THREE MAJOR DIFFICULTIES IN THE LEARNING OF DEMONSTRATIVE GEOMETRY

By

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Three Major Difficulties in the Learning of Demonstrative Geometry

By ROLLAND R. SMITH

PART I
ANALYSIS OF ERRORS

CHAPTER I
PURPOSE AND METHOD

EFFICIENT and successful teaching of demonstrative geometry in the senior high school requires on the part of the teacher much more than a knowledge of the subject matter. The young person who goes into the geometry classroom after leaving college with honors in mathematics is not necessarily a good teacher. Unless he has been forewarned in one way or another, he is likely to resort to the lecture method which his professors have used in college and then find to his surprise that his pupils have learned little. He may have taken courses in which he studied the general laws of learning as applied to pupils of high school age, but even so he will have difficulty in translating his knowledge to fit the specific requirements of the classroom. Part of his training may have been to observe the work of a highly efficient, successful, and artistic teacher whom he may try to imitate. He will find, however, that he has not been keen enough to grasp the meaning and purpose of many of the techniques. Not knowing before hand how a group of pupils will react to a given situation, he fails to see when and how the experienced teacher has avoided pitfalls by introducing many details of development not necessarily needed in the finished product but indispensable to the learning process. Before he can become adept in preparing a course of study or planning his everyday lessons, he needs to know what difficulties pupils will have with the many component tasks which when integrated fulfill the desired aim. A teacher can plan a skillful development only when he has reached a point where he can predict within reasonable limits what the reactions of a group of pupils will be.

A teacher cannot sit in an armchair and by reasoning alone tell how pupils will react to the many situations of the classroom. One who has taught for many years will inevitably know more about pupils' difficulties and the way to remedy, minimize, and obviate them than one who has never taught. But unless he has consciously put his mind to the study of these difficulties and has sufficient background to get meaning from the study he will have missed one of the best methods of improving his teaching. The first step in devising methods to overcome pupils' difficulties is to find out what the difficulties are.

The basis for such a study is classroom experience with pupils of various types. For the best results, however, experience alone is insufficient. In order to get the most insight from a study of pupils' learning difficulties one must set up typical situations relevant to the end in view, observe and analyze the responses of individuals. Only when results are recorded systematically are they in shape for careful analysis. And when the learning process requires a rather lengthy period of time as it does in the introduction to demonstrative geometry, the process of setting up situations, observing reactions, and recording the results must continue for the same length of time. A single test or even a series of tests at the end of a learning period will show errors which can be tabulated as to type. A series of tests over a period of time while learning is progressing will give a better insight into the learning process and may lead the investigator to a more basic classification of errors than might otherwise be apparent.

This study is in two parts. Part I is an investigation of pupils' learning difficulties in demonstrative geometry over a period of fifty consecutive teaching days—from the first meeting of the classes through the study of *congruence* and *parallel lines*. During this time, tests were given nearly every day. All the errors were marked and the percentage of pupils making each type of error was recorded. As the study progressed, the writer learned that the number of errors which could be classified under three headings—namely, (1) those due to unfamiliarity with geometric figures, (2) those due to not sensing the meaning of the *if-then relationship*, and (3) those due to a meager understanding of the meaning of proof which far exceeded all other errors. Errors under these headings persisted throughout the study. In the final check of errors by means of a test on "originals" almost all of the errors fell into this classification. For these reasons, the writer believes that these types of error indicate the major learning difficulties in the beginning of demonstrative geometry and has, therefore, restricted the study to a discussion of these three difficulties.

The lack of space given to these three headings in textbooks and the literature on the teaching of geometry suggests that teachers are unaware of the fact that pupils' difficulties are so basic. Suggestions on the teaching of geometry fall short of revealing the fundamental errors. As stated above, the first step toward a method of teaching that will take care of the fundamental weaknesses of the pupils is to become aware of those weaknesses. It is the purpose of Part I of this study, therefore, to take this first step; that is, to identify and analyze the serious learning difficulties that pupils have in connection with (1) geometric figures, (2) the meaning of the if-then relationship, and (3) the meaning of proof.

The study involved all the pupils taking the first course in demonstrative geometry during the fall of 1932 in Classical High School, Springfield, Massachusetts. The number of pupils was 114. All the pupils were members of the 10A class, 10A referring to the second semester of the tenth school year. They were divided into five groups according to their previous record in mathematics, the author teaching two of them in successive periods. The remaining classes were taught by three other experienced teachers. All classes met four times a week for a period of sixty minutes.

Contrary to custom, all the pupils in the five classes studied the same topics and subtopics day by day. So far as was possible, with four teachers involved, the method of teaching was the same in all classes. Bulletins giving detailed information as to topics to be covered each day and methods to be used were given to the teachers, who were cautioned to deal on any given day with only the topics outlined for that day. Necessarily, the pace of the better classes had to be slackened to meet the requirements of the slower groups. On account of the large number of tests given, the pace was even slower than that ordinarily required for the poorest classes.

The first reading of the test papers was done by the teacher of each class. At this time a cross was placed against each error. The papers were then given to the writer. He analyzed the errors as to type and placed code marks against them to indicate the type of error. The results were then tabulated according to individual pupils and finally according to groups as shown in the following chapters.

The final tabulations give the percentage of pupils making errors in any given case according to four groups: Group A, 30 pupils with I.Q.'s (Terman) from 125 to 146; Group B, 50 pupils with I.Q.'s from 111 to 125; Group C, 34 pupils with I.Q.'s from 90 to 110; and Group T, 114 pupils consisting of Groups A, B, and C together. The median for Group A was 131; for Group B, 117; for Group C, 106; and for Group, T, 116. The mean of I.Q.'s for Group T was 117.9 with a standard deviation of 11.7.

No extended statistical treatment is required to fulfill the purpose of Part I. In every case the percentage of pupils making a particular type of error is given. We are not so much interested in making comparison of the percentages as we are in noting the separate percentages. When a large percentage of the pupils make a certain type of error we see that a learning difficulty and a teaching problem are involved. Comparisons between percentages have been made only where a series of them clearly show a trend or where the difference between two percentages is so large as to be obviously significant. We are not interested in finding out how much more difficult one thing is than another so much as in discovering what is difficult.

Part II is a description of methods devised to help pupils with their difficulties in connection with the three types of errors already mentioned and a study of the effect of these methods on a group of students with the same mean I.Q. and same standard deviation as Group T. The experimentation for Part II was done in the fall of 1938. The method of procedure will be described in full later.

Chapter II

COMPLEX FIGURES

Geometric figures are the materials used in the study of demonstrative geometry in the senior high school. They constitute the vehicle which carries the logic of the course. The demonstrations made, the conclusions drawn, all concern geometric figures. Efficient reasoning in geometry presupposes, therefore, a familiarity with such figures. One of the arguments for using geometry to develop methods of reasoning is that the materials with which the pupils reason are simple as compared to those necessary in history, literature, or economics. At the same time we should not be so blinded by this statement that we do not see the need of making sure that pupils are sufficiently familiar with the figures.

It is the tendency in textbooks to define a term, perform a construction, or prove a theorem with the use of as simple a figure as possible and to expect the pupil to apply what he has learned (without help) to more complex figures. We propose to show in this chapter that many pupils do not make the generalization readily.

For the purpose of this study we define a complex figure as one different from the one in which a given term is defined, a given construction is practiced, or a given theorem introduced. A figure may be complicated by adding a line or lines or by turning it about. Thus, if perpendicular lines are illustrated where defined with a figure showing a horizontal and vertical line, the figure is considered complex if the lines are rotated so that they are oblique to the horizontal. Likewise, it would be complex with respect to *perpendicular* if it involved a triangle with an altitude.

We shall study pupils' reactions to complex figures under three headings: *Constructions*, *Meaning of Terms*, and *Recognition of Theorems.*

Constructions

In order to discover pupils' reactions to constructions in complex figures, they were first shown how to make a construction in a simple situation and were then given practice with it. Then, without further training, they were given construction problems which required the application of what they had learned to a more complex situation. For one fundamental construction problem this procedure was followed by an explanation and a second test given to ascertain the effect of the explanation.

The tests, tabulations of results, and discussions follow.

Bisecting Lines

Test 1[1] was given on the third day, after the pupils had learned how to bisect horizontal straight lines and had practiced to the point of mastery. The purpose of the test was to discover whether the ability to bisect a horizontal straight line could be used in bisecting the sides of a triangle.

Test 1

1. Bisect AB.

2. Bisect the sides of triangle ABC.

TABLE 1

Percentage of Pupils Making Errors on Each Exercise of Test 1

Exercise	Percentage in Group[2]			
	A	B	C	T
1	0	0	0	0
2	0	4	15	6

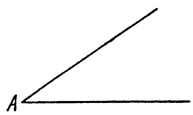
There were no errors on Ex. 1 showing that, for the time being at least, all the pupils could bisect a horizontal straight line correctly. The transfer from Ex.1 to Ex. 2 was 100% in case of the upper I.Q. group (Group A), but dropped somewhat in the lower groups. In the middle group (Group B) 4% of the pupils were in error and in the lowest group (Group C) 15%. We may consider the transfer as high, since 94% of the entire group were correct on Ex. 2.

Bisecting Angles

Test 2 was given on the fifth day. Pupils had practiced bisecting angles with the vertex at the left and the opening at the right, but had not bisected any other angles. The purpose of the test was similar to that of Test 1; namely, to discover whether the ability to bisect an angle given in a special position could be used in bisecting the angles of a triangle.

Test 2

1. Bisect $\angle A$.



2. Bisect the angles of triangle ABC.

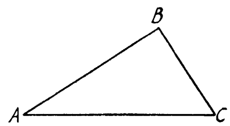


TABLE 2

Percentage of Pupils Making Errors on Each Exercise of Test 2

Exercise	Percentage in Group			
	A	B	C	T
1	0	0	0	0
2	10	12	23	15

The following types of error were made on Ex. 2 of Test 2.

1. Bisection of $\angle A$ correct, bisection of other angles confused.

2. Bisection of all angles confused.

3. Bisection of two angles correct. Bisection of third angle confused.

The third type of error accounted for two-thirds of all errors. Confusion showed

[1] Tests are numbered and recorded not in the order given, but in an order that furthers an understanding of the discussion. Only those tests which have a direct bearing on the topics selected are recorded.

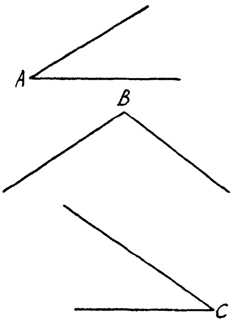
[2] Group A comprised 30 pupils with I.Q.'s from 126–146; Group B, 50 with I.Q.'s from 111–125; and Group C, 34 with I.Q.'s from 90 to 110. Group T was a combination of Groups A, B, and C (see page 3).

in not drawing the first arc across the sides of the angle to be bisected. The arc was sometimes drawn across one side of one angle and one side of another angle. The fact that the line did not appear to bisect the angle seemed to cause no anxiety.

As soon as Test 2 was finished, Test 2a was given to ascertain whether the difficulty in Ex. 2 of Test 2 was caused by the different positions of the angles.

Test 2a

Bisect angles A, B, and C.



Since all the pupils were correct on the three exercises, it may be concluded that the difficulty was not the position of the angles. The cause of difficulty was obviously the confusion caused by the many arcs and lines. Pupils had not generalized the method of bisecting an angle sufficiently to make sure that their first arc in each case intersected the two sides of the angle being bisected. This conclusion is borne out by the fact that after pupils had discussed the method of bisecting an angle in general; that is, so that it applied to no particular figure, the errors were almost entirely eliminated.

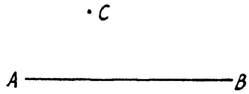
Constructing Perpendiculars

The next test (on the construction of perpendiculars) was given on the eighth day after pupils had practiced with Exs. 1–4 of the Test. In this test we find more varied complications of figures than in Tests 1 or 2 and consequently greater variations in results. It was given after Test 6 (see page 9) but before Test 6 was discussed.

Test 3

1. Construct a line perpendicular to AB at C.

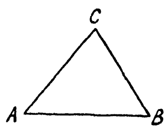
2. Construct a line perpendicular to AB from C.



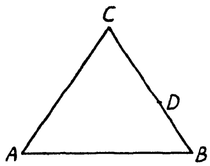
3. Construct a line perpendicular to AB at A.

4. Construct a line perpendicular to AB from C.

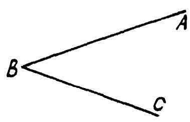
5. Construct a line from B perpendicular to AC.



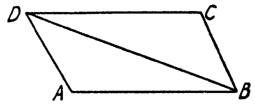
6. Construct a line from D perpendicular to AC.



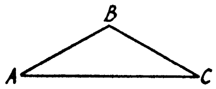
7. At B construct a line perpendicular to AB and another perpendicular to BC.



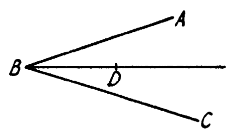
8. From *C* construct a line perpendicular to *DB*.



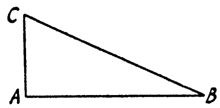
9. From *A*, *B*, and *C* construct lines perpendicular to the opposite sides of the triangle.



10. From *D* construct a line perpendicular to *AB*.



11. Bisect angle *C* and continue the bisector until it meets *AB*. Call the point of intersection *D*. From *D* construct a line perpendicular to *CB*.



12. Construct an isosceles triangle. From any point on the base of the triangle construct lines perpendicular to the equal sides.

TABLE 3

Percentage of Pupils Making Errors on Each Exercise of Test 3

Exercise	Percentage in Group			
	A	B	C	T
1	0	0	0	0
2	0	0	0	0
3	0	2	0	1
4	3	0	0	1
5	10	12	15	12
6	13	10	12	11
7	10	12	24	15
8	3	8	12	8
9	13	26	26	23
10	7	10	12	10
11	17	20	26	21
12	20	36	59	39

The following types of error were made.

1. Perpendicular drawn to the wrong line.
2. Perpendicular at or from the wrong point.
3. Wrong method of construction resulting in a line obviously not perpendicular.
4. Incomplete.
5. Angles bisected in Ex. 9.
6. Perpendicular bisectors of the sides constructed in Ex. 9.

There were no errors in Exs. 1 and 2 and the number of errors in Exs. 3 and 4 was negligible. These were the practice exercises. In these only one point and one line were involved. In the others, although the point and the line were explicitly stated, many pupils could not dissociate them from the rest of the figure and were confused. The reader should note how a very little change in the situation (the figures) causes a change in the results. For example, Ex. 7 requires the pupil do twice what he has already done correctly in Ex. 3, the main difference being the fact that when he is constructing the perpendicular to *AB* he must ignore *BC*, and when constructing the perpendicular to *BC* he must ignore *AB*. Nevertheless 15% of the pupils could not do Ex. 7 correctly. In Ex. 9, where there were three perpendiculars to construct and two of them required the extension of a line, 23% of the pupils were in error. (A very few of these were pupils who did not follow directions as stated in the preceding list.) This was in spite of the fact that they had demonstrated their ability to take care of the elements of this exercise by doing Exs. 2 and 4 correctly.

In the twelfth exercise we see what is likely to happen when the problem is further complicated by being stated in words without an accompanying figure. Thirty-nine per cent (59% in Group C) of the pupils could not do this exercise.

In subsequent class work it was found that the errors due to the complexity of the figures could be almost entirely eliminated by generalizing the method of construction. By analysis of the problem pupils were led to see that in every case the point of the compasses should be placed on the given point and the first arc drawn so that it would intersect the given line twice. If the line was not sufficiently long so that the arc would intersect it twice, it had to be extended. (See page 40.)

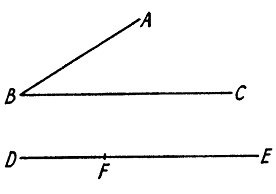
Constructing an Angle Equal to a Given Angle

The next two tests concern the construction of an angle equal to a given angle. In these we meet a new complication. Pupils were confronted for the first time with the necessity of thinking of the order of drawing lines in constructing a complex figure. The reader will see that this complication caused more errors than were found in any preceding test.

Test 4 was given on the tenth day. Pupils were familiar with angles and with the reading of angles and had practiced making an angle equal to a given angle with all lines in the position as shown in Ex. 1 until all but 4% (see Table 4) had mastered the construction.

Test 4

1. Construct an angle equal to angle *ABC* using *F* as vertex and *FE* as one side.



2. Construct a figure like the one below so that angle 1 will be equal to angle 2. In constructing the figure what line did you draw first? What line did you draw next?

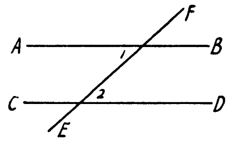


TABLE 4

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 4

Exercise	Percentage in Group			
	A	B	C	T
1	0	6	3	4
2	63	80	71	73

The actual construction required in Ex. 2 was the same as that in Ex. 1, but the pupils were under the necessity of drawing the lines in the correct order. An analysis of the results shows that 36% drew the lines in an impossible order for completion of the construction, 29% were entirely confused and 8% had wrong constructions for the angles.

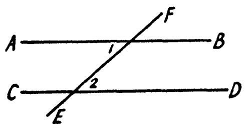
On the following day, the construction of an angle equal to a given angle was discussed from the general point of view and Ex. 2 of Test 4 was discussed as follows.

How many lines are there? What angles are we required to make equal? Suppose I draw *AB* and *CD* first and then draw *EF*. The angles 1 and 2 are already made and I have not constructed them equal. (This was illustrated at the board.) This shows that the order of drawing the lines is important. In order to construct an angle equal to another, you must first draw one of them. In this case draw either $\angle 1$ or $\angle 2$ before you try to construct the other.

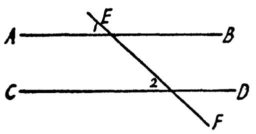
The discussion was left here. Pupils were not shown how to construct the figure. Test 5 was then given to discover the reactions of pupils after the kind of discussion reported above.

Test 5

1. Construct a figure like the one below right, so that $\angle 1$ will equal $\angle 2$. What line did you draw first? What line did you draw next?



2. Construct a figure like the one below right, so that $\angle 1$ will equal $\angle 2$. What line did you draw first? What line did you draw next?



3. Given triangle ABC. Construct another triangle DEF so that DE will equal AB, EF will equal BC, and $\angle E$ will equal $\angle B$.

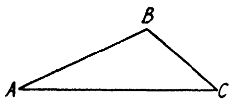


TABLE 5

Percentage of Pupils Making Errors on Each Exercise of Test 5

Exercise	Percentage in Group			
	A	B	C	T
1	23	34	33	31
2	37	48	56	47
3	73	80	85	80

The effect of the very brief discussion of Ex. 1 is seen in the comparison of the number of pupils who did this exercise incorrectly in Test 4 and in Test 5. In Test 4, 73% of the pupils were wrong. In Test 5 the corresponding per cent was 31. Only 3% drew the lines in the wrong order as compared to 36% in Test 4. There was little evidence of complete confusion as in the preceding test. The most frequent error was that of making the wrong angles equal (10%).

A very slight change in the problem from Ex. 1 to Ex. 2 caused the number of pupils in error to jump from 31% to 47%. In Ex. 2, 4% drew the lines in the wrong order and 15% made the wrong angles equal. And a seemingly very simple exercise (Ex. 3) so far as construction is concerned caused nearly double the number of errors (80%). In this exercise 29% were entirely confused as to method of procedure (the order of drawing the lines was again the probable cause), and 40% were wrong because they made either two angles and the included side or three sides of one equal to the corresponding parts of the other instead of two sides and the included angle.

The tendency (shown in Ex. 3) of many pupils to construct a figure, not according to directions, but so that it looks right in the end will be discussed in connection with the if-then relationship in the next chapter.

From the results of the foregoing tests we see that mere ability to perform a construction in a particular situation is not necessarily sufficient to assure ability to make the same construction in a slightly different situation. A small change in the situation causes difficulty. Pupils are confused by additional lines or the moving about of a figure. They do not dissociate the essential and particular parts to which they should give attention from the parts of a figure they should disregard. And when it comes to a figure which requires analysis to know what parts are to be drawn first, what second, and so on, the difficulties (if instruction is not given) are overwhelming.

Meaning of Terms

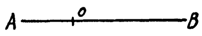
In order to discover pupils' reactions to the meaning of terms in complex figures the procedure employed in connection with constructions was used. Pupils were trained as to the meaning of certain terms to the point of mastery (or nearly so) with a simple figure and then tested on the meaning of these same terms using complex figures. The results of four tests are given here: one on *perpendiculars;* two on the terms *two sides and the included angle* and *two angles and the included side;* and one on *alternate interior angles* in connection with *parallel lines.*

Perpendiculars

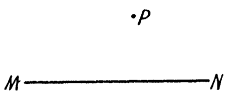
TEST 6

Use rulers only.

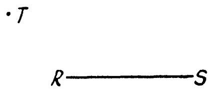
1. Draw a line perpendicular to AB at O.



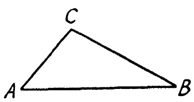
2. Draw a line perpendicular to MN from P.



3. Draw a line perpendicular to RS from T.

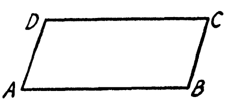


4. Draw a line from C perpendicular to AB.

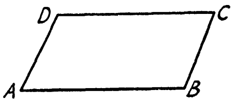


5. Draw lines perpendicular to AB at A and at B.

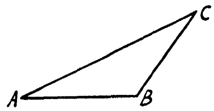
6. Draw a line from D perpendicular to AB.



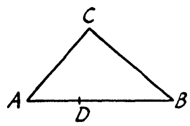
7. Draw a line from C perpendicular to AB.



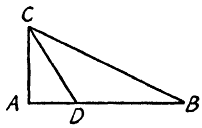
8. From B draw a line perpendicular to AC.



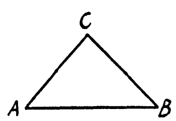
9. From D draw a line perpendicular to BC.



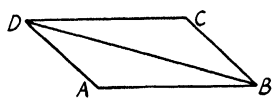
10. From D draw a line perpendicular to BC.



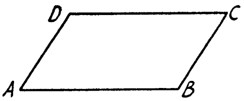
11. From A, B, and C draw lines perpendicular to the opposite sides.



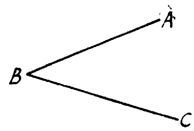
12. From C draw a line perpendicular to DB.



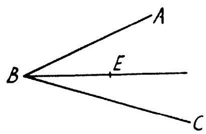
13. At B draw a line perpendicular to BC.



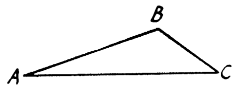
14. At B draw a line perpendicular to AB and another perpendicular to BC.



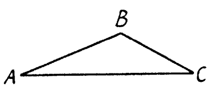
15. From E draw a line perpendicular to AB.



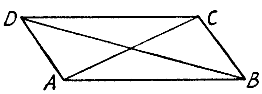
16. From C draw a line perpendicular to AB.



17. From A, B, and C draw lines perpendicular to the opposite sides.



18. From the ends of AC. draw lines perpendicular to DB.



This test was given on the sixth day (see the first footnote on page 4) after the meaning of *perpendicular* had been discussed and illustrated as in Exs. 1–3. Pupils had practiced drawing perpendiculars to horizontal lines as in these same three exercises. The purpose of the test was to discover whether the ability to draw perpendiculars to horizontal lines as in the practice exercises could be used in drawing perpendiculars in complex figures.

TABLE 6

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 6

Exercise	Percentage in Group			
	A	B	C	T
1	0	0	0	0
2	0	0	0	0
3	3	0	0	1
4	3	0	9	4
5	3	2	6	4
6	3	10	6	7
7	7	2	3	4
8	10	12	15	12
9	7	26	21	19
10	3	21	18	16
11	7	14	12	11
12	7	26	24	20
13	37	48	47	45
14	20	38	36	32
15	10	36	29	27
16	23	40	65	43
17	27	60	65	52
18	37	54	41	46

The practice exercises 1–3 required in each case a perpendicular to a horizontal line, and there was no other line to cause confusion. Exs. 4–7 also required perpendiculars to horizontal lines, but there were slight complications. The increase in the number of errors (to 4 and 7%) was not great but was consistent. In Exs. 8–12 pupils were asked to draw perpendiculars to lines that were not horizontal and immediately the number of errors increased considerably. The percentages of pupils in error on these exercises ranged from 11 to 20. The most frequent error was to draw a perpendicular to the wrong line or to draw it so that it was obviously not perpendicular. In the next two exercises, 13 and 14, we find a new complication. The perpendiculars were to be drawn at a point on a line where a second line met it. The percentage in error jumped to 45 in the thirteenth exercise and 32 in the fourteenth. Most of the errors on Ex. 15 were due to the fact that pupils drew a line perpendicular to the horizontal line *BE* instead of to *AB*. Exs. 16 and 17 required the extension of a line before the perpendicular could be drawn just as did Ex. 3. However, these two exercises required the drawing of perpendiculars to lines that were not horizontal and the figures contained lines which had to be ignored while drawing a particular perpendicular. While the percentage of pupils in error on Ex. 3 was only 1, the percentage on Ex. 16 was 43 and on Ex. 17 was 52. Ex. 18 was included for comparison with Ex. 12. The figure for the latter had two diagonals instead of one, and the language of the directions for it required more careful reading. Otherwise the two exercises were nearly the same. Yet the percentage in error on the eighteenth exercise was 46 as compared to 20 on the twelfth.

Choosing s.a.s. and a.s.a.

The next two tests, 7 and 8 as recorded here, were given on the forty-fifth day. Pupils had been proving triangles congruent for several days by means of the *two sides and the included angle* (s.a.s.) and *two angles and the included side* (a.s.a.) relations, but had had no specific practice in choosing these combinations, and had had no work at all with overlapping triangles. The purpose of Test 7 was to discover to what extent pupils would make errors in choosing s.a.s. and a.s.a. in figures of varying degrees of complexity, particularly in figures containing overlapping triangles.

Before the tests were given, a triangle *ABC* was drawn on the board, and the following exercises discussed.

Fill in the blanks so that the results will be—

1. s.a.s. of $\triangle ABC$. $AB, \cdots, BC$.
2. a.s.a. of $\triangle ABC$. $\angle A, \cdots, \angle C$.
3. a.s.a. of $\triangle ABC$. $\cdots, BC, \cdots$
4. s.a.s. of $\triangle ABC$. $\cdots, \angle C, \cdots$

Teachers were asked to continue the practice until they were reasonably sure that all pupils understood what they were expected to do.

Test 7

Fill in the blanks so that the results will be—

1. s.a.s. of $\triangle ABD$ (Fig. 1). $BD, \angle 3, \cdots$
2. s.a.s. of $\triangle DBC$ (Fig. 1). $BC, \angle 2, \cdots$
3. a.s.a. of $\triangle ABD$ (Fig. 1). $\angle A, \cdots, \angle 1$.
4. a.s.a. of $\triangle BDC$ (Fig. 1). $\angle 2, \cdots, \angle 4$.

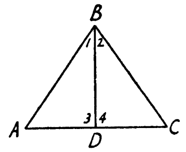


Fig. 1.

5. s.a.s. of $\triangle ABC$ (Fig. 2). $BC, \cdots, AC$.
6. a.s.a. of $\triangle ACD$ (Fig. 2). $\angle 2, \cdots, \angle 4$.
7. s.a.s. of $\triangle ACD$ (Fig. 2). $\cdots, \angle 4, \cdots$
8. a.s.a. of $\triangle ACD$ (Fig. 2). $\cdots, AC, \cdots$

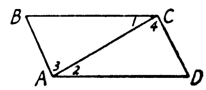


Fig. 2.

9. s.a.s. of $\triangle COD$ (Fig. 3). $CD, \cdots, DO$.
10. a.s.a. of $\triangle AOB$ (Fig. 3). $\angle 3, \cdots, \angle 2$.
11. s.a.s. of $\triangle AOB$ (Fig. 3). $\cdots, \angle 2, \cdots$
12. a.s.a. of $\triangle COD$ (Fig. 3). $\angle 1, DC, \cdots$

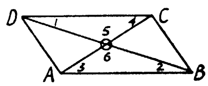


Fig. 3.

13. s.a.s. of $\triangle ACE$ (Fig. 4). $AC, \cdots, CE$.
14. a.s.a. of $\triangle BCD$ (Fig. 4). $\cdots, CD, \cdots$

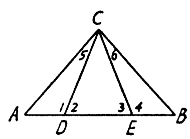


Fig. 4.

15. s.a.s. of $\triangle CAE$ (Fig. 5). $AC, \cdots, AE$.
16. s.a.s. of $\triangle CAD$ (Fig. 5). $AC, \angle A, \cdots$
17. a.s.a. of $\triangle ACD$ (Fig. 5). $\cdots, AC, \cdots$
18. a.s.a. of $\triangle ACE$ (Fig. 5). $\cdots, AC, \cdots$
19. s.a.s. of $\triangle AEB$ (Fig. 5). $\cdots, \angle B, \cdots$
20. s.a.s. of $\triangle CDB$ (Fig. 5). $\cdots, \angle B, \cdots$

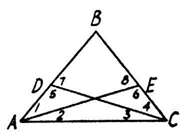


Fig. 5.

TABLE 7

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 7

Exercise	Percentage in Group			
	A	B	C	T
1	3	2	3	3
2	0	0	3	1
3	0	4	12	5
4	0	4	6	4
5	10	10	16	11
6	0	6	6	4
7	0	4	9	4
8	3	4	12	6
9	3	6	12	7
10	0	0	6	2
11	3	2	6	4
12	13	16	12	14
13	3	28	41	25
14	3	34	47	30
15	10	10	21	13
16	3	4	24	10
17	23	32	68	40
18	17	26	56	32
19	3	8	18	10
20	0	4	15	6

Exs. 13–20 involved overlapping triangles. The percentages of pupils in error on these exercises ranged from 6 to 40, half of which were 25 or higher as compared with a range from 1 to 14 of which all but two were 7 or below on the first twelve exercises. Throughout the test the exercises requiring a choice of sides caused fewer errors than those requiring the choice of angles.

TEST 8

1. Trace the three sides of triangle *BCD* with a colored pencil. Fill in the blanks so that the result will be a.s.a. of triangle *BCD*. (Fig. 1)

· · ·, *CD*, · · ·

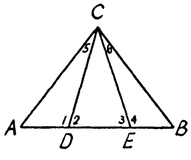


FIG. 1.

2. Trace the three sides of triangle *ACE* with a colored pencil. Fill in the blank so that the result will be s.a.s. of triangle *ACE*. (Fig. 2)

AC, · · ·, *CE*

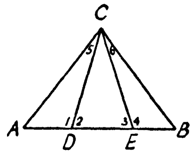


FIG. 2.

3. Trace the three sides of triangle *ACE* with a colored pencil. Fill in the blanks so that the result will be a.s.a. of triangle *ACE*. (Fig. 3)

· · · , *AC*, · · ·

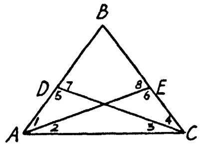


FIG. 3.

4. Trace the three sides of triangle *AEB* with a colored pencil. Fill in the blanks so that the result will be s.a.s. of triangle *AEB*. (Fig. 4).

· · · , ∠*B*, · · ·

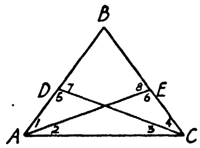


FIG. 4.

TABLE 8

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 8

Exercise	Percentage in Group			
	A	B	C	T
1	3	18	32	18
2	3	21	29	19
3	7	14	29	17
4	0	4	3	3

Except for the tracings with colored pencils, Exs. 1, 2, 3, and 4 of Test 8 were identical respectively with Exs. 14, 13, 18, and 19 of Test 7. Comparison of the results of these two sets of exercises shows that the use of colored pencils was effective.

Pupils could isolate a particular triangle when it was brought out by being traced with a colored pencil more easily than when not. The percentages of pupils in error on these exercises in Test 7 (with no tracing) were 30, 25, 32, and 10; on Test 8 (with tracing) the corresponding percentages were consistently lower, being 18, 19, 17, and 3. Even so, the percentages in error continued to be higher than in those exercises of Test 7 which did not involve overlapping triangles.

Again as in Test 7, it was found easier to choose a side (see Ex. 4) than to chose angles.

Alternate Interior Angles

The next test (Test 9) was given on the forty-ninth day. Pupils knew the definition of alternate interior angles, but had had experience with them only in case of a figure involving two parallel lines and a transversal (see figure for Ex. 1).

Test 9

Read carefully. If the lines are parallel as indicated in each of the following exercises, what *alternate interior angles* are equal? (There may be more than one pair.) Supply numbers in the angles if you need them.

1. (Fig. 1). If AB is parallel to CD,

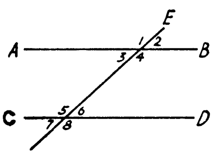


Fig. 1.

2. (Fig. 2). If AB is parallel to CD,
3. (Fig. 2). If AD is parallel to BC,

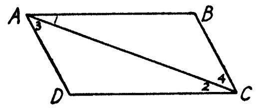


Fig. 2.

4. (Fig. 3). If DE is parallel to AC,

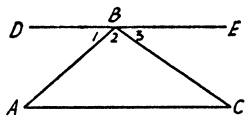


Fig. 3.

5. (Fig. 4). If AB is parallel to CD,

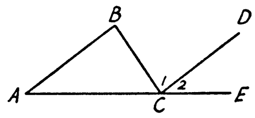


Fig. 4.

6. (Fig. 5). If AD is parallel to BC,
7. (Fig. 5). If AB is parallel to CD,

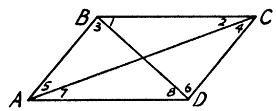


Fig. 5.

8. (Fig. 6). If AB is parallel to CF,

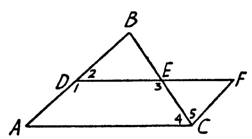


Fig. 6.

9. (Fig. 7). If AE is parallel to BD,

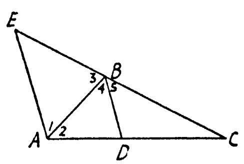


Fig. 7.

10. (Fig. 8). If BC is parallel to AF,

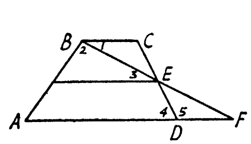


Fig. 8.

TABLE 9

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 9

Exercise	Percentage in Group			
	A	B	C	T
1	3	6	6	5
2	10	24	38	25
3	7	24	38	24
4	14	38	44	33
5	17	24	35	25
6	7	38	47	32
7	7	38	53	34
8	63	62	82	68
9	23	48	53	43
10	63	78	97	80

Typical errors on each exercise with the percentage of pupils making them, beginning with Ex. 2, are listed below.

Ex. 2. $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, 15%.

Ex. 3. $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, 14%.

Ex. 4. $\angle 1 = \angle A$ only, 7%; $\angle 3 = \angle C$ only 5%; $\angle 1 = \angle C$, $\angle 3 = \angle A$, 5%; $\angle 1 = \angle 3$, 4%.

Ex. 5. $\angle 1 = \angle A$, 4%; $\angle A = \angle 2$, 4%; none, 3%; $\angle 1 = \angle 2$, 3%; $\angle B = \angle C$, 3%.

Ex. 6. Four pairs of angles equal, 14%.

Ex. 7. Four pairs of angles equal, 10%.

Ex. 8. $\angle B = \angle 5$ only, 48%; $\angle 2 = \angle F$ only, 5%.

Ex. 9. $\angle 3 = \angle 2$, $\angle 1 = \angle 4$, 25%; none, 5%.

Ex. 10. $\angle 1 = \angle F$ only, 55%.

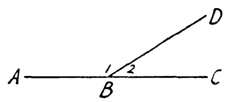
If we consider as correct those exercises where one pair of angles was given correctly and the others not mentioned, the percentages of pupils in error would read as follows for Group T: Ex. 4 (21%), Ex. 6 (26%), Ex. 7 (29%), Ex. 8 (15%), and Ex. 10 (25%). The errors of omission on Exs. 8 and 10 far outnumbered the errors of commission.

The figure of Ex. 1 was that with which the definition of alternate interior angles was given. Only 5% of the pupils made errors on this exercise. There was a decided increase in the number of pupils making errors on all the other exercises, as will be seen by a mere glance at the results recorded in Table 9.

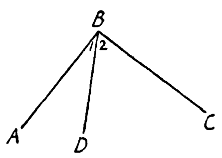
Recognition of the Application of Theorems in Complex Figures

We have already shown that even though a pupil may be able to perform a given construction in a simple figure he may not be able to perform the same construction in a complex figure and that terms which have meaning to him in a simple figure may not be clear to him in a complex figure. In this section of the chapter we propose to show that a similar conclusion may be drawn in regard to a pupil's recognition of the application of theorems in complex figures.

The following test (Test 10) was given on the twenty-sixth day. Pupils were familiar with the proposition: When one straight line meets another so as to form adjacent angles these angles are supplementary, in connection with a figure as in Ex. 1, but with no other figure. Before the test was given the following two figures were placed on the board, discussed from the point of view of the theorem, and left there while the pupils took the test.



$\angle 1$ is supplementary to $\angle 2$.



No.

The discussion of the two figures was as follows: In the first figure we have one straight line meeting another so as to form adjacent angles 1 and 2, so we write under it, "$\angle 1$ is supplementary to $\angle 2$." In the second figure we have adjacent angles, but they are not formed by one straight line meeting another. Hence under this figure, we write, "No."

Test 10

If one straight line meets another so as to form adjacent angles, these angles are supplementary.

In some of the figures on this paper, it is possible to apply the above theorem *one or more times*. In some of the figures it is not possible to apply the theorem. When it is possible, tell what angles are supplementary. When it is not possible, write the word "No."

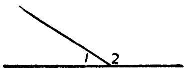


Fig. 1.

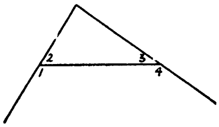


Fig. 2.

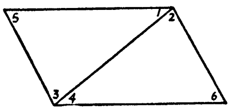


Fig. 3.

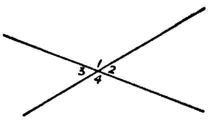


Fig. 4.

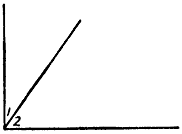


Fig. 5.

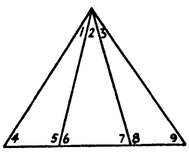


Fig. 6.

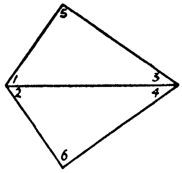


Fig. 7.

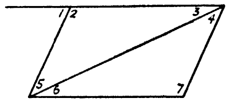


Fig. 8.

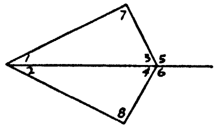


Fig. 9.

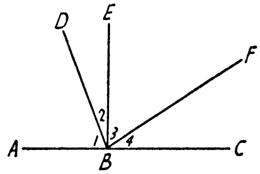


Fig. 10.

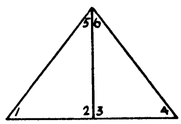


Fig. 11.

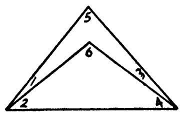


Fig. 12.

Overlapping angles caused serious difficulty in Ex. 10. In this exercise 82% of Group C and 67% of Group T were in error. The most frequent error was: ∠1 is supplementary to ∠2, ∠2 to ∠3, and ∠3 to ∠4 (30%). Exs. 2, 6, 9, and 11 caused appreciable difficulty, especially with

TABLE 10

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 10

Exercise	Percentage in Group			
	A	B	C	T
1	0	0	0	0
2	7	4	15	8
3	0	4	0	2
4	0	0	0	0
5	0	2	0	1
6	7	10	12	10
7	0	0	3	1
8	0	4	3	3
9	3	4	9	5
10	53	64	82	67
11	0	4	9	4
12	0	2	6	3

Group *C*. The most frequent errors were: Ex. 2, "No"; Ex. 6, "No"; Ex. 9, ∠3 is supplementary to ∠4 and ∠5 to ∠6. In Ex. 11 there were miscellaneous errors such as No, ∠5 is supplementary to ∠6, and ∠1 is supplementary to ∠4. Except for Ex. 10, however, the transfer was high.

The following test (Test 11) was given on the thirty-first day. The purpose was to discover to what extent pupils would recognize applications of the given three statements in the given figures. Exs. 1,2, and 3 were practice exercises.

TEST 11

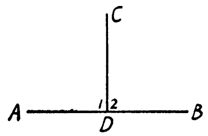
Read the three statements written below. Then look at each figure (and the given statement, if there is one), and decide whether any one of the three statements applies to it. If one of the statements does apply to a given figure, tell what angles are therefore equal and tell which statement applies. If the statements do not apply, write the word "No." Assume that all the lines are straight.

1. *A bisector divides an angle into two equal angles.*

2. *The adjacent angles at the foot of a perpendicular are equal.*

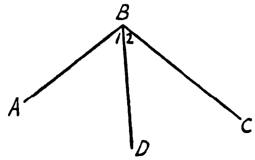
3. *If two straight lines intersect, the opposite angles are equal.*

Examples. In the figure below you are told that *CD* is perpendicular to *AB*. Statement 2 applies and ∠1=∠2. You should therefore write under the figure, "∠1=∠2 (statement 2)."



CD⊥*AB*.

In the figure below none of the three statements applies. You should therefore write under the figure the word "No."



Proceed similarly with the figures below.

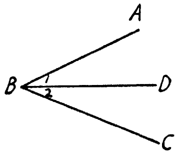


FIG. 1.
CD bisects ∠*B*.

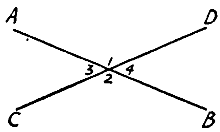


FIG. 2.

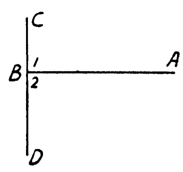


FIG. 3.
AB⊥*CD*.

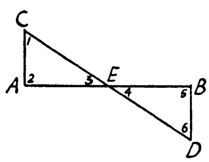


FIG. 4.

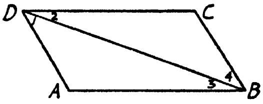


FIG. 5.

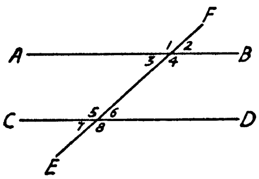


FIG. 6.

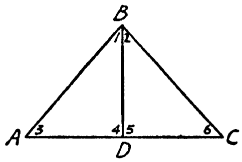


FIG. 7.
$BD \perp AC$.

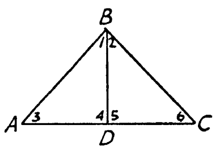


FIG. 8.
BD bisects $\angle B$.

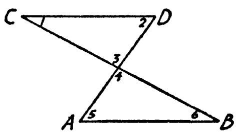


FIG. 9.

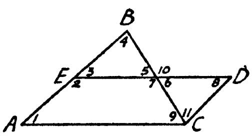


FIG. 10.

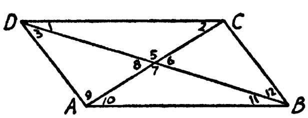


FIG. 11.

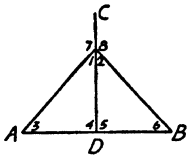


FIG. 12.
CD bisects *AB*.

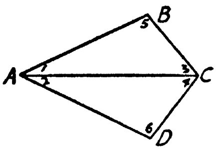


FIG. 13.
$AB = AD$.

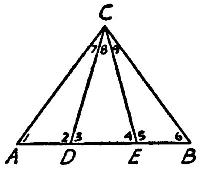


FIG. 14.

TABLE 11

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 11

Exercise	Percentage in Group			
	A	B	C	T
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0
4	3	22	18	16
5	10	32	24	24
6	7	12	24	14
7	7	8	15	10
8	23	22	38	27
9	3	18	6	11
10	3	22	21	17
11	10	22	24	19
12	53	82	79	73
13	23	50	41	40
14	13	14	6	11

Typical errors were as follows:

Ex. 4. Angles chosen equal because they appeared equal in the figure, statement 1 or 2 given as reason, 8%. "No," 6%. None of the 16% in error recognized the opposite angles.

Ex. 5. Angle bisectors assumed by 19%.

Ex. 6. "No," 7%. $\angle 1 = \angle 2$, etc., for various reasons, 7%.

Ex. 7. $\angle 1 = \angle 2$, $\angle 4 = \angle 5$ (statement 2). 7%.

Ex. 8. $\angle 4 = \angle 5$, $\angle 1 = \angle 2$ (statement 2), 10%. $\angle 4 = \angle 5$ (statement 1 or 2), 10%. "No," 3%.

Ex. 9. "No," 7%.

Ex. 10. "No," 10%. Omitted, 4%.

Ex. 11. Assumed angle bisectors, 8%. "No," 6%.

Ex. 12. $\angle 4 = \angle 5$ (statement 2), 26%. $\angle 1 = \angle 2$ (statement 1), 14%. $\angle 4 = \angle 5$ (statement 1), 13%. Several pairs equal (statement 1), 14%.

Ex. 13. Assumed angle bisectors, 32%. Omitted, 6%.

Ex. 14. Omitted, 10%.

Most of the errors on this test were due to a cause not under discussion in this chapter. They were due to the pupils' assuming equality of angles from the appearance of the figure and their lack of understanding that they might correctly draw conclusions only from the data. We shall therefore return to this test in Chapter IV. For our present purposes we should see that five figures (Exs. 4, 6, 9, 10, and 11) in addition to the second figure called for the recognition of opposite angles made by two intersecting lines and that the number of pupils in error on these exercises ranged from 11 to 19 per cent. This in spite of the fact that there were no errors on Ex. 1.

Test 12 was given on the forty-third day after pupils had been using *two sides and the included angle* and *two angles and the included side* in proving triangles congruent for several days. They had not, however, been cautioned against combinations that were neither of these. The purpose of the test was to discover to what extent pupils would realize that certain given combinations of sides and angles in triangles were neither *two sides and the included angle* nor *two angles and the included side*.

Test 12

Read the hypothesis of each exercise, then basing your answers only on the two theorems following, answer the questions. (Use the symbols s.a.s. = s.a.s. and a.s.a. = a.s.a.)

If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another, the triangles are congruent.

If two angles and the included side of one triangle are equal respectively to two angles and the included side of another, the triangles are congruent.

1. (Use Fig. 1.) Hyp. $AB = BC$, $BD = BD$. Are the triangles congruent? If so, why?
2. (Use Fig. 1.) Hyp. $AB = BC$, $BD = BD$, $\angle A = \angle C$. Are the triangles congruent? If so, why?

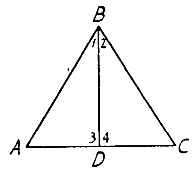


Fig. 1.

3. (Use Fig. 1.) Hyp. $AD = DC$, $BD = BD$, $\angle 3 = \angle 4$. Are the triangles congruent? If so, why?
4. (Use Fig. 1.) Hyp. $\angle 1 = \angle 2$, $\angle A = \angle C$, $BD = BD$. Are the triangles congruent? If so; why?
5. (Use Fig. 2.) Hyp. $\angle 1 = \angle 2$, $\angle 3 = \angle 4$, $BD = BD$. Are the triangles congruent? If so, why?
6. (Use Fig. 2.) Hyp. $AD = BC$, $BD = BD$, $\angle 3 = \angle 4$. Are the triangles congruent? If so, why?

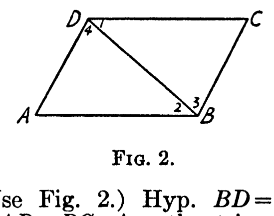


Fig. 2.

7. (Use Fig. 2.) Hyp. $BD = BD$, $\angle 1 = \angle 2$, $AD = BC$. Are the triangles congruent? If so, why?
8. (Use Fig. 2.) Hyp. $\angle 1 = \angle 4$, $\angle 3 = \angle 2$, $BD = BD$. Are the triangles congruent? If so, why?
9. (Use Fig. 3.) Hyp. $\angle 1 = \angle 6$, $\angle 4 = \angle 2$, $AB = DC$. Are triangles AOB and DOC congruent? If so, why?

10. (Use Fig. 3.) Hyp. $AO=OC$, $AB=DC$, $\angle 1=\angle 6$. Are triangles AOB and DOC congruent? If so, why?

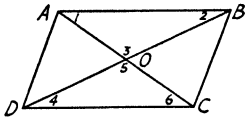


Fig. 3.

11. (Use Fig. 3.) Hyp. $\angle 1=\angle 6$, $\angle 3=\angle 5$, $AB=DC$. Are triangles AOB and DOC congruent? If so, why?

12. (Use Fig. 4.) Hyp. $\angle 1=\angle 2$, $\angle A=\angle C$, $BD=BE$. Are triangles ABD and CBE congruent? If so, why?

13. (Use Fig. 4.) Hyp. $AD=EC$, $BD=BE$, $\angle 4=\angle 5$. Are triangles ABD and CBE congruent? If so, why?

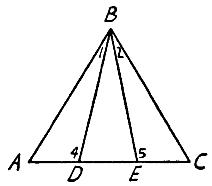


Fig. 4.

14. (Use Fig. 4.) Hyp. $\angle ABE=\angle CBD$, $AB=BC$ and $\angle A=\angle C$. Are triangles ABE and CBE congruent? If so, why?

TABLE 12

Percentage of Pupils Making Errors on Each Exercise of Test 12

Exercise	Percentage in Group			
	A	B	C	T
1	3	6	12	7
2	13	26	44	28
3	3	6	3	4
4	7	26	44	26
5	7	6	9	7
6	10	8	9	9
7	10	20	50	26
8	23	6	9	12
9	0	10	6	6
10	3	6	3	4
11	3	18	44	22
12	7	26	53	29
13	0	4	9	4
14	20	30	12	22

The high percentages in error (except in the case of Ex. 14) were on exercises in which the triangles could not be proved congruent by using only the given propositions and the given data (exs. 2, 4, 7, 11, and 12). Pupils who made errors on these exercises were not sufficiently alert to note that combinations of given sides and angles were neither *two sides and the included angle* nor *two angles and the included side*.

On the eighth exercise 12% said "No," probably due to the fact that they had already worked with many exercises in which $\angle 1=\angle 2$ and $\angle 3=\angle 4$. A change in the usual situation confused them. Note that Group A did the poorest work on this exercise.

The fourteenth exercise involved overlapping triangles. For this reason, evidently the number of errors was considerably higher than on the other exercises where the triangles were congruent under the given conditions. Judging from the number of pupils who said that triangles were congruent in exercises where they were not (under the conditions) it may well be that many guessed correctly on this exercise and the number confused by the overlapping triangles is reported too low. On 14% of the papers there was not a single "No" except on the first exercise.

We have now investigated pupils' reactions to complex figures from three points of view—in constructions, with regard to the meaning of terms, and in connection with the recognition of theorems. In all three respects we have come to the same conclusion. Changes in slight details in the figures affect the pupils' responses. Ability in connection with a simple figure does not insure ability in connection with a complex figure even though the complication may be slight. The assumption that pupils necessarily carry over what they have learned in connection with a simple figure and apply it without help to a complex figure is false. (See Chapter IX, "Transfer of Training.")

Chapter III

THE IF-THEN RELATIONSHIP

The if-then relationship is fundamental to postulational thinking which the pupil usually meets for the first time formally in demonstrative geometry. All propositions of geometry can be put in the if-then form. In fact, one mathematical philosopher defines mathematics as "the class of all propositions of the form, *P* implies *Q*";[1] that is, if *P* is true, then *Q* is true. The concept is so fundamental that it defies explanation in terms simpler than itself. We may explain that the statement —If two sides of a triangle are equal, the angles opposite those sides are equal—means that if you construct a triangle by making two sides equal, then the angles opposite those sides will be equal without further work on your part. But even so we are using the same sentence structure, simply making the statement in more concrete form. Furthermore, if we do this we are telling the truth but not the whole truth, for the statement holds whether we construct a figure for it or not. We might say as is said in the definition above—Two equal sides in a triangle implies two equal angles opposite those sides—but then we are explaining in terms less understandable than those originally given.

The literature on the teaching of geometry is replete with statements concerning the failure of pupils to grasp the logic of geometry. It is often said that pupils begin to memorize without understanding because they do not "know what it is all about." It is not hard to understand why pupils have difficulty when many of them have no inner feeling for the meaning of the if-then relationship. When they believe, as is shown (see Chapter VII), that the two converse statements. *If two sides of a triangle are equal the angles opposite those sides are equal,* and *If two angles of a triangle are equal, the sides opposite those angles are equal,* mean little more than the bald statement, *In an isosceles triangle two sides and two angles are equal,* they have little basis for beginning geometry with understanding.

That many pupils do not understand the logical implication of the if-then relationship before it is developed in the geometry class, and that the growth of the concept is slow after development is begun, will be shown by the results of the tests recorded in this chapter.

Constructing a Figure According to Data

The first test recorded in this chapter, Test 13, although not seemingly a direct test of pupils' understanding of the if-then relationship, nevertheless has a direct bearing upon it. Pupils were asked to construct the figure in the test making *AB* and *ED* perpendicular to *BD* and *C* the middle point of *BD*. If they bisected *BD*, that would be making *C* the middle point of *BD* directly. If, however, instead of bisecting *BD*, they made *AB* equal to *ED*, that would be making *C* the middle point of *BD* indirectly. The second method could have been proved correct provided the pupils had had knowledge of congruence, but they did not have this knowledge. The first method puts the statement, *C* is the middle point of *BD*, in the category of given conditions with respect to the figure; the second method puts it in the conclusion.

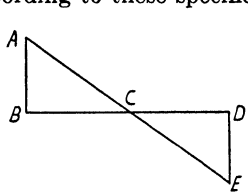
We do not claim that pupils misunderstood the difference between these two categories at this stage for they had not been discussed. We do claim that the large number of pupils who made the construction by the second method indicates very definitely that a fundamental teaching problem is involved. If a pupil makes a construction not according to the given conditions, but according to some other method that will make the figure look right in the end, he will argue that he is correct, and he can show measurements in

[1] Russell, Bertrand, *The Principles of Mathematics*, Allen and Unwin, Ltd., London, 1937, p. 1.

substantiation of his argument. The teacher knows that a fundamental principle is at stake—the difference between hypothesis and conclusion. The pupil does not know it, he does not even sense it. Therein lies the problem.

TEST 13

In the figure, $AB \perp BD$ and $ED \perp BD$. C is the middle point of BD. Construct the figure according to these specifications.



This test was given on the twelfth day. The purpose of the test was to discover whether the pupils would construct the figure directly according to the given conditions or indirectly by some other method. Pupils had had several days' practice with construction exercises, but no preparation toward the specific purpose of this test.

TABLE 13

PERCENTAGE OF PUPILS MAKING ERRORS ON TEST 13

Type of Error	Percentage in Group			
	A	B	C	T
1	33	44	41	40
2	0	2	6	3
3	0	8	12	7

The types of error indicated by the numbers 1, 2, and 3 are as follows:

1. Errors in connection with the bisector.
2. Errors in connection with the perpendiculars.
3. Other errors.

The following list shows further analysis of the errors under each of the three headings, together with the percentage of pupils in Group T making each type of error.

1(a). BD not bisected, instead AB made equal to DE, 34%.
(b). BD not bisected, AB not made equal to DE, 2%.
(c). Attempted bisection of BD incorrect, 4%.
2(a). Construction of perpendiculars incorrect, 1%.
(b). Attempt to construct perpendiculars from A and E (unknown points), 2%.
3(a). Attempt to make $\angle A = \angle E$ in addition to the correct construction, 4%.
(b). AE not through C, 2%.
(c). Entirely confused, 1%.

It is in the type of error numbered 1(a) that we are interested here. About one third of all the pupils thought that they were correct in making $AB = ED$ instead of making $BC = CD$. The distinction between hypothesis and conclusion is very definitely involved in this error.

Constructing a Figure According to the Conditions of an If-Then Sentence

On the same day (the twelfth) the following exercises (Test 14) were given as a more direct check of the pupils' understanding of the meaning of the if-then relationship.

TEST 14

1. Using rulers and compasses, construct a figure to test the truth of the following statement:

   If two sides of a triangle are equal, then the angles opposite those sides are equal.

2. Using rulers and compasses, construct a figure to test the truth of the following statement:

   If two angles of a triangle are equal, then the sides opposite those angles are equal.

Caution: Do not draw arcs on a figure *after* you have constructed it. If you wish to test the equality of any lines or angles after you have constructed a figure, do so with ruler or protractor.

Pupils knew how to construct a triangle with two sides equal. They also knew how to construct a triangle with two angles equal. They had not been asked to do an exercise like those in Test 14 before. The purpose of the test was to discover whether, without training, pupils would sense the meaning of the if-then relationship sufficiently to construct the figures for these two converse theorems differently. The first exercise requires the pupil to construct two sides equal, and the second, two angles equal in order that the triangles be made according to the given conditions.

TABLE 14

Percentage of Pupils Making Errors on Test 14

Percentage in Group			
A	B	C	T
37	40	50	42

The types of error and the percentage of pupils in Group T making each type follow:

1. Made angles equal in both figures, 12%.
2. Made sides equal in both figures, 7%.
3. Constructed one figure correctly, drew the other without construction, 10%.
4. Attempt to make both the sides and the angles equal, 4%.
5. Figure omitted for one of the exercises, 9%.

Without training, 58% of the pupils constructed two sides equal in the first exercise and two angles equal on the second exercise, thus showing an understanding of the meaning of the if-then relationship sufficient to construct figures for these exercises correctly. This leaves 42%, who judging from the types of errors made, and the simplicity of the task required, showed little understanding of the relationship.

On the following day (the thirteenth) Test 14 was discussed in the following manner. The first exercise means that if you make two sides of a triangle equal, then the angles opposite those sides will turn out to be equal whether you wish them to be equal or not. You have no control over the result. When you constructed a figure to test the truth of this exercise, should you have made two sides or two angles equal? You should have made two sides equal, and then should have measured the angles with a protractor to see if the angles were equal. The second exercise was discussed in like manner.

On the fourteenth day, the day following the preceding discussion, Test 15 was given. It was of the same type as Test 14. The purpose was to discover the reactions of the pupils as a result of the explanation given. It will be noted that the figures required were more complex than those in Test 14.

Test 15

1. Using ruler and compasses, construct a figure to test the truth of the following statement:

   If a line bisects the vertex angle of an isosceles triangle, it is perpendicular to the base.

2. Using ruler and compass, construct a figure to test the truth of the following statement:

   If a line is drawn from the vertex of an isosceles triangle perpendicular to the base, it bisects the vertex angle.

TABLE 15

Percentage of Pupils Making Errors on Test 15

Percentage in Group			
A	B	C	T
30	34	50	38

The types of error and the percentages of pupils making each type of error were as follows:

1. Bisected the vertex angle in both exercises, 12%.
2. Constructed perpendicular bisector of line in second exercise, 12%.
3. Made two base angles equal in both exercises, 7%.
4. Constructed perpendicular to base on both exercises, 3%.
5. Compasses not used on the second exercise, 4%.

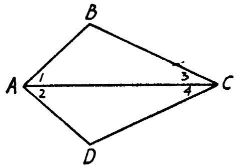
It is difficult to make a valid comparison of the results of Test 14 and Test 15. The exercises of the second are obviously more complex than those of the first, and we have shown in the preceding chapter that a pupil may be confused by a complication of figures. However, comparison of the two tests is not the important thing here. It is important to note that a large percentage of the pupils made errors even after the meaning of the if-then relationship had been explained concretely in connection with the construction of a figure. The usual procedure in geometry is to go ahead at once with the use of propositions of the if-then form on the tacit assumption that the underlying meaning is clear. Our evidence is against this assumption. The concept requires development over a period of time, and if pupils do not grasp its meaning readily in the concrete setting described here, the chances of their understanding it in the much more abstract setting of formal demonstration (without even a more careful development) certainly cannot be greater. (For a development of the meaning of the if-then relationship, see Chapter VII.)

That the growth of the concept requires careful attention over a period of time is confirmed by the results of the following test (Test 16). This test was given on the eighteenth day, six days after the subject was first introduced, and explanations similar to that after Test 14 had been given on each of these days.

TEST 16

Construct a figure like the one below to test the truth of each of the following exercises:

1. If $AB=AD$ and $\angle 1=\angle 2$, then $\angle 3=\angle 4$.



2. If $\angle 1=\angle 2$ and $\angle 3=\angle 4$, then $AB=AD$.

TABLE 16

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 16

Exercise	Percentage in Group			
	A	B	C	T
1	3	16	21	14
2	3	8	15	9

The types of error and the percentage of pupils in Group T making each type follow:

On Ex. 1. (1) A converse construction, 12%;
(2) Faulty construction, 2%.

On Ex. 2. (1) A converse construction, 6%;
(2) Faulty construction, 3%.

We find that at least 12% of the pupils (perhaps more since the 6% wrong on the second exercise may not be among the 12% wrong on the first exercise) persisted in constructing the figure contrary to the given conditions.

Choosing Hypothesis and Conclusion from a Verbal Statement

Discussion of the meaning of the if-then relationship was discontinued from this point to the thirty-eighth day. During the interval the meaning of deduction and the use of the axioms came under discussion. The procedure for explaining the meaning of the if-then relationship described in this chapter contains the germ of the method subsequently used in Part II. At this time, however, the method was not perfected and was not used consistently as it was in later classes. That there were many pupils who did not make the connection between the work already done and the procedure in drawing a figure and writing the hypothesis and conclusion in terms of the figure when a verbal statement in the if-then form is given may be seen from the table following Test 17 (given on the thirty-eighth day).

TEST 17

Draw a neat figure and write the hypothesis and conclusion for each of the following exercises:

1. If a line bisects the vertex angle of an isosceles triangle, it bisects the base also.
2. If lines are drawn from any point on the perpendicular bisector of a line to the extremities of the line, they are equal.
3. If, at any point on the bisector of an angle, a line is drawn perpendicular to the bisector and extended to meet the sides of the angle, it (the perpendicular) is divided into two equal parts by the bisector.

Pupils had a working knowledge of the meaning of hypothesis and conclusion for they had for several days been proving exercises in which the hypothesis and conclusion were given explicitly in terms of a figure. As a matter of fact, they had proved the very exercises of this test when given in this way. There had been no discussion of the procedure in drawing a figure and writing the hypothesis and conclusion in terms of that figure when an exercise was given in the form of a verbal statement.

TABLE 17

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 17

Exercise	Percentage in Group			
	A	B	C	T
1	37	60	76	59
2	13	42	50	37
3	13	42	68	42

The types of error were as follows:

1. Hypothesis incomplete.
2. Hypothesis incomplete with irrelevant additions.
3. Hypothesis complete with irrelevant additions.
4. Wrong conclusion.
5. A combination of (1) and (4).
6. A combination of (2) and (4).
7. A combination of (3) and (4).
8. Wrong figure.

The distribution of errors according to type showed no general tendency in one direction. The greatest number of errors was recorded under type 6, but this type covered so many particular kinds of error (really meaning confusion) that it does not indicate any specific cause of the errors.

The results recorded in Table 17 can lead us toward only one conclusion. Fully half of the pupils did not have a feeling for the meaning of the if-then relationship sufficiently accurate to allow them to write correctly the hypothesis and conclusion in terms of a figure. It would seem that mere guesses prompted the reactions rather than any feeling for the logical implications. Pupils saw the words of the propositions and understood them singly. Then, not knowing what to do about it, they inserted the relationships indicated (and others not indicated) indiscriminately under the headings hypothesis and conclusion.

CHAPTER IV

THE MEANING OF PROOF

THE PROOF of a proposition in geometry is deductive. Only what is implied by the data may be assumed concerning a figure. The reasoning proceeds by means of syllogistic thinking from the data to the final statement. Each intermediate conclusion becomes the basis for further deductions until the final conclusion is reached. No statement may be used as a reason unless it has been previously agreed upon. No deduction can be made correctly unless the conditions of the reason given have been completely fulfilled. A conclusion when drawn must agree explicitly with the conclusion of the authority.

All this must be assimilated consciously or unconsciously by a pupil before he can make a proof in geometry correctly—whether he writes the proof formally or thinks about it informally. Obviously a mere explanation of these points as in the preceding paragraph, or even a lengthy expansion of the ideas, would be insufficient. They are too abstract, too far removed from the pupils' experience to make *telling* effective.

A large percentage of pupils, when they begin to study geometry have little conception of what it means to draw a conclusion from a general statement, and a specific application of it. They will draw a conclusion when the conditions are not fulfilled, and they will draw irrelevant conclusions. They do not realize that they are restricted as to data and reasons they may use, but are prone to make inferences from the appearance of a figure, or to use a reason which they have manufactured. The evidence in this chapter shows that these are basic difficulties in the learning of demonstrative geometry.

The discussion continues under the headings, Deduction, Meaning of Hypothesis, Acceptable Reasons, and Proofs of Exercises.

Deduction

Deduction is seen in its simplest formal aspect in the simple syllogism which consists of three statements—a major premise, a minor premise, and a conclusion. The major premise is a general statement. The minor premise is a specific statement which fulfills the conditions of the major premise. The conclusion merely repeats the conclusion of the major premise with respect to some particular person or thing named in the minor premise.

When a pupil makes a correct deduction in geometry, he either does it by chance or memory, or he follows consciously or unconsciously the three steps of a simple syllogism. That many pupils have little realization of what it means to make a deduction is shown in the results of the following test (Test 18).

Test 18

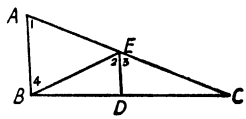
Read the directions carefully. In some exercises below, a third statement follows logically after the first two statements. In some of the exercises, a third statement does not follow logically. Whenever it is possible, write the third statement. If a third statement does not follow logically write the word "No."

I. 1. Any student who has room 323 for a home room is a senior.
   2. Mary is a student and has room 323 for a home room.
   3. Therefore, · · ·

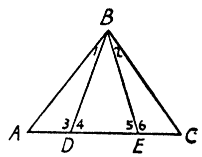
II. 1. Any student in this school who arrives after 8:30 A.M. is marked tardy.
   2. Henry, a student in this school, arrived this morning at 8:32.
   3. Therefore, · · ·

III. 1. All horses have four feet.
   2. This animal has four feet.
   3. Therefore, · · ·

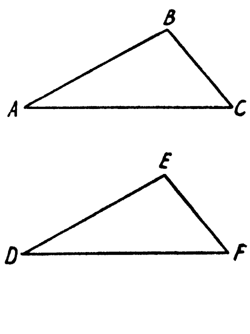
IV. 1. If two sides of a triangle are equal, the angles opposite the equal sides are equal.
   2. Angles 1 and 4 are the angles opposite the equal sides AE and BE in the triangle AEB.
   3. Therefore, · · ·



V. 1. If two sides of a triangle are equal, the angles opposite the equal sides are equal.
   2. In triangle DBE, $DB = BE$.
   3. Therefore, · · ·

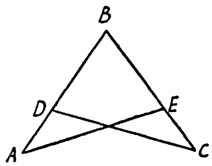


VI. 1. If two triangles have three sides of one equal to three sides of the other, they are equal.
   2. $AB = DE$ and $BC = EF$.
   3. Therefore, · · ·



VII. 1. Things equal to the same thing are equal to each other.
2. $a=b$ and $c=d$.
3. Therefore, · · ·

VIII. 1. If two sides of a triangle are equal, the angles opposite the equal sides are equal.
2. $AB=BC$.
3. Therefore, · · ·



Test 18 was given on the fifteenth day without preparation so far as the meaning of deduction is concerned. The theorem stated in Exs. IV, V, and VIII was familiar because of the work in constructions and the if-then relationship (see Test 14, page 21). The proposition in Ex. VI and the axiom in Ex. VII had not been mentioned in class. The purpose of the test was to discover whether the pupils, without training, would know when a deduction can be made from a given general statement, and when it cannot be made, and whether they could make correct deductions. The results of the test are shown in Table 18, and a detailed analysis of the errors made by Group T follows the table.

TABLE 18

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 18

Exercise	Percentage in Group			
	A	B	C	T
I	7	4	3	4
II	0	2	0	1
III	23	16	24	20
IV	17	32	21	25
V	20	28	41	30
VI	50	68	91	70
VII	30	58	71	54
VIII	53	72	97	75

A list of errors made by the pupils of Group T, with the percentage of pupils making each error, follows:

Exercise I.

Mary is a student, 1%.
"No," 3%.

Exercise II.

"No," 1%.

Exercise III.

This is a horse, 13%.
I do not know, 5%.
This animal has four feet, 2%.

Exercise IV.

$AE=BE$. 11%.
Triangle AEB is isosceles, 5%.
"No," 5%.
The triangles are equal, 1%.
The sides opposite $\angle$s 2 and 3 are equal, 1%.
Omitted, 2%.

Exercise V.

Omitted, 6%.
Several pairs of angles equal, 15%.
DBE is isosceles, 2%.
"No," 5%.
$\angle A=\angle C$, 1%.
$AB=BC$, 1%.

Exercise VI.

The triangles are equal, 32%.
$AC=DF$, 24%.
$\angle B=\angle E$, 1%.
I do not know, 5%.
The three angles are equal, 3%,
That does not make the triangles equal, the angles must be equal also, 1%.
Omitted, 4%.

Exercise VII.

$a=b$, $c=d$, 10%.
$a=c$, 12%.
$ab=cd$, 10%.
I do not know, 10%.
Omitted, 4%.
$b=d$, 2%.
$a=b$, $b=c$, $c=d$, 2%.
Things are equal to each other, 1%.
All the sides and angles are equal, 1%.
Things equal to the same thing are equal to each other, 2%.

Exercise VIII.

$\angle A = \angle C$, 58%.
Some pair of sides given equal, 5%.
I do not know, 3%.
It is an isosceles triangle, 2%.
$\angle D = \angle E$, 4%.
Omitted, 2%.
$\angle A = \angle B = \angle C$, 1%.

The first three exercises dealt with very simple everyday things. There was this difference, however, between the first two exercises and the third exercise. While a conclusion could be drawn correctly in the first two exercises, because the conditions of the general statement in each were fulfilled, no conclusion could be drawn correctly in the third exercise. This made a decided difference in the number of errors. The number of errors on the first two exercises was very small, but on the third exercise 20% of the pupils failed (13% of them saying, "Therefore it is a horse"). We are led to this conclusion, therefore—that while nearly all the pupils could make a deduction correctly in an everyday situation when there is no complication (the conditions were fulfilled), many were not sufficiently aware of what is involved to respond correctly when the conditions were not fulfilled.

The remaining exercises were geometric. In each case, the percentage of pupils who made errors was greater than the percentage in Exs. I, II, or III. The range of percentages was from 25 to 75. Exs. VI, VII, and VIII, in which the conditions of the general statement were not fulfilled, and the pupils should have written "No" meaning "No conclusion possible," caused by far the greatest number of errors.

Ex. IV was identical in form with Exs. I and II. The first premise was a familiar statement. The minor premise fulfilled the conditions of the first premise so explicitly that not even a glance at the figure was necessary in order to draw the correct conclusion. Yet 25% made errors on this exercise, as compared to 4% and 1% respectively on Exs. I and II.

Ex. V had the same first premise as Ex. IV, but in order to draw a conclusion pupils had to choose the angles opposite the given equal sides. On this exercise 30% made errors, 16% choosing the angles incorrectly.

The greatest number of errors was made on Ex. VIII, which again used the same first premise as Ex. IV. Most of the errors here, however, can be attributed to the complexity of the figure (see Chapter II). More than half of the pupils (58%) thought of the figure as a triangle with $\angle A$ and $\angle C$ opposite the equal sides and 4% said that $\angle D = \angle E$. This still leaves 13% who made other errors.

Of all the exercises, Ex. VI required the kind of reasoning most like that in the first simple exercises in congruence. In such exercises, pupils need to show that two sides and the included angle, or two angles and the included side, or three sides of one triangle, are equal to the corresponding parts of another triangle before they can draw a conclusion that the triangles are congruent. In Ex. VI we have one part of such an exercise isolated. The first premise called for the fulfillment of three conditions before a conclusion could be drawn. Only two of these conditions were fulfilled. Yet nearly three-fourths of the pupils (70%) did write a conclusion. What is more, less than half of these (32%) wrote the conclusion called for by the major premise. Many (24%) wrote as the conclusion the missing part of the conditions ($AC = DF$).

The tendency of many pupils to write as a conclusion something entirely irrelevant or something suggested by separate words or phrases of the major or minor premise is shown in the results of all the exercises except the first two. In Ex. III, 2% of the pupils made errors of this type. In Ex. IV, 18%. In Ex. V, 3%. In Ex. VI, 29%. In Ex. VIII, 7%. In Ex. VII, the showing in this respect was the worst. Although no conclusion could be drawn correctly, 54% of the pupils wrote a conclusion. The variety of conclusions written (see list) show a rather complete confusion of ideas.

What strikes us most is the diversity of answers showing that pupils do not know when the conditions of a statement have been fulfilled nor what conclusion they should draw when the conditions have been fulfilled. They have no clear notion that they must hold themselves to the given statements and draw conclusions entirely on the basis of these statements. Anything that is suggested to them by the words or phrases or the figure gives them a lead as to what to write. They use their imaginations instead of reasoning from the given statements.

As a result of this test, we are led to the following conclusions:

(1) Although pupils can make a deduction when simple everyday situations without any complications are given, they cannot necessarily do so when there are complications even in such simple situations.

(2) Pupils need to learn that a conclusion can be drawn only when all the conditions are fulfilled.

(3) They must learn to analyze a statement to find out what the conditions are that must be fulfilled, and they must be able to see whether all the conditions have been fulfilled.

(4) They must learn that when the conditions are fulfilled the only conclusion that can be drawn is the one stated in the general statement (first premise). Pupils certainly are not ready to make demonstrations in geometry until they have mastered these things.

This test was discussed on the seventeenth day, two days after the test was given. Pupils were shown what was wrong with their papers and the right answers were given. At that time we had no general method of discussing the concepts involved. The general method was developed later (see Chapter VIII). That the explanation given was not sufficient to carry over to an unusual situation will be seen from the results of the following test (Test 19) given on the nineteenth day.

Test 19

Write the conclusion if there is one; otherwise write "No."

I. 1. All girls have blue hair.
2. Mary is a girl.
3. Therefore, · · ·

II. 1. If two sides of a triangle are equal, the angles opposite those sides are right angles.
2. In this triangle, $AB = BC$.
3. Therefore, · · ·

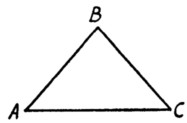


TABLE 19

Percentage of Pupils Making Errors on Each Exercise of Test 19

Exercise	Percentage in Group			
	A	B	C	T
1	3	6	6	5
2	40	56	68	55

The types of error and the percentage of pupils making each type follow.

Ex. 1. "No." 5%.

Ex. 2. "No." 30%.

Stated that the first statement was false, and went no further, 9%.

Wrote a wrong conclusion, 14%.

Omitted, 2%.

Examples of the wrong conclusion written for the second exercise are as follows:

(1) $\angle A = \angle C$,

(2) the third angle is a right angle,

(3) $\angle A = \angle C$, but they could not be right angles,

(4) then it could not be a triangle, and

(5) it is an isosceles triangle.

In both exercises, the first premise was a false statement. On the first exercise, however, only 5% of the pupils made errors. Pupils could probably conceive of a world in which all the girls have blue hair and therefore had little trouble in drawing a correct conclusion. They could not conceive of an isosceles triangle containing two right angles. What they had learned from the discussion of the preceding test was probably inhibited by the impossible situation from the concrete point of view. The result was that 55% either said that there was no conclusion, or wrote wrong conclusions as illustrated above. They had not yet learned that deduction is a sort of intellectual game in which the given statements must be followed explicitly.

The evidence so far recorded in this chapter leads the writer to believe that the cause of the logical errors so often made by pupils in proving exercises at the beginning of demonstrative geometry is fundamental. It goes back to a very hazy notion of the meaning of deduction. This statement seems only natural and might be granted without evidence except that writers of textbooks have not been sufficiently aware of this fundamental weakness to include a careful development of the meaning of deduction.

Meaning of Hypothesis

In making deductions concerning a geometric figure, we are allowed to assume with respect to it only those things that are explicitly given. These given things constitute the hypothesis. From these, and these alone, may we proceed, and by means of accepted reasons and repeated deductions arrive at a conclusion.

Such restrictions are contrary to the pupil's experience before he begins demonstrative geometry. He is used to judging from appearances. It is natural for him to take the total situation—intuitive notions, appearance of symmetry, estimations of size—as a background for his arguments in geometry. To be held to the implications of only a part of a total situation is a new experience. He has to learn the meaning of hypothesis.

Again we have a fundamental concept which is not made clear by mere telling. When Test 11 (see page 16) was given, pupils had already been proving with success simple exercises involving definitions, axioms, and supplementary angles. In all of these exercises, they were of course restricted to the hypothesis with respect to a figure. In spite of this, when they were tested in a way somewhat different from their immediate experience, they showed tendencies to draw conclusions from the appearance of the figures. (See list of errors on page 17. Note especially the errors on Exs. 12 and 13.)

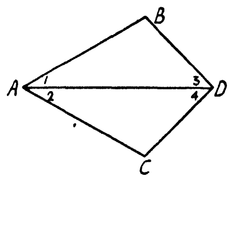
On the thirty-fourth day, just before beginning the work in congruence, another test (Test 20) was given to discover pupils' reactions to the meaning of hypothesis. As has already been said, the terms *hypothesis* and *conclusion* had been used in connection with simple exercises involving axioms and definitions, and also in exercises involving supplementary angles.

The purpose of the test was to discover whether the pupils would realize, without being specifically told in connection with this test that they must hold to the hypothesis and previously accepted propositions in choosing equal sides and angles. The only applicable authority in addition to the hypothesis was the postulate: *Any quantity is equal to itself.* This postulate had been discussed and accepted just previous to the test. It was illustrated by the use of a figure like that in the test. Pupils had been told that if they were asked if any sides of triangle ABD were equal to sides of triangle ACD, they could say that $AD = AD$, and that the reason would be: "Any quantity is equal to itself (abbreviation, *identity*)."

TEST 20

Look at the hypothesis and conclusion of the following exercises, then answer the questions.

1. Hyp. $AB = AC$, $\angle 1 = \angle 2$.
   Con. $\angle 3 = \angle 4$.
2. Hyp. $\angle 1 = \angle 2$, $\angle 3 = \angle 4$.
   Con. $\angle B = \angle C$.



Are there any lines or angles in one triangle which are equal to lines or angles in the other triangle of each exercise? If so, tell which ones they are and tell why they are equal.

TABLE 20

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 20

Exercise	Percentage in Group			
	A	B	C	T
1	30	60	57	51
2	23	56	59	48

The distribution of errors follows:

1. Some of the correct statements omitted, no additions, correct reasons. (Ex. 1, 5%; Ex. 2, 6%.)
2. All the correct statements given, reasons correct, wrong statements in addition. (Ex. 1, 17%; Ex. 2, 17%.)
3. All the correct statements given, wrong reasons, no additions. (Ex. 1, 2%; Ex. 2, 2%.)
4. All the correct statements given, wrong reasons, wrong statements added. (Ex. 1, 4%; Ex. 2, 0%.)
5. Some of the correct statements omitted, wrong reasons, no additions. (Ex. 1, 3%; Ex. 2, 3%.)
6. Some of the correct statements omitted, correct reasons for the right statements, wrong statements added. (Ex. 1, 11%; Ex. 2, 6%.)
7. Some of the correct statements omitted, wrong reasons, wrong statements added. (Ex. 1, 10%; Ex. 2, 11%.)
8. Omitted. (Ex. 1, 0%; Ex. 2, 3%.)

Examples of errors of the above types follow:

Type 1

Ex. 1. $AB=AC$ (hyp.); $AD=AD$ (identity)

Ex. 2. $AD=AD$ (identity)

Type 2

Ex. 1. $AD=AD$ (identity); $AB=AC$, $\angle 1=\angle 2$ (hyp.); $\angle 3=\angle 4$ (parts of a bisected angle); $BD=DC$ (results of subtracting equals from equals).

Ex. 2. $AD=AD$ (identity); $\angle 1=\angle 2$, $\angle 3=\angle 4$, $\angle B=\angle C$ (hyp.).

Type 3

Ex. 1. $AD=AD$ (identity); $AB=AC$ (they are equal to the same line); $\angle 1=\angle 2$ (parts of a bisected angle).

Ex. 2. $AD=AD$ (identity); $\angle 1=\angle 2$, $\angle 3=\angle 4$ (parts of a bisected angle).

Type 4

Ex. 1. $AD=AD$ (identity); $AB=AC$, $BD=DC$, $\angle 1=\angle 2$, $\angle 3=\angle 4$. (A bisector divides an angle into two equal parts. Therefore, all lines and angles of the angle will be equal.)

Type 5

Ex. 1. $AD=AD$ (identity); $\angle 1=\angle 2$ (parts of a bisected angle).

Ex. 2. $\angle 3=\angle 4$ (parts of a bisected line).

Type 6

Ex. 1. $AD=AD$ (identity); $AB=AC$ (hyp.), $BD=CD$ (equals subtracted from equals).

Ex. 2. $\angle 1=\angle 2$, $\angle 3=\angle 4$ (hyp.); $\angle B=\angle C$ (conclusion).

Type 7

Ex. 1. $AB=AC$ (hyp.); $\angle 1=\angle 2$, $\angle 3=\angle 4$ (angles equal to the same angle), $BD=DC$ (equal to the same line).

Ex. 2. $\angle 1=\angle 2$, $\angle 3=\angle 4$ (parts of a bisected angle); $\angle B=\angle C$ (equal to the same angle).

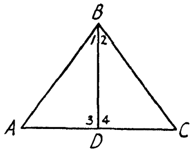
Another test, recorded here as Test 21, was given for the same purpose—to discover whether pupils had grasped the meaning of hypothesis. The test follows.

TEST 21

Cross out the parts of the following statements that are not necessarily true.

1. If BD bisects $\angle B$, then I know for certain that

$$AB=BC$$
$$\angle 1=\angle 2$$
$$\angle 3=\angle 4$$
$$\angle A=\angle C$$
$$AD=DC$$



2. If BD is perpendicular to AC, then I know for certain that $AB=BC$ (then followed the same statements as in the first exercise).
3. If BD bisects AC, then I know for certain that $AB=BC$ (etc. as in Ex. 1).
(The figure was repeated for Exs. 2 and 3.)

Test 21 was given on the thirty-sixth day. Pupils had been working with exercises requiring them to hold to given conditions from the twentieth day on. In fact, just preceding this test they had proved triangles congruent in a figure like the one here, using as hypothesis the kind of data in this test. (For pupils' success in this preceding work, see Test 24 and Table 24, page 34.)

TABLE 21

PERCENTAGE OF PUPILS MAKING ERRORS ON EACH EXERCISE OF TEST 21

Exercise	Percentage in Group			
	A	B	C	T
1	3	20	33	19
2	3	12	35	17
3	10	40	53	36

A more detailed analysis of the errors follows:

Ex. 1. All five statements left as true, 2%.
Statements 1 and 2 left, 3%.
Statements 1 and 5 left, 6%.
Three statements left as true, 8%.

Ex. 2. All five statements left as true, 0%.
Two statements left, 10%. (6% left statements 3 and 5.)
Three statements left, 4%.
Four statements left, 3%.

Ex. 3. All five statements left as true, 4%.
Two statements left as true, 22%. (18% left statements 3 and 5.)
Three statements left as true, 6%.
Four statements left, 3%.

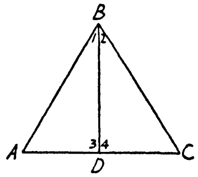
The fact that so few pupils left as true all five statements in each exercise argues for the conclusion that some attempt was made to adhere to the given facts and not to draw conclusions entirely from the appearance of the figure. In the third exercise, 18% were evidently confused between *bisector* and *perpendicular bisector.* Not so many (6%) were confused between *perpendicular* and *perpendicular bisector* in the second exercise. Even though the pupils had for some time been proving exercises deductively, many did not fully appreciate the meaning of hypothesis. When confronted with a novel situation, they did not restrict themselves to the data, but drew conclusions from the appearance of the figure.

The next test (Test 22) was given as soon as Test 21 had been completed. It was intended as a teaching device. Its primary purpose was to show those pupils who had made errors in the preceding test that the conditions of each of the first three exercises were not sufficient to determine the figure, that the figure could be drawn with various shapes and still adhere to the conditions. It was to show them objectively why they were wrong in drawing their conclusions. The test itself, however, failed in this purpose. The given figures were so potent that more pupils missed the possibility of changing them than made errors in the preceding test. The point was not made clear until the test was discussed. The test and the results are recorded here to show that the characteristics of a particular figure may have more effect upon a pupil's reasoning than the conditions given in connection with it.

TEST 22

1. If the only thing you know about this figure is that it is a triangle ABC, and that BD must bisect $\angle B$, can you draw

the figure so that AD does not equal DC? If so, draw it. If not, tell why not.



2. If the only thing you know about this figure is that it is a triangle ABC, and that BD must be perpendicular to AC, can you draw the figure so that AB is not equal to BC? If so, draw it. If not, tell why not.
3. If the only thing you know about this figure is that it is a triangle ABC, and that BD must bisect AC, can you draw the figure so that AB is not equal to BC? If so, draw it. If not, tell why not.
4. If the only thing you know about this figure is that it is a triangle ABC, and that AB must equal BC and BD must bisect $\angle B$, can you draw the figure so that AD does not equal DC? If so, draw it. If not, tell why not.

TABLE 22

Percentage of Pupils Making Errors on Each Exercise of Test 22

Exercise	Percentage in Group			
	A	B	C	T
1	13	32	39	29
2	13	32	35	28
3	47	52	68	55
4	43	50	59	51

On Ex. 1, 20% said "No," 7% drew the figure incorrectly, and 2% omitted the exercise.

On Ex. 2, 18% said "No," and 10% drew the figure incorrectly.

On Ex. 3, 38% said "No," 13% drew the figure incorrectly, and 4% omitted the exercise.

On Ex. 4, only 3% said that the figure could be redrawn. The other 48% in error were so counted because their reasons were wrong. This exercise will be discussed in the next section, entitled "Acceptable Reasons."

Examples of reasons given on Exercise 1. The parts of a bisected angle are equal. If you bisect "correct" and draw the lines straight, it will have to come out equal. If BD bisects $\angle B$ it also bisects AC; no matter how large $\angle B$ is AC always corresponds. One of the angles in the triangle is bisected thus making $\angle 1 = \angle 2$. Each time you enlarge or decrease $\angle 1$ and $\angle 2$ you enlarge or decrease AD and DC. It is an isosceles triangle, and it will always be in the middle of AC.

Examples of reasons given on Exercise 2. If the perpendicular is correct and the lines straight, it will have to be equal. You can make the perpendicular any which way and AB will equal BC. When a line is bisected it is divided into two equal parts, so AB will always be equal to BC. Any point on BD connected with A and C will make equal lines; the triangles are congruent.

Examples of reasons given on Exercise 3. The bisector makes AB equal to BC. AB and BD meet at the same point to construct the triangle. AB and BC must make the same angle with AD and DC in order to meet at the same point to form a triangle, so AB will equal BC. One line bisecting an angle will naturally be perpendicular to the other side, making the remaining sides equal.

Acceptable Reasons

In proving a statement in geometry we are restricted in our reasons to the hypothesis, definitions, the unproved propositions, and the propositions we have proved up to that point. To one who has studied mathematics for some time these restrictions seem natural enough. To the beginner in demonstrative geometry they seem more or less absurd. Until the concept is built up, the beginner will give any kind of reason which seems useful to him, whether it has been previously accepted or not. "It has to be," "It looks so," "It cannot be any other way," "If those two lines go up evenly, the angles have to be equal," "Any one can see that,"—these are reasons that often seem justifiable to a beginner. He does not restrict himself to the hypothesis and propositions which have been previously accepted, but argues from the total situation and anything in his experience, intuitive or otherwise, which comes to his mind. (See also page 52 and following pages.)

Reasons given in answer to the questions of Test 22 indicate the tendency of pupils to give "made up" reasons. We have already given examples of these reasons for Exs. 1, 2, and 3. We shall now discuss the answers to Ex. 4.

Unlike the data for the first three exercises, the data for this exercise determined the figure. It could not be drawn so that AD would not equal DC. Most of the pupils saw this. Only 3% of them said that it could be redrawn. But in spite of the fact that a large part of the class could have proved the triangles congruent correctly if they had been asked specifically to do so (see Test 24, page 34), it did not occur to 48% of them to mention the congruence of the triangles as the most acceptable reason for their answer. Twenty-five per cent of the pupils gave as a reason, "AD will always 'turn out' equal to DC so long as we keep to the given conditions," 23% gave manufactured reasons. They reverted to intuitions instead of using the principles of deduction, which they had been using in their formal proofs.

Examples of the manufactured reasons are as follows:

The two parts of AC came out equal; it could not help it. You could not have equal sides if you drew AD unequal to DC. If BD bisects $\angle B$, then the line must go through the middle of AC. When an angle is formed, and then it is bisected, and both sides of the angle are equal, the base must meet both sides at A and C. The bisection of the angle and the fact that $AB = AC$ makes $AD = DC$. It is an isosceles triangle, therefore BD will always be the middle point of AC. It always happens to come out with AD and DC equal. When you connect the limits of these two lines, it will cut off an equal distance on both lines making an isoscelestriangle, and naturally when the bisector of angle B meets this line, it will bisect the base of the triangle making $AD = DC$. It just cannot be done, my brain tells me that.

When pupils do not grasp the fact that their reasons in deductive proofs are restricted to those which have been accepted either by assumption or proof, they are likely to miss the underlying purpose of a deductive science. Up to this time in their experience a "proof" has been something to convince them of the truth of a statement. In elementary geometry a "proof" is not necessarily for this purpose. Many of the theorems proved could easily be accepted intuitively. (It is for this reason that some advocate the postulation of all such statements in a first course in deductive geometry.) The real purpose is to show how these theorems are dependent upon previously accepted propositions, and how they grow out of the previous statements by means of deduction and make one large logical whole. The study of demonstrative geometry has more meaning when this point is made clear by specific development.

Not so difficult for pupils to realize as the restriction of reasons just discussed is the fact that the list of reasons is increased by the proof of theorems. Nevertheless this is a point that should receive some attention. As soon as a theorem has been proved, it may be used as an authority in succeeding proofs. If this were not so, each proof would have to be reduced to the postulates and proofs would become increasingly cumbersome. Some pupils grasp this situation readily, others do not grasp it so readily, as shown by the results of the following test (Test 23).

Test 23 was given on the forty-first day immediately after the proof of the theorem, "If two sides of a triangle are equal, the angles opposite those sides are equal," had been discussed in class. The statement was made, "We may now accept this theorem and use it in proofs," but it was not emphasized, and the method of using it was not shown. The purpose of the test was to see what percentage of the pupils would use the theorem in their proofs. Previous to this time they had become used to using four theorems, (1) When one straight line meets another so as to form adjacent angles, the angles are supplementary, (2) Supplements of the same angle or of equal angles are equal, (3) Complements of the same angle or of equal angles are equal, and (4) When two straight lines intersect, the opposite angles are equal. All other reasons had been postulates or definitions.

Test 23

Prove the following exercise.
Hyp. $AB=BC$, $DACE$ is a straight line.
Con. $\angle 3=\angle 4$.

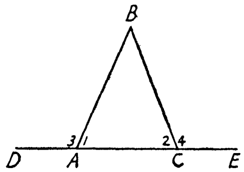


TABLE 23

Percentage of Pupils Who Proved the Exercise in Test 23 Correctly

Column 1 shows the total percentage, Column 2 shows the percentage of pupils who used the new theorem, and Column 3 the percentage of pupils who proved the exercise by means of congruent triangles.

Group	1	2	3
A	100	57	43
B	76	54	22
C	68	12	56
T	80	42	38

Of the 80% who proved the exercise correctly, about half (42%) used the new theorem, and about half (38%) proved it without the new theorem. Of the 20% in error, all of whom bisected angle B and used the congruent triangle method, 13% said that $\angle 3$ and $\angle 4$ were corresponding parts of congruent triangles (an error under the heading of "complex figures").

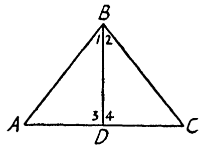
Proofs of Exercises

In order to show the types of errors made by pupils in formal proofs, we shall record the results of two tests. The first test, Test 24, consisted of two exercises which had been discussed in class; the second, Test 25, contained four "originals," each containing some element that was new to the pupils.

Test 24 was given on the thirty-sixth day. Pupils had had the work on the preceding day as shown in Test 20, page 29, had continued with the development proving these triangles congruent, and their corresponding parts equal, and had finally discussed the two exercises which appear in the present test.

Test 24

1. Hyp. BD bisects $\angle B$, $AB=BC$.
   Con. $\angle A=\angle C$.



2. Hyp. $AB \perp DE$, AB bisects DE.
   Con. $\angle 1=\angle 2$.

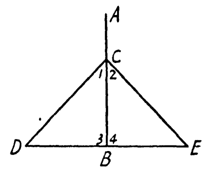


TABLE 24

Percentage of Pupils Making Errors on Each Exercise of Test 24

Exercise	Percentage in Group			
	A	B	C	T
1	3	12	12	10
2	20	28	41	30

In spite of the fact that these exercises had already been discussed in class, 10% of the pupils made errors on the first exercise, and 30% on the second. The errors were distributed as follows:

Errors due to "complex figure"; Ex. 1 (1%), Ex. 2 (7%).

Errors due to weakness in deduction; Ex. 1 (5%), Ex. 2 (20%).

Confused (multiplicity of errors); Ex. 1 (4%), Ex. 2 (3%).

The errors due to weakness in deduction can be further distributed as follows:

Unacceptable reasons: Ex. 1 (2%), Ex. 2 (0%).

Unfulfilled conditions; Ex. 1 (0%), Ex. 2 (13%).

Wrong conclusion from a stated reason, Ex. 1 (3%), Ex. 2 (7%).

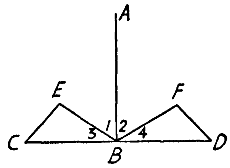
Although the number of pupils making errors may seem small, particularly in the first exercise, we cannot be at all sure from the results of this test that the pupils who made no errors had a full understanding of the proofs. Memory played a large role. Tests 21 and 22 (see pages 30 and 31) were given immediately after Test 24. They involved figures of the same type as those in Test 24 and the same type of hypothesis. Yet the percentages of pupils in error on these two subsequent tests were larger than those on Test 24. Teachers should beware of the false assumption that because a pupil can reproduce a proof once seen, he understands it. At the same time we see that some pupils persist in the types of error we have discussed, even when they are reproducing a proof.

On the forty-sixth day, four days before the conclusion of the study, a final full period test was given on congruent triangles. The purpose of the test was to discover the types of errors made on original exercises. Each of the four exercises on the test required original thinking in some particular. Pupils had done an exercise in which they were required to use complementary angles in order to prove triangles congruent, but had never had an exercise where it was necessary to prove triangles congruent, and then make use of complementary angles as in Ex. 1. They had not proved triangles congruent in connection with overlapping triangles as in Ex. 2. The use of the axiom, "If equals are added to equals, the sums are equal," had never been used in connection with lines in an exercise where triangles were to be proved congruent as in Ex. 3. In Ex. 4, the necessity of proving two sets of triangles congruent was new.

Test 25

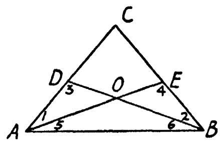
1. Prove the following exercise:

Hyp. AB is the perpendicular bisector of CD. $CE=DF$, $\angle C=\angle D$.
Con. $\angle 1=\angle 2$.



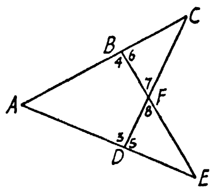
2. Prove the following exercise:

Hyp. $AC=BC$, $AD=BE$.
Con. $AE=BD$.



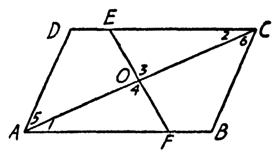
3. Prove the following exercise:

Hyp. $AB=AD$, $BC=DE$.
Con. $\angle C=\angle E$.



4. Prove the following exercise:

Hyp. $AB=DC$, $AD=BC$, EF is a straight line passing through O, the middle point of the diagonal AC.
Con. $FO=OE$.



Suggestion: It is sometimes necessary to prove that more than one pair of triangles are congruent before you can arrive at your conclusion.

Pupils were allowed sixty minutes for the test. At the end of twenty minutes they were requested to begin the second exercise whether they had finished the first or not. Similarly, at the end of the next ten minutes, they were asked to begin the third exercise. Thus, all pupils tried at least three exercises.

The percentages of pupils making errors are recorded in the following table (Table 25).

TABLE 25

Percentage of Pupils Making Errors on Each Exercise of Test 25

Exercise	Percentage in Group			
	A	B	C	T
1	40	50	73	54
2	33	36	62	43
3	53	72	77	68
4	23	40	74	46

The errors in Group T were distributed as follows:

	Ex. 1	Ex. 2	Ex. 3	Ex. 4
1. Errors due to complexity of figure	18%	3%	24%	14%
2. Conditions unfulfilled	26%	22%	7%	14%
3. Unacceptable reasons	0%	1%	4%	2%
4. Wrong conclusion from a stated reason	0%	0%	3%	0%
5. Overdetermined construction line	0%	5%	10%	0%
6. Worthless (many errors)	5%	1%	0%	0%
7. Could not find a method	5%	11%	19%	14%
8. No method after proving the first pair of triangles equal in Ex. 4	—	—	—	3%

We see, therefore, that the statement made early in this study, that errors due to complexity of figure, and errors due to not understanding the meaning of proof, persist and account for the largest number of errors committed. The errors of omission such as those numbered 7 and 8 are beyond the scope of this study. They require an entirely different kind of analysis. The error numbered 5 is one that creeps in in certain particular kinds of exercises. It is not an error typical of all exercises. Note that errors due to not understanding the meaning of the if-then relationship to the extent of not being able to write the hypothesis and conclusion correctly have been avoided in this test. The hypothesis and conclusion of each exercise was given explicitly so that this error would not appear.

Throughout the study there is little, if any, evidence that the lower I.Q. groups make different kinds of errors from those made by the upper groups. In almost every case some pupils of the upper groups made mistakes which were made by the lower groups. It is true, however, that fewer pupils in the upper groups made mistakes. Only in a very few unexplainable cases did Group A make more mistakes than Group B or Group C.

By carrying this experiment over a period of fifty days, watching, recording, analyzing, and classifying pupils' reactions, we have come to the conclusion that changes in the teaching of geometry are necessary to take care of pupils' difficulties. Their difficulties are fundamental. Halfway measures will not suffice. In order that pupils may pursue the study of demonstrative geometry with understanding, and realize the aims for which it is taught, methods must be devised to help them not so much with specific difficulties as with the three fundamental difficulties we have discussed.

Three Major Difficulties in the Learning of Demonstrative Geometry

By Rolland R. Smith

Part II

DESCRIPTION AND EVALUATION OF METHODS USED TO REMEDY ERRORS

Chapter V

PROCEDURE IN PART II

During the interval between the analysis of the results in Part I, and the beginning of the experiment for Part II of this study, methods of dealing with the three fundamental difficulties were devised and tried out on several successive classes. Elements of the methods which were awkward or inefficient were dropped and replaced by others. Finally, a complete course of study based on the new methods was written.

In the fall of 1938 a study was made to show that the new methods actually did what they were intended to do; that is, to help pupils with their difficulties. The usual method of procedure would be to set up two comparable groups, teach one group by the new methods and the other by the old methods, and then compare the groups. This was not possible here. All the mathematics teachers in the school had been working with enthusiasm trying out the new methods. None of them could have gone back to the spirit of the old methods. Besides, we did not consider it a fair proposition to teach any group of pupils by any but the best methods we knew. It was necessary, therefore, to make use of a different procedure.

The new experiment was set up in such a way that Group T of the 1932 study might be considered as the control group. From three classes 74 pupils (Group D) with the same mean I.Q., and same standard deviation as Group T, were chosen. This equated the groups so far as I.Q.'s are concerned. On each day of the 1938 experiment the teachers and pupils covered the same subject matter as on the corresponding day in 1932. By this is meant that day by day they took exercise for exercise, figure for figure, definition for definition, theorem for theorem, etc., in the new study as in the old. The only exceptions to this general rule are explicitly shown as the discussion proceeds. The members of Group D were younger, on the average, than the members of Group T, because geometry was now begun in the 10B (the first semester of the tenth year) class instead of the 10A (the second semester) class. The pupils of Group T had had a year and a half of algebra while Group D had had only one year. The method of dealing with the subject matter in 1938 was different from the methods used in 1932.

What evidence we have (see page 38) leads us to believe that Group D was certainly no better at the start than Group T. Since the I.Q.'s were equated, the same subject matter was used day by day, and, since age and mathematical experience were in favor of Group T instead of Group D, any superiority shown by Group D may be attributed to the methods.

It should be understood that the necessity for making valid comparisons between the results of Groups T and D made it impossible to follow the new course of study except for its methods. The new course is different from the old, not only in methods, but also in choice of exercises and the order of introduction of the subject matter. In many cases it wanders from the purely geometric in order to apply what has been learned to non-geometric situations. Because of the limitations of the experiment, the best that could be done was to superimpose the new methods on the old subject matter.

It should be understood too that we are not making a comparison of methods which attempts to get at meanings with traditional methods that disregarded meanings. The course of study used in 1932 was written in an attempt to get at meanings. The beginnings of the methods used in 1938 are to be found in the 1932 study. The comparison is between a good course, judged by the standards of the day, and a better course.

Tests 1, 2, and 3 were given to Group D, using the same methods as those used with Group T. In every case, Group D had poorer or equal results. We conclude, therefore, that Group D was certainly no better than Group T at the start.

The results of these three tests are shown in Table 26.

In Test 3, Exs. 9, 11, and 12 were different in the case of Group D from those given to Group T (see Test 3, page 5, and Test 3a, page 40).

TABLE 26

COMPARISON OF THE PERCENTAGES OF PUPILS IN GROUPS T AND D MAKING ERRORS ON TESTS 1, 2, AND 3

Test and Exercise	Percentage of Pupils Making Errors		Difference
	T	D	
Test 1			
Bisecting lines			
1	0	0	0
2	6	9	−3
Test 2			
Bisecting angles			
1	0	0	0
2	15	20	−5
Test 3			
Constructing Perpendiculars			
1	0	1	−1
2	0	1	−1
3	1	3	−2
4	1	1	0
5	12	24	−12
6	11	22	−11
7	15	22	−7
8	8	9	−1
10	10	14	−4

The same tests were given throughout to Group D as to Group T. However, only a few key tests have been chosen in order to make comparisons. These comparisons, together with a description of the new methods used, are given in succeeding chapters.

CHAPTER VI

ANALYZING FIGURES

IN CHAPTER II we gave evidence to show that one of the basic difficulties in the learning of demonstrative geometry is the inability of many pupils to transfer to a complex figure skills which they had acquired in connection with a simple figure. We found that slight changes in the figure affected the pupils' responses. Ability in connection with a simple figure did not insure ability in connection with a complex figure even though the complication was slight.

Methods devised to minimize the difficulties in connection with figures were based on the assumption that pupils at first see the figures as wholes. When they are required to make a construction in a complex figure, the lines and points to which they should give attention do not stand apart from the other points and lines. The application of a definition or a theorem to a complex figure is made confusing because the particular configuration, which should be dissociated from the rest of the figure, is obscured by the picture as a whole.

To remedy this defect, analysis and generalization appear necessary. When pupils learn how to construct a line perpendicular to a horizontal line from a point not on the line, their attention is concentrated upon this particular figure, and the particular letters used to designate the point and the line. They do not necessarily analyze the essential procedure and then generalize it. They are more likely to think in particular, "With C as center, draw arcs on AB," than to think in general, "With the given point as center, draw arcs on the given line." Then, when they are confronted with the problem of constructing the altitudes of an obtuse triangle, they are confused by the larger number of points and lines, and the necessity of extending certain lines. They do not know which points to use as centers nor where to draw the arcs.

Similarly, when pupils learn to recognize alternate interior angles in a figure consisting of two parallel lines and a transversal, they cannot necessarily transfer this ability so that they can recognize alternate interior angles in a parallelogram with diagonals drawn. There are parts of the first figure that are not essential—for one thing the transversal not only meets the parallels, but goes beyond them in both directions. This is not true in the second figure. Such unessential details as this, coupled with the fact that the purely essential parts have not been sufficiently analyzed, cause confusion in the more complex figure.

In order to apply a theorem to a complex figure, pupils must see in that figure the particular configuration which comprises the essential relationships of the theorem. The natural tendency to see a figure as a whole conspires against this analysis. Teachers of geometry, therefore, are faced with the problem of devising methods which will help pupils to see the essential parts of a figure, to analyze them so that they will see them in a more complex figure, dissociating the relevant from the irrelevant. Any method that will foster the attitude of analysis and generalization should prove effective.

Constructions

The following two illustrations will indicate the method used in this study to lessen the difficulties in connection with constructions. The purpose of the method was to generalize the procedure of a construction so that it would apply to any figure rather than to a particular figure.

After pupils had learned how to bisect angle BAC in a particular figure, they were asked the following questions:

1. Where do you first put the point of the compass?
2. Where do you draw the first arc?
3. Where do you then put the point of the compass?
4. What do you do next?

The tendency at first was to answer in terms of the particular figure. The answers to the first two questions, for example, were: (1) on A, and (2) so that it intersects AB and AC. Pupils were then told that in order to make use of this analysis in bisecting angles in other figures, they should answer the questions in general terms without mentioning the particular letters. The following answers were then given.

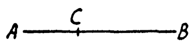
1. On the vertex of the angle I am to bisect.
2. So that it intersects the sides of the angle I am bisecting.
3. In two places—at the points where the first arc crosses the sides of the angles.
4. I draw arcs that intersect within the angle.

Drawing the bisector from the vertex to the intersection of these two arcs was not mentioned in this generalization, since this step seemed to cause no difficulty.

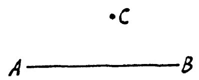
Pupils were then drilled upon the answers to these questions. Since they had previously been given practice in designating the vertices and sides of angles in complex figures, this analysis and generalization of the procedure in bisecting an angle served as a bridge to allow transfer from the simple figure with which they had acquired the skill to more complex figures.

In connection with the construction of perpendiculars, even more generalization was possible. Pupils learned to construct perpendiculars in the following four cases, and then were shown that one general method applies to all of them.

1. At C construct a line perpendicular to AB.



2. From C construct a line perpendicular to AB.



3. At A construct a line perpendicular to AB.

4. From C construct a line perpendicular to AB.

In Exercises 3 and 4, the necessity of extending the line was discussed.

Then, as in the case of bisecting an angle, the procedure was analyzed and generalized by means of the following questions and answers:

1. In each case where do you first put the point of the compass? *Answer:* On the point mentioned.

2. Where do you draw the arc? *Answer:* On the line mentioned.

3. How many times must the arc cross the line? *Answer:* Twice.

4. If the arc does not cross the line twice, what do you do? *Answer:* Extend the line.

The rest of the construction was not analyzed, since it seemed to cause no difficulty.

In order to determine whether this method was effective, the following procedure was used:

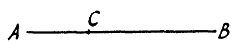
For a home assignment on the seventh day, pupils in Group D were asked to study the method of constructing a line perpendicular to a given line at a point on the line and from a point not on the line. The textbook gave the method using figures like those in Exs. 1 and 2 above. In class the next day practice was given with these particular figures. Test 3a was then given. After the test the analysis and generalization indicated above was given and drilled on for five minutes only. Then the test was given again.

As a control group for this particular test, 74 pupils (the same number as in Group D) with the same range of I.Q.'s, the same mean and standard deviation, were chosen from three classes. (We shall call this Group E.) The same thing was done in Group E as in Group D, except that the pupils in Group E were given five minutes of practice on the four exercises (the first four in the test), instead of the analysis and generalization.

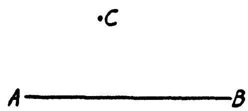
The results are shown in Table 27.

Test 3a

1. Construct a line perpendicular to AB at C.



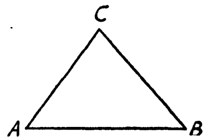
2. Construct a line perpendicular to AB from C.



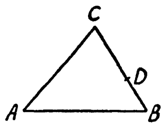
3. Construct a line perpendicular to AB at A.

4. Construct a line perpendicular to AB from C.

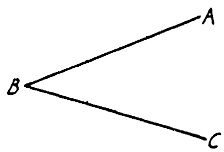
5. Construct a line from B perpendicular to AC.



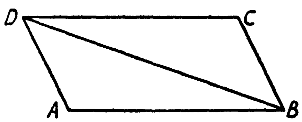
6. Construct a line from D perpendicular to AC.



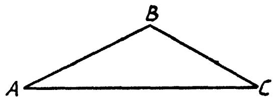
7. From B construct a line perpendicular to AB and another perpendicular to BC.



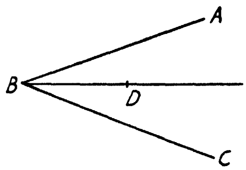
8. From C construct a line perpendicular to DB.



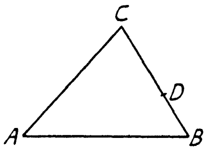
9. From A construct a line perpendicular to BC. From B construct a line perpendicular to AC. From C construct a line perpendicular to AB.



10. From D construct a line perpendicular to AB.



11. At D construct a line perpendicular to BC.



12. At D construct a line perpendicular to AD.

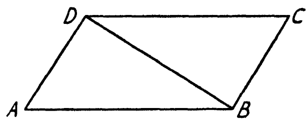


TABLE 27

COMPARISON OF THE IMPROVEMENT IN GROUPS D AND E IN TWO TRIALS OF TEST 3A

Group		Number of Pupils Making Errors on Exercise											
		1	2	3	4	5	6	7	8	9	10	11	12
D	First trial	1	1	2	1	18	16	16	7	25	10	10	13
	Second trial	0	0	0	0	5	4	3	1	11	3	5	3
	Difference (d_1)	1	1	2	1	13	12	13	6	14	7	5	10
E	First trial	0	1	3	2	19	15	13	9	28	10	10	13
	Second trial	0	0	2	1	13	11	5	5	19	8	7	7
	Difference (d_2)	0	1	1	1	6	4	8	4	9	2	3	6
d_1-d_2		1	0	1	0	7	8	5	2	5	5	2	4

Improvement is indicated in both groups. Greater improvement was shown in Group D in every exercise except Exs. 2 and 4, and in both of these exercises the pupils were nearly perfect in the first trial. The greater improvement in Group D is statistically significant as shown in the footnote.[1] Although the greater improvement in Group D is small it should be considered in light of the fact that the method of analysis and generalization had been used for a very short time only, five minutes.

One other procedure was tried in connection with the construction problems. The results of Test 4 given to Group T (see page 7) indicated lack of awareness that the order in which lines are drawn is of importance. For this reason, Test 4a was given to Group D, instead of Test 4, to discover the effect of more specific directions. Ex. 1 is the same on both tests. In Ex. 2 of Test 4a, pupils were told in what order to draw the lines. Comparison of the results is shown in Table 28.

TEST 4a

Ex. 2. Reproduce the figure below as follows: Draw a horizontal line AB, and place a point G on it. Construct an angle equal to angle 1, thus getting the line EGF. Place a point H on EF so that GH will equal GH in the given figure. Construct an angle equal to angle 2, thus getting the line CD.

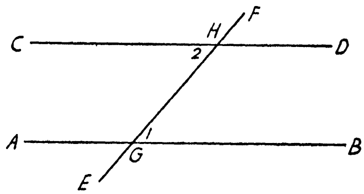


TABLE 28

COMPARISON OF THE PERCENTAGES OF PUPILS IN GROUP T MAKING ERRORS ON TEST 4 WITH THE PERCENTAGES OF PUPILS IN GROUP D MAKING ERRORS ON TEST 4A

Exercise	Percentage of Pupils Making Errors		Difference (d)	S.E. of Diff. (σ)	d/σ
	T	D			
1	4.4	2.7	1.7	2.8	.6
2	72.8	18.9	53.9	7.5	7.2

The difference in the results on Ex. 2 is striking. Difficulties can be avoided by giving careful directions as in Ex. 2 of Test 4a. To be sure pupils must finally be able to analyze a procedure sufficiently to do an exercise as stated in Ex. 2 of Test 4, but it is wise not to heap too many difficulties upon them at the start.

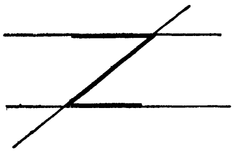
Application of Definitions and Theorems

Whenever a term is defined, it should be illustrated in complex figures as well as in a simple figure. When a proposition is proved, or otherwise accepted, pupils should have practice in seeing its application in complex figures. The teacher should find methods that will focus the attention of the pupils upon the particular parts of a figure which it is necessary for them to see. Colored crayons may be used to make applications stand out. Particular parts of figures may be drawn separately at the side of the complex figure. The entire figure can be drawn on the blackboard, and then the non-essential parts erased in order to show the essential parts. Some peculiarity of the situation such as the zigzag always formed by the alternate interior angles made by two parallels and a transversal may be mentioned explicitly. In every case, practice should be

[1] We may consider the twelve differences as twelve different observations of the same thing in testing the assumption that differences between the groups are due only to chance.

$$\Sigma(d_1 - d_2) = 40$$
$$M = 3.33$$
$$\Sigma(d_1 - d_2)^2 = 215$$
$$s^2 = \frac{215}{12} - (3.33)^2 = 6.84$$
$$\sigma_M = .789$$
$$t = \frac{M}{\sigma_M} = 4.22$$

For eleven degrees of freedom $t_{.01} = 3.106$ and the likelihood of exceeding $t = 3.8$ is .003. We see, therefore, that $t = 4.22$ may be considered significant.



given in seeing the application in a number of complex figures under the supervision of the teacher before there can be any great probability that all the pupils will be able to make the applications by themselves.

We shall record here the results of two tests given to show the effectiveness of some of the methods mentioned above.

On the thirty-first day Test 11 (see page 16) on the recognition of opposite angles formed by two intersecting straight lines was given to Group D. It was given originally to Group T immediately after the equality of the opposite angles had been shown in a figure consisting only of two intersecting lines. In the case of Group D, we went one step further. The teachers pointed out explicitly the two intersecting lines in the figure for Ex. 4, asked pupils to compare that part of the figure with the figure for Ex. 2, and to tell what angles were therefore equal.

Comparison of the results is shown in Table 29. The figures in which the opposite angles occurred were for Exs. 2, 4, 6, 9, 10, and 11. The differences are not statistically significant, except in one case.[2] However, they are consistently in one direction, and for this reason we may put some reliance in them. The small differences bear out the contention that practice should be given under the supervision of the teacher in the application of a theorem to several figures before pupils are asked to make the application on their own initiative.

[2] The likelihood of a difference of per cents exceeding two standard deviations by chance is about .05 and of exceeding two and a half standard deviations is about .01. Most authorities would not consider a difference significant unless it reaches at least two standard deviations, and many prefer two and a half standard deviations.

TABLE 29

COMPARISON OF THE PERCENTAGES OF PUPILS IN GROUPS T AND D MAKING ERRORS ON TEST 12

Exercise	Percentage of Pupils Making Errors		Difference (d)	S.E. of Diff. (σ)	d/σ
	T	D			
2	0	0	0	0	—
4	15.8	5.4	10.4	4.8	2.2
6	14.0	6.8	7.2	4.7	1.5
9	10.5	4.1	6.4	4.0	1.6
10	16.7	6.8	9.9	5.0	2.0
11	19.3	4.1	15.2	5.0	3.0

On the forty-ninth day, Test 9 (see page 13), was given to the pupils of Group D. When the test was given to Group T, the members of this group knew the definition of alternate interior angles, but had had experience with them only in a figure consisting of two parallels and a transversal. In addition to this, Group D had been shown the zigzags by means of colored crayon, and had discussed Exs. 2 and 3 of the test, making use of colored crayon and noting the zigzags. Comparison of the results (see Table 30) shows that Group D was not so successful with Ex. 1, the figure with which the definition was given in both groups. But in spite of this handicap, Group D was unquestionably better on all the other exercises with the possible exception of Ex. 10.

TABLE 30

COMPARISON OF THE PERCENTAGES OF PUPILS IN GROUPS T AND D MAKING ERRORS ON EACH EXERCISE OF TEST 9

Exercise	Percentage of Pupils Making Errors		Difference (d)	S.E. of Diff. (σ)	d/σ
	T	D			
1	5.3	6.8	−1.5	3.5	−4.2
2	24.6	1.4	23.2	5.4	4.3
3	23.7	2.5	21.2	5.4	3.9
4	33.4	6.8	26.6	6.3	4.3
5	25.4	0	25.4	5.4	4.7
6	32.4	12.2	20.2	6.4	3.2
7	34.2	13.5	20.7	6.5	3.2
8	68.3	50.0	18.3	6.8	2.7
9	43.0	5.4	37.6	6.5	5.8
10	79.8	67.5	12.3	6.4	1.9

If we consider as correct those exercises where one pair of angles was given correctly, and the others not mentioned, the percentages of pupils in error would read as follows for Group D: Ex. 4 (3%), Ex. 6 (3%), Ex. 7 (6%), Ex. 8 (3%), and Ex. 10 (7%). Considering these percentages and comparing them with the corresponding percentages given on page 14 (they were 21, 26, 29, 15, and 25 respectively), we get even a better picture of the improvement. In the case of Group D, the errors of commission were found to be almost negligible in number.

Other specific methods of dealing with figures are suggested by Test 6, page 9; Test 7, page 11; Test 8, page 12; Test 10, page 15, and Test 12, page 18. Once a teacher is aware of the fact that pupils have difficulties with complex figures, the remedy is not hard to discover. In order to minimize the difficulties, he should make definite provision in his lesson plans for practice with figures, not only when a new topic is presented, but also later whenever it appears necessary.

Chapter VII

DEVELOPING THE MEANING OF THE IF-THEN RELATIONSHIP

From the results of the tests discussed in Chapter III we came to the conclusion that many pupils beginning demonstrative geometry have very little understanding of the logical implications of a proposition stated in the if-then form. For additional evidence toward the same conclusion, we gave the following test to Group D on the twelfth day.

Test 14a

What is the difference in meaning, if any, between the first two statements below?

1. If two sides of a triangle are equal, the angles opposite those sides are equal.
2. If two angles of a triangle are equal. the sides opposite those angles are equal.

Does either one of the statements above mean the same thing as the statement below? If so, which one?

3. In an isosceles triangle, two sides and two angles are equal.

Out of the seventy-four pupils in the group, sixty-one said there was no difference in meaning between the first two statements, six gave uninterpretable answers, and only seven gave correct answers. Fifty-five said that the third statement meant the same as "both the others," four said that it meant the same as 1 and 2 together, twelve the same as 1, and one the same as 2. Only two said that it was not the same as either. It is obvious that the group as a whole had very little conception of the difference in meaning between the first two statements. The realization of the difference in meaning involves an understanding of the if-then relationship.

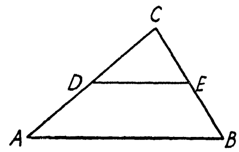
It will be granted that pupils know what we mean when we say, "If you do excellent work in geometry, I will give you an A." But there is a difference between this and the geometric situation. This is dynamic. There is an element of time in it. The condition comes first in point of time. The conclusion comes afterward. There is also a causal element. Fulfilling the conditions in the if-part of the sentence in a sense causes the conclusion. The geometric situation is static. For either statement (1) If two sides of a triangle are equal, the angles opposite those sides are equal, or (2) If two angles of a triangle are equal, the sides opposite those angles are equal, the pupil sees the same figure in the textbook. What he sees is a triangle with two sides and two angles equal. So far as this static figure is concerned, there is no thought of the equal sides coming first and the equal angles coming afterwards, or vice versa. They are both there at the same time. Nor does it occur to him that the equal sides may be, in a sense, the cause of the equal angles, or the equal angles the cause of the equal sides. For these reasons, the real meaning of the if-then relationship in a geometric proposition is hidden, and may not become clear in time for pupils to begin proofs with understanding. It should not be surprising to us that so many pupils flounder in the early stages of demonstrative geometry. When a pupil does not sense what it means, "If this is so, then that is so," the work he does must indeed be confusing and meaningless to him.

In order to develop the meaning of the if-then relationship, the author has introduced into the geometric situation the time and causal elements which are lacking. If a pupil constructs a figure according to certain conditions, he finds that other relationships result. Although both sets of relationships are true of the completed figure, they separate themselves into two categories. The pupil has control over one set. He made them so. He has no control over the other set. They came as a result of what he did. The given conditions were put into the figure first. The other relationships came afterward. Once a pupil sees that the relationships in a figure can be put into these two categories, the concept may be associated with the corresponding if-then statement. When the difference between the two categories is clear, the meaning of the if-then relationship has taken on concrete meaning.

The germ of this method was used with Group T (see page 22), but since its details had not been clearly worked out at that time, it was not used consistently, and had little effect. The method as used with Group D is described completely in the following paragraphs.

The development was begun on the thirteenth day with the discussion of the following exercises.

1. We have experiments in geometry just as we have experiments in science. We do certain things to a figure, and certain other things result. Here we shall bisect two sides of a triangle, connect the points of bisection, and see what will happen.



(a) Draw a line AB 8 cm. long.

(b) Complete triangle ABC with any dimensions you choose.

(c) Bisect AC and call the midpoint D.

(d) Bisect BC and call the midpoint E.

(e) Draw a straight line from D to E.

(f) Measure DE. How long is it? Do your classmates have the same result, even though the dimensions of their triangles are different?

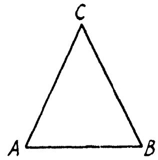
(g) Did you plan to make DE equal to 4 cm.?

(h) Did you know that DE is 4 cm. before you completed the figure, or did you learn this after you completed it?

(i) If you did the same work again accurately, with AB equal to 8 cm., could you make DE any length but 4 cm.?

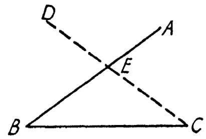
The important thing for you to learn from this experiment is that *if* you do certain things in making a figure, *then* certain other things inevitably result. Here you draw a triangle with a base 8 cm. long. You bisected the other two sides, and drew a straight line joining the midpoints of these sides. You had control over all these things; but you had no control over the *length* of the line connecting the midpoints. *Once you have done certain things you have no control over the result.*

2. Draw angle ACB, making CA equal to CB. Draw AB. You now have triangle ABC. Measure $\angle A$ and $\angle B$. What do you note about these angles? Did you plan to make $\angle A$ equal to $\angle B$? If you did this same work again accurately, could you bring about a result such that $\angle A$ would not be equal to $\angle B$?



In the figure you have just made $CA = CB$ and $\angle A = \angle B$. Which of these two equalities did you control? Over which did you have no control? We can say, then, that the equality of $\angle A$ and $\angle B$ is a result of the equality of CA and CB.

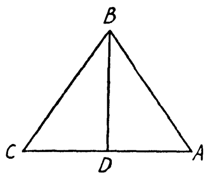
3. Draw an acute angle ABC. Then make an angle BCD equal to $\angle ABC$, as indicated by the dotted line. You now have a triangle BCE. Measure BE and CE.



In the figure you have just made $\angle B = \angle C$ and $BE = CE$. Which of these two equalities did you have under your control? In your figure would you say that the equality of $\angle B$ and $\angle C$ is a result of the equality of BE and CE, or vice versa?

4. Draw an angle ABC. Then, using B as center, mark off BA equal to BC. What kind of triangle is $\triangle ABC$? Bisect $\angle ABC$, and continue the bisector until it meets AC at D.

Measure AD and DC. Is AC bisected?



In the figure you have just made, $AB = BC$, BC bisects $\angle B$, and BD bisects AC. Which of these three things were under your control, and which was the result of what you did?

5. Draw $\angle ABC$ and then, using B as center, mark off BA equal to BC. Draw AC. Bisect AC, thus finding the midpoint D. Draw BD.

Measure $\angle ABD$ and $\angle CBD$. Is $\angle B$ bisected?

In the figure you have just made, $AB = BC$, BD bisects $\angle B$, and BD bisects AC. Which of these three things did you have under your control, and which was a result of what you did?

Note that the two figures you have made look very much alike, but that they were made differently. In the first of these two exercises, the *given conditions* (the part over which you had control) are $AB = BC$ and BD bisects $\angle B$. The *conclusion* is BD bisects AC. In the second exercise, the given conditions are $AB = BC$ and BD bisects AC. The conclusion is BD bisects $\angle B$.

The conditions and conclusion of an exercise may be stated briefly in an if-then sentence, the conditions being in the if-part. The conditions and conclusions of Exs. 4 and 5 may be stated thus:

(1) *If* $AB = BC$ and BD bisects $\angle B$, *then* BD bisects AC.

(2) *If* $AB = BC$ and BD bisects AC, *then* BD bisects $\angle B$.

The reader can see from the exercises themselves what the objective is. The first exercise introduces the distinction between given conditions and conclusion without mentioning the words. In the completed figure AB is 8 cm. (approximately, of course), and DE is 4 cm., but there can be a difference in attitude toward the lengths of the two lines. The pupil made AB 8 cm. long. He planned it that way. DE "turned out" to be 4 cm. He did not even know that it was 4 cm. until he measured it.

The next two exercises give practice with the same concept. By constructing the figures, pupils can see for themselves the equalities over which they have control, and distinguish them from the equalities over which they have no control, but which are a result of what they did. The use of converses such as have been employed here is effective in clarifying the distinction. The figures when completed look very much the same, but they were made differently.

The last two exercises carry the development to its conclusion. Now that the distinction between the parts of a figure over which the pupil has control, and parts which are a result of what he has done, has been made clear, the words *given conditions* and *conclusion* are associated with the two categories. Finally, the method of stating the distinction by an if-then sentence is introduced. Since the fundamental concept is clear, the method of stating it will be understood.

During the following days at least one construction exercise a day, exercises which had been discussed as constructions only with Group T, was treated according to the method described here. At the end of each discussion, pupils were asked to state the exercise as an if-then sentence.

When it came time to introduce the words *hypothesis* and *conclusion*, they were associated with the two categories already developed. The hypothesis refers to what is given, the parts of the figure over which the pupil would have control if he constructed it. Conclusion refers to what is to be proved, the parts of a figure over which he does not have control, the parts which result from what has already been done.

As a test of the efficiency of this method of developing the meaning of the if-then relationship, Test 17 (see page 23), was given to Group D. This was done on the thirty-eighth day under the same conditions as it was given to Group T, except that in the case of the latter group, the meaning of the if-then relationship had not been carefully developed. The test required the drawing of a figure and the writing of the hypothesis and conclusion in terms of that figure from three verbal statements. In neither group had such an exercise been required or discussed before. The superiority of Group D is shown in Table 31.

TABLE 31

Comparison of the Percentages of Pupils in Groups T and D Making Errors on Each Exercise of Test 18

Exercise	Percentage of Pupils Making Errors		Difference (d)	S.E. of Diff. (σ)	d/σ
	T	D			
1	58.6	10.8	47.8	7.3	6.6
2	36.8	16.2	20.6	6.8	3.1
3	42.1	17.6	24.6	7.0	3.5

All the differences in this table are statistically significant (see footnote to page 43). It appears certain that pupils who, without specific training for the kind of task required in these exercises, can draw a figure and write the hypothesis and conclusion in terms of a figure, understand the meaning of the if-then relationship. The drop in the number of errors is quite striking. Even among those who made errors there was not the confusion shown by Group T. For example, there were no irrelevant relationships brought in for Ex. 1, only two pupils made such additions for Ex. 2, and only three for Ex. 3.

Chapter VIII

DEVELOPING THE MEANING OF PROOF

Developing the meaning of proof is not a simple matter that can be done in a day or two. There are too many details for the pupils to assimilate. The evidence presented in Chapter IV led us to conclude that many pupils have no conception of what it means to draw a conclusion from a general statement, and an application of it. They wrote conclusions when the conditions were not fulfilled. They omitted conclusions when the conditions were fulfilled. Often the conclusions were entirely foreign to anything in the given statements. Nor did they sense the restrictions necessary in a deductive science. In drawing conclusions concerning geometric figures, they did not hold to the data but assumed relationships from appearances. In their proofs they often used "manufactured reasons," statements that had not been accepted. It was clearly evident that they did not know "what it was all about."

It will be granted that these pupils had been making informal deductions for many years. But in all those deductions, they had probably never been so definitely restricted to verbal logic. They could bring to bear anything they chose—appearances, intuition, any of their senses, hearsay, prejudices, or what not. They made their deductions from the total situation or from any part of it that impressed them. Now for the first time, probably, they were required to attend to a definite part of the total situation. They must not assume anything concerning a figure merely from the appearance. They must hold themselves to the data. They must not think in terms of intuitive notions which had been growing up with them over a period of years. They must think only in terms of a restricted set of propositions. They had, therefore, not only to form new habits, but to inhibit old ones.

The methods used with Group D to develop the meaning of proof will be discussed under the headings Deduction, Acceptable Reasons, and Formal Proofs.

Deduction

The aim in the development of the meaning of deduction was to show the pupils the following: (1) a simple deduction involves three statements, a general statement, an application of it to some particular person or thing, and a conclusion; (2) no conclusion may be drawn concerning the specific person or thing unless there is an application, in other words, an exact fulfillment of the conditions, of the general statement; and (3) if the conditions are fulfilled there is always a conclusion. Besides developing the understanding of these facts, we wished to foster the habit of looking for the conditions which are to be fulfilled, to develop skill in detecting just what the conditions are so that they may know when they have been fulfilled, and finally to have them check each conclusion to see whether it is the conclusion called for by the reason they are using.

The development was begun on the fifteenth day with the giving of Test 18 (see page 25) on syllogisms. The results of this test are shown in Table 32.

TABLE 32

COMPARISON OF THE PERCENTAGES OF PUPILS IN GROUPS T AND D MAKING ERRORS ON EACH EXERCISE OF TEST 18

Exercise	Percentage of Pupils Making Errors		Difference (d)	S.E. of Diff. (σ)	d/σ
	T	D			
I	4.4	2.5	1.9	2.8	.7
II	.9	1.4	−.5	1.6	−.3
III	20.1	9.4	10.7	5.5	2.0
IV	25.2	24.3	.9	6.4	1.1
V	29.8	12.2	17.6	6.3	2.8
VI	70.0	50.0	20.0	7.2	2.8
VII	54.3	21.6	32.7	7.4	4.6
VIII	75.3	62.2	13.1	6.8	1.9

We learn two things from a study of this table in which only three out of the eight differences are statistically significant, and one of the differences is negative. First, that the development of the if-then relationship, which had been going on since the twelfth day, had some effect, but very little, on the pupils' ability to carry through these syllogisms correctly. And secondly, that at the beginning of the development of the meaning of proof Group D was not much, if at all, superior to Group T.

The development was continued on the sixteenth day. The method used will be shown by quoting the directions given to the teachers conducting the experiment.

"To the pupils. Suppose you know that any student who has room 323 for a home room is a senior; that one morning you meet a new student, stop to talk to him, find out that his name is John Stuart, and in the course of the conversation learn that his home room is 323. What thought would come immediately to your mind? (Have the pupils write this statement so that you can see how many are right. They will probably all be right. Tell them that this is the way that the intelligent human mind works.)

"To the pupils. What you have done is to draw a *conclusion* from two statements. Written formally, it would look like this. (Then write the following on the board.)

1. If any student has home room 323, he (or she) is a senior.
2. John Stuart is a student and has home room 323.
3. Therefore, John Stuart is a senior.

"The third statement is called a *deduction* from the first two statements. We have *deduced* the third statement from the first two statements. We did not come to the conclusion by any means other than a thought process.

"Leave the first syllogism on the board, and write the following:

1. If any student has home room 323, he (or she) is a senior.
2. Mary Smith is a student and has home room 322.
3. Therefore, . . .

"Have the pupils write their conclusion. More than likely most of them will write, 'She is not a senior.' Discuss the matter with them. Show that there might easily be an overflow, and that some seniors may have room 322 as a home room. From what is stated, we do not know whether Mary is a senior or not. However, write the conclusion, 'She is not a senior' on the board, and put a cross beside it to show that it is wrong.

"There is no possible conclusion in this case. A deduction cannot be made. What I have written on the board is an example of wrong deductive reasoning.

"Leave the first two syllogisms on the board, and write the following:

1. If any student has home room 323, he (or she) is a senior.
2. Frank Allen has home room 323.
3. Therefore, . . .

"Have the pupils write their conclusion. Most will say, 'He is a senior.' Perhaps some one will say that Frank may be the teacher, and therefore there can be no conclusion. Write the conclusion, 'He is a senior' on the board, and put a cross beside it.

"Leave these syllogisms on the board, and write the following:

1. Any child in this town under twelve years of age found on a public street after 9:00 P.M. is immediately sent home.
2. John Smith, a child eleven years old, was found on June Street at ten o'clock.
3. Therefore, John Smith was immediately sent home.

"Ask. Is this an example of good deductive thinking? Point out that it is not. Probably many of your pupils will do this for you. We do not know whether it was A.M. or P.M., and we do not know whether June Street is a public street. Put a cross against it.

"We have seen, from studying the situation involved in each of these exercises, which of our conclusions were correct and which were not. Now we should analyze each to see if we can tell, by giving attention to the statements alone, when we can draw a conclusion correctly and when we cannot. Could we have told by careful reading of the statements that we could make a deduction in the first exercise and could not in the remaining exercises?

"Discussion of the first syllogism. Have the pupils note that the second statement *fits* the if-part of the first statement exactly. Note that the third statement *fits* the then-part of the first statement exactly. When this occurs, we have an example of good deductive reasoning.

"What are the *conditions* of the first statement? (*Answer:* The person must be a student, and he must have home room 323.) There are two conditions.

"Are these two conditions fulfilled?

(*Answer:* Yes. The second statement says that John Stuart is a student and that he has home room 323.)

"What is the conclusion of the first sentence? (*Answer:* He is a senior.)

"Is this carried out exactly in the third statement? (*Answer:* Yes. It says that John Stuart is a senior.)

"Summary. The second statement must fit the if-part of the first statement exactly (it must fulfill its conditions). The third statement must fit the then-part of the first statement (that is, it must carry out the conclusion exactly as it is stated).

"Now go on to the second syllogism. Does the second statement fit the if-part of the first statement. (*Answer:* No.) Why not? How many conditions are there to be fulfilled? (Two. See above.) Are they both fulfilled? Which one is fulfilled, and which one is not fulfilled? Can we make a deduction in this case?

"Treat the third syllogism in the same way, then repeat the summary.

"Treat the fourth syllogism in the same way, except to show that the conditions are found in the subject, and all its modifiers, of the first sentence and the conclusion in the predicate.

"Emphasize the summary."

As already stated, it is the tendency of pupils, prior to their formal work in geometry, to make their deductions, using the total situation or whatever part of it impresses them. The preceding syllogisms were chosen to make use of this method of reasoning. It was possible to show pupils that they were wrong in their conclusions, particularly in the second and third exercises, by discussing with them the concrete situation involved. No careful analysis of the statements themselves was necessary to show the pupils their errors. The analysis of the statements, the concept of conditions and their fulfillment, the realization that the statements themselves held the key to the correctness or incorrectness of the conclusions, the introduction of the terms to be used came afterward. Pupils could understand these abstract things because they had first seen the incorrectness of their conclusions concretely.

On the seventeenth day, the following summary was given:

1. A deduction consists of three statements, (a) a general statement, (b) a specific statement which is an application of the general statement, and (c) a conclusion.

2. No conclusion can be drawn from the first two statements unless the second statement fulfills exactly the conditions of the first statement; that is, *fits* exactly the if-part of the first statement.

3. If the second statement fulfills the conditions of the first statement, then there always is a conclusion and this conclusion must follow exactly the then-part of the first sentence.

Then the test on syllogisms (Test 18) was discussed. In each case the general statement was changed to the if-then form, and pupils were asked to consider each exercise in the light of the summary given above. In the cases where no conclusion could be drawn, pupils were asked to change the second statement so that a conclusion could be drawn. Then they made the necessary conclusion.

Pupils were cautioned that they must not use their imaginations in making these deductions. They must not bring in extraneous matter. They must concentrate on the given sentences and restrict their thinking to what is explicitly stated.

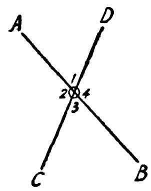
On the eighteenth day, the following exercises were discussed as applications of the preceding development.

Study the examples given here and tell whether they illustrate right or wrong deductive reasoning.

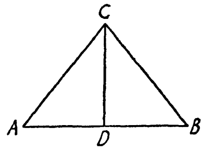
I. 1. If two straight lines intersect, the opposite angles are equal.

2. AB and CD in the figure below are straight lines intersecting at O.

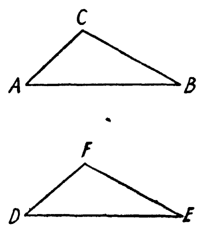
3. Therefore, $\angle 1 = \angle 3$ and $\angle 2 = \angle 4$.



II. 1. If a line bisects the vertex angle of an isosceles triangle, it is perpendicular to the base.
2. In the figure below, $AC=BC$ and CD bisects $\angle C$.
3. Therefore, CD bisects AB.



III. 1. If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another triangle, the triangles are congruent.
2. $AB=DE$, $\angle B=\angle E$, $BC=EF$.
3. Therefore, $\triangle ABC \cong \triangle DEF$.



IV. 1. If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, they are congruent.
2. $AB=DE$, $AC=DF$, and $\angle C=\angle F$.
3. Therefore, $\triangle ABC \cong \triangle DEF$.

The postulates concerning congruence had previously been accepted after inductive treatment (see page 53).

Teachers were asked to comment on Ex. IV above as follows: It looks at first glance as if the conditions were fulfilled. We have two sides and an angle of one triangle equal to two sides and an angle of the other. But when we look at the figure we see that the angle is not included between the sides. One of the conditions is not fulfilled, and so there can be no conclusion. It should be clear that the only reason that we look at the figure is to see the relative positions of the lines, points, angles, etc.—to see where they are. We do not look at the figure to see whether the lines or angles are equal.

In order to test the effectiveness of the development to this point, Test 19 (see page 28) was given on the nineteenth day. In each of the two exercises the first statement was false, but its conditions were explicitly fulfilled, and so a conclusion was necessary. No such exercise had been previously discussed. In order to make the correct responses, a pupil would have to inhibit his natural tendency to reason intuitively, make use of what he had learned about deduction, and follow the words of the given statements without recourse to other considerations. The results are shown in Table 33.

TABLE 33

Comparison of the Percentages of Pupils in Groups T and D Making Errors on Each Exercise of Test 20

Exercise	Percentage of Pupils Making Errors		Difference (d)	S.E. of Diff. (σ)	d/σ
	T	D			
1	5.3	4.1	1.2	3.2	.4
2	55.2	39.2	16.0	7.5	2.1

The number of pupils in either group making errors on the first exercise is so small that comparison is rather meaningless. The difference in the second exercise is on the border line of statistical significance. It is probably safe to say that the recorded difference is not entirely due to chance, and yet at the same time we get an insight from this table as to how difficult the abstract notions of deduction are for pupils beginning demonstrative geometry.

The test was discussed as follows. If the conditions of the general statement are fulfilled, there *must be* a conclusion and the conclusion must be the conclusion called for by the then-part of the sentence. Of course, the first statements of these two exercises are foolish. Yet, at the same time, their conditions were fulfilled. What we are trying to do is to show you what it means to make a deduction. We gave these foolish statements in order to emphasize the method, and to help you to remember it better. Assuming that the first statements are true (and that is what you are asked to do), there must be a conclusion to these two exercises.

Acceptable Reasons

Early in the course in demonstrative geometry, the pupil is confronted with the question, "Why?" He may be asked, "Why are these triangles congruent?" The teacher knows the meaning of *why* in this case; the pupil usually does not know it. He may have been asked a short time before, "Why are you unprepared?" and he told *how it came about* that he did not do his home lesson. He is to be pardoned, then, if he interprets the teacher's question to mean, "How did it happen that these triangles are congruent?" An appropriate answer would be, "The artist who made the drawings for the book was instructed to make them congruent." Then again, he may interpret the question to mean, "For what purpose are these triangles congruent?" The answer to this question he must confess he does not know. A better question would be, "What do you know about these triangles to make you sure that they are congruent?" Even then, the pupil may not give the correct response.

The difficulty lies deeper than the form of the question. The teacher means, "What postulate, theorem, or definition, previously accepted in this course, the conditions of which have been fulfilled, can you cite as an authority for the conclusion you have just reached?" This question would be explicit, but the meaning would not be clear to the pupil. He has made many informal inferences during his life, but he has not been restricted as to his premises. Now, perhaps for the first time, he is restricted to a very narrow set of acceptable reasons. He should be made aware of this limitation before he is required to make formal proofs in geometry.

Pupils can be shown that not all evidence is allowed in court proceedings. For example, a lie detector may be used prior to a trial in the hope that indirectly some evidence admissible in court may be found, but the direct evidence may not be admitted. If a lawyer should attempt to bring in such direct evidence, the judge would say in effect, "Your evidence (reasons) may be convincing and seem reasonable to you, but according to the law of the land I may not listen to you." It is, therefore, ruled out of court. This is only one example of the limitations placed upon lawyers in pleading their cases. There are many other examples. The lawyer must play the game according to the rules.

After a brief introduction like that just given, the pupil is prepared to hear that in the arguments he will henceforth give in geometry there are certain rules of good deductive thinking he must follow. One of them concerns the restriction in reasons he may use. He is ready for a discussion as to what constitutes an acceptable reason.

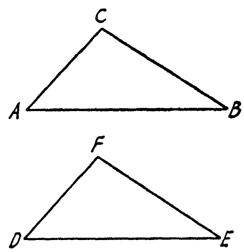
At the beginning of deduction in geometry, the only acceptable reasons (in addition to the particular hypothesis and construction of the given exercise) are definitions, postulates, and axioms.[1] Inferences from definitions should have been part of the pupil's work from the very beginning of his geometry course in connection with constructions and study of

[1] The writer makes the distinction between *axioms* and *postulates* that is current in secondary school teaching. An axiom is an assumption concerning quantities in general. A postulate is an assumption in the particular field of geometry. There is some psychological advantage in the distinction, but it is not necessary.

figures; hence at this stage it remains only to state explicitly that a definition is at all times acceptable in a deductive argument.

The following illustration will indicate how the postulates were introduced with Group D. First, the definition of congruent triangles was given as "triangles which have three sides of one equal to three sides of the other, and three angles of one equal to three angles of the other." Then followed construction work and discussion as shown below.

Draw any triangle ABC. Construct a line DE equal to AB, then construct $\angle D$ equal to $\angle A$ and DF equal to AC (see the figure below). Draw EF. Measure BC and EF, $\angle C$ and $\angle F$, $\angle B$ and $\angle D$. You see that three sides and three angles of one triangle are equal respectively to three sides and three angles of the other. Triangle DEF, according to the definition of *congruent,* is therefore congruent to triangle ABC.



1. How many sides of one triangle did you construct equal to sides of the other? How many angles of one did you construct equal to angles of the other?

Note that although you made only two sides and one angle of one triangle equal two sides and one angle of the other, the result was two triangles with three sides and three angles of one equal to three sides and three angles of the other.

The angle must be in a definite position with respect to the two sides—it must be the angle formed by the two sides. The angle made by two sides of a triangle is said to be *included* between these two sides.

2. What conditions were fulfilled to make these two triangles congruent? (Two sides and the included angle of one were made equal respectively to ....) Write as an if-then sentence the relationship illustrated in this construction exercise.

This was done on the seventeenth day.

Similarly, other geometric "facts" were introduced until we had the following list. Pupils wrote this list under the heading *postulates* in their notebooks.

1. If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, they are congruent.
2. If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, they are congruent.
3. If two triangles have three sides of one equal to three sides of the other, they are congruent.
4. Any quantity is equal to itself (identity).
5. If one straight line meets another, so as to form adjacent angles, these angles are supplementary.
6. If two adjacent angles form a right angle, they are complementary.
7. If two angles are supplementary to the same angle or to equal angles, they are equal.
8. If two angles are complementary to the same angle or to equal angles, they are equal.
9. If two straight lines intersect, the opposite angles are equal.
10. All right angles and all straight angles are equal.

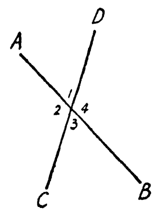
Axioms were also introduced by inductive methods.

As theorems were proved, another list was begun and added to. Pupils always had before them these lists of propositions which were "acceptable reasons." If they used others, they knew they were going against the rules of deduction, and would be "called for a foul."

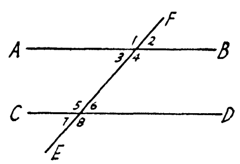
In order to show pupils how these

axioms and postulates are used, to prepare them for the later use of "Why?", questions were first given in a form which would tend to elicit the correct response. The following are illustrations:

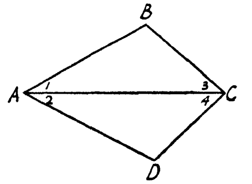
1. Since $\angle 2$ and $\angle 4$ are both supplementary to $\angle 1$ in this figure, they are equal. Which of the statements in your list would you give as an authority for this equality. (The list contained only items 1–8 at this point.)



2. In the figure AB and CD are straight lines intersected by line EF. Suppose you are told that $\angle 5$ is supplementary to $\angle 3$, and are asked to show that $\angle 5 = \angle 1$. Are $\angle 5$ and $\angle 1$ supplementary to the same angle? What is your authority for saying that $\angle 5 = \angle 1$?



3. If you were told that $\angle 1$ and $\angle 2$ are each 30° and AB and AD are each 2 cm., what could you find out about this figure



without further measurements? You know that $AB = AD$ and $\angle 1 = \angle 2$. You see that AC is the same line in both triangles. How many sides of one triangle do you know to be equal to sides of the other? How many angles? Is the angle included between the two sides? Have you sufficient information now to know that the triangles are congruent? What authority in your list would you give as a reason that they are congruent? If the triangles are congruent, what sides and angles do you know to be equal in addition to those you knew about at first?

4. Construct an isosceles triangle and bisect the vertex angle. It appears that the base is also bisected. Explain by means of congruent triangles why this has to be true.

During the time of this development, pupils were constantly reminded that the purpose of our work was to see how these new things grew out of the postulates. They readily accepted the restrictions as to reasons. When some pupil gave a manufactured reason as an authority, he was at once called to account by his classmates. Because of the kind of development given, he was convinced of his error as soon as it was called to his attention.

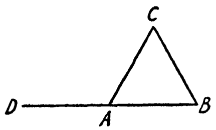
As a check on the efficiency of this method of showing that reasons must be restricted to those which have been accepted, Test 22 (see page 31) was given to Group D. The fourth exercise is the one in which we are interested here. When the test was given to Group T, 48% of the pupils gave unacceptable reasons. Only 20% of Group D gave such reasons. Fourteen per cent of Group D gave no reason rather than give one not on the list when they could not seem to decide upon the correct reason. Sixty per cent of Group D were entirely correct on this exercise as compared to 49% of Group T.

Formal Proofs

All the development so far described was made use of in the discussion of formal proofs. Figures were studied in connection with the postulates, and later with the theorems. Knowledge of the meaning of the if-then relationship was used in connection with the writing of the hypothesis and conclusion from a verbal statement, and in understanding the implications of the reasons given. The development of the meaning of deduction was used as a basis of checking proofs to see whether they were correct. And, of course, pupils were allowed to give only acceptable reasons.

The first formal proof was introduced on the twentieth day by means of the following exercise. The bulletin given to the teachers for this day is reproduced to show the method in detail.

"Suppose we are given this figure in which we are told that $AC=AD$ and $BC=AD$ and are asked to show without measuring that triangle ABC is isosceles. That is, we are asked to prove that if. . . . (Have pupils give this if-then statement in terms of the figure.)



"Discuss the proof informally, holding the pupils accountable for acceptable reasons.

"The proof written out in form would look like this.

Hypothesis. $AC=AD$, $BC=AD$.
Conclusion. $\triangle ABC$ is isosceles.
Proof.

Statements	Reasons
1. $AC=AD$, $BC=AD$.	1. Hyp.
2. $AC=BC$.	2. If two things are equal to the same thing, they are equal to each other.
3. $\triangle ABC$ is isoceles.	3. If a triangle has two equal sides, it is isosceles.

"Now let us analyze this proof. The hypothesis refers to what is given concerning the figure, the part that is put into it, the part over which we would have control if we constructed it, the if-part of the sentence.

"The conclusion refers to the result, the part over which you have no control, the things which happen because of the hypothesis, the things that happen because of the things which were put into the figure. The conclusion contains what you are to prove.

"We know that we are restricted as to the reasons we may use. For the present we can use only the hypothesis, any definition we know, the axioms and postulates in our list. This is one of the most important rules in the game. If we use any others, we shall be called for a foul. From time to time we shall add to the list of reasons we may use.

"Drill. What are the three things we can use for reasons?

"Now look at the reason column. Have we used any 'unacceptable' reasons? What is the first reason? (Hyp.) What is the second reason? (An axiom.) What is the third reason? (A definition. It tells what an isosceles triangle is.)

"The first step in checking a proof is to see whether all the reasons are acceptable.

"The second step in checking a proof is to see whether the conditions of each reason have been fulfilled. (Do this now.)

"The first reason is 'Hyp.' This is the reason for statement 1. Is statement 1 found in the hypothesis? Is it actually given as you say it is?

"What are the conditions of the second reason? (We must have two things equal to the same thing.) Do we have two things equal to the same thing, and have we written that we have them? Yes, in statement 1.

"What are the conditions of the third reason? (We must have a triangle with two equal sides.) Do we have two equal sides, and have we written that we have them? (Yes, in statement 2.)

"The third step in checking a proof is to see whether we have followed exactly the conclusion, the then-part, of each reason.

"What is the then-part of reason 2? (They are equal to each other.) What does 'they' refer to? Does the statement opposite this reason; that is, statement 2, follow this then-part exactly?

"Do the same thing with reason 3.

"Write the proof in syllogistic form as follows, to show pupils explicitly that the reasoning is exactly the same as that with which they dealt in the development of deduction.

I. 1. If two things are equal to the same thing, they are equal to each other.
   2. $AC=AD$, $BC=AD$ (Hyp.)
   3. Therefore, $AC=BC$.

II. 1. If a triangle has two equal sides, it is isosceles.
   2. $AC=BC$ (just proved).
   3. Therefore $\triangle ABC$ is isosceles."

From this point on there was nothing fundamentally new in the development of the meaning of proof. New subject matter was introduced, new postulates accepted, theorems were proved and listed as acceptable reasons. The method of analysis was carefully developed, but since it was discussed in the same way with Group D as with Group T, it does not affect the comparisons we are making, and so will not be reported here.

At first all exercises were simple. Many of them were discussed according to the method just described. It was the aim to show that each proof could be divided into separate syllogisms and each one checked to see if it was good reasoning. At least once a day for several days a proof was checked by the use of the following three questions.

1. Are all the reasons acceptable ones?
2. Are all the conditions of each reason fulfilled? (In case the reason was "Hyp." this question was interpreted to mean, "Is the statement actually in the hypothesis?")
3. Is the conclusion of each reason followed exactly?

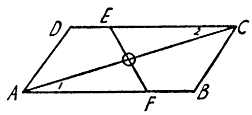
The way in which the check works can be seen by using it on the following "proof" taken from a pupil's paper.

Hyp. $AB=DC$, $AD=BC$, EF is a straight line passing through O, the middle point of AC.

Con. $FO=OE$.

Proof.

Statements	Reasons
1. $AB=DC, AD=BC$	1. Hyp.
2. $AC=AC$	2. Identity.
3. $\triangle ABC \cong \triangle ADC$.	3. s.s.s. = s.s.s.
4. $\angle 1=\angle 2$.	4. Equal angles formed by a diagonal.
5. $AO=CO$.	5. Hyp.
6. $\triangle AOF \cong \triangle EOC$.	6. a.s.a. = a.s.a.
7. $FO=OE$.	7. C.p.c.t.e.



Are all the reasons acceptable ones? A glance at the list shows that the reason for statement 4 is not acceptable. This is a danger signal. Perhaps the statement is correct, and an acceptable reason can be found. Perhaps the statement is wrong, and another method must be sought. In this case, we can find the proper reason. *Corresponding parts of congruent triangles are equal.* All the other reasons are acceptable.

Are the conditions of each reason fulfilled? We find no trouble until we get to reason 6. How many conditions are there? Four. (1) One angle of one triangle must be equal to one angle of the other, (2) another angle of one triangle must be equal to another angle of the other, (3) a side of one triangle must be equal to a side of the other triangle, and (4) the side in each case must be included between the two angles. A check shows that we are missing one angle. Can we find it?

Is the conclusion of each reason followed exactly? Yes. For example, identity means, "Any quantity is equal to itself," and we have stated $AC=AC$. The conclusion of reason 6 is, "The triangles are congruent," and we have stated "$\triangle AOF \cong \triangle EOC$."

Pupils should realize that this method of checking does not tell them *what* is right. It merely shows them *when* something is wrong. They can know by this method of procedure, when they are right and when they are wrong. The danger signal should lead them to further analysis. They should realize that in case they cannot find a proof, it is better to leave a blank page than to fill it with errors. The first would show merely that they were not clever enough to find a proof; the second that they do not know what it means to make a demonstration.

We wish to reiterate that throughout the work with Group D, the same subject matter was discussed day by day as with Group T. The same textbook was used, the same exercises were discussed in class, the same exercises were given for home assignment. What was different was the method of approach. Whatever evidence we have shows that Group D was certainly no better mathematically than Group T at the start. The superiority of Group T, then, in later work we attribute to the difference in method.

By the time that the final test on originals was given (Test 25, see page 35), Group D was doing work superior in every way to that of Group T. This was shown not only in their written work, but even more so in the discussion of their proofs. We have no way of showing here the improvement in the discussions for we had no tests in which Group T did this kind of thing. We can, however, show the superiority of Group D over Group T in the results of Test 25, which are given in the following table (Table 34).

TABLE 34

COMPARISON OF THE PERCENTAGES OF PUPILS IN GROUPS T AND D MAKING ERRORS ON EACH EXERCISE OF TEST 25

Exercise	Percentage of Pupils Making Errors		Difference (d)	S.E. of Diff. (σ)	d/σ
	T	D			
1	54.4	17.6	36.8	7.3	5.0
2	42.9	17.6	25.3	7.0	3.6
3	68.4	39.3	29.1	7.4	3.9
4	45.6	31.0	14.6	7.3	2.5

The tests in this study were constructed on the assumption that geometry is taught in the senior high school to show a pupil what a demonstration is, and to give him a notion of what is meant by postulational thinking. The results of the tests have shown a multiplicity of errors, but we have been able to group most of these under a few main headings. The remedial methods presented here do not assure a pupil's ability to prove a difficult original exercise, but do give him a method of checking a proof once it is made either by himself or by another.

To understand a proof, to be able to check its validity is one thing. To be able to make a proof when the situation involves new factors is another. Obviously ability to do the one must precede ability to do the other, and the clearer the understanding of the former, the greater the chance of success in the latter. In some cases, pupils may never get far beyond the earlier stage which is one of appreciation, but they have then certainly fulfilled one of the fundamental aims in teaching geometry. To be able to understand what a demonstration is, and whether a proof is correct, should be of value to large numbers of pupils who would not profit by an extended course in geometry where the aim is to develop ability to do more or less difficult originals. Many pupils, who by traditional methods have been unable even the reach the appreciation stage, may be helped by the use of the careful development of meanings outlined in this study.

Chapter IX

TRANSFER OF TRAINING

In this study there have been many instances of transfer of training as well as the lack of it. The purpose of this chapter is to place the study against a background of the theories of transfer and to discuss the reactions of the pupils in terms of those theories.

Theories of Transfer

Geometry was taught for centuries as a mental discipline. "Just as we have legs for walking, eyes for seeing, and a tongue for talking—different parts of the body for different bodily functions—so it was supposed there might be 'parts' in the soul with functions or duties of their own."[1] The mind was supposed to have separate parts for such functions as sensation, memory, imagination, will power, and reasoning. These faculties could be trained just as muscles are hardened by suitable practice and when trained could be used without loss of efficiency in any field of endeavor. Geometry was the subject, par excellence, for training the reasoning power.

Faculty psychology goes back to the time of Plato and even before. The location of different mental functions in different parts of the body was attempted by several of the Pythagorean philosophers. The scholastics of the Middle Ages gave it shape and made a dogma of it. It remained supreme until the beginning of the present century even though Locke and Herbart made some protests against it in the eighteenth century.

In this country the attack against faculty psychology began with James' arguments against phrenology. It was he also who carried on the first experimental work to determine whether transfer is a reality. The death blow was given by Thorndike and Woodworth in 1901. Their conclusion was, "Improvement in any single mental function rarely brings about equal improvement in any other function, no matter how similar, for the working of every mental function-group is conditioned by the nature of the data in each particular case."[2]

Since this experiment by Thorndike and Woodworth, many others of similar nature have been made.[3] In each of these experiments the conclusions were much the same. Although in most of them some transfer took place, they "clearly demonstrated that, whether the processes were simple or complex, the effects of special training were transferred to a far more limited degree than previous educationists had supposed."[4] For this reason, among others, the theory of mental faculties and formal discipline was abandoned.

It could not be denied, however, that in most of the experiments transfer from one mental task to another was found in a measurable degree. Consequently, the problem of later investigators has been, not whether transfer takes place but under what circumstances it occurs.

It was early seen that when the amount of transfer was great, there was a close similarity between the two tasks and that when the similarity decreased the amount of transfer also decreased. This observa-

[1] Burt, Cyril, "Historical Note on Faculty Psychology," *Secondary Education with Special Reference to Grammar Schools and Technical High Schools*, Report of the Consultative Committee on Secondary Education with Special Reference to Grammar Schools and Technical High Schools (London: His Majesty's Stationery Office, 1939), p. 429.

[2] Thorndike, E. L. and Woodworth, R. S., "Influence of Improvement in One Mental Function upon the Efficiency in Other Functions," *Psychological Review*, VIII (1901), 250.

[3] Summarized by Blair, Vevia, "The Present Status of 'Disciplinary Values' in Education," *The Reorganization of Mathematics in Secondary Education*, A Report by the National Committee on Mathematical Requirements (The Mathematical Association of America, Inc., 1923), pp. 89–96.

[4] Burt, Cyril, *Op. Cit.*, p. 434.

tion led to the theory of identical elements as summarized in the following quotation. "A change in one mental function alters any other only so far as the two functions have as factors identical elements. The change in the second function is in amount that due to the change in the elements common to it and the first."[5]

This early theory of transfer had as its background "association" psychology which emphasized the one-to-one relation between specific stimuli and specific response. It explained transfer in a purely mechanical fashion. When there were identical elements in the external situation there was the probability of transfer, otherwise not. And yet it has been established that such things as ideals, attitudes, and generalizations transfer to situations where the identical elements are not apparent. It appears, therefore, that the identical element theory explains transfer in only a limited number of cases. The experiments of Lashley point toward a lesser emphasis upon the one-to-one relation between stimuli and responses as an explanation of mental reactions.[6]

Later investigators have liberalized this theory to allow for identical elements which are subjective as well as objective. The common element may be a method or an ideal which has been freed from a particular situation and so usable by conscious effort on subject matter of a different kind. It is generally agreed that transfer may be most effectively assured by methods of teaching which make the methods or ideals learned during the training period clearly conscious and freed of their context. Says Burt, "The common elements may be elements of (1) material, (2) method, (3) ideal. A common element is more likely to be usable if the learner becomes clearly conscious of its nature and its general applicability."[7]

Of nearly the same import is the following conclusion of Jordan made after analyzing several experiments on transfer. "It seems fair to conclude, therefore, that the development of meaning and understanding in material learned brings about a condition very conducive to transfer to functions similar in material and procedure."[8]

In opposition to the identical elements theory, Judd has taken the view that the most important instances of transfer are due to what he calls *generalized experience.* He has championed the view that particular ideas and methods may by suitable teaching be generalized so that they are no longer associated only with the material with which they are obtained. The method of teaching is all important according to this theory. "The type of training which pupils receive is determined by the method of presentation and by the degree to which self-activity is induced rather than by the content." Also, "Any subject which emphasizes particular items of knowledge and does not stimulate generalizations is educationally barren." And again, "It is the nature of generalizations and abstractions that they extend beyond the particular experience in which they originate."[9]

Stress is given to the development of meaning and understanding and the freeing of concepts from the context in which they are developed. In this connection, Bode says, "Human beings have the capacity of detaching the thing thus suggested or indicated (meaning) and treating it independently, apart from the specific setting." And a little later, "The great advantage in detaching meanings in this way is that they become more readily available in a variety of situations."[10]

[5] Thorndike, E. L., *Educational Psychology* (New York: Teachers College, Columbia University, 1903), p. 90.

[6] Lashley, Karl S., *Brain Mechanisms and Intelligence* (Chicago: University of Chicago Press, 1929).

[7] Burt, Cyril, *Formal Training,* Report of a committee appointed by the British Association and presented at Bristol, 1930. P. 4.

[8] Jordan, A. M., *Educational Psychology* (New York: Henry Holt and Co., 1933), p. 225.

[9] Judd, C. H., *Psychology of Secondary Education* (Boston: Ginn and Co., 1928).

[10] Bode, Boyd H., *Fundamentals of Education* (New York: The Macmillan Co., 1921), p. 152.

The most recent explanation of transfer of training is that given by the investigators in gestalt psychology. They conclude that the most significant things in our environment are not elements but the relations between these elements. There can be generalities in the forms of awareness, tendencies, attitudes, mental sets, and dispositions based upon relationships and not upon the specific nature of any particular situation or element in it. These can be considered general since they can be obtained from any one of many situations which are similar only in the relations involved and can be transferred to other situations which contain these same relationships. What transfers is a *pattern of experience*.[11]

All these theories of transfer may be confusing to the teacher of geometry. Yet there is a core of sameness in them that points toward a workable basis for teaching. Of one thing we can be sure. No great amount of transfer is automatic. If we wish the teaching of geometry to result in anything more than the learning of specific things in geometry, we must know what we are striving for and then search for suitable methods of teaching. The fact that there are identical elements in geometric reasoning and deductive reasoning elsewhere will not necessarily result in automatic transfer from the one situation to the others. The pupil may not note the similarity. He may not be sufficiently conscious of the method of reasoning in geometry to see the similarity. The relationships involved may not be seen clearly and they may not be freed from their context. The pupil may experience geometric reasoning but not reasoning in general. If we hope for transfer we must teach for it. Meanings must be developed. The methods of reasoning must be brought to the level of awareness. The logical relations developed should be made available in non-geometric situations by using them in a variety of contexts.

[11] Commins, W. D., *Principles of Educational Psychology* (New York: The Ronald Press, 1937), p. 432.

Application of the Theories of Transfer to this Study

The transfer shown in connection with constructions and the recognition of particular parts of geometric figures may well be explained by the presence of identical elements. After learning how to bisect a horizontal straight line, only 6% of Group T and 9% of Group D (see Table 26, page 38) were unable to bisect the sides of a given triangle correctly. The two tasks are obviously very much alike in detail and the transfer was large. It is important to note, however, that there were some who could not perform the second task in spite of the great similarity to the first. The identical elements did not insure transfer.

As shown in the results of further tests (see particularly Test 3, page 5 and Table 3, page 6) the differences between the practice exercises and the test exercises became increasingly more potent than the likenesses as the exercises became more complex. Many of the pupils did not react to the identical elements that were present. Explaining in terms of the theories of transfer, we might say that the methods of construction had not been generalized, they had not been freed from the particular figure with which they had been learned. Or we might say that the methods themselves had not been made clear as procedures or patterns of experience. They had been learned as isolated mechanical operations but the essential relationships had not been seen.

When the procedure was analyzed and generalized as shown on page 40, not only was the pattern made clearer and brought to consciousness but it was freed from a particular figure. The transfer was therefore significantly greater in Group D than in Group E which did not have the analysis and generalization (see Table 27, page 41).

Much the same things can be said in connection with the pupils' recognition of the application of terms and theorems in a complex figure. For example, Table 9, page 14 (showing the results of a test in recognizing alternate interior angles) indicates transfer from a simple figure to complex figures without training. The identical elements were, therefore, potent with some pupils. With others they were not sufficient to cause transfer. Pointing out the zig-zag (page 43) showed the pupils more clearly the essential relation between the lines and the transversal in forming alternate interior angles and gave them a pattern to apply in other figures. Table 30 (page 43) shows the improvement after this was done.

Transfer was shown also in connection with the if-then relationship and the meaning of proof. But here we cannot explain with any great satisfaction on the basis of identical elements. The situations are much more complex. It seems more nearly adequate to explain it in terms of meanings, attitudes and ideals; on the basis of bringing the concepts to the level of awareness and generalizing them; or from the point of view of patterns of experience.

It was assumed that the errors made in choosing hypothesis and conclusion from verbal statements were due to the lack of understanding of the if-then relationship. Accordingly, a method was introduced to develop its meaning. Every effort was made to show the pupils of Group D (Group T did not have this training) what is meant by conditions and conclusion in connection with geometric figures. These concepts were associated with the if-part and the then-part of if-then sentences. Pupils were made aware of the fact that they were studying, not the particular sentences under discussion, but the meaning of the if-then relationship in general. By emphasizing the contrast between conditions and conclusion in connection with various geometric figures, the concept stood out from any particular context. It became a pattern of experience in its own right. Consequently, when the pupils of Group D were asked to apply the meaning of the if-then relationship to a new situation—choosing hypothesis and conclusion from a verbal statement, the transfer was greater than in the case of Group T (see Table 31, page 47).

The important point to note in the development of proof described in this study is that the reasoning involved in the geometric proofs was studied as a procedure. "Unless time is set apart for a study of the reasoning processes," says Blackhurst, "no experience with reasoning can be generated. It is not only possible but highly probable that the pupil who has reasoned through all the material of a present day geometry has not in the least added to his experience with reasoning. In other words he has experienced geometry but not reasoning."[12]

In this development, the pupils first learned the meaning of deduction through the medium of syllogisms employing non-geometric material. The process of deduction was analyzed so that the pattern of procedure could be seen. The procedure was then put into words; that is, generalized. When geometric proofs were made, the pupils were shown that these proofs were successive applications of the principles they had learned in connection with syllogisms. They were also shown a method of checking their proofs which again used this procedure. In other words each proof was not an isolated experience but an application of an ever widening pattern.

No attempt has been made in this study to ascertain the effect of improvement in geometry upon the improvement in reasoning outside the field of geometry. The assumption has been that the first step in insuring transfer from geometry to other fields is to improve the reasoning in geometry itself and that, as Mursell says, "when any ability is most intelligently taught and organized for its own sake, it is thereby taught and organized in such a way as will facilitate transfer."[13]

[12] Blackhurst, J. H., "The Educational Values of Logical Geometry," THE MATHEMATICS TEACHER, XXXII (April, 1939), 164.

[13] Mursell, James L., *The Psychology of Secondary School Teaching* (New York: W. W. Norton and Co., 1932), p. 104.

Chapter X

SUMMARY

The purpose of this study was (1) to discover, analyze, and classify errors made by pupils beginning the study of demonstrative geometry and (2) to devise methods of teaching which by making meanings clearer would lessen the number of errors. Consequently the study is in two parts corresponding to the double purpose. The experimentation for Part I was carried through in the fall of 1932 by means of diagnostic tests given over a period of fifty consecutive teaching days to 114 pupils with I.Q.'s ranging from 90 to 146 with a mean I.Q. of 117.9 and standard deviation of 11.7. Almost all of the errors could be classified under three headings, (1) those due to unfamiliarity with figures, (2) those due to not sensing the meaning of the if-then relationship, and (3) those due to meager understanding of the meaning of proof, and the study is restricted to a discussion of these three types of error. The experimentation for Part II was carried on in the fall of 1938 after the methods devised had been tried out on several successive classes. The group used in 1938 consisted of 74 pupils with a distribution of I.Q.'s comparable to that of the original group.

In 1932 the experiment involved five classes and four teachers. Bulletins were issued daily describing in detail the topics and exercises to be covered and the methods to be used. In 1938, three classes and three teachers were involved. In order to keep all factors except method constant in the two experiments, the bulletins given out in 1932 were followed faithfully day by day so far as topics and exercises were concerned. The new factor in the 1938 experiment was the teaching methods which had been developed to deal with the previously mentioned three types of error. The results of certain key tests recorded in Part II showed that these methods caused significant improvement.

The following sections contain a brief summary of the preceding chapters. They consist of the conclusions reached and selected illustrations of the evidence which led to the conclusions.

Complex Figures

This study shows that *even though a pupil may know the meaning of a term, perform a construction, or apply a theorem in connection with a simple figure, he may or may not react correctly to these same things in a complex figure.* After learning how to bisect a horizontal straight line, 6% of the pupils could not bisect the sides of a triangle correctly (page 4). Although all the pupils could bisect a single angle in a given position, 15% could not bisect the angles of a triangle (page 4). They could draw a perpendicular to a line when only a point and a line were involved but as the figure was made more complex the number of pupils making errors reached as high as 52% (page 10). After showing their ability to recognize opposite angles made by two intersecting lines when only the two lines were given, 16% of the pupils did not recognize the opposite angles when the ends of these lines were connected to form two triangles (page 17). They could recognize two sides and the included angle and two angles and the included side in a triangle but as many as 40% made errors when the figure involved overlapping triangles (page 12). When all but 5% of the group were able to identify alternate interior angles in a figure consisting only of two parallel lines and a transversal, a slight complication caused 25% to make errors and further complications increased the number of pupils making errors to as much as 80% (page 14).

The methods devised to minimize the difficulties in connection with figures were based on the assumption that pupils at first see the figures as wholes and that any procedure which would help them to see the essential parts and dissociate these from the irrelevant would prove effective. Construction problems were analyzed and generalized so that the method would apply to any figure and not merely to a particular figure (page 39). Terms and propositions were illustrated in complex figures as well as in simple figures (page 42) and whenever possible some peculiarity of a configuration like the zig-zags accompanying alternate interior angles was pointed out (page 43). That these methods proved effective is shown by the results recorded in Tables 27, 29, and 30 (pages 41 and 43).

The If-Then Relationship

The if-then relationship is fundamental to postulation thinking. Lack of a clear understanding of its meaning is a real handicap to students beginning demonstrative geometry. *Many students do not understand the logical implication of the if-then relationship before it is developed in the geometry class, and the growth of the concept is slow after development is begun.*

Out of 74 pupils in the 1938 group, 61 said that there was no difference between the converse statements. *If two sides of a triangle are equal, the angles opposite those sides are equal,* and *If two angles of a triangle are equal, the sides opposite those angles are equal.* And 55 of them said that those two statements together meant the same thing as *In an isosceles triangle two sides and the two angles opposite them are equal* (page 44). When asked to construct figures to test the truth of the two converse statements just stated only 58% of the 1932 group sensed the meaning of the if-then relationship well enough to make two sides equal for the first statement and two angles equal for the second (page 22). Difficulties of this kind persisted in the case of fully one tenth of the group even after six days' practice (page 23). A multiplicity of errors was made by 59% of the pupils in drawing a figure and writing the hypothesis and conclusion in terms of that figure for a rather simple verbal statement (page 24).

The meaning of the if-then relationship was developed concretely with the 1938 group by means of constructions. Pupils saw that when they did certain things in making a figure certain other things resulted. They learned to feel the difference in category between the relationships they put into a figure—the things over which they had control—and the relationships which resulted without any action on their part. Finally the difference in these two categories was associated with the difference between given conditions and conclusion, between the if-part and the then-part of a sentence (page 45 ff). Because of this development the pupils in 1938 showed significant superiority over the 1932 group in their ability to write the hypothesis and conclusion in terms of a figure when given a proposition (page 47).

Meaning of Proof

The meaning of proof was discussed under four headings, Deduction, Meaning of Hypothesis, Acceptable Reasons, and Proofs of Exercises. *It was found that a large percentage of pupils, when they begin the study of demonstrative geometry, have little conception of what is involved in making a formal deduction.* An analysis of a test on syllogisms resulted in the following conclusions (page 28).

(1) Although pupils can make a deduction when simple everyday situations without any complications are given, they cannot necessarily do so when there are complications even in such simple situations.

(2) Pupils need to learn that a conclusion can be drawn only when all the conditions are fulfilled.

(3) They must learn to analyze a statement to find out what the conditions are that must be fulfilled, and they must be able to see whether all the conditions have been fulfilled.

(4) They must learn that when the conditions are fulfilled the only conclusion

that can be drawn is the one stated in the general statement (first premise). Pupils certainly are not ready to make demonstrations in geometry until they have mastered these things.

Pupils do not readily grasp the concept of holding to the data. Without careful training they will give more weight to the appearance of a figure than to the hypothesis. In a multiple choice test (Test 21, page 30) in which only one of five choices followed from the data but in which all five choices appeared true in the figure, a large number of pupils made errors (36% on one exercise). When this test was given, the pupils had been working for sixteen days, with exercises requiring them to hold to given conditions.

Pupils are handicapped in their geometric reasoning by most of their previous experience in intuitive thinking concerning things in general. It has been their experience to make deductions using any part of the total situation that appeals to them. In geometry they are definitely restricted in their reasoning to the data, definitions, and propositions assumed or previously proved. *In order to understand the significance of a proof in geometry pupils must learn at the beginning of their formal work that they are restricted as to the reasons they may use.* Left to their own devices pupils will give reasons made up to suit the occasion (page 32).

In the final test on original exercises (Test 25, page 35), there were only a few errors which could not be classified under the three headings discussed here. No conclusion could be drawn as to the type which caused the most errors, the kind of error depending upon the exercise (see distribution of errors, page 36).

The 1938 group of pupils was given a careful development of the meaning of proof. Deduction was introduced by means of syllogisms the correctness of which could first be checked by studying the concrete situation. The syllogisms were then analyzed to discover means by which the statements themselves gave the cue to their rightness or wrongness. The result of this discussion was the following summary concerning deductions (page 50).

1. A deduction consists of three statements, (a) a general statement, (b) a specific statement which is an application of the general statement, and (c) a conclusion.

2. No conclusion can be drawn from the first two statements unless the second statement fulfills exactly the conditions of the first statement: that is, *fits* exactly the if-part of the first statement.

3. If the second statement fulfills the conditions of the first statement, then there always is a conclusion and this conclusion must follow exactly the then-part of the first statement.

Postulates were introduced inductively and listed (page 52 ff). Theorems were listed as they were proved. Pupils were led to see that the purpose of a proof was to see how new theorems grew out of the postulates and theorems. If, therefore, they used reasons not on their lists they were going contrary to the purpose of a proof.

The necessity of holding definitely to the data was shown by drawing special and general figures with the same data (page 31). Pupils could see that many different shaped figures could be drawn with the same hypothesis and that the only things true of all of them were those resulting from what was specifically given.

When the first formal proofs were introduced they were shown to be successive applications of the principles learned in connection with syllogisms (page 55 ff). Since pupils had learned a method of checking the correctness of a syllogism, they now had a method of checking a geometric proof. They were urged to check each proof by asking themselves the following three questions (page 56).

1. Are all the reasons acceptable ones?
2. Are all the conditions of each reason fulfilled?
3. Is the conclusion of each reason followed exactly?

If the answers to all these questions was in the affirmative the pupil could be reasonably sure that his proof was correct. If at least one answer was negative, something was wrong and further thought required.

The emphasis throughout was on the method of procedure and only secondarily on the geometry. To be sure, the geometry had to be learned and organized because it was the basis of the reasoning. But the aim was never to allow the details of geometry to obscure the analysis of the thought processes.

Because of this careful discussion of the meaning of proof, the 1938 group showed significant superiority over the 1932 group in proving original exercises (see Table 34, page 57).

In Chapter IX, the methods introduced in this study were shown to be in accordance with the laws of the transfer of training. These methods should be, therefore, not only the foundation for better results in geometry itself but helpful in building up a course of study which will allow for the carryover of the fundamental procedures to other fields.