that is (because AZ and ZY are proportional), as the rectangle of AZ into ZY, that is, as the area AZYa; and the sum of all the mean hourly motions from the beginning will be as the sum of all the areas OYZA, that is, as the area NAZ. But the greatest AZY^t is equal to the rectangle of the arc *ha* into the radius of the circle; and therefore the sum of all these rectangles in the whole circle will be to the like sum of all the greatest rectangles as the area of the whole circle to the rectangle of the whole circumference into the radius, that is, as 1 to 2. But the hourly motion corresponding to that greatest rectangle was $16^{\circ} 16^{\text{th}} 27^{\text{th}} 42^{\text{v}}$ and this motion in the complete course of the sidereal year, $365^{\text{d}}. 6^{\text{h}}. 9^{\text{m}}$, amounts to $39^{\circ} 38' 7'' 50'''$; and therefore the half of the $19^{\circ} 49' 3'' 55'''$, is the mean motion of the nodes corresponding to the whole circle. And the motion of the nodes, in the time while the sun is carried from N to A, is to $19^{\circ} 49' 3'' 55'''$ as the area NAZ to the whole circle.

Thus it would be if the node was after every hour drawn back again to its former place, that so, after a complete revolution, the sun at the year's end would be found again in the same node which it had left when the year began. But, because of the motion of the node in the meantime, the sun must needs meet the node sooner; and now it remains that we compute the abbreviation of the time. Since, then, the sun, in the course of the year, travels 360 degrees, and the node in the same time by its greatest motion would be carried $39^{\circ} 38' 7'' 50'''$, or 39.6355 degrees; and the mean motion

of the node in any place N is to its mean motion in its quadratures as AZ^2 to AT^2 ; the motion of the sun will be to the motion of the node in N as $360 AT^2$ to $39.6355AZ^2$; that is, as $9.0827646AT^2$ to AZ^2 . Therefore, if we



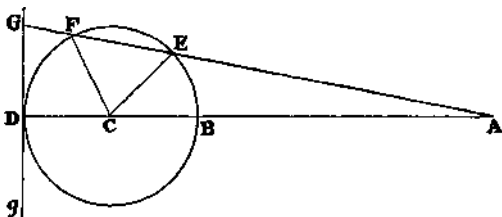
suppose the circumference NAn of the whole circle to be divided into little equal parts, such as Aa , the time in which the sun would describe the little arc Aa , if the circle was quiescent, will be to the time in which it would describe the same arc, supposing the circle together with the nodes to be revolved about the centre T , inversely as $9.0827646AT^2$ to $9.0827646AT^2 + AZ^2$; for the time is inversely as the velocity with which the little arc is described, and this velocity is the sum of the velocities of both sun and node. If, therefore, the sector NTA represent the time in which the sun by itself, without the motion of the node, would describe the arc NA , and the indefinitely small part ATa of the sector represent the little moment of the time in which it would describe the least arc Aa ; and (letting fall aY perpendicular upon Nn) if in AZ we take dZ of such length that the rectangle of dZ into ZY may be to the least part ATa of the sector as AZ^2 to $9.0827646AT^2 + AZ^2$; that is to say, that dL may be to $\frac{1}{2}AZ$ as AT^2 to $5.0827646AT^2 + AZ^2$; the rectangle of dZ into ZY will represent the decrement of the time arising from the motion of the node, while the arc Aa is described; and if the curve $NdGn$ is the locus where the point d is always found, the curvilinear area NdZ will be as the whole decrement of time

while the whole arc NA is described; and, therefore, the excess of the sector NAT above the area NdZ will be as the whole time. But because the motion of the node in a less time is less in proportion to the time, the area AaYZ must also be diminished in the same proportion; which may be done by taking in AZ the line eZ of such length, that it may be to the length of AZ as, AZ^2 to $9.0827646AT^2 + AZ^2$; for so the rectangle of eZ into ZY will be to the area AZYa as the decrement of the time in which the arc ha is described to the whole time in which it would have been described, if the node had been quiescent; and, therefore, that rectangle will be as the decrement of the motion of the node. And if the curve NeFn is the locus of the point e, the whole area NeZ, which is the sum of all the decrements *of that motion*, will be as the whole decrement *thereof* during the time in which the arc AN is described; and the remaining area NAc will be as the remaining motion, which is the true motion of the node, during the time in which the whole arc NA is described by the joint motions of both sun and node. Now the area of the semicircle is to the area of the figure NeF« found by the method of infinite series nearly as 793 to 60. But the motion corresponding or proportional to the whole circle was $19^{\circ} 49' 3'' 55'''$; and therefore the motion corresponding to double the figure Nefn is $1^{\circ} 29' 58'' 2'''$, which taken from the former motion leaves $18^{\circ} 19' 5'' 53'''$, the whole motion of the node with respect to the fixed stars in the interval between two of its conjunctions with the sun; and this motion subtracted from the annual motion of the sun, 360° , leaves $341^{\circ} 40' 54'' 7'''$, the motion of the sun in the interval between the same conjunctions. But as this motion is to the annual motion 360° , so is the motion of the node but just now found $18^{\circ} 19' 5'' 53'$ to its annual motion, which will therefore be $19^{\circ} 18' 1'' 23'''$; and this is the mean motion of the nodes in the sidereal year. By astronomical tables, it is $19^{\circ} 21' 21'' 50'''$. The difference is less than $1/300$ part of the whole motion, and seems to arise from the eccentricity of the moon's orbit, and its inclination to the plane of the ecliptic. By the eccentricity of this orbit the motion of the nodes is too much accelerated; and, on the other hand, by the inclination of the orbit, the motion of the nodes is somewhat retarded, and reduced to its just velocity.

PROPOSITION XXXIII. PROBLEM XIV

To find the true motion of the nodes of the moon.

In the time which is as the area $NTA - WZ$ (in the preceding fig.) that motion is as the area $NA^\wedge$, and hence is given; but because the calculus is too difficult, it will be better to use the following construction of the Problem. About the centre C , with any radius CD , describe the circle $BEFD$;



produce DC to A so as AB may be to AC as the mean motion to half the mean true motion when the nodes are in their quadratures (that is, as $19^\circ 18' 1'' 23'''$ to $19^\circ 49' 3'' 55'''$; and therefore BC is to AC as the difference of those motions $0^\circ 31' 2'' 32'''$ to the latter motion $19^\circ 40' 3'' 55'''$, that is, as 1 to 38%0). Then through the point D draw the indefinite line Gg , touching the circle in D ; and if we take the angle BCE , or BCF , equal to the double distance of the sun from the place of the node, as found by the mean motion, and drawing AE or AF cutting the perpendicular DG in G , we take another angle which shall be to the whole motion of the node in the interval between its syzygies (that is, to $9^\circ 11' 3''$) as the tangent DG to the whole circumference of the circle BED , and add this *last* angle (for which the angle DAG may be used) to the mean motion of the nodes, while they are passing from the quadratures to the syzygies, and subtract it from their mean motion while they are passing from the syzygies to the quadratures, we shall have their true motion; for the true motion so found will nearly agree with the true motion which comes out from assuming the times as the area $NTA - N^\wedge Z$, and the motion of the node as the area $NA^\wedge$; as anyone who chooses to examine and make the computations will find: and this is the semimenstrual equation of the motion of the nodes. But there is also

a menstrual equation, but which is by no means necessary for finding of the moon's latitude; for since the variation of the inclination of the moon's orbit to the plane of the ecliptic is liable to a twofold inequality, the one semimenstrual, the other menstrual, the menstrual inequality *of this variation*, and the menstrual equation of the nodes, so moderate and correct each other, that in computing the latitude of the moon both may be neglected.

COR. From this and the preceding Proposition it appears that the nodes are quiescent in their syzygies, but regressive in their quadratures, by an hourly motion of $16^{\text{h}} 10^{\text{th}} 26^{\text{iv}}$; and that the equation of the motion of the nodes in the octants is $1^{\circ} 30'$; all of which exactly agree with the phenomena of the heavens.

SCHOLIUM

Mr. *Machin*, Professor *Gresham*, and Dr. *Henry Pemberton*, separately found out the motion of the nodes by a different method. Mention has been made of this method in another place. Their papers, which I have seen, contained two Propositions, and exactly agreed with each other in both of them. Mr. *Machin's* paper, coming first to my hands, I shall here insert.

THE MOTION OF THE MOON'S NODES

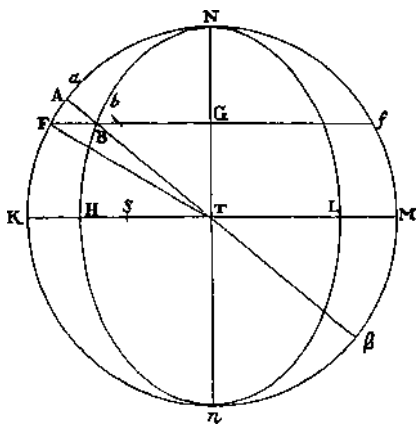
"PROPOSITION I

"The mean motion of the sun from the node is defined by a geometric mean proportional between the mean motion of the sun and that mean motion with which the sun recedes with the greatest swiftness from the node in the quadratures.

"Let T be the earth's place, N the line of the moon's nodes at any given time, KT a perpendicular thereto, TA a right line revolving about the centre with the same angular velocity with which the sun and the node recede from each other, in such sort that the angle between the quiescent right line N and the revolving line TA may be always equal to the distance of the places of the sun and node. Now if any right line TK be divided into parts TS and SK , and those parts be taken as the mean hourly motion of the sun to the mean hourly motion of the node in the quadratures, and there be taken the right line TH , a mean proportional between the part TS and the whole TK , this right line will be proportional to the sun's mean motion from the node.

"For let there be described the circle $NK \cup M$ from the centre T and with the radius TK , and about the same centre, with the semiaxes TH and TN , let there be described an ellipse $NHnL$; and in the time in which the sun recedes from the node through the arc Ntf , if there be drawn the right line Tba , the area of the sector $NTtf$ will be the exponent of the sum of the motions of the sun and node in the same time. Let, therefore, the extremely small arc ah be that which the right line Tba , revolving according to the aforesaid law, will uniformly describe in a given interval of time, and the extremely small sector TAa will be as the sum of the velocities with which the sun and node are carried two different ways in that time. Now the sun's velocity is almost uniform, its inequality being so small as scarcely to pro-

duce the least inequality in the mean motion of the nodes. The other part of this sum, namely, the mean quantity of the velocity of the node, is increased in the recess from the syzygies in a squared ratio of the sine of its distance from the sun (by Cor., Prop. xxxi of this Book), and, being greatest in its quadratures with the sun in K, is in the same ratio to the sun's velocity as SK to TS, that is, as (the difference of the squares of TK and TH, or) the rectangle KHM to TH². But the ellipse NBH divides the sector ATa, the exponent of the sum of these two velocities, into two parts ABba and



Btb, proportional to the velocities. For, produce BT to the circle in P, and from the point B let fall upon the greater axis the perpendicular BG, which being produced both ways may meet the circle in the points F and f, and because the space *ABba* is to the sector *TBb* as the rectangle *ABP* is to BT" (that rectangle being equal to the difference of the squares of TA and TB, because the right line A? is equally cut in T, and unequally in B), therefore when the space *ABba* is the greatest of all in K, this ratio will be the same as the ratio of the rectangle K H M to HT². But the greatest mean velocity of the node was shown above to be in that very ratio to the velocity of the sun; and therefore in the quadratures the sector *ATa* is divided into parts

proportional to the velocities. And because the rectangle KHM is to HT^2 as FBf to BG^2 , and the rectangle $AB\beta$ is equal to the rectangle FBf , therefore the little area $ABba$, where it is greatest, is to the remaining sector TBb as the rectangle $AB\epsilon$ is to BG^2 . But the ratio of these little areas always was as the rectangle $AB\epsilon$ to BT^2 ; and therefore the little area $ABba$ in the place A is less than its correspondent little area in the quadratures in the squared ratio of BG to BT , that is, in the squared ratio of the sine of the sun's distance from the node. And therefore the sum of all the little areas $ABba$, namely, the space ABN , will be as the motion of the node in the time in which the sun hath been going over the arc NA since he left the node; and the remaining space, namely, the elliptic sector NTB , will be as the sun's mean motion in the same time. And because the mean annual motion of the node is that motion which it performs in the time that the sun completes one period of its course, the mean motion of the node from the sun will be to the mean motion of the sun itself as the area of the circle is to the area of the ellipse; that is, as the right line TK to the right line TH , which is a mean proportional between TK and TS ; or, which comes to the same, as the mean proportional TH to the right line TS .

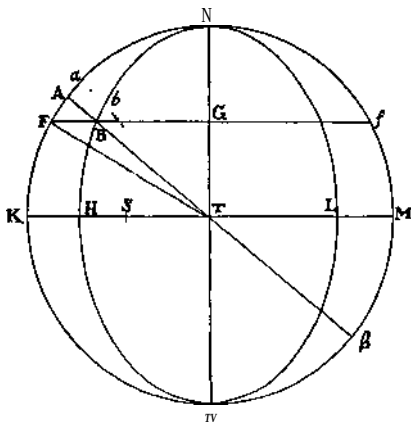
" PROPOSITION II

"The mean motion of the moon's nodes being given, to find their true motion.

"Let the angle A be the distance of the sun from the mean place of the node, or the sun's mean motion from the node. Then if we take the angle B , whose tangent is to the tangent of the angle A as TH to TK , that is, as the square root of the ratio of the mean hourly motion of the sun to the mean hourly motion of the sun from the node, when the node is in the quadrature, that angle B will be the distance of the sun from the node's true place. For join FT , and, by the demonstration of the last Proposition, the angle FTN will be the distance of the sun from the mean place of the node, and the angle ATN the distance from the true place, and the tangents of these angles are between themselves as TK to TH .

"COR. Hence the angle FTA is the equation of the moon's nodes; and the sine of this angle, where it is greatest in the octants, is to the radius as KH

is to $TK + TH$. But the sine of this equation in any other place A is to the greatest sine as the sine of the sums of the angles $FTN + ATN$ is to the



radius; that is, nearly as the sine of double the distance of the sun from the mean place of the node (namely, $2FTN$) to the radius.

"SCHOLIUM

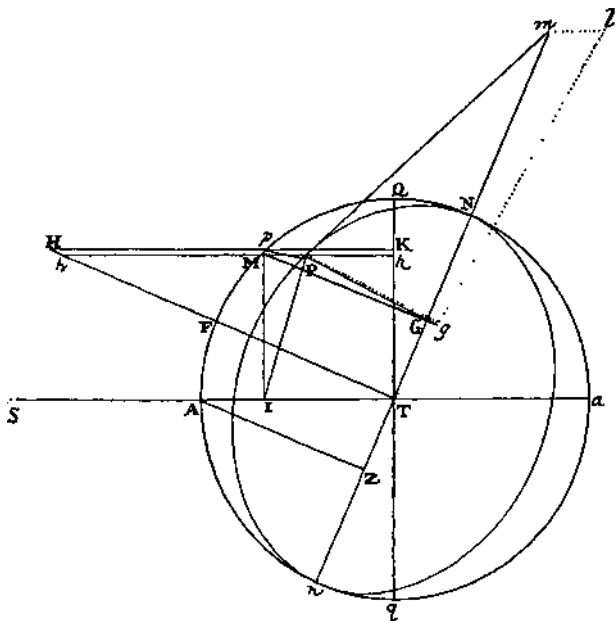
"If the mean hourly motion of the nodes in the quadratures be $16'' 16''' 37^{IV} 42^V$, that is, in a whole sidereal year, $39^0 38' 7'' 50'''$, TH will be to TK as the square root of the ratio of the number 9.0827646 to the number 10.0827646, that is, as 18.6524761 to 19.6524761. And, therefore, TH is to HK as 18.6524761 to 1; that is, as the motion of the sun in a sidereal year to the mean motion of the node $19^0 18' 1'' 23^{2/3}'''$.

"But if the mean motion of the moon's nodes in 20 Julian years is $86^0 50' 16''$, as is obtained from the observations made use of in the theory of the moon, the mean motion of the nodes in one sidereal year will be $19^0 20' 1'' 58'''$ and TH will be to HK as 360^0 to $19^0 20' 31'' 58'''$; that is, as 18.61214 to 1: and from hence the mean hourly motion of the nodes in the quadratures will come out $16'' 18''' 48^{IV}$. And the greatest equation of the nodes in the octants will be $i^0 29' 57''$."

PROPOSITION XXXIV. PROBLEM XV

To find the hourly variation of the inclination of the moon's orbit to the plane of the ecliptic.

Let A and a represent the syzygies; Q and q the quadratures; N and n the nodes; P the place of the moon in its orbit; p the orthographic projection of that place upon the plane of the ecliptic; and mTl the momentary motion of the nodes as above. If upon Tm we let fall the perpendicular PG ,



and joining pG we produce it till it meet 17 in g , and join also Pg , the angle PGp will be the inclination of the moon's orbit to the plane of the ecliptic when the moon is in P ; and the angle Vgp will be the inclination of the same after a small moment of time is elapsed; and therefore the angle GPg

will be the momentary variation of the inclination. But this angle $G?g$ is to the angle GTg as TG to PG and Vp to PG conjointly. And, therefore, if for the momnt of time we assume an hour, since the angle $GT\#$ (by Prop, xxx) is to the angle $33'' 10'''$ as $IT \cdot PG \cdot AZ$ to AT' , the angle GPE (or the hourly variation of the inclination) will be to the angle $33'' 10''' 33^{IV}$ as $IT \cdot AZ \cdot TG \cdot \frac{Pp}{PG}$ to AT' . Q.E.I.

And thus it would be if the moon were uniformly revolved in a circular orbit. But if the orbit is elliptical, the mean motion of the nodes will be diminished in proportion of the less axis to the greater, as we have shown above; and the variation of the inclination will be also diminished in the same proportion.

COR. 1. Upon Nn erect the perpendicular TF , and let pM be the hourly motion of the moon in the plane of the ecliptic; upon QT let fall the perpendiculars pK , Mk , and produce them till they meet TF in H and h ; then IT will be to AT as Kk to Mp ; and TG to Hp as TZ to AT ; and, therefore, $IT \cdot TG$ will be equal to $\frac{Kk \cdot Hp \cdot TZ}{Mp}$, that is, equal to the area $HpMh$ multiplied into the ratio $\frac{TZ}{Mp}$; and therefore the hourly variation of the inclination will be to $33'' 10''' 33^{IV}$ as the area $HpMh$ multiplied into $AZ \cdot \frac{TZ}{Mp} \cdot \frac{Pp}{PG}$ is to AT' .

COR. 11. And, therefore, if the earth and nodes were after every hour drawn back from their new and instantly restored to their old places, so that their situation might continue given for a whole periodic month together, the whole variation of the inclination during that month would be to $33'' 10''' 33^{IV}$ as the aggregate of all the areas $HpMh$, generated in the time of one revolution of the point P (with due regard in summing to their proper signs $+$ $-$), multiplied into $AZ \cdot TZ \cdot \frac{Pp}{PG}$ to $Mp \cdot AT'$; that is, as the whole circle $QAqa$ multiplied into $AZ \cdot TZ \cdot \frac{Pp}{PG}$ to $Mp \cdot AT'$, that is, as the circumference $QAqa$ multiplied into $AZ \cdot TZ \cdot \frac{Pp}{PG}$ to $2Mp \cdot AT'^2$.

COR. HI. And, therefore, in a given position of the nodes, the mean hourly variation, from which, if uniformly continued through the whole month, that menstrual variation might be generated, is to $33'' 10''' 33^{iv}$ as $AZ \cdot TZ$ $\frac{Pp}{PG}$ is to $2AT^2$, or as $Pp \cdot \frac{AZ \cdot TZ}{\frac{1}{2}AT}$ is to $PG \cdot 4AT$; that is (because Pp is to PG as the sine of the aforesaid inclination to the radius, and $\frac{AZ \cdot TZ}{\frac{1}{2}AT}$ to $4AT$ as the sine of double the angle KTn to four times the radius), as the sine of the same inclination multiplied into the sine of double the distance of the nodes from the sun to four times the square of the radius.

COR. iv. Seeing the hourly variation of the inclination, when the nodes are in the quadratures, is (by this Prop.) to the angle $33'' 10''' 33^{iv}$ as $IT \cdot AZ \cdot TG$ $\frac{Pp}{PG}$ is to AT^2 , that is, as $\frac{IT \cdot TG}{\frac{1}{2}AT} \cdot \frac{Pp}{PG}$ to $2AT$, that is, as the sine of double the distance of the moon from the quadratures multiplied into $\frac{Pp}{PG}$ is to twice the radius, the sum of all the hourly variations during the time that the moon, in this situation of the nodes, passes from the quadrature to the syzygy (that is, in the space of 177VG hours) will be to the sum of as many angles $33'' 10''' 33^{iv}$ or 5878'', as the sum of all the sines of double the distance of the moon from the quadratures multiplied into $\frac{Pp}{PG}$ is to the sum of as many diameters; that is, as the diameter multiplied into $\frac{Pp}{PG}$ is to the circumference; that is, if the inclination be $5^0 1'$, as $7 \cdot 0000$ is to 22, or as 278 to 10000. And, therefore, the whole variation, composed out of the sum of all the hourly variations in the aforesaid time, is 163'', or $2 43''$.

PROPOSITION XXXV. PROBLEM XVI

To a given time to find the inclination of the moons orbit to the plane of the ecliptic.

Let AD be the sine of the greatest inclination, and AB the sine of the least. Bisect BD in C; and round the centre C, with the radius BC, describe the circle BGD. In AC take CE in the same proportion to EB as EB to twice BA. And if to the time given we set off the angle AEG equal to double the

distance of the nodes from the quadratures, and upon AD let fall the perpendicular GH, AH will be the sine of the inclination required.

For GE^2 is equal to

$$GH^2 + HE^2 = BHD + HE^2 = HBD + HE^2 - BH^2 - HBD + BE^2 -$$

$$2BH \cdot BE = BE^2 + 2EC \cdot BH = 2EC \cdot AB + 2EC \cdot BH = 2EC \cdot AH;$$

wherefore, since $2EC$ is given, GE'' will be as AH . Now let AEG represent double the distance of the nodes from the quadratures, in a given moment



of time after, and the arc Gg , on account of the given angle GEg , will be as the distance GE . But Hh is to Gg as GH to GC , and, therefore, Hh is as the rectangle $GH \cdot Gg$, or $GH \cdot GE$, that is, as $\frac{GH}{GE} = r \cdot GE^2$, or $\frac{1}{GE} \cdot TA H$; that is, as

AH and the sine of the angle AEG conjointly. If, therefore, in any one case, AH be the sine of inclination, it will increase by the same increments as the sine of inclination doth (by Cor. m of the preceding Prop.), and therefore will always continue equal to that sine. But when the point G falls upon either point B or D , AH is equal to this sine, and therefore remains always equal thereto. Q.E.D.

In this demonstration I have supposed that the angle BEG , representing double the distance of the nodes from the quadratures, increaseth uniformly; for I cannot descend to every minute circumstance of inequality. Now suppose that BEG is a right angle, and that Gg is in this case the hourly increment of double the distance of the nodes from the sun, then (by Cor. in of the last Prop.) the hourly variation of the inclination in the same case will be to $33'' 10''' 33''''$ as the rectangle of AH , the sine of the inclination, into the sine of the right angle BEG , double the distance of the nodes from the sun, is to four times the square of the radius; that is, as AH , the sine of the mean inclination, is to four times the radius; that is, seeing the mean inclination is about $5^\circ 8\frac{1}{2}'$, as its sine 896 is to 40000, the quad-

ruptle of the radius, or as 224 to 10000. But the whole variation corresponding to BD, the difference of the sines, is to this hourly variation as the diameter BD is to the arc Gg, that is, conjointly as the diameter BD to the semicircumference BGD, and as the time of $2079\frac{7}{10}$ hours, in which the node proceeds from the quadratures to the syzygies, is to one hour, that is, as 7 to 11, and $2079\frac{7}{10}$ to 1. Therefore, compounding all these proportions, we shall have the whole variation BD to $33'' 10''' 33^{IV}$ as $224 \cdot 7 \cdot 2079\frac{7}{10}$ is to 110000, that is, as 29645 to 1000; and from thence that variation BD will come out $16' 23\frac{1}{2}''$.

And this is the greatest variation of the inclination, abstracting from the situation of the moon in its orbit; for, if the nodes are in the syzygies, the Inclination suffers no change from the various positions of the moon. But if the nodes are in the quadratures, the inclination is less when the moon is in the syzygies than when it is in the quadratures by a difference of $2' 43''$, as we showed (Cor. iv of the preceding Prop.); and the whole mean variation BD, diminished by $1 \cdot 21\frac{1}{2}''$, the half of this excess, becomes $15' 2''$, when the moon is in the quadratures; and, increased by the same, becomes $17' 45''$ when the moon is in the syzygies. If, therefore, the moon be in the syzygies, the whole variation in the passage of the nodes from the quadratures to the syzygies will be $17' 45''$; and, therefore, if the inclination be $5'' 17' 20''$, when the nodes are in the syzygies, it will be $4^\circ 59' 35''$ when the nodes are in the quadratures and the moon in the syzygies. The truth of all this is confirmed by observations.

Now if the inclination of the orbit should be required when the moon is in the syzygies, and the nodes anywhere between them and the quadratures, let AB be to AD as the sine of $4^\circ 59' 35''$ is to the sine of $5^\circ 17' 20''$, and take the angle AEG equal to double the distance of the nodes from the quadratures; and AH will be the sine of the inclination desired. To this inclination of the orbit the inclination of the same is equal, when the moon is 90° distant from the nodes. In other situations of the moon, this menstrual inequality, to which the variation of the inclination is subject in the calculus of the moon's latitude, is balanced, and in a manner taken off, by the menstrual inequality of the motion of the nodes (as we said before), and therefore may be neglected in the computation of the said latitude.

SCHOLIUM

By these computations of the lunar motions I was desirous of showing that by the theory of gravity the motions of the moon could be calculated from their physical causes. By the same theory I moreover found that the annual equation *of* the mean motion of the moon arises from the varying dilatation which the orbit of the moon suffers from the action of the sun according to Cor. vi, Prop, LXVI, Book i. The force of this action is greater in the penginean sun, and dilates the moon's orbit; in the apogean sun it is less, and permits the orbit to be again contracted. The moon moves slower in the dilated and faster in the contracted orbit; and the annual equation, by which this inequality is regulated, vanishes in the apogee and perigee of the sun. In the mean distance of the sun from the earth it rises to about $n' 50''$; in other distances of the sun it is proportional to the equation of the sun's centre, and is added to the mean motion of the moon, while the earth is passing from its aphelion to its perihelion, and subtracted while the earth is in the opposite semicircle. Taking for the radius of the great orbit 1000, and $16\frac{7}{8}$ for the earth's eccentricity, this equation, when of the greatest magnitude, by the theory of gravity comes out $11' 49''$. But the eccentricity *of* the earth seems to be somewhat greater, and with the eccentricity this equation will be augmented in the same proportion. Suppose the eccentricity $16\frac{11}{12}$, and the greatest equation will be $11' 51''$.

Further; I found that the apogee and nodes of the moon move faster in the perihelion of the earth, where the force of the sun's action is greater, than in the aphelion thereof, and that inversely as the cube of the ratio of the earth's distance from the sun; and hence arise annual equations *of* those motions proportional to the equation of the sun's centre. Now the motion of the sun varies inversely as the square of the earth's distance from the sun; and the greatest equation of the centre which this inequality generates is $1^{\circ} 56' 20''$, corresponding to the above-mentioned eccentricity *of* the sun, $16\frac{17}{12}$. But if the motion of the sun had been inversely as the cube of the distance, this inequality would have generated the greatest equation $2^{\circ} 54' 30''$; and therefore the greatest equations which the inequalities of the motions of the moon's apogee and nodes do generate are to $2^{\circ} 54' 30''$ as the mean diurnal motion of the moon's apogee and the mean diurnal motion

of its nodes are to the mean diurnal motion of the sun. Hence the greatest equation of the mean motion of the apogee comes out $19' 43''$, and the greatest equation of the mean motion of the nodes $9' 24''$. The former equation is added, and the latter subtracted, while the earth is passing from its perihelion to its aphelion, and contrariwise when the earth is in the opposite semicircle.

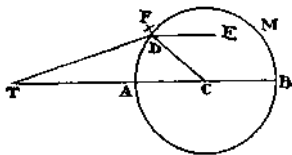
By the theory of gravity I likewise found that the action of the sun upon the moon is somewhat greater when the transverse diameter of the moon's orbit passes through the sun than when the same is perpendicular upon the line which joins the earth and the sun; and therefore the moon's orbit is somewhat larger in the former than in the latter case. And hence arises another equation of the moon's mean motion, depending upon the situation of the moon's apogee in respect of the sun, which is greatest when the moon's apogee is in the octants of the sun, and vanishes when the apogee arrives at the quadratures or syzygies; and it is added to the mean motion while the moon's apogee is passing from the quadrature of the sun to the syzygy, and subtracted while the apogee is passing from the syzygy to the quadrature. This equation, which I shall call the semiannual, when greatest in the octants of the apogee, arises to about $3' 45''$, so far as I could determine from the phenomena: and this is its quantity in the mean distance of the sun from the earth. But it is increased and diminished inversely as the cube of the sun's distance, and therefore is nearly $3' 34''$ when that distance is greatest, and $3' 56''$ when least. But when the moon's apogee is without the octants, it becomes less, and is to its greatest amount as the sine of double the distance of the moon's apogee from the nearest syzygy or quadrature is to the radius.

By the same theory of gravity, the action of the sun upon the moon is somewhat greater when the line of the moon's nodes passes through the sun than when it is at right angles with the line which joins the sun and the earth; and hence arises another equation of the moon's mean motion, which I shall call the second semiannual; and this is greatest when the nodes are in the octants of the sun, and vanishes when they are in the syzygies or quadratures; and in other positions of the nodes is proportional to the sine of double the distance of either node from the nearest syzygy or quadrature. And it is added to the mean motion of the moon, if the sun is

behind the node which is nearest to him, and is subtracted, if forward; and in the octants, where it is of the greatest magnitude, it arises to 47" in the mean distance of the sun from the earth, as I find from the theory of gravity. In other distances of the sun, this equation, greatest in the octants of the nodes, is inversely as the cube of the sun's distance from the earth; and therefore in the sun's perigee it comes to about 49", and in its apogee to about 45".

By the same theory of gravity, the moon's apogee goes forwards at the greatest rate when it is either in conjunction with or in opposition to the sun, but in its quadratures with the sun it goes backwards; and the eccentricity comes, in the former case, to its greatest quantity; in the latter, to its least, by Cor. VII, vm, and ix, Prop, LXVI, Book I. And those inequalities, by the Corollaries we have named, are very great, and generate the principle which I call the semiannual equation of the apogee; and this semiannual equation in its greatest quantity comes to about $12^0 18'$, as nearly as I could determine from the phenomena. Our countryman, *Horrox*, was the first who advanced the theory of the moon's moving in an ellipse about the earth placed in its lower focus. Dr. *Halley* improved the notion, by putting the centre of the ellipse in an epicycle whose centre is uniformly revolved about the earth; and from the motion in this epicycle the mentioned inequalities in the progress and regress of the apogee, and in the quantity of eccentricity, do arise. Suppose the mean distance of the moon from the earth to be divided into 100000 parts, and let T represent the earth, and TC the moon's mean eccentricity of 5505 such parts. Produce TC to B, so as CB may be the sine of the greatest semiannual equation $12^0 18'$ to the radius

TC; and the circle BDA described about the centre C, with the radius CB, will be the epicycle spoken of, in which the centre of the moon's orbit is placed, and revolved according to the order of the letters BDA. Set off the angle BCD equal to twice the annual argument, or twice the distance of the sun's true place from the place



of the moon's apogee once corrected, and CTD will be the semiannual equation of the moon's apogee, and TD the eccentricity of its orbit, tending to the place of the apogee now twice corrected. But, having the moon's

mean motion, the place of its apogee, and its eccentricity, as well as the longer axis of its orbit 200000, from these data the true place of the moon in its orbit, together with its distance from the earth, may be determined by the methods commonly known.

In the perihelion of the earth, where the force of the sun is greatest, the centre of the moon's orbit moves faster about the centre C than in the aphelion, and that inversely as the cube of the sun's distance from the earth. But, because the equation of the sun's centre is included in the annual argument, the centre of the moon's orbit moves faster in its epicycle BDA, inversely as the square of the sun's distance from the earth. Therefore, that it may move yet faster, inversely as the distance, suppose that from D, the centre of the orbit, a right line DE is drawn, tending towards the moon's apogee once corrected, that is, parallel to TC; and set off the angle EDF equal to the excess of the aforesaid annual argument above the distance of the moon's apogee from the sun's perigee forwards; or, which comes to the same thing, take the angle CDF equal to the complement of the sun's true anomaly to 360^0 ; and let DF be to DC as twice the eccentricity of the great orbit to the sun's mean distance from the earth, and the sun's mean diurnal motion from the moon's apogee to the sun's mean diurnal motion from its own apogee conjointly, that is, as $33\frac{3}{4}$ to 1000, and $52' 27'' 16'''$ to $59' 8'' 10'''$ conjointly, or as 3 to 100; and imagine the centre of the moon's orbit placed in the point F to be revolved in an epicycle whose centre is D, and radius DF, while the point D moves in the circumference of the circle DABD: for by this means the centre of the moon's orbit comes to describe a certain curved line about the centre C, with a velocity which will be almost inversely as the cube of the sun's distance from the earth, as it ought to be.

The calculus of this motion is difficult, but may be rendered easier by the following approximation. Assuming, as above, the moon's mean distance from the earth of 100000 parts, and the eccentricity TC of 5505 such parts, the line CB or CD will be found $1172\frac{3}{4}$, and DF $35\frac{1}{4}$ of those parts; and this line DF at the distance TC subtends the angle at the earth, which the removal of the centre of the orbit from the place D to the place F generates in the motion of this centre; and double this line DF in a parallel position, at the distance of the upper focus of the moon's orbit from the earth, sub-

tends at the earth the same angle as DF did before, which that removal generates in the motion of this upper focus; but at the distance of the moon from the earth this double line 2DF at the upper focus, in a parallel position to the first line DF, subtends an angle at the moon, which the said removal generates in the motion of the moon, which angle may be therefore called the second equation of the moon's centre; and this equation, in the mean distance of the moon from the earth, is nearly as the sine of the angle which that line DF contains with the line drawn from the point F to the moon, and when greatest amounts to $2' 25''$. But the angle which the line DF contains with the line drawn from the point F to the moon is found either by subtracting the angle EDF from the mean anomaly of the moon, or by adding the distance of the moon from the sun to the distance of the moon's apogee from the apogee of the sun; and as the radius is to the sine of the angle thus found, so is $2' 25''$ to the second equation of the centre: to be added, if the fore-mentioned sum be less than a semi-circle; to be subtracted, if greater. And from the moon's place in its orbit thus corrected, its longitude may be found in the syzygies of the luminaries.

The atmosphere of the earth to the height of 35 or 40 miles refracts the sun's light. This refraction scatters and spreads the light over the earth's shadow; and the dissipated light near the limits of the shadow dilates the shadow. On this account, to the diameter of the shadow, as it comes out by the parallax, I add 1 or $1 \frac{1}{3}$ minutes in lunar eclipses.

But the theory of the moon ought to be examined and proved from the phenomena, first in the syzygies, then in the quadratures, and last of all in the octants; and whoever pleases to undertake the work will find it not amiss to assume the following mean motions of the sun and moon at the *Royal Observatory of Greenwich*, to the last day of *December* at noon, in the year 1700, o.s.: the mean motion of the sun $V3\ 20^\circ\ 43'\ 40''$, and of its apogee $S\ y^\circ\ 44'\ 30''$; the mean motion of the moon $\llcorner\ 15^\circ\ 21'\ 00''$; of its apogee, $X.\ 8^\circ\ 20'\ 00''$; and of its ascending node $Si\ 27^\circ\ 24'\ 20''$; and the difference of meridians between the Observatory at *Greenwich* and the *Royal Observatory at Paris*, o.^h. $9^m. 20^s$.: but the mean motion of the moon and of its apogee are not yet obtained with sufficient accuracy.

PROPOSITION XXXVI. PROBLEM XVII

To find the force of the sun to move the sea.

The sun's force ML or PT to disturb the motions of the moon, was (by Prop, xxv), in the moon's quadratures, to the force of gravity with us, as 1 to 638092.6; and the force T M - L M or 2PK in the moon's syzygies is double that quantity. But, descending to the surface of the earth, these forces are diminished in proportion of the distances from the centre of the earth, that is, in the proportion of $60\frac{1}{2}$ to 1; and therefore the former force on the earth's surface is to the force of gravity as 1 to 38604600; and by this force the sea is depressed in such places as are 90 degrees distant from the sun. But by the other force, which is twice as great, the sea is raised not only in the places directly under the sun, but in those also which are directly opposed to it; and the sum of these forces is to the force of gravity as 1 to 12868200. And because the same force excites the same motion, whether it depresses the waters in those places which are 90 degrees distant from the sun, or raises them in the places which are directly under and directly opposed to the sun, the aforesaid sum will be the total force of the sun to disturb the sea, and will have the same effect as if the whole was employed in raising the sea in the places directly under and directly opposed to the sun, and did not act at all in the places which are 90 degrees removed from the sun.

And this is the force of the sun to disturb the sea in any given place, where the sun is at the same time both vertical, and in its mean distance from the earth. In other positions of the sun, its force to raise the sea is directly as the versed sine of double its altitude above the horizon of the place, and inversely as the cube of the distance from the earth.

COR. Since the centrifugal force of the parts of the earth, arising from the earth's diurnal motion, which is to the force of gravity as 1 is to 289, raises the waters under the equator to a height exceeding that under the poles by 85472 *Paris* feet, as above, in Prop, xix, the force of the sun, which we have now shown to be to the force of gravity as 1 is to 12868200, and therefore is to that centrifugal force as 289 to 12868200, or as 1 to 44527, will be able to raise the waters in the places directly under and directly opposed to the sun to a height exceeding that in the places which are 90 degrees removed from

the sun only by one *Paris* foot and 1131/30 inches; for this measure is to the measure of 85472 feet as 1 to 44527.

PROPOSITION XXXVII. PROBLEM XVIII

To find the force of the moon to move the sea.

The force of the moon to move the sea is to be deduced from its ratio to the force of the sun, and this ratio is to be determined from the ratio of the motions of the sea, which are the effects of those forces. Before the mouth of the river *Avon*, three miles below *Bristol*, the height of the ascent of the water in the vernal and autumnal syzygies of the luminaries (by the observations of *Samuel Sturmy*) amounts to about 45 feet, but in the quadratures to 25 only. The former of those heights arises from the sum of the aforesaid forces, the latter from their difference. If, therefore, S and L are supposed to represent respectively the forces of the sun and moon while they are in the equator, as well as in their mean distances from the earth, we shall have $L + S$ to $L - S$ as 45 to 25, or as 9 to 5.

At *Plymouth* (by the observations of *Samuel Colepress*) the tide in its mean height rises to about 16 feet, and in the spring and autumn the height thereof in the syzygies may exceed that in the quadratures by more than 7 or 8 feet. Suppose the greatest difference of those heights to be 9 feet, and $L + S$ will be to $L - S$ as $25\frac{1}{2}$ to $11\frac{1}{2}$, or as 41 to 23; a proportion that agrees well enough with the former. But because of the great tide at *Bristol*, we are rather to depend upon the observations of *Sturmy*; and, therefore, till we procure something that is more certain, we shall use the proportion of 9 to 5.

But because of the reciprocal motions of the waters, the greatest tides do not happen at the times of the syzygies of the luminaries, but, as we have said before, are the third in order after the syzygies; or (reckoning from the syzygies) follow next after the third approach of the moon to the meridian of the place after the syzygies; or rather (as *Sturmy* observes) are the third after the day of the new or full moon, or rather nearly after the twelfth hour from the new or full moon, and therefore fall nearly upon the forty-third hour after the new or full moon. But in this port they come to pass about the seventh hour after the approach of the moon to the meridian of the

place; and therefore follow next after the approach of the moon to the meridian, when the moon is distant from the sun, or from opposition with the sun by about 18 or 19 degrees forwards. So the summer and winter seasons come not to their height in the solstices themselves, but when the sun is advanced beyond the solstices by about a tenth part of its whole course, that is, by about 36 or 37 degrees. In like manner, the greatest tide is raised after the approach of the moon to the meridian of the place, when the moon has passed by the sun, *or the opposition thereof*, by about a tenth part of the whole motion from *one greatest tide to the next following greatest tide*. Suppose that distance about 18 degrees; and the sun's force in this distance of the moon from the syzygies and quadratures will be of less moment to augment and diminish that part of the motion of the sea which proceeds from the motion of the moon than in the syzygies and quadratures themselves in the proportion of the radius to the cosine of double this distance, or of an angle of 37 degrees; that is, in the ratio of 10000000 to 7986355; and, therefore, in the preceding analogy, in place of S we must put 0.7986355S.

But further, the force of the moon in the quadratures must be diminished, on account of its declination from the equator; for the moon in those quadratures, or rather in 18V2 degrees past the quadratures, declines from the equator by about $23^{\circ} 13'$; and the force of either luminary to move the sea is diminished as it declines from the equator nearly as the square of the cosine of the declination; and therefore the force of the moon in those quadratures is only 0.8570327L; hence we have L.4 0.7986355S to 0.8570327L-0.7986355S as 9 to 5.

Further yet, the diameters of the orbit in which the moon should move, setting aside the consideration of eccentricity, are one to the other as 69 to 70; and therefore the moon's distance from the earth in the syzygies is to its distance in the quadratures, other things being equal, as 69 to 70; and its distances, when 18MJ degrees advanced beyond the syzygies, where the greatest tide was excited, and when 18V2 degrees passed by the quadratures, where the least tide was produced, are to its mean distance as 69.098747 and 69.897345 to 69V2. But the force of the moon to move the sea varies inversely as the cube of its distance; and therefore its forces, in the greatest and least of those distances, are to its force in its mean distance as 0.9830427 and

1.017522 is to 1. From this we have 1.017522L - 0.7986355s to 0.9830427 - 0.8570327L - 0.7986355S as 9 to 5; and S to L as 1 to 4.4815. Therefore, since the force of the sun is to the force of gravity as 1 to 12868200, the moon's force will be to the force of gravity as 1 to 2871400.

COR. 1. Since the waters attracted by the sun's force rise to the height of 1 foot and $11\frac{1}{30}$ inches, the moon's force will raise the same to the height of 8 feet and $7\frac{5}{12}$ inches; and the joint forces of both will raise the same to the height of 10V2 feet; and when the moon is in its perigee to the height of $12\frac{1}{2}$ feet, and more, especially when the wind sets the same way as the tide. And a force of that amount is abundantly sufficient to produce all the motions of the sea, and agrees well with the ratio of those motions; for in such seas as lie free and open from east to west, as in the *Pacific* sea, and in those tracts of the *Atlantic* and *Ethiopic* seas which lie without the tropics, the waters commonly rise to 6, 9, 12, or 15 feet; but in the *Pacific* sea, which is of a greater depth, as well as of a larger extent, the tides are said to be greater than in the *Atlantic* and *Ethiopic* seas; for, to have a full tide raised, an extent of sea from east to west is required of no less than 90 degrees. In the *Ethiopic* sea, the waters rise to a less height within the tropics than in the temperate zones: because of the narrowness of the sea between *Africa* and the southern parts of *America*. In the middle of the open sea the waters cannot rise without falling together, and at the same time, upon both the eastern and western shores, when, notwithstanding, in our narrow seas, they ought to fall on those shores by alternate turns; upon this account there is commonly but a small flood and ebb in such islands as lie far distant from the continent. On the contrary, in some ports, where to fill and empty the bays alternately the waters are with great violence forced in and out through shallow channels, the flood and ebb must be greater than ordinary; as at *Plymouth* and *Chepstow Bridge* in *England*, at the mountains of *St. Michael*, and the town of *Avranches*, in *Normandy*, and at *Cambaia* and *Pegu* in the *East Indies*. In these places the sea is hurried in and out with such violence as sometimes to lay the shores under water, sometimes to leave them dry for many miles. Nor is this force of the influx and efflux to be stopped till it has raised and depressed the waters to 30, 40, or 50 feet and above. And a like account is to be given of long and shallow channels or straits, such as the *Magellanic* straits, and those channels which environ

England. The tide in such ports and straits, by the violence of the influx and efflux, is augmented greatly. But on such shores as lie towards the deep and open sea with a steep descent, where the waters may freely rise and fall without that precipitation of influx and efflux, the ratio of the tides agrees with the forces of the sun and moon.

COR. 11. Since the moon's force to move the sea is to the force of gravity as 1 to 2871400, it is evident that this force is inappreciable in statical or hydrostatical experiments, or even in those of pendulums. It is in the tides only that this force shows itself by any sensible effect.

COR. in. Because the force of the moon for moving the sea is to the like force of the sun as 4.4815 to 1, and those forces (by Cor. xiv, Prop, LXVI, Book 1) are as the densities of the bodies of the sun and moon and the cubes of their apparent diameters conjointly, the density of the moon will be to the density of the sun directly as 4.4815 to 1, and inversely as the cube of the moon's diameter to the cube of the sun's diameter; that is (seeing the mean apparent diameters of the moon and sun are $31' 16\frac{1}{2}''$, and $32' 12''$), as 4891 to 1000. But the density of the sun was to the density of the earth as 1000 to 4000; and therefore the density of the moon is to the density of the earth as 4891 is to 4000, or as n to 9. Therefore the body of the moon is more dense and more earthly than the earth itself.

COR. iv. And since the true diameter of the moon (from the observations of astronomers) is to the true diameter of the earth as 100 to 365, the mass of matter in the moon will be to the mass of matter in the earth as 1 to 39.788.

COR. v. And the accelerative gravity on the surface of the moon will be about three times less than the accelerative gravity on the surface of the earth.

COR. VI. And the distance of the moon's centre from the centre of the earth will be to the distance of the moon's centre from the common centre of gravity of the earth and moon as 40.788 to 39.788.

COR. vii. And the mean distance of the centre of the moon from the centre of the earth will be (in the moon's octants) nearly $60\frac{2}{3}$ of the greatest semidiameters of the earth; for the greatest semidiameter of the earth was 19658600 *Parts* feet, and the mean distance of the centres of the earth and

moon, consisting of $60\frac{1}{2}$ such semidiameters, is equal to 1187379440 feet. And this distance (by the preceding Cor.) is to the distance of the moon's centre from the common centre of gravity of the earth and moon as 40.788 to 39.788; which latter distance, therefore, is 1158268534 feet. And since the moon, in respect of the fixed stars, performs its revolution in $27^d. 7^h. 43^{\frac{1}{2}}m.$, the versed sine of that angle which the moon in a minute of time describes is 12752341 to the radius 1000,000000,000000; and as the radius is to this versed sine, so are 1158268534 feet to 14.7706353 feet. The moon, therefore, falling towards the earth by that force which retains it in its orbit, would in one minute of time describe 14.7706353 feet; and, if we augment this force in the proportion of $178^{\frac{2}{10}}\frac{1}{10}$ to $177^{\frac{2}{10}}\frac{1}{10}$, we shall have the total force of gravity at the orbit of the moon, by Cor., Prop. 111; and the moon falling by this force, in one minute of time would describe 14.8538067 feet. And at the 60th part of the distance of the moon from the earth's centre, that is, at the distance of 197896573 feet from the centre of the earth, a body falling by its weight, would, in one second of time, likewise describe 14.8538067 feet. And, therefore, at the distance of 19615800, which compose one mean semidiameter of the earth, a heavy body would describe in falling 15.11175, or 15 feet, 1 inch, and $4\frac{1}{11}$ lines, in the same time. This will be the descent of bodies in the latitude of 45 degrees. And by the foregoing table, to be found under Prop. xx, the descent in the latitude of *Pans* will be a little greater by an excess of about $\frac{1}{2}$ parts of a line. Therefore, by this computation, heavy bodies in the latitude of *Parts* falling in a vacuum will describe 15 *Pans* feet, 1 inch, $4\frac{2}{13}$ lines, very nearly, in one second of time. And if the gravity be diminished by taking away a quantity equal to the centrifugal force arising in that latitude from the earth's diurnal motion, heavy bodies falling there will describe in one second of time 15 feet, 1 inch, and $1\frac{1}{2}$ lines. And with this velocity heavy bodies do really fall in the latitude of *Pans*, as we have shown above in Prop. iv and xix.

COR. VIII. The mean distance of the centres of the earth and moon in the syzygies of the moon is equal to 60 of the greatest semidiameters of the earth, subtracting only about one 30th part of a semidiameter; and in the moon's quadratures the mean distance of the same centres is $60\frac{1}{2}$ such semidiameters of the earth; for these two distances are to the mean distance of the moon in the octants as 69 and 70 to $69\frac{1}{2}$, by Prop. xxvni.

COR. IX. The mean distance of the centres of the earth and moon in the syzygies of the moon is 60 mean semidiameters of the earth, and a 10th part of one semidiameter; and in the moon's quadratures the mean distance of the same centres is 61 mean semidiameters of the earth, subtracting one 30th part of one semidiameter.

COR. x. In the moon's syzygies its mean horizontal parallax in the latitudes of 0, 30, 38, 45, 52, 60, 90 degrees is $57' 20''$, $57' 16''$, $57' 14''$, $57' 12''$, $57' 10''$, $57' 8''$, $57' 4''$, respectively.

In these computations I do not consider the magnetic attraction of the earth, whose quantity is very small and unknown: if this quantity should ever be found out, and the measures of degrees upon the meridian, the lengths of isochronous pendulums in different parallels, the laws of the motions of the sea, and the moon's parallax, with the apparent diameters of the sun and moon, should be more exactly determined from phenomena: we should then be enabled to bring this calculation to a greater accuracy.

PROPOSITION XXXVIII. PROBLEM XIX

To find the figure of the moon's body.

If the moon's body were fluid like our sea, the force of the earth to raise that fluid in the nearest and remotest parts would be to the force of the moon by which our sea is raised in the places under and opposite to the moon as the accelerative gravity of the moon towards the earth is to the accelerative gravity of the earth towards the moon, and the diameter of the moon is to the diameter of the earth conjointly; that is, as 39.788 to 1, and 100 to 365 conjointly, or as 1081 to 100. Therefore, since our sea, by the force of the moon, is raised to $8\frac{3}{4}$ feet, the lunar fluid would be raised by the force of the earth to 93 feet; and upon this account the figure of the moon would be a spheroid, whose greatest diameter produced would pass through the centre of the earth, and exceed the diameters perpendicular thereto by 186 feet. Such a figure, therefore, the moon possesses, and must have had from the beginning. Q.E.I.

COR. Hence it is that the same face of the moon always is turned toward the earth; nor can the body of the moon possibly rest in any other position, but would return always by a libratory motion to this situation; but those

librations, however, must be exceedingly slow, because of the weakness of the forces which excite them; so that the face of the moon, which should be always directed to the earth, may, for the reason assigned in Prop, xvn, be turned towards the other focus of the moon's orbit, without being immediately drawn back, and turned again towards the earth.

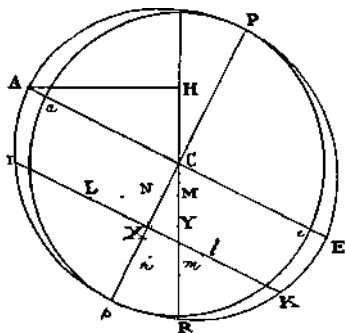
LEMMA I

// APEp represent the earth uniformly dense, marked with the centre C, the poles P, p, and the equator AE; and if about the centre C, with the radius CP, we suppose the sphere Papc to be described', and QR to denote the plane on which a right line, drawn from the centre of the sun to the centre of the earth, stands at right angles, and further suppose that the several particles of the whole exterior earth PapAPcpK, without the height of the said sphere, endeavor to recede towards this side and that side from the plane QR, every particle by a force proportional to its distance from that plane: I say, in the first place, that the whole force and efficacy of all the particles that are situated in AE, the circle of the equator, and disposed uniformly without the globe, encompassing the same after the manner of a ring, to wheel the earth about its centre, is to the whole force and efficacy of as many particles in that point A of the equator which is at the greatest distance from the plane QR, to wheel the earth about its centre with a like circular motion as is 1 to 2. And that circular motion will be performed about an axis lying in the common section of the equator and the plane QR.

For let there be described from the centre K, with the diameter IL, the semicircle INL. Suppose the semicircumference INL to be divided into innumerable equal parts, and from the several parts N to the diameter IL let fall the sines NM. Then the sums of the squares of all the sines NM will be equal to the sums of the squares of the sines KM, and both sums together will be equal to the sums of the squares of as many semidiameters KN; and therefore the sum of the squares of all the sines NM will be but half so great as the sum of the squares of as many semidiameters KN.

Suppose now the circumference of the circle AE to be divided into the like number of little equal parts, and from every such part F a perpendicular FG to be let fall upon the plane QR, as well as the perpendicular AH

For let IK be any lesser circle parallel to the equator AE, and let LI be any two equal particles in this circle, situated without the sphere *Vape*; and if upon the plane QR, which is at right angles with a radius drawn to the sun, we let fall the perpendiculars LM, *Im*, the total forces by which these particles recede from the plane QR will be proportional to the perpendiculars LM, *Im*. Let the right line L/ be drawn parallel to the plane *Yape*, and bisect the same in X; and through the point X draw N« parallel to the plane QR, and meeting the perpendiculars LM, *Im*, in N and *n*; and upon the plane QR let fall the perpendicular XY. And the contrary forces of the particles L and / to wheel about the earth contrariwise are as LM MC,



and Im mC ; that is, as LN $MC + NM$ MC , and In $mC - nm$ mC ; or LN $MC + NM$ MC , and LN $mC - NM$ mC , and LN $Mm - NM$ $(MC + raC)$, the difference of the two, is the force of both taken together to turn the earth round. The positive part of this difference LN Mm , or $2LN$ NX , is to $2AH$ HC , the force of two particles of the same size situated in A , as LX'' to AC^* ; and the negative part NM $(MC + wC)$, or $2XY$ CY , is to $2AH$ HC , the force of the same two particles situated in A , as CX'' to AC^2 . And therefore the difference of the parts, that is, the force of the two particles L and I , taken together, to wheel the earth about, is to the force of two particles, equal to the former and situated in the place A , to turn in like manner the earth round, as $LX' - CX''$ is to AC'' . But if the circumference IK of the circle IK is supposed to be divided into an infinite number of little equal parts L , all the LX^2 will be to the like number of IX^J as 1 to 2 (by Lem. i.); and to the same number of AC''' as IX''' is to $2AC'''$; and the same number of CX'' to as many $AC^{>}$ as $2CX^J$ is to $2AC''$. Therefore the united forces of all the particles in the circumference of the circle IK are to the joint forces of as many particles in the place A as $IX^J - 2CX''$ is to

$2AC^2$; and therefore (by Lem. 1) to the united forces of as many particles in the circumference of the circle AE as $IX^2 - 2CX^2$ is to AC^2 .

Now if Pp , the diameter of the sphere, is conceived to be divided into an infinite number of equal parts, upon which a like number of circles IK are supposed to stand, the matter in the circumference of every circle IK will be as IX^2 ; and therefore the force of that matter to turn the earth about will be as IX^2 into $IX^2 - 2CX^2$; and the force of the same matter, if it was situated in the circumference of the circle AE, would be as IX^2 into AC^2 . And therefore the force of all the particles of the whole matter situated without the sphere in the circumferences of all the circles is to the force of the like number of particles situated in the circumference of the greatest circle AE as all the IX^2 into $IX^2 - 2CX^2$ is to as many IX^2 into AC^2 ; that is, as all the $AC^2 - CX^2$ into $AC^2 - 3CX^2$ to as many $AC^2 - CX^2$ into AC^2 ; that is, as all the $AC^4 - 4AC^2 \cdot CX^2 + 3CX^4$ to as many $AC^4 - AC^2 \cdot CX^2$; that is, as the whole fluent quantity, whose fluxion is $AC^4 - 4AC^2 \cdot CX^2 + 3CX^4$, is to the whole fluent quantity, whose fluxion is $AC^4 - AC^2 \cdot CX^2$; and, therefore, by the method of fluxions, as $AC^4 \cdot CX - \frac{1}{2}AC^2 \cdot CX^3 + \frac{3}{2}CX^5$ is to $AC^4 \cdot CX - \frac{1}{2}AC^2 \cdot CX^3$; that is, if for CX we write the whole Cp , or AC , as $\frac{1}{2}AC^3$ is to $\frac{3}{2}AC^3$; that is, as 2 is to 5. Q.E.D.

LEMMA III

The same things still supposed, I say, in the third place, that the motion of the whole earth about the axis above named arising from the motions of all the particles, will be to the motion of the aforesaid ring about the same axis in a ratio compounded of the ratio of the matter in the earth to the matter in the ring; and the ratio of three squares of the quadrantal arc of any circle to two squares of its diameter, that is, in the ratio of the matter to the matter, and of the number 925275 to the number 1000000.

For the motion of a cylinder revolved about its quiescent axis is to the motion of the inscribed sphere revolved together with it as any four equal squares are to three circles inscribed in three of those squares,¹ and the motion of this cylinder is to the motion of an exceedingly thin ring surrounding both sphere and cylinder in then common contact as double the matter in the cylinder is to triple the matter in the ring; and this motion of the ring, uniformly continued about the axis of the cylinder, is to the uniform

[¹ Appendix, Note 47.]

motion of the same about its own diameter performed in the same periodic time as is the circumference of a circle to double its diameter.

HYPOTHESIS II

If the other parts of the earth were taken away, and the remaining ring was carried alone about the sun in the orbit of the earth by the annual motion, while by the diurnal motion it was in the meantime revolved about its own axis inclined to the plane of the ecliptic by an angle of 23½ degrees, the motion of the equinoctial points would be the same, whether the ring were fluid, or whether it consisted of a hard and rigid matter.

PROPOSITION XXXIX. PROBLEM XX

To find the precession of the equinoxes.

The middle hourly motion of the moon's nodes in a circular orbit, when the nodes are in the quadratures, was $16'' 35''' i6^{iv} \wedge 6^v$; the half of which, $8'' 17''' 38^{iv} i8^v$ (for the reasons above explained), is the mean hourly motion of the nodes in such an orbit, which motion in a whole sidereal year becomes $20^\circ 11' 46''$. Since, therefore, the nodes of the moon in such an orbit would be yearly transferred $20^\circ 11' 46''$ backwards, and, if there were more moons, the motion of the nodes of every one (by Cor. xvi, Prop, LXVI, Book 1) would be as its periodic time, if upon the surface of the earth a moon was revolved in the time of a sidereal day, the annual motion of the nodes of this moon would be to $20^\circ 11' 46''$ as $23^h. 56^m.$, the sidereal day, is to $27^d. 7^h. 43^m.$, the periodic time of our moon, that is, as 1436 is to 39343. And the same thing would happen to the nodes of a ring of moons encompassing the earth, whether these moons did not mutually touch each the other, or whether they were molten, and formed into a continued ring, or whether that ring should become rigid and inflexible.

Let us, then, suppose that this ring is in quantity of matter equal to the whole exterior earth *PapAPepE*, which lies without the sphere *Vape* (see fig., Lem. 11); and because this sphere is to that exterior earth as aC^2 is to AC^2 — aC that is (seeing *PC* or *aC* the least semidiameter of the earth is to *AC* the greatest semidiameter of the same as 229 is to 230), as 52441 is to 459; if this ring encompassed the earth round the equator, and both together were revolved about the diameter of the ring, the motion of the ring

(by Lcm. in) would be to the motion of the inner sphere as 459 to 52441 and 1000000 to 925275 conjointly, that is, as 4590 to 485223; and therefore the motion of the ring would be to the sum of the motions of both ring and sphere as 4590 is to 489813. Therefore, if the ring adheres to the sphere, and communicates its motion to the sphere, by which its nodes or equinoctial points recede, the motion remaining in the ring will be to its former motion as 4590 is to 489813; on account of which the motion of the equinoctial points will be diminished in the same ratio. Therefore, the annual motion of the equinoctial points of the body, composed of both ring and sphere, will be to the motion **20° 11' 46'' as 1436 to 39343 and 4590 to 489813** conjointly, that is, as 100 to 292369. But the forces by which the nodes of a number of moons (as we explained above), and therefore by which the equinoctial points of the ring recede (that is, the forces 3IT, in fig., Prop. xxx), are in the several particles as the distances of those particles from the plane QR; and by these forces the particles recede from that plane: and therefore (by Lem. 11) if the matter of the ring was spread all over the surface of the sphere, after the fashion of the figure *VapAFepE*, in order to make up that exterior part of the earth, the total force or power of all the particles to wheel about the earth round any diameter of the equator, and therefore to move the equinoctial points, would become less than before in the proportion of 2 to 5. Therefore the annual regress of the equinoxes now would be **to 20° 11' 46'' as 10 is to 73092; that is, would be 9'' 56''' 50^v.**

But because the plane of the equator is inclined to that of the ecliptic, this motion is to be diminished in the ratio of the sine 91706 (which is the cosine of $23\frac{1}{4}$ degrees) to the radius 100000; and the remaining motion will now be 9'' 7''' 20^{iv}, which is the annual precession of the equinoxes arising from the force of the sun.

But the force of the moon to move the sea was to the force of the sun nearly as 4.4815 to 1; and the force of the moon to move the equinoxes is to that of the sun in the same proportion. Whence the annual precession of the equinoxes proceeding from the force of the moon comes out 40'' 52''' 52^{iv}, and the total annual precession arising from the united forces of both will be 50'' 00''' 12^{iv}, the amount of which motion agrees with the phenomena; for the precession of the equinoxes, by astronomical observations, is about 50'' yearly.

If the height of the earth at the equator exceeds its height at the poles by more than $17\frac{1}{6}$ miles, the matter thereof will be more rare near the surface than at the centre; and the precession of the equinoxes will be augmented by the excess of height, and diminished by the greater rarity.

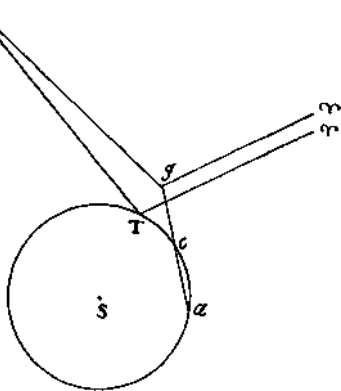
And now we have described the system of the sun, the earth, moon, and planets, it remains that we add something about the comets.

L E M M A I V

The comets are more remote than the moon, and are in the regions of the planets.

As the comets were placed by astronomers beyond the moon, because they were found to have no diurnal parallax, so their annual parallax is a convincing proof of their descending into the regions of the planets; for all the comets which move in a direct course according to the order of the signs, about the end of their appearance become more than ordinarily slow or retrograde, if the earth is between them and the sun; and more than ordinarily swift, if the earth is approaching to a heliocentric opposition with them; on the other hand, those which move against the order of the signs, towards the end of their appearance appear swifter than they ought to be, if the earth is between them and the sun; and slower, and perhaps retrograde, if the earth is in the other side of its orbit. And these appearances proceed chiefly from the diverse situations which the earth acquires in the course of its motion, after the same manner as it happens to the planets, which appear sometimes retrograde, sometimes more slowly, and sometimes more swiftly, progressive, according as the motion of the earth falls in with that of the planet, or is directed in the contrary way. If the earth move the same way with the comet, but, by an angular motion about the sun, so much swifter that right lines drawn from the earth to the comet converge towards the parts beyond the comet, the comet seen from the earth, because of its slower motion, will appear retrograde; and even if the earth is slower than the comet, the motion of the earth being subtracted, the motion of the comet will at least appear retarded; but if the earth tends the contrary way to that of the comet, the motion of the comet will from thence appear accelerated; and from this apparent acceleration, or retardation, or

would have arrived in the V time of the last observation, if it had continued to move uniformly in the right line ac. Therefore, if we draw g parallel to T T', and make the angle TgV equal to the angle TQG, this angle g V will be equal to the longitude of the comet seen from the place g, and the angle TVg will be the parallax which arises from the earth's being transferred from the place q into the



place T; and therefore V will be the place of the comet in the plane of the ecliptic. And this place V is commonly lower than the orbit of Jupiter.

The same thing may be deduced from the incurvation of the way of the comets; for these bodies move almost in great circles, while their velocity is great; but about the end of their course, when that part of their apparent motion which arises from the parallax bears a greater proportion to their whole apparent motion, they commonly deviate from those circles, and when the earth goes to one side, they deviate to the other; and this deflection, because of its corresponding with the motion of the earth, must arise chiefly from the parallax; and the quantity thereof is so considerable, as, by my computation, to place the disappearing comets a good deal lower than Jupiter. Hence it follows that when they approach nearer to us in their perigees and perihehons they often descend below the orbits of Mars and the inferior planets.

The near approach of the comets is further confirmed from the light of their heads; for the light of a celestial body, illuminated by the sun, and receding to remote parts, diminishes as the fourth power of the distance; namely, as the square, on account of the increase of the distance from the sun, and as another square, on account of the decrease of the apparent diameter. Therefore, if both the quantity of light and the apparent diameter

of a comet are given, its distance will be given also, by taking the distance of the comet to the distance of a planet directly as their diameters and inversely as the square root of their lights. Thus, in the comet of the year 1682, Mr. *Flamsteed* observed with a telescope of 16 feet, and measured with a micrometer, the least diameter of its head, 2' 00"; but the nucleus or star in the middle of the head scarcely amounted to the tenth part of this measure, and therefore its diameter was only 11" or 12"; but in the light and splendor of its head it surpassed that of the comet in the year 1680, and might be compared with the stars of the first or second magnitude. Let us suppose that Saturn with its ring was about four times more lucid; and because the light of the ring was almost equal to the light of the globe within, and the apparent diameter of the globe is about 21", and therefore the united light of both globe and ring would be equal to the light of a globe whose diameter is 30", it follows that the distance of the comet was to the distance of Saturn inversely as 1 to $\sqrt{4}$, and directly as 12" to 30"; that is, as 24 to 30, or 4 to 5. Again; the comet in the month of *April*, 1665, as *Hewelcle* informs us, excelled almost all the fixed stars in splendor, and even Saturn itself, as being of a much more vivid color; for this comet was more lucid than that other which had appeared about the end of the preceding year, and had been compared to the stars of the first magnitude. The diameter of its head was about 6'; but the nucleus, compared with the planets by means of a telescope, was plainly less than Jupiter; and sometimes judged less, sometimes judged equal, to the globe of Saturn within the ring. Since, then, the diameters of the heads of the comets seldom exceed 8' or 12', and the diameter of the nucleus or central star is but about a tenth or perhaps fifteenth part of the diameter of the head, it appears that these stars are generally of about the same apparent magnitude with the planets. But since their light may be often compared with the light of Saturn, yea, and sometimes exceeds it, it is evident that all comets in their perihelions must either be placed below or not far above Saturn; and they are much mistaken who remove them almost as far as the fixed stars; for if it were so, the comets could receive no more light from our sun than our planets do from the fixed stars.

So far we have gone, without considering the obscuration which comets suffer from that plenty of thick smoke which encompasses their heads, and through which the heads always show dull, as through a cloud. But the

more a body is obscured by this smoke, the nearer must it be allowed to come to the sun, that it may vie with the planets in the quantity of light which it reflects. Hence it is probable that the comets descend far below the orbit of Saturn, as we proved before from their parallax. But, above all, the thing is evinced from their tails, which must be due either to the sun's light reflected by a smoke arising from them, and dispersing itself through the ether, or to the light of their own heads. In the former case, we must shorten the distance of the comets, lest we be obliged to allow that the smoke arising from their heads is propagated through such a vast extent of space, and with such a velocity and expansion as will seem altogether incredible; in the latter case, the whole light of both head and tail is to be ascribed to the central nucleus. But, then, if we suppose all this light to be united and condensed within the disk of the nucleus, certainly the nucleus will by far exceed Jupiter itself in splendor, especially when it emits a very large and lucid tail. If, therefore, under a less apparent diameter, it reflects more light, it must be much more illuminated by the sun, and therefore much nearer to it; and the same argument will bring down the heads of comets sometimes within the orbit of Venus, viz., when, being hid under the sun's rays, they emit such huge and splendid tails, like beams of fire, as sometimes they do; for if all that light was supposed to be gathered together into one star, it would sometimes exceed not one Venus only, but a great many such united into one.

Lastly, the same thing is inferred from the light of the heads, which increases in the recess of the comets from the earth towards the sun, and decreases in their return from the sun towards the earth. Thus, the comet of the year 1665 (by the observations of *Hewelcke*), from the time that it was first seen, was always losing of its apparent motion, and therefore had already passed its perigee; but yet the splendor of its head was daily increasing, till, being hid under the sun's rays, the comet ceased to appear. The comet of the year 1683 (by the observations of the same *Hewelcke*), about the end of *July*, when it first appeared, moved at a very slow rate, advancing only about 40 or 45 minutes in its orbit in a day's time; but from that time its diurnal motion was continually upon the increase, till *September 4*, when it arose to about 5 degrees; and therefore, in all this interval of time, the comet was approaching to the earth. This is likewise proved from the diam-

eter of its head, measured with a micrometer; for, on *August 6*, *Hewelcle* found it only 6' 5", including the coma, which, on *September 2*, he observed to be 9' 7"; and therefore its head appeared far less about the beginning than towards the end of the motion, though about the beginning, because nearer to the sun, it appeared far more lucid than towards the end, as the same *Hewelcle* declares. Therefore in all this interval of time, on account of its recess from the sun, it decreased in splendor, notwithstanding its approach towards the earth. The comet of the year 1618, about the middle of *December*, and that of the year 1680, about the end of the same month, did both move with their greatest velocity, and were therefore then in their perigees, but the greatest splendor of their heads was seen two weeks before, when they had just got clear of the sun's rays, and the greatest splendor of their tails a little earlier, when yet nearer to the sun. The head of the former comet (according to the observations of *Cysat*), on *December 1*, appeared greater than the stars of the first magnitude; and, on *December 16* (then in the perigee), it was diminished but little in magnitude, but much diminished in the splendor and brightness of its light. On *January 7*, *Kepler*, being uncertain about the head, left off observing. On *December 12*, the head of the latter comet was seen and observed by Mr. *Flamsteed*, when but 9 degrees distant from the sun, which is scarcely to be done in a star of the third magnitude. On *December 15* and *17*, it appeared as a star of the third magnitude, its luster being diminished by the brightness of the clouds near the setting sun. On *December 26*, when it moved with the greatest velocity, being almost in its perigee, it was less than the mouth of *Pegasus*, a star of the third magnitude. On *January 3*, it appeared as a star of the fourth. On *January 9*, as one of the fifth. On *January 13*, it was hid by the splendor of the moon, then in her increase. On *January 25*, it was scarcely equal to the stars of the seventh magnitude. If we compare equal intervals of time, taken on one side of the perigee and then on the other, we shall find that the head of the comet, which at both intervals of time was far, but yet equally removed from the earth, and should therefore have shone with equal splendor, appeared brightest on the side of the perigee towards the sun, and disappeared on the other. Therefore, from the great difference of light in the one situation and in the other, we conclude the great vicinity of the sun and comet in the former, for the light of comets tends to be reg-

ular, and to appear greatest when the heads move fastest, and are therefore in their perigees, except so far as it is increased by their nearness to the sun.

COR. 1. Therefore the comets shine by the sun's light, which they reflect.

COR. 11. From what has been said, we may likewise understand why comets are so frequently seen in that region in which the sun is, and so seldom in the other. If they were visible in the regions far above Saturn, they would appear more frequently in the parts opposite to the sun; for such as were in those parts would be nearer to the earth, whereas the presence of the sun must obscure and hide those that appear in the region in which he is. Yet, looking over the history of comets, I find that four or five times more have been seen in the hemisphere towards the sun than in the opposite hemisphere; besides, without doubt, not a few, which have been hid by the light of the sun: for comets descending into our parts neither emit tails, nor are so well illuminated by the sun as to reveal themselves to our naked eyes, until they have come nearer to us than Jupiter. But the far greater part of that spherical space, which is described about the sun with so small a radius, lies on that side of the earth which faces the sun; and the comets in that greater part are commonly more strongly illuminated, for they are for the most part nearer to the sun.

COR. HI. Hence also it is evident that the celestial spaces are void of resistance; for though the comets are carried in oblique paths, and sometimes contrary to the course of the planets, yet they move every way with the greatest freedom, and preserve their motions for an exceeding long time, even where contrary to the course of the planets. I am out in my judgment, if they are not a sort of planets revolving in orbits returning into themselves with a continual motion; for the opinion of some writers, that they are no other than meteors, an opinion based on the continual changes that happen to their heads, seems to have no foundation; for the heads of comets are encompassed with huge atmospheres, and the lowermost parts of these atmospheres must be the densest; and therefore it is in the clouds only, not in the bodies of the comets themselves, that these changes are seen. Thus the earth, if it were viewed from the planets, would, without all doubt, shine by the light of its clouds, and the solid body would scarcely appear through the surrounding clouds. Thus also the belts of Jupiter are formed in the clouds of that planet, for they change their position to each other, and the

solid body of Jupiter is hardly to be seen through them; and much more must the bodies of comets be hid under their atmospheres, which are both deeper and thicker.

PROPOSITION XL. THEOREM XX

That the comets move in some of the conic sections, having their foci in the centre of the sun, and by radii drawn to the sun describe areas proportional to the times.

This Proposition appears from Cor. i, Prop, xiii, Book i, compared with Prop, VIII, xii, and xiii, Book iii.

COR. I. Hence if comets revolve in orbits returning into themselves, the orbits will be ellipses; and their periodic times will be to the periodic times of the planets as the n th power of their principal axes. And therefore the comets, which for the most part of their course are more remote than the planets, and upon that account describe orbits with greater axes, will require a longer time to finish their revolutions. Thus if the axis of a comet's orbit was four times greater than the axis of the orbit of Saturn, the time of the revolution of the comet would be to the time of the revolution of Saturn, that is, to 30 years, as 4^4 (256) is to 1, and would therefore be 240 years.

COR. n. But their orbits will be so near to parabolas, that parabolas may be used for them without sensible error.

COR. in. And, therefore, by Cor. vn, Prop, xvi, Book 1, the velocity of every comet will always be to the velocity of any planet, supposed to be revolved at the same distance in a circle about the sun, nearly as the square root of double the distance of the planet from the centre of the sun to the distance of the comet from the sun's centre. Let us suppose the radius of the great orbit, or the greatest semidiameter of the ellipse which the earth describes, to consist of 100000000 parts; and then the earth by its mean diurnal motion will describe 1720212 of those parts, and $71675\frac{1}{2}$ by its hourly motion. And therefore the comet, at the same mean distance of the earth from the sun, with a velocity which is to the velocity of the earth as $\sqrt{2}$ to 1, would by its diurnal motion describe 2432747 parts, and $101364\frac{1}{2}$ parts by its hourly motion. But at greater or less distances both the diurnal and hourly motion will be to this diurnal and hourly motion inversely as the square root of the distances, and is therefore given.¹

¹ Appendix, Note 48]

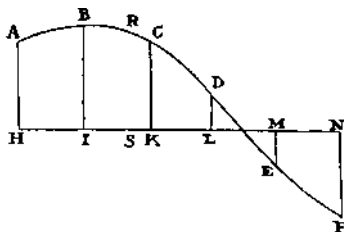
COR. IV. Therefore if the latus rectum of the parabola is four times the radius of the great orbit, and the square of that radius is supposed to consist of 100000000 parts, the area which the comet will daily describe by a radius drawn to the sun will be 1216373M2 parts, and the hourly area will be 50682¼ parts. But, if the latus rectum is greater or less in any ratio, the diurnal and hourly area will be less or greater inversely as the square root of that ratio.

L E M M A V

To find a curved line of the parabolic kind which shall pass through any given number of points?

Let those points be A, B, C, D, E, F, &c, and from the same to any right line HN, given in position, let fall as many perpendiculars AH, BI, CK, DL, EM, FN, &c.

b	$2b$	$3b$	$4b$	$5b$
c	$2c$	$3c$	$4c$	
d	$2d$	$3d$		
e	$2e$			
f				



CASE i. If HI, IK, KL, &c, the intervals of the points H, I, K, L, M, N, &c, are equal, take b , ib ,

$3b$, ajb , $5b$, &c, the first differences of the perpendiculars AH, BI, CK, &c; their second differences, c , $2c$, y , qc , &c.; their third, d , $2d$, $3d$, &c, that is to say, so as AH - BI may be = b , BI - CK = $2b$, CK - DL = $3b$, DL + EM = $4b$, - EM + FN = $5b$, &c.; then $b-2b-c$, &c, and so on to the last difference, which is here f . Then, erecting any perpendicular RS, which may be considered as an ordinate of the curve required, in order to find the length of this ordinate, suppose the intervals HI, IK, KL, LM, &c, to be units, and let AH = a , - HS = p , $\sqrt{2p}$ into - IS = q , $1/3q$ into + SK = r , $\sqrt{r}$ into + SL = s , $\sqrt{s}$ into + SM = t proceeding in this manner, to ME, the last perpendicular but one, and prefixing negative signs before the terms HS, IS, &c, which lie from S towards A; and positive signs before the terms SK, SL, &c, which lie on the other side of the point S; and, observing well the signs, RS will be = $a + bp + cq + dr + es + ft, + \&c.$

[¹ Appendix, Note 49]

CASE 2. But if HI, IK, &c, the intervals of the points H, I, K, L, &c, are unequal, take $b, 2b, 3b, 4b, 5b, \&c.$, the first differences of the perpendiculars AH, BI, CK, &c, divided by the intervals between those perpendiculars; $c, 2c, 3c, 4c, \&c.$, their second differences, divided by the intervals between every two; $d, 2d, 3d, \&c.$, their third differences, divided by the intervals between every three; $e, 2e, \&c.$, their fourth differences, divided by the intervals between every four; and so forth; that is, in such manner, that b may be $= \frac{AH - BI}{HI}$, $2b = \frac{BI - CK}{IK}$, $3b = \frac{CK - DL}{KL}$, &c., then $c = \frac{b - 2b}{HK}$, $2c = \frac{2b - 3b}{IL}$, $3c = \frac{3b - 4b}{KM}$, &c., then $d = \frac{c - 2c}{HL}$, $2d = \frac{2c - 3c}{IM}$, &c. And those differences being found, let AH be $= a$, $-HS = p$, p into $-IS = q$, q into $+SK = r$, r into $+SL = s$, s into $+SM = t$; proceeding in this manner to ME, the last perpendicular but one; and the ordinate RS will be $= a + bp + cq + dr + es + ft + \&c.$

COR. Hence the areas of all curves may be nearly found; for if some number of points of the curve to be squared are found, and a parabola be supposed to be drawn through those points, the area of this parabola will be nearly the same with the area of the curvilinear figure proposed to be squared: but the parabola can be always squared geometrically by methods generally known.

LEMMA VI

Certain observed places of a comet being given, to find the place of the same at any intermediate given time.

Let HI, IK, KL, LM (in the preceding fig.) represent the times between the observations; HA, IB, KC, LD, ME, five observed longitudes of the comet; and HS the given time between the first observation and the longitude required. Then if a regular curve ABCDE is supposed to be drawn through the points A, B, C, D, E, and the ordinate RS is found out by the preceding Lemma, RS will be the longitude required.

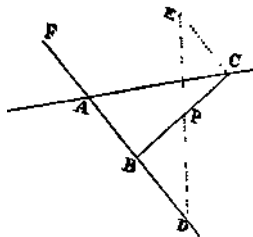
By the same method, from five observed latitudes, we may find the latitude at a given time.

If the differences of the observed longitudes are small, let us say 4 or 5 degrees, then three or four observations will be sufficient to find a new longitude and latitude; but if the differences are greater, as of 10 or 20 degrees, five observations ought to be used.

LEMMA VII

Through a given point P to draw a right line BC, whose parts PB, PC, cut off by two right lines AB, AC, given in position, may be one to the other in a given ratio.

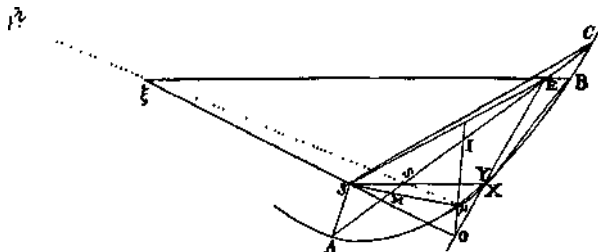
From the given point P suppose any right line PD to be drawn to either of the right lines given, as AB; and produce the same towards AC, the other given right line, as far as E, so as PE may be to PD in the given ratio. Let EC be parallel to AD. Draw CPB, and PC will be to PB as PE to PD. Q.E.F.



LEMMA VIII

Let ABC be a parabola, having its focus in S. By the chord AC bisected in I cut off the segment ABCI, whose diameter is In and vertex M. In IM produced take MO equal to one-half of IM. Join OS, and produce it to i, so that Si may be equal to 2SO. Now, supposing a comet to revolve in the arc CBA, draw EB, cutting AC in E: I say, the point E will cut off from the chord AC the segment AE, nearly proportional to the time.

For if we join EO, cutting the parabolic arc ABC in Y, and draw MX touching the same arc in the vertex M, and meeting EO in X, the curvilinear



area AEXMA will be to the curvilinear area ACYMA as AE to AC; and, therefore, since the triangle ASE is to the triangle ASC in the same ratio,

the whole area ASEXMA will be to the whole area ASCYMA as AE is to AC. But, because £O is to SO as 3 to 1, and EO to XO in the same ratio, SX will be parallel to EB; and, therefore, joining BX, the triangle SEB will be equal to the triangle XEB. Therefore, if to the area ASEXMA we add the triangle EXB, and from the sum subtract the triangle SEB, there will remain the area ASBXMA, equal to the area ASEXMA, and therefore in the ratio to the area ASCYMA as AE to AC. But the area ASBYMA is nearly equal to the area ASBXMA; and this area ASBYMA is to the area ASCYMA as the time of description of the arc AB is to the time of description of the whole arc AC; and, therefore, AE is to AC nearly in the proportion of the times. Q.E.D.

COR. When the point B falls upon the vertex M of the parabola, AE is to AC accurately in the proportion of the times.

SCHOLIUM

If we join M£ cutting AC in 8, and in it take *in* in proportion to MB as 27MI to 16 MM, and draw *Bn*, this *Bn* will cut the chord AC, in the ratio of the times, more accurately than before; but the point *n* is to be taken beyond or on this side the point £, according as the point B is more or less distant from the principal vertex of the parabola than the point M.

LEMMA IX

AI²
The right lines IM and MM, and the length —[^], are equal among themselves.

For 4SM is the latus rectum of the parabola belonging to the vertex M.

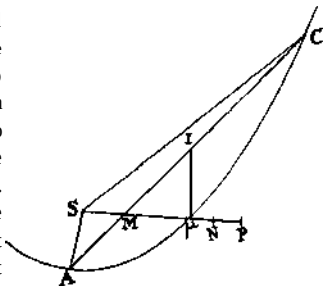
LEMMA X¹

Produce SM to N and P, so that MN may be one-third of MI, and SP may be to SN as SN to SM; and in the time that a comet would describe the arc AMC, // it was supposed to move always forwards with the velocity which it has in a height equal to SP, it would describe a length equal to the chord AC.

For if the comet with the velocity which it hath in M was in the said time supposed to move uniformly forwards in the right line which touches the

[¹ Appendix, Note 50]

parabola in M, the area which it would describe by a radius drawn to the point S would be equal to the parabolic area ASCuA; and therefore the space contained under the length described in the tangent and the length SM would be to the space contained under the lengths AC and SM as the area ASCMA is to the triangle ASC, that is, as SN to SM. Therefore AC is to the length described in the tangent as SM to SN. But since the velocity of the comet in the height SP (by Cor. vi, Prop. xvi, Book i) is to the velocity of the same in the height SM, inversely as the square root of SP to SM, that is, in the ratio of SM to SN, it follows that the length described with this velocity will be to the length in the same time described in the tangent, as SM to SN. Therefore since AC, and the length described with this new velocity, are in the same proportion to the length described in the tangent, they must be equal between themselves. Q.E.D.



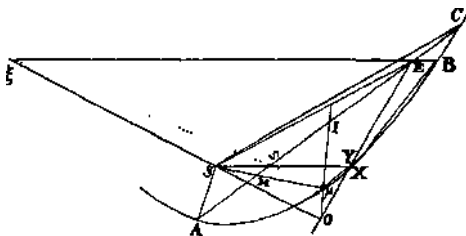
COR. Therefore a comet, with that velocity which it hath in the height SM $4 - \frac{2}{3}M$, would in the same time nearly describe the chord AC

LEMMA XI

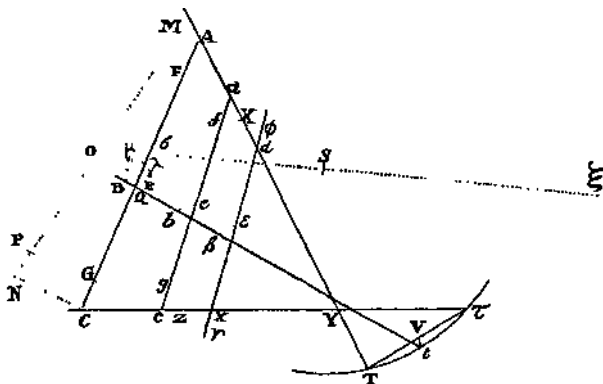
// a comet void of all motion was let fall from the height SN, or $SM + \frac{1}{2}IM$, towards the sun, and was still impelled to the sun by the same force uniformly continued by which it was impelled at first, the same, in one-half of that time in which it might describe the arc AC in its own orbit, would in descending describe a space equal to the length IM.

For in the same time that the comet would require to describe the parabolic arc AC, it would (by the last Lemma), with that velocity which it hath in the height SP, describe the chord AC; and, therefore (by Cor. vii, Prop. xvi, Book i), if it was in the same time supposed to revolve by the force of its own gravity in a circle whose semidiameter was SP, it would

Let S represent the sun; T, /, T three places of the earth in the earth's orbit; TA, tB, TC three observed longitudes of the comet; V the time be-



tween the first observation and the second; W the time between the second and the third; X the length which in the whole time V f W the comet might describe with that velocity which it has in the mean distance of the



earth from the sun, which length is to be found by Cor. in, Prop, XL, Book HI; and /V a perpendicular upon the chord TT. In the mean observed longitude /B take at pleasure the point B, for the place of the comet in the plane of the ecliptic; and from thence, towards the sun S, draw the line BR, which may be to the perpendicular /V as the product of SB and Sr is to the cube

of the hypotenuse of the right-angled triangle whose sides are SB and the tangent of the latitude of the comet in the second observation to the radius /B. And through the point E (by Lem. VII) draw the right line AEC, whose parts AE and EC, terminating in the right lines TA and TC, may be one to the other as the times V and W: then A and C will be nearly the places of the comet in the plane of the ecliptic in the first and third observations, if B was its place rightly assumed in the second.

Upon AC, bisected in I, erect the perpendicular I/. Through B imagine the line B* drawn parallel to AC. Imagine the line Si drawn, cutting AC in A, and complete the parallelogram zIAn. Take lo equal to 3IA; and through the sun S imagine the line o£ drawn equal to 3S0 4- 3/A. Then, canceling the letters A, E, C, I, from the point B towards the point £, imagine the new line BE drawn, which may be to the former BE as the square of the ratio of the distance BS to the quantity $Su + \frac{1}{2}/A$. And through the point E draw again the right line AEC by the same rule as before; that is, so that its parts AE and EC may be one to the other as the times V and W between the observations. Thus A and C will be the places of the comet more accurately.

Upon AC, bisected in I, erect the perpendiculars AM, CN, 10, of which AM and CN may be the tangents of the latitudes in the first and third observations, to the radii TA and TC. Join MN, cutting IO in O. Draw the rectangular parallelogram /Uu, as before. In IA produced take ID equal to $Su + -2/3k$. Then in MN, towards N, take MP, which may be to the above-found length X as the square root of the ratio of the mean distance of the earth from the sun (or of the semidiameter of the earth's orbit) to the distance OD. If the point P fall upon the point N; A, B, and C will be three places of the comet, through which its orbit is to be described in the plane of the ecliptic. But if the point P falls not upon the point N, in the right line AC take CG equal to NP, so that the points G and P may be on the same side of the line NC.

By the same method as the points E, A, C, G were found from the assumed point B, from other points *b* and *ß* assumed at pleasure, find out the new points *e*, *a*, *c*, *g*, and *c*, *a*, *K*, *Y*. Then through G, *g*, and Y draw the circumference of a circle Ggy, cutting the right line TC in Z: and Z will be one place of the comet in the plane of the ecliptic. And in AC, *ac*, *a**, taking AF, *af*, *a<l>*, equal respectively to CG, *eg*, KY; through the points F, *f*, and *<t>*,

draw the circumference of a circle Ff , cutting the right line AT in X ; and the point X will be another place of the comet in the plane of the ecliptic. And at the points X and Z , erecting the tangents of the latitudes of the comet to the radii TX and TZ , two places of the comet in its own orbit will be determined. Lastly, if (by Prop, xix, Book 1) to the focus S a parabola is described passing through those two places, this parabola will be the orbit of the comet. Q.E.I.

The demonstration of this construction follows from the preceding Lemmas, because the right line AC is cut in E in the proportion of the times, by Lemma vn, as it ought to be by Lemma vm; and BE , by Lemma xi, is a portion of the right line BS or $B\mathcal{E}$ in the plane of the ecliptic, intercepted between the arc ABC and the chord AEC ; and MP (by Cor., Lcm. x) is the length of the chord of that arc, which the comet should describe in its proper orbit between the first and third observations, and therefore is equal to MN , providing B is a true place of the comet in the plane of the ecliptic.

But it will be convenient to assume the points B , b , P , not at random, but nearly true. If the angle AQt , at which the projection of the orbit in the plane of the ecliptic cuts the right line $/B$, is roughly known, at that angle with Bt draw the line AC , which may be to KT as the square root of the ratio of SQ to St ; and, drawing the right line SEB so as its part EB may be equal to the length Vt , the point B will be determined, which we are to use for the first time. Then, canceling the right line AC and drawing anew AC according to the preceding construction, and, moreover, finding the length MP , in $/B$ take the point b , by this rule, that, if TA and TC intersect each other in Y , the distance Yb may be to the distance YB in a ratio compounded of the ratio of MP to MN , and the square root of the ratio of SB to Sb . And by the same method you may find the third point P , if you please to repeat the operation the third time; but if this method is followed, two operations generally will be sufficient; for if the distance BZ happens to be very small, after the points F , f , and G , g , are found, draw the right lines Yf and Gg , and they will cut TA and TC in the points required, X and Z .

EXAMPLE

Let the comet of the year 1680 be proposed. The following table shows the motion thereof, as observed by *Vlamsteed*, and calculated afterwards

by him from his observations, and corrected by Dr. *Halley* from the same observations.

	Time		Sun's longitude	Comet's	
	Apparent	True		Longitude	Latitude north
	h m	h m s	° ' "	° ' "	° ' "
1680, Dec. 12	4 46	4 46 0	♊ 1 51 23	♊ 6 32 30	8 28 0
21	6 32 $\frac{1}{2}$	6 36 59	11 06 44	♋ 5 08 12	21 42.13
24	6 12	6 17 52	14 09 26	18 49 23	25 23 5
26	5 14	5 20 44	16 09 22	28 24 13	27 00 52
29	7 55	8 03 02	19 19 43	♋ 13 10 41	28 09 58
30	8 02	8 10 26	20 21 09	17 38 20	28 11 53
1681, Jan. 5	5 51	6 01 38	26 22 18	♋ 8 48 53	26 15 7
9	6 49	7 00 53	♋ 0 29 02	18.44 04	24 11 56
10	5 54	6 06 10	1 27 43	20 40 50	23 43 52
13	6 56	7 08 55	4 33 20	25 59 48	22 17 28
25	7 44	7 58 42	16 45 36	♌ 9 35 0	17 56 30
30	8 07	8 21 53	21 49 58	13 19 51	16 42 18
Feb. 2	6 20	6 34 51	24 46 59	15 13 53	16 04 1
5	6 50	7 04 41	27 49 51	16 59 06	15 27 3

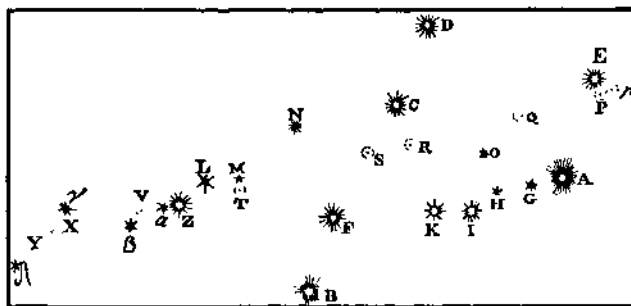
To these you may add some observations of mine.¹

	Apparent time	Comet's	
		Longitude	Latitude north
	h m	° ' "	° ' "
1681, Feb. 25	8 30	♌ 26 18 35	12 46 46
27	8 15	27 04 30	12 36 12
Mar. 1	11 0	27 52 42	12 23 40
2	8 0	28 12 48	12 19 38
5	11 30	29 18 0	12 03 16
7	9 30	♍ 0 4 0	11 57 0
9	8 30	0 43 4	11 45 52

These observations were made by a telescope of 7 feet, with a micrometer and threads placed in the focus of the telescope; by these instruments we

[¹ Appendix, Note 51]

determined the positions both of the fixed stars among themselves, and of the comet in respect of the fixed stars. Let A represent the star of the fourth magnitude in the left heel of Perseus (*Bayer's* θ), B the following star of the third magnitude in the left foot (*Bayer's* ι), C a star of the sixth magni-



tude (*Bayer's* ι) in the heel of the same foot, and D, E, F, G, H, I, K, L, M, N, O, Z, α , β , γ , δ other smaller stars in the same foot; and let p, P, Q, R, S, T, V, X represent the places of the comet in the observations above set down; and, reckoning the distance AB of $80\frac{1}{2}$ parts, AC was $52\frac{1}{4}$ of those parts; BC, $58\frac{3}{4}$; AD, $57\frac{1}{12}$; BD, $82\frac{1}{11}$; CD, $23\frac{3}{8}$; AE, $29\frac{1}{4}$; CE, $57\frac{1}{2}$; DE, $49\frac{11}{12}$; AI, $27\frac{1}{12}$; BI, $52\frac{1}{6}$; CI, $36\frac{1}{12}$; DI, $53\frac{3}{11}$; AK, $38\frac{3}{8}$; BK, 43; CK

The fixed stars	Their longitudes	Their latitude north	The fixed stars	Their longitudes	Their latitude north
	$\circ$ $^{\circ}$ $'$ $''$	$\circ$ $'$ $''$		$\circ$ $^{\circ}$ $'$ $''$	$\circ$ $'$ $''$
A	γ 36 41 50	12 8 36	I,	γ 29 33 34	12 7 48
B	28 40 23	11 17 54	M	29 18 54	12 7 20
C	27 58 30	12 40 25	N	28 48 29	12 31 9
E	26 27 17	12 52 7	Z	29 44 48	11 57 13
F	28 28 37	11 52 22	α	29 52 3	11 55 48
G	26 56 8	12 4 58	β	π 0 8 23	11 48 56
H	27 11 45	12 2 1	γ	0 40 10	11 55 18
I	27 25 2	11 53 11	δ	1 3 20	11 30 42
K	27 42 7	11 53 26			

$31\frac{5}{8}$; FK, 29; FB, 23; FC, $36\frac{1}{4}$; AH, $18\frac{6}{7}$; DH, $50\frac{7}{8}$; BN, $46\frac{5}{12}$; CN, $31\frac{1}{3}$; 3L, $45\frac{5}{12}$; NL, $31\frac{1}{4}$. HO was to HI as 7 to 6, and, produced, did pass between the stars D and E, so as the distance of the star D from this right line was $\frac{1}{8}$ CD. LM was to LN as 2 to 9, and, produced, did pass through the star H. Thus were the positions of the fixed stars determined in respect to one another.

Mr. Pound has since observed a second time the positions of these fixed stars amongst themselves, and obtained their longitudes and latitudes according to the preceding table (p. 509).

The positions of the comet to these fixed stars were observed to be as follows:

Friday, *February* 25, o.s., at $8\frac{1}{2}^h$. P.M., the distance of the comet in *p* from the star E was less than $\frac{2}{3}$ AE, and greater than $\frac{1}{3}$ AE, and therefore nearly equal to $\frac{2}{3}$ AE; and the angle *ApE* was a little obtuse, but almost right. For from A, letting fall a perpendicular on *pE*, the distance of the comet from that perpendicular was $\frac{1}{3}$ *pE*.

The same night, at $9\frac{1}{2}^h$, the distance of the comet in P from the star E was greater than $\frac{1}{3}$ AE, and less than $\frac{1}{2}$ AE, and therefore nearly equal to $\frac{1}{2}$ of AE, or $\frac{2}{3}$ AE. But the distance of the comet from the perpendicular st fall from the star A upon the right line PE was $\frac{1}{6}$ PE.

Sunday, *February* 27, $8\frac{1}{4}^h$. P.M., the distance of the comet in Q from the star O was equal to the distance of the stars O and H; and the right line QO produced passed between the stars K and B. I could not, by reason of interring clouds, determine the position of the star to greater accuracy.

Tuesday, *March* 1, n^h . P.M., the comet in R lay exactly in a line between the stars K and C, so as the part CR of the right line CRK was a little greater than $\frac{1}{3}$ CK, and a little less than $\frac{1}{3}$ CK + $\frac{1}{8}$ CR, and therefore = $\frac{1}{3}$ CK + $\frac{1}{8}$ CR, or $\frac{16}{24}$ CK.

Wednesday, *March* 2, 8^h . P.M., the distance of the comet in S from the star I was nearly $\frac{1}{3}$ FC; the distance of the star F from the right line CS produced was $\frac{1}{2}$ FC; and the distance of the star B from the same right line was five times greater than the distance of the star F; and the right line NS produced passed between the stars H and I five or six times nearer to the

Saturday, *March 5*, 11 $\frac{1}{2}$ ^h. P.M., when the comet was in T, the right line MT was equal to $\frac{1}{2}$ ML, and the right line LT produced passed between B and F four or five times nearer to F than to B, cutting off from BF a fifth or sixth part thereof towards F; and MT produced passed on the outside of the space BF towards the star B four times nearer to the star B than to the star F. M was a very small star, scarcely to be seen by the telescope; but the star L was greater, and of about the eighth magnitude.

Monday, *March 7*, 9 $\frac{1}{2}$ ^h. P.M., the comet being in V, the right line Va produced did pass between B and F, cutting off, from BF towards F, $\frac{1}{10}$ of BF, and was to the right line V β as 5 to 4. And the distance of the comet from the right line *af* was $\frac{1}{2}$ V β .

Wednesday, *March 9*, 8 $\frac{1}{2}$ ^h. P.M., the comet being in X, the right line yX was equal to $\frac{1}{4}$ y δ ; and the perpendicular let fall from the star B upon the right line yX was $\frac{2}{5}$ of y δ .

The same night, at 12^h., the comet being in Y, the right line YY was equal to $\frac{1}{3}$ of y δ , or a little less, as perhaps "u\ of Y<§ and a perpendicular let fall from the star 8 on the right line YY was equal to about $\frac{1}{6}$ or $\frac{1}{7}$ y δ . But the comet, being then extremely near the horizon, was scarcely discernible, and therefore its place could not be determined with the same certainty as in the foregoing observations.

From these observations, by constructions of figures and calculations, I deduced the longitudes and latitudes of the comet; and Mr. *Pound*, by correcting the places of the fixed stars, has determined more correctly the places of the comet, which correct places are set down above. Though my micrometer was none of the best, yet the errors in longitude and latitude (as derived from my observations) scarcely exceed one minute. The comet (according to my observations), about the end of its motion, began to decline sensibly towards the north, from the parallel which it described about the end of *February*.

Now, in order to determine the orbit of the comet from the observations above described, I selected those three which *Ylamsteed* made (*Dec. 21*, *Jan. 5*, and *Jan. 25*); from which I found *St* of 9842.1 parts, and *Vt* of 455, supposing the semidiameter of the earth's orbit contains 10000. Then, for the first observation, assuming *tB* of 5657 of those parts, I found *SB* 9747, *BE*

X 8528.4, PM 8450, MN 8475, NP 25; from this, by the second operation, I Obtained the distance *tb* 5640; and by this operation I at last deduced the distances TX 4775 and TZ 11322. From these values, determining the orbit, I found its descending node in 35, and ascending node in VS $10^{\circ} 53'$; the inclination of its plane to the plane of the ecliptic $61^{\circ} 20'$, the vertex thereof (or the perihelion of the comet) distant from the node $8^{\circ} 38'$, and in $11^{\circ} 43'$, with latitude $7^{\circ} 34'$ south; its latus rectum 236.8; and the diurnal area described by a radius drawn to the sun 93585, supposing the square of the semidiameter of the earth's orbit 100000000; that the comet in this orbit moved directly according to the order of the signs, and on Dec. 8^d. 00^h. 04TM. \ M . was in the vertex or perihelion of its orbit. All this I determined by scale and compass, and the chords of angles, taken from the table of natural sines, in a pretty large figure, in which, to wit, the radius of the earth's orbit (consisting of 10000 parts) was equal to 16V3. inches of an *English* foot.

Lastly, in order to discover whether the comet did truly move in the orbit so determined, I investigated its places in this orbit partly by arithmetical operations, and partly by scale and compass, to the times of some of the observations, as may be seen in the following table:

The Comet's							
	Distance from sun	Longitude computed	Latitude computed	Longitude observed	Latitude observed	Difference longitude	Difference latitude
Dec. 12	2792	$106^{\circ} 32'$	$8^{\circ} 18\frac{1}{2}'$	$106^{\circ} 31\frac{1}{2}'$	$8^{\circ} 26'$	+1	- $7\frac{1}{2}'$
29	8403	$113^{\circ} 13\frac{2}{3}'$	28 00	$113^{\circ} 11\frac{3}{4}'$	$28^{\circ} 10\frac{1}{2}'$	+2	- $10\frac{1}{2}'$
Feb. 5	16669	$117^{\circ} 00'$	$15^{\circ} 29\frac{2}{3}'$	$116^{\circ} 59\frac{2}{3}'$	$15^{\circ} 27\frac{2}{3}'$	+0	+ $2\frac{1}{4}'$
Mar 5	21737	$119^{\circ} 19\frac{3}{4}'$	12 4	$119^{\circ} 20\frac{1}{2}'$	$12^{\circ} 3\frac{1}{2}'$	-1	+ $1\frac{1}{4}'$

But afterwards Dr. *Halley* did determine the orbit to a greater accuracy by an arithmetical calculus than could be done by graphic operations; and, retaining the place of the nodes in S and VS $10^{\circ} 53'$, and the inclination of the plane of the orbit to the ecliptic $61^{\circ} 20'$, as well as the time of the comet's being in perihelion, Dec. 8^d. 00^h. 04TM. he found the distance of the perihelion from the ascending node measured in the comet's orbit $9^{\circ} 20'$, and the latus rectum of the parabola 2430 parts, supposing the mean distance of the sun from the earth to be 100000 parts; and from these data by

an accurate arithmetical calculus, he computed the places of the comet to the times of the observations as given in the table on this page.

This comet also appeared in the *November* before, and at *Coburg*, in *Saxony*, was observed by Mr. *Gottfried Kirch*, on the 4th of that month, on the 6th and nth, o.s.; from its positions to the nearest fixed stars observed with sufficient accuracy, sometimes with a two-foot, and sometimes with a ten-foot telescope; from the difference of longitudes of *Coburg* and *London*, 11° ; and from the places of the fixed stars observed by Mr. *Pound*, Dr. *Hcdley* has determined the places of the comet as follows:

Nov. 3, $17^{11}.2^m$., apparent time at *London*, the comet was in $Si\ 29^{\circ} 51'$, with $i^{\circ} 17' 45''$ latitude north.

Nov. 5, $15^{11}.58^m$., the comet was in $Ti\ 3^{\circ} 23'$, with $i^{\circ} 6'$ latitude north.

Nov. 10, $i6^h.31^{111}$., the comet was equally distant from two stars in $\langle SL$, which are designated *O* and *T* in *Bayer*; but it had not quite touched the right line that joins them, but was very little distant from it. In *Flamsteed's*

True time	The Comet's			Places in	
	Distance from the sun	Longitude computed	Latitude computed	Longitude	Latitude
d h m		α β γ	α β γ	β γ	β γ
<i>Dec.</i> 12 4 46	28028	γ 6 29 25	8 26 0 bor.	-3 5	-2 0
21 6 37	61076	α 5 6 30	21 43 20	-1 42	+1 7
24 6 18	70008	18 48 20	25 22 40	-1 3	-0 25
26 5 20	75576	28 22 45	27 1 36	-1 28	+0 44
29 8 3	84021	γ 13 12 40	28 10 10	+1 59	+0 12
30 8 10	86661	17 40 5	28 11 20	+1 45	-0 33
<i>Jan.</i> 5 6 1 $\frac{1}{2}$	101440	γ 8 49 49	26 15 15	+0 56	+0 8
9 7 0	110959	18 44 36	24 12 54	+0 32	+0 58
10 6 6	113162	20 41 0	23 44 10	+0 10	+0 18
13 7 9	120000	26 0 21	22 17 30	+0 33	+0 2
25 7 59	145370	γ 9 33 40	17 57 55	-1 20	+1 25
30 8 22	155303	13 17 41	16 42 7	-2 10	-0 11
<i>Feb.</i> 2 6 35	160951	15 11 11	16 4 15	-2 42	+0 14
5 7 4 $\frac{1}{2}$	166686	16 58 55	15 29 13	-0 41	+2 0
25 8 41	202570	26 15 46	12 48 0	-2 49	+1 10
<i>Mar.</i> 5 11 39	216205	29 18 35	12 5 40	+0 35	+2 14

catalogue this star θ was then in $\text{rfl } 14^{\circ}15'$, with $\text{i}^{\circ}41'$ lat. north nearly, and $\text{T in TIE } 17^{\circ}3\frac{1}{2}'$ with $\text{o}^{\circ}34'$ lat. south; and the middle point between those stars was $\text{TIR } 15^{\circ}39\frac{1}{4}'$, with $\text{o}^{\circ}33\frac{1}{2}'$ lat. north. Let the distance of the comet from that right line be about $10'$ or $12'$; and the difference of the longitude of the comet and that middle point will be $7'$; and the difference of the latitude nearly $7\frac{1}{2}''$; and thence it follows that the comet was in $\text{W } 15^{\circ}32'$, with about $26'$ lat. north.

The first observation from the position of the comet with respect to certain small fixed stars had all the exactness that could be desired; the second also was accurate enough. In the third observation, which was the least accurate, there might be an error of $6'$ or $7'$, but hardly greater. The longitude of the comet, as found in the first and most accurate observation, being

True time	Longitude observed	Latitude N observed	Longitude computed	Latitude computed	Errors in Longitude	Errors in Latitude
<i>Nov</i>	<i>d h m</i>	<i>u r s</i>	<i>u r s</i>	<i>u r s</i>	<i>r e</i>	<i>r e</i>
3 16 47	$\odot$ 29 51 0	1 17 45	$\odot$ 29 51 22	1 17 32 N	+0 22	-0 13
5 15 37	Π 3 23 0	1 6 0	Π 3 24 32	1 6 9	+1 32	+0 9
10 16 18	15 32 0	0 27 0	15 33 2	0 25 7	+1 2	-1 53
16 17 00			$\simeq$ 8 16 45	0 53 7 S		
18 21 34			18 52 15	1 26 54		
20 17 0			28 10 36	1 53 35		
23 17 5			Π 13 22 42	2 29 0		
<i>Dec</i>						
12 4 46	Ψ 6 32 30	8 28 0	Ψ 6 31 20	8 29 6 N	-1 10	+1 6
21 6 37	$\simeq$ 5 8 12	21 42 13	$\simeq$ 5 6 14	21 44 42	-1 58	+2 29
24 6 18	18 49 23	25 23 5	18 47 30	25 23 35	-1 53	+0 30
26 5 21	28 24 13	27 0 52	28 21 42	27 2 1	-2 31	+1 9
29 8 3	χ 13 10 41	28 9 58	χ 13 11 14	28 10 38	+0 33	+0 40
30 8 10	17 38 0	28 11 53	17 38 27	28 11 37	+0 7	-0 16
<i>Jan</i>						
5 6 1 $\frac{1}{2}$	Υ 8 48 53	26 15 7	Υ 8 48 51	26 14 57	-0 2	-0 10
9 7 1	18 44 4	24 11 56	18 43 51	24 12 17	-0 13	+0 21
10 6 6	20 40 50	23 43 32	20 40 23	23 43 25	-0 27	-0 7
13 7 9	25 59 48	22 17 28	26 0 8	22 16 32	+0 20	-0 56
25 7 59	Σ 9 35 0	17 56 30	Σ 9 34 11	17 56 6	-0 49	-0 24
30 8 22	13 19 51	16 42 18	13 18 28	16 40 5	-1 23	-2 13
<i>Feb</i>						
2 6 35	15 13 53	16 4 1	15 11 59	16 2 17	-1 54	-1 54
5 7 4 $\frac{1}{2}$	16 59 6	15 27 3	16 59 17	15 27 0	+0 11	-0 3
25 8 41	26 18 35	12 46 46	26 16 59	12 45 22	-1 36	-1 24
<i>Mar.</i>						
1 11 10	27 52 42	12 23 40	27 51 47	12 22 28	-0 55	-1 12
5 11 39	29 18 0	12 3 16	29 20 11	12 2 50	+2 11	-0 26
9 8 38	Υ 0 43 4	11 45 52	Υ 0 42 43	11 45 35	-0 21	-0 17

computed in the aforesaid parabolic orbit, comes out $Si\ 29^0\ 30'\ 22''$, its latitude north $i^0\ 25'\ 7''$, and its distance from the sun $n\ 5546$.

Moreover, Dr. *Halley*, observing that a remarkable comet had appeared four times at equal intervals of 575 years (that is, in the month of *September* after *Julius Caesar* was killed; *An. Chr.* 531, in the consulate of *Lampadius* and *Orestes*; *An. Chr.* 1106, in the month of *February*; and at the end of the year 1680; and that with a long and remarkable tail, except when it was seen after *Caesar's* death, at which time, by reason of the inconvenient situation of the earth, the tail was not so conspicuous), set himself to find out an elliptic orbit whose greater axis should be 1382957 parts, the mean distance of the earth from the sun containing 10000 such; in this orbit a comet might revolve in 575 years; and, placing the ascending node in $S\ 2^0\ 2'$, the inclination of the plane of the orbit to the plane of the ecliptic in an angle of $61^0\ 6'\ 48''$, the perihelion of the comet in this plane in $tf\ 22^0\ 44'\ 25''$, the equal time of the perihelion $Dec.\ 7^d.\ 23^h.\ 9^m.$, the distance of the perihelion from the ascending node in the plane of the ecliptic $9^0\ 17'\ 35''$, and its conjugate axis 18481.2, he computed the motions of the comet in this elliptic orbit. The places of the comet, as deduced from the observations, and as arising from computation made in this orbit, may be seen in the preceding table (p. 514).

The observations of this comet from the beginning to the end agree as perfectly with the motion of the comet in the orbit just now described as the motions of the planets do with the theories from whence they are calculated, and by this agreement plainly evince that it was one and the same comet that appeared all that time, and also that the orbit of that comet is here rightly defined.

In the foregoing table we have omitted the observations of *Nov.* 16, 18, 20, and 23, as not sufficiently accurate, for at those times several persons had observed the comet. *Nov.* 17, o.s., *Ponthio* and his companions, at 6^h . in the morning at *Rome* (that is, $5^h.\ 10^m.$ at *London*), by threads directed to the fixed stars, observed the comet in $= 8^0\ 30'$, with latitude $0^0\ 40'$ south. Their observations may be seen in a treatise which *Ponthio* published concerning this comet. *Celso*, who was present, and communicated his observations in a letter to *Cassini*, saw the comet at the same hour in $= 8^0\ 30'$, with latitude $0^0\ 30'$ south. It was likewise seen by *Gallet* at the same hour at

Avignon (that is, at 5^h. 42^m. morning at *London*) in $\approx 8^\circ$ without latitude. But by the theory the comet was at that time in $\approx 8^\circ 16' 45''$, and its latitude was $0^\circ 53' 7''$ south.

Nov. 18, at 6^h. 30^m. in the morning at *Rome* (that is, at 5^h. 40^m. at *London*), *Ponthio* observed the comet in $\approx 13^\circ 30'$, with latitude $1^\circ 26'$ south; and *Cellio* in $\approx 13^\circ 30'$, with latitude $1^\circ 00'$ south. But at 5^h. 30^m. in the morning at *Avignon*, *Gallet* saw it in $\approx 13^\circ 00'$, with latitude $1^\circ 00'$ south. In the University of *La Fleche*, in *France*, at 5^h. in the morning (that is, at 5^h. 9^m. at *London*), it was seen by *Ango*, in the middle between two small stars, one of which is the middle of the three which lie in a right line in the southern hand of *Virgo*, *Bayer's* ψ ; and the other is the outmost of the wing, *Bayer's* θ . Hence the comet was then in $\approx 12^\circ 46'$ with latitude $50'$ south. And I was informed by *Dr. Halley*, that on the same day at *Boston* in *New England*, in the latitude of $42\frac{1}{2}^\circ$, at 5^h. in the morning (that is, at 9^h. 44^m. in the morning at *London*), the comet was seen near $\approx 14^\circ$, with latitude $1^\circ 30'$ south.

Nov. 19, at 4^h $\frac{1}{2}$. at *Cambridge*, the comet (by the observation of a young man) was distant from *Spica* $\mathfrak{M}$ about 2° towards the northwest. Now the *Spike* was at that time in $\approx 19^\circ 23' 47''$, with latitude $2^\circ 1' 59''$ south. The same day, at 5^h. in the morning, at *Boston* in *New England*, the comet was distant from *Spica* $\mathfrak{M}$ 1° , with the difference of $40'$ in latitude. The same day, in the island of *Jamaica*, it was about 1° distant from *Spica* $\mathfrak{M}$. The same day, *Mr. Arthur Storer*, at the river *Patuxent*, near *Hunting Creek*, in *Maryland*, in the confines of *Virginia*, in lat. $38\frac{1}{2}^\circ$, at 5 in the morning (that is, at 10^h. at *London*), saw the comet above *Spica* $\mathfrak{M}$, and very nearly joined. with it, the distance between them being about $\frac{3}{4}$ of one degree. And from these observations compared, I conclude, that at 9^h. 44^m. at *London* the comet was in $\approx 18^\circ 50'$, with about $1^\circ 25'$ latitude south. Now by the theory the comet was at that time in $\approx 18^\circ 52' 15''$, with $1^\circ 26' 54''$ lat. south.

Nov. 20, *Montenari*, Professor of Astronomy at *Padua*, at 6^h. in the morning at *Venice* (that is, 5^h. 10^m. at *London*), saw the comet in $\approx 23^\circ$, with latitude $1^\circ 30'$ south. The same day, at *Boston*, it was distant from *Spica* $\mathfrak{M}$ by about 4° of longitude east, and therefore was in $\approx 23^\circ 24'$, nearly.

Nov. 21, *Ponthio* and his companions, at 7^h $\frac{1}{4}$. in the morning, observed the comet in $\approx 27^\circ 50'$, with latitude $1^\circ 16'$ south; *Cellio*, in $\approx 28^\circ$; *Ango*

at 5^h. in the morning, in $\simeq 27^{\circ} 45'$; *Montenari* in $\simeq 27^{\circ} 51'$. The same day, in the island of *Jamaica*, it was seen near the beginning of $\mathfrak{M}$, and of about the same latitude with *Spica* $\mathfrak{M}$, that is, $2^{\circ} 2'$. The same day, at 5^h. morning, at *Ballasore*, in the *East Indies* (that is, at 11^h. 20^m. of the night preceding at *London*), the distance of the comet from *Spica* $\mathfrak{M}$ was taken $7^{\circ} 35'$ to the east. It was in a right line between the *Spike* and the *Balance*, and therefore was then in $\simeq 26^{\circ} 58'$, with about $1^{\circ} 11'$ lat. south; and after 5^h. 40^m. (that is, at 5^h. in the morning at *London*), it was in $\simeq 28^{\circ} 12'$ with $1^{\circ} 16'$ lat. south. Now by the the theory the comet was then in $\simeq 28^{\circ} 10' 36''$, with $1^{\circ} 53' 35''$ lat. south.

Nov. 22, the comet was seen by *Montenari* in $\mathfrak{M} 2^{\circ} 33'$; but at *Boston* in *New England* it was found in about $\mathfrak{M} 3^{\circ}$, and with almost the same latitude as before, that is, $1^{\circ} 30'$. The same day, at 5^h. morning at *Ballasore*, the comet was observed in $\mathfrak{M} 1^{\circ} 50'$; and therefore at 5^h. morning at *London*, the comet was in $\mathfrak{M} 3^{\circ} 5'$, nearly. The same day at 6^{1/2}^h. in the morning at *London*, *Dr. Hooke* observed it in about $\mathfrak{M} 3^{\circ} 30'$, and that in the right line which passeth through *Spica* $\mathfrak{M}$ and *Cor Leonis*; not, indeed, exactly, but deviating a little from that line towards the north. *Montenari* likewise observed, that this day, and some days after, a right line drawn from the comet through *Spica* passed by the south side of *Cor Leonis* at a very small distance therefrom. The right line through *Cor Leonis* and *Spica* $\mathfrak{M}$ did cut the ecliptic in $\mathfrak{M} 3^{\circ} 46'$ at an angle of $2^{\circ} 51'$; and, if the comet had been in this line and in $\mathfrak{M} 3^{\circ}$, its latitude would have been $2^{\circ} 26'$; but since *Hooke* and *Montenari* agree that the comet was at some small distance from this line towards the north, its latitude must have been somewhat less. On the 20th, by the observation of *Montenari*, its latitude was almost the same with that of *Spica* $\mathfrak{M}$, that is, about $1^{\circ} 30'$. But by the agreement of *Hooke*, *Montenari*, and *Ange*, the latitude was continually increasing, and therefore must now, on the 22^d, be sensibly greater than $1^{\circ} 30'$; and, taking a mean between the extreme limits but now stated, $2^{\circ} 26'$ and $1^{\circ} 30'$, the latitude will be about $1^{\circ} 58'$. *Hooke* and *Montenari* agree that the tail of the comet was directed towards *Spica* $\mathfrak{M}$, declining a little from that star towards the south according to *Hooke*, but towards the north according to *Montenari*; and, therefore, that declination was scarcely sensible; and the tail, lying nearly

parallel to the equator, deviated a little from the opposition of the sun towards the north.

Nov. 23, o.s., at 5^h. morning, at *Nuremberg* (that is, at 4½^h. at *London*), *Mr. Zimmerman* saw the comet in $\text{m } 8^{\circ} 8'$, with $2^{\circ} 31'$ south lat., its place being obtained by taking its distances from fixed stars.

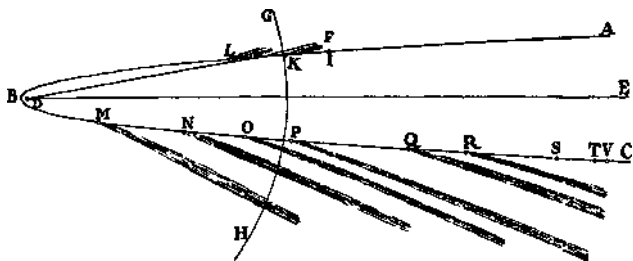
Nov. 24, before sunrise, the comet was seen by *Montenari* in $\text{m } 12^{\circ} 52'$ on the north side of the right line through *Cor Leonis* and *Spica m*, and therefore its latitude was somewhat less than $2^{\circ} 38'$; and since the latitude, as we said, by the concurring observations of *Montenari*, *Ango*, and *Hooke*, was continually increasing, therefore, it was now, on the 24th, somewhat greater than $1^{\circ} 58'$; and, taking the mean quantity, may be reckoned $2^{\circ} 18'$, without any considerable error. *Ponthio* and *Gallet* will have it that the latitude was now decreasing; and *Cellio*, and the observer in *New England*, that it continued the same, viz., of about 1° , or $1\frac{1}{2}^{\circ}$. The observations of *Ponthio* and *Cellio* are rougher, especially those which were made by taking the azimuths and altitudes; as are also the observations of *Gallet*. Those are better which were made by taking the position of the comet to the fixed stars by *Montenari*, *Hooke*, *Ango*, and the observer in *New England*, and sometimes by *Ponthio* and *Cellio*. The same day, at 5^h. morning, at *Ballasore*, the comet was observed in $\text{m } 11^{\circ} 45'$; and, therefore, at 5^h. morning at *London*, was in $\text{m } 13^{\circ}$, nearly. And, by the theory, the comet was at that time in $\text{m } 13^{\circ} 22' 42''$.

Nov. 25, before sunrise, *Montenari* observed the comet in $\text{m } 17\frac{3}{4}^{\circ}$, nearly; and *Cellio* observed at the same time that the comet was in a right line between the bright star in the right thigh of *Virgo* and the southern scale of *Libra*; and this right line cuts the comet's way in $\text{m } 18^{\circ} 36'$. And, by the theory, the comet was in $\text{m } 18\frac{1}{4}^{\circ}$, nearly.

From all this it is plain that these observations agree with the theory, so far as they agree with one another; and by this agreement it is made clear that it was one and the same comet that appeared all the time from *Nov. 4* to *Mar. 9*. The path of this comet did twice cut the plane of the ecliptic, and therefore was not a right line. It did cut the ecliptic not in opposite parts of the heavens, but in the end of *Virgo* and beginning of *Capricorn*, including an arc of about 98° ; and therefore the way of the comet did very much deviate from the path of a great circle; for in the month of *Nov*, it declined

at least 3^0 from the ecliptic towards the south; and in the month of *Dec.* following it declined 29^0 from the ecliptic towards the north; the two parts of the orbit in which the comet descended towards the sun, and ascended again from the sun, declining one from the other by an apparent angle of above 30^0 , as observed by *Montenari*. This comet traveled over nine signs, namely, from the last deg. of $\text{£}1$ to the beginning of tt , beside the sign of *SI*, through which it passed before it began to be seen; and there is no other theory by which a comet can go over so great a part of the heavens with a regular motion. The motion of this comet was very unequable; for about the 20th of *Nov.* it described about 5^0 a day. Then its motion being retarded between *Nov.* 26 and *Dec.* 12, to wit, in the space of $15^{\wedge}$ days, it described only 40^0 . But the motion thereof being afterwards accelerated, it described near 5^0 a day, till its motion began to be again retarded. And the theory which justly corresponds with a motion so unequable, and through so great a part of the heavens, which observes the same laws with the theory of the planets, and which accurately agrees with accurate astronomical observations, cannot be otherwise than true.

And, thinking it would not be improper, I have given in the annexed figure, plotted in the plane of the curve, a true representation of the orbit which this comet described, and of the tail which it emitted in several places. In this drawing ABC represents the orbit of the comet, D the sun,



DE the axis of the orbit, DF the line of the nodes, GH the intersection of the sphere of the earth's orbit with the plane of the comet's orbit, I the place of the comet *Nov.* 4, *Ann.* 1680; K the place of the same *Nov.* 11; L the place of the same *Nov.* 19; M its place *Dec.* 12; N its place *Dec.* 21; O its

place *Dec.* 29; P its place *Jan.* 5 following; Q its place *Jan.* 25; R its place *Feb.* 5; S its place *Feb.* 25; T its place *March* 5; and V its place *March* 9. In determining the length of the tail, I made the following observations.

Nov. 4 and 6, the tail did not appear; *Nov.* n, the tail just began to show itself, but did not appear above $V2^\circ$ long through a 10-foot telescope; *Nov.* ly, the tail was seen by *Ponthio* more than 15^0 long; *Nov.* 18, in *New England*, the tail appeared 30° long, and directly opposite to the sun, extending itself to the planet Mars, which was then in W, $9^0 54'$; *Nov.* 19, in *Maryland*, the tail was found 15^0 or 20^0 long; *Dec.* 10 (by the observation of Mr. *Flamsteed*), the tail passed through the middle of the distance intercepted between the tail of the serpent of Ophiuchus and the star $\$$ in the south wing of Aquila, and did terminate near the stars A, u, b in *Bayer's* tables. Therefore the end of the tail was in VS 19° , with latitude about $34\frac{3}{4}^\circ$ north; *Dec.* 11, it ascended to the head of Sagitta (*Bayer's* a, /?), terminating in VS $26^\circ 43'$, with latitude $38^\circ 34'$ north; *Dec.* 12, it passed through the middle of Sagitta, nor did it reach much farther; terminating in $\mathcal{E} 4^0$, with latitude $42\frac{1}{2}^\circ$ north, nearly. But these things are to be understood of the length of the brighter part of the tail; for, with a more faint light, observed, too, perhaps, in a serener sky, at *Rome*, *Dec.* 12, $5^h. 40^{TM}$, by the observation of *Ponthio*, the tail arose to 10° above the rump of the Swan, and the side thereof towards the west and towards the north was 45' distant from this star. But about that time the tail was 3^0 broad towards the upper end; and therefore the middle thereof was $2^0 15'$ distant from that star towards the south, and the upper end was X in 22^0 , with latitude 61° north; and thence the tail was about 70^0 long; *Dec.* 21, it extended almost to Cassiopeia's Chair, equally distant from $\mathcal{E}$ and from Schedir, so as its distance from either of the two was equal to the distance of the one from the other, and therefore did terminate in T 24° , with latitude $47\frac{1}{2}^\circ$; *Dec.* 29, it reached to a contact with Scheat on its left, and exactly rilled up the space between the two stars in the northern foot of Andromeda, being 54^0 in length; and therefore terminated in b 19^0 , with 35° of latitude; *Jan.* 5, it touched the star n in the breast of Andromeda on its right side, and the star M of the girdle on its left; and, according to our observations, was 40° long; but it was curved, and the convex side thereof lay to the south; and near the head of the comet it made an angle of 4^0 with the circle which passed through the sun and the comet's

head; but towards the other end it was inclined to that circle in an angle of about 10° or n° ; and the chord of the tail contained with that circle an angle of 8° . *Jan.* 13, the tail terminated between Alamech and Algol, with a light that was sensible enough; but with a faint light it ended over against the star K in Perseus' side. The distance of the end of the tail from the circle passing through the sun and the comet was $3^0 50'$; and the inclination of the chord of the tail to that circle was $8M2^0$. *Jan.* 25 and 26, it shone with a faint light to the length of 6° or 7° ; and, for a night or two after, when there was a very clear sky, it extended to the length of 12^0 , or somewhat more, with a light that was very faint and very hardly to be seen; but the axis thereof was exactly directed to the bright star in the eastern shoulder of Auriga, and therefore deviated from the opposition of the sun towards the north by an angle of 10° . Lastly, *Feb.* 10, with a telescope I observed the tail 2° long; for that fainter light which I spoke of did not appear through the glasses. But *Ponthio* writes, that, on *Feb.* 7, he saw the tail 12^0 long. *Feb.* 25, the comet was without a tail, and so continued till it disappeared.

Now if one reflects upon the orbit described, and duly considers the other appearances of this comet, he will be easily satisfied that the bodies of comets are solid, compact, fixed, and durable, like the bodies of the planets; for if they were nothing else but the vapors or exhalations of the earth, of the sun, and other planets, this comet, in its passage by the neighborhood of the sun, would have been immediately dissipated; for the heat of the sun is as the density of its rays, that is, inversely as the square of the distance of the places from the sun. Therefore, since on *Dec.* 8, when the comet was in its perihelion, the distance thereof from the centre of the sun was to the distance of the earth from the same as about 6 to 1000, the sun's heat on the comet was at that time to the heat of the summer-sun with us as 1000000 to 36, or as 28000 to 1. But the heat of boiling water is about three times greater than the heat which dry earth acquires from the summer-sun, as I have tried; and the heat of red-hot iron (if my conjecture is right) is about three or four times greater than the heat of boiling water. And therefore the heat which dry earth on the comet, while in its perihelion, might have received from the rays of the sun, was about 2000 times greater than the heat of red-hot iron. But by so fierce a heat, vapors and exhalations, and every volatile matter, must have been immediately consumed and dissipated.

This comet, therefore, must have received an immense heat from the sun, and retained that heat for an exceeding long time; for a globe of iron of an inch in diameter, exposed red-hot to the open air, will scarcely lose all its heat in an hour's time; but a greater globe would retain its heat longer in the ratio of its diameter, because the surface (in proportion to which it is cooled by the contact of the ambient air) is in that ratio less in respect of the quantity of the included hot matter; and therefore a globe of red-hot iron equal to our earth, that is, about 40000000 feet in diameter, would scarcely cool in an equal number of days, or in above 50000 years. But I suspect that the duration of heat may, on account of some latent causes, increase in a yet less ratio than that of the diameter; and I should be glad that the true ratio was investigated by experiments.

It is further to be observed, that the comet in the month of *December*, just after it had been heated by the sun, did emit a much longer tail, and much more splendid, than in the month of *November* before, when it had not yet arrived at its perihelion; and, universally, the greatest and most fulgent tails always arise from comets immediately after their passing by the neighborhood of the sun. Therefore the heat received by the comet conduces to the greatness of the tail: from this, I think I may infer, that the tail is nothing else but a very fine vapor, which the head or nucleus of the comet emits by its heat.

But we have had three several opinions about the tails of comets: for some will have it that they are nothing else but the beams of the sun's light transmitted through the comets' heads, which they suppose to be transparent; others, that they proceed from the refraction which light suffers in passing from the comet's head to the earth; and, lastly, others, that they are a sort of cloud or vapor constantly rising from the comets' heads, and tending towards the parts opposite to the sun. The first is the opinion of such as are yet unacquainted with optics; for the beams of the sun are seen in a darkened room only in consequence of the light that is reflected from them by the little particles of dust and smoke which are always flying about in the air; and, for that reason, in air impregnated with thick smoke, those beams appear with great brightness, and impress the eye more strongly; in a yet finer air they appear more faint, and are less easily discerned; but in the heavens, where there is no matter to reflect the light, they can never be seen

at all. Light is not seen as it is in the beam, but as it is thence reflected to our eyes; for vision can be produced in no other way than by rays falling upon the eyes; and, therefore, there must be some reflecting matter in those parts where the tails of the comets are seen: for otherwise, since all the celestial spaces are equally illuminated by the sun's light, no part of the heavens could appear with more splendor than another. The second opinion is liable to many difficulties. The tails of comets are never seen variegated with those colors which commonly are inseparable from refraction; and the distinct transmission of the light of the fixed stars and planets to us is a demonstration that the ether or celestial medium is not endowed with any refractive power: for, as to what is alleged, that the fixed stars have been sometimes seen by the *Egyptians* environed with a coma, because that has but rarely happened, it is rather to be ascribed to a casual refraction of clouds; and so the radiation and scintillation of the fixed stars to the refractions both of the eyes and air; for, upon laying a telescope to the eye, those radiations and scintillations immediately disappear. By the tremulous agitation of the air and ascending vapors, it happens that the rays of light are alternately turned aside from the narrow space of the pupil of the eye; but no such thing can have place in the much wider aperture of the object glass of a telescope; and hence it is that a scintillation is occasioned in the former case, which ceases in the latter; and this cessation in the latter case is a demonstration of the regular transmission of light through the heavens, without any perceptible refraction. But, to obviate an objection that may be made from the appearing of no tail in such comets as shine but with a faint light, as if the secondary rays were then too weak to affect the eyes, and for that reason it is that the tails of the fixed stars do not appear, we are to consider, that by the means of telescopes the light of the fixed stars may be augmented above an hundredfold, and yet no tails are seen; that the light of the planets is yet more copious without any tail; but that comets are seen sometimes with huge tails, when the light of their heads is but faint and dull. For so it happened in the comet of the year 1680, when in the month of *December* it was scarcely equal in light to the stars of the second magnitude, and yet emitted a notable tail, extending to the length of 40° , 50^0 , 60° , or 70° , and upwards; and afterwards, on the 27th and 28th of *January*, when the head appeared but as a star of the 7th magnitude, yet the tail (as we said above), with a

light that was clearly perceptible, though faint, was stretched out to 6° or 7° in length, and with a languishing light that was more difficult to see, even to 12° , and upwards. But on the 9th and 10th of *February*, when to the naked eye the head appeared no more, through a telescope I viewed the tail of 2° in length. But further: if the tail was due to the crumbling of the celestial matter, and did deviate from the opposition of the sun, according to the figure of the heavens, that deviation in the same places of the heavens should be always directed towards the same parts. But the comet of the year 1680, *December 28^d. 8½^h. P.M.* at *London*, was seen in $8^\circ 41'$, with latitude north $28^\circ 6'$; while the sun was in $V3\ 18^\circ 26'$. And the comet of the year 1577, *December 29A*, was in $X\ 8^\circ 41'$, with latitude north $28^\circ 40'$, and the sun, as before, in about $VS\ 18^\circ 26M$ n both cases the situation of the earth was the same, and the comet appeared in the same place of the heavens; yet in the former case the tail of the comet (as well by my observations as by the observations of others) deviated from the opposition of the sun towards the north by an angle of 4% degrees; whereas in the latter there was (according to the observations of *Tycho*) a deviation of 21 degrees towards the south. The crumbling, therefore, of the heavens being thus disproved, it remains that the phenomena of the tails of comets must be derived from some reflecting matter.

And that the tails of comets do arise from their heads, and tend towards the parts opposite to the sun, is further confirmed from the laws which the tails observe: As that, lying in the planes of the comets' orbits which pass through the sun, they constantly deviate from the opposition of the sun towards the parts which the comets' heads in their progress along these orbits have left. That to a spectator, placed in those planes, they appear in the parts directly opposite to the sun; but, as the spectator recedes from those planes, their deviation begins to appear, and daily becomes greater. That the deviation, other things being equal, appears less when the tail is more oblique to the orbit of the comet, as well as when the head of the comet approaches nearer to the sun, especially if the angle of deviation is estimated near the head of the comet. That the tails which have no deviation appear straight, but the tails which deviate are likewise bended into a certain curvature. That this curvature is greater when the deviation is greater; and is more sensible when the tail, other things being equal, is longer; for in the

shorter tails the curvature is hardly to be perceived. That the angle of deviation is less near the comet's head, but greater towards the other end of the tail; and that because the convex side of the tail regards the parts from which the deviation is made, and which lie in a right line drawn out infinitely from the sun through the comet's head. And that the tails that are long and broad, and shine with a stronger light, appear more resplendent and more exactly defined on the convex than on the concave side. Upon these accounts it is plain that the phenomena of the tails of comets depend upon the motions of their heads, and by no means upon the places of the heavens in which their heads are seen; and that, therefore, the tails of comets do not proceed from the refraction of the heavens, but from their own heads, which furnish the matter that forms the tail. For, as in our air, the smoke of a heated body ascends either perpendicularly if the body is at rest, or obliquely if the body is moved obliquely, so in the heavens, where all bodies gravitate towards the sun, smoke and vapor must (as we have already said) ascend from the sun, and either rise perpendicularly if the smoking body is at rest, or obliquely if the body, in all the progress of its motion, is always leaving those places from which the upper or higher parts of the vapor had risen before; and that obliquity will be least where the vapor ascends with most velocity, namely, near the smoking body, when that is near the sun. But, because the obliquity vanes, the column of vapor will be incurvated; and because the vapor in the preceding side is something more recent, *that is, has ascended something more late from the body*, it will therefore be somewhat more dense on that side, and must on that account reflect more light, as well as be better defined. I add nothing concerning the sudden uncertain agitation of the tails of comets, and their irregular figures, which authors sometimes describe, because they may arise from the mutations of our air, and the motions of our clouds, in part obscuring those tails; or, perhaps, from parts of the Milky Way which might have been confounded with and mistaken for parts of the tails of the comets as they passed by.

But that the atmospheres of comets may furnish a supply of vapor great enough to fill so immense spaces, we may easily understand from the rarity of our own air; for the air near the surface of our earth possesses a space 850 times greater than water of the same weight; and therefore a cylinder

of air 850 feet high is of equal weight with a cylinder of water of the same breadth, and but one foot high. But a cylinder of air reaching to the top of the atmosphere is of equal weight with a cylinder of water about 33 feet high: and, therefore, if from the whole cylinder of air the lower part of 850 feet high is taken away, the remaining upper part will be of equal weight with a cylinder of water 32 feet high: and from thence (and by the hypothesis, confirmed by many experiments, that the compression of air is as the weight of the incumbent atmosphere, and that the force of gravity is inversely as the square of the distance from the centre of the earth) proceeding by calculation, by Cor., Prop, xxii, Book 11, I found, that, at the height of one semidiameter of the earth, reckoned from the earth's surface, the air is more rare than with us in a far greater ratio than that of the whole space within the orbit of Saturn to a spherical space one inch in diameter; and therefore, if a sphere of our air of but one inch in thickness was equally rarefied with the air at the height of one semidiameter of the earth from the earth's surface, it would fill all the regions of the planets to the orb of Saturn, and far beyond it. Therefore, since the air at greater distances is immensely rarefied, and the coma or atmosphere of comets is ordinarily about ten times higher, reckoning from their centres, than the surface of the nucleus, and the tails rise yet higher, they must therefore be exceedingly rare; and though, on account of the much thicker atmospheres of comets, and the great gravitation of their bodies towards the sun, as well as of the particles of their air and vapors towards each other, it may happen that the air in the celestial spaces and in the tails of comets is not so vastly rarefied, yet from this computation it is plain that a very small quantity of air and vapor is abundantly sufficient to produce all the appearances of the tails of comets; for that they are, indeed, of a very notable rarity appears from the shining of the stars through them. The atmosphere of the earth, illuminated by the sun's light, though but of a few miles in thickness, quite obscures and extinguishes the light not only of all the stars, but even of the moon itself; whereas the smallest stars are seen to shine through the immense thickness of the tails of comets, likewise illuminated by the sun, without the least diminution of their splendor. Nor is the brightness of the tails of most comets ordinarily greater than that of our air, an inch or two

in thickness, reflecting in a darkened room the light of the sunbeams let in by a hole of the window shutter.

And we may pretty nearly determine the time spent during the ascent of the vapor from the comet's head to the extremity of the tail, by drawing a right line from the extremity of the tail to the sun, and marking the place where that right line intersects the comet's orbit; for the vapor that is now in the extremity of the tail, if it has ascended in a right line from the sun, must have begun to rise from the head at the time when the head was in the point of intersection. It is true, the vapor does not rise in a right line from the sun, but, retaining the motion which it had from the comet before its ascent, and compounding that motion with its motion of ascent, arises obliquely; and, therefore, the solution of the Problem will be more exact, if we draw the line which intersects the orbit parallel to the length of the tail; or rather (because of the curvilinear motion of the comet) diverging a little from the line or length of the tail. And by means of this principle T found that the vapor which, *January* 25, was in the extremity of the tail, had begun to rise from the head before *December* 11, and therefore had spent in its whole ascent 45 days; but that the whole tail which appeared on *December* 10 had finished its ascent in the space of the two days then elapsed from the time of the comet's being in its perihelion. The vapor, therefore, about the beginning and in the neighborhood of the sun rose with the greatest velocity, and afterwards continued to ascend with a motion constantly retarded by its own gravity; and the higher it ascended, the more it added to the length of the tail; and while the tail continued to be seen, it was made up of almost all that vapor which had risen since the time of the comet's being in its perihelion; nor did that part of the vapor which had risen first, and which formed the extremity of the tail, cease to appear, till its too great distance, as well from the sun, from which it received its light, as from our eyes, rendered it invisible. Whence also it is that the tails of other comets which are short do not rise from their heads with a swift and continued motion, and soon after disappear, but are permanent and lasting columns of vapors and exhalations, which, ascending from the heads with a slow motion of many days, and partaking of the motion of the heads which they had from the beginning, continue to go along together with them through the heavens. From this again we have another argument proving the celestial spaces to

be free, and without resistance, since in them not only the solid bodies of the planets and comets, but also the extremely rare vapors of comets' tails, maintain their rapid motions with great freedom, and for an exceeding long time.

Kepler ascribes the ascent of the tails of the comets to the atmospheres of their heads; and their direction towards the parts opposite to the sun to the action of the rays of light carrying along with them the matter of the comets' tails; and without any great incongruity we may suppose, that, in so free spaces, so fine a matter as that of the ether may yield to the action of the rays of the sun's light, though those rays are not able sensibly to move the gross substances in our parts, which are clogged with so palpable a resistance. Another author thinks that there may be a sort of particles of matter endowed with a principle of levity, as well as others are with a power of gravity; that the matter of the tails of comets may be of the former sort, and that its ascent from the sun may be owing to its levity; but, considering that the gravity of terrestrial bodies is as the matter of the bodies, and therefore can be neither more nor less in the same quantity of matter, I am inclined to believe that this ascent may rather proceed from the rarefaction of the matter of the comets' tails. The ascent of smoke in a chimney is due to the impulse of the air with which it is entangled. The air rarefied by heat ascends, because its specific gravity is diminished, and in its ascent carries along with it the smoke which floats in it; and why may not the tail of a comet rise from the sun after the same manner? For the sun's rays do not act upon the mediums which they pervade otherwise than by reflection and refraction; and those reflecting particles heated by this action, heat the matter of the ether which is involved with them. That matter is rarefied by the heat which it acquires, and because, by this rarefaction, the specific gravity with which it tended towards the sun before is diminished, it will ascend therefrom, and carry along with it the reflecting particles of which the tail of the comet is composed. But the ascent of the vapors is further promoted by their circumgyration about the sun; in consequence thereof they endeavor to recede from the sun, while the sun's atmosphere and the other matter of the heavens are either altogether quiescent, or are only moved with a slower circumgyration derived from the rotation of the sun. And these are the causes of the ascent of the tails of the comets in the neighbor-

hood of the sun, where their orbits are bent into a greater curvature, and the comets themselves are plunged into the denser and therefore heavier parts of the sun's atmosphere: upon which account they do then emit tails of an huge length; for the tails which then arise, retaining their own proper motion, and in the meantime gravitating towards the sun, must be revolved in ellipses about the sun in like manner as the heads are, and by that motion must always accompany the heads, and freely adhere to them. For the gravitation of the vapors towards the sun can no more force the tails to abandon the heads, and descend to the sun, than the gravitation of the heads can oblige them to fall from the tails. They must by their common gravity either fall together towards the sun, or be retarded together in their common ascent therefrom; and, therefore (whether from the causes already described, or from any others), the tails and heads of comets may easily acquire and freely retain any position one to the other, without disturbance or impediment from that common gravitation.

The tails, therefore, that rise in the perihelion positions of the comets will go along with their heads into far remote parts, and together with the heads will either return again from thence to us, after a long course of years, or rather will be there rarefied, and by degrees quite vanish away; for afterwards, in the descent of the heads towards the sun, new short tails will be emitted from the heads with a slow motion; and those tails by degrees will be augmented immensely, especially in such comets as in their perihelion distances descend as low as the sun's atmosphere; for all vapor in those free spaces is in a perpetual state of rarefaction and dilatation; and from hence it is that the tails of all comets are broader at their upper extremity than near their heads. And it is not unlikely but that the vapor, thus continually rarefied and dilated, may be at last dissipated and scattered through the whole heavens, and by little and little be attracted towards the planets by its gravity, and mixed with their atmosphere; for as the seas are absolutely necessary to the constitution of our earth, that from them, the sun, by its heat, may exhale a sufficient quantity of vapors, which, being gathered together into clouds, may drop down in rain, for watering of the earth, and for the production and nourishment of vegetables; or, being condensed with cold on the tops of mountains (as some philosophers with reason judge), may run down in springs and rivers; so for the conservation of the

seas, and fluids of the planets, comets seem to be required, that, from their exhalations and vapors condensed, the wastes of the planetary fluids spent upon vegetation and putrefaction, and converted into dry earth, may be continually supplied and made up; for all vegetables entirely derive their growths from fluids, and afterwards, in great measure, are turned into dry earth by putrefaction; and a sort of slime is always found to settle at the bottom of putrefied fluids; and hence it is that the bulk of the solid earth is continually increased; and the fluids, if they are not supplied from without, must be in a continual decrease, and quite fail at last. I suspect, moreover, that it is chiefly from the comets that spirit comes, which is indeed the smallest but the most subtle and useful part of our air, and so much required to sustain the life of all things with us.

The atmospheres of comets, in their descent towards the sun, by running out into the tails, are spent and diminished, and become narrower, at least on that side which regards the sun; and in receding from the sun, when they less run out into the tails, they are again enlarged, if *Hewelcke* has justly marked their appearances. But they are seen least of all just after they have been most heated by the sun, and on that account then emit the longest and most resplendent tails; and, perhaps, at the same time, the nuclei are environed with a denser and blacker smoke in the lowermost parts of their atmosphere; for smoke that is raised by a great and intense heat is commonly the denser and blacker. Thus the head of that comet which we have been describing, at equal distances both from the sun and from the earth, appeared darker after it had passed by its perihelion than it did before; for in the month of *December* it was commonly compared with the stars of the third magnitude, but in *Not/ember* with those of the first or second; and such as saw both appearances have described the first as of another and greater comet than the second. For, *November* 19, this comet appeared to a young man at *Cambridge*, though with a pale and dull light, yet equal to *Spica Virginis*; and at that time it shone with greater brightness than it did afterwards. And *Montenari*, *November* 20, o.s., observed it larger than the stars of the first magnitude, its tail being then 2 degrees long. And Mr. *Storer* (by letters which have come into my hands) writes, that in the month of *December*, when the tail appeared of the greatest bulk and splendor, the head was but small, and far less than that which was seen in the month of

November before sun rising; and, conjecturing at the cause of the appearance, he judged it to proceed from the existence of a greater quantity of matter in the head at first, which was afterwards gradually spent.

And, for the same reason, I find, that the heads of other comets, which did put forth tails of the greatest bulk and splendor, have appeared but obscure and small. For in *Brazil*, *March* 5, 1668, N.S., 7^h. P.M., *Valentin Estancel* saw a comet near the horizon, and towards the southwest, with a head so small as scarcely to be discerned, but with a tail above measure splendid, so that the reflection thereof from the sea was easily seen by those who stood on the shore; it looked like a fiery beam extended 23 degrees in length from the west to south, almost parallel to the horizon. But this excessive splendor continued only three days, decreasing apace afterwards; and while the splendor was decreasing, the bulk of the tail increased: also in *Portugal* it is said to have taken up one-quarter of the heavens, that is, 45 degrees, extending from west to east with a very notable splendor, though the whole tail was not seen in those parts, because the head was always hid under the horizon: and from the increase of the bulk and decrease of the splendor of the tail, it appears that the head was then in its recess from the sun, and had been very near to it in its perihelion, as the comet of 1680 was. And we read, in the *Saxon Chronicle*, of a like comet appearing in the year 1106, *the star whereof was small and obscure* (as that of 1680), *but the splendor of its tail was very bright, and like a huge fiery beam stretched out in a direction between the east and north*, as *Hewelme* has it also from *Simeon*, the monk of *Durham*. This comet appeared in the beginning of *February*, about the evening, and towards the southwest part of heaven; from this, and from the position of the tail, we infer that the head was near the sun. *Matthew Paris* says, // *was distant from the sun by about a cubit, from three o'clock*^ (rather six) *till nine, putting forth a long tail*. Such also was that resplendent comet described by *Aristotle*, Book 1, *Meteor*. 6. *The head whereof could not be seen, because it had set before the sun, or at least was hid under the sun's rays; but next day it was seen as well as might be; for, having left the sun but a very little way, it set immediately after it. And the scattered light of the head, obscured by the too great splendor* (of the tail) *did not yet appear. But afterwards* (as *Aristotle* says) *when the splendor* (of the tail) *had diminished, (the head of) the comet recovered its*

native brightness; and the splendor (of its tail) reached now to a third part of the heavens (that is, to 60°). This appearance was in the winter season (an. 4, Olymp. 101), and, rising to Orion's girdle, it there vanished away. It is true that the comet of 1618, which came out directly from under the sun's rays with a very large tail, seemed to equal, if not to exceed, the stars of the first magnitude; but then, abundance of other comets have appeared yet greater than this, that put forth shorter tails; some of which are said to have appeared as big as Jupiter, others as big as Venus, or even as the moon.

We have said, that comets are a sort of planets revolved in very eccentric orbits about the sun; and as, in the planets which are without tails, those are commonly less which are revolved in lesser orbits, and nearer to the sun, so in comets it is probable that those which in their perihelion approach nearer to the sun are generally of less magnitude, that they may not agitate the sun too much by their attractions. But as to the transverse diameters of their orbits, and the periodic times of their revolutions, I leave them to be determined by comparing comets together which after long intervals of time return again in the same orbit. In the meantime, the following Proposition may give some light in that inquiry.

PROPOSITION XLII. PROBLEM XXII

To correct a comet's orbit found as above.

OPERATION I. Assume that position of the plane of the orbit which was determined according to the preceding Proposition; and select three places of the comet, deduced from very accurate observations, and at great distances one from the other. Then suppose A to represent the time between the first observation and the second, and B the time between the second and the third; but it will be convenient that in one of those times the comet be in its perigee, or at least not far from it. From those apparent places find, by trigonometric operations, the three true places of the comet in that assumed plane of the orbit; then through the places found, and about the centre of the sun as the focus, describe a conic section by arithmetical operations, according to Prop, xxi, Book 1. Let the areas of this figure which are terminated by radii drawn from the sun to the places found be D and E; namely, D the area between the first observation and the second, and E the area between the second and third; and let T represent the whole time in

which the whole area $D + E$ should be described with the velocity of the comet found by Prop, xvi, Book i.

OPER. 2. Retaining the inclination of the plane of the orbit to the plane of the ecliptic, let the longitude of the nodes of the plane of the orbit be increased by the addition of $20'$ or $30'$, which call P . Then from the aforesaid three observed places of the comet let the three true places be found (as before) in this new plane; as also the orbit passing through those places, and the two areas of the same described between the two observations, which call d and e ; and let t be the whole time in which the whole area $d + e$ should be described.

OPER. 3. Retaining the longitude of the nodes in the first operation, let the inclination of the plane of the orbit to the plane of the ecliptic be increased by adding thereto $20'$ or $30'$, which call Q . Then from the aforesaid three observed apparent places of the comet let the three true places be found in this new plane, as well as the orbit passing through them, and the two areas of the same described between the observation, which call 3 and ϵ ; and let T be the whole time in which the whole area $3 + \epsilon$ should be described.

Then taking C to 1 as A to B ; and G to 1 as D to E ; and g to 1 as d to e ; and Y to 1 as 8 to c ; let S be the true time between the first observation and the third; and, observing well the signs $+$ and $-$, let such numbers m and n be found out as will make $2G - 2C = mG - mg + nG - ny$; and $2T - 2S = mX - mt + nT - nr$. And if, in the first operation, I represents the inclination of the plane of the orbit to the plane of the ecliptic, and K the longitude of either node, then $I + nQ$ will be the true inclination of the plane of the orbit to the plane of the ecliptic, and $K + mP$ the true longitude of the node. And, lastly, if in the first, second, and third operations, the quantities R , r , and p , represent the parameters of the orbit, and the quantities $\frac{1}{L}$, $\frac{1}{l}$, $\frac{1}{\lambda}$, the transverse diameters of the same, then $R + mr - mR + np - nR$ will be the true parameter, and $\frac{1}{L + ml - m\lambda + n\lambda - nL}$ will be the true trans-

verse diameter of the orbit which the comet describes; and from the transverse diameter given the periodic time of the comet is also given. Q.E.I. But the periodic times of the revolutions of comets, and the transverse

diameters of their orbits, cannot be accurately enough determined but by comparing comets together which appear at different times. If, after equal intervals of time, several comets are found to have described the same orbit, we may thence conclude that they are all but one and the same comet revolved in the same orbit; and then from the times of their revolutions the transverse diameters of their orbits will be given, and from those diameters the elliptic orbits themselves will be determined.

To this purpose the orbits of many comets ought to be computed, supposing those orbits to be parabolic; for such orbits will always nearly agree with the phenomena, as appears not only from the parabolic orbit of the comet of the year 1680, which I compared above with the observations, but likewise from that of the notable comet which appeared in the year 1664 and 1665, and was observed by *Hewelcke*, who, from his own observations, calculated the longitudes and latitudes thereof, though with little accuracy. But from the same observations *Dr. Halley* did again compute its places; and from those new places determined its orbit, finding its ascending node in π $21^{\circ} 13' 55''$; the inclination of the orbit to the plane of the ecliptic $21^{\circ} 18' 40''$; the distance of its perihelion from the node, estimated in the comet's orbit, $49^{\circ} 27' 30''$, its perihelion in $\odot$ $8^{\circ} 40' 30''$, with heliocentric latitude south $16^{\circ} 01' 45''$; the comet to have been in its perihelion *November* 24^d. 11^h. 52^m. P.M. equal time at *London*, or 13^h. 8^m. at *Dantzick*, O.S.; and that the latus rectum of the parabola was 410286 of such parts as the sun's mean distance from the earth is supposed to contain 100000. And how nearly the places of the comet computed in this orbit agree with the observations, will appear from the table calculated by *Dr. Halley* (p. 535).

In *February*, the beginning of the year 1665, the first star of Aries, which I shall hereafter call γ , was in Υ $28^{\circ} 30' 15''$, with $7^{\circ} 8' 58''$ north lat.; the second star of Aries was in Υ $29^{\circ} 17' 18''$, with $8^{\circ} 28' 16''$ north lat.; another star of the seventh magnitude, which I call A, was in Υ $28^{\circ} 24' 45''$, with $8^{\circ} 28' 33''$ north lat. The comet *Feb.* 7^d. 7^h. 30^m. at *Paris* (that is, *Feb.* 7^d. 8^h. 37^m. at *Dantzick*), O.S., made a triangle with those stars γ and A, which was right-angled in γ ; and the distance of the comet from the star γ was equal to the distance of the stars γ and A, that is, $1^{\circ} 19' 46''$ of a great circle; and therefore in the parallel of the latitude of the star γ it was $1^{\circ} 20' 26''$. Therefore if from the longitude of the star γ there be subtracted the longitude $1^{\circ} 20'$

Apparent time at Danzack	The observed distances of the comet from		The observed places		The places computed to the orbit
<i>December</i> d h m					
3 18 29½	The Lion's heart	46 24 20	Long. $\approx$ 7 01 00	$\approx$ 7 1 29	
	The Virgin's spike	22 52 10	Lat S. 21 39 0	21 38 50	
4 18 1½	The Lion's heart	46 2 45	Long. $\approx$ 6 15 0	$\approx$ 6 16 5	
	The Virgin's spike	23 52 40	Lat S. 22 24 0	22 24 0	
7 17 48	The Lion's heart	44 48 0	Long. $\approx$ 3 6 0	$\approx$ 3 7 33	
	The Virgin's spike	27 56 40	Lat S. 25 22 0	25 21 40	
17 14 43	The Lion's heart	51 15 15	Long. $\approx$ 2 56 0	$\approx$ 2 56 0	
	Orion's right shoulder	45 43 30	Lat S. 49 25 0	49 25 0	
19 9 25	Procyon	35 13 50	Long. $\approx$ 28 40 30	$\approx$ 28 43 0	
	Bright star of Whale's jaw	52 56 0	Lat S. 45 48 0	45 46 0	
20 9 53½	Procyon	40 49 0	Long. $\approx$ 13 03 0	$\approx$ 13 5 0	
	Bright star of Whale's jaw	40 04 0	Lat S. 39 54 0	39 53 0	
21 9 9½	Orion's right shoulder	26 21 25	Long. $\approx$ 2 16 0	$\approx$ 2 18 30	
	Bright star of Whale's jaw	29 28 0	Lat S. 33 41 0	33 39 40	
22 9 0	Orion's right shoulder	29 47 0	Long. $\approx$ 24 24 0	$\approx$ 24 27 0	
	Bright star of Whale's jaw	20 29 30	Lat S. 27 45 0	27 46 0	
26 7 58	The bright star of Aries	23 20 0	Long. $\approx$ 9 0 0	$\approx$ 9 2 28	
	Aldebaran	26 44 0	Lat S. 12 36 0	12 34 13	
27 6 45	The bright star of Aries	20 45 0	Long. $\approx$ 7 5 40	$\approx$ 7 8 45	
	Aldebaran	28 10 0	Lat S. 10 23 0	10 23 13	
28 7 39	The bright star of Aries	18 29 0	Long. $\approx$ 5 24 45	$\approx$ 5 27 52	
	Pollux	29 37 0	Lat S. 8 22 50	8 23 37	
31 6 45	Andromeda's girdle	30 48 10	Long. $\approx$ 2 7 40	$\approx$ 2 8 20	
	Pollux	32 53 30	Lat S. 4 13 0	4 16 25	
<i>Jan. 1665</i> 7 7 37½	Andromeda's girdle	25 11 0	Long. $\approx$ 28 24 47	$\approx$ 28 24 0	
	Pollux	37 12 25	Lat N. 0 54 0	0 53 0	
13 7 0	Andromeda's head	28 7 10	Long. $\approx$ 27 6 54	$\approx$ 27 6 39	
	Pollux	38 55 20	Lat N. 3 6 50	3 7 40	
24 7 29	Andromeda's girdle	20 32 15	Long. $\approx$ 26 29 15	$\approx$ 26 28 50	
	Pollux	40 5 0	Lat N. 5 25 50	5 26 0	
<i>February</i> 7 8 37			Long. $\approx$ 27 4 46	$\approx$ 27 24 55	
			Lat N. 7 3 29	7 3 15	
22 8 46			Long. $\approx$ 28 29 46	$\approx$ 28 29 58	
			Lat N. 8 12 36	8 10 25	
<i>March</i> 1 8 16			Long. $\approx$ 29 18 15	$\approx$ 29 18 20	
			Lat N. 8 36 26	8 36 12	
7 8 37			Long. $\approx$ 0 2 48	$\approx$ 0 2 42	
			Lat N. 8 56 30	8 56 56	

26'', there will remain the longitude of the comet Υ $27^{\circ} 9' 49''$. *M. Auzout*, from this observation of his, placed the comet in Υ $27^{\circ} 0'$, nearly; and, by the drawing in which *Dr. Hooke* delineated its motion, it was then in Υ $26^{\circ} 59' 24''$. I place it in Υ $27^{\circ} 4' 46''$, taking the middle between the two extremes.

From the same observations, *M. Auzout* made the latitude of the comet at that time 7° and $4'$ or $5'$ to the north; but he had done better to have made it $7^{\circ} 3' 29''$, the difference of the latitudes of the comet and the star Υ being equal to the difference of the longitude of the stars Υ and A .

February 22^d. 7^h. 30^m. at *London*, that is, February 22^d. 8^h. 46^m. at *Dantzick*, the distance of the comet from the star A , according to *Dr. Hooke's* observation, as was delineated by himself in a scheme, and also by the observations of *M. Auzout*, delineated in like manner by *M. Petit*, was a fifth part of the distance between the star A and the first star of Aries, or $15' 57''$; and the distance of the comet from a right line joining the star A and the first of Aries was a fourth part of the same fifth part, that is, $4'$; and therefore the comet was in Υ $28^{\circ} 29' 46''$, with $8^{\circ} 12' 36''$ north lat.

March 1, 7^h. 0^m. at *London*, that is, March 1, 8^h. 16^m. at *Dantzick*, the comet was observed near the second star in Aries, the distance between them being to the distance between the first and second stars in Aries, that is, to $1^{\circ} 33'$, as 4 to 45 according to *Dr. Hooke*, or as 2 to 23 according to *M. Gottignies*. And, therefore, the distance of the comet from the second star in Aries was $8' 16''$ according to *Dr. Hooke*, or $8' 5''$ according to *M. Gottignies*; or, taking a mean between both, $8' 10''$. But, according to *M. Gottignies*, the comet had gone beyond the second star of Aries about a fourth or a fifth part of the space that it commonly went over in a day, to wit, about $1' 35''$ (in which he agrees very well with *M. Auzout*); or, according to *Dr. Hooke*, not quite so much, as perhaps only $1'$. Therefore if to the longitude of the first star in Aries we add $1'$, and $8' 10''$ to its latitude, we shall have the longitude of the comet Υ $29^{\circ} 18'$, with $8^{\circ} 36' 26''$ north lat.

March 7, 7^h. 30^m. at *Paris*, that is, March 7, 7^h. 37^m. at *Dantzick*, from the observations of *M. Auzout*, the distance of the comet from the second star in Aries was equal to the distance of that star from the star A , that is, $52' 29''$; and the difference of the longitude of the comet and the second star in Aries was $45'$ or $46'$, or, taking a mean quantity, $45' 30''$; and therefore

the comet was in $8^{\circ} 2' 48''$. From the drawing constructed by M. *Petit*, based on the observations of M. *Auzout*, *Hewel* determined the latitude of the comet $8^{\circ} 54'$. But the engraver did not rightly trace the curvature of the comet's way towards the end of the motion; and *Hevehus*, in the drawing of M. *Auzout's* observations which he constructed himself, corrected this irregular curvature, and so made the latitude of the comet $8^{\circ} 55' 30''$. And, by further correcting this irregularity, the latitude may become $8^{\circ} 56'$, or $8^{\circ} 57'$.

This comet was also seen *March* 9, and at that time its place must have been in $8^{\circ} 18'$, with $9^{\circ} 3V2'$ north lat., nearly.

This comet appeared for three months, in which space of time it traveled over almost six signs, and in one of the days described almost 20 degrees. Its course did very much deviate from a great circle, bending towards the north, and its motion towards the end from retrograde became direct; and, notwithstanding that its course was so uncommon, yet by the table it appears that the theory, from beginning to end, agrees with the observations no less accurately than the theories of the planets usually do with the observations of them; but we are to subtract about $2'$ when the comet was swiftest, which we may effect by taking off $12''$ from the angle between the ascending node and the perihelion, or by making that angle $49^{\circ} 27' 18''$. The annual parallax of both these comets (this and the preceding) was very conspicuous, and by its quantity demonstrates the annual motion of the earth in the earth's orbit.

This theory is likewise confirmed by the motion of that comet, which in the year 1683 appeared retrograde, in an orbit whose plane contained almost a right angle with the plane of the ecliptic, and whose ascending node (by the computation of Dr. *Halley*) was in $W 23^{\circ} 23'$; the inclination of its orbit to the ecliptic $8^{\circ} 11'$; its perihelion in $X 25^{\circ} 29' 30''$; its perihelion distance from the sun 56020 of such parts as the radius of the earth's orbit contains 100000; and the time of its perihelion was *July* $2^d. 3^h. 50^m$. And the places thereof, computed by Dr. *Halley* in this orbit, are compared with the places observed by Mr. *Flamsteed*, in the following table (p. 538).

This theory is yet further confirmed by the motion of that retrograde comet which appeared in the year 1682. The ascending node of this (by Dr. *Halley's* computation) was in $8 21^{\circ} 16' 30''$; the inclination of its orbit

1683 Equatorial time	Sun's place	Comet's longitude computed	Latitude north computed	Comet's longitude observed	Latitude north observed	Difference longitude	Difference latitude
d h m	° ' "	° ' "	° ' "	° ' "	° ' "	° ' "	° ' "
July 13.12.55	♌ 1 02 30	♌ 3 05 42	29 28 13	♌ 3 6 42	29 28 20	+1 00	+0 07
15.11.15	2 53 12	11.37 48	29 34 0	11 39 43	29 34 50	+1 55	+0 50
17 10 20	4 45 45	10 7 6	29 33 30	10 8.40	29 34 0	+1 34	+0 30
23 13 40	10 38.21	5 10 27	28 51 42	5 11.30	28 50 28	+1 03	-1 14
25.14 5	12 35 28	3 27 53	24 24 47	3 27 0	28 23 40	-0 53	-1 7
31 9 42	18 09 22	♌ 27 55 3	26 22 52	♌ 27 54 24	26 22 25	-0 39	-0 27
31 14 55	18 21 53	27 41 7	26 16 57	27 41. 8	26 14 50	+0 1	-2 7
Aug. 2 14 56	20 17 16	25 29 32	25 16 19	25 28 46	25 17 28	-0 46	+1. 9
4 10 49	22 02 50	23 18 20	24 10 49	23.16 55	24 12 19	-1 25	+1 30
6 10 9	23 56 45	20 42 23	22 47 5	20 40 32	22 49 5	-1 51	+2 0
9 10 26	26 50 52	16 7 57	20 6 37	16 5.55	20 6 10	-2 2	-0 27
15 14 1	♍ 2 47 13	3 30 48	11 37.33	3 26 18	11 32 1	-4 30	-5 32
16 15 10	3 48 2	0 43 7	9 34.16	0 41 55	9 34 13	-1 12	-0 3
18.15.44	5 45 33	♌ 24 52 53	5 11 15	♌ 24 49 5	5 9 11	-3 48	-2 4
		South	South				
22 14.44	9 35 49	11. 7.14	5 16 58	11 07 12	5 16.58	-0 2	-0. 3
23 15 52	10 36 48	7 2 18	8 17 9	7 1 17	8 16 41	-1 1	-0 28
26 16 2	13 31 10	♍ 24 45 31	16 38 0	♍ 24 44 00	16 38 20	-1 31	+0 20

to the plane of the ecliptic $17^{\circ} 56' 00''$; its perihelion in $\approx 2^{\circ} 52' 50''$; its perihelion distance from the sun 58328 parts, of which the radius of the earth's orbit contains 100000; the equal time of the comet's being in its perihelion *September 4^d. 7^h. 39^m*. And its places determined from Mr. *Flamsteed's* observations, are compared with its places computed from our theory in the following table:

1682 App time	Sun's place	Comet's longitude computed	Latitude north computed	Comet's longitude observed	Latitude north observed	Difference longitude	Difference latitude
d h m	° ' "	° ' "	° ' "	° ' "	° ' "	° ' "	° ' "
Aug. 19.16.38	♍ 7 0 7	♌ 18 14 28	25 50 7	♌ 18 14 40	25 49 55	-0 12	+0 12
20 15 38	7 55 52	24 46 23	26 14 42	24 46 22	26 12 52	+0 1	+1 50
21. 8 21	8 36 14	29 37 15	26 20 3	29 38 02	26 17 37	-0 47	+2 26
22 8. 8	9 33 55	♍ 6 29 53	26 8 42	♍ 6 30 3	26 7 12	-0 10	+1 30
29 08 20	16 22 40	♌ 12 37 54	18 37 47	♌ 12 37 49	18 34 5	+0 5	+3 42
30 7 45	17 19 41	15 36 1	17 26 43	15 35 18	17 27 17	+0 43	-0 34
Sept 1 7 33	19 16 9	20 30 53	15 13 0	20 27 4	15 9 49	+3 49	+3 11
4 7 22	22 11 28	25 42 0	12 23 48	25 40 58	12 22 0	+1 2	+1 48
5 7 32	23 10 29	27 0 46	11 33 08	26 59 24	11 33 51	+1 22	-0 43
8 7 16	26 5 58	29 58 44	9 26 46	29 58 45	9 26 43	-0 1	+0 3
9 7 26	27 5 9	♍ 0 44 10	8 49 10	♍ 0 44 4	8 48 25	+0 6	+0 45

This theory is also confirmed by the retrograde motion of the comet that appeared in the year 1723. The ascending node of this comet (according to the computation of Mr. *Bradley*, Savihan Professor of Astronomy at *Oxford*) was in $T\ 14^{\circ}16'$, the inclination of the orbit to the plane of the ecliptic $49^{\circ}59'$. Its perihelion was in $8\ 12^{\circ}15'20''$, its perihelion distance from the sun 998651 parts, of which the radius of the earth's orbit contains 1000000, and the equal time of its perihelion *September* $16^d. 16^h. 10^m$. The places of this comet computed in this orbit by Mr. *Bradley*, and compared with the places observed by himself, his uncle Mr. *Pound*, and Dr. *Halley*, may be seen in the following table.

1723 Equatorial time	Comet's longitude observed	Latitude north observed	Comet's longitude computed	Latitude north computed	Difference longitude	Difference latitude
d h m	o i v	o i v	o i v	o i v	"	"
<i>Oct.</i> 9 8 5	$\approx 7\ 22\ 15$	5 2 0	$\approx 7\ 21\ 26$	5 2 47	+49	-47
10 6 21	6 41 12	7 44 13	6 41 42	7 43 18	-50	+55
12 7 22	5 39 58	11 55 0	5 40 19	11 54 55	-21	+5
14 8 57	4 59 49	14 43 50	5 0 37	14 44 1	-48	-11
15 6 35	4 47 41	15 40 51	4 47 45	15 40 55	-4	-4
21 6 22	4 2 32	19 41 49	4 2 21	19 42 3	+11	-14
22 6 24	3 59 2	20 8 12	3 59 10	20 8 17	-8	-5
24 8 2	3 55 29	20 55 18	3 55 11	20 55 9	+18	+9
29 8 56	3 56 17	22 20 27	3 56 42	22 20 10	-25	+17
30 6 20	3 58 9	22 32 28	3 58 17	22 32 12	-8	+16
<i>Nov.</i> 5 5 53	4 16 30	23 38 33	4 16 23	23 38 7	+7	+26
8 7 6	4 29 36	24 4 30	4 29 54	24 4 40	-18	-10
14 6 20	5 2 16	24 48 46	5 2 51	24 48 16	-35	+30
20 7 45	5 42 20	25 24 45	5 43 13	25 25 17	-53	-32
<i>Dec.</i> 7 6 45	8 4 13	26 54 18	8 3 55	26 53 42	+18	+36

From these examples it is abundantly evident that the motions of comets are no less accurately represented by our theory than the motions of the planets commonly are by the theories of them; and, therefore, by means of this theory, we may enumerate the orbits of comets, and so discover the periodic time of a comet's revolution in any orbit; hence, at last, we shall have the transverse diameters of their elliptic orbits and their aphehan distances.

That retrograde comet which appeared in the year 1607 described an orbit whose ascending node (according to Dr. *Halley's* computation) was in $8^{\circ} 21'$; and the inclination of the plane of the orbit to the plane of the ecliptic $17^{\circ} 2'$; whose perihelion was in $\text{£}^{\circ} 2^{\circ} 16'$; and its perihelion distance from the sun 58680 of such parts as the radius of the earth's orbit contains 100000; and the comet was in its perihelion *October* 16^d. 3^h. 50^m.; which orbit agrees very nearly with the orbit of the comet which was seen in 1682. If these were not two different comets, but one and the same, that comet will finish one revolution in the space of 75 years; and the greater axis of its orbit will be to the greater axis of the earth's orbit as $\sqrt{75}$ to 1, or as 1778 to 100, nearly. And the aphelion distance of this comet from the sun will be to the mean distance of the earth from the sun as about 35 to 1; from these data it will be no hard matter to determine the elliptic orbit of this comet. But these things are to be supposed on condition, that, after the space of 75 years, the same comet shall return again in the same orbit. The other comets seem to ascend to greater heights, and to require a longer time to perform their revolutions.

But, because of the great number of comets, of the great distance of their aphelions from the sun, and of the slowness of their motions in the aphelions, they will, by their mutual gravitations, disturb each other; so that their eccentricities and the times of their revolutions will be sometimes a little increased, and sometimes diminished. Therefore, we are not to expect that the same comet will return exactly in the same orbit, and in the same periodic times: it will be sufficient if we find the changes no greater than may arise from the causes just spoken of.

And hence a reason may be assigned why comets are not comprehended within the limits of a zodiac, as the planets are; but, being confined to no bounds, are with various motions dispersed all over the heavens; namely, to this purpose, that in their aphelions, where their motions are exceedingly slow, receding to greater distances one from another, they may suffer less disturbance from their mutual gravitations: and hence it is that the comets which descend the lowest, and therefore move the slowest in their aphelions, ought also to ascend the highest.

The comet which appeared in the year 1680 was in its perihelion less distant from the sun than by a sixth part of the sun's diameter; and because of

its extreme velocity in that proximity to the sun, and some density of the sun's atmosphere, it must have suffered some resistance and retardation; and therefore, being attracted somewhat nearer to the sun in every revolution, will at last fall down upon the body of the sun. Nay, in its aphelion, where it moves the slowest, it may sometimes happen to be yet further retarded by the attractions of other comets, and in consequence of this retardation descend to the sun. So fixed stars, that have been gradually wasted by the light and vapors emitted from them for a long time, may be recruited by comets that fall upon them; and from this fresh supply of new fuel those old stars, acquiring new splendor, may pass for new stars. Of this kind are such fixed stars as appear on a sudden, and shine with a wonderful brightness at first, and afterwards vanish by little and little. Such was that star which appeared in Cassiopeia's Chair; which *Cornells Gemma* did not see upon the 8th of *Not/ember*, 1572, though he was observing that part of the heavens upon that very night, and the sky was perfectly serene; but the next night (*Not/ember* 9) he saw it shining much brighter than any of the fixed stars, and scarcely inferior to Venus in splendor. *Tycho Brake* saw it upon the nth of the same month, when it shone with the greatest lustre; and from that time he observed it to decay by little and little; and in 16 months' time it entirely disappeared. In the month of *November*, when it first appeared, its light was equal to that of Venus. In the month of *December*, its light was a little diminished, and was now become equal to that of Jupiter. In *January*, 1573, it was less than Jupiter, and greater than Sirius, and about the end of *February* and the beginning of *March* became equal to that star. In the months of *April* and *May* it was equal to a star of the second magnitude; in *June*, *July*, and *August*, to a star of the third magnitude; in *September*, *October*, and *November*, to those of the fourth magnitude; in *December* and *January*, 1574, to those of the fifth; in *February* to those of the sixth magnitude; and in *March* it entirely vanished. Its color at the beginning was clear, bright, and inclining to white; afterwards it turned a little yellow; and in *March*, 1573, it became ruddy, like Mars or Aldebaran; in *May* it turned to a kind of dusky whiteness, like that we observe in Saturn; and that color it retained ever after, but growing always more and more obscure. Such also was the star in the right foot of Serpentarius, which

Kepler's scholars first observed *September* 30, o.s., 1604, with a light exceeding that of Jupiter, though the night before it was not to be seen; and from that time it decreased by little and little, and in 15 or 16 months entirely disappeared. Such a new star appearing with an unusual splendor is said to have moved *Hipparchus* to observe, and make a catalogue of, the fixed stars. As to those fixed stars that appear and disappear by turns, and increase slowly and by degrees, and scarcely ever exceed the stars of the third magnitude, they seem to be of another kind, which revolve about their axes, and, having a light and a dark side, show those two different sides by turns. The vapors which arise from the sun, the fixed stars, and the tails of the comets, may meet at last with, and fall into, the atmospheres of the planets by their gravity, and there be condensed and turned into water and humid spirits; and from thence, by a slow heat, pass gradually into the form of salts, and sulphurs, and tinctures, and mud, and clay, and sand, and stones, and coral, and other terrestrial substances.

GENERAL SCHOLIUM

The hypothesis of vortices is pressed with many difficulties. That every planet by a radius drawn to the sun may describe areas proportional to the times of description, the periodic times of the several parts of the vortices should observe the square of their distances from the sun; but that the periodic times of the planets may obtain the $\frac{2}{3}$ th power of their distances from the sun, the periodic times of the parts of the vortex ought to be as the $\frac{3}{2}$ th power of their distances. That the smaller vortices may maintain their lesser revolutions about Saturn, Jupiter, and other planets, and swim quietly and undisturbed in the greater vortex of the sun, the periodic times of the parts of the sun's vortex should be equal; but the rotation of the sun and planets about their axes, which ought to correspond with the motions of their vortices, recede far from all these proportions. The motions of the comets are exceedingly regular, are governed by the same laws with the motions of the planets, and can by no means be accounted for by the hypothesis of vortices; for comets are carried with very eccentric motions through all parts of the heavens indifferently, with a freedom that is incompatible with the notion of a vortex.

Bodies projected in our air suffer no resistance but from the air. Withdraw the air, as is done in Mr. *Boyle's* vacuum, and the resistance ceases; for in this void a bit of fine down and a piece of solid gold descend with equal velocity. And the same argument must apply to the celestial spaces above the earth's atmosphere; in these spaces, where there is no air to resist their motions, all bodies will move with the greatest freedom; and the planets and comets will constantly pursue their revolutions in orbits given in kind and position, according to the laws above explained; but though these bodies may, indeed, continue in their orbits by the mere laws of gravity, yet they could by no means have at first derived the regular position of the orbits themselves from those laws.

The six primary planets are revolved about the sun in circles concentric with the sun, and with motions directed towards the same parts, and almost

in the same plane. Ten moons are revolved about the earth, Jupiter, and Saturn, in circles concentric with them, with the same direction of motion, and nearly in the planes of the orbits of those planets; but it is not to be conceived that mere mechanical causes could give birth to so many regular motions, since the comets range over all parts of the heavens in very eccentric orbits; for by that kind of motion they pass easily through the orbs of the planets, and with great rapidity; and in their aphelions, where they move the slowest, and are detained the longest, they recede to the greatest distances from each other, and hence suffer the least disturbance from their mutual attractions. This most beautiful system of the sun, planets, and comets, could only proceed from the counsel and dominion of an intelligent and powerful Being. And if the fixed stars are the centres of other like systems, these, being formed by the like wise counsel, must be all subject to the dominion of One; especially since the light of the fixed stars is of the same nature with the light of the sun, and from every system light passes into all the other systems: and lest the systems of the fixed stars should, by their gravity, fall on each other, he hath placed those systems at immense distances from one another.

This Being¹ governs all things, not as the soul of the world, but as Lord over all; and on account of his dominion he is wont to be called *Lord God παντοκράτωρ*, or *Universal Ruler*; for *God* is a relative word, and has a respect to servants; and *Deity* is the dominion of God not over his own body, as those imagine who fancy God to be the soul of the world, but over servants. The Supreme God is a Being eternal, infinite, absolutely perfect; but a being, however perfect, without dominion, cannot be said to be Lord God; for we say, my God, your God, the God of *Israel*, the God of Gods, and Lord of Lords; but we do not say, my Eternal, your Eternal, the Eternal of *Israel*, the Eternal of Gods; we do not say, my Infinite, or my Perfect: these are titles which have no respect to servants. The word God* usually signifies *Lord*; but every lord is not a God. It is the dominion of a spiritual being which constitutes a God: a true, supreme, or imaginary dominion

¹ Appendix, Note 52]

*Dr Potoc\ derives the Latin word *Deus* from the Arabic *du* (in the oblique case *dt*), which signifies *Lord*. And in this sense princes are called *gods*, *Psal.* lxxxii. vcr. 6, and *John* x. ver. 35. And *Moses* is called a *god* to his brother *Aaron*, and a *god* to *Pharaoh* (*Exod.* IV. ver. 16; and vn. vcr. 1). And in the same sense the souls of dead princes were formerly, by the Heathens, called *gods*, but falsely, because of their want of dominion.

makes a true, supreme, or imaginary God. And from his true dominion it follows that the true God is a living, intelligent, and powerful Being; and, from his other perfections, that he is supreme, or most perfect. He is eternal and infinite, omnipotent and omniscient; that is, his duration reaches from eternity to eternity; his presence from infinity to infinity; he governs all things, and knows all things that are or can be done. He is not eternity and infinity, but eternal and infinite; he is not duration or space, but he endures and is present. He endures forever, and is everywhere present; and, by existing always and everywhere, he constitutes duration and space. Since every particle of space is *always*, and every indivisible moment of duration is *everywhere*, certainly the Maker and Lord of all things cannot be *never* and *nowhere*. Every soul that has perception is, though in different times and in different organs of sense and motion, still the same indivisible person. There are given successive parts in duration, coexistent parts in space, but neither the one nor the other in the person of a man, or his thinking principle; and much less can they be found in the thinking substance of God. Every man, so far as he is a thing that has perception, is one and the same man during his whole life, in all and each of his organs of sense. God is the same God, always and everywhere. He is omnipresent not *virtually* only, but also *substantially*; for virtue cannot subsist without substance. In him* are all things contained and moved; yet neither affects the other: God suffers nothing from the motion of bodies; bodies find no resistance from the omnipresence of God. It is allowed by all that the Supreme God exists necessarily; and by the same necessity he exists *always* and *everywhere*. Whence also he is all similar, all eye, all ear, all brain, all arm, all power to perceive, to understand, and to act; but in a manner not at all human, in a manner not at all corporeal, in a manner utterly unknown to us. As a blind man has no idea of colors, so have we no idea of the manner by which the all-wise God perceives and understands all things. He is utterly void of all body and bodily figure, and can therefore neither be seen, nor heard, nor touched; nor ought he to be worshiped under the representation of any

* This was the opinion of the Ancients. So *Pythagoras*, in *Cicer de Nat Dror* lib 1 *Thules*, *Anaxagoras*, *Virgil*, *Gorg* lib iv vcr. 220, and *Aeneid*, lib vi. vcr 721 *Philo Allegor*, at the beginning of lib. 1 *Aratus*, in his *Phaenom* at the beginning. So also the sacred writers as *St Paul*, *Act** wn \tr 27, 28 *St John's Gosp.* chap xiv ver 2 *Moses*, in *Deut* iv ver 30, and x vcr 14 *David*, *Psal* exxxv, vcr 7, 8, 9 *Solomon*, 1 *Kings* viii ver. 27 *Job*, xxv vcr 12, 13, 14 *Jeremiah*, xxi ver 25, 24 *The Idolaters* supposed the sun, moon, and stars, the souls of men, and other parts of the world, to be parts of the Supreme God, and therefore to be worshiped, but erroneously

corporeal thing. We have ideas of his attributes, but what the real substance of anything is we know not. In bodies, we see only their figures and colors, we hear only the sounds, we touch only their outward surfaces, we smell only the smells, and taste the savors; but their inward substances are not to be known either by our senses, or by any reflex act of our minds: much less, then, have we any idea of the substance of God. We know him only by his most wise and excellent contrivances of things, and final causes ^j¹ we admire him for his perfections; but we reverence and adore him on account of his dominion: for we adore him as his servants; and a god without dominion, providence, and final causes, is nothing else but Fate and Nature. Blind metaphysical necessity, which is certainly the same always and everywhere, could produce no variety of things. All that diversity of natural things which we find suited to different times and places could arise from nothing but the ideas and will of a Being necessarily existing. But, by way of allegory, God is said to see, to speak, to laugh, to love, to hate, to desire, to give, to receive, to rejoice, to be angry, to fight, to frame, to work, to build; for all our notions of God are taken from the ways of mankind by a certain similitude, which, though not perfect, has some likeness, however. And thus much concerning God; to discourse of whom from the appearances of things, does certainly belong to Natural Philosophy.

Hitherto we have explained the phenomena of the heavens and of our sea by the power of gravity, but have not yet assigned the cause of this power. This is certain, that it must proceed from a cause that penetrates to the very centres of the sun and planets, without suffering the least diminution of its force; that operates not according to the quantity of the surfaces of the particles upon which it acts (as mechanical causes used to do), but according to the quantity of the solid matter which they contain, and propagates its virtue on all sides to immense distances, decreasing always as the inverse square of the distances.² Gravitation towards the sun is made up out of the gravitations towards the several particles of which the body of the sun is composed; and in receding from the sun decreases accurately as the inverse square of the distances as far as the orbit of Saturn, as evidently appears from the quiescence of the aphelion of the planets; nay, and even to the remotest aphelion of the comets, if those aphelions are also quiescent.

[¹ Appendix, Note 53]

[² Appendix, Note 54]

But hitherto I have not been able to discover the cause *of* those properties of gravity from phenomena, and I frame no hypotheses;¹ for whatever is not deduced from the phenomena is to be called an hypothesis; and hypotheses, whether metaphysical or physical, whether of occult qualities or mechanical, have no place in experimental philosophy. In this philosophy particular propositions are inferred from the phenomena, and afterwards rendered general by induction. Thus it was that the impenetrability, the mobility, and the impulsive force *of* bodies, and the laws of motion and of gravitation, were discovered. And to us it is enough that gravity does really exist, and act according to the laws which we have explained, and abundantly serves to account for alt the motions of the celestial bodies, and of our sea.²

And now we might add something concerning a certain most subtle spirit which pervades and lies hid in all gross bodies; by the force and action of which spirit the particles of bodies attract one another at near distances, and cohere, if contiguous; and electric bodies operate to greater distances, as well repelling as attracting the neighboring corpuscles; and light is emitted, reflected, refracted, inflected, and heats bodies; and all sensation is excited, and the members of animal bodies move at the command of the will, namely, by the vibrations of this spirit, mutually propagated along the solid filaments of the nerves, from the outward organs of sense to the brain, and from the brain into the muscles. But these are things that cannot be explained in few words, nor are we furnished with that sufficiency of experiments which is required to an accurate determination and demonstration of the laws by which this electric and elastic spirit operates.

[¹ Appendix, Note 55] [² Appendix, Note 56]

THE SYSTEM OF THE WORLD¹

[I.] *The matter of the heavens is fluid.*

It was the ancient opinion of not a few, in the earliest ages of philosophy, that the fixed stars stood immovable in the highest parts of the world; that under the fixed stars the planets were carried about the sun; that the earth, as one of the planets, described an annual course about the sun, while by a diurnal motion it was in the meantime revolved about its own axis; and that the sun, as the common fire which served to warm the whole, was fixed in the centre of the universe.

This was the philosophy taught of old by *Philolaus*, *Aristarchus* of *Samos*, *Plato* in his riper years, and the whole sect of the *Pythagoreans*; and this was the judgment of *Anaxtmander*, more ancient still; and of that wise king of the *Romans*, *Numa Pompilius*, who, as a symbol of the figure of the world with the sun in the centre, erected a round temple in honor of *Vesta*, and ordained perpetual fire to be kept in the middle of it.

The *Egyptians* were early observers of the heavens; and from them, probably, this philosophy was spread abroad among other nations; for from them it was, and the nations about them, that the *Greeks*, a people more addicted to the study of philology than of Nature, derived their first, as well as soundest, notions of philosophy; and in the Vestal ceremonies we may yet trace the ancient spirit of the *Egyptians*; for it was their way to deliver their mysteries, that is, their philosophy of things above the common way of thinking, under the veil of religious rites and hieroglyphic symbols.

It is not to be denied that *Anaxagoras*, *Democritus*, and others, did now and then start up, who would have it that the earth possessed the centre of the world, and that the stars were revolved towards the west about the earth quiescent in the centre, some at a swifter, others at a slower rate.

However, it was agreed on both sides that the motions of the celestial bodies were performed in spaces altogether free and void of resistance. The whim of solid orbs was of a later date, introduced by *Eudoxus*, *Cahppus*,

[¹ Appendix, Note 57]

and *Aristotle*; when the ancient philosophy began to decline, and to give place to the new prevailing fictions of the *Greeks*.

But, above all things, the phenomena of comets can by no means tolerate the idea of solid orbits. The *Chaldeans*, the most learned astronomers of their time, looked upon the comets (which of ancient times before had been numbered among the celestial bodies) as a particular sort of planets, which, describing eccentric orbits, presented themselves to view only by turns, once in a revolution, when they descended into the lower parts of their orbits.

And as it was the unavoidable consequence of the hypothesis of solid orbits, while it prevailed, that the comets should be thrust into spaces below the moon; so, when later observations of astronomers restored the comets to their ancient places in the higher heavens, these celestial spaces were necessarily cleared of the incumbrance of solid orbits.

[2.] *The principle of circular motion in free spaces.*

After this time, we do not know in what manner the ancients explained the question, how the planets came to be retained within certain bounds in these free spaces, and to be drawn off from the rectilinear courses, which, left to themselves, they should have pursued, into regular revolutions in curvilinear orbits. Probably it was to give some sort of satisfaction to this difficulty that solid orbs had been introduced.

The later philosophers pretend to account for it either by the action of certain vortices, as *Kepler* and *Descartes*; or by some other principle of impulse or attraction, as *Borelli*, *Hooe*, and others of our nation; for, from the laws of motion, it is most certain that these effects must proceed from the action of some force or other.

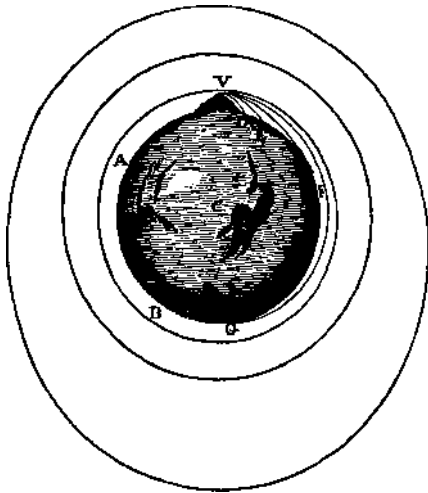
But our purpose is only to trace out the quantity and properties of this force from the phenomena (p. 192), and to apply what we discover in some simple cases as principles, by which, in a mathematical way, we may estimate the effects thereof in more involved cases; for it would be endless and impossible to bring every particular to direct and immediate observation.

We said, *in a mathematical way*, to avoid all questions about the nature or quality of this force, which we would not be understood to determine by any hypothesis; and therefore call it by the general name of a centripetal

force, as it is a force which is directed towards some centre; and as it regards more particularly a body in that centre, we call it circumsolar, circumterrestrial, circumjovial; and so in respect of other central bodies.

[3.] *The action of centripetal forces.*

That by means of centripetal *forces* the planets may be retained in certain orbits, we may easily understand, if we consider the motions of projectiles (pp. 2-4); for a stone that is projected is by the pressure of its own weight forced out of the rectilinear path, which by the initial projection alone it should have pursued, and made to describe a curved line in the air; and



through that crooked way is at last brought down to the ground; and the greater the velocity is with which it is projected, the farther it goes before it falls to the earth. We may therefore suppose the velocity to be so increased, that it would describe an arc of 1, 2, 5, 10, 100, 1000 miles before it arrived at the earth, till at last, exceeding the limits of the earth, it should pass into space without touching it.

Let AFB represent the surface of the earth, C its centre, VD, VE, VF the curved lines which a body would describe, if projected in an horizontal direction from the top of an high mountain successively with more and more velocity (p. 418); and, because the celestial motions are scarcely retarded by the little or no resistance of the spaces in which they are performed, to keep up the parity of cases, let us suppose either that there is no air about the earth, or at least that it is endowed with little or no power of resisting; and for the same reason that the body projected with a less velocity describes the lesser arc VD, and with a greater velocity the greater arc VE, and, augmenting the velocity, it goes farther and farther to F and G, if the velocity was still more and more augmented, it would reach at last quite beyond the circumference of the earth, and return to the mountain from which it was projected.

And since the areas which by this motion it describes by a radius drawn to the centre of the earth are (by Prop. 1, Book 1, *Pnncip. Math.*) proportional to the times in which they are described, its velocity, when it returns to the mountain, will be no less than it was at first; and, retaining the same velocity, it will describe the same curve over and over, by the same law.

But if we now imagine bodies to be projected in the directions of lines parallel to the horizon from greater heights, as of 5, 10, 100, 1000, or more miles, or rather as many semidiameters of the earth, those bodies, according to their different velocity, and the different force of gravity in different heights, will describe arcs either concentric with the earth, or variously eccentric, and go on revolving through the heavens in those orbits just as the planets do in their orbits.

[4.] *The certainty of the proof.*

As when a stone is projected obliquely, that is, any way but in the perpendicular direction, the continual deflection thereof towards the earth from the right line in which it was projected is a proof of its gravitation to the earth, no less certain than its direct descent when suffered to fall freely from rest; so the deviation of bodies moving in free spaces from rectilinear paths, and continual deflection therefrom towards any place, is a sure indication of the existence of some force which from all quarters impels those bodies towards that place.

And as, from the supposed existence of gravity, it necessarily follows that all bodies about the earth must press downwards, and therefore must either descend directly to the earth, if they are let fall from rest, or at least continually deviate from right lines towards the earth, if they are projected obliquely; so, from the supposed existence of a force directed to any centre, it will follow, by the like necessity, that all bodies upon which this force acts must either descend directly to that centre, or at least deviate continually towards it from right lines, if otherwise they should have moved obliquely in these right lines.

And how from the motions given we may infer the forces^A or from the forces given we may determine the motions, is shown in the first two Books of our *Principles of Philosophy*.

If the earth is supposed to stand still, and the fixed stars to be revolved in free spaces in the space of 24 hours, it is certain the forces by which the fixed stars are held in their orbits are not directed to the earth, but to the centres of those orbits, that is, of the several parallel circles, which the fixed stars, declining to one side and the other from the equator, describe daily; also that by radii drawn to the centres of the orbits the fixed stars describe areas exactly proportional to the times of description. Then, because the periodic times are equal (by Cor. in, Prop, iv, Book 1), it follows that the centripetal forces are as the radii of the several orbits, and that they will continually revolve in the same orbits. And the like consequences may be drawn from the supposed diurnal motion of the planets.

That forces should be directed to no body on which they physically depend, but to innumerable imaginary points in the axis of the earth, is an hypothesis too incongruous. It is more incongruous still that those forces should increase exactly in proportion of the distances from this axis; for this is an indication of an increase to immensity, or rather to infinity; whereas the forces of natural things commonly decrease in receding from the fountain from which they flow. But, what is yet more absurd, neither are the areas described by the same star proportional to the times, nor are its revolutions performed in the same orbit; for as the star recedes from the neighboring pole, both areas and orbits increase; and from the increase of the area it is demonstrated that the forces are not directed to the axis of the earth. And this difficulty (Cor. 1, Prop. 11, Book 1) arises from the twofold motion

that is observed in the fixed stars, one diurnal round the axis of the earth, the other exceedingly slow round the axis of the ecliptic. And the explication thereof requires a composition of forces so involved and so variable, that it is hardly to be reconciled with any physical theory.

[5.] *Centripetal forces are directed to the individual centres of the planets.*

That there are centripetal forces actually directed to the bodies of the sun, of the earth, and other planets, I thus infer.

The moon revolves about our earth, and by radii drawn to its centre (p. 406) describes areas nearly proportional to the times in which they are described, as is evident from its velocity compared with its apparent diameter; for its motion is slower when its diameter is less (and therefore its distance greater), and its motion is swifter when its diameter is greater.

The revolutions of the satellites of Jupiter about that planet are more regular (p. 401); for they describe circles concentric with Jupiter by uniform motions, as exactly as our senses can perceive.

And so the satellites of Saturn are revolved about this planet with motions nearly (p. 402) circular and uniform, scarcely disturbed by any eccentricity hitherto observed.

That Venus and Mercury are revolved about the sun, is demonstrable from their moon-like appearances (p. 403); when they shine with a full face, they are in those parts of their orbits which in respect of the earth lie beyond the sun; when they appear half full, they are in those parts which lie over against the sun; when horned, in those parts which lie between the earth and the sun; and sometimes they pass over the sun's disk, when directly interposed between the earth and the sun.

And Venus, with a motion almost uniform, describes an orbit nearly circular and concentric with the sun.

But Mercury, with a more eccentric motion, makes remarkable approaches to the sun, and goes off again by turns; but it is always swifter as it is near to the sun, and therefore by a radius drawn to the sun still describes areas proportional to the times.

Lastly, that the earth describes about the sun, or the sun about the earth, by a radius from the one to the other, areas exactly proportional to the times,

is demonstrable from the apparent diameter of the sun compared with its apparent motion.

These are astronomical experiments; from which it follows, by Prop. i, II, in, in the first Book of our *Principles*, and their Corollaries (pp. 40-45), that there are centripetal forces actually directed (either accurately or without considerable error) to the centres of the earth, of Jupiter, of Saturn, and of the sun. In Mercury, Venus, Mars, and the lesser planets, where experiments are wanting, the arguments from analogy must be allowed in their place.

[6.] *Centripetal forces decrease inversely as the square of the distances from the centres of the planets.*

That those forces (pp. 40-45) decrease as the inverse square of the distances from the centre of every planet, appears by Cor. vi, Prop. iv, Book 1; for the periodic times of the satellites of Jupiter are one to another (pp. 401-402), as the $\frac{3}{2}$ th power of their distances from the center of this planet.

This proportion has been long ago observed in those satellites; and Mr. *Flamsteed*, who had often measured their distances from Jupiter by the micrometer, and by the eclipses of the satellites, wrote to me, that it holds to all the accuracy that possibly can be discerned by our senses. And he sent me the dimensions of their orbits taken by the micrometer, and reduced to the mean distance of Jupiter from the earth, or from the sun, together with the times of their revolutions, as follows:

The greatest elongation of the satellites from the centre of Jupiter as seen from the sun				The periodic times of their revolutions			
	<i>l</i>	<i>m</i>	<i>n</i>	<i>d</i>	<i>h</i>	<i>m</i>	<i>s</i>
1st	1	48	or 108	1	18	28	36
2d	3	01	or 181	3	13	17	54
3d	4	46	or 286	7	03	59	36
4th	8	13½	or 493½	16	18	5	13

Hence the $\frac{3}{2}$ th power of the distances may be easily seen. For example: the 16^d. 18^h. 05^m. 13^s. is to the time 1^d. 18^h. 28^m. 36^s. as $493\frac{1}{2}'' \sqrt{493\frac{1}{2}''}$ to $108'' \sqrt{108''}$, neglecting those small fractions which, in observing, cannot be certainly determined.

Before the invention of the micrometer, the same distances were determined in semidiameters of Jupiter thus:

<i>Distance of the</i>	1st	2d	3d	4th
<i>By Galileo</i>	6	10	16	28
<i>By Simon Marius</i>	6	10	16	26
<i>By Cassini</i>	5	8	13	23
<i>By Borelli, more exactly</i>	$5\frac{2}{3}$	$8\frac{2}{3}$	14	$24\frac{2}{3}$

After the invention of the micrometer:

<i>Distance of the</i>	1st	2d	3d	4th
<i>By Townley</i>	5 51	8 78	13 47	24 72
<i>By Flamsteed</i>	5 31	8 85	13 98	24 23
<i>More accurately by the eclipses</i>	5 578	8 876	14 159	24 903

And the periodic times of those satellites, by the observations of Mr. *Flamsteed*, are $1^d 18^h 28^m 36^s$ | $3^d 13^h 17^m 54^s$ | $7^d 3^h 59^m 36^s$ | $16^d 18^h 5^m 13^s$, as above.

And the distances thence computed are 5.578 | 8.878 | 14.168 | 24.968, accurately agreeing with the distances by observation.

Cassini assures us (pp. 403-404) that the same proportion is observed in the circumsaturnal planets. But a longer course of observations is required before we can have a certain and accurate theory of those planets.

In the circumsolar planets, Mercury and Venus, the same proportion holds with great accuracy, according to the dimensions of their orbits, as determined by the observations of the best astronomers.

[7.] *The superior planets revolve about the sun, and the radii drawn to the sun describe areas proportional to the times.*

That Mars is revolved about the sun is demonstrated from the phases which it shows, and the proportion of its apparent diameters (pp. 403--406); for from its appearing full near conjunction with the sun, and gibbous in its quadratures, it is certain that it surrounds the sun.

And since its diameter appears about five times greater when in opposition to the sun than when in conjunction therewith, and its distance from the earth is inversely as its apparent diameter, that distance will be about five times less when in opposition to than when in conjunction with the sun; but in both cases its distance from the sun will be nearly the same with the distance which is inferred from its gibbous appearance in the quadratures. And as it encompasses the sun at almost equal distances, but in respect of the earth is very unequally distant, so by radii drawn to the sun it describes areas nearly uniform; but by radii drawn to the earth, it is sometimes swift, sometimes stationary, and sometimes retrograde.

That Jupiter, in a higher orbit than Mars, is likewise revolved about the sun, with a motion nearly uniform, as well in distance as in the areas described, I infer thus.

Mr. *Flamsteed* assured me, by letters, that all the eclipses of the innermost satellite which hitherto have been well observed do agree with his theory so nearly, as never to differ therefrom by two minutes of time; that in the outermost the error is little greater; in the outermost but one, scarcely three times greater; that in the innermost but one the difference is indeed much greater, yet so as to agree as nearly with his computations as the moon does with the common tables; and that he computes those eclipses only from the mean motions corrected by the equation of light discovered and introduced by Mr. *Romer*. Supposing, then, that the theory differs by a less error than that of 2 from the motion of the outmost satellite as hitherto described, and taking as the periodic time $16^d \cdot 18^h \cdot 5^m \cdot 13^s$ to 2 in time, so is the whole circle of 360° to the arc $1^\circ 48''$, the error of Mr. *Flamsteed's* computation, reduced to the satellite's orbit, will be less than $1^\circ 48''$; that is, the longitude of the satellite, as seen from the centre of Jupiter, will be determined with a less error than $1^\circ 48''$. But when the satellite is in the middle of the shadow, that longitude is the same with the heliocentric longitude of Jupiter; and, therefore, the hypothesis which Mr. *Flamsteed* follows, viz., the *Copernican*, as improved by *Kepler*, and (as to the motion of Jupiter) lately corrected by himself, rightly represents that longitude within a less error than $1^\circ 48''$; but by this longitude, together with the geocentric longitude, which is always easily found, the distance of Jupiter from the sun is

determined; which must, therefore, be the very same with that which the hypothesis exhibits. For that greatest error of 1' 48" that can happen in the heliocentric longitude is almost insensible, and quite to be neglected, and perhaps may arise from some yet undiscovered eccentricity of the satellite; but since both longitude and distance are rightly determined, it follows of necessity that Jupiter, by radii drawn to the sun, describes areas so conditioned as the hypothesis requires, that is, proportional to the times.

And the same thing may be concluded of Saturn from his satellite, by the observations of Mr. *Huygens* and Dr. *Halley*; though a longer series of observations is yet wanting to confirm the thing, and to bring it under a sufficiently exact computation.

[8.] *The force which controls the superior planets is not directed to the earth, but to the sun.*

For if Jupiter was viewed from the sun, it would never appear retrograde nor stationary, as it is seen sometimes from the earth, but always to go forwards with a motion nearly uniform (p. 405). And from the very great inequality of its apparent geocentric motion, we infer (by Book 1, Prop. in, Cor. iv) that the force by which Jupiter is turned out of a rectilinear course, and made to revolve in an orbit, is not directed to the centre of the earth. The same argument holds good for Mars and for Saturn. Another centre of these forces is therefore to be looked for (by Book 1, Prop. n and in, and the Corollaries of the latter), about which the areas described by radii intervening may be uniform; and that this is the sun, we have proved already in Mars and Saturn nearly, but accurately enough in Jupiter. It may be alleged that the sun and planets are impelled by some other force equally and in the direction of parallel lines; but by such a force (by Cor. vi of the Laws of Motion) no change would happen in the situation of the planets one to another, nor any sensible effect follow: but our business is with the causes of sensible effects. Let us, therefore, neglect every such force as imaginary and precarious, and of no use in the phenomena of the heavens; and the whole remaining force by which Jupiter is impelled will be directed (by Book 1, Prop. III, Cor. 1) to the centre of the sun.

[9.] *The circumsolar force decreases in all planetary spaces inversely as the square of the distances from the sun.*

The distances of the planets from the sun come out the same, whether, with *Tycho*, we place the earth in the centre of the system, or the sun with *Copernicus*: and we have already proved that these distances are true in Jupiter.

Kepler and *Boulliau* have, with great care (p. 404), determined the distances of the planets from the sun; and hence it is that their tables agree best with the heavens. And in all the planets, in Jupiter and Mars, in Saturn and the earth, as well as in Venus and Mercury, the cubes of their distances are as the squares of their periodic times; and therefore (by Cor. vi, Prop. iv, Book 1) the centripetal circumsolar force throughout all the planetary regions decreases as the inverse square of the distances from the sun. In examining this proportion, we are to use the mean distances, or the transverse semiaxes of the orbits (by Prop. xv, Book 1), and to neglect those little fractions, which, in defining the orbits, may have arisen from the insensible errors of observation, or may be ascribed to other causes which we shall afterwards explain. And thus we shall always find the said proportion to hold exactly; for the distances of Saturn, Jupiter, Mars, the earth, Venus, and Mercury, from the sun, obtained from the observations of astronomers, are, according to the computation of *Kepler*, as the numbers 951000, 519650, 152350, 100000, 72400, 38806; by the computation of *Boulliau*, as the numbers 954198, 522520, 152350, 100000, 72398, 38585; and from the periodic times they come out 953806, 520116, 152399, 100000, 72333, 38710. Their distances, according to *Kepler* and *Boulliau*, scarcely differ by any sensible quantity, and where they differ most the distances calculated from the periodic times fall in between them.

[10.] *The circumterrestrial force decreases inversely as the square of the distances from the earth. This is shown on the hypothesis that the earth is at rest.*

That the circumterrestrial force likewise decreases as the inverse square of the distances, I infer thus.

The mean distance of the moon from the centre of the earth, is, in semidiameters of the earth, according to *Ptolemy*, *Kepler* in his *Ephemerides*,

oulliau, *Hewelckc*, and *Riccioli*, 59; according to *Flamsteed*, 59%; according to *Tycho*, $56\frac{1}{2}$; to *Vendelin*, 60; to *Copernicus*, $60\frac{1}{3}$; to *Kircher*, 62V2 pp. 407-409).

But *Tycho*, and all that follow his tables of refraction, making the refractions of the sun and moon (altogether against the nature of light) to exceed those of the fixed stars, and that by about four or five minutes in the horizon, did thereby augment the horizontal parallax of the moon by about the same number of minutes; that is, by about the 12th or 15th part of the whole parallax. Correct this error, and the distance will become 60 or 61 semidiameters of the earth, nearly agreeing with what others have determined.

Let us, then, assume the mean distance of the moon 60 semidiameters of the earth, and its periodic time in respect of the fixed stars $27^d. 7^h. 43^m.$, as astronomers have determined it. And (by Cor. vi, Prop, iv, Book 1) a body revolved in our air, near the surface of the earth supposed at rest, by means of a centripetal force which should be to the same force at the distance of the moon inversely as the squares of the distances from the centre of the earth, that is, as 3600 to 1, would (excluding the resistance of the air) complete a revolution in $1^h. 24^m. 27^s.$

Suppose the circumference of the earth to be 123249600 *Paris* feet, as has been determined by the late mensuration of the *French* (p. 425), then the same body, deprived of its circular motion, and falling by the impulse of the same centripetal force as before, would, in one second of time, describe $15\frac{1}{12}$ *Paris* feet.

This we infer by a calculus formed upon Prop, xxxvi of Book 1, and it agrees with what we observe in all bodies about the earth. For by the experiments of pendulums, and a computation based thereon, *Huygens* hath demonstrated that bodies falling by all that centripetal force with which (of whatever nature it is) they are impelled near the surface of the earth, do, in the second of time, describe $15\frac{1}{12}$ *Parts* feet.

[i.] *The same proved on the hypothesis that the earth moves.*

But if the earth is supposed to move, the earth and moon together (by Cor. rv of the Laws of Motion, and Prop, LVII, Book 1) will be revolved about their common centre of gravity. And the moon (by Prop, LX, Book 1) will in the same periodic time, $27^d 7^h. 43^m.$, with the same circumterrestrial

force diminished as the inverse square of the distance, describe an orbit whose semidiameter is to the semidiameter of the former orbit, that is, to 60 semidiameters of the earth, as the sum of both the bodies of the earth and moon is to the first of two mean proportionals between this sum and the body of the earth; that is, if we suppose the moon (on account of its mean apparent diameter $31V2$) to be about $\frac{1}{2}$ of the earth, as 43 to H ($42 \cdot 43^2$), or as about 128 to 127. And therefore the semidiameter of the orbit, that is, the distance between the centres of the moon and earth, will in this case be $60V2$ semidiameters of the earth, almost the same with that assigned by *Copernicus*, which the *Tychonic* observations by no means disprove; and, therefore, the squared ratio of the decrement of the force holds good in this distance. I have neglected the increment of the orbit which arises from the action of the sun as inconsiderable; but if that is subtracted, the true distance will remain about 60% semidiameters of the earth.

[12.] *The decrease of the forces inversely as the square of the distances from the earth and planets is proved also from the eccentricities of the planets and the very slow motion of the apsides.*

But further (p. 407), this proportion of the decrement of the forces is confirmed from the eccentricity of the planets, and the very slow motion of their apsides; for (by the Corollaries of Prop. XLV, Book 1) in no other proportion could the circumsolar planets once in every revolution descend to their least and once ascend to their greatest distance from the sun, and the places of those distances remain immovable. A small error from the squared ratio would produce a motion of the apsides considerable in every revolution, but in many enormous.

But now, after innumerable revolutions, hardly any such motion has been perceived in the orbits of the circumsolar planets. Some astronomers affirm that there is no such motion; others reckon it no greater than what may easily arise from the causes hereafter to be assigned, which is of no moment in the present question.

We may even neglect the motion of the moon's apse (pp. 406-407), which is far greater than in the circumsolar planets, amounting in every revolution to three degrees; and from this motion it is demonstrable that the circumterrestrial force decreases in no less than the inverse square, but far less than

the inverse cube of the distance; for if the square were gradually changed into the cube, the motion of the apse would thereby increase to infinity; and, therefore, by a very small mutation, would exceed the motion of the moon's apse. This slow motion arises from the action of the circumsolar force, as we shall afterwards explain. But excluding this cause, the apse or apogee of the moon will be fixed, and the squared ratio of the decrease of the circumterrestrial force in different distances from the earth will accurately take place.

[13.] *The intensity of the forces directed to the individual planets. The powerful circumsolar force.*

Now that this proportion has been established, we may compare the forces of the several planets among themselves (p. 406).

In the mean distance of Jupiter from the earth, the greatest elongation of the outermost satellite from Jupiter's centre (by the observations of Mr. *Flamsteed*) is 8' 13"; and therefore the distance of the satellite from the centre of Jupiter is to the mean distance of Jupiter from the centre of the sun as 124 to 52012, but to the mean distance of Venus from the centre of the sun as 124 to 7234; and their periodic times are $13\frac{3}{4}^d$. and $224\frac{2}{3}$.; and from hence (according to Cor. 11, Prop. iv, Book 1), dividing the distances by the squares of the times, we infer that the force by which the satellite is impelled towards Jupiter is to the force by which Venus is impelled towards the sun as 442 to 143; and if we diminish the force by which the satellite is impelled according to the inverse square of the distance 124 to 7234, we shall have the circumjovial force in the distance of Venus from the sun to the circumsolar force by which Venus is impelled as 100 to 143 or as 1 to 1100; therefore at equal distances the circumsolar force is 1100 times greater than the circumjovial.

And, by the like computation, from the periodic time of the satellite of Saturn $15^d. 22^h$. and its greatest elongation from Saturn, while that planet is in its mean distance from us, 3' 20", it follows that the distance of this satellite from Saturn's centre is to the distance of Venus from the sun as 92% to 7234; and from thence that the absolute circumsolar force is 2360 times greater than the absolute circumsaturnal.

[14.] *The small circumterrestrial force.*

From the regularity of the heliocentric and irregularity of the geocentric motions of Venus, of Jupiter, and the other planets, it is evident (by Cor. iv, Prop. III, Book 1) that the circumterrestrial force, compared with the circumsolar, is very small.

Riccioh and *Vendehn* have each tried to determine the sun's parallax from the moon's dichotomies observed by the telescope, and they agree that it does not exceed half a minute.

Kepler, from *Tycho*'s observations and his own, found the parallax of Mars insensible, even in opposition to the sun, when that parallax is somewhat greater than the sun's.

Flamsteed attempted the same parallax with the micrometer in the perigeon position of Mars, but never found it above 25"; and hence concluded the sun's parallax at most 10".

From this it follows that the distance of the moon from the earth bears no greater proportion to the distance of the earth from the sun than 29 to 10000; nor to the distance of Venus from the sun than 29 to 7233.

From these distances, together with the periodic times, by the method above explained, it is easy to infer that the absolute circumsolar force is at least 229400 times greater than the absolute circumterrestrial force.

And though we were only certain, from the observations of *Ricciolt* and *Vendehn*, that the sun's parallax was less than half a minute, yet from this it will follow that the absolute circumsolar force exceeds the absolute circumterrestrial force 8500 times.

[15.] *Apparent diameters of the planets.*

By similar computations I happened to discover an analogy, that is observed between the forces and the bodies of the planets; but, before I explain this analogy, the apparent diameters of the planets in their mean distances from the earth must be determined.

Mr. *Flamsteed* (p. 402), by the micrometer, measured the diameter of Jupiter 40" or 41"; the diameter of Saturn's ring 50"; and the diameter of the sun about 32' 13".

But the diameter of Saturn is to the diameter of the ring, according to Mr. *Huygens* and Dr. *Halley*, as 4 is to 9; according to *Gallet*, as 4 is to 10;

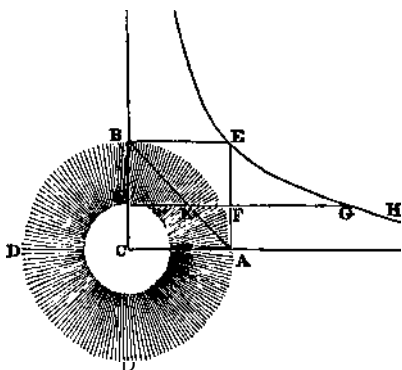
and according to *Hooke* (by a telescope of 60 feet), as 5 to 12. And from the middle ratio, 5 to 12, the diameter of Saturn's body is inferred to be about 21".

[16.] *Correction of the apparent diameters.*

Such as we have said are the apparent magnitudes; but, because of the unequal refrangibility of light, all lucid points are dilated by the telescope, and in the focus of the object glass cover a circular space whose breadth is about the 50th part of the aperture of the glass.

It is true, that towards the circumference the light is so diffuse as hardly to be visible; but towards the middle, where the light is of greater intensity, and is sensible enough, it makes a small lucid circle, whose breadth varies according to the splendor of the lucid point, but is generally about the 3d, or 4th, or 5th part of the breadth of the whole.

Let ABD represent the circle of the whole light; PQ the small circle of the denser and clearer light; C the centre of both; CA, CB semidiameters of the greater circle containing a right angle at C; ACBE the square comprehended under these semidiameters; AB the diagonal of that square; EGH an hyperbola with the centre C and asymptotes CA, CB; PG a perpendicular erected from any point P of the line BC, and meeting the hyperbola in G, and the right



lines AB, AE, in K and F: and the density of the light in any place P will, by my computation, be as the line FG, and therefore at the centre infinite, but near the circumference very small. And the whole light within the small circle PQ is to the whole without as the area of the quadrilateral figure CAKP is to the triangle PKB. And we are to understand the small circle

PQ to be there terminated, where FG, the intensity of the light, begins to be less than what is required for visibility.

Hence it was, that, at the distance of 191382 feet, a fire of 3 feet in diameter, through a telescope of 3 feet, appeared to M. *Ptcard* of 8" in breadth, when it should have appeared only of 3" 14''; and hence it is that the brighter fixed stars appear through the telescope as of 5" or 6" in diameter, and that with a good full light; but with a fainter light they appear to run out to a greater breadth. Hence, likewise, it was that *Hewelcke*, by diminishing the aperture of the telescope, did cut off a great part of the light towards the circumference, and brought the disk of the star to be more distinctly defined, which, though hereby diminished, did yet appear as of 5" or 6" in diameter. But Mr. *Huygens*, only by clouding the eyeglass with a little smoke, did so effectually extinguish this scattered light, that the fixed stars appeared as mere points, void of all appreciable breadth. Hence also it was that Mr. *Huygens*, from the breadth of bodies interposed to intercept the whole light of the planets, reckoned their diameters greater than others have measured them by the micrometer; for the scattered light, which could not be seen before for the stronger light of the planet, when the planet is hid, appears in every way the farther spread. Lastly, it is for this reason that the planets appear so small when projected on the disk of the sun, being lessened by the dilated light. For to *Hewelcke*, *Gallet*, and Dr. *Halley*, Mercury did not seem to exceed 12" or 15"; and Venus appeared to Mr. *Crabtree* only 1 3"; to *Horrox* but 1 12"; though by the mensurations of *Hewelcke* and *Huygens* without the sun's disk, it ought to have been seen at least 1 24". Thus the apparent diameter of the moon, which in 1684, a few days both before and after the sun's eclipse, was measured at the Observatory of *Pans* 31' 30", in the eclipse itself did not seem to exceed 30' or 30' 05"; and therefore the diameters of the planets are to be diminished when without the sun, and to be augmented when within it, by some seconds. But the errors seem to be less than usual in the micrometer measurements. So from the diameter of the shadow, determined by the eclipses of the satellites, Mr. *Flamsteed* found that the semidiameter of Jupiter was to the greatest elongation of the outermost satellite as 1 to 24.903. Therefore since that elongation is 8' 13", the diameter of Jupiter will be 39M2"; and, rejecting the scattered light, the diameter found by the micrometer 40" or 41" will

be reduced to $39\frac{1}{2}$; and the diameter of Saturn 21" is to be diminished by the like correction, and to be reckoned 20", or something less. But (if I am not mistaken) the diameter of the sun, because of its stronger light, is to be diminished somewhat more, and to be reckoned about 32', 32' 6".

[17.] *Why some of the planets are more, others less dense, and the forces in all are proportional to the quantities of matter.*

That bodies so different in magnitude should come so near to a proportionality with their forces, is not without some mystery (pp. 416-417).

It may be that the remoter planets, for want of heat, have not those metallic substances and ponderous minerals with which our earth abounds; and that the bodies of Venus and Mercury, as they are more exposed to the sun's heat, are also harder baked, and more compact.

For, from the experiment of the burning-glass, we see that the heat increases with the density of light; and this density increases inversely as the square of the distance from the sun; whence the sun's heat in Mercury is proved to be sevenfold its heat in our summer seasons. But with this heat our water boils; and those heavy fluids, quicksilver and the spirit of vitriol, gently evaporate, as I have tried by the thermometer; and therefore there can be no fluids in Mercury but what are heavy, and able to bear a great heat, and from which substances of great density may be formed.

And why not, if God has placed different bodies at different distances from the sun, so that the denser bodies always possess the nearer places, and each body enjoys a degree of heat suitable to its condition, and proper for its constitution? From this consideration it will best appear that the weights of all the planets are one to another as their forces.

But I should be glad if the diameters of the planets were more accurately measured; and that may be done, if a lamp, set at a great distance, is made to shine through a circular hole, and both the hole and the light of the lamp are so diminished that the image may appear through the telescope just like the planet, and may be defined by the same measure: then the diameter of the hole will be to its distance from the objective glass as the true diameter of the planet to its distance from us. The light of the lamp may be diminished by the interposition either of pieces of cloth, or of smoked glass.

[18.] *Another analogy between forces and bodies attracted is shown in the heavens.*

Of kin to the analogy we have been describing is another observed between the forces and the bodies attracted (pp. 411-414). Because the action of the centripetal force upon the planets decreases inversely as the square of the distance, and the periodic time increases as the %th power of the distance, it is evident that the actions of the centripetal force, and therefore the periodic times, would be equal in equal planets at equal distances from the sun; and in equal distances of unequal planets the total actions of the centripetal force would be as the bodies of the planets; for if the actions were not proportional to the bodies to be moved, they could not equally retract these bodies from the tangents of their orbs in equal times: nor could the motions of the satellites of Jupiter be so regular, if it was not that the circumsolar force was equally exerted upon Jupiter and all its satellites in proportion of their several weights. And the same thing is to be said of Saturn in respect of its satellites, and of our earth in respect of the moon, as appears from Cor. 11 and 111, Prop, LXV, Book 1. And, therefore, at equal distances, the actions of the centripetal force are equal upon all the planets in proportion to their bodies, or to the quantities of matter in their several bodies; and for the same reason must be the same upon all the particles of the same size of which the planet is composed; for if the action was greater upon some sort of particles than upon others, in proportion to their quantity of matter, it would be also greater or less upon the whole planets, not in proportion to the quantity only, but likewise to the sort of matter more copiously found in one and more sparingly in another.

[19.] *It is found also in terrestrial bodies.*

In such bodies as are found on our earth of very different sorts, I examined this analogy with great care (pp. 351-352).

If the action of the circumterrestrial force is proportional to the bodies to be moved, it will (by the second Law of Motion) move them with equal velocity in equal times, and will make all bodies, let fall, to descend through equal spaces in equal times, and all bodies, hung by equal threads, to vibrate in equal times. If the action of the force was greater, the times would be less; if that was less, these would be greater.

But it has been long ago observed by others, that (allowance being made for the small resistance of the air) all bodies descend through equal spaces in equal times; and, by the help of pendulums, that equality of times may be observed with great exactness.

I tried the thing in gold, silver, lead, glass, sand, common salt, wood, water, and wheat. I provided two equal wooden boxes. I filled the one with wood, and suspended an equal weight of gold (as exactly as I could) in the centre of oscillation of the other. The boxes, hung by equal threads of n feet, made a couple of pendulums perfectly equal in weight and figure, and equally exposed to the resistance of the air: and, placing the one by the other, I observed them to play together forwards and backwards for a long while, with equal vibrations. And therefore (by Cor. i and vi, Prop. xxiv, Book n) the quantity of matter in the gold was to the quantity of matter in the wood as the action of the motive force upon all the gold to the action of the same upon all the wood; that is, as the weight of the one to the weight of the other.

And by these experiments, in bodies of the same weight, one could have discovered a difference of matter less than the thousandth part of the whole.

[20.I *The agreement of those analogies.*

Since the action of the centripetal force upon the bodies attracted is, at equal distances, proportional to the quantities of matter in those bodies, reason requires that it should be also proportional to the quantity of matter in the body attracting.

For all action is mutual, and (pp. 13, 26, by the third Law of Motion) makes the bodies approach one to the other, and therefore must be the same in both bodies. It is true that we may consider one body as attracting, another as attracted; but this distinction is more mathematical than natural. The attraction resides really in each body towards the other, and is therefore of the same kind in both.

[21.] *Their coincidence.*

And hence it is that the attractive force is found in both. The sun attracts Jupiter and the other planets; Jupiter attracts its satellites; and, for the same reason, the satellites act as well one upon another as upon Jupiter, and all the planets mutually one upon another.

And though the mutual actions of two planets may be distinguished and considered as two, by which each attracts the other, yet, as those actions are between both, they do not make two but one operation between two terms. Two bodies may be attracted each to the other by the contraction of a cord interposed. There is a double cause of action, namely, the disposition of both bodies, as well as a double action so far as the action is considered as upon two bodies; but as between two bodies, it is but one single one. It is not one action by which the sun attracts Jupiter, and another by which Jupiter attracts the sun; but it is one action by which the sun and Jupiter mutually endeavor to approach each the other. By the action with which the sun attracts Jupiter, Jupiter and the sun endeavor to come nearer together (by the third Law of Motion); and by the action with which Jupiter attracts the sun. Likewise Jupiter and the sun endeavor to come nearer together. But the sun is not attracted towards Jupiter by a twofold action, nor Jupiter by a twofold action towards the sun; but it is one single intermediate action, by which both approach nearer together.

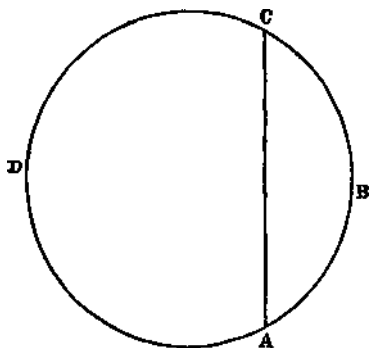
Thus iron draws the loadstone (p. 26), as well as the loadstone draws the iron; for all iron in the neighborhood of the loadstone draws other iron. But the action between the loadstone and iron is single, and is considered as single by the philosophers. The action of iron upon the loadstone is, indeed, the action of the loadstone between itself and the iron, by which both endeavor to come nearer together: and so it manifestly appears; for if you remove the loadstone, the whole force of the iron almost ceases.

In this sense it is that we are to conceive one single action to be exerted between two planets arising from the conspiring natures of both; and this action standing in the same relation to both, if it is proportional to the quantity of matter in the one, it will be also proportional to the quantity of matter in the other.

[22.] *The forces due to relatively very small bodies are inappreciable.*

Perhaps it may be objected that, according to this philosophy (p. 415), all bodies should attract one another, contrary to the evidence of experiments in terrestrial bodies; but I answer, that the experiments in terrestrial bodies do not count; for the attractions of homogeneous spheres near their surfaces are (by Prop, LXXII, Book 1) as their diameters. Hence a sphere of

one foot in diameter, and of a like nature to the earth, would attract a small body placed near its surface with a force 20000000 times less than the earth would do if placed near its surface; but so small a force could produce no sensible effect. If two such spheres were distant but by T4 of an inch, they



would not, even in spaces void of resistance, come together by the force of their mutual attraction in less than a month's time; and lesser spheres will come together at a rate yet slower, namely, in the proportion of their diameters. Nay, whole mountains will not be sufficient to produce any sensible effect. A mountain of an hemispherical figure, three miles high, and six broad, will not, by its attraction, draw

the pendulum two minutes out of the true perpendicular; and it is only in the great bodies of the planets that these forces are to be perceived, unless we may reason about smaller bodies in manner following.

[23.] *The forces directed towards all terrestrial bodies are proportional to their quantities of matter.*

Let ABCD (p. 26) represent the globe of the earth cut by any plane AC into two parts ACB and ACD. The part ACB bearing upon the part ACD presses it with its whole weight; nor can the part ACD sustain this pressure and continue unmoved, if it is not opposed by an equal contrary pressure. And therefore the parts equally press each other by their weights, that is, equally attract each other, according to the third Law of Motion; and, if separated and let go, would fall towards each other with velocities inversely as the bodies. All this we may try and see in the loadstone, whose attracted part does not propel the part attracting, but is only stopped and sustained thereby.

Suppose now that ACB represents some small body on the earth's surface; then, because the attractions of this particle, and of the remaining part

ACD of the earth towards each other, are equal, but the attraction of the particle towards the earth (or its weight) is as the matter of the particle (as we have proved by the experiment on pendulums), the attraction of the earth towards the particle will likewise be as the matter of the particle; and therefore the attractive forces of all terrestrial bodies will be as their several quantities of matter.

[24.] *Shown that the same forces are directed towards celestial bodies.*

The forces (p. 413), which are as the matter in terrestrial bodies of all forms, and therefore are not mutable with the forms, must be found in all sorts of bodies whatsoever, celestial as well as terrestrial, and be in all proportional to their quantities of matter, because among all there is no difference of substance, but of modes and forms only. But in the celestial bodies the same thing is likewise proved thus. We have shown that the action of the circumsolar force upon all the planets (reduced to equal distances) is as the matter of the planets; that the action of the circumjovial force upon the satellites of Jupiter observes the same law; and the same thing is to be said of the attraction of all the planets towards every planet: but it follows from this (by Prop, LXIX, Book 1) that their attractive forces are as their several quantities of matter.

[25.] *The forces decrease from the surfaces of the planets outwardly, inversely as the square of the distances, and inwardly, directly as the distances from the centres.*

As the parts of the earth attract one another so do those of all the planets. If Jupiter and its satellites were brought together, and formed into one globe, without doubt they would continue to attract one another as before. And, on the other hand, if the body of Jupiter was broken into more globes, surely these would no less attract one another than they do the satellites now. From these attractions it is that the bodies of the earth and all the planets assume a spherical figure, and their parts cohere, and are not dispersed through the ether. But we have before proved that these forces arise from the universal nature of matter (p. 416), and that, therefore, the force of any whole globe is made up of the several forces of all its parts. And from

this it follows (by Cor. II, Prop, LXXIV, Book i) that the force of every particle decreases inversely as the square of the distance from that particle; and (by Prop, LXXIII and LXXV, Book i) that the force of an entire globe, reckoning from the surface outwards, decreases inversely as the square of the distance, but reckoning inwards, decreases simply as the first power of the distances from the centres, if the matter of the globe be uniform. And when the matter of the globe, reckoning from the centre towards the surface, is not uniform (pp. 415—417), nevertheless, the decrease from the surface outwards is inversely as the square of the distance (by Prop, LXXVI, Book 1), provided that difformity is similar in places round about at equal distances from the centre. And two such globes will (by the same Proposition) attract one the other with a force decreasing inversely as the square of the distance between their centres.

[26.] *The intensities of the forces and the resulting motions in individual cases.*

Therefore the absolute force of every globe is as the quantity of matter which the globe contains; but the motive force by which every globe is attracted towards another, and which, in terrestrial bodies, we commonly call their weight, is as the content under the quantities of matter in both globes divided by the square of the distance between their centres (by Cor. iv, Prop, LXXVI, Book 1), to which force the quantity of motion, by which each globe in a given time will be carried towards the other, is proportional. And the accelerative force, by which every globe according to its quantity of matter is attracted towards another, is as the quantity of matter in that other globe divided by the square of the distance between the centres of the two (by Cor. II, Prop, LXXVI, Book 1); to which force, the velocity by which the attracted globe will, in a given time, be carried towards the other is proportional. And from these principles well understood, it will now be easy to determine the motions of the celestial bodies among themselves.

[27.] *All the planets revolve around the sun.*

From comparing the forces of the planets one with another, we have above seen that the circumsolar does more than a thousand times exceed

all the rest; but by the action of a force so great it is unavoidable that all bodies within, nay, and far beyond, the bounds of the planetary system must descend directly to the sun, unless by other motions they are impelled towards other parts: nor is our earth to be excluded from the number of such bodies; for certainly the moon is a body *of* the same nature with the planets, and subject to the same attractions with the other planets, seeing it is by the circumterrestrial force that it is retained in its orbit. But that the earth and moon are equally attracted towards the sun, we have above proved; we have likewise before proved that all bodies are subject to the said common laws of attraction. Nay, supposing any of those bodies to be deprived of its circular motion about the sun, by having its distance from the sun, we may find (by Prop. xxxvi, Book i) in what space of time it would in its descent arrive at the sun; namely, in half that periodic time in which the body might be revolved at one-half of its former distance; or in a space of time that is to the periodic time of the planet as 1 to $4\sqrt{2}$ as that Venus in its descent would arrive at the sun in the space of 40 days, Jupiter in the space of two years and one month, and the earth and moon together in the space of 66 days and 19 hours. But, since no such thing happens, it must needs be, that those bodies are moved towards other parts (p. 3), nor is every motion sufficient for this purpose. To hinder such a descent, a due proportion of velocity is required. And hence depends the force of the argument drawn from the retardation of the motions of the planets. Unless the circumsolar force decreased as the square of their increasing slowness, the excess thereof would force those bodies to descend to the sun; for instance, if the motion (other things being equal) was retarded by one-half, the planet would be held in its orbit by one-fourth of the former circumsolar force, and by the excess of the other three-fourths would descend to the sun. And therefore the planets (Saturn, Jupiter, Mars, Venus, and Mercury) are not really retarded in their perigees, nor become really stationary, or regressive with slow motions. All these are but apparent, and the absolute motions, by which the planets continue to revolve in their orbits, are always direct, and nearly uniform. But that such motions are performed about the sun, we have already proved; and therefore the sun, as the centre of the absolute motions, is quiescent. For we can by no means allow quiescence to the earth, lest the planets in their perigees should indeed be truly retarded, and become truly

stationary and regressive, and so for want of motion should descend to the sun. But further; since the planets (Venus, Mars, Jupiter, and the rest) by radii drawn to the sun describe regular orbits, and areas (as we have shown) nearly and to sense proportional to the times, it follows (by Prop. in, and Cor. in, Prop, LXV, Book i) that the sun is moved with no notable force, unless perhaps with such as all the planets are equally moved with, according to their several quantities of matter, in parallel lines, and so the whole system is transferred in right lines. Reject that translation of the whole system, and the sun will be almost quiescent in the centre thereof. If the sun was revolved about the earth, and carried the other planets round about itself, the earth ought to attract the sun with a great force, but the circumsolar planets with no force producing any sensible effect, which is contrary to Cor. in, Prop, LXV, Book i. Add to this, that if hitherto the earth, because of the gravitation of its parts, has been placed by most authors in the lowermost region of the universe; now, for better reason, the sun possessed of a centripetal force exceeding our terrestrial gravitation a thousand times and more, ought to be depressed into the lowermost place, and to be held for the centre of the system. And thus the true disposition of the whole system will be more fully and more exactly understood.

[28.] *The common centre of gravity of the sun and all the planets is at rest and the sun moves with a very slow motion. Explanation of the solar motion.*

Because the fixed stars are quiescent one in respect of another (pp. 419-420), we may consider the sun, earth, and planets, as one system of bodies carried hither and thither by various motions among themselves; and the common centre of gravity of all (by Cor. iv of the Laws of Motion) will either be quiescent, or move uniformly forwards in a right line: in which case the whole system will likewise move uniformly forwards in right lines. But this is an hypothesis hardly to be admitted; and, therefore, setting it aside, that common centre will be quiescent: and from it the sun is never far removed. The common centre of gravity of the sun and Jupiter falls on the surface of the sun; and though all the planets were placed towards the same parts from the sun with Jupiter, the common centre of the sun and all of them would scarcely recede twice as far from the sun's centre; and, there-

fore, though the sun, according to the different situations of the planets, is variously agitated, and always wandering to and fro with a slow motion of libration, yet it never recedes one entire diameter of its own body from the quiescent centre of the whole system. But from the weights of the sun and planets above determined, and the situation of all among themselves, their common centre of gravity may be found; and, this being given, the sun's place at any supposed time may be obtained.

[29.] *Nevertheless, the planets revolve in ellipses having foci at the centre of the sun; and the radii drawn to the sun describe areas proportional to the times.*

About the sun, thus librated, the other planets are revolved in elliptic orbits (p. 421), and by radii drawn to the sun, describe areas nearly proportional to the times, as is explained in Prop, LXV, Book 1. If the sun were quiescent, and the other planets did not act mutually one upon another, their orbits would be elliptic, and the areas exactly proportional to the times (by Prop, xi, and Cor., Prop, LXVIII, Book 1). But the actions of the planets among themselves, compared with the actions of the sun on the planets, are of no moment, and produce no sensible errors. And those errors are less in revolutions about the sun agitated in the manner but now described than if those revolutions were made about the sun quiescent (by Prop, LXVI, Book 1, and Cor., Prop, LXVIII, Book 1), especially if the focus of every orbit is placed in the common centre of gravity of all the lower included planets; viz., the focus of the orbit of Mercury in the centre of the sun; the focus of the orbit of Venus in the common centre of gravity of Mercury and the sun; the focus of the orbit of the earth in the common centre of gravity of Venus, Mercury, and the sun; and so of the rest. And by this means the foci of the orbits of all the planets, except Saturn, will not be sensibly removed from the centre of the sun, nor will the focus of the orbit of Saturn recede sensibly from the common centre of gravity of Jupiter and the sun. And therefore astronomers are not far from the truth, when they reckon the sun's centre the common focus of all the planetary orbits. In Saturn itself the error thence arising does not exceed i' 45". And if its orbit, by placing the focus thereof in the common centre of gravity of Jupiter and the sun, shall hap-

pen to agree better with the phenomena, from thence all that we have said will be further confirmed.

[30.] *The sizes of the orbits and the motion of their aphelions and nodes.*

If the sun was quiescent, and the planets did not act one on another, the aphelions and nodes of their orbits would likewise (by Prop. 1, xi, and Cor., Prop, XIII, Book 1) be quiescent. And the longer axes of their elliptic orbits would (by Prop, xv) be as the cube roots of the squares of their periodic times: and therefore from the given periodic times would be also given. But those times are to be measured not from the equinoctial points, which are movable, but from the first star of Aries. Put the semiaxis of the earth's orbit 100000, and the semiaxes of the orbits of Saturn, Jupiter, Mars, Venus, and Mercury, from their periodic times, will come out 953806, 520116, 152399, 72333, 38710, respectively. But from the sun's motion every semiaxis is increased (by Prop, LX, Book 1) by about one-third of the distance of the sun's centre from the common centre of gravity of the sun and planet (pp. 424-428). And from the actions of the exterior planets on the interior, the periodic times of the interior are somewhat protracted, though scarcely by any sensible quantity; and their aphelions are transferred (by Cor. vi and vn, Prop, LXVI, Book 1) by very slow motions forwards. And on the like account the periodic times of all, especially of the exterior planets, will be prolonged by the actions of the comets, if any such there are, without the orbit of Saturn, and the aphelions of all will be thereby carried forwards. But from the progress of the aphelions the regress of the nodes follows (by Cor. xi, XIII, Prop, LXVI, Book 1). And if the plane of the ecliptic is quiescent, the regress of the nodes (by Cor. xvi, Prop, LXVI, Book 1) will be to the progress of the aphelion in every orbit as the regress of the nodes of the moon's orbit to the progress of its apogee nearly, that is, as about 10 to 21. But astronomical observations seem to confirm a very slow progress of the aphelions, and a regress of the nodes in respect of the fixed stars. And hence it is probable that there are comets in the regions beyond the planets, which, revolving in very eccentric orbits, quickly fly through their perihelion parts, and, by an exceedingly slow motion in their aphelions, spend almost their whole time in the regions beyond the planets; as we shall afterwards explain more fully.¹

[31.] *From the foregoing principles are derived all the lunar motions thus far noted by astronomers.*

The planets thus revolved about the sun (pp. 433-435) may at the same time carry others revolving about themselves as satellites or moons, as appears by Prop, LXVI, Book 1. But from the action of the sun our moon must move with greater velocity, and, by a radius drawn to the earth, describe an area greater for the time; it must have its orbit less curved, and therefore approach nearer to the earth in the syzygies than in the quadratures, except so far as the motion of eccentricity hinders those effects. For the eccentricity is greatest when the moon's apogee is in the syzygies, and least when the same is in the quadratures; and hence it is that the perigeon moon is swifter and nearer to us, but the apogean moon slower and farther from us, in the syzygies than in the quadratures. But further, the apogee has a progressive and the nodes a regressive motion, both unequal. For the apogee is more swiftly progressive in its syzygies, more slowly regressive in its quadratures, and by the excess of its progress above its regress is yearly transferred forwards; but the nodes are quiescent in their syzygies, and most swiftly regressive in their quadratures. But further, still, the greatest latitude of the moon is greater in its quadratures than in its syzygies; and the mean motion swifter in the aphelion of the earth than in its perihelion. More inequalities in the moon's motion have not hitherto been taken notice of by astronomers; but all these follow from our principles in Cor. ii-xm, Prop, LXVI, Book 1, and are known really to exist in the heavens. And this may be seen in that most ingenious, and if I mistake not, of all the most accurate, hypothesis of Mr. *Horrox*, which Mr. *Flamsteed* has fitted to the heavens; but the astronomical hypotheses are to be corrected in the motion of the nodes; for the nodes admit the greatest equation or prosthaphaeresis in their octants, and this inequality is most conspicuous when the moon is in the nodes, and therefore also in the octants; and hence it was that *Tycho*, and others after him, referred this inequality to the octants of the moon, and made it menstrual; but the reasons by us adduced prove that it ought to be referred to the octants of the nodes, and to be made annual.

[32.] *Several irregularities of motion are deduced, not hitherto observed.*

Besides those inequalities taken notice of by astronomers (pp. 434, 473-475), there are yet some others, by which the moon's motions are so disturbed, that hitherto by no law could they be reduced to any certain regulation. For the velocities or hourly motions of the apogee and nodes of the moon, and their equations, as well as the difference between the greatest eccentricity in the syzygies and the least in the quadratures, and that inequality which we call the variation, in the progress of the year are augmented and diminished (by Cor. xiv, Prop, LXVI, Book 1) as the cube of the sun's apparent diameter. Besides that, the variation is mutable nearly as the square of the time between the quadratures (by Cor. 1 and 11, Lem. x, and Cor. xvi, Prop, LXVI, Book 1); and all those inequalities are somewhat greater in that part of the orbit which faces the sun than in the opposite part, but by a difference that is scarcely or not at all perceptible.

[33.] *The distance of the moon from the earth at a given time.*

By a computation (p. 444), which for brevity's sake I do not describe, I also find that the area which the moon by a radius drawn to the earth describes in the several equal moments of time is nearly as the sum of the number 237710, and the versed sine of the double distance of the moon from the nearest quadrature in a circle whose radius is unity; and therefore that the square of the moon's distance from the earth is as that sum divided by the hourly motion of the moon. Thus it is when the variation in the octants is in its mean quantity; but if the variation is greater or less, that versed sine must be augmented or diminished in the same ratio. Let astronomers try how exactly the distances thus found will agree with the moon's apparent diameters.

[34.] *The motions of the satellites of Jupiter and Saturn derived from the motions of our moon.*

From the motions of our moon we may derive the motions of the moons or satellites of Jupiter and Saturn (p. 433); for the mean motion of the nodes of the outermost satellite of Jupiter is to the mean motion of the nodes of our moon in a proportion compounded of the square of the periodic time of the earth about the sun to the periodic time of Jupiter about the sun, and the simple proportion of the periodic time of the satellite about

Jupiter to the periodic time of our moon about the earth, by Cor. xvi, Prop. LXVI, Book i: and therefore those nodes, in the space of a hundred years, are carried $8^{\circ} 24'$ backwards, or forwards. The mean motions of the nodes of the inner satellites are to the (mean) motion of (the nodes of) the outermost as their periodic times to the periodic time of this, by the same Corollary, and are thence given. And the motion of the apse of every satellite orbit forwards is to the motion of its nodes backwards, as the motion of the apogee of our moon is to the motion of its nodes, by the same Corollary, and is thence given. The greatest equations of the nodes and line of the apsides of each satellite orbit are to the greatest equations of the nodes and the line of the apsides of the moon respectively as the motion of the nodes and line of the apsides of the satellites' orbits in the time of one revolution of the first equations is to the motion of the nodes and apogee of the moon in the time of one revolution of the last equations. The variation of a satellite seen from Jupiter is to the variation of our moon in the same proportion as the whole motions of their nodes respectively, during the times in which the satellite and our moon (after parting from) are revolved (again) to the sun, by the same Corollary; and therefore in the outermost satellite the variation does not exceed $5'' 12'''$. From the small amount of those inequalities, and the slowness of the motions, it happens that the motions of the satellites are found to be so regular, that the more modern astronomers either deny all motion of the nodes, or affirm them to be very slowly regressive.

[35.] *The planets rotate around their own axes uniformly with respect to the stars; these motions are well adapted for the measurement of time* (P.423).

While the planets are thus revolved in orbits about remote centres, in the meantime they make their several rotations about their proper axes: the sun in 26 days; Jupiter in $9\text{h } 56^{\text{m}}$; Mars in $242/3^{\text{h}}$; Venus in 23^{h} ; and that in planes not much inclined to the plane of the ecliptic, and according to the order of the signs, as astronomers determine from the spots or maculae that by turns present themselves to our sight in their bodies; and there is a like revolution of our earth performed in 24^{11} ; and those motions are neither accelerated nor retarded by the actions of the centripetal forces, as appears by Cor. xxii, Prop. LXVI, Book 1; and therefore of all others they are the most

uniform and most fit for the measurement of time; but those revolutions are to be reckoned uniform not from their return to the sun, but to some fixed star: for as the position of the planets to the sun is non-uniformly varied, the revolutions of those planets from sun to sun are rendered non-uniform.

[36.] *In like manner the moon rotates around its axis by diurnal motion; hence arises its libration.*

In like manner is the moon revolved about its axis by a motion most uniform in respect of the fixed stars, viz., in $27^d. y^h. 43^m$, that is, in the space of a sidereal month; so that this diurnal motion is equal to the mean motion of the moon in its orbit; upon this account the same face of the moon always turns towards the centre about which this mean motion is performed, that is, the exterior focus of the moon's orbit, nearly; and hence arises a deflection of the moon's face from the earth, sometimes towards the east, and other times towards the west, according to the position of the focus towards which it is turned; and this deflection is equal to the equation of the moon's orbit, or to the difference between its mean and true motions; and this is the moon's libration in longitude: but it is likewise affected with a libration in latitude arising from the inclination of the moon's axis to the plane of the orbit in which the moon is revolved about the earth; for that axis retains the same position to the fixed stars, nearly, and hence the poles present themselves to our view by turns, as we may understand from the example of the motion of the earth, whose poles, by reason of the inclination of its axis to the plane of the ecliptic, are by turns illuminated by the sun. To determine exactly the position of the moon's axis to the fixed stars, and the variation of this position, is a problem worthy of an astronomer.

[37.] *The precession of the equinoxes and the libratory motion of the axes of the earth and planets.*

By reason of the diurnal revolutions of the planets, the matter which they contain endeavors to recede from the axis of this motion, and hence the fluid parts rising higher towards the equator than about the poles (p. 424), would lay the solid parts about the equator under water, if those parts did not rise also (pp. 424, 429); upon this account the planets are somewhat thicker about the equator than about the poles; and their equinoctial points (p. 433) thence become regressive; and their axes, by a motion of nutation,

twice in every revolution, librate towards their ecliptics, and twice return again to their former inclination, as is explained in Cor. XVIII, Prop, LXVI, Book 1; and hence it is that Jupiter, viewed through very long telescopes, does not appear altogether round (p. 428), but having its diameter that lies parallel to the ecliptic somewhat longer than that which is drawn from north to south.

[38.] *The ocean must flow twice and ebb twice, each day, and the highest water occurs at the third hour after the approach of the luminaries to the meridian of the place.*

And from the diurnal motion and the attractions (pp. 435-440) of the sun and moon our sea ought twice to rise and twice to fall every day, as well lunar as solar (by Cor. xix, xx, Prop, LXVI, Book 1), and the greatest height of the water to happen before the sixth hour of either day and after the twelfth hour preceding. By the slowness of the diurnal motion the flood is retracted to the twelfth hour; and by the force of the motion of reciprocation it is protracted and deferred till a time nearer to the sixth hour. But until that time will be more accurately determined by the phenomena, why should we not choose the middle between those extremes, and conjecture the greatest height of the water to happen at the third hour?² In this manner the water will rise all that time in which the force of the luminaries to raise it is greater, and will fall all that time in which their force is less; namely, from the ninth to the third hour when that force is greater, and from the third to the ninth when it is less. The hours I reckon from the approach of each luminary to the meridian of the place, as well under as above the horizon; and by the hours of the lunar day I understand the twenty-fourth parts of that time which the moon spends before it comes about again by its apparent diurnal motion to the meridian of the place which it left the day before.

[39.] *The tide is greatest in the syzygies of the luminaries and least in their quadratures, and at the third hour after the moon reaches the meridian; outside of the syzygies and quadratures the tide deviates somewhat from that third hour towards the third hour after the solar culmination.*

But the two motions which the two luminaries raise will not appear distinct, but will make a certain mixed motion. In the conjunction or oppo-

sition of the luminaries their forces will be conjoined, and bring on the greatest flood and ebb. In the quadratures the sun will raise the waters which the moon depresses, and depress the waters which the moon raises; and from the difference of their forces the smallest of all tides will follow. And because (as experience tells us) the force of the moon is greater than that of the sun, the greatest height of the water will happen about the third lunar hour. Outside of the syzygies and quadratures, the greatest tide which by the *single force of the moon* should take place at the third lunar hour, and by the single force of the sun at the third solar hour, by the combined forces of both must happen at an intermediate time that approaches nearer to the third hour of the moon than to that of the sun; and, therefore, while the moon is passing from the syzygies to the quadratures, during which time the third hour of the sun precedes the third of the moon, the greatest tide will precede the third lunar hour, and that by the greatest interval a little after the octants of the moon; and by like intervals the greatest tide will follow the third lunar hour, while the moon is passing from the quadratures to the syzygies.

[40.] *The tides are greatest when the luminaries are nearest the earth.*

But the effects of the luminaries depend upon their distances from the earth; for when they are less distant their effects are greater, and when more distant their effects are less, and that as the third power of their apparent diameters. Therefore it is that the sun in the winter time, being then in its perigee, has a greater effect, and makes the tides in the syzygies somewhat greater, and those in the quadratures somewhat less, other things being equal, than in the summer season; and every month the moon, while in the perigee, raises greater tides than at the distance of fifteen days before or after, when it is in its apogee. Hence it comes to pass that two highest tides do not follow one the other in two immediately succeeding syzygies.

[41.] *The tides are greatest about the equinoxes.*

The effect of either luminary likewise depends upon its declination or distance from the equator; for if the luminary were placed at the pole, it would constantly attract all the parts of the waters, without any increase or remission of its action, and could cause no reciprocation of motion; and,

therefore, as the luminaries decline from the equator towards either pole, they will by degrees lose their force, and on this account will excite lesser tides in the solstitial than in the equinoctial syzygies. But in the solstitial quadratures they will raise greater tides than in the quadratures about the equinoxes; because the effect of the moon, then situated in the equator, most exceeds the effect of the sun; therefore the greatest tides take place in those syzygies, and the least in those quadratures, which happen about the time of both equinoxes; and the greatest tide in the syzygies is always succeeded by the least tide in the quadratures, as we find by experience. But because the sun is less distant from the earth in winter than in summer, it comes to pass that the greatest and least tides more frequently appear before than after the vernal equinox, and more frequently after than before the autumnal.

[42.] *Without the equator the floods are alternately greater and less.*

Moreover, the effects of the luminaries depend upon the latitudes of places. Let *ApeP* represent the earth on all sides covered with deep waters; C its centre; P, *p* its poles; AE the equator; F any place without the equator; F/ the parallel of the place; D*d* the correspondent parallel on the other side of the equator; L the place which the moon possessed three hours before; H the place of the earth directly under it; *h* the opposite place; K, *k* the places at 90 degrees distance; CH, *Ch* the greatest heights of the sea from the centre of the earth; and CK, *Ck* the least heights; and if with the axes H*h*, K*k* an ellipse is described, and by the revolution of that ellipse about its longer axis H*h* a spheroid *HPKhpk* is formed, this spheroid will nearly represent the figure of the sea; and CF, *Cf*, CD, *Cd* will represent the sea in the places F, *f*, D, *d*. But further; if in the said revolution of the ellipse any point N describes the circle NM, cutting the parallels F*f*, D*d*, in any places R, T, and the equator AE in S, CN will represent the height of the sea in all those places R, S, T, situated in this circle. Wherefore, in the diurnal revolution of any place F the greatest flood will be in F, at the third hour after the approach of the moon to the meridian above the horizon; and afterwards the greatest ebb in Q, at

the third hour after the setting of the moon; and then the greatest flood in *f*, at the third hour after the approach of the moon to the meridian under the horizon; and, lastly, the greatest ebb in *Q*, at the third hour after the rising of the moon; and the latter flood in *f* will be less than the preceding flood in *F*. For the whole sea is divided into two huge and hemispherical floods, one in the hemisphere *KHkC* on the north side, the other in the opposite hemisphere *KHkC*, which we may therefore call the northern and the southern floods: these floods, being always opposite the one to the other, come by turns to the meridians of all places after the interval of twelve lunar hours; and, seeing the northern countries partake more of the northern flood, and the southern countries more of the southern flood, thence arise tides alternately greater and less in all places without the equator in which the luminaries rise and set. But the greater tide will happen when the moon declines towards the vertex of the place, about the third hour after the approach of the moon to the meridian above the horizon; and when the moon changes its declination, that which was the greater tide will be changed into a lesser; and the greatest difference of the floods will fall out about the times of the solstices, especially if the ascending node of the moon is about the first of Aries. So the morning tides in winter exceed those of the evening, and the evening tides exceed those of the morning in summer; at *Plymouth* by the height of one foot, but at *Bristol* by the height of fifteen inches, according to the observations of *Colepress* and *Sturmy*.

[43.] *By the persistence of the impressed motion the difference of the tides is reduced and the greatest may be the third flood after the syzygies in a month.*

But the motions which we have been describing suffer some alteration from that force of reciprocation which the waters [having once received] retain a little while by their inertia; hence it comes to pass that the tides may continue for some time, though the actions of the luminaries should cease. This power of retaining the impressed motion lessens the difference of the alternate tides, and makes those tides which immediately succeed after the syzygies greater, and those which follow next after the quadratures less. And hence it is that the alternate tides at *Plymouth* and *Bristol* do not differ much more one from the other than by the height of a foot, or of

fifteen inches; and that the greatest tides of all at those ports are not the first but the third after the syzygies.

And, besides, all the motions are retarded in their passage through shallow channels, so that the greatest tides of all, in some straits and mouths of rivers, are the fourth, or even the fifth, after the syzygies.

[44.] *The motion of the sea may be retarded by obstructions in the bed.*

It may also happen that the greatest tide may be the fourth or fifth after the syzygies, or occur even later, because the motions of the sea are retarded in passing through shallow places towards the shores; for so the tide arrives at the western coast of *Ireland* at the third lunar hour, and an hour or two after at the ports in the southern coast of the same island; as also at the islands *Cassiterides*, commonly called *Sorhngs*; then successively at *Falmouth*, *Plymouth*, *Portland*, the isle of *Wight*, *Winchester*, *Dover*, the mouth of the *Thames*, and *London Bridge*, spending twelve hours in this passage. But further; the propagation of the tides may be obstructed even by the channels of the ocean itself, when they are not of sufficient depth, for the flood happens at the third lunar hour in the *Canary* islands; and at all those western coasts that he towards the Atlantic ocean, as of *Ireland*, *France*, *Spain*, and all *Africa*, to the *Cape of Good Hope*, except in some shallow places, where it is impeded, and takes place later; and in the straits of *Gibraltar*, where, by reason of a motion propagated from the Mediterranean sea, it flows sooner. But, passing from those coasts over the breadth of the ocean to the coasts of *America*, the flood arrives first at the most eastern shores of *Brazil*, about the fourth or fifth lunar hour; then at the mouth of the river of the *Amazons* at the sixth hour, but at the neighboring islands at the fourth hour; afterwards at the islands of *Bermudas* at the seventh hour, and at port *St. Augustin* in *Florida* at seven and a half. And therefore the tide is propagated through the ocean with a slower motion than it should be according to the course of the moon; and this retardation is very necessary, that the sea at the same time may fall between *Brazil* and *New France*, and rise at the *Canary* islands, and on the coasts of *Europe* and *Africa*, and *vice versa*: for the sea cannot rise in one place but by filling in another. And it is probable that the *Pacific* sea is agitated by the same laws; for in the coasts of *Chili* and *Peru* the highest flood is said to happen at the

third lunar hour. But with what velocity it is thence propagated to the eastern coasts of *Japan*, the *Philippine* and other islands adjacent to *China*, I have not yet learned.

[45.] *From the obstructions of beds and shores, various phenomena arise, such as that the sea may flow only once every day.*

Further, it may happen (p. 439) that the tide may be propagated from the ocean through different channels towards the same port, and may pass quicker through some channels than through others, in which case the same tide, divided into two or more succeeding one another, may compound new motions of different kinds. Let us suppose one tide to be divided into two equal tides; the former thereof precedes the other by the space of six hours, and happens at the third or twenty-seventh hour from the approach of the moon to the meridian of the port. If the moon at the time of this approach to the meridian were in the equator, every six hours alternately there would arise equal floods, which, meeting with as many equal ebbs, would so balance one the other, that, for that day, the water would stagnate, and remain quiet. If the moon then declined from the equator, the tides in the ocean would be alternately greater and less, as was said; and from hence two greater and two lesser tides would be alternately propagated towards that port. But the two greater floods would make the greatest height of the waters to occur in the middle time between both, and the greater and lesser floods would make the waters to rise to a mean height in the middle time between them; and in the middle time between the two lesser floods the waters would rise to their least height. Thus in the space of twenty-four hours the waters would come, not twice, but once only to their greatest, and once only to their least height; and their greatest height, if the moon declined towards the elevated pole, would happen at the sixth or thirtieth hour after the approach of the moon to the meridian; and when the moon changed its declination, this flood would be changed into an ebb.

Of all this we have an example in the port of *Batshaw*, in the kingdom of *Tunquin*, in the latitude of 20° 50' north. In that port, on the day which follows after the passage of the moon over the equator, the waters stagnate; when the moon declines to the north, they begin to flow and ebb, not twice, as in other ports, but once only every day; and the flood happens at the set-

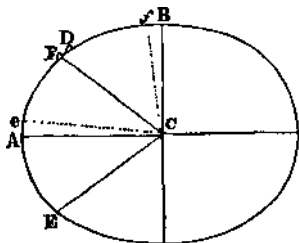
ting, and the greatest ebb at the rising of the moon. This tide increases with the declination of the moon till the seventh or eighth day; then for the seventh or eighth day following it decreases at the same rate as it had increased before, and ceases when the moon changes its declination. After this the flood is immediately changed into an ebb; and thenceforth the ebb happens at the setting and the flood at the rising of the moon, till the moon again changes its declination. There are two inlets from the ocean to this port: one more direct and short between the island *Hainan* and the coast of *Quantung*, a province of *China*; the other round about between the same island and the coast of *Cochim*; and through the shorter passage the tide is propagated more rapidly to *Batshaw*.

[46.] *The times of the tides in channels are more irregular than in the ocean.*

In the channels of rivers the influx and reflux depends upon the current of the rivers, which obstructs the ingress of the waters from the sea, and promotes their egress to the sea, making the ingress later and slower, and the egress sooner and faster; and hence it is that the reflux is of longer duration than the influx, especially far up the rivers, where the force of the sea is less. So *Sturmy* tells us, that in the river *Avon*, three miles below *Bristol*, the water flows only five hours, but ebbs seven; and without doubt the difference is yet greater above *Bristol*, as at *Caresham* or the *Bath*. This difference does likewise depend upon the quantity of the flux and reflux; for the more vehement motion of the sea near the syzygies of the luminaries more easily overcoming the resistance of the rivers, will make the ingress of the water to happen sooner and to continue longer, and will therefore diminish this difference. But while the moon is approaching to the syzygies, the rivers will be more plentifully filled, their currents being obstructed by the greatness of the tides, and therefore will somewhat more retard the reflux of the sea a little after than a little before the syzygies. Upon this account the slowest tides of all will not happen in the syzygies, but precede them a little; and I observed above that the tides before the syzygies are also retarded by the force of the sun; and from both causes conjoined the retardation of the tides will be both greater and sooner before the syzygies. All this I find to be so, by the tide tables which *Flamsteed* has composed from a great many observations.

[47.] *From a greater and deeper ocean arise greater tides, and these are greater at the coasts of continents than at islands in mid-ocean, and greater still in shallow bays which open with wide inlets to the sea.*

By the laws we have been describing, the times of the tides are governed; but the greatness of the tides depends upon the greatness of the seas. Let



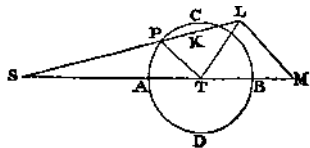
C represent the centre of the earth, EADB the oval figure of the seas, CA the longer semiaxis of this oval, CB the shorter erected at right angles upon the former, D the middle point between A and B, and ECF or eCf the angle at the centre of the earth, subtended by the breadth of the sea that terminates in the shores E, F, or e, f . Now, supposing that the point A is in the middle between the points E,

F, and the point D in the middle between the points e, f , if the difference of the heights CA, CB represent the quantity of the tide in a very deep sea surrounding the whole earth, the excess of the height CA above the height CE or CF will represent the quantity of the tide in the middle of the sea EF, terminated by the shores E, F; and the excess of the height Ce above the height Cf will nearly represent the quantity of the tide on the shores f of the same sea. Hence it appears that the tides are far less in the middle of the sea than at the shores; and that the tides at the shores are nearly as EF (pp. 481-482), the breadth of the sea not exceeding a quadrantal arc. And hence it is that near the equator, where the sea between *Africa* and *America* is narrow, the tides are far less than towards either side in the temperate zones, where the seas are extended wider; or on almost all the shores of the Pacific sea, as well towards *America* as towards *China*, and within as well as without the tropics; and that in islands in the middle of the sea they scarcely rise higher than two or three feet, but on the shores of great continents are three or four times greater, and above, especially if the motions propagated from the ocean are by degrees contracted into a narrow space, and the water, to fill and empty the bays alternately, is forced to flow and ebb with great violence through shallow places; as *Plymouth* and *Chepstow*

Bridge in England, at the mount of St. Michael and town of At/ranches in Normandy, and at Cambaia and Pegu in the East Indies. In these places, the sea, hurried in and out with great violence, sometimes lays the shores under water, sometimes leaves them dry, for many miles. Nor is the force of the influx and efflux to be broken till it has raised or depressed the water to forty or fifty feet and more. Thus also long and shallow straits that open to the sea with mouths wider and deeper than the rest of their channel (such as those about *Britain* and the *Magellanic Straits* at the eastern entry) will have a greater flood and ebb, or will more increase and remit their course, and therefore will rise higher and be depressed lower. On the coast of *South America* it is said that the *Pacific* sea in its reflux sometimes retreats two miles, and gets out of sight of those that stand on shore. Hence in these places the floods will be also higher; but in deeper waters the velocity of influx and efflux is always less, and therefore the ascent and descent is so too. Nor in such places is the ocean known to ascend to more than six, eight, or ten feet. The quantity of the ascent I compute in the following manner.

[48.] *The force of the sun to disturb the motions of the moon, computed from the foregoing principles.*

Let S represent the sun, T the earth (pp. 440-441), P the moon, PADB the moon's orbit. In SP take SK equal to ST and SL to SK in the squared ratio of SK to SP. Parallel to PT draw LM; and, supposing the mean quantity of the circumsolar force directed towards the earth to be represented by the distance ST or SK, SL will represent the quantity thereof directed towards the moon. But that force is compounded of the parts SM, LM; of which the force LM and that part of SM which is represented by TM, do disturb the motion of the moon (as appears from Prop. XLVI, Book 1, and its Corollaries). So far as the earth and moon are revolved about their common centre of gravity, the earth will be liable to the action of the like forces. But we may refer the sums, as well of the forces as of the motions, to the moon, and represent the sums of the forces



by the lines TM and ML , which are proportional to them. The force LM , in its mean quantity, is to the force by which the moon may be revolved in an orbit, about the earth quiescent, at the distance PT , in the squared ratio of the moon's periodic time about the earth, to the earth's periodic time about the sun (by Cor. XVII, Prop, LXVI, Book i); that is, in the squared ratio of $x^f. 7^h. 43^{TM}$. to $365^{\wedge} 6^{l*}. g^m$.; or as 1000 to 178725, or 1 to $i78^{2\%}$. The force by which the moon may revolve in its orbit about the earth at rest, at the distance PT of $60V2$ semidiameters of the earth, is to the force by which it may revolve in the same time at the distance of 60 semidiameters as $60V2$ to 60; and this force is to the force of gravity with us as 1 to 60 60, nearly; and therefore the mean force ML is to the force of gravity at the surface of the earth as $1 \cdot 60V2$ is to $60 \cdot 60 \cdot i78^{29/40}$, or 1 to 638092.6. Hence the force TM will be also given from the ratio of the lines TM , ML . And these are the forces of the sun, by which the moon's motions are disturbed.

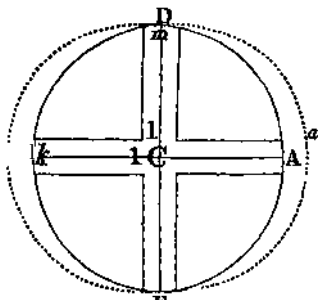
[49.] *Computation of the attraction of the sun which moves the sea.*

If from the moon's orbit (p. 478) we descend to the earth's surface, those forces will be diminished in the ratio of the distances $60V2$ and 1; and therefore the force LM will then become 38604600 times less than the force of gravity. But this force acting equally everywhere upon the earth, will scarcely produce any change on the motion of the sea, and therefore may be neglected in the explanation of that motion. The other force TM , in places where the sun is vertical, or in their nadir, is triple the quantity of the force ML , and therefore but 12868200 times less than the force of gravity.

[50.] *Computation of the height of the tides under the equator due to solar attraction.*

Suppose now $ADBE$ to represent the spherical surface of the earth, $aDbE$ the surface of the water overspreading it, C the centre of both, A the place to which the sun is vertical, B the place opposite; D , E places at 90 degrees distance from the former; $ACEmlk$ a right-angled cylindric canal passing through the earth's centre. The force TM in any place is as the distance of the place from the plane DE , on which a line from A to C stands at right

angles, and therefore in the part of the canal which is represented by $EClm$ is zero, but in the other part $AClk$ is as the gravity at the several heights; for in descending towards the centre of the earth, gravity is (by Prop, LXXIII, Book i) everywhere as the height; and therefore the force TM drawing the water upwards will diminish its gravity in the leg $AClk$ of the canal in a given ratio: upon this account the water will ascend in

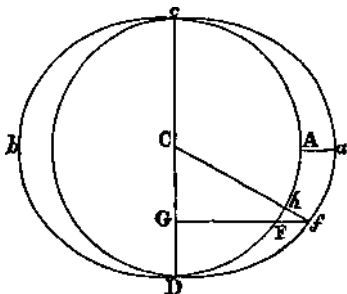


this leg, till its diminished gravity is compensated by its greater height; nor will it rest in an equilibrium till its total gravity becomes equal to the total gravity in $EClm$, the other leg of the canal. Since the gravity of every particle is as its distance from the earth's centre, the weight of the whole water in either leg will increase as the square of the height; and therefore the height of the water in the leg $AClk$ will be to the height thereof in the leg $ClmE$ as the square root of the ratio of the number 12868201 to 12868200, or in the ratio of the number 25623053 to the number 25623052, and the height of the water in the leg $EClm$ is to the difference of the heights, as 25623052 is to 1. But the height in the leg $EClm$ is of 19615800 *Paris* feet, as has been lately found by the measurement of the *French*; and, therefore, by the preceding proportion, the difference of the heights comes out $91/5$ inches of the *Paris* foot; and the sun's force will make the height of the sea at A to exceed the height of the same at E by 9 inches. And though the water of the canal $ACEmlJ$ (be supposed to be frozen into a hard and solid consistency, yet the heights thereof at A and E , and all other intermediate places, would still remain the same.

[51.] *Computation of the height of the tide at the parallels of latitude, due to solar attraction.*

Let ha (in the following figure) represent that excess of height of nine inches at A , and hf the excess of height at any other place h ; and upon DC

let fall the perpendicular $/G$, meeting the globe of the earth in F ; and since the distance of the sun is so great that all the right lines drawn thereto may be considered as parallel, the force TM in any place $/$ will be to the same force in the place A as the sine FG to the radius AC . And, therefore, since those forces tend to the sun in the direction of parallel lines, they will generate the parallel heights Ff , Aa , in the same ratio; and therefore the figure of the water $Dfacb$ will be a spheroid made by the revolution of an ellipse about its longer axis ab . And the perpendicular height fh will be to the oblique height Ff as fG to fC , or as FG to AC : and therefore the height fh is to



the height Aa in the squared ratio of FG to AC , that is, in the ratio of the versed sine of double the angle $DC/$ to double the radius, and is therefore given. And hence at the different moments of time during the apparent revolution of the sun about the earth we may infer the proportion of the ascent and descent of the waters at any given place under the equator, as well as of the diminution of that ascent and descent, whether arising from the latitude of places or from the sun's declination; viz., that on account of the latitude of places, the ascent and descent of the sea is in all places diminished as the squares of the cosines of latitude; and on account of the sun's declination, the ascent and descent under the equator is diminished as the squares of the cosine of declination. And in places without the equator the half-sum of the morning and evening ascents (that is, the mean ascent) is diminished nearly in the same ratio.

[52.] *The ratio of the tides under the equator in syzygies and quadratures, due to the joint attraction of sun and moon.*

Let S and L respectively represent the forces of the sun and moon placed in the equator, and at their mean distances from the earth; R the radius; T and V the versed sines of double the complements of the sun's and moon's

declinations to any given time; D and E the mean apparent diameters of the sun and moon: and, supposing F and G to be their apparent diameters at that given time, their forces to raise the tides under the equator will be, in the syzygies, $\frac{VG^3}{2RE^3} L + \frac{TF^3}{2RD^3} S$; in the quadratures, $\frac{VG^3}{2RE^3} L - \frac{TF^3}{2RD^3} S$. And if the same ratio is likewise observed under the parallels, from observations accurately made in our northern climates we may determine the proportion of the forces L and S; and then by means of this rule predict the quantities of the tides to every syzygy and quadrature.

f 53.] *Computation of the lunar attraction causing tides and the height of the water resulting therefrom.*

At the mouth of the river Avon, three miles below Bristol (pp. 479-482), in spring and autumn, the whole ascent of the water in the conjunction or opposition of the luminaries (by the observation of *Sturmy*) is about 45 feet, but in the quadratures only 25. Because the apparent diameters of the luminaries are not here determined, let us assume them in their mean quantities, as well as the moon's declination in the equinoctial quadratures in its mean quantity, that is, $23\frac{1}{2}^0$; and the versed sine of double its complement will be 1682, supposing the radius to be 1000. But the declinations of the sun in the equinoxes and of the moon in the syzygies are zero, and the versed sines of double the complements are each 2000. Hence those forces become $L + S$ in the syzygies, and $\frac{1682}{2000} L - S$ in the quadratures, respectively proportional to the heights of the tides of 45 and 25 feet, or of 9 and 5 paces. And, therefore, multiplying the extremes and the means, we have $5L + 5S = \frac{15138}{2000} L - 9S$, or $L = \frac{28000}{5138} S = 5\frac{5}{11} S$.

But further, I remember to have been told that in summer the ascent of the sea in the syzygies is to the ascent thereof in the quadratures as about 5 to 4. In the solstices themselves it is probable that the proportion may be somewhat less, as about 6 to 5; hence it would follow that L is $= 5\frac{1}{5} S$ [for then the proportion is $\frac{1682}{2000} L + \frac{1682}{2000} S : L - \frac{1682}{2000} S = 6 : 5$]. Till we can more certainly determine the proportion from observation let us assume $L = 5\frac{1}{3} S$;

and since the heights of the tides are as the forces which excite them, and the force of the sun is able to raise the tides to the height of nine inches, the moon's force will be sufficient to raise the same to the height of four feet. And if we allow that this height may be doubled, or perhaps tripled, by that force of reciprocation which we observe in the motion of the waters, and by which their motion once begun is kept up for some time, there will be force enough to generate all that quantity of tides which we really find in the ocean.

[54.] *These forces of the sun and moon can hardly be perceived, except by the tides which they raise in the sea.*

Thus we have seen that these forces are sufficient to move the sea. But, so far as I can observe, they will not be able to produce any other effect sensible on our earth ^j¹ for since the weight of one grain in 4000 is not sensible in the nicest balance; and the sun's force to move the tides is 12868200 less than the force of gravity; and the sum of the forces of both moon and sun, exceeding the sun's force only in the ratio of 6¹/_{Ai} to 1, is still 2032890 times less than the force of gravity; it is evident that both forces together are 500 times less than what is required perceptibly to increase or diminish the weight of any body in a balance. And, therefore, they will not sensibly move any suspended body; nor will they produce any sensible effect on pendulums, barometers, bodies swimming in stagnant water, or in the like statical experiments. In the atmosphere, indeed, they will excite such a flux and reflux as they do in the sea, but with so small a motion that no sensible wind will be thence produced.

[55.] *The moon is about six times denser than the sun.*

If the effects of both moon and sun in raising the tides (p. 484), as well as their apparent diameters, were equal among themselves, their absolute forces would (by Cor. xrv, Prop, LXVI, Book 1) be as their magnitudes. But the effect of the moon is to the effect of the sun as about 5MB is to 1; and the moon's diameter is less than the sun's in the ratio of 31½ to 321/5, or of 45 to 46. Now the force of the moon is to be increased directly as the ratio of the effect, and inversely as the cube of the ratio of the diameter. Hence the force of the moon compared with its magnitude will be to the force of the

[¹ Appendix, Note 59.]

sun compared with its magnitude in the ratio compounded of $5\frac{1}{4}$ to 1, and inversely as the cube of the ratio of 45 to 46, that is, in the ratio of about $5\frac{7}{10}$ to 1. And therefore the moon, in respect of the magnitude of its body, has an absolute centripetal force greater than the sun in respect of the magnitude of its body in the ratio of $5\frac{7}{10}$ to 1, and is therefore more dense in the same ratio.

[56.] *The moon is denser than our earth in the ratio of about 3 to 2.*

In the time of $27^{\text{h}} 43^{\text{m}}$, in which the moon makes its revolution about the earth, a planet may be revolved about the sun at the distance of 18.954 diameters of the sun from the sun's centre, supposing the mean apparent diameter of the sun to be 32M; and in the same time the moon may be revolved about the earth at rest, at the distance of 30 of the earth's diameters. If in both cases the number of diameters was the same, the absolute circumterrestrial force would (by Cor. 11, Prop, LXXII, Book 1) be to the absolute circumsolar force as the magnitude of the earth to the magnitude of the sun. Because the number of the earth's diameters is greater in the ratio of 30 to 18.954, the body of the earth will be less in that ratio cubed, that is, in the ratio of $3^{\frac{28}{10}}$ to 1. Therefore the earth's force, for the magnitude of its body, is to the sun's force, for the magnitude of its body, as $3^{\frac{28}{10}}$ to 1; and consequently the earth's density to the sun's will be in the same ratio. Since, then, the moon's density is to the sun's density as $5\frac{7}{10}$ to 1, the former will be to the earth's density as $5\frac{7}{10}$ to $3^{\frac{28}{10}}$, or as 23 to 16. Therefore, since the moon's magnitude is to the earth's magnitude as about 1 to $41\frac{1}{2}$, the moon's absolute centripetal force will be to the earth's absolute centripetal force, as about 1 to 29, and the quantity of matter in the moon is to the quantity of matter in the earth in the same ratio. And hence the common centre of gravity of the earth and moon is more exactly determined than hitherto it has been; from the knowledge of this we may now infer the moon's distance from the earth with greater accuracy. But I would rather wait till the ratio of the bodies of the moon and earth one to the other is more exactly defined from the phenomena of the tides, hoping that in the meantime the circumference of the earth may be measured from more distant stations than anybody has yet employed for this purpose.

[57.] *On the distance of the stars.*

Thus I have given an account of the system of the planets. As to the fixed stars, the smallness of their annual parallax proves them to be removed to immense distances from the system of the planets: that this parallax is less than one minute is most certain; and from this it follows that the distance of the fixed stars is above 360 times greater than the distance of Saturn from the sun. Those who consider the earth one of the planets, and the sun one of the fixed stars, may remove the fixed stars to yet greater distances by the following arguments. From the annual motion of the earth there would happen an apparent transposition of the fixed stars, one in respect of another, almost equal to their double parallax; but the greater and nearer stars, in respect of the more remote, which are only seen by the telescope, have not hitherto been observed to have the least motion. If we should suppose that motion to be only less than 20", the distance of the nearer fixed stars would exceed the mean distance of Saturn by above 2000 times. Again, the disk of Saturn, which is only 17" or 18" in diameter, receives only about $\frac{1}{2100000000}$ of the sun's light; for so much less is that disk than the whole spherical surface of the orb of Saturn. Now if we suppose Saturn to reflect about $\frac{1}{6}$ of this light, the whole light reflected from its illuminated hemisphere will be about $\frac{1}{1200000000}$ of the whole light emitted from the sun's hemisphere; and, therefore, since light is rarefied inversely as the square of the distance from the luminous body, if the sun was 10000 $\sqrt{42}$ times more distant than Saturn, it would yet appear as lucid as Saturn now does without its ring, that is, somewhat more lucid than a fixed star of the first magnitude. Let us, therefore, suppose that the distance from which the sun would shine as a fixed star exceeds that of Saturn by about 100000 times, and its apparent diameter will be 7^v 16^v and its parallax arising from the annual motion of the earth 13^{IV} : and so great will be the distance, the apparent diameter, and the parallax of the fixed stars of the first magnitude, in bulk and light equal to our sun. Some may, perhaps, imagine that a great part of the light of the fixed stars is intercepted and lost in its passage through so vast spaces, and upon that account pretend to place the fixed stars at nearer distances; but at this rate the remoter stars could be scarcely seen. Suppose, for example, that $\frac{3}{4}$ of the light is lost in its passage from the nearest fixed stars to us; then $\frac{3}{4}$ will be lost twice in its passage through a

double space, thrice through a triple, and so forth. And, therefore, the fixed stars that are at a double distance will be 16 times more obscure, viz., 4 times more obscure on account of the diminished apparent diameter; and, again, 4 times more on account of the lost light. And, by the same argument, the fixed stars at a triple distance will be $9 \times 4 = 36$, or 144 times more obscure; and those at a quadruple distance will be $16 \times 4 \times 4 = 256$, or 1024 times more obscure; but so great a diminution of light is no ways consistent with the phenomena and with that hypothesis which places the fixed stars at different distances.

[58.] *That comets, when they come into view, are nearer than Jupiter is shown from their parallax in longitude.*

The fixed stars being, therefore, at such vast distances from one another (pp. 491-493), can neither attract each other perceptibly, nor be attracted by our sun. But the comets must unavoidably be acted on by the circumsolar force; for as the comets were placed by astronomers beyond the moon, because they were found to have no diurnal parallax, so their annual parallax is a convincing proof of their descending into the regions of the planets. For all the comets which move in a direct course, according to the order of the signs, about the end of their appearance become more than ordinarily slow, or retrograde, if the earth is between them and the sun; and more than ordinarily swift if the earth is approaching to a heliocentric opposition with them. Whereas, on the other hand, those comets which move against the order of the signs, towards the end of their appearance, appear swifter than they ought to be, if the earth is between them and the sun; and slower, and perhaps retrograde, if the earth is in the other side of its orbit. This is occasioned by the motion of the earth in different situations. If the earth go the same way with the comet, with a swifter motion, the comet becomes retrograde; if with a slower motion, the comet becomes slower, however; and if the earth move the contrary direction, it becomes swifter; and by ascertaining the differences between the slower and swifter motions, and the sums of the more swift and retrograde motions, and comparing them with the situation and motion of the earth from whence they arise, I found, by means of this parallax, that the distances of the comets at the time they cease to be visible to the naked eye are always less than the distance of Saturn, and generally even less than the distance of Jupiter.

[59.] *This is shown by the parallax in latitude.*

The same thing may be inferred from the curvature of the path of the comets (p. 493). These bodies go on nearly in great circles while their motion continues swift; but about the end of their course, when that part of their apparent motion which arises from the parallax bears a greater proportion to their whole apparent motion, they commonly deviate from those circles; and when the earth goes to one side, they deviate to the other; and this deflection, since it corresponds with the motion of the earth, must arise chiefly from the parallax; and the quantity thereof is so considerable, as, by my computation, to place the disappearing comets a good deal closer than Jupiter. Hence it follows, that, when they approach nearer to us in their perigees and perihelions, they often are within the orbits of Mars and the inferior planets.

[60.] *This is shown otherwise by the parallax.*

Moreover, this nearness of the comets is confirmed by the annual parallax of the orbit, so far as the same is obtained pretty nearly by the supposition that the comets move uniformly in right lines. The method of computing the distance of a comet according to this hypothesis from four observations (first attempted by *Kepler*, and perfected by Dr. *Wallis* and Sir *Christopher Wren*) is well known; and the comets reduced to this regularity generally pass through the middle of the planetary region. So the comets of the years 1607 and 1618, as their motions are defined by *Kepler*, passed between the sun and the earth; that of the year 1664 within the orbit of Mars; and that in 1680 within the orbit of Mercury, as its motion was defined by Sir *Christopher Wren* and others. By a like rectilinear hypothesis, *Hewelc* placed all the comets about which we have any observations within the orbit of Jupiter. It is a false notion, therefore, and contrary to astronomical calculation, which some have entertained who, from the regular motion of the comets, either remove them into the regions of the fixed stars, or deny the motion of the earth; whereas their motions cannot be reduced to perfect regularity, unless we suppose them to pass through the regions near the earth. And these are the arguments drawn from the parallax, so far as it can be determined without an exact knowledge of the orbits and motions of the comets.

[61.] *From the light of their heads it is shown that comets descend as far as the orbit of Saturn.*

The near approach of the comets is further confirmed from the light of their heads (pp. 493-497); for the light of a celestial body, illuminated by the sun, and receding to remote parts, is diminished inversely as the fourth power of the distance; namely, as the second power, on account of the increase of the distance from the sun; and as another second power, on account of the decrease of the apparent diameter. Hence it may be inferred, that Saturn being at a double distance, and having its apparent diameter nearly half that of Jupiter, must appear about 16 times more obscure; and that, if its distance were 4 times greater, its light would be 256 times less; and therefore would be hardly perceivable to the naked eye. But now the comets often equal Saturn's light, without exceeding him in their apparent diameters. So the comet of the year 1668, according to Dr. *Hoove's* observations, equaled in brightness the light of a fixed star of the first magnitude; and its head, or the star in the middle of the coma, appeared, through a 15-foot telescope, as lucid as Saturn near the horizon; but the diameter of the head was only 25"; that is, almost the same with the diameter of a circle equal to Saturn and his ring. The coma or hair surrounding the head was about ten times as broad; namely, 4%'. Again, the least diameter of the hair of the comet of the year 1682, observed by *Mr. Flamsteed* with a 16-foot tube, and measured with the micrometer, was 2' 0"; but the nucleus, or star in the middle, scarcely possessed the tenth part of this breadth, and was therefore only 11" or 12" broad; but the light and clearness of its head exceeded that of the year 1680, and was equal to that of the stars of the first or second magnitude. Moreover, the comet of the year 1665, in *April*, as *Hewelcle* informs us, exceeded almost all the fixed stars in splendor, and even Saturn itself, as being of a much more vivid color; for this comet was more lucid than that which appeared at the end of the foregoing year, and was compared to the stars of the first magnitude. The diameter of the coma was about 6'; but the nucleus, compared with the planets by means of a telescope, was plainly less than Jupiter, and was sometimes thought less, sometimes equal to the body of Saturn within the ring. To this breadth add that of the ring, and the whole face of Saturn will be twice as great as that of the comet, with a light not at all more intense; and therefore the comet was

nearer to the sun than Saturn. From the ratio of the nucleus to the whole head found by these observations, and from its breadth, which seldom exceeds 8' or 12', it appears that the stars of the comets are most commonly of the same apparent magnitude as the planets; but that their light may be compared oftentimes with that of Saturn, and sometimes exceeds it. And hence it is certain that in their perihelions their distances can scarcely be greater than that of Saturn. At twice that distance, the light would be four times less, and, on account of its dim paleness, would be as much inferior to the light of Saturn as the light of Saturn is to the splendor of Jupiter: this difference would be easily observed. At a distance ten times greater, their bodies must be greater than that of the sun; but their light would be 100 times fainter than that of Saturn. And at distances still greater, their bodies would far exceed the sun; but, being in such dark regions, they could be no longer visible. It is certainly impossible to place the comets in the middle regions between the sun and fixed stars, accounting the sun as one of the fixed stars; for certainly they would receive no more light there from the sun than we do from the greatest of the fixed stars.

[62.] *They descend far below the orbit of Jupiter and sometimes below the orbit of the earth.*

So far we have gone without considering the obscuration which comets suffer from that plenty of thick smoke which encompasses their heads, and through which the heads always show dull as through a cloud; for the more a body is obscured by this smoke, the nearer must it necessarily approach the sun, that it may vie with the planets in the quantity of light which it reflects; hence it is probable that the comets descend far below the orbit of Saturn, as we proved before from their parallax. But, above all, the thing is confirmed from their tails, which must be due either to the sun's light reflected from a smoke arising from them, and dispersing itself through the ether, or to the light of their own heads.

In the former case we must shorten the distance of the comets, lest we be obliged to allow that the smoke arising from their heads is propagated through such a vast extent of space, and with such a velocity of expansion, as will seem altogether incredible; in the latter case the whole light of both

head and tail must be ascribed to the central nucleus. But, then, if we suppose all this light to be united and condensed within the disk of the nucleus, certainly the nucleus will by far exceed Jupiter itself in splendor, especially when it emits a very large and lucid tail. If, therefore, under a less apparent diameter, it reflects more light, it must be much more illuminated by the sun, and therefore much nearer to it. So the comet that appeared *December* 12 and 15, o.s., 1679, at the time it emitted a very shining tail, whose splendor was equal to that of many stars like Jupiter, if their light were dilated and spread through so great a space, was, as to the magnitude of its nucleus, less than Jupiter (as Mr. *Flamstced* observed), and therefore was much nearer to the sun: nay, it was even less than Mercury. For on the 17th of that month, when it was nearer to the earth, it appeared to *Cassini* through a 35-foot telescope a little less than the globe of Saturn. On the 8th of this month, in the morning, Dr. *Halley* saw the tail, appearing broad and very short, and as if it rose from the body of the sun itself, at that time very near its rising. Its form was like that of an extraordinary bright cloud; nor did it disappear till the sun itself began to be seen above the horizon. Its splendor, therefore, exceeded the light of the clouds till the sun rose, and far surpassed that of all the stars together, as yielding only to the immediate brightness of the sun itself. Neither Mercury, nor Venus, nor the moon itself, are seen so near the rising sun. Imagine all this dilated light collected together, and crowded into the orbit of the comet's nucleus which was less than Mercury; by its splendor, thus increased, becoming so much more conspicuous, it will vastly exceed Mercury, and therefore must be nearer to the sun. On the 12th and 15th of the same month, this tail, extending itself over a much greater space, appeared rarer; but its light was still so vigorous as to become visible when the fixed stars were hardly to be seen, and soon after to appear like a fiery beam shining in a wonderful manner. From its length, which was 40 or 50 degrees, and its breadth of 2 degrees, we may compute what the light of the whole must be.

[63.] *This is confirmed from the signal splendor of their tails in the vicinity of the sun.*

This near approach of the comets to the sun is confirmed from the situation they are seen in when their tails appear most resplendent; for when

the head passes by the sun, and lies hid under the solar rays, very bright **and** shining tails, like fiery beams, are said to issue from the horizon; but afterwards, when the head begins to appear, and is got farther from the sun, that splendor always decreases, and turns by degrees into a paleness like to that of the Milky Way, but much brighter at first; after that vanishing gradually. Such was that resplendent comet described by *Aristotle*, Book I, *Meteor.* 6. "The head thereof could not be seen, because it set before the sun, or at least was hid under the sun's rays; but the next day it was barely visible; for, having left the sun but a very little way, it set immediately after it; and the scattered light of the head obscured by the too great splendor [of the tail] did not yet appear. But afterwards [says *Aristotle*], when the splendor of the tail was now diminished, [the head of] the comet recovered its native brightness. And the splendor of its tail reached now to a third part of the heavens [that is, to 60°]. It appeared in the winter season, and, rising to Orion's girdle, there vanished away." Two comets of the same kind are described by *Justin*, Book xxxvn, which, according to him, "shone so bright, that the whole heaven seemed to be on fire; and by their greatness filled up a fourth part of the heavens, and by their splendor exceeded that of the sun." By these last words is intimated a near position of these bright comets to each other and to the rising or setting sun. We may add to these the comet of the year n o i or 1106, "the star of which was small and obscure [like that of 1680]; but the splendor arising from it was extremely bright, reaching like a fiery beam to the east and north," as *Hewelcke* has it from *Simeon*, the monk of *Durham*. It appeared at the beginning of *February* about the evening in the southwest. From this and from the situation of the tail we may infer that the head was near the sun, *Matthew Paris* says, "it was about one cubit from the sun; from the third [or rather the sixth] to the ninth hour sending out a long stream of light." The comet of 1264, in *July*, or about the solstice, preceded the rising sun, sending out its beams with a great light towards the west as far as the middle of the heavens; and at the beginning it ascended a little above the horizon: but as the sun went forwards it retired every day farther from the horizon, till it passed by the very middle of the heavens. It is said to have been at the beginning large and bright, having a large coma, which decayed from day to day. It is described in *Append. Matth. Vans, Hist. Ang.* after this manner: "*An. Christi*

1265, there appeared a comet so wonderful, that none then living had ever seen the like; for, rising from the east with a great brightness, it extended itself with a great light as far as the middle of the hemisphere towards the west." The Latin original being somewhat barbarous and obscure, it is here subjoined. *Ab oriente enim cum magno fulgore surgens, usque ad medium hemisphaem versus occidentem, omnia perlucide pertrahebat.*

"In the year 1401 or 1402, the sun being got below the horizon, there appeared in the west a bright and shining comet, sending out a tail upwards, in splendor like a flame of fire, and in form like a spear, darting its rays from west to east. When the sun was sunk below the horizon, by the lustre of its own rays it enlightened all the borders of the earth, not permitting the other stars to show their light, or the shades of night to darken the air, because its light exceeded that of the others, and extended itself to the upper part of the heavens, flaming," &c, *Hist. Byzant. Due. Mich. Nepot.* From the situation of the tail of this comet, and the time of its first appearance, we may infer that the head was then near the sun, and went farther from him every day; for that comet continued for three months. In the year 1527, *August* n, about 4 in the morning, there was seen almost throughout *Europe* a terrible comet in *Leo*, which continued flaming an hour and a quarter every day. It rose from the east, and ascended to the south and west to a prodigious length. It was most conspicuous to the north, and its cloud (that is, its tail) was very terrible; having, according to the fancies of the vulgar, the form of an arm a little bent holding a sword of a vast magnitude. In the year 1618, in the end of *November*, there began a rumor, that there appeared about sunrise a bright beam, which was the tail of a comet whose head was yet concealed within the brightness of the solar rays. On *November* 24, and from that time, the comet appeared with a bright light, its head and tail being extremely resplendent. The length of the tail, which was at first 20 or 30 deg., increased till *December* 9, when it arose to 75 deg., but with a light much fainter and more dilute than at the beginning. In the year 1668, *March* 5, N.S., about 7 in the evening, *Valentin Estancel*, being in *Brazil*, saw a comet near the horizon in the southwest. Its head was small, and scarcely discernible, but its tail extremely bright and refulgent, so that the reflection of it from the sea was easily seen by those who stood upon the shore. This great splendor lasted but three days, decreasing very remark-

ably from that time. The tail at the beginning extended itself from west to south, and in a situation almost parallel to the horizon, appearing like a shining beam 23 deg. in length. Afterwards, the light decreasing, its magnitude increased till the comet ceased to be visible; so that *Cassint*, at *Bologna*, saw it (*March* 10, 11, 12) rising from the horizon 32 deg. in length. In *Portugal* it is said to have taken up a fourth part of the heavens (that is, 45 deg.), extending itself from west to east with a notable brightness; though the whole of it was not seen, because the head in this part of the world always lay hid below the horizon. From the increase of the tail it is plain that the head receded from the sun, and was nearest to it at the beginning, when the tail appeared brightest.

To all these we may add the comet of 1680, whose wonderful splendor at the conjunction of the head with the sun was above described. But so great a splendor argues the comets of this kind to have really passed near the fountain of light, especially since the tails never shine so much in their opposition to the sun; nor do we read that fiery beams have ever appeared there.

[64.] *From the light of the heads it appears, under otherwise the same conditions, how much greater this light is in the vicinity of the sun.*

Lastly, the same thing is inferred (pp. 495-497) from the light of the heads increasing in the recess of the comets from the earth towards the sun, and decreasing in their return from the sun towards the earth; for so the last comet of the year 1665 (by the observation of *Hewelclfe*), from the time that it was first seen, was always losing of its apparent motion, and therefore had already passed its perigee; yet the splendor of its head was daily increasing, till, being hid by the sun's rays, the comet ceased to appear. The comet of the year 1683 (by the observation of the same *Hewelclfe*), about the end of *July*, when it first appeared, moved at a very slow rate, advancing only about 40 or 45 minutes in its orbit in a day's time. But from that time its diurnal motion was continually upon the increase till *September* 4, when it arose to about 5 degrees; and therefore in all this interval of time the comet was approaching to the earth. Which is likewise proved from the diameter of its head measured with a micrometer; for, *August* 6, *Hewelcke* found it only 6' 5", including the coma; which, *September* 2, he observed

to be 9' 7". And therefore its head appeared far less about the beginning than towards the end of its motion, though about the beginning, because nearer to the sun, it appeared far more lucid than towards the end, as the same *Hewelcke* declares. Therefore in all this interval of time, on account of its recess from the sun, it decreased in splendor, notwithstanding its access towards the earth. The comet of the year 1618, about the middle of *December*, and that of the year 1680, about the end of the same month, did both move with their greatest velocity, and were therefore then in their perigees; but the greatest splendor of their heads was seen two weeks before, when they had just got clear of the sun's rays; and the greatest splendor of their tails a little more early, when yet nearer to the sun. The head of the former comet, according to the observations of *Cysat*, *December* 1, appeared greater than the stars of the first magnitude; and, *December* 16 (being then in its perigee), of a small magnitude, and the splendor or clearness was much diminished. *January* 7, *Kepler*, being uncertain about the head, left off observing. *December* 12, the head of the last comet was seen and observed by *Vlamsteed* at the distance of 9 degrees from the sun, which a star of the third magnitude could hardly have been. *December* 15 and 17, the same appeared like a star of the third magnitude, its splendor being diminished by the bright clouds near the setting sun. *December* 26, when it moved with the greatest swiftness, and was almost in its perigee, it was inferior to *Os Pegasi*, a star of the third magnitude. *January* 3, it appeared like a star of the fourth; *January* 9, like a star of the fifth. *January* 13, it disappeared, by reason of the brightness of the moon, which was then in its increase. *January* 25, it was scarcely equal to the stars of the seventh magnitude. If we take equal times on each hand of the perigee, the heads placed at remote distances would have shone equally before and after, because of their equal distances from the earth. That in one case they shone very bright, and in the other vanished, is to be ascribed to the nearness of the sun in the first case, and his distance in the other; and from the great difference of the light in these two cases we infer its great nearness in the first of them; for the light of the comets tends to be regular, and to appear greatest when their heads move the swiftest, and are therefore in their perigees, except so far as it is increased by their nearness to the sun.

[65.] *The same is confirmed by the large number of comets seen in the region of the sun.*

From these things I at last discovered why the comets frequent so much the region of the sun. If they were to be seen in the regions a great way beyond Saturn, they must appear oftner in those parts of the heavens that are opposite to the sun; for those which are in that situation would be nearer to the earth, and the interposition of the sun would obscure the others: but, looking over the history of comets, I find that four or five times more have been seen in the hemisphere towards the sun than in the opposite hemisphere; besides, without doubt, not a few which have been hid by the light of the sun; for comets descending into our parts neither emit tails, nor are so well illuminated by the sun, as to discover themselves to our naked eyes, till they are nearer to us than Jupiter. But the far greater part of that spherical space, which is described about the sun with so small a radius, lies on that side of the earth which regards the sun, and the comets in that greater part are more strongly illuminated, being for the most part nearer to the sun: besides, from the remarkable eccentricity of their orbits, it comes to pass that their lower apsides are much nearer to the sun than if their revolutions were performed in circles concentric to the sun.

[66.] *This is confirmed also by the greater magnitude and splendor of the tails after the conjunction of the heads with the sun than before.*

Hence also we understand why the tails of the comets, while their heads are descending towards the sun, always appear short and rare, and are seldom said to have exceeded 15 or 20 *deg.* in length; but in the recess of the heads from the sun often shine like fiery beams, and soon after reach to 40, 50, 60, 70 *deg.* in length, or more. This great splendor and length of the tails arises from the heat which the sun communicates to the comet as it passes near it. And thence, I think, it may be concluded, that all the comets that have had such tails have passed very near the sun.

[67.] *The tails arise from the atmospheres of the comets.*

We may conclude from the preceding results that the tails arise from the atmospheres of the heads (pp. 522-525): but we have had three several opinions about the tails of comets; for some will have it that they are nothing else but the beams of the sun's light transmitted through the com-

ets' heads, which they suppose to be transparent; others, that they proceed from the refraction which light suffers in passing from the comet's head to the earth; and, lastly, others, that they are a sort of cloud or vapor constantly rising from the comets' heads, and tending towards the parts opposite to the sun. The first is the opinion of such as are yet unacquainted with optics; for the beams of the sun are not seen in a darkened room, but in consequence of the light that is reflected from them by the little particles of dust and smoke which are always flying about in the air; and hence it is that in air impregnated with thick smoke they appear with greater brightness, and are more faintly and more difficultly seen in a finer air; but in the heavens, where there is no matter to reflect the light, they are not to be *seen* at all. Light is not seen as it is in the beams, but as it is thence reflected to our eyes; for one cannot see, except by rays falling upon the eyes, and therefore there must be some reflecting matter in those parts where the tails of comets are seen; and so the argument turns upon the third opinion; for that reflecting matter can be nowhere found but in the place of the tail, because otherwise, since all the celestial spaces are equally illuminated by the sun's light, no part of the heavens could appear with more splendor than another. The second opinion is liable to many difficulties. The tails of comets are never seen variegated with those colors which ever tend to be inseparable from refraction; and the distinct transmission of the light of the fixed stars and planets to us is a demonstration that the ether or celestial medium is not endowed with any refractive power. For as to what is alleged that the fixed stars have been sometimes seen by the *Egyptians* environed with a coma because that has but rarely happened, it is rather to be ascribed to a casual refraction of clouds, as well as the radiation and scintillation of the fixed stars to the refractions both of the eyes and air; for upon applying a telescope to the eye, those radiations and scintillations immediately disappear. By the tremulous agitation of the air and ascending vapors, it happens that the rays of light are alternately turned aside from the narrow space of the pupil of the eye; but no such thing can have place in the much wider aperture of the object glass of a telescope; and hence it is that a scintillation is occasioned in the former case which ceases in the latter; and this cessation in the latter case is a demonstration of the regular transmission of light through the heavens without any perceptible refraction. But, to

obviate an objection that may be made from the appearing of no tail in such comets as shine but with a faint light, as if the secondary rays were then too weak to affect the eyes, and for this reason it is that the tails of the fixed stars do not appear, we are to consider that by the means of telescopes the light of the fixed stars may be augmented above an hundredfold and yet no tails are seen; that the light of the planets is yet more copious without any tail, but that comets are seen sometimes with huge tails when the light of their heads is but faint and dull; for so it happened in the comet of the year 1680, when in the month of *December* it was scarcely equal in light to the stars of the second magnitude, and yet emitted a notable tail, extending to the length of 40, 50, 60, or 70 degrees, and more; and afterwards, on the 27th and 28th of *January*, the head appeared only as a star of the seventh magnitude; but the tail (as was said above), with a light that was clearly perceptible, though faint, was stretched out to 6 or 7 degrees in length, and with a languishing light that was more difficult to see, even to 12 degrees and more. But on *February* 9th and 10th, when to the naked eye the head appeared no more, I saw through a telescope the tail of 2 degrees in length. But further, if the tail was due to the refraction of the celestial matter, and did deviate from the opposition of the sun, as the figure of the heavens requires, that deviation, in the same places of the heavens, should be always directed towards the same parts: but the comet of the year 1680, *December* 28^d. 8½^h. P.M. at *London*, was seen in *Pisces*, 8° 41', with latitude north 28° 6', while the sun was in *Capricorn* 18° 26'. And the comet of the year 1577, *December* 29, was in *Pisces* 8° 41', with latitude north 28° 40'; and the sun, as before, in about *Capricorn* 18° 26'. In both cases the situation of the earth was the same, and the comet appeared in the same place of the heavens; yet in the former case the tail of the comet (as well by my observations as by the observations of others) deviated from the opposition of the sun towards the north by an angle of 4½ degrees, whereas in the latter there was (according to the observation of *Tycho*) a deviation of 21 degrees towards the south. The refraction, therefore, of the heavens being thus disproved, it remains that the phenomena of the tails of comets must be explained by some reflecting matter. That vapors sufficient to fill such immense spaces may arise from the comets' atmospheres, may be easily understood from what follows.

[68.] *The air and vapors are extremely rare in celestial spaces, and a very small amount of vapor may be sufficient to explain all the phenomena of the tails of comets.*

It is well known that the air near the surface of our earth possesses a space about 1200 times greater than water of the same weight; and therefore a cylindric column of air 1200 feet high is of equal weight with a cylinder of water of the same breadth, and but one foot high. But a cylinder of air reaching to the top of the atmosphere is of equal weight with a cylinder of water about 33 feet high; and therefore if from the whole cylinder of air the lower part of 1200 feet high is taken away, the remaining upper part will be of equal weight with a cylinder of water 32 feet high. Therefore, at the height of 1200 feet, or two furlongs, the weight of the incumbent air is less, and consequently the rarity of the compressed air greater, than near the surface of the earth in the ratio of 33 to 32. And, having this ratio, we may compute the rarity of the air in all places whatsoever (by the help of Cor., Prop, XXII, Book 11), supposing the expansion thereof to be inversely proportional to its compression; and this proportion has been proved by the experiments of *Hooke* and others. The result of the computation I have set down in the following table, in the first column of which you have the height of the air in miles, whereof 4000 make a semidiameter of the

AIR'S

Height	Compression	Expansion
0	33	1
5	17 8515	1 8486
10	9 6717	3 4151
20	2 852	11 571
40	0 2525	136 83
400	0 xvii 1224	26956 xv
4000	0 cv 4465	73907 cu
40000	0 cxcii 1628	20263 clxxxix
400000	0 ccx 7895	41798 ccvii
4000000	0 ccxii 9878	33414 ccix
Infinite	0 ccxii 6041	54622 ccix

earth; in the second the compression of the air, or the incumbent weight; in the third its rarity or expansion, supposing gravity to decrease inversely as the square of the distances from the earth's centre. And the Latin numeral characters are here used for certain numbers of ciphers, as o.xvii 1224 for 0.00000000000000001224, and 26956 xv for 26956000000000000000.

But from this table it appears that the air, in proceeding upwards, is rarefied in such manner, that a sphere of that air which is nearest to the earth, of but one inch in diameter, if dilated with that rarefaction which it would have at the height of one semidiameter of the earth, would fill all the planetary regions as far as the sphere of Saturn, and a great way beyond; and at the height of ten semidiameters of the earth would fill up more space than is contained in the whole heavens on this side the fixed stars, according to the preceding computation of their distance. And though, by reason of the far greater thickness of the atmospheres of comets, and the great quantity of the circumsolar centripetal force, it may happen that the air in the celestial spaces, and in the tails of comets, is not so vastly rarefied, yet from this computation it is plain that a very small quantity of air and vapor is abundantly sufficient to produce all the appearances of the tails of comets; for that they are indeed of a very notable rarity appears from the shining of the stars through them. The atmosphere of the earth, illuminated by the sun's light, though but of a few miles in thickness, obscures and extinguishes the light not only of all the stars, but even of the moon itself; whereas the smallest stars are seen to shine through the immense thickness of the tails of comets, likewise illuminated by the sun, without the least diminution of their splendor.

[69.] *In what manner the tails may arise from the atmospheres of their heads.*

Kepler ascribes the ascent of the tails of comets to the atmospheres of their heads, and their direction towards the parts opposite to the sun, to the action of the rays of light carrying along with them the matter of the comets' tails; and without any great incongruity we may suppose that, in so free spaces, so fine a matter as that of the ether may yield to the action of the rays of the sun's light, though those rays are not able sensibly to move the gross substances in our parts, which are clogged with so palpable a re-

sistance. Another author thinks that there may be a sort of particles of matter endowed with a principle of levity, just as others are with a power of gravity; that the matter of the tails of comets may be of the former sort, and that its ascent from the sun may be due to its levity; but, considering that the gravity of terrestrial bodies varies as the matter of the bodies, and therefore can be neither more nor less in the same quantity of matter, I am inclined to believe that this ascent may rather proceed from the rarefaction of the matter of the comets' tails. The ascent of smoke in a chimney is due to the impulse of the air with which it is entangled. The air rarefied by heat ascends, because its specific gravity is diminished, and in its ascent carries along with it the smoke with which it is engaged. And why may not the tail of a comet rise from the sun after the same manner?⁷ For the sun's rays do not act any way upon the mediums which they pervade but by reflection and refraction; and those reflecting particles heated by this action, heat the matter of the ether which is involved with them. That matter is rarefied by the heat which it acquires, and because by this rarefaction the specific gravity, with which it tended towards the sun before, is diminished, it will ascend therefrom like a stream, and carry along with it the reflecting particles of which the tail of the comet is composed; the impulse of the sun's light, as we have said, promoting the ascent.

[70.] *That the tails arise from these atmospheres is evident from their divers appearances.*

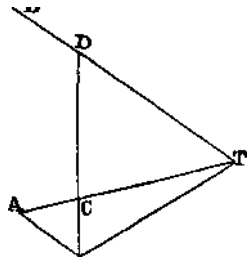
But that the tails of comets do arise from their heads (p. 524), and tend towards the parts opposite to the sun, is further confirmed from the laws which the tails observe; for, lying in the planes of the comets' orbits which pass through the sun, they constantly deviate from the opposition of the sun towards the parts which the comets' heads in their progress along those orbits have left; and to a spectator placed in those planes they appear in the parts directly opposite to the sun; but as the spectator recedes from those planes, their deviation begins to appear, and daily becomes greater. And the deviation, other things being equal, appears less when the tail is more oblique to the orbit of the comet, as well as when the head of the comet approaches nearer to the sun. Further, the tails which have no deviation appear straight, but the tails which deviate are likewise bent into a certain

curvature; and this curvature is greater when the deviation is greater, and is more perceptible when the tail, other things being equal, is longer; for in the shorter tails the curvature is hardly to be perceived. And the angle of deviation is less near the comet's head, but greater towards the other end of the tail, and that because the lower side of the tail regards the parts from which the deviation is made, and which lie in a right line drawn out infinitely from the sun through the comet's head. And the tails that are longer and broader, and shine with a stronger light, appear more resplendent and more exactly defined on the convex than on the concave side. Upon these accounts it is plain that the phenomena of the tails of comets depend upon the motions of their heads, and by no means upon the places of the heavens in which their heads are seen; and that, therefore, the tails of the comets do not proceed from the refraction of the heavens, but from their own heads, which furnish the matter that forms the tail; for as in our air the smoke of a heated body ascends either perpendicularly, if the body is at rest, or obliquely if the body is moved obliquely, so in the heavens, where all the bodies gravitate towards the sun, smoke and vapor must (as we have already said) ascend from the sun, and either rise perpendicularly, if the smoking body is at rest, or obliquely, if the body, in the progress of its motion, is always leaving those places from which the upper or higher parts of the vapors had risen before. And that obliquity will be less where the vapor ascends with greater velocity, namely, near the smoking body, when that is near the sun; for there the force of the sun by which the vapor ascends is stronger. But because the obliquity is varied, the column of vapor will be incurvated; and because the vapor in the preceding side is something more recent, that is, has ascended somewhat later from the body, it will therefore be somewhat more dense on that side, and must on that account reflect more light, as well as be better defined, the vapor on the other side languishing by degrees, and vanishing.

[71.] *That comets sometimes come within the orbit of Mercury is shown by their tails.*

But it is none of our present business to explain the causes of the appearances of Nature. Let those things which we have last said be true or false,

we have at least made out, in the preceding discourse, that the rays of light are directly propagated from the tails of comets in right lines through the heavens, in which those tails appear to the spectators wherever placed; and consequently the tails must ascend from the heads of the comets towards the parts opposite to the sun. And from this principle we may determine anew the limits of their distances in the following manner. Let S represent the sun, T the earth, ST A the elongation of a comet from the sun, and ATB the apparent length of its



tail; and since the light is propagated from the extremity of the tail in the direction of the right line TB, that extremity must lie somewhere in the line TB. Suppose it in D, and join DS cutting TA in C. Then, since the tail is always stretched out towards the parts nearly opposite to the sun, and therefore the sun, the head of the comet, and the extremity of the tail, lie in a right line, the comet's head will be found in C. Parallel to TB draw SA, meeting the line TA in A, and the comet's head C must necessarily be found between T and A, because the extremity of the tail lies somewhere in the infinite line TB; and all the lines SD which can possibly be drawn from the point S to the line TB must cut the line TA somewhere between T and A. Therefore the distance of the comet from the earth cannot exceed the interval TA, nor its distance from the sun the interval SA beyond, or ST on this side the sun. For instance: the elongation of the comet of 1680 from the sun, *Dec.* 12, was 9° , and the length of its tail 35° at least. If, therefore, a triangle TSA is made, whose angle T is equal to the elongation 9° , and angle A equal to ATB, or to the length of the tail, viz., 35° , then SA will be to ST, that is, the limit of the greatest possible distance of the comet from the sun to the semidiameter of the earth's orbit, as the sine of the angle T is to the sine of the angle A, that is, as about 3 to 11. And therefore the comet at that time was less distant from the sun than by $\frac{1}{11}$ of the earth's distance from the sun, and hence either was within the orbit of Mercury, or between that orbit and the earth. Again, on *Dec.* 21, the elongation of the comet from the sun was $32\frac{2}{3}^{\circ}$, and the length of its tail 70° . There-

fore, as the sine of 32° is to the sine of 70° , that is, as 4 is to 7, so was the limit of the comet's distance from the sun to the distance of the earth from the sun, and consequently the comet had not then got without the orbit of Venus. Dec. 28, the elongation of the comet from the sun was 55° , and the length of its tail 56° ; and therefore the limit of the comet's distance from the sun was not yet equal to the distance of the earth from the same, and consequently the comet had not then got without the earth's orbit. But from its parallax we find that its egress from the orbit happened about Jan. 5, and also that it had descended far within the orbit of Mercury. Let us suppose it to have been in its perihelion Dec. the 8th, when it was in conjunction with the sun; and it will follow that in the journey from its perihelion to its exit out of the earth's orbit it had spent 28 days; and consequently that in the 26 or 27 days following, in which it ceased to be further seen by the naked eye, it had scarcely doubled its distance from the sun; and by limiting the distances of other comets by the like arguments, we come at last to this conclusions—that all comets, during the time in which they are visible by us, are within the compass of a spherical space described about the sun as a centre, with a radius double, or at most triple, the distance of the earth from the sun.

[72.] *The comets move in conic sections having one focus in the centre of the sun, and by their radii drawn to that centre describe areas proportional to the times.*

And hence it follows that the comets, during the whole time of their appearance unto us, being within the sphere of activity of the circumsolar force, and therefore agitated by the impulse of that force, will (by Cor. 1, Prop, XIII, Book 1, for the same reason as the planets) be made to move in conic sections that have one focus in the centre of the sun, and, by radii drawn to the sun, to describe areas proportional to the times; for that force is propagated to an immense distance, and will govern the motions of bodies far beyond the orbit of Saturn.

[73.] *These conic sections are near to parabolas. This is inferred from the velocity of the comets.*

There are three hypotheses about comets (p. 498); for some will have it that they are generated and perish as often as they appear and vanish;

others, that they come from the regions of the fixed stars, and are seen by us in their passage through the system of our planets; and, lastly, others will have it, that they are bodies continually revolving about the sun in very eccentric orbits. In the first case, the comets, according to their different velocities, will move in conic sections of all sorts; in the second, they will describe hyperbolas, and in either of the two will frequent indifferently all quarters of the heavens, as well those about the poles as those towards the ecliptic; in the third, their motions will be performed in ellipses very eccentric, and very nearly approaching to parabolas. But (if the law of the planets is observed) their orbits will not much decline from the plane of the ecliptic; and, so far as I could hitherto observe, the third case obtains; for the comets do, indeed, chiefly frequent the zodiac, and scarcely ever attain to a heliocentric latitude of 40° . And that they move in orbits very nearly parabolic, I infer from their velocity; for the velocity with which a parabola is described is everywhere to the velocity with which a comet or planet may be revolved about the sun in a circle at the same distance in the ratio of $y/2$ to 1 (by Cor. vn, Prop. xvi, Book 1); and, by my computation, the velocity of comets is found to be much the same. I examined the thing by inferring successively the velocities from the distances, and the distances both from the parallaxes and the phenomena of the tails, and never found the errors of excess or defect in the velocities greater than what might have arisen from the errors in the distances computed after that manner. But I likewise made use of the reasoning that follows.

[74.] *The length of time in which comets describing parabolic orbits pass through the sphere of the earth's orbit.*

Supposing the radius of the earth's orbit to be divided into 1000 parts: let the numbers in the first column of Table 1 (p. 616) represent the distance of the vertex of the parabola from the sun's centre, expressed by those parts; and a comet in the times expressed in col. 2, will pass from its perihelion to the surface of the sphere which is described about the sun as a centre with the radius of the earth's orbit; and in the times expressed in col. 3, 4, and 5, it will double, triple, and quadruple its distance from the sun.

TABLE I

The distance of a comet's perihelion from the sun's centre	The time of a comet's passage from its perihelion to a distance from the sun equal to			
	The radii of the earth's orbit	To its double	To its triple	To its quadruple
	d h m	d h m	d h m	d h m
0	27 11 12	77 16 28	142 17 14	219 17 30
5	27 16 07	77 23 14		
10	27 21 00	78 06 24		
20	28 06 40	78 20 13	144 03 19	221 08 54
40	29 01 32	79 23 34		
80	30 13 25	82 04 56		
160	33 05 29	86 10 26	153 16 08	232 12 20
320	37 13 46	93 23 38		
640	37 09 49	105 01 28		
1280		106 06 35	200 06 43	297 03 46
2560			147 22 31	300 06 03

The time of a comet's ingress into the sphere of the earth's orbit, or of its egress from the same, may be inferred from its parallax, but with more expedition by the following:

TABLE II

The apparent elonga- tion of a comet from the sun	Its apparent diurnal motion in its own orbit		Its distance from the earth in parts, of which the radius of the earth's orbit contains 1000
	Direct	Retrgressive	
60°	2° 18'	00° 20'	1000
65	2 33	00 35	845
70	2 55	00 57	684
72	3 07	01 09	618
74	3 23	01 25	551
76	3 43	01 45	484
78	4 10	02 12	416
80	4 57	02 49	347
82	5 45	03 47	278
84	7 18	05 20	209
86	10 27	08 19	140
88	18 37	16 39	70
90	Infinite	Infinite	00

[75.] *The velocity with which the comets of 1680 passed through the sphere of the earth's orbit.*

The ingress of a comet into the sphere of the earth's orbit, or its egress from the same, happens at the time of its elongation from the sun, expressed in col. 1, against its diurnal motion. So in the comet of 1681, *Jan. 4*, o.s., the apparent diurnal motion in its orbit was about $3^0 5'$, and the corresponding elongation $712/3^\circ$; and the comet had acquired this elongation from the sun *Jan. 4*, about six in the evening. Again, in the year 1680, *Nov. 11*, the diurnal motion of the comet that then appeared was about 4° ; and the corresponding elongation 79° happened *Nov. 10*, a little before midnight. Now at the times named these comets had arrived at an equal distance from the sun with the earth, and the earth was then almost in its perihelion. But the first table is fitted to the earth's mean distance from the sun assumed of 1000 parts; and this distance is greater by such an excess of space as the earth might describe by its annual motion in one day's time, or the comet by its motion in 16 hours. To reduce the comet to this mean distance of 1000 parts, we add those 16 hours to the former time, and subtract them from the latter; and thus the former becomes *Jan. 4ⁱ. 10^h*. afternoon; the latter *Nov. 10*, about six in the morning. But from the tenor and progress of the diurnal motions it appears that both comets were in conjunction with the sun between *Dec. 7* and *Dec. 8*; and from thence to *Jan. 4^d. 10^h*. afternoon on one side, and to *Nov. 10^d. 6^h*. of the morning on the other, there arc about 28 days. And so many days (by Table 1) the motions in parabolic orbits do require.

[76.] *These comets were not two, but one and the same comet; it is more accurately determined in what orbit and with what velocity the comet traversed the heavens.*

But though we have hitherto considered those comets as two, yet, from the coincidence of their penhehons and agreement of their velocities, it is probable that in effect they were but one and the same; and if so, the orbit of this comet must have either been a parabola, or at least a conic section very little differing from a parabola, and at its vertex almost in contact with the surface of the sun. For (by Table 11) the distance of the comet from the earth, *Nov. 10*, was about 360 parts, and *Jan. 4*, about 630. From which dis-

tances, together with its longitudes and latitudes, we infer the distance of the places in which the comet was at those times to have been about 280: the half of which, viz., 140, is an ordinate to the comet's orbit, cutting off a portion of its axis nearly equal to the radius of the earth's orbit, that is, to 1000 parts. And, therefore, dividing the square of the ordinate 140 by 1000, the segment of the axis, we find the latus rectum 19.6, or in a round number 20; the fourth part whereof, 5, is the distance of the vertex of the orbit from the sun's centre. But the time corresponding to the distance of 5 parts in Table 1 is 27. $16^h. y^m$. In this time, if the comet moved in a parabolic orbit, it would have been carried from its perihelion to the surface of the sphere of the earth's orbit described with the radius 1000, and would have spent the double of that time, viz., $55^d. 8\frac{1}{4}^h$. in the whole course of its motion within that sphere: and so in fact it did; for from Nov. $10^d. 6^h$. of the morning, the time of the comet's ingress into the sphere of the earth's orbit, to Jan. $4^d. 10^h$. afternoon, the time of its egress from the same, there are $55^d. 16^h$. The small difference of $7\frac{3}{4}^h$. in this rough way of computing is to be neglected, and perhaps may arise from the comet's motion being to a small degree slower, as it must have been if the true orbit in which it was carried was an ellipse. The middle time between its ingress and egress was Dec. $8^d. 2^h$. of the morning; and therefore at this time the comet ought to have been in its perihelion. And accordingly that very day, just before sunrise, Dr. *Halley* (as we said) saw the tail short and broad, but very bright, rising perpendicularly from the horizon. From the position of the tail it is certain that the comet had then crossed over the ecliptic, and got into north latitude, and therefore had passed by its perihelion, which lay on the other side of the ecliptic, though it had not yet come into conjunction with the sun; and the comet being at this time between its perihelion and its conjunction with the sun, must have been in its perihelion a few hours before; for in so near a distance from the sun it must have been carried with great velocity, and have apparently described almost half a degree every hour.

[77'] *It is shown by additional examples with what velocity the comets move.*

By like computations I find that the comet of 1618 entered the sphere of the earth's orbit Dec. 7, towards sunset; but its conjunction with the

sun was *Nov.* 9, or 10, about 28 days intervening, as in the preceding comet; for from the size of the tail of this, in which it was equal to the preceding, it is probable that this comet likewise did come almost into a contact with the sun. Four comets were seen that year, of which this was the last. The second, which made its first appearance *Oct.* 31, in the neighborhood of the rising sun, and was soon after hidden under the sun's rays, I suspect to have been the same as the fourth, which emerged out of the sun's rays about *Nov.* 9. To these we may add the comet of 1607, which entered the sphere of the earth's orbit *Sept.* 14, o.s., and arrived at its perihelion distance from the sun about *Oct.* 19, 35 days intervening. Its perihelion distance subtended an apparent angle at the earth of about 23 degrees, and was therefore of 390 parts. And to this number of parts about 34 days correspond in Table 1. Further, the comet of 1665 entered the sphere of the earth's orbit about *March* 17, and came to its perihelion about *April* 16, 30 days intervening. Its perihelion distance subtended an angle at the earth of about seven degrees, and therefore was of 122 parts: and corresponding to this number of parts, in Table 1 we find 30 days. Again; the comet of 1682 entered the sphere of the earth's orbit about *Aug.* 11, and arrived at its perihelion about *Sept.* 16, being then distant from the sun by about 350 parts, to which, in Table 1, belong $33\frac{1}{2}$ days. Lastly; that memorable comet of *Johann Midlers*, which in 1472 was carried through the circumpolar parts of our northern hemisphere with such rapidity as to describe 40 degrees in one day, entered the sphere of the earth's orbit *Jan.* 21, about the time that it was passing by the pole, and, hastening from thence towards the sun, was hid under the sun's rays about the end of *February*; whence it is probable that 30 days, or a few more, were spent between its ingress into the sphere of the earth's orbit and its perihelion. Nor did this comet truly move with more velocity than other comets, but owed the greatness of its apparent velocity to its passing by the earth at a near distance.

[78.] // *is proposed to determine the orbits of comets.*

It appears, then, that the velocity of comets (p. 505), so far as it can be determined by these rough ways of computing, is that very velocity with which parabolas, or ellipses near to parabolas, ought to be described; and

therefore the distance between a comet and the sun being given, the velocity of the comet is nearly given. And hence arises this Problem.

PROBLEM

The relation between the velocity of a comet and its distance from the sun's centre being given, the comet's orbit is required.

If this Problem were resolved, we should thence have a method of determining the orbits of comets to the greatest accuracy; for if that relation be twice assumed, and from thence the orbit be twice computed, and the error of each orbit be found from observations, the assumption may be corrected by the rule of false position, and a third orbit may thence be found that will exactly agree with the observations. And by determining the orbits of comets after this method, we may come, at last, to a more exact knowledge of the parts through which those bodies travel, of the velocities with which they are carried, what sort of orbits they describe, and what are the true magnitudes and forms of their tails according to the various distances of their heads from the sun; whether, after certain intervals of time, the same comets do return again, and in what periods they complete their several revolutions. But the Problem may be resolved by determining, first, the hourly motion of a comet, at a given time, from three or more observations, and then deriving the orbit from this motion. And thus the determination of the orbit, depending on one observation, and its hourly motion at the time of this observation, will either confirm or disprove itself; for the conclusion that is drawn from the motion only of an hour or two and a false hypothesis, will never agree with the motions of the comets from beginning to end. The method of the whole computation is this.

LEMMA I

To cut two right lines OR, TP, given in position, by a third right line RP, so that TRP may be a right angle; and, if another right line SP is drawn to any given point S, the product obtained from multiplying this line SP, by the square of the right line OR terminated at a given point O, may be a given magnitude.

It is done graphically thus. Let the given magnitude of the product be $M^2 N$; from any point r of the right line OR erect the perpendicular rp

For the given coefficients

$$\frac{M^2N}{OR^3} - SP, \frac{2TR}{TP} \cdot \frac{M^2N}{OR^3} - \frac{BP}{SP}, \frac{3TR^2}{TP^2} \cdot \frac{M^2N}{OR^3} - \frac{SB^2}{2SP^2},$$

putting $F, \frac{F}{G}, \frac{F}{GH}$, and carefully observing the signs, we find

$$F + \frac{F}{G}e + \frac{F}{GH}ee = 0, \text{ and } e + \frac{ee}{H} = -G.$$

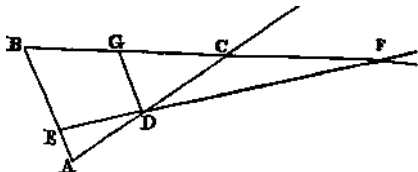
Hence, neglecting the very small term $\frac{e^2}{H}$, e comes out equal to $-G$. If the error $\frac{e^2}{H}$ is not negligible, take $-G - \frac{G^2}{H} = e$.

And it is to be observed that here a general method is hinted at for solving the more intricate sort of problems, as well by trigonometry as by arithmetic, without those involved computations and resolutions of affected equations which hitherto have been in use.

LEMMA II

To cut three right lines given in position by a fourth right line that shall pass through a point assigned in any of the three, and so that its intercepted parts shall be in a given ratio one to the other.

Let AB, AC, BC be the right lines given in position, and suppose D to be the given point in the line AC. Parallel to AB draw DG meeting BC in



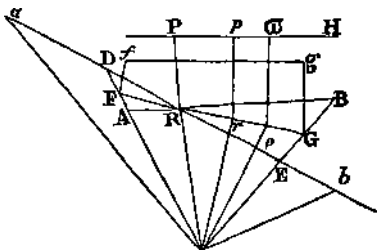
G; and, taking GF to BG in the given ratio, draw FDE; and FD will be to DE as FG to BG. Q.E.F.

By trigonometry thus. In the triangle CGD all the angles and the side CD are given, and from thence its remaining sides are found; and from the given ratios the lines GF and BE are also given.

LEMMA III

To find and represent graphically the hourly motion of a comet to any given time.

From observations of the best credit, let three longitudes of the comet be given, and, supposing ATR , RTB to be their differences, let the hourly motion be required at the time of the middle observation TR . By Lem. n, draw the right line ARB , so as its intercepted parts, AR , RB ,



may be as the times between the observations; and if we suppose a body in the whole time to describe the whole line AB with an equal motion, and to be in the meantime viewed from the place T, the apparent motion of that body about the point R will be nearly the same as that of the comet at the time of the observation TR.

The same more accurately.

Let Ta , Tb be two longitudes given at a greater distance on one side and on the other; and by Lem. n draw the right line aRb so that its intercepted parts aR , Rb may be as the times between the observations aTR , RTb . Suppose this to cut the lines TA , TB , in D and E ; and since the error of the inclination TRa increases nearly as the square of the time between the observations, draw FRG , so that either the angle DRF may be to the angle ARF , or the line DF to the line AF , as the square of the ratio of the whole time between the observations aTB to the whole time between the observations ATB , and use the line thus found FG in place of the line AB found above.

It will be convenient that the angles ATR , RTB , aTA , BTb be no less than ten or fifteen degrees, the times corresponding no greater than eight or twelve days, and the longitudes taken when the comet moves with the greatest velocity; for thus the errors of the observations will bear a less ratio to the differences of the longitudes.

LEMMA IV

To find the longitudes of a comet at any given times.

It is done by taking in the line FG the distances Rr, Rp, proportional to the times, and drawing the lines TV, Tp. The way of working by trigonometry is manifest.

LEMMA V

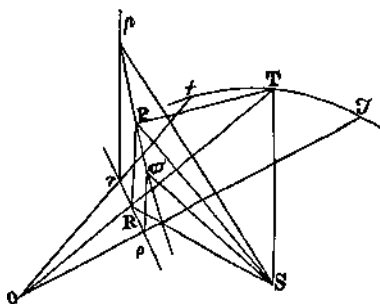
To find the latitudes.

On TF, TR, TG, as radii, at right angles erect Ff, RP, Gg, tangents of the observed latitudes; and parallel to fg draw PH. The perpendiculars rp, od, meeting PH, will be the tangents of the sought latitudes to Tr and Tp as radii.

PROBLEM I

From the assumed ratio of the velocity to determine the orbit of a comet.

Let S represent the sun; t, T, T three places of the earth in its orbit at equal distances; p, P, & as many corresponding places of the comet in its orbit, so that the distances interposed between place and place may answer



to the motion of one hour; pr, PR, wp perpendiculars let fall on the plane of the ecliptic, and rRp the vestige of the orbit in this plane. Join Sp, SP, SO, SR, ST, tr, TR, TP, TP, and let tr, rp meet in O, TR will nearly converge to the same point O, or the error will be inconsiderable. By the premised Lemmas the angles rOR, ROp are given, as well

as the ratios pr to tr, PR to TR, and up to Tp. The figure / T T O is likewise given both in magnitude and position, together with the distance ST, and the angles STR, PTR, STP. Let us assume the velocity of the comet in the place P to be to the velocity of a planet revolved about the sun in a circle,

at the same distance SP , as V is to 1; and we shall have a line pPW to be determined, of this condition, that the space $p\&$, described by the comet in two hours, may be to the space $V\ tr$ (that is, to the space which the earth describes in the same time multiplied by the number V) as the square root of the ratio of ST , the distance of the earth from the sun, to SP , the distance of the comet from the sun; and that the space pV , described by the comet in the first hour, may be to the space $P\&$, described by the comet in the second hour, as the velocity in p to the velocity in P ; that is, as the square root of the ratio of the distance SP to the distance Sp , or in the ratio of iSp to $SP + Sp$; for in this whole work I neglect small fractions that can produce no perceptible error.

In the first place, then, as mathematicians, in the resolution of affected equations, are wont, for the first step, to assume the root by conjecture, so, in this analytical operation, I judge of the sought distance TR as I best can by conjecture. Then, by Lcm. n I draw rp , first supposing rR equal to Rp , and again (after the ratio of SP to Sp is discovered) so that rR may be to Rp as $2SP$ to $SP + Sp$, and I find the ratios of the lines pu , rp , and OR , one to the other. Let M be to $V\ tT$ as OR to pw ; and because the square of pu is to the square of $V\ tr$ as ST to SP , we shall have, from this, OR^2 to M^2 as ST to SP , and therefore the product $OR^m\ SP$ equal to the given product $M^2\ ST$; whence (supposing the triangles STP , PTR to be now placed in the same plane) TR , TP , SP , PR will be given, by Lem. 1. All this I do, first by graphic procedure in a rough and hasty way; then by a new graph with greater care; and, lastly, by an arithmetical computation. Then I proceed to determine the position of the lines rp , pu with the greatest accuracy, together with the nodes and inclination of the plane Spw to the plane of the ecliptic; and in that plane Spw I describe the orbit in which a body let go from the place P in the direction of the given right line pu would be carried with a velocity that is to the velocity of the earth as pw to $V\ tr$. Q.E.F.

PROBLEM II

To correct the assumed ratio of the velocity and the orbit thence found.

Take an observation of the comet about the end of its appearance, or any other observation at a very great distance from the observations used before,

and find the intersection of a right line drawn to the comet, in that observation with the plane *Spu*, as well as the comet's place in its orbit at the time of the observation. If that intersection happens in this place, it is a proof that the orbit was rightly determined; if otherwise, a new number *V* is to be assumed, and a new orbit is to be found; and then the place of the comet in this orbit at the time of that probatory observation, and the intersection of a right line drawn to the comet with the plane of the orbit, are to be determined as before; and by comparing the variation of the error with the variation of the other quantities, we may conclude, by the Rule of Three, how far those other quantities ought to be varied or corrected, so as the error may become as small as possible. And by means of these corrections we may have the orbit exactly, providing the observations upon which the computation was founded were exact, and that we did not err much in the assumption of the quantity *V*; for if we did, the operation is to be repeated till the orbit is exactly enough determined. Q.E.F.

[END OF THE SYSTEM OF THE WORLD.]

AN HISTORICAL AND EXPLANATORY APPENDIX

BY

FLORIAN CAJORI

I. * Frontispiece. *Portrait of Newton*. The photogravure has been made from a portrait of Newton, which has been gummed in volume 2 of a large work entitled *Heads in Taille Douce* (p. 128). This volume is in the Pepys Library at Cambridge. The Masters and Fellows of Magdalene College graciously consented to have the portrait photographed for reproduction in the present edition of Newton's *Prinapia*. J. Edleston¹ gives an engraving prepared from this same portrait: but the portrait here shown is a photographic reproduction. The original drawing is in India ink. As to the year when it was made, Edleston concludes (p. xix): "In assigning, therefore, the date of the portrait to the period of a few years on either side of 1691, we shall not perhaps be very wide of the truth. If this supposition be well-founded, this portrait may be considered as the most interesting of all the known portraits of our philosopher, as representing him at a time of his life the least remote from those memorable eighteen months which it cost him to produce the great work that has immortalized his name."

¹ J. Edleston, *Correspondence of Sir Isaac Newton and Professor Cotes*, London, 1850, frontispiece.

2. *Facsimile of the title page of the first edition of the Prinapia*. A close approach to the date when Newton made alterations in this page may be obtained from the following considerations. Newton's changes in the title page indicate that he was president of the Royal Society of London, but they do not indicate that he had been knighted. In the second edition of the *Prinapia*, 1713, his knighthood appears in the words "Auctore Isaaco Newtono, Equite Aurato." We know that Newton was elected president of the Royal Society on Nov. 30, 1703; he was knighted Jan. 16, 1705. Therefore the alterations on the title page must have been made in the interval between these two dates. This conclusion is in conformity with a remark of Flamsteed to Pound,¹ Nov. 15, 1704, "The book [Newton's *Opticas*] makes no noise in town, as the *Prinapia* did, which I hear he is preparing again for the press with necessary corrections." The alterations were not printed.

¹ Edleston, *op cit.*, p. xv.

* The numbers refer to corresponding footnotes in the text.

3 (p. xvii). *Preface to the First Edition of the Principia*. This Preface in the **first** edition has no date and lacks the author's signature. The signature "Is. Newton" and the date "Dabam Cantabrgiae, e Collegio S. Trinitatis, Mail 8. 1686" first appear in the second edition, 1713. The preface to the first edition of Newton's *Opticas*, 1704, bears no date, while in the second edition, 1718, the date "April 1, 1704" is added. Probably Newton came to recognize the importance of dates in the course of his bitter controversy with Leibniz on the invention of the calculus.

4 (p. xix). *Alterations and corrections made in preparing the second edition of the Princepta*. The statement of changes indicated in Newton's short Preface may be supplemented by the following remarks of Ball:¹ "I possess in manuscript a list of the additions and variations made in the second edition; the changes are very numerous, in fact I find that of the 494 (i.e., 510-16) pages in the first edition 397 are more or less modified in the second edition. The most important alterations are the new preface by Cotes; the propositions on the resistance of fluids, book 11. section vii. props. 34-40; the lunar theory in book in.; the proposition on the precession of the equinoxes, book 111. prop. 39; and the propositions on the theory of comets, book in. props. 41,42."

In preparing copy for the second edition of the *Principia*, Cotes took great care to remove errors and imperfections. Newton wrote to him on Oct. 11, 1709: "I would not have you be at the trouble of examining all the Demonstrations in the Principia. Its impossible to print the book wth out some faults and if you print by the copy sent you, correcting only such faults as occur in reading over the sheets to correct them as they are printed off, you will have labour more then it's fit to give you."² In 1713, after the second edition had appeared from the press, Newton sent Cotes a list of errata, perhaps intending it to be printed as a table of errata To this Cotes replied, Dec. 22, 1713:³ "I observe You have put down about 20 Errata besides those in my Table—I believe You will not be surprnz'd if I tell You I can send You 20 more as considerable, which I have casually observ'd, and which seem to have escap'd You • and I am far from thinking these forty are all that may be found out, notwithstanding that I think the Edition to be very correct. I am sure it is much more so than the former, which was carefully enough printed; for besides Your own corrections and those I acquainted You with whilst the Book was printing, I may venture to say I made some Hundreds, with which I never acquainted You."

Certain changes occurring in the second edition of the *Principia* are mentioned in Notes 3,19,24,26,27,29, 30, 39,42,45.

¹ W W R. Ball, *An Essay on Newton's Prinapta*, London, 1893, p. 74

² Edleston, *op cit*, p. 5.

³ Edleston, *op at*, pp. 167, 168

5 (p. xx). *Cotes's Preface to the Second Edition of the Principia*. It was at the suggestion of Richard Bentley, Master of Trinity College in Cambridge, that Cotes wrote this Preface. "I have S^r Isaac's Leave," wrote Bentley, "to remind you of what You and I were talking of, An alphabetical Index, and a Preface in your own Name; If you please to draw them up ready for y^e press, to be printed after my Return to Cambridg, You will oblige Yours R Bentley."¹

Cotes wrote to Newton on Feb. 18, 1712-3, about the Preface: "I think it will be proper besides the account of the Book and its improvements, to add something more particularly concerning the manner of Philosophizing made use of and wherein it differs from that of Descartes and Others, I mean in first demonstrating the Principle it employs. This I would not only assert but make evident by a short deduction of the Principle of Gravity from the Phaenomena of Nature in a popular way that it may be understood by ordinary readers and may serve at y^e same time as a specimen to them of the Method of ye whole Book."-' Then follows a detailed plan which was afterwards somewhat modified. Newton himself prepared a short Preface which made it unnecessary for Cotes to enter into a recital of the "improvements" in the second edition of the *Prinapia*. Cotes's Preface is therefore confined to "the manner of philosophizing" and an examination of the objections of Leibniz (without mentioning his name) and of the system of vortices. Leibniz, in a letter (April 9, 1716) written under excitement, calls the Preface "pleine d'aigreur."

As stated, the primary object of the Preface was to combat Descartes' theory of vortices. The need of such discussion, twenty-six years after the first appearance of Newton's *Prtnicipia*, indicates the great popular attachment to the views of Descartes. Not only was his theory of vortices generally held at this time (1713) on the European continent, but also in England. Cartesian cosmology invaded England soon after Descartes' publication of his theory in 1644. Henry More, of Christ's College, Cambridge, one of the first fellows of the Royal Society of London, in his earlier years had been in correspondence with Descartes and an admirer of his. More's friend, Joseph Glanvill, of Exeter College, Oxford, also a fellow of the Royal Society, wrote appreciatively of Descartes' vortices. The writings of Robert Boyle teem with references to Descartes, "the most acute modern philosopher," yet in Boyle there is only one reference that I could find, to the Cartesian theory of vortices, and that reference was "without allowing this hypothesis to be more than not very improbable."³ Robert Hooke wrote in criticism of some aspects of the vortex theory.*

Descartes' theory of vortices received a popular exposition in the famous textbook on physics, written in French by Rohault. A Swiss physician, Theophilc Bonet, made a Latin translation of this text, which appeared in Geneva in 1674 and in London in 1682. Thus England began to use this well written textbook

five years before the publication of Newton's *Principia*. The profound divergence of the mechanics of Rohault and Newton stands out glaringly in Rohault's statement that motion in a circle is as natural as in a straight line. The Cartesian doctrine had elements of popular strength. The non-mathematician could understand it. Everyone had seen chips of wood whirled about in eddies of rivers. Everyone had seen a minute whirlwind raise the dust in tiny cyclones. Planets moved like pieces of wood in eddies. These mental pictures carried conviction. On the contrary, Newton's law of inverse squares in gravitational attraction meant nothing to one not accustomed to mathematical thinking.⁵ British mathematicians like Halley, David and James Gregory, Keill, Whiston, Cotes, Taylor, Robert Smith, and Saunderson favored Newton's doctrines. Newton himself lectured at Cambridge, certainly as late as 1687,⁶ but the details relating to his activity as a lecturer are exceedingly meager. After 1692 he had a long illness. In 1696 he was appointed Warden of the Mint. He was succeeded in the Lucasian Chair at Cambridge about 1701 by Whiston, who lectured on Newtonian philosophy. From these facts alone one might infer that Newton's system easily displaced Cartesianism in British universities. But such was not the fact; the Cartesian system displayed wonderful vitality, even in Cambridge. For about forty years after the first publication of Newton's *Principia* the French system maintained a foothold in England. I offer a few facts in support of this statement. The essayist, Joseph Addison, of Magdalen College, Oxford, delivered an oration in 1693, six years after the publication of Newton's *Principia*, in which he praises Descartes, "who had bravely asserted the truth" against the followers of Aristotle.⁷ Whiston⁸ refers to David Gregory's teaching Newton at Edinburgh, "while we at Cambridge, poor wretches, were ignominiously studying the fictitious hypotheses of the Cartesian." I have already referred to the publication in England in 1682 of Rohault's physics, containing a popular exposition of Descartes' system. Fifteen years later, in 1697, a new translation of that book into Latin appeared from the pen of Samuel Clarke, of Cams College, Cambridge, whom Whewell describes as a "friend and disciple of Newton." While the translation was in progress, Whiston spoke his mind to Clarke on the fitness of such a translation in the following terms:⁹ "Since the youth of the university must have, at present, some System of Natural Philosophy for their studies and exercises; and since the true system of Sir Isaac Newton's was not yet made easy enough for the purpose, it is not improper, for their sakes, yet to translate and use the system of Rohault... but that as soon as Sir Isaac Newton's Philosophy came to be better known, that only ought to be taught, and the other dropped." It should be added that Rohault's was reputed to be by far the best treatise of that time on physics in general. Clarke's translation, in better Latinity, played an important role as a textbook, in both English and American colleges. John Playfair¹⁰ says that this new and elegant

translation contained additional notes, in which Clarke explained the views of Newton, so that the notes contained virtually a refutation of the text, avoiding, however, all appearance of controversy. Thus, continues Playfair, "the Newtonian Philosophy first entered the University of Cambridge, under the protection of the Cartesian." Playfair's statement needs emendation in one respect. Clarke's edition of Rohault, as printed in 1697, did not contain the additions as footnotes, but as annotations at the end of the volume; they are shorter than in the later editions and refer to ancient writers, and do not refute Descartes' theory of vortices. Clarke's refutation came at a later date. Four editions of Clarke's Latin translation appeared. The third, issued in 1710, differs from the first in having the notes not at the end of the volume, but at the bottom of the pages as footnotes, and greatly enlarged. This third edition (perhaps also the second of 1703, which I have not seen) contains a new annotation which relates to Descartes' vortices and points out conclusively that these vortices do not explain the facts of observation. They do not explain the motion of comets which cut the orbital planes of the planets at all angles; they would make a planet move fastest when farthest from the sun, while as a matter of fact it moves slowest when in that position. On this subject, there is given a long quotation from Newton's *Prtnapta*. The popularity of Clarke's later editions of Rohault may be due largely to the footnotes. Taken as a whole, the text was acceptable to followers of Newton as well as to those of Descartes. Both sides were fairly presented. Professor Playfair directs attention to the fact that tutors in colleges, whose instructions "constitute the real and efficient system" in a British university, sometimes held different views from those of the professors. Thus Professor Keill introduced in his lectures Newtonian philosophy at Oxford, but the Oxford tutors "were not cast in that mold till long afterwards." Ball states that "at Cambridge until recently professors only rarely put themselves into contact with or adapted their lectures for the bulk of the students____Accordingly if we desire to find to whom the spread of a general study of the Newtonian philosophy was immediately due, we must look not to Newton's lectures or writings, but among proctors, moderators, or college tutors who had accepted his doctrines."¹¹ Clarke's edition of Rohault suited therefore the needs of tutors, whichever of the two opposing scientific views they favored. That in 1723 Rohault's text was by no means discredited in England is evident from the appearance of an English translation of Clarke's edition, with notes. Other editions of this translation appeared as late as 1729 and 1735. According to Hodlay's life of Samuel Clarke, Rohault was still the Cambridge textbook in 1730, three years after the death of Newton and forty-three years after the appearance of Newton's *Principia*. It looks as if two different practices of instruction had been carried on for many years without open controversy between the two factions, one favoring Descartes as expounded by Rohault, the other favoring

Newton as expounded in Clarke's footnotes, in Whiston's lectures published in 1710 and 1716, and in the teaching of Richard Laughton, a noted tutor at Clare Hall in Cambridge. Desaguliers,¹¹ who moved from Oxford to London in 1713, informs us that "he found the Newtonian philosophy generally received among persons of all ranks and professions, and even among the ladies by the help of experiments." Somewhat at variance with this statement is that of Voltaire,¹⁸ who visited England in 1727 and declared that though Newton survived the publication of the *Prtnapia* more than forty years, yet at the time of his death he had not above twenty followers in England. But Voltaire¹⁴ said also: "A Frenchman who arrives in London finds a great alteration in philosophy, as in other things. He left the world full, he finds it empty. At Paris you see the universe composed of vortices of subtle matter, in London we see nothing of the kind."

On the European continent, the vortices of Descartes enjoyed a longer life. Attempts were made by Huygens, Perrault, Johann II Bernoulli, and others to remove some of the glaring defects in the original theory of vortices, but by the middle of the eighteenth century the Newtonian system had gained complete ascendancy.

Cotes's Preface is of historical importance in other respects. It is interpreted as advocating the theory of "action at a distance" (see Note 8), and the theory that gravity is an innate property of matter (see Note 6).

¹ Edleston, *op cit*, p 148.

² Edleston, *op cit*, pp 151, 154

³ *Worlds of the Honourable Robert Boyle*, vol 5, London, 1772, p 403.

⁴ Robert Hooke, *Micrographia*, London, 1665, pp 60, 61

⁵ On the difficulty of understanding the *Principia*, see Ball, *op cit*, pp 114-116

⁶ Edleston, *op cit*, p. xcvi

⁷ D. Brewster's *Memoirs of Sir Isaac Newton*, vol 1, ed 2, Edinburgh, 1860, pp. 291, 292

⁸ Whiston's *Memoirs of His Own Life*, p. 36, quoted by Brewster, *op cit*, vol 1, p 291

⁹ Brewster, *op at*, vol 1, p 295

¹⁰ J Play fair, "Dissertation Fourth," in *Encyclopaedia Britannica*, ed 8, vol 1, pp 609, 610, quoted by Brewster, *op at*, vol 1, pp. 290, 291

¹¹ W W R Ball, *History of the Study of Mathematics at Cambridge*, Cambridge, 1889, p 74

¹² I T Desaguliers, *Physico-Mechanical Lectures*, London, 1717, quoted by W Whewell, *History of the Inductive Sciences*, vol i, cd 3, New York, 1875, p 426

¹³ F M A Voltaire, quoted by Brewster, *op at*, vol 1, p 290.

¹⁴ Voltaire, *Elements de la philosophic de Newton*, 1783; *CEuvres*, vol 31, 1785, quoted by Whewell, *op. cit*, vol 1, 1875, p. 431

6 (p. xxi). Cotes's Preface. *The nature of gravity*. Cotes's words may have contributed to a misunderstanding of the views of Newton. Cotes says "that the attribute of gravity was found in all bodies" and that "gravity must have a place among the primary qualities of all bodies"; he refers to "the nature of gravity in earthly bodies." In expressions of this sort it might seem implied that gravity is an inherent property of matter. Phrases in Newton's *Principia* (1687) appear to carry a similar implication. Newton says (Book i, Prop, LX): "If two bodies . . . attracting each other with forces inversely proportional to the square of

their distance"; (Book I, Prop, LXIX) "the absolute forces of the attracting bodies"; (Book I, Prop, LXXII) "the attraction of one corpuscle towards the several particles of one sphere"; (Book I, Prop, LXXV) "the attraction of every particle is inversely as the square of its distance from the centre of the attracting sphere"; (Book I, Prop. LXXVII) "let now the corpuscle P attract the sphere"; (Book III, Prop, v) "Jupiter and Saturn... by their mutual attractions sensibly disturb each other's motions." In these expressions, the "bodies" or the "corpuscles" are represented as active, as "attracting." They are not passive like a chip of wood carried about by an eddy in a pool, or like a planet passively swept through space by a Cartesian vortex. It was easy, therefore, to jump to the inference that in the Newtonian theory, gravity was an innate, inherent property of matter. Indeed, such an interpretation was made by writers on the European continent, for example by Huygens, Lalande, Bordas-Demouhyn and others,¹ and has been generally held by astronomers and physicists. Thus, after the publication of the *Principia* in 1687, Huygens forthwith abandoned the explanation of planetary motion by Descartes' theory of vortices, and published his adherence to Newton's celestial mechanics. But Huygens did not accept the view that gravitation was an innate property of matter, a view which he attributed to Newtonian philosophy. On this point Huygens rejected what he interpreted to be the tenet of Newton, and continued his adhesion to the tenet of Descartes.²

While readers of the first edition of the *Principia* had some justification in attributing to Newton the view that gravity was an innate property of matter, they were nevertheless mistaken. In the first edition Newton had made no explicit declaration on this point. We know now that before publishing his great book, as early as Feb. 28, 1678-9, in a letter to Robert Boyle,³ he speculated on the "cause of gravity" and endeavored to explain attraction by the action of an "aether," consisting of "parts differing from one another in subtilty by indefinite degrees." (See Note 55.) It is evident that Newton was no more a believer in gravity as an innate property of bodies than was Descartes. But readers of the first edition of the *Principia* had no means of knowing this. His letter to Boyle was not then made public.

Even Bendej, a great friend and admirer of Newton's, at first entertained the wrong idea of his attitude; in letters to Bentley of 1692-3, Newton strongly opposed the doctrine that gravity was an innate property of matter and also the doctrine of "action at a distance." These letters, like that to Boyle, were not printed until many years later, and could therefore not immediately influence scientific opinion generally. In a letter to Bendej,⁴ Newton wrote:

"You some times speak of gravity as essential and inherent to matter. Pray, do not ascribe that notion to me; for the cause of gravity is what I do not pretend to know, and therefore would take more time to consider of it."

In another letter Newton wrote:

"It is inconceivable, that inanimate brute matter, should, without the mediation of something else, which is not material, operate upon and affect other matter without mutual contact, as it must be, if gravitation, in the sense of Epicurus, be essential and inherent in it. And this is one reason why I desired you would not ascribe innate gravity to me. That gravity should be innate, inherent, and essential to matter, so that one body may act upon another at a distance through a vacuum, without the mediation of any thing else, by and through which their action and force may be conveyed from one to another, is to me so great an absurdity, that I believe no man, who has in philosophical matters a competent faculty of thinking, can ever fall into it. Gravity must be caused by an agent acting constantly according to certain laws; but whether this agent be material or immaterial, I have left to the consideration of my readers."

In the second edition of the *Principia* (1713) Newton made his position clearer by three additions to the text of 1687. In the Scholium following Prop, LXIX of Book 1, Newton says: "I here use the word *attraction* in general for any endeavor whatever, made by bodies to approach each other, whether that endeavor arise from the action of the bodies themselves, as tending to each other or agitating each other by spirits emitted; or whether it arises from the action of the ether or of the air, or of any medium whatever, whether corporeal or incorporeal, in any manner impelling bodies placed therein towards each other." Here he maintains an agnostic attitude. In Book 111, when discussing the Rules of Reasoning in Philosophy, he adds: "All bodies whatsoever are endowed with a principle of mutual gravitation____Not that I affirm gravity to be essential to bodies: by their *vis insita* I mean nothing but their inertia." Finally, in the General Scholium at the end of the *Princtpia*, he said, "I do not frame hypotheses" on the nature of gravity. This was the proper attitude for him to take in a work like the *Princtpia*. To Boyle he described his notions on this subject to be "so indigested" that he was "not well satisfied" with them.

More positive than in the *Principia* was Newton's statement in the "Advertisement" to the second edition of his *Opticas*, July 16,1717: "And to shew that I do not take Gravity for an Essential Property of Bodies, I have added one Question [Query 31] concerning its Cause, chusing to propose it by way of a Question, because I am not yet satisfied about it for want of Experiments."

Not only is it a mistake to attribute the doctrine that gravity is an innate quality of bodies to Newton, but it seems to be also a mistake to attribute it to Cotes, notwithstanding some of the phrases that I have quoted from his Preface. That it is a mistake appears from the correspondence between Cotes and Samuel Clarke. Cotes submitted to Clarke his draft of the Preface to the second edition of the *Principia*. He writes to Clarke:⁸ "I return You my thanks for Your corrections

of the Preface, and particularly for Your advice in relation to that place where I seem'd to assert Gravity to be Essential to Bodies. I am fully of Your mind that it would have furnish'd matter for Cavilling, and therefore I struck it out immediately upon D^r Cannon's mentioning Your Objection to me, and so it never was printed—My design in that passage was not to assert Gravity to be essential to Matter, but rather to assert that we are ignorant of the Essential propertys of Matter and that in respect to our Knowledge Gravity might possibly lay as fair a claim to that Title as the other Propertys which I mention'd. For I understand by Essential propertys such propertys without which no others belonging to the same substance can exist: and I would not undertake to prove that it were impossible for any of the other Properties of Bodies to exist without even Extension."

The question of the nature of gravity has aroused new interest with the advent of Einstein's general theory of relativity, according to which gravity is looked upon not as innate to bodies, but rather as some modification of space. According to Einstein, the earth produces in its surroundings a gravitational held, which, acting on the apple, brings about its motion of fall. In Einstein's gravitational field," in general, a ray of light is propagated curvilinearly. The difference between the new and the old physics is stated by Eddington thus: "Einstein's law of gravitation controls a geometrical quantity curvature in contrast to Newton's law which controls a mechanical quantity force."⁷

¹ Edleston, *op cit*, p 159

² *Traite de la lumiere*, par C H D Z, Leyden, 1690, pp 125-180; *Discours de la cause de la pesanteur* As early as 1669 Huygens read before the Pans academy a speculation on the cause of gravity based on a modification of Cartesian vortices. He did not publish on this subject before 1690. When Newton's *Principia* appeared in 1687, Huygens at once accepted Newton's centripetal force varying inversely as the square of the distance, because motions in the solar system were explained with great success by this law. But Huygens rejected Newton's idea that particles of matter of all bodies attract each other, because he could not see how such attraction could be explained on any mechanical principle. Edleston (*op cit*, pp xxxi, lix) makes the interesting statement that the only time Newton and Huygens met, in 1689, at a meeting of the Royal Society of London, Huygens talked on the cause of gravity, while Newton discussed double refraction in Iceland crystals—each of the two great physicists discoursing on the topic most intimately associated with the other. For details, see also F Rosenberger, *haac Newton und seine physikalischen Principien*, Leipzig, 1895, pp 234-248.

³ *Isaaci Newtoni Opera* (Horsley's ed.), vol 4, 1782, pp 385-394

⁴ *Worlds of Richard Bentley*, vol. 3, London, 1838, pp. 210, 211. Letter of Newton to Bentley, "Trinity College, Jan. 17, 1692-3."

⁵ Edleston, *op. at*, pp. 150, 159

⁶ A Einstein, *Relativity, the Special and General Theory*, tr. R W Lawson, New York, 1921, pp 75, 88

⁷ A S Eddington, *The Nature of the Physical World*, New York, 1929, p 133.

7 (p. xxx). Cotes's Preface. Cotes's term for the earth's orbit is *orbis magnus* (the great orbit). It is a term frequently used also by Newton to designate the earth's orbit in its annual revolution around the sun. The term was introduced by Copernicus (*De revolutionibus orbium caelestium*, Lib. 1, Cap. x) and was used by Rheticus, Kepler, and others. The path of the earth was called the "great orbit," not, of course, because of its dimension, for the orbits of the superior plan-

ets are greater, but because of its great importance to the practical astronomer, who must take cognizance of it, in explaining the apparent motions of the sun and planets. In all parts of the *Principia* and the *System of the World* where the term *orbis magnus* occurs, I have substituted for it the expression "earth's orbit." I may add that Newton himself uses the name "earth's orbit" in his *Opticas*, Book II, Part III, Prop. xi.

8 (p. xxxi). Cotes's Preface. *Action at a distance*. The doctrine of "action at a distance" in gravitational attraction has been wrongly ascribed to Newton; it is more properly due to Cotes, who, in his Preface to the Second Edition of the *Principia*, argues against Descartes' theory of vortices. Cotes does not use the phrase "action at a distance," nor does he explicitly advocate the view that celestial spaces are void. He does argue that if a celestial fluid exists it "has no inertia, because it has no resisting force." The implicating sentences of his Preface read as follows: "Those who would have the heavens filled with a fluid matter, but suppose it void of any inertia, do indeed in words deny a vacuum, but allow it in fact. For since a fluid matter of that kind can not be distinguished from empty space, the dispute is now about names and not the nature of things." Samuel Clarke is more definite. In one of the footnotes to his later editions of Rohault he refers explicitly to "that immense Space which is void of all matter."

In Note 6 *supra* I quoted from Newton's letters to Bentley passages relating to gravity, where he says: "That one body may act upon another at a distance through a vacuum, without the mediation of any thing else, by and through which their action and force may be conveyed from one to another, is to me so great an absurdity, that I believe no man, who has in philosophical matters a competent faculty of thinking, can ever fall into it."

Maxwell¹ says: "We find in his 'Optical Queries' and in his letters to Boyle, that Newton had very early made the attempt to account for gravitation by means of the pressure of a medium, and that the reason he did not publish these investigations 'proceeded from hence only, that he found he was not able, from experiment and observation, to give a satisfactory account of this medium, and the manner of its operation in producing the chief phenomena of Nature.'..²

"And when the Newtonian philosophy gained ground in Europe, it was the opinion of Cotes rather than that of Newton that became most prevalent, till at last Boscovich propounded his theory, that matter is a congeries of mathematical points, each endowed with the power of attracting or repelling the others according to fixed laws. In his world, matter is unextended, and contact is impossible. He did not forget, however, to endow his mathematical points with inertia."

Although the phrase "action at a distance" appears very simple, it is subtle on closer inspection and some physicists have pointed out "how weak are the grounds on which we deny principal action at a distance."⁸

An important event in the history of the doctrine "action at a distance" was the appearance of Maxwell's electromagnetic theory of light, in which it was held that electromagnetic disturbances travel with *finite* velocities. Previously, electric and magnetic attraction and repulsion had been assumed to take place instantaneously.

The element of time has come to be considered also in gravitation. The phrase "action at a distance," instead of being used in the old sense with reference to the nonexistence of a medium intervening between attracting masses, is employed since the advent of the theory of relativity to indicate an *instantaneous* action at a distance.⁴ In place of an agent we now consider the time of action. But even now the view of Newton is misrepresented. Newtonian action at a distance is spoken of as "immediate action." Newton, on the other hand, postulates an agent and gives it time to act. To be sure, in his calculations of gravitational attractions, he assumes, as a necessary approximation (having no experimental data on the speed of propagation of gravitational action), that the action is instantaneous, but not so in his talks on gravity. In a letter to Boyle⁵ he considers the cause of gravitation between two approaching bodies. They "make the ether between them begin to rarefy"; and again,* in his hypotheses on light, he says, "So may the gravitating attraction of the earth be caused by the continual condensation of some other such like ethereal spirit ... in such a way ... as to cause it [this spirit] from above to descend with great celerity for a supply; in which descent it may bear down with it the bodies it pervades, with force proportional to the superficies of all their parts it acts upon."

¹ J. C. Maxwell, *Proceedings of the Royal Institution of Great Britain*, vol. 7, 1873-1875, London, pp. 48, 49

² C. Maclaunn's *Account of Sir Isaac Newton's Philosophical Discoveries*, London, 1748

³ A. Schuster, *The Progress of Physics, 1875-1908*, Cambridge, 1911, p. 37

⁴ It is of interest that, in one place, Laplace made the assumption that the transmission of gravity is not instantaneous, and he found that in order to produce the known effects in the secular acceleration of the moon, gravity must travel seven million times faster than light. The moon, with its subtle orbital inequalities, has in this problem, as in others, displayed a treacherous behavior. Laplace's calculation has been found to be incomplete and his velocity of gravity to be illusory (Sec Laplace, *Mecanique celeste*, Livre x, the close of Chap vii.)

⁵ *Isaac Newton Opera*, op cit, vol. 4, p. 385.

⁶ S. P. Rigaud, *Historical Essay on the First Publication of Newton's Principia*, Oxford, 1838, Appendix, pp. 69, 70

9 (p. xxxv). *The alterations and additions made in the third edition of the Principia* are indicated in Newton's Preface to that edition only in a general way. A detailed list was prepared by the astronomer J. C. Adams, of Pembroke College, Cambridge, and printed in David Brewster's *Memoirs ... of Sir Isaac Newton* (ed. 2), vol. 2, Edinburgh, 1860, Appendix No. xxx, pp. 414-419.

Certain changes occurring in the third edition of the *Principia* are mentioned in Notes 11, 19, 26, 29, 33, 39, 42.

10(p. 1). *Translations of the Principia made by Motte and Thorp*. In revising Motte's translation of Cotes's Preface and of the *Principia* of Newton, use has been made of Robert Thorp's translation into English (ed. 2, London, 1802) of Cotes's Preface and the first book of the *Principia*. Occasional aid has been derived also from J. Ph. Wolfers' translation of the *Principia* into German, 1872. The geometrical figures of the *Principia* are taken from the third edition (1726).

Andrew Motte's translation of the *Principia*, from Latin into English, was made in 1729, from the third edition (1726).

11(p. 1). Definition 1 of the *Principia*, *Quantity of matter, or mass*. Newton does not define density. His definition of mass, as the product of density and volume, has been variously appraised. Mach¹ says: "As regards the concept of mass, we remark first that Newton's formulation which defines mass as the quantity of matter of a body, determined by the product of volume and density, is unfortunate. Since we can define density only as the mass of unit volume, the circle is obvious." But it is not easy to believe that Newton was guilty of an *argumentum in circulo* so manifest. Crew² holds that "in the time of Newton, density and specific gravity were employed as synonymous, and the density of water was taken arbitrarily to be unity. The three fundamental units employed... were therefore density, length, time, instead of our mass, length, time. On such a system, it is both natural and logically permissible to define mass in terms of density."

Newton gives a definition of equal densities of bodies in a later passage in the *Principia* (Book in, Prop. vi, Cor. iv), where he says: "If all the solid particles of all bodies are of the same density, and cannot be ranfied without pores, then a void, space, or vacuum must be granted. By bodies of the same density, I mean those, whose inertias are in the proportion of their bulks." It is to be observed, also, that in this passage Newton does not say that the small solid particles, which he assumes to be of the same density, are all of the same size. If all were assumed to be of the same size, then the densities of bodies would be proportional to the numbers of such small particles in equal volumes. Hoppe attributes this latter concept of density to Newton, and claims that it is found earlier in the writings of Francois Lubin, John Kepler, Pierre Gassendi and Robert Boyle.³

But Newton's corpuscular idea, as described in his *Opticas*, goes against Hoppe's interpretation of Newton. In his *Opticas* (third edition, 1721, pp. 375-376), he says: "It seems probable to me, that God in the beginning formed matter in solid, massy, hard, impenetrable, moveable particles, of such sizes and figures, and with such other properties and in such proportion to space, as most conduced to the end for which He formed them; and that these primitive particles, being solids, are incomparably harder than any porous bodies compounded of them; even so very hard, as never to wear or break in pieces: no ordinary power being able to divide what God himself made one in the first creation."

In the use of the concept of mass, as distinguished from weight, Newton has forerunners who perceived the difference between mass and weight more or less clearly. Crew finds the earliest quantitative conception of this idea in Huygens' discussion of centripetal force, in 1673, which was fully discussed in his posthumous *De vi centrifuga*, 1703. Huygens states that when particles move with equal speeds along equal circles, the centripetal forces arc to each other as "the weights of the particles" or as their "solid quantities"—*sicut mobdium gravitates sen quantitas solidas*. Here the "solid quantity" indicates mass. Hoppe claims the concept of mass for Kepler, who designates it by the word *moles* as in the following quotation from Kepler's *Astronomia nova* (1609): "If two stones were removed to any part of the world, near each other but outside the field of force of a third related body, then the two stones, like two magnetic bodies, would come together at some intermediate place, each approaching the other through a distance proportional to the mass [*moles*] of the other."⁴

¹ E. Mach, *Die Mechanik in ihrer Entwicklung* (ed. 8), Leipzig, 1921, chap. 2, § 3, p. 188.

² H. Crew, *The Rise of Modern Physics*, Baltimore, 1928, p. 124.

³ E. Hoppe, *Archiv für Geschichte der Mathematik, der Naturwissenschaften und der Technik*, n. s., vol. 11, 1929, pp. 354-361. For further statements of Newton on the constitution of matter, consult Sir Isaac Newton, 1727-1927, *A Bicentenary Evaluation of His Work*, Baltimore, 1928, pp. 224, 225.

⁴ T. Kepler, Introduction to *Astronomia nova*, 1609, *Opera omnia* (ed. Ch. F. Schönbacher), vol. 3, p. 151, *Kepler's Neue Astronomie*, übersetzt von Max Caspar, München-Berlin, 1929, p. 26.

12 (p. 1). Book i, Definition n. *Quantity of motion*, as the expression is used in the *Principia*, is equivalent to the term *momentum* in more modern mechanics, and is measured by the product of mass and velocity.

13 (p. 6). Scholium following Definition viii. *Absolute motion and absolute time*. Newton pointed out that "the parts of that immovable space, in which those [absolute] motions are performed, do by no means come under the observation of our senses." But he adds, "yet the thing is not altogether desperate; for we have some arguments to guide us, partly from the apparent motions, which are the differences of the true motions; partly from the forces, which are the causes and effects of the true motions, etc." In the light of more recent thought the question arises, in connection with rectilinear motion, whether the existence of "apparent" or relative motion, as revealed to our senses, necessarily carries with it the existence of absolute motion, as vaguely suggested by Newton. Or is it not possible that relative motion is the only rectilinear motion that exists? Take automobiles A, B, and C. Suppose B gains on A with a velocity of 10 kilometers per hour, while C, traveling along the same straight road, in the same direction, gains on A with a velocity of 15 kilometers per hour. From the relative velocity of 5 kilometers per hour, which is the difference of the velocities 15 and 10, we cannot ascertain the velocity of A; A may be at rest on the road, or moving. Of importance in this argument is the circumstance that, from the velocity of A, or from

its state of rest on the road, the inference cannot be drawn by syllogism that such velocity or rest is absolute. "Absolute motion," says Newton, "is the translation of a body from one absolute place into another," and "absolute rest is the continuance of the body in the same part of... immovable space." The existence of absolute motion or rest cannot be established merely from the existence of relative motion or rest. In our illustration of automobile motion, we know that the road itself is in motion, being carried by the earth in its orbit, and so on. Thus, we are forced back to Newton's own admission, that there is no way of bringing absolute motion or absolute space under the observation of the senses. Newton does not mention a universal ether in his discussion of absolute motion, but he might have argued, as has been done since, that motion through such an ether constitutes absolute motion. Here two remarks come to mind: the existence of such an hypothetical ether has been denied in the eighteenth and twentieth centuries; the motion through this ether cannot be said to proceed "under the observation of the senses."

More convincing is Newton's remark on absolute rotation. Two globes are kept by a cord at a given distance apart and are revolved about their common centre of gravity. From the tension of the cord the angular velocity may be determined. Here we have a rotation, resulting from a dynamical experiment more or less familiar through sense-perception, which makes no reference to terrestrial, solar, or stellar positions, and seems therefore absolute. It is absolute in somewhat the same sense as Foucault's pendulum may be said to establish the earth's absolute rotation. If this view is correct, then Newtonian dynamics dealt with a rotation which was truly absolute, nevertheless empirical. Does it not follow, one is tempted to ask, that the space in which absolute rotation takes place must itself be absolute? Newton does not draw such an inference, but commentators have declared that the absoluteness of rotation and acceleration compelled Newton to recognize that space could not be relative; for, otherwise, space would have a dual structure, relative for rectilinear motion, absolute for rotation.

Remarks similar to those which I applied to absolute rectilinear motion bear on the discussion of absolute time. It would seem to follow, therefore, that the existence of absolute rectilinear motion and of absolute time are postulates made in Newtonian mechanics; they are not based on experimental evidence and may therefore be said to be metaphysical. There appears to be no *a priori* argument against acceptance as a foundation in mechanics of concepts, some of which are observable and others unobservable or metaphysical. The two types of concepts might form a perfectly solid and coherent structure which yields results in accord with observational data, to a degree of accuracy lying within the probability of experimental error. Indeed, Newton's assumptions satisfied this test in the scientific developments extending over a period of two hundred years. During that

time, astronomy and physics made tremendous strides forward. Celestial mechanics flourished; so did engineering and physical science.

On esthetic grounds or on grounds of mistrust of metaphysics, it might be said that an empirical science should be based only on observable phenomena. Religious fears caused Bishop Berkeley, in his *Principles of Human Knowledge* (1710) and in his *Analyst* (1734), to object to absolute space. More recently the desirability of a purely empirical foundation was stressed by Ernst Mach in his *Die Mechanik*.¹

In the nineteenth century the researches of Faraday and Maxwell on electromagnetism led to experimental results which could only be explained on the assumption of the existence of relative motion. A moving magnet gives rise to a magnetic field and induces an electric current in neighboring conductors which it passes. This is the fundamental phenomenon in dynamos generating currents. Can the velocity of the magnet be considered absolute? "Absolute motion," according to Newton, is "translation of a body from one absolute place to another"; now "place" is absolute when the "space" is absolute, and "absolute space" exists "in its own nature, without regard to anything external." Now a magnet, if in absolute motion, "without regard to anything external" (not even a neighboring conductor which it passes), could not generate an electric current. If, instead of a moving magnet, we consider a moving charge of electricity, similar remarks apply. Plainly, electromagnetic phenomena invoke velocities that are "relative." Such considerations did not, however, rule out "absolute velocity" from physical science, for other phenomena might need the concept of absoluteness, and it was not yet recognized that all atoms and therefore all matter are really electrical.

A more serious situation arose near the close of the nineteenth century. The luminiferous ether of Newton, Huygens, and Hooke in the seventeenth century, which had been discarded by most scientists in the eighteenth century, was reinstated in the nineteenth century. The prevailing belief was that this ether was stagnant, and that the earth could move through it without dragging the ether along. In the minds of many, this stagnant ether constituted a fundamental frame of reference in the explanation of absolute motion. But the stagnant ether was not altogether satisfactory and a few physicists, such as G. G. Stokes, advocated an ether which is dragged as is water by a moving ship. Could this question be settled by experiment? To answer this question, Michelson and Morley in 1887 performed the now famous experiment² at Cleveland, Ohio, which Michelson is reported to have called an "unfortunate experiment," for it did not yield itself to satisfactory treatment in the old Newtonian mechanics. If the earth did not drag the ether, there would be an ether wind, the so-called "ether drift." The result of the test showed no such "drift," so that, as interpreted at that time, the earth in the Cleveland cellar dragged the ether along with it. Such a result had not been

expected; it seemed to indicate properties of the ether which it was impossible to reconcile with properties required to explain other known phenomena, such as Bradley's aberration of light and the rectilinear path of vertical rays. For nearly twenty years this experiment was a cloud in the scientific firmament.

Perhaps the nature of the Michelson and Morley experiment may be brought to mind best by the statement that, just as a man swimming upstream a given distance and back again requires more time than if swimming in still water, so a ray of light traveling a given distance against an ether wind and back again requires more time than if the ether had been at rest with respect to the apparatus. It is assumed that the swimmer (ray of light) moves always with the same velocity relative to the water (ether). But Michelson and Morley's delicate interferometer indicated no difference of time: hence the inference that there was no "ether drift."

In 1892, G. F. Fitzgerald⁸ of Dublin and H. A. Lorentz⁴ of Leyden, independently, made the audacious and seemingly arbitrary assumption that a moving body contracts along the line of its motion. A yardstick is shorter when moving in the direction of its length than when it is at rest. On this assumption the Michelson and Morley experiment could be explained, even though the ether was not moving with the earth. But physicists in general did not derive much contentment from this contraction theory. Twelve years passed and then Albert Einstein, at that time in Zurich, advanced his special relativity theory.⁶ He built this theory on purely observational foundations, which should explain and coordinate all known phenomena of light, particularly the Michelson and Morley experiment. That trouble-maker, the nineteenth-century luminiferous ether, he cast aside as being purely hypothetical. He discarded also Newton's rectilinear *absolute* motion as having no observational basis. He felt justified in postulating that the velocity of light in a vacuum is constant and independent of the motion of its source. This independence was shown later to exist by Willem de Sitter,⁶ by observations on double stars. The second assumption of Einstein was the "principle of relativity" in the restricted sense: If relative to one coordinate system, a second is a uniformly moving coordinate system devoid of rotation, then natural phenomena run their course with respect to the second system according to the same general laws as with respect to the first system. In the dynamics of this theory, the velocity of light plays a leading role. A train is traveling on a rectilinear railroad track. Lightning has struck the rails at two places A and B far distant from each other. A man on the track, who happened to be at the midpoint M of the distance AB, perceives the two flashes of lightning at the same time and calls them simultaneous. Let M' be the midpoint of the distance AB on the moving train. Will an observer on the train, placed at M', find the two flashes simultaneous? No! For he is traveling on the train toward B, and there-

fore is moving toward the beam of light coming from B, and away from the beam coming from A. Hence the observer on the train comes to the conclusion that the flash B took place before the one at A. Thus, events simultaneous with reference to the railroad track were not simultaneous with reference to the train. Simultaneity is relative. Every reference body or coordinate system has its own particular time. The statement of the time of an event is not independent of the state of motion of the body of reference; it is not absolute. But in the Newtonian physics a statement of time was given an absolute significance. Einstein's special theory of relativity yields mathematical results in agreement with the Fitzgerald-Lorentz contraction. This is not strange, for all three physicists aimed to make provision for the phenomena revealed by the Michelson and Morley experiment. Lorentz also established equations relating to distances and times of a coordinate system C (the uniformly moving train), expressed in terms of the coordinate system C (the rectilinear railroad track). These equations, known as the "Lorentz transformations,"⁷ fit into Einstein's special theory of relativity. I give below in parallel columns the values x', y', z', t' of an event with respect to the coordinate system C when the values x, y, z, t of the same event with respect to C are given. C moves with respect to C with a uniform velocity v . The velocity of light in a vacuum is represented by c . The axes of the two systems C and C are respectively parallel. We assume for simplicity the event localized on the x-axis

The Newton Transformations

$$x' = x - vt$$

$$y' = y$$

$$z' = z$$

$$t' =$$

$$=$$

$$t$$

The Lorentz Transformations

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Comparing the two sets of equations, one sees that the more recent is much more complicated. Relativity affords an example of a theory which has grown far more involved in consequence of being founded upon purely empirical data. The two systems merge into one when coordinate systems have relative velocities v that are infinitesimal as compared with the velocity of light. It is this fact that enabled the Newtonian mechanics to represent planetary motions to a high de-

grce of approximation and success. For remarks on Einstein's general theory of relativity of 1915, see Note 6 on the Nature of Gravity.

¹ Mach, *op. at.*, pp. 216-237.

² A. A. Michelson and E. W. Morley, in *Silliman's Journal*, ser. 3, vol. 34, 1887, p. 333.

³ *Scientific Writings of G. F. Fitzgerald*, Dublin, 1902, pp. lx, 562; O. Lodge, *Philos. Trans.*, A, vol. 184, London, 1894, p. 749.

⁴ H. A. Lorentz, *Verlagen d. Zittingen d. K. Academic van Wetenschappen*, Amsterdam, vol. 1, 1892, p. 74.

⁵ Einstein, *op. at.*, contains a popular exposition.

⁶ W. de Sitter, *Physikalische Zeitschrift*, vol. 14, 1913, pp. 429, 1267.

⁷ Lorentz, *loc. at.* For a simple derivation of the Lorentz transformation, see Einstein, *op. at.*, Appendix I, p. 139.

14 (p. 13). *Laws of Motion*. Because of their importance, I reproduce here the three laws in the original Latin:

Lex I (in editions of 1687 and 1713). Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.

Lex I (in edition of 1726). Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus illud a viribus impressis cogitur statum suum mutare.

Lex II. Mutationem motis proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur.

Lex III. Actioni contrariam semper et aequalcm esse reactionem: sive corporum duorum actiones in se mutuo semper esse aequales et in partes contrarias dirigi.

The first law is frequently called the "law of inertia." Students of relativity point out that we do not know of any body in Nature which is at absolute rest, not on the earth nor on the sun or stars; that there is rest only with respect to some system of coordinates. Einstein gave a critical examination of what he called a "Gahleon system of coordinates," a system in which the law of inertia holds relative to it, and in which no gravitational field exists which is therefore not rigidly attached to the earth. "The visible fixed stars are bodies for which the law of inertia certainly holds to a high degree of approximation____The laws of mechanics of Galileo-Newton can be regarded valid only for a Gahleon system of coordinates."¹ See Note 13.

¹ Einstein, *op. at.*, pp. 12, 13.

I give a few additional references to the laws of motion—Samuel Horsley (the editor of Newton's *Opera*, 1779-1785) gave a metaphysical discussion of Newton's laws of motion, which is printed in *Lord Monboddo and Some of His Contemporaries* (ed. William Knight), London, 1900, pp. 281-284, 298, 302-305.

J. C. Maxwell, *Matter and Motion*, London, 1876.

Sir W. Thomson and P. G. Tait, *Treatise on Natural Philosophy*, Part i, Cambridge, 1879.

K. Pearson, *Grammar of Science*, London, 1900, pp. 321-327, 533-536; Appendix, Notes 1 and 2.

E. Mach, *Die Mechanik in ihrer Entwicklung*, ed 8, Leipzig, 1921.

E. Freundlich, *Grundlagen der einsteinschen Gravitations Theorie*, Berlin, 1920, p. 42.

A. S. Eddington, *Space, Time, and Gravitation*, Cambridge, 1920, Chap. ix.

15 (pp. 13,45). Second Law of Motion. *Force*. By Newton's second Definition, "quantity of motion" (momentum) arises "from the velocity and quantity of matter conjointly," that is, from mv . By Newton's second Law of Motion, "change of motion," that is, change in the quantity of motion, "is proportional to the motive force impressed." Thus, we have "change of motion" as the measure of the force which produces it. Thus arose the measurement of force by the product of mass and acceleration. This concept of force has played a fundamental role in mechanics from the time of Newton to the close of the nineteenth century. It will continue to play a basic role in mechanics involving velocities that are small in comparison to the velocity of light. But as a concept in general cosmological mechanics it has faded into the background. A most far-reaching experimental result was obtained in 1901 by W. Kaufmann, namely, that the mass of an electron increases rapidly as its speed nears the velocity of light.¹ The invariance of mass in Newtonian mechanics was thus shown to be incorrect. (See Note n.) The Newtonian force of gravitational attraction between two bodies varies as the product of their masses, and inversely as the square of the distance. This force was rendered ambiguous by recent research, because (1) mass depends on velocity and (2) distance, according to the theory of relativity, depends upon the location of the observer. Einstein's gravitational theory of 1915 undermines the belief in the reality of gravitation as a "force." But his theory of 1915 does not include similar treatment of electromagnetic forces. Generalizations of Einstein's theory of 1915, to embrace also electromagnetic forces, were made in somewhat different ways by H. Weyl² in 1918, by Eddington³ in 1921, and by Einstein⁴ himself in 1929.

¹ W. Kaufmann, *Gothinger Nachrichten*, Nov. 8, 1901, see also the volumes for 1902 and 1903.

² H. Weyl, *Sitzungsberichte der Preuss. Akademie d. Wissensch.*, Phys.-Math. Klasse, 1918, p. 465.

³ E. Eddington, *Proceedings of the Royal Society of London*, A 99, 1921, p. 104.

⁴ Einstein, "Zur einheitlichen Feldtheorie," *Sitzungsberichte der Preuss. Akademie d. Wissensch.*, Phys.-Math. Klasse, 1929, 1.

16 (pp. 21,36). Book 1, Scholium and Lemma xi. *Obsolete mathematical expressions and notations*. In Newton's Latin editions of the *Principia*, as well as in Motte's translation into English, there occur certain mathematical expressions which are no longer used in mathematics and are therefore not immediately understood by a reader familiar only with modern phraseology. I have altered the translation by substituting for the old, corresponding modern terminology. Most frequent of the obsolete terms are "duplicate ratio," "subduplicate ratio," "triplicate ratio," "subtriplicate ratio," "sesquiphcate ratio," "subsesquiphcate ratio," "sesquialteral ratio." For these I have used, respectively, the terms "square of the ratio," "square root of the ratio," "cube of the ratio," "cube root of the ratio," " $3/2$ th power of the ratio," " n th power of the ratio," "ratio of 3 to 2." In a

few rare occurrences, the old usage of the term "proportion" corresponds to the modern "ratio."

I have discarded the vinculum as the sign of aggregation and have introduced round parentheses instead. In some places where the reasoning is unusually involved, I have introduced the modern notation for proportion, $a : b = c : d$, in place of the rhetorical form used in the *Principia*. For the sign of multiplication ($\times$) regularly occurring in the original *Principia*, I have introduced the dot ($\cdot$), placed halfway up the height of the short letters to distinguish it from the decimal point, which is placed on the lower level. In a few places I have employed the solidus in printing fractions.

17 (p. 22). Book 1, Scholium after the Laws of Motion. *Wren and Mariotte*. The reader may wish to consult P. G. Tait's "Note on a Singular Passage in the *Principia*," *Proceedings of the Royal Society of Edinburgh*, vol. 13, 1886, pp. 72-78, dealing with questions of priority.

18 (p. 63). Book 1, Prop, xvi, Cor. iv. By the *mean distance* from the focus of an ellipse is meant the major semiaxis or the distance SB from the focus to an extremity of the minor axis BC. This distance is also the radius of the circle in question. The proof of Cor. iv follows immediately from Cor. vi, Prop, iv, as indicated in the text. The last two sentences in Cor. iv show the consistency of this result with Prop, xvi, according to which the velocity in the elliptic orbit is to that

in the circular orbit as——: —, where L and l are the latera recta of ellipse
BC SB

and circle, respectively. But $BC : SB = V(SB \cdot L) : V(SB \cdot l)$. Hence the velocities as derived from Prop, xvi are inversely as the square roots of the equal distances SB, and are therefore themselves equal.

19 (p. 76). Book 1, Lemma xvii. *Trapezium*. Newton used the word "trapezium" in the Euclidean sense as any quadrilateral except a square or parallelogram

20 (p. 90). Book 1, Lemma xxii. *Geometry of comes*. For a statement of the role that this Lemma and other parts on the geometry of conies given in the first Book of the *Principia* play in the history of synthetic geometry, consult J. J. Milne, "Newton's Contribution to the Geometry of Conies," in *Isaac Newton, 1642-1727*, London, 1927, pp. 96-114; also Ernst Kötter, *Die Entwicklung der synthetischen Geometrie, von Monge bis auf Staudt*, Leipzig, 1901, pp. 8, 9, 30, 39, 41, 109, and in other places indicated in the alphabetical index under "Newton." Consult also Michel Chasles, *Aperçu historique sur l'origine et le développement des méthodes en géométrie*; R. H. Graham, "Newton's Influence on Modern Geometry," *Nature*, vol. 42, 1890, pp. 139-142; C. Taylor, "The *Principia* and Modern Geometry," *Cambridge Philosophical Society Proceedings*, vol. 3, 1880, pp. 359, 360.

Newton's Lemma xxii aims mainly to transfer certain points in the figure to infinity.

21 (p. no). Book i, Lemma xxviii. *Oval figures*. Brougham and Routh¹ point out that in this Lemma Newton tried to prove that no figure oval in form, or no continuous curve confined to the finite part of a plane and returning into itself, is capable of definite quadrature. In other words, Newton endeavored to show that the area of such a curve cannot be expressed in terms of rational numbers or of irrationals arising from the solution of algebraic equations with rational coefficients, but can be expressed only in transcendental numbers such as Π , which arises as a factor in the area of a circle. Brougham and Routh state that the conclusiveness of Newton's reasoning has been questioned and they cite, as an example, the curve,

$$y^m = x^{(n-1)^m} (a^2 - x^n),$$

where m and n are even positive integers; this curve satisfies Newton's specifications, nevertheless admits of exact quadrature. Consider, for simplicity, the special statement when $m = n - 2$ and a is a positive integer. The curve $y^2 = x^2(a^2 - x^2)$

has, for the segment $0 < x < a$, the area,

$$A = \int_0^a 2y \, dx = -\frac{1}{3}(a^2 - x^2)^{3/2} \Big|_0^a = \frac{1}{3}a^3 - \frac{1}{3}(a^2 - x^2)^{3/2}.$$

Removing the radical, one obtains an algebraic equation of the second degree with respect to the required area A . Hence the area of this segment, though in general irrational, is not transcendental for algebraic values of x . If $x = a$, one has as the area of the entire oval $V\sqrt{cf}$, which is rational.

Newton's passage, "The equation by which any one intersection of two lines is found at the same time exhibits all their intersections by as many roots, and therefore rises to as many dimensions as there are intersections," is open to objection in its application to the spiral, a transcendental curve.

¹ H Brougham and E J Routh, *Analytical View of Sir Isaac Newton's Principia*, London, 1855, pp 72-74. See also Ball, *Essay*, p 83

22 (p. 113). Book i, Prop, xxxi and Scholium. *Kepler's problem*. Newton deals here with a famous problem first arising in John Kepler's *Astronomia nova*,¹ 1609. It calls for the determination of x in the equation

$$x - e \sin x = z,$$

where e and z are given. The astronomer J. C. Adams described in an article" two rapid methods of approximation, and then points out, "that the first method explained above is exactly equivalent to that given by Newton in the *Prinapia* at pp. 101,102 of the second edition, and at pp. 109, no [first part of the Scholium] of the third edition, when Newton's expressions are put into the modern analytical

form. None of the subsequent authors, however, mention this method as being Newton's, the unusual form in which Newton's solution is given having, no doubt, caused them to overlook it. In the first edition of the *Principia* a modification of the method is given which was, I have no doubt, intended by Newton to be equivalent to the second method given above, but by some inadvertence, instead of the denominator of hx' being

$$1 - e \cos(x' + \frac{1}{2}e \sin x')$$

when expressed in the above notation, he takes it to be what is equivalent to

$$1 - e \cos(x + \frac{1}{2}e \sin x'),$$

which is only true for the first approximation when x_0 is taken = z "

The first method referred to is described by Adams thus:

"The equation to be solved by successive approximations is

$$x - e \sin x = z,$$

where z is the known mean anomaly, e the eccentricity, and x the eccentric anomaly to be determined.

"Suppose x_0 to be an approximate value of x , found whether by estimation, by graphical construction, or by a previous rough calculation, and let

$$x_0 - e \sin x_0 = z_0.$$

Then if

$$\delta x_0 = \frac{z - z_0}{1 - e \cos x_0}$$

and

$$x' = x_0 + \delta x_0,$$

x' will be a much more approximate value of x than x_0 .

"Similarly, if we put

$$x' - e \sin x' = z'$$

and if

$$\delta x' = \frac{z - z'}{1 - e \cos x'}$$

and

$$x'' = x' + \delta x'$$

x'' will be a much more approximate value of x than x' , and so on."

Kepler's problem has engaged the attention of many astronomers; 123 papers are listed in a bibliography of the problem in the *Bulletin Astronomique*, Paris, vol. 17, 1900, pp. 37-47. Several tables for solving the problem are extant, the best of which is J. J. Astrand's *Hulfstafeln zur leichten und genaue Auflosung des Kepler'schen Problemes*, Leipzig, 1890.

¹ *Opera*, vol. 3, p. 411; *Kepler's Neue Astronomic*, pp. 358, 414, 52 *

² J. C. Adams, "On Newton's Solution of Kepler's Problem," *Monthly Notices of the Royal Astronomical Society*, London, vol. 43, 1882, pp. 43-49.

23 (p. 141). Book 1, Prop. XLV, Cor. 11. *Motion of the apsides in orbits*. In the second and third editions of the *Principia* occurs the sentence, "The apse of the moon is about twice as swift" (*Apsis lunae est duplo velocior arciter*), that is, twice as swift as the angle $1^\circ 31' 28''$, which marks the forward motion of the apse in one revolution of the body in its elliptical orbit. It is noteworthy that this sen-

fence, "The apse of the moon is about twice as swift," does not occur in the first edition of the *Principia*, and that the angle given there is $1^{\circ} 31' 14''$. We have here the celebrated discrepancy between the computed and the observed motion of the apsides of the moon. The observed value is about double the computed value. The computation is based upon the addition of $1/7357.45^{\text{th}}$ part of the centripetal force, by which the earth attracts the moon, this addition being caused by the solar attraction upon the moon.

The moon problem commanded the attention of Newton for many years.¹ About 1694 he obtained some observational data from the astronomer Flamsteed at Greenwich, for the purpose of verifying the results of his mathematical inquiries, but desired more. Flamsteed, on the contrary, was more inclined to hold back his observations for a while until all should appear in book form, as a monument of his labors as an observational astronomer. This conflict of interests, combined with Newton's irritability at that time in personal matters, and Flamsteed's sensitiveness because of serious physical weakness, led to the regrettable controversy between these two great men, in which many hard words were used.²

The discrepancy between theory and observation in the motion of the lunar apsides seems to have been overcome by Newton, but nowhere in any edition of the *Principia* does he explicitly give the explanation, although in Book I, Prop. xxv, he gives figures which would make the annual progression of the line of the apse nearly what it actually is. There, Newton estimates that the total disturbing force of the sun on the moon is to the centripetal force of the earth in the ratio of 1 to $178^{29/40}$; this ratio yields fairly close results for the apsidal motion.

That Newton had overcome the chief difficulty in the motion of the lunar line of the apsides follows also from his remark in Book I, Prop. XXIII: "But the motions of the apsides thus found must be diminished in the proportion of 5 to 9, or of about 1 to 2, on account of a cause which I cannot here stop to explain" (*Diminui tamen debet motus augis sic inventus in ratione 5 ad 9 vel 1 ad 2 archer ob causam quam hic exponere non vacat*). Pemberton, who edited the third edition of the *Principia*, suggested that there be added "a brief hint at the principle whence the precept contained in this line was deduced," but Newton did not act on this suggestion. Adams remarks that if Newton had complied with this suggestion, "all the difficulties connected with the motion of the moon's apogee would have been avoided."⁸

In the *Portsmouth Collection of Books and Papers, Cambridge*, 1888, pp. xi-xiii, xxvi-xxx, and in section 1, division ix, numbers 7, 12, there are elaborations⁴ which according to Ball did not seem to have satisfied Newton, if one may judge by the alterations made in the manuscript, but which nevertheless yield as the mean annual motion of the apogee $38^{\circ} 51' 52''$, while the observed motion was $40^{\circ} 41' 30''$.

Astronomers at that time had no means of knowing what results, if any, Newton had reached privately; all they could know were the published results as found in the *Principia*. The motion of the lunar apsides stood out as a problem not fully accounted for, on Newton's law of gravitation. Nearly twenty years passed before any substantial advance was made, but, when finally under way, it came in a sensational manner. The French mathematician, Alexis Claude Clairaut, in 1752 gained a prize of the St. Petersburg Academy for a paper on the theory of the moon, which explained the motion of the apsides.⁶ He was the first to apply modern analysis to the theory of the moon. At first the motion of the apsides seemed to Clairaut wholly inexplicable by Newton's law. He started boldly to advance a new hypothesis regarding gravitation, by assuming the centripetal force

to be not $\text{aj} \frac{1}{r^2}$, but as $\frac{1}{r^2} + \frac{1}{r^3}$. But before proceeding far on this new theory he took

the precaution to carry his calculation, using Newton's law, to a higher degree of approximation, and, behold, in 1749, he reached results agreeing with observation. The agreement was rightly looked upon as a brilliant triumph for the Newtonian theory. A full investigation of the problem was given by Laplace in his *Mecanique celeste*?

Of interest is a personal remark of Newton's, in connection with lunar theory; he told Halley that the theory of the moon made his head ache and kept him awake so often that he would think of it no more.⁷ But we have shown that Newton did continue to give attention to the irregularities of the moon. Lunar theory has caused headaches to astronomers of later date. In our own century, such moon specialists as E. W. Brown and W. de Sitter have found unexpected and unexplained shifts of the moon from its predicted positions. The moon's motion has not yet been exactly explained by gravitational theory. It has been conjectured that the cause of the differences between the prediction and observation lies in the variation of the earth's rate of rotation, probably caused by slight periodic changes in the shape of the earth.⁸

¹ For a historical paper by S. B. Gaythorpe, "On Horrocks's Treatment of the Eviction and the Equation of the Centre, with a Note on the Elliptic Hypothesis of Albert Curtz and its Correction by Boulliau and Newton," containing several references to the different editions of Newton's *Prinapta*, see *Monthly Notices of the Royal Astronomical Society*, London, vol. 85, 1925, pp. 858-865.

² See F. Baily, *An Account of the Revd. John Flamsteed*, London, 1835, pp. 69, 217, Brewster, *op at* pp. 163-183.

³ Brewster, *op. at.*, vol. 2, p. 418.

⁴ See also Ball, *Essay*, pp. 85, 109.

⁵ Clairaut, *Theorie de la lune*, Petersbourg, 1752, second edition, Paris, 1765, where, on page 5, are given historical notes. For Clairaut's earlier papers see *Memoires de l'Academie des sciences* for the year 1745, Paris, 1749, p. 577, also a still earlier paper on pp. 329-364, and a criticism of this by Buffon on pp. 493-500, Clairaut's reply to Buffon on pp. 529-548 and Buffon's replication, pp. 551-552. Clairaut wrote on the orbit of the moon in these *Memoires* for the year 1748, Paris, 1752, pp. 421-440. See also Brougham and Routh, *Analytical View of Newton's Prinapia*, pp. 89-98; E. W. Brown, *An Introductory Treatise on the Lunar Theory*, Cambridge, 1896, pp. 2, 6, 127, 237, 238.

⁶ P. S. Laplace, *Mecanique celeste*, 1799-1825, Tome in, Livrc VII, pp. 169-303, (*Euvres de Laplace*, Tome III, Paris, 1878, pp. 181-323).

⁷ Brewster, *op. cit.*, p. 108.

⁸ Consult P. A. Hansen, "Darlegung der theor. Berechnung d in den Mondtafeln angewandten Storungen," *Abhandl. d. math.-phys. Classe der k sachsich Gesellschaft d Wissenschaften*, Leipzig, vol. 6, 1864, vol. 7, 1865, Simon Newcomb, *Astronomical papers for the use of the American Ephemeris*, vol. 9, Pt. I, p. 18; papers in the *Monthly Notices of the Royal Astronomical Society*, London, by E. W. Brown, vol. 73, 1913, p. 692, vol. 74, 1914, pp. 156, 362, 396, 424, 552, vol. 75, 1915, p. 506; and by A. C. D. Crommehn, vol. 83, 1923, p. 359. Also the article, "Moon," by J. Jackson in the *Encyclopaedia Britannica*, ed. 14.

24 (p. 167). Book 1, Prop. LX. *Two mean proportionals*. If $a : x : x : y : y : b$, then x and y are said to be "two mean proportionals" between a and b | x is said to be the "first" of the two mean proportionals. We get $x^2 = ay$ and $y^2 = bx$. Eliminating y , $x = 3(a^2b)$. If $a = S4 - P$ and $b = S$, then, in Prop. LX, the principal axis of P's ellipse when S moves, is to the principal axis of P's ellipse when S is stationary (the periodic time of P remaining the same), as $S + P : y : (S + P)^2 S$.

25 (p. 197). Book 1, Prop. LXXV; also Book in, Prop. vm. *Attraction between spheres*. Newton's determination of the attraction of a solid sphere on an external point is more fully explained, by the aid of modern analysis, in a paper by A. N. Kriloff in the *Monthly Notices of the Royal Astronomical Society*, London, vol. 85, 1925, p. 571.

The theorem that two spheres attract each other by a force varying inversely as the square of the distance between their centers has been shown by James Pierpont to hold also in elliptic space. See Pierpont's paper in the *Bulletin of the American Mathematical Society*, vol. 35, 1929, pp. 351-356.

26 (p. 228). Book 1, Prop. xcvi. *Velocity of light*. Newton arrives at the conclusion that the velocity of light in a denser medium is greater than in a rarer medium. But Jean Leon Foucault proved in 1850 that the velocity of light is less in water than in air. This was looked upon by nineteenth-century physicists as an *experimentum crucis* which definitely disproved Newton's emission theory. The twentieth century led to a revival of corpuscular theories of light, and Foucault's experiment came to be no longer considered decisive. Newton had assumed in the *Principia* that the corpuscle of light is "attracted or impelled perpendicularly towards either of those mediums, and not agitated or hindered by any other force." To arrive at a result in accordance with Foucault's experiment, it is only necessary to assume that the corpuscle meets with a different reception. As Alex. Wood¹ pointed out, it may be that the flying corpuscle is not attracted at all; it may be that when it reaches the surface of the water, the component of its velocity parallel to the surface is *diminished* by action of a frictional kind, while the component of its velocity perpendicular to the surface remains unchanged. According to such an assumption, the velocity of the corpuscle in water would be less than in air, as demanded by experiment. In 1924 the theory of "wave mechanics" was advanced, which endeavors to unify the corpuscular theory and the

wave theory of light. For an explanation of Foucault's experiment on the theory of "wave mechanics," see Ludwig Flamm in *Die Naturwissenschaften*, vol. 15, July 15, 1927.

¹ A. Wood, *In Pursuit of Truth*, London, 1927, p. 47.

27 (p. 230). Prop. xevi, Scholium. *Diffraction of light*. The phenomenon which Newton describes as "inflection" of light is now called "diffraction" of light.

28 (p. 236). Book II, Prop. II. *Resistance proportional to velocity*. If, for the successive equal time intervals, the velocities are v_1, v_2, v_3, v_4 , etc., and the resistances are cv_1, cv_2, cv_3, cv_4 , etc., then, $v_2 = v_1 - cv_1$, $v_3 = v_2 - cv_2$, $v_4 = v_3 - cv_3$, etc. Therefore,

$$1 : c = v_1 : v_1 - v_2 = v_2 : v_2 - v_3 = v_3 : v_3 - v_4 = \text{etc.},$$

and by Lem. 1, Book II, $v_1 : v_2 = v_2 : v_3 = v_3 : v_4 = \text{etc.}$,

and

$$v_1 : v_2 = v_1 v_2 : v_2 v_3 = 1 : c^2,$$

$$v_1 : v_4 = v_1 v_2 v_3 : v_2 v_3 v_4 = 1 : c^3, \text{ etc.}$$

Also,

$$v_2 = v_1(1 - c), v_3 = v_1(1 - c)^2, \dots, v_{n+1} = v_1(1 - c)^n.$$

29 (p. 247). Book II, Prop. VII. *Resistance of spheres*. Let the initial velocity of a body be V and its subsequent velocity v , also let the initial resistance be R and its subsequent resistance be r , and t the time. For a second body, let the corresponding symbols be V_1, v_1, R_1, r_1 , and t_1 . By supposition,

$$V^2 : v^2 = R : r$$

$$V_1^2 : v_1^2 = R_1 : r_1$$

$$t : t_1 = \frac{V}{R} : \frac{V_1}{R_1}.$$

Then the loss in velocity, $-dv$, of the first body in time, dt , is

$$-dv = r dt = \frac{Rv^2}{V^2} dt.$$

Dividing by v^3 and integrating,

$$v^{-1} = \frac{R}{V^2} t + C.$$

When $t = 0$, $v = V$; hence, $C = V^{-1}$.

Simplifying,

$$v = \frac{V^2}{Rt + V}.$$

Hence the loss of velocity is,

$$V - v = \frac{VRt}{Rt + V}.$$

For the second body,

$$V_1 - v_1 = \frac{V_1 R_1 t_1}{R_1 t_1 + V_1} = \frac{V_1 R t}{R t + V}.$$

Therefore the losses in velocities of the two bodies in times t and t_1 are to each other as their initial velocities V and V_1 .

To prove the second part, relating to the spaces s , use the value given above for v , in $ds = vdt$; integrating,

$$s = \frac{V^2}{R} \left\{ \log (Rt + V) - \log V \right\}.$$

Writing the corresponding formula for the second body, and simplifying, one obtains

$$s : s_1 = Vt : V_1 t_1. \quad \text{Q.E.D.}$$

30 (p. 249). Book 11, Lemma 11. *Fixed infinitesimals*. For the benefit of readers interested in the early history of the calculus in England, I quote certain parts of Augustus De Morgan's article "On the Early History of Infinitesimals in England," published in the *Philosophical Magazine*, ser. 4, vol. 4, 1852, pp. 321-330. De Morgan proves that in Newton's early papers and in the first edition of the *Principia*, infinitely small quantities (that is, fixed infinitesimals) are freely used: "Up to the year 1704, and so far as algebraical calculus was concerned, Newton himself used infinitely small quantities; and nothing else in any document yet published. The prime and ultimate ratios, or limits, appear in the *Principia*, but are abandoned in those places in which fluxions are alluded to. I proceed to establish these assertions in detail." I premise that a fluxion, as used by Newton, is a velocity, or a time-derivative. The fluxion which Newton denoted by $\dot{x}$ is now written $\frac{dx}{dt}$. In the *Principia* Newton's dot-notation for fluxions does not occur.

I quote further from De Morgan: "In the first edition of the *Principia* (1687) the description of the fluxions is founded on infinitesimals, and in the second (1713) this foundation is somewhat altered. In the first, moments are *infinitely small quantities*; in the second, it is not clear what else they are. As in the following extract from the first edition, with its substitute in the second:

First Edition (Book 11, Lemma 11)

"Cave tamen intellexens particulas finitas. Momenta, quam primum fimitae sunt magnitudinis, destnunt esse momenta. Fimri emm repugnat altquatenus perpetuo eorum incremento vel decremento. Intelhgenda sunt principia jam-jam nascentia finitarum magnitudinum."

Second Edition (ditto)

"Cave tamen intellexeris particulas finitas. Particulae fimitae non sunt momenta sed quantitates ipsae ex momentis gemtae. Intelhgenda sunt principia jam-jam nascentia finitarum magnitudinum."

I venture the following translations:

But take care not to look upon finite particles as such. Moments, as soon as they are of finite magnitude, cease to be moments To be given finite bounds is in some measure contrary to their continuous increase or decrease. We are to conceive them as the just nascent principles of finite magnitudes.

But take care not to look upon finite particles as such. Finite particles are not moments, but the very quantities generated by the moments. We are to conceive them as the just nascent principles of finite magnitudes.

De Morgan comments: "Through the difficulty of the phrases in both extracts this much distinctly appears, that in the first edition the moments, or momentaneous increments, are infinitely small quantities: and this is what I assert____ Newton commenced with the infinitesimal system in as absolute form as did Leibniz, so far as infinitely small quantities of the first order are concerned. Further than these he did not go; and the early distinction between the systems of the two is this, that Newton, holding to the conception of the velocity or fluxions, used the infinitely small increment as a means of determining it; while, with Leibniz, the relation of the infinitely small increments is itself the object of determination."

In 1704 appeared Newton's *Quadratura curvarum* with an Introduction explaining the foundations of fluxions. Here Newton tried to avoid all use of infinitely small constants. We behold in this publication the high-water mark of Newton's efforts to place the doctrine of fluxions upon a thoroughly logical basis. He said, "The very smallest errors in mathematical matters are not to be neglected." De Morgan remarks, "In 1704 Newton in the *Quadratura curvarum* renounced and abjured the infinitely small quantity; but he did it in a manner which would lead one to suppose that he had never held it."¹

In all three editions of the *Pnncipia* which appeared within Newton's lifetime, quantities that are infinitely small in comparison to others, are dropped. See, for example, Book 1, Prop, xxxix, Cor. III; Prop, XLV, Exam. 2 and 3; also Book 11, Prop, x and xv.

¹ For fuller discussion see F. Cajon, *A History of the Conceptions of Limits and Fluxions in Great Britain*, Chicago, 1919, pp 2-32.

31 (p. 250). Book 11, Lemma 11. *Fluxions in the Principia*. At this place also in Book 1, Lemmas 1, 11, xi (Scholium), Newton makes a brief general statement of the principles of fluxions and fluents (that is, of the differential and integral calculus), but he does not give his notation. No one is known to have obtained from the *Prinapta* alone a working knowledge of fluxions. It is generally supposed that he deduced many of his theorems in the *Principia* with the aid of his theory of fluxions and fluents, and afterwards translated his results into the syn-

thetic form. It is interesting to note that in the *Principia* itself there is a confirmation of this supposition. In Lemma 11 of Book III, he twice gives "a fluent quantity, whose fluxion is ____"

32 (p. 251). Book 11, Lemma 11, Cor. 1. If A, B, C, D, E are *continually proportional*, then, by definition,

$$A : B = B : C = C : D = D : E.$$

Suppose C is given, and suppose $D = CX$, then

$$A = CX^{-2}, B = CX^{-1}, D = CX, E = CX^2.$$

The moments are

$$a = 2CX^{-2}x = -2A \cdot \frac{x}{X}$$

$$b = -CX^{-1}x = -B \cdot \frac{x}{X}$$

$$d = Cx = D \cdot \frac{x}{X}$$

$$e = 2CXx = 2E \cdot \frac{x}{X}.$$

Hence, $a : b : d : e = -2A : -B : D : 2E$.

33 (P. 251). Book 11, Prop vii, Scholium *Newton and Leibniz on the invention of the calculus*. In the first two editions of the *Principia*, the Scholium to Prop vn is wholly different from the Scholium given in the third edition (1726) and translated by Motte. These changes are important in the history of the controversy on the priority and independence of the invention of the differential calculus by Newton and Leibniz. The Scholium in the first edition was as follows:

"In letters which went between me and that most excellent geometer, G. W. Leibniz, ten years ago, when I signified that I was in the knowledge of a method of determining maxima and minima, of drawing tangents, and the like, and when I concealed it in transposed letters involving this sentence (*Data acquatione quotcunque; fluentes quantitates involvente, fluxiones invenire, et vice versa*; that is, Having any given equation involving never so many flowing quantities, to find the fluxions, and vice versa) that most distinguished man wrote back that he had also fallen upon a method of the same kind, and communicated his method, which hardly differed from mine, except in his forms of words and symbols."

It seems to me that Newton's statement, "which hardly differed from mine, except in the forms of words and symbols," overstates the identity of the Newtonian and Leibnizian systems; there was a difference in the fundamental concepts. Newton, at the time when he wrote the *Principia*, used infinitely small quantities for the purpose of finding *fluxions* (velocities or time-derivative); Leibniz made infinitely small quantities themselves the fundamental concepts, in his *differentials*

of different orders. In 1713, when the second edition of the *Principia* appeared, the controversy on the invention of the calculus was well under way; the *Cotnamerium epistolicum* of John Collins, stating the English side of the dispute, had appeared in 1712. Nevertheless, the Scholium of the first edition of the *Principia*, quoted above, was allowed to stand in the second edition, the addition of the words "and the concept of the generation of quantities" (*et idea generationis quantitatum*) excepted, so that the last clause now read: "which hardly differed from mine, except in the forms of words and symbols, and the concept of: the generation of quantities." This addition appears to me to add greater precision to the statement.

The difference between Newton and Leibniz in "the concept of the generation of quantities" was greater in 1704 when Newton's *Quadratura curvarum* appeared in print than in 1687. In the Introduction to this publication, Newton said: "I consider mathematical quantities in this place not as consisting of very small parts, but as described by a continuous motion." Thus Newton's exposition of fundamentals in 1704 was somewhat different from that in 1687; no use was made of infinitely small constant quantities in 1704. Therefore Newton's addition to the Scholium in the *Principia* of 1713, "and the concept of the generation of quantities," was very much to the point.

As the controversy progressed, Leibniz, in a letter (April 9, 1716) to an Italian priest, A. S. Conti, then residing in London, reminded Newton of the admission he had made in the Scholium, which Newton seemed now desirous of disavowing. Leibniz died in 1716. Soon after, on November 14, 1716, Newton published in Raphson's *Fluxions* (1716, date of tide, 1715) the following statement: "He [Leibniz] pretends that in my book of *Principia* I allowed him the invention of the *Calculus differentials*, independently of my own; and that to attribute this invention to myself, is contrary to my own Knowledge. But in the paragraph there referred unto I do not find one word to this purpose." With reference to Newton's admission in the Scholium, De Morgan remarks that Newton was weak enough, "first to deny the plain and obvious meaning, and secondly, to omit it entirely from the third edition of the *Principia*."

When in 1726 the third edition of the *Principia* was issued, there appeared in place of the Scholium of the previous editions the new Scholium, which has been printed in all later editions of the *Principia*. This new Scholium makes no mention of Leibniz. On the controversy over the invention of the calculus, see De Morgan, *Essays on the Life and Work of Newton* (ed. P. E. B. Jourdain), Chicago, 1914; M. Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. 3, ed. 2, Leipzig, chap. 92, pp. 233-261; also other histories of mathematics.

34 (P. 257). Book II, Prop. x. *Corrections by Johann and Nicolaus Bernoulli*. On October 14, 1712, Newton wrote to Cotes: "There is an error in the tenth

Proposition of the second Book, Probl. III, which will require the reprinting of about a sheet and an half! I was told of it since I wrote to you, and am correcting it. I will pay the charge of reprinting it, and send it to you as soon as I can make it ready."¹

Edleston gives additional information on that matter. This error in the first edition of the *Principia* arose in finding the value of the resistance to the motion of a projectile in the air; it was pointed out to Newton by Nicolaus Bernoulli (1687-1759), the nephew of Johann Bernoulli (1667-1748); Nicolaus was on a visit to England during September and October, 1712. Newton's result, when the curve described is a circle, had been previously shown to be erroneous by Johann Bernoulli in a letter to Leibniz of August, 1710, and in a communication to the French Academy, published in its *Memoires* for 1711, Paris, 1714, pp. 50-56.² Newton sent to Johann Bernoulli, by Nicolaus, a copy of his *Analysts per quantitatum series, fluxiones, ac differentias*, published in 1711, and proposed him (Johann Bernoulli) as a member of the Royal Society, into which he was accordingly elected on the first of the December following.

¹ Edleston, *op. cit.*, p. 142.

² See also the second edition of the *Histoire de l'Académie royale des sciences*, *Memoires*, année 1711, pp. 50-54; and the paper by Nicolaus Bernoulli, pp. 54-56.

35 (P- 333)- Third edition, Book 11, Prop, xxxiv, Scholium. *Surface of least resistance*. In the first edition of the *Principia* there are nine Corollaries to Prop, xxxm; in the second and third editions, only six Corollaries, and the sixth is rewritten. After Prop, xxxiii, up to Prop, XLI, there is much revision in the second edition, and the results of new experiments are introduced. Prop, xxxv of the first edition becomes Prop, xxxiv of the second and third editions. The Scholium following Prop, xxxv in the first edition, dealing with the surface of least resistance, is the same (unimportant minute changes excepted) as the Scholium which follows Prop, xxxiv in the second and third editions.

In this Scholium Newton gives without proof the geometrical conditions which a surface of revolution must satisfy, that it may move with the least resistance in the direction of its axis through a resisting medium. This is the earliest problem of the kind that is now solved by the calculus of variations. In the Portsmouth Collection,¹ there is the draft of a letter to a correspondent at Oxford, probably David Gregory, and written about 1694, in which Newton explains his method of solution. I reproduce that letter, making only slight changes in notation such as $(Nn)^2$ for " Nn^2 ," $\frac{dq}{dt}$ for " $\dot{q}$," and the use of parentheses in place of the vinculum.

"Sir, I now thank you heartily both for the very kind visit you made me here and for the errata you gave me notice of in my book.... The figure which feels

the least resistance in the Schol. of Prop. xxxiv, Lib. n, is demonstrable by these steps.

" i . If upon BM be erected infinitely narrow parallelograms $BGhb$ and $MNom$ and their distance Mb and altitudes MN , BG be given, and the semi-sum of their bases $\frac{Mm+Bb}{2}$ be also given and called S and their semi-difference $\frac{Mm-BG}{2}$

be called x : and if the lines BG , bh , MN , mo , butt upon the curve $nNgG$ in the points n , N , g , and G , and the infinitely little lines on and hg be equal to one another and called C , and the figure $mnNgGB$ be turned about its axis BM to generate a solid, and this solid move uniformly in water from M to B according to the direction of its axis BM : the summ of the resistances of the two surfaces generated by the infinitely little lines Gg , Nn shall be least when $(gG)^*$ is to $(»N)^*$ as $BG \times Bb$ to $MN \times Mm$.

"For the resistances of the surfaces generated by the revolution of Gg and Nn are as $\frac{BG}{(Gg)^2}$ and $\frac{MN}{(Nn)^2}$, that is, if $(Gg)^2$ and $(Nn)^2$ be called p and q , as $\frac{BG}{p}$

and $\frac{MN}{q}$ and their sum $\frac{BG}{p} + \frac{MN}{q}$ is least when the fluxion thereof

$$-\frac{BG}{p^2} \cdot \frac{dp}{dt} - \frac{MN}{q^2} \cdot \frac{dq}{dt} \text{ is nothing, or } -\frac{BG}{p^2} \cdot \frac{dp}{dt} = \frac{MN}{q^2} \cdot \frac{dq}{dt}.$$

"Now $p = (Gg)^2 = (Bb)^2 + (gh)^2 = s^2 - 2sx + x^2 + c^2$ and therefore

$$\frac{dp}{dt} = -2s \frac{dx}{dt} + 2x \frac{dx}{dt},$$

and by the same argument $\frac{dq}{dt} = 2s \frac{dx}{dt} + 2x \frac{dx}{dt}$ and therefore

$$\frac{BG \times \left(2s \frac{dx}{dt} - 2x \frac{dx}{dt} \right)}{p^2} = \frac{MN \left(2s \frac{dx}{dt} + 2x \frac{dx}{dt} \right)}{q^2},$$

or $\frac{BG(s-x)}{p^2} = \frac{MN(s+x)}{q^2}$ and thence p^2 is to q^2 as $BG(s-x)$ to $MN(s+x)$,

that is, $(gG)^4$ to $(nN)^4$ as $BG \times Bb$ to $MN \times Mm$.

"2. If the curve line $DnNgG$ be such that the surface of the solid generated by its revolution feels the least resistance of any solid with the same top and bottom BG and CD , then the resistance of the two narrow annular surfaces generated by the revolution of the [infinitely little lines «N and] Gg is less than if the intermediate solid $bgNM$ be removed [along CB without altering Mb , until bg comes] to BG [supposing as before that on is equal to hg and by consequence is the least that can be], and therefore $(gG)^*$ is to $(nN)^4$ as $BG [x Bb$ is to $MN \times Mm]$.

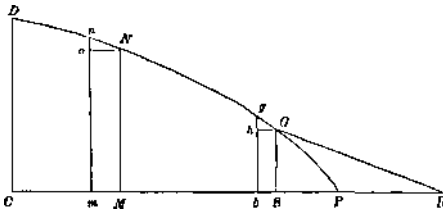
"*{Also if] $g\dot{h}$ be equal to $\dot{h}G$ so that the angle [$g\dot{C}h$ is 45^{deg}] then will $4(B\dot{b})^4$ be [to $(nN)^4$ as $BG \times B\dot{b}$ is to] $MN \times M\dot{m}$, and by consequence $4(BG)^4$ is to $(GR)^4$ as $(BG)^2$ is to $MN \times BR$ or $4(BG)^2 \times BR$ is to $(GR)^2$ [as GR to MN].

"•If the altitude of the frustum of the cone spoken of in the preceding paragraph be infinitely small, the semi-angle of the cone becomes equal to 45° . Hence when the total resistance is a minimum, the curve meets the extreme ordinate GB at an angle of 45° .

"Whence the proposition to be demonstrated easily follows."

The foregoing is Newton's draft of the letter, kept in the Portsmouth Collection, as amended in some defective parts of the manuscript by the interpolation of words placed in brackets. These interpolations were made by the editors of the *Catalogue* of the Portsmouth Collection. The brackets as printed in the *Catalogue* are not all properly inserted; I have followed the placing of them as determined by Oskar Bolza of Freiburg in Baden, who had secured from Arthur Berry of Cambridge (England) a facsimile of the original manuscript.

Bolza made an intensive study of Newton's solution² and arrived at the conclusion that the words "until bg comes," inserted by the editors of the *Catalogue* of the Portsmouth Collection, are not quite satisfactory. I reproduce here in translation part of Bolza's article in which the figure and the assumptions made in Newton's letter are followed:



"Then displace the arc Ng by an infinitely small distance K parallel to the axis CB to the new position $N'g'$ and draw the straight lines $N'n$, Gg . Then, in general, for any arc A lying in the plane of the figure, we designate by (A) the resistance of the surface generated by the revolution of A about the axis BC , we obtain

$$(DnN'g'GB) > (DnNgGB) \dots\dots\dots (1)$$

Since

$$(N'g') = (Ng),$$

there follows from (1)

$$(nN') + (g'G) > (nN) + (gG)^1 \dots\dots\dots (2)$$

But the expression $(nN') + (g'G)$, being a function of K , must have a minimum for $K = 0$. This is an ordinary problem in minima which Newton solves in the first part of his proof by the procedure still common at the present time, it being

only necessary to remark that in the expressions for («N) and (gG) he neglects at the outset the magnitude *hg-on* as infinitely small in comparison with BG and MN.

"By solving the minimal problem, Newton obtains the relation

$$\frac{BG \cdot Bb}{(Gg)^4} = \frac{MN \cdot Mm}{(Nn)^4} \dots \dots \dots (3)$$

which, according to modern phrasing, is nothing else than the well-known first integral of the Eulerian differential equation for the variation problem before us. For, if one designates

$$\begin{array}{ll} \angle gGh = S_0, & \angle nN_0 = S, \\ \cot g S_0 = q_0, & \cot g S = q, \\ BG = y_0, & MN = y, \end{array}$$

and expresses the distance *Bb*, *Gg*, *Mm*, *Nn* in terms of *hg=on*, then equation (3) becomes

$$\frac{yq}{(1+q^2)^2} = \frac{y_0q_0}{(1+q_0^2)^2} \dots \dots \dots (4)$$

or

$$\frac{yq}{(1+q^2)^2} = \text{constant}, \dots \dots \dots (5)$$

and this is in fact the first integral of the Eulerian differential equation for our problem,

$$\int_{y_0}^{y_1} \frac{y dy}{1+q^2} = \text{extreme} \dots \dots \dots (6).$$

But Newton advances an important step by determining the slope of the curve at the point G and thereby the constant of integration in (5).

"For in the Scholium referred to above, he had given the following theorem: To obtain the frustum of the cone of given radius OC of the base and of given altitude OD which meets the least resistance in the direction of the axis OD, bisect the altitude OD in Q, produce OQ to S so that QS = QC. Then S is the vertex of the required frustum of the cone.

"When OD becomes infinitely small, then angle SCO approaches the value 45° . From this Newton concludes, what appears as self-evident on the conception of a curve as a polygon of an infinite number of infinitely small sides, that for the solid of revolution of least resistance, the tangent at the point G makes with the axis BC an angle of 45° , that therefore $\phi_0 = 45^\circ$. Equation (4) becomes therefore

$$\frac{yq}{(1+q^2)^2} = \frac{y_0}{4} \dots \dots \dots (7)$$

But this is the theorem in Newton's *Principia*:

$$\frac{MN}{GB} = \frac{(GR)^3}{4BR \cdot (GB)^2} \dots \dots \dots (8)$$

(where GR is parallel to the tangent to the curve DG, at N), translated from geometrical to analytical form.

"Interesting in the calculus of variations is the peculiar mode of variation of the curve DG, used by Newton, which reminds one of the thoughts of Du Bois-Reymond³ on the fundamental lemma of the calculus of variations.

"The success of this mode of variation lies in the absence of the variable x under the integral (6)."

Bolza proceeds to show how this process may be extended to a more general problem in the calculus of variations.

An elaborate discussion of the solid of least resistance, including much historical material, is found in an article written by A. R. Forsyth.⁴

³A *Catalogue of the Portsmouth Collection of Books and Papers Written by or Belonging to Sir Isaac Newton*, Cambridge, 1888, pp xxi-xxii

²Oskar Bolza, in *Bibliotheca mathematica*, ser. 3, vol. 13, 1913, pp. 146-149.

³Du Bois-Reymond, in *Mathematische Annalen*, vol. 15, 1879, p. 313

* A. R. Forsyth, in *Isaac Newton, 1642-1727, A Memorial Volume*, ed W. J. Grinstead, London, 1927, pp-75-86

36 (p. 337). Third edition, Book 11, Prop, xxxvi. *Water running out of small orifices*. Of this difficult problem, Brougham and Routh state:¹ The investigation of this question, as given by Newton in his first edition, was very erroneous. He had totally neglected the contraction of the vein after the fluid had passed the orifice; hence he had deduced that the velocity of the efflux was that due to *half* the height of the water in the vessel. This mistake he afterwards corrected, but the investigation still remains open to very serious objections."

¹ *Op. cit.*, p. 264, see also p. 380.

37 (pp. 361, 362). Book 11, Prop, XL, Scholium, Exper. 13. *Time and angular measure*. In the measure of time, 1 sec. = 60 thirds; 1 third = 60 fourths. In angular measure, $1^\circ = 60'$, $1' = 60''$, $1'' = 60'''$, $1''' = 60^{iv}$.

38 (p. 378). Book 11, Prop, XLVIII-L. *Velocity of sound*. In Newton's theoretical deduction of the velocity of sound, he came to the conclusion that the velocity varied directly as the square root of the "elastic force" and inversely as the square root of the "density of the medium," and obtained as the velocity through air 979 feet per second, the experimental value being about 1142 feet. Newton threw out conjectures as to the cause of the discrepancy between the experimental and theoretical values, which rested on assumptions far from convincing. The true explanation was given, over a century later, by Laplace, who showed that, in respect of air, the "elastic force" must be multiplied by the factor 1.41, which is the ratio

of the specific heat of air at constant pressure and the specific heat at constant volume. By this correction, due allowance is made for the changes in elasticity arising from the heat compression and the cold rarefaction.

Notwithstanding Newton's failure to consider these variations of elasticity, his theoretical discussion is a masterpiece of theoretical deduction. Few have recognized this fact more fully than Laplace, who completed the theory. Laplace first published the correction, without demonstration, in the *Annales de physique et de chimie*, vol. 3, 1816, pp. 238-241, and remarked, "La maniere dont il y parvient est un des traits les plus remarquables de son genie." In his *Mecanique celeste*, 1798-1825, Livre xii, p. 95, he says: "Sa theorie, quoique imparfaite, est un monument de son genie."¹

¹ Consult also I Todhunter and K. Pearson, *A History of the Theory of Elasticity*, vol. 1, 1886, pp 161, 162, remarks on Newton's treatment of elastic bodies are found on pp 13, 14, 26, 93.

39 (p. 402). Book in, Phenomenon 1. *Huygenian telescope*. Newton refers here and in Book in, Prop, xix, to an instrument, a "123-foot telescope" used by the Reverend James Pound in 1719. Such long telescopes were used, before the invention of the achromatic lens by Dollond about the middle of the eighteenth century. They were designed by Christiaan Huygens, and several were constructed by him and his brother Constantijn. By increasing the length of the telescope, the blurring effect of images with colored edges was greatly reduced and at the same time the magnifying power was increased. Newton states in his *Opticas*, Book 1, Part 1, Prop, VII, Exp. 16, that "a telescope of 64 feet in length, with an aperture of $2\frac{1}{3}$ inches, magnifies about 120 times, with as much distinctness as one of a foot in length, with $\frac{1}{3}$ of an inch aperture, magnifies 15 times." Instruments were made as long as 300 feet. The magnifying power of a telescope is measured by the focal length of the object glass, divided by the focal length of the eyepiece. As it was not found possible at that time to construct serviceable eyepieces of very short focal length, the focal length of the objective was increased. Huygens presented to the Royal Society of London three lenses having focal lengths of 123, 170, and 210 feet, respectively. An account of a recent examination of these lenses is given in *Nature*, vol. 123, 1929, p. 655, also p. 575. It being impractical to construct telescopic tubes of such great lengths, Huygens discarded their use and adopted for nocturnal observations the "aerial" telescope. The objective was mounted on a high pole and set in line with the eyepiece by means of cords, but it was found difficult to get the eye and object glasses "married" or brought parallel to each other, and at the same time in line with the object to be observed.

The Royal Society of London loaned the 123-foot objective to the Reverend James Pound, who mounted it in Wanstead Park, Essex, on a Maypole which had been removed from the Strand and which had been procured by Newton himself

for mounting this telescope. Verses were affixed to the pole, by a local wit, which began:

Once I adorned the Strand
But now have found
My way to pound
In Baron Newton's land.

40 (p. 407). Book in, Prop. iv. *The earth-moon test of the law of gravitation*. Except the experimental constants used, this computation given in the *Prinapta* is probably the same as Newton's first computation of 1665 or 1666. This famous Prop. iv is the same in the three editions of the *Prinapta* (1687, 1713, 1726), except the list of astronomical authorities cited. New in the third edition is the Scholium which follows Prop. iv. In the first two editions there was no scholium at this place.

It is well known that in 1665 or 1666 Newton made his first test of the validity of the law of gravitation, and that he did not announce the law until about twenty years later. What was the cause of this long delay? The old reply to this question was that in 1665 or 1666 Newton had a very inaccurate value for the size of the earth, namely 60 miles to a degree of latitude, that, consequently, his computed and experimental values for the intensity of gravity on the earth's surface did not agree, and that the validity of the law did not appear until J. Picard's measurements of the earth in France, which yielded the value of 69 V3 miles to a degree of latitude. This explanation of Newton's delay in announcing the law of gravitation has recently been shown to be incorrect. It has been shown that fairly accurate values for the size of the earth were known since Snell's measurements of 1617 (yielding 66 $\frac{2}{3}$ English statute miles to a degree of latitude) and Norwood's of 1636 (yielding 69M such miles).

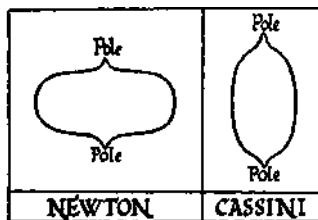
Did Newton in 1666 know Snell's or Norwood's values for the size of the earth? Strangely, Newton nowhere in his writings or letters mentions any particular numerical value previous to the appearance of his *Prinapta* (1687) in which he adopted the determination of Picard. Nevertheless, it is now definitely settled that Newton was familiar with Snell's value at least as early as 1672, and very probably in 1666. That Newton knew this value in 1672 is evident from his edition of Varen's *Geography*, published in that year. Varen gives in that book a small table of distances at which a mountain peak of known altitude can be seen at sea, disregarding refraction. Varen's computation is erroneous and is corrected by Newton. *That correction was made by the use of Snell's value*. Thus, a mountain height of one German mile could be seen at a distance of 29 $\frac{1}{4}$ German miles according to Varen, and of 41 $\frac{1}{2}$ German miles according to Newton.¹

What really delayed Newton for twenty years were theoretical questions of great difficulty relating to the attraction of a sphere upon an external point—a problem which he did not solve before 1684 or 1685, and which was first explained

in the *Principia*, Book i, Prop, LXXV and LXXVI, and Corollaries. The facts which bear on this question are fully set forth in an article in *Sir Isaac Newton, 1727-1927*, Baltimore, 1928, pp. 127-187. See also the *Principia*, Book III, Prop. vm.

¹ See *Mathematical Gazette*, vol. 14, 1929, p. 415.

41 (p. 427). Book in, Prop. xix. *Figure of the earth*. In France Jacques Cassini was misled, by somewhat inaccurate geodetic measurements taken over a comparatively short meridian, to the belief that the earth was elongated at the poles. The opposing theories of Newton and Cassini are strikingly shown in the following old caricature:



Later geodetic measurements made in Lapland by Maupertuis and Clairaut, when taken with those previously made in France, afforded data extending over a much longer meridian distance and showed a flattening of the earth at the poles, as demanded by Newton's theory of gravitation. Voltaire wittily designated Maupertuis as "Aplatisseur du monde et de Cassini." Later Voltaire and Maupertuis became involved in a heroic-comic controversy, and Voltaire (in his *Discours sur la moderation*) wrote,

Vouz avez confirme dans des lieux pleins d'ennui
Ce que Newton connut sans sortir de chez lui.

Detailed historical information is found in Isaac Todhunter's *Theories of Attraction and Figure of the Earth*, London, 1873, and in A. D. Butterfield's *History of the Determination of the Figure of the Earth from Arc Measurements*, Worcester, Mass., 1906.

On theoretical grounds, Newton found the diameter of the earth at the equator to be to the diameter from pole to pole as 230 is to 229, on the supposition that the matter of the earth was all "uniformly dense." The compression of the earth is accordingly $(230-229) \div 230$, or $1/230$. The latest experimental determination of the earth's compression, derived from pendulum observations on the intensity of gravity, yields the value $1/297.4$. See *United States Coast and Geodetic Survey, Its Work, Methods and Organization*, Special Publication No. 23, Washington, 1928, p. 127.

42 (p. 430). Book in, Prop. xx. *The toise*. 1 toise = 1.949+ meters = 6.394+ English feet.

43. (p. 433). Book in, Prop, XXII, xxv-xxxv. *The problem of three bodies*. In studying the lunar orbit under the influence of the earth and sun, the attractive forces acting according to the law of gravitation, Newton originated a particular statement of a famous general problem which came to be known in celestial mechanics as the "problem of three bodies." While the determination of the motion of two bodies considered as particles, when their mutual attractions are the only force considered, is a comparatively simple problem in mechanics, the addition of a third body gives rise to unexpected difficulties. No perfectly general solution of this problem has yet been found. Associated with the early history of the "problem of three bodies" after Newton are the names of D'Alembert, Clairaut, Euler, Lagrange, Laplace.¹

¹ For an outline history of this problem, including the nineteenth century, see E. O. Lovett, "The Problem of Three Bodies," *Science*, vol. 29, 1909, pp. 81-91.

44 (p. 435). Book in, Prop. xxvi. *A satellite of Jupiter*. By the "outmost satellite" of Jupiter is meant the outmost moon of the four moons discovered by Galileo; it is called Calisto. There are now nine moons of Jupiter known. One of them is inside the four moons found by Galileo; the remaining four moons of recent discovery have their orbits more remote from Jupiter than Calisto.

45 (p. 439). Book in, Prop. xxiv. *Time of the highest tide*. That the highest tide does not occur at the syzygy, but about three days later, Newton attributes to the "force of reciprocation" which the water once moved retains a little while by its inertia. But Laplace showed (*Mecanique celeste*, Livre xin, Chap. 1) that this is not true, that the greatest full tides should occur exactly at the syzygies, and the least exactly at the quadratures, and that therefore the explanation of the delay must be sought in other circumstances; it is caused by friction.¹

¹ Brougham and Routh, *op. cit.*, p. 293; see also p. 397.

The reader will find much information in an article by J. Proudman, "Newton's Work on the Theory of the Tides," in *Isaac Newton, 1642-1727* ..., pp. 87-95.

46 (p. 445). Book in, Prop, XXVIII. The "radii" in question are like Ad in the figure of Lemma xi, Book 1, the curvilinear angle dAB being the angle of contact. If gA or jB be taken as the radius of curvature, we have very nearly

$(gb + bd)^2 = gA + Ad^2$, or the curvature $\frac{1}{gA} = \frac{2bd}{Ad^2}$. If bd be taken as the evanes-

cent tangent of the angle of contact, then the curvatures of two curves, for equal infinitesimal values of Ad , are to each other as the tangents of those angles.

47 (p. 488). Book in, Lemma 111. By "motion" is meant here *momentum*, mv . Assuming, for simplicity, unit angular velocity and unit density, Newton's first

statement in the proof, that the sphere and the circumscribed cylinder have momentums in the ratio of 3rc:16, may be shown as follows:

The momentum of a material circular area of radius R , rotating about an axis perpendicular to it and passing through its center, is (taking r as the variable radius of a flat ring, $2\pi r$ in circumference, and dr in width),

$$2\pi \int_0^R r^2 dr = \frac{2}{3}\pi R^3.$$

The momentum of the cylinder in question is therefore $\frac{2}{3}\pi R^3$.

As for the rotating sphere, take any section of it perpendicular to the axis of rotation, the momentum of which is $\frac{2}{3}\pi R^3 \sin^3 \alpha$ times an infinitesimal thickness $R \sin \alpha \cdot d\alpha$, or $\frac{2}{3}\pi R^4 \sin^3 \alpha \cdot d\alpha$, α being the angle between R and the abscissa. The momentum of the whole sphere is therefore

$$\frac{2}{3}\pi R^4 \int_0^{\frac{\pi}{2}} \sin^3 \alpha \cdot d\alpha = \frac{1}{2}\pi R^4.$$

We have, therefore,

momentum of the sphere : momentum of the cylinder =

$$\frac{1}{2}\pi R^4 : \frac{2}{3}\pi R^3 = 3\pi : 16, \dots\dots\dots (1)$$

Newton's second statement in his proof, that the momentums of the cylinder and thin ring, both taken about the axis of the cylinder, are as double the matter in the cylinder to triple the matter in the ring, is evident from the proportion,

$$\frac{2}{3}\pi R^3 : 2\pi R^2 \cdot dR = 2(2\pi R^3) : 3(2\pi R \cdot dR), \dots (2)$$

To obtain the momentum of this ring about its own diameter, multiply its infinitesimal arc, $R \cdot d\alpha$, by the distance, $R \cos \alpha$, of that arc from the vertical diameter, and the momentum of the ring about its diameter becomes

$$4R^3 \cdot dR \int_0^{\frac{\pi}{2}} \cos \alpha \cdot d\alpha = 4R^3 \cdot dR.$$

Hence, Newton's third statement in the proof is evident from the proportion,

$$2\pi R^3 \cdot dR : 4R^3 \cdot dR = 2\pi R : 4R, \dots\dots\dots (3)$$

Multiplying together corresponding terms of the proportions (1), (2), and (3), and simplifying, we obtain,

momentum of the sphere : momentum of the ring about its diameter =

$$\pi^3 R^3 : 32 \cdot dR = 3\pi^2 (\frac{2}{3}\pi R^3) : 32(2\pi R \cdot dR) = \\ .925275(\frac{2}{3}\pi R^3) : 1000000(2\pi R \cdot dR).$$

48 (p. 498). Book in, Props, XL-XLII. *Comets*. Some additions and changes occur in the second and third editions. Thus, new in the second and third editions, are what follows Cor. in of Prop, XL, also the Corollary and Scholium of

Lemma VIII, the data of the comet of 1681 following "Nov. 20" in Prop. xu, and finally the part of Prop. XLII which follows the "Q.E.I." The third edition differs from the second in the closing part of Prop. XLII.

49 (p. 499). Book in, Lemma v, following Prop. XL. *Newton's formulas of interpolation*. This famous Lemma deals with these formulas. Newton discusses this subject in four other places: (1) in his *Methodus differentialis*, published in 1711; (2) in a letter of May 8, 1675, to John Smith, giving detailed instructions for the calculation of certain tables; (3) in a manuscript first published in 1927, contained in the Portsmouth Collection of Newtonian manuscripts; (4) in his letters of June 13, 1676, and October 24, 1676, first published in the *Commercium epistolicum* edited by John Collins in 1712. All the contributions of Newton to the theory of interpolation are brought together in a booklet by Duncan C. Fraser, under the title, *Newton's Interpolation Formulas*, London, 1927.¹ Fraser reproduces the *Methodus differentialis* and the manuscript in the Portsmouth Collection, each in facsimile, also in an English translation. He presents much interesting historical material. Newton's interpolation process, as explained in some modern treatises and called "Newton's formula of interpolation" or "Newton's original formula of interpolation," is not Newton's general formula, but only his special formula for values of the argument that are equidistant. This attachment of Newton's name to the special use, without mention of his more general treatment, is seen, for example, in Simon Newcomb, *Logarithmic and other Mathematical Tables*, New York [1882], p. 62; in Herbert L. Rice, *The Theory and Practice of Interpolation*, Lynn, Mass., 1899, pp. 41-61; in H. C. Plummer, *An Introductory Treatise on Dynamical Astronomy*, Cambridge (England), 1918, p. 325. On the contrary, in T. N. Thiele, *Interpolationsrechnung*, Leipzig, 1909, p. 3, there is an explanation of "divided differences" which lead to Newton's general formula, and this explanation is applicable to values of the argument which are not necessarily equidistant.

¹ Fraser's booklet is reprinted from the *Journal of the Institute of Actuaries*, vol. 51, pp. 77-106 (October, 1918) and pp. 311-232 (April, 1919); vol. 58, pp. 53-95 (March, 1927). A more recent contribution to this subject by Fraser is a detailed article, entitled "Newton and Interpolation," in *Isaac Newton, 1642-1727* ..., pp. 43-69.

50 (p. 502). Book in, Lemma x. A theorem of Newton's. KrilofP has drawn attention to the fact that the relation between the time of description of an arc of a parabolic orbit of comets, the two radii vectores drawn to the extremities of that arc, and the chord, is called in modern treatises on theoretical astronomy "Euler's theorem,"² "Lambert's theorem,"⁸ or "Euler-Lambert's theorem," even though Lagrange, in his *Mécanique analytique*, Pt. 11, § vii, p. 26, pointed out that it is only the analytical expression of Newton's Lemma x in Book iii. KrilofI

shows that the required algebraic transformations can be performed in a very simple manner.

¹ *Monthly Notices of the Royal Astronomical Society*, London, vol. 84, 1924, p. 392.

² L. Euler, *Miscellanea Berlinensia*, vol. 7, 1743, "Determinatione orbitae cometæ anno 1742 observati."

⁸ J. H. Lambert, *Nouveaux memoires de l'Academie royale des sciences et belles-lettres*, année 1771, Berlin, 1773, pp. 352-364.

51 (p. 508). Book III, Prop. XLI, Example. *Newton's astronomical observations*. While it is generally known that Newton carried on experiments on light, which were of high importance, also on falling bodies and the motion of pendulums in resisting media, it is not widely known that he refers in this place in the *Principia* to astronomical observations which he himself had made in 1681. He used a "telescope of 7 feet," apparently not a reflector.

52 (p. 544). Book III, General Scholium to Prop. XLII. *Newton's idea of God*. In the first edition of Newton's *Principia* (1687) no statement is made on the nature of God. Nevertheless, criticism was passed upon the *Principia*, on theological grounds, by two prominent thinkers, Bishop Berkeley, who in 1710 published his *Principles of Human Knowledge*, and Leibniz, who on February 10, 1711, wrote a letter to Hartsoeker, a Dutch physician at Diisseldorf, which was published on May 5, 1712, in the *Memoirs of Literature*, a weekly sold in London.¹

Berkeley attacks Newton's exposition in the *Principia* of the notion of absolute space, absolute time, and absolute motion. "This celebrated author," says Berkeley, "holds there is an *absolute Space*, which, being unperceivable to sense, remains in itself similar and immoveable." As to absolute motion, "I must confess it does not appear to me that there can be any motion other than *relative*, so that to conceive motion, there must be at least conceived two bodies—But the chief advantage arising from it [relative space as advocated by Berkeley] is that we are freed from that dangerous dilemma, ... of thinking either Real Space is God, or else that there is something besides God which is eternal, uncreated, infinite, indivisible, unmutable. Both which may justly be thought pernicious and absurd notions."² Thus the absolute space, time, and motion of Newton was attacked as an atheistic conception.

Leibniz, the second distinguished critic, does not mention Newton or the *Principia* in his letter to Hartsoeker, but the reference is obvious. Leibniz says:

"Thus the ancients and moderns who avow that gravity is an *occult quality*, are right if they mean thereby that there is a certain mechanism unknown to them, by which bodies are impelled toward the center of the earth. But if their notion is that this transpires without any mechanism, by a simple *primitive property* [qualite primitive], or by a law of God which brings about this effect without using any intelligible means [moyens intelligibles], then it is a senseless occult

quality, which is so very occult that it can never be cleared up, even though a Spirit, not to say God himself, were endeavoring to explain it."

Neither Newton nor Cotes, who edited the second edition (1713) of the *Princeptia*, made a direct reference to Berkeley. But Cotes, writing to Newton on March 18, 1713, refers to the letter of Leibniz to Hartsoeker, and says: "I think it will be proper [to] add somethings by which your Book may be cleared from some prejudices which have been industriously laid against it. As that it deserts Mechanical causes, is built upon Miracles, and recurrts to Occult qualities. That You may not think unnecessary to answer such Objections You may be pleased to consult a Weekly Paper called *Memoires of Literature* and sold by Ann Baldwin in Warwick-Lane___ You will find a very extraordinary Letter of Mr Leibnitz to Mr Hartsoeker which will confirm what I have said. I do not propose to mention Mr. Leibnitz's name, twere better to neglect him, but the Objections I think may very well be answered and even retorted upon the maintainers of Vortices."³ Cotes made a spirited reply to Leibniz toward the end of his Preface to the Second Edition of the *Prinapia*.

That Newton was misinterpreted by Leibniz with reference to the nature of gravity, is evident. Newton did not believe in action at a distance without the aid of an intervening medium. (See Note 6.) That he was interested in theological questions even before he wrote the *Princeptia*, is evident from his annotations of Henry More's book, *On the Prophet Daniel and the Apocalypse*, found in Newton's own copy of this book, "ex dono Reverendi Authons." These annotations have not yet received due attention in the study of Newton's theological and cosmological concepts at different periods of his career. Referring to the *Princeptia*, Newton said in a letter to Richard Bentley (Dec. 10, 1692), "When I wrote my treatise about our system, I had an eye on such principles as might work with considering men for the belief of a Deity; and nothing can rejoice me more than to find it useful for that purpose."

And so, in 1713, twenty-six years after the first appearance of the *Princeptia*, Newton, then seventy-one years old, prepared the famous General Scholium, printed at the end of the second edition of the *Princeptia*.

The sentence, "And thus much concerning God; to discourse of whom from the appearances of things, does certainly belong to Natural Philosophy," is of interest for two reasons. First, it was not in the original draft of the General Scholium; Newton added it, as the second edition was going through the press.⁴ Secondly, Newton here gives his justification for treating of this subject in the *Princeptia*. To obtain an idea of God "from the appearances of things, does certainly belong to Natural Philosophy."

In the third edition (1726) of the *Princeptia*, there are six additions or changes made to the General Scholium. These alterations are as follows:

(1) New in the 1726 edition is the sentence, "And lest the systems of the fixed stars should, by their gravity, fall on each other, he hath placed those systems at immense distances from one another."

(2) New in the 1726 edition following the words "God of Israel" are the words, "the God of Gods, and Lord of Lords."

(3) In the 1726 edition the passage, "the Eternal of Gods; we do not say, my Infinite, or my Perfect," takes the place of "we do not say, my Infinite, your Infinite, the Infinite of Israel; we do not say my Perfect, your Perfect, the Perfect of Israel," in the second edition.

(4) In the 1726 edition, "He is not eternity and infinity" takes the place of "He is not eternity or infinity" in the second edition

(5) The four sentences in the 1726 edition, "Every soul that has perception is ... God is the same God, always and everywhere," are an addition to the Scholium as given in the second edition.

(6) The four sentences in the 1726 edition, "For we adore him as his servants; ... by a certain similitude which, though not perfect, has some likeness, however," are an addition to the Scholium in the second edition. But substantially this addition Newton prepared long before 1726, in fact, only six months after the publication of the second edition in 1713. It is given in a list of corrections and additions⁵ to the second edition, which he sent to Cotes; but this was not printed at the time.

It is to be noted that Newton's idea of God in the second (1713) edition of the *Principia* is drawn largely from the "appearances of things," while the interpolations printed in the third edition (1726) are taken more particularly from the "ways of mankind."

¹ This letter of Leibniz' is found in Leibniz, *Opera omnia*, vol. 2, Geneva, 1768, Pt. 11, p. 60; *Philosophische Schriften von Leibniz* (ed. C. I. Gerhardt), vol. 3, Berlin, 1887, p. 519. See also Brewster, *Memoirs of Sir Isaac Newton*, vol. 2, chap. 22, pp. 219-222, Edlcston, *Correspondence of Sir Isaac Newton and Professor Cotes*, p. 153. Leibniz attacks the theory of gravity also in his *Essais de theodicee sur la bonte de Dieu*, 1710.

² G. Berkeley, *Principles of Human Knowledge*, Part 1, Par. 111, 117.

³ Edleston, *op. at.*, p. 153.

* See Edleston, *op. at.*, p. 155.

⁵ Edleston, *op. at.*, p. 165.

53(p. 546). Book in, General Scholium to Prop. XLII. *Final causes*. Newton used the term "final cause" as did Aristotle, who distinguished four kinds of cause: "material," "formal," "efficient," and "final"; Aristotle's final cause was the purpose, aim, or end for which a thing is made. Thus: "For to sojourn in the land, is cause we come to thee"; a pen is made to serve in writing.

54 (p. 546). Book in, General Scholium to Prop. XLII. *Statement of the law of gravitation*. The modern general statement of this law, namely, that "every particle of matter attracts every other particle with a force varying directly as the

product of their masses and inversely as the square of the distance between them," is **nowhere found in the *Principia*** and the *System of the World*. Perhaps the nearest approach to it is given in the General Scholium at the end of the *Principia*, where Newton states that "gravity... operates... according to the quantity of the solid matter which they [the sun and planets] contain, and propagates its virtue on all sides to immense distances, decreasing always as the inverse square of the distances." For other approximations to the modern form of statement, see Book 1, Prop, LXXVI, Cor. 111 and iv, and the *System of the World*, Par. 26.

55 (P. 547). Book in, General Scholium to Prop. XLII. *Newton's use of hypotheses*. "I frame no hypotheses" (*hypotheses non fingo*) is the expression of Newton, occurring in the General Scholium at the end of his *Prtnapta*—an expression frequently quoted to indicate his contempt for reckless speculation and his absolute reliance upon observation and experiment. No doubt readers of Newton's *Principia*, of his early published papers on light, and of his *Opticas*, will be puzzled by this absolute declaration, "hypotheses non fingo," for surely Newton himself framed many hypotheses—as many, perhaps, as any other scientist of note. How is this statement of his position to be reconciled with his actual practice?

In the first place, it should be noted that Newton does not advance "hypotheses non fingo" as a general proposition, applying to all his scientific endeavor; it is used by him in connection with a public statement relating to that special, that difficult and subtle subject, the real nature of gravitation, which was mysterious then and has remained so to our day. Moreover, this "hypotheses non fingo" is to be taken, not as his private practice, nor his individual habit of thought, but as the position which he took in public print, on the occasion of placing before the scientific world the positive results of his mathematical thinking, which were primarily based on observation and experimentation. Newton's "hypotheses non fingo" disrupted from its context is a complete misrepresentation of Newton.

An examination of the various passages in Newton's writings, relating to the use of hypotheses, discloses the rule that experimental facts must invariably take precedence over any hypothesis in conflict with them. Secondly, hypotheses which seem incapable of verification by experiment are to be viewed with suspicion. In any event, one should observe the distinction between exact experimental results and mere suggestions derived from hypotheses.

It may be of some interest to give in chronological order some of the leading hypotheses advanced by Newton, as well as his comments made at different times of his career, on the proper use of hypotheses.

In 1666, when about twenty-four years of age, Newton cogitated on gravity, framed the hypothesis of its reaching to the moon and of its variation inversely

as the square of the distance; he then proceeded to test it by comparing the implications of that hypothesis with the results of actual experiment.¹

In 1672 were published Newton's experiments on the dispersion of light.² To explain why the length of the spectrum obtained from light entering the room through a small circular opening was five times its width, Newton set up different hypotheses, or "suspensions" as he called them, but removed every one of them as untenable, and was finally led to his *experimentum crucis*. One of these untenable hypotheses was that the solar rays after leaving the prism forming the spectrum moved in curved lines, some more curved than others—"I had often seen a Tennis-ball, struck with an Oblique Racket, describe such a Curve Line. ... If the Rays of Light should possess Globular Bodies, and by their Oblique passage, out of one Medium into another, acquire a Circulating Motion, they ought to feel the greater resistance from the ambient Aether, on that side, where the motions conspire, and thence be continually bowed to the other. But notwithstanding this plausible ground of suspicion... I could observe no such Curvity in them." In this research, the young man of thirty indulged with perfect freedom in the formulation of hypotheses. Two of them were no other than the corpuscular theory of light and that of the luminiferous ether! These were hypotheses as bold and far-reaching as have ever been advanced in physics. Nor were they of the type that are readily verifiable by experiment. However, his sub-hypothesis of the curved path of these light particles was easily eliminated through experimental test.

In 1672, in reply to criticisms made by Robert Hooke, Newton explained how Hooke's hypothesis that light is caused by pulses or, better, by wave motions of the ether, could be used to explain the phenomena of spectral colors, and the colors of thin plates. But Newton found Hooke's hypothesis unable to explain the fact that light travels in straight lines.⁸ If light consisted of vibrations, he argued, it would, like sound, spread "into the Shadowed Medium." We see here that Newton has no hesitation in entertaining the hypothesis of the wave motion of light in the ether.

However, toward the close of Newton's reply to Hooke there appears the first foreshadowing of impatience with hypotheses. Hooke would explain the spectrum caused by a prism, not by the different refrangibility of the rays, but by the splitting and rarefying of ethereal pulses. Hooke's hypothesis was one of pulses in the ether, rather than of actual waves in the ether. Newton considers Hooke's hypothesis, but finds it insufficient, and finally says, "But whatever be the Advantages or Disadvantages of this Hypothesis, I hope I may be excused from taking it up, since I do not think it needful to explicate my Doctrine by any Hypothesis at all." Farther on Newton says:⁴

"You see therefore how much it is beside the business in hand, to dispute about Hypotheses. For which reason I shall now, in the last place, proceed to Abstract the difficulties in the *Animadversor's Discourse*, and without having regard to any Hypothesis, consider them in general Terms. And they may be reduced to these three Queries:

"1. Whether the unequal Refractions made without respect to any inequality of Incidence, be caused by the different Refrangibility of several Rays; or by the splitting, breaking, or dissipating the same Ray into diverging Parts?⁷

"2. Whether there be more than two sorts of Colours?

"3. Whether Whiteness be a Mixture of all Colours⁵"

The interesting revelation in this last quotation is that Newton dismisses the two major hypotheses of light as wave motion and as flying particles, in order to consider three minor hypotheses, more easily disposed of by experimental test, but nevertheless hypotheses, although he does not designate them by that name.

When another critic, Gaston Pardies, Professor of Mathematics in the Parisian College of Clermont, called the refrangibility of light a hypothesis, Newton replied with emphasis and warmth, that his theory "seemed to contain nothing else than certain properties of light, which I have discovered and regard it not difficult to prove; and if I had not perceived them to be true, I would have preferred to reject them as futile and inane speculation, rather than acknowledge them as hypotheses."⁵

In reply to further objections of Pardies, Newton says.⁶

"For the best and safest method of philosophizing seems to be, first diligently to investigate the properties of things and establish them by experiment, and then to seek hypotheses to explain them. For hypotheses ought to be fitted merely to explain the properties of things and not attempt to predetermine them except in so far as they can be an aid to experiments. If any one offers conjectures about the truth of things from the mere possibility of hypotheses, I do not see how any thing certain can be determined in any science; for it is always possible to contrive hypotheses, one after another, which are found rich in new tribulations. Wherefore I judged that one should abstain from considering hypotheses as from a fallacious argument, and that the force of their opposition must be removed, that one may arrive at a maturer and more general explanation." This passage is important; it is the fullest single statement that Newton has made on the use of hypotheses. He recognizes the role they can be made to play in research, and he places the proper restrictions to their use.

In 1673, in a reply to criticisms of Christiaan Huygens, Newton said:⁷ "Nor is it easier to frame an Hypothesis by assuming only two Original Colours, rather than an indefinite Variety; unless it be easier to suppose that there are but two Figures, Sizes, and Degrees of Velocity or Force of the Aethereal Corpuscles or

Pulses, rather than an indefinite Variety; which certainly would be a harsh Supposition____ But to examine how Colours may be explained Hypothetically, is beside my purpose. I never intended to show wherein consists the Nature and Difference of Colours, but... to leave it to others to Explicate by Mechanical Hypotheses, the Nature and Difference of these Qualities; which I take to be no difficult matter." Here, for a moment, Newton allows himself the luxury of considering ethereal corpuscles or pulses of different forms, sizes, and velocities; he does not reprimand himself for it, nor state that he is indulging in unscientific procedure, but finally he decides to leave these hypotheses to others. He saw that often the same experimental fact could be explained by different hypotheses,⁸ hence the futility of arguing about them in heated discussion.

My quotations from Newton suggest the motive which induced him to take a stand against the use of hypotheses, namely, the danger of becoming involved in disagreeable controversies. This guess is confirmed by Newton's own statement contained in a communication of December 9, 1675, to the Royal Society:⁹ "I had formerly purposed never to write any hypothesis of light and colours, fearing it might be a means to engage me in vain disputes: but I hope a declared resolution to answer nothing, that looks like a controversy, unless possibly at my own time upon some by-occasion, may defend me from that fear." And then Newton proceeds to elaborate more fully his explanations of refraction and reflection of light, involving corpuscles of different sizes, causing in their flight compressions and rarefactions of the ether at a refracting surface. This is his famous hypothesis of "fits" of easy reflections and refractions, which is of interest to twentieth-century physicists as bearing considerable resemblance to the modern theory of "wave mechanics." Newton could no more dispense with hypotheses in his own cogitations than an eagle can dispense with flight. Nor did Newton succeed in avoiding controversy.

The years 1672-6 were years of conflict, and Newton became more and more disinclined to make any of his speculations public. By nature Newton was very sensitive to criticism. He loathed controversy.

Did his disagreeable experiences of 1672-6 stop his private indulgence in the practice of framing hypotheses? Not at all. His mind remained as free, bold, and imaginative as ever. To friends like Robert Boyle, he felt free to disclose his innermost thoughts, both in conversation and by letter. To be convinced of this fact one needs only to read Newton's letter¹⁰ to Boyle on the cause of gravitation, written on February 28, 1678/9. (See also Note 6.) This was about eight years before the publication of the *Principia*. On the very topic which in the *Principia* led to the declaration, "hypotheses non fingo," there was in this letter to Boyle a very revelry of un-verifiable hypotheses. But he puts forth the caution, "my notions

about things of this kind are so indigested, that I am not well satisfied myself with them." I quote:

"And, first, I suppose that there is diffused through all space an aethereal substance, capable of contraction and dilation, strongly elastic; and, in a word, much like air in all respects, but far more subtle____When two bodies, moving toward one another, come nearer together, I suppose the aether between them to grow rarer than before.... When two bodies, approaching one another, come so near together as to make the aether between them begin to rarefy, they will begin to have a reluctance from being brought nearer together, and an endeavour to recede from one another,... but at length, when they come so near together, that the excess of pressure of the external aether, which surrounds the bodies, about that of the rarefied aether, which is between them, is so great, as to overcome the reluctance which the bodies have from being brought together, then will that excess of pressure drive them with violence together____Now, hence I conceived it is, chiefly, that a fly walks on water without wetting its feet, and consequently without touching the water";

"I shall set down one conjecture more,... it is about the cause of gravity. For this end I will suppose aether to consist of parts differing from one another in subtility by indefinite degrees: that in the pores of bodies, there is less of the grosser aether in proportion to the finer, than in open spaces; and consequently, that in the great body of the earth there is much less of the grosser aether, in proportion to the finer, than in the regions of the air: and that... from the top of the air to the surface of the earth, and again from the surface of the earth to the centre thereof, the aether is insensibly finer and finer. Imagine, now, any body suspended in the air, or lying on the earth; and the aether being, by the hypothesis, grosser in the pores which are in the upper parts of the body, than in those which are in the lower parts; and that grosser aether, being less apt to be lodged in those pores, than the finer aether below; it will endeavour to get out, and give way to the finer aether below, which cannot be, without the bodies descending to make room above for it to go out into."

These speculations on gravity were too extravagant to find a place in a work like the *Principia*. Hence Newton's exercise of control over himself in his public declaration, "hypotheses non fingo." He pressed the distinction between hypotheses and experimental law, which he himself carefully observed. In his *Principia*, Book I, Rule 3, "of reasoning in [natural] philosophy," he says,¹¹ "We are certainly not to relinquish the evidence of experiments for the sake of dreams and vain fictions of our own devising; nor are we to recede from the analogy of Nature, which is wont to be simple, and always consonant to itself."

The classic pronouncement of Newton against the improper use of hypotheses is found at the end of his *Principia*: "Whatever is not deduced from the phe-

nomena, is to be called an hypothesis; and hypotheses, whether metaphysical or physical, whether of occult qualities or mechanical, have no place in experimental philosophy. In this philosophy particular propositions are inferred from the phenomena, and afterwards rendered general by induction. Thus it was that the impenetrability, the mobility, and the impulsive force of bodies, and the laws of motion and of gravitation, were discovered."

In 1704 was published Newton's book entitled *Opticas*. It contained some earlier papers and also some new matter. The book could not avoid hypotheses, but an effort was made to exclude the more fanciful of them from the main body of the work. The parts which seemed to Newton over-highly speculative, or which represented investigations left incomplete, were introduced at the close in the form of "Queries." In these Queries Newton apparently allowed himself full freedom of speech. They afford a wonderful insight into the operations of Newton's mind on questions of physical inquiry.

In general, we may say that Newton gave his scientific imagination free scope and made very extensive use of hypotheses. But hypotheses were always scrupulously dethroned whenever they were found to be in irreconcilable conflict with experimental fact. In his publications Newton frequently avoided full statement of his hypotheses, hoping thereby to escape much-dreaded controversies.

¹ The test was most probably in general outline the same as the one given in the *Principia*, Book III, Prop. iv.

² *Philosophical Transactions Abridged* (J Lowthrop), vol. 1, London, 1705, pp. 128-130.

³ *Philosophical Transactions Abridged*, vol. 1, 1705, p. 146; *Philosophical Transactions*, No. 88, 1672, p. 5086.

⁴ *Philosophical Transactions Abridged*, vol. 1, 1705, p. 148, *Philosophical Transactions*, No. 88, 1672, p. 5086.

⁵ *Philosophical Transactions Abridged*, vol. 1, 1705, p. 141; *Philosophical Transactions*, No. 84, 1672, p. 4091.

⁶ *Uaaci Newton Opera*, vol. 4, pp. 314, 315 Newton's letter to Oldenburg, June 2, 1672 This part of the letter was not printed in the *Philosophical Transactions*

⁷ *Philosophical Transactions Abridged*, vol. 1, 1705, p. 157; *Philosophical Transactions*, No. 97, 1673, p. 6108.

⁸ *Philosophical Transactions Abridged*, vol. 1, 1705, p. 145, *Philosophical Transactions*, No. 88, 1672, p. 5086.

⁹ T Birch, *History of the Royal Society of London*, vol. 3, 1757, p. 248.

¹⁰ *Isaac Newton Opera*, vol. 4, pp. 385-394.

¹¹ Present volume, pp. 398-399.

56 (p. 547). Book in, General Scholium to Prop. XLII. *Causality*, Questions of cause and effect have interested philosophers of all times—Aristotle, Hume, Kant, Mill. Among the earliest scientists to instill confidence in the existence of a comprehensive system of causality in physical phenomena was Newton. Kepler's three laws of planetary motion were brilliant discoveries, but they did not satisfy the requirements of causality. It was Newton who showed that the three apparently independent laws were the logical consequence of one fundamental law of Nature. At the opening of Book in of the *Principia*, Newton discusses relations of cause and effect in the Rules of Reasoning in Philosophy.

But in his *Principia* Newton did not venture to make suggestions on the cause of gravity. (See Note 6.) Such ultra-speculative writing is found in his letters to Richard Bentley and Robert Boyle (see Note 55 on Hypotheses), but would have been inappropriate in the *Principia*, a work aiming at close adherence to experimental facts and rigid mathematical deduction from them. Hence, Newton's famous passage in the General Scholium at the end of the second and third editions of the *Principia*, "Hitherto I have not been able to discover the cause of those properties of gravity from phenomena, and I frame no hypotheses." Relations of cause and effect have engaged the attention of scientists since the time of Newton. Famous is Robert Mayer's dictum—*causa aequat effectum*—a metaphysical tenet which guided him in arriving at the principle of the conservation of energy.

While Newton endeavored to subject the phenomena of physical Nature to mechanical law, he realized more fully than did his immediate followers that he had not succeeded in doing so altogether. In fact, he did not believe that there was a "world machine" which kept on running according to the law of gravitation without supervision of God,¹ but rather that irregularities in the solar system caused by action of planets and comets on each other are regulated by God whenever emergencies arise. "This most beautiful system of the sun, planets, and comets," says Newton in his *Principia* (Book in, Prop, XLII, General Scholium), "could only proceed from the counsel and dominion of an intelligent and powerful Being." The sufficiency of mechanical law to explain the mechanical phenomena of the solar system was entertained by Laplace, the great successor of Newton in elaborating mathematically the consequences of the law of gravitation. Laplace said in his *Essai philosophique sur les probabilités*:² "If an intelligence, for one given instant, recognizes all the forces which animate Nature, and the respective positions of the things which compose it, and if that intelligence is also sufficiently vast to subject these data to analysis, it will comprehend in one formula the movements of the largest bodies of the universe as well as those of the minutest atom: nothing will be uncertain to it, and the future as well as the past will be present to its vision. The human mind offers in the perfection which it has been able to give to astronomy, a modest example of such an intelligence." In these words we have a bold assertion of the belief in determinism, causal relation. Laplace believed that on the law of gravitation he and others had succeeded in showing what Newton did not do, namely, that the solar system is stable. In answer to a question whether it was true that in his *Mecanique celeste* he had never mentioned the Creator, Laplace told Napoleon, "Je n'avais pas besoin de cette hypothese-la." Laplace had indeed succeeded in explaining, for example, all the inequalities in the observed motions of Jupiter and Saturn by Newton's law; every difficulty had led him to a new triumph.³

But even in classical mechanics Laplace's proof of the stability of the solar system is no longer considered conclusive. The nineteenth century came to demand greater mathematical rigor than had the eighteenth century. K. Weierstrass, in the last years of his life, gave much attention to this question of the stability of the solar system. So did H. Poincaré. But thus far no rigorous answer has been given. Referring to this problem, H. F. Baker said at the meeting of the British Association in 1913: "For those who can make pronouncements in regard to this I have a feeling of envy; for their methods, as yet, I have a quite other feeling. The interest of this problem alone is sufficient to justify the craving of the pure mathematician for powerful methods and unexceptional rigour."⁴

Laplace's determinism has lost its place in more recent physical science on still other grounds, namely, the role that statistical science and probability have come to play in some of the most fundamental inquiries. These sciences relate to the behavior of large assemblies. We acquire a new electric globe. How long will it last? We do not know. It cannot be predicted mathematically. But we do know that, by a statistical study of large numbers of that make of globes, the mean life of such a globe is a certain number of days. There is no individual relation of causality in this probable duration of our new globe. We are dealing here with a statistical mean, in which the ordinary relation of cause and effect does not prevail. If the glass were heated in the ordinary manner above its melting point, the relation of causality would hold directly, for we are certain the glass would melt. According to Kant, the law of causality is, "Everything that happens (begins to be) presupposes something upon which it follows according to some rule." No such rule exists for the length of service of our individual globe.

Eddington says, "In recent times some of the greatest triumphs of physical prediction have been furnished by admittedly statistical laws which do not rest upon a basis of causality."⁵ According to modern views, the second law of thermodynamics⁶ has a statistical foundation. The absence of determinism is seen in present-day atomic physics. Planck's derivation of his original quantum theory⁷ rested on probability. The new quantum mechanics developed by W. Heisenberg⁸ in 1927 emphasizes indeterminism, or the principle of uncertainty in microscopic processes. He points out that we cannot simultaneously measure the position and the velocity of an individual particle. Thus we cannot find both the position and the velocity of an electron, in an exact sense, at any one moment. Suppose we had a microscope so powerful as to reveal to us atomic and intra-atomic events. An isolated electron could not be seen, for we cannot see an object unless it emits or reflects light. But if light comes from an electron, the electron has been interacting with a light quantum or photon. This interaction changes the velocity of the electron just as surely as the velocity of a billiard ball is changed when hit by another, or the position of a gun changes when it recoils from the

emission of a projectile. In the act of sending out light, the electron jumps. **The** ray of light may tell where the electron was, but it does not tell with what velocity the electron was moving, before the jump. Stating the principle somewhat more generally, if we can measure where the microscopic particle is, we cannot tell how fast it is moving, and if by certain effects we can measure how fast it is moving, we cannot at the same time tell where it is. Prediction of its status for the future cannot be made with exactness, but only with limited accuracy, expressed in terms of a certain probability.

This principle of indeterminism cuts at the root of the older physics, in which position and velocity were fundamental. Laplace's general determinism falls to ruins. According to this principle of indeterminism, we must abandon the conception of a chain of precise causal relations connecting the history of the physical universe, both past and future, with the exact state of the universe at a given instant.

Future research may afford important modification of this principle. Meanwhile, Eddington points out that this principle provides the believer in free will with an escape from the conclusions of the mechanistic philosophy of Laplace and the nineteenth-century classical science.

Great have been the advances since Newton. But the march of science is not always unidirectional. After two hundred years of wandering, we are returning to the teaching of Newton in some of our general viewpoints. Newton saw no conflict between religion and science. Present-day thought, as Eddington remarks, tends to remove some of the obstacles to a reconciliation of religion and science.

¹ Consult A. I. Snow, *Matter and Gravity in Newton's Philosophy*, Oxford, 1926, p. 204.

² *CEuvres completes*, Tome vn, p. vi.

³ Laplace, *Mecanique celeste*, Livre xv, Chap. 1, p. 324.

⁴ H. F. Baker, *Report of the British Association for 1913*, London, 1914, p. 371.

⁵ Eddington, *The Nature of the Physical World*, p. 298.

⁶ Consult H. Weyl, *Philosophie der Mathematik und Naturwissenschaften*, Munchen, 1927, pp. 154, 155.

⁷ Consult W. Schottky, "Das Kausalproblem der Quantentheorie als eine Grundfrage der modernen Naturforschung überhaupt," in *Die Naturwissenschaften*, vol. 9, 1921, pp. 492-496; E. Mach, *Analyse der Empfindungen*, ed. 3, 1902, Chap. v.

⁸ W. Heisenberg of Copenhagen in *Zeitschrift für Physik*, vol. 43, 1927, pp. 172-198. For comments on Heisenberg, see N. Bohr in *Die Naturwissenschaften*, vol. 16, 1928, p. 246.

57 (p. 549). *Translator of the System of the World*. An English translation of the *System of the World* was published in 1728, but the name of the translator was withheld. There is a long Latin passage in the *System of the World* which is virtually identical with a Latin passage in Book in of the *Principia*. The translations of these passages into English are so nearly alike, that they most probably are attributable to one and the same translator. Consider, in particular, the critical clause in that passage (*Principia*, Book in, Prop. XLI, Example; *System of the World*, Par. 37), "Nam quod dicitur Fixas ab Aegyptiis comatas nonnun-

quam visas fuisse." The translation of "comatas" caused trouble and was rendered in both books by the use of three words, "coma or capillitium," the whole clause being translated, "For as to what is alleged that the fixed stars have been sometimes seen by the Egyptians environed with a coma or capillitium." Such singular coincidence in a free translation makes it virtually certain that both books are put into English by the same translator—namely, Andrew Motte. In this revision of Motte's translation I have omitted the words "or capillitium."

58 (p. 576). *System of the World*, Paragraph 30. *A remote planet predicted?* J. Ph. Wolfers, in his German edition of Newton's *Principia*, Berlin, 1872, p. 659, finds this passage (particularly the part "Hence it is probable that there are comets... which ... spend almost their whole time in regions beyond the planets") extremely interesting, as pointing to the existence of the planet Uranus, which at that time had not been observed, and which was first seen by William Herschel in 1781. Newton considered planets and comets to be closely related heavenly bodies, a view quite in accordance with modern ideas.

59 (P- 594)- *System of the World*, Paragraph 54. *Terrestrial tidal effects*. Tidal effects on land, which Newton considered insensible, were detected by Michelson and Gale in the solid earth, on the grounds of the Yerkes Observatory, in 1919, by the application of very delicate methods of observing interference fringes of monochromatic light. The results were used in a determination of the rigidity of the earth.¹

¹ A. A. Michelson and Henry G. Gale, *Physical Review*, ser. 2, vol. 15, 1920, p. 144; *Science*, n.s., vol. 50, 1919, p. 327; also vol. 39, 1914, p. 927.

