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# A TREATISE ON ELECTRICITY

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# A TREATISE ON ELECTRICITY

BY

F. B. PIDDUCK

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# PREFACE

THE principal changes in this edition correspond to the advance of the science in the last ten years, except that the experiments of Compton did not seem ripe for discussion at the time the manuscript was completed. A separate chapter has been assigned to Röntgen rays, the chapters on electric oscillations and electricity in gases have been revised, and the last chapter rewritten so as to present a short account of recent atomic theory. Vector analysis has been used to lend conciseness to the theory of the electromagnetic field: a complete symbolic calculus divorced from Cartesian analysis is now generally considered a mistake, but physicists find a vectorial treatment interesting, if it does not go too far.

My best thanks are due to Prof. C. T. R. Wilson and Prof. J. Zenneck for illustrations used already in the first edition of this book, and to Prof. E. V. Appleton, Dr F. W. Aston, the Editors of the *Philosophical Magazine*, Prof. M. Siegbahn and Mr L. B. Turner for similar kindness in connexion with the present edition. Dr J. Chadwick, Capt. P. P. Eckersley and Dr C. D. Ellis have given me much valuable information, and I am indebted to Prof. J. S. Townsend for permission to utilise the facilities of his laboratory. Blocks for the illustrations have been kindly supplied by Messrs Edward Arnold, Messrs Baird and Tatlock, the Cambridge and Paul Instrument Co., Messrs Longmans, Green and Co., Messrs Nalder Bros., Messrs Newton and Wright, Messrs W. G. Pye and Co., Messrs Taylor and Francis, the British Westinghouse Co. and the Weston Electrical Instrument Co.

F. B. P.

*September*, 1924.

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CORRIGENDUM

p. 220, lines 11 and 12 from below, *read* "$\kappa_2$ of air under normal conditions, and $\kappa_3$ of the substance under examination."

# CHAPTER I

## MATHEMATICAL INTRODUCTION

**1. Right-handed coordinate axes.** If $Ox$, $Oy$, $Oz$ are the positive directions of a set of rectangular axes in three dimensions, and $Ox$, $Oy$ are fixed, the third line $Oz$ may be drawn at right angles to the plane $xOy$ in two directions. Let the axes be moved so that $Ox$ points due east, and $Oy$ due north. Then the axes are called *right-handed* if $Oz$ points vertically upwards. Fig. 1 shows

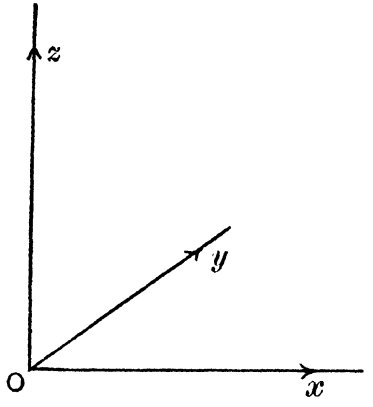



Fig. 1

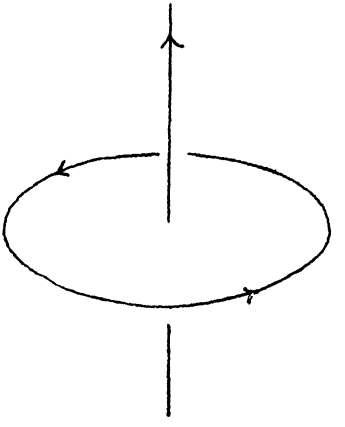


Fig. 2

a right-handed set in perspective, the axis of $y$ pointing away from the observer. Only right-handed axes are used in this book.

Imagine an ordinary (right-handed) corkscrew to be pointed along $Ox$ and turned so as to travel in the positive direction of $Ox$. A quarter-turn will then bring $Oy$ to the position $Oz$, and similarly a quarter-turn about $Oy$ brings $Oz$ to $Ox$, and a quarter-turn about $Oz$ brings $Ox$ to $Oy$.

Thus right-handed axes are closely related to what is called the *right-handed screw law*, associating every directed line with a direction of rotation round it (Fig. 2).

**2. Vectors.** The ordinary positive or negative quantity of algebra, regarded as having magnitude without direction, is called a *scalar*. A *vector* may be described briefly as a directed magnitude. It must be capable of being represented completely by a line of

suitable length drawn in a suitable direction, and must further obey the "parallelogram law"; i.e. the joint effect of two vectors represented by the lines $PA$, $PB$ must be a vector represented by $PD$ the diagonal of the parallelogram drawn with $PA$, $PB$ as adjacent sides. The velocity and acceleration of a particle, and the force acting on it, are vectors.

Vectors are denoted by thick or Clarendon letters $\mathbf{u}$, $\mathbf{r}$, $\mathbf{E}$, etc., the corresponding ordinary letters $u$, $r$, $E$, etc. being used for the magnitudes of the vectors. The magnitude of a vector is always taken as positive.

If $\mathbf{u}$ is a vector, $-\mathbf{u}$ is defined to be a vector of the same magnitude in a reverse direction. More generally, if $c$ is any scalar, $c\mathbf{u}$ is a vector of magnitude $|c|\,\mathbf{u}$, parallel to $\mathbf{u}$ if $c$ is positive and in the opposite direction if $c$ is negative. Thus

$$-(-\mathbf{u}) = +\mathbf{u}, \quad c(-\mathbf{u}) = -c\mathbf{u}, \quad -c(-\mathbf{u}) = c\mathbf{u},$$

just as if $c$ and $\mathbf{u}$ were both scalars.

Composition of vectors is represented by the sign of addition: thus $\mathbf{u}+\mathbf{v}$ is the resultant of $\mathbf{u}$ and $\mathbf{v}$, and $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$. By $\mathbf{u}-\mathbf{v}$ we mean $\mathbf{u}+(-\mathbf{v})$, the vector sum of $\mathbf{u}$ and $-\mathbf{v}$. The equation $\mathbf{w} = \mathbf{u}+\mathbf{v}$ expresses the same fact as $\mathbf{w}-\mathbf{u} = \mathbf{v}$; the former that $\mathbf{w}$ is the resultant of $\mathbf{u}$ and $\mathbf{v}$ and the latter that $\mathbf{v}$ is the resultant of $\mathbf{w}$ and $\mathbf{u}$ reversed. Thus we can transfer any term in a vector equation from one side to the other, as in ordinary algebra. Any number of vectors can be added in succession, and the sum represents the same vector in whatever order they are taken. If $c$ is any scalar, the resultant of $c\mathbf{u}$ and $c\mathbf{v}$ is $c$ times the resultant of $\mathbf{u}$ and $\mathbf{v}$, or

$$c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}.$$

These results show that addition, subtraction and multiplication by scalars are performed by the rules of ordinary algebra.

The vector $\lambda\mathbf{u}+\mu\mathbf{v}$ may be identified with any vector in the plane of $\mathbf{u}$ and $\mathbf{v}$ by suitable choice of the scalars $\lambda$, $\mu$. Similarly $\lambda\mathbf{u}+\mu\mathbf{v}+\nu\mathbf{w}$ may be identified with any vector in space by choice of $\lambda$, $\mu$, $\nu$. Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be three vectors, each of unit magnitude, parallel to a set of coordinate axes $Ox$, $Oy$, $Oz$. If $P$ is the point $(x, y, z)$, the vector $OP$ (Fig. 3) is

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z,$$

and any other vector **u** may be written

$$\mathbf{u} = \mathbf{i}u_x + \mathbf{j}u_y + \mathbf{k}u_z,$$

where $u_x$, $u_y$, $u_z$ are the rectangular components of **u**. Space is often saved by writing a vector in this form, instead of specifying its components. Addition of vectors implies addition of their components.

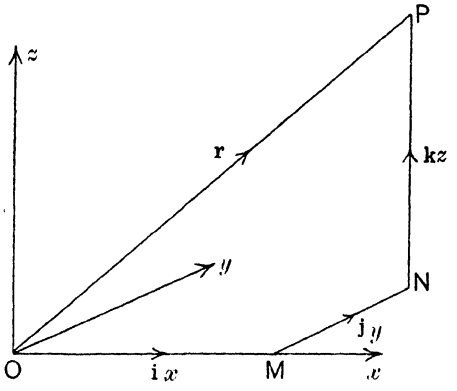



Fig. 3

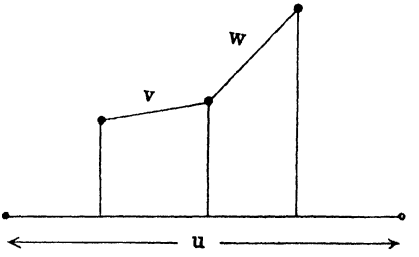



Fig. 4

**3. Scalar product of two vectors.** If **u**, **v** are two vectors represented by the lines $PA$, $PB$, which make an angle $\theta$ with each other, the quantity $PA \,.\, PB \cos\theta$ is called the *scalar product* of **u** and **v** and denoted by **uv**. It is the product of the magnitude of either vector and the projection of the other on it. The equation $\mathbf{uv} = 0$ states that, if neither **u** nor **v** is of zero magnitude, **u** and **v** are at right angles to each other. If **u**, **v**, **w** are any three vectors,

$$\mathbf{u}(\mathbf{v} + \mathbf{w}) = \mathbf{uv} + \mathbf{uw} \quad \ldots\ldots\ldots\ldots (1).$$

Fig. 4 suggests a proof of this theorem, which need not be written out. Further,

$$\mathbf{uv} = \mathbf{vu} \text{ and } \mathbf{u} \,.\, c\mathbf{v} = c\mathbf{uv},$$

$c$ being any scalar. The scalar product combines with addition, and with scalars, by the ordinary rules of algebra. By repeated application of (1) we have for any number of vectors

$$(\mathbf{u} + \mathbf{v} + \ldots)(\mathbf{a} + \mathbf{b} + \ldots) = \mathbf{ua} + \mathbf{ub} + \mathbf{va} + \mathbf{vb} + \ldots \quad (2),$$

all possible pairs being taken on the right. The product **uu** is

written $\mathbf{u}^2$, and is the square of the magnitude $u$ of the vector. The formulae

$$(\mathbf{u}+\mathbf{v})^2 = \mathbf{u}^2 + \mathbf{v}^2 + 2\mathbf{uv}, \quad (\mathbf{u}-\mathbf{v})^2 = \mathbf{u}^2 + \mathbf{v}^2 - 2\mathbf{uv}$$

express the magnitudes of the sum and difference of two vectors in terms of the separate magnitudes and the scalar product. The latter is equivalent to the cosine formula of a plane triangle.

We shall often, in what follows, use physical illustrations of vector formulae. If $\mathbf{F}$ is a force and $\mathbf{u}$ the displacement of its point of application, the work done is $\mathbf{Fu}$. Scalar products only appear in geometry after we have learnt to regard areas as vectors. A plane area is completely specified in size and direction in space by a vector at right angles to its plane and of magnitude equal to the area. Let $\mathbf{S}$ be a vector area, and let all its points be given a displacement $\mathbf{u}$ (Fig. 5). The area traces out an oblique cylinder whose base is $S$ and whose height is the projection of $\mathbf{u}$ on the normal. The volume traced out is therefore the scalar product $\mathbf{uS}$, or again $u_n S$, where $u_n$ is the component of $\mathbf{u}$ normal to the plane.

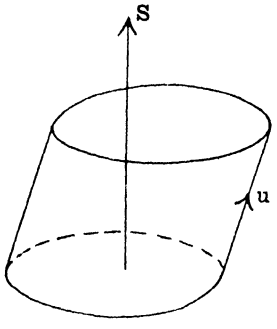


Fig. 5

If $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are trirectangular unit vectors (Art. 2),

$$\left.\begin{aligned}\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 &= 1\\ \mathbf{jk} = \mathbf{ki} = \mathbf{ij} &= 0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(3).$$

Hence from (2) the scalar product of two vectors

$$\mathbf{u} = \mathbf{i}u_x + \mathbf{j}u_y + \mathbf{k}u_z, \quad \mathbf{v} = \mathbf{i}v_x + \mathbf{j}v_y + \mathbf{k}v_z$$

is

$$\mathbf{i}^2u_xv_x + \mathbf{ij}u_xv_y + \mathbf{ik}u_xv_z + \mathbf{ji}u_yv_x + \mathbf{j}^2u_yv_y + \mathbf{jk}u_yv_z + \mathbf{ki}u_zv_x + \mathbf{kj}u_zv_y + \mathbf{k}^2u_zv_z,$$

or
$$\left.\begin{aligned}\mathbf{uv} &= u_xv_x + u_yv_y + u_zv_z.\\ \text{In particular}\quad \mathbf{u}^2 &= u_x^2 + u_y^2 + u_z^2,\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(4)$$

the ordinary Cartesian expression. The components of a vector are themselves scalar products, the scalar products with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

**4. Vector product of two vectors.** Let $PA$, $PB$ (Fig. 6) represent the vectors $\mathbf{u}$, $\mathbf{v}$ respectively, $PB$ being shown receding from the observer, and let a line $PC$ be drawn at right angles to the plane $APB$ and of length $PA \,.\, PB \sin\theta$. Then $PC$ represents a vector called the *vector product* of $\mathbf{u}$ and $\mathbf{v}$, and denoted by $V\mathbf{uv}$. Its magnitude is the area of the parallelogram having $PA$, $PB$ as

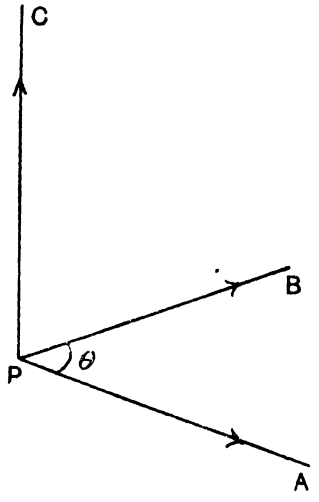



Fig. 6

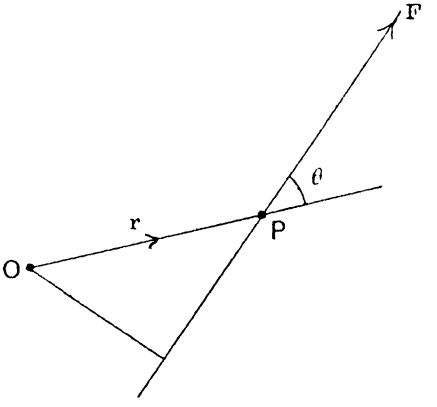



Fig. 7

adjacent sides. The positive direction of $V\mathbf{uv}$ is such that a small right-handed turn round it brings $PA$ towards $PB$. If $PA$ and $PB$ are interchanged, the direction of $PC$ changes. Thus $V\mathbf{vu} = -V\mathbf{uv}$, and in particular $V\mathbf{uu} = 0$. The vector product of $\mathbf{u}$ and $\mathbf{v}$ is not altered by compounding with $\mathbf{u}$ any vector parallel to $\mathbf{v}$, or *vice versâ*, for this alters neither the plane nor the area.

The vector product is wanted to describe moments in mechanics. If a force $\mathbf{F}$ (Fig. 7) acts at a point $P$ of a rigid body, it can be transferred to $O$ by introducing a couple whose moment is $rF\sin\theta$ and whose axis is perpendicular to the plane of the paper. The couple is represented completely by the vector $V\mathbf{rF}$.

Changing the sign of $\mathbf{u}$ or $\mathbf{v}$ changes that of their vector product. More generally the vector product of $c_1\mathbf{u}$ and $c_2\mathbf{v}$ is $c_1c_2V\mathbf{uv}$, where $c_1$ and $c_2$ are any two scalars. The vector product has also the property

$$V\mathbf{u}(\mathbf{v}+\mathbf{w}) = V\mathbf{uv} + V\mathbf{uw} \quad \ldots\ldots\ldots(5)$$

analogous to (1), the sum on the right being the vector sum of the two vectors $V\mathbf{uv}$ and $V\mathbf{uw}$. It is sufficient to prove the theorem when $\mathbf{v}$ and $\mathbf{w}$ are at right angles to $\mathbf{u}$; for compounding with $\mathbf{v}$ or $\mathbf{w}$ a vector parallel to $\mathbf{u}$ does the same to $\mathbf{v}+\mathbf{w}$ and leaves (5) unaltered. We can also take $\mathbf{u}$ as a *unit* vector. In this reduced case $V\mathbf{uv}$ is simply the vector $\mathbf{v}$ turned through a right angle round $\mathbf{u}$, and similarly for $\mathbf{w}$. The parallelogram of composition of $V\mathbf{uv}$ and $V\mathbf{uw}$ is therefore that of $\mathbf{v}$ and $\mathbf{w}$ turned through a right angle, which proves the theorem. Similarly $V(\mathbf{u}+\mathbf{v})\mathbf{w} = V\mathbf{uw} + V\mathbf{vw}$. Repeated application of these formulae gives

$$V(\mathbf{u}+\mathbf{v}+\ldots)(\mathbf{a}+\mathbf{b}+\ldots) = V\mathbf{ua} + V\mathbf{ub} + V\mathbf{va} + \ldots \quad (6),$$

where however we cannot change the order of the factors in any term, unless the sign is changed.

The vector products of the trirectangular unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are

$$\left.\begin{aligned} V\mathbf{ii} = V\mathbf{jj} = V\mathbf{kk} &= 0 \\ V\mathbf{jk} = -\,V\mathbf{kj} &= \mathbf{i} \\ V\mathbf{ki} = -\,V\mathbf{ik} &= \mathbf{j} \\ V\mathbf{ij} = -\,V\mathbf{ji} &= \mathbf{k} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (7),$$

and the vector product of the two vectors

$$\mathbf{u} = \mathbf{i}u_x + \mathbf{j}u_y + \mathbf{k}u_z, \quad \mathbf{v} = \mathbf{i}v_x + \mathbf{j}v_y + \mathbf{k}v_z$$

is

$$V\mathbf{uv} = \mathbf{i}(u_y v_z - u_z v_y) + \mathbf{j}(u_z v_x - u_x v_z) + \mathbf{k}(u_x v_y - u_y v_x) \ldots (8).$$

**5. Formulae for three vectors.** The vector product, being itself a vector, can be combined with a third vector to form scalar and vector products.

(1) If $\phi$ is the angle between $\mathbf{w}$ and the vector product $V\mathbf{uv}$ (Fig. 8), the scalar product $(V\mathbf{uv})\mathbf{w}$ is $w\cos\phi$ times the area of the parallelogram $APB$, that is the volume of the parallelepiped with $PA$, $PB$, $PC$ as adjacent edges. This is equal, by symmetry, to the scalar product of $\mathbf{u}$ and $V\mathbf{vw}$, or

$$(V\mathbf{uv})\mathbf{w} = \mathbf{u}(V\mathbf{vw}) \ldots . (9).$$

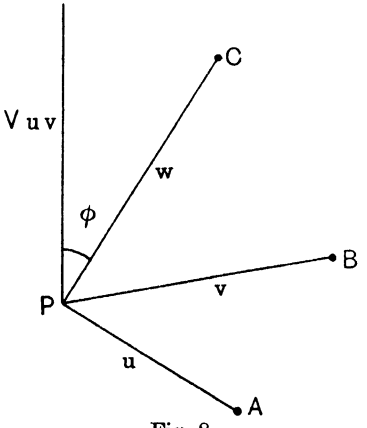


Fig. 8

The theorem follows also from (4) and (8), each expression being equal to

$$\begin{vmatrix} u_x, & u_y, & u_z \\ v_x, & v_y, & v_z \\ w_x, & w_y, & w_z \end{vmatrix}.$$

(2) The vector product of $\mathbf{w}$ and $V\mathbf{uv}$ can be transformed as follows. Writing $\mathbf{r}$ for $V\mathbf{uv}$, the vector $V\mathbf{wr}$ is perpendicular to $\mathbf{r}$, and therefore lies in the plane of $\mathbf{u}$ and $\mathbf{v}$. Hence $V\mathbf{wr} = \lambda\mathbf{u} + \mu\mathbf{v}$, where $\lambda$ and $\mu$ are scalars. Form the scalar product of each side with $\mathbf{w}$. The left-hand side is $\mathbf{w}(V\mathbf{wr}) = V(\mathbf{ww})\,\mathbf{r} = 0$, so that $\lambda\mathbf{uw} + \mu\mathbf{vw} = 0$. Thus $\lambda = \kappa\mathbf{vw}$ and $\mu = -\kappa\mathbf{uw}$, or

$$V\mathbf{w}V\mathbf{uv} = V\mathbf{wr} = \kappa\{(\mathbf{vw})\,\mathbf{u} - (\mathbf{uw})\,\mathbf{v}\},$$

where $\kappa$ is some scalar. Since each side of the equation contains the magnitudes of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ to the first degree, $\kappa$ is a pure number. Putting $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{w} = \mathbf{j}$, the left-hand side becomes $V\mathbf{jk} = \mathbf{i}$ and the right-hand side becomes $\kappa\mathbf{i}$. Hence $\kappa = 1$, so that

$$V\mathbf{w}V\mathbf{uv} = (\mathbf{vw})\,\mathbf{u} - (\mathbf{uw})\,\mathbf{v} \quad \ldots\ldots\ldots(10).$$

The result can be verified from (4) and (8).

**6. Differentiation and integration of vectors.** Let a vector $\mathbf{u}$ have an assigned magnitude and direction for any assigned time $t$. At time $t'$, $\mathbf{u}$ will have become a new vector $\mathbf{u}'$. The limit to which the vector $\dfrac{\mathbf{u}' - \mathbf{u}}{t' - t}$ tends, as $t'$ tends to $t$, is called the *derivate* of $\mathbf{u}$ with respect to $t$ and written $\dfrac{d\mathbf{u}}{dt}$. If $\mathbf{r}$ is the vector from a fixed origin $O$ to a point $P$ (Fig. 9), $\mathbf{r}' - \mathbf{r}$ is the vector $PP'$ and $d\mathbf{r}/dt$ represents the *velocity* of $P$ in magnitude and direction. The infinitesimal vector $\mathbf{r}' - \mathbf{r} = d\mathbf{r}$ is the displacement of $P$ in the infinitesimal time $t' - t = dt$. The second derivate of $\mathbf{r}$ is the acceleration of $P$, and so on.

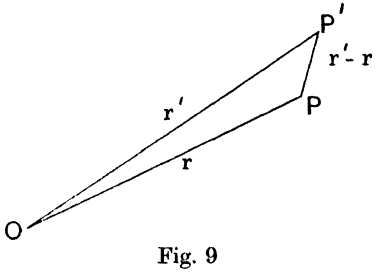


Fig. 9

Since differentiation is subtraction followed by a scalar division,

the rules for vector differentiation resemble those for scalar quantities. Thus

$$\frac{d}{dt}(\mathbf{u}+\mathbf{v}) = \frac{d\mathbf{u}}{dt} + \frac{d\mathbf{v}}{dt},$$

where the right-hand side denotes composition of the vectors $d\mathbf{u}/dt$, $d\mathbf{v}/dt$. In particular, if $\mathbf{u} = \mathbf{i}u_x + \mathbf{j}u_y + \mathbf{k}u_z$,

$$\frac{d\mathbf{u}}{dt} = \mathbf{i}\frac{du_x}{dt} + \mathbf{j}\frac{du_y}{dt} + \mathbf{k}\frac{du_z}{dt}.$$

We can also integrate vectors with respect to a scalar variable. Let $P$ be a point inside a solid body, and let the matter in the neighbourhood of $P$ be acted on by a known force $\mathbf{F}$ per unit volume. If $d\tau$ is an element of volume near $P$, the force on the element is $\mathbf{F}\,d\tau$, and the total force on the body is $\int_\tau \mathbf{F}\,d\tau$ extended through the body, where the sign $\int$ stands for summation of the infinitesimal quantities $\mathbf{F}\,d\tau$, each one of which is a vector. A vector like $\mathbf{F}$, given in magnitude and direction at all points of a region, is called a *vector function of position.*

Let a point $P$ move from $A$ to $B$ along an arc of a curve in space under the action of the force $\mathbf{F}$, and let $d\mathbf{s}$ be an element of arc of the curve, considered as a vector. The work done by $\mathbf{F}$ in the displacement $d\mathbf{s}$ is $\mathbf{F}\,d\mathbf{s}$, and the total work done is $\int_A^B \mathbf{F}\,d\mathbf{s}$, the limit of a sum of infinitesimal scalar products. This integral is called the *line-integral* of the vector $\mathbf{F}$, and in defining $\int_A^B \mathbf{F}\,d\mathbf{s}$ it is necessary to specify not only the end points $A$, $B$ but also the curve joining them. $\int_A^B d\mathbf{s}$ itself is the vector $AB$, in this case independent of the joining path. It is a vector integral, like the more general integral $\int W\,d\mathbf{s}$, where $W$ is a scalar magnitude given for all points $P$, e.g. the density of a heterogeneous solid body at $P$. Such a scalar is called a *scalar function of position.*

A general idea of a vector function of position is afforded by a strained solid. If $P$ is a point inside an elastic body which is slightly deformed, the displacement $\mathbf{u}$ of $P$ is a small, continuous vector function of position. Let $S$ be a finite part of a surface

drawn in the body. Each element of $S$ is regarded as a vector area and denoted by $d\mathbf{S}$. When the body is strained, the element $d\mathbf{S}$ traces out a volume $\mathbf{u}d\mathbf{S}$ (Art. 3), and the whole sheet $S$ traces out a volume $\int \mathbf{u}d\mathbf{S}$, the limit of a sum of infinitesimal scalar products. Such an integral is called the *surface-integral* or *flux* of the vector $\mathbf{u}$.

**7. Gradient of a scalar function of position.** Let $U$ be the value of a scalar function of position at $P$, and $U'$ its value at a neighbouring point $P'$. Then the limit of $\dfrac{U'-U}{PP'}$, as $P'$ approaches $P$ indefinitely from a given direction, is called the *derivate* of $U$ in that direction, and denoted by $\partial U/\partial s$. To find how $\partial U/\partial s$ varies with direction, suppose that a surface $S$ is drawn, at all points of which $U$ has the same value as at $P$, and a surface $S'$ at all points of which $U$ has the slightly greater value $U'$. Let the normal to $S$ at $P$ meet $S'$ at $T$. The surfaces can be represented approximately by their tangent planes at $S$ and $T$. If a line $PP'$ is drawn making an angle $\theta$ with $PT$ and cutting the surface $S'$ at $P'$, $PP' = PT \sec\theta$ approximately. Thus

$$\frac{U'-U}{PP'} = \frac{U'-U}{PT}\cos\theta.$$

Passing to the limit and writing $R$ for the (positive) derivate in the direction $ST$, we have for any other direction

$$\frac{\partial U}{\partial s} = R\cos\theta,$$

where $\theta$ is the acute angle between $d\mathbf{s}$ and $ST$. A similar proof holds when $\theta$ is obtuse.

Hence there is one and only one direction which gives the derivate its greatest positive value $R$. We define the *gradient* of $U$ to be a vector parallel to this direction, and of magnitude equal to $R$. The derivate of $U$ in any other direction is the resolved part of the gradient in that direction. Hence if $U$ is given as a function of three rectangular coordinates $x, y, z$, the $x$-component of the gradient is $\partial U/\partial x$, and the gradient itself is the vector

$$\mathbf{i}\frac{\partial U}{\partial x} + \mathbf{j}\frac{\partial U}{\partial y} + \mathbf{k}\frac{\partial U}{\partial z}.$$

The gradient of a scalar function $U$ will be denoted by $\mathbf{D}U$. It may be regarded as the effect of a vector operator

$$\mathbf{D} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \quad \ldots\ldots\ldots\ldots\ldots (11)$$

on $U$, though this symbolic interpretation is not necessary for our immediate purpose.

When $P$ receives a small vector displacement $d\mathbf{s}$, $U$ is changed by an amount $(\partial U/\partial s)\, ds = R\cos\theta\, ds$, the scalar product of $\mathbf{D}U$ and $d\mathbf{s}$. Hence

$$dU = \mathbf{D}U\, d\mathbf{s} \quad \ldots\ldots\ldots\ldots\ldots (12).$$

Integrating this equation from $A$ to $B$ along any curve, we have

$$\int_A^B \mathbf{D}U\, d\mathbf{s} = \int_A^B dU = U_B - U_A.$$

Hence the line-integral of the gradient of $U$ along any curve is the increase of $U$ in passing from one end to the other.

**8. Divergence of a vector function of position.** Let $P$ be a point inside an elastic body, and let the body be deformed so that $P$ receives a small displacement $\mathbf{u}$, a continuous vector function of position. Volumes in the neighbourhood of $P$ are altered in the ratio $1 + Q$ to 1, where $Q$ is a scalar function of position, of the same order as $\mathbf{u}$. We call $Q$ the *divergence* of the vector $\mathbf{u}$, and write it div $\mathbf{u}$.

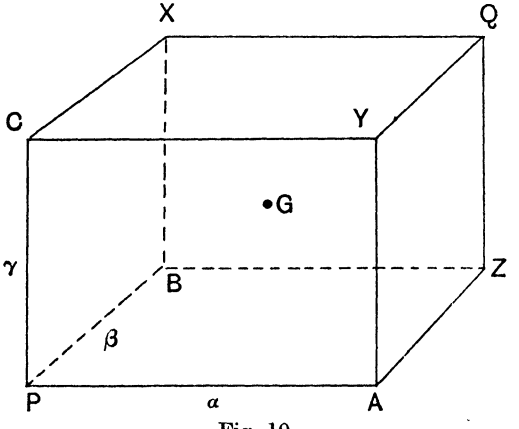


Fig. 10

If $S$ is a small closed surface surrounding $P$, the particles on the surface receive displacements $\mathbf{u}$, and the volume of $S$ increases by $\int \mathbf{u}\, d\mathbf{S}$, from Art. 6. It follows that div $\mathbf{u}$ is the limit of the ratio

$$\frac{\text{flux of } \mathbf{u} \text{ outwards through } S}{\text{volume enclosed by } S} \quad \ldots\ldots\ldots (13)$$

as $S$ shrinks indefinitely round $P$, the physical interpretation showing that the limit is the same for all shapes of $S$.

To find div $\mathbf{u}$ when $\mathbf{u} = \mathbf{i}u_x + \mathbf{j}u_y + \mathbf{k}u_z$, let $(x, y, z)$ be the coordinates of the centre $G$ of the parallelepiped $PQ$ in Fig. 10, and let the sides of the parallelepiped parallel to $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be $\alpha$, $\beta$, $\gamma$ respectively, $\alpha$, $\beta$ and $\gamma$ being small. We need only consider the displacements normal to each face. The mean displacement over the face $PBXC$ is $u_x - \frac{1}{2}\alpha\partial u_x/\partial x$, which contributes an amount

$$-\left(u_x - \tfrac{1}{2}\alpha\frac{\partial u_x}{\partial x}\right)\beta\gamma$$

to the flux of $\mathbf{u}$. The opposite face $QYAZ$ contributes an amount

$$\left(u_x + \tfrac{1}{2}\alpha\frac{\partial u_x}{\partial x}\right)\beta\gamma,$$

giving altogether $\alpha\beta\gamma\partial u_x/\partial x$, or $\tau\partial u_x/\partial x$, where $\tau$ is the enclosed volume. The faces $PAYC$, $QXBZ$ contribute $\tau\partial u_y/\partial y$, and the faces $PAZB$, $QXCY$ contribute $\tau\partial u_z/\partial z$. Adding and dividing by $\tau$ we have

$$\operatorname{div}\mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \quad \ldots\ldots\ldots\ldots (14).$$

Now let $S$ be a *finite* region of space, the points of which receive displacements $\mathbf{u}$. The volume increases by $\int \mathbf{u}\,d\mathbf{S}$ as before. Each volume element $d\tau$, moving so as to embrace the same particles, increases by $\operatorname{div}\mathbf{u}\,d\tau$, so that the total increase of volume is also represented by $\int \operatorname{div}\mathbf{u}\,d\tau$ extended through the interior. Hence we have the *theorem of flux*, that for any vector $\mathbf{u}$

$$\int_S \mathbf{u}\,d\mathbf{S} = \int_\tau \operatorname{div}\mathbf{u}\,d\tau,$$

where $S$ is any closed surface and $\tau$ the region enclosed by it, the vector $d\mathbf{S}$ pointing everywhere away from $\tau$. Sufficient conditions of validity are that $\mathbf{u}$ should be continuous in magnitude and direction at all points on or inside $S$, and div $\mathbf{u}$ finite.

Let $P$ be a point in a moving fluid, and let the density of the fluid near $P$ be $\rho$ (a scalar function of position) and the velocity $\mathbf{v}$ (a vector function of position). Both density and velocity are supposed to vary with the time. Let $S$ be a finite part of a surface

described in the fluid, and supposed not to move with it. The particles on a surface element $d\mathbf{S}$ move through a distance $\mathbf{v}\,dt$ in time $dt$, so that $\rho\mathbf{v}\,d\mathbf{S}\,dt$ grams of fluid cross $d\mathbf{S}$ in time $dt$. The mass of fluid crossing $S$ per second is therefore the *flux of the vector* $\rho\mathbf{v}$ *through* $S$. If $S$ is a small closed surface enclosing a space $\tau$, the flux is $\tau$ div $(\rho\mathbf{v})$ from (13). Since the mass originally inside is $\rho\tau$, the rate of loss can also be written $-\dfrac{\partial\rho}{\partial t}\tau$, so that we must have at all points

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{v}) = 0.$$

This is a general equation connecting the density and velocity of a moving fluid, called the *equation of continuity*, the name referring not to mathematical continuity but to the absence of holes in the fluid, tacitly assumed in the proof. It is used in Art. 195.

**9. Curl of a vector function of position.** Let us calculate the line-integral of a vector round a small closed plane curve $C$ near a point $P$. Let $Q$ be a (variable) point on the curve whose coordinates relative to $P$ are $(x', y', z')$, so that $PQ = \mathbf{i}x' + \mathbf{j}y' + \mathbf{k}z'$. If

$$\mathbf{u} = \mathbf{i}u_x + \mathbf{j}u_y + \mathbf{k}u_z$$

is the value of the vector at $P$, its value at $Q$ is, to the first order in $x', y', z'$,

$$\begin{aligned}&\mathbf{i}\left(u_x + x'\frac{\partial u_x}{\partial x} + y'\frac{\partial u_x}{\partial y} + z'\frac{\partial u_x}{\partial z}\right)\\ &+\mathbf{j}\left(u_y + x'\frac{\partial u_y}{\partial x} + y'\frac{\partial u_y}{\partial y} + z'\frac{\partial u_y}{\partial z}\right)\\ &+\mathbf{k}\left(u_z + x'\frac{\partial u_z}{\partial x} + y'\frac{\partial u_z}{\partial y} + z'\frac{\partial u_z}{\partial z}\right),\end{aligned}$$

where $u_x$, $\partial u_x/\partial x$, ... are calculated at $P$. The vector element of arc is

$$d\mathbf{s} = \mathbf{i}\,dx' + \mathbf{j}\,dy' + \mathbf{k}\,dz'.$$

Forming the scalar product and integrating, the line-integral of $\mathbf{u}$ round $C$ is

$$\begin{aligned}&u_x\int dx' + \frac{\partial u_x}{\partial x}\int x'dx' + \frac{\partial u_x}{\partial y}\int y'dx' + \frac{\partial u_x}{\partial z}\int z'dx'\\ &+u_y\int dy' + \frac{\partial u_y}{\partial x}\int x'dy' + \frac{\partial u_y}{\partial y}\int y'dy' + \frac{\partial u_y}{\partial z}\int z'dy'\\ &+u_z\int dz' + \frac{\partial u_z}{\partial x}\int x'dz' + \frac{\partial u_z}{\partial y}\int y'dz' + \frac{\partial u_z}{\partial z}\int z'dz'.\end{aligned}$$

Terms like $\int dx'$ and $\int x'dx'$ taken round the contour vanish. $\int y'dz'$ is the area of projection of $C$ on the plane of $yz$. Let $\mathbf{S}$ be the small vector area enclosed by $C$, and let its direction be derived from the direction in which $C$ is traversed in the line-integral by the right-handed screw law, as shown in Fig. 11. The projection of $\mathbf{S}$ on the plane of $yz$ is equal to the scalar product $\mathbf{iS}$, so that

$$\int y'dz' = \pm\, \mathbf{iS}.$$

Both quantities are positive in the figure, so that the positive sign must be taken. Since

$$\int y'dz' + \int z'dy' = 0, \quad \int z'dy' = -\,\mathbf{iS},$$

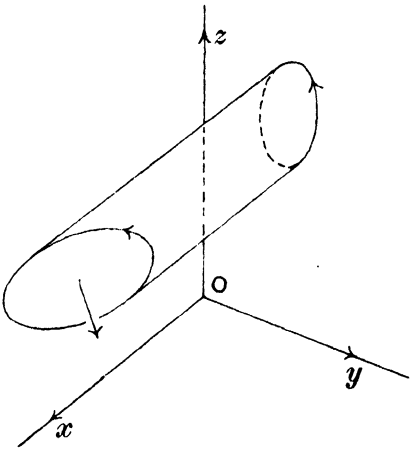



Fig. 11

and similarly for the other integrals. The line-integral of $\mathbf{u}$ is therefore

$$\mathbf{iS}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) + \mathbf{jS}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) + \mathbf{kS}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right).$$

The vector

$$\mathbf{i}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \quad \ldots\ldots (15)$$

is called the *curl* of $\mathbf{u}$ and written curl $\mathbf{u}$. Thus the line-integral of $\mathbf{u}$ round a small plane contour is the scalar product of curl $\mathbf{u}$ and $\mathbf{S}$, where $\mathbf{S}$ is the small vector area. In particular, if a small circle is drawn with centre $P$, and $L$ is the line-integral of $\mathbf{u}$ round the circle, divided by the area, there is one and only one direction of the plane for which $L$ has its greatest positive value. If this is $R$, curl $\mathbf{u}$ is a vector of magnitude $R$ normal to the plane.

Let $C$ be a finite closed curve (not necessarily plane), $S$ an unclosed surface having $C$ as its rim. Cut up $S$ into a large number of elementary areas $d\mathbf{S}$, to each of which the foregoing theory can be applied, and add the results. The internal boundaries contribute nothing to $\Sigma \int \mathbf{u}\, d\mathbf{s}$, since each element $d\mathbf{s}$ of such a boundary is

traversed twice in opposite directions, as is clear from Fig. 12. Hence $\Sigma \int \mathbf{u} d\mathbf{s}$ is simply $\int \mathbf{u} d\mathbf{s}$ taken round the outer edge $C$, or

$$\int_C \mathbf{u} d\mathbf{s} = \int_S \text{curl } \mathbf{u} d\mathbf{S}.$$

Thus *the line-integral of a vector round a closed curve $C$ in space is equal to the flux of its curl through any unclosed surface $S$ having $C$ as its rim*, the direction of integration round $C$ being derived from the general direction of the vectors $d\mathbf{S}$ by the right-handed screw law. This is *Stokes' theorem*. Sufficient conditions of validity are that $\mathbf{u}$ should be continuous and curl $\mathbf{u}$ finite, at all points on $C$ or $S$.

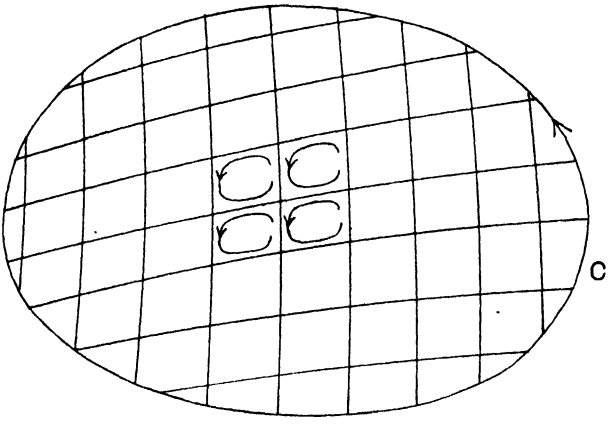


Fig. 12

Curl and divergence are closely related to each other. A vector whose divergence vanishes everywhere is said to be *solenoidal*. Let $\mathbf{u}$ be a solenoidal vector, and let $S$ and $S'$ be two surfaces with the same rim $C$. $S$ and $S'$ together form a closed surface to which the theorem of flux can be applied; but since the vector areas of one surface must be reversed to make them point away from the interior the theorem gives

$$\int_S \mathbf{u} d\mathbf{S} - \int_{S'} \mathbf{u} d\mathbf{S} = 0.$$

Hence the flux is the same for any two surfaces having a given rim, and we can speak unequivocally of the flux of a solenoidal vector *through a circuit* $C$. The curl of a vector is always solenoidal, for the divergence is, from (14) and (15),

$$\frac{\partial}{\partial x}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right),$$

which vanishes identically. Stokes' theorem expresses the flux of curl $\mathbf{u}$ directly in terms of $C$.

A vector whose curl vanishes everywhere is the gradient of some scalar function of position. For if $APB$ and $AQB$ are two paths from $A$ to $B$, $APBQA$ is a closed curve to which Stokes' theorem can be applied, so that

$$\int_{APB} \mathbf{u}\,d\mathbf{s} = \int_{AQB} \mathbf{u}\,d\mathbf{s}.$$

Thus $\int \mathbf{u}\,d\mathbf{s}$ depends only on the terminal points, and if its value is $U_B - U_A$, then $\mathbf{u} = \mathbf{D}U$ (Art. 7).

**10. Laplace's operator.** The divergence of the gradient

$$\mathbf{D}U = \mathbf{i}\frac{\partial U}{\partial x} + \mathbf{j}\frac{\partial U}{\partial y} + \mathbf{k}\frac{\partial U}{\partial z}$$

of a scalar function $U$ is

$$\operatorname{div} \mathbf{D}U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \Delta U \quad \ldots\ldots\ldots (16),$$

where

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is called Laplace's operator. It converts one scalar function of position into another. $\Delta U$ is the same whatever rectangular unit axes are taken in (16), and can be interpreted independently of coordinates as follows.

Let a sphere of small radius $r$ be drawn with $P$ as centre. Take six points whose vectors relative to $P$ are $\pm r\mathbf{i}$, $\pm r\mathbf{j}$, $\pm r\mathbf{k}$. The values of $U$ at the first two points are

$$U \pm r\frac{\partial U}{\partial x} + \tfrac{1}{2}r^2\frac{\partial^2 U}{\partial x^2} \pm \ldots.$$

the mean of which is $U + \frac{1}{2}r^2\partial^2 U/\partial x^2$ to the second order in $r$. Hence the mean value of $U$ at all the six points exceeds that at $P$ by approximately $\frac{1}{6}r^2\Delta U$; and since movement of the rectangular frame $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ sweeps out the whole sphere, the same is true of the mean value of $U$ over the whole sphere. Thus we can define $\Delta U$ as the limit of the ratio

$$\frac{\text{(mean value of } U \text{ over a sphere of radius } r \text{ round } P) - \text{(value at } P)}{\frac{1}{6}r^2}$$

as $r \to 0$.

The theorem of flux applied to $\mathbf{D}U$ gives

$$\int_S \mathbf{D}U d\mathbf{S} = \int_\tau \Delta U d\tau \quad \ldots\ldots\ldots\ldots (17).$$

The theorem is useful in finding $\Delta U$ when $U$ is a spherically symmetrical scalar function, i.e. a function only of the distance of $P$ from a fixed point $O$. The gradient of $U$ is a radial vector of magnitude $dU/dr$ (more accurately, the component radially outwards is $dU/dr$). Apply (17) to the space between two spheres of radii $r$ and $r + dr$ with $O$ as centre. The flux through the inner boundary is $4\pi r^2 dU/dr$, and that through the outer boundary is $4\pi r^2 dU/dr + 4\pi d(r^2 dU/dr)$, giving in all a flux

$$4\pi \frac{d}{dr}\left(r^2 \frac{dU}{dr}\right) dr$$

outwards from the annulus. The volume of the annulus is $4\pi r^2 dr$, and therefore

$$\Delta U = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dU}{dr}\right) = \frac{d^2U}{dr^2} + \frac{2}{r}\frac{dU}{dr} \quad \ldots\ldots\ldots (18).$$

In the same way, if $U$ is an axially symmetrical function, depending only on the distance $r$ from $P$ to a fixed line,

$$\Delta U = \frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} \quad \ldots\ldots\ldots\ldots\ldots (19).$$

If $U$ and $V$ are two scalar functions of position, $U\mathbf{D}V$ is the vector

$$\mathbf{i}U\frac{\partial V}{\partial x} + \mathbf{j}U\frac{\partial V}{\partial y} + \mathbf{k}U\frac{\partial V}{\partial z},$$

and

$$\operatorname{div}(U\mathbf{D}V) = U\Delta V + \mathbf{D}U\mathbf{D}V$$

from (14) and (16). Hence

$$\operatorname{div}(U\mathbf{D}V - V\mathbf{D}U) = U\Delta V - V\Delta U.$$

Applying the theorem of flux we have *Green's reciprocal theorem*, namely that

$$\int_S (U\mathbf{D}V - V\mathbf{D}U)\, d\mathbf{S} = \int_\tau (U\Delta V - V\Delta U)\, d\tau.$$

If the vector $U\mathbf{D}V - V\mathbf{D}U$ vanishes at infinity so that its magnitude is of an order smaller than $1/r^2$, the flux tends to zero as $S$ is taken

larger and larger, and then $\int (U\Delta V - V\Delta U)\, d\tau$, taken over the whole of space, vanishes.

Laplace's operator $\Delta$ is a scalar operator like $d/dt$, and can be applied to vectors as well as scalars. $\Delta\mathbf{u}$ can be defined either as $\mathbf{i}\Delta u_x + \mathbf{j}\Delta u_y + \mathbf{k}\Delta u_z$ or as the limit of

$$\frac{(\text{mean value of } \mathbf{u} \text{ over a sphere of radius } r \text{ round } P) - (\text{value at } P)}{\frac{1}{6}r^2}$$

as $r \to 0$, the sums (or integrals) implied in the term *mean* being vector sums.

The operation *curl*, applied twice to a vector $\mathbf{u}$, gives a vector whose $x$-component is

$$\frac{\partial}{\partial y}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u_x.$$

This is also the $x$-component of the vector $\mathbf{D}\,\mathrm{div}\,\mathbf{u} - \Delta\mathbf{u}$, so that

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} = \mathbf{D}\,\mathrm{div}\,\mathbf{u} - \Delta\mathbf{u} \quad \ldots\ldots\ldots(20).$$

Similarly if $U$ is a scalar function and $\mathbf{u}$ a vector function of position, the curl of $U\mathbf{u}$ is the sum of $U$ curl $\mathbf{u}$ and the vector product of $\mathbf{D}U$ and $\mathbf{u}$, or

$$\mathrm{curl}\, U\mathbf{u} = U\,\mathrm{curl}\,\mathbf{u} + V\mathbf{D}U, \mathbf{u} \quad \ldots\ldots\ldots(21).$$

**11. The equation of wave-motion.** The partial differential equation

$$\Delta\theta = \frac{1}{c^2}\frac{\partial^2\theta}{\partial t^2}$$

implies that if the quantity $\theta$ is at time $t = 0$ distinct from zero within a certain region of space, this "disturbed" region advances in all directions with velocity $c$. For this reason the equation is called the "equation of wave-motion," $c$ being the velocity of propagation of all the waves.

A general proof of the theorem would occupy too much space. We consider only two cases. Firstly, let $\theta$ depend on $x$ and $t$ only. Then

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\theta}{\partial t^2}.$$

The integral of this equation is $\theta = f(x - ct) + \phi(x + ct)$, where $f$ and $\phi$ are arbitrary functions. The term $f(x - ct)$ is unchanged as long as $x - ct$ is, and therefore represents motion with velocity $c$ along the axis of $x$. Similarly the other term represents motion back along the axis of $x$, without change of type.

Secondly, let $\theta$ depend only on $r$ and $t$, where $r$ is the distance of the point $P$ from the origin. The equation of wave-motion becomes, from equation (18),

$$\frac{\partial^2\theta}{\partial r^2} + \frac{2}{r}\frac{\partial\theta}{\partial r} = \frac{1}{c^2}\frac{\partial^2\theta}{\partial t^2},$$

or

$$\frac{\partial^2(r\theta)}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2(r\theta)}{\partial t^2}.$$

Thus $r\theta = f(r - ct) + \phi(r + ct)$, representing waves diverging and converging radially with velocity $c$. The solution for waves travelling outwards is

$$\theta = \frac{1}{r} f\left(t - \frac{r}{c}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (22).$$

Still more special solutions of the equation of wave-motion are found by giving particular forms to the function $f$. Equation (22) is useful in the theory of electric waves (Art. 197).

**12. Bessel's functions of order zero.** The function

$$J_0(x) = 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{1^2 \cdot 2^2}\left(\frac{x}{2}\right)^4 - \frac{1}{1^2 \cdot 2^2 \cdot 3^2}\left(\frac{x}{2}\right)^6 + \ldots \quad (23)$$

is called a Bessel's function of order zero. It is a particular case of the general Bessel's function

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r}$$

which, however, we shall not use. The values of $J_0(x)$ and its derivative $J_0'(x)$ have been tabulated and may be regarded as known functions. Fig. 13 shows the curve $y = J_0(x)$ from $x = 0$ to $x = 25$.

We can show from equation (23) that $J_0(x)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0,$$

which is known as Bessel's differential equation. It is generally more convenient to consider, instead of $J_0(x)$, the quantity $J_0(kr)$, regarded as a function of $r$, and call it $J_0$ for short. It is then necessary to distinguish between $J_0'$, which is $\frac{dJ_0(kr)}{d(kr)}$, and $\frac{dJ_0}{dr}$.

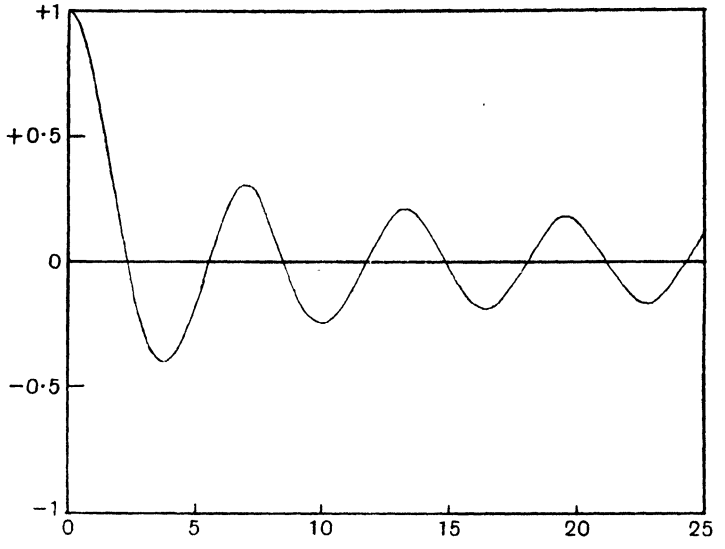



Fig. 13

The differential equation becomes

$$r\frac{d^2J_0}{dr^2} + \frac{dJ_0}{dr} + k^2rJ_0 = 0 \quad \ldots\ldots\ldots\ldots \quad (24).$$

Writing this in the form $k^2rJ_0 = -\frac{d}{dr}\left(r\frac{dJ_0}{dr}\right)$ and integrating, we have

$$\int_0^r rJ_0\,dr = -\frac{r}{k^2}\frac{dJ_0}{dr} = -\frac{r}{k}J_0' \quad \ldots\ldots\ldots \quad (25).$$

A second useful formula is

$$(k'^2 - k^2)\int_0^r rJ_0(kr)\,J_0(k'r)\,dr$$
$$= rJ_0(k'r)\frac{dJ_0(kr)}{dr} - rJ_0(kr)\frac{dJ_0(k'r)}{dr} \quad \ldots(26).$$

To verify this we notice that

$$\frac{d}{dr}\left[rJ_0(k'r)\frac{dJ_0(kr)}{dr} - rJ_0(kr)\frac{dJ_0(k'r)}{dr}\right]$$

$$= J_0(k'r)\frac{dJ_0(kr)}{dr} - J_0(kr)\frac{dJ_0(k'r)}{dr}$$

$$+ rJ_0(k'r)\frac{d^2J_0(kr)}{dr^2} - rJ_0(kr)\frac{d^2J_0(k'r)}{dr^2}$$

$$= -k^2rJ_0(k'r)J_0(kr) + k'^2rJ_0(kr)J_0(k'r) \text{ from (24).}$$

The result (26) follows at once. These equations are used in discussing the distribution of high-frequency currents in wires (Art. 191).

# CHAPTER II

## PERMANENT MAGNETISM

**13. Magnets.** There are a number of bodies which are capal of attracting iron to a greater or less degree. These are called *magnets*. The mineral magnetite, which contains considerable quantities of the iron oxides FeO and $Fe_2O_3$, is a natural magnet; other substances can be made into magnets, or *magnetised*. Of pure metals, iron, cobalt and nickel alone show this property to a marked degree; iron is the best of the three, the relative efficacy of the three metals being roughly in the ratio 3 : 2 : 1. An iron rod can be magnetised very easily by winding it round with cotton-covered copper wire until the wire forms a spiral covering it closely from end to end, and then joining the ends of the wire to the terminals of an electric battery. While the current is flowing the rod is strongly magnetised, and a portion of its power remains when the current is cut off. Care must be taken to pass a sufficiently strong current or to wind a sufficiently large number of turns of wire round the rod.

The capacity of iron to retain its magnetism after the current is cut off may be called its residual power; and the extent of this depends very much on the treatment the iron has previously received. Wrought iron is used where residual magnetism is a disadvantage, for it loses nearly all the magnetism given it by the current under the influence of vibration or shock. For permanent magnets, on the other hand, the qualities required are high residual power and great resistance to demagnetisation by external magnets or by rough usage; and this is secured by using hard steel. As many standard measuring instruments depend on the constancy of the magnets they contain, special attention is paid to this property. The magnets are often made of tungsten steel, hardened from a temperature of about 800° C. After magnetisation they are exposed to shock and change of temperature, so that whatever magnetism remains is not disturbed by the further slight disturbances met with in ordinary work. This process is known as "artificial ageing."

Iron and steel magnets when heated to a bright red heat lose their magnetisation, which they do not regain on cooling.

A convenient way of ascertaining whether a body is a magnet or not is to dip it into iron filings. The filings do not adhere to all parts of the body, so that the magnetic property is not equally diffused throughout it. If a long knitting-needle is magnetised by a current in a uniform and closely wound spiral, filings will only adhere to it in the neighbourhood of the ends. The two points round which they cluster are called the *poles* of the magnet. The two poles have however essentially distinct properties, which may be examined as follows. Magnetise two knitting-needles $AB$, $A'B'$

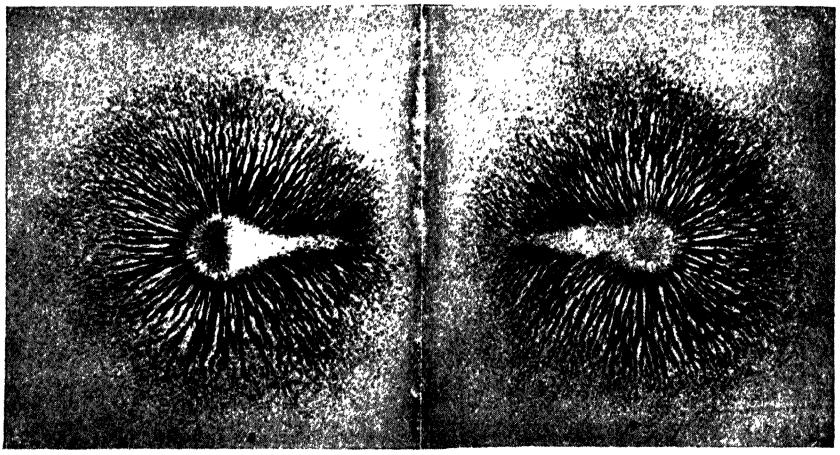


Fig. 14

similarly, so that the end $A$ corresponds to $A'$ and $B$ to $B'$. It is found that $A$ repels $A'$ and $B$ repels $B'$, while $A$ attracts $B'$ and $B$ attracts $A'$. Like poles repel, unlike poles attract.

The ideal magnet is regarded as possessing two point-poles, approximately realised when a long steel bar is magnetised. Robison showed in 1769 that ball-ended magnets give a better approximation, and such magnets are often used in simple experiments. If the ends of a steel rod are screwed into steel balls of about a centimetre diameter, and the whole magnetised, the magnetic property is found to reside almost entirely in the steel balls. This is shown on dipping such a magnet into iron filings, when hardly any of the filings adhere to the connecting steel rod.

Moreover, the magnetic power is almost uniformly distributed over the spheres, as may be seen from Fig. 14, which represents the distribution of filings sprinkled on a card held over a ball-ended magnet. We thus acquire the notion of some influence emanating radially from the poles in all directions, which will shortly be made more precise.

A body magnetised at random may deviate considerably from the ideal of a body with two point-poles, but it is not desirable to reject the idea of poles. A body of this kind is regarded as an assemblage of like and unlike poles variously located inside it. We cannot isolate the opposite poles in any way: for if a bar-magnet is broken a new pole is formed at each of the fresh ends, and each half after separation behaves like a new bar-magnet.

**14. Induced magnetisation.** A magnet always attracts a piece of ordinary iron, although the like poles of two magnets repel one another. The explanation of this can be found by a simple

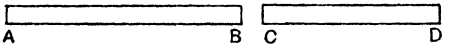



Fig. 15

experiment. Lay a steel magnet $AB$ (Fig. 15) in the same straight line with a rod $CD$ of wrought iron, previously unmagnetised. As long as $AB$ is there the rod $CD$ will attract iron filings, which will adhere to it mainly at the two ends, but the power of attracting such filings is almost entirely lost when $AB$ is removed. This shows that the rod $CD$ has temporarily become a magnet, and the rod is said to be magnetised by induction. We can test the nature of the induced poles by observing whether a particular end, say $D$, repels or attracts one end of a magnet similar to $AB$. The pole induced at $C$ in the wrought iron rod is found to be similar to the pole $A$ of the original magnet, while the pole at $D$ is like the pole $B$. The process of attraction therefore consists primarily in the magnetisation of the rod by induction, and then $B$ attracts the unlike pole $C$, the other poles having less effect on account of their being further off.

If a pole of a magnet is brought very near a weak pole of the same kind attraction may result, instead of the weak repulsion

observed at moderate distances. Induced magnetism may thus overpower the ordinary or residual kind. Magnetisation is always induced in a magnet to some extent when other magnets are near, although the induced effects are relatively much greater with wrought iron than with hardened steel. With steel magnets induced magnetism is normally negligible in comparison with the permanent magnetism, and the magnetic properties of good magnets remain unaltered over considerable periods of time. For the present we need only consider magnets of invariable quality, which are called *permanent magnets*. We shall form an estimate in Art. 109 of the extent to which induction takes place in a magnet under the conditions of ordinary use.

**15. Strength of a magnetic pole.** The important question arises as to what number can be assigned to the pole of a permanent magnet which shall express its power of being attracted or repelled by other poles. The question is a theoretical one, since we cannot experiment, except approximately, on ideal poles: also it is not convenient to measure the forces between magnets directly. To obtain some notion of the strength of an ideal pole we may consider an ideal experiment to be performed, merely for the sake of definition and without implying that it would be easy to realise the theoretical conditions in practice.

Suppose that we have any number of magnets distributed in a region in any manner. Such a region, that is a region of magnetic attractions and repulsions, is called a *magnetic field*. If a new pole is introduced and placed at an assigned point $P$ of the field, it will in general be acted on by a certain force due to the magnets in the field: and we shall assume that the line of action of this force is the same whatever pole is placed at $P$. Suppose now that we place a number of poles in succession at $P$, and observe the magnitudes $F_1, F_2, F_3, \ldots$ of the forces acting on them, regard being had to the sign of the forces: then the *strengths* of the various poles are defined to be in the ratios $F_1 : F_2 : F_3$, etc. We may thus select any of the poles and call it the *unit pole*, or pole of unit strength, and the strength of any other pole is then known when the force on it is measured.

If the above definition is to be useful it must give to a particular

pole of a permanent magnet the same strength under all conditions. We shall have to postulate further that the ratios of the forces $F_1, F_2, F_3, \ldots$ are independent of the particular field used in their determination.

In order to determine pole-strengths in absolute measure, the strength + 1 is assigned to the chosen unit pole, and other strengths determined by the above definition. We have only to imagine a field produced by a single magnetic pole to see that poles which are *like* the chosen unit pole have positive strengths, while the poles which are *unlike* it have negative strengths. We shall for this reason speak in future of *positive* and *negative* magnetic poles, and in saying that the strength of a particular pole is $m$ we understand that $m$ has sign as well as magnitude.

**16. Magnetic force and magnetic potential.** Common observation shows that magnets exert more force on one another when they are near together than when far apart, so that a magnetic pole in general experiences different forces when placed at different points of a magnetic field. The field at a point $P$ is completely specified by the mechanical force $\mathbf{H}$ with which it acts on a unit pole placed at $P$, and the vector $\mathbf{H}$ is called the *magnetic force* at $P$. It follows that a pole of strength $m$ placed at the same point would experience a force $m\mathbf{H}$. The nomenclature is awkward, so that the reader must remember that the term *magnetic force* is always used in the above precise sense, and never loosely for the force acting on any magnetic pole.

If $\mathbf{H}$ is the same at all points the field is said to be uniform.

The mutual forces between magnetic poles will be assumed to be of the type known in mechanics as *conservative*, and the potential energy of a unit pole when lying at a point $P$ in a field is called the *magnetic potential* at $P$. Thus if the magnetic potential at $P$ is $\Omega$, exactly that amount of work will be done by the forces of the field when a unit pole is taken from $P$ to an assigned point. The assigned point is conveniently taken at an infinite distance from all the magnets in the field.

A magnetic field is specified completely when $\Omega$ is given at all points of it. Let a unit pole at the point $P$ be given a small displacement represented by the vector $d\mathbf{s}$. The forces of the field do

work equal to the scalar product $\mathbf{H}d\mathbf{s}$ (Art. 3), and this must be accounted for by the decrease $-d\Omega$ of potential energy. Hence

$$d\Omega = -\mathbf{H}d\mathbf{s}.$$

Comparing this with the general vector equation (12) we have therefore

$$\mathbf{H} = -\mathbf{D}\Omega \ldots\ldots\ldots\ldots\ldots\ldots (27).$$

Thus the magnetic force is the negative gradient of the magnetic potential. In Cartesian form

$$\mathbf{H} = -\mathbf{i}\frac{\partial\Omega}{\partial x} - \mathbf{j}\frac{\partial\Omega}{\partial y} - \mathbf{k}\frac{\partial\Omega}{\partial z}.$$

If two fields of strengths $H$ and $H'$ are superposed at right angles to each other the resultant field is of magnitude $(H^2 + H'^2)^{\frac{1}{2}}$ and makes an angle $\theta$ with the direction of $\mathbf{H}$, where $\tan\theta = H'/H$.

**17. A magnet in a uniform magnetic field.** The tendency of a compass-needle to set in a fixed direction shows that there is a natural magnetic field at points on the earth's surface. This field, in the absence of special disturbing circumstances, may be taken as uniform over the small space occupied by an ordinary room. Theoretically, we might examine the effect of the earth's field on a magnet by supporting the latter at its centre of gravity, and thus eliminating the effects of gravitation. This is not convenient, and in practice it is better to examine the horizontal and vertical forces separately. The ordinary pivoted compass-needle is supported somewhat above its centre of gravity, and is thus not very sensitive to vertical fields. On the other hand, it turns freely round in a horizontal plane, so that we may begin by considering the horizontal component only of the earth's magnetic field, that is the magnetic force in a plane parallel to the earth's surface.

Regarding a magnet as an assemblage of ideal poles, let $m$ be the strength of one of them. When the magnet is placed in a uniform field $\mathbf{H}$ the pole is acted on by a force $m\mathbf{H}$ in a fixed direction, and the resultant mechanical force on the whole magnet is $\mathbf{H}\Sigma m$. The conditions are realised experimentally when a magnet is floated on a cork in water, $\mathbf{H}$ representing the horizontal component of the earth's magnetic field, and vertical forces being

practically inoperative for reasons similar to those advanced for the compass-needle. Such a floating magnet tends to point approximately north and south, but it has no tendency to move bodily in any direction. The resultant force on the whole magnet in the uniform field **H** must therefore vanish, i.e. $\Sigma m = 0$. Hence we have the important result that the total strength of the poles of any magnet is zero, or the magnet is compounded of positive and negative poles to equal amounts.

Let $A$, $B$ be the poles of a simple bar-magnet, of strengths $m$ and $-m$ respectively, and let the distance between the poles be $\delta$. When the magnet lies in a uniform field **H** with $BA$ making an angle $\theta$ with the direction of **H**, the forces on the magnet are $m\mathbf{H}$ at each end as shown in Fig. 16, and these forces form a couple of magnitude $mH\delta \sin\theta$ tending to decrease $\theta$. The product $m\delta$ of the positive pole and the distance between the two poles is called the *moment* of a bar-magnet and denoted by $M$. Hence the couple on a bar-magnet in a uniform field is $MH \sin\theta$, and in a given field the couple will be the same for two different magnets if only they have the same moment $M$.

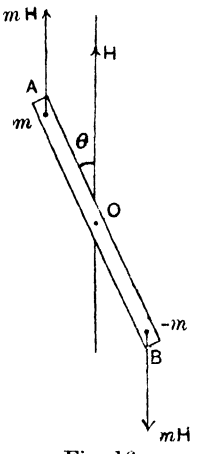


Fig. 16

In defining the strength of magnetic poles it was supposed that any pole might be taken as a unit and given the strength + 1. This indeterminacy will be removed later when we have considered the law of action of one pole on another at various distances apart: but in any case it is convenient to call the north end of a compass-needle a *positive* pole. The line joining the poles of a bar-magnet is called its *axis*, and the positive direction of the axis is that of a line proceeding from the negative to the positive pole. Hence when a bar-magnet is used as a compass the axis coincides with the direction of the earth's horizontal component, since the stable position of rest of the magnet in Fig. 16 is clearly given by $\theta = 0$. This direction is known as *magnetic north*.

In a non-uniform magnetic field a magnet will tend to set in a certain direction, but will in general be acted on by a resultant force as well.

**18. The magnetometer., Inverse square law of magnetic poles.** The magnetometer is essentially an instrument for the comparison of magnetic fields. One form (Fig. 17) consists of a large compass-box let into a board about a metre long, the box being moveable about its centre and containing a short bar-magnet. A long double pointer of non-magnetic material is attached to the magnet and projects at right angles to the magnet so as to move over a graduated scale round the circumference of the box. A graduated groove for holding magnets is cut in the board and points towards the centre of the compass-box. To adjust the magnetometer the compass-box is first turned till the zero line of its scale points

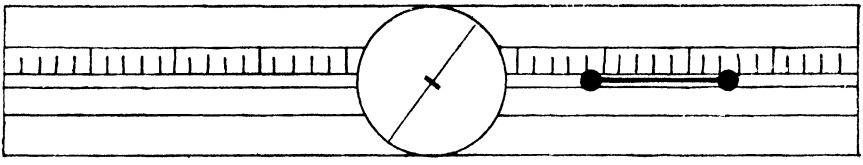


Fig. 17

along the groove, and the whole instrument can then be turned round until the pointers read zero. It is clear that in this position, the normal position for the magnetometer, the groove points magnetically east and west.

We can demonstrate the law of action of magnetic poles with a magnetometer as follows:

(1) *The magnetic force due to a single pole is directed radially outwards from it.* Turn the magnetometer through 90° from its normal position, so that the groove points magnetically north and south. The needle will now point to 90° on the scale. The reading is unaltered when a bar-magnet is laid in the groove, and this holds good even if one end of the magnet is raised from the groove, provided that it remains magnetically north of the compass-needle. Hence neither pole exerts any east and west component of magnetic force on the needle. It may happen that the needle suddenly swings through 180° and comes to rest pointing in the opposite direction, since the magnetic force due to the bar-magnet may overpower the earth's field; but the two fields are accurately parallel.

(2) *The magnetic force at an assigned distance from a pole is proportional to the strength of the pole.* This is a matter of definition.

Let $P$ be the pole to be examined, its strength being $m$, and imagine a unit pole $A$ to be placed in the assigned position. The pole $A$ exerts on $P$ a force proportional to $m$, and this force is equal and opposite to the force exerted by $P$ on $A$. The latter is however the magnetic force due to the pole $P$.

(3) *The magnetic force at distance r from a given pole is proportional to* $1/r^2$. Place the magnetometer in its normal position and lay a ball-ended magnet in the groove (Fig. 17). If the poles are distant $r$ and $r+l$ from the compass-needle, they would, on the above law, produce a magnetic field $H'$ proportional to $\frac{1}{r^2} - \frac{1}{(r+l)^2}$, and in an east and west direction. We have, from Art. 16, $H' = H \tan\theta$, where $H$ is the horizontal component of the earth's field and $\theta$ the observed deflexion of the needle. Thus for a given magnet placed in various positions along the groove we should have $\frac{1}{r^2} - \frac{1}{(r+l)^2}$ proportional to $\tan\theta$, i.e. $\frac{r^2(r+l)^2}{2r+l} \tan\theta = \text{const.}$ In an ordinary experiment the constancy is found to hold good to about 2 per cent., which is as near as could be expected.

A more direct method is to fix a vertical groove directly above the compass-needle, and get rid of the effect of one pole by resting it in this groove, while the other pole lies in the original groove at varying distances $r$ from the centre. With this arrangement it is easy to verify that $\tan\theta$ is inversely proportional to $r^2$.

It follows from what has been said that the magnetic force at distance $r$ from a pole of strength $m$ is $\lambda m/r^2$, where $\lambda$ is some number. Hence two poles of strengths $m$, $m'$ distant $r$ apart repel each other with a force of $F$ dynes, where $F = \lambda mm'/r^2$. By suitable choice of the *unit pole*, hitherto left indefinite, we can make the constant $\lambda$ equal to unity. For if the unit pole is taken to be that (north-seeking) pole which repels an exactly similar pole distant one centimetre away with a force of one dyne, we have $F = 1$ when $m = m' = r = 1$. Hence $\lambda = 1$.

With these units the magnetic force due to a single pole $m$ at distance $r$ is $m/r^2$, and field-strengths are now measured in terms of a perfectly definite unit, namely the field which would exist one

centimetre away from a unit pole. This unit field is called a field of one *gauss*. The magnetic potential at distance $r$ from a pole $m$, or the work done by the field when a unit pole is taken away to infinity, is $\int_r^\infty \frac{m}{r^2}\,dr = \frac{m}{r}$.

The most general magnetic field may be regarded as produced by a number of poles distributed in a certain way. The magnetic potential $\Omega$ at a point $P$ is then given by the equation

$$\Omega = \Sigma\,(m/r) \quad \ldots\ldots\ldots\ldots\ldots\ldots (28),$$

where $r$ is the distance of $P$ from a particular pole of strength $m$ and $\Sigma$ indicates summation over all the poles present.

Instead of being concentrated in points, magnetic poles may be distributed continuously through space in such a way that an element of volume $d\tau$ contains a total strength of pole $\rho\,d\tau$. Then $\rho$ is called the volume-density of magnetism, and the magnetic potential of such a distribution at a point $P$ is $\Omega = \int \frac{\rho\,d\tau}{r}$, where $r$ is the distance from $P$ to $d\tau$ and the integration extends over the whole volume containing poles. Similarly we may have poles spread in layers over surfaces. The strength of pole per unit area near a point is called the surface-density of magnetism at that point and denoted by $\sigma$. The resulting magnetic potential is then $\Omega = \int \frac{\sigma\,dS}{r}$, where the surface-integral extends over the surface or surfaces carrying the magnetic layers.

## 19. Measurement of the horizontal magnetic force of the earth and the moment of a magnet.

PART I. *Measurement of $M/H$.* The magnetometer is set up in its normal position with the groove pointing east and west, and a magnet laid in the groove as shown in Fig. 17. Let $d$ be the distance of the centre of the magnet from the small compass-needle, $d \pm \delta$ the distances of the poles of the magnet. Then if $\pm m$ are the pole-strengths, the magnet exerts a magnetic force in the east and west direction of magnitude

$$H' = \frac{m}{(d-\delta)^2} - \frac{m}{(d+\delta)^2} = \frac{2Md}{(d^2-\delta^2)^2},$$

where $M$ is the moment of the magnet. If $\theta$ is the deflexion, $H' = H \tan\theta$, so that

$$\frac{M}{H} = \frac{(d^2 - \delta^2)^2}{2d} \tan\theta \ldots\ldots\ldots\ldots (29).$$

With ball-ended magnets $2\delta$ is the distance between the centres of the spherical ends. The distribution of magnetism in cylindrical magnets is not so simple, but consistent results are obtained with *short* cylindrical magnets by assuming that the distance between the poles is five-sixths the length of the magnet. It is advisable to read both ends of the double pointer on the compass-needle and also to take readings with the magnet turned over. Further, to correct for a possible error in fixing the compass-box of the magnetometer in the middle of the scale, the readings are repeated with the magnet on the opposite side of the compass-needle. The value of $\theta$ is thus a mean derived from eight observations.

PART II. *Measurement of MH.* The magnetometer is removed and the magnet $M$ suspended horizontally from a torsionless silk fibre, so as to oscillate round a vertical axis through its centre $O$ (Fig. 16). When the needle is swinging the earth's field exerts a couple of magnitude $MH \sin\theta$ on it in the direction tending to decrease $\theta$, so that

$$I \frac{d^2\theta}{dt^2} = - MH \sin\theta,$$

where $I$ is the moment of inertia of the magnet about the axis of rotation. For small oscillations $\sin\theta$ may be replaced by $\theta$, and we have $Id^2\theta/dt^2 + MH\theta = 0$. The motion is then simple harmonic, and of periodic time $T = 2\pi\,(I/MH)^{\frac{1}{2}}$. Hence if $T$ is observed the value of $MH$ is given by

$$MH = \frac{4\pi^2 I}{T^2} \ldots\ldots\ldots\ldots\ldots\ldots (30).$$

The moment of inertia $I$ is calculated from the form of the magnet. Thus for a cylinder of mass $m$, length $2l$ and radius $a$,

$$I = m\,(\tfrac{1}{3}l^2 + \tfrac{1}{4}a^2).$$

Similarly for a ball-ended magnet consisting of two spheres of

radius $b$ with centres $2\delta$ apart, joined by a rod whose radius $a$ is small compared with its length,

$$I = m\frac{a^2(\delta - b)^3 + 4b^3(\delta^2 + \frac{2}{5}b^2)}{3a^2(\delta - b) + 4b^3}.$$

We obtain $M$ and $H$ separately by solving equations (29) and (30).

**20. Magnetic field of a small bar-magnet.** Let us have a small bar-magnet $OA$, with poles $-m$, $m$ at $O$, $A$ respectively. Take $O$ as origin and $OA$, the positive direction of the axis of the magnet, as axis of $z$. If $P$ is the point $(x, y, z)$,

$$OP = r = (x^2 + y^2 + z^2)^{\frac{1}{2}}.$$

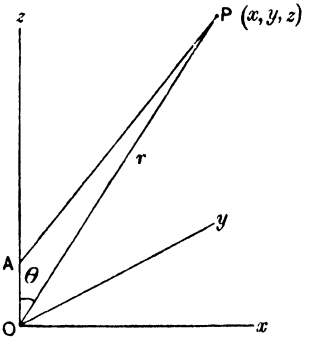


Fig. 18

If $\delta$ is the distance between the poles, $AP$ is obtained from $OP$ by writing $z - \delta$ for $\delta$, i.e. $\frac{1}{AP} = \frac{1}{r} - \delta\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$ correct to the first order in $\delta$. The magnetic potential at $P$ due to the magnet is thus

$$\Omega = \frac{m}{AP} - \frac{m}{OP} = m\left\{\frac{1}{r} - \delta\frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right\} - \frac{m}{r}.$$

Hence
$$\Omega = -M\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = \frac{Mz}{r^3} = \frac{M\cos\theta}{r^2} \quad \ldots\ldots\ldots (31),$$

where $M = m\delta$ is the moment of the magnet. Equation (27) gives

$$\left.\begin{aligned}\mathbf{H} &= \mathbf{i}M\frac{\partial^2}{\partial x\partial z}\left(\frac{1}{r}\right) + \mathbf{j}M\frac{\partial^2}{\partial y\partial z}\left(\frac{1}{r}\right) + \mathbf{k}M\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right)\\ &= \mathbf{i}\frac{3Mxz}{r^5} + \mathbf{j}\frac{3Myz}{r^5} + \mathbf{k}M\left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right)\end{aligned}\right\} \ldots\ldots (32).$$

The following method, due to Gauss, enables us to determine the law of action of magnetic poles *ab initio* by measurements with a small bar-magnet on the magnetometer.

The magnet is first placed magnetically east of the compass-box as shown in Fig. 19, with the magnetometer in its normal position. Let the strengths of the poles of the magnet be $\pm m$, and let $r \pm \delta$ be their distances from the compass-needle. If the

magnetic force at distance $r$ from a pole $m$ is $mf(r)$, the small bar-magnet will produce at the centre of the compass-needle a magnetic force $mf(r-\delta) - mf(r+\delta)$ in a westerly direction. Expanding to the first order in $\delta$, the magnitude of the force

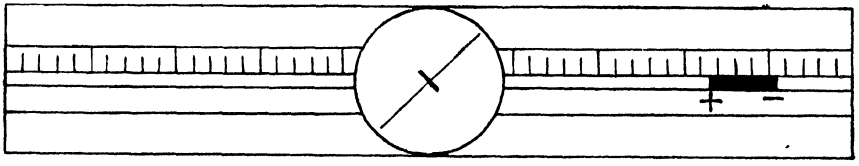


Fig. 19

becomes $-2m\delta f'(r) = -Mf'(r)$, where $M$ is the moment. Hence the compass-needle will be deflected to the west through an angle $\theta_1$ given by

$$-Mf'(r) = H \tan\theta_1.$$

Now let the magnetometer be turned through a right angle and the magnet laid *across* the groove at the same distance $r$ from the needle as before, as shown in Fig. 20. The positive pole of the magnet now produces at the centre a magnetic force practically equal to $mf(r)$, and it is easy to see that the two poles neutralise each other except as regards the easterly component, which is

$$2mf(r)\frac{\delta}{r} = \frac{Mf(r)}{r}.$$

Hence in this position the needle is deflected to the east through an angle $\theta_2$ given by

$$Mf(r)/r = H \tan\theta_2.$$

It is found that $\tan\theta_1 = 2\tan\theta_2$ within the limits of experimental error. Hence

$$f'(r) = -2f(r)/r,$$

or

$$\frac{df}{f} + 2\frac{dr}{r} = 0.$$

Integrating, we have $r^2 f = \text{constant}$, so that $f(r)$ varies as $1/r^2$.

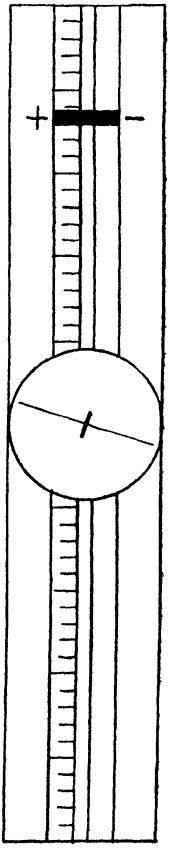


Fig. 20

**21. Terrestrial magnetism.** The quantities determining the earth's magnetic field at any place are

(1) the *horizontal component* $H$,

(2) the *declination*, or acute angle in a horizontal plane between magnetic and true north,

(3) the *dip*, or angle $\theta$ at which the axis of a compass-needle supported at its centre of gravity dips downwards.

In the latter case the direction of the needle is that of the *resultant* field of the earth. In addition to the horizontal component $H$ we have a component $V$ vertically downwards, and $V$ is clearly given by the equation

$$V = H \tan \theta.$$

To measure the *declination* at any place we require a magnetic and an astronomical observation, the former to fix the position of the magnetic meridian and the latter that of true north. An accurate determination can be made with an instrument called the declination compass. Thanks to the magnetic surveys carried out by Rücker, Thorpe and Walker, the declination at any given place can be estimated fairly accurately without measurement.

A useful instrument for rough surveying work is the *prismatic compass* (Fig. 21). In this instrument the compass-needle carries a graduated card or a light ring of non-magnetic metal divided into degrees. The observer looks through the vertical slit shown on the left of the figure, and holds the instrument so that the object is exactly behind the vertical thread shown on the right. By means of a right-angled prism covering the lower half of the slit he can also see the graduations on the divided scale, and it is easy to observe the position of the vertical thread among the scale divisions. The graduations are conveniently arranged so that the reading gives at once the angle between the magnetic meridian and the direction of the object from the observer. In this way angles can

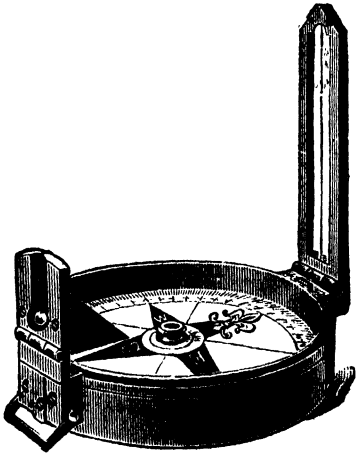


Fig. 21

be measured to a fraction of a degree, and for absolute azimuths we only want the local declination, which can be found with fair accuracy from ordnance maps.

The *dip* is determined by means of the dip-circle, which is essentially a magnet capable of moving round a horizontal axis through its centre of gravity: such a magnet will make an angle $\theta$ with the magnetic north when the plane of the dip-circle is in the magnetic meridian.

The following are the approximate values of the magnetic elements in London at the present time:

Horizontal component, 0·184.
Declination, 14° W.
Dip, 67°.
Vertical component, 0·43.

All these elements are subject to large changes in time. It would appear that the changes are more or less periodic, going through a complete cycle in about 1000 years, though exact observations have not been made over this range. In addition to these secular changes there are smaller diurnal and other variations, indicating a direct or indirect dependence on the position of the heavenly bodies. Lastly, irregular variations frequently occur, particularly in arctic and antarctic regions, which have been called magnetic storms.

A fuller account of terrestrial magnetism will be found in the *Dictionary of Applied Physics*, vol. 2, pp. 528–561.

Magnetic elements, as measured in a laboratory, do not usually correspond to the earth's field alone, since iron causes great disturbance, particularly with girders running north and south. The horizontal component in parts of the Electrical Laboratory, Oxford, is as low as 0·1 gauss, so that standard magnetic observations would be out of the question. There is also a good deal of disturbance in laboratories from electric mains and generators. A magnetic observatory must be made entirely of non-magnetic material, and situated at a considerable distance from railway lines, generating stations, and other sources of disturbance. Modern electrical instruments do not however depend on the strength of the earth's field at the place where they are situated, but on fields artificially produced and of known strength.

**22. Gauss' theorem.** The flux of magnetic force outwards through a closed surface $S$ in a magnetic field is $4\pi$ times the sum of the strengths of all the magnetic poles inside $S$.

Consider first of all the case of a positive pole $m$ placed at an

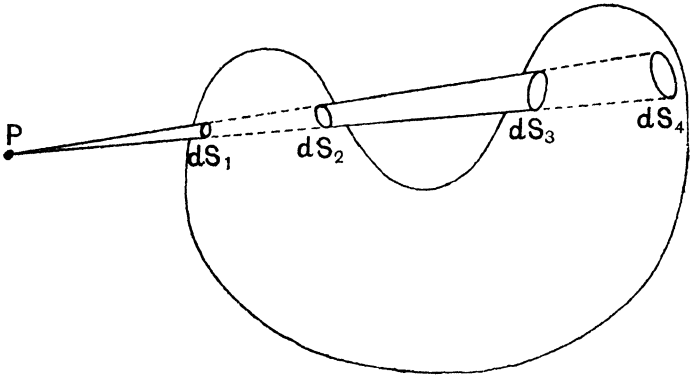



Fig. 22

external point $P$ (Fig. 22). Describe a small cone of solid angle $d\omega$ round $P$, cutting the surface in an even number of surface elements (in the figure $dS_1$, $dS_2$, $dS_3$, $dS_4$). The magnetic force at a point distant $r$ from $P$ is $\mathbf{H} = m/r^2$ radially outwards, so that the scalar product $\mathbf{H}d\mathbf{S}$ is $m/r^2$ times the projection of the area perpendicular to the ray, that is $\pm\, m\,d\omega$. At points of entry like $dS_1$ and $dS_3$, $\mathbf{H}$ makes an obtuse angle with $d\mathbf{S}$, so that $\mathbf{H}d\mathbf{S}$ is negative and equal to $-m\,d\omega$. At points of exit like $dS_2$ and $dS_4$, $\mathbf{H}d\mathbf{S}$ is positive and equal to $m\,d\omega$. If $P$ is an external point the points of entry and exit occur in pairs, and the contribution of the small cone to $\int \mathbf{H}d\mathbf{S}$ is zero. Hence the integral is itself zero.

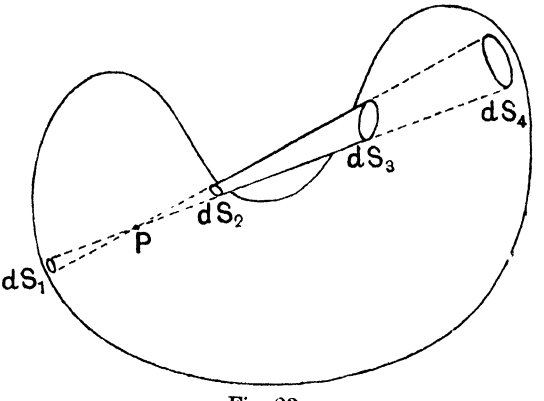



Fig. 23

If $P$ is an internal point (Fig. 23), there are always two more

points of exit than entry, and the cone contributes $2m d\omega$ to the integral. Now in tracing out the complete solid angle $4\pi$ each elementary cone is counted in twice. Hence $\int d\omega$ must be taken equal to $2\pi$ only, and $\int \mathbf{H} d\mathbf{S} = 4\pi m$ .....(33).

The result can be generalised to cover any distribution of magnetic poles. Each internal pole contributes $4\pi m$ to the integral, and each external pole nothing. Hence we have the complete theorem as enunciated above.

**23. Lines and tubes of force.** An *equipotential surface* is one at every point of which the potential is the same. In a given magnetic field $\Omega$ is a given scalar function of position, and the equipotential surfaces are given by $\Omega = C$, where $C$ is some constant. By giving different values to $C$ we get a set of equipotential surfaces.

Starting from a point $P$ in a magnetic field imagine a very short line $PQ$ drawn in the direction of the resultant magnetic force at $P$. From $Q$ a similar line $QR$ is drawn, and so on. In the limit the curved line $PQR\ldots$, which marks at every point the direction of the resultant magnetic force, is called a *line of force*. For a single pole the lines of force are straight, pointing from or to the pole according as its strength is positive or negative, and the same thing occurs in the immediate neighbourhood of an infinitely small pole in any magnetic field.

It follows from Art. 7, since $\mathbf{H}$ is the gradient of $\Omega$, that $\mathbf{H}$ is normal at any point to the equipotential surface through the point. Thus lines of force cut equipotential surfaces at right angles everywhere.

The equations of equipotential surfaces and lines of force can be obtained in simple cases. Consider for example the field produced by two equal and opposite poles, which we may take to lie at the points $(-a, 0, 0)$ and $(a, 0, 0)$ respectively. If the pole-strengths are $m$ and $-m$, the potential at the point $(x, y, z)$ is

$$\Omega = m\{(x+a)^2 + y^2 + z^2\}^{-\frac{1}{2}} - m\{(x-a)^2 + y^2 + z^2\}^{-\frac{1}{2}}.$$

Confining our attention to the sections of equipotential surfaces by a plane through the poles, for example the plane $z = 0$, the equipotential surfaces are given by

$$m\{(x+a)^2 + y^2\}^{-\frac{1}{2}} - m\{(x-a)^2 + y^2\}^{-\frac{1}{2}} = C.$$

The lines of force are found by the condition that they cut the equipotential curves at right angles. Eliminating the constant $C$ from the last equation by differentiation we have for the differential equation of all the equipotential curves

$$\{(x+a)^2+y^2\}^{-\frac{3}{2}}\left\{x+a+y\frac{dy}{dx}\right\}-\{(x-a)^2+y^2\}^{-\frac{3}{2}}\left\{x-a+y\frac{dy}{dx}\right\}=0.$$

Replacing $dy/dx$ by $-dx/dy$ gives the proper direction of the tangent to the line of force through the point $(x, y)$: hence we have the differential equation of the lines of force in the form

$$\frac{(x+a)\frac{dy}{dx}-y}{\{(x+a)^2+y^2\}^{\frac{3}{2}}}=\frac{(x-a)\frac{dy}{dx}-y}{\{(x-a)^2+y^2\}^{\frac{3}{2}}},$$

which may be written

$$\frac{d\left(\frac{x+a}{y}\right)}{\left\{1+\left(\frac{x+a}{y}\right)^2\right\}^{\frac{3}{2}}}=\frac{d\left(\frac{x-a}{y}\right)}{\left\{1+\left(\frac{x-a}{y}\right)^2\right\}^{\frac{3}{2}}}.$$

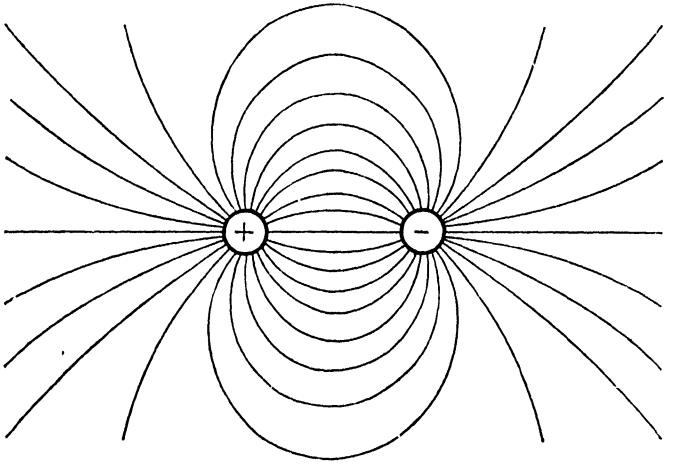



Fig. 24

Integrating, we have the equation of the lines of force, namely

$$\frac{x+a}{y}\left\{1+\left(\frac{x+a}{y}\right)^2\right\}^{-\frac{1}{2}}-\frac{x-a}{y}\left\{1+\left(\frac{x-a}{y}\right)^2\right\}^{-\frac{1}{2}}=\text{const.},$$

or $(x+a)\{(x+a)^2+y^2\}^{-\frac{1}{2}}-(x-a)\{(x-a)^2+y^2\}^{-\frac{1}{2}}=\text{const.}$

The lines of force are shown in Fig. 24. Similarly Fig. 25 shows the lines of force due to two equal magnetic poles of the same sign.

Diagrams of this kind are very instructive, as they show at a glance the state of a field at every point. Further, we shall see in

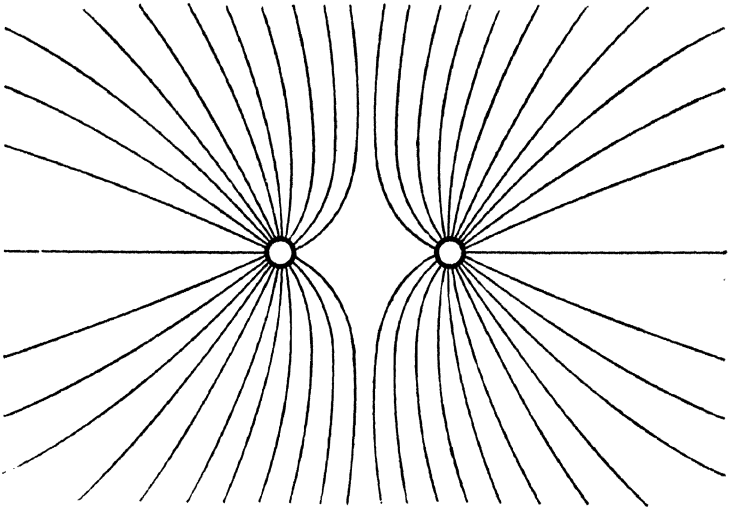


Fig. 25

the next article that the *strength* of the field is also indicated by the closeness with which the lines crowd together at different points of their length.

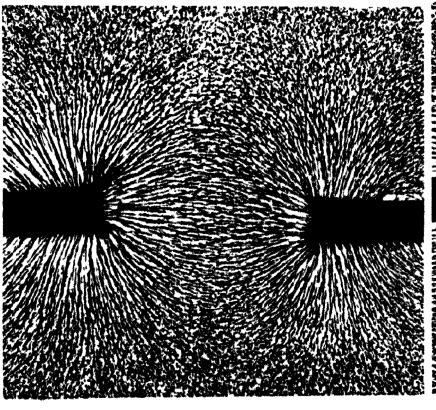


Fig. 26

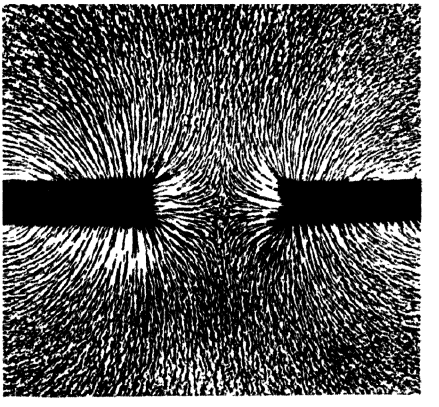


Fig. 27

Lines of force are exhibited experimentally by means of iron filings, which when sprinkled on paper or glass in a fairly strong magnetic field tend to orientate themselves along the lines of the

field. Permanent diagrams can be made by using stiff paraffined paper, which is tapped to make the filings settle into position, and gently warmed so as to melt the wax and fix the filings.

Fig. 26 is made by using opposite poles of two similar magnets, and may be compared with the theoretical diagram Fig. 24. Similarly Fig. 27 corresponds to Fig. 25.

The lines of force which start out from points on a closed curve form a tubular surface called a *tube of force*. Fig. 28 shows the tube formed by lines of force starting out normally from the perimeter of a small element of surface $dS$ near $P$. Let $dS'$ be the cross-section of the tube near another point $P'$, and let $H$, $H'$ be the resultant magnetic forces at $P$, $P'$ respectively. The immediate neighbourhood will be supposed to be devoid of magnetic poles, so that we can apply Gauss' theorem to the truncated tube in the figure. The tubular sides contribute nothing to the flux of magnetic force, since **H** is everywhere perpendicular to $d\mathbf{S}$. The ends contribute $-H\,dS$ and $H'dS'$ respectively, and since the total flux vanishes $H\,dS = H'\,dS'$.

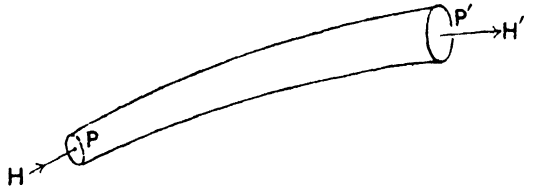


Fig. 28

Thus for a small tube of force the product of the resultant field and the cross-section of the tube is constant. Lines of force therefore crowd together in the strongest parts of the field.

If we consider a closed circuit $C$ in a part of space in which there are no magnetic poles, the flux of magnetic force through $C$ can be specified in terms of $C$ alone, i.e. it is the same for all unclosed surfaces $S$ having $C$ as their rim. (This is another way of saying that **H** is solenoidal; cf. Art. 9.) For a given tube of force will contribute nothing to the flux of force through $C$ if it does not pass through $C$, but if it does interlace $C$ it gives a definite contribution to the flux irrespective of the surface $S$ used to calculate it. For this reason the phrase "number of tubes of force passing through a circuit" is often used instead of "flux of force through a circuit": but we shall adhere to the latter phrase as expressing more precisely what is meant.

**24. Potential energy of a small magnetised body in any field.** We now consider a small body, magnetised as irregularly as we please, placed at a point $O$ in a magnetic field. Let $Q$ be a neighbouring point in the body, at which a specimen pole $m$ is located, $\mathbf{s}$ being the vector $OQ$. If $\Omega$ is the magnetic potential of the external field at $O$, the potential at $Q$ is $\Omega + \mathbf{s}\mathbf{D}\Omega$ from Art. 7, to the first order of small quantities. Writing $\mathbf{H} = -\mathbf{D}\Omega$ for the magnetic force at $O$, the potential energy of the pole $m$ is $m\Omega - m\mathbf{s}\mathbf{H}$. Hence if $\Sigma$ denotes summation over all the poles $Q$ in the small body, the potential energy of the body in the field is

$$\Omega\Sigma m - \Sigma m\mathbf{s}\mathbf{H}.$$

The first term vanishes since $\Sigma m = 0$. Writing

$$\mathbf{M} = \Sigma m\mathbf{s} \quad \ldots\ldots\ldots\ldots\ldots\ldots (34),$$

the required potential energy is the scalar product $-\mathbf{M}\mathbf{H}$. The vector $\mathbf{M}$ is called the *magnetic moment* of the small body, and its line of action the *magnetic axis*. These definitions agree, in the case of a simple bar-magnet, with those adopted previously.

To find the magnetic potential which the body itself produces at an external point $P$, we have only to suppose the field to be that of a unit pole placed at $P$. If $\mathbf{r}$ is the vector $OP$, $\mathbf{H}$ is a vector of magnitude $1/r^2$ along $PO$, that is $-\mathbf{r}/r^3$. Hence the magnetic potential of the body at $P$ is $\mathbf{M}\mathbf{r}/r^3$, in agreement with (31). The small body is therefore magnetically equivalent in all respects to a certain bar-magnet, and we have found its moment and axis for any assigned distribution of poles in the body.

If the field in which the body is imbedded is uniform, potential energy can only be changed by turning the body, and thus it is acted on by a couple, which is represented completely by the vector product $V\mathbf{M}\mathbf{H}$. To prove this, observe that the pole $m$ at $Q$ is acted on by a force $m\mathbf{H}$, which is equivalent to the same force at $O$ together with a couple $mV\mathbf{s}\mathbf{H}$ (Art. 4). The force vanishes since $\Sigma m = 0$, and the vector sum of the couples is $V\mathbf{M}\mathbf{H}$.

**25. Poisson's analysis of magnetism.** The method of iron filings shows that a magnet continues to show magnetic properties at both its ends, however often it is broken (Fig. 29). It is thus natural to suppose that the smallest parts of a magnetised

body are themselves magnets, a medium of this kind being said to be *polarised*. The magnetic state of a volume-element $d\tau$ of a magnetised body is specified when its magnetic moment is given in magnitude and direction. The magnetic moment *per unit volume*

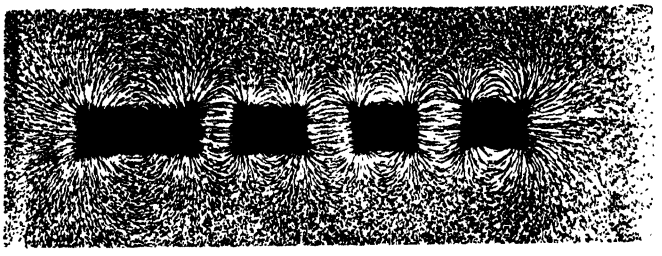


Fig. 29

near any point is a vector **I** called the *intensity of magnetisation*. Thus the magnetic moment of the volume-element $d\tau$, in the sense of Art. 24, is $\mathbf{I}d\tau$. The relation of **I** to the local magnetic force is an experimental matter which will occupy us in Ch. VII.

Let a rod be magnetised uniformly in the direction of its length (Fig. 30). Since each element is polarised the positive and negative poles in the interior cancel one another, and there is no free magnetism. At the ends, however, the compensation ceases, the positive poles being left at one end and the negative poles at the other. Thus we have a distribution of positive magnetism over the right-hand face and of negative magnetism over the left-hand face. Let $\pm\sigma$ be the pole-strength per unit area, which we may call the surface-density of magnetism. If $l$ is the length and $A$ the cross-section of the rod, the pole-strengths at the ends are $\pm\sigma A$, and the magnetic moment $\sigma lA$. Since the volume is $lA$ the magnetic moment is by definition equal to $IlA$, so that $\sigma = I$. Hence in a uniformly magnetised rod the surface-density of magnetism on the flat ends is equal to $I$. If one end is cut off obliquely, so that its normal makes an angle $\theta$ with **I**, a similar argument shows that $\sigma = I\cos\theta$. Hence if a body is uniformly magnetised, the surface-density of magnetism is equal to the

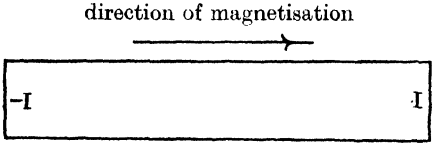



Fig. 30

normal component of **I**, and the volume-density of magnetism is zero.

The conclusion as to $\sigma$ is not invalidated when **I** is not uniform, since $\sigma$ depends solely on the local value of **I**. Hence we have always

$$\sigma = I_n \quad \ldots\ldots\ldots\ldots\ldots\ldots (35).$$

A simple example shows that the volume-density is not zero in general. If $I$ in Fig. 30 increases from left to right, the two surface layers together possess a positive charge, which must be compensated by a negative volume charge. General magnetisation means that **I** is an arbitrary vector function of position. Let $\tau$ be a small volume cut from the body. From (35) the surface element $d\mathbf{S}$ of $\tau$ carries a pole-strength equal to the scalar product $\mathbf{I}d\mathbf{S}$, so that the total surface charge is $\int \mathbf{I}d\mathbf{S}$ extended over the surface, which is $\tau$ div **I** from the theorem of flux. The compensating volume charge must therefore be $-\tau$ div **I**, or the volume-density is

$$\rho = -\text{div } \mathbf{I} \quad \ldots\ldots\ldots\ldots\ldots\ldots (36).$$

The distributions (35) and (36) are known as Poisson's distributions. If **I** is a constant vector, $\rho = 0$ as before.

Uniform magnetisation cannot be realised accurately in a short rod, as the magnetism is thrown back more or less from the ends (cf. Art. 19). Further, the magnetic force due to the distribution near the ends tends to oppose the magnetisation of the rod, since it acts in a direction opposite to **I**.

# CHAPTER III

## ELECTROSTATICS

**26. Electrical attractions and repulsions.** It has long been known that substances such as amber, glass, resin when rubbed are capable of attracting light bodies. These attractions are called *electrical*, and the rubbed body is said to be *electrified*, or to carry a *charge of electricity*, while a body in its normal state is called *neutral* by way of distinction. To study these effects, suspend a light metallic or other carrier from a silk fibre so that it can hold a glass rod in a horizontal position. If the rod is rubbed vigorously with a silk handkerchief and placed in the carrier, it will be seen to be attracted by the handkerchief when the latter is brought near. Two similarly prepared glass rods, on the other hand, repel one another. The glass rod is in this case said to be *positively* and the silk handkerchief *negatively* electrified, the terms positive and negative being used at present only in a descriptive sense. Now prepare an ebonite rod by rubbing it with flannel, and suspend it as before. Two similarly prepared rods repel one another and either attracts the flannel, as in the previous experiment. But the ebonite rod will *attract* a glass rod, and be *repelled* by the silk handkerchief with which the latter has been rubbed. We must thus suppose that an ebonite rod becomes charged negatively by friction with flannel, and that two negative charges repel one another. We can now see the significance of the terms positive and negative: for if we assign a + sign to the phenomenon of repulsion and a − sign to that of attraction we see that + acting on + gives +, − acting on − gives +, while + acting on − gives −. This is in formal analogy with the rule of multiplication of signs in algebra; an analogy which will be fully developed later.

Other substances may be used instead of glass and ebonite, but we never obtain any evidence of the existence of other kinds of electrification than those already named, because a rubbed body which repels a positively charged body is always found to attract a negatively charged one, and *vice versâ*. A convenient way of testing electrification is to prepare a number of pith balls by coating them with aluminium foil and suspending them from silk

fibres. Pith balls that touch an electrified glass rod acquire positive electrification: that is, they repel one another and the glass rod, and are attracted by the silk handkerchief, while they attract pith balls that have been in contact with the ebonite rods. The effects with well-dried materials are more marked than those obtained directly with the rods, and in addition we perceive that the force between two small charged bodies acts along the line joining them.

A pith ball that has not been electrified is always *attracted* by an electrified body, whether positively or negatively charged. As in magnetism, however, this phenomenon is more complex than the original one of the repulsions and attractions of charged bodies. It is considered further in Art. 31.

**27. The electroscope. Conductors and insulators.** Simple electrical experiments are more conveniently performed by means of the *electroscope* than by either of the preceding methods.

This instrument consists essentially of a brass rod with a knob at the upper end, the lower end dipping into a metal case with glass sides (Fig. 31), the rod and case being separated from each other by a stopper of ebonite. Two strips of thin gold or aluminium foil are attached to the lower end of the rod. When the knob is touched with an electrified glass or ebonite rod the leaves diverge on account of the mutual repulsion of the electrification on them, and being very light they are very easily deflected. The amount of divergence may be indicated by a graduated scale attached to the central rod or case.

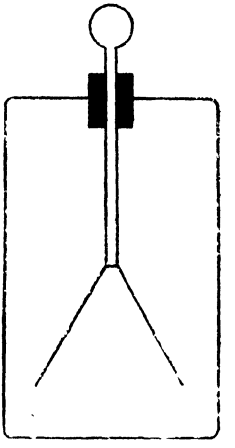


Fig. 31

If the electroscope is charged and left to itself, the leaves remain in the same position, at any rate for a considerable time. The same is true of two pith balls suspended from dry silk fibres and left repelling or attracting one another in air. Thus the air, ebonite, and silk have the property of being able to resist the loss of electrification by a body: such substances are called *insulators*. On the other hand, if the knob of a charged electroscope is touched the leaves collapse immediately, showing that electrification can escape through or into the human

body. Substances allowing the passage of electricity are called *conductors*. This view of the removal of electricity by passage through a conductor is supported by a simple experiment. If an ebonite rod is held in the hand and placed in contact with the knob of an electroscope the leaves do not collapse, because the electrification cannot pass through the insulator to the hand, although it would pass freely through the body if it once got so far. If the ebonite rod is replaced by one of metal, however, the leaves fall at once, showing that metals are conductors of electricity.

The division of bodies into conductors and insulators is not absolutely strict, as there is a continuous gradation between the two classes. Thus the leaves of an electroscope whose knob is touched by a wooden rod held in the hand collapse gradually, and at different rates depending on the dryness and nature of the wood. It must not, however, be thought that the distinction between conductors and insulators is an unreal one, for it is possible to find substances which approximate very closely to the ideal of both kinds. Thus, for example, a well-designed electroscope with ebonite, sulphur, or amber insulation will retain some charge for at least a week, while if metallic contact is made between the knob and the case the leaves collapse in the merest fraction of a second —the time taken for the electricity to disappear is, in fact, unmeasurably small.

In the following table we have endeavoured to place common substances in the order of their insulating power, but the exact order is somewhat uncertain and depends on the physical state of the substance.

| | |
|---|---|
| Insulators | Air (best)<br>Amber and amber preparations<br>Sulphur<br>Freshly cut ebonite<br>Paraffin wax<br>Quartz<br>Dry glass<br>Wood soaked in melted paraffin wax |
| Semi-Insulators | Wood<br>Oils |

| | |
|---|---|
| Conductors | Water |
| | Salt solutions |
| | Carbon |
| | Mercury |
| | Metals (best) |

Air is an admirable insulator under ordinary conditions, but it has its limits of resistance like all other insulators, and breaks down under high electric force with the passage of a *spark*. The conditions for a spark are favourably realised when conductors with sharp or pointed edges are present, a fact which is utilised in the construction of lightning-conductors, and also in influence machines (Art. 32). The hot air in the neighbourhood of flames is also a deficient insulator.

Defective insulation can often be traced to a conducting layer on the surface, as for example when moisture deposits on glass. Ebonite becomes oxidised when exposed to strong light and loses its insulating power; and a conducting layer of grease is deposited on its surface if it is fingered. This can be obviated by making a narrow saw-cut round a piece of ebonite required for electrostatic experiments, leaving a fresh surface not likely to be contaminated.

The class of bodies described as semi-insulators cannot be used in electrostatic experiments: but we shall see that they often suffice for work with electric batteries (Art. 67). Liquids also can never be classed as good insulators, because they *convect* electricity away by their motion, although they do not all *conduct* in the strict sense.

The conducting power of the human body and of metals explains a phenomenon which misled early investigators, namely that metals when rubbed do not appear to be electrified. That this is not the case is easily shown by rubbing a metal rod held in an insulating handle, when it is found to be electrified just like other bodies. Otherwise the electricity is conducted away as soon as it is produced by friction.

**28. The electrical theory of matter.** It will be convenient to describe here that view of the constitution of matter which holds

the field at present, without offering any evidence in support of it. Matter, positive and negative electricity are all atomic in structure; that is, they exist in portions either indistinguishable from each other or of only a small number of different kinds. It is supposed that the *hydrogen atom* consists of two parts; a *proton* or atom of positive electricity containing nearly all the mass of the atom, and an *electron* or atom of negative electricity whose mass is about 2000 times less. The electron revolves normally about the proton in an orbit whose diameter is about $10^{-8}$ cm. Proton and electron are much smaller than this, probably at least 50,000 times as small, so that the atom has a very open structure, comparable to one marble rotating about another at a distance of half a mile. Atoms of heavier elements consist similarly of a *nucleus* of closely packed protons and electrons, and *circumnuclear electrons* in orbits as before. Thus a helium nucleus consists of four protons and two electrons, with two circumnuclear electrons; a gold nucleus of about 200 protons and 120 electrons. In the *molecule* two or more atoms share their circumnuclear electrons and become more or less firmly knit together. We are not concerned here with the details of the theory (which are still highly problematical), but with the insight which it affords into the behaviour of electrified bodies in bulk. We regard insulators as substances whose atoms do not readily lose or gain circumnuclear electrons, though they may be stimulated to do so by friction. Thus when a glass rod is rubbed with a silk handkerchief the surface atoms of the glass give up electrons to the silk, and the glass is positively electrified. The action is limited to those molecules which come into rubbing contact, but it is not so with conductors. We may suppose that metallic atoms have but a weak hold on their outermost electrons in the solid state, so that some of them move away without remaining permanently attached to any particular atom. If the conductor is insulated, i.e. separated by an insulator from other conductors, the electrons remain on it as they cannot pass through the insulator and also cannot under normal conditions escape into the air.

The electrical theory of matter affords a simple and perfectly definite specification of *amounts* of electricity. The neutral atom or molecule contains altogether the same number of protons and

electrons. For the convenience of assigning to a neutral atom the electrical charge zero we say that each proton carries a charge of $q$ units of positive electricity (or carries a charge $+q$), and each electron a charge $-q$, where the unit need not yet be specified. A body which contains $n$ more protons than electrons is said to carry a charge $nq$; if $n$ more electrons than protons, a charge $-nq$. The mutual repulsion of positively charged bodies is explained by supposing that protons repel one another. Let the force of repulsion at distance $r$ be $q^2f(r)$ dynes. A neutral atom or molecule, as a whole, does not repel or attract either a proton or an electron; the compensation can only be brought about (if they act independently) by an electron *attracting* a proton with a force of $q^2f(r)$ dynes, and *repelling* another electron with the same force.

Consider now a small body carrying a charge $+e$ at distance $r$ from a second small body carrying a charge $+e'$. The former can be regarded as containing $m$ protons and the latter $n$ protons, where $e = mq$ and $e' = nq$. The force between the bodies is repulsive and of magnitude $mnq^2f(r) = ee'f(r)$. It is easy to see that the statement holds when one or both of $e, e'$ are negative, provided that an attraction is regarded as a negative repulsion. The statement gives the attractions and repulsions quantitatively in terms of the charges in the bodies, being more far-reaching than the rule of signs in Art. 26.

Coulomb showed experimentally that the force between two small charged bodies is inversely proportional to the square of the distance between them, i.e. that $f(r) \propto 1/r^2$. We shall not however make this assumption till we have described a more convenient and accurate method of verifying it. The nuclear theory of matter is resumed in Art. 223, and forms the basis of most of the work from there onwards.

**29. Electric force and electric potential.** The description of the electric field now follows on the same lines as that of the magnetic field (Art. 16). The *electric force* at any point is the force which would act on a unit charge placed at the point. The precise unit is left at present unsettled. If $\mathbf{E}$ is the electric force at a point $P$ in an electric field, the force on a charge $e$ placed there is $e\mathbf{E}$. The resultant electric force at distance $r$ from a charge $e$ is, with the notation of the last article, $ef(r)$.

The *electric potential* is the work that would be done by the forces in the field when a unit charge is moved to infinity from the point considered, i.e. the potential energy of a unit charge at the point. At distance $r$ from a charge $e$ the potential is

$$V = e \int_r^\infty f(r)\, dr.$$

The electrical potential vanishes at infinite distances from the charges in the field.

When a number of charges are present the potential is obtained by summation of the separate potentials due to each. A charged body contains a large number of electrons which, though quite distinct, are close together. We do not usually wish to know the exact position of each, but only the average number per cubic centimetre or per square centimetre of a layer. We may thus consider charges extended through regions of space with *volume-density* $\rho$ or spread over surfaces with *surface-density* $\sigma$, and then the potential of the distribution is obtained by integration (cf. Art. 34).

A difficulty arises in the definition of potential from the fact that electricity can move about in conducting bodies. Electricity settles down in conductors in a way which depends on the other charges present in the field. Thus when a unit charge is taken away from any point $P$ to infinity the distribution of electricity in the field is changed, and the work actually done will differ from that which would have been performed if the charges had been fixed in position before the unit charge was moved. It is the latter work which defines the potential at $P$. In other words, in calculating potentials the disturbing effect of the unit charge is neglected. One way of avoiding this reservation would be to say that, if work $W$ is actually done on a charge $e$ moved from $P$ to infinity, then the potential at $P$ is the limit of the ratio $W/e$ when $e$ is indefinitely diminished.

The electric force $\mathbf{E}$ at any point is the negative gradient of the electric potential, or

$$\mathbf{E} = -\mathbf{D}V \quad \ldots\ldots\ldots\ldots\ldots\ldots (37).$$

The surfaces $V = \text{const.}$ are called *equipotential surfaces.*

We shall now prove a certain reciprocal theorem connecting charge and potential in two distributions. Let charges $e_1, e_2, \ldots e_n$ be placed at the points $P_1, P_2, \ldots P_n$, and let $V_1$ be the potential

at $P_1$ of all the charges except $e_1$, $V_2$ the potential at $P_2$ of all the charges except $e_2$, and so on. Then

$$V_1 = e_2 f(r_{12}) + e_3 f(r_{13}) + \ldots + e_n f(r_{1n}),$$

$$V_2 = e_1 f(r_{21}) + e_3 f(r_{23}) + \ldots + e_n f(r_{2n}),$$

and so on. Let $e_1', e_2', \ldots e_n'$ be any other charges placed at the same points, $V_1', V_2', \ldots V_n'$ the corresponding potentials. The sum $e_1'V_1 + e_2'V_2 + \ldots + e_n'V_n$ involves $r_{12}$ in the form

$$(e_1'e_2 + e_2'e_1) f(r_{12}),$$

and similarly for any other mutual distance. Hence

$$e_1'V_1 + e_2'V_2 + \ldots + e_n'V_n = \Sigma (e_p'e_q + e_q'e_p) f(r_{pq})$$

extended over all possible pairs of numbers $p$, $q$. By symmetry this is equal to $e_1V_1' + e_2V_2' + \ldots + e_nV_n'$. Hence

$$\Sigma eV' = \Sigma e'V.$$

This is known as Gauss' reciprocal theorem.

**30. Potential of a conductor.** All points on or inside a conductor on which electricity is at rest will now be shown to be at the same potential. If two neighbouring points $P$, $Q$ on or inside the conductor were at different potentials, the potential of $P$ being the higher, electricity would tend to flow from $P$ to $Q$ because there would be an electric force from $P$ towards $Q$, and in a conductor there is nothing to prevent the passage of charge in these circumstances. The common potential of all points of the conductor is called the *potential of the conductor*. It follows that the bounding surface of a conductor is an equipotential surface of the charges in the field. This conclusion is independent of any particular law of force between electric charges.

If two charged conductors are joined by a wire held in an insulating handle, electricity flows from the one at higher to the one at lower potential. This process ceases when the potentials have become equal, but if a battery were provided to maintain the potentials at their original values it would go on indefinitely (cf. Art. 57). In this chapter we shall however be concerned with electricity at rest.

**31. Electrical influence.** If an electrified ebonite rod is brought near to one end of an elongated conductor the other end repels a pith ball that has previously been touched by the rod, while the near end of the conductor attracts it. Thus negative electricity collects at the far end and positive electricity on the part near the rod, the middle parts being comparatively neutral. These effects are said to be due to *electrical influence.*

An easy explanation can be given on the electrical theory of matter. The circumnuclear electrons of conductors are not all

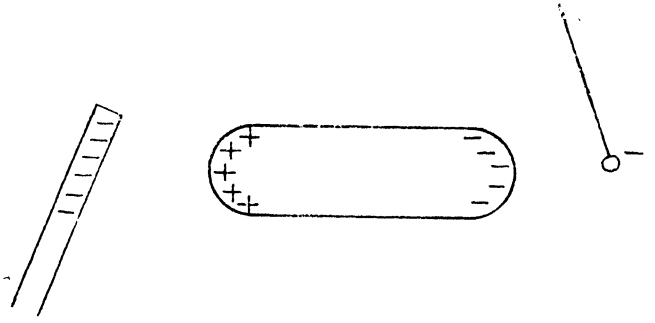


Fig. 32

bound to any particular atom, but some of them can move about in the metal, even in the neutral state. When a negatively charged rod is brought near, the electrons are repelled by the rod, and tend to congregate in the remote parts of the conductor, leaving positive charge on the parts near the rod. (The same distribution would occur if electrons had a positive charge, so that the experiment proves nothing about the sign of the electronic charge.)

Electrical influence is shown very well with an electroscope. Thus if an electroscope is charged positively the leaves begin to diverge still further when a positively charged body approaches the knob, from the induced increase of positive charge on the leaves. A negative charge reduces the divergence of the leaves as it comes near. If the knob is touched while the positive charge is near it, the leaves collapse, since the positive charge originally on them tends to go even further away from the influencing charge, that is into the body of the experimenter. There remains only the negative charge attracted to the neighbourhood of the knob. Now suppose the hand to be taken away and the influencing charge

subsequently removed. The negative charge will spread over the whole of the knob and leaves, and the latter will diverge, though with the opposite kind of electricity from that of the external charge.

We have thus a means of charging an electroscope with electricity of either sign by use of a charge of one sign only. If the rod had been removed before the hand was taken away there would have been no deflexion of the leaves. This illustrates the fact that the order of operations in experiments depending on influence is not immaterial, quite different effects being obtained by doing things in a different order. The human body in these experiments plays the part of a large conductor practically free from charge, and the same results would be obtained by connecting the knob to a large uncharged metal conductor, or to a gas-pipe having contact with the soil. A connexion of this kind is known as *earthing* a conductor, or putting it to earth.

Although in theoretical reasoning potential is defined to be zero at infinite distances, yet we can only recognise *differences* of potential, so that we can take any convenient conductor to be at zero potential. It is conventional to take an earthed body to be at zero potential. An earthed conductor is sometimes said to be uninsulated.

We can now understand the attraction which a charged body exerts on neutral bodies. The first effect of a positive charge is to bring negative charge to the nearer part of the neutral body and repel positive charge to the remote parts, and since on the whole the negative charge is nearer the attraction exceeds the repulsion.

**32. Influence machines.** Machines have been devised for the continuous production of electricity depending on the fundamental principle of friction, but these do not work regularly, and have been superseded by machines working by electric influence. Of the various types of influence machine the Wimshurst machine (Fig. 33) has been found the most efficient. Two circular plates of glass or ebonite are placed parallel to one another on an axle, and arranged so that they rotate in opposite directions on turning a handle. The outer surfaces of the plates carry a number of thin sheets of metal

cut into the form of sectors. At the level of the centre of the plates are fixed two double combs of metal enclosing the plates at the end without touching them, with the teeth pointing towards the sectors. The combs are fixed to insulating stands and also to a pair of brass knobs whose distance apart can be altered. On

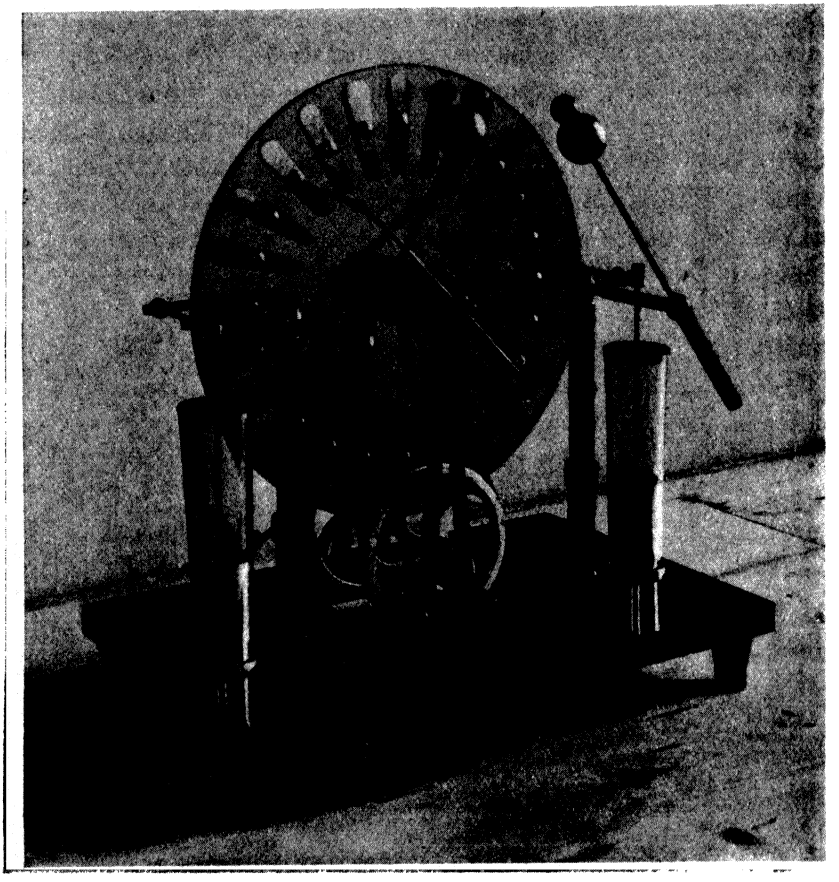


Fig. 33

either side a brass rod is fixed to the axle, carrying at each end a metallic brush which touches each sector as it passes. These rods are inclined at angles of 45° to the horizontal as shown.

The mode of action of the machine is most easily explained with a diagram in which the two discs are imagined to be replaced by two coaxial cylinders revolving one inside the other. In Fig. 34

$X$, $Y$ represent the combs with their knobs attached, and $A$, $B$, $C$, $D$ the four brushes at the ends of the two brass rods.

Imagine to begin with that a particular sector $P$ carries a negative charge, all the other sectors being uncharged. As the discs rotate $P$ comes opposite to the brush $A$, and at the same time the sector originally at $Q$ is in contact with the brush. The negative charge on $P$ induces a positive charge on the sector and

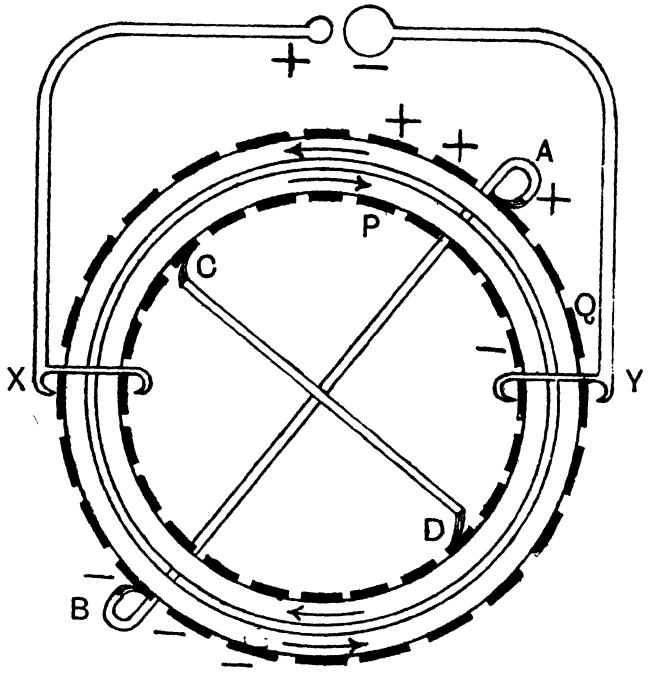



Fig. 34

brush $A$, and this requires a negative charge to reside on the opposite brush $B$ and its sector. The inductive effect of $P$ is still appreciable when it is some distance from the brush $A$. Hence as the discs rotate further, other sectors passing $A$ have positive charges induced on them, the opposite process occurring at the other extremity of the diameter. The figure shows the charges present just before the original sector comes in front of the comb $Y$. The sharp points of the teeth discharge this sector more or less completely: but the three upper sectors on the outer disc carry their positive charge with them until they come opposite the brush

$C$, where they induce negative charges on the sectors passing $C$, causing positive charges to reside on the sectors passing $D$. Negative charges are subsequently induced on the sectors of the outer disc opposite $B$, thus continuing the supply of negative electricity to the sectors in this region. A little consideration shows that the reinforcement of the charges present continually increases, and an ever-increasing number of sectors is brought into play. Ultimately all the sectors carry charges: on the inner disc the charge is negative in the part $CYD$ and positive in the part $DXC$, while on the outer disc we have positive charge along $AXB$ and negative along $BYA$. The charges on the sectors between $Y$ and $D$, $Y$ and $A$, $X$ and $C$, and $X$ and $B$ are however comparatively small because these sectors have passed a discharging comb and have not yet arrived at a brush. As regards the collection of the charge on the knobs, consider the sectors passing the comb $X$. Both sets are positively charged, and induce negative electricity on the teeth of the comb, leaving positive electricity on the knob connected to $X$. The negative charge on the comb escapes from the teeth and neutralises the positive charge on the sector, so that the conductor $X$ is continually acquiring an excess of positive charge. The opposite charge is being accumulated on $Y$, and this may go on until a spark passes between the knobs themselves. The electrical energy of the separation of the charges of course comes out of the increased work required to turn the handle against the mutual attraction of the charges on the sectors.

The slight initial charges required to start the action are generally present, and the machine is usually self starting. If not, an electrified rod may be held near the sectors opposite one brush. When driven at a constant speed with an auxiliary motor the influence machine gives a very constant source of high potential

**33. Cavendish's experimental proof of the law of force.** A crucial test of the law of force between electric charges is afforded by an experiment first performed by Cavendish in 1773, and repeated with refinements and modifications under Maxwell's direction about a hundred years later. The experiment can be carried out in the laboratory as follows.

Take two large tin boxes $S_1$, $S_2$ (Fig. 35) and place $S_1$ inside $S_2$

and insulated from it by means of paraffin blocks. A small hole cut in the box $S_2$ is fitted with an ebonite plug in which are bored three small cups $A$, $B$, $C$ to contain mercury. Wires soldered to $S_1$, $S_2$ and ending in the mercury-cups make contact with $B$, $A$ respectively. A wire dipping in $C$ leads to a sensitive electroscope, or better to one terminal of a quadrant electrometer (Art. 48). We can make metallic connexion between $B$ and $A$, or between $B$ and $C$, by means of a short wire passing sideways through an ebonite rod and bent round at the ends as shown in the figure.

Initially $B$ is joined to $A$ and the electroscope discharged. In this state $S_1$ and $S_2$ form a single compound conductor of which one part lies completely within the other. The compound conductor is now charged to a high potential with a small influence machine, and the electricity settles down in its position of equilibrium on the two conductors. On removing the key from the cups the charges on $S_1$ and $S_2$ are insulated from one another. The outer conductor $S_2$ is now earthed and the key put into the cups $B$, $C$, thus connecting the inner conductor $S_1$ with the electroscope. It is found that no deflexion of the electroscope occurs, showing that there is no charge on $S_1$.

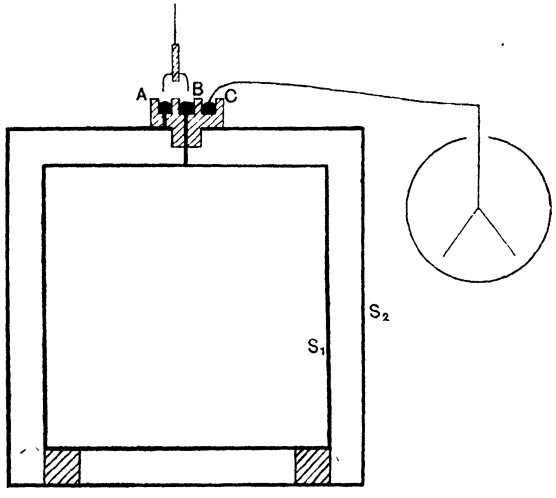


Fig. 35

This was verified by Cavendish and Maxwell when $S_1$ and $S_2$ are spheres, and by Cavendish in the case of one cube inside another. We conclude that whenever one conductor lies entirely inside another, and the two are joined and charged with electricity, none of the charge resides on the inner conductor. When the two shells are joined so as to form one conductor, all points of both are at the same potential. None of this potential is due to the inner conductor since it is entirely devoid of charge; and since the inner

conductor may have any shape whatever it follows that the electricity on the outer shell is sufficient by itself to produce constant potential at *all* points inside it. In particular, an electrified spherical shell gives constant potential at all internal points. The only law of force consistent with this can be shown to be that of the inverse square as follows.

Let the potential at distance $r$ from a charge $e$ be written in the form $e\phi'(r)/r$, where $\phi$ is some function to be determined. Let $R$

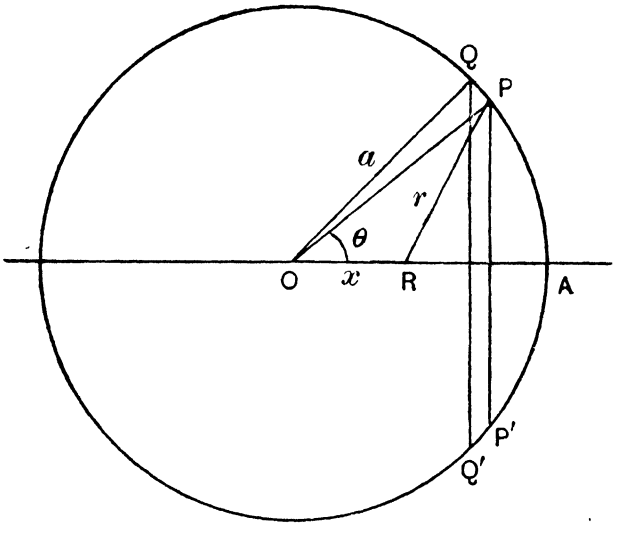



Fig. 36

(Fig. 36) be a point distant $x$ from the centre $O$ of an electrified spherical shell of radius $a$, and let $PP'$, $QQ'$ represent two neighbouring planes perpendicular to $ORA$, such that the angle $AOP = \theta$ and $AOQ = \theta + d\theta$. The electricity must be spread uniformly over the shell, say to amount $\sigma$ per unit area. If $RP = r$ the potential at $R$ of the part of the shell intercepted between the two planes is $2\pi\sigma a^2 \sin\theta\, d\theta\, \phi'(r)/r$, since the area of the intercept is $2\pi a^2 \sin\theta\, d\theta$ and every point of it is at distance $r$ from $R$. Regarding $R$ as a fixed point and $\theta$ as the variable, the equation

$$r^2 = a^2 + x^2 - 2ax\cos\theta$$

when differentiated gives $r\,dr = ax\sin\theta\, d\theta$. Hence the potential of the small intercept at $R$ is

$$\frac{2\pi\sigma a}{x}\phi'(r)\,dr.$$

The extreme limits of $r$ are $a - x$ and $a + x$, so that the potential of the whole electrified shell at $R$ is given by

$$V = \frac{2\pi\sigma a}{x}\int_{a-x}^{a+x}\phi'(r)\,dr = \frac{2\pi\sigma a}{x}\{\phi(a+x) - \phi(a-x)\}.$$

Since $V$ is to be independent of $x$, $\phi(a+x) - \phi(a-x) = \lambda x$, where $\lambda$ is some constant. Differentiating twice, we have

$$\phi''(a+x) - \phi''(a-x) = 0,$$

showing that $\phi''(x) = B$, a constant. Hence $\phi'(x) = Bx + C$, and the potential at distance $r$ from a charge $e$ is of the form $e(B + C/r)$. Now by hypothesis the potential vanishes when $r$ is infinite, so that $B = 0$. Thus $V = eC/r$, and the electric force is

$$E = -dV/dr = eC/r^2,$$

which proves the result.

Maxwell showed that if the law of force was the inverse $p$th power, then $p$ could not differ from 2 by more than 1/20,000 without giving rise to an observable charge on the inner conductor in his experiment. Evidence will be given in Art. 250 that two nuclei repel each other according to Coulomb's law down to distances of the order of $10^{-11}$ cm.

**34. Electrical units.** We now see that two charges $e$, $e'$ distant $r$ apart in air repel one another with a force of $F$ dynes, where $F = kee'/r^2$ and $k$ is some constant. Evidently $k$ cannot be determined until we have settled the unit in which $e$ and $e'$ are to be measured.

The unit charge of electricity is taken to be that positive charge which repels an equal positive charge distant one centimetre away with a force of one dyne. This amount of electricity is known as the *electrostatic unit of charge*. With these units $F = 1$ when $e = e' = r = 1$, so that $k = 1$. Hence the repulsion between two charges $e$, $e'$ in general is $ee'/r^2$ dynes, an attraction being regarded as usual as a negative repulsion. The electric force at distance $r$ from a charge $e$ is

$$E = e/r^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (38)$$

radially outwards, and the potential

$$V = e/r \quad \ldots\ldots\ldots\ldots\ldots\ldots (39).$$

In the case of a volume-distribution the last formula is replaced by $V = \int \frac{\rho d\tau}{r}$, where $r$ is the distance of the point where $V$ is calculated from $d\tau$, and the integration extends over the volume containing charges. Similarly for surface-distributions we have $V = \int \frac{\sigma dS}{r}$.

It will appear that volume-distributions do not exist when electricity is at rest on conductors. We shall consider them, however, because such a distribution is the most general kind, and a surface-distribution can be regarded as a thin volume-distribution of great volume-density.

**35. Electric force due to an infinite charged plane.** Let $RO$ be the perpendicular let fall from $R$ on the plane (Fig. 37), and let $PP'$, $QQ'$ be two circles in the plane with centre $O$ and radii

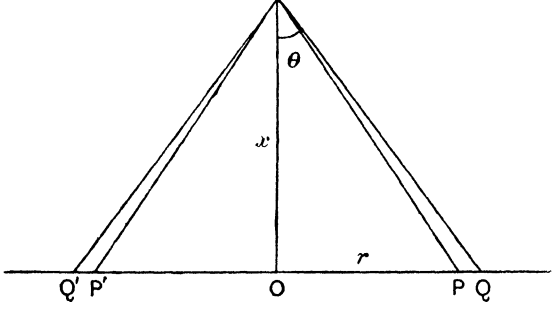



Fig. 37

$r$, $r + dr$ respectively. If $OR = x$ and the angle $ORP = \theta$, then $r = x \tan \theta$, so that the area of the ring between the circles is

$$2\pi r dr = 2\pi x^2 \tan \theta \sec^2 \theta d\theta.$$

Each part of the ring is at a distance $x \sec \theta$ from $R$ and acts radially along the line joining it to $R$. The resolved part along $OR$, which is evidently the direction of the resultant electric force, is obtained by multiplying by $\cos \theta$. Hence the electric force at $R$ due to the small ring is

$$2\pi\sigma x^2 \tan \theta \sec^2 \theta d\theta \cos \theta \div (x^2 \sec^2 \theta) = 2\pi\sigma \sin \theta d\theta,$$

where $\sigma$ is the surface-density. Integrating from $\theta = 0$ to $\theta = \frac{1}{2}\pi$, the electric force due to the whole disc becomes

$$E = 2\pi\sigma \quad \ldots\ldots\ldots\ldots\ldots\ldots (40)$$

in the direction $OR$.

Thus the electric force is the same at all points on one side of the plane. The direction of the force changes on crossing the plane, so that the force undergoes a discontinuous change of magnitude $4\pi\sigma$. Since the force does not become infinite, however, there is no discontinuity in the potential as we cross the sheet.

**36. Gauss' theorem. Superficiality of charge.** Since the laws of magnetism and electrostatics are formally identical, the proof of Gauss' theorem (Art. 22) may be transferred simply to electrostatics, and we have:

The flux of electric force outwards through any closed surface $S$ in an electric field, that is the surface integral $\int \mathbf{E}\,d\mathbf{S}$ taken over the whole surface, is $4\pi$ times the total charge inside $S$.

It follows from Gauss' theorem that the electricity at rest on a conductor is confined to a very thin layer near its surface, and is not diffused throughout the volume. If $S$ is any surface described entirely on the body of the conductor, there is no electric force at any point of $S$, so that $\int \mathbf{E}\,d\mathbf{S} = 0$. Hence $S$ contains no charge, and since $S$ is quite arbitrary there can be no charge at any point definitely inside the conductor. This conclusion is strikingly exemplified by the experiment of Cavendish already mentioned.

It appears at first sight surprising that electricity can be spread into a thin superficial layer: but electrons are much smaller than the diameter of the atom and have plenty of room even in a very thin layer. The electric "fluid," in fact, must be regarded as almost indefinitely compressible on account of the smallness of the electrons.

A simple example of Gauss' theorem is to prove that *the electric force due to a charged sphere is the same, at all external points, as if the charge was collected at the centre of the sphere.* Let $\mathbf{E}$ be the electric force at an external point distant $r$ from the centre, and describe a sphere $S$ of radius $r$ concentric with the conductor. It is obvious, from symmetry, that $\mathbf{E}$ is directed along the radius and coincides with its normal component at points on $S$. Applying

Gauss' theorem to the surface $S$, we have $4\pi r^2 E = 4\pi e$, or $E = e/r^2$. The electric force is therefore that due to a charge $e$ concentrated at the centre.

The following result, though not important in electrostatics, throws light on a certain theoretical point (see next article). A sphere of radius $a$ contains electricity distributed uniformly throughout the volume with volume-density $\rho$. It is required to find the electric force at any point $P$. If $P$ is an external point distant $r$ from the centre $O$, Gauss' theorem gives as before

$$4\pi r^2 E = 4\pi \cdot \tfrac{4}{3}\pi\rho a^3,$$

or

$$E = 4\pi\rho a^3/3r^2.$$

For an internal point, the charge inside a sphere of radius $r$ is not $\frac{4}{3}\pi\rho a^3$, but $\frac{4}{3}\pi\rho r^3$. Hence $E = \frac{4}{3}\pi\rho r = er/a^3$, so that $\mathbf{E} = e\mathbf{r}/a^3$, where $\mathbf{r} = OP$. The electric force is parallel and proportional to $OP$. The potential at $P$ is $V = \text{const.} - er^2/2a^3$, and at the surface it takes the value $e/a$. Hence at internal points

$$V = \frac{3e}{2a} - \frac{er^2}{2a^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots (41).$$

**37. Laplace's equation and Poisson's equation.** Consider a distribution of electricity in which the volume-density $\rho$ is given at all points. Let $S$ be a small surface surrounding the point $P$. The flux of electric force outwards through $S$ is $4\pi\rho\tau$ from Gauss' theorem, $\tau$ being the enclosed volume. It is equal to $\tau$ div $\mathbf{E}$ from the theorem of flux; hence

$$\text{div}\,\mathbf{E} = 4\pi\rho \quad \ldots\ldots\ldots\ldots\ldots\ldots (42).$$

Since $\mathbf{E} = -\mathbf{D}V$, $\text{div}\,\mathbf{E} = -\Delta V$, and we have *Poisson's equation*

$$\Delta V = -4\pi\rho \quad \ldots\ldots\ldots\ldots\ldots\ldots (43).$$

In particular, if there is no electricity in the neighbourhood of the point $P$ which we are considering, $\rho = 0$ and

$$\Delta V = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (44).$$

This is known as *Laplace's equation.*

The subject can also be approached with advantage from the opposite point of view, in which Laplace's equation is general and Poisson's equation a modification caused by indefinitely near approach of electricity to the point $P$. The potential at $P$ due to

a charge $e$ at $O$ is the spherically symmetrical scalar function $e/r$, and equation (18) on p. 16 gives

$$\Delta V = e\left\{\frac{d^2}{dr^2}\left(\frac{1}{r}\right) + \frac{2}{r}\frac{d}{dr}\left(\frac{1}{r}\right)\right\} = 0.$$

The effect of superposing any number of charges at different points is to add the separate potentials, leading to Laplace's equation in general. But the proof would not apply without further consideration to charges indefinitely near to the point $P$. It is obvious, therefore, that the term $-4\pi\rho$ in Poisson's equation arises from the electricity in the immediate vicinity of the point considered. A special example will make this clear.

Let $S$ (Fig. 38) be a surface containing a distribution of electricity of uniform volume-density $\rho$, $O$ a point inside $S$. Draw a sphere $S_0$ of radius $a$ with $O$ as centre, and let $V_0$ be the potential at the point $P$ of the electricity inside $S_0$, $V_1$ that of the electricity between $S_0$ and $S$. Then the total potential at the point $P$ is $V = V_0 + V_1$. Since the electricity between $S_0$ and $S$ is nowhere infinitely near to $P$, $\Delta V_1 = 0$. The charge inside $S_0$ is $e = \frac{4}{3}\pi\rho a^3$, so that $V_0 = 2\pi\rho a^2 - \frac{2}{3}\pi\rho r^2$ from (41). Equation (18) now shows that $\Delta V_0 = -4\pi\rho$ as we should expect. It is $V_0$ that contributes the whole of $\Delta V$, notwithstanding that $V_0$ itself tends to zero with $a$.

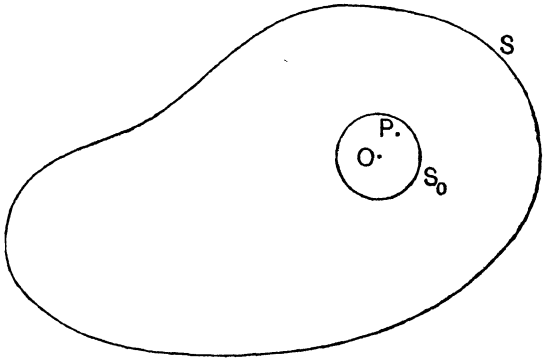


Fig. 38

It follows from Laplace's equation that the potential cannot be a maximum or a minimum at a point not occupied by electricity. For if $P$ is such a point, the mean value of $V$ on a small sphere with $P$ as centre is equal to $V_P$ to the second order (Art. 10), and hence $V$ must exceed $V_P$ at some points on the sphere and be less than $V_P$ at other points. The result is known as Earnshaw's theorem. It follows that a small charged body introduced into an electrostatic field cannot rest in stable equilibrium. An electro-

static field may possess points of no electric force, but we can always find small displacements from those positions which lead to instability. If electrostatic laws held everywhere, electrons could not remain at rest in separated positions within the atom.

**38. General principles of electrostatics.** The science of electrostatics, which seeks to determine theoretically the equilibrium of electricity on conductors, either isolated or under external influence, has been developed mathematically to a considerable extent. We can illustrate the principles on which the development is based by a special case. Consider an insulated sphere under the influence of a negative charge $e$ at a point outside it. We know that there will be positive charge induced on the part of the sphere nearest the charge $e$, and negative charge on the remote portions (Art. 31). The effect of the induced charges is to produce, inside the conductor, an electric field opposed to that of the charge $e$, and the equilibrium distribution is *that which exactly neutralises the impressed field at all internal points*. That is, knowing only that the charges reside on the surface, we have to determine the surface-density at every point so that the total potential due to all causes is constant inside the conductor. A problem in electrostatics is thus not immediately soluble, since the potential depends on the distribution of the charge and the latter is not given beforehand. The process of solution is essentially tentative, although general methods have been devised for certain classes of problems. One of these methods is described in Art. 46.

**39. Force just outside a conductor. Stress per unit area.** The electric force just outside a conductor is $4\pi\sigma$, where $\sigma$ is the surface-density in the neighbourhood, and is directed along the normal to the conductor.

Take an element of area $PQ$ on the conductor (Fig. 39), its linear dimensions being of the first order of small quantities. Let $R$, $S$ be two points whose distance from the element is of the second order of small quantities, $R$ being inside the conductor and $S$ in air. The calculation of the electric force at $R$, $S$ is conveniently divided into two parts, namely that of the charge on $PQ$ and that of the charge on the rest of the conductor and any other electricity

present. Evidently the latter is nearly the same for the points $R$, $S$, while the former is reversed. The element $PQ$ acts on both $R$ and $S$ like an infinite electrified plane, so that the electric force at $R$ due to $PQ$ is $2\pi\sigma$ directed along the inward normal. The total electric force at $R$ must vanish since $R$ is inside the conductor: thus the electric force due to the remaining charges is $2\pi\sigma$ along the outward normal. At the point $S$ the forces due to both causes are $2\pi\sigma$ along the outward normal, altogether $4\pi\sigma$ in that direction. This result is known as Coulomb's law. If the electric force at a point just outside a conductor placed in any electric field is known, the surface-density in the vicinity can also be found.

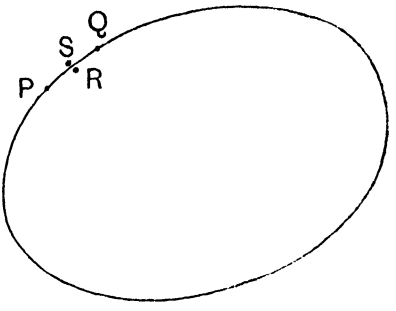


Fig. 39

If $A$ is the small area of the element $PQ$, the charge on the element is $\sigma A$. The electric force at points on $PQ$ due to the charges in the field other than that on $PQ$ itself is $2\pi\sigma$. Hence the charge on the element $PQ$ is acted on by a force $2\pi\sigma^2 A$ due to the remaining charges; i.e. the surface of a charged conductor in any field experiences a stress of magnitude $2\pi\sigma^2$ per unit area, directed at every point along the outward normal to the surface. If the distribution of charge is known we can calculate from this formula the forces acting on the conductors in the field.

**40. Lines and tubes of force.** The theory of these is similar to that in Art. 23. Recapitulating their properties in free space we have:

(1) The direction of a line of force is that of the resultant electric force at every point. Thus lines of force run from high to low potential.

(2) The cross-section of a small tube of force at any point of its length varies inversely as the resultant electric force.

The following theorem gives a relation between the tubes of force and the charges in the field:

(3) Lines of force start out from positive charges on conductors

and end on negative charges on conductors of lower potential. The amounts of electricity at the ends of a tube of force are equal and opposite.

The first part of this theorem follows from equation (37). To prove the second, let a small tube of force start from a place where the surface-density is $\sigma$ and end where it is $\sigma'$. Cut off the tube by surfaces parallel to the surface of the conductor and just outside it (Fig. 40), and consider the truncated tube so formed. The normal component of electric force on $dS$ is $-4\pi\sigma$ since the normal to the truncated tube is opposite to the normal to the conductor. Similarly at the end $dS'$. Thus, since the truncated tube encloses no charge Gauss' theorem gives

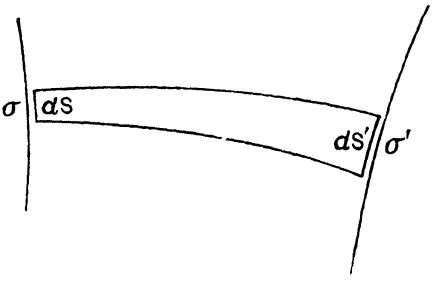


Fig. 40

$$4\pi\sigma\, dS + 4\pi\sigma' dS' = 0, \quad \text{or} \quad \sigma\, dS = -\sigma' dS'.$$

Moreover, the constant product of the electric force and cross-section of the tube at any point is equal to $4\pi$ times the charge from which the tube started out.

The result (3) follows for any finite tube by addition. Another easy proof would have been to produce the tubes into the conductors at both ends and then close the ends and apply Gauss' theorem.

Just as lines of magnetic force can be exhibited to the eye by means of iron filings, so lines of electric force due to two conductors can be shown by the tendency of small elongated bodies to set along the lines of force. Several methods have been devised for making such diagrams, but it is not very easy to obtain good effects. The following will be found fairly satisfactory.

A piece of stiff, smooth cardboard is stained black and well dried, and tinfoil is then pasted on one side to represent the conductors in the field. The conductors are joined to the terminals of a Wimshurst machine by spring clips, and the machine is rotated uniformly by means of an auxiliary motor. The place of the iron filings in magnetic charts is taken by small crystals of a substance which crystallises in prismatic form, for example oxalic

acid. These are sprinkled uniformly on the cardboard out of a pepper castor. As the crystals fall they set along the lines of electric force provided that the electric field is strong enough, and

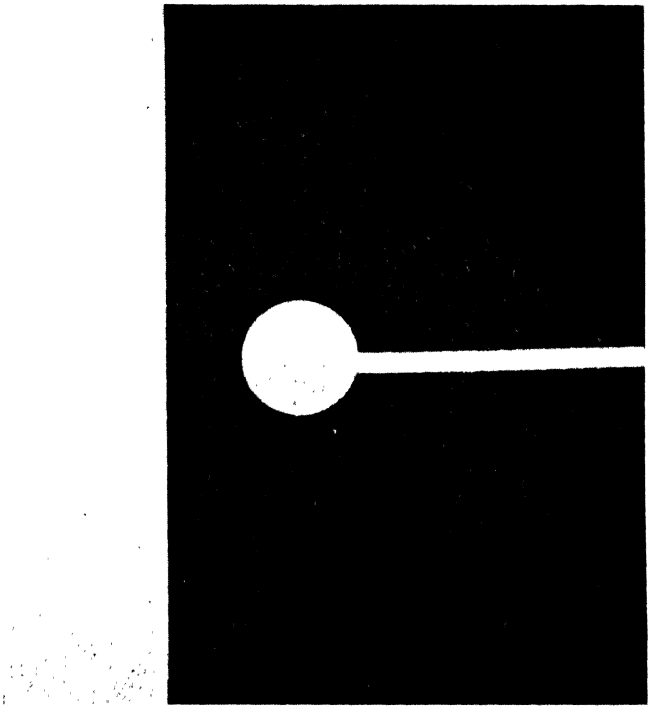


Fig. 41

the diagrams are then photographed. Fig. 41, showing the lines of force between a sphere and plane, has been obtained in this way. The appearance of the diagrams is improved by cutting out pieces of tinfoil of the right shape and pasting them over the negative so as to make the parts representing the conductors opaque.

**41. Capacity of a conductor.** If a charge $e$ is required to raise an isolated conductor to a potential $V$, a charge $ne$ will raise it to a potential $nV$. Thus the ratio $e/V$ depends only on the shape and size of the conductor, and not on its charge. Writing $e/V = C$, $C$ is called the *capacity* of the conductor.

Since the charge on a conductor charged to potential $V$ is $CV$, conductors of large capacity can receive considerable charges without having their potentials greatly raised. Such a conductor is analogous to a water-reservoir of large area, which can hold large quantities of water without having recourse to great depths.

For a sphere of radius $a$, $e/V = a$, so that the capacity of a sphere is equal to its radius. Thus a concrete unit of capacity is afforded by a sphere of radius one centimetre. Such a unit is called the *electrostatic unit of capacity*, but it is often convenient to say that a conductor has a capacity of so many centimetres.

Another capacity which can be calculated is that of a circular cylinder whose length $l$ is large compared with its radius $a$. The distribution of electricity on the cylinder will be nearly uniform in the middle, and only appreciably disturbed at distances from the ends comparable with the radius. Thus if $O$ is the middle point of the axis (Fig. 42) the potential at $O$ will be approximately that of a uniform layer of surface-density $\sigma$ over the curved surface, the electricity on the ends being neglected. The charge on the strip between planes distant $z$ and $z + dz$ from $O$ is $2\pi\sigma a\,dz$, giving rise to potential $2\pi\sigma a\,dz\,(a^2 + z^2)^{-\frac{1}{2}}$ at $O$. Hence the total potential at $O$, i.e. the potential of the cylinder, is given by

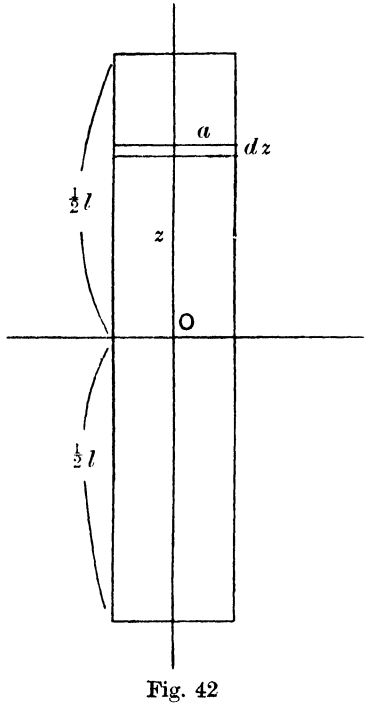


Fig. 42

$$V = \int_{-\frac{1}{2}l}^{\frac{1}{2}l} \frac{2\pi\sigma a\,dz}{(z^2 + a^2)^{\frac{1}{2}}} = 4\pi\sigma a \sinh^{-1}\frac{l}{2a} = 4\pi\sigma a \log_e \left\{\frac{l}{2a} + \left(1 + \frac{l^2}{4a^2}\right)^{\frac{1}{2}}\right\}$$

$$= 4\pi\sigma a \log_e \frac{l}{a} \text{ approximately.}$$

Also the charge on the cylinder is approximately equal to $2\pi\sigma la$. Hence

$$C = \frac{e}{V} = \frac{l}{2\log_e (l/a)} = \frac{0{\cdot}217l}{\log_{10} (l/a)}.$$

For example, an isolated wire 1 metre long and 2 mm. in diameter has a capacity of 7·2 cm. When the wire is joined to other pieces of apparatus, however, the distribution of charge is altered and the total charge on the wire is not given by the above formula.

**42. Electrical screening.** If an insulated electroscope is surrounded by a wire-gauze cage connected to earth, no movement of the leaves can be produced by the most powerful electric charges outside the cage. The screening effect depends on the

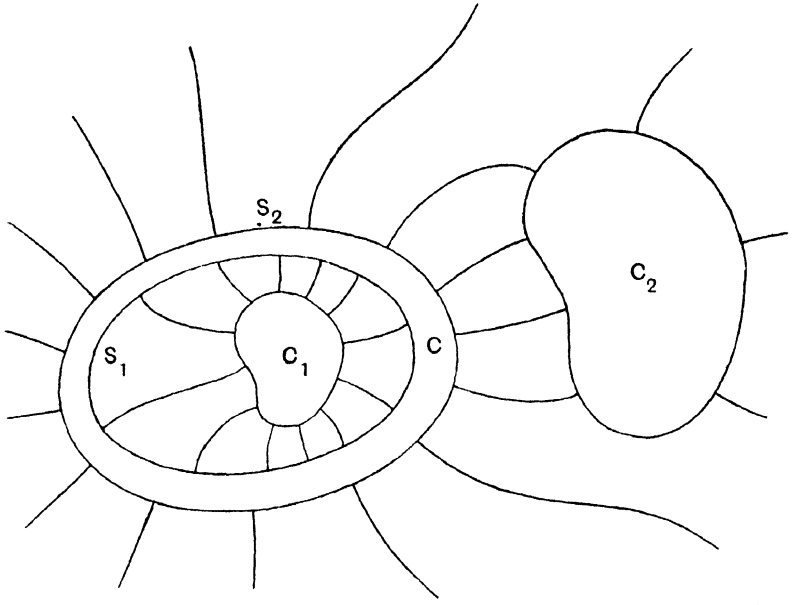



Fig. 43

fact that the cage is very nearly a closed conductor surrounding the electroscope. It can be shown that the screening from outside action is complete in the case of one conductor entirely surrounding another.

Of three conductors $C_1$, $C$, $C_2$, insulated from one another (Fig. 43), let $C$ completely enclose $C_1$ and be kept at a fixed potential $V$. The figure shows the conductors, with a sketch of the lines of force passing between them: $S_1$ is the inner surface of $C$ and $S_2$ the outer. First imagine that $S_2$ is removed to infinity, $C_2$ being absent, so that $C$ becomes an infinite conductor with a cavity

$S_1$. There will be a certain distribution of electricity on $C_1$ and $S_1$ corresponding to a given charge on $C_1$ when $C$ is at zero potential. Secondly, suppose that the whole of the interior of $C$ is filled up with metal. Then there will be a second distribution of electricity on $C_2$ and $S_2$ corresponding to a given charge on $C_2$ when $C$ is maintained at potential $V$. Let the two distributions be now superposed. The first distribution gives zero potential everywhere outside $S_1$, while the second gives potential $V$ everywhere inside $S_2$. Hence in the combined distribution we have potential $V$ in the space between $S_1$ and $S_2$, and also constant potential over $C_1$ and $C_2$. The combined distribution is therefore the actual distribution. But since the two distributions are independent of one another, the actual distribution on $C_1$ and $S_1$ is not affected when $C_2$ is moved about, or when the charge of $C_2$ is altered, nor is the potential at any point inside $C$ changed in any way. Similarly $C_1$ has no effect on the electrostatic phenomena outside the conductor $C$.

This theory explains a celebrated experiment of Faraday, which consists in fixing a long metal cylinder to the knob of an electroscope and lowering small charged bodies into it by means of a silk fibre. A charged insulated body makes the leaves diverge, since the body induces an equal and opposite charge on the walls of the cylinder and repels a charge of its own sign on to the electroscope. It is found that the divergence of the leaves is independent of the position of the body inside, provided the body is fairly inside the cylinder, and that it does not alter even if the body touches the wall of the cylinder.

A second experiment of the same kind is to verify that equal and opposite charges are produced by friction, as required by the electrical theory of matter. Let a silk handkerchief be tucked into the cylinder, and a glass rod then thrust inside and turned round vigorously. No divergence of the leaves of the electroscope is observed, so that the total charge generated is zero. The sensitiveness of the experiment may be tested by withdrawing the glass rod: the negative charge on the silk then causes a large divergence to take place.

The principle of screening is constantly used in the construction of electrical instruments, in the special form that, since a closed

conductor screens perfectly, a nearly closed one is nearly perfect. Wires leading to electrometers are screened from outside influence by passing them down the centre of earthed brass tubes fitted with ebonite plugs, the instrument itself being screened by means of a brass case connected to earth. The electroscope in Fig. 31 is screened to some extent by its case.

**43. Condensers.** While it is easy to measure a difference of potential accurately, the direct measurement of electric charge presents great difficulties. It is thus advantageous to make the measurement of charge depend on that of potential; and this might be done with the aid of conductors of known form and capacity, the charge on which would be known in terms of their potentials. There are two difficulties in the way, namely that to obtain a large enough capacity large conductors would have to be used, and that isolated conductors are considerably affected by external influence. These difficulties are obviated by the proper use of *condensers*.

A condenser consists essentially of two conductors, all or nearly all the lines of force from one of which (called the *inner plate*) end on the other. Evidently one way of securing this is to have the latter conductor enclosing the former almost entirely, an arrangement which also protects the inside plate from outside influence. The *capacity* of the condenser is the ratio of the charge on the inner plate to the difference of potential between the two plates.

Condensers are used extensively in electrostatic measurements and in wireless telegraphy. If the potential-differences are of the order of 100 electrostatic units (30,000 volts) the insulation of the air will give way: in this case it may be necessary to employ compressed air between the plates. At these potentials insulation generally is difficult.

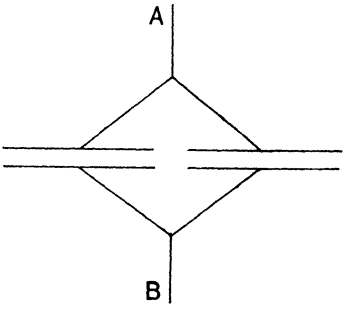


Fig. 44

We can obtain a condenser whose capacity is the sum of two given condensers by joining them *in parallel* (Fig. 44). This is done by joining

the outside plates together with a wire, and also the inside plates, as shown symbolically in the figure. Let $C$, $C'$ be the capacities of the two condensers, $V$ the difference of potential between the "terminals" $A$, $B$ when the condensers are in parallel. Then the charge on the inner plate of the first condenser is $CV$, and on the second $C'V$. In the compound condenser the total charge is $(C + C')\ V$, so that the capacity is $C + C'$.

A large capacity is conveniently built up by taking a number of smaller condensers and joining them in parallel, when we get a capacity which is the sum of those of the separate condensers. It is also possible to divide a large difference of potential between several condensers by joining the inner plate of each condenser to the outer plate of the next.

**44. Parallel plate and cylindrical condensers. Guard-rings.** Two parallel plates brought close together form a condenser, and then it is immaterial which plate we regard as the inner one. Suppose that we have two plates $C_1$, $C_2$, each of area $A$, so great in comparison with the distance $d$ apart that the plates may be

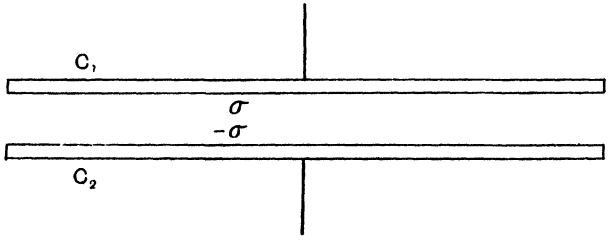



Fig. 45

considered as infinite. The lines of force then run practically straight across between the plates, and the surface-density is uniform except near the edges. Further, if the plates are joined at the back to other apparatus, nearly all the electricity lies on the opposing surfaces of the plates and there is very little at the back. If $\pm\ \sigma$ are the surface-densities, the charge on the positive plate is $\sigma A$. Each plate gives rise to an electric force of magnitude $2\pi\sigma$, and it is easy to see that the separate forces assist one another between the plates and are in opposite directions everywhere else. Hence there is an electric field of strength $4\pi\sigma$ between the plates,

and no force at the back. The difference of potential between the plates is $4\pi\sigma d$, so that the capacity is

$$C = \frac{A}{4\pi d} \text{ centimetres} \quad \ldots\ldots\ldots\ldots (45).$$

Thus if $d$ is made small the capacity can be made large without the use of large plates. Parallel plate condensers are sometimes constructed with three plates, the two outer ones being joined together to a single terminal. The capacity is then $A/2\pi d$, and the combination screens the inner plate.

Fig. 46 shows a diagram of the lines of force between the plates of a condenser, prepared by the method previously described. It

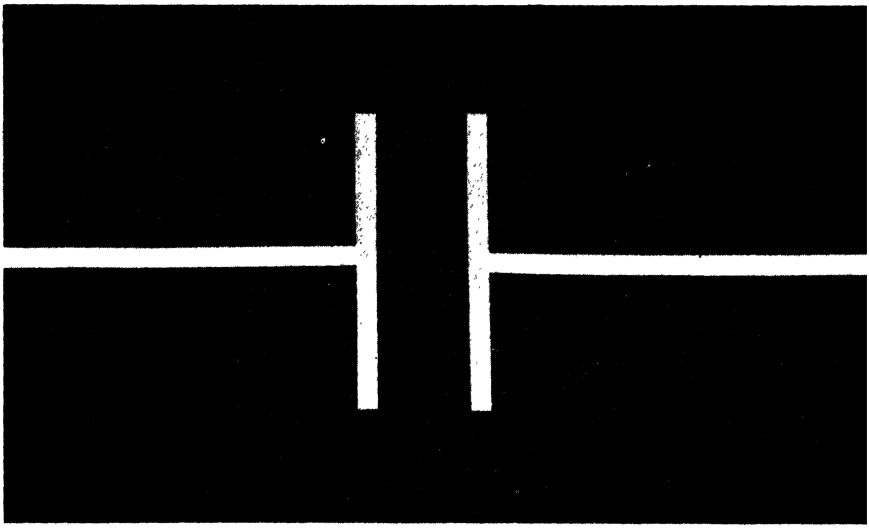


Fig. 46

will be noticed that the lines of force pass straight across between the plates even when the latter are quite small, but that they spread out near the edges. The irregular arrangement of the crystals at the back of the plates shows that there is very little electric force there.

The spreading out of the lines of force near the edges makes the capacity of an actual parallel plate condenser somewhat greater than that calculated from equation (45). The *guard-ring* is a device due to Lord Kelvin for avoiding this uncertainty. The top plate is composite, consisting of a circular disc $C_1$ (Fig. 47) surrounded by

a concentric ring $C_3$ separated from it by a narrow gap. When $C_1$ and $C_3$ are at the same potential the lines of force from $C_2$ pass very nearly straight across to $C_1$, and the charge on the inner plate $C_1$ is $AV/4\pi d$, where $A$ is the area of $C_1$ and $V$ the difference of

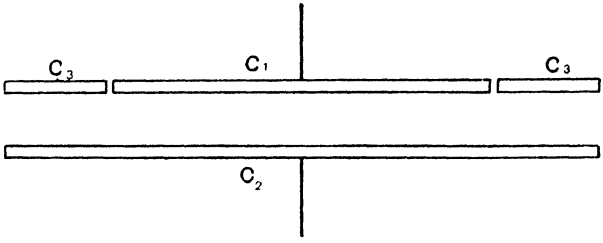



Fig. 47

potential between $C_1$ and $C_2$. In using such a condenser the plates $C_1$, $C_3$ must be at the same potential at the time of making the experiment, otherwise there will still be an error due to the edges.

Another form of condenser consists of two coaxial cylinders $C_1$, $C_2$ (Fig. 48), whose length is large compared with the radii. The "plates" $C_1$, $C_2$ are of brass tubing. Let $a$ be the *outside* radius of $C_1$, $b$ the *inside* radius of $C_2$, $l$ the length, and let $E$ be the electric force at distance $r$ from the centre, where $a < r < b$. Imagine a cylinder drawn with radius $r$ and closed by two planes perpendicular to the axis and distant one centimetre apart. In applying Gauss' theorem to this cylinder we get no contribution to $\int \mathbf{E}\,d\mathbf{S}$ from the ends, since the electric force is everywhere radial. On the curved surface $\mathbf{E}$ and $d\mathbf{S}$ are parallel, so that if $q$ is the charge per unit length of the inner tube, $2\pi rE = 4\pi q$, or $E = 2q/r$. The difference of potential between the plates is therefore $V = 2q \log (b/a)$, and the charge on the inner plate approximately $e = lq$. Hence the capacity of the condenser is

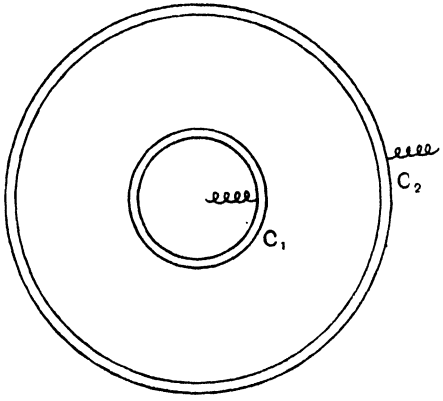



Fig. 48

$$C = \frac{l}{2 \log_e (b/a)} = \frac{0{\cdot}217l}{\log_{10} (b/a)} \text{ centimetres} \ldots\ldots (46).$$

The total charge on the outer cylinder is not necessarily equal to that on the inner, as there may be charges present on the outermost surface of all.

Fig. 49 shows a convenient form of cylindrical condenser for laboratory work. Here $C_1$ is the inner tube, $C_2$ the outer tube, and $C_3C_3$ are short cylinders separated from $C_1$ by narrow air-gaps.

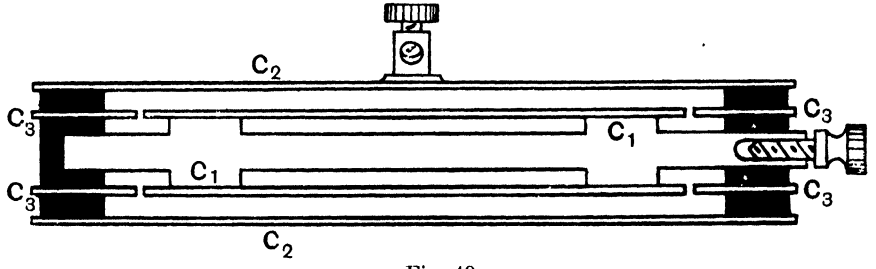



Fig. 49

When $C_1$ and $C_3$ are at the same potential the lines of force pass nearly straight across from $C_2$ to $C_1$, and we can make use of equation (46) to calculate the charge on $C_1$. In use the end rings $C_3C_3$ are usually earthed, and observations are made at a time when the inner plate is at zero potential (cf. Art. 74).

**45. Theory of a system of conductors.** The condenser is a particular arrangement of two, or with a guard-ring three, conductors. We have now to consider the principles involved when there are $n$ conductors at potentials $V_1, V_2, \ldots V_n$, the total charges being $e_1, e_2, \ldots e_n$. Since the potential at any point of a field only involves the charges to the first degree, there must be a set of linear equations connecting $e_1, e_2, \ldots$ with $V_1, V_2, \ldots$. Hence

$$e_1 = q_{11}V_1 + q_{12}V_2 + \ldots + q_{1n}V_n,$$
$$e_2 = q_{21}V_1 + q_{22}V_2 + \ldots + q_{2n}V_n,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$e_n = q_{n1}V_1 + q_{n2}V_2 + \ldots + q_{nn}V_n.$$

Here $q_{11}$ is the total charge on the first conductor when its potential is unity and that of all the other conductors zero; and then $q_{1r}$ is the charge on the $r$th conductor, and so on. The coefficients $q_{rr}$ are called *coefficients of capacity*, and $q_{rs}$, with $r$ and $s$ unequal, *coefficients of influence*. The coefficients have several properties, the most important being the reciprocal relation $q_{rs} = q_{sr}$. Gauss'

reciprocal theorem $\Sigma eV' = \Sigma e'V$ (Art. 29) can be extended at once to the case in which $e$ is the total charge on a conductor and $V$ the corresponding potential. Taking as the two distributions those in which

(1) all the $V$'s vanish except $V_r$,
(2) all the $V'$'s vanish except $V_s'$;

we have $e_s V_s' = e_r' V_r$. But by definition we have

$$e_s = q_{sr} V_r \text{ and } e_r' = q_{rs} V_s'.$$

Hence

$$q_{rs} = q_{sr}.$$

The conductors, when charged, act on one another with forces derived from a potential energy $W$, which we call the energy of the system. If the potentials had been $kV_1, kV_2, \ldots kV_n$ instead of $V_1, V_2, \ldots V_n$, the charges would have been $ke_1, ke_2, \ldots ke_n$. A small extra charge $e_1 dk$ can be brought up and placed on conductor 1 by expending work $e_1 dk \,.\, kV_1$, and similarly for the others, so that the work done against the mutual attractions and repulsions in a small increment $dk$ of $k$ is $(e_1 V_1 + e_2 V_2 + \ldots + e_n V_n)\, k dk$. The total work done in assembling the system is found by integrating from $k = 0$ to $k = 1$, so that

$$\begin{aligned} W &= \tfrac{1}{2} (e_1 V_1 + e_2 V_2 + \ldots + e_n V_n) \\ &= \tfrac{1}{2} (q_{11} V_1^2 + q_{22} V_2^2 + \ldots + 2q_{12} V_1 V_2 + \ldots), \end{aligned}$$

all possible squares and products being included in the sum.

The coefficients $q$ depend on the form and position of the conductors, and are only known in a few simple cases, since the electrostatic problem of the distribution of electricity over the surfaces has to be solved before they can be calculated. In particular, the coefficient $q_{rr}$ is not the same as the capacity of the $r$th conductor when it alone is present in the field. We can illustrate this theory from Fig. 49. The general equations are

$$\begin{aligned} e_1 &= q_{11} V_1 + q_{12} V_2 + q_{13} V_3, \\ e_2 &= q_{21} V_1 + q_{22} V_2 + q_{23} V_3, \\ e_3 &= q_{31} V_1 + q_{32} V_2 + q_{33} V_3. \end{aligned}$$

Take $V_1 = V_3 = 0$, $V_2 = 1$. Then $e_1 = q_{12}$, $e_2 = q_{22}$, $e_3 = q_{32}$. Since $C_3$ acts as a guard-ring to $C_1$ we have, from the last article,

$$e_1 = -\frac{l}{2 \log_e (b/a)}.$$

Hence $$q_{12} = -\frac{l}{2\log_e (b/a)}.$$

The other coefficients $q_{11}, q_{22}, q_{33}, q_{23}, q_{31}$ are virtually unknown.

If $C_3$ is at zero potential, but $C_1$ at potential $V_1$, we should have $e_1 = q_{11}V_1 + q_{12}V_2$. Thus the charge on $C_1$ would be incorrectly calculated from the formula $e_1 = \frac{l}{2\log_e (b/a)}(V_1 - V_2)$ unless $C_1$ and $C_3$ were at the same potential, and the error could not be estimated precisely, since $q_{11}$ is unknown. This illustrates the improper use of guard-rings.

**46. Two-dimensional problems in electrostatics.** The distribution of electricity on infinitely long parallel cylindrical conductors can be found in many cases by the method of *conjugate functions*. Taking the axis of $z$ parallel to the generators of the cylinders, the problem reduces to finding a solution of the equation $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 = 0$ which shall make $V$ constant over two given curves, namely the cross-sections of the cylinders, e.g. by the plane $z = 0$. Real quantities $U$ and $V$ such that $U + iV$ is a function of $x + iy$ only are said to be conjugate functions of $x$ and $y$. It is well known that in this case

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \quad \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y},$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

Hence the curves $U = \text{const.}$, $V = \text{const.}$ cut at right angles everywhere, and $V$ may be taken as the potential in some electrostatic problem. The curves $V = \text{const.}$ are the equipotential curves, or sections of the equipotential surfaces, and $U = \text{const.}$ the lines of force.

By making different assumptions as to the form of the function we obtain the solution of various problems. If

$$U + iV = -i \log (x + iy),$$

$$U = \theta \text{ and } V = -\log r$$

in polar coordinates. The lines of force are radii, the equipotential curves concentric circles. We have therefore the field of a single charged cylinder, or of a line uniformly electrified.

Two equal and opposite electrified lines, at the points $A\ (-c, 0)$ and $B\ (c, 0)$ respectively, make

$$U + iV = -i \log \frac{x + iy + c}{x + iy - c},$$

or

$$U = \theta_1 - \theta_2, \quad V = \log \frac{r_2}{r_1},$$

where $r_1$, $\theta_1$ are polar coordinates of the point $P\ (x, y)$ with $A$ as pole and $r_2$, $\theta_2$ its polar coordinates with $B$ as pole. Since the angle $BPA$ is $\theta_2 - \theta_1 = -U$, the lines of force are circles through $A$ and $B$, and the equipotential curves the coaxial circles with $A$ and $B$ as limiting points. In analytical form

$$V = -\tfrac{1}{2} \log \frac{(x + c)^2 + y^2}{(x - c)^2 + y^2},$$

so that the equipotential $V$ is the circle

$$x^2 + y^2 + 2xc \coth V + c^2 = 0 \quad \ldots\ldots\ldots(47),$$

whose centre is the point $(-c \coth V, 0)$ and whose radius is $c$ cosech $V$.

Let it be required to determine the distribution of electricity on two parallel conducting circular cylinders of radius $a$ whose axes

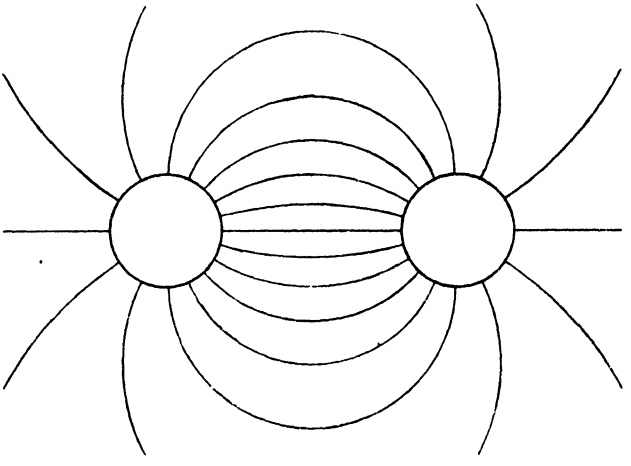


Fig. 50

are at distance $2h$ apart, the cylinders carrying equal and opposite charges. Taking as origin a point half-way between their axes, we can identify the cross-sections with the equipotential $V = \pm V_0$ by writing

$$h = c \coth V_0, \quad c \operatorname{cosech} V_0.$$

Hence $$c = (h^2 - a^2)^{\frac{1}{2}}, \quad \cosh V_0 = h/a \quad \ldots\ldots\ldots(48).$$

The limiting points $(\pm c, 0)$ being known, we can draw the lines of force at once. Fig. 50 shows a set corresponding to equidistant values of $U$, with $h = 1{\cdot}73$ cm. and $a = 0{\cdot}66$ cm. The pair of cylinders, like any other system of two conductors, is a condenser when the charges are equal and opposite, as all the lines of force from one cylinder end on the other. In order to find its capacity we have only to find the charge per unit length of either conductor. This is determined by the following general theorem.

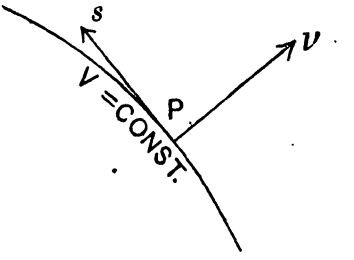


Fig. 51

Let $\nu$ (Fig. 51) be the normal to an equipotential, $s$ the direction of the tangent, $E$ the resultant electric force at $P$. The direction-cosines of $s$ are easily seen to be $\left(-\frac{1}{E}\frac{\partial V}{\partial y}, \frac{1}{E}\frac{\partial V}{\partial x}\right)$, so that

$$\left(\frac{\partial U}{\partial s}\right)_{V \text{ const.}} = -\frac{1}{E}\frac{\partial V}{\partial y}\frac{\partial U}{\partial x} + \frac{1}{E}\frac{\partial V}{\partial x}\frac{\partial U}{\partial y}$$

$$= -\frac{1}{E}\left\{\left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right\} = -\frac{E^2}{E} = -E.$$

If the equipotential considered is the surface of a conductor, $E = 4\pi\sigma$. Hence, disregarding signs, we have

$$\sigma = \frac{1}{4\pi}\frac{\partial U}{\partial s}.$$

The charge per unit length of the conductor between the generators through the points $P$, $Q$ is therefore given by

$$e = \int_P^Q \sigma ds = \frac{U_Q - U_P}{4\pi} \quad \ldots\ldots\ldots\ldots(49).$$

Hence if $U$ is found in terms of the position of a point on the cross-section, the charge between any two points is known. Since the lines of force in Fig. 50 are drawn for equidistant values of $U$ they divide the conductors into regions carrying equal charges, thus showing graphically the greater concentration of charge on the parts of the cylinders which are nearest together. $U$ increases by $2\pi$ in going once round one of the cross-sections, so that the total charge per unit length of either cylinder in the problem just

treated is $\frac{1}{2}$ from equation (49). Since the difference of potential between the cylinders is $2V_0$, the capacity per unit length is

$$\frac{1}{4V_0} = \frac{1}{4\cosh^{-1}(h/a)}.$$

This formula will be practically applicable to two finite cylinders whose length is great in comparison with the distance apart of their axes. Writing $l$ for the former and $d = 2h$ for the latter, the capacity is

$$C = \frac{l}{4\cosh^{-1}\left(\frac{d}{2a}\right)} = \frac{0{\cdot}109l}{\log_{10}\left\{\frac{d}{2a} + \left(\frac{d^2}{4a^2} - 1\right)^{\frac{1}{2}}\right\}} \text{ cm.} \quad \ldots(50).$$

For example, if we take a length of wire of 2 mm. diameter, covered with cotton insulation $\frac{1}{2}$ mm. thick, cut it in two and lay the halves alongside one another so as to touch everywhere, we have a condenser whose capacity is 13 cm. per metre of wire used. The capacity of long coils of wire wound back on themselves is often considerable*. In case the wires are at a great distance apart compared with their diameters, equation (50) becomes

$$C = \frac{0{\cdot}109l}{\log_{10}(d/a)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(51).$$

**47. Sensitive electroscopes.** Electroscopes are very sensitive to charge on account of the small capacity of the gold leaf, by virtue of which a small charge produces considerable difference of potential between the leaves and the case. They are not, however, usually sensitive to potential itself. This sensitiveness can be obtained by improved design and by reading the deflexion of the leaves with a microscope. C. T. R. Wilson designed a very delicate electroscope, the essential parts of which are shown in Fig. 52. The external apparatus is connected to a mercury cup $C$, leading down a quartz tube to the gold leaf $L$. An insulated plate $P$ is placed opposite $L$, so that its distance from $L$ can be adjusted by means of a screw. The whole is surrounded by a metal case connected to earth, a small window being left for microscopical observation of the gold leaf. The case is mounted on a stand so that it can be tilted at various angles to the horizon. In use the

* The capacity of a 500 ohm coil in a resistance box may amount to 100 cm., i.e. if cut in two the coil would form a condenser of this capacity.

plate $P$ is maintained at a constant potential of about 200 volts ($\frac{2}{3}$ electrostatic unit) from the terminal $T$. When the gold leaf is connected to earth it is attracted more or less to the plate, and on having its potential raised by a small amount the leaf moves to the right or left according as its potential is of the same or opposite sign to that of the plate $P$, the object of tilting being to diminish

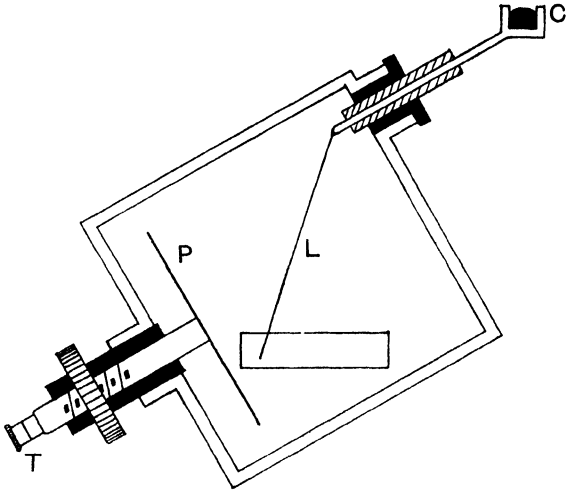



Fig. 52

the effect of gravity on the gold leaf. If the angle made by the plate with the horizon is small, or if the potential of the plate is large, the equilibrium of the leaf may even be unstable, the greatest sensitiveness being obtained with the least margin of stability.

In the *string electroscope* two small plates are set up parallel to one another and a fine quartz fibre is stretched between them on insulating supports. The fibre is silvered to make it conduct, and maintained at a high potential by a battery of small cells. If the two plates are at different potentials the fibre is pulled aside and the deflexion is read off with a microscope focussed on the fibre. This instrument is quick in action and useful for observing transient phenomena, such as the entry of radioactive $\alpha$ particles into a metallic vessel (Art. 248).

**48. Electrometers.** An electrometer is an instrument for the accurate measurement of differences of potential. Some electro-

scopes fulfil this condition, but it is usual to reserve the name electrometer for instruments of one of the two following types.

Kelvin's *absolute electrometer* measures the pull between the plates of a parallel plate condenser. If $A$ is the area of a plate fitted with a guard-ring, the thrust exerted on it is $2\pi\sigma^2 A$, where $\sigma$ is the surface-density. If $d$ is the distance between the plates and $V_1$, $V_2$ their potentials, then $V_1 - V_2 = 4\pi\sigma d$. Thus the thrust on the plate is

$$\frac{A\,(V_1 - V_2)^2}{8\pi d^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (52).$$

It is clearly immaterial which plate is taken as positive.

Fabry and Perot designed a very sensitive absolute electrometer, in which the plates are separated by three springs, variations of the distance being observed by interference fringes. Their electrometer was capable of measuring the difference of potential between the terminals of an ordinary electric battery, that is, differences of potential of the order of 1/200th of an electrostatic unit. The ordinary form is only suited for high potentials of the order of 10 electrostatic units.

Absolute electrometers are not suited for ordinary laboratory work, for which the best type is the *quadrant electrometer*, also due to Lord Kelvin, and modified by Dolezalek (Fig. 53). The moveable part of the instrument is a so-called "needle" made of paper covered with tinfoil and cut out in order to decrease the weight. This is suspended from the central rod of the instrument by a fibre of quartz, which is very strong and true in its elastic properties. If the fibre is not silvered it can be made conducting by dipping it every now and then in a conducting liquid, for example a solution of calcium chloride. The needle is charged up with a battery of small accumulators. The quadrants are four insulated pieces of brass so shaped that they fit close together and form a box nearly enclosing the needle. A light mirror is fixed just above the needle so as to show the angle turned through by reflexion of the light from a lamp on to a graduated scale. The electrometer is enclosed in a brass case connected to earth to guard against outside influence, and pierced only with small holes for the terminals and a window for use with the lamp and scale.

Fig. 54 gives a top view of the needle, the upper half of the

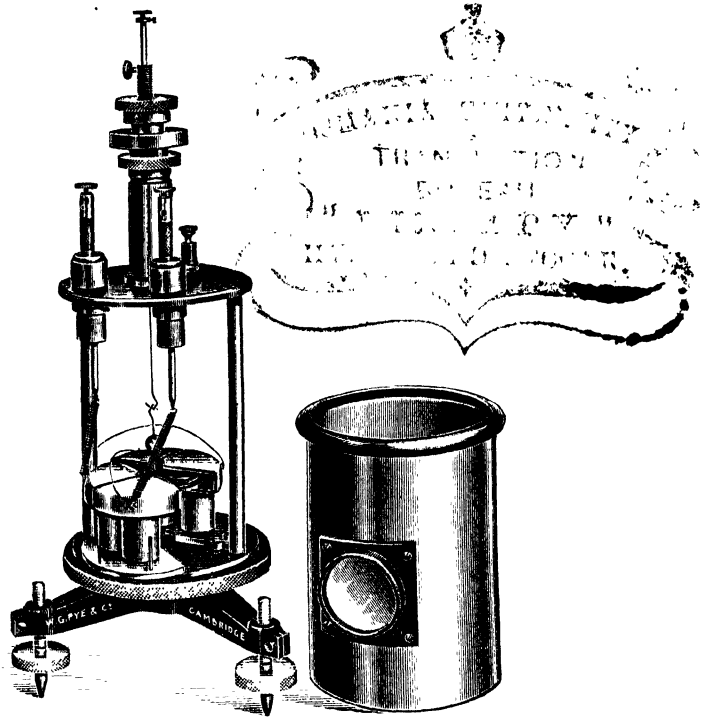



Fig. 53

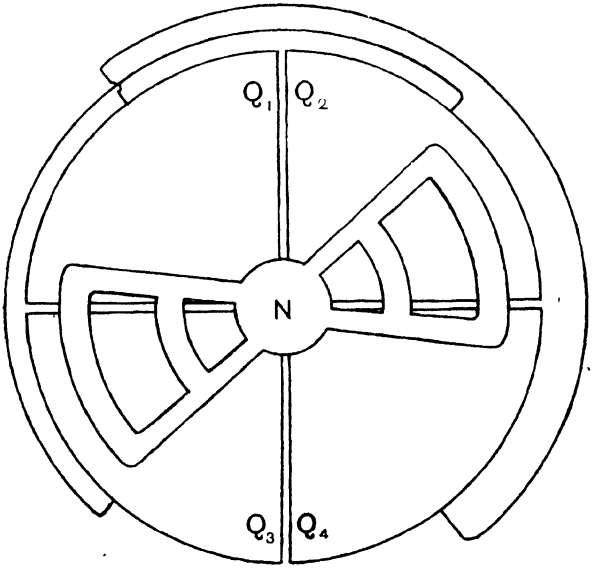



Fig. 54

quadrants being removed in order to show it. The quadrants $Q_1Q_4$, $Q_2Q_3$ are joined together, and each pair connected to a terminal of the instrument.

In the ordinary (or *heterostatic*) method of using an electrometer one pair of quadrants is earthed, and the needle is put at a potential of the order of 50 volts ($\frac{1}{6}$th of an electrostatic unit). An electrometer is a complicated system of three conductors (neglecting the case), and no simple formula will express its behaviour in all conditions. It may, however, be treated plausibly as follows. Let the pair of quadrants $Q_1Q_4$ be at potential zero, the pair $Q_2Q_3$ at potential $v$, and the needle at a potential $V$ large compared with $v$. These conductors are taken as the conductors 1, 2, 3 in the theory of Art. 45, so that the electrical potential energy is

$$\tfrac{1}{2}(q_{22}v^2 + q_{33}V^2 + 2q_{23}vV),$$

where $q_{22}$, $q_{33}$ and $q_{23}$ depend on the angular deflexion $\theta$ of the needle from its normal position of rest. The couple tending to increase $\theta$ is therefore

$$-\frac{1}{2}\left(\frac{dq_{22}}{d\theta}v^2 + \frac{dq_{33}}{d\theta}V^2 + 2\frac{dq_{23}}{d\theta}vV\right).$$

Retaining only two terms in the expansion of $dq_{22}/d\theta$, ... in powers of $\theta$, the electrical couple for small angular deflexions becomes

$$(\alpha + \beta\theta)\,v^2 + (\gamma + \delta\theta)\,V^2 + (\epsilon + \zeta\theta)\,vV,$$

where $\alpha$, $\beta$, ... are constants. The fibre exerts a couple $k\theta$ in the opposite direction, so that equilibrium occurs when

$$(\alpha + \beta\theta)\,v^2 + (\gamma + \delta\theta)\,V^2 + (\epsilon + \zeta\theta)\,vV = k\theta,$$

or

$$\theta = \frac{\alpha v^2 + \gamma V^2 + \epsilon vV}{k - \beta v^2 - \delta V^2 - \zeta vV} \quad \ldots\ldots\ldots\ldots (53).$$

Suppose that the electrometer is set up so that the needle lies normally as much over $Q_1$ as $Q_3$, and as much over $Q_2$ as $Q_4$. $\theta$ will then vanish when $v = 0$, for all values of $V$. Hence $\gamma = 0$. Of the remaining terms $\epsilon vV$ may be expected to predominate over $\alpha v^2$, and $\delta V^2$ over $\beta v^2$ and $\zeta vV$. Hence we should have approximately

$$\theta = \frac{\epsilon vV}{k - \delta V^2},$$

or

$$\theta = \frac{Vv}{A + BV^2} \quad \ldots\ldots\ldots\ldots (54),$$

where $A = k/\epsilon$ and $B = -\delta/\epsilon$. The angular deflexion $\theta$ does not exceed 7° in practice, so that no great error is committed in regarding $\theta$ as the deflexion of the spot of light on the scale.

Experiment shows that the deflexion is nearly proportional to $v$ for an assigned moderately high potential $V$ (Art. 65), but the sensitiveness $\theta/v$ increases with $V$ up to a certain point and afterwards decreases. The law of variation is approximately that of formula (54), with positive values of $A$ and $B$. The maximum sensitiveness of an electrometer was discovered by J. Hopkinson in 1885, and explained by Gouy as above. The electrical couple at angle $\theta$, whether the needle is in equilibrium or not, is approximately $\delta\theta V^2 + \epsilon v V$, or

$$\frac{k}{A}(vV - BV^2\theta).$$

If the quadrants are connected together ($v = 0$), we have therefore a couple $-kBV^2\theta/A$ which assists the restoring couple $-k\theta$, giving in all a couple proportional to $(A + BV^2)\,\theta$. The time of a small swing of the needle should therefore be proportional to $(A+BV^2)^{-\frac{1}{2}}$. This law was verified by Gouy over a large range.

The couple $BV^2$, which is inimical to sensitiveness, can be reduced by making the air-gap between the quadrants very small. In the limiting case of an infinitely narrow gap, the quadrants enclose the needle entirely, and the charge on the needle is independent of $\theta$ when $v = 0$. Hence $q_{33}$ is independent of $\theta$, and $B = 0$. Experiment shows that the potential of the needle for maximum sensitiveness, which is theoretically $(A/B)^{\frac{1}{2}}$, is less for large air-gaps than for small air-gaps, and may be as low as 40 volts.

A quadrant electrometer is more sensitive to potential than any electroscope. Ordinary patterns will detect a difference of potential of about 1/100 volt (1/30,000 of an electrostatic unit), so that small charges can be measured in spite of its considerable capacity (about 50 cm.; see Art. 74). Metal suspensions are often used when the greatest sensitiveness is not wanted.

F. A. and A. F. Lindemann and T. C. Keeley designed a small quadrant electrometer in which a light needle is suspended between two fibres of silica, and its motion observed with a microscope. A sensitiveness of 450 scale divisions per volt can be obtained, with

an electrostatic capacity of 2 cm. String electroscopes are also sensitive, and have small capacity.

**49. Contact potential. Piezo-electricity.** Fig. 55 represents diagrammatically a parallel-plate condenser with one plate connected to an electrometer and the other to earth. The other pair of quadrants is earthed as usual, while $A$ is earthed to begin with and then insulated. Movement of $B$ towards $A$ does not affect the electrometer as long as the plates are of the same metal. With *different* metals, however, if we start with the plates close together and draw them quickly apart, the spot of light moves away, returning when the plates are moved back again. Since this is exactly what would happen if $AB$ was a charged condenser, we are led to suppose that two plates of different metals take up different potentials when joined to a common conductor. There is, in fact, a definite difference of potential for every pair of metals, since all motion of the spot of light can be prevented by applying a suitable potential to $B$ (Art. 74). The natural difference of potential of two metals is called their *contact potential-difference*, defined more accurately as the difference of potential at a point just outside $A$ and a point just outside $B$, when $A$ and $B$ are joined together at the back.

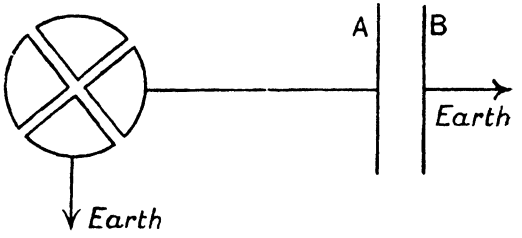


Fig. 55

Contact potentials are additive, i.e. the contact potential of two metals $A$, $C$ is the sum of the contact potentials of $A$, $B$ and of $B$, $C$, where $B$ is any other metal. Contact potentials can be determined with an electrometer whose quadrants are of different metals, as was done originally by Lord Kelvin.

The property of insulators which corresponds most nearly to the contact potential of metals is their behaviour during friction. The electroscope or quadrant electrometer will detect the smallest charges produced by friction. It is found that while a silk handkerchief will acquire a negative charge by friction against glass, it

is positively charged when rubbed against metals or ebonite. More generally, substances can be arranged in a series such that each when rubbed by the next acquires positive electricity, the following being the general order: Catskin, glass, wool, silk, metals, sealing-wax, ebonite, sulphur.

The effect is additive, i.e. any member of the series is positively electrified by friction with one later, and negatively electrified by friction with one earlier than itself.

J. and P. Curie discovered in 1880 that crystals devoid of a centre of symmetry, such as tourmaline and quartz, become electrified on the application of stress. This phenomenon is called piezo-electricity, and consists in the production of a polarisation analogous to that of dielectrics, the direction depending on the external forces applied. Quartz has been studied most closely. No polarisation is produced when a quartz crystal is compressed along its optic axis, but is produced by compression in any other direction. If the crystal while under stress is passed through the flame of a Bunsen burner so as to remove the surface layers attendant on polarisation (Art. 52), and the stress then removed, the crystal is found to possess free charges at every point of its surface equal and opposite to those originally produced.

It has been known for at least 200 years that a crystal of tourmaline becomes electrically polar when heated, and the polarity persists in the parts into which a heated crystal is separated when broken. These *pyro-electric* effects depend on alteration of form caused by rise of temperature, i.e. they are really piezo-electric in nature, the temperature playing only a secondary *rôle*.

**50. Dielectrics.** We have hitherto regarded an insulator merely as something which prevents the passage of charge from one conductor to another: but the insulator plays a very definite part in electrical experiments, as is shown by the following. Take a parallel plate condenser and join one plate to one terminal of a quadrant electrometer in the ordinary way. Begin with the outer plate to earth, and discharge the electrometer and inner plate by connecting them temporarily to earth with a wire. If the outer plate is now raised to a definite positive potential there will be a negative charge induced on the inner plate, and a positive charge repelled

to the electrometer, the latter showing a deflexion. If the experiment is repeated with a slab of glass or ebonite between the plates of the condenser the deflexion is much greater than before, showing that more charges reside on the plates for a given applied potential. In other words, the capacity of a condenser is increased when a solid insulator replaces the air between the plates.

This effect was discovered by Cavendish about 1770, and rediscovered by Faraday. In view of the important part played by insulators Faraday gave them the name of *dielectrics*, and this is generally used when we wish to draw attention to the electrical phenomena occurring in the insulators themselves. Cavendish and Faraday showed that the capacity of a condenser was increased in a constant ratio when another dielectric was substituted for air; this ratio is called the *dielectric constant* (or specific inductive capacity) of the dielectric, and denoted by $K$. This property is turned to good account in constructing condensers of large capacity in small compass, which can be done with tinfoil and mica, the mica keeping the plates at a constant small distance apart and also increasing the capacity in the ratio of about 6 to 1.

Strictly speaking, air itself is a dielectric, though its properties differ but slightly from those of a perfect vacuum. However, it is convenient once for all to take a vacuum as the standard in electricity, and assign to it the dielectric constant unity. In other words, the electrostatic unit of charge is defined to be that which repels an equal charge placed one centimetre away *in vacuo* with a force of one dyne. This will be adopted henceforth.

The experimental methods of measuring dielectric constants are described in the next chapter: the following table gives their values for certain common insulators.

*Dielectric constants ($K$), referred to vacuum.*

| | | | | |
|---|---|---|---|---|
| Ebonite | 2·7 | Alcohol | 26 | |
| Glass | 5 to 7 | Paraffin | 2·1 | |
| Gutta-percha | 2·5 | Petroleum | 2·1 | |
| Mica | 6 | Water | 81 | |
| Paraffin wax | 2·3 | Air | 1·00059 | at 0° C. and 760 mm. pressure |
| Quartz | 4·4 to 4·6 | Hydrogen | 1·00026 | |
| Sulphur | 3·8 to 4 | Carbon dioxide | 1·00096 | |

The results for gases show how little error is committed in taking $K = 1$ for them.

**51. Various forms of condenser.** Many practical condensers are made with solid or liquid dielectrics between the plates for reasons already explained. Glass and mica will insulate at potentials where air would allow a spark to pass between the plates.

The *Leyden jar* is a bottle with a wide neck covered outside and inside with tinfoil. Contact with the inside is made by springs attached to a brass rod carrying a screw terminal, which dips into the bottle through a wooden stopper. The outside of the jar is covered with shellac in order to diminish the leakage of charge over the surface. The capacity is of the order of 1000 cm. Small Leyden jars are fitted to influence machines to store the electricity as it is formed and so strengthen the spark; but the insulation of a Leyden jar does not suffice for electrostatic measurements.

Fig. 56 shows an accurate variable condenser. By turning the handle a greater or less area of the moveable vanes comes opposite the fixed vanes, and the capacity is changed accordingly. Variable condensers can be immersed in oil to improve the insulation at high potentials, or to increase the capacity. They must be calibrated for accurate work, i.e. the capacity corresponding to any position of the pointer must be determined by comparison with a standard condenser.

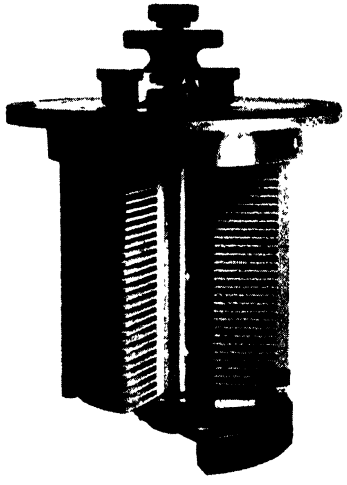


Fig. 56

Electrical principles are strikingly illustrated by placing an electroscope across the plates of a charged variable condenser. The divergence of the leaves increases when the capacity is reduced and decreases when the capacity is increased, for the charge is constant and the difference of potential inversely proportional to the capacity.

Condensers whose capacity is of the order of a million centimetres are made with tinfoil and mica or paraffined paper, a large

number of sheets being connected in parallel. For example, let us calculate the number of sheets of tinfoil 20 cm. square required in the construction of a condenser of a million centimetres capacity, using mica insulation 0·2 mm. thick. As the sheets of tinfoil are laid alternately the capacity is $nKA/2\pi d$, where there are $n$ sheets each of area $A$ connected to each terminal. Taking $K = 6$, this gives $n = 52$, so that about a hundred sheets would be required altogether.

Fig. 57 shows the method of mounting large condensers for laboratory use. All the coatings on one side are joined together to

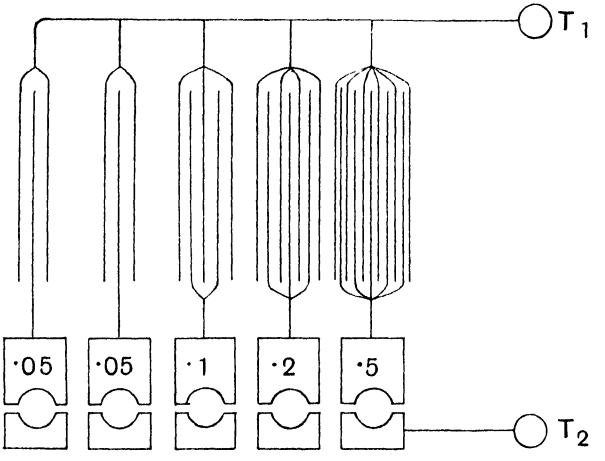



Fig. 57

one screw terminal $T_1$, while the others are brought in sets to separate lugs, which can be connected by plugs to a collecting bar, and hence to the terminal $T_2$. The values of the separate capacities are marked on the lugs. Large condensers are also made by taking two long strips of paper and two of paper coated chemically with tin, the strips being rolled up and subsequently impregnated with paraffin wax. Such condensers are compact and inexpensive, but do not insulate as well as mica condensers.

Commercial condensers often give a residual discharge; that is, they continue to discharge slightly for some time after their plates have been connected together.

**52. Theory of dielectrics.** The point of theoretical importance in the discovery of Cavendish and Faraday is that the proportional increase of capacity is the same for all possible shapes of condenser.

For the same charges on the plates the difference of potential with a given dielectric is $1/K$ times its value for a vacuum, i.e. the electric force at any point is $1/K$ times its value *in vacuo*. Hence we have the fundamental rule that the electric force at distance $r$ from a charge $e$ immersed in a homogeneous dielectric is

$$E = \frac{e}{Kr^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (55).$$

A satisfactory theory of dielectrics was first propounded by Kelvin on the lines of a corresponding theory of magnetisation due to Poisson. Translated into modern notation this theory is as follows. When a nucleus atom is subjected to electric force, the protons tend to move in the direction of the electric force and the electrons in the opposite direction. Thus an atom which is normally like the

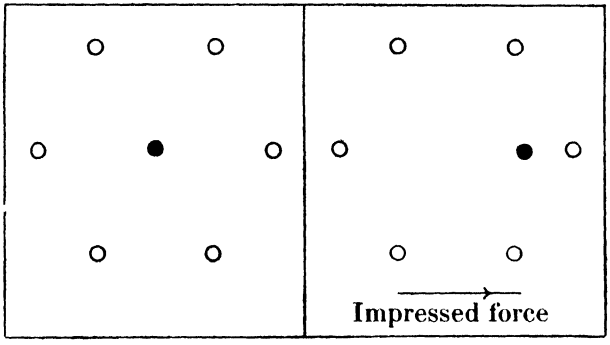



Fig. 58

left-hand diagram of Fig. 58 (in which the black spot represents the nucleus and the white spots circumnuclear electrons) tends to a configuration more like that on the right, positive electricity tending to one end of the atom and negative electricity to the other. An atom so modified is an *electric doublet*, or electric analogue of the small magnetised body of Art. 24, the vector $\Sigma e\mathbf{s}$ taking the place of $\Sigma m\mathbf{s}$ and being the *electric moment* of the atom. We need not assume that the electrons are stationary, for the electric force will alter the orbits of moving electrons and deflect them on the average in the same direction as if they were at rest. Let us postulate, without inquiring further into the mechanism, that there is, when averaged over a small interval of time, an *electric moment per unit volume*, or *polarisation* $\mathbf{P}$, at every point, the analogue of

the magnetisation **I** of a magnetised body. The theory of Art. 25 can now be applied entire. The distribution is equivalent, on the whole, to a volume distribution of electricity of volume-density $\rho$ and a distribution of surface-density $\sigma$ over the boundary, where

$$\rho = -\operatorname{div} \mathbf{P}, \quad \sigma = P_n \quad \ldots\ldots\ldots\ldots (56).$$

The potential at an external point $A$ is given by

$$V = \int \frac{\rho \, d\tau}{r} + \int \frac{\sigma \, dS}{r} \quad \ldots\ldots\ldots\ldots\ldots (57),$$

where $r$ is the distance of $A$ from the element of volume or surface considered.

More difficult considerations arise at a point *inside* the dielectric. In particular, the vector called the *electric force* takes on a more theoretical aspect. We must dismiss from our minds the idea of **E** as the force on a unit charge imbedded, say, in a hole in the dielectric. This point of view gives rise to great difficulty. We define the electric force as that due to the surface-distribution $\sigma$ over the external boundary and the volume-distribution $\rho$, supposed mathematically continuous. There is an electric potential $V$ at internal points, given by (57), and **E** is the negative gradient of $V$. From (42)

$$\operatorname{div} \mathbf{E} = 4\pi\rho = -4\pi \operatorname{div} \mathbf{P},$$

or

$$\operatorname{div} (\mathbf{E} + 4\pi\mathbf{P}) = 0 \quad \ldots\ldots\ldots\ldots (58).$$

We shall show that the behaviour of dielectrics is completely explained by supposing that the polarisation at any point is parallel and proportional to the electric force. Write

$$\mathbf{P} = k\mathbf{E} \quad \ldots\ldots\ldots\ldots\ldots (59).$$

For the sake of generality we shall suppose that the factor $k$ is a general scalar function of position, i.e. that the properties of the dielectric vary from point to point. Substituting from (59) in (58) and writing

$$K = 1 + 4\pi k \quad \ldots\ldots\ldots\ldots (60),$$

we have

$$\operatorname{div} K\mathbf{E} = 0,$$

so that $K\mathbf{E}$ is a solenoidal vector.

The symbol $K$ has been used because it is identical with the dielectric constant as previously defined. To verify this, and gain insight into the action of dielectrics, let us find from the theory the

electric force at distance $r$ from a charge $e$ immersed in a homogeneous dielectric.

Draw a small sphere $S_0$ of radius $r_0$ with $e$ as centre (Fig. 59), and imagine the dielectric inside the sphere removed to make room for the charge $e$. Let $V$ be the potential at a point in the dielectric at distance $r$ from $e$. Since $K$ is now constant, $\text{div}\,\mathbf{E} = 0$, or $\Delta V = 0$. From equation (18)

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0,$$

or

$$d^2(rV)/dr^2 = 0,$$

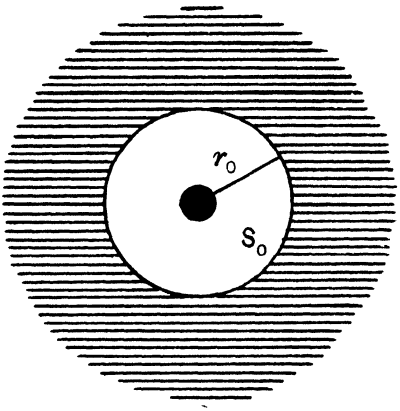



Fig. 59

and $V$ vanishes at infinity. Hence $V = \alpha/r$, where $\alpha$ is some constant, and the corresponding electric force at any point in the dielectric is $\alpha/r^2$ radially outwards. The polarisation is $P = k\alpha/r^2$ radially outwards. The volume-density $\rho$ of Poisson's theory vanishes from equation (56), and the surface-density on the inner surface is $-k\alpha/r_0^2$. The total charge thus arising on $S_0$ is therefore $-4\pi k\alpha$, which acts on external points as if concentrated at the centre of the sphere. The potential $V$ at a point in the dielectric, therefore, is what would arise from a charge $e - 4\pi k\alpha$ at the centre of the sphere, or $V = (e - 4\pi k\alpha)/r$. Hence $\alpha = e - 4\pi k\alpha$, or

$$\alpha = \frac{e}{1 + 4\pi k} = \frac{e}{K}.$$

The electric force at a point distant $r$ from $e$ in the dielectric is thus $e/Kr^2$, confirming our statement that $K$ is the dielectric constant. The removal of matter inside the sphere $r_0$ is a mathematical device to allow us to visualise things clearly. The result is independent of $r_0$, and we can allow the dielectric to approach the charge $e$ as closely as we please.

Thus the reduction of the electric force from $e/r^2$ to $e/Kr^2$ in a dielectric is due to an induced charge on the contiguous surface of the dielectric, the magnitude of the charge being $-e(1 - 1/K)$. Since $K = 1 + 4\pi k$, we should expect the dielectric constants of

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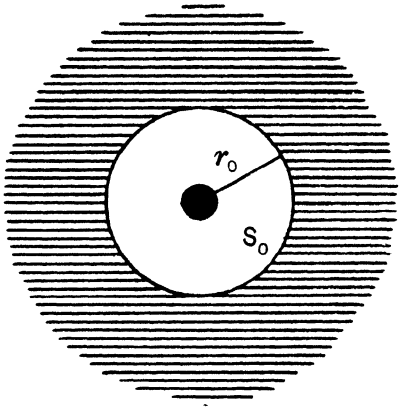



Fig. 59

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all substances to exceed unity, and this is found to be the case. Gases, which contain but few molecules per unit volume, have small values of $k$, and $K$ differs by very little from unity. Substances with large values of $k$ are those which allow a large polarisation for weak electric forces, i.e. those whose electrons are but weakly bound to the atom. We should thus expect conductors to behave like dielectrics of infinite dielectric constant, and we find in fact that those results in the theory of dielectrics which retain their meaning when $K = \infty$ hold for conductors. For example, putting $K = \infty$ in the dielectric just considered, we have an induced charge $-e$ on $S_0$, as we should expect. This charge of course screens the inner charge completely at external points, while the screening for dielectrics is not perfect.

**53. Condenser with a partial layer of dielectric.** Let $d$ be the distance between the plates of a parallel-plate condenser (Fig. 60) and $A$ the area of either plate, the linear dimensions being supposed large in comparison with $d$. Let there be a slab of dielectric of thickness $h$ between the plates, the rest of the space being occupied by air, for which $K$ may be taken as unity. The polarisation being uniform and of magnitude $P$, there is no volume-charge in the dielectric, but only two surface-charges of density $\pm P$. Let $X$ be the electric force in the air-spaces and $Y$ that in the dielectric, $\pm \sigma$ the surface-densities on the plates. The electric force due to the surface-layers is found from Art. 35, and thus we have

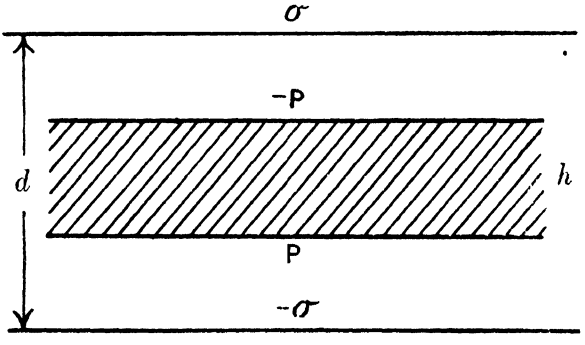


Fig. 60

$$X = 4\pi\sigma, \quad Y = 4\pi(\sigma - P).$$

Since $P = kY$ by definition, $Y = 4\pi\sigma/K$. The difference of potential between the plates of the condenser is

$$X(d - h) + Yh = 4\pi\sigma(d - h + h/K),$$

and the charge on the top plate is $\sigma A$. The capacity of the condenser is therefore

$$\frac{KA}{4\pi\{Kd-(K-1)h\}},$$

which reduces to $A/4\pi d$ when $h=0$, and to $KA/4\pi d$ when $h=d$. The electric force inside the dielectric is $1/K$ times that in the air-gap. This is a particular case of a general theorem which will be given in the next article.

**54. Boundary conditions at the surface of two dielectrics.** Let $C$ be the surface separating two dielectrics $K_1$, $K_2$ (Fig. 61),

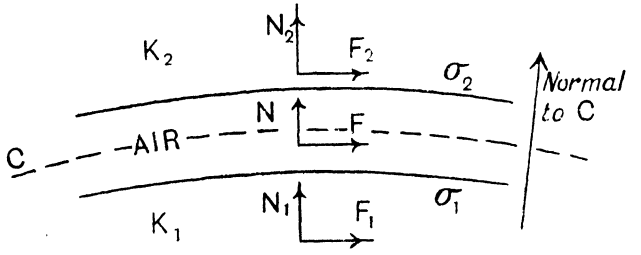



Fig. 61

and let $F_1$, $N_1$ be the tangential and normal components of electric force just inside $K_1$, $F_2$ and $N_2$ the same quantities just inside $K_2$. In order to gain some insight into the effects where two dielectrics meet, imagine a thin air-gap to be left between them, its position coinciding very nearly with the surface of separation $C$. Let $F$, $N$ be the components of electric force in the air-gap.

(1) *The tangential component of electric force is the same on either side of the surface of separation.* On crossing a surface-layer of surface-density $\sigma$ there is a sudden change of $4\pi\sigma$ in the normal component of electric force (Art. 35), but no change in the tangential component, since the surface-layer acts practically as an infinite charged plane as regards points very near to it on either side. Thus $F_1=F=F_2$.

(2) *The product of dielectric constant and normal component of electric force is continuous in crossing the surface of separation.* Let $\sigma_1$, $\sigma_2$ be the surface-densities of Poisson's distribution on the two dielectrics. Then from what has just been said we have

$$N=N_1+4\pi\sigma_1, \quad N_2=N+4\pi\sigma_2.$$

Thus $$N_1 + 4\pi\sigma_1 = N_2 - 4\pi\sigma_2,$$

and we also have $\sigma_1 = k_1 N_1, \quad \sigma_2 = -k_2 N_2.$

Hence $$N_1(1 + 4\pi k_1) = N_2(1 + 4\pi k_2),$$

or $$K_1 N_1 = K_2 N_2.$$

(3) *The electric force just outside a conductor immersed in a dielectric is* $4\pi\sigma/K$. Imagine a thin air-gap to be removed between the conductor and the dielectric (Fig. 62). The electric force in the air-gap is $E = 4\pi\sigma$ at right angles to the surface of the conductor; and the preceding theorems show that the electric force $E'$ just inside the dielectric has no tangential component, and that $KE' = E$. Hence $E' = 4\pi\sigma/K$ normal to the conductor.

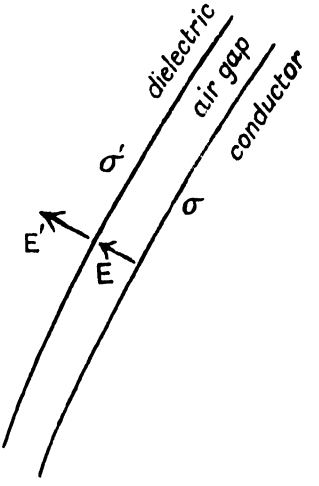



Fig. 62

**55. Electric force and electric induction.** It is of some importance, in view of later developments, to inquire what happens when the polarisation is not proportional to the electric force. We now require two vectors **P**, **E**, not in general parallel to each other, to express the state of the medium at any point. Equation (58) is independent of any special hypothesis. Thus if we define a vector

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} \quad \ldots\ldots\ldots\ldots\ldots\ldots (61),$$

$$\text{div}\,\mathbf{D} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (62).$$

The vector **D** is called the *electric induction*. **D** and **E** have important characteristic properties. **D** is always solenoidal, but not in general the gradient of any scalar function. **E** is always a gradient, but not in general solenoidal. The difference in the behaviour of **D** and **E** is lost in the homogeneous isotropic dielectric of Art. 52, where **D** is a constant multiple of **E**.

The boundary conditions of Art. 54 are easily generalised. We find that the *tangential component of electric force* is continuous in crossing the boundary of two media, and so is the *normal component of electric induction*. The normal component of electric force and the tangential component of electric induction, on the other hand, undergo in general a sudden change.

# CHAPTER IV

## ELECTRIC CURRENTS

**56. Electricity in motion.** When two conductors at different potentials are joined by a wire, there is a redistribution of charge which brings them to a common potential. The movement of electricity along the wire is called an electric current.

In this chapter we shall deal mostly with currents in metallic conductors, and we observe that though static charges are confined to the surface of conductors, electric currents flow normally through the interior (Art. 66), and only in one exceptional case over the surface (Art. 190). The electric current produced by joining two conductors is momentary, and it is immaterial at this stage whether positive electricity moves and negative electricity remains at rest, or *vice versâ*, or whether both move. It will be best to make the more general assumption at first.

Suppose now that two conductors have been joined by a wire, and we are considering a particular cross-section of the wire at time $t$. In a small interval of time $dt$ a certain charge $+ de_1$ of positive electricity will pass the section in one direction, and a charge $- de_2$ of negative electricity will pass it in the opposite direction under the driving electric force, making a total transfer $de$, where $$de = de_1 - (- de_2) = de_1 + de_2.$$

The rate at which electricity is being carried over in a given direction is called the *current* in that direction, and denoted by the letter $i$. Thus $i = de/dt$. An "electrostatic" unit of current is implied, consisting in the passage of one electrostatic unit of charge per second. In the case just considered $i$ is not constant, all that is known being that it begins and ends by being zero.

When electricity passes along a wire it produces magnetic and thermal effects which it does not produce at rest (Arts. 58, 70), but the method of discharge is not suitable for their study, partly because the amount of electricity which passes is small, but chiefly because the current varies. Constant currents are produced in principle by voltaic cells (Arts. 57, 63), and in practice by accumulators (Arts. 63, 169) or electric generators (Art. 143). A current

is at any moment the same at all cross-sections of a wire, provided that there is no accumulation of electricity anywhere along the wire.

**57. The voltaic cell.** Differences of potential, as described hitherto, have been produced by friction or electrical influence. These methods produce relatively small charges at high potentials. We now describe a method of producing large charges at low potentials, the necessary energy being derived from chemical sources. Let a plate of copper and a plate of zinc be partly immersed in a vessel containing dilute sulphuric acid. They will then, when joined to the terminals of a quadrant electrometer, produce a deflexion; and the potentials which are produced cannot be dismissed simply as contact potentials, since the additive law mentioned in Art. 49 is not obeyed.

The earliest observations in this subject are those of Galvani, published in 1791. He noticed that a frog's leg, attached to an iron hook so that the nerve touched the metal, contracted suddenly when touched with a piece of metal in contact with the hook. Since the same contractions could be produced with an electrical machine they were considered to be electrical in nature. It was reserved for Volta to point out the essence of Galvani's discovery, namely that the production of electricity did not depend on the presence of animal preparations, but rather on that of certain liquids. Replacing the frog's leg by acidulated water, he devised in 1800 the combination of metal and acid already described, called the *voltaic cell*. If the copper plate of one cell is joined to the zinc of the other, the two form a compound cell or *electric battery*, and if the remaining plates are joined to the terminals of an electrometer the difference of potential is found to be twice as great as before. Cells so joined "in series" are used to obtain large differences of potential, cells joined "in parallel" (i.e. copper to copper and zinc to zinc) to obtain more current.

Other metals and liquids can be used in the construction of a voltaic cell, with varying potentials, of the order of a few hundredths of an electrostatic unit. The difference of potential between the plates of a simple cell falls after it has been delivering current for any length of time. The cell is then said to be *polarised*. For this

reason a simple cell is practically useless, and recourse is had to more complicated forms. One of these is the Daniell cell, shown in Fig. 63. A stout zinc rod or cylinder is placed in zinc sulphate solution contained in a porous pot of unglazed earthenware. Outside the pot is a larger vessel containing a solution of copper sulphate, into which dips a piece of sheet copper bent into the form of a cylinder. The action of the cell will be discussed in Art. 172: it will give current without becoming unduly polarised and can be used for simple experiments such as the following.

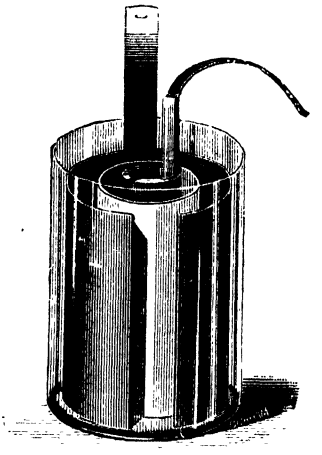


Fig. 63

Let the copper and zinc (the *poles*, as they are called) of a Daniell cell be connected to an electrometer, and the deflexion observed. If the poles are joined by wire (say 3 metres of S.W.G. 36 copper wire), the wire will become warm and the deflexion of the electrometer will fall. An electric current, therefore, heats a wire in which it flows. As regards the deflexion, we may regard the wire as constantly tending to reduce the difference of potential between the poles to zero, while the chemical action strives to maintain it at its original value, with an intermediate result.

If the copper pole of a Daniell cell is joined to a terminal $A$ of an electrometer and the zinc pole to a terminal $B$, the spot of light is deflected in the same way as if $A$ was charged up slightly from a glass rod rubbed with silk. Thus the copper is at a higher potential than the zinc, and is called the positive pole of the cell. In the connecting wire, or *external circuit*, therefore, the current flows from the copper to the zinc, and the method can obviously be applied to any other cell.

**58. Magnetic action of currents. The tangent galvanometer.** Experiments with voltaic cells show at once that a current produces a magnetic field. The discovery dates from 1820, when Oersted found that a wire carrying a current would deflect a compass-needle near it. The laws of the action of currents on

magnets are discussed in Arts. 78–81, and we shall only consider here as much as is necessary for our immediate purpose.

If a straight wire is placed over a compass-needle pointing north and south, and a current passes along the wire from south to north (Fig. 64), the positive pole of the needle is deflected from north

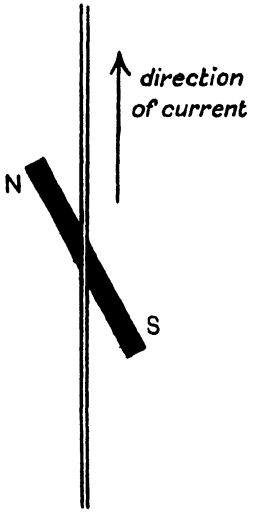



Fig. 64

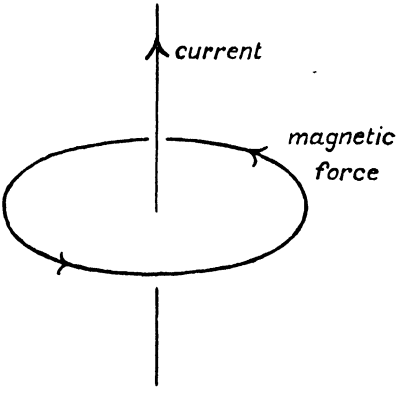



Fig. 65

towards the west. This effect may be described concisely by an appropriate *right-handed screw law* of electric currents; namely that the magnetic force at any point due to a straight current is, at least approximately, in the direction in which the point would be moved by a small right-handed twist round the positive direction of the current (Fig. 65).

If a wire is doubled back on itself, the parts being insulated from one another, no magnetic effect is produced at points a large distance off compared with the distance between the two parts. We can, therefore, neutralise the magnetic effect of any wire by providing it with a return wire near to it. If a straight wire carrying a given current produces a certain deflexion to the west when lying vertically over a compass-needle and pointing magnetically north, turning the wire through two right angles produces an equal easterly deflexion. Thus the magnetic field changes sign with that of the current, and will be shown in the next article to be proportional to it.

A simple compass-needle suffices to detect electric currents:

instruments designed for more accurate measurement are called *galvanometers*. Of these the fundamental one is the *tangent galvanometer* (Fig. 66), consisting of a circular coil of wire of one or more turns, moveable about a vertical axis, with a small compass-needle at the centre of the circle. The compass-needle is fitted with two pointers and lies in a box with a graduated scale like that used in magnetometers. The scale is marked so that the line of zeros is perpendicular to the plane of the coil of wire. Thus the pointers read zero when the plane of the coil is in the magnetic meridian and no current is flowing. Our first experiment is to set the plane of the coil magnetically east and west, so that the pointers read 90°, and apply a current, when the needle is found either to remain at rest or to turn through two right angles. A circular coil therefore produces, at its centre, a magnetic force perpendicular to its plane (cf. Art. 81), since the force due to the coil would not otherwise assist or directly oppose the earth's horizontal field.

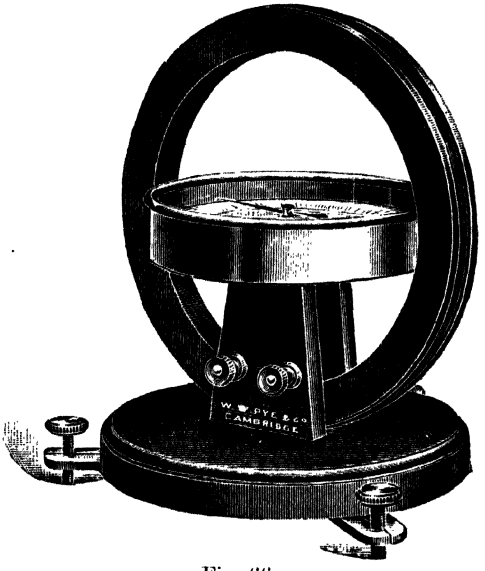


Fig. 66

The tangent galvanometer measures primarily the magnetic force $F$ due to the current at the centre of the circular coil, and its normal position when in use is with the plane of the coil in the magnetic meridian, so that $F$ acts east and west. If $H$ is the earth's magnetic field, a deflexion of $\theta$ to the west means that $F$ acts to the west and is of magnitude $H \tan \theta$ (cf. Art. 18). If therefore we can find the connexion between $F$ and the current we shall be in a position to measure currents with the instrument. This question is discussed in the next article.

**59. Electromagnetic unit of current.** We can prove experimentally that the magnetic force due to a current is proportional

to the strength of the current. (The relation is not self-evident: thus the heating effect of a current is proportional to its square.) Two similar insulated coils of wire are wound close together round a wooden frame as shown in the figure, and a compass-box is fixed with the centre of the needle in the plane of the coils. The same current is sent (1) through one coil only, (2) through both coils in the same direction, (3) through both coils in opposite directions, with the coils in the magnetic meridian. The equality of current in the three cases is tested with a tangent galvanometer at a considerable distance from the coils: evidently the tangent galvanometer will on any theory always give the same deflexion while the same current passes through it. The following table gives the readings in an actual experiment:

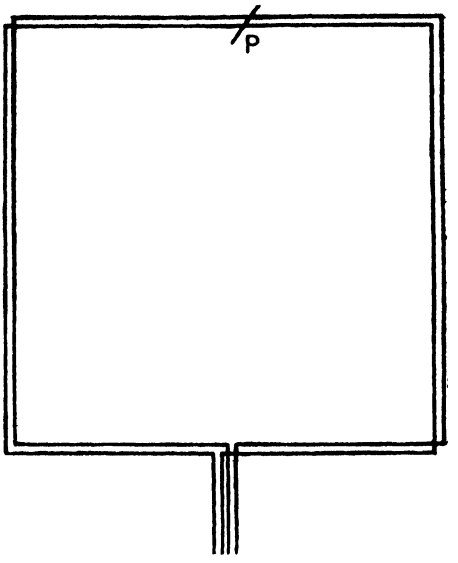


Fig. 67

| Deflexion $\theta_1$ with current in one coil only | Deflexion $\theta_2$ with current in same direction in both coils | Deflexion $\theta_3$ with current in the two coils opposed | $\frac{\tan \theta_2}{\tan \theta_1}$ |
|---|---|---|---|
| 16° 24′ | 30° 6′ | 0° 6′ | 1·97 |
| 20° 24′ | 36° 36′ | 0° 12′ | 1·99 |
| 23° 24′ | 41° 6′ | 0° 6′ | 2·01 |
| 27° 36′ | 46° 18′ | 0° 6′ | 2·00 |
| 31° 48′ | 51° 6′ | −0° 12′ | 2·00 |
| 35° 36′ | 55° 12′ | 0° 6′ | 2·00 |
| 39° 24′ | 58° 42′ | 0° 0′ | 2·00 |
| 42° 54′ | 61° 30′ | −0° 24′ | 1·98 |

The smallness of the numbers in the third column proves the absence of magnetic effect of wires bent back on themselves. Now when the current flows in both coils in the same direction the amount of electricity passing a given point $P$ of the composite wire (Fig. 67) is twice as much as when the current flows in one wire only;

that is, the current by definition is twice as great as before. We should therefore have $\tan\theta_2 = 2\tan\theta_1$, and the numbers in the last column confirm this relation.

Returning to the tangent galvanometer, we have to find how the magnetic force $F$ at the centre of the coils depends on the current $i$, the number of turns $n$, and the radius $r$ of the coils. A current $i$ circulating in $n$ wires close together is equivalent to a current $ni$ in a single wire, since the passage of charge is $n$ times as great. Hence $F$ can only be given by an expression of the form $F = nif(r)$. To determine $f$ we require a special experiment with a tangent galvanometer having two sets of coils, one of a single turn of radius $r$ and the other of a double turn of radius $2r$, the windings being arranged so that the current is in opposite directions in the two sets. No deflexion will be observed, showing that $F$ is unaltered when $n$, $r$ are changed to $2n$, $2r$ respectively, and therefore depends on the ratio $n/r$ only. Hence $F = kni/r$, where $k$ is some quantity which is the same for all instruments. The constant $k$ required to give $F$ in gausses is now perfectly definite, since $i$ is the number of electrostatic units of charge passing per second. Its value has been shown to be about $2 \times 10^{-10}$; but as it is not convenient to describe the experiments at this stage we shall define a new unit of current (and incidentally a new unit of charge) by giving $k$ an arbitrary value, leaving the question of the relation between the two units for later examination. It is most convenient to take $k$ equal to $2\pi$, and if this is done $i$ is measured in what are called *electromagnetic units of current*. Thus the electromagnetic unit of current is such that it would give a field of $2\pi$ gausses at the centre of a circular loop of wire of radius 1 cm. With this understanding the deflexion $\theta$ of a tangent galvanometer used in the customary way is given by $2n\pi i/r = H\tan\theta$, or

$$i = \frac{rH}{2n\pi}\tan\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (63).$$

The quantity $rH/2n\pi$ is called the constant of the tangent galvanometer. Hence if $H$ is known we have a method of measuring currents in electromagnetic units.

The value $k = 2\pi$ is chosen because it gets rid of an inconvenient factor in the theory; for in a tangent galvanometer of one turn the magnetic force at the centre is $2\pi i/r$ for a length $2\pi r$ of wire, that

is $i/r^2$ per centimetre, so that the effect of a given length of wire is expressed in its simplest form (cf. Art. 96).

Since an electric charge is detected by its repulsions, and the most important effect of a current is its magnetic effect, it is convenient to suppose charge measured in electrostatic units and current in electromagnetic units. This will be understood, in the absence of qualifications, throughout the book. The ratio of the electromagnetic to the electrostatic unit of current will be denoted by $c$, so that a current $i$ in electrostatic units becomes $i/c$ when measured in electromagnetic units. We shall describe later (Art. 142) experiments by which $c$ has been shown to have the value $3 \times 10^{10}$ very nearly, and we shall use this value in numerical calculations.

**60. The derived system of electromagnetic units.**

(1) *The derived electromagnetic unit of current* (*ampere*). Both the electromagnetic and electrostatic systems of units have been fixed on purely theoretical grounds without reference to the commonly occurring values of the magnitudes in question. A practical unit should be more or less commensurate with the quantities that are normally measured. Most electromagnetic units are of inconvenient size, and therefore a third system of units, called the "derived system of electromagnetic units," has been devised. The unit of current on this system is one-tenth of the electromagnetic unit; and like all the derived units it is given a name, being called the *ampere*. Thus a current of $i$ electromagnetic units is equal to $10i$ amperes or to $ci$ electrostatic units, so that an ampere is very nearly $3 \times 10^9$ electrostatic units of current. The reader should become familiar with the magnitude of currents in common use. Thus the current passing through a 32 candle-power electric lamp on a 100 volt circuit is of the order of $\frac{1}{3}$ ampere; but electrical effects must not be taken as depending solely on the current, since a lamp of the same candle-power designed for a 200 volt circuit takes only half the current (cf. Art. 70). Currents between $10^{-3}$ and $10^{-7}$ amperes are "galvanometer currents" requiring some detection, while more than 5 amperes begins to heat the wires ordinarily used in a laboratory inconveniently.

(2) *The electromagnetic unit of charge and the derived unit*

(*coulomb*). The charge transferred by a constant current $i$ in time $t$ is $e = it$, or $\int i\,dt$ if the current is variable. The units of charge in different systems are thus related to one another in the same way as those of current. The charge conveyed by a steady current of one ampere flowing for one second is called a *coulomb*. Hence a charge $e$ measured in electrostatic units $= e/c$ in electromagnetic units $= 10e/c$ coulombs, and 1 coulomb is $3 \times 10^9$ electrostatic units of charge.

(3) *The electromagnetic unit of potential and the derived unit* (*volt*). Like the electrostatic unit, the electromagnetic unit of potential is defined so that two points are at unit difference of potential when one erg of work is required to move a unit charge from one point to the other. Let $P$, $Q$ be two points whose potential difference is one electrostatic unit. Since the electromagnetic unit of charge is $c$ electrostatic units, the work done in moving a charge of one E.M. unit from $P$ to $Q$ is $c$ ergs, and therefore the difference of potential between $P$ and $Q$ is $c$ electromagnetic units. Hence the electrostatic unit of potential is $c$ electromagnetic units, and a potential $V$ measured in electrostatic units $= cV$ in electromagnetic units.

The derived unit of potential is taken as $10^8$ times the absolute E.M. unit, and is called the *volt*. The electrostatic unit of potential is thus $c/10^8 = 300$ volts, a convenient thing to remember. If we multiply charge in coulombs by potential difference in volts we do *not* obtain work in ergs, but to do so we must first convert to absolute E.M. units and then form the product.

The difference of potential between two points is often called the electromotive force (E.M.F.) between them; a term in general use, though open to objection. We shall use it for the difference of potential between the terminals of a cell or other source of electricity. The potential differences occurring in current electricity are usually small compared with those produced by electrostatic means. It is an easy matter to charge up a body by friction to 1000 volts; the E.M.F. of electric lighting circuits is from about 100 to 250 volts, while that of the Daniell cell is only 1·09 volts. There is, however, no difficulty in measuring small potentials by electrostatic means. The quadrant electrometer when used heterostatically gives a deflexion of 80 scale divisions (mm.) or more for

a potential difference of one volt between the quadrants, when the needle is charged to 50 volts.

(4) *The electromagnetic unit of capacity. Farads and microfarads.* The capacity of an isolated conductor in any system of units is the ratio of its charge to its potential, potential being defined so that it vanishes at infinity. Suppose that we have a conductor of capacity 1 cm. at a potential of 1 E.S.U. In electromagnetic units its charge would be $1/c$ and its potential $c$, so that its capacity would be $1/c^2$. Hence the electromagnetic unit of capacity is $9 \times 10^{20}$ cm.

The derived unit of capacity is called the *farad*, and is the capacity of a conductor which requires one coulomb to raise its potential by one volt. In practice we need only consider condensers: a condenser has a capacity of one farad if, when one plate is earthed, one coulomb given to the other raises its potential by one volt. We can show as above that the farad is equal to $10^{-9}$ electro magnetic units; it is however an inconveniently large unit, and the microfarad*, which is one-millionth part of a farad, is used instead. We have

$$1 \text{ microfarad} = 10^{-15} \text{ E.M. units} = 900{,}000 \text{ cm.}$$

The microfarad is itself a large unit: the condensers used in wireless telegraphy (Fig. 56) have capacities of the order of 1 millimicrofarad or 900 cm., and are commonly expressed in millimicrofarads.

The relation of the various units is given in the Appendix, together with a table of notation and other units to be introduced hereafter.

**61. The magnetic effect of a moving charge.** Since an electric current is simply the passage of electric charge, a charged conductor when moved should itself affect a magnet in its neighbourhood. Such an effect was observed by Rowland in 1876: the most accurate quantitative experiments are those of Eichenwald, which we shall now describe.

A thin insulating disc $A$ (Fig. 68) was mounted on an axle, and on it was pasted a circular strip of tinfoil $BCD$ extending nearly

* The prefix *micro-* when applied to the various units means one millionth part. Other prefixes are *milli-*, meaning one thousandth part, and *meg-*, meaning a million times.

round its circumference. One end $B$ of the strip was permanently connected to an insulated brass slip ring on the axle, and could be charged up by means of a brush $E$ pressing against the slip ring. The disc was surrounded by a fixed earthed metallic case $F$ nearly enclosing it, but with an aperture which enabled the end $D$ of the tinfoil strip to be earthed at will. The magnet was suspended above the disc by a torsion fibre, and shielded from electrostatic influence. It consisted of a glass rod $G$ carrying at

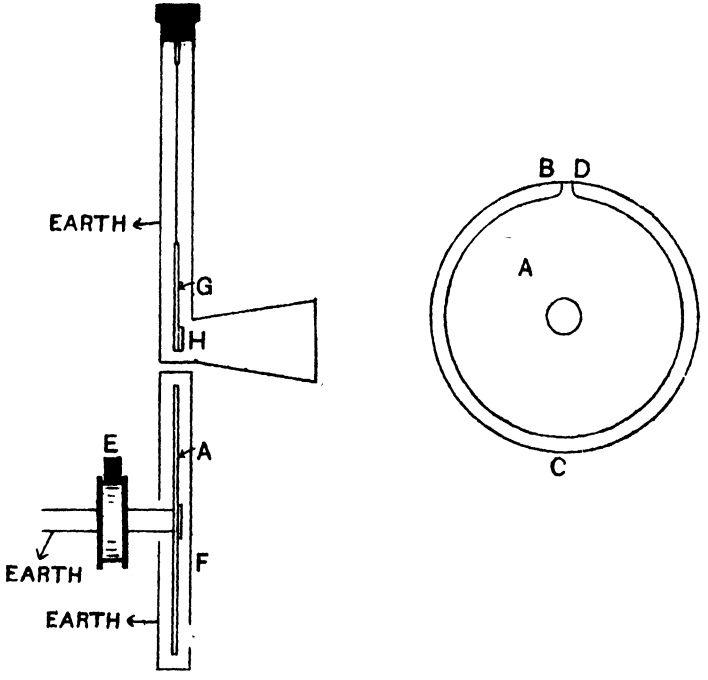



Fig. 68

each end a short horizontal magnet, the axes of the two magnets being turned in opposite directions so as to neutralise the effect of the earth's magnetic field. The position of the magnet was read off by reflexion with a lamp and scale, using the small mirror $H$ attached to the magnet.

In the first experiment the disc was charged to potential $V$ by means of the brush and rotated $n$ times per second, and the small deflexion $\theta$ read. If $C$ is the electrostatic capacity of the condenser formed by $A$ and $F$, a charge $CV$ passes $n$ times a second, and

this is equivalent to a current of $nCV$ electrostatic units. Hence we should have $nCV = k\theta$, where $k$ is a certain constant. The constancy of $nCV/\theta$ in Eichenwald's experiments is shown in the following table:

| $n$ | $C$ | $V$ | $\theta$ | $nCV/\theta$ |
|---|---|---|---|---|
| 77 | 39·0 | 1875 | 5·8 | $0{\cdot}97 \times 10^6$ |
| 77 | 50·9 | 3750 | 14·1 | 1·04 |
| 77 | 50·9 | 5000 | 19·2 | 1·02 |
| 15 | 45·8 | 6250 | 4·1 | 1·05 |
| 30 | 45·5 | 6250 | 8·6 | 0·99 |
| 77 | 45·5 | 6250 | 21·4 | 1·02 |
| 150 | 50·2 | 6250 | 45·5 | 1·03 |

The end $D$ was next put to earth and a current of $i$ electromagnetic units passed through the strip from a battery, the disc remaining at rest. This current is equivalent to $ic$ electrostatic units, where $c$ is the ratio of the units. Hence we should have $ic = k\phi$, where $\phi$ is the deflexion produced. The value of $k$ is known from the preceding experiments, and thus $c$ can be found. Eichenwald found values ranging from $2{\cdot}86 \times 10^{10}$ to $3{\cdot}16 \times 10^{10}$, mean $3{\cdot}01 \times 10^{10}$. This agrees very well with the accepted value $3 \times 10^{10}$ (Art. 142), and thus the results are entirely conclusive as to the magnetic effect of moving charges.

**62. Electrolysis.** The currents considered so far have moved through metals, or been carried along with them as in the last article. The passage of electricity through conducting liquids is different, in that chemical action is involved. This phenomenon is known by the name of *electrolysis*, and liquids that conduct in this way are called *electrolytes*. The current is conveyed to and removed from the liquid by wires or plates dipping into it and called *electrodes*, the one at higher potential being called the *anode* and the one at lower potential the *cathode*. The whole apparatus is called a *voltameter*. Pure water is a bad conductor, the best electrolytes being acids and salt solutions.

When the current passes constituents of the acid or salt appear at the electrodes, where they are sometimes set free. In other cases secondary actions take place depending on the material of the electrode. The following are typical instances.

(1) $A$ and $B$ (Fig. 69) represent two platinum electrodes joined to a battery or other source of potential and dipping into pure hydrochloric acid. A current passes and bubbles of hydrogen appear at the cathode $B$, chlorine at the anode $A$. These bubbles rise and are set free, and it is found that the volumes of hydrogen and chlorine, if collected separately, are equal. Here the sole chemical action is the liberation of the constituents of the acid.

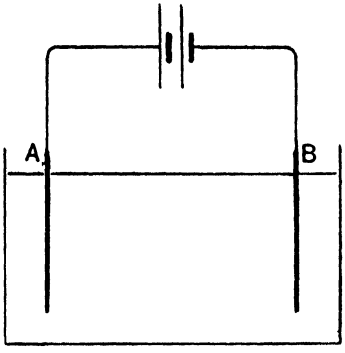


Fig. 69

(2) More complex phenomena occur when $A$ and $B$ are copper plates dipping into a solution of copper sulphate. When current flows, copper reaches the cathode and is deposited there, while the radicle $SO_4$ reaches the anode and combines with it to form copper sulphate, which enters the solution. The final effect is that one electrode gains copper and the other loses it, the solution remaining unaltered.

The quantitative laws governing these actions were first enunciated by Faraday in 1833. They are

(1) For any given substance the amount set free at an electrode is proportional to the quantity of electricity which passes, regardless of the time taken to do so;

(2) The amount of a substance which is set free by a given quantity of electricity is proportional to its chemical equivalent weight. By chemical equivalent weight is meant the atomic or molecular weight divided by the valency.

We can verify Faraday's first law by means of three voltameters containing the same electrodes and electrolyte. If the current divides after passing through the first voltameter, part passing through the second voltameter and the rest through the third, the amount set free altogether from the second and third voltameters is the same as that set free from the first. Similarly if any number of voltameters are arranged in series so that the same current flows through each, Faraday's second law can be verified by measuring the amounts of the various substances liberated (cf. Art. 164).

The first law is important to us now, because if the amount of a substance liberated by one coulomb has once been measured, we can apply this knowledge to find how many coulombs have passed in a circuit in a given time, and thus to measure currents. For theoretical purposes currents are defined by their magnetic effects, as already explained, and the *true ampere* will always be defined in this way. But the experiments necessary to determine a current in true amperes (Art. 93) are elaborate and difficult, while great accuracy can easily be obtained by electrolysis. It is therefore better to take the results of certain standard experiments and redefine the ampere for practical purposes by its electrolytic effect. At an international conference held at London in 1908 the *international ampere* was defined as the unvarying current which, when passed through a solution of nitrate of silver in water, in accordance with an authorised specification, deposits silver at the rate of 1·11800 milligrams per second.

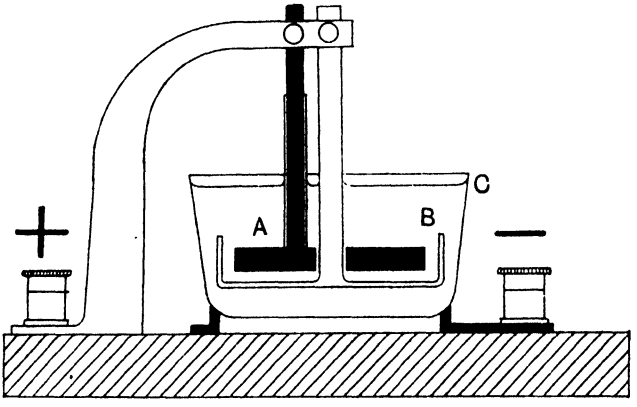



Fig. 70

A voltameter for determining, with the greatest accuracy, the amount of silver deposited by the passage of one coulomb, or the electrochemical equivalent of silver as it is called, is shown in Fig. 70. The anode *A* is a perforated disc, contained in a shallow glass basin *B* supported by a glass rod which passes through the centre of the disc. The basin catches particles which may become detached from the anode. A glass lid (not shown) lets down to isolate the anode liquid when removing it from the bowl. The cathode *C* is a platinum bowl which also serves to hold the electro-

lyte, a 15 per cent. solution of silver nitrate. The mass of silver deposited on the bowl is determined after a known current has passed for a known time.

Experiments on the electrochemical equivalent of silver were made by T. Mather and F. E. Smith in 1906–7, and by Rosa, Dorsey and Miller in 1912, great attention being paid to purity of the electrolyte. The current in the former experiments was measured with the Ayrton-Jones current weigher (Art. 93). The electrochemical equivalent was found to be 1·11827 milligrams per coulomb. Faraday's first law was subjected to a searching test, for it was found that the same number was obtained, within the limits of experimental error, for several forms of voltameter and for currents differing in the ratio of 16 to 1, while the strength of the solution could vary from 1·5 to 50 parts of silver nitrate per 100 parts of solution without affecting the value appreciably. The experiments of Rosa, Dorsey and Miller were carried out with an electrodynamometer more akin to Kelvin's current balance (Art. 93), and gave the value 1·11800. At present, therefore, we cannot say that the electro-chemical equivalent of silver is known to be more than 1 part in 3000.

Currents produced by electrostatic methods can also be shown to cause electrolysis. We require a small voltameter with electrodes of fine platinum wire, and a rather large influence machine with a large condenser connected across its terminals. A sensitive tangent galvanometer included in the circuit will also be deflected, showing that currents taken from an influence machine produce appreciable magnetic effects.

**63. Polarisation. Cells and accumulators.** When a circuit containing a voltameter is closed a comparatively large current may flow at first, but it will diminish and finally become constant. This may be shown to be due to the formation of an E.M.F. between the electrodes tending to oppose the applied E.M.F., for if the voltameter is taken out of circuit and the electrodes joined to the terminals of an electrometer a deflexion occurs which remains steady for some time. The voltameter itself will now act like a cell, and if the cell in Fig. 69 is replaced by a galvanometer a current will flow in a direction opposite to that of the original current.

The voltameter is then said to be polarised. The effect disappears more or less rapidly when the electrodes are joined by a thick wire, or short-circuited as it is called. Polarisation is due to some change in the surface of one or both electrodes caused by the passage of the current: it does not occur in the electrolysis of copper sulphate with copper electrodes, where the anode loses copper and the cathode gains it, without either electrode undergoing any change of composition.

When a voltaic cell is closed through an external circuit the current which circulates causes electrolysis inside the cell, and the polarisation of cells is of the same nature as that now described. The current in the interior of a cell flows from the negative to the positive pole; for example in the Daniell cell from the zinc to the copper. The effect on the electrodes of that cell is that copper is deposited on the positive pole and zinc dissolves into the zinc sulphate, so that the surfaces dipping into the liquids are not altered, at any rate for some time.

The *Leclanché cell* (Fig. 71) is convenient if weak currents are required at irregular intervals, as it requires very little attention and keeps well. The positive pole is a stout rod of hard gas-carbon closely packed with manganese dioxide and contained in a porous pot. Outside the pot is a saturated solution of ammonium chloride, in which dips a zinc rod forming the negative pole. When a current passes, the layer of hydrogen which tends to form on the positive pole is oxidised by the manganese dioxide, but as this takes place slowly there is considerable polarisation for large currents. The E.M.F. on open circuit is about 1·4 volts, falling rapidly on the passage of a current and recovering when left to itself. The so-called dry cell is the same in principle, but the electrolyte and depolariser are thick pastes.

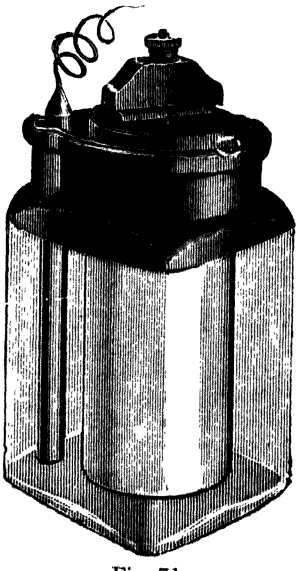


Fig. 71

In many voltameters the effect of passing a current is reversible; that is, the chemical changes accompanying polarisation are re-

versed when the voltameter is removed from the circuit and its electrodes joined by a wire. Such a voltameter acts like a cell which has to be made (or charged, as it is called), and the energy expended in charging is stored up as chemical energy which can be partially recovered on discharge. Thus the voltameter can be used to store electrical energy: this is the principle of the *accumulator*, which is the most useful source of electricity for laboratory work.

The simplest form of accumulator, invented by Planté about 1879, consists of two lead plates immersed in dilute sulphuric acid and charged up for some time. The normal E.M.F. of the lead accumulator is from 2 to 2·1 volts, and it is not advisable to use it when the E.M.F. falls much below 2 volts. The amount of current obtainable depends on the size of the plates and the treatment they have received. The following simple experiment shows the behaviour of a very small cell whose plates were 10 cm.$^2$ in area. It was charged with a current of 2 amperes flowing for 5 minutes, and the E.M.F. was 2·2 volts at first, falling rapidly to 2·1 volts and then remaining steady. On short-circuiting for one minute the E.M.F. fell to 1 volt, but recovered to 1·5 volts in one minute more. Further short-circuiting reduced the E.M.F. still more.

Accumulators are made up in various forms giving E.M.F.'s from 2 volts upwards (Art. 169). A battery of small accumulators in

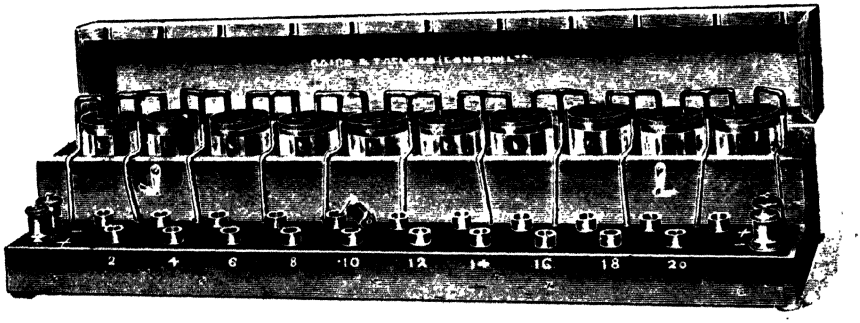



Fig. 72

series is a convenient source of high potential when but little current is required. Fig. 72 shows a battery of 10 cells yielding about 20 volts.

**64. Standard cells.** It is very desirable to have a concrete realisation of potential difference in the shape of cells whose E.M.F. remains constant. The Daniell cell satisfies this condition fairly well. Standard cells are devised to give the utmost constancy of E.M.F. at a given temperature, when delivering minute currents only.

The *Weston normal cell* (Fig. 73) is very reliable when set up with pure materials, and also very little affected by temperature. The E.M.F.'s of such cells differ by only one or two parts in 100,000, and remain constant for years. Cells which are accurate to about one part in a thousand may be constructed as follows.

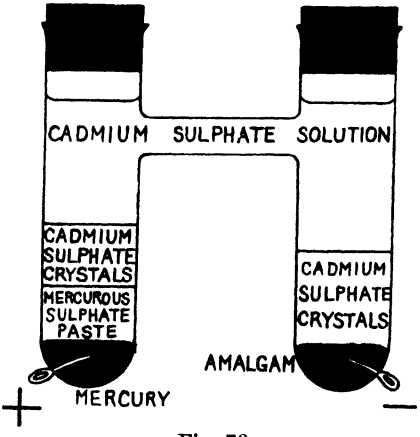


Fig. 73

(1) *The vessel.* The cell is prepared in an H-shaped glass vessel in which are fused two fine platinum wires. Rinse the vessel with dilute nitric acid and then with distilled water.

(2) *Preparation of the saturated solution of cadmium sulphate.* Take about 10 cm.$^3$ of commercially pure cadmium sulphate crystals and add 25 cm.$^3$ of distilled water. Prepare a saturated solution of cadmium sulphate by rubbing the ingredients together in a mortar for 10 minutes.

(3) *Preparation of the mercurous sulphate paste.* Take about 5 cm.$^3$ of commercially pure mercurous sulphate, add a little mercury and rinse it in a beaker with a little dilute sulphuric acid (1 part of acid to 6 parts of water by volume), afterwards rinsing twice with distilled water and once with a little of the cadmium sulphate solution for some time. Then pour off the solution.

Grind 1 cm.$^3$ of cadmium sulphate to a fine powder in a mortar together with 2 cm.$^3$ of the mercurous sulphate just prepared and $\frac{1}{4}$ cm.$^3$ of pure mercury. Add calcium sulphate solution if necessary till the whole forms a thin paste.

(4) *Preparation of the amalgam.* Weigh out about 6 gm. of commercially pure cadmium, and to each gram take $\frac{1}{2}$ cm.$^3$ of pure

mercury. Heat the mixture to about 60° C. in a beaker over a water bath, and while still warm rinse with dilute sulphuric acid and then with distilled water.

(5) *Filling the cell.* Pour enough of the warm amalgam to cover the platinum wire into one limb of the H-vessel with a warmed pipette, and let it solidify. Into the other limb pour mercury to cover the platinum wire, and over it a layer of the mercurous sulphate paste about 1 cm. deep, using a pipette so that none of the paste adheres to the sides of the tube. Add the saturated solution of cadmium sulphate up to the level of the cross tube, and afterwards put some crystals of cadmium sulphate into each arm. The corks, previously soaked in distilled water, can now be inserted and the cell sealed.

A Weston cell short-circuited for one minute was found by F. E. Smith to have recovered its E.M.F. within one part in 10,000 one minute afterwards. Short-circuiting for five days affects the cell seriously for some days, but it recovers completely in six weeks. As actually used, standard cells are not called on to yield any but very small currents for short spaces of time.

*E.M.F. of Weston cells.*

| Temp. ° C. | E.M.F. in volts | Temp. ° C. | E.M.F. in volts | Temp. ° C. | E.M.F. in volts |
|---|---|---|---|---|---|
| 10 | 1·0186 | 17 | 1·0184 | 24 | 1·0181 |
| 11 | 1·0186 | 18 | 1·0184 | 25 | 1·0181 |
| 12 | 1·0186 | 19 | 1·0183 | 26 | 1·0180 |
| 13 | 1·0185 | 20 | 1·0183 | 27 | 1·0180 |
| 14 | 1·0185 | 21 | 1·0183 | 28 | 1·0180 |
| 15 | 1·0185 | 22 | 1·0182 | 29 | 1·0179 |
| 16 | 1·0184 | 23 | 1·0182 | 30 | 1·0179 |

For most purposes the E.M.F. of the Weston cell may be taken as 1·018 at all ordinary temperatures. The measurement of E.M.F. in absolute measure depends on principles yet to be explained (Art. 141).

**65. Ohm's law.** We now consider more closely the relation between the current and the E.M.F. in a circuit. The first step is to satisfy ourselves experimentally that the deflexion of a quadrant

electrometer as ordinarily used, that is the difference between the actual reading on the scale and the reading when the quadrants are both to earth, is proportional to the potential $v$ of the pair of quadrants which is not earthed. Let two different cells give deflexions $\theta_1$ and $\theta_2$ when applied separately to the terminals of an electrometer; then the law of proportionality will be verified if the two in series give a deflexion $\theta_1 + \theta_2$. Testing an ordinary electrometer in this way, we find that the law holds all over the scale to within 1 per cent., provided that we take the mean of the deflexions with the cells joined in opposite ways to the quadrants.

Having now means for measuring current strength such as a tangent galvanometer, and having also the quadrant electrometer to give the difference of potential between any two points, it is possible to examine the way in which the current set up in a conducting wire depends on the potential difference between its two ends.

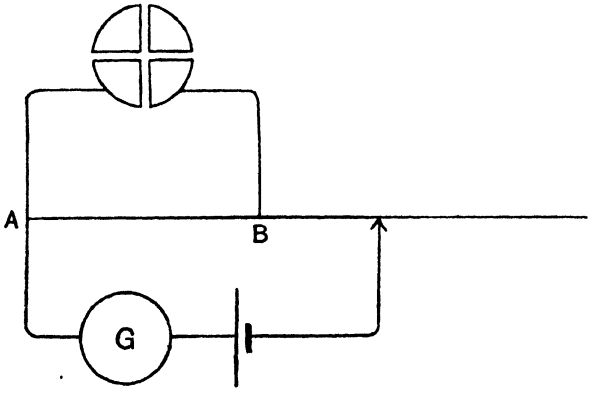



Fig. 74

The principles involved in the passage of currents through conductors were not clearly grasped in the early days of electrical science. By 1827, however, G. S. Ohm had perceived that the ideas of current and electrical pressure, previously confused, were distinct though mutually dependent, and enunciated the law bearing his name. The idea of potential was not yet completely specified, so that the form in which Ohm's law was given differed somewhat from the form in which it can now be stated, which is as follows: If $A$, $B$ are two points on a wire at constant temperature, the

difference of potential between $A$ and $B$ is proportional to the current, other things being equal.

The wire $AB$ to be tested is prolonged, and connexion made with a moveable contact to a cell and a tangent galvanometer (Fig. 74). The ends of the wire are joined to the quadrants of an electrometer, both instruments being provided with reversing keys. By moving this contact about we can get various currents in $AB$ and observe also the corresponding differences of potential between the ends of the wire. The following is the result of an actual experiment:

| Deflexion $\theta$ of tangent galvanometer | Deflexion $E$ of electrometer in arbitrary units | $\frac{E}{\tan\theta}$ |
|---|---|---|
| 15° 30′ | 24·0 | 86·8 |
| 23° 12′ | 37·8 | 88·2 |
| 33° | 56·6 | 87·4 |
| 43° | 78·3 | 87·0 |
| 50° 6′ | 105·0 | 87·8 |

The figures in the last column show no systematic variation, and therefore give a sufficient verification of Ohm's law.

A much more accurate verification can be made by indirect methods, that is by assuming the law and testing the deductions from it. Chrystal showed in 1876 that a deviation from Ohm's law of one part in ten million could have been easily detected if it had existed. Small discrepancies have since been observed at high current-densities.

If the experiment of Fig. 74 is repeated with a wire of different material, or of different length or cross-section, the ratio of the difference of potential to the current, i.e. $V/i$, will still be a constant, though the constants will be different in each case. Thus the current $i$ flowing between two points which differ in potential by $V$ is given by $V/i = R$, where $R$ depends on the material, length, cross-section and temperature of the wire. It is independent of $V$ and $i$ except in so far as the current may cause a change of temperature of the wire. This constant $R$ is called the *resistance* of the wire between the two points considered.

The derived electromagnetic unit of resistance is called the *ohm*, and a wire is said to have a resistance of 1 ohm if a current of 1 ampere flows when the difference of potential between its ends

is 1 volt. The ampere being $\frac{1}{10}$ of the absolute unit and the volt $10^8$ absolute units, it follows that the equation $i = V/R$ can be used unaltered in the absolute system if we take as the absolute unit of resistance $10^{-9}$ ohms. Large resistances are conveniently measured in megohms, a megohm being a million ohms.

A 32 candle-power glow-lamp for a 100 volt circuit has a resistance of 300 ohms when hot; a small electric bell 3 or 4 ohms. The flexible copper wire used for electric light connexions has a resistance of about 1 ohm for every 40 metres, and that of solid copper wire of 1 mm. diameter is much the same.

It will appear that electrolytes have real resistances, and that the same is true of the interior of a voltaic cell (Arts. 170, 172). But while the internal resistance of a standard cell may be 500 ohms, that of a medium-sized accumulator is only a small fraction of an ohm, and can be neglected in experiments with several hundred ohms in circuit. The behaviour of electrolytes is however not simple; Ohm's law is not satisfied in the sense of the current being proportional to the applied potential, but only after subtracting the E.M.F. of polarisation. Ohm's law fails with certain crystals used in wireless telegraphy (Art. 180), and in the conduction of electricity through gases (Ch. XII).

**66. Resistances in series and parallel.** Two resistances $AB$, $BC$, having the end $B$ of the first joined to the end $B$ of the second (Fig. 75) are said to be *in series*, and if $A$ and $C$ are joined to a battery the current flows first through $AB$ and then through $BC$. Let $R_1$ and $R_2$ be the resistances of the two coils, $V_1$, $V_2$ and $V_3$ the potentials of $A$, $B$, $C$ respectively when a current is passing, and $i$ the current. Then applying Ohm's law to the part $AB$ we have $V_1 - V_2 = R_1 i$. Similarly $V_2 - V_3 = R_2 i$, and on addition

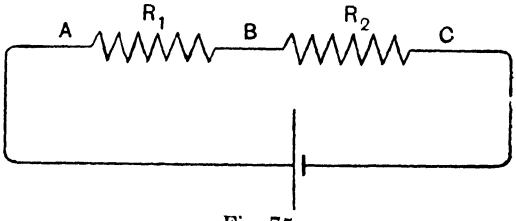


Fig. 75

$$V_1 - V_3 = (R_1 + R_2)\, i.$$

Now if the whole coil acts like one of resistance $R$ we shall have $V_1 - V_3 = Ri$: hence $R = R_1 + R_2$. Thus the resistance equivalent

to two coils in series is obtained by adding the separate resistances, and so for any number.

Since a uniform wire of length $l$ centimetres may be regarded as $l$ pieces 1 cm. long in series, it follows that its resistance is proportional, *caeteris paribus*, to its length $l$.

Two resistances joined up as in Fig. 76 are said to be *in parallel*. The current $i$ flowing from the battery divides at $A$, part $i_1$ going through $R_1$ and the rest $i_2$ through $R_2$. Since there is no accumulation of electricity at any point of the wire $i = i_1 + i_2$. If $V_1$, $V_2$ are the potentials of $A$ and $B$ respectively, Ohm's law applied to the resistance $R_1$ gives $V_1 - V_2 = R_1 i_1$, and similarly $V_1 - V_2 = R_2 i_2$.

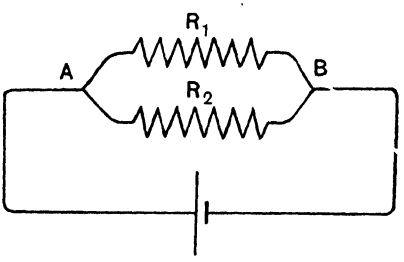


Fig. 76

Hence $$i = (V_1 - V_2)\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

But $i = (V_1 - V_2)/R$, where $R$ is the equivalent resistance of the branched circuit. Thus

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}, \text{ or } R = \frac{R_1 R_2}{R_1 + R_2} \quad \ldots\ldots\ldots (64).$$

The equivalent resistance is less than either of the separate resistances $R_1$, $R_2$. It is often necessary to calculate the fraction of the main current $i$ which passes through each branch, and $R_1 i_1 = R_2 i_2$ from the above. Hence the current divides in the inverse ratio of the resistances. The current through $R_1$ is $\frac{R_2 i}{R_1 + R_2}$ and that through $R_2$ is $\frac{R_1 i}{R_1 + R_2}$.

If there are any number of resistances in parallel we find instead of (64) the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots \quad \ldots\ldots\ldots\ldots (65).$$

When a current flows in a long straight wire the cross-sections must be equipotentials, for otherwise lateral conduction would take place. The longitudinal flow is not affected by dividing the

wire into a large number of similar and parallel filaments. Since the fall of potential down each filament is the same, and the resistances equal, the currents in the filaments are equal, so that a steady current is *distributed uniformly over the cross-section of a wire.* An approximate experimental verification is found by observing that the film of copper deposited on the cathode, in the electrolysis of copper sulphate, is appreciably uniform in the central parts.

If $n$ filaments, each of resistance $R$, go to the whole wire, the resistance of the wire is $R/n$ from (65), and is therefore inversely proportional to the sectional area. Hence the resistance of a uniform wire is proportional to its length and inversely proportional to its sectional area, provided that the temperature remains constant. It is therefore given by $R = kl/a$, where $l$ is the length, $a$ the area of the cross-section, and $k$ is a constant.

The constant $k$ at a given temperature is called its *specific resistance* and is the resistance of a cube of metal of side 1 cm. for currents normal to one face. The best conductors are those whose resistances are lowest when drawn into a wire of given size, that is, those for which $k$ is least. The quantity $1/k$, which is a measure of the conducting power of the material, is called the *conductivity* and denoted by $\sigma$. The specific resistance and conductivity of substances are intrinsic properties independent of the dimensions of the wires into which they are drawn. Measurements of the conductivity of solid metals give discrepancies which cannot be explained as errors of observation, but are due to differences in physical state. Thus the specific resistance of a metal is increased by cold-drawing and diminished by annealing. Impurities generally increase the resistance, especially that of gold and platinum.

**67. Variation of resistance with temperature.** Fig. 77 shows the variation of the specific resistance of certain metals with temperature. The specific resistance of a pure metal is roughly proportional to the absolute temperature, but becomes very small near the absolute zero. Kamerlingh Onnes, who studied the effect of impurities in gold, concluded that its resistance practically vanishes at 10° abs. He also discovered the phenomenon of *superconduction,* or sudden fall of resistance between 2° and 6° abs., which occurs with mercury, tin, lead and thallium. The results for

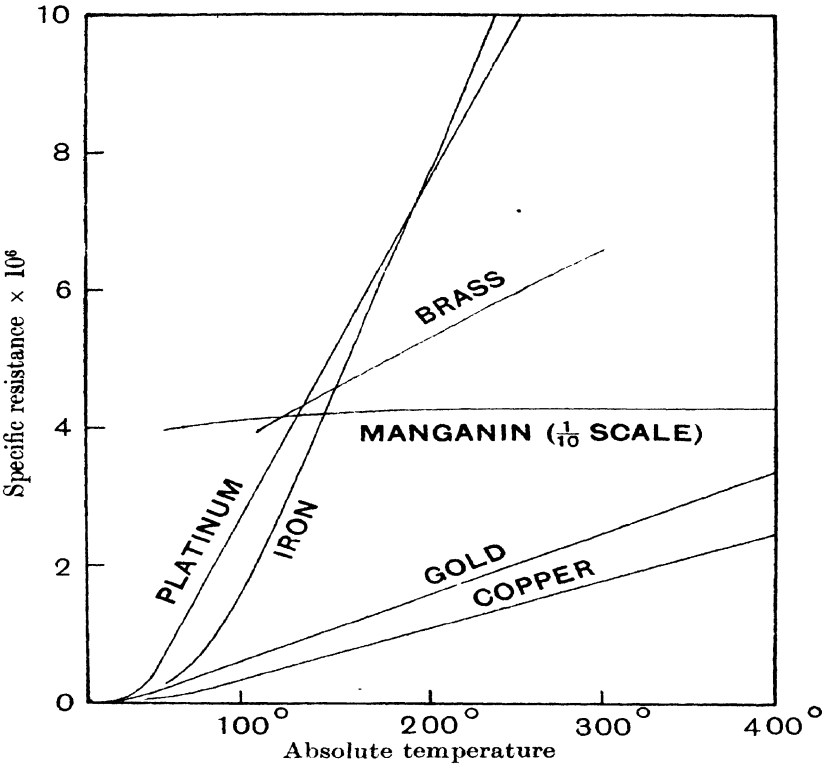

10
8
6
4
2
0
Specific resistance × 10⁶
BRASS
MANGANIN (1/10 SCALE)
PLATINUM
IRON
GOLD
COPPER
100°
200°
300°
400°
Absolute temperature

Fig. 77

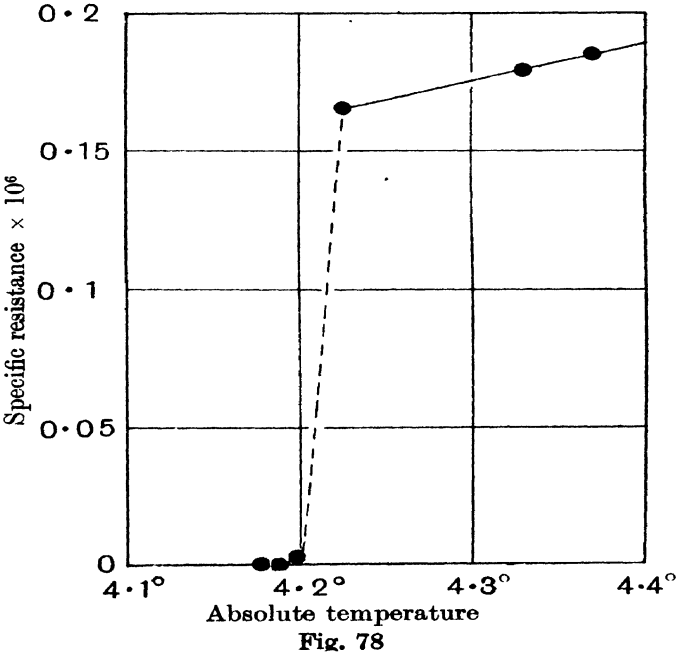

0·2
0·15
0·1
0·05
0
Specific resistance × 10⁶
4·1°
4·2°
4·3°
4·4°
Absolute temperature

Fig. 78

mercury are shown in Fig. 78. The specific resistance of (solid) mercury falls continuously down to a temperature of 4·23° abs., and seems to be tending to a small finite limit. It is, however, negligibly small (less than $10^{-6}$ of its value at 0° C.) at 4·18° abs., the fall being extremely rapid, and possibly discontinuous. Lead is super-conducting below 6° abs. Its low resistance below that temperature leads to remarkable consequences, which are described in Art. 125.

The *temperature coefficient*, or fractional rise of resistance per degree centigrade, specifies the behaviour of a metal over a small range of temperature. In the following approximate table the numbers in the last column are the mean temperature coefficients between 0° C. and 100° C.

| Metal | Specific resistance at 18° C. (ohms per cm. cube) | Conductivity at 18° C. | Temperature coefficient of resistance |
|---|---|---|---|
| Silver .. | $1{\cdot}65 \times 10^{-6}$ | $6{\cdot}06 \times 10^{5}$ | 0·00400 |
| Copper (drawn) | $1{\cdot}78 \times 10^{-6}$ | $5{\cdot}62 \times 10^{5}$ | 0·00428 |
| Gold .. .. | $2{\cdot}4 \times 10^{-6}$ | $4{\cdot}2 \times 10^{5}$ | 0·0038 |
| Aluminium .. | $3{\cdot}0 \times 10^{-6}$ | $3{\cdot}3 \times 10^{5}$ | 0·0038 |
| Zinc .. .. | $6{\cdot}1 \times 10^{-6}$ | $1{\cdot}6 \times 10^{5}$ | 0·0037 |
| Platinum .. | $1{\cdot}16 \times 10^{-5}$ | $8{\cdot}62 \times 10^{4}$ | 0·00367 |
| Nickel .. | $1{\cdot}2 \times 10^{-5}$ | $8{\cdot}3 \times 10^{4}$ | 0·0062 |
| Iron .. .. | $1{\cdot}4 \times 10^{-5}$ | $7{\cdot}1 \times 10^{4}$ | 0·0062 |
| Lead .. .. | $2{\cdot}1 \times 10^{-5}$ | $4{\cdot}8 \times 10^{4}$ | 0·0043 |
| Mercury .. | $9{\cdot}41 \times 10^{-5}$ | $1{\cdot}062 \times 10^{4}$ | 0·0009 |
| Bismuth .. | $1{\cdot}2 \times 10^{-4}$ | $8{\cdot}3 \times 10^{3}$ | 0·0042 |

Specific resistances are reduced to absolute electromagnetic units by multiplying by $10^9$, conductivities by dividing by $10^9$.

The curves for brass and manganin (Fig. 77) illustrate the behaviour of alloys, the temperature coefficient of which is generally smaller than for pure metals. Manganin (84 per cent. copper, 12 per cent. manganese, 4 per cent. nickel) has a very small temperature coefficient (about 0·00001) and is used in resistance boxes and in wire standards of resistance, the correction for change of temperature being almost negligible. It has a small thermo-electric effect against copper (Ch. VI), thus avoiding disturbance due to unequal temperatures in the circuit. German silver (temp. coeff. about 0·0004) is useful if great constancy is not essential.

The specific resistances of indifferent conductors are given in the following table, but the numbers are only approximate, and for the best insulators include to some extent the effects of surface-leakage.

| *Substance* | *Specific resistance* (ohms per cm. cube) |
|---|---|
| Graphite .. .. .. .. | 0·003 |
| 10 % solution of sodium chloride .. | 8 |
| Tap water .. .. .. .. | 1500 |
| Ordinary distilled water .. .. | $1{\cdot}5 \times 10^5$ |
| Pure water .. .. .. .. | $2{\cdot}5 \times 10^8$ |
| Oils .. .. .. .. .. | Of the order $10^{12}$ |
| Paraffin wax, quartz, ebonite, sulphur | ,, $10^{15}$ |

It appears from these numbers that the criterion of good insulation is much less stringent in current electricity than in electrostatic work. All that is wanted in current electricity is that practically all the current in a circuit shall be carried by the metallic wire and that none leaks away through the insulating sheath. If we have a copper wire and a "wire" of ebonite of the same size in parallel, the current divides in the ratio of the specific resistances (Art. 66); that is, in the ratio $10^{21}$ to unity. Even if the available cross-section of ebonite was a billion times that of the copper no appreciable part of the current would be carried by the insulator. It is thus not surprising that substances like wood and oil can safely be regarded as insulators for these purposes.

The resistance of *selenium* is much reduced when light falls on it. It is used in nearly all methods of transmitting pictures telegraphically.

**68. Rheostats and resistance boxes.** The *sliding rheostat* is a convenient instrument for adjusting the current in a circuit. If consists of a long coil of German silver wire wound in grooves on a slate cylinder, with its ends connected to two screw terminals. A third terminal is connected with a slider by means of which more or less of the coil may be inserted between it and either of the other terminals, and the adjustment of resistance is practically continuous. Fig. 79 shows a rheostat for currents of the order of 10 amperes. Here one terminal is joined to one end of a continuous

coil of thick wire interrupted at intervals by heavy brass studs, and the other terminal (not visible) is joined inside to the moveable handle, carrying a brass slider which moves over the studs. The adjustment proceeds in a number of stages. A danger with all rheostats is that of inadvertently allowing more current to pass than the rheostat, or rest of the circuit, can safely carry. Rough adjustment of current suffices for charging accumulators, in experiments with electric generators, and so forth, and then a tray of glow-lamps is useful, one or more lamps being joined in parallel between fixed terminals (cf. Fig. 221).

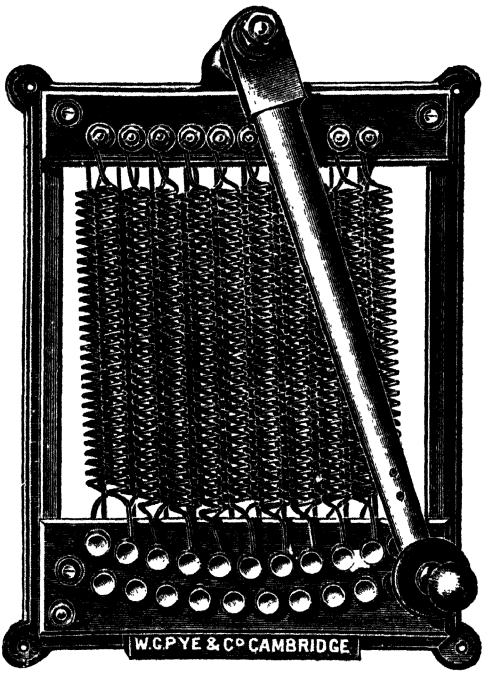



Fig. 79

An arrangement of coils known as a resistance box is generally used for accurate laboratory work. Fig. 80 shows the

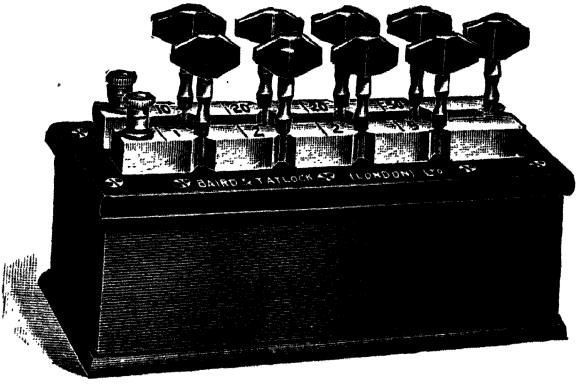


Fig. 80

completed instrument, Fig. 81 the arrangement of the coils. Each coil consists of a manganin wire doubled back on itself and wound

on an insulating spool, and its ends are fastened to the lower sides of the heavy brass strips which form the top of the box. When a brass plug is placed in a conical hole between two strips the resistance of the corresponding coil is cut out since nearly all the current passes through the plug. The resistances of the coils are adjusted accurately to their marked values, and by combining them properly any required number of ohms can be put in the circuit (in Fig. 80 from 1 to 100). It is necessary in using resistance boxes to press the plugs in firmly, particularly with coils of high resistance, otherwise variations will occur in the current. Many troubles can be traced to insufficient contact between parts of a circuit. The coils of a well-made resistance box should be correct to about one part in 2000.

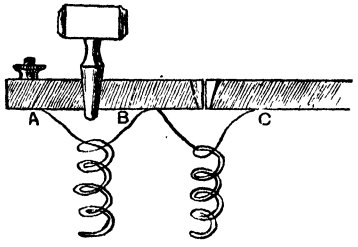


Fig. 81

Standard resistances are made as nearly as possible an exact number of ohms, and are used immersed in a water or paraffin bath at a known temperature. It will appear shortly that all that is done in ordinary laboratory experiments is to *compare* resistances: thus all the resistances in a laboratory may be supposed known in terms of a single coil, which we may call the laboratory standard. The relation of this standard to the true ohm as defined in Art. 65 is a matter requiring special investigation (see Art. 140).

**69. Galvanometers, ammeters and voltmeters.** Mention may here be made of certain laboratory instruments for measuring current and potential, deferring a detailed description until the principles of their action have been discussed (Arts. 86, 87). The most sensitive *galvanometers* are read by the reflexion method with a lamp and scale: many in current use will measure a current of one-millionth of an ampere. A modern galvanometer is independent of external magnetic fields such as that of the earth, and may therefore be fitted with a pointer moving over a scale graduated so as to give the current directly in absolute measure. Fixed-scale instruments of this kind for measuring large currents are called *ammeters*.

The accuracy of the scale-readings of an ammeter may be checked by the electrolytic deposition of copper as follows.

The electrolyte is a 25 per cent. solution of copper sulphate to which 1 per cent. of sulphuric acid has been added. The electrodes are copper plates, the cathode being removeable and smoothed off at the edges. A circuit is set up containing the ammeter, a rheostat and a key in addition to the cell and voltameter. If it is desired to test the accuracy of (say) the reading 50 on the ammeter scale, first take a trial observation to adjust the current roughly to its right value. Cut off the current and clean the cathode thoroughly with sand-paper, then dip it in dilute sulphuric acid and afterwards in distilled water and alcohol, drying it *very gently* over a Bunsen flame. Weigh the cathode accurately, put it in its place, and turn on the current. Use the rheostat to keep the deflexion of the ammeter as near to 50 as possible, and cut the current off after a measured time. Dip the cathode successively in dilute sulphuric acid, distilled water and alcohol, then dry gently and weigh as before. If $m$ is the mass of copper deposited and $t$ the time of the experiment in seconds, the true current $i$ in amperes is given by

$$m = 3{\cdot}294 \times 10^{-4}\, it.$$

The reading of the ammeter is compared with this.

A *voltmeter* indicates directly the difference of potential between its terminals. The usual type consists of a sensitive fixed-scale galvanometer in series with a high resistance. At present, however, we may confine our attention to the *electrostatic voltmeter*, which in principle is the same as an electrometer in which the needle is joined to one pair of quadrants. An electrometer so used is said to be connected *idiostatically*, as it works without extraneous charging of the needle. Let $V$ be the difference of potential between the two pairs of quadrants, and let us neglect contact potential (such as arises between the needle and the quadrants). The theory of Art. 48 makes it probable that the deflexion is given approximately by

$$\theta = \frac{V^2}{P + QV^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (66),$$

where $P$ and $Q$ are constants; compare equation (53). The deflexion will, therefore, be proportional at first to the *square* of the applied

voltage, so that the instrument is relatively insensitive to small voltages. Contact potential makes the deflexions for voltages $+ V$ and $- V$ unequal, and can be put in evidence in this way. Electrostatic voltmeters reading below 1000 volts are, in essence, robust electrometers with a number of needles fixed to the same axis. One pair of quadrants can be suppressed and the needle deflected by attraction of the other pair, as shown in Fig. 82, which represents an instrument designed by Lord Kelvin for voltages up to 10,000, the motion of the needle being controlled by gravity.

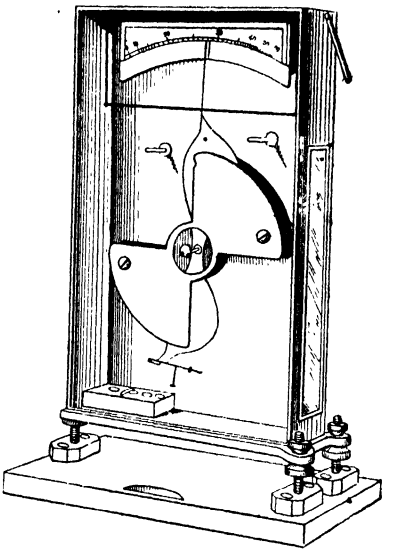


Fig. 82

**70. Heat produced by currents.** Let $V$ be the difference of potential between two points $A$, $B$ of a circuit, and in time $dt$ let a charge $de_1$ of positive electricity pass from $A$ to $B$ and a charge $- de_2$ of negative electricity from $B$ to $A$. The work done on the positive electricity is $Vde_1$ and on the negative $- V(- de_2)$; in all $V(de_1 + de_2) = Vidt$ by Art. 56. The work done in driving the current is therefore $Vi$ ergs per second as far as the part $AB$ of the circuit is concerned, and this work must reappear as heat developed in $AB$. For steady currents the heat developed in time $t$ is therefore $Vit$ measured in work units: if the current is not steady it is expressed by $\int Vidt$. True electromagnetic units are understood here. For steady currents $V = Ri$ by Ohm's law, and the heat developed is $Ri^2/J$ calories per second, where $J = 4{\cdot}18 \times 10^7$ is the mechanical equivalent of heat and $R$, $i$ are in absolute electromagnetic units. Thus the rate of heat-development is proportional to the square of the current.

A simple experiment is to find the mechanical equivalent of heat by electrical methods. We require a coil of fine wire immersed in water in a calorimeter, with thick leads which do not become appreciably heated. First weigh the calorimeter and coil when dry, and also when containing enough water to cover the fine part of

the wire. Take the air-temperature $\alpha$ and the initial temperature $\theta_0$ of the water. Set up a circuit containing the calorimeter, an ammeter and a key, together with a cell of E.M.F. sufficient to cause a rise in the temperature of the calorimeter of at least half a degree per minute. Pass a current through the coil for about 20 minutes, reading the ammeter each minute and stirring the water in the calorimeter constantly. Note the final temperature $\theta_1$ and the time $t$ of the experiment in seconds. After stopping the current observe the falling temperature each minute for 10 minutes, stirring as before. In this way we find the average fall of temperature $\omega$ per minute on account of radiation at the temperature $\theta_1$. The mean temperature during the rise is $\frac{1}{2}(\theta_0 + \theta_1)$, so that we may expect radiation to go on during the rise at an average rate

$$\omega \{\tfrac{1}{2}(\theta_0 + \theta_1) - \alpha\}/(\theta_1 - \alpha)$$

per minute, assuming that the rate of radiation is proportional to the difference of temperature between the calorimeter and the air. The radiation correction to be applied to $\theta_1$ is therefore approximately

$$\phi = \frac{\omega t}{60} \frac{\frac{1}{2}(\theta_0 + \theta_1) - \alpha}{\theta_1 - \alpha}.$$

If $i$ is the mean current in amperes and $R$ the (known) resistance of the coil in ohms, the heat developed in the coil is $10^7 Ri^2/J$ calories per second. Hence if $W$ is the water value of the calorimeter and $m$ the mass of the contained water, the mechanical equivalent $J$ is given by

$$\frac{10^7 Ri^2}{J} t = (m + W)(\theta_1 - \theta_0 + \phi).$$

The values of $J$ obtained electrically agree with those derived from mechanical experiments, thus justifying the application of the principle of energy to these phenomena. Further, the experiments afford a fresh test of Ohm's law. For since the same value of $J$ can be obtained with widely different currents, it follows that the rate of development of heat is proportional to the square of the current, so that $V$ is proportional to $i$.

The development of heat by electric currents is put to laboratory use in *fuses*, or short wires of fusible metal inserted in a circuit, which become heated and melt when more than a certain current

passes through them, thus protecting the ammeters or other electrical apparatus in the circuit from excessive current. That the action of fuses is directly due to heat can be shown by cooling a fuse wire in water, when it will carry many times its normal current.

The unit of work in the derived system of electromagnetic units is the *joule*, named in honour of J. P. Joule, who first showed that the heating effect of currents is in accordance with the conservation of energy*. One joule is $10^7$ ergs, and is the work done on one coulomb moving through a difference of potential of one volt. The *watt* is the derived unit of power, and denotes the performance of work at the rate of one joule or $10^7$ ergs per second. It follows that if a current of $i$ amperes is flowing in a circuit, the rate at which work is being converted into heat in a part $AB$ of the circuit is $Vi$ watts, where $V$ is the difference of potential between $A$ and $B$ in volts.

Since the mechanical equivalent of heat, in terms of the calorie at 15° C., is $4{\cdot}18 \times 10^7$, we have

$$1 \text{ calorie} = 4{\cdot}18 \text{ joules}.$$

The joule is a small unit of energy: the usual technical unit is the *Board of Trade unit*, or *kilowatt-hour*. This is the energy consumed when 1000 watts are used for one hour, so that

$$1 \text{ kilowatt-hour} = 3{\cdot}6 \times 10^6 \text{ joules}.$$

The cost of electrical energy for lighting purposes is generally between 6*d*. and 10*d*. per kilowatt-hour. This amount of energy will keep a 16 candle-power lamp alight at normal brilliancy for about 60 hours.

**71. The Wheatstone's bridge.** The arrangement of four resistances $R_1$, $R_2$, $R_3$, $R_4$ shown in Fig. 83 was used by Christie in 1833, and more extensively by Wheatstone in 1843. $A$ and $C$ are joined to the poles of a battery, $B$ and $D$ to a galvanometer $G$.

The special advantage of Wheatstone's bridge is that the current in $BD$ is made zero, and therefore it is possible to compare resistances accurately without depending on the uniformity of the scale-readings of galvanometers. We shall show that, if the resistances $R_1$, $R_2$, $R_3$, $R_4$ in the arms of the bridge are adjusted so that no

* The $Ri^2/J$ law is often called Joule's law.

current flows in the galvanometer whether the battery in $AC$ is on or off, they satisfy the condition

$$R_1/R_2 = R_3/R_4 \quad \ldots\ldots\ldots\ldots\ldots\ldots (67).$$

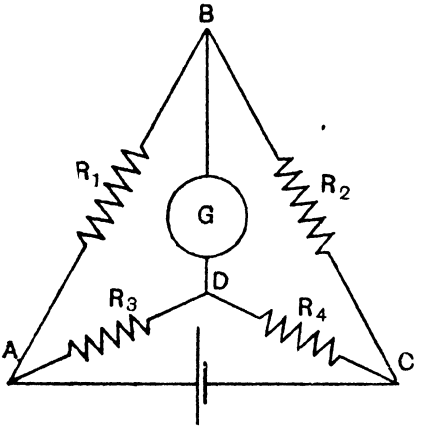



Fig. 83

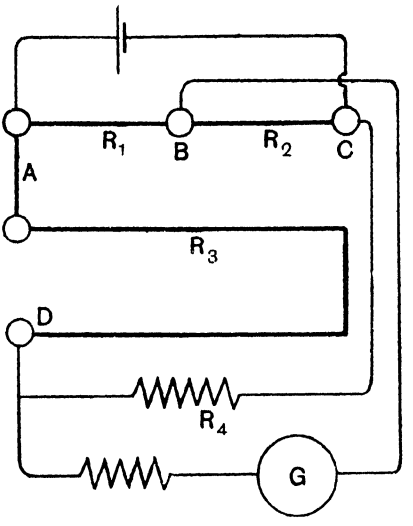



Fig. 84

For if there is no current in $BD$ the current in $AB$ has the same strength $i$ as that in $BC$. Similarly the currents in $AD$ and $DC$ have the same strength $j$. Then applying Ohm's law to the four arms in

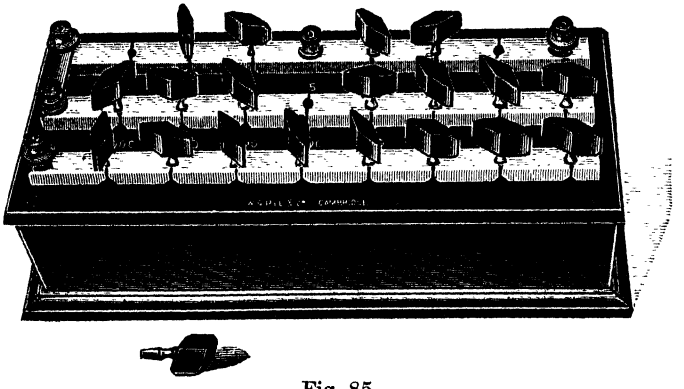


Fig. 85

succession we have $A - B = R_1 i$, $B - C = R_2 i$, $A - D = R_3 j$, $D - C = R_4 j$, where the letters $A, B, C, D$ are used for the potentials at the corners of the bridge. But $B = D$ since no current passes in $BD$. Hence $R_1 i = R_3 j$ and $R_2 i = R_4 j$, from which the above result follows. Thus if we have three known resistances $R_1$, $R_2$, $R_3$,

equation (67) enables us to measure any other resistance $R_4$ in terms of them, and great accuracy is obtainable by using a sensitive galvanometer. Hardly less important is that the ratio of the currents in $ABC$ and $ADC$ is known with the same accuracy. A resistance box should be inserted in the galvanometer circuit and a key in both this and the battery circuit.

The so-called Post Office box (Fig. 85) is a Wheatstone's bridge ready for the measurement of unknown resistances. The method of using the box to measure a resistance $R_4$ will be evident from the diagram (Fig. 84) which shows its relation to the theoretical form of bridge.

The following example shows how a single observer can find the variation of the resistance of a coil of wire with temperature. The coil employed was of copper, wound on a hollow spool inside a thin-walled glass tube so that it could be placed in a water bath, and a thermometer was passed down the spool to indicate the temperature of the coil. A Post Office box was used in which $R_1 = 100$, $R_2 = 1$, and balance was obtained when the coil was cold with $R_3 = 108$. Thus the resistance of the coil when cold was 1·08 ohms. When $R_3$ was altered to 113 the balance of the bridge was destroyed and the galvanometer deflected, but the balance could be recovered by heating up the coil slowly until the galvanometer returned to zero. The thermometer was read when this occurred, and thus the temperature at which the resistance of the coil is 1·13 ohms was known. Proceeding similarly with $R_3 = 118$ and so on, we can find the variation up to the temperature of boiling water.

The principle of the Wheatstone's bridge is very important and is extensively used. One application is to compare, with great accuracy, the resistances of two coils $A$, $B$ known to be nearly equal, for example two one-ohm coils from a resistance box. Let $C$, $D$ be two other resistances, also nearly equal. Then a Wheatstone's bridge made up with the four coils is nearly balanced, and exact balance can be obtained by shunting $A$ with a comparatively large resistance $a$ and $B$ with a resistance $b$. Then

$$\frac{\frac{1}{A}+\frac{1}{a}}{\frac{1}{B}+\frac{1}{b}} = \frac{D}{C}.$$

Next interchange $A$ and $B$ and balance again with shunt $a'$ and $b'$ respectively. Then

$$\frac{\frac{1}{A}+\frac{1}{a'}}{\frac{1}{B}+\frac{1}{b'}}=\frac{C}{D},$$

from which we have

$$\left(\frac{1}{A}+\frac{1}{a}\right)\left(\frac{1}{A}+\frac{1}{a'}\right)=\left(\frac{1}{B}+\frac{1}{b}\right)\left(\frac{1}{B}+\frac{1}{b'}\right).$$

With well-made coils $A$ and $B$ will not differ by more than one part in 1000, and $A/a$, $B/b$, etc. are of the same order of magnitude. Hence neglecting terms of the order $10^{-6}$ we have

$$\frac{1}{B^2}-\frac{1}{A^2}=\frac{1}{A}\left(\frac{1}{a}+\frac{1}{a'}\right)-\frac{1}{B}\left(\frac{1}{b}+\frac{1}{b'}\right).$$

If $A$ and $B$ are both nearly equal to $A_0$, this gives approximately

$$A-B=\frac{A_0^2}{2}\left(\frac{1}{a}+\frac{1}{a'}-\frac{1}{b}-\frac{1}{b'}\right),$$

so that the difference of the resistances of $A$, $B$ is known. In order to obtain the ratio of the resistances correct to the order $10^{-6}$ it is only necessary that $A_0$ and the shunting resistances should be known to one part in 1000 of their value, for we have

$$\frac{A-B}{A_0}=\frac{A_0}{2}\left(\frac{1}{a}+\frac{1}{a'}-\frac{1}{b}-\frac{1}{b'}\right),$$

and the terms on the right are already small of the order $10^{-3}$.

Further information on resistance and its measurement will be found in the *Dictionary of Applied Physics*, vol. 2, pp. 700–726.

**72. Comparison of electromotive forces. The potentiometer.** An accumulator will deliver a current for a considerable time without suffering any appreciable change in its electromotive force, but standard cells are not constructed to give more than the minutest current. It is thus necessary to have some means of comparing the electromotive forces of two cells without requiring both of them to deliver steady currents. This is supplied by the compensation method due in principle to Poggendorff and du Bois-Reymond.

Let $R_1$, $R_2$ (Fig. 86) represent two resistance boxes connected in series with an accumulator of electromotive force $V$. A "shunt" circuit containing another cell $v$, a resistance box and sensitive galvanometer, is connected to the ends of the resistance $R_1$ and can be put into operation by means of a key. The resistances $R_1$, $R_2$ are adjusted so that no current passes through the galvanometer. Let $i$ be the current in the main circuit, $A$, $B$, $C$ the potentials of the corresponding points in the diagram. Then $A - B = R_1 i$ and $V = A - C = (R_1 + R_2)\, i$, and since no current flows in the shunt circuit $v = R_1 i$. Hence

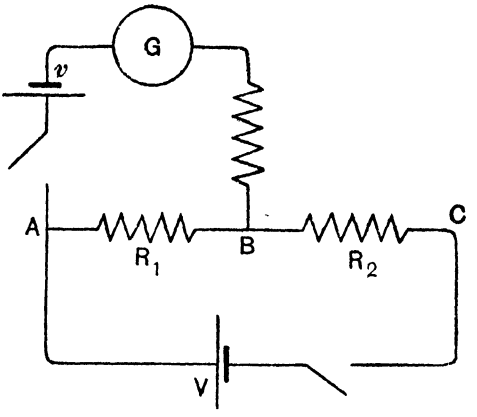


Fig. 86

$$\frac{v}{V} = \frac{R_1}{R_1 + R_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (68).$$

The E.M.F. of the cell $V$ in the main circuit must exceed that in the shunt circuit. It is convenient to keep $R_1 + R_2$ at a fixed value by using two resistance boxes of the same pattern with only half the full number of plugs, and arranging so that the plugs in one box correspond to gaps in the other, and *vice versâ*.

If $v$ is a standard cell, the E.M.F. $V$ of the accumulator is accurately known from equation (68), and then the E.M.F. of any other cell can be found by substituting it for the standard cell in the shunt circuit. Moreover, a voltmeter can be attached to the poles of $V$ during the experiment, and its reading can be compared with the true E.M.F. given by equation (68). We can thus calibrate a voltmeter at the points on its scale corresponding to 2, 4, 6, ... volts approximately.

The arrangement of the two resistances $R_1$, $R_2$ is also useful as a potential-divider or *potentiometer*; for the difference of potential between $A$ and $B$ is $R_1/(R_1 + R_2)$ times the E.M.F. of the cell $V$ applied to $AC$, and $AB$ can be applied to keep any two parts of an apparatus at an adjustable difference of potential. A sliding

potentiometer giving adjustment in small stages is useful in electrostatics and in the conduction of electricity through gases. A number of equal coils of wire are soldered to brass strips arranged

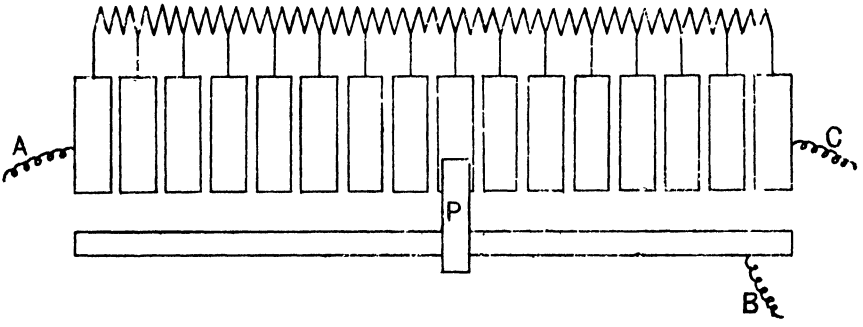



Fig. 87

alongside one another as shown in Fig. 87, the two end strips $A$, $C$ acting as terminals. A sliding brass piece $P$ moves over the segments and also slides on a brass rod from which connexion is taken off at $B$. If there are 50 coils of (say) 20 ohms each, and a 2 volt accumulator is applied to $AC$, the difference of potential between $A$ and $B$ may be quickly adjusted to 1/50 volt, 2/50 volt, etc., at will.

### 73. Other applications of Ohm's law.

(1) *Standardisation of a sensitive galvanometer by shunting.* It is often necessary to find the current in a sensitive galvanometer which corresponds to a given deflexion; or the current per scale division if this is the same all over the scale.

An accumulator $V$ is used, and a circuit set up as in Fig. 88. The resistances $R$ and $S$ should be considerable (at least 100 ohms) and $X$ is either 1 ohm or $\frac{1}{10}$ ohm. $R$ or $S$ is then adjusted until the right deflexion is obtained. Neglecting the internal resistance of the accumulator, it is not difficult to show that the current $i$ through the galvanometer is given by

$$V = i\left\{R + S + G + \frac{R}{X}(S + G)\right\},$$

where $G$ is the resistance of the galvanometer.

If $X$ is small compared with $R$ and $S$,

$$i = \frac{VX}{R(S + G)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (69).$$

This approximate formula is more easily arrived at as follows. Since $S + G$ is merely a large shunt to $X$, the equivalent resistance of the parallel circuits is nearly equal to $X$, so that the difference of potential between $A$ and $B$ is approximately $VX/R$. Applying Ohm's law to the galvanometer circuit then gives equation (69). The E.M.F. of the accumulator may be read on a voltmeter if necessary, but it often happens that the same cell is used in another part of the experiment and that $V$ cancels out.

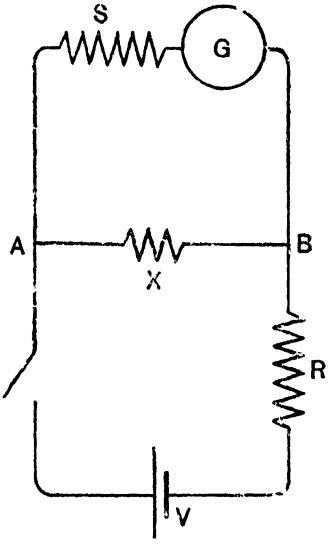



Fig. 88

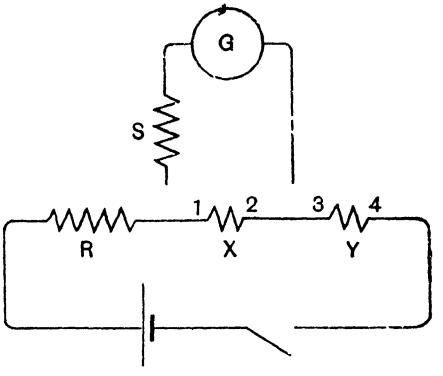



Fig. 89

(2) *Measurement of a small resistance.* Let 12 be the terminals of the small resistance $X$ which it is desired to measure (Fig. 89), 34 those of a standard small resistance $Y$, for example the $\frac{1}{10}$th ohm coil of a resistance box. Set up a circuit as shown with an additional resistance $R$ sufficient to prevent the passage of an excessive current, leaving the galvanometer circuit free to connect to any points in the main circuit. First join the galvanometer circuit to 12 and adjust $S$ for a certain deflexion $\theta$ of the galvanometer. Since $S + G$ is large in comparison with $X$ the current in the galvanometer may be taken as $Xi/(S + G)$, where $i$ is the current in the main circuit. Now disconnect the galvanometer circuit from 12 and connect it to 34, changing the resistance $S$ to the value $S'$ necessary to make the deflexion $\theta$ once more. Then

since the galvanometer currents are the same in the two cases we have

$$\frac{X}{S+G} = \frac{Y}{S'+G} \quad \ldots\ldots\ldots\ldots\ldots\ldots (70),$$

giving $X$.

**74. Experiments with the quadrant electrometer.** The following experiments give more trouble than that, for example, of Art. 65, since the electrometer is insulated and liable to gain or lose charge by leakage. Insulating plugs of ebonite should be scraped occasionally with sand-paper and protected as far as possible from light and dust, but may nevertheless leak in damp weather, so that observations should be taken as soon as possible after insulating the electrometer. Connections of any length are made by wires passing down earthed brass tubes, as a guard against electrostatic influence.

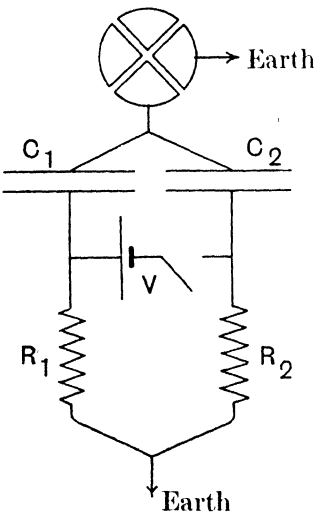


Fig. 90

(1) *Comparison of capacities.* First insulate the electrometer with the key off. Then adjust the ratio of the resistances $R_1$, $R_2$ till there is no deflexion on pressing down the key. On pressing down the key the outside plates of the condensers, previously at zero potential, take up the potentials $R_1 V/(R_1 + R_2)$ and $-R_2 V/(R_1 + R_2)$ respectively. Since the electrometer is not deflected the inner plates are still at zero potential, so that the charges on the inner plates are

$$\frac{R_1 C_1 V}{R_1 + R_2} \text{ and } -\frac{R_2 C_2 V}{R_1 + R_2}.$$

But there is no charge on the electrometer, and initially no charge on either of the plates. Hence the actual charges on the plates must be equal and opposite, i.e.

$$R_1 C_1 = R_2 C_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (71).$$

If a condenser of known capacity is taken as $C_1$, the capacity of another condenser $C_2$ is found from the last equation. It will sometimes happen that one of the condensers leaks, so that on

pressing down the key the electrometer moves off continuously. In this case we should look for a sudden jump of the needle on making the circuit, and adjust the resistances until it vanishes, as nearly as we can judge.

(2) *Measurement of the charge required to produce a given deflexion of the quadrant electrometer.*

We require a condenser of known capacity $C$. First insulate the electrometer (Fig. 91) with the battery circuit open, and then close the circuit, obtaining a steady deflexion of $\theta$ scale divisions. If $V$, $v$ are the potentials of the outside and inside plates of the condenser, the charge on the electrometer, being equal and opposite to the charge on the inner plate of the condenser, is $e = C(V - v)$. Here $V = R_1 V_0/(R_1 + R_2)$. To find $v$ connect the electrometer directly to the point $M$ and reproduce the deflexion $\theta$ with resistances $S_1$ and $S_2$ instead of $R_1$ and $R_2$. Then

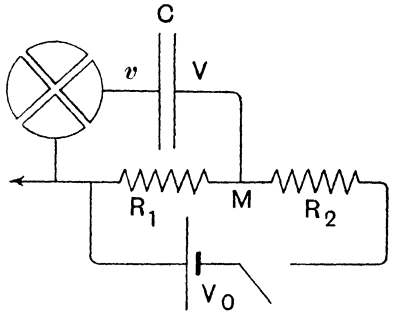


Fig. 91

$$v = S_1 V_0/(S_1 + S_2).$$

It is convenient to make $R_1 + R_2 = S_1 + S_2$ as suggested in Art. 72, in which case the charge on the electrometer corresponding to the deflexion $\theta$ is

$$e = \frac{C(R_1 - S_1) V_0}{R_1 + R_2} \quad \ldots\ldots\ldots\ldots\ldots (72).$$

$V_0$ is read on a voltmeter and reduced to electrostatic units by dividing by 300, and then $e$ is obtained in electrostatic units.

It is usually found that the charge is proportional to the deflexion if the latter is not too large, and then the charge per scale-division is $C(R_1 - S_1) V_0/(R_1 + R_2)\theta$. The effective capacity of the electrometer at a given deflexion is the ratio of $e$ to $v$. This is the quantity

$$c = \frac{C(V - v)}{v} = \frac{C(R_1 - S_1)}{S_1} \quad \ldots\ldots\ldots (73).$$

It is of the order of 50 cm., not varying much with deflexion, but increasing with the potential of the needle.

(3) *Measurement of large resistance with a condenser and an electrometer or electrostatic voltmeter.* We shall describe a method suitable for resistances of the order of 1 megohm. For larger resistances a smaller condenser is used.

Charge up the condenser $C$ (about 10 microfarads) to (say) 150 volts by bringing a wire from an influence machine near to a wire extended from the condenser circuit, the other terminal of the influence machine being earthed. The potential of the inner plate of the condenser is read on an electrostatic voltmeter. Having observed this, press down the key for about 10 seconds and then insulate and wait for the deflexion to become steady again. Let $V$ be the potential at time $t$. The current is $-CdV/dt$, and Ohm's law gives $V = -RCdV/dt$, so that $t = RC \log_e (V_0/V)$. Hence if the potential falls from $V_0$ to $V_1$ in $t$ seconds,

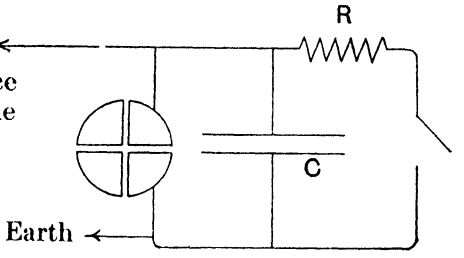


Fig. 92

$$R = \frac{t}{2{\cdot}302C \log_{10} (V_0/V_1)} \quad \ldots\ldots\ldots\ldots (74),$$

giving $R$ in ohms when $C$ is in farads. The insulation in this experiment must be good.

(4) *Measurement of contact potential.* The plates whose contact difference of potential it is required to measure are placed in a metal case so that they can slide in and out while remaining parallel to one another (Fig. 93), and the case is put to earth to protect the inner plate from electrostatic disturbances. It is possible to adjust the ratio $R_1/R_2$ so that the electrometer is not affected by pulling the right-hand plate away from a short distance in front of the other (cf. Art. 49). The applied

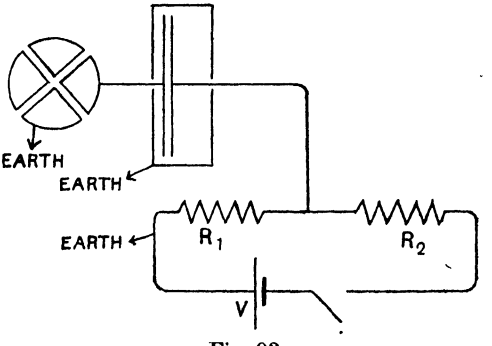


Fig. 93

potential, which is $-R_1V/(R_1+R_2)$, is then equal and opposite to the natural contact potential of the right-hand plate over the left-hand plate. This latter potential is therefore $R_1V/(R_1+R_2)$, positive in the figure since a long stroke denotes the positive pole of a cell.

The magnitude of the contact potential depends very much on the state of the opposing surfaces. The following table, due to Anderson and Bowen, gives the values for freshly-cleaned surfaces.

*Contact Potential Differences (volts).*

| | |
|---|---|
| Aluminium – copper .. | + 0·83 |
| Zinc – copper .. .. | + 0·76 |
| Lead – copper .. .. | + 0·55 |
| Brass – copper .. .. | + 0·15 |
| Gold – copper .. .. | – 0·11 |
| Silver – copper .. .. | – 0·13 |

The additive law of Art. 49 is always found to hold within the limits of experimental error.

**75. Measurement of dielectric constants.** Dielectric constants of well-insulating solids may be measured by using two parallel-plate condensers of identical dimensions and comparing their capacities as in the last article. The two condensers may be combined into a single composite condenser with a common middle plate (with guard-ring), having air on one side and the solid dielectric on the other. As the induced charges will leak on to the dielectric in time, it is advisable to close the battery circuit only momentarily.

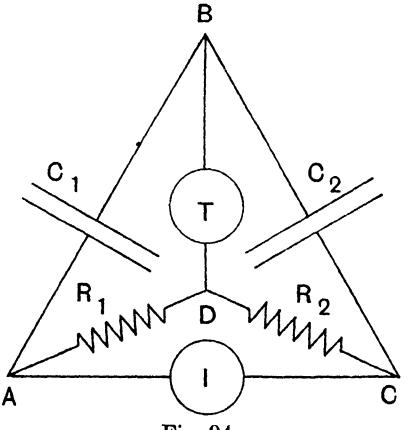


Fig. 94

The electrometric method fails with badly-insulating solids and with liquids, but capacities can always be compared by the following method, which does not differ in principle from the preceding one. The two condensers $C_1$, $C_2$ form a Wheatstone's bridge with two resistances $R_1$, $R_2$ (Fig. 94). The battery is replaced by a small

induction coil $I$ (see Art. 137) which gives intermittent currents alternating in direction, and the galvanometer by a telephone receiver $T$, which is a sensitive detector for these currents. The interrupted current does not give a very pure note, and it is more pleasant to use a high-frequency alternator (Art. 154) or an oscillating triode (Art. 185) instead of the induction coil. The resistances $R_1$, $R_2$ are of the order of 10,000 ohms, preferably liquid (distilled water in cells with platinised platinum electrodes). It is not convenient to vary the resistances, so that one of the condensers must be adjustable. The adjustment is made till there is no sound in the telephone, i.e. till there is no current in the arm $BD$ whatever changes occur in the arm $AC$. Then if $i$ is the current in the arms $AB$, $BC$ and $j$ that in the arms $AD$, $DC$, we have

$$A - B = \frac{1}{C_1}\int i\,dt, \quad B - C = \frac{1}{C_2}\int i\,dt,$$
$$A - D = R_1 j, \qquad D - C = R_2 j,$$

where $A$, $B$, $C$, $D$ stand for the potentials at the corresponding points. Since $B = D$, this gives

$$R_1 C_1 = R_2 C_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (75).$$

Suppose that the condensers have plates of the same area $A$, and that one or both of the distances $d_1$, $d_2$ are adjustable with micrometer screws. Then if $C_1$ is the air-condenser,

$$C_1 = A/4\pi d_1, \quad C_2 = KA/4\pi d_2.$$

Hence equation (75) gives $R_1/d_1 = KR_2/d_2$, so that $K$ is known.

This method is not applicable to gases, for which $K$ differs by very little from unity. The dielectric constants of gases can be measured by the following method, due to Boltzmann.

A parallel-plate condenser is placed under a bell-jar connected to a mercury manometer and an air-pump, and the bell-jar is exhausted. One plate of the condenser is joined to an electrometer by a wire leading out through the bell-jar, the other to a source of high potential consisting of a battery of cells one pole of which is earthed. Suppose that the electrometer is initially earthed and the outside plate at potential $V_0$. On insulating there is then a charge $CV_0$ on the inner plate, where $C$ is the capacity of the condenser. Air is now admitted to the bell-jar, and the electrometer shows a

slight deflexion $\theta$. The potential of the electrometer is $\lambda\theta$, where $\lambda$ is some constant. There is thus a charge $KC(V_0 - \lambda\theta)$ on the inner plate of the condenser and $c\lambda\theta$ on the electrometer, where $c$ is the capacity of the electrometer, making $KC(V_0 - \lambda\theta) + c\lambda\theta$ altogether on the insulated system. A second experiment, with exhausted bell-jar, consists in first insulating the electrometer and then changing the potential of the outside plate of the condenser from $V_0$ to $V_0 + V$, producing a deflexion $\phi$. Since the charge on the insulated system is the same in all three cases, we have

$$CV_0 = KC(V_0 - \lambda\theta) + c\lambda\theta = C(V_0 + V - \lambda\phi) + c\lambda\phi.$$

It follows from these equations that

$$(K-1)CV_0 = (KC - c)\lambda\theta, \text{ and } CV = (C - c)\lambda\phi.$$

Hence

$$\frac{(K-1)V_0}{V} = \frac{KC - c}{C - c} \cdot \frac{\theta}{\phi}.$$

Since $K$ differs by very little from unity, the last equation is practically equivalent to $K - 1 = V\theta/V_0\phi$.

Boltzmann satisfied himself that the deflexions observed were really due to the difference of the dielectric constants of air and vacuum, and not to charges produced by friction when the air is let in. In his experiments $V_0$ was of the order of 300 volts, $V$ 1 volt, and $\theta/\phi$ of the order 1/10.

**76. Platinum resistance thermometry.** The resistance of a coil of wire is an index of its temperature or that of any vessels in which it may be immersed. Platinum can be used between temperatures of $-200°$ C. and 1000° C. Suppose we have a coil of platinum immersed in a vessel, and that its resistance is $R$. If $R_0$ is the resistance when immersed in melting ice and $R_{100}$ in boiling water, the "platinum temperature" $t$ is given by the equation

$$\frac{t}{100} = \frac{R - R_0}{R_{100} - R_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots (76).$$

Callendar showed that the relation of $t$ to the corresponding true temperature $\theta$ (thermodynamic temperature with 0° and 100° fixed points) is given accurately by

$$\theta - t = \frac{\delta}{10000}\theta(\theta - 100) \quad \ldots\ldots\ldots\ldots (77),$$

where $\delta$ is a constant for a particular specimen of wire, and always

nearly equal to 1·5. To determine $\delta$ we require an observation of the boiling-point of sulphur, which was shown by Callendar and Griffiths to be at $\theta = 444{\cdot}53°$ C. For low-temperature work the boiling-point of oxygen (− 182·5° C.) is used instead.

The resistance of the platinum coil is measured on a Wheatstone's bridge, with the connexions of Fig. 95. The thermometer itself, shown diagrammatically in the figure, consists of a platinum coil wound on mica in a glass or porcelain tube. Uncertainty as to the temperature of the leads in the upper part of the tube is avoided by using a pair of dummy leads, identical with the true leads except that they are not attached to a coil. The leads are of copper or platinum and are kept apart by discs of mica inserted in the tube, which also serve to hinder convection currents. The difference of the resistances of the coil and dummy leads is measured by a simple device due to Carey Foster, in which the two fixed arms of the bridge are kept equal. The point of contact $C$ of the galvanometer slides along a stretched wire $XY$, and adjustment is made till the galvanometer shows no deflexion when the battery is on. The resistances in the arms $AD$, $DB$ are equal, so that if $x$ and $y$ are the resistances of the two segments of the wire, the condition of balance is

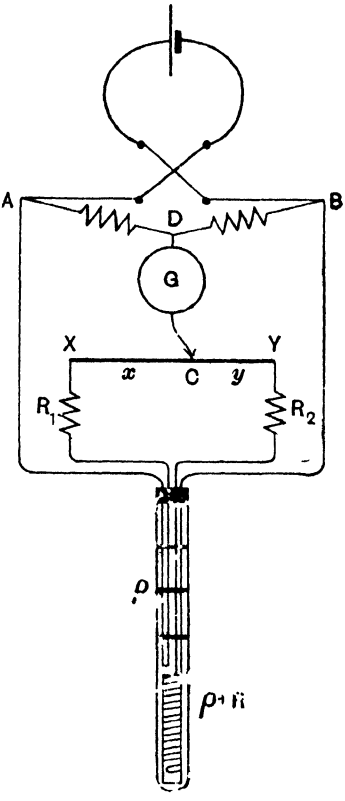


Fig. 95

$$\rho + R_1 + x = \rho + R + R_2 + y,$$

where $\rho$ is the resistance of the dummy leads and $\rho + R$ of the thermometer coil with its leads. Hence

$$R = (R_1 - R_2) + (x - y) \quad \ldots\ldots\ldots\ldots (78),$$

which gives $R$, and then $\theta$ is known from equations (76) and (77). Several values can be given to $R_1 - R_2$ on the same bridge by means

of plug contacts, so that the range of variation of $R$ may greatly exceed that obtainable from the bridge-wire alone. As it is difficult to obtain a wire of any prescribed resistance, the bridge-wire is often shunted so that the range afforded by the wire corresponds to some particular rise of temperature, for example a rise of 10 platinum degrees. The heating of the platinum coil by the battery current has to be allowed for in the most accurate work.

Langley's *bolometer* is an instrument for the measurement of radiant heat depending on the variation of resistance of a blackened platinum strip. In Tucker's *hot wire microphone* the impact of sound waves on a fine platinum wire destroys the balance of a Wheatstone's bridge, of which the wire forms one arm.

**77. Currents in solid conductors.** We now show how currents are specified when they flow in solid conductors which cannot be even approximately regarded as wires. Consider a solid metallic cylinder in which the flow is uniform over the cross-section and parallel to the axis of the cylinder. The current per unit area of the cross-section is called the *current-density*. We may define current-density more generally as follows.

To fix ideas, suppose that only the electrons move, that each carries a charge of $e$ electrostatic units, and that their average velocity near a point $P$ of space is $\mathbf{v}$, a vector function of position. The stream of electrons is practically equivalent to a moving electric fluid of electric volume-density $\rho = ne$, where $n$ is the number of electrons per cm.$^3$ near $P$. From Art. 8 the charge flowing through a surface-element $d\mathbf{S}$ drawn in the conductor is $\rho\mathbf{v}d\mathbf{S}$ electrostatic units per second, or $\frac{\rho}{c}\mathbf{v}d\mathbf{S}$ electromagnetic units of current. The vector

$$\mathbf{u} = \frac{\rho\mathbf{v}}{c} \quad \ldots\ldots\ldots\ldots\ldots\ldots (79)$$

is called the current-density at the point $P$. It is parallel to the mean velocity of the stream, and of magnitude equal to the current through a small transverse area, divided by the area. The current through the element is the scalar product

$$\mathbf{u}d\mathbf{S} = u_n dS \quad \ldots\ldots\ldots\ldots\ldots\ldots (80),$$

where $u_n$ is the component of $\mathbf{u}$ normal to the surface. The current through a finite surface $S$ is equal to the flux of $\mathbf{u}$ through $S$.

If protons move, or if there are several groups of moving electrons, equation (80) still holds, but (79) is generalised into

$$\mathbf{u} = \frac{1}{c}(\rho\mathbf{v} + \rho'\mathbf{v}' + \ldots).$$

The direction of flow at every point of an isotropic conductor (that is, one whose properties are the same in all directions round a point) is that of the impressed electric force. Imagine a small cylinder $PQ$ of cross-section $\alpha$ and length $l$ described with its axis parallel to the electric force $\mathbf{E}$ (Fig. 96). Then the cylinder may be regarded as a short piece of wire carrying a current $u\alpha$, the difference of potential between its ends being $lE$. The resistance is $l/\sigma\alpha$ where $\sigma$ is the conductivity, so that Ohm's law gives

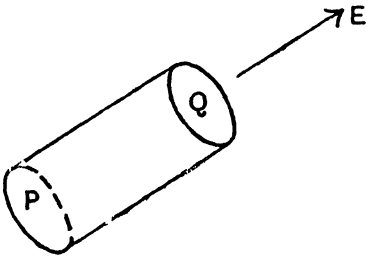


Fig. 96

$$lE = \frac{l}{\sigma\alpha}u\alpha, \quad \text{or} \quad \mathbf{u} = \sigma\mathbf{E} \ldots\ldots\ldots\ldots (81).$$

The current-density is therefore parallel and proportional to the electric force. This form of Ohm's law corresponds more closely than the previous one to Ohm's original ideas on conduction, which proceeded on an analogy between electric and thermal conduction. The electric force in this formula must be measured in electromagnetic units, true or derived.

The conditions that hold good while crossing the boundary separating two different conductors in contact are

(1) the tangential component of electric force is continuous,

(2) the normal component of current-density is continuous.

To prove (1), let $AB$ (Fig. 97) represent the surface of separation, and let a rectangle $PQRS$ be drawn whose sides $PQ$, $RS$ parallel to the surface are of the first order of small quantities, $PS$ and $RQ$ being of the second order. Let $F$, $F'$ be the tangential components of electric force at $P$, $S$ respectively. By the principle of energy,

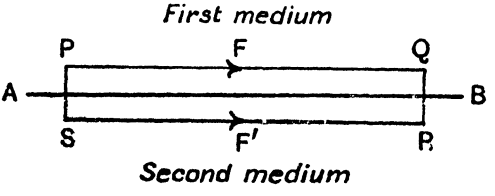


Fig. 97

no work is done by the forces of the field when a unit charge is taken round $PQRS$. Hence $F \cdot PQ - F' \cdot PQ = 0$, or $F = F'$. To prove (2), imagine a surface-element $dS$ drawn on the interface, and let $u_n$, $u_n'$ be the normal components of current-density in the first and second media respectively. The charge entering (or leaving) the first medium through $dS$ is $u_n dS$, and that leaving (or entering) the second medium $u_n' dS$, per second. Hence $u_n = u_n'$, since there can be no accumulation of charge on the boundary.

| S.W.G. | Diameter in mm. | Copper | | Manganin | German silver |
|---|---|---|---|---|---|
| | | Safe current (amperes) | Ohms per metre | Ohms per metre | Ohms per metre |
| 12 | 2·64 | 15·0 | ·0032 | ·077 | ·041 |
| 14 | 2·03 | 9·8 | ·0054 | ·131 | ·070 |
| 16 | 1·63 | 6·8 | ·0083 | ·204 | ·109 |
| 18 | 1·22 | 4·2 | ·0148 | ·361 | ·193 |
| 20 | ·914 | 2·6 | ·0260 | ·645 | ·345 |
| 22 | ·711 | 1·7 | ·0435 | 1·07 | ·57 |
| 24 | ·559 | 1·1 | ·070 | 1·73 | ·92 |
| 26 | ·457 | ·7 | ·105 | 2·58 | 1·38 |
| 28 | ·376 | ·5 | ·155 | 3·82 | 2·02 |
| 30 | ·315 | ·4 | ·222 | 5·45 | 2·90 |
| 32 | ·274 | ·3 | ·293 | 7·18 | 3·83 |
| 34 | ·234 | ·2 | ·404 | 9·90 | 5·27 |
| 36 | ·193 | ·15 | ·590 | 14·5 | 7·74 |
| 38 | ·152 | ·1 | ·950 | 23·2 | 12·4 |
| 40 | ·122 | ·06 | 1·48 | 36·3 | 19·4 |
| 42 | ·102 | ·05 | 2·10 | 53·4 | 27·8 |
| 44 | ·081 | ·03 | 3·30 | 81·7 | 43·5 |
| 46 | ·061 | ·02 | 5·90 | 145·5 | 77·4 |

The current in an electric main must not be so great as to cause a dangerous rise of temperature in it. It is easy to prove that the rate of development of heat in a solid conductor is $\sigma E^2$ per cm.$^3$ per second, measured in ergs if $\sigma$ and $E$ are measured in true electromagnetic units. In derived units the heat developed is $\sigma E^2$ watts, or $\sigma E^2/4{\cdot}18$ calories per cm.$^3$ per second, which may be written $u^2/4{\cdot}18\sigma$, where $u$ is the current-density in amperes per cm.$^2$ Obviously $u$ must not exceed a certain amount unless there is some special adjunct for removing the heat, such as a water bath. A current-density of 200 amperes to the square centimetre is a safe limit with large copper cables, but larger current-densities

can be used with small wires, in which the radiating surface is greater in proportion to the volume.

The size of wires in Great Britain is usually specified in terms of standard wire gauge (S.W.G.) numbers, and cables are made up of separate wires by stranding. Thus a 7/22 cable means one composed of 7 strands of wire of No. 22 S.W.G. The table on p. 145 gives the metric dimensions of the S.W.G. wires, and the resistances per metre in copper, manganin and German silver.

# CHAPTER V

## MAGNETIC EFFECT OF CURRENTS

**78. Straight currents.** In this chapter we shall study closely the laws of the action of currents on magnets and of magnets on currents, beginning with a thin straight wire carrying a current. Iron filings strewn on a flat surface encircling the wire tend to set along circles with the wire as centre (Fig. 98). This suggests that the lines of magnetic force due to the current are circles round the wire, the magnetic force being related to the current by the right-handed screw rule (cf. Art. 58).



Fig. 98

Quantitative results are obtained by using a compass-box with a graduated circular scale, such as those on magnetometers. If the wire is set in a vertical position magnetically east or west of the needle, either no deflexion is observed or else the needle turns through two right angles. Thus the magnetic force at any point due to the current is accurately at right angles to the radius vector from the wire to the point. A horizontal wire in the plane of the needle may tend to lift it somewhat at one end, but does not cause a scale deflexion in any position, so that there is no component of magnetic force

| $r$ | $\theta$ | $F$ in arbitrary units | $rF$ |
|---|---|---|---|
| 8·05 | 61° 30′ | 1·842 | 148 |
| 10·4 | 55° 20′ | 1·446 | 150 |
| 12·9 | 49° 5′ | 1·154 | 148 |
| 15·4 | 44° | 0·966 | 149 |
| 20·4 | 35° 45′ | 0·720 | 147 |
| 22·9 | 32° 30′ | 0·637 | 145 |

parallel to the wire. Now suppose that the wire is placed in a vertical position either magnetically north or south of the needle, so as to produce a magnetic force $F$ at the centre of the needle. Since this is magnetically east and west $F = H \tan\theta$, where $\theta$ is the deflexion of the needle. In this way we find that $F$ is inversely proportional at any point to the perpendicular distance $r$ of the

point from the straight wire. The table on p. 147 gives the result of a series of observations. Seven or eight feet of straight vertical wire is required for the experiment, which otherwise gives no trouble.

Since $F$ varies directly as the current and inversely as $r$, we may write

$$F = \frac{2\lambda i}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (82),$$

where $\lambda$ is a constant, as yet undetermined, depending on the unit chosen for the measurement of current.

To find the magnetic potential of a straight current, if it exists, take cylindrical co-ordinates $(r, \theta, z)$ in space, and let the infinitely long wire coincide with the axis of $z$, the positive direction of current being the positive direction of the axis. If $P$ is the point $(r, \theta, z)$ and $Q$ a near point $(r + dr, \theta + d\theta, z + dz)$, the work done by the magnetic force of the current when a unit pole is moved from $P$ to $Q$ is

$$0 \,.\, dr + \frac{2\lambda i}{r} \,.\, r\,d\theta + 0 \,.\, dz = 2\lambda i\, d\theta.$$

Hence there is a magnetic potential, whose value at $P$ is

$$\Omega = \text{const.} - 2\lambda i\, \theta \quad \ldots\ldots\ldots\ldots (83).$$

The magnetic potential differs in one essential respect from that due to a distribution of magnets—it is not single-valued. Quite apart from the arbitrary constant, $\Omega$ is indeterminate since $\theta$ might equally well be written $\theta \pm 2\pi$, $\theta \pm 4\pi$, etc. The magnetic potential decreases by $4\pi\lambda i$ whenever the point $P$ describes a curve like that shown in the figure, where the current $i$ is supposed to flow vertically upwards from the paper. The process of going once round the current in a right-handed screw direction is described as *interlacing the current positively*, so that the work done by the current on a unit magnetic pole in one positive interlace is $4\pi\lambda i$. When the pole describes a curve which does not interlace the current, $\theta$ resumes its original value without increase of $2\pi$, and no work is done.

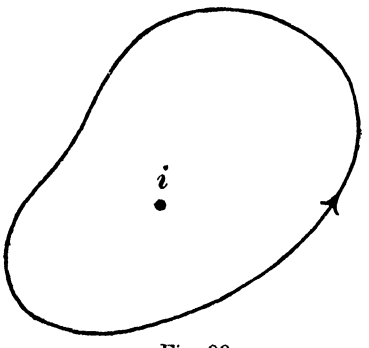


Fig. 99

**79. Magnetic force of a small plane circuit.** The general theory of the magnetic action of currents can be built up as follows. Let $C$ (Fig. 100) be a circuit of any form, and suppose it to be covered with an unclosed surface $S$ which is cut up into a large number of small areas. A current $i$ flowing round $C$ is equivalent, as regards transport of electricity, to currents $i$ flowing round the boundaries of each of the areas, as shown by the arrows; for then the line of separation of two areas is traversed by equal and opposite currents. We shall assume that the total magnetic force is the vector sum of the magnetic forces due to the separate currents; hence we know it if we can determine the magnetic force of a small plane circuit at a relatively distant point*. This is a matter partly for theory and partly for experiment. We know (Art. 59) that the magnetic force at an assigned point is proportional to the current. It is also proportional to the area of the current; for if, for example, the four circuits marked with arrows in Fig. 100 are equal in area, each produces nearly the same magnetic force at a point $P$, and the four circuits together produce four times the magnetic force, the total area being also four

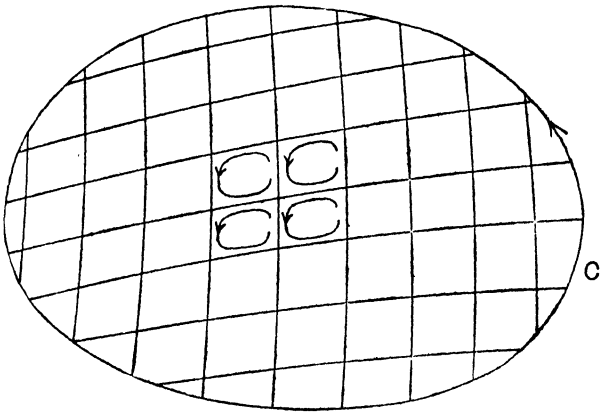


Fig. 100

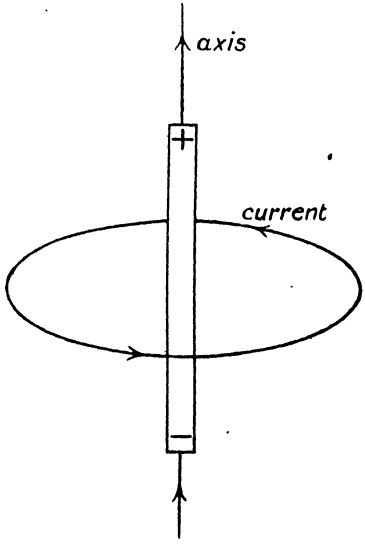


Fig. 101

* The surface $S$ can always be drawn so as not to pass near the point at which the magnetic force is to be calculated.

times as great, which could not happen on any other law than proportionality of magnetic force and area. We now show by experiment that a small current behaves at a distance like *a small bar-magnet whose axis is perpendicular to its plane* (Fig. 101).

To show this set up, magnetically east and west of a magnetometer needle, a small bar-magnet $M$ (Fig. 102), fixed horizontally

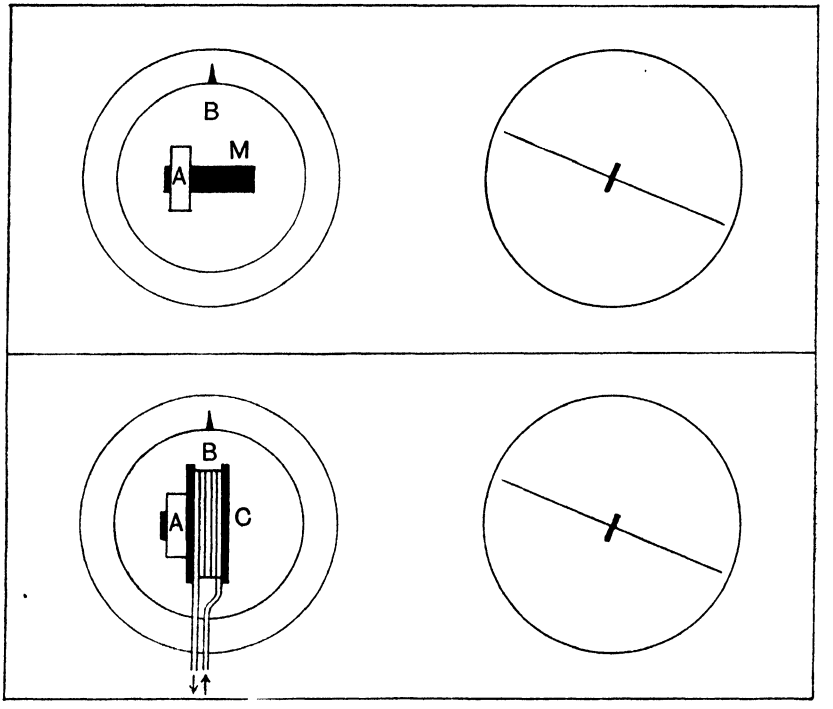



Fig. 102

into a hole in a vertical wooden block $A$, attached to a turning table $B$. The table $B$ carries a pointer moving over a graduated scale, and is set first so that the axis of the magnet points towards the magnetometer needle. The deflexion of the needle is observed, and also when the table is turned through various angles $\theta$ from its original position. The magnet is then replaced by a closely wound coil $C$ on a wooden core, furnished with a peg of the same diameter as the magnet, so that it can fit into the same hole. The plane of the coil is perpendicular to the axis of the magnet, and the current in it is adjusted till the deflexion in the position $\theta = 0$ is the same as it was for the magnet. It is now found that the equality subsists

approximately for all values of $\theta$, as shown in the following table, which gives the results of an experiment with two coils.

*Distance from coil or magnet to needle, 25 cm.*

| $\theta$ | Magnet 6 cm. long, 1·2 cm. in diameter | Circular coil, 72 turns, mean area 27·2 cm.², current 2·55 amps. | Rectangular coil, 72 turns, mean area 29·6 cm.², current 2·39 amps. |
|---|---|---|---|
| −90° | −0° 6′ | −0° 18′ | −0° 18′ |
| −80° | 3° 24′ | 3° 39′ | 3° 39′ |
| −70° | 6° 48′ | 7° 15′ | 7° 15′ |
| −60° | 9° 57′ | 10° 42′ | 10° 27′ |
| −50° | 13° 6′ | 13° 48′ | 13° 39′ |
| −40° | 15° 42′ | 16° 36′ | 16° 18′ |
| −30° | 18° 42′ | 18° 54′ | 18° 48′ |
| −20° | 20° 27′ | 20° 45′ | 20° 42′ |
| −10° | 22° 18′ | 22° 15′ | 22° 18′ |
| 0° | 23° 15′ | 23° 15′ | 23° 15′ |
| 10° | 23° 39′ | 23° 39′ | 23° 36′ |
| 20° | 23° 21′ | 23° 33′ | 23° 27′ |
| 30° | 22° 15′ | 22° 39′ | 22° 33′ |
| 40° | 20° 15′ | 21° 3′ | 20° 48′ |
| 50° | 17° 39′ | 18° 33′ | 18° 21′ |
| 60° | 14° 0′ | 15° 18′ | 14° 48′ |
| 70° | 9° 45′ | 10° 24′ | 10° 48′ |
| 80° | 5° 6′ | 5° 33′ | 5° 30′ |
| 90° | 0° 24′ | 0° 18′ | 0° 6′ |

A similar agreement is found when the magnet and coils are magnetically south of the needle. Since the potential of a small magnet at any point $P$ is, in the notation of Art. 20, $M \cos\theta/r^2$, that of a small circuit of area $A$ carrying a current $i$ is $\lambda i A \cos\theta/r^2$, or $-\lambda i\phi$, where $\phi = -A\cos\theta/r^2$ is the solid angle subtended by the circuit at $P$. We take the negative sign because it assists the memory to consider the solid angle as positive when the current, as seen from $P$, appears to circulate round in the right-handed screw direction. The same constant $\lambda$ is used, here and in equation (83), for a reason which will appear presently.

**80. Circuits of any form.** A few qualitative observations will help us to appreciate the general theory. Fig. 103 shows the lines of magnetic force due to a circular current at points in a plane through its axis. We observe the tendency of the lines of magnetic

force to interlace the circuit, which acts like a girdle binding them together in a sheaf. The lines are nearly circles near the wire, as

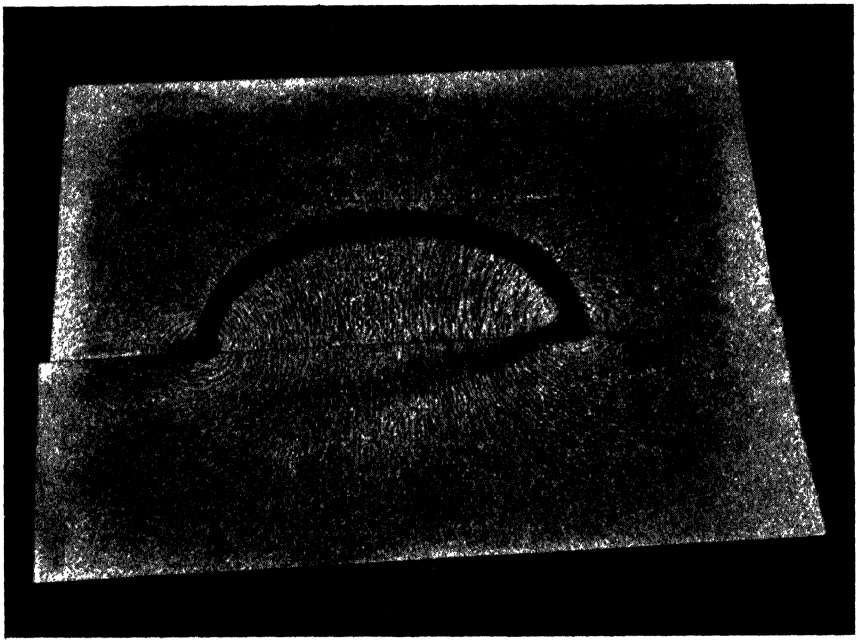


Fig. 103

we should expect from Art. 78. It can be shown, by means of a small compass-needle, that the lines of force pass through the circuit in the general direction shown in Fig. 104; that is, they interlace the circuit positively in the sense in which the term is used in Art. 78.

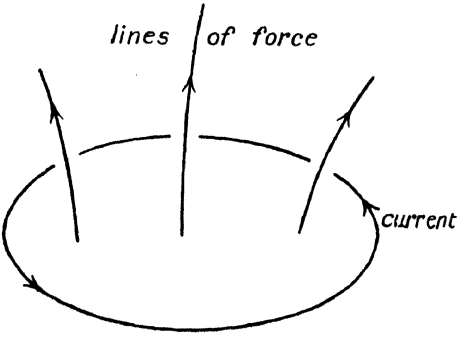



Fig. 104

Resuming now the argument at the beginning of Art. 79, each of the elementary circuits in Fig. 100 produces at any point $P$ a magnetic potential $-\lambda i\phi$, and therefore the total magnetic potential is $-\lambda i\phi$, where $\phi$ is the solid angle subtended by the whole circuit at $P$.

The formula $$\Omega = -\lambda i\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots (84)$$ is therefore general. The convention as to sign has already been explained, and is illustrated in Fig. 105, where *acbd* is the circuit and *ACBD* its projection from $P$ on a circle of unit radius with $P$ as centre, the solid angle being represented by the shaded area in the figure, and there positive. Equation (84) agrees with, and includes, equation (83); for though we speak of an infinite straight current, the circuit must be closed by wires at a great distance, and we can assume these to lie in the plane corresponding to $y = 0$ in Fig. 99, when a little consideration shows that the solid angle subtended by the circuit at $P$ is $2\theta$.

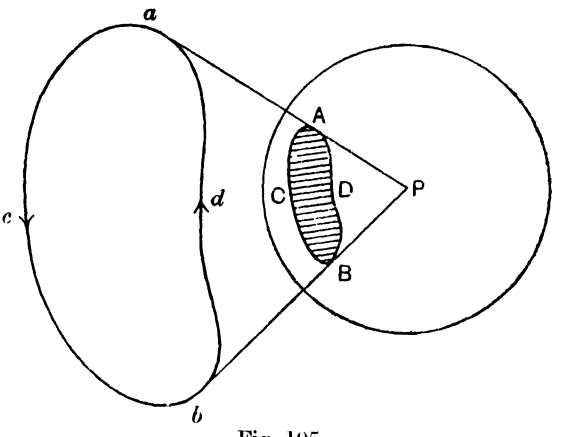


Fig. 105

Returning to the general formulae, $\phi$ vanishes when $P$ is at infinity since the circuit, if finite, subtends no solid angle at $P$. Thus no constant need be added to $\Omega$, if we suppose that one of its values vanishes at infinity. As $P$ moves in towards the circuit from the position of Fig. 105, the solid angle increases. If $P$ describes a closed curve which does not interlace the circuit, the initial increase of the shaded area is followed by a decrease, and $\Omega$ resumes its original value; but if $P$ describes a closed curve which interlaces the circuit positively, the shaded area can increase until it has covered *the whole unit sphere and its original area as well*, so that the magnetic potential increases in this case by $4\pi\lambda i$. The result of Art. 78, that work $4\pi\lambda i$ is done by the current on a unit pole is one positive interlace, is therefore general.

## 81. Verifications of the theory.

(1) *The magnetic force due to a plane circuit of any form at a point in its plane is at right angles to the plane.* This is proved by laying the circuit vertically with its plane magnetically east and

west and observing a compass-needle whose centre is in the plane. If $P$ is any point in the plane of the circuit and inside it, the solid angle at $P$ has the constant value $2\pi$. Hence no work is done on a unit pole in moving it about inside the circuit, i.e. there should be no tangential component of magnetic force. Similarly for points in the plane and outside the circuit.

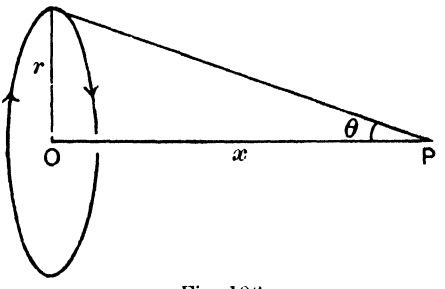


Fig. 106

(2) *Magnetic force at points on the axis of a circular wire carrying a current.*

If $O$ is the centre and $P$ a point on the axis distant $x$ from $O$ (Fig. 106), the solid angle subtended by the circular wire at $P$ is $2\pi(1 - \cos\theta)$, where $\tan\theta = r/x$, $r$ being the radius of the circle. Hence

$$\Omega = -2\pi\lambda i\{1 - x(r^2 + x^2)^{-\frac{1}{2}}\}.$$

The magnetic force is $-\partial\Omega/\partial x$, that is a force

$$F = 2\pi\lambda i\, r^2 (r^2 + x^2)^{-\frac{3}{2}} \quad \ldots\ldots\ldots\ldots (85)$$

directed along $PO$.

To verify this set up a circular coil in the magnetic meridian and find the deflexions $\delta$ for various distances $x$ of the centre of a compass-needle from the centre of the coil, with the compass-box magnetically east or west. The following numbers derived from an experiment show that $F$, or $\tan\delta$, varies as $(r^2 + x^2)^{-\frac{3}{2}}$.

$r = 12{\cdot}8$ cm.

| $x$ | $\delta$ | $(r^2+x^2)^{\frac{3}{2}} \tan\delta$ |
|---|---|---|
| 0 | 53° 12′ | 2800 |
| 8·15 | 39° 24′ | 2870 |
| 11·5 | 29° 6′ | 2830 |
| 14·1 | 22° 15′ | 2820 |
| 17·1 | 16° | 2800 |
| 20·1 | 11° 39′ | 2850 |
| 23·1 | 8° 48′ | 2850 |

Fig. 107 shows the lines of force at points in a plane through the axis. It may be compared with the filings diagram in Fig. 103.

Putting $x = 0$ in equation (85) we find that a circular coil of radius $r$ produces at its centre a magnetic force $2\pi\lambda i/r$. Comparing this with the discussion of Art. 59, on which the definition of the

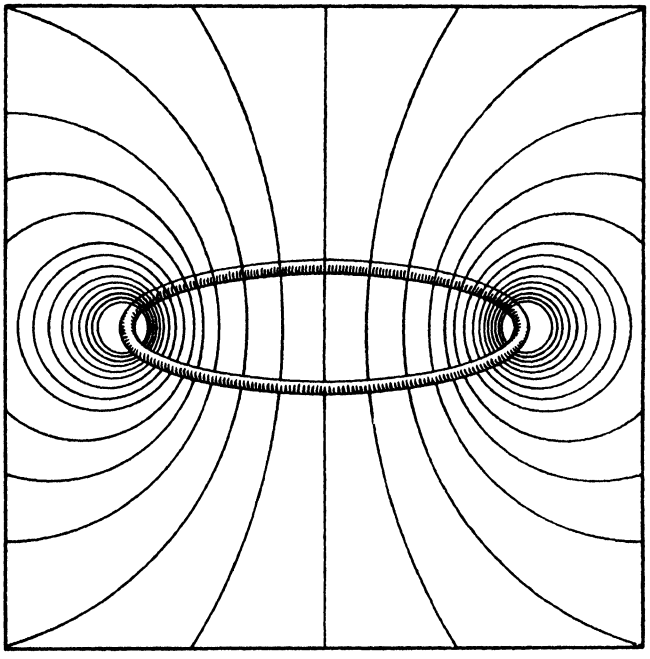


Fig. 107

electromagnetic unit of current was based, we see that we have to put $\lambda = 1$ to make the formulae applicable to currents measured in true electromagnetic units. With this understanding the theory of the magnetic action of currents is complete, and contained in the statement that the magnetic potential of a current at any point is

$$\Omega = -i\phi \quad \ldots\ldots\ldots\ldots\ldots\ldots (86).$$

The work done by the field on a unit pole when the latter interlaces the circuit once positively is $4\pi$ times the current.

We can now give the precise relation of a small plane current to its equivalent magnet (Art. 79). The equivalent magnetic moment is

$$M = iA \quad \ldots\ldots\ldots\ldots\ldots\ldots (87),$$

where $i$ is the current in true electromagnetic units and $A$ the area of the circuit. If there are $n$ turns close together, of mean area $A$, and the current is in amperes, the moment is $niA/10$.

**82. Magnetic force due to a long cylindrical wire.** Let $O$ be the centre of a cross-section of the wire (Fig. 108), $OA = a$ its radius, the current being supposed to flow vertically upwards in the figure. The magnetic force at a point $P$ distant $r$ from $O$ will always be perpendicular to $OP$; for since the current is distributed symmetrically round $O$, similar filaments on opposite sides of $OP$ neutralise one another as regards the radial component of magnetic force. If $H$ is the magnitude of the force at an *external* point $P$, $2\pi rH$ is the work done in taking a unit magnetic pole once round the current in a positive direction.

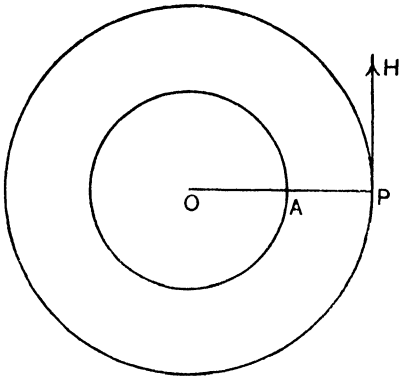


Fig. 108

This is equal to $4\pi i$, where $i$ is the total current in the whole wire. Thus $H = 2i/r$, which is the same as if the current $i$ was concentrated along the axis of the wire. In verifying the law of the inverse distance (Art. 78) we need not, therefore, trouble to choose thin wires.

The force at an *internal* point $P$ can also be found, since we know that the current is distributed uniformly over the cross-section (Art. 66). When $r < a$, a circle of radius $r$ interlaces a current of only $r^2i/a^2$: hence we have now $H = 2ri/a^2$, showing that the magnetic force is proportional to the distance of $P$ from the central line of the wire. Thus the magnetic force of a long wire increases as we approach it, has its greatest value $2i/a$ on the surface, and then decreases uniformly towards the centre.

**83. Solenoids. Helmholtz coils.** The magnetic action of currents affords the most convenient means of producing a magnetic field of known strength. Take a ring of non-magnetic material, and wind insulated wire closely over it so as to cover it uniformly. A ring so wound is called an endless solenoid: for theoretical purposes it may be replaced by a set of coils each carrying the same current and spaced uniformly after the manner shown in Fig. 109, but closer together.

To find the magnetic force at any point $P$, drop a perpendicular $PO$ on the axis. Two coils placed symmetrically on opposite sides of the plane through $OP$ and the axis give a resultant magnetic force perpendicular to that plane: and since we may suppose all the coils arranged in pairs in this way, the total magnetic force **H** at $P$ is also perpendicular to the plane, as shown in the figure. Let $n$ be the number of turns of wire on the ring, $i$ the current in true electromagnetic units.

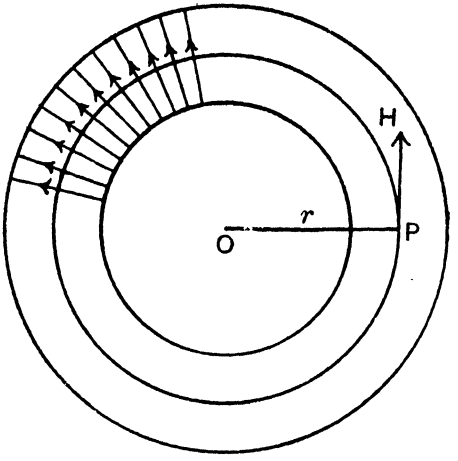


Fig. 109

Imagine that a unit pole is carried round a circle, perpendicular to the axis, with $O$ as centre and radius $OP = r$. If $P$ is within the ring all the $n$ turns are interlaced in one revolution, so that the work done by the magnetic force is $4\pi ni$. Thus $2\pi rH = 4\pi ni$, or $H = 2ni/r$. The lines of magnetic force inside the ring are, therefore, circles round the axis, and the magnetic force varies inversely as the distance from the axis. If the cross-section of the ring is small compared with its radius, the magnetic force is appreciably constant.

If $P$ is in the space outside the ring no work is done in taking the unit pole round the circle, since no currents are interlaced. Hence the theoretical solenoid gives no magnetic force outside.

The magnetic force inside a long straight solenoid, which is practically realised by winding insulated wire closely round a long straight core, may be deduced from the preceding results. Consider a solenoid in the form of a ring of radius $r$, large compared with the cross-section. The number of turns of wire is $n$ along the whole length $2\pi r$ of the circumference, i.e. the number of turns per centimetre is given by $m = n/2\pi r$. If we make this substitution in the expression for $H$ and then make $r$ infinite, we find that the magnetic force inside an infinite straight solenoid is $4\pi mi$, where $m$ is the number of turns per centimetre. The force is constant and parallel to the solenoid, whatever the shape of the cross-section of the latter.

The formula becomes

$$H = \frac{4\pi mi}{10} \quad \ldots\ldots\ldots\ldots\ldots\ldots (88)$$

if $i$ is to be in amperes and $H$ in gausses. It will hold approximately for any solenoid whose length is great in comparison with the dimensions of its cross-section. We can find an exact expression for the magnetic force at points on the axis of a finite solenoid of circular cross-section, as follows.

Let $2l$ be the length of the solenoid, $a$ its radius and $O$ its centre (Fig. 110). Taking the axis of the solenoid as axis of $x$, let us find

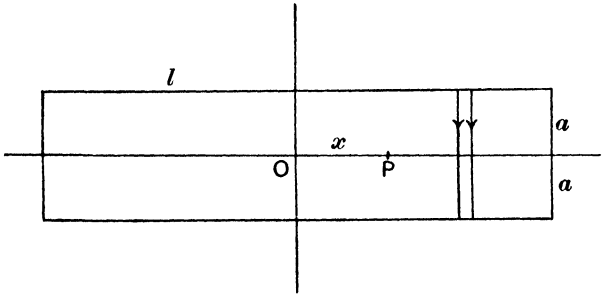
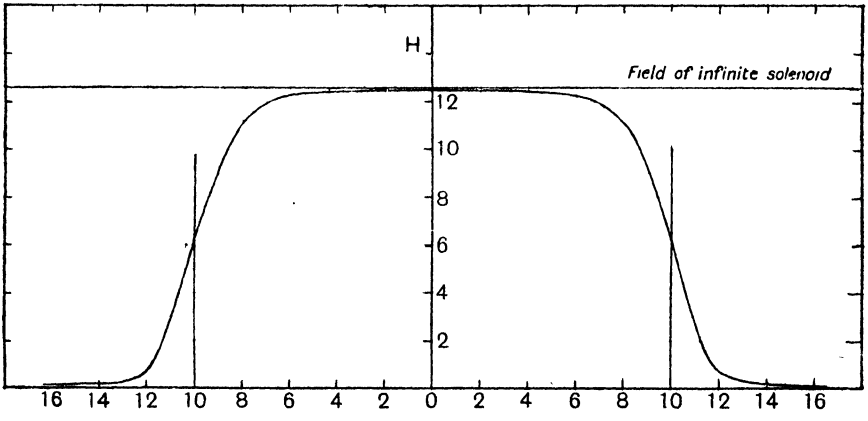



Fig. 110




Length 20 cm., radius 1 cm., 10 turns per cm., current 1 ampere.

Fig. 111

the magnetic force at $P$, where $OP = x$. Consider the action of the group of windings between the abscissae $\xi$ and $\xi + d\xi$. Supposing the coils very close together, the number of turns of wire is $m\,d\xi$. From Art. 81 each turn produces at $P$ a magnetic force

$$2\pi i a^2 \{a^2 + (\xi - x)^2\}^{-\frac{3}{2}}:$$

hence the total magnetic force of the whole solenoid at $P$ is

$$H = 2\pi mi a^2 \int_{-l}^{l} \{a^2 + (\xi - x)^2\}^{-\frac{3}{2}} d\xi$$
$$= 2\pi mi \left[ \frac{l + x}{\{a^2 + (l + x)^2\}^{\frac{1}{2}}} + \frac{l - x}{\{a^2 + (l - x)^2\}^{\frac{1}{2}}} \right].$$

Fig. 111 shows the variation of magnetic force along the axis of a particular solenoid, calculated from the last formula. Thus the magnetic field inside a solenoid whose length is ten times its diameter remains constant to about 10 per cent. over four-fifths of the length.

Nearly uniform magnetic fields are also produced by a disposition due to Helmholtz. Fig. 112 represents a section of two equal coaxial circular circuits $AB$, $CD$ by a plane through the axis. Let

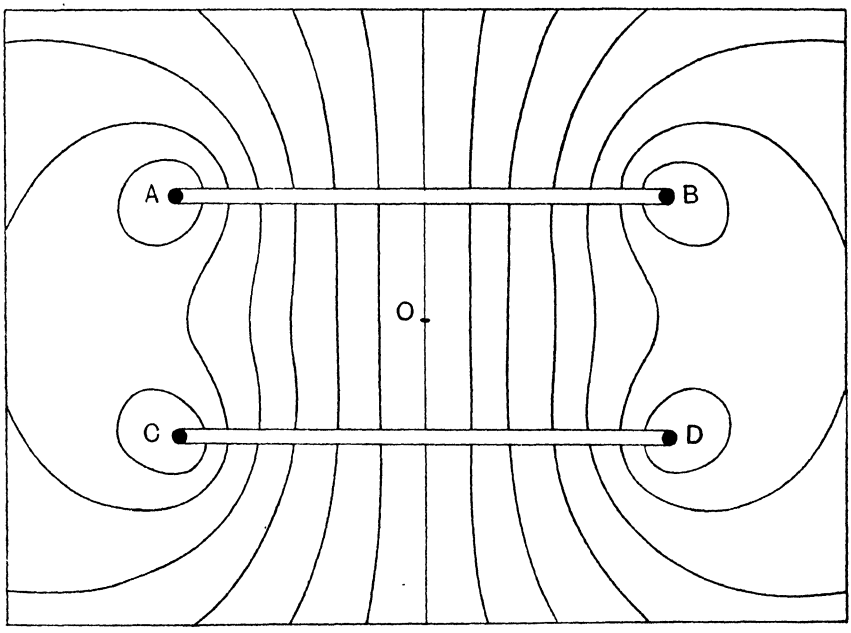



Fig. 112

$a$ be the radius of each, $2b$ the distance between the planes, $O$ the point on the axis mid-way between the planes. If a current $i$ flows in the same direction in both circuits, the magnetic force at a point on the axis distant $x$ from $O$ is

$$2\pi ia^2 [\{a^2 + (b - x)^2\}^{-\frac{3}{2}} + \{a^2 + (b + x)^2\}^{-\frac{3}{2}}]$$
$$= \frac{4\pi ia^2}{(a^2 + b^2)^{\frac{3}{2}}} \left[ 1 + \frac{3\,(4b^2 - a^2)}{2\,(a^2 + b^2)^2} x^2 \right]$$

to the third order in $x$. The terms in $x^2$ vanish if $a = 2b$, or if the circuits are placed at a distance apart equal to the radius of either. The field near $O$ is then nearly uniform, and the lines of force nearly straight, as shown in Fig. 112. The disposition is useful when we wish to have easy access to an apparatus which is to be placed in a nearly uniform field.

**84. Action of magnets on currents.** The mechanical force exerted by currents on magnets implies reaction of the magnets on the wires carrying the current. Striking experiments can be devised to show these effects.

Fig. 113 shows an arrangement for producing continuous rotation of a wire round a magnet. A metal cylinder is partly filled with paraffin wax, by which a magnet is supported vertically inside it, and the cylinder is then filled with mercury until it nearly covers the magnet. A straight piece of copper wire is suspended by a short piece of thinner wire from a clamp, and dips in the mercury as shown in the figure. When a current passes through the apparatus in the direction shown, the straight wire moves continually round in the counter-clockwise direction, while on reversing the current it moves with equal speed in the opposite direction.

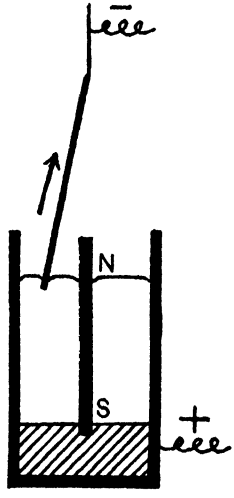


Fig. 113

Another experiment is made with a light battery and a small coil of several turns. Connect the coil to the battery, fix the coil vertically, and then float both of them on a small board lying on water. The plane of the coil will tend to set magnetically east and west in the earth's magnetic field. This is, in essence, the same as *de la Rive's floating battery.*

A very interesting phenomenon is that known as the electromagnetic rotation of a conducting liquid. Take a shallow cylindrical copper vessel, and pour in it paraffin wax followed by a thin layer of mercury. Contact is made by a central dipping electrode and a wire attached to the vessel. By this means a radial current is sent through the mercury layer. When a bar-magnet is held vertically

over the central electrode and near the mercury, the latter will rotate quickly in the vicinity of the electrode, and more slowly in the outer parts. Its motion can be seen from the movement of dust-particles on the surface. If the north pole of the magnet is nearer the liquid, and the current flows radially outwards, the rotation as seen from above is in the counter-clockwise direction.

Whatever the forces are which act on the parts of a wire carrying a current in a magnetic field, they are at all points perpendicular to the wire. To prove this it is only necessary to make part of the circuit moveable in the direction of its length, for example a straight copper rod sliding through two very slightly larger holes, or lying in two **V**-shaped grooves. Such a rod will not move in the direction of its length, no matter what magnetic field it is placed in.

**85. Potential energy of a current in a magnetic field.** The potential energy of a unit pole at any point being $-i\phi$, that of a pole of strength $m$ is $-mi\phi$, where $\phi$ is the solid angle under which the current is seen from the point. Now consider the flux of magnetic force through the circuit. The pole sends out lines of force equally in all directions, the total flux for a solid angle $4\pi$ being $4\pi m$ by Gauss' theorem. That is, if we write $N = m\phi$ for the flux of magnetic force through the current-carrying circuit due to the pole, the potential energy is $-iN$. In this expression the strength of the pole has disappeared, explicitly at any rate. The flux of force through the circuit for any number of poles is got by adding the fluxes due to the separate poles: hence the potential energy of a system consisting of a current and any distribution of poles is $-iN$, where $N$ is the flux of magnetic force through the current due to the poles, $N$ being reckoned as positive when lines of force thread the current in the positive direction, as shown in Fig. 104.

This theorem enables us to find the force and couple on a current in any magnetic field. The forces are statically equivalent to the reversed resultant of the forces exerted by the current on the poles in the field. The motion of a rigid circuit, such as the floating battery, is covered by the rule that the current tries to embrace as many lines of force as possible: for since it moves so as to make its potential energy a minimum, its final position makes $N$ a maximum.

As an example let us find the forces exerted by a uniform magnetic field $H$ on a plane circuit carrying a current $i$. Let the circuit consist, in the first instance, of a single turn of area $A$, its plane making an angle $\theta$ with the direction of $H$, as shown in Fig. 114. The flux of force through the circuit being $HA \sin\theta$, the potential energy of the current in the position $\theta$ is

$$W = -iHA \sin\theta.$$

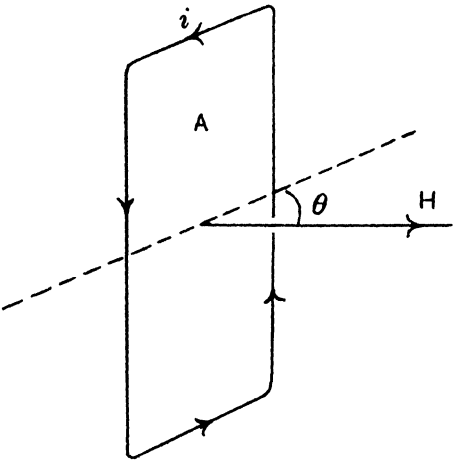


Fig. 114

The circuit can be moved parallel to itself without altering $W$. Hence the field exerts no resultant force on it, but only a couple tending to increase $\theta$, whose magnitude is

$$-\frac{dW}{d\theta} = iHA \cos\theta \quad \ldots\ldots\ldots\ldots\ldots(89).$$

If the current is left to itself it will take up the position $\theta = \frac{1}{2}\pi$ normal to $H$. If, on the other hand, it is constrained by torsion so that $\theta$ is small, the couple is approximately $iHA$. The couple is $n$ times as large when there are $n$ windings close together, each of area $A$ and carrying a current $i$.

The principle of the moving coil is used in many electrical measuring instruments (Arts. 86, 87, 89, 133). The extension of the theory from discrete to continuous distributions of poles is considered in Art. 119.

**86. The moving-coil galvanometer.** Although the definition of current in electromagnetic units has been based on the tangent galvanometer, that instrument suffers from great practical disadvantage. To get an absolute measurement of current we must know the horizontal component of the earth's magnetic field in the place in which the galvanometer is to be used, as the presence of iron in modern buildings makes it impossible to take $H$ from the standard observation of a survey. The galvanometer constant is

seriously affected by external magnetic fields of the order of 0·01 gausses, frequently produced by currents or magnets. For example, a long vertical wire carrying a current of 10 amperes at a distance of 10 metres from a tangent galvanometer would produce a 1 per cent. error in its readings.

For these reasons galvanometers are largely constructed on the principle of the moving coil. Such galvanometers (Fig. 115) were introduced by Lord Kelvin. A rectangular coil, of several hundred turns of fine wire, is suspended from a fixed pillar by a fine strip of phosphor-bronze, and hangs between the poles of a permanent steel magnet. There is also a solid core of soft iron fixed inside the coil. The action of this will be better understood after we have considered the theory of induced magnetism (Arts. 117–9): the core concentrates the lines of force, and they pass almost radially in the air-gap. Fig. 116 gives a vertical view of the coil and a sketch of the lines of magnetic induction.

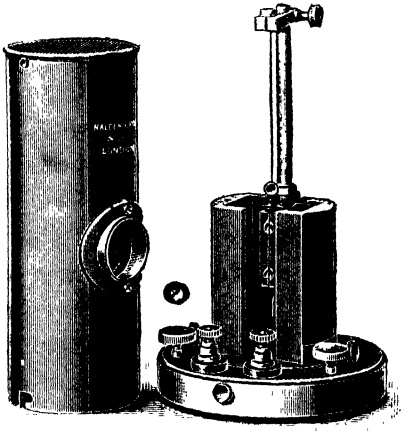


Fig. 115

The connexions to the coil are made through the suspension and through a fine wire leading to the lower end of the coil. The field in the air-gap is of the order of 300 gausses, so that the galvanometer is hardly affected by external magnetic fields. The coil is set so that its plane is approximately parallel to the line $NS$ when no current is flowing. An ordinary galvanometer of this type will measure currents of the order of one-millionth of an ampere, while

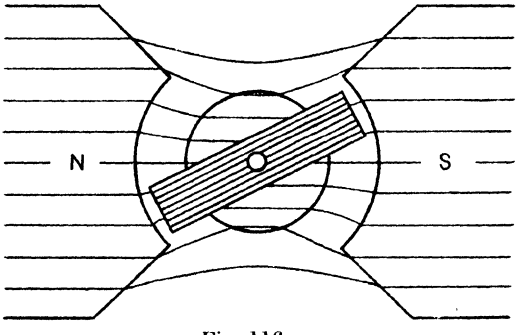



Fig. 116

very sensitive instruments will give a deflexion for about $10^{-12}$ amperes.

Since the lines of force in the air-gap are nearly radial, the couple on the coil is almost independent of the angle $\theta$ which its plane makes with the line $NS$, and is proportional to the current $i$, say $\lambda i$. The restraining couple due to the twisting of the suspension is equal to $\mu\theta$, where $\mu$ is a constant, and the coil takes up a position for which these couples balance. Hence $\lambda i = \mu\theta$, or $i = k\theta$, where $k$ is written for $\mu/\lambda$. The quantity $k$ is called the galvanometer constant, and it is usually understood to be expressed so that the formula gives $i$ in amperes. The value of $k$ is found by standardising, as in Art. 73.

The sensitiveness of the galvanometer can be increased by making the moving coil long and narrow, and by bringing the poles of the magnet close together. Galvanometers with a large number of turns are sensitive to current, but on account of their greater resistance they are not necessarily most sensitive to potential. Both kinds of sensitiveness are required on occasion, the former to detect the discharge current of small condensers (Art. 142), and the latter to measure small alternating potentials (Art. 107).

Galvanometers are mostly made *dead-beat*, that is, so as to come to rest in their proper position as soon as possible. The method of ensuring this is explained in Art. 130. Galvanometers with the opposite property of swinging as long as possible without coming to rest are said to be *ballistic*.

A moving-coil galvanometer when tested with a tangent galvanometer will sometimes show about 5 per cent. deviation from proportionality of scale-deflexion to current. It is therefore advisable, after having obtained a deflexion in any experiment, to standardise the galvanometer at the same deflexion.

For other galvanometers reference may be made to the *Dictionary of Applied Physics*, vol. 2, pp. 366–77.

**87. Ammeters and voltmeters.** The basis of most instruments for measuring both current and potential is a *sensitive moving-coil galvanometer*, with a pointer moving over a fixed scale. This scale is graduated so as to read directly in convenient small units. A common type is the so-called millivoltmeter, whose

moving parts are shown in Fig. 117. The wire suspension is done away with, and the coil is mounted between agate bearings, so that it can be carried about freely. The torsion couple is produced by two light spiral springs above and below the coil, which also serve as the wires leading the current in and out of the instrument.

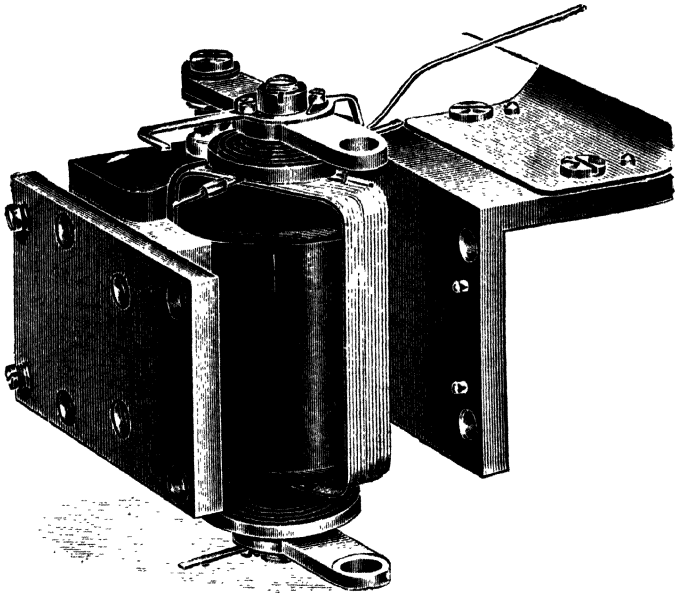


Fig. 117

The millivoltmeter is so called because each division of the scale corresponds to a difference of potential of one-thousandth of a volt between the terminals.

A sensitive galvanometer of this type can be shunted by a small resistance so as to form an ammeter, and placed in series with a large resistance so as to form a voltmeter for measuring higher voltages. In the former case let $R$ be the resistance of the shunt, $G$ that of the galvanometer. Then an incoming current $i$ divides in the ratio $R$ to $G$, $Ri/(R + G)$ going through the galvanometer and the rest through the shunt. If $R$ is small most of the current goes through the shunt, and the main current may be very high without giving much deflexion of the galvanometer. The same fixed-scale instrument can be used with different shunts in order to give different ranges of measurement.

The use and value of fixed-scale instruments dates from the discovery of methods of producing really permanent steel magnets. There is practically no limit to the current that can be measured in this way. All that is wanted is a shunt of sufficiently low resistance that will carry the current without undue rise of temperature. Fig. 118 shows a 15 ampere shunt consisting of a strip of manganin.

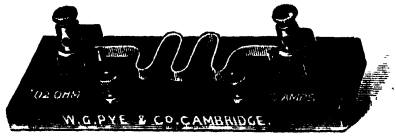


Fig. 118

If a galvanometer is placed in *series* with a resistance $S$, only the fraction $G/(G + S)$ of the total drop of potential occurs between the terminals of the instrument. Thus a millivoltmeter of 1 ohm resistance and of 100 scale-divisions, reading up to $\frac{1}{10}$ volt, may be made to read up to 10 volts by placing a coil of 99 ohms in series with it.

Fig. 119 shows the terminals and internal connexions of a combined ammeter and voltmeter. The reader will find it interesting to calculate the resistances $R$ and $T$ required, say, to make the instrument read as an ammeter up to 1 ampere (terminals $+$, $-A$) and as a voltmeter up to 10 volts (terminals $+$, $-V$), given that the full deflexion with the terminals $+$ and $-G$ corresponds to a current of 1/100 ampere and that both $S$ and $G$ are 1 ohm.

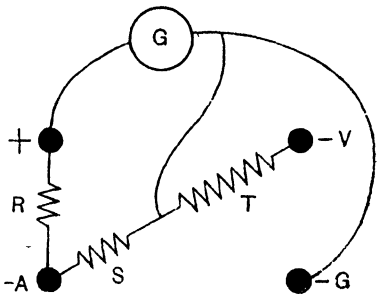


Fig. 119

**88. The string galvanometer.** For following the changes of current and voltage occurring in electrical machinery, and for medical cardiography, we require a galvanometer which shall be at once sensitive and quick to take up its final deflexion. This is attained in the *string galvanometer* invented by Einthoven, the principle of which is shown in Fig. 120. $CC$ is a fine quartz fibre, silvered to make it conduct, and carrying the current to be measured. It lies between the poles $NS$ of a strong electro-magnet, the poles being brought close to the fibre and perforated to allow the movement of the latter to be observed with a microscope $ED$. A current

passing down the fibre from above to below experiences a mechanical force in the direction of the arrow $a$, causing a deflexion of the fibre.

The instrument shares with the moving-coil galvanometer the advantage of being independent of external magnetic fields. The

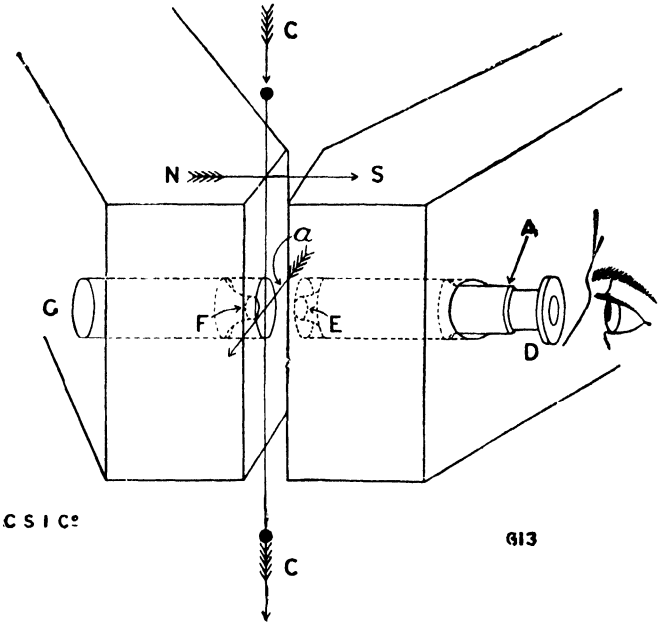



Fig. 120

tension of the fibre makes the zero very steady, and also renders the readings practically independent of levelling. The quick response is a consequence of the lightness of the fibre, which also allows the air to damp it so as to make it nearly dead-beat. Currents of $10^{-12}$ ampere can be detected with low tension on the fibre and a magnification of 750 diameters. Increasing the tension of the fibre increases the quickness of response with a corresponding loss of sensitiveness.

**89. The ballistic galvanometer.** While galvanometers giving steady deflexions should be dead-beat, transient currents such as those produced by the discharge of condensers are measured with the ballistic galvanometer. When a charge of $e$ coulombs is sent suddenly through the galvanometer, the spot of light will begin to swing and move to and fro through a gradually decreasing arc with

the original position as centre. The gradual decay of the swing, or *damping*, should be as small as possible. In the absence of damping the spot of light would swing through a distance $\tau$ on either side of the zero position; $\tau$ is called the *throw* of the galvanometer. The fundamental formula of the ballistic galvanometer is

$$e = \frac{T}{2\pi} k\tau \quad \ldots\ldots\ldots\ldots\ldots\ldots (90),$$

where $k$ is the galvanometer constant and $T$ the time of swing of the coil:

To prove this, we have to extend the theory of Art. 86 by taking into account the angular inertia of the moving coil. Let $I$ be its moment of inertia about a line coinciding with the suspending wire. Then the total couple tending to increase the deflexion $\theta$ is $\lambda i - \mu\theta$, and the equation of motion of the coil in general is

$$I\frac{d^2\theta}{dt^2} = \lambda i - \mu\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (91),$$

damping being neglected.

If the coil starts from rest in the position $\theta = 0$ we have by integration

$$I\frac{d\theta}{dt} = \lambda\int_0^t i\,dt - \mu\int_0^t \theta\,dt.$$

The integral $\int_0^t i\,dt$ is the total charge $e$ that has passed through the galvanometer from time 0 to time $t$. If this charge passes through quickly, the coil has no time to move, so that $\int_0^t \theta\,dt$ is negligible. The last equation then shows that the coil acquires a sudden angular velocity $\omega$, given by

$$I\omega = \lambda e \quad \ldots\ldots\ldots\ldots\ldots\ldots (92).$$

When the charge has once passed, the coil swings without any current flowing in it. The motion of the coil in this latter phase is regulated by the equation

$$I\frac{d^2\theta}{dt^2} + \mu\theta = 0,$$

with the condition that $\theta = 0$ and $d\theta/dt = \omega$ when $t = 0$. The corresponding equation of energy is

$$I\left(\frac{d\theta}{dt}\right)^2 + \mu\theta^2 = \text{constant}.$$

Applying this to the instants (1) when $t = 0$, (2) at the first stopping place, for which $\theta = \tau$ and $d\theta/dt = 0$, we have

$$I\omega^2 = \mu\tau^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (93).$$

Further, the time of the simple harmonic oscillation of the coil is given by

$$T = 2\pi\,(I/\mu)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (94).$$

Combining the equations (92), (93) and (94) we have

$$e = \frac{I\tau}{\lambda}\left(\frac{\mu}{I}\right)^{\frac{1}{2}} = \frac{\mu\tau}{\lambda}\left(\frac{I}{\mu}\right)^{\frac{1}{2}} = \frac{T}{2\pi}\,k\tau \quad \ldots\ldots\ldots (95),$$

since $k = \mu/\lambda$. This equation gives $e$ in terms of constants $T$, $k$ which can be found by experiment.

To allow for small damping assume a frictional couple proportional to the angular velocity. Then the equation of motion of the coil swinging freely is

$$\frac{d^2\theta}{dt^2} + 2\kappa\frac{d\theta}{dt} + n^2\theta = 0,$$

where $\kappa$ is a small constant specifying the amount of damping. If we neglect $\kappa^2$ the solution is

$$\theta = e^{-\kappa t}\,(A\cos nt + B\sin nt).$$

In the case considered $\theta = 0$ when $t = 0$, so that $A = 0$. Since the damping in a few swings is small, the early stages of the motion are represented by $\theta = B\,(1 - \kappa t)\sin nt$; and since $\kappa = 0$ gives a swing of amplitude $B$, $B$ is the throw corrected for damping. Let $\theta_0$ be the first throw, $\theta_1$ the next *on the same side*. These correspond to times $nt = \frac{1}{2}\pi$ and $\frac{5}{2}\pi$ approximately, so that

$$\theta_0 = B\left(1 - \frac{\kappa\pi}{2n}\right) \text{ and } \theta_1 = B\left(1 - \frac{5\kappa\pi}{2n}\right).$$

Thus

$$B - \theta_0 = \tfrac{1}{4}\,(\theta_0 - \theta_1),$$

i.e. the correction to true throw is got by adding one-quarter of the difference between the observed throws. Hence the true throw $\tau$ for use in equation (90) is given by

$$\tau = \theta_0 + \tfrac{1}{4}\,(\theta_0 - \theta_1) \quad \ldots\ldots\ldots\ldots (96).$$

Thus in an experiment in which the zero reading on the scale is (say) 24·2, if the galvanometer swings to the right and first comes to rest in the position 48·3, the next reading on the right being 47·4, $\tau = 48{\cdot}5 - 24{\cdot}2 = 24{\cdot}3$. If $k = 10^{-6}$ and $T = 4$ seconds, the total charge passing through the galvanometer is $1{\cdot}55 \times 10^{-5}$ coulombs.

**90. Measurement of a large capacity.** The ballistic throw produced by the discharge can be measured with a sensitive galvanometer if the condenser is of the order of a microfarad. We require a two-way key $PQR$ such that $Q$ can be joined to $P$ or $R$

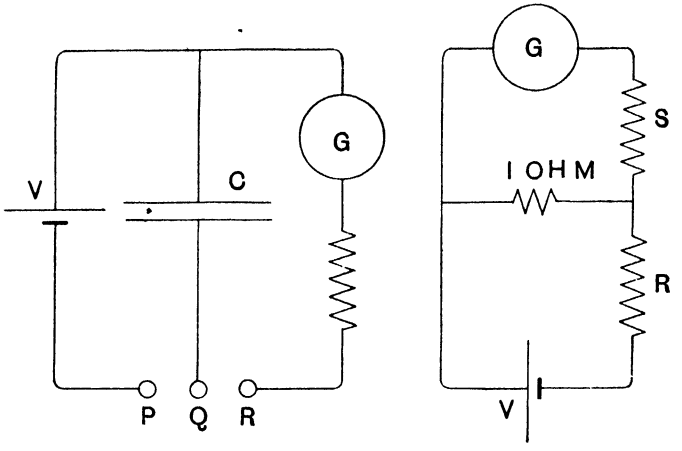



Fig. 121

at will. Start with $PQ$ joined, thus charging the condenser to potential $V$ (Fig. 121). When the key is pressed down so as to disconnect $PQ$ and join $QR$, a charge of $CV$ coulombs is sent suddenly through the galvanometer, $C$ being expressed in farads and $V$ in volts. Let $\tau$ be the ballistic throw, corrected for damping. Then

$$CV = \frac{T}{2\pi} k\tau.$$

The time of swing $T$ can be observed with a stop-watch. It is safest to have a resistance of several thousand ohms in the galvanometer circuit. This does not affect the ballistic throw, for the charge sent through is always the same, and it is only when $R$ is of the order of 100,000 ohms that the discharge begins to take place slowly (Art. 74).

We can eliminate $V$ and $k\tau$ by using the same cell and galvanometer in the standardising experiment, shown in the second part of Fig. 121.

Shunt through a small resistance $X$ (about 1 ohm), obtaining the same steady deflexion $\tau$ as the original throw, by adjusting the

resistance $R$. If $S + G$ is considerably greater than unity (at least 100 ohms), we have approximately

$$\frac{VX}{R} = (S + G)\, k\tau.$$

Dividing these equations,

$$C = \frac{XT}{2\pi R\,(S + G)} \quad \ldots\ldots\ldots\ldots\ldots (97),$$

giving $C$ in farads when $X$, $R$, $S$ and $G$ are in ohms.

**91. Mechanical forces between currents. Mutual inductance.** Let there be two circuits of any form, carrying currents $i$, $i'$ respectively, and let the flux of magnetic force through the second circuit due to unit current in the first be $M$. Then the flux through the second circuit where the first carries current $i$ is $N = iM$. The potential energy of the second circuit in the magnetic field of the first, that is the potential energy of the system, is

$$-i'N = -ii'M.$$

The symmetry of this result shows that $M$ is also the flux of magnetic force through the first circuit due to a unit current in the second. $M$ is called the *mutual inductance* of the two circuits: it evidently depends only on their form and relative position.

A general method of calculating $M$ is given in Art. 97. We now calculate the force between a current $i$ in a circular wire of centre $O$ and radius $a$ and a current $i'$ in a parallel circular wire of small radius $b$ whose centre $P$ is on the axis of the first circle. Let $OP = x$, and let the currents circulate in the directions shown. The magnetic force at $P$ due to the large circuit has been found to be $2\pi i a^2 (a^2 + x^2)^{-\frac{3}{2}}$ directed along $PO$. Since the lines of force thread the second circuit positively, the potential energy is obtained by multiplying by $-i' . \pi b^2$. Thus the potential energy of the system is $W = -2\pi^2 i i' a^2 b^2 (a^2 + x^2)^{-\frac{3}{2}}$. The force tending

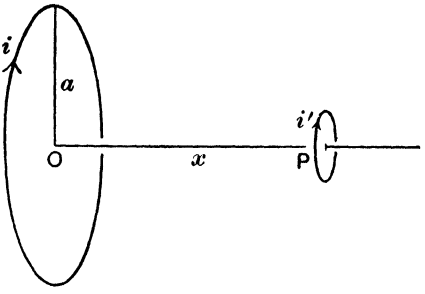


Fig. 122

to increase $x$ is therefore

$$-\frac{dW}{dx} = -6\pi^2 ii' a^2 b^2 x (a^2 + x^2)^{-\frac{5}{2}}.$$

The circuits therefore attract one another when the currents are in the same direction, and repel one another when in opposite directions. It is easy to show that the force of attraction is greatest when $x = \frac{1}{2}a$.

The first accurate observations of the mechanical forces between currents were those of W. Weber (1846). Two circular coils of many turns of wire were placed with their planes at right angles to one another, and one of the coils was supported by a bifilar suspension. By observing the small deflexions caused by a current passing through both coils in series, Weber was able to verify that the couple on the moveable coil was proportional to the square of the current. In a second experiment Weber placed the fixed coil in various positions with respect to the other, and showed that the observed couples agreed accurately with those to be expected on theoretical grounds.

**92. Self-inductance of a circuit.** By analogy with the mutual inductance of two circuits we may define the *self-inductance* $L$ of a single wire $C$ as the flux of magnetic force through it due to a unit current in $C$ itself. Care is however necessary in defining and calculating self-inductances, because if $C$ is taken as a geometrical line $L$ is infinite, in contrast with mutual inductances where the circuits can safely be taken as lines. For actual wires, of finite cross-section, $L$ is defined as a mean magnetic flux as follows.

If $dS$ is an element of cross-section of the wire, the electricity passing through $dS$ traces out a tubular region of space which we may call a current-filament. Let unit current flow in the whole wire, and imagine the wire cut up into a large number $n$ of filaments carrying equal currents. Then $L$ is the mean of the magnetic fluxes through the filaments, i.e. if $N$ is the flux through one of the filaments due to the current in the whole wire,

$$L = \frac{\Sigma N}{n},$$

where the summation extends over all the filaments. Writing $di = 1/n$ for the current in a filament, this becomes

$$L = \int N\, di \text{ where } \int di = 1.$$

It is not assumed in this definition that the current is uniformly distributed over the cross-section.

The self-inductance $L$ of a wire determines the potential energy of the mutual electromagnetic actions between its constituent current-filaments. Suppose in the first place that unit current is flowing in the wire. Then the potential energy of the filament $di$ is $-Ndi$, and on integration the potential energy would appear to be $-L$. But in this way the force between each pair of filaments is counted in twice, so that the potential energy is only $-\frac{1}{2}L$. When a current $i$ is flowing the force between each pair of filaments is $i^2$ times as great as before, and the potential energy in general is therefore $-\frac{1}{2}Li^2$. This quantity is essentially negative, as we should expect since two filaments of the wire (as it is not difficult to show from general principles) will always attract each other.

The theoretically simplest self-inductance is that of a solenoid whose length $l$ is great in comparison with its diameter. If $A$ is the area of cross-section of the core and $n$ the number of turns, the magnetic force inside the solenoid for unit current is $4\pi n/l$ except in the immediate neighbourhood of a winding. Slight disturbances of the field occur here, but the field always remains of the same order of magnitude; for if the wires are closely packed the radius of the cross-section of each is $l/2n$, and the magnetic force just outside one of the wires due to the unit current in the wire itself is $2 \div (l/2n) = 4n/l$. This finite disturbance only occurs in a narrow belt near the windings, so that $N$ can safely be calculated on the assumption of a uniform interior field. Hence the flux through a single turn is $4\pi nA/l$, and the total flux $4\pi n^2A/l$, neglecting the effect of the ends. Thus

$$L = 4\pi n^2 A/l \quad \ldots\ldots\ldots\ldots\ldots\ldots (98)$$

absolute electromagnetic units. The formula gives $L = 79{,}000$ E.M.U. for the solenoid of Fig. 111.

A solenoid is easier to treat than a simple circuit or loop of wire, because the magnetic force of the latter is much greater near the wire than elsewhere. We shall resume the calculation of self-inductances in Ch. VIII, since their chief importance is in connexion with electromagnetic induction.

**93. Electrodynamometers.** An electrodynamometer is an instrument for measuring the forces between two circuits carrying currents. One form consists of a fixed coil placed vertically, and within it a moveable coil suspended like the coil of a moving-coil galvanometer from a torsion fibre. The two coils are initially at right angles. If currents flow in both coils, there is a couple on the moveable coil due to the magnetic field of the fixed coil, and the instrument shows a deflexion. The moveable coil is brought back to its original position by turning the torsion head. The couple required to do this is proportional to the angle which the torsion head has turned through: it balances the couple exerted by the fixed coil on the moveable coil in its original position, and is therefore proportional to the product of the currents in the two coils. If the two coils are joined in series and the same current $i$ passes through them, the couple is proportional to $i^2$. Electrodynamometers, when once standardised, can be used to measure a current irrespective of external magnetic fields, and are then known as current-balances.

Lord Kelvin designed an accurate form of current-balance known by his name, the principle being shown in Fig. 123. *A*, *B*, *C*, *D*

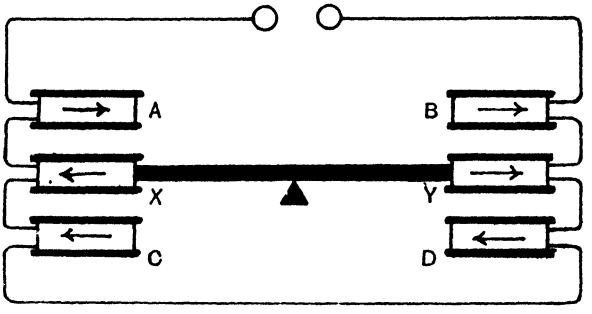



Fig. 123

represent four fixed horizontal coils wound so that the current circulates in them in the directions shown. The moveable coils *X*, *Y* are joined in series with the fixed coils, but arranged so that the current in *Y* is in the opposite direction to that in *X*. A horizontal beam connects *X* to *Y* and is supported somewhat after the manner of the beam of a balance. It has a graduated scale in front, on which a rider of known weight can slide. When a current

flows through the balance, $X$ is repelled by $A$ and attracted by $C$, while $Y$ is attracted by $B$ and repelled by $D$, all four forces combining to give a couple in the counter-clockwise direction in the figure, which can be balanced by known weights.

Fig. 124 shows an actual instrument. The central coils are not visible, but only the triangular trough attached to the right-hand

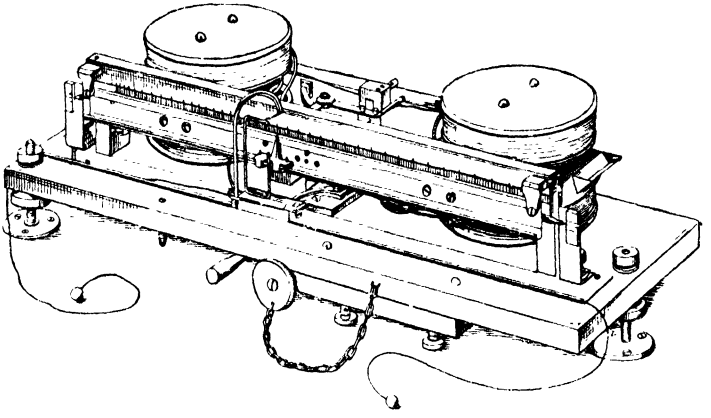


Fig. 124

coil, which corresponds to $Y$ in Fig. 123. A known weight is first placed in the trough, with no current flowing, and the rider, which is of the same weight, is placed at zero on the left of the scale. A counterpoise lever is then adjusted till the beam is in its correct position, as shown by two marks opposite the projections at its ends. Current is applied and the rider is moved to the right till the beam is again in its normal position, the current being then read off on a scale opposite the pointer. Connexion with the moveable coils is made through the suspension of the beam, so that the latter may swing as freely as possible.

The greatest sensitiveness is obtained by placing the coils so that the couple exerted is a maximum. If the coils $X$, $Y$ were small in comparison with the fixed coils, and if the latter consisted of a single turn each, we have seen in Art. 91 that the most favourable position is for $X$ to be at a distance from $A$ and $C$ equal to half the radius of either coil. The best distance is of the same order in Kelvin's balance.

If the dimensions of an electrodynamometer are accurately measured, the forces can be calculated theoretically for given

currents, so that an electrodynamometer will measure current in absolute electromagnetic units. One of the most perfect absolute electrodynamometers is that designed by W. E. Ayrton and J. V. Jones, the principle of which is illustrated by Fig. 125.

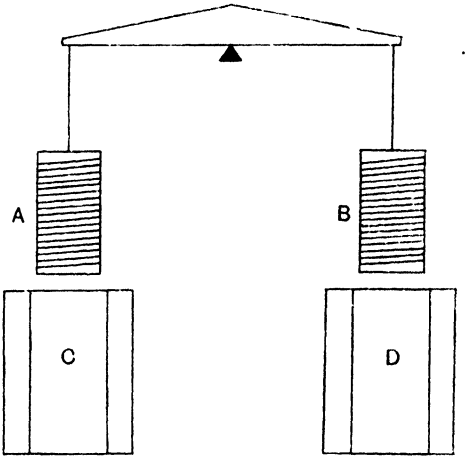


Fig. 125

Two coils $A$, $B$, wound on marble cylinders, are suspended from the beam of a sensitive balance. $C$, $D$ represent hollow cylinders also wound with wire, which can be raised so as to surround the coils $A$, $B$. The winding is such that $A$ repels $C$ and $B$ attracts $D$ when a current passes in series through all the coils. The couple is then exactly compensated by adding weights on a scale pan hanging from the left of the beam. This current-balance was used in finding the electro-chemical equivalent of silver, and was considered to be accurate to 1 part in 50,000.

**94. Action of a magnetic field on an element of current.** We shall now show that the force exerted by a magnetic field on a closed circuit can be analysed into forces acting on its elements of length according to a definite law; and although we cannot really deal with the separate parts of a closed circuit, this representation of the forces will be useful.

Let $\mathbf{H}$ be the magnetic force in the neighbourhood of an element $d\mathbf{s}$ of the circuit. When the element is given a small displacement $\mathbf{u}$, it traces out a small parallelogram whose area is precisely the vector area $V d\mathbf{s}, \mathbf{u}$ (Fig. 126). Suppose for the moment that unit current is flowing in the wire. The increase in the flux of magnetic force through the circuit, in so far as it arises from $d\mathbf{s}$, is the flux through the small parallelogram, that is the scalar product $\mathbf{H}\,(V d\mathbf{s}, \mathbf{u})$ of $\mathbf{H}$ and $V d\mathbf{s}, \mathbf{u}$. (The displacement in the figure being such that the current tends to embrace more lines of force, the

positive sign must be taken.) Hence the elementary work done in the displacement is $-\mathbf{H}(V d\mathbf{s}, \mathbf{u})$. The vector formula (9) on p. 6 enables us to transform this expression into a scalar product with $\mathbf{u}$, namely

$$(-V\mathbf{H}, d\mathbf{s})\,\mathbf{u} = (V d\mathbf{s}, \mathbf{H})\,\mathbf{u}.$$

In other words, the work done is the same as if each element $d\mathbf{s}$ was acted on by a force $V d\mathbf{s}, \mathbf{H}$. The virtual displacement $\mathbf{u}$ need not be the same at all points, provided that the circuit is moved as a rigid body, so that the mutual actions of its various parts do no work. Hence when a current $i$ flows in the circuit, the resultant force and couple on the circuit are the same as if each element $d\mathbf{s}$ was acted on by a force equal to *i times the vector product of* $d\mathbf{s}$ *and* $\mathbf{H}$.

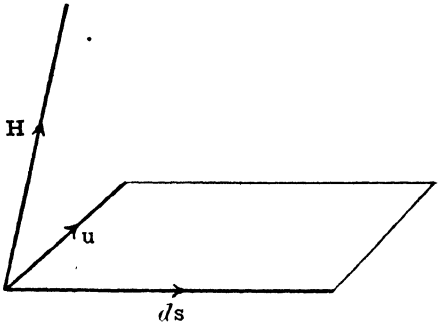


Fig. 126

This rule gives an easy explanation of the actions of magnets on currents described in Art. 84. Thus in the electromagnetic rotation of liquids, a current radially outwards calls forth at every point a mechanical force tending to turn the carriers of electricity round the axis.

If we regard an electric current as a flow of electrons, and also assume that the elementary law holds good for each separate part of a circuit, we have a rule for the action of a magnetic field on a moving charge, that a charge of $e$ electrostatic units moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{H}$ experiences a force $e/c$ *times the vector product of* $\mathbf{v}$ *and* $\mathbf{H}$. To prove this, let $P$, $Q$ (Fig. 127) represent two sections distant $ds$ from each other. Let $n$ be the number of free electrons per cm.$^3$, $A$ the cross-section of the wire and $\mathbf{v}$ the velocity of drift. The amount of electricity passing $P$ per second is $neAv$ electrostatic units, so that the current $i$ measured electromagnetically is equal to $neAv/c$. The total force exerted by the magnetic field on the electrons between

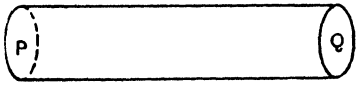


Fig. 127

$P$ and $Q$ is $ids H \sin\theta = neAvds H \sin\theta/c$, where $\theta$ is the angle between $\mathbf{H}$ and $d\mathbf{s}$ or $\mathbf{v}$. The number of electrons concerned is $nAds$. Hence the force on each is $evH \sin\theta/c$, or $(e/c)\, V\mathbf{vH}$. The view of electromagnetic action thus indicated is that every electron moving in a magnetic field is acted on by a definite force, which it transfers by collision to the atoms of the metal, the total force reappearing as the mechanical force on the element of the wire. The electrons are driven to the side of the wire, but only to a very slight extent (Art. 258).

**95. Equations of the electromagnetic field.** We now consider steady currents distributed continuously through space. Let $\mathbf{H}$ be the magnetic force and $\mathbf{u}$ the current-density at any point. When a unit pole is taken round a circuit $C$ in the field, the magnetic force does work on it equal to $4\pi$ times the current enclosed; for each filament of current which is interlaced contributes $4\pi i$ and each filament which is not interlaced contributes nothing (Arts. 80, 81). The current enclosed is the flux of $\mathbf{u}$ through $C$, and the work is the line-integral of $\mathbf{H}$. Hence

$$4\pi \int_S \mathbf{u}\, d\mathbf{S} = \int_C \mathbf{H}\, d\mathbf{s} = \int_S \operatorname{curl} \mathbf{H}\, d\mathbf{S}$$

by Stokes' theorem, $S$ being any surface with $C$ as rim. Since $C$ and $S$ may be anywhere we must have

$$4\pi\mathbf{u} = \operatorname{curl} \mathbf{H} \quad \ldots\ldots\ldots\ldots\ldots\ldots (99).$$

This is a general equation connecting the magnetic force and current-density at any point of the field, when there is no magnetisable matter anywhere. It implies that $\mathbf{u}$ is solenoidal, a condition that is always fulfilled in steady flow.

Since the line-integral of magnetic force is not in general zero, there is no magnetic potential. On the other hand, the lines of magnetic force are re-entrant curves, and it is reasonable to suppose that the magnetic force is solenoidal. Let us try to solve the vector equation (99) for $\mathbf{H}$, on the assumption that

$$\operatorname{div} \mathbf{H} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (100).$$

Assume provisionally that

$$\mathbf{H} = \operatorname{curl} \mathbf{A} \quad \ldots\ldots\ldots\ldots\ldots\ldots (101),$$

where $\mathbf{A}$ is a vector to be determined. Then equation (100) is satisfied identically, and (99) gives

$$4\pi\mathbf{u} = \text{curl curl } \mathbf{A} = \mathbf{D} \text{ div } \mathbf{A} - \Delta\mathbf{A}$$

from equation (20). The last equation is not tractable unless we assume further that div $\mathbf{A} = 0$, when it becomes

$$\Delta\mathbf{A} = -4\pi\mathbf{u}.$$

Writing $\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z$ and $\mathbf{u} = \mathbf{i}u_x + \mathbf{j}u_y + \mathbf{k}u_z$ we have three equations of Poisson's type to determine $A_x$, $A_y$, $A_z$, namely

$$\Delta A_x = -4\pi u_x, \quad \Delta A_y = -4\pi u_y, \quad \Delta A_z = -4\pi u_z.$$

A particular solution is found by taking, at a point $P$,

$$A_x = \int \frac{u_x d\tau}{r}, \quad A_y = \int \frac{u_y d\tau}{r}, \quad A_z = \int \frac{u_z d\tau}{r},$$

where $u_x$, $u_y$, $u_z$ are the components of current-density at a point $Q$ distant $r$ from $P$, $d\tau$ is an element of volume at $Q$, and integration is over the whole of space. The latter equations are resumed in the vector equation

$$\mathbf{A} = \int \frac{\mathbf{u} d\tau}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots (102),$$

where the integral on the right is a vector integral (Art. 6), each volume element $d\tau$ contributing an infinitesimal vector $\mathbf{u}\,d\tau/r$. The vector $\mathbf{A}$ is called the *vector potential of magnetic force.*

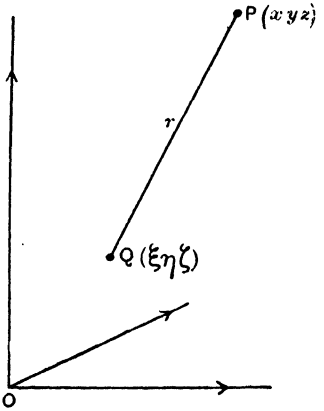


Fig. 128

To show that $\mathbf{A}$ is really solenoidal, let $P$ be the point $(x, y, z)$ (Fig. 128), and $Q$ the point $(\xi, \eta, \zeta)$. Instead of calculating $A_x$ at $P$ let us calculate it at a neighbouring point $(x + h, y, z)$, and consider the contribution of the volume-element near the point $(\xi + h, \eta, \zeta)$. Since $u_x$ is replaced by $u_x + h\partial u_x/\partial\xi$ and $r$ is the same as before, $A_x$ becomes

$$\int \left(u_x + h\frac{\partial u_x}{\partial \xi}\right)\frac{d\tau}{r} = A_x + h\int \frac{\partial u_x}{\partial \xi}\frac{d\tau}{r}.$$

The expression must be equal to $A_x + h\partial A_x/\partial x$ to the first order in $h$, so that

$$\frac{\partial A_x}{\partial x} = \int \frac{\partial u_x}{\partial \xi} \frac{d\tau}{r}.$$

Adding to the two similar equations we have

$$\operatorname{div} \mathbf{A} = \int \operatorname{div} \mathbf{u} \frac{d\tau}{r},$$

where div **u** on the right is calculated at $Q$. But div **u** vanishes everywhere, so that div **A** = 0.

When **A** is known the magnetic force is given by equation (101). The solution thus obtained is not unique, since we can clearly superpose the magnetic field of any permanent magnetic distribution that may be present. The above solution, which gives **H** = 0 when **u** = 0 everywhere, may however fairly be described as giving the magnetic field *due to the currents*, and we shall neglect the subsidiary terms.

Thus the work law, helped by certain natural assumptions, determines the magnetic force of any distribution of currents in space. Applying the formulae to the linear circuit we shall obtain results equivalent to those already used.

**96. Magnetic field of a linear current. Ampère's law.** Let $Q$ be a point on a closed circuit $C$, and let a thin wire, whose cross-section is $A$, coincide with $C$ and carry a current $i$. The current-density vanishes outside the wire, and is of magnitude $u = i/A$ inside. The volume-element $d\tau = A\,ds$, where $ds$ is an element of arc at $Q$ (Fig. 129), so that $u\,d\tau = i\,ds$. Regarded as vectors, **u** and $d\mathbf{s}$ are parallel, so that we have the vector equation $\mathbf{u}\,d\tau = i\,d\mathbf{s}$, and (102) gives, for the vector potential at $P$,

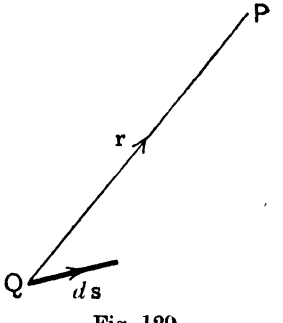


Fig. 129

$$\mathbf{A} = i \int \frac{d\mathbf{s}}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots (103),$$

a vector integral round $C$, each element $d\mathbf{s}$ contributing $\frac{i}{r}\,d\mathbf{s}$. The curl of the latter vector is, from equation (21), $i$ times the vector product of $\mathbf{D}(1/r)$ and $d\mathbf{s}$, since $d\mathbf{s}$ is independent of the position

of $P$. $\mathbf{D}(1/r)$ is a vector of magnitude $1/r^2$ along $PQ$, that is $-\mathbf{r}/r^3$, where $\mathbf{r} = QP$. Hence the element $d\mathbf{s}$ contributes to $\mathbf{H}$ an amount $i/r^3$ times the vector product of $d\mathbf{s}$ and $\mathbf{r}$. This is Ampère's elementary law, sufficient but not necessary to explain the laws of electromagnetism. The elementary magnetic force at $P$ in Fig. 129 is vertically upwards from the paper, in the sense of the right-handed screw law, which is thus confirmed. A special case of the law was used by anticipation in Art. 59.

Ampère's law affords the easiest means of calculating the magnetic force of jointed circuits. Thus if we have a square circuit of side $2a$, the magnetic force at a point on a line through the centre parallel to one side, and distant $x$ from the centre, should be

$$\frac{2i}{a(a-x)}\{(a-x)^2+a^2\}^{\frac{1}{2}}+\frac{2i}{a(a+x)}\{(a+x)^2+a^2\}^{\frac{1}{2}}.$$

In verifying such a formula experimentally we shall be indirectly confirming the electromagnetic laws of Art. 95. The following table gives the result of such a comparison.

*Square circuit, side* 44·6 *cm.*

| $x$ | Ratio of magnetic force to force at the centre | |
|---|---|---|
| | Obs. | Calc. |
| 8·3 | 1·04 | 1·03 |
| 14·2 | 1·07 | 1·07 |
| 19·7 | 1·15 | 1·15 |
| 22·5 | 1·22 | 1·22 |
| 25·4 | 1·30 | 1·31 |
| 28·6 | 1·43 | 1·46 |
| 31·7 | 1·66 | 1·68 |

The magnetic effect of a moving charge has been described in Art. 61, and in Art. 94 we showed that the action of a magnetic field on it is found by replacing the element of current $i\,d\mathbf{s}$ by $e\mathbf{v}/c$. Transferring this analogy to Ampère's law we should therefore expect a charge $e$ moving with uniform velocity $\mathbf{v}$ to produce at any point $P$ a magnetic force

$$\frac{e}{cr^3}V\mathbf{vr},$$

where $\mathbf{r}$ is the vector from $e$ to $P$.

The connexion between Ampère's law and the law $\Omega = -i\phi$ will be established by proving it for a small plane circuit. Let $C$ be such a circuit of vector area $\mathbf{S}$. The vector potential at any point $P$ is given by equation (103). Forming the scalar product with a constant vector $\mathbf{a}$, we have

$$\mathbf{aA} = i\int_C \frac{\mathbf{a}d\mathbf{s}}{r},$$

and the integral on the right is, from Art. 9, the scalar product of $\mathbf{S}$ with $\mathrm{curl}_1(\mathbf{a}/r)$, the suffix 1 being used to indicate that $Q$ is the variable and $P$ is kept fixed for the moment. From equation (21)

$$\mathrm{curl}_1\left(\frac{\mathbf{a}}{r}\right) = V\mathbf{D}_1\frac{1}{r}, \mathbf{a}$$

so that

$$\mathbf{aA} = i\mathbf{S}\left(V\mathbf{D}_1\frac{1}{r}, \mathbf{a}\right) = i\left(V\mathbf{S}, \mathbf{D}_1\frac{1}{r}\right)\mathbf{a}$$

from (9). Since this holds for any constant vector $\mathbf{a}$,

$$\mathbf{A} = iV\mathbf{S}, \mathbf{D}_1\frac{1}{r}.$$

The gradient $\mathbf{D}_1(1/r)$ is equal and opposite to the gradient $\mathbf{D}(1/r)$, in which $P$ is the variable and $Q$ is kept fixed. Hence the vector potential of the small circuit at $P$ is

$$\mathbf{A} = iV\mathbf{D}\frac{1}{r}, \mathbf{S}.$$

Taking a vector origin in the area and the unit vector $\mathbf{k}$ normal to the plane, $\mathbf{S} = \mathbf{k}S$ and

$$\mathbf{D}\frac{1}{r} = \mathbf{i}\frac{\partial}{\partial x}\frac{1}{r} + \mathbf{j}\frac{\partial}{\partial y}\frac{1}{r} + \mathbf{k}\frac{\partial}{\partial z}\frac{1}{r}.$$

Thus

$$\begin{aligned}\mathbf{A} &= iS\left[V\mathbf{ik}\frac{\partial}{\partial x}\frac{1}{r} + V\mathbf{jk}\frac{\partial}{\partial y}\frac{1}{r} + V\mathbf{kk}\frac{\partial}{\partial z}\frac{1}{r}\right]\\ &= iS\left[\mathbf{i}\frac{\partial}{\partial y}\frac{1}{r} - \mathbf{j}\frac{\partial}{\partial x}\frac{1}{r}\right]\end{aligned}$$

from equation (7) on p. 6. From equation (15),

$$\begin{aligned}\mathrm{curl}\,\mathbf{A} &= iS\left[\mathbf{i}\frac{\partial^2}{\partial x\partial z}\frac{1}{r} + \mathbf{j}\frac{\partial^2}{\partial y\partial z}\frac{1}{r} - \mathbf{k}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{1}{r}\right]\\ &= iS\left[\mathbf{i}\frac{\partial^2}{\partial x\partial z}\frac{1}{r} + \mathbf{j}\frac{\partial^2}{\partial y\partial z}\frac{1}{r} + \mathbf{k}\frac{\partial^2}{\partial z^2}\frac{1}{r}\right].\end{aligned}$$

Thus

$$\mathbf{H} = iS\mathbf{D}\left(\frac{\partial}{\partial z}\frac{1}{r}\right),$$

showing that there is a magnetic potential

$$\Omega = -iS\frac{\partial}{\partial z}\frac{1}{r}.$$

This is $-i$ times the solid angle subtended by the circuit at $P$, which proves the theorem.

The present theory shows that the formula $\Omega = -i\phi$ is the only one consistent with the work law in its widest sense, and also shows the conditions under which a magnetic potential exists. When currents are distributed through space, the magnetic force is not derivable from a potential; but when the currents are located in limited tubular regions of space, a magnetic potential exists at points not in these regions. The peculiarity of this potential is that it is multiple-valued, permitting different work to be done on a unit pole when the latter is moved round a circuit which does or does not interlace the tubular regions.

**97. Neumann's formula for mutual inductance.** Let $Q$ in Fig. 129 be a point on a circuit $C$ carrying unit current, and let $\mathbf{s}$ be the vector from a fixed origin to $Q$, $\mathbf{s}'$ the vector to the variable point $P$. The magnetic force at $P$ is $\mathbf{H} = \text{curl } \mathbf{A}$, where $\mathbf{A} = \int_C \frac{d\mathbf{s}}{r}$. The flux of magnetic force through a second closed circuit $C'$ is $\int_{S'} \mathbf{H} d\mathbf{S}'$, where $S'$ is any surface having $C'$ as rim. Thus

$$M = \int_{S'} \mathbf{H} d\mathbf{S}' = \int_{C'} \mathbf{A} d\mathbf{s}'$$

by Stokes' theorem. Substituting for $\mathbf{A}$ we have Neumann's formula

$$M = \int_C \int_{C'} \frac{d\mathbf{s}\, d\mathbf{s}'}{r} \quad \ldots\ldots\ldots\ldots\ldots (104),$$

the element of the integral being $1/r$ times the scalar product of the elements $d\mathbf{s}$, $d\mathbf{s}'$ of the two circuits, and integration being over all possible combinations of the two.

**98. Mutual inductance of two parallel circles on the same axis.** Consider first the case in which the circles are in the same plane, and therefore concentric. Let $a$, $b$ be the radii, and take

an element $ds$ at $P$ on the circle of radius $a$ and one of length $ds' = b\,d\psi$ on the circle of radius $b$. Then the elements are inclined at an angle $\psi$, and $r^2 = a^2 + b^2 - 2ab\cos\psi$. The elementary scalar product being $ds\,ds'\cos\psi$, the total contribution of $ds$ to $M$ is

$$ds\int\frac{\cos\psi\,ds'}{r} = ds\int_0^{2\pi}\frac{b\cos\psi\,d\psi}{(a^2+b^2-2ab\cos\psi)^{\frac{1}{2}}}.$$

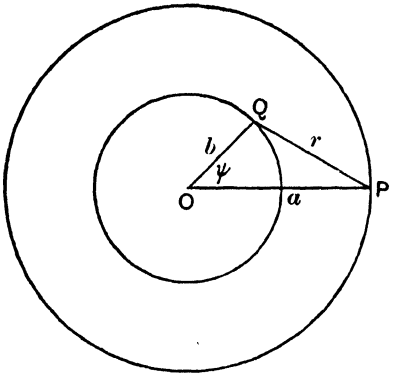



Fig. 130

The mutual inductance of the two circles is obtained by writing the whole circumference $2\pi a$ of the outer circle instead of $ds$, so that

$$M = 4\pi ab\int_0^{\pi}\frac{\cos\psi\,d\psi}{(a^2+b^2-2ab\cos\psi)^{\frac{1}{2}}}.$$

Writing $\psi = \pi - 2\phi$ we have

$$M = -8\pi ab\int_0^{\frac{\pi}{2}}\frac{\cos 2\phi\,d\phi}{(a^2+b^2+2ab\cos 2\phi)^{\frac{1}{2}}}.$$

This may be reduced to elliptic integrals by writing

$$k^2 = \frac{4ab}{(a+b)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(105).$$

Thus we find after a little reduction that

$$M = \frac{8\pi ab}{a+b}\int_0^{\frac{\pi}{2}}\frac{(2\sin^2\phi - 1)\,d\phi}{(1-k^2\sin^2\phi)^{\frac{1}{2}}},$$

or

$$M = 4\pi(ab)^{\frac{1}{2}}\left[\left(\frac{2}{k}-k\right)K - \frac{2}{k}E\right] \quad \ldots\ldots(106),$$

where $K$ and $E$ are complete elliptic integrals of the first and second kind to modulus $k$.

If the circles are no longer concentric but lie in parallel planes distant $x$ apart, with the line joining their centres perpendicular to both planes, equation (106) still holds, but (105) is replaced by

$$k^2 = \frac{4ab}{(a+b)^2+x^2} \quad \ldots\ldots\ldots\ldots\ldots (107).$$

The elliptic integrals have all been tabulated, and hence the quantity in square brackets is known in terms of $k$. Writing

$$k = \sin\gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots (108),$$

$M/(ab)^{\frac{1}{2}}$ can be tabulated for various values of $\gamma$, as was done by Maxwell. A table is given at the end of this article.

In the case of concentric circles the calculation is simplified by observing that

$$\cos\gamma = \frac{a-b}{a+b} \quad \ldots\ldots\ldots\ldots\ldots\ldots (109),$$

which follows readily from (105) and (108). The following are two examples:

(1) $a = 16{\cdot}025$ cm., $b = 10{\cdot}550$ cm., $\gamma = 78^\circ\, 6'$, $M = 168{\cdot}61$ absolute E.M. units.

(2) $a = 16{\cdot}025$ cm., $b = 1{\cdot}750$ cm., $\gamma = 36^\circ\, 35'$, $M = 3{\cdot}79$ absolute E.M. units.

Mutual inductance calculations are used in finding resistance in absolute measure (Art. 140). Similarly absolute measurements of current depend on a calculation of the force between two circuits. Fortunately one such calculation, of the pull between a finite circular solenoid and *one* turn of another coaxial solenoid, makes use of Maxwell's tables.

Let $M$ be the mutual inductance of the single turn and a section of the solenoid distant $x$ from one end. If $l$ is the length of the solenoid and $n$ the total number of turns, the number of turns between sections $x$ and $x + dx$ is $n\,dx/l$. The pull between the single turn and this group of $n\,dx/l$ turns is

$$ii'\frac{dM}{dx}\frac{n\,dx}{l} = \frac{nii'}{l}\,dM.$$

Hence the total pull is

$$\frac{nii'}{l}\int dM = \frac{nii'}{l}(M_1 - M_2),$$

where $M_1$ is the mutual inductance of the single turn and one circular end of the solenoid, and $M_2$ the mutual inductance between it and the other end of the solenoid.

---

*Mutual inductance of circles.*

| $\gamma$ | $M/(ab)^{\frac{1}{2}}$ | $\gamma$ | $M/(ab)^{\frac{1}{2}}$ | $\gamma$ | $M/(ab)^{\frac{1}{2}}$ |
|---|---|---|---|---|---|
| 5° 0′ | ·00166 | 8° 18′ | ·00753 | 11° 36′ | ·02068 |
| 6 | ·00175 | 24 | ·00780 | 42 | ·02124 |
| 12 | ·00185 | 30 | ·00809 | 48 | ·02179 |
| 18 | ·00196 | 36 | ·00839 | 54 | ·02234 |
| 24 | ·00207 | 42 | ·00871 | 12 0 | ·02291 |
| 30 | ·00219 | 48 | ·00900 | 6 | ·02350 |
| 36 | ·00231 | 54 | ·00929 | 12 | ·02409 |
| 42 | ·00244 | 9 0 | ·00961 | 18 | ·02469 |
| 48 | ·00256 | 6 | ·00994 | 24 | ·02531 |
| 54 | ·00270 | 12 | ·01028 | 30 | ·02594 |
| 6 0 | ·00284 | 18 | ·01063 | 36 | ·02657 |
| 6 | ·00299 | 24 | ·01097 | 42 | ·02721 |
| 12 | ·00314 | 30 | ·01132 | 48 | ·02785 |
| 18 | ·00329 | 36 | ·01169 | 54 | ·02851 |
| 24 | ·00346 | 42 | ·01205 | 13 0 | ·02919 |
| 30 | ·00362 | 48 | ·01243 | 6 | ·02988 |
| 36 | ·00378 | 54 | ·01282 | 12 | ·03057 |
| 42 | ·00396 | 10 0 | ·01322 | 18 | ·03128 |
| 48 | ·00413 | 6 | ·01362 | 24 | ·03199 |
| 54 | ·00432 | 12 | ·01402 | 30 | ·03273 |
| 7 0 | ·00450 | 18 | ·01444 | 36 | ·03348 |
| 6 | ·00470 | 24 | ·01488 | 42 | ·03422 |
| 12 | ·00491 | 30 | ·01532 | 48 | ·03498 |
| 18 | ·00511 | 36 | ·01576 | 54 | ·03575 |
| 24 | ·00533 | 42 | ·01620 | 14 0 | ·03653 |
| 30 | ·00554 | 48 | ·01666 | 6 | ·03732 |
| 36 | ·00578 | 54 | ·01714 | 12 | ·03814 |
| 42 | ·00603 | 11 0 | ·01762 | 18 | ·03897 |
| 48 | ·00627 | 6 | ·01811 | 24 | ·03980 |
| 54 | ·00651 | 12 | ·01861 | 30 | ·04065 |
| 8 0 | ·00676 | 18 | ·01913 | 36 | ·04151 |
| 6 | ·00701 | 24 | ·01964 | 42 | ·04237 |
| 12 | ·00726 | 30 | ·02014 | 48 | ·04325 |

*Mutual inductance of circles (continued).*

| $\gamma$ | $M/(ab)^{\frac{1}{2}}$ | $\gamma$ | $M/(ab)^{\frac{1}{2}}$ | $\gamma$ | $M/(ab)^{\frac{1}{2}}$ |
|---|---|---|---|---|---|
| 14° 54′ | ·04415 | 19° 54′ | ·106606 | 24° 54′ | ·212613 |
| 15 0 | ·04504 | 20 0 | ·108254 | 25 0 | ·215272 |
| 6 | ·04596 | 6 | ·109923 | 6 | ·217955 |
| 12 | ·04690 | 12 | ·111607 | 12 | ·220661 |
| 18 | ·04784 | 18 | ·113306 | 18 | ·223390 |
| 24 | ·04880 | 24 | ·115028 | 24 | ·226141 |
| 30 | ·04976 | 30 | ·116766 | 30 | ·228915 |
| 36 | ·05074 | 36 | ·118524 | 36 | ·231718 |
| 42 | ·05174 | 42 | ·120295 | 42 | ·234540 |
| 48 | ·05274 | 48 | ·122086 | 48 | ·237388 |
| 54 | ·05377 | 54 | ·123894 | 54 | ·240260 |
| 16 0 | ·05480 | 21 0 | ·125724 | 26 0 | ·243154 |
| 6 | ·05585 | 6 | ·127575 | 6 | ·246076 |
| 12 | ·05690 | 12 | ·129440 | 12 | ·249024 |
| 18 | ·05798 | 18 | ·131324 | 18 | ·251995 |
| 24 | ·05907 | 24 | ·133226 | 24 | ·254984 |
| 30 | ·06017 | 30 | ·135149 | 30 | ·258005 |
| 36 | ·06129 | 36 | ·137092 | 36 | ·261053 |
| 42 | ·06242 | 42 | ·139051 | 42 | ·264121 |
| 48 | ·06356 | 48 | ·141031 | 48 | ·267219 |
| 54 | ·06472 | 54 | ·143033 | 54 | ·270340 |
| 17 0 | ·06590 | 22 0 | ·145049 | 27 0 | ·273486 |
| 6 | ·06709 | 6 | ·147086 | 6 | ·276657 |
| 12 | ·06830 | 12 | ·149145 | 12 | ·279856 |
| 18 | ·06952 | 18 | ·151226 | 18 | ·283081 |
| 24 | ·07075 | 24 | ·153324 | 24 | ·286327 |
| 30 | ·07199 | 30 | ·155442 | 30 | ·289603 |
| 36 | ·07325 | 36 | ·157581 | 36 | ·292907 |
| 42 | ·07452 | 42 | ·159739 | 42 | ·296237 |
| 48 | ·07581 | 48 | ·161919 | 48 | ·299594 |
| 54 | ·07712 | 54 | ·164119 | 54 | ·302980 |
| 18 0 | ·07845 | 23 0 | ·166340 | 28 0 | ·306390 |
| 6 | ·07978 | 6 | ·168582 | 6 | ·309829 |
| 12 | ·08114 | 12 | ·170842 | 12 | ·313293 |
| 18 | ·08250 | 18 | ·173128 | 18 | ·316782 |
| 24 | ·08388 | 24 | ·175432 | 24 | ·320303 |
| 30 | ·08529 | 30 | ·177758 | 30 | ·323849 |
| 36 | ·08671 | 36 | ·180104 | 36 | ·327424 |
| 42 | ·08814 | 42 | ·182474 | 42 | ·331027 |
| 48 | ·08959 | 48 | ·184866 | 48 | ·334660 |
| 54 | ·09105 | 54 | ·187278 | 54 | ·338323 |
| 19 0 | ·09253 | 24 0 | ·189711 | 29 0 | ·342006 |
| 6 | ·09403 | 6 | ·192165 | 6 | ·345720 |
| 12 | ·09554 | 12 | ·194643 | 12 | ·349470 |
| 18 | ·09707 | 18 | ·197143 | 18 | ·353243 |
| 24 | ·09862 | 24 | ·199666 | 24 | ·357046 |
| 30 | ·100175 | 30 | ·202211 | 30 | ·360876 |
| 36 | ·101761 | 36 | ·204780 | 36 | ·364738 |
| 42 | ·103358 | 42 | ·207363 | 42 | ·368631 |
| 48 | ·104972 | 48 | ·209978 | 48 | ·372550 |

*Mutual inductance of circles (continued).*

| $\gamma$ | $M/(ab)^{\frac{1}{2}}$ | $\gamma$ | $M/(ab)^{\frac{1}{2}}$ | $\gamma$ | $M/(ab)^{\frac{1}{2}}$ |
|---|---|---|---|---|---|
| 29° 54′ | ·376499 | 34° 54′ | ·615354 | 39° 54′ | ·950244 |
| 30 0 | ·380477 | 35 0 | ·621033 | 40 0 | ·958093 |
| 6 | ·384488 | 6 | ·626747 | 6 | ·965991 |
| 12 | ·388530 | 12 | ·632503 | 12 | ·973940 |
| 18 | ·392599 | 18 | ·638296 | 18 | ·981936 |
| 24 | ·396697 | 24 | ·644128 | 24 | ·989984 |
| 30 | ·400830 | 30 | ·650002 | 30 | ·998081 |
| 36 | ·404990 | 36 | ·655913 | 36 | 1·006230 |
| 42 | ·409184 | 42 | ·661863 | 42 | 1·014428 |
| 48 | ·413410 | 48 | ·667858 | 48 | 1·022678 |
| 54 | ·417663 | 54 | ·673886 | 54 | 1·030978 |
| 31 0 | ·421950 | 36 0 | ·679958 | 41 0 | 1·039329 |
| 6 | ·426268 | 6 | ·686070 | 6 | 1·047731 |
| 12 | ·430617 | 12 | ·692224 | 12 | 1·056187 |
| 18 | ·434999 | 18 | ·698418 | 18 | 1·064696 |
| 24 | ·439413 | 24 | ·704650 | 24 | 1·073255 |
| 30 | ·443861 | 30 | ·710924 | 30 | 1·081865 |
| 36 | ·448338 | 36 | ·717240 | 36 | 1·090529 |
| 42 | ·452849 | 42 | ·723597 | 42 | 1·099249 |
| 48 | ·457392 | 48 | ·729997 | 48 | 1·108021 |
| 54 | ·461968 | 54 | ·736438 | 54 | 1·116846 |
| 32 0 | ·466577 | 37 0 | ·742921 | 42 0 | 1·125726 |
| 6 | ·471220 | 6 | ·749446 | 6 | 1·134657 |
| 12 | ·475894 | 12 | ·756014 | 12 | 1·143646 |
| 18 | ·480604 | 18 | ·762626 | 18 | 1·152688 |
| 24 | ·485347 | 24 | ·769279 | 24 | 1·161785 |
| 30 | ·490123 | 30 | ·775979 | 30 | 1·170937 |
| 36 | ·494934 | 36 | ·782719 | 36 | 1·180150 |
| 42 | ·499780 | 42 | ·789501 | 42 | 1·189416 |
| 48 | ·504661 | 48 | ·796330 | 48 | 1·198735 |
| 54 | ·509574 | 54 | ·803204 | 54 | 1·208113 |
| 33 0 | ·514520 | 38 0 | ·810120 | 43 0 | 1·217547 |
| 6 | ·519504 | 6 | ·817080 | 6 | 1·227036 |
| 12 | ·524520 | 12 | ·824086 | 12 | 1·236585 |
| 18 | ·529576 | 18 | ·831137 | 18 | 1·246192 |
| 24 | ·534663 | 24 | ·838232 | 24 | 1·255856 |
| 30 | ·539786 | 30 | ·845373 | 30 | 1·265578 |
| 36 | ·544948 | 36 | ·852562 | 36 | 1·275360 |
| 42 | ·550143 | 42 | ·859794 | 42 | 1·285198 |
| 48 | ·555375 | 48 | ·867072 | 48 | 1·295098 |
| 54 | ·560642 | 54 | ·874397 | 54 | 1·305054 |
| 34 0 | ·565949 | 39 0 | ·881769 | 44 0 | 1·315071 |
| 6 | ·571290 | 6 | ·889186 | 6 | 1·325151 |
| 12 | ·576665 | 12 | ·896653 | 12 | 1·335288 |
| 18 | ·582079 | 18 | ·904165 | 18 | 1·345488 |
| 24 | ·587529 | 24 | ·911726 | 24 | 1·355747 |
| 30 | ·593020 | 30 | ·919329 | 30 | 1·366068 |
| 36 | ·598545 | 36 | ·926984 | 36 | 1·376450 |
| 42 | ·604110 | 42 | ·934688 | 42 | 1·386893 |
| 48 | ·609712 | 48 | ·942444 | 48 | 1·397398 |

*Mutual inductance of circles (continued).*

| $\gamma$ | $M/(ab)^{\frac{1}{2}}$ | $\gamma$ | $M/(ab)^{\frac{1}{2}}$ | $\gamma$ | $M/(ab)^{\frac{1}{2}}$ |
|---|---|---|---|---|---|
| 44° 54′ | 1·407968 | 49° 54′ | 2·023803 | 54° 54′ | 2·846006 |
| 45 0 | 1·418598 | 50 0 | 2·038034 | 55 0 | 2·864981 |
| 6 | 1·429291 | 6 | 2·052351 | 6 | 2·884069 |
| 12 | 1·440051 | 12 | 2·066748 | 12 | 2·903266 |
| 18 | 1·450874 | 18 | 2·081230 | 18 | 2·922574 |
| 24 | 1·461762 | 24 | 2·095797 | 24 | 2·941994 |
| 30 | 1·472712 | 30 | 2·110445 | 30 | 2·961525 |
| 36 | 1·483725 | 36 | 2·125177 | 36 | 2·981170 |
| 42 | 1·494806 | 42 | 2·139995 | 42 | 3·000929 |
| 48 | 1·505954 | 48 | 2·154900 | 48 | 3·020802 |
| 54 | 1·517167 | 54 | 2·169889 | 54 | 3·040792 |
| 46 0 | 1·528443 | 51 0 | 2·184968 | 56 0 | 3·060898 |
| 6 | 1·539789 | 6 | 2·200130 | 6 | 3·081119 |
| 12 | 1·551202 | 12 | 2·215380 | 12 | 3·101456 |
| 18 | 1·562680 | 18 | 2·230719 | 18 | 3·121913 |
| 24 | 1·574225 | 24 | 2·246148 | 24 | 3·142488 |
| 30 | 1·585840 | 30 | 2·261665 | 30 | 3·163183 |
| 36 | 1·597524 | 36 | 2·277274 | 36 | 3·183998 |
| 42 | 1·609271 | 42 | 2·292970 | 42 | 3·204937 |
| 48 | 1·621077 | 48 | 2·308754 | 48 | 3·225992 |
| 54 | 1·632980 | 54 | 2·324631 | 54 | 3·247170 |
| 47 0 | 1·644941 | 52 0 | 2·340602 | 57 0 | 3·268474 |
| 6 | 1·656968 | 6 | 2·356664 | 6 | 3·289905 |
| 12 | 1·669069 | 12 | 2·372817 | 12 | 3·311460 |
| 18 | 1·681238 | 18 | 2·389063 | 18 | 3·333137 |
| 24 | 1·693479 | 24 | 2·405403 | 24 | 3·354941 |
| 30 | 1·705793 | 30 | 2·421839 | 30 | 3·376876 |
| 36 | 1·718177 | 36 | 2·438370 | 36 | 3·398936 |
| 42 | 1·730637 | 42 | 2·454995 | 42 | 3·421124 |
| 48 | 1·743166 | 48 | 2·471716 | 48 | 3·443444 |
| 54 | 1·755769 | 54 | 2·488532 | 54 | 3·465894 |
| 48 0 | 1·768440 | 53 0 | 2·505444 | 58 0 | 3·488473 |
| 6 | 1·781195 | 6 | 2·522459 | 6 | 3·511186 |
| 12 | 1·794021 | 12 | 2·539568 | 12 | 3·534033 |
| 18 | 1·806920 | 18 | 2·556777 | 18 | 3·557014 |
| 24 | 1·819895 | 24 | 2·574083 | 24 | 3·580129 |
| 30 | 1·832946 | 30 | 2·591490 | 30 | 3·603380 |
| 36 | 1·846071 | 36 | 2·608996 | 36 | 3·626768 |
| 42 | 1·859274 | 42 | 2·626608 | 42 | 3·650294 |
| 48 | 1·872554 | 48 | 2·644319 | 48 | 3·673957 |
| 54 | 1·885907 | 54 | 2·662131 | 54 | 3·697759 |
| 49 0 | 1·899338 | 54 0 | 2·680046 | 59 0 | 3·721703 |
| 6 | 1·912852 | 6 | 2·698064 | 6 | 3·745787 |
| 12 | 1·926444 | 12 | 2·716186 | 12 | 3·770017 |
| 18 | 1·940111 | 18 | 2·734412 | 18 | 3·794386 |
| 24 | 1·953859 | 24 | 2·752743 | 24 | 3·818899 |
| 30 | 1·967687 | 30 | 2·771182 | 30 | 3·843559 |
| 36 | 1·981595 | 36 | 2·789727 | 36 | 3·868363 |
| 42 | 1·995584 | 42 | 2·808378 | 42 | 3·893315 |
| 48 | 2·009653 | 48 | 2·827138 | 48 | 3·918417 |

*Mutual inductance of circles (continued).*

| $\gamma$ | $M/(ab)^{\frac{1}{2}}$ | $\gamma$ | $M/(ab)^{\frac{1}{2}}$ | $\gamma$ | $M/(ab)^{\frac{1}{2}}$ |
|---|---|---|---|---|---|
| 59° 54′ | 3·943666 | 64° 54′ | 5·421532 | 69° 54′ | 7·451115 |
| 60 0 | 3·969066 | 65 0 | 5·455983 | 70 0 | 7·499113 |
| 6 | 3·994618 | 6 | 5·490649 | 6 | 7·547449 |
| 12 | 4·020317 | 12 | 5·525536 | 12 | 7·596130 |
| 18 | 4·046174 | 18 | 5·560646 | 18 | 7·645153 |
| 24 | 4·072183 | 24 | 5·595977 | 24 | 7·694527 |
| 30 | 4·098348 | 30 | 5·631535 | 30 | 7·744254 |
| 36 | 4·124669 | 36 | 5·667322 | 36 | 7·794338 |
| 42 | 4·151148 | 42 | 5·703333 | 42 | 7·844780 |
| 48 | 4·177783 | 48 | 5·739578 | 48 | 7·895588 |
| 54 | 4·204583 | 54 | 5·776057 | 54 | 7·946760 |
| 61 0 | 4·231539 | 66 0 | 5·812771 | 71 0 | 7·998308 |
| 6 | 4·258658 | 6 | 5·849716 | 6 | 8·050230 |
| 12 | 4·285939 | 12 | 5·886906 | 12 | 8·102533 |
| 18 | 4·313386 | 18 | 5·924333 | 18 | 8·155211 |
| 24 | 4·340998 | 24 | 5·962006 | 24 | 8·208281 |
| 30 | 4·368777 | 30 | 5·999921 | 30 | 8·261743 |
| 36 | 4·396721 | 36 | 6·038082 | 36 | 8·315598 |
| 42 | 4·424836 | 42 | 6·076494 | 42 | 8·369854 |
| 48 | 4·453121 | 48 | 6·115157 | 48 | 8·424513 |
| 54 | 4·481575 | 54 | 6·154071 | 54 | 8·479581 |
| 62 0 | 4·510204 | 67 0 | 6·193241 | 72 0 | 8·535060 |
| 6 | 4·539005 | 6 | 6·232671 | 6 | 8·590958 |
| 12 | 4·567985 | 12 | 6·272356 | 12 | 8·647276 |
| 18 | 4·597138 | 18 | 6·312306 | 18 | 8·704018 |
| 24 | 4·626471 | 24 | 6·352521 | 24 | 8·761194 |
| 30 | 4·655979 | 30 | 6·393001 | 30 | 8·818806 |
| 36 | 4·685669 | 36 | 6·433752 | 36 | 8·876855 |
| 42 | 4·715544 | 42 | 6·474772 | 42 | 8·935351 |
| 48 | 4·745597 | 48 | 6·516067 | 48 | 8·994302 |
| 54 | 4·775835 | 54 | 6·557639 | 54 | 9·053704 |
| 63 0 | 4·806260 | 68 0 | 6·599488 | 73 0 | 9·113567 |
| 6 | 4·836872 | 6 | 6·641619 | 6 | 9·173900 |
| 12 | 4·867673 | 12 | 6·684032 | 12 | 9·234702 |
| 18 | 4·898661 | 18 | 6·726735 | 18 | 9·295978 |
| 24 | 4·929841 | 24 | 6·769723 | 24 | 9·357741 |
| 30 | 4·961217 | 30 | 6·813005 | 30 | 9·419994 |
| 36 | 4·992783 | 36 | 6·856578 | 36 | 9·482734 |
| 42 | 5·024548 | 42 | 6·900450 | 42 | 9·545978 |
| 48 | 5·056507 | 48 | 6·944623 | 48 | 9·609731 |
| 54 | 5·088669 | 54 | 6·989098 | 54 | 9·673991 |
| 64 0 | 5·121026 | 69 0 | 7·033873 | 74 0 | 9·738770 |
| 6 | 5·153588 | 6 | 7·078964 | 6 | 9·804079 |
| 12 | 5·186352 | 12 | 7·124362 | 12 | 9·869919 |
| 18 | 5·219322 | 18 | 7·170072 | 18 | 9·936286 |
| 24 | 5·252501 | 24 | 7·216104 | 24 | 10·00320 |
| 30 | 5·285883 | 30 | 7·262452 | 30 | 10·07068 |
| 36 | 5·319477 | 36 | 7·309127 | 36 | 10·13871 |
| 42 | 5·353281 | 42 | 7·356128 | 42 | 10·20731 |
| 48 | 5·387300 | 48 | 7·403454 | 48 | 10·27647 |

*Mutual inductance of circles (continued).*

| $\gamma$ | $M/(ab)^{\frac{1}{2}}$ | $\gamma$ | $M/(ab)^{\frac{1}{2}}$ | $\gamma$ | $M/(ab)^{\frac{1}{2}}$ |
|---|---|---|---|---|---|
| 74° 54′ | 10·34622 | 80° 0′ | 14·90704 | 85° 0′ | 23·15395 |
| 75 0 | 10·41655 | 6 | 15·02247 | 6 | 23·40052 |
| 6 | 10·48748 | 12 | 15·13925 | 12 | 23·65241 |
| 12 | 10·55902 | 18 | 15·25738 | 18 | 23·90984 |
| 18 | 10·63115 | 24 | 15·37691 | 24 | 24·17308 |
| 24 | 10·70392 | 30 | 15·49786 | 30 | 24·44234 |
| 30 | 10·77731 | 36 | 15·62026 | 36 | 24·71791 |
| 36 | 10·85134 | 42 | 15·74414 | 42 | 25·00006 |
| 42 | 10·92601 | 48 | 15·86953 | 48 | 25·28912 |
| 48 | 11·00133 | 54 | 15·99647 | 54 | 25·58540 |
| 54 | 11·07731 | 81 0 | 16·12498 | 86 0 | 25·88926 |
| 76 0 | 11·15397 | 6 | 16·25511 | 6 | 26·20108 |
| 6 | 11·23131 | 12 | 16·38688 | 12 | 26·52127 |
| 12 | 11·30933 | 18 | 16·52033 | 18 | 26·85026 |
| 18 | 11·38807 | 24 | 16·65552 | 24 | 27·18855 |
| 24 | 11·46750 | 30 | 16·79246 | 30 | 27·53665 |
| 30 | 11·54766 | 36 | 16·93120 | 36 | 27·89511 |
| 36 | 11·62855 | 42 | 17·07179 | 42 | 28·26458 |
| 42 | 11·71018 | 48 | 17·21427 | 48 | 28·64571 |
| 48 | 11·79256 | 54 | 17·35868 | 54 | 29·03922 |
| 54 | 11·87571 | 82 0 | 17·50507 | 87 0 | 29·44593 |
| 77 0 | 11·95963 | 6 | 17·65349 | 6 | 29·86672 |
| 6 | 12·04434 | 12 | 17·80399 | 12 | 30·30260 |
| 12 | 12·12985 | 18 | 17·95663 | 18 | 30·75463 |
| 18 | 12·21616 | 24 | 18·11146 | 24 | 31·22405 |
| 24 | 12·30330 | 30 | 18·26853 | 30 | 31·71218 |
| 30 | 12·39128 | 36 | 18·42790 | 36 | 32·22057 |
| 36 | 12·48011 | 42 | 18·58963 | 42 | 32·75093 |
| 42 | 12·56979 | 48 | 18·75380 | 48 | 33·30520 |
| 48 | 12·66035 | 54 | 18·92046 | 54 | 33·88560 |
| 54 | 12·75181 | 83 0 | 19·08970 | 88 0 | 34·49465 |
| 78 0 | 12·84417 | 6 | 19·26157 | 6 | 35·13533 |
| 6 | 12·93744 | 12 | 19·43614 | 12 | 35·81099 |
| 12 | 13·03165 | 18 | 19·61351 | 18 | 36·52565 |
| 18 | 13·12681 | 24 | 19·79376 | 24 | 37·28401 |
| 24 | 13·22294 | 30 | 19·97697 | 30 | 38·09171 |
| 30 | 13·32005 | 36 | 20·16323 | 36 | 38·95555 |
| 36 | 13·41816 | 42 | 20·35263 | 42 | 39·88383 |
| 42 | 13·51728 | 48 | 20·54529 | 48 | 40·88686 |
| 48 | 13·61744 | 54 | 20·74129 | 54 | 41·97762 |
| 54 | 13·71865 | 84 0 | 20·94075 | 89 0 | 43·17285 |
| 79 0 | 13·82093 | 6 | 21·14378 | 6 | 44·49455 |
| 6 | 13·92429 | 12 | 21·35051 | 12 | 45·97255 |
| 12 | 14·02877 | 18 | 21·56105 | 18 | 47·64863 |
| 18 | 14·13438 | 24 | 21·77555 | 24 | 49·58404 |
| 24 | 14·24113 | 30 | 21·99414 | 30 | 51·87366 |
| 30 | 14·34906 | 36 | 22·21696 | 36 | 54·67647 |
| 36 | 14·45818 | 42 | 22·44419 | 42 | 58·29054 |
| 42 | 14·56851 | 48 | 22·67598 | 48 | 63·38499 |
| 48 | 14·68008 | 54 | 22·91251 | 54 | 72·09483 |
| 54 | 14·79293 | | | | |

*Mutual inductance of circles (continued).*

| $\gamma$ | $M/(ab)^{\frac{1}{2}}$ | $\gamma$ | $M/(ab)^{\frac{1}{2}}$ | $\gamma$ | $M/(ab)^{\frac{1}{2}}$ |
|---|---|---|---|---|---|
| 89° 0′ | 43·1728 | 89° 19′ | 47·9513 | 89° 37′ | 55·2141 |
| 1 | 43·3840 | 20 | 48·2614 | 38 | 55·7738 |
| 2 | 43·5983 | 21 | 48·5792 | 39 | 56·3586 |
| 3 | 43·8167 | 22 | 48·9058 | 40 | 56·9718 |
| 4 | 44·0393 | 23 | 49·2402 | 41 | 57·6143 |
| 5 | 44·2649 | 24 | 49·5840 | 42 | 58·2905 |
| 6 | 44·4945 | 25 | 49·9379 | 43 | 59·0106 |
| 7 | 44·7291 | 26 | 50·3025 | 44 | 59·7778 |
| 8 | 44·9676 | 27 | 50·6769 | 45 | 60·5927 |
| 9 | 45·2127 | 28 | 51·0634 | 46 | 61·4582 |
| 10 | 45·4601 | 29 | 51·4624 | 47 | 62·3821 |
| 11 | 45·7141 | 30 | 51·8737 | 48 | 63·3850 |
| 12 | 45·9726 | 31 | 52·2998 | 49 | 64·4926 |
| 13 | 46·2370 | 32 | 52·7412 | 50 | 65·7189 |
| 14 | 46·5072 | 33 | 53·1986 | 51 | 67·0718 |
| 15 | 46·7832 | 34 | 53·6723 | 52 | 68·5662 |
| 16 | 47·0652 | 35 | 54·1639 | 53 | 70·2280 |
| 17 | 47·3533 | 36 | 54·6765 | 54 | 72·0948 |
| 18 | 47·6486 | | | | |

# CHAPTER VI

## THERMOELECTRICITY

**99. The Seebeck effect.** It is a matter of everyday observation in a laboratory that no current flows in a circuit that does not contain an electric battery, provided that the temperature is the same throughout. That this is not true when the temperature is different in different parts was first shown by Seebeck in 1821. He found that a current flowed round a circuit composed of two different metals, if one of the junctions was hotter than the other. The direction of the current depends on the pair of metals as well as on the temperatures of the junctions: in a copper-iron circuit in which one junction is at the freezing-point and the other at the boiling-point of water, the current flows from copper to iron across the hot junction and from iron to copper across the cold junction.

The electromotive force in the circuit, which is of the order of $10^{-3}$ volt for 100° difference of temperature, is most conveniently measured by a potentiometer method, as shown in Fig. 131. The hot junction is placed in a hole bored in a massive iron cylinder $C$, and insulated from it by wrapping the wires round with asbestos. The cylinder is heated in a sand-bath over a Bunsen burner. Its temperature rises gradually, and is measured with a thermometer inserted in a second hole similar to the first. In this way observations can be made up to a temperature of about 300°. The cold junction is immersed in a mixture of ice and water at 0° C., protected from the radiation of the cylinder by a sheet of asbestos.

Let $v$ be the thermal E.M.F. and $V$ the E.M.F. of the auxiliary battery. No current will pass through the galvanometer when both keys are down, provided that $v = XV/(X + R)$. Since $X$ is small in comparison with $R$ ($X$ being most conveniently $\frac{1}{10}$th ohm and $R$ of the order of 100 ohms) we have approximately $v = XV/R$. It is, however, difficult to maintain a balance, as the temperature will not remain constant over the time required for an observation. It is best, after obtaining a rough balance, to diminish $R$ slightly, thus throwing the galvanometer spot off the zero. The spot creeps back to zero as the cylinder heats up, and the temperature is read

at the moment when the deflexion is zero. The process is then repeated with a smaller value of $R$, and so on.

Two metals are necessary to produce thermoelectric currents. Thus if the terminals of a sensitive galvanometer are joined by a

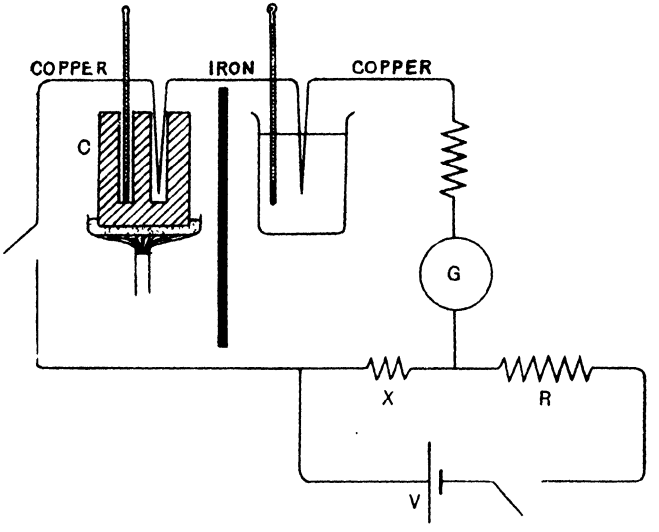



Fig. 131

long bare copper wire, no deflexion occurs when a Bunsen burner plays on the wire at any point. Similarly the total E.M.F. in a thermoelectric circuit consisting of two given metals depends only on the temperatures of the junctions, and not on the temperatures of all the parts of the circuit. This principle allows us to keep the left-hand beaker in Fig. 131 at 100° C. and the right-hand beaker at 0° C. and thereby measure the E.M.F. of the circuit for these temperatures of the junctions, without keeping the galvanometer and the resistance $X$ also at 0°.

**100. Laws of the thermoelectric circuit.** Let $A$, $B$ be two metals, the temperatures of the hot and cold junctions being $\theta_1$ and $\theta_2$ respectively. The thermal voltage $V$ will be reckoned as positive when the current flows from $B$ to $A$ across the hot junction and from $A$ to $B$ across the cold junction. We shall use the symbol $(A, B)$ for $V$ when it is desired to specify the metals used, and $(\theta_1, \theta_2)$ when we consider more particularly the temperatures of the junctions. The laws of the thermoelectric circuit are:

(1) For a given pair of metals, the E.M.F. for temperatures $\theta_1$ and $\theta_3$ is the sum of the E.M.F.'s for $\theta_1$ and $\theta_2$ and for $\theta_2$ and $\theta_3$, or symbolically, $(\theta_1, \theta_3) = (\theta_1, \theta_2) + (\theta_2, \theta_3)$.

(2) For given temperatures of the junctions, the E.M.F. of two metals $A$, $C$ is the sum of the E.M.F.'s of $A$, $B$ and of $B$, $C$, $B$ being any third metal. Symbolically,

$$(A, C) = (A, B) + (B, C).$$

The first law can be verified by direct measurement, for example with a single copper-iron circuit and three beakers maintained at 0°, 50°, and 100° C. Its utility is obvious: instead of finding the E.M.F. of a given pair of metals for all temperatures of both junctions, we have only to make observations for various temperatures of the hot junction, keeping the cold junction at any convenient temperature such as 0° C., and then the thermal voltages in other cases come out as differences. The second law can also be verified directly, or more neatly by a null method. It is found that no current flows in the circuit of Fig. 132 if the metals are joined in the order $ABCABC$, and the junctions made alternately hot and cold. The circuit is clearly equivalent to three thermoelectric circuits with metals $(B, C)$, $(C, A)$ and $(A, B)$ connected in series, so that we have

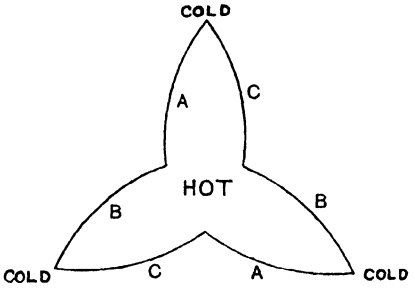


Fig. 132

$$(B, C) + (C, A) + (A, B) = 0,$$

which is law (2) above, since $(C, A) = -(A, C)$.

Combining the two laws we have the following rule for finding the thermal E.M.F. of any circuit for any temperatures of the junctions. Select a convenient temperature (e.g. 0° C.) for the cold junction and a standard metal (e.g. lead), and measure the thermal E.M.F. for various metals at different temperatures of the hot junction. If $A_\theta$ is this E.M.F. for a metal $A$ against lead, with temperature $\theta$ of the hot junction, the E.M.F. of a thermoelectric circuit of metals $A$, $B$ and temperatures $\theta_1$, $\theta_2$ is given by

$$V = A_{\theta_1} - A_{\theta_2} - (B_{\theta_1} - B_{\theta_2}) \ldots\ldots\ldots\ldots (110).$$

**101. Dependence of thermal E.M.F. on temperature.** Fig. 133 shows the thermal voltages of various metals against lead, one junction being maintained at 0° and the other at various temperatures. The figure is constructed from experiments by Noll,

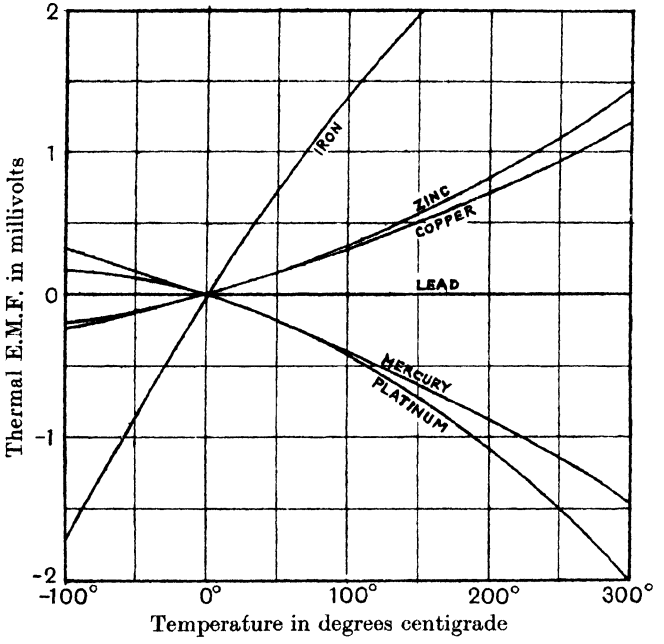



Fig. 133

Dewar and Fleming, in which great care was taken to obtain pure materials: but considerable discrepancies sometimes arise which are partly accounted for by differences in the treatment to which the metals have been subjected. The range of accurate observation is not sufficient to decide the form of the curves in the figure: they may be represented approximately by parabolas whose axes are vertical. This is equivalent to expressing the thermal E.M.F. $V$ for a temperature of $t°$ centigrade in the form $V = at + bt^2$: but the formula is not rigidly true.

The laws of the thermoelectric circuit given in the last article show that if the temperatures $(0°, t°)$ of the junctions are prescribed we can arrange the metals in a *thermoelectric series* such that each has a positive E.M.F. when used with the next one in the series. For small values of $t$ the series is as follows:

Antimony, iron, zinc, copper, silver, lead, mercury, platinum-rhodium, platinum, nickel, bismuth.

It was found by Cumming in 1823 that the order of the metals is not invariable, but can be changed by the use of higher temperatures for the hot junction. This phenomenon is known as *thermoelectric inversion*. For example, iron is positive with respect to copper for temperatures less than 550° C., but negative above this temperature.

The choice of metals in a thermal junction depends on the use to which it is put. An antimony-bismuth couple is used for great sensitiveness: it gives a thermal voltage of about $1{\cdot}2 \times 10^{-3}$ per degree, which is about twelve times that of an iron-copper circuit. Iron and constantan is another sensitive combination. Thermal junctions used at low temperatures have to be calibrated against a hydrogen or helium thermometer. At high temperatures, on the other hand, the chief desideratum is high melting-point. Thus a circuit of platinum and 10 per cent. platinum-rhodium alloy can be used up to 1500° C., where it gives a potential of about 16 millivolts.

The thermoelectric behaviour of iron is irregular. Belloc found that $dV/d\theta$ for an iron-platinum circuit fell until the temperature rose to 380° C., then rose to a maximum value for a temperature of 830°, and fell to a second minimum value at 950°, after which it rose again. The three bends in the curve were very sharp, the two highest being near the transition temperatures of iron (Art. 116).

**102. The Peltier effect.** The question as to the source of the energy of thermoelectric currents was partly settled by the discovery of Peltier (1834) that heat is in general absorbed or liberated when a current crosses the junction of two metals. This can be shown with the apparatus of Fig. 134. The two junctions $A$, $B$ of a composite circuit are enclosed in two glass bulbs joined by a tube containing a thread of mercury, the whole forming a differential air-thermometer. The Joule heat of the current in the wires is the same in both bulbs and has no effect on the mercury thread, but nevertheless the thread of mercury moves in one direction on turning on the current, and in the other direction when the current is reversed.

In a copper-iron circuit, heat is developed when the current flows from iron to copper and absorbed when the current flows from copper to iron. Thus in a thermoelectric circuit, in which the current flows from copper to iron across the hot junction, heat is abstracted from the hot junction by the passage of the current and given up to the cold junction, so that the thermoelectric circuit develops electrical power at the expense of the heat supplied, and acts like a heat engine. This point of view will be developed in Art. 104.

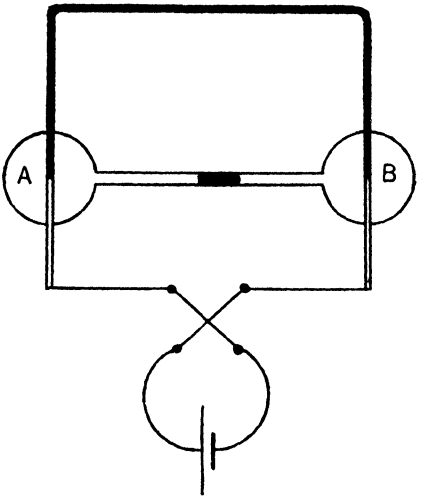


Fig. 134

If heat is absorbed by a current flowing across a junction, an equal amount of heat is liberated in the same time by an equal current in the opposite direction. More generally, the heat absorbed per second by a current $i$ flowing from a metal $B$ to a metal $A$ is found to be proportional to the current $i$, and equal to $\Pi i$, where $\Pi$ is called the Peltier coefficient of $A$ with respect to $B$. $\Pi$ depends on the temperature. The case of liberation of heat is included in the formula if we agree that negative absorption means liberation: this convention will be adhered to throughout.

The proportionality of heating effect and current can be shown roughly by means of the air-thermometer described, using only small movements of the thread of mercury. Exact measurement is difficult because of the Joule heat developed by the mere passage of the current through the wires. The amounts of heat to be measured are also very small: for example, that developed by 1 ampere in a copper-iron junction in 18 months would just suffice to boil an ordinary-sized kettle.

Careful experiments were made by Jahn in 1888, in which the Peltier heats of various metals against copper were determined in absolute measure. His plan was to enclose the junction in a Bunsen ice calorimeter. The Joule heat was considerably greater than the

Peltier heats, and had to be allowed for by reversing the current. The proportionality of Peltier effect and current was verified accurately.

**103. The Kelvin effect.** In 1851 Lord Kelvin showed that the Peltier absorption of heat was not the only effect of its kind, but that there must be thermal effects depending on the passage of a current along a single unequally heated conductor. He found that heat was liberated when an electric current passed from hot to cold in copper, and absorbed when it passed from cold to hot. The effects in iron were exactly opposite.

The Kelvin effect in copper can be shown by the following method due to Callendar, depending on the variation of resistance of the metal with temperature (Fig. 135). $ACB$ is a very thin wire fastened into thicker wire at the end, the middle point being soldered to another wire attached to a sensitive low-resistance galvanometer. Copper wires $AD$, $BD$ complete a low-resistance Wheatstone's bridge made entirely of copper, and $A$, $B$ are connected to a battery through a resistance and a reversing key. Let a current be sent through the bridge so as to pass from $A$ to $B$ in both branches, and suppose that the bridge is accurately balanced with an adjustable shunt $S$ of low resistance placed across $BD$ or $AD$. On reversing the current the balance is upset in a direction corresponding to an increase in the resistance of $AC$. The wires $AD$, $BD$ are not appreciably heated by the current, but the thin wire $ACB$ does become hot. Thus there is an upward temperature-gradient from $A$ to $C$ and a downward gradient from $C$ to $B$ along the branch. Assuming for the moment that heat is liberated in passing from hot to cold in copper, there is originally absorption of heat in $AC$ and liberation in $CB$, in addition to the heat caused by resistance. On reversing the current heat begins to be liberated in

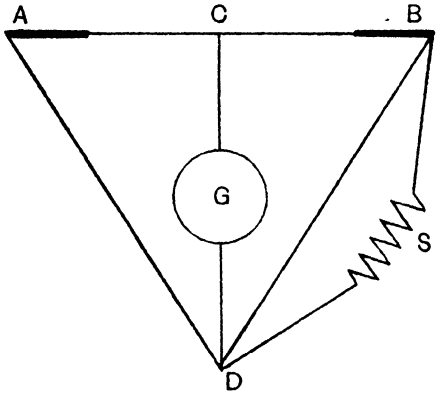


Fig. 135

$AC$ and absorbed in $CB$, so that the temperature of $AC$ on the whole begins to rise and that of $CB$ begins to fall. This causes an increase in the resistance of $AC$ and a decrease in that of $CB$, which destroys the balance of the bridge in the direction actually observed. Thus the assumption as to the sign of the effect in copper is verified. The bridge should be lagged with cotton-wool in order to prevent local changes of temperature, which disturb the galvanometer.

The laws of the Kelvin effect may be expressed in terms of a coefficient $\sigma$, sometimes called the specific heat of electricity, but which we shall call the *coefficient of the Kelvin effect*. It is defined as the heat absorbed per second when a current of 1 ampere flows in the given metal from one point of the metal to another whose temperature is 1° higher, over and above the heat developed according to Joule's law. The Kelvin effect has been found to be proportional to the current: thus when a current of $i$ amperes flows for time $t$ from one point to another differing from it by the small temperature $d\theta$ the heat absorbed is $\sigma i t d\theta$.

The determination of $\sigma$ in absolute measure is very difficult on account of the smallness of the heat to be measured, and the disturbing effects of Joule heat, thermal conductivity and emissivity in the metal. King used the method of Fig. 135 to measure $\sigma$ for copper, and Berg compensated the heat of the Kelvin effect by the Joule heat due to the resistance of the metal.

**104. Theory of the thermoelectric circuit.** We shall now develop the theory of the thermoelectric circuit considered as an engine for converting heat into electrical work. The work done in carrying the electric charges round the thermoelectric circuit is accounted for by the Peltier absorption and liberation of heat at the hot and cold junctions and the Kelvin effect in the two metals. Both these thermal effects are *reversible**, that is, they are changed in sign and unaltered in magnitude when the current flows in the opposite direction. The irreversible heat developed on account of resistance depends on the square of the current, but for small currents it is negligible in comparison with the Peltier and Kelvin effects, both of which are proportional to the first power of the current. We can now apply the laws of thermodynamics to the

* For the conditions of reversibility of a heat engine, see Buckingham's *Thermodynamics*, pp. 94–5, 113–4.

thermoelectric circuit considered as a reversible heat engine, as was first done by Lord Kelvin.

Let a small current of $i$ amperes pass round the circuit for a time $t$, the hot junction being at absolute temperature $\theta$ and the cold junction at temperature $\theta_0$. The Kelvin coefficients for metals $A$ and $B$ are taken as $\sigma$ and $\sigma'$ respectively at temperature $\theta$. In time $t$ the work done in conveying charge round the circuit is $Vit$ joules, or $10^7\ Vit$ ergs. If $J$ is the mechanical equivalent of heat this requires that $\frac{10^7 Vit}{J}$ calories should be abstracted in the form of Peltier and Kelvin heat. Heat of amount $\Pi it$ calories is absorbed at the hot junction and $\Pi_0 it$ rejected to the vessel containing the cold junction: further, during the passage of the current from hot to cold in metal $A$ the heat absorbed is $-it\int_{\theta_0}^{\theta}\sigma d\theta$, while heat of amount $it\int_{\theta_0}^{\theta}\sigma' d\theta$ is absorbed when the current flows from cold to hot in metal $B$. Hence applying the first law of thermodynamics and dividing out by $it$ we have

$$\frac{10^7 V}{J} = \Pi - \Pi_0 - \int_{\theta_0}^{\theta}(\sigma - \sigma')\, d\theta.$$

The second law of thermodynamics states that if $dQ$ is a small element of heat taken up at any part of the cycle at temperature $\theta$, then $\int dQ/\theta = 0$, where the integral means summation over all the elements of heat taken up from outside in the whole operation, heat liberated being regarded as negative. Thus

$$\frac{\Pi}{\theta} - \frac{\Pi_0}{\theta_0} = \int_{\theta_0}^{\theta}\frac{\sigma - \sigma'}{\theta} d\theta.$$

Differentiating the last two equations with respect to $\theta$, we have

$$\frac{10^7}{J}\frac{dV}{d\theta} = \frac{d\Pi}{d\theta} - (\sigma - \sigma') \quad \ldots\ldots\ldots(111),$$

and

$$\frac{d}{d\theta}\left(\frac{\Pi}{\theta}\right) = \frac{\sigma - \sigma'}{\theta} \quad \ldots\ldots\ldots\ldots(112).$$

Eliminating $\sigma - \sigma'$ we have

$$\frac{10^7}{J}\frac{dV}{d\theta} = \frac{d\Pi}{d\theta} - \theta\frac{d}{d\theta}\left(\frac{\Pi}{\theta}\right),$$

or

$$\Pi = \frac{10^7}{J}\theta\frac{dV}{d\theta} \quad \ldots\ldots\ldots\ldots(113).$$

This value when substituted in (112) gives further

$$\sigma - \sigma' = \frac{10^7}{J}\,\theta\,\frac{d^2V}{d\theta^2} \quad \ldots\ldots\ldots\ldots\ldots (114).$$

$V$ is supposed to be measured in volts, and the factor $10^7/J$ has the value $1/4{\cdot}18 = 0{\cdot}239$.

**105. Verifications of the theory.** The equations (113) and (114) give a means of calculating the Peltier and Kelvin coefficients from observations on thermal voltage at various temperatures. The coefficients should also be measured for the same specimens of metal, as slight impurities or differences in physical state have a great effect on thermoelectric properties. The following results are from Jahn's experiments already mentioned:

Temperature 0° C. = 273 abs.

| Metals | $dV/d\theta$ in volts per degree | $\frac{10^7}{J}\,\theta\,\frac{dV}{d\theta}$ | Peltier coefficient calories per second per ampere |
|---|---|---|---|
| Silver-copper | $+ 2{\cdot}12 \times 10^{-6}$ | $+ 1{\cdot}38 \times 10^{-4}$ | $+ 1{\cdot}15 \times 10^{-4}$ |
| Iron-copper | $+ 11{\cdot}28 \times 10^{-6}$ | $+ 7{\cdot}35 \times 10^{-4}$ | $+ 8{\cdot}79 \times 10^{-4}$ |
| Platinum-copper | $- 1{\cdot}40 \times 10^{-6}$ | $- 0{\cdot}912 \times 10^{-4}$ | $- 0{\cdot}889 \times 10^{-4}$ |
| Zinc-copper | $+ 1{\cdot}51 \times 10^{-6}$ | $+ 0{\cdot}997 \times 10^{-4}$ | $+ 1{\cdot}62 \times 10^{-4}$ |
| Cadmium-copper | $+ 2{\cdot}64 \times 10^{-6}$ | $+ 1{\cdot}72 \times 10^{-4}$ | $+ 1{\cdot}71 \times 10^{-4}$ |
| Nickel-copper | $- 20{\cdot}03 \times 10^{-6}$ | $- 13{\cdot}1 \times 10^{-4}$ | $- 12{\cdot}1 \times 10^{-4}$ |

The agreement of the numbers in the last column confirms equation (113), except in the case of copper and zinc. The necessity for working with the same specimens, when special precautions are not taken to secure purity, is seen from the fact that Jahn's values for $dV/d\theta$ differ from those of more recent experiments by much more than the differences of the last two columns in the above table.

Equation (114) has not been verified so accurately, for several reasons. The Kelvin effect is exceedingly difficult to measure, determinations of it and the thermal voltage have not been made on the specimens, and the values of $d^2V/d\theta^2$ are themselves uncertain. In the last column of the following table values of $d^2V/d\theta^2$ at 100° C. have been used which are the mean of those of several observers. The values of $\sigma$ are taken chiefly from Cermak and Berg.

Kelvin coefficients at 100° C. = 373 abs.

| | $\sigma$ in calories per coulomb per degree | | $\sigma - \sigma'$ observed | $\sigma - \sigma'$ calculated from (114) |
|---|---|---|---|---|
| Copper | $+ 4{\cdot}7 \times 10^{-7}$ | Copper-lead | $+ 3{\cdot}7 \times 10^{-7}$ | $+ 7{\cdot}1 \times 10^{-7}$ |
| Iron | $-29 \times 10^{-7}$ | Iron-lead | $-30 \times 10^{-7}$ | $-27 \times 10^{-7}$ |
| Mercury | $- 5{\cdot}5 \times 10^{-7}$ | Mercury-lead | $- 6{\cdot}5 \times 10^{-7}$ | $- 7{\cdot}7 \times 10^{-7}$ |
| Platinum | $-22 \times 10^{-7}$ | Platinum-lead | $-23 \times 10^{-7}$ | $-22 \times 10^{-7}$ |
| Zinc | $+ 8 \times 10^{-7}$ | Zinc-lead | $+ 7 \times 10^{-7}$ | $+13 \times 10^{-7}$ |
| Cadmium | $+24 \times 10^{-7}$ | Cadmium-lead | $+23 \times 10^{-7}$ | $+28 \times 10^{-7}$ |
| Lead | $+ 1 \times 10^{-7}$ | | | |

Considering these difficulties we must regard the agreement of the numbers in the last column as sufficient.

**106. Thermopiles.** We can use the thermoelectric circuit as a source of potential, arranging thermocouples in series as shown in Fig. 136. Such an arrangement is called a thermopile, and considerable voltages can be developed: thus if the hot junctions of a constantan-manganin pile of 100 couples are maintained at 550° C. and the cold junctions at 0° the E.M.F. is about 5 volts.

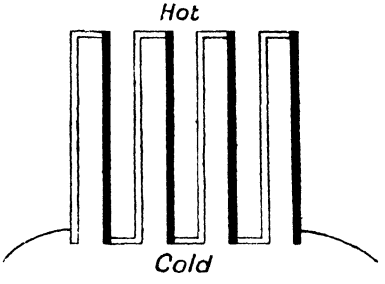


Fig. 136

The thermopile has a low efficiency as a heat engine; that is, the electrical work obtainable from it is but a small fraction of the heat supplied. To illustrate this, consider the constantan-manganin pile just mentioned, supposing that the wire used throughout is of 2 mm. diameter, the length of each segment of wire being 20 cm. We shall neglect the radiation of heat by the hot metals to begin with, and compare the electrical work that can be extracted from the pile with the heat required to maintain the temperatures constant against conduction from the hot to the cold junctions.

The temperature-gradient $-d\theta/dx$ in each wire is of the order 30 degrees per cm., and the cross-section of each wire is

$$\pi/100 \text{ cm.}^2 = 1/30 \text{ (say)}.$$

Hence the flow of heat down each wire is approximately $K$ calories per second, where $K$ is the thermal conductivity of the metal. The

value of $K$ for both constantan and manganin being about 1/20, the total flow of heat from the hot to the cold junction by way of the 200 connecting strips of wire is about 10 calories per second.

Since the specific resistance of either metal is about $4{\cdot}5 \times 10^{-5}$ the total internal resistance of the battery is $4000 \times 4{\cdot}5 \times 10^{-5} \times (100/\pi)$ ohms, i.e. about 5 ohms. If the resistance of the external circuit is $R$ ohms the current is $i = 5/(R + 5)$ amperes, and the energy utilised outside the battery per second is

$$Ri^2 \text{ joules} = \frac{Ri^2}{4{\cdot}18} \text{ calories} = \frac{6R}{(R+5)^2} \text{ calories, approximately.}$$

This has the maximum value 3/10 when $R = 5$. Hence the efficiency of this thermopile, considered as a heat engine, can in no case exceed 3 per cent. The heat radiated from the wires may amount to several hundred calories per second. It follows that the thermopile must always be a very inefficient source of electrical power.

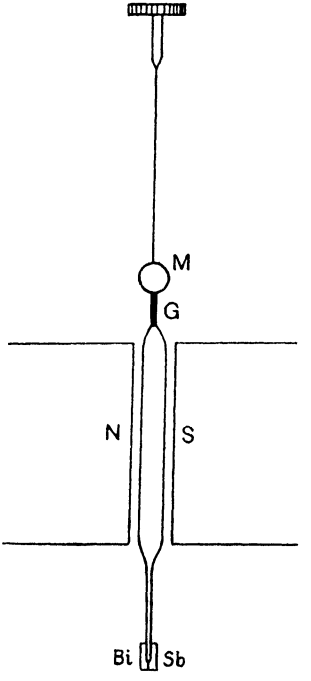


Fig. 137

The thermopile is, however, of great use for measuring either high temperatures or small differences of temperature, and is in many ways more convenient than a resistance thermometer. If the detecting instrument is a galvanometer of comparatively high resistance there is every advantage in multiplying the number of junctions up to a certain point, i.e. until the resistance of the thermopile is comparable with that of the galvanometer. The greatest sensitiveness can nevertheless be obtained with a *single* junction, as in Boys' *radio-micrometer*, the principle of which is shown in Fig. 137. The antimony-bismuth thermocouple is connected to a small loop of wire lying between the poles of a magnet like the coil of a sensitive moving-coil galvanometer. The loop is attached to a fine glass rod $G$ carrying a mirror $M$, and the whole is suspended from a fine quartz fibre. Radiation falling on the couple, or on a blackened copper disc attached to it, sends a

current round the loop, which is thereby turned through an angle. The low resistance of the whole loop more than compensates the disadvantage of having only one junction.

**107. Thermal galvanometers.** We shall see in Arts. 131 and 133 that alternating currents give no reading on an ordinary moving-coil galvanometer. The heat produced may however be used in all cases to actuate a thermoelectric couple. Fig. 138 shows a device invented by Klemenčič, and subsequently improved. A fine wire $AB$, carrying the current to be measured, is soldered at the centre to two wires $ac$, $bc$ of different metals, forming a thermoelectric couple which deflects a galvanometer attached to the terminals $ab$. The sensitiveness depends on the temperature to which the junction will rise; hence it can be increased considerably by placing the whole instrument in a vacuum (as was first done by Lebedew), whereby the radiation of heat from the wires is diminished.

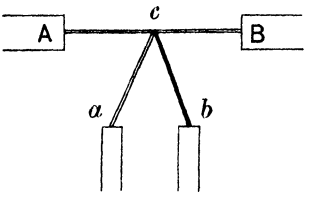


Fig. 138

In Duddell's *thermo-galvanometer* (Fig. 139) the current passes through a fine filament placed under the junction of a radio-micrometer. The filament, or "heater," is usually a platinised quartz fibre of from 4 to 100 ohms resistance: its distance from the junction can be adjusted by a screw shown at $F$ in the figure. The steadiness of the zero depends on maintaining reasonably constant temperature near the thermal junction, and this is secured by closing the chamber containing the heater and couple with a heavy block $E$ of copper, the whole instrument being contained in a second copper case provided with a glass window. Readings are made with lamp and scale, the small square mirror being shown at $H$, while the instrument is made portable by means of the clamp $B$ which grips the moveable loop.

A less sensitive but very convenient instrument on the same principle is Duddell's thermo-ammeter, in which the thermocouple is attached to a moving coil mounted between bearings after the manner of a millivoltmeter. The instrument is graduated so as to read directly in milliamperes.

An obvious difficulty in the way of attaining great sensitiveness with thermal instruments is that their deflexion is proportional to the square of the current, so that if the current is halved the

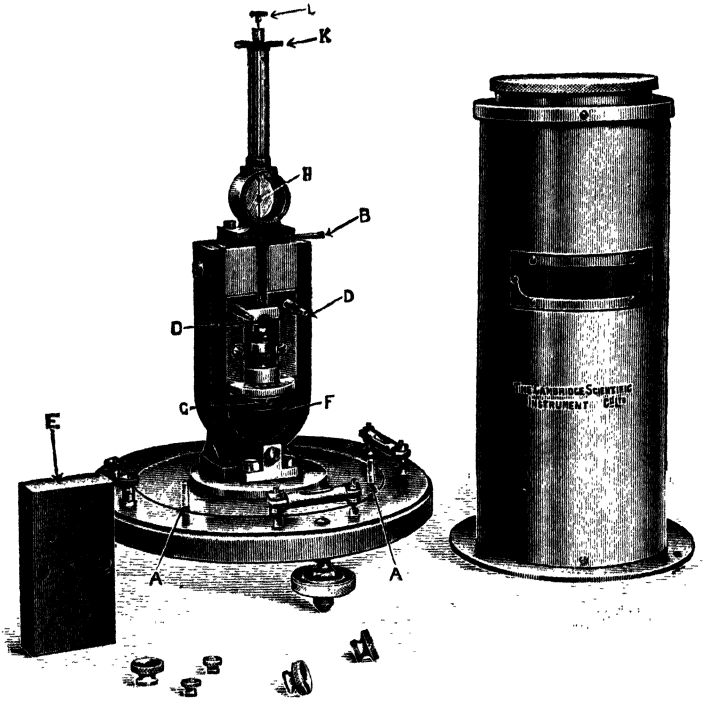



Fig. 139

deflexion is reduced to one-quarter. Duddell's thermo-galvanometer will detect currents of $10^{-6}$ ampere with a resistance of the order of 100 ohms, but thermal galvanometers are generally considered sensitive if a current of 1 milliampere yields an appreciable deflexion. It is difficult to give exact data since the sensitiveness depends on the direct-current galvanometer used with it.

# CHAPTER VII

## INDUCED MAGNETISM

**108. Variation of magnetisation with impressed magnetic force.** We have hitherto regarded all magnets as practically permanent, neglecting the change in their magnetism which occurs whenever they are placed in magnetic fields. We have now to discuss the magnetic properties of iron and other substances more

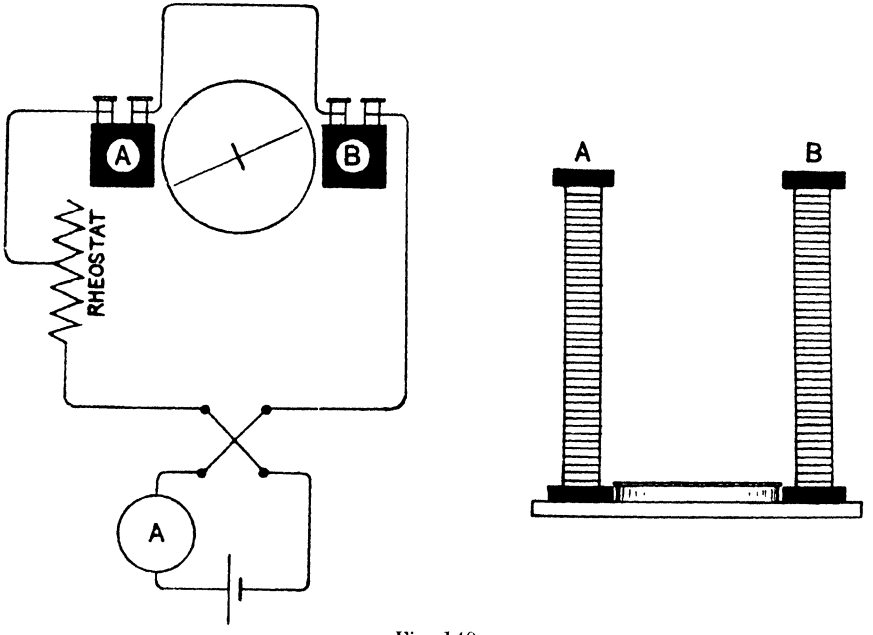



Fig. 140

generally, and find out what circumstances determine the strength of magnets. The first step is to find how the intensity of magnetisation $I$ of a body varies when the field $H$ in which it lies is changed. The experiment could hardly be carried out without solenoids, by which we can produce uniform fields of known strength, variable at will. The following is a simple experiment on a thin wire of magnetic material (Fig. 140).

$A$ and $B$ are two similar vertical solenoids placed magnetically east and west of a compass-box, $A$ being fixed and $B$ moveable in a groove pointing towards the compass-needle. The solenoid

$A$ is the magnetising solenoid, while $B$ serves to compensate the magnetic effect of the solenoid itself, leaving the magnetised wire only to affect the compass-needle. To effect compensation a strong current is first of all passed through both solenoids in the same direction, and $B$ moved in its groove until there is no deflexion of the needle.

The wire to be tested is now passed down the centre of the fixed solenoid $A$, so that its lower end is on a level with the needle. The rheostat in circuit should have considerable resistance in addition to large carrying-capacity (say 200 ohms, capable of carrying 2 amperes continuously), so that we can alter the current over a large range. The position of the reversing key shows at once what the sign of the current is at any time; the positive direction of current will be taken to be that which produces a magnetising force vertically downwards. If $H$ is the magnitude of this force, $n$ the number of turns of the solenoid and $l$ its length,

$$H = \frac{4\pi ni}{10l} \quad \ldots\ldots\ldots\ldots\ldots\ldots (115),$$

$i$ being measured in amperes.

Suppose that $A$ is to the west of the compass-needle, and that the deflexion of the needle is $\theta$, measured from north towards east. If $I$ is the corresponding intensity of magnetisation and $a$ the radius of the wire, this deflexion is due to the poles $\pm \pi a^2 I$ at the ends of the wire. Thus if $r$ is the distance of the lower pole from the needle and $h$ the earth's horizontal component,

$$\pi a^2 I \left\{ \frac{1}{r^2} - \frac{r}{(r^2 + l^2)^{\frac{3}{2}}} \right\} = h \tan \theta.$$

Hence $I$ is known if $h$ is known. We can avoid measuring $h$ by making a single subsidiary observation. When the experiment is finished take away the wire and the moveable solenoid, and observe the deflexion $\alpha$ caused by a known current $i_0$ in $A$ alone. Let $b$ be the mean radius of the cross-section of the *solenoid*; then from equation (87) the magnetic effect of $A$ is that of a bar-magnet whose total moment is $ni_0\pi b^2/10$, and whose poles are therefore $\pm ni_0\pi b^2/10l$, provided that $r$ is large compared with $b$. Thus

$$\frac{ni_0\pi b^2}{10l} \left\{ \frac{1}{r^2} - \frac{r}{(r^2 + l^2)^{\frac{3}{2}}} \right\} = h \tan \alpha.$$

Dividing the last two equations we have

$$I = \frac{nb^2 i_0}{10la^2 \tan\alpha} \tan\theta \quad \ldots\ldots\ldots\ldots (116).$$

Equations (115) and (116) give $H$ and $I$ from observations of $i$ and $\theta$. The method is not very accurate on account of the finite length of the experimental solenoids, but is sufficiently so for most purposes.

In making the observations we first cut off the current and pass the wire, previously demagnetised, down the centre of $A$. A weak positive current $i$ is then passed through the solenoids. As the resistance is slowly cut out the intensity of magnetisation will rise as shown by the line $OAP$ in Fig. 141. By the time the point $P$ is reached $I$ has become nearly constant. The iron is then said to be *saturated*, and $P$ may be called a saturation point.

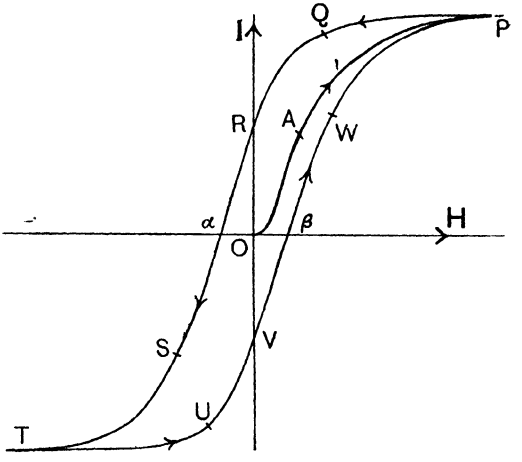
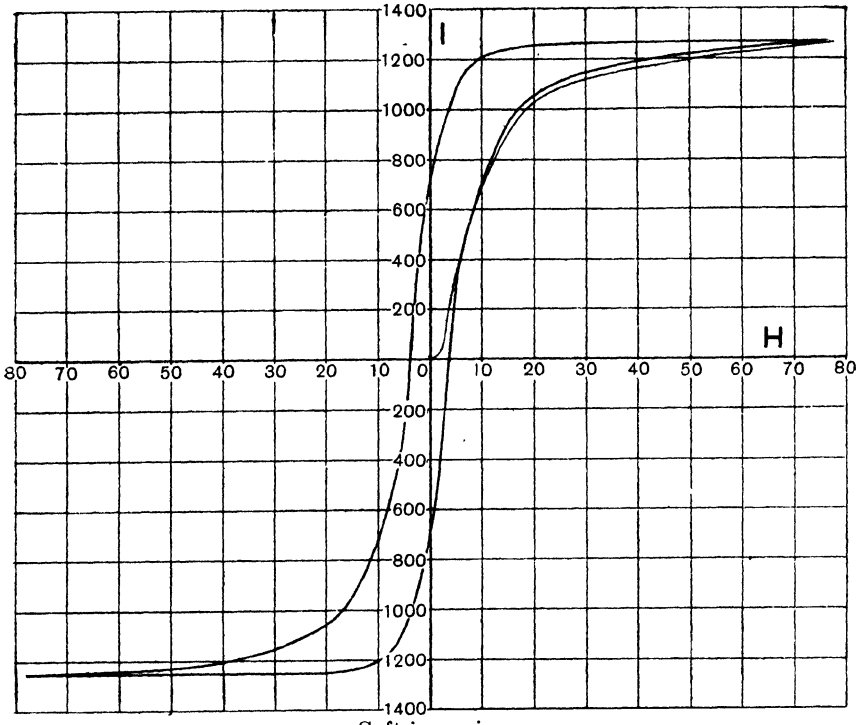


Fig. 141

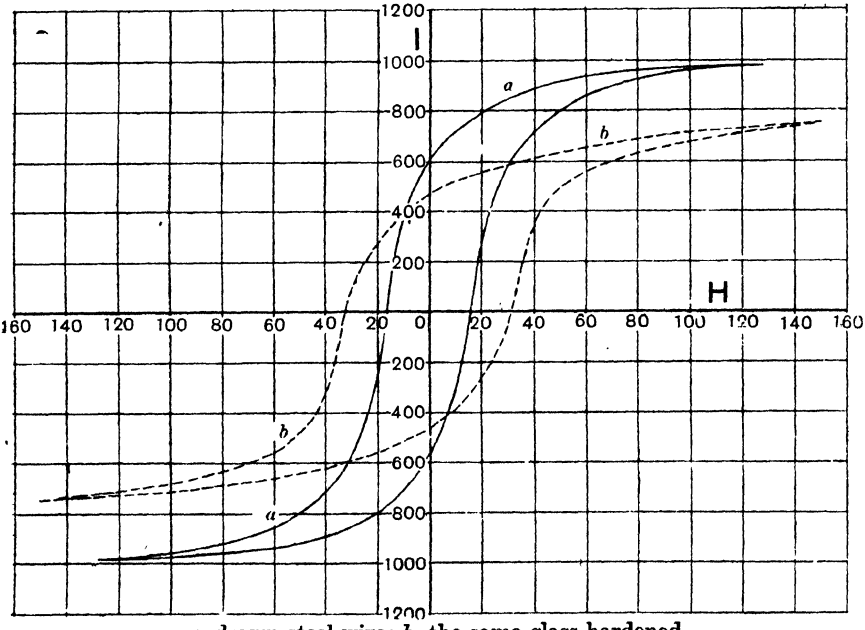
It is convenient to arrange that the point $P$ is reached when the rheostat is short-circuited. If we begin to put resistance in the circuit again by means of the rheostat, the magnetisation does not resume its original value for any given current, but is consistently greater. At the point $R$ the current is zero and may next be reversed. If, after reversal, resistance is cut out of the circuit continuously, the magnetisation falls constantly as shown in the line $R\alpha ST$, changing sign at $\alpha$ and approaching negative saturation at $T$. This latter point is the exact counterpart of $P$, and the remaining part $TUV\beta WP$ of the curve is treated in the same way as the first half.

Fig. 142 shows a curve obtained in this way with a soft iron wire. In Fig. 143 the continuous curve $a$ is drawn for a steel rod (bicycle spoke), the curve $b$ showing the considerable effect caused when




Soft iron wire

Fig. 142




*a*, drawn steel wire; *b*, the same glass-hardened

Fig. 143

the spoke is glass-hardened by plunging it suddenly into cold water after heating it to a bright red heat.

**109. Hysteresis.** The preceding experiments show that the magnetisation of iron is not in general proportional to the magnetising force. Indeed, we cannot say what $I$ is at all in a given field $H$ without further information, since the behaviour of every specimen depends on the previous magnetic forces to which it has been subjected as well as on that acting at the moment. The curves of the last article show the characteristic tendency of magnetisation to lag behind the impressed magnetic force: thus along $PQR$ in

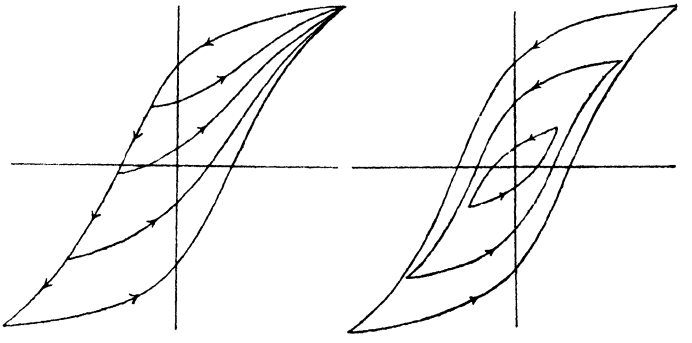


Fig. 144

Fig. 141 the magnetisation tends to retain the large amount it has acquired at $P$, while the lowness of the ascending branch $\beta WP$ shows that the iron has some difficulty in ridding itself of the effects of its previous negative saturation. The phenomenon, first described by Warburg in 1881, was named *hysteresis* by Ewing, who examined it in detail.

A great variety of hysteresis curves can be drawn by varying the manner in which the magnetic force is applied, some of these being shown roughly in the figure. The first diagram of Fig. 144 shows the effect of beginning to increase the magnetising current again before arriving at negative saturation, the second some cyclic curves obtained by varying the magnetic force continuously between fixed limits, the limits being too small to give the normal hysteresis loop. The abnormal curves lie in general within the original saturation loop, though certain curves may just leave it, for example the initial curve in Fig. 142.

The hysteresis loop furnishes an exact definition of certain terms to which no precise meaning has previously been attached. The length *OR* in Fig. 141 is a measure of the *residual magnetism* of the rod, representing the magnetisation when the magnetic force has been removed after saturation. Similarly $O\alpha$, the magnetic force in the opposite direction necessary to remove the magnetism altogether from the rod, is called the *coercive force*. It is a measure of the power of the specimen to resist the abstraction of its magnetism when placed in a magnetic field.

The hysteresis curve throws light on the permanence of ordinary magnets. Since the magnetic forces to which magnets are normally subjected do not exceed one gauss, no serious loss of magnetisation is to be expected in a substance like tungsten steel, with a coercive force of about 60 gausses. The magnetism may, however, weaken slowly as a result of vibration or shock.

The hysteresis curve for soft iron (Fig. 142) has been traced from an experiment in which care was taken not to shake the specimen. Soft iron wire is very sensitive to shock, particularly in the early stages of magnetisation, and can be demagnetised effectively by striking it against a table.

Ewing introduced the convenient method of demagnetising iron by subjecting it to an alternating magnetic field whose amplitude diminishes steadily to zero. A rod can be demagnetised in the solenoid used to determine its ($H$, $I$) curve, with the aid of a small alternating-current generator and a rheostat of considerable range. An alternating current is first sent through the solenoid whose maximum value is sufficient for saturation: the current is gradually brought as low

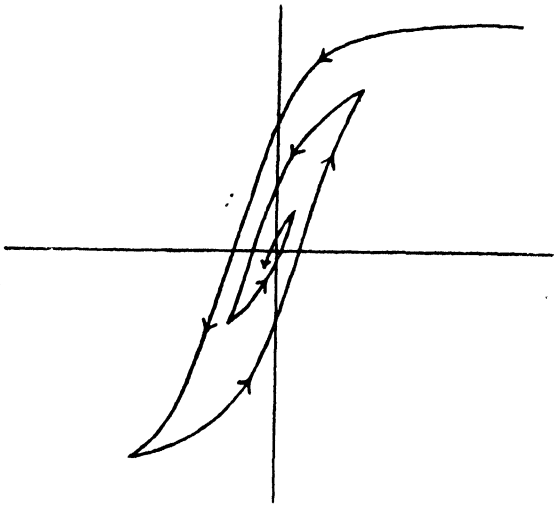


Fig. 145

as possible by means of the rheostat, and the rod should then be free from magnetism. Fig. 145 illustrates the changes undergone by the iron in this process. As the amplitude of magnetic force is reduced the hysteresis loops gradually diminish and finally vanish into the origin, leaving the iron in a state in which $I$ and $H$ are both zero. It might perhaps be thought that the same effect could be produced in a single cycle, e.g. by choosing the point of reversal in the left-hand diagram of Fig. 144 so that the return loop passes through the origin. After this treatment, however, the iron still shows the effect of its previous magnetic history, in that it responds differently to positive and negative magnetising forces.

Iron takes a little time to reach its final magnetisation after a change. The effect is hardly noticeable in the cyclic method of Art. 108.

**110. Work of magnetisation.** In discussing the energy relations of hysteresis we need not probe deeply into atomic structure, but only postulate the existence of so much magnetic moment per atom or molecule.

Let us have a rod of length $l$ and cross-section $A$ uniformly magnetised by a field $H$ to intensity $I$ (Fig. 146). The equivalent surface-distributions are $\pm AI$ as shown. When the magnetisation $I$ is changed to $I + dI$, $H$ remaining unaltered, work is done equal to that done in transferring pole-strength $A\,dI$ from one end of the rod to the other, that is $lAH\,dI$.

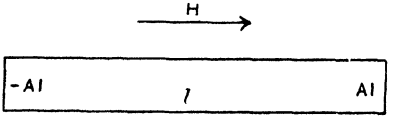


Fig. 146

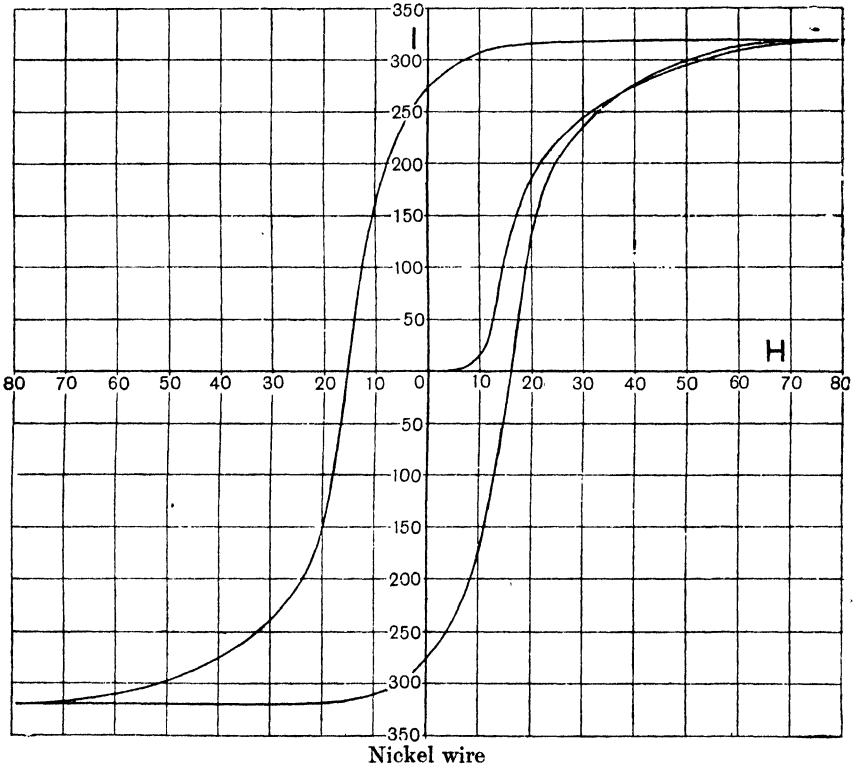
No work is done on the intramolecular forces of the body when $H$ increases to $H + dH$ without change of $I$, since the moment of the molecular magnets is unchanged. Work would be done merely on the current or external magnet producing the field. Hence when the force and magnetisation are changed from $H$ and $I$ to $H + dH$ and $I + dI$, work $lAH\,dI$ is done on the magnetised rod. In taking the rod round a complete hysteresis cycle the work done is $lA \int H\,dI$. Since $lA$ is the volume of the rod, the work per unit volume is equal to $\int H\,dI$, the area of the hysteresis loop.

If the rod is taken repeatedly round the cycle by means of an

alternating current, the work reappears as heat and the temperature of the rod rises until checked by radiation. Thus when magnetic substances are taken round hysteresis cycles by external power, part of the power is lost in heating the substances. Such losses are called *hysteresis losses*.

Hysteresis losses can be computed by measuring the ($H$, $I$) loop with a planimeter, a process which gives the loss corresponding to any assigned maximum value of $H$. The energy dissipated in a single cycle in soft iron is of the order of 10,000 ergs per cubic centimetre, that in steel being about ten times as large (p. 216).

**111. Magnetic properties of iron, nickel and cobalt.** The fact that specimens of iron and steel prepared in different ways




Nickel wire

Fig. 147

have very different magnetic qualities is utilised in various ways. Where iron is taken round its hysteresis cycle many times a second, as in the alternate current transformer (Art. 158), the chief de-

sideratum is low hysteresis loss, combined with great magnetisability. This condition is fulfilled by pure annealed iron, the small area of whose hysteresis curve is seen from Fig. 142. The presence of small quantities of other elements may greatly affect the hysteresis loss and coercive force, leaving the maximum magnetisation much the same. For example, an iron alloy containing silicon, patented

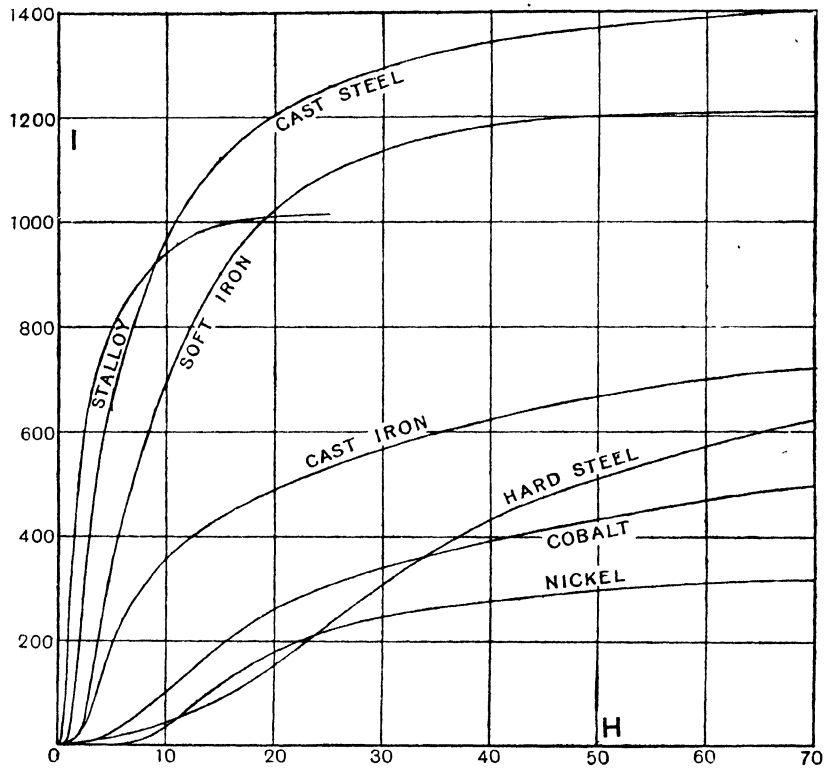



Fig. 148

under the name of Stalloy, has a hysteresis loss per cm.$^3$ of only 3000 ergs per cycle, together with great magnetisability.

A hysteresis curve of small area means either low coercive force or low residual magnetism, or both, so that materials suitable for the cores of transformers are entirely unsuited for making permanent magnets. Hard tungsten-steel, or iron alloyed with from 3 to 8 per cent. of tungsten, has a coercive force of about 60 gausses. Its hysteresis loss per cycle is about 25 times that in soft iron.

Magnetism is not in general an atomic phenomenon, since the

properties of an alloy may be quite different from those of its constituents. An alloy containing 12 per cent. of manganese is practically non-magnetic, the intensity of magnetisation being given approximately by the equation $I = \frac{1}{30}H$. There is no residual magnetism, even after exposure to the strongest fields. Small additions of nickel increase the magnetisability of iron, but further additions reduce it, until an alloy of 68 per cent. of iron and 32 per cent. of nickel is also practically non-magnetic.

The $(H, I)$ curve for a soft nickel wire is shown in Fig. 147, the curve being similar to that of steel. The maximum magnetisation

*Magnetisation of strongly magnetic substances.*

| | Intensity of magnetisation $I$ | | Magnetic induction $B$ | | Coercive force | $\int IdH$, energy loss per cycle, erg/cm.$^3$ |
|---|---|---|---|---|---|---|
| | maximum | residual | maximum | residual | | |
| Cast steel .. | 1400 | 700 | 18000 | 9000 | 2 | 20000 |
| Tungsten steel, hardened .. | 1300 | 800 | 16000 | 10000 | 60 | 260000 |
| Soft iron .. | 1200 | 900 | 15000 | 11000 | 2 | 10000 |
| Hard steel .. | 1000 | 700 | 12000 | 9000 | 40 | 120000 |
| Transformer iron | 1000 | 600 | 12000 | 8000 | 1·5 | 8000 |
| Silicon transformer iron .. | 1000 | 500 | 12000 | 6000 | 0·7 | 3000 |
| Cobalt .. .. | 700 | 250 | 9000 | 3000 | 10 | 20000 |
| Nickel .. .. | 400 | 200 | 5000 | 2500 | 10 | 12000 |
| Heusler alloy: Cu 61·5 %, Mn 23·5 %, Al 15 % .. | 300 | 200 | 4000 | 2500 | 8 | 8000 |

that nickel can acquire varies from 300 to 400; that is, about one-third that of iron. The coercive force is comparatively high, ranging from 10 to 20.

Cobalt is intermediate in magnetic properties between iron and nickel, and can acquire a magnetisation of about 700. The coercive force is much the same as for nickel.

The comparative behaviour of iron, nickel and cobalt is shown in Fig. 148, which shows the magnetisation curve obtained on the first application of the magnetic force. The range of $H$ is not great enough to show the whole of the curve for hard steel, the magnetisation of which rises under high force to almost the amount acquired

by soft iron. The table on p. 216 gives some approximate numerical data concerning the various substances hitherto mentioned, at ordinary temperatures. The meaning and importance of the "magnetic induction" $B$ is explained in Arts. 117, 129; the values are given here for reference.

Heusler observed in 1903 that certain alloys of copper, manganese and aluminium, all of which are practically non-magnetisable by themselves, had strong magnetic qualities. For example, an alloy containing 61·5 per cent. of copper, 23·5 per cent. of manganese, and 15 per cent. of aluminium, has magnetic properties resembling those of nickel (see table).

**112. Electromagnets.** The simplest form of electromagnet consists of an iron or steel rod wound uniformly with wire. Such

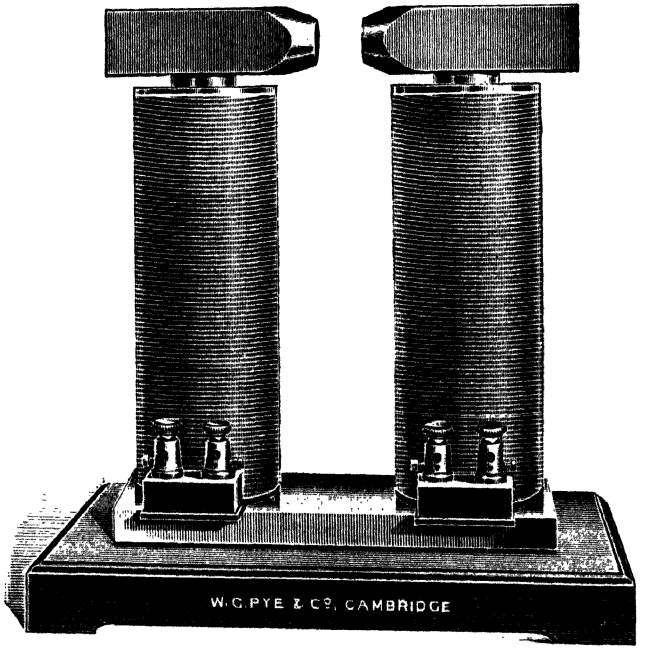



Fig. 149

a rod acts like a permanent magnet, only more strongly, as the saturation intensity of magnetisation can be maintained instead of the residual intensity.

Electromagnets are used in laboratories to produce strong and

tolerably uniform magnetic fields. A common type is shown in Fig. 149. Two soft iron pillars are firmly fixed in a strong iron base, a solenoid of many turns of thick wire being slipped over each. In this form the electromagnet is essentially a horse-shoe magnet kept excited by external current. Strong magnetic fields are formed between the poles if the latter are brought close together, and this

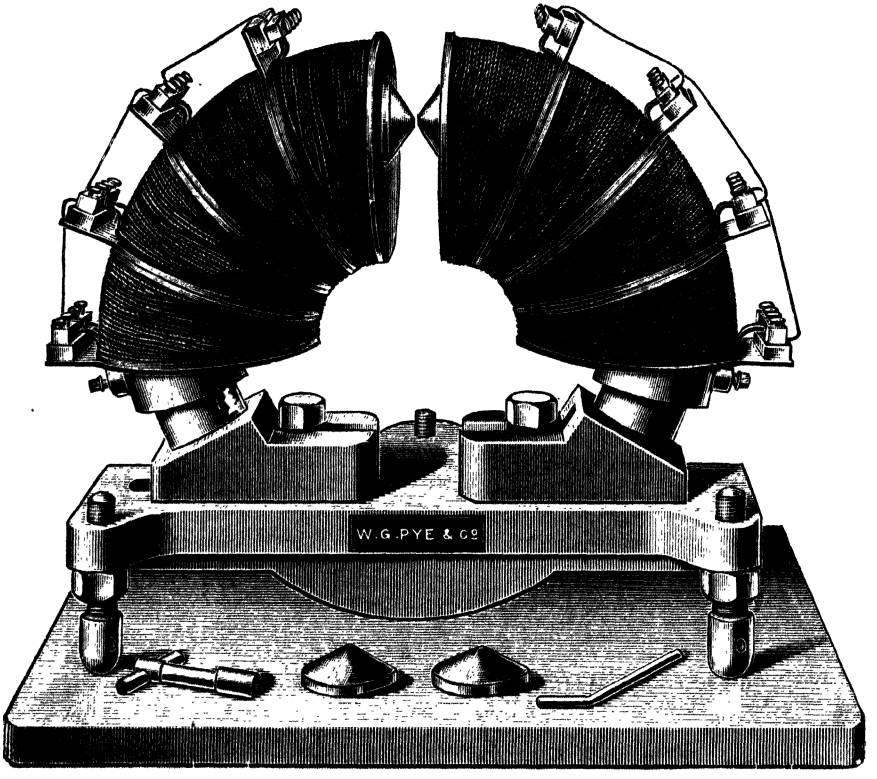



Fig. 150

is effected by means of two masses of soft iron, called pole-pieces, which lie on the fixed pillars. The strength of the field varies from 2,000 to 10,000 gausses, being greatest with large magnetising currents and conical pole-pieces close together. Fig. 150 shows an electromagnet of du Bois' type, capable of giving fields of 40,000 to 50,000 gausses.

It is difficult to produce a magnetic field which is strong as well as uniform. In order to see what can be done without iron, con-

**114. Paramagnetism and diamagnetism.** Although the substances already mentioned are the only ones known with strong magnetic properties, all bodies are magnetic to some degree. In view of this fact, it is as well to define the unit magnetic pole at the outset as that which repels a similar pole distant 1 cm. away *in vacuo* with a force of 1 dyne. The magnetisability of air, though small, is measurable, and has to be taken into account in determinations of the magnetic properties of other substances.

Weakly magnetic substances behave in a simple manner in a magnetic field: the intensity of magnetisation $I$ is proportional to $H$, and there is no hysteresis. For isotropic bodies the directions coincide, and we can write

$$\mathbf{I} = \kappa \mathbf{H} \quad \ldots\ldots\ldots\ldots\ldots\ldots (118),$$

where $\kappa$ is a constant characteristic of the body, though depending of course on its physical state and temperature. $\kappa$ is called the *susceptibility*. If it is positive the body is said to be *paramagnetic*: but there are also substances, called *diamagnetic* substances, which are magnetised by induction in a direction diametrically opposite to that of the impressed field.

Curie's method of determining $\kappa$ consists in measuring the forces acting on small pieces of the substance when placed in a non-homogeneous magnetic field. The principles involved are illustrated by the following example. Suppose that we have a small hollow vessel capable of containing the solid, liquid or gas to be examined, the outside volume being $v_1$ and the inside volume $v_2$. Let $\kappa_1$ be the susceptibility of the material of the vessel, $\kappa_2$ of the substance under examination, and $\kappa_3$ of air under normal conditions. Suppose that the mechanical forces on the vessel are observed when it is (1) exhausted, (2) filled with air at normal temperature and pressure, and (3) filled with the substance, and let these be denoted by $F_1$, $F_2$, and $F_3$ respectively. In case (1) there is a force on the vessel arising from a volume $v_1 - v_2$ of a substance of susceptibility $\kappa_1$; but since a volume $v_1$ of air is displaced its effect has to be subtracted. Hence equation (117) gives

$$F_1 = \{(v_1 - v_2)\,\kappa_1 H - v_1 \kappa_2 H\} \times (\text{magnitude of } \mathbf{D}H),$$

or, writing $1/\lambda$ for $H$ times the magnitude of $\mathbf{D}H$,

$$(v_1 - v_2)\,\kappa_1 - v_1 \kappa_2 = \lambda F_1 \quad \ldots\ldots\ldots\ldots (119).$$

The results of the second and third experiments are similarly expressed by

$$(v_1 - v_2)\,\kappa_1 - v_1\kappa_2 + v_2\kappa_2 = \lambda F_2 \quad \ldots\ldots\ldots (120),$$

and

$$(v_1 - v_2)\,\kappa_1 - v_1\kappa_2 + v_2\kappa_3 = \lambda F_3 \quad \ldots\ldots\ldots (121).$$

Equations (119), (120) and (121) give $\kappa_1$, $\kappa_2$ and $\kappa_3$ in terms of measured quantities. In a single experiment with a small solid body we are really measuring the difference of the susceptibilities of the body and air, so that some such process as the above is necessary to obtain absolute values.

In Curie's experiments the substance to be examined was contained in a bulb $C$ attached to a delicate torsion balance and lying between the poles of an electromagnet (Fig. 151). The magnetising coils are turned through a small angle, and $C$ lies on the line of symmetry a little away from the strongest part of the field. The magnetic force at points on the line of symmetry is parallel to the axis of $y$ but variable in the direction of the axis of $x$. Curie chose the point at which $H\,dH/dx$ had its maximum value, in order that the force on the body might be only slightly altered by a small change in its position. The following table gives the approximate value of $\kappa$ for certain substances (mostly at ordinary temperatures).

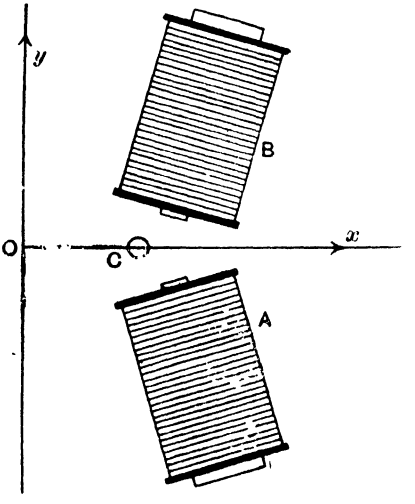



Fig. 151

| Values of $10^7\,\kappa$ | | | |
|---|---|---|---|
| Paramagnetic substances | | Diamagnetic substances | |
| Solid oxygen.. | 4200 | Bismuth .. | −130 |
| Platinum .. | 200 | Mercury .. | − 26 |
| Liquid oxygen | 30 | Silver.. .. | − 15 |
| Aluminium .. | 18 | Quartz .. | − 12 |
| Oxygen .. | 1·6 | Lead .. .. | − 11 |
| Air .. .. | 0·32 | Glass .. .. | − 9 |
| Carbon dioxide | 0·17 | Water .. | − 7·2 |
| | | Copper .. | − 7 |
| | | Hydrogen .. | − 0·24 |
| | | Argon .. | − 0·11 |

Iron itself acts like a true paramagnetic substance if demagnetised and only subjected to very small magnetic forces, the susceptibility being about 7.

The work of Curie, continued by Honda, allows one to survey the magnetism of the elements generally; and though there are many anomalies which show that magnetism often depends on aggregates larger than the molecule, a relation to the periodic table is clear. The strongest paramagnetic substances are the group of metals from titanium to nickel, and the group of rare earths, where atomic theory postulates an inner set of electrons in process of development with another set in arrested development outside it (Arts. 223, 276). The triads Ru, Rh, Pd and Os, Ir, Pt, also associated with the development of an inner set, form another group. Other atoms tend to be diamagnetic; the three strongest (P, Sb, Bi) have their inner layers complete and five electrons in their outermost layer. Dynamical explanation of these facts has hitherto been lacking.

**115. Magnetic liquids and magnetic crystals.** Wiedemann showed that solutions of iron salts are paramagnetic and that, when allowance is made for the diamagnetism of the water of solution, the susceptibility is proportional to the mass of iron in solution. The constant of proportionality has one value for ferrous salts and another for ferric salts, independent of the nature of the salt. We are therefore dealing with the specific magnetic properties of the ions $\overset{++}{\mathrm{Fe}}$ and $\overset{+++}{\mathrm{Fe}}$ (Art. 166); that is, iron atoms from which two and three electrons respectively have been removed from the outermost layer (Art. 223).

Quincke found that a paramagnetic liquid, contained in a capillary tube, rises in the tube when a strong magnetic field is applied near the meniscus. The effect is explained by the theory of Art. 113. Let $A$ be the cross-section of the tube, and consider the forces on the slice of liquid between heights $z$ and $z + dz$ above the normal surface. If $\rho$ is the density and $p$ the pressure at height $z$, the downward force due to gravity is $g\rho A\,dz$ and the thrust due to the difference of pressure on the faces is $A\,dp$. The upward force due to the magnetic field is, from equation (117),

$$\tau I \frac{dH}{dz} = A\,dz \,.\, \kappa H \,.\, \frac{dH}{dz}.$$

Hence for equilibrium

$$\kappa A H\, dH = g\rho A\, dz + A\, dp,$$

which gives on integration

$$\tfrac{1}{2}\kappa H^2 = g\rho z + p + \text{const.}$$

Let $T$ be the surface-tension and $a$ the radius of the meniscus. Then if $h_1$ is the height of the meniscus, and the surface of the main mass of liquid is in a region of no magnetic force,

$$\tfrac{1}{2}\kappa H^2 = g\rho h_1 - 2T/a,$$

where $H$ now denotes the magnetic force opposite the meniscus. If $h_2$ is the height of the meniscus when there is no magnetic field,

$$0 = g\rho h_2 - 2T/a.$$

Subtracting and writing $h = h_1 - h_2$ for the rise of level produced by the field, we have

$$\kappa H^2 = 2g\rho h.$$

Thus $\kappa$ can be found by measuring $h$ in known fields, as was done by Quincke for many liquids. The susceptibility of the air must be allowed for as before.

The intensity of magnetisation in crystals is not in general parallel to the magnetic force. A circular disc cut from a crystal in general experiences a couple when placed in a uniform magnetic field in the plane of the disc, the couple being a measure of the component of magnetisation perpendicular to the field. The behaviour of magnetite, examined by Weiss and Quittner, is complex. Discs cut parallel to a plane of cubic symmetry are magnetically isotropic in weak fields, but at 300 gausses the crystal can be magnetised most easily at an angle of 45° to a direction of cubic symmetry, and there are oblique components of magnetisation. The mineral pyrrhotite is analysed approximately as FeS, but different specimens differ considerably, a large crystal being formed apparently by the twinning of smaller crystals of simpler magnetic behaviour. The elementary crystal, according to Weiss, has three mutually perpendicular planes of symmetry. Along one of the normals it is hardly magnetic at all. Along a second $I$ is nearly proportional to $H$ up to $H = 7000$, after which it tends quickly to its saturation value of 47. The saturation intensity along the third normal is the same, but it is reached at $H = 700$.

**116. Effect of temperature on magnetisation.** It has been known for over 100 years that white-hot iron is but weakly magnetic, if at all. J. Hopkinson showed in 1889 that the loss of magnetism is practically complete at 760° C. Fig. 152, from experiments by Curie, shows how the intensity of magnetisation varies

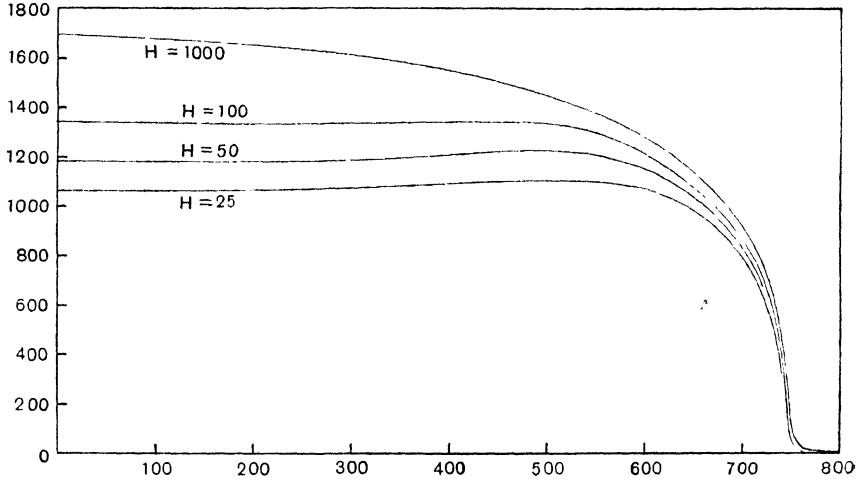



Fig. 152

with temperature for different magnetic forces: the value taken for $I$ is the mean of those for ascent and descent on the normal hysteresis curve. The fall of magnetisation near 750° C. is very quick, and the form of the $(H, I)$ curve also changes near this point. At 740° C. saturation is reached at about 400 gausses, but $I/H$ is nearly constant up to 1300 gausses at 760° C., showing that the iron has become paramagnetic.

The susceptibility of paramagnetic iron falls more and more slowly with increasing temperature up to 900° C., then rapidly for a few degrees near 900° C., and afterwards slowly up to 1280° C. There is a discontinuous, or nearly discontinuous, rise at this point, and a subsequent fall up to 1370° C., the highest temperature used by Curie. The critical points were re-measured by Honda and others. Iron is said to be in the $\alpha$ or ferromagnetic state below 760° C., in the $\beta$ state between 760° C. and 900° C., in the $\gamma$ state between 900° C. and 1280° C., and in the $\delta$ state above. As it cools it gives out abnormal heat at three temperatures a little below the critical points, the lag being possibly analogous to that of a super-cooled liquid.

The effect of temperature on magnetic substances in general was examined by Curie with the apparatus of Fig. 151, in which the bulb $C$ was surrounded by an electric furnace giving temperatures up to 1370° C. Similar experiments were performed by Honda. Their results are expressed in terms of the mass-susceptibility, or magnetic moment per gram in a field of one gauss. There is on the whole a fairly well-marked difference between paramagnetic and diamagnetic substances. Many diamagnetic substances (water, quartz, diamond, silicon, phosphorus, sulphur, arsenic and gold) have constant or nearly constant mass-susceptibility. The (negative) mass-susceptibilities of bismuth and antimony become numerically much smaller on fusion, and are constant in the liquid state: tin is paramagnetic when solid and diamagnetic when liquid. The (positive) mass-susceptibilities of molybdenum, tungsten and osmium are nearly constant, those of indium and manganese rise considerably with increase of temperature, and those of platinum, palladium, oxygen and most magnetic salts fall. Curie showed that the mass-susceptibilities of palladium and oxygen are inversely proportional to the absolute temperature, and Wiedemann's experiments prove the same for magnetic salts over a small range of temperature.

The magnetisation of iron is much affected by stress. Similarly magnetisation of an iron rod changes its length, as was first observed by Joule. The rod is lengthened by small applied magnetic forces, most so ($2\frac{1}{2}$ parts in a million) when $H = 80$. There is a contraction of about 5 parts in a million in very strong fields.

**117. Magnetic force and magnetic induction.** The laws of magnetism and statical electricity being formally identical, we can take over here the considerations applicable to electric force inside a dielectric, in the generalised form of Art. 55, no assumption of proportionality of $\mathbf{I}$ and $\mathbf{H}$ being admissible in general. The *magnetic force* inside a magnetised body is defined, purely theoretically, as that due to the surface-distribution $\sigma$ of equation (35) and the volume-distribution $\rho$ of equation (36). There is therefore a magnetic potential $\Omega$ at all internal points, and

$$\mathbf{H} = -\mathbf{D}\Omega \quad \ldots\ldots\ldots\ldots\ldots\ldots (122).$$

We can adapt Poisson's equation and obtain

$$\text{div } \mathbf{H} = 4\pi\rho = -4\pi \text{ div } \mathbf{I},$$

or

$$\text{div } (\mathbf{H} + 4\pi\mathbf{I}) = 0 \quad \ldots\ldots\ldots\ldots (123).$$

The vector

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{I} \quad \ldots\ldots\ldots\ldots\ldots (124)$$

is called the *magnetic induction*, and the definition applies to the most general case, when **H** and **I** are neither parallel nor proportional to each other.

In a vacuum, where $\mathbf{I} = 0$, **B** is identical with **H**; and this is practically true for air, since $I$ is only 0·0013 in a field of 40,000 gausses. The vectors **B** and **H** have characteristic properties which are very important on the theory. Thus from equation (123) **B** is always solenoidal, but is not in general a gradient. On the other hand, **H** is always a gradient, but is not in general solenoidal.

**B** can be interpreted simply on the electrical theory of matter, in which magnetism is ascribed to orbits in the atoms. Represent the average effect of an orbit by a current $i$ flowing in a small circuit of area $A$ in a definite plane. If there are $n$ atoms per cm.$^3$, each with a similar orbit, $I = niA$ from Art. 81. Let a unit cube be placed so that two of its faces are parallel to the plane of the orbits, and let a perpendicular line cut these faces in $P$ and $Q$. The solid angle subtended at $P$ by the vast majority of circuits (all except those near $P$) is zero or a multiple of $4\pi$, and similarly for $Q$. For any one such circuit, therefore, the line-integral of magnetic force along the straight line from $P$ to $Q$, which is the mean component of magnetic force parallel to $PQ$, is $4\pi i$ if $PQ$ pierces the circuit, and zero if it does not. Since the sum of the areas of the circuits inside the cube is $nA$, $PQ$ will pierce on the average $nA$ circuits. Hence, though the magnetic force normal to the orbits varies greatly from point to point, its mean value, taken through the whole cube, is $4\pi inA = 4\pi I$. A similar argument shows that there is no mean magnetic force parallel to the orbits. Hence the mean value, in a vectorial sense, of the magnetic force due to all the orbits is $4\pi\mathbf{I}$. The magnetic induction at any point is *the mean magnetic force in a small region of space surrounding the point*, **H** from the impressed field and $4\pi\mathbf{I}$ from the atomic orbits. The analogy with dielectrics fails here, for the mean electric force near a point in a dielectric is **E**, and not **D**.

The importance of the vector **B** is in connexion with electromagnetic induction (Art. 121). It can be measured without previous knowledge of **I** (Art. 129). In iron, where **I** is large in comparison with **H**, **B** is nearly equal to $4\pi\mathbf{I}$.

A *line of induction* is a line giving at every point the direction of **B** at that point, just as a line of force gives the direction of **H**. We can imagine tubes of induction drawn to fill the magnetic field (cf. Art. 23). The property of lines of magnetic force there proved, that the product of the resultant magnetic force and cross-section

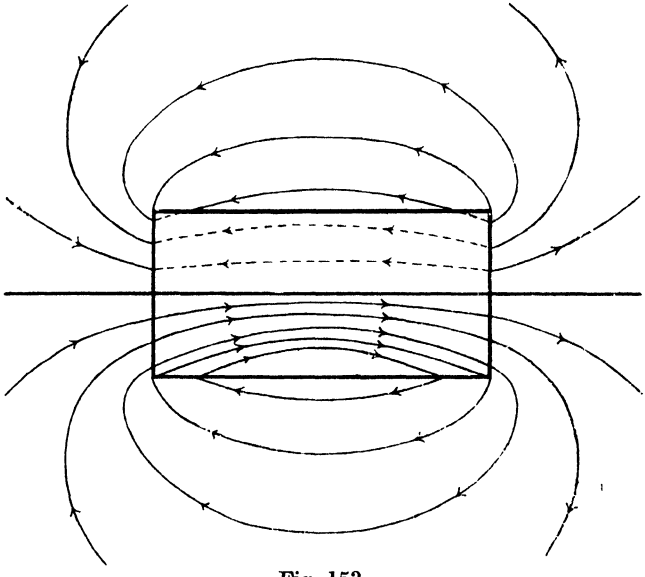


Fig. 153

of a small tube is constant, fails when we pass inside a magnetised body. Lines of magnetic force, in fact, begin or end on Poisson's distributions. On the other hand, if a small tube of induction is drawn the product of $B$ and the cross-section of the tube is constant, since this expresses the solenoidal property enjoyed by **B** even inside magnetised bodies. Thus lines of induction never begin or end, but circulate in closed curves.

Fig. 153 is a rough sketch of the lines of force and lines of induction for a short bar-magnet magnetised from left to right. Only one half of each diagram is shown, the top giving the lines of force and the bottom the lines of induction. In the space outside

the magnet the lines of force and induction coincide. Inside, we must remember that **H** is the force due to the impressed field (if any) and Poisson's magnetic distributions, so that the lines of force and induction run for the most part in opposite directions. This is due to remanence: the lines of force and induction would coincide everywhere if **B** was proportional to **H**.

When the material is in the form of a long rod the lines of magnetic force due to the ends of the rod hardly pass down the rod at all, so that the impressed field is the actual field. Thus with long rods we get the full magnetic force magnetising the rod, while in short rods the magnetic force due to the ends has a demagnetising effect.

**118. Conditions at the boundary of two media. Magnetic screening.** Bearing in mind the analogy of electric and magnetic force and induction, we prove at once the following boundary conditions at the surface of separation of two different magnetic media (cf. Arts. 54, 55):

(1) The tangential component of magnetic force is continuous;

(2) The normal component of magnetic induction is continuous, on crossing the boundary.

These conditions necessitate refraction of both lines of force and lines of induction in crossing the boundary. They also give definite boundary conditions which correspond to the condition of constant potential all over a conductor in electrostatics, enabling us sometimes to find the magnetic force theoretically.

We shall illustrate this by considering a spherical shell of paramagnetic material in a uniform magnetic field, which explains more or less the behaviour of a soft iron shell under the same conditions. Take the centre of the shell as origin, and the direction of the field as axis of a system of polar coordinates in space. The inner radius of the shell is taken as $b$, the outer radius as $a$, and the susceptibility of the material of the shell $\kappa$. The magnetic potential will be represented by different analytical expressions in the three regions: let it be $\Omega_1$ inside the shell, $\Omega_2$ in the material of the shell, and $\Omega_3$ outside. Evidently $\Omega_1$ and $\Omega_3$ satisfy Laplace's equation, i.e. $\Delta\Omega_1 = \Delta\Omega_3 = 0$. $\Omega_2$ also satisfies Laplace's equation: for since $\mathbf{I} = \kappa\mathbf{D}\Omega$, $\mathbf{B} = -(1 + 4\pi\kappa)\,\mathbf{D}\Omega$, and the solenoidal

condition $\operatorname{div} \mathbf{B} = 0$ gives $\Delta\Omega_2 = 0$. The condition of continuity of the tangential component of magnetic force at the inner surface gives

$$\left(\frac{\partial\Omega_1}{\partial\theta}\right)_{r=b} = \left(\frac{\partial\Omega_2}{\partial\theta}\right)_{r=b} \quad \ldots\ldots\ldots\ldots(125).$$

Similarly the continuity of the normal component gives

$$\left(\frac{\partial\Omega_1}{\partial r}\right)_{r=b} = (1 + 4\pi\kappa)\left(\frac{\partial\Omega_2}{\partial r}\right)_{r=b} \quad \ldots\ldots\ldots(126).$$

The conditions holding at $r = a$ are similarly

$$\left(\frac{\partial\Omega_2}{\partial\theta}\right)_{r=a} = \left(\frac{\partial\Omega_3}{\partial\theta}\right)_{r=a} \quad \ldots\ldots\ldots\ldots(127)$$

and

$$(1 + 4\pi\kappa)\left(\frac{\partial\Omega_2}{\partial r}\right)_{r=a} = \left(\frac{\partial\Omega_3}{\partial r}\right)_{r=a} \quad \ldots\ldots\ldots(128).$$

Finally for large distances from the origin the field has to tend to uniformity, i.e.

$$\lim_{r\to\infty} \Omega_3 = -Hz = -Hr\cos\theta \ldots\ldots\ldots(129),$$

where an arbitrary constant can be put on the right if desired. The mathematical problem is to find expressions for $\Omega_1$, $\Omega_2$, $\Omega_3$ which satisfy Laplace's equation and the conditions (125)–(129). Try a solution of the form

$$\Omega_1 = Ar\cos\theta, \quad \Omega_2 = Cr\cos\theta + \frac{D\cos\theta}{r^2}, \quad \Omega_3 = -Hr\cos\theta + \frac{E\cos\theta}{r^2},$$

where $\Omega_1$ has only one term since it is finite at the origin. The first term in each expression represents a uniform field, the second term in $\Omega_2$ and $\Omega_3$ that of a small magnet at the origin, so that Laplace's equation is satisfied. Condition (129) is also fulfilled, and conditions (125)–(128) if $A$, $C$, $D$, $E$ satisfy the equations

$$A = C + D/b^3, \quad A = (1 + 4\pi\kappa)(C - 2D/b^3),$$

$$C + D/a^3 = -H + E/a^3, \quad (1 + 4\pi\kappa)(C - 2D/a^3) = -H - 2E/a^3,$$

which are sufficient to determine the constants.

Fig. 154 shows the lines of induction for the value $\kappa = 1$, $b/a$ being taken as 2/3. The theory is not really applicable in such cases, because such high magnetisability is only found in *ferromagnetic* substances, i.e. substances for which $I$ is not proportional to $H$. The results are however qualitatively true for soft iron under

moderate magnetising forces, and give us some idea of the trend of the lines of induction *inside* the metal, which are not directly observable. The point to be especially noticed is that lines of induction tend to leave the cavity and crowd in the material of

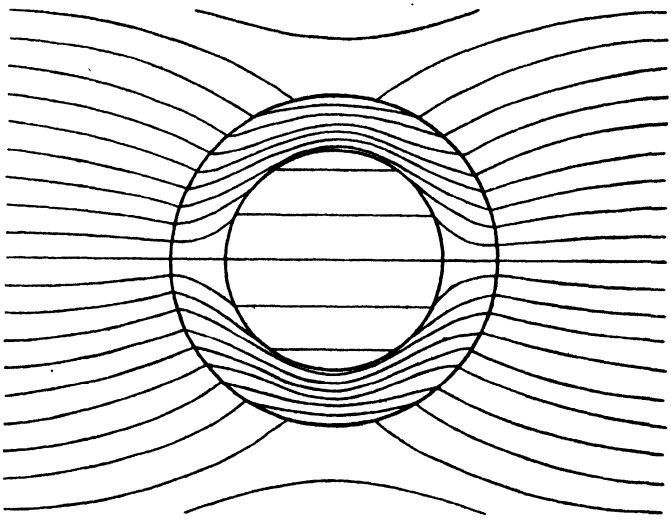


Fig. 154

the shell. Remembering that the density of lines of induction (in air) is a measure of the strength of the field, we see that the magnetic force inside the cavity is small compared with the impressed field. Thus soft iron acts as a screen against external magnetic action, but the screening power is not so perfect as that possessed by an earthed conductor in electrostatics. An iron sphere designed for a magnetic screen must be well annealed, demagnetised before use, and not exposed to very great magnetic forces. Galvanometers have sometimes been enclosed in soft iron spheres to protect them from disturbance.

**119. Properties of currents in a magnetic medium.** Let $C$ be a circuit carrying a current $i$, in a region containing magnetisable bodies of any kind. The magnetic force consists of two parts; a part $\mathbf{H}_0$ due to the current and $\mathbf{H}_1$ due to Poisson's distributions in the magnetic bodies. The line-integral of $\mathbf{H}_0$ round a closed curve $C'$ drawn in the field is $4\pi$ times the current interlaced by $C'$, and

the line-integral of $\mathbf{H}_1$ is zero since $\mathbf{H}_1$ is derived from a single-valued potential. Hence the line-integral of $\mathbf{H}$ itself round $C'$ is $4\pi$ times the current interlaced by $C'$, whether magnetic matter is present or not. The theorem does not hold in general when $\mathbf{B}$ is substituted for $\mathbf{H}$.

It is $\mathbf{B}$, on the other hand, and not $\mathbf{H}$, that determines the mechanical forces on the current. We shall show that the potential energy of the current $C$ is $-iN$, where $N$ is the flux of magnetic induction through $C$ in the positive sense.

Let the circuit $C$ intersect the plane of Fig. 155 in $C_1$ and $C_2$, and let $A$ be a finite magnetic body which does not interlace $C$, so that $C$ can be covered with a cap $S$ which does not cut any magnetic matter. The potential energy of the circuit in the field of $A$, considered as an assemblage of poles as in Art. 85, is $-iN$, where $N$ is the flux of magnetic force through $S$. Since magnetic force and magnetic induction coincide on $S$, $N$ is equal to the flux of magnetic induction through $S$, and therefore to the flux of magnetic induction through any cap on $C$, whether it intersects magnetic matter or not (cf. Art. 9). If the magnetic body interlaces $C$, we can always cut it up into two parts, to each of which the foregoing proof applies, and hence the theorem is true in general. Thus when the circuit $C$ is moved in such a way that $N$ changes, the current remaining constant, work $i$ times the increase of $N$ is done on it by the field.

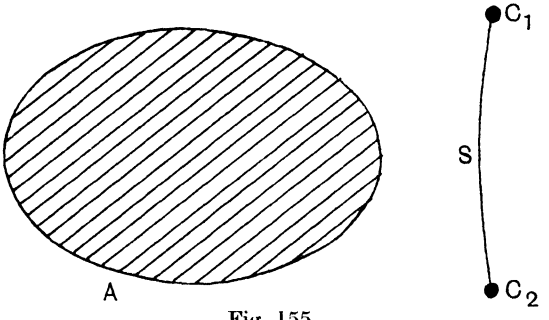


Fig. 155

The concentration of lines of magnetic induction in iron cores is utilised to increase the mechanical forces on a circuit slipped over these cores, as in the moving-coil galvanometer.

# CHAPTER VIII

## INDUCTION OF CURRENTS

**120. Faraday's experiments.** In 1831 Faraday showed that currents could be produced in closed circuits not containing any battery by making or breaking other circuits in their vicinity, and by the motion of circuits and magnets relative to one another. The qualitative laws of the phenomenon were completely unravelled by Faraday in two memoirs*, which well repay careful study on account of the exhaustive nature of the experiments and the admirable clearness with which they are described. Faraday first prepared two solenoids of many turns of wire on the same core, insulated from one another, one being connected to a battery and the other to a sensitive galvanometer. There was not the slightest deflexion of the galvanometer while a steady current flowed through the battery circuit; but on first closing the circuit the galvanometer showed a momentary deflexion, and again in the opposite direction when the current was cut off. These temporary induced currents were found to be capable of magnetising steel needles. As regards the direction of the induced currents, imagine the solenoids straightened out so as to form parallel wires *A*, *B* (Fig. 156). Then on closing the battery circuit (or *primary*) the current in the galvanometer circuit (or *secondary*) was in the opposite direction as shown in the figure: on cutting off the current in the primary that induced in the secondary was in the direction of the original primary current. Precisely similar effects were obtained on moving the circuits relatively to each other. Thus if the wire *A*, carrying a constant current, approached the wire *B* it induced a current in the opposite direction as long as the motion continued, the current being reversed when the wire receded.

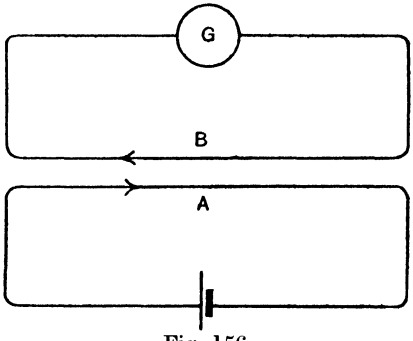


Fig. 156

* Faraday, "Experimental Researches in Electricity," Ser. 1, *Phil. Trans.*, vol. 122, p. 125 (Nov. 24th, 1831); Ser. 2, *Phil. Trans.*, vol. 122, p. 163 (Jan. 12th, 1832).

Greater effects were obtained with smaller currents by winding the primary and secondary helices over the same closed iron ring (Fig. 157). To elucidate the effect of the ring Faraday wound two solenoids on a hollow pasteboard cylinder, in which metallic cores could be inserted. A copper core had no effect on the magnitude of the induced current, while an iron core produced a great increase,

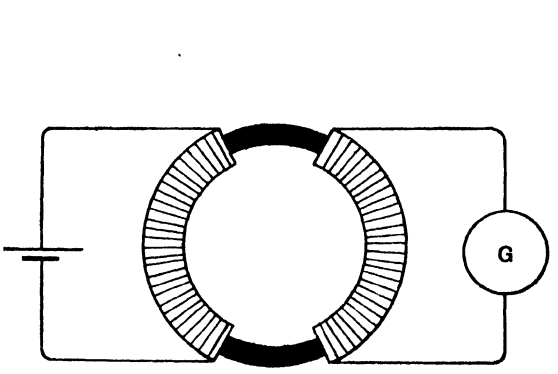



Fig. 157

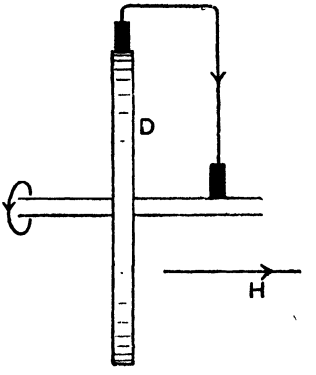



Fig. 158

but not so much as a complete ring. Further, currents were induced by suddenly inserting a bar-magnet into the core, the direction of the momentary current in the solenoid being related to that of the axis of the magnet by the *left-handed* screw law. An equal and opposite deflexion occurred on suddenly removing the magnet.

It thus appears that the effects obtained from currents and magnets are similar in kind, the only requisite being the presence of a magnetic field. It occurred to Faraday that the earth's magnetic field might be utilised, and he succeeded in obtaining currents by turning a coil of wire suddenly through two right angles in the earth's field.

Faraday was able to show the induction of currents in solid masses of metal moving in a magnetic field and thus to construct the first machine for the mechanical generation of electricity. In one experiment a metal disc $D$ (Fig. 158) rotated in a magnetic field, and conducting brushes were pressed against the axle and circumference. On joining these brushes by a wire a steady current was found to flow in the wire from the circumference inwards, so that the current in the disc itself was radially outwards. The effect was greatest when the resultant field of the earth was perpendicular to the disc, and vanished when the disc was in the magnetic meridian, suggesting

that the current is proportional to the component $H$ of magnetic force perpendicular to the plane of the disc. Faraday also obtained currents from the rotation of a bar-magnet in a similar manner, by making connexion with the side and end of the magnet respectively. Here the magnet acts at once as the producer of the field and as the moving conductor.

By using wires of different metal in the ordinary induction experiments Faraday concluded that the currents were in the ratio of the conducting powers of the metals, and that the "force" (electromotive force of induction) was, *caeteris paribus*, independent of the nature of the wire. The condition necessary for the induction of currents was clearly enunciated by Faraday, namely that there should be a cutting of "magnetic curves" by the circuit; that is, that there should be a change in the flux of magnetic induction through the circuit.

**121. The law of electromagnetic induction.** The exact law of electromagnetic induction was first enunciated by F. E. Neumann in 1845. Consider a single circuit $C$ of self-inductance $L$ (Art. 92) carrying a current $i$ at time $t$, in the presence of a system of magnetic poles. Suppose, for the sake of generality, that it contains also a cell of E.M.F. $V$. From what we know of the mechanism of the voltaic cell it appears likely that the energy changes in the cell during the passage of a given charge round the circuit are always the same, so that in time $dt$ work $Vi\,dt$ is done on the current by the cell. This work is expended in three ways:

(1) In heating the circuit. It appears likely, on any theory of conduction, that work $Ri^2dt$ is used up in this way.

(2) When the relative position of the circuit and the magnetic system changes, the forces between the circuit and the system do work $i\,dN$, where $N$ is the flux of magnetic induction through the circuit due to the magnetic system (Arts. 85, 119).

(3) When the current is changing the work done by the forces between the elementary filaments of the current is $d\,(\frac{1}{2}Li^2) = Li\,di$, as long as the circuit is undeformed. Hence

$$Vi\,dt = Ri^2dt + i\,dN + Li\,di,$$

or

$$Ri = V - \frac{dN}{dt} - L\frac{di}{dt} \quad \ldots\ldots\ldots\ldots (130).$$

Now $Li$ is in a certain sense the flux of magnetic force (or induction) through the circuit due to the current itself, being the mean flux through elementary filaments carrying equal currents. Hence the total flux due to all causes, including the current itself, is $N + Li$, and writing this briefly $N$, we have the following statement of the law of electromagnetic induction:

*When the flux of magnetic induction through a circuit changes, the currents induced are such as arise from an electromotive force equal to the rate of decrease of the flux.*

In this enunciation the flux of induction is supposed to have sign as well as magnitude (cf. Art. 85), and electromotive force is reckoned as positive when it tends to send a current round the circuit in the standard direction. The directions of the currents in Faraday's experiments are all correctly accounted for by this theory. The case of the rotating disc (Fig. 158) is not so clear: it is considered further in Art. 139.

When there is more than one circuit considerations of energy do not suffice to determine the laws, as only one equation is obtained in this way. We shall assume that the law of electromagnetic induction here enunciated holds for each circuit separately; that is, the induced E.M.F. is $-dN/dt$, where $N$ is the flux of magnetic induction due to all causes, inclusive of the circuits present in the field.

In the case of two circuits let $L_1$, $L_2$ be the self-inductances, $M$ the mutual inductance, $R_1$, $R_2$ the resistances and $i_1$, $i_2$ the currents. The flux of magnetic induction through the first circuit is $L_1 i_1$ due to itself and $Mi_2$ due to the second circuit, so that the E.M.F. of electromagnetic induction is $-L_1 di_1/dt - M di_2/dt$. If in addition electromotive forces $V_1$, $V_2$ are acting in the two circuits, the equations for the currents are

$$\left.\begin{aligned} L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} + R_1 i_1 &= V_1 \\ M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} + R_2 i_2 &= V_2 \end{aligned}\right\} \quad \ldots\ldots\ldots (131).$$

The importance of self and mutual inductance will now be understood, as their effect is always felt with changing currents, and for very rapid changes the currents are determined more by inductance than by resistance.

**122. Verifications of the theory.** If a sensitive ballistic galvanometer is available, some of the laws of electromagnetic induction can be verified in the laboratory by a method due to Felici. $A$ (Fig. 159) is a coil of about 20 turns of thick wire mounted on a board, the dimensions being about $50 \times 40$ cm. The coil $X$ consists of a single turn of wire stretched round four small brass screws at the corners, so that it can be replaced by another wire of different material or diameter, but of the same configuration. $B$ represents a small solenoid of thick wire in three layers, each layer having about 10 turns to the centimetre; and $Y$ is a small secondary coil of about 20 turns sliding inside $B$, and wound on a wooden core of about 2 cm.$^2$ sectional area. The solenoid $C$ and core $Z$, to which we shall refer later, are similar to $B$ and $Y$.

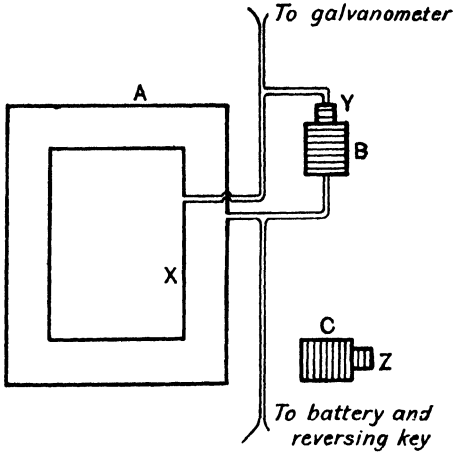


Fig. 159

To obtain a fair test of the theory, we must make sure that there is no flux of induction through the leads, and also that the three pairs of coils do not act inductively on one another. The latter condition is practically secured by the disposition of the coils in the figure, and the former by making all the connexions of flexible wire whose two strands are twisted round one another, shown in the figure by the parallel lines near together.

The core $Y$ is first removed from $B$ and a current of about 10 amperes passed through $A$ and $B$ in series. On reversing there will be a throw of the galvanometer due to the inductive effect of $A$ on $X$. If $Y$ is now inserted the effect of $B$ on it is added, and the throw is greater or less according as the two electromotive forces assist or counteract each other. We can arrange that the latter shall be the case and also adjust the position of $Y$ inside $B$ so that exact compensation occurs and the galvanometer is not affected by any changes in the battery circuit. The pair of coils

$C$, $Z$ is then adjusted in the same way (the connexions not being shown), and we are now ready to make the following experiments.

(1) With the connexions of the figure, replace the wire $X$ by one of the same configuration but different material, or different cross-section. The galvanometer still shows no deflexion. Hence the E.M.F. of induction is independent of the nature and cross-section of the wire forming the circuit $X$, as we should expect from the theory.

(2) Interchange the connexions of $A$ and $X$; so that $X$ is in the battery circuit and $A$ in the galvanometer circuit. No throw occurs on making or reversing the current, showing that the inductive effect of $A$ on $X$ for a particular current is equal to that of $X$ on $A$.

(3) Connect the primary coils $B$, $C$ in parallel so that the current through $A$ divides between $B$ and $C$, and connect the three secondary coils $X$, $Y$, $Z$ in series. Again there is no throw of the galvanometer under any circumstances. It follows that the inductive effects of $B$ on $Y$ and of $C$ on $Z$, which are equal for the same currents, are the same when the current is divided between them. Hence the inductive effect is proportional to the current.

**123. Calculation of self-inductance.** A convenient rule can be given for the self-inductance of a wire of circular cross-section bent into the form of a curve whose radius of curvature is large compared with the radius of the cross-section, a few isolated points or corners excepted. Consider in the first place a circular circuit (Fig. 160), $C$ being the line of centres of the cross-sections and $C_1$ the inner rim of the wire. If $P$ is any point inside the circle $C_1$ it is easy to show that the magnetic force at $P$ can be calculated without serious error by concentrating the current in the wire along its axis $C$.

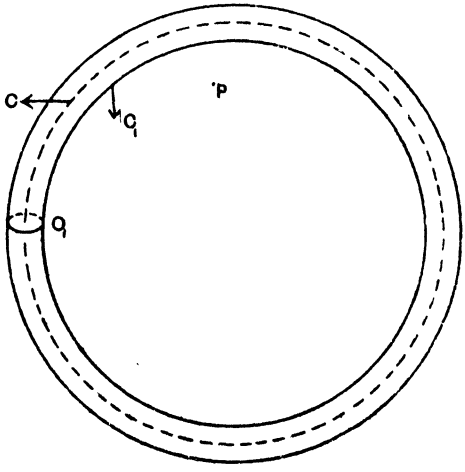


Fig. 160

This is evidently true if $P$ is not near the wire: and if $P$ is close to the wire the greater part of the magnetic force arises from the neighbouring part of the circuit, which acts on it practically like a long straight wire carrying a current. The magnetic force is again, from Art. 82, the same as if the current $i$ was concentrated along the line of centres.

Now take a particular cross-section of the wire as shown in Fig. 161, and let $dS$ be an element of area distant $r$ from the centre, the circuit $C_1$ meeting the cross-section at $O_1$. Let unit current flow in the wire, and let $M$ be the mutual inductance of $C$ and $C_1$. The flux of magnetic force through $C_1$ is $M$, and the flux through the filament on $dS$, which we call $N$, exceeds $M$ by the flux through the band-like space between that filament and $C_1$. As before, the magnetic force inside is practically all due to the neighbouring part of the current, being therefore $2\rho/a^2$ at distance $\rho$ from $O$, where $a$ is the radius of the cross-section. Let $l$ be the total length of the wire. Then since the lines of magnetic force at points in the cross-section are circles with $O$ as centre, all lines of force passing between $dS$ and $O_1$ pass between $R$ and $O_1$, and the flux of force between the filament on $dS$ and $C_1$ is equal to

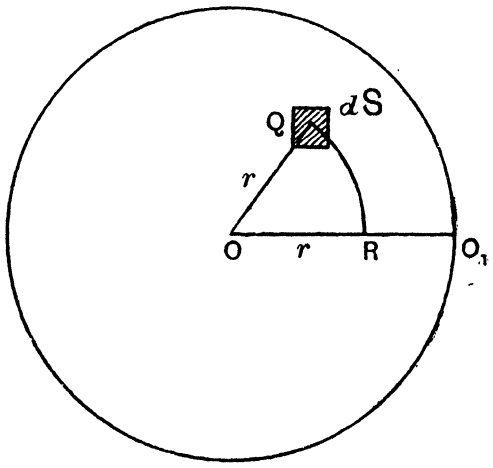


Fig. 161

$$l\int_r^a \frac{2\rho}{a^2}\,d\rho = \frac{l(a^2-r^2)}{a^2}.$$

Hence

$$N = M + \frac{l(a^2-r^2)}{a^2}.$$

Referring to the definition of the self-inductance given in Art. 92 and remembering that $di = dS/\pi a^2$ since the unit current is spread uniformly over the cross-section of the wire, we have

$$L = \frac{1}{\pi a^2}\int N\,dS = \frac{M+l}{\pi a^2}\int dS - \frac{l}{\pi a^4}\int r^2\,dS = M + l - \frac{l}{\pi a^4}\int_0^a 2\pi r^3\,dr.$$

Hence $L = M + \frac{1}{2}l$. The method of reasoning is evidently general and applies to circuits of any shape, and we have the following rule of calculation:

Draw the line of centres of the cross-sections of the wire and also draw a line parallel to it in the surface of the wire. Calculate the mutual inductance of these two circuits and add to it half the total length of the wire. The result is the self-inductance of the circuit.

The unit implied in these calculations is the absolute electromagnetic unit, or centimetre, since self-inductance has the dimensions of a length. The unit of inductance in the derived system is the *henry*, or $10^9$ cm. A circuit of inductance one henry is such that an E.M.F. of one volt is developed when the current is changing at the rate of one ampere per second: for then $di/dt = 1/10$ absolute E.M.U., and if $L = 10^9$ the E.M.F. is $10^8$ E.M.U. $= 1$ volt. The henry is a rather large unit, and small inductances are expressed in millihenrys ($10^6$ cm.) or centimetres.

## 124. Self-inductance of special circuits.

(1) *Rectangular circuit.* Using the result of the last article, $M$ is found by Neumann's formula (Art. 97). We begin by finding the value of $\iint ds\,ds'/r$ for two equal and parallel lines of length $l$ distant $h$ apart. Writing $J$ for this double integral, we have

$$J = \int_0^l \int_0^l \frac{dx\,dy}{\{(x-y)^2 + h^2\}^{\frac{1}{2}}} = \int_0^l dx \left[\sinh^{-1}\frac{y-x}{h}\right]_{y=0}^{y=l}$$

$$= \int_0^l \left[\sinh^{-1}\frac{l-x}{h} + \sinh^{-1}\frac{x}{h}\right] dx = 2\int_0^l \sinh^{-1}\frac{x}{h}\,dx.$$

The integral is evaluated by the substitution $x = h \sinh\theta$, giving

$$J = 2\left[-(l^2+h^2)^{\frac{1}{2}} + l\log\frac{l + (l^2+h^2)^{\frac{1}{2}}}{h} + h\right].$$

Considering the two rectangles in Fig. 162, which corresponds to Fig. 160, we see that $M$ receives no contribution from mutually perpendicular sides, but that

$$M = \iint \pm\, ds\,ds'/r$$

for pairs of parallel sides only. Let the central line $C$ be a rectangle $a$ by $b$, and let $r$ be the (small) radius of the cross-section of the

wire. Then the contribution of $PQ$ and $R'S'$ to $M$, which is negative, is

$$2\left[(a^2+b^2)^{\frac{1}{2}} - a - b\log\frac{b+(a^2+b^2)^{\frac{1}{2}}}{a}\right].$$

The contribution of $PQ$ and $P'Q'$ is

$$2\left[r - (b^2+r^2)^{\frac{1}{2}} + b\log\frac{b+(b^2+r^2)^{\frac{1}{2}}}{r}\right],$$

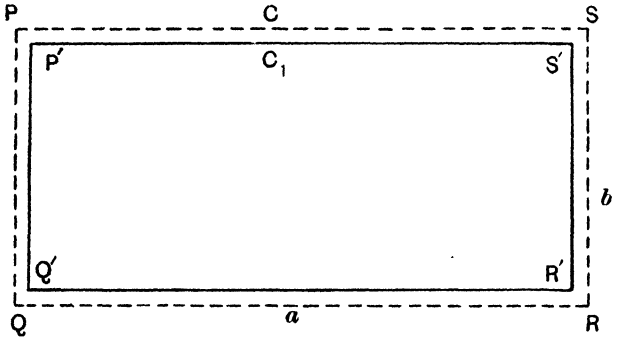



Fig. 162

which is approximately $2\left\{-b + b\log\frac{2b}{r}\right\}$, since $r$ is small. The sum of these terms gives the part of $M$ arising from the side $PQ$, and dealing similarly with the other sides we find

$$M = 4\left\{-2a - 2b + 2(a^2+b^2)^{\frac{1}{2}} + a\log(2a/r) + b\log(2b/r) - a\log\frac{a+(a^2+b^2)^{\frac{1}{2}}}{b} - b\log\frac{b+(a^2+b^2)^{\frac{1}{2}}}{a}\right\}.$$

The self-inductance is obtained by adding $a+b$. If we write $d$ for the diagonal of the rectangle, $c = a+b$ for the semi-perimeter, we find a convenient formula due to Mascart, namely

$$L = 9{\cdot}210\left\{c\log_{10}\frac{2ab}{r} - a\log_{10}(a+d) - b\log_{10}(b+d)\right\} - 7c + 8d \text{ centimetres} \quad \ldots.(132).$$

Thus the self-inductance of a square circuit of side one metre, made of round-sectioned wire 1 mm. in diameter, is 5660 cm., or $5{\cdot}66 \times 10^{-5}$ henry.

(2) *Circular coil of wire of one turn.* Let $a$ be the radius of the circle into which the wire is bent, $r$ that of the cross-section of the

wire itself. We have first to find an approximate expression for the mutual inductance of two concentric circles of radii $a$ and $a - r$, when $r$ is small. The formula for any radii is given in Art. 98. Here $k^2 = 4ab/(a+b)^2$ tends to unity; and if we write $k'$ for $(1-k^2)^{\frac{1}{2}}$ we have $k' = (a-b)/(a+b) = r/2a$ nearly. The elliptic integral $E$ tends to unity, while $K$ tends to infinity. We have

$$K = \int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2\phi)^{-\frac{1}{2}} d\phi$$

$$= \int_0^{\frac{\pi}{2}} (k'^2 + k^2\cos^2\phi)^{-\frac{1}{2}} d\phi = \int_0^{\frac{\pi}{2}} (k'^2 + k^2\sin^2\phi)^{-\frac{1}{2}} d\phi$$

$$= \int_0^{\frac{\pi}{2}} (k'^2 + \sin^2\phi)^{-\frac{1}{2}} d\phi \text{ approximately.}$$

Since $k'$ is small, the quantity under the integral becomes large when $\phi = 0$. We shall therefore split up the range of integration into two parts, namely 0 to $\alpha$ and $\alpha$ to $\frac{1}{2}\pi$, where $\alpha$, though small, is large compared with $k'$. Thus

$$K = \int_0^{\alpha} (k'^2 + \sin^2\phi)^{-\frac{1}{2}} d\phi + \int_{\alpha}^{\frac{\pi}{2}} (k'^2 + \sin^2\phi)^{-\frac{1}{2}} d\phi$$

$$= \int_0^{\alpha} (k'^2 + \phi^2)^{-\frac{1}{2}} d\phi + \int_{\alpha}^{\frac{\pi}{2}} \frac{d\phi}{\sin\phi} = \sinh^{-1}\left(\frac{\alpha}{k'}\right) - \log\tan\tfrac{1}{2}\alpha$$

$$= \log(2\alpha/k') - \log(\tfrac{1}{2}\alpha) = \log(4/k') = \log(8a/r).$$

Hence $$M = 4\pi a\,(K - 2E) = 4\pi a\left(\log_e \frac{8a}{r} - 2\right),$$

and thus

$$L = 4\pi a\left(\log_e \frac{8a}{r} - \frac{7}{4}\right) = a\left(28{\cdot}94 \log_{10}\frac{a}{r} + 4{\cdot}14\right) \text{centimetres} \ldots (133).$$

For example, a circular ring of radius 2 cm., of wire 2 mm. in diameter, has a self-inductance of about 83·5 cm.: this gives us an idea of the order of magnitude of the self-inductance of small loops ($10^{-7}$ henry). The effect of thinning down the wire is considerable: thus if the wire was a filament of diameter 1/10 mm., the self-inductance of the same loop would be 158·8 cm.

**125. Growth of current on closing a circuit.** Let the self-inductance and resistance of the circuit be $L$ and $R$ respectively, and suppose that an E.M.F. of constant magnitude $V$ is applied at time $t = 0$. Then the equation for the current at any subsequent time is

$$L\frac{di}{dt} + Ri = V \quad \ldots\ldots\ldots\ldots\ldots\ldots (134).$$

The solution of this is

$$i = \frac{V}{R} + Ae^{-Rt/L},$$

where $A$ is a constant. Since $i = 0$ when $t = 0$, $A = -V/R$, and

$$i = \frac{V}{R}(1 - e^{-Rt/L}) \quad \ldots\ldots\ldots\ldots\ldots\ldots (135).$$

Thus the current creeps up to its final value $V/R$ gradually, and the larger the ratio $R/L$ the greater will be the rapidity of approach to this value.

Fig. 163 shows the effect of applying 5 volts to a circuit of

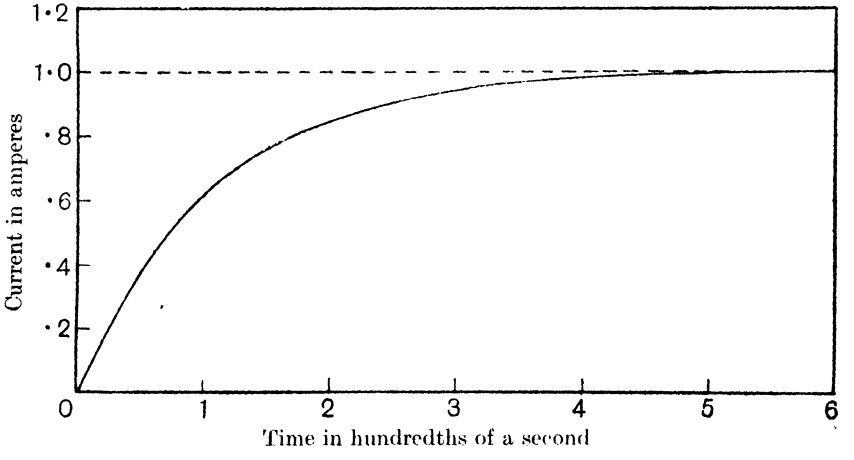



Fig. 163

resistance 5 ohms and inductance 50 millihenrys. The current is practically steady after one-twentieth of a second.

The time of attaining the steady state may be much greater if the coil is wound on an iron core or ring, as the changing induction in the core is added to that due to the current in the circuit. The effect is roughly equivalent to an increase in the self-inductance of the coil, but as the flux of induction is no longer proportional

to the current we cannot assign any particular self-inductance to the coil. The current in a large electromagnet takes several seconds to attain its final value. The time of attaining an approximately steady state is comparable with $L/R$. Kamerlingh Onnes found that it is very large in a circuit composed of lead in the super-conducting state (Art. 67). A current induced in the circuit fell by less than 1 per. cent. per hour, corresponding to a value of $L/R$ of the order of 4 days. We have thus the remarkable phenomenon of a current continuing to flow for several days without applied electromotive force.

Induction effects in resistance coils can be much reduced by bending the wires back on themselves before winding them on the bobbins (Fig. 81), or better by having several such double-wound sections in series.

The effect which occurs on breaking a circuit must be distinguished from what happens on closing it. A spark generally passes on opening a circuit containing an electromagnet or a large inductance, and a shock can be felt if the terminals of the coil are touched with the hands. In the absence of inductance the current would fall to zero in a time comparable with that taken by light to travel round the circuit. This however gives rise to a large E.M.F. of electromagnetic induction in the circuit, and it is easy to see that this acts in a direction favouring the continuance of the current. If the self-inductance is great enough the insulation of the air may break down, and the current continue for a short time after the key is pulled up.

**126. Measurement of mutual and self-inductance.** The following methods are for comparatively large inductances (from 10 millihenrys upwards).

(1) *Measurement of the mutual inductance of two coils.*

The primary circuit (Fig. 164) contains, in addition to the cell and coil, a reversing key, a resistance box and an accurate small resistance $X$ (1 ohm or 1/10 ohm).

Let currents $i_1$, $i_2$ flow in the two circuits at any time. Then since the secondary contains no battery,

$$M\frac{di_1}{dt} + L_2\frac{di_2}{dt} + (R+G)\,i_2 = 0,$$

where $G$ is the resistance of the galvanometer. Applying the equation to the short interval of time occupied in reversal of the

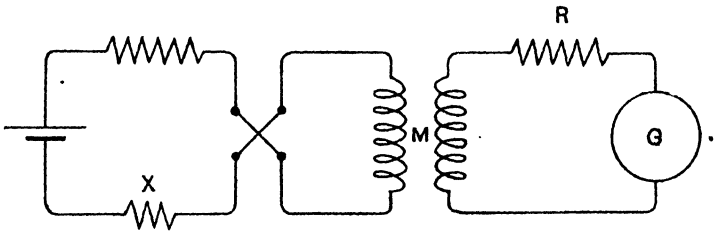



Fig. 164

primary current from $i$ to $-i$, and integrating with respect to $t$ over that interval, we have

$$2Mi + [L_2 i_2]_{\text{limits}} + (R + G) \int i_2 dt = 0.$$

But $i_2$ vanishes both initially and finally, and $\int i_2 dt$ is equal to $e$ the total charge sent through the galvanometer. Hence we have from the theory of the ballistic galvanometer

$$2Mi = (R + G) \frac{T}{2\pi} k\tau,$$

where $\tau$ is the corrected throw (Art. 89) and $T$ the time of swing of the moving coil.

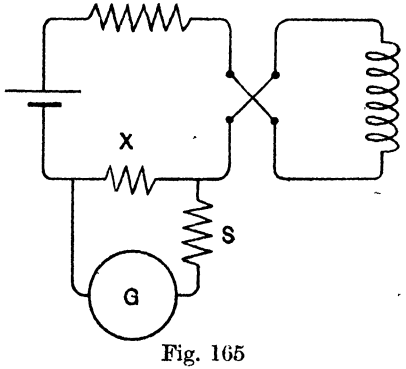



Fig. 165

The second part of the experiment consists in disconnecting the galvanometer from the secondary and shunting it across the small resistance $X$, adjusting the resistance $S$ in series with it till there is a steady deflexion $\tau$ equal to the preceding ballistic throw (Fig. 165). Then $Xi = (S + G)\, k\tau$, approximately. Dividing the last two equations, we have

$$M = \frac{X}{2} \frac{R + G}{S + G} \frac{T}{2\pi} \quad \ldots\ldots\ldots\ldots\ldots\ldots (136).$$

Variable standards of mutual inductance can be obtained consisting of one fixed and one moveable coil, the mutual inductance being known in each position.

(2) *Measurement of self-inductance.*

The coil $L$ is placed in the arm $AB$ of a Wheatstone's bridge in series with a box whose resistance can be increased by a small amount $X$ (say 1/10 ohm), the other arms being adjustable and non-inductive (Fig. 166).

First balance the bridge accurately for steady currents. Let $\tau$ be the ballistic throw of the galvanometer on reversing the current, $\theta$ the steady deflexion caused by making the small increase $X$ in the resistance of $AB$. Then

$$L = \frac{X}{2}\frac{\tau}{\theta}\frac{T}{2\pi} \quad \ldots\ldots\ldots\ldots\ldots\ldots (137).$$

The proof depends on the fact that an E.M.F. $V$ introduced into the arm $AB$ of the bridge in any way gives rise to a current $\lambda V$ in the galvanometer, where $\lambda$ is some constant. If $i_1$ is the current in $AB$ at any time during reversal,

$$V = L di_1/dt$$

and a current $\lambda L di_1/dt$ flows through the galvanometer. The total charge passing through the galvanometer during reversal is therefore $e = 2\lambda Li$, where $i$ is the initial current in $AB$ and $-i$ the final current. Hence

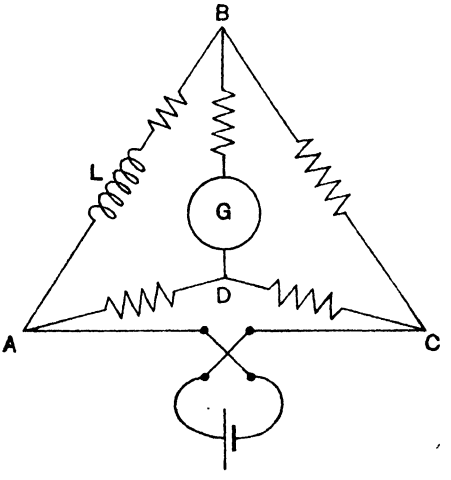


Fig. 166

$$2\lambda Li = \frac{T}{2\pi} k\tau.$$

A small increase $X$ in the resistance of $AB$ gives rise to an E.M.F. $Xi$ in the arm $AB$ and a current $\lambda Xi$ through the galvanometer. Thus

$$\lambda Xi = k\theta,$$

and (137) follows by division of the last two equations.

(3) *Comparison of the self-inductance of a coil with the capacity of a condenser.*

The self-inductance $L$ is set up in one arm $AD$ of a Wheatstone's bridge, and the capacity $C$ (of the order of a microfarad) placed as

a shunt across the opposite arm $BC$ (Fig. 167). The bridge is first balanced for steady currents, and the balance must be made very exactly, if necessary by shunting round about 5 ohms of one of the resistances with an auxiliary box. On reversing there will in

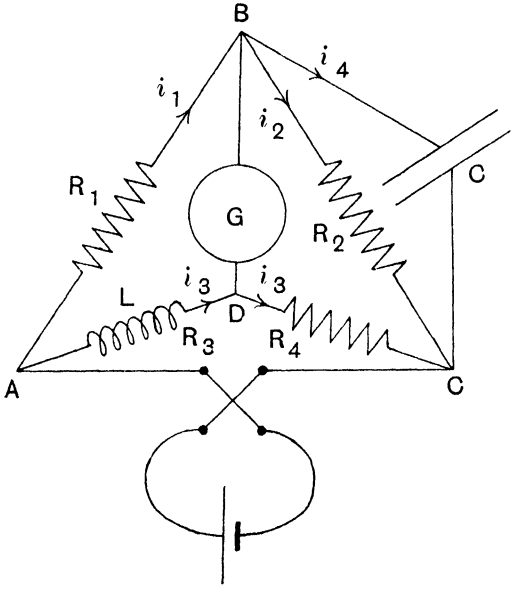



Fig. 167

general be a ballistic throw, but this can be made to disappear by suitable choice of the absolute values of the resistances, always keeping their ratios so that the bridge is balanced for steady currents.

Let us find the condition that no current should pass through the galvanometer whatever occurs in the battery circuit. The currents in the arms being as shown in the figure, we have

$$L\frac{di_3}{dt} + R_3 i_3 = V_A - V_D = V_A - V_B = R_1 i_1,$$

$$V_B - V_C = R_2 i_2 = R_4 i_3, \quad i_4 = C\frac{d}{dt}(V_B - V_C) = CR_4\frac{di_3}{dt}.$$

Substituting in the first equation, we have

$$Li_4 = CR_4(R_1 i_1 - R_3 i_3) = CR_1 R_4 i_1 - CR_2 R_3 i_2 = CR_1 R_4 (i_1 - i_2).$$

But since $i_4 = i_1 - i_2$ we have

$$L = CR_1 R_4 = CR_2 R_3 \quad \ldots\ldots\ldots\ldots (138).$$

This gives $L$ when $C$ is known.

**127. Measurement of magnetic fields with the ballistic galvanometer.** The arrangements are somewhat different according as we wish to measure (1) fields of the order 5000 gausses, such as those produced between the poles of an electromagnet, (2) weak fields such as that of the earth. Fig. 168 shows the connexions for performing both measurements in one circuit. *MM* represents the electromagnet, through which a known current can be sent. A small coil of about 100 turns of thin wire lies between its poles, held loosely in a clamp so that it can be knocked out of the region of strong magnetic force. There is also in circuit an *earth inductor*, or large coil capable of being turned through two right angles round an axis. The coil should have about 1000 turns, and a diameter of at least 20 cm. *G* is a sensitive ballistic galvanometer and *R* a resistance sufficiently large to render the damping of the galvanometer small (cf. Art. 130).

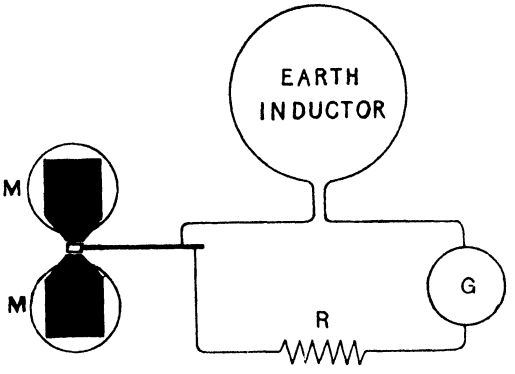


Fig. 168

The electromagnet, galvanometer and ammeter should be at some distance from the earth inductor in order to avoid disturbing the earth's field more than necessary. With the electromagnet cut off first set the plane of the earth inductor magnetically east and west by the aid of a compass-needle. Then turn the coil of the inductor quickly through two right angles, producing a ballistic throw $\tau_1$. Let $A$ be the mean area of the coil, $m$ the number of turns, and $R$ the total resistance in the circuit, inclusive of the two coils and the galvanometer. If $N$ is the flux of magnetic force through the circuit at any time and $L$ its self-inductance, the current $i$ is given by

$$L\frac{di}{dt} + Ri = -\frac{dN}{dt}.$$

Initially $N = mAH$, where $H$ is the earth's horizontal component, and finally $N = -mAH$. Hence we have by integration with respect to $t$

$$2mAH = R\frac{T}{2\pi}k\tau_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (139).$$

where $T$ is the time of swing and $k$ the galvanometer constant. A similar observation of the throw $\tau_2$ when the earth inductor is supported horizontally gives the vertical component $V$, by the formula

$$2mAV = R\frac{T}{2\pi}k\tau_2 \quad \ldots\ldots\ldots\ldots\ldots (140).$$

Obtain a throw $\tau_3$ by knocking out the small coil quickly from between the poles of the electromagnet. If $B$ is the mean area of the small coil, $n$ the number of its turns, the field $H'$ between the poles is given by

$$nBH' = R\frac{T}{2\pi}k\tau_3 \quad \ldots\ldots\ldots\ldots\ldots (141).$$

The galvanometer constant $k$ being determined by standardising (Art. 73), $H$, $V$ and $H'$ are known from the last three equations. If everything is measured, as usual, in derived electromagnetic units, the field-strengths will have to be multiplied by $10^8$ to bring them to gausses.

There is a general theorem covering the use of the ballistic galvanometer in all these cases, namely that if the flux of magnetic induction through the circuit due to external causes changes from $N_1$ to $N_2$, the total charge passing round the circuit is

$$e = (N_1 - N_2)/R \quad \ldots\ldots\ldots\ldots\ldots (142).$$

For we have $L\,di/dt + dN/dt + Ri = 0$, and $i$ vanishes both initially and finally. Hence on integration we find

$$(N_1 - N_2) + R\int i\,dt = 0,$$

which is equivalent to (142).

**128. Induction with permanent magnets.** The law of electromagnetic induction enables us to find out how the flux of induction through a small coil varies as the coil is moved about, and hence to find the value of $B$ at various points in the field of a magnet. We shall consider the variation of $B$ along the axis of a cylindrical bar-magnet. A small coil of $n$ turns and resistance $R$ is wound on a bobbin sliding over the magnet, and connected in series with a resistance box and ballistic galvanometer so that the total resistance of the circuit is $R + G$. Let $A$ be the area of the cross-section of the magnet, and let a ballistic throw $\tau$ occur when the coil is suddenly moved along from a place where

the magnetic induction has the value $B_0$ to one where it has the value $B_1$. Then

$$nA\,(B_1 - B_0) = \frac{T}{2\pi}(R + G)\,k\tau \quad \ldots\ldots\ldots(143).$$

The magnet to be tested is held in a clamp with the exploring coil slipped over it, and a wooden rod of the same diameter is held in the same line with the magnet with another clamp. In this way the measurement of $B$ (which then becomes $H$) can be extended beyond the end of the magnet until its further changes are inappreciable. When the galvanometer is once standardised, equation (143) gives the difference in the values of $B$ at any two points; and since $B$ vanishes at a great distance from the magnet it is known at all points along the magnet. In practice, the values thus obtained are not quite accurate since the coil is slightly larger than the magnet and also has an appreciable length. But they indicate a continuous falling off of the flux of induction as we move away from the centre of the magnet, as we should expect from our knowledge of the distribution of the lines of magnetic induction (Art. 117). The following table gives the result of an experiment on a cast steel magnet (length 20 cm., diameter 6·5 mm.). If derived units are used throughout the measurements, the values of $B$ obtained from equation (143) must be multiplied by $10^8$ to reduce them to true electromagnetic units, in which they are usually expressed.

| Distance from centre of magnet (centimetres) | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|---|
| $B$ | 6570 | 6360 | 5750 | 4640 | 2910 | 440 | 30 |

If $B_1 - B_0$ is known beforehand, an experiment of this kind gives the constant $\frac{T}{2\pi}k$ characteristic of the ballistic galvanometer. The principle of Hibbert's *magnetic standard* for this purpose is shown in Fig. 169. The outer shell is a piece of cast iron $CC$ in cup form. A steel rod $R$ is firmly screwed into this frame, and carries a cast iron plate $PP$ separated from the shell only by a narrow annular gap. The whole is magnetised by a current sent round the steel rod, so that lines of magnetic induction run up the rod and

across the narrow gap, returning through the outer shell. A coil is then wound horizontally on a sliding cylinder, so that it can fall through the gap after having been raised to a definite position. The value of $B_1 - B_0$ is thus always the same for a particular instrument.

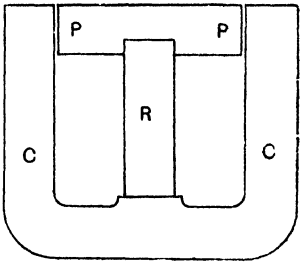


Fig. 169

Ballistic galvanometers can also be standardised by means of two coils of known mutual inductance, the current in one being reversed. An observation of the initial current is then required as well. The method of the magnetic standard is the simplest both in theory and practice.

**129. The ($H$, $B$) curve for an iron ring.** It is now evident that the value of $B$ in any given case can be ascertained experimentally without previous knowledge of that of $I$. We proceed to show how to take off a hysteresis curve for a specimen of iron, showing the relation of $B$ to the magnetising force $H$. The specimen to be examined is made in the form of a ring so that there are no disturbing ends, the magnetising coil consisting of a solenoid wound completely over it. Let $m$ be the number of turns, $r$ the mean radius of the ring. Then the magnetising force for a current of $i$ amperes or $i/10$ electromagnetic units is, from Art. 83,

$$H = mi/5r \quad \ldots\ldots\ldots\ldots\ldots\ldots (144).$$

Comparatively thin rings should be used in order that the magnetic force may not vary much over the cross-section.

The primary circuit contains, in addition to an ammeter and

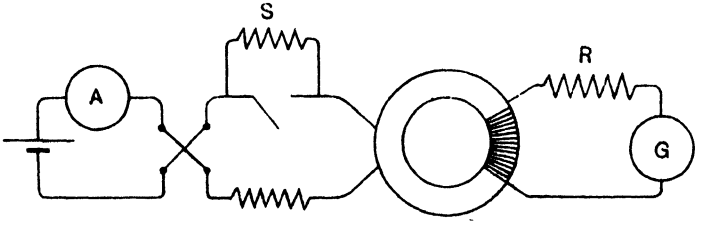


Fig. 170

rough adjustable resistance, a resistance box $S$ capable of carrying high currents and shunted across a key (Fig. 170). The secondary

consists of $n$ turns of wire wound over the ring, in series with a resistance box $R$ and ballistic galvanometer. Let $A$ be the cross-section of the core, $\tau$ the ballistic throw consequent on a sudden change of induction from $B_0$ to $B_1$. Then the change of induction in absolute units is given by

$$nA\,(B_1 - B_0) = 10^8 \frac{T}{2\pi}\,(R + G)\,k\tau \ldots\ldots\ldots (145).$$

The method of obtaining the points on the curve is best understood by reference to the complete curve, supposed already drawn (Fig. 171). We can tell whether $H$ is positive or negative by a glance at the reversing key, while the direction of the ballistic throw tells us whether $B$ is increasing or decreasing. Send first of all enough current through the solenoid to saturate the iron (point $P$). Then the throw produced by reversal (from $P$ to $Q$) gives $2B_P$ if we suppose the curve symmetrical. Starting from $P$ we can find the change of induction produced by a sudden diminution of the current from its saturation value. Place a resistance in the shunt box $S$ (Fig. 170) and knock out the key. Read the throw and the ammeter (point $\alpha$). Since $B_\alpha - B_P$ is known and also $B_P$, we can find $B_\alpha$. Reverse the current with the key, and again observe the throw. Since $B_\beta - B_\alpha$ is known and also $B_\alpha$, $B_\beta$ is known. Proceed similarly to the point $Q$ by pressing down the shunt key, then to $\gamma$ by knocking it out, to $\delta$ by reversal, and back to $P$. This gives a complete set of six points corresponding to a particular current, and any other set can be found by taking other resistances in the shunt circuit. Check the accuracy of the experiment by adding up the throws

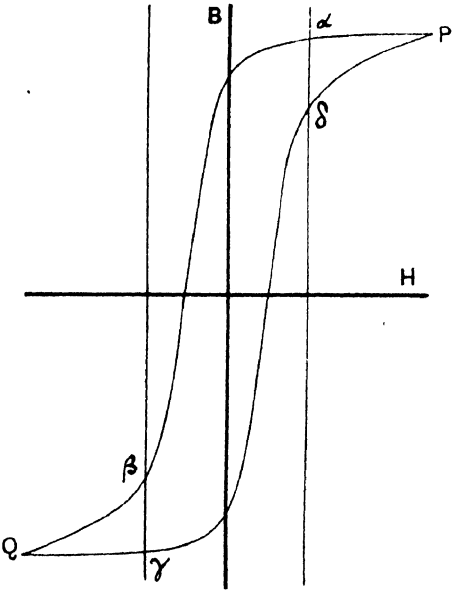
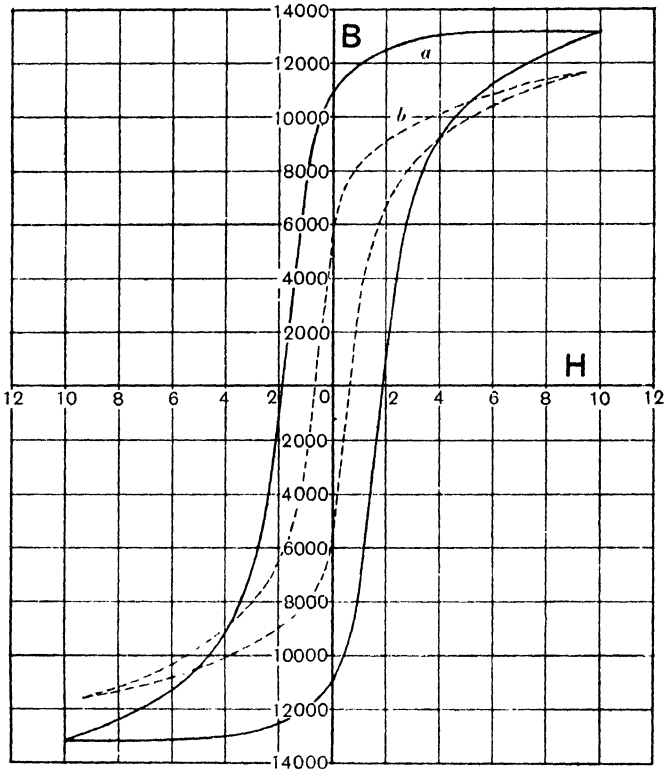


Fig. 171

from $P$ to $Q$ and $Q$ to $P$, which should give the same throw as that obtained by direct reversal of the saturation current.

Fig. 172 shows the hysteresis curves for specimens of soft iron




($a$) soft iron; ($b$) silicon transformer iron.

Fig. 172

and silicon transformer iron*. The maximum values of $B$ and the residual magnetism of various materials have already been given (Art. 111).

**130. Damping of galvanometers by induction.** When a ballistic moving-coil galvanometer is contained in a circuit, the movement of the coil induces currents in the circuit, and the action of the field on these currents tends to bring the coil to rest. Let the coil be of $n$ turns on a core of cross-section $A$, making small swings in a uniform magnetic field **H**. Let $I$ be the moment of

* I am indebted to Mr F. Stroud for this curve.

inertia of the coil about the suspending fibre, $\theta$ the small angle which the plane of the coil makes with **H** at time $t$, $\mu\theta$ the torsional couple due to the fibre, $f d\theta/dt$ a retarding couple due to air friction. If $N$ is the total flux of magnetic force through the coils, the couple on the coil is $i\,dN/d\theta$ due to the magnetic field, and its equation of motion is

$$I\frac{d^2\theta}{dt^2} = -\mu\theta - f\frac{d\theta}{dt} + i\frac{dN}{d\theta}.$$

The effect of self-induction in the coil is generally negligible, and $Ri = -dN/dt$, where $R$ is the resistance of the whole circuit. Since $N$ is approximately equal to $nAH\theta$, we have

$$I\frac{d^2\theta}{dt^2} + f\frac{d\theta}{dt} + \mu\theta = inAH = -\frac{n^2A^2H^2}{R}\frac{d\theta}{dt}.$$

The effect, therefore, of closing the galvanometer through a circuit of resistance $R$ is to increase the damping coefficient $f$ by an amount $n^2A^2H^2/R$, which is large when $R$ is small. The moving-coil galvanometer, which is by nature fairly ballistic, is made dead-beat by the addition of an extra winding, short-circuited and insulated from the main coil. A galvanometer which has no such addition comes to rest quickly enough when its terminals are short-circuited. Fig. 173 shows a way of connecting a ballistic galvanometer to a circuit, by means of a two-way key which allows $B$ to be joined to $A$ or $C$ at will. When $BC$ is joined the galvanometer is short-circuited for reading the zero, and when $AB$ is joined it is in circuit. The larger the resistance in circuit the less the damping, and considerable resistance may be wanted if we are to apply the damping correction of Art. 89 correctly. In measuring the time of swing of a galvanometer it is best to set the coil moving through a large arc and then leave the oscillations to go on on open circuit.

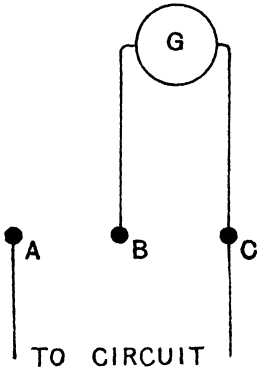


Fig. 173

An interesting application of the principle of electromagnetic damping occurs in the Grassot Fluxmeter, which is essentially a moving-coil galvanometer with a very weak restraining fibre, so that the predominant force on the coil is that due to the fixed magnets. Any change in the flux magnetic induction $N$ through

the external circuit sets the coil in motion, and the electromagnetic damping subsequently brings it to rest in a new position. It can be shown that the angle turned through by the coil is approximately proportional to the total change of flux. The equation of motion of the coil is $Id^2\theta/dt^2 = inAH$. On integration with respect to $t$ we have $\int i\,dt = 0$, since $d\theta/dt$ vanishes both initially and finally. The law of electromagnetic induction gives

$$\frac{dN}{dt} + L\frac{di}{dt} + nAH\frac{d\theta}{dt} + Ri = 0,$$

and integrating once more with respect to $t$ we have

$$N_2 - N_1 = nAH\,(\theta_1 - \theta_2),$$

where $\theta_1$, $\theta_2$ are the initial and final deflexions.

**131. Alternating currents and potentials.** The behaviour of coils with self and mutual induction is conveniently studied with currents varying periodically with the time, generally called alternating currents. The simplest alternating current, called a *sinoidal* current, is given by an expression of the form $i = I\cos(pt - \alpha)$, $I$ being called the *amplitude*, and similarly the simplest alternating potential is given by

$$v = V\cos(pt - \alpha).$$

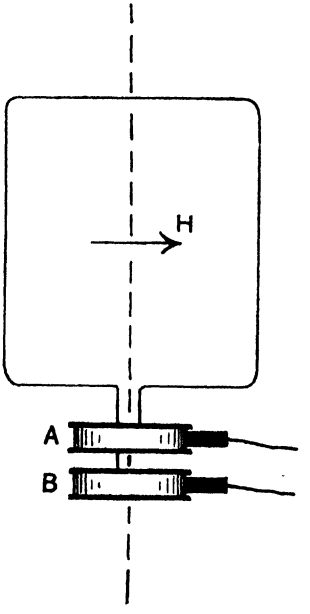


Fig. 174

The principle of the generation of alternating potentials will be understood from Fig. 174. A coil of wire, capable of rotating about an axis, is rigidly connected to two *slip-rings* $A$, $B$ on the same axle. These are solid rings of copper fastened to the ends of the coil and insulated from one another. Suppose now that the whole rotates with uniform angular velocity $\omega$ in a uniform transverse field $H$. If the plane of the coil makes an angle $\omega t$ with the direction of $H$ at time $t$, the flux of force through it is $N = HA\sin\omega t$, where $A$ is the area of the loop, and the E.M.F. of electromagnetic induction

$-\omega HA \cos \omega t$. This is therefore the difference of potential between $A$ and $B$ on open circuit. Strips of carbon (called brushes) can be pressed against the slip-rings and attached to wires for utilising the alternating potential generated in this way.

If the field is not uniform the potential difference is periodic, but not sinoidal. We shall define an important quantity in connexion with such currents and potentials. An alternating current is said to have the *effective value* $I$ when its heating effect in a given coil is the same as that of a steady current $I$. If $i$ is the value of the alternating current at time $t$, the heating effect in a coil of resistance $R$ in time $dt$ is $Ri^2dt$, and in a whole period $R\int_0^T i^2dt$. This must however be equal to $RI^2T$ by definition. Hence the effective current $I$ is given by

$$I^2 = \frac{1}{T}\int_0^T i^2dt \quad \ldots\ldots\ldots\ldots\ldots\ldots (146),$$

and for this reason is often called the root-mean-square (R.M.S.) value of the current. Similarly the effective value of an alternating E.M.F. is given by

$$V^2 = \frac{1}{T}\int_0^T v^2dt \quad \ldots\ldots\ldots\ldots\ldots\ldots (147).$$

If $i$ is a mixture of "harmonics" of different orders (Art. 156) of the form

$$i = a_0 + a_1 \cos pt + b_1 \sin pt + \ldots + a_n \cos npt + b_n \sin npt + \ldots,$$

$$I^2 = a^2 + \tfrac{1}{2}(a_1^2 + b_1^2) + \ldots + \tfrac{1}{2}(a_n^2 + b_n^2) + \ldots.$$

We shall see shortly that the effective values of alternating currents and voltages can be read off directly on alternate current measuring instruments. The effective value of a sinoidal current of amplitude $I$ is $I/\sqrt{2}$, and similarly for the voltage. Hence an alternating current voltmeter applied to a sinoidal E.M.F. of amplitude 100 only indicates 70·7.

### 132. Examples of alternating current circuits.

(1) *Circuit containing self-inductance and resistance.* Let an E.M.F. $V \cos pt$ be applied to a circuit of self-inductance $L$ and resistance $R$. Then the current $i$ is given by the equation

$$L\frac{di}{dt} + Ri = V \cos pt \quad \ldots\ldots\ldots\ldots (148).$$

The complementary function is $Ae^{-Rt/L}$, and diminishes rapidly with increase of $t$. Hence after some time the current will be given entirely by the particular integral

$$i = V\frac{R\cos pt + Lp\sin pt}{L^2p^2 + R^2} \quad \ldots\ldots\ldots(149).$$

This may be written $i = I\cos(pt - \alpha)$ .....(150),

where $$I = \frac{V}{(L^2p^2 + R^2)^{\frac{1}{2}}}, \quad \tan\alpha = \frac{Lp}{R} \quad \ldots\ldots\ldots(151).$$

Any given stage in the current $i$ occurs later than the corresponding stage in the applied voltage by a time $t$ given by $pt = \alpha$: the angle $\alpha$ is called the *lag* of the current behind the voltage.

The work expended in a complete period $T$ is $\int_0^T vi\,dt$, where $v$ is the instantaneous value of the applied voltage. Hence the work performed per second by the applied voltage, or the *power*, is

$$W = \frac{1}{T}\int_0^T vi\,dt \quad \ldots\ldots\ldots(152).$$

The formula applies to any periodic current, whether sinoidal or not. Here $v = V\cos pt$ and $i$ is given by equation (150), so that

$$W = VI\int_0^T \cos pt\cos(pt - \alpha)\,dt = \tfrac{1}{2}VI\cos\alpha.$$

If $V$ and $I$ are understood to mean effective values, the last equation is simply $W = VI\cos\alpha$ .....(153).

It is easy to show that the work $W$ is entirely expended in heating the circuit, and that none is done in the long run against electromagnetic induction. Equation (151) shows that inductive circuits allow less current to pass than non-inductive circuits of equal resistance. The quantity $(L^2p^2 + R^2)^{\frac{1}{2}}$, which determines the amplitude of current for a given voltage, is called the *impedance* of the circuit, and increases with the frequency.

(2) *Circuit containing capacity and resistance.* Let $C$ be the capacity, $R$ the resistance. Then if $e$ is the charge on the outer plate of the condenser in Fig. 175, the current is $i = -de/dt$ and the difference of potential between the plates $e/C$. Thus

$$V_A - V_B = -\frac{e}{C}, \qquad V_B - V_C = Ri = -R\frac{de}{dt},$$

and $$V_A - V_C = V\cos pt.$$

By addition,
$$RC\frac{de}{dt} + e = -VC\cos pt \quad \ldots\ldots\ldots(154).$$

The particular integral of this equation is
$$e = -\frac{VC}{R^2C^2p^2+1}(\cos pt + RCp\sin pt),$$
so that
$$i = \frac{VCp}{R^2C^2p^2+1}(RCp\cos pt - \sin pt).$$

This can be written in the form $i = I\cos(pt-\alpha)$, where
$$I = \frac{VCp}{(R^2C^2p^2+1)^{\frac{1}{2}}}, \quad \tan\alpha = -\frac{1}{RCp} \quad \ldots\ldots(155).$$

The angle $\alpha$ is essentially negative, so that the current *leads* the applied E.M.F. The impedance of the circuit is $(R^2 + 1/C^2p^2)^{\frac{1}{2}}$, which decreases as the frequency increases. At frequency 50 a condenser of 1 microfarad capacity has an impedance of 3000 ohms, but at frequency $10^6$ one of 0·16 ohm only. The condenser may let through considerable current at this frequency (by electrical influence, not by conduction).

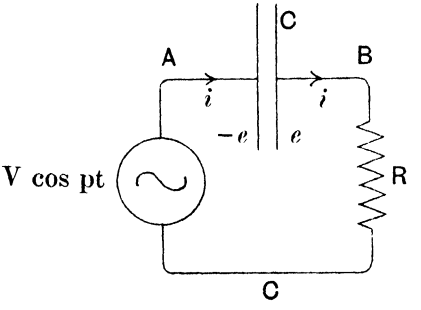


Fig. 175

(3) *Circuit containing capacity, self-inductance and resistance.* This case includes (1) and (2). If the coil in Fig. 175 has self-inductance $L$ as well as resistance $R$, the equation $V_B - V_C = Ri$ is replaced by $V_B - V_C = L\,di/dt + Ri$, and (154) by
$$LC\frac{d^2e}{dt^2} + RC\frac{de}{dt} + e = -VC\cos pt \quad \ldots\ldots(156).$$

The particular integral is $i = I\cos(pt-\alpha)$ as before, where
$$I = \frac{V}{\left\{\left(Lp - \frac{1}{Cp}\right)^2 + R^2\right\}^{\frac{1}{2}}}, \quad \tan\alpha = \frac{Lp - \frac{1}{Cp}}{R} \ldots(157).$$

The discussion of the free and forced currents in the system will occupy us in Ch. XI.

**133. Alternating current ammeters, voltmeters and wattmeters.** One type of alternating current ammeter is constructed on the principle of the electrodynamometer (Art. 93). Suppose that the permanent magnets of a moving-coil galvanometer are replaced by a fixed coil of wire, and that the same current $i$ flows through the fixed and moving coils. Let $I$ be the moment of inertia of the moving coil about its axis, $\mu\theta$ the restraining couple due to the suspension for a deflexion $\theta$, $\lambda i^2$ the couple due to the current $i$ passing through the coils. The equation of motion of the coil is

$$I\frac{d^2\theta}{dt^2} = \lambda i^2 - \mu\theta \quad \ldots\ldots\ldots\ldots (158).$$

Integrate the last equation from $t = 0$ to $t = T$. The left-hand side vanishes since the motion is periodic, and after division by $T$ we have

$$\frac{\lambda}{T}\int_0^T i^2 dt = \frac{\mu}{T}\int_0^T \theta\, dt.$$

The coefficient of $\lambda$ is $I^2$, where $I$ is the effective value of the alternating current; and the coefficient of $\mu$ is $\bar{\theta}$ the mean deflexion of the coil. Hence $\lambda I^2 = \mu\bar{\theta}$. In practice the moment of inertia is so great that the small oscillations about the mean position are inappreciable. Hence we may replace $\bar{\theta}$ by the observed deflexion $\theta$ and write

$$\lambda I^2 = \mu\theta.$$

Equation (158) shows that we should obtain the same equation if a *steady* current $I$ passed through the coils. Thus *if an electrodynamometric ammeter is graduated with direct current, it will read effective values when used on alternating current.*

One type of alternating current voltmeter consists of an electrodynamometer in series with a large non-inductive resistance, and the preceding remarks apply without modification. It is obvious from the definition that all thermal ammeters and voltmeters read effective values on alternating current circuits: and almost all accurate alternate current instruments belong to one or other of these types. An ordinary moving-coil galvanometer cannot be used with alternating currents: for its mean deflexion is proportional to $\int_0^T i\,dt$, so that it only measures the direct-current component, which is usually zero.

The *wattmeter* is an instrument giving the power consumed in

any part of a circuit, for example a coil, by a single reading. If $v$ is the difference of potential at time $t$ between the ends of the coil and $i$ the current in the circuit, this power is $\frac{1}{T}\int_0^T vi\,dt$ watts. The ends of the coil are joined to the "voltage terminals" of the wattmeter (Fig. 176), which go to a moving coil mounted between bearings like that of a millivoltmeter, and the "current terminals" are placed in the main circuit so that the current $i$ flows through a fixed coil taking the place of the permanent magnets. The current in the moving coil is proportional to $v$ at any time, and the couple between the two coils proportional to $vi$. A simple extension of the preceding theory shows that the deflexion of the moving coil is determined entirely by the quantity $\frac{1}{T}\int_0^T vi\,dt$; that is, by the power consumed in the coil. The instrument is graduated with direct current, using steady voltages $v$ and currents $i$, applied from independent sources. For the practical use of the wattmeter see Arts. 157, 158. The theory does not allow for self-inductance in the voltage coil.

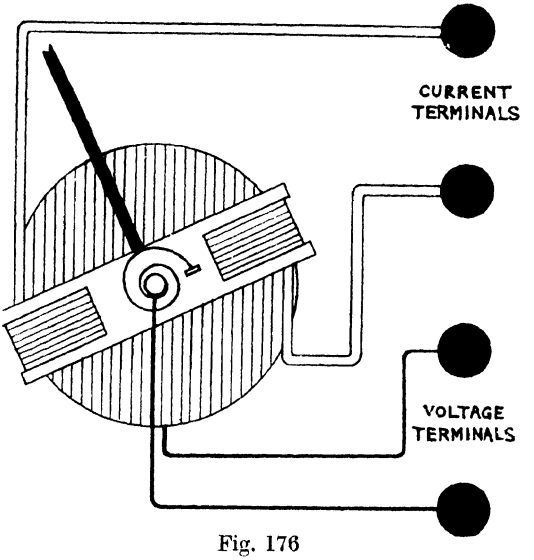


Fig. 176

The *telephone receiver* and the *vibration galvanometer* are used to detect small alternating currents. The principle of the former, which was invented by Graham Bell in 1877, is shown in Fig. 177. The current from the terminals $T$, $T$ passes through two coils wound over a horse-shoe magnet $MM$. Just in front of the magnet is a diaphragm $D$ of sheet iron. A changing current in the coils causes a fluctuation in the force with which the diaphragm is attracted, so that the latter is set into vibration. The instrument is most pleasant to use when the frequency of the impressed current

is near the middle of the musical scale. Low notes are difficult to pick up when there is noise in the room.

One form of vibration galvanometer consists of a ballistic moving-coil galvanometer of high natural period, which can be adjusted to be equal to that of the alternating current. A lamp and scale is used, and a band of light is seen on the scale, whose width corresponds to a greater or less amplitude of vibration of the moving coil. When the width is least, the current is either zero or a minimum. The vibration galvanometer is selective, and can be used when the applied alternating E.M.F. is far from sinoidal.

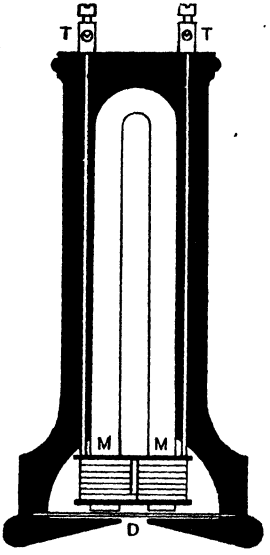


Fig. 177

**134. Measurement of inductance by alternating currents.** The following method is suitable for a coil $L$ of low resistance, and inductance 10 millihenrys or more. The source of alternating current is a rotary converter (Art. 154), which gives a fairly sinoidal E.M.F. The connexions are shown in Fig. 178. The self-inductance of the alternate-current ammeter $A$ is assumed to be small in comparison with that of the coil. Let $R$ be the sum of the resistances of the coil and ammeter. Then if $V, I$ are the readings of the voltmeter and ammeter respectively, and $2\pi/p$ is the periodic time of the simple harmonic E.M.F.,

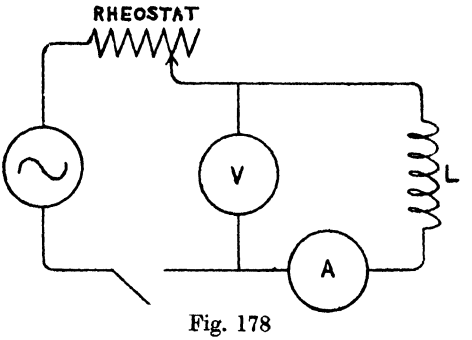


Fig. 178

$$V^2 = I^2 (L^2p^2 + R^2),$$

for equation (151) holds when effective values are used instead of amplitudes. After measuring $V$ and $I$ replace the source of alternating potential by an accumulator, and adjust the rheostat till the ammeter reading is the same as before. If $V_0$ is the new voltmeter reading, $V_0^2 = R^2I^2$. Hence

$$L^2p^2I^2 = V^2 - V_0^2 \quad \ldots\ldots\ldots\ldots (159).$$

The frequency $p/2\pi$ of the alternating current is either equal to, or a known multiple of, the number of revolutions of the rotary converter per second; or it may be measured on a frequency-meter (Art. 154). Hence $p$ is known, and equation (159) gives $L$ in henrys.

Mutual inductances can also be measured in this way. If two coils of self-inductance $L_1$, $L_2$ are joined in series, they form a coil of self-inductance $L_1 + L_2 + 2M$, for the flux through the first coil due to unit current is $L_1 + M$ and that through the second coil is $L_2 + M$. If one of the coils is turned over without changing the connexions, the self-inductance becomes $L_1 + L_2 - 2M$. Hence two measurements with the compound coil give $M$ and $L_1 + L_2$, and a comparison of the latter number with that obtained from the separate coils checks the accuracy of the experiment.

Alternate current measurements are not so accurate as those with direct current, for several reasons. In the first place, there is nothing in alternating current work to compare in steadiness with the current given by an accumulator. Alternating currents always vary slightly in frequency or amplitude and the readings of the instruments are never quite steady. Direct current galvanometers are sensitive, so that they take very little current when used as voltmeters and have a low resistance when shunted as ammeters. Electrodynamometric instruments require more current, and low-reading A.C. voltmeters often take an appreciable fraction of the current in the circuit. Low-reading wattmeters also absorb appreciable power, which makes the power actually consumed in a coil somewhat less than that indicated by the wattmeter. Electrodynamometers are by no means free from self-induction, and for high frequencies the readings of thermal galvanometers are more reliable.

**135. The generalised Wheatstone's bridge.** We shall here consider a generalisation of Wheatstone's bridge (Fig. 179) which covers all cases of experimental application. The current is alternating and the detector across $BD$ a telephone or vibration galvanometer. Let the arm $AB$ contain a condenser of capacity $C_1$ in series with a coil of self-inductance $L_1$ and resistance $R_1$, and further let the condenser have a finite internal resistance $\rho_1$, as it would if

filled with a conducting liquid. If no current flows in $BD$, $i_1 = i_2 = I$ (say), $i_3 = i_4 = J$. Let all the variable quantities be proportional to $e^{ipt}$, where the real part of the expressions is taken when wanted for physical interpretation. Let $A$, $B$, $C$, $D$ denote the potentials at the four corners of the bridge, $\pm e_1$, $\pm e_2$, $\pm e_3$, $\pm e_4$ the charges on the plates of the condensers at time $t$.

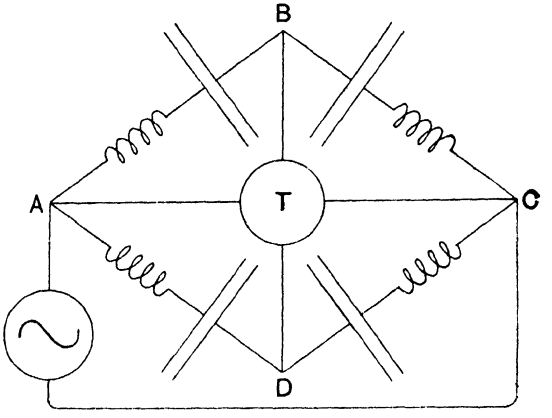


Fig. 179

The difference of potential between the plates of $C_1$ being $e_1/C_1$, the internal current in the condenser is $e_1/\rho_1 C_1$. Hence

$$I = de_1/dt + e_1/\rho_1 C_1,$$

and

$$A - B = L_1 \frac{dI}{dt} + R_1 I + \frac{e_1}{C_1}.$$

Replacing the operator $d/dt$ by $ip$, we have

$$e_1 = \frac{I}{\frac{1}{\rho_1 C_1} + ip}.$$

so that

$$A - B = I\left[iL_1 p + R_1 + \frac{1}{\frac{1}{\rho_1} + iC_1 p}\right] \quad \ldots\ldots (160).$$

The three similar equations can easily be written down. Remembering that $B = D$ we therefore have

$$\frac{iL_1 p + R_1 + \dfrac{1}{\frac{1}{\rho_1} + iC_1 p}}{iL_2 p + R_2 + \dfrac{1}{\frac{1}{\rho_2} + iC_2 p}} = \frac{iL_3 p + R_3 + \dfrac{1}{\frac{1}{\rho_3} + iC_3 p}}{iL_4 p + R_4 + \dfrac{1}{\frac{1}{\rho_4} + iC_4 p}}.$$

The conditions of balance* are obtained by equating real and imaginary parts of this equation, after cross-multiplication, to zero.

The results are simpler in special cases. The case in which there are no condensers can be treated either independently or by putting $\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0$, which gives

$$R_1 R_4 - R_2 R_3 = p^2 (L_1 L_4 - L_2 L_3),$$

and

$$R_1 L_4 + R_4 L_1 = R_2 L_3 + R_3 L_2.$$

These conditions will not be satisfied for all frequencies unless either $L_1/R_1 = L_2/R_2$, $L_3/R_3 = L_4/R_4$, or $L_1/R_1 = L_3/R_3$, $L_2/R_2 = L_4/R_4$, so that the bridge cannot be balanced by adjusting the resistances alone. In general all that can be obtained by a single adjustment is a *minimum* sound in the telephone.

An examination of the order of magnitude of the terms in equation (160) will tell us exactly what is being measured on the Wheatstone's bridge in the presence of various disturbing causes. For example, in measuring the resistance of electrolytes (Art. 170) the liquid is enclosed in what is really a parallel-plate condenser, so that we wish the term $1/\rho_1$ to be large in comparison with $C_1 p$, and at the same time $\rho_1$ to be large in comparison with both $L_1 p$ and $R_1$, so that $A - B$ shall be approximately equal to $\rho_1 I$. If, on the contrary, $1/\rho_1$ was small in comparison with $C_1 p$, we should be measuring the capacity of the condenser when filled with the liquid, that is the dielectric constant of the liquid, instead of its resistance. An example will make this clear. Let $C_1$ consist of two parallel plates 5 cm. in diameter, distant 2 cm. apart, and let $L_1 = 10^{-7}$ henry, $R_1 = 10^{-2}$ ohm (the order of the irreducible inductance and resistance of the connecting leads). If the condenser is filled with distilled water of conductivity $10^{-6}$ (cf. Art. 171), $\rho_1 = 1{\cdot}02 \times 10^5$ ohms, and since the dielectric constant of water is about 80, $C_1 = 7 \times 10^{-11}$ farads. When $p = 500$, $1/\rho_1$ is still large compared with $C_1 p$, and experiments on the Wheatstone's bridge would measure $\rho_1$. As $p$ rises the two terms become of the same order of magnitude, and the capacity term finally predominates for frequencies of the order $10^6$ (electric oscillations).

* The reader should notice why there are *two* conditions. If the calculation of the current in $BD$ had been performed quite generally by the ordinary methods, it would have appeared in the form $P \cos pt + Q \sin pt$. The conditions $P = Q = 0$ are therefore necessary if it is to vanish. The amplitude of current in all such cases is of the form $(P^2 + Q^2)^{\frac{1}{2}}$.

**136. The alternating current transformer.** The simplest type of alternating current transformer is Faraday's ring (Fig. 157) consisting of two coils wound on the same iron core. An alternating current flowing in the primary will induce currents in the secondary and give rise to an E.M.F. which in general differs from that applied to the primary. The iron core makes nearly all the lines of induction due to the primary current pass through the secondary, and *vice versâ*; but as the theory would be complicated by hysteresis we shall consider the following ideal circuit instead. Let two endless solenoids be closely interwound on the same ring so that the lines of force circulate only in the interior. Let $n_1$, $n_2$ be the number of turns of the two solenoids. The fluxes through the two solenoids, when unit current flows through one of them, are in the ratio $n_1 : n_2$, so that $L_1/M = M/L_2 = n_1/n_2$. Hence

$$L_1 L_2 = M^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (161).$$

This is the condition that no lines of induction shall escape between the two coils, or the condition of no magnetic leakage, as it is called.

Let an E.M.F. $V \cos pt$ be applied to the primary as shown in Fig. 180. The resistances of the two coils $L_1$, $L_2$ will be assumed

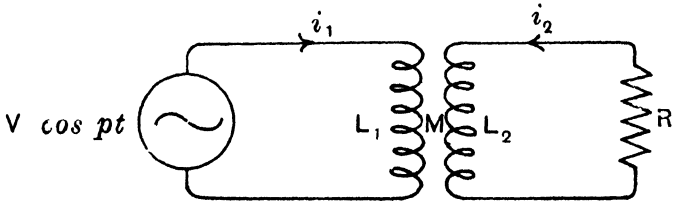



Fig. 180

to be negligible. Let the secondary circuit be completed by a non-inductive resistance $R$. The equations for the currents are

$$\left.\begin{aligned} L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} &= V \cos pt \\ M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} &= -Ri_2 \end{aligned}\right\} \quad \ldots\ldots\ldots (162).$$

Multiply the first equation by $M$ and the second by $L_1$. The left-hand sides become equal by virtue of (161), and we have

$$Ri_2 = -\frac{M}{L_1} V \cos pt \quad \ldots\ldots\ldots\ldots (163).$$

Now $Ri_2$ is the difference of potential between the terminals of the secondary coil, and $M/L_1 = n_2/n_1$. Hence the E.M.F. developed in the secondary is the fraction $n_2/n_1$ of that applied to the primary. If the secondary has many turns the E.M.F. developed in it may greatly exceed $V \cos pt$; if few turns it will be greatly less. We have thus a simple means of *transforming* alternating E.M.F.'s by electromagnetic induction, and this is the chief reason for the use of alternating currents in electrical practice.

The current in the secondary differs by a phase of 180° from the E.M.F. applied to the primary for all values of $R$. The primary current is found by substituting from (163) in the first of the equations (162) and integrating. Thus

$$i_1 = \frac{V}{L_1 R p}(L_2 p \cos pt + R \sin pt) \ldots\ldots\ldots (164).$$

For small values of $R$ the primary current lags nearly a quarter of a period behind the voltage, but the phase-difference diminishes as more and more current is taken off the secondary. This may be expressed, if desired, by saying that the effective inductance of the primary is diminished by the presence of the secondary.

**137. The induction coil.** The problem of obtaining high-voltage electricity from an ordinary battery was first successfully solved by Ruhmkorff, who, in 1851, produced sparks longer than those of an influence machine. His apparatus, which is the same in principle as that previously used by du Bois-Reymond, is called the *induction coil*. It depends on the fact that a large E.M.F. is induced in a secondary coil of many windings when the current in a neighbouring primary coil is suddenly cut off. Fig. 181 shows an induction coil for sparks of medium length, and Fig. 182 the internal connexions, with the exception of the secondary coil. The battery is connected to two terminals marked $P$ and $N$, and the primary circuit contains a reversing key $R$. The primary coil consists of about 300 turns of thick copper wire wound on an iron core $AB$ composed of a bundle of soft iron wires, the resistance of the coil being of the order of $\frac{1}{3}$ ohm. The hammer $H$ is a short iron cylinder attached to a stiff spring, carrying on the remote side a stout piece of platinum $P$. A similar piece $Q$ is fixed on the end of a screw so that its distance from $P$ is adjustable between certain

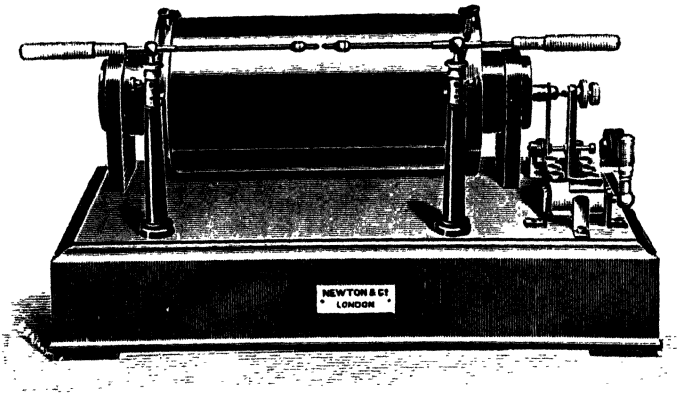



Fig. 181

limits, and the position of the hammer can also be adjusted by a screw $S$ which pulls it away from the core. A condenser $C$, of the order of 1 microfarad, is built into the base of the instrument and connected as a shunt across the platinum "break" $PQ$. The secondary coil consists of about 15 miles of thin copper wire having a resistance of from ten to fifteen thousand ohms. The different layers of the coil must be most carefully insulated from one another, which can be done by interposing a paper disc covered with shellac between each successive pair of layers. The secondary is enclosed in a thin ebonite cylinder and its ends brought out to two terminals to which pointed conductors are attached for testing the length of spark.

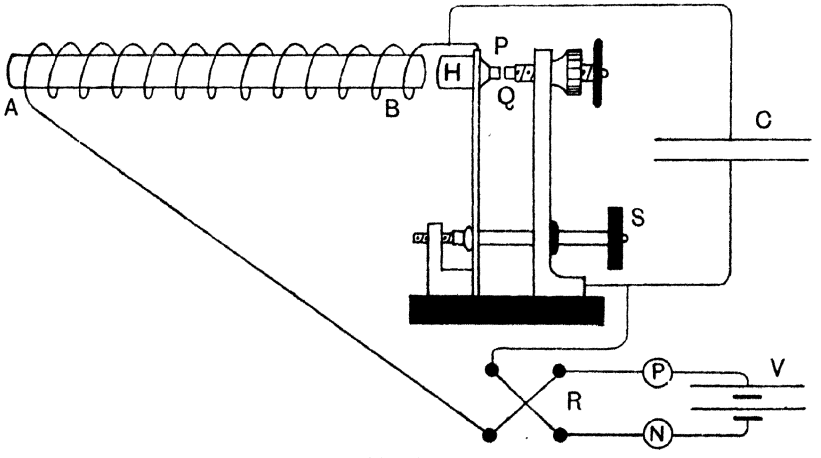



Fig. 182

In using an induction coil it is best to begin with the screw $S$ as far back as possible so that the hammer is near the core of the primary. The platinum point $Q$ is screwed up till it is about one millimetre from $P$, and the hammer then brought forward slowly by means of the screw $S$. The coil will begin to act as soon as $P$ and $Q$ touch, and slight sparking will take place at the break. Excessive sparking is undesirable as it destroys the platinum contacts, and occurs when the initial distance between $P$ and $Q$ is too great. The platinum contacts will become rough and uneven with use, and should be smoothed at intervals with a fine file. The E.M.F. applied depends on the use to which the coil is put. Thus 6–8 volts suffice for vacuum tube discharges and soft Röntgen ray bulbs, more for spark gaps at atmospheric pressure, particularly with condensers in parallel: but 16 volts is the most that should be put across the terminals proper.

The action of the coil is as follows. When $P$ and $Q$ touch, the circuit is completed and a current flows through the primary, rising to a maximum somewhat as shown in Fig. 163. The iron core $AB$ becomes magnetised and attracts the hammer $H$, thereby breaking the circuit. The attraction disappears with the current, the hammer springs back and again makes contact with $Q$, and so on.

When the circuit is broken part of the current continues to flow for a short time across $PQ$ as a spark, and a momentary current also flows into the condenser $C$, charging it up. The subsequent current in the condenser circuit is oscillatory, and the primary and secondary circuits influence each other (cf. Art. 182). In this way the core is effectively demagnetised in a short time after the breaking of the circuit. The change of induction on "break" is more rapid than that on "make," resulting in a much greater E.M.F. in the secondary.

If the secondary coil is closed by a circuit obeying Ohm's law, alternating currents flow in the circuit. For if $R$ is the total resistance of the secondary and $N$ the flux of magnetic induction through it due to all causes, the current at any time is given by $dN/dt + Ri = 0$. On integration between limits 0 and $T$, where $T$ is the time of oscillation of the hammer spring, we find $\int i\,dt = 0$, since $N$ resumes its original value. Hence the total charge passing any point of the secondary in a period vanishes, so that the current

must alternate in direction. This does not necessarily hold when the coil is used to give spark discharges or to work Röntgen ray bulbs. The E.M.F. induced on "make" does not suffice to break down the insulation of the air, and current flows only on "break." The current is then practically unidirectional, and can be reversed by the key *R*.

The hammer interrupter will not carry heavy currents, for which a mercury interrupter is used, on the principle of Fig. 183. A cylinder *C* has a nozzle at the lower end dipping into mercury, and also a side-tube *T* some way up. The nozzle is shaped so that when *C* is rotated rapidly the mercury rises and is ejected at *T*. The stream falls on a metal plate *P* bent in the form of a sector of a circle, so that the current between *C* and *P* is interrupted regularly whenever the cylinder reaches a certain position. The cylinder and reservoir are enclosed in an air-tight case filled with coal-gas. This interrupter can be worked off the electric mains, in series with a high-current rheostat.

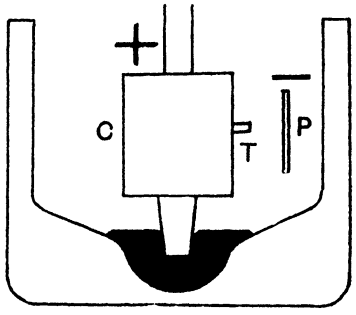


Fig. 183

**138. Elihu Thomson's experiments.** Let an iron rod, made up of a bundle of iron wires bound together, be inserted in a hollow solenoid so as to protrude from one end (Fig. 184), and an alternating current sent through the solenoid. If the rod is fixed vertically, an aluminium ring sliding over it will rise on applying the current and hang freely in the air. The position of equilibrium will be stable, for the ring will return to its position when either lifted or raised. Further, if the ring is held down by hand it will become in a very few seconds too hot to hold. The phenomenon ceases when the ring is cut across.


Fig. 184

These facts are all in accordance with theory and afford an interesting example of electromagnetic induction. The electromagnet gives a periodic field which, if of sine form, has the same phase at all points, the direc-

tion and amplitude varying from point to point. Let $L$ be the self-inductance and $R$ the resistance of the ring, and let the flux of magnetic induction through it at time $t$ be $N \cos pt$, where $N$ depends on the position of the ring. If $i$ is the current,

$$L\frac{di}{dt} + Ri = Np \sin pt,$$

giving
$$i = -\frac{LNp^2}{L^2p^2 + R^2}\cos pt + \frac{RNp}{L^2p^2 + R^2}\sin pt.$$

Since $i$, other things being equal, is proportional to $N$, the potential energy $W$ of the circuit is not $-iN$ but $-\frac{1}{2}iN$ (cf. Art. 45). Thus

$$W = \frac{LN^2p^2}{2(L^2p^2 + R^2)}\cos^2 pt - \frac{RNp}{2(L^2p^2 + R^2)}\cos pt \sin pt.$$

The ring, unless it is very light, will only respond to the average force on it, and this is derived from the potential energy

$$\frac{LN^2p^2}{4(L^2p^2 + R^2)},$$

the second term contributing nothing to the mean. In the absence of other forces the ring will be driven to the *weakest* parts of the field, where $N$ is least. If $z$ is the height of the ring in Fig. 184, the lifting force on it is

$$\frac{Lp^2}{2(L^2p^2 + R^2)}N\frac{dN}{dz}.$$

The mean rate of development of heat is the mean value of $Ri^2$ during a period, or

$$\frac{RN^2p^2}{2(L^2p^2 + R^2)}.$$

As an example consider an aluminium ring of 4 cm. diameter, the diameter of the cross-section being 4 mm. Then $L = 66$ cm., and $R = 3 \times 10^{-4}$ ohms $= 3 \times 10^5$ E.M. units. If the core is of soft iron, fully magnetised for the maximum current and of diameter 3 cm., $N = 100{,}000$ inside the solenoid. Suppose that the core protrudes 20 cm. from the solenoid, and that the distribution of magnetic induction along it is similar to that of the bar-magnet in Art. 128. The greatest value of $N\,dN/dz$ is half-way up, where it is about $3{\cdot}2 \times 10^8$, $N$ being about 80,000. Thus the greatest lifting force on the ring is about 2640 dynes, or 2·7 gm. weight.

Since the mass of the ring is 1·6 gm., it should be lifted, but a ring which is much thinner will not be lifted. The rate of development of heat is $2{\cdot}4 \times 10^8$ ergs or 5·7 calories per second. Since the specific heat of aluminium is 0·21, the temperature of the ring would rise about 17° per second until checked by radiation.

**139. Electronic theory of induced currents.** The electrical theory of matter throws much light on the induction of currents in moving conductors. Let $P$ be a point on a closed circuit $C$ of thin wire, $\mathbf{H}$ the magnetic force at $P$ and $PQ = d\mathbf{s}$ an element of arc. Let $\mathbf{u}$ be the velocity of the wire at $P$, $\mathbf{v}$ the mean velocity of the electrons relative to the wire. The total velocity of the electrons is $\mathbf{u} + \mathbf{v}$, and $\mathbf{H}$ acts on an electron near $P$ with a force

$$\frac{e}{c} V(\mathbf{u} + \mathbf{v})\mathbf{H} = \frac{e}{c} V\mathbf{u}\mathbf{H} + \frac{e}{c} V\mathbf{v}\mathbf{H}.$$

Since $e/c$ is the charge on the electron in electromagnetic units, there is an *apparent electric force* (in electromagnetic units) equal to

$$V\mathbf{u}\mathbf{H} + V\mathbf{v}\mathbf{H}.$$

The total E.M.F. in the circuit is therefore equal to the line-integral

$$\int_C (V\mathbf{u}\mathbf{H})\, d\mathbf{s} + \int_C (V\mathbf{v}\mathbf{H})\, d\mathbf{s}.$$

The element of the first integral is

$$-V(\mathbf{H}\mathbf{u})\, d\mathbf{s} = -\mathbf{H}(V\mathbf{u}\, d\mathbf{s})$$

from equation (9), and the second element $-\mathbf{H}(V\mathbf{v}\, d\mathbf{s})$, which vanishes since $\mathbf{v}$ is parallel to $d\mathbf{s}$. Hence the total E.M.F. induced in the circuit $C$ is

$$-\int_C \mathbf{H}(V\mathbf{u}\, d\mathbf{s}).$$

If the element $PQ$ (Fig. 185) moves to $P'Q'$ in time $dt$, the area $PP'Q'Q$ is exactly represented by $V\mathbf{u}\, dt, d\mathbf{s} = dt\, V\mathbf{u}\, d\mathbf{s}$, and the flux of magnetic force through it is its scalar product with $\mathbf{H}$. Thus the element $PQ$ is increasing the flux of magnetic force at the rate $\mathbf{H}(V\mathbf{u}\, d\mathbf{s})$, so that the integral $\int_C \mathbf{H}(V\mathbf{u}\, d\mathbf{s})$ is the total rate of increase of flux. Hence the E.M.F. is $-dN/dt$, as it should be.

The theory will not explain the currents induced in a stationary circuit by a variable magnetic field. This phenomenon shows that

a variable magnetic field is accompanied by an electric field, and receives its natural interpretation in terms of the ether (Art. 195).

Returning to moving circuits, let us now calculate the E.M.F. in Faraday's rotating disc (Fig. 158).

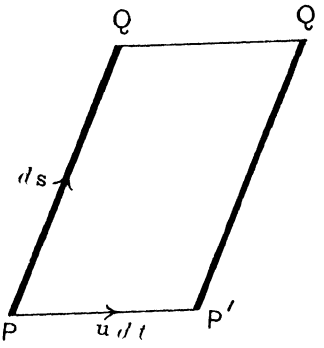


Fig. 185

Let $\omega$ be the angular velocity of the disc, $H$ the magnetic force at distance $r$ from the axis, supposed perpendicular to the plane of the disc. The field is not necessarily uniform, but will be assumed to be symmetrical about the axis, so that $H$ is a function of $r$ only. The average velocity of a group of electrons distant $r$ from the axis is $r\omega$ in a direction perpendicular to the radius vector, and hence the average mechanical force exerted by the field on an electron of the group is $Her\omega/c$, which is directed radially inwards since the charge on an electron is negative. In other words, there is an apparent electric force, or force per unit charge, of magnitude $Hr\omega/c$ in electrostatic units, or $Hr\omega$ electromagnetic units, radially outwards. If the disc is insulated there will be an accumulation of electrons near the axis, until the apparent electric force is compensated at every point by the electrostatic field due to the accumulated charges. A wire connecting the rim and axle will allow electrons to flow back again to the rim, so that the direction of the current is correctly accounted for by the theory.

The work done in moving a unit charge from the axle to the rim, or the difference of potential between the brushes in Fig. 158, is $\omega \int_c^b H r\,dr$ in electromagnetic units, where $b$ is the radius of the rim and $c$ that of the axle. Now the flux of magnetic force through the space between the rim and axle is $N = 2\pi \int_c^b H r\,dr$, and $\omega = 2\pi n$, where $n$ is the number of revolutions of the disc per second. Hence the induced E.M.F. is

$$V = nN \text{ electromagnetic units} \quad \ldots\ldots\ldots(165).$$

The effect of magnets in retarding the motion of neighbouring metallic masses, or of having their own motion retarded by their

presence, was examined by several experimenters prior to 1830. Thus Arago found in 1825 that a magnet suspended over a metal disc tends to follow the motion of the disc when the latter is rotated. Babbage and Herschel, who repeated Arago's experiments, found that the magnet ultimately came to rest making an angle $\theta$ with the meridian, where $\sin\theta$ was proportional to the angular velocity of the disc. The order of magnitude of the effect in different metals was also found to be that of the conductivities of the metals. The deflexions were much reduced when the continuity of the disc was destroyed by radial cuts extending nearly to the centre.

These effects were first adequately explained by Faraday in the two papers already cited, in which they were ascribed to currents induced in the metals by the relative motion of the disc and magnet. The electric *forces* are determined by the strength of the magnet and the speed of rotation; but the *currents* are smallest in discs of low conductivity, and their circulation is also hindered by radial cuts.

The heating effect of currents in metallic masses or eddy currents as they are called, was first observed by Foucault in 1855. Let a solid cylinder rotate in a field perpendicular to the axis (Fig. 186). The velocity at a point $P$ $(r, \theta)$ of a cross-section is $r\omega$ at right

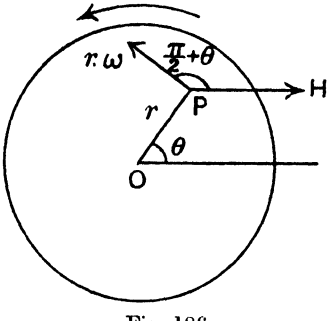



Fig. 186

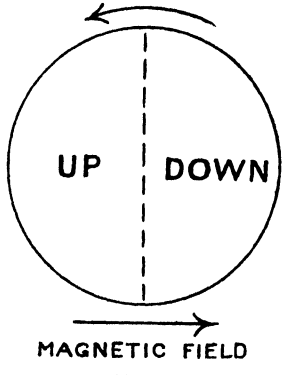



Fig. 187

angles to the radius vector, so that an electron at $P$ would experience a force $-er\omega H \sin(\frac{1}{2}\pi + \theta)$ parallel to the $z$-axis. Thus the induced electric force is of magnitude $-r\omega H \cos\theta$, measured vertically upwards. Fig. 187 shows roughly the way in which the currents

would move in the central sections of the cylinder, if the mutual influence of the currents themselves could be neglected. These currents are developed very much less in a thin disc, where there is but little opportunity for continuous circulation, the electric force simply tending to drive electrons towards the two flat faces. Hence if a metal cylinder is built up of discs parallel to the axis and insulated from each other, it can be rotated in a transverse magnetic field without giving rise to heavy eddy currents. This device is used in the armatures of electric generators, where excessive loss of energy would otherwise occur.

**140. Absolute measurement of resistance.** The resistance of a given coil can be determined in absolute electromagnetic units by a method due to L. Lorenz (1873), depending on the use of

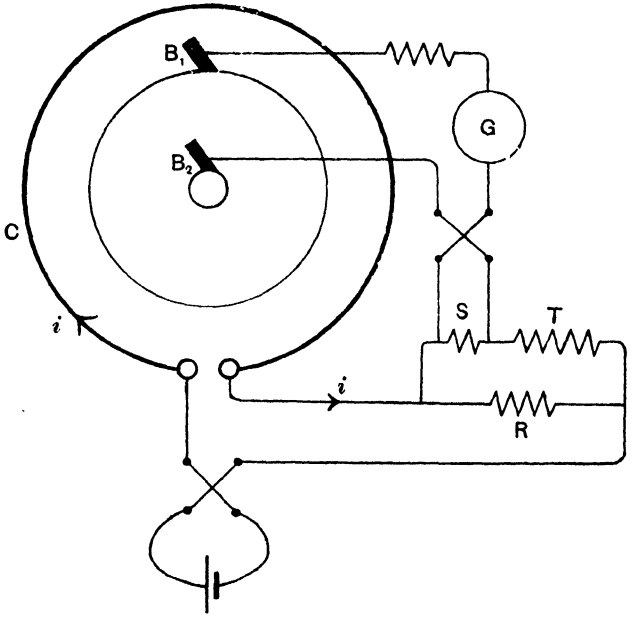



Fig. 188

Faraday's rotating disc. The following method is accurate enough for a laboratory experiment (Fig. 188).

$C$ is a circular coil of a large number $m$ of turns concentric with the rotating disc, $B_1$ and $B_2$ brass brushes making contact with the rim and axle of the disc respectively. $R$ is a standard coil whose

resistance is required in absolute electromagnetic units, $S$ and $T$ other resistances whose magnitudes are known in terms of $R$. The disc can be driven at a rapid and uniform speed of $n$ revolutions per second by a belt from a small electric motor, and is fitted with a revolution counter.

The galvanometer will be deflected when the disc spins, even when no current flows in the coil $C$, from flux due to the earth's magnetic field or from stray fields. This deflexion is taken as the zero and $T$ adjusted till it does not alter on reversing the current in the battery circuit. The fall of potential down $S$ due to the battery current is then equal to that generated by the rotation of the disc in the field of the coil $C$, so that equation (165) gives

$$\frac{RSi}{R+S+T} = nN.$$

Let $M_1$ be the mutual inductance of the coil $C$ and the rim of the disc, calculated for a single turn in the mean position of the turns of $C$, $M_2$ the same calculated for $C$ and the axle of the disc. Then $N = mi\,(M_1 - M_2)$, so that

$$R\frac{S}{R+S+T} = mn\,(M_1 - M_2) \quad \ldots\ldots\ldots(166).$$

$M_1$ and $M_2$ can be found from formulae already given (Art. 98); and since the ratios of $R$, $S$, $T$ are known $R$ is given by this equation in absolute electromagnetic units.

As far as the principle of the method is concerned, we might suppress the shunt circuit $S + T$ and shunt the Lorenz disc directly across the resistance $R$, in which case equation (166) becomes simply $R = mn\,(M_1 - M_2)$. But with an apparatus of ordinary size the requisite value of $R$ would be very small (of the order 1/1000 ohm), and also the only means of obtaining a balance would be to adjust the speed of rotation, which is not always convenient.

Other methods have been used to determine resistances in absolute measure, all depending on the principle of electromagnetic induction. Thus the ordinary method of measuring a mutual inductance consists in comparing it with a resistance $X$, in terms of which other resistances $R$, $S$, $G$ are supposed to be known (cf. equation (136)). But whereas in the laboratory $X$ would be taken as known in ohms, we might just as well calculate the

mutual inductance theoretically from Neumann's formula or otherwise, and use the experiment to determine $X$ in absolute measure.

The older experiments seemed to show that the ohm is the resistance, at 0° C., of a column of mercury of cross-section 1 mm.$^2$ and of length nearly equal to 106·3 cm. In practice the diameter of the containing tube of the mercury resistance would itself be determined by weighing a column of mercury, so that it is expedient to define by weight instead of by cross-section. The international conference held in London in 1908 defined the *international ohm* as the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice, 14·4521 gm. in mass, of a constant cross-sectional area and of a length 106·300 cm. It had previously been found that resistance standards consisting of mercury columns in glass were far more reliable than wire coils.

Of all these methods that of Lorenz is generally considered the best. The chief difficulties that have to be overcome are as follows.

(1) Secular changes in wire standards of resistance. While carefully constructed mercury standards can be relied on to about two parts in 100,000, the best resistance coils are liable to alter in time, the change in some cases exceeding one part in 5000. This seems to be chiefly due to the effect of moisture on the shellac used to cover the wire. Hermetically sealed coils are more reliable, but all coils should be compared frequently with mercury standards.

(2) Effects due to rapid variations of terrestrial magnetism.

(3) Thermoelectric effects at the sliding contacts of the brushes. Such effects are liable to sudden changes which, like the foregoing, render it difficult to obtain an accurate balance.

(4) The difficulty of keeping the angular velocity of the disc sufficiently uniform.

In F. E. Smith's experiments two discs were used, the currents in the two sets of coils circulating in opposite directions, so that the effect of terrestrial magnetism is eliminated. Thermoelectric effects were much reduced by this arrangement and by the use of a special form of brush resembling a violin bow, lubricated with petrol. The edges of the discs were placed in a region of zero magnetic force, so that a slight uncertainty as to their diameters should not affect the mutual inductance between them and the coils. The axle carried with it a commutator which charged and

discharged a condenser in one arm of a Wheatstone's bridge (cf. Art. 142); and the speed of rotation could be kept uniform by keeping the bridge balanced during a run.

Smith found that the international ohm is equal to 1·00052 absolute ohms, with an estimated probable error of 4 in the last place. It thus appears that the true ohm may be defined, to an accuracy of about one part in 25,000, as the resistance of a mercury column of mass 14·4446 grams and length 106·245 cm., the area of cross-section being nearly 1 mm.$^2$ as before.

A second accurate method of finding absolute resistance is to calculate the self-inductance of a coil theoretically, compare it with a capacity as in Art. 126, and place the condenser in a Wheatstone's bridge with commutator, as in Art. 142. The theory is contained in equations (138) and (172). Combining them we have $L$ expressed in terms of a frequency and resistances whose ratios are known with great accuracy, and thus the absolute standard is known. Grüneisen and Giebe found the international ohm to be 1·00051 times the absolute ohm, in close agreement with Smith's result.

**141. International and true electrical units. E.M.F. in absolute measure.** The true electrical units are defined in a general way without reference to the feasibility of any tests that may be implied. Thus the ampere is specified by its magnetic effect. The definition is then practically useful, as it amounts to supposing that all currents are measured with an electrodynamometer of known dimensions. The true volt is the difference of potential between two points $A$, $B$ such that $10^7$ ergs of work are required to move one coulomb from $A$ to $B$. Definition by energy is awkward, and in practice we are driven to take the ohm as the second fundamental unit, defining the volt as the difference of potential between $A$ and $B$ when the resistance is one ohm and a current of one ampere is flowing. Fortunately, the ohm can be determined in absolute measure independently of all other electrical units, as we have shown in the last article.

The *international* system of units recommended by the London Conference (1908) aims at being realisable in practice, and at the same time sufficiently near to the true units for all but the most

accurate experiments. The international ampere is probably correct to about one part in 4000, while the international ohm is too large by about one part in 2000. These errors are too small to affect ordinary experiments, and the international units have been adopted legally for regulating the sale of electric power. The important point to notice is that the international units themselves can be reproduced with great accuracy. For example, the researches of Smith, Mather and Lowry showed that the amounts of silver deposited by the same current in different silver voltameters agree to about one part in 100,000 if proper precautions are taken, so that a current can be measured in international amperes to this order of accuracy. A slightly smaller accuracy is attained in the construction of mercury standards of resistance.

The London Conference defined the *international volt* as the difference of potential which, when applied to a conductor whose resistance is one international ohm, would produce a current of one international ampere.

We have now to complete the account of standard cells given in Art. 64, by showing how their E.M.F. is determined in absolute measure. A circuit is set up containing a silver voltameter and a standard resistance of (say) one international ohm, together with a battery and a regulating resistance. A Weston cell is placed in series with a key and a sensitive galvanometer so as to form a potentiometer circuit connected to the terminals of the standard resistance. The current is adjusted till the galvanometer shows no deflexion, and is then measured by means of the silver voltameter. If its magnitude is $V$ international amperes, then the E.M.F. of the Weston cell is $V$ international volts.

The London Conference recommended the following formula for the E.M.F. of the Weston cell at temperatures between 0° and 40° C. in terms of its E.M.F. $E_{20}$ at 20° C.:

$$E_t = E_{20} - 0{\cdot}0000406\,(t-20)^2 - 0{\cdot}00000095\,(t-20)^2 + 0{\cdot}00000001\,(t-20)^3 \ldots (167).$$

Experiments carried out at Washington in 1910, by representatives of the four principal standardising laboratories of the world, gave the value 1·0183 international volts for the E.M.F. of the Weston normal cell at 20°. This value is probably correct to

about one part in 50,000, and is now universally adopted for standardising purposes. In the case of a cell whose E.M.F. at 20° differs from the normal value, formula (167) may still be used with the proper value of $E_{20}$.

Using the most recent value for the international ohm and ampere in terms of the true units, the E.M.F. of the Weston cell at 20° comes out at 1·0188 true volts.

Further information on the subject of Arts. 140 and 141 will be found in the *Dictionary of Applied Physics*, vol. 2, pp. 211–273.

**142. Ratio of the electromagnetic to the electrostatic unit of charge.** The absolute magnitude of the electromagnetic units having been placed on a firm basis, we can now consider how these units are related to the electrostatic system.

An approximate determination of the ratio of the units can be made by charging and discharging a condenser repeatedly after the manner of Art. 90. For this purpose the alternate connexion of $Q$ with $P$ and $R$ is made with a rotating commutator, whose mode of action will be understood from Fig. 189, $P$, $Q$ and $R$ being three brushes pressing against a rotating segmented wheel. In this way the condenser $C$ is discharged $n$ times per second through a dead-beat galvanometer $G$ after having been charged up to potential $V$, electrostatic disturbance being avoided by joining one of the terminals of the galvanometer to the case. Let $\theta$ be the steady deflexion. Then $nCV = k\theta$, since charge $nCV$ passes through the galvanometer per second. The experiment is performed in succession with two parallel plate condensers of equal perimeter and separation. Subtracting eliminates the edge correction and the capacity of the leads in the experiment, and then

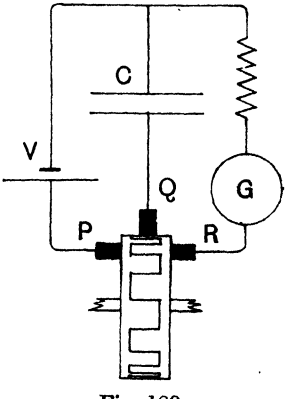


Fig. 189

$$n\,(C_1 - C_2)\,V = k\,(\theta_1 - \theta_2),$$

where $C_1 - C_2$ can be calculated in electrostatic units from the dimensions of the condensers.

The galvanometer is then standardised as in Art. 73, using the

same cell $V$ and adjusting the resistances for a steady deflexion $\theta_1 - \theta_2$. With the rotation of that article,

$$VX = R\,(S + G)\,k\,(\theta_1 - \theta_2).$$

The last two equations give on division

$$C_1 - C_2 = \frac{X}{nR\,(S + G)} \quad \ldots\ldots\ldots\ldots (168).$$

If derived units are used throughout, this equation gives $C_1 - C_2$ in farads. By reduction $C_1 - C_2$ can be found in true electromagnetic units, and since it is known in electrostatic units the quantity

$$c^2 = \frac{\text{electromagnetic unit of capacity}}{\text{electrostatic unit of capacity}}$$

is found. Hence $c$ is known.

Maxwell devised an accurate null method for measuring the capacity in electromagnetic units, which dispenses with the standardisation of the galvanometer. The condenser is placed in the arm $BC$ of a Wheatstone's bridge (Fig. 190). The terminals $P$, $Q$, $R$ press against a rotating commutator, so that $Q$ makes contact alternately with $P$ and $R$. The phenomena occurring during a single period of vibration are of two kinds:

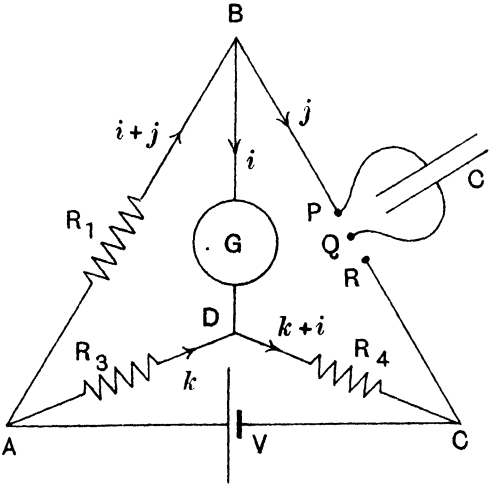


Fig. 190

(1) When $QR$ is joined the condenser begins to be charged up, and rapidly acquires its final charge corresponding to the difference of potential $v_0$ which would exist between $B$ and $C$ in the absence of the arm $BC$.

(2) When $PQ$ is joined the condenser is discharged and steady currents flow through the arms of the bridge just as if $BC$ was absent.

The stage (1) differs from (2) in superposing a sudden discharge

of electricity on the steady current that would flow in the absence of the condenser. We shall find the condition that the total charge passing through the galvanometer in a single period is zero, i.e. that the galvanometer should show no deflexion.

Let the period (1) last from time $t = 0$ to $t = T$, the currents at any intermediate time $t$ being as shown in the figure. Let $v$ be the difference of potential between $B$ and $C$ at time $t$. Then

$$j = C\,dv/dt, \quad Gi + R_4(k + i) = v,$$
$$R_3 k + R_4(k + i) = V, \quad R_1(i + j) = R_3 k - Gi.$$

Writing $I$, $J$, $K$ for $\int_0^T i\,dt$, $\int_0^T j\,dt$, $\int_0^T k\,dt$ respectively, we have on integration

$$J = Cv_0, \quad R_3 K + R_4(K + I) = VT, \quad R_1(I + J) = R_3 K - GI.$$

Hence
$$I = \frac{R_3 VT - (R_3 + R_4) R_1 Cv_0}{(R_1 + G)(R_3 + R_4) + R_3 R_4} \qquad \ldots\ldots\ldots (169).$$

Let the stage (2) last from time $t = T$ to $t = 1/n$, where $n$ is the frequency of the tuning fork. The currents in this stage are given by $j = 0$ and

$$Gi + R_4(k + i) = v_0, \quad R_3 k + R_4(k + i) = V, \quad R_3 k = (R_1 + G)\, i.$$

Hence
$$i = \frac{R_3 V}{(R_1 + G)(R_3 + R_4) + R_3 R_4} \qquad \ldots\ldots\ldots (170),$$

and
$$\frac{v_0}{V} = \frac{R_3(R_4 + G) + R_4(R_1 + G)}{(R_1 + G)(R_3 + R_4) + R_3 R_4} \qquad \ldots\ldots\ldots (171).$$

It follows from (169) and (170) that the total charge passing through the galvanometer in the period $1/n$ is

$$I + i\left(\frac{1}{n} - T\right) = \frac{R_3 V/n - (R_3 + R_4) R_1 Cv_0}{(R_1 + G)(R_3 + R_4) + R_3 R_4}.$$

Hence the galvanometer will show no deflexion if

$$R_3 V = n(R_3 + R_4) R_1 Cv_0,$$

or, using (171), if

$$C = \frac{R_3}{nR_1(R_3 + R_4)} \frac{(R_1 + G)(R_3 + R_4) + R_3 R_4}{R_3(R_4 + G) + R_4(R_1 + G)} \qquad \ldots (172).$$

Thus the capacity is measured in electromagnetic units in terms of the resistances.

Thomson and Searle modified the method so as to permit of the use of a condenser with a guard-ring.

A beautiful method of determining $c$ by measurement of electromotive force was used by Fabry and Perot. By means of their sensitive absolute electrometer (Art. 48) they found the E.M.F. of a standard cell at 0° C. to be $4{\cdot}8443 \times 10^{-3}$ electrostatic units, the E.M.F. of the cell being 1·4491 volts $= 1{\cdot}4491 \times 10^{7}$ E.M.U. The ratio

$$c = \frac{\text{electrostatic unit of potential}}{\text{electromagnetic unit of potential}}$$

was thus found to be $2{\cdot}9913 \times 10^{10}$.

The following table gives some of the experimental values of the ratio of the units:

| | |
|---|---|
| Thomson and Searle (1890) | $2{\cdot}996 \times 10^{10}$ |
| Abraham (1892) .. .. | 2·991 |
| Hurmuzescu (1896) .. | 3·001 |
| Fabry and Perot (1898) .. | 2·991 |
| Rosa and Dorsey (1907) .. | 2·997 |

Hence although it is probable that $c$ is less than $3 \times 10^{10}$, the difference is small and not yet known with sufficient accuracy to be worth taking into account.

# CHAPTER IX

## APPLIED ELECTRICITY

**143. Mechanical generation of electricity.** The use of primary batteries on a large scale is, for various reasons, quite out of the question; and recourse is had to the generation of currents by electromagnetic induction. The simplest apparatus for generating direct currents is shown diagrammatically in Fig. 191.

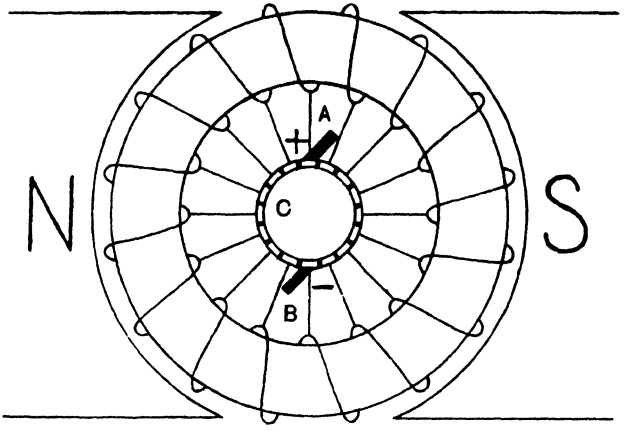



Fig. 191

An iron ring lies between the poles of a magnet, and is fixed on an axle so that it can be made to revolve at any desired speed. The axle also carries a *commutator* $C$, consisting of a number of copper strips parallel to the axle and insulated from it and from each other. The ring is wound uniformly with a continuous length of wire, and contacts are taken off at regular intervals (in Fig. 191 at every single turn) to the separate segments of the commutator. The whole system of ring, wires and commutator is rigidly attached to the axle and rotates with it. The *brushes* $A$, $B$ are two pieces of carbon pressed against the commutator at opposite points of its circumference, but fixed in space so that the commutator slides under the brushes when the machine is running. Wires lead from the brushes to carry off the currents generated.

Fig. 192 shows roughly the way in which the lines of magnetic induction pass through the ring from left to right. The E.M.F.

generated by electromagnetic induction can be found for any given distribution of the lines as follows.

Let the distribution be specified by the flux of magnetic induction across an assigned cross-section of the ring: thus if $N$ is the flux

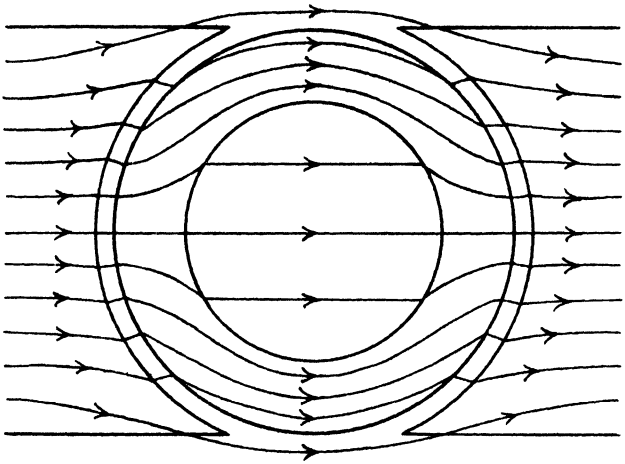


Fig. 192

across a section making an angle $\theta$ with the horizontal, we take $N = f(\theta)$, a given function of $\theta$, which evidently changes sign twice as we pass round the ring. Let $P$, $Q$ (Fig. 193) represent

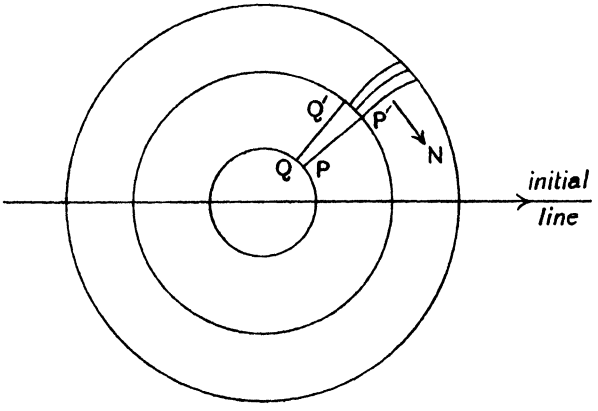



Fig. 193

points on two adjacent segments of the commutator, joined to the windings on the ring by wires $PP'$, $QQ'$ respectively. The angle which $PP'$ makes with the horizontal at time $t$ will be

denoted by $\theta$ or $\omega t$ according as it is desired to call attention to the actual position of $P$ in space or the position of a particular commutator segment at time $t$.

The number of commutator segments is usually considerable (at least 20), and the theory is simplified by supposing the number to be infinitely great. We shall further assume that the windings on the ring are very close together like those on a solenoid. Writing $d\theta$ for the infinitesimal angle between $PP'$ and $QQ'$, $P'$ and $Q'$ may be regarded as the ends of a flat and nearly closed coil of $m\,d\theta/2\pi$ turns, where $m$ is the number of turns of wire on the whole ring. The flux of magnetic induction through this small coil at time $t$ is

$$\nu = \frac{Nm\,d\theta}{2\pi} = \frac{m\,d\theta}{2\pi} f(\omega t),$$

since $N = f(\theta)$.

Electromagnetic induction causes an E.M.F. inside the ring which drives the electric charges on the ring into certain definite positions, as in an electrostatic problem. At a stationary point outside the ring (such as one of the brushes) there is no electromagnetic induction, but only the effect of the accumulated charges. Hence what we are really interested in is the potential at any point of the ring *due to the charges on the ring*. Let this be $V$ at the point $P'$. Applying the law of electromagnetic induction to the small coil $Q'P'$, we have

$$dV - \frac{d\nu}{dt} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (173),$$

since no current flows in the coil. Hence

$$dV = \frac{m\omega\,d\theta}{2\pi} f'(\omega t) = \frac{m\omega\,d\theta}{2\pi} f'(\theta).$$

Integrating,

$$V = \text{const.} + \frac{m\omega f(\theta)}{2\pi} = \text{const.} + \frac{mN\omega}{2\pi}.$$

The arbitrary constant is unimportant, so that the distribution of potential is completely determined. If $n$ is the number of revolutions of the ring per second, $\omega = 2\pi n$, and

$$V = \text{const.} + mnN \quad \ldots\ldots\ldots\ldots\ldots (174).$$

In Fig. 191 the potential is greatest at the top, opposite the brush $A$, and least opposite the brush $B$.

Equation (174) is easily adapted to the ideal case, not realisable with an iron ring, in which the lines of magnetic induction pass straight across from left to right, so that the ring is situated in a region of uniform **B**. If $A$ is the area of cross-section of the ring, $f(\theta) = AB \sin \theta$, and

$$V = \text{const.} + mnAB \sin \theta \quad \ldots\ldots\ldots (175).$$

Since $\theta = \omega t$ this may equally well be written

$$V = \text{const.} + mnAB \sin \omega t \quad \ldots\ldots\ldots (176).$$

The double form of the expression helps us to see the part played by the commutator of the machine. If we fix our attention on a particular segment of the commutator and follow it round as it moves, the potential will alternate in time in accordance with the law (176). A fixed brush does not follow the segment round, but picks out a definite position in space corresponding to a particular value of $\theta$, and takes up the potential of the segments as they pass it. Equation (175), or more generally (174), thus gives the *steady* potential acquired by a brush placed in the position $\theta$.

The brushes are naturally placed in such a position that their difference of potential is as great as possible. If $N_{\max}$ is the greatest value of $N$ for any cross-section, the opposite section will have the flux $-N_{\max}$, and if the brushes are placed at these points the E.M.F. developed is

$$V_0 = 2mnN_{\max} \text{ true electromagnetic units}$$
$$= \frac{2mnN_{\max}}{10^8} \text{ volts} \quad \ldots\ldots\ldots\ldots\ldots\ldots (177).$$

Consider for example a ring of cross-section 150 cm.$^2$, rotating 25 times a second in a field such that the greatest value of $B$ is 15,000. If there are 100 turns of wire on the ring, $m = 100$, $n = 25$, and $N = 2{\cdot}25 \times 10^6$. Hence $V_0 = 112$ volts.

**144. The drum armature.** The term *armature* is used generally for the part of a direct-current generator from which the currents are drawn, in contrast with the *field magnets* which furnish the field of force. The ring armature of Fig. 191 is used merely for didactic purposes, and not now found in practice. The effective flux of induction is confined to the comparatively small cross-section of the ring itself, whereas it is desirable to use the much greater flux

obtainable from a solid core lying between the poles of the field-magnets. The latter object is attained in the *drum armature* introduced by v. Hefner-Alteneck in 1872. The principle of the drum armature, illustrated diagrammatically in Fig. 194, is to wind a

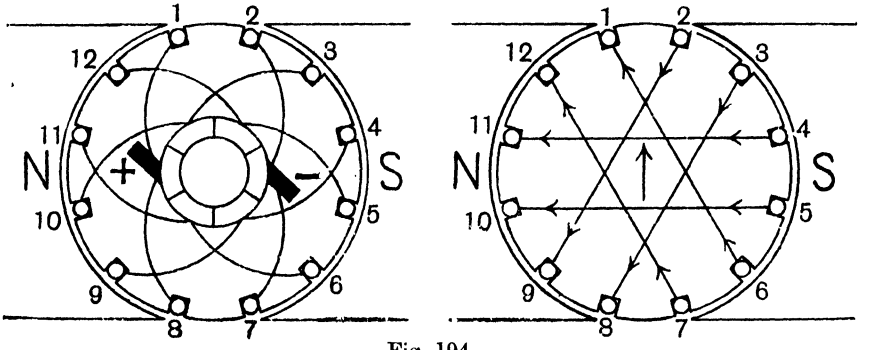



Fig. 194

solid cylindrical core uniformly with wire somewhat after the manner in which wool is wound into a ball. The small circles marked 1 to 12 represent copper wires or bars lying parallel to the axle, and let into slots in the core. The commutator, here shown as having six segments only, is rigidly fixed to the axle as before, and is joined to the numbered wires by connexions shown in the left-hand figure. The right-hand figure shows the way in which the wires are joined up at the back, and represents a view from the same end as before, i.e. it is what we should see at the back if the armature core was transparent.

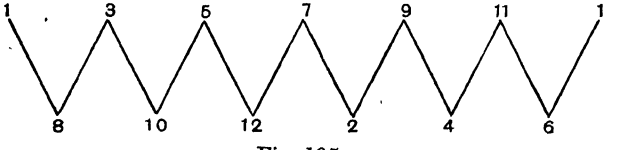



Fig. 195

The drum winding gives a closed circuit returning on itself. The wire may be supposed to begin at the nearest end of the bar 1, to proceed to the commutator and to the bar 8, thence to the back and to 3. Coming to the front it proceeds to 10 *viâ* a commutator segment, thence to 5, and so on. The complete cycle is shown in Fig. 195, where the top line includes bars traversed from back to front and the bottom line those traversed from front to back*.

* The arrows in Fig. 194 have a special significance. See Art. 148.

The essential identity of ring and drum armatures may be seen by considering the case in which the number of turns of wire is very great. The section 1 8 3 of the drum winding, for example, may be considered as a single nearly closed flat coil, since the ends 1, 3 are now very near together. This is connected to a commutator segment like the top turn of the ring winding in Fig. 191, the only difference being that the corresponding segment of the commutator is displaced through an angle of 90° in the clockwise direction. Hence, with drum armatures, the line of brushes is nearly parallel to the direction of the magnetic field.

The formulae of the last article therefore apply to the drum winding also, $N$ now denoting the maximum flux of induction through any cross-section of the whole core, which is much greater than that for a ring occupying the same space.

If the iron armature was solid its motion in a magnetic field would give rise to eddy currents, which would heat the armature

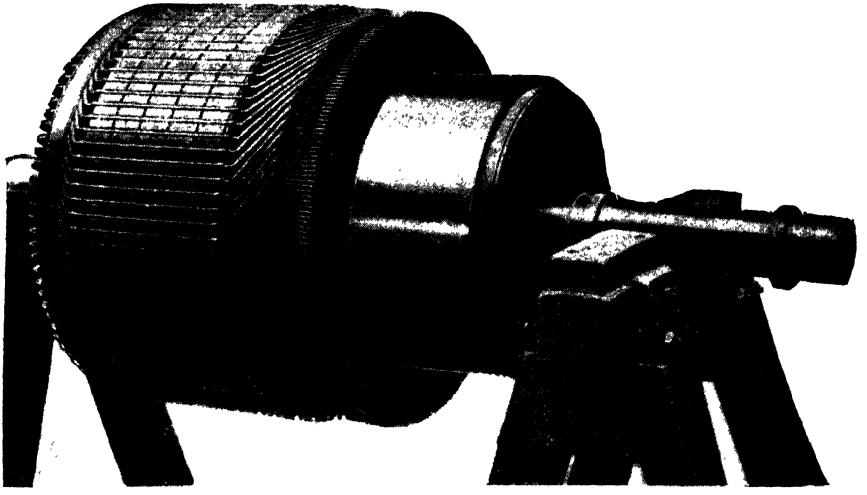


Fig. 196

and absorb power which might otherwise be saved. For this reason the cores of armatures are *laminated*; that is, formed of thin iron discs insulated from one another.

Fig. 196 shows a drum armature completely wound and with commutator attached.

**145. Field-magnets. Saturation curve.** Permanent magnets suffice for weak generators, e.g. for motor ignition. Larger machines contain electromagnets excited by external current. A fundamental experiment is to find how the difference of potential between the brushes (on open circuit) varies with the current used to excite the field-magnets. Fig. 197 gives a diagrammatic representation of the generator and of the connexions for this experiment. It is convenient to have wires from the brushes brought out to two

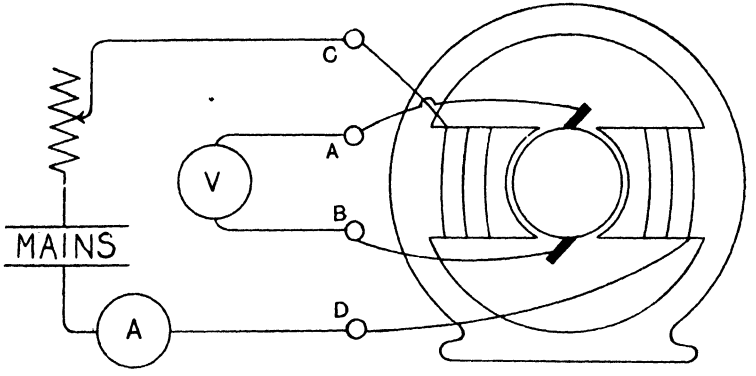



Fig. 197

insulated screw terminals $A$, $B$ on the case of the machine, while the terminals $C$, $D$ are joined inside to the magnetising coils, which are themselves connected across by a wire not shown in the figure. The magnetising current is supplied from the electric mains or from accumulators, and regulated by a rheostat capable of carrying the required current at every stage. The terminals $A$, $B$ are joined to a suitable voltmeter, and readings of the ammeter and voltmeter are taken with the machine running at constant speed. Fig. 198 shows a curve taken off from a certain small generator. Its general shape is readily predicted: for the field-magnets and the armature core constitute a nearly continuous iron or steel bar wound with wire somewhat after the manner of a solenoid. The "magnetising force" is therefore roughly proportional to the current in the coils, while equation (177) shows that the brush voltage is proportional to the flux of induction across the armature core. The $(i, V)$ curve of Fig. 198 should therefore resemble the $(H, B)$ curve of a specimen of iron: the bend in the curve is however less pronounced. There will always be enough residual magnetism left over from previous

experiments to prevent the curve from passing through the origin. The curve with decreasing current is but slightly higher than that obtained on ascending, as the complete hysteresis cycle is not described.

Although Fig. 197 is purely diagrammatic, it does in fact represent a favourable shape for the field-magnets and shell of the machine. The field-magnet windings should be near the armature, so that the flux of magnetic induction through the latter should

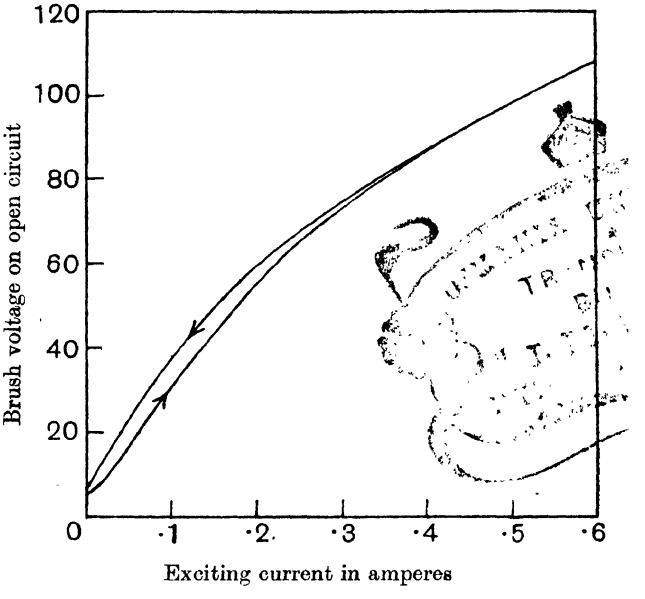



Fig. 198

be as great as possible. Some old patterns of generator are very unfavourable in this respect, as many lines of magnetic induction which might have passed through the core are allowed to crowd into other masses of iron in the vicinity.

The ($i$, $V$) curve of the present article is known as the *saturation curve* (or *magnetic characteristic*) of the machine, and is useful in helping us to predict its performance under various conditions. An electric motor is used to drive the machine, and we can modify the experiment by keeping the exciting current constant and verifying that $V$ is proportional to the speed of revolution.

**146. Shunt, series, and compound generators.** Machines which have their field-magnets excited electrically are known as *generators* (or *dynamos*) to distinguish them from magneto-electric machines with permanent magnets. In the early stages of electrical engineering the exciting current was supplied by a few primary batteries, and later by a small auxiliary magneto-electric machine. An important advance was that made by S. Hjorth about 1851, and by W. v. Siemens in 1867, who suggested using the currents produced by the generator for the excitation of its own field-magnets.

Self-exciting generators are mainly of two kinds, *shunt-wound* and *series-wound* generators. These are shown diagrammatically in Fig. 199, where the field-magnets are drawn as a separate coil $M$ for the sake of clearness. The wires at the bottom lead to the external circuit, and the names of the machines are derived from the fact, which is obvious from the diagrams, that the field-magnets are shunted across the armature in one case and are in series with it in the other. If the generator is provided with four separate terminals as shown in Fig. 197 it may be arranged as a shunt or series generator at will: thus the machine is shunt-wound if $AC$ and $BD$ are joined and the current taken from $A$ and $B$, while a series winding is obtained by joining $A$ to $C$ and using $B$ and $D$ as terminals for the external circuit.

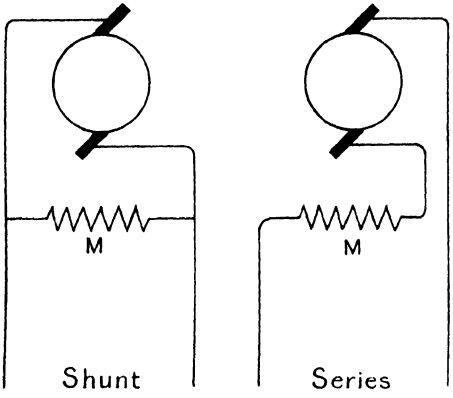


Fig. 199

These generators depend on residual magnetism to start them in the first place. Consider a shunt-wound machine on open circuit. The residual magnetism of the field-magnets first gives a small E.M.F. This sends a small current round the field-magnets, which may reinforce or destroy the original magnetism. If the former, the increased field causes a rise on the brush voltage, which again strengthens the field-magnets, and so on. The result is that after a few revolutions of the machine the field-magnets are saturated

and the generator is developing its full brush voltage. If on the other hand the first weak current sent through the field-magnets is in the direction of demagnetisation, the process will go on until the magnetisation is all removed, and the generator will not work at all. The reader can verify this latter fact himself, by taking a shunt-wound generator and reversing the connexions of the field-magnets.

The *compound* generator has both shunt and series windings on the field-magnets. Its function is explained in the next article.

**147. Characteristic curves of generators.** We have hitherto supposed that no current is taken off the generator to any external circuit. But if the armature terminals are connected through an external resistance, a current flows round the circuit, and this in general will change the brush voltage. Take the magneto-electric machine of Fig. 191, and assume for the present that the currents in the armature do not disturb the impressed magnetic field. If $i$ is the current from $A$ to $B$ in the external circuit, a current $i$ flows inside the armature from $B$ to $A$, and this is divided so that a current $\frac{1}{2}i$ flows down the elementary coil $P'Q'$ considered in Art. 143. If $dR$ is the resistance of $P'Q'$, equation (173) becomes

$$dV - \frac{dv}{dt} = -\tfrac{1}{2}i\,dR, \text{ or } dV = \frac{m\omega f'(\theta)\,d\theta}{2\pi} - \tfrac{1}{2}i\,dR.$$

This may be written $dV = dV_0 - \frac{1}{2}i\,dR$, where $V_0$ corresponds to zero current. Hence if $V$ is the actual brush voltage and $V_0$ the brush voltage on open circuit,

$$V = V_0 - Ri \quad \ldots\ldots\ldots\ldots\ldots\ldots (178),$$

where $R = \frac{1}{2}\int dR$ is the resistance of the armature considered as a divided circuit connecting $B$ and $A$.

The behaviour of a generator when supplying current is shown by a curve connecting the external current $i$ with the difference of potential $V$ between the brushes. This curve is called the *external characteristic*, or simply the *characteristic* of the generator, and it is usually understood that the speed is kept the same for all the currents. The characteristic of a separately excited generator would be a straight line if it were not that the currents in the armature disturb the magnetic field (armature reaction). In practice, the

brush voltage at high currents is lower than that calculated from equation (178). The calculated and observed voltages for a certain small generator are given in Fig. 200, the difference between the two curves showing the effect of armature reaction.

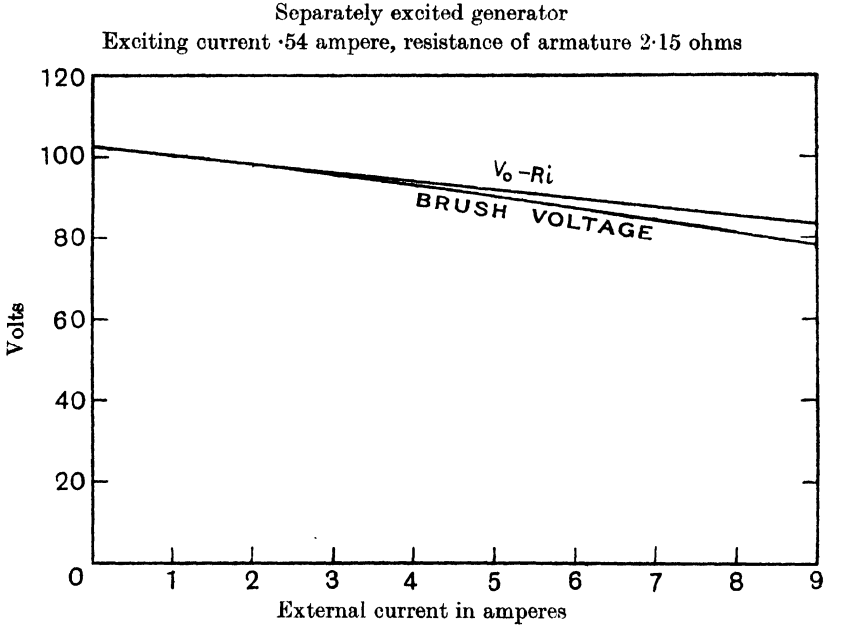



Fig. 200

Fig. 201 shows the connexions for taking off the characteristic of a shunt-wound generator, $A$, $B$ being the terminals of the armature and $C$, $D$ of the field-magnets. Since any fall of brush voltage now diminishes the current in the field-magnets, the brush voltage falls more rapidly than before as the load increases. If the armature resistance is sufficiently high it may even happen that the generator will not yield more than a certain current, brush voltage and current falling together when the external resistance is reduced below a certain amount (Fig. 202). Peculiar results are found for the return half of the curve obtained by increasing the

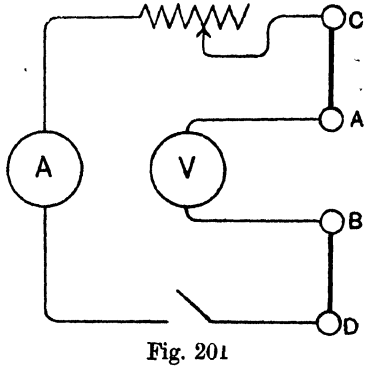



Fig. 201

external resistance from a small value to infinity, the effect of hysteresis being particularly noticeable here. These curves are of theoretical interest (Art. 148), but cannot be traced fully with machines of modern design. The armature resistance of such machines is so low that the voltage falls only a few per cent. up to full load, the theoretical maximum current being far greater than the windings would carry.

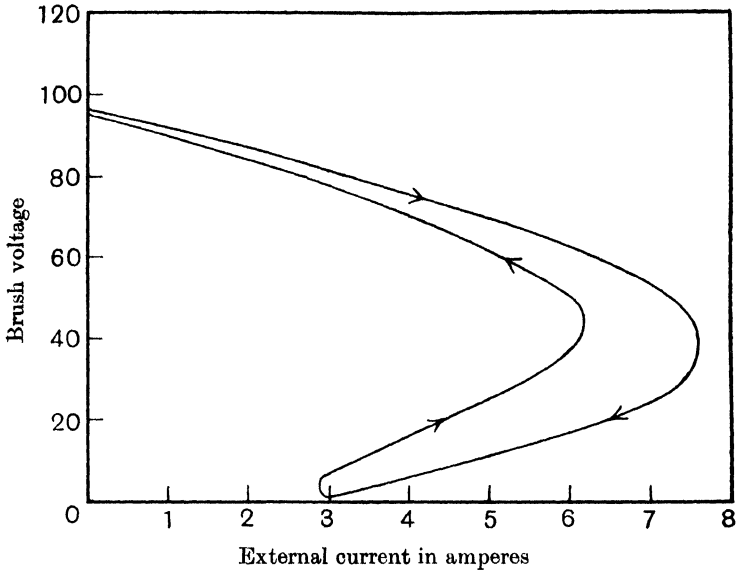



Fig. 202

In the series generator the whole current passes through the field-magnets. Hence there is no excitation on open circuit, and the terminal voltage has the low value resulting from remanent magnetism. The excitation rises as more and more current is taken off, and with it the terminal voltage: but the increase ceases when the field-magnets are saturated. The terminal voltage thus rises to a maximum and then decreases. In the compound generator the initial rise of excitation due to the series winding compensates the drop of terminal voltage that would occur with the simple shunt machine, and the terminal voltage can be made nearly independent of the current over a large range.

The compound generator is useful for small lighting installations, but it is by no means generally employed in generating stations,

because it is found easy to regulate the voltage of simple shunt machines with a resistance in the field-magnet circuit. The series generator, on account of the great variation of terminal voltage with current, is rarely used.

**148. Theory of direct-current generators.** We have seen that the variation of terminal voltage with current depends on three factors, namely

(1) loss of potential due to the resistance of armature, or armature and field-magnets;

(2) change of potential due to alteration in the excitation of the field-magnets;

(3) armature reaction.

The effect of the first two causes can be determined graphically from the saturation curve and the resistances of the armature and field-magnets. We shall neglect hysteresis and consider the generator to which Figs. 198 and 200 refer, in shunt winding.

Let $R$ be the resistance of the armature, $S$ the resistance of the

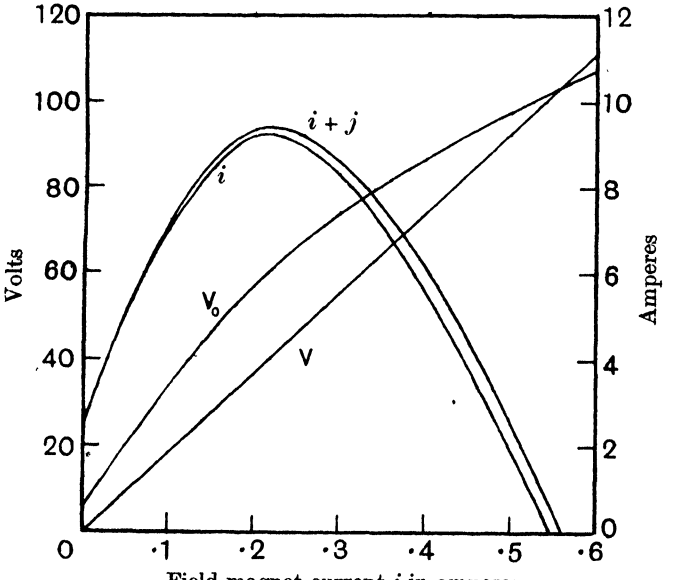



Fig. 203

field-magnets and $j$ the current through them, $i$ the current in the external circuit. The curve $V_0$ in Fig. 203 is the mean of the

ascending and descending curves in Fig. 198. The brush voltage $V$ is evidently given by $V = Sj$, and is therefore represented by a straight line in the diagram. The current in the armature is $i + j$ since the armature supplies current to the field-magnets as well as to the external circuit, so that $V = V_0 - R(i + j)$. Hence $i + j = (V_0 - V)/R$ and can be plotted on the figure for various values of $j$, and the curve representing $i$ as a function of $j$ may also be drawn. If we now pick off values of $i$ and plot them against the corresponding values of $V$, we have the theoretical form of the shunt characteristic (Fig. 204). Comparison with

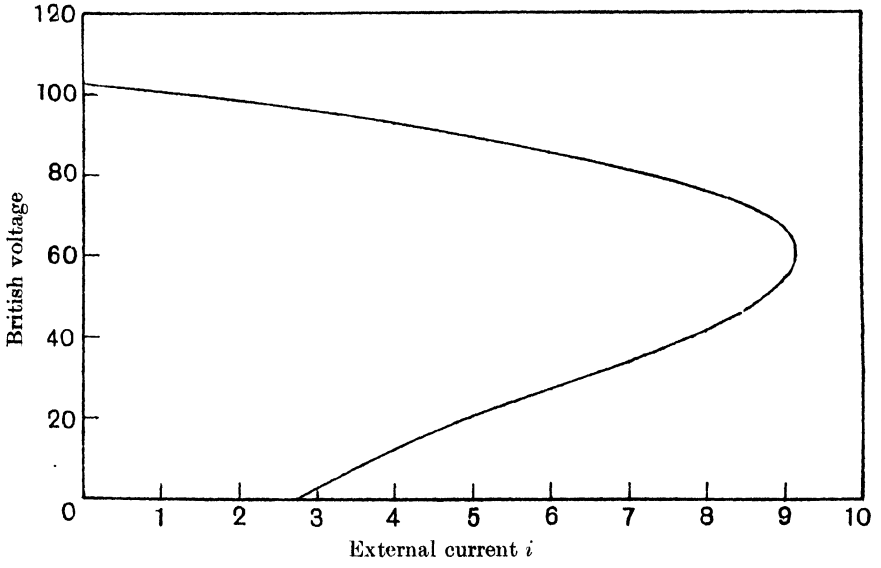



Fig. 204

Fig. 202 shows that the chief features of the characteristic are correctly represented. We should not expect perfect agreement, because we have neglected armature reaction and also because it is not easy to ascertain the resistance of the armature under the conditions of running. An appreciable part of the resistance always occurs where the carbon brushes press on the commutator.

The theory of the series generator is simpler, because the external current $i$ is the same as that through the field-magnets. If $V_0$ is the voltage corresponding to the current $i$ in the saturation curve and $V$ the terminal voltage (difference of potential between

the ends of the external circuit, not that between the brushes), we have $V = V_0 - (R + S)i$.

The general effect of armature reaction is easily predicted. Consider the drum armature of Fig. 194, when a current is taken

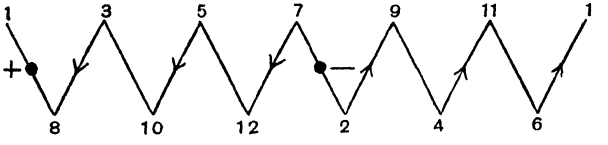



Fig. 205

from the brushes. The complete cycle of the armature winding is shown in Fig. 205, where we have shown the position of the brushes at the time being and the direction of the currents in the armature coils. These directions have already been shown on the right-hand part of Fig. 194. Remembering that this diagram represents the coils at the back of the armature, it is easy to see that the armature currents give rise to a transverse magnetic field pointing vertically upwards, as shown by the central arrow. The field is thus twisted round in a counter-clockwise direction, and the E.M.F. will fall on this account alone, quite apart from the loss of voltage due to resistance (cf. Fig. 200).

Since the brushes always short-circuit a part of the armature when passing from one segment of the commutator to the next (cf. Fig. 191), sparking will occur if the difference of potential is too great. For this reason it is usual to *advance* the brushes of a generator slightly, i.e. to move them round in the direction of rotation.

**149. The direct-current motor.** Suppose that the armature of a magneto-electric machine (Fig. 206) is held fast, and a current sent through it from $A$ to $B$ by means of an external battery, $A$ being attached to the positive pole of the battery. The two halves of a ring armature can be considered as two solenoids bent over until their ends meet, so that the armature coils may for all intents and purposes be replaced by two magnetic poles SN at the ends of the solenoids (cf. Art. 108). The N pole of the field-magnets repels the N pole in the armature and attracts the S pole. Similarly

the S pole of the field-magnets exerts attractions and repulsions on the armature poles, and the four forces produce, in **Fig.** 206,

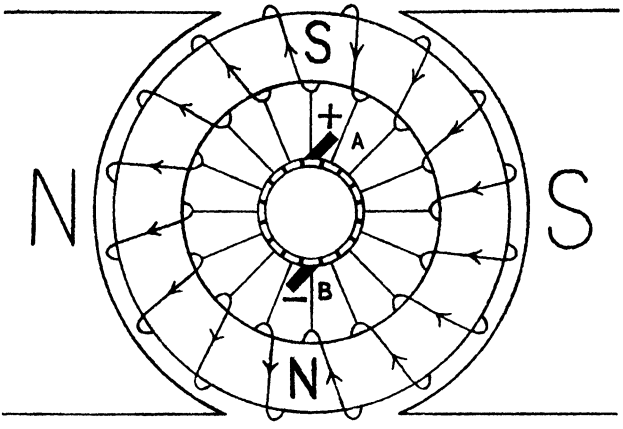



Fig. 206

a couple tending to turn the armature in a counter-clockwise direction. If the armature is free to move it will do so and may be made to perform mechanical work. This is the principle of the electric motor.

It is clear from what has been said that motors and generators are essentially the same in construction, and indeed every generator can be made to run as a motor without structural alteration. We proceed to find an expression for the couple exerted on the armature, or the *torque* as it is called, when a current $i$ flows through it. Consider the elementary coil in Fig. 193, in which current $\frac{1}{2}i$ now flows from $Q'$ to $P'$. The potential energy of this coil in the magnetic field is $-\frac{1}{2}iN$ per turn, and the couple on it is $\frac{1}{2}i\,dN/d\theta$ per turn. Since there are $m\,d\theta/2\pi$ turns, the couple on the elementary coil is $im\,dN/4\pi$.

When the brushes are placed in the position of maximum flux the total couple on the coils of the right-hand half of the armature is $imN_{\max}/2\pi$; and since there is an equal couple on the left-hand half the torque $T_0$ is given by

$$T_0 = imN_{\max}/\pi \quad \ldots\ldots\ldots\ldots\ldots\ldots (179).$$

If $i$ is measured in true electromagnetic units $T_0$ comes out in absolute C.G.S. units; but it is more instructive to measure $i$ in

amperes and $T$ in kilogram-metres (moment due to the weight of 1 kilogram at a distance of 1 metre). Equation (179) is then replaced by

$$T_0 = \frac{3{\cdot}24}{10^{10}} imN_{\max} \quad \ldots\ldots\ldots\ldots (180).$$

For example, if a current of 50 amperes is sent through the armature of the magneto-electric machine considered in Art. 143, $T_0 = 3{\cdot}6$ kilogram-metres. In these formulae $N_{\max}$ is the maximum flux due to the field-magnets only, neglecting any change in the magnetism of the ring caused by the armature current. The formulae apply equally to the drum armature.

The electric motor has the remarkable property of self-regulation; that is, it takes only just enough current from the mains to perform the work on hand. In order to obtain insight into the process of regulation, let us suppose that the motor is running steadily at a speed of $n$ revolutions per second, and that its rotation is resisted by a couple $T$ arising from external braking and from friction or other dissipative causes in the motor itself. Since there is no angular acceleration $T$ must be equal to the couple $T_0$ arising from the mechanical action of the magnetic field on the current $i$ in the armature, so that

$$T = imN_{\max}/\pi \quad \ldots\ldots\ldots\ldots (181).$$

If $A$ and $B$ are connected to the terminals of a battery of potential $V_0$, and $R$ is the resistance of the armature, we cannot conclude that $i = V_0/R$, because the mere rotation of the armature makes it develop an E.M.F. just as if it was the armature of a generator. This E.M.F. is, from Art. 143,

$$V = 2mnN_{\max} \quad \ldots\ldots\ldots\ldots (182),$$

and tends to drive a current through the armature from $B$ to $A$, i.e. to oppose the applied potential $V_0$. Hence

$$Ri = V_0 - V = V_0 - 2mnN_{\max} \quad \ldots\ldots\ldots (183).$$

The E.M.F. $V$ is called the back E.M.F. of the motor, and must be taken into consideration in any theory of the action of motors.

Neglecting for the present the effect of armature currents on $N$ (motor armature reaction), we have the two equations (181) and

(183) to determine $i$ and $n$ when the applied voltage and the resisting couple are given. As $T$ increases, $i$ rises, from equation (181). Equation (183) then shows that $n$ must decrease, and with it the back E.M.F. Hence with a motor whose field-magnets have constant excitation, an increase in the load causes a rise in the armature current and a drop in the speed: this drop will be smaller the smaller the resistance of the armature. In modern machines it does not amount to more than a few per cent. at full load: the limit to the load is determined rather by the heating of the armature coils with increasing current than by any marked slowing down of the motor.

Returning to the magneto-electric machine whose data are given in Art. 143, suppose that the armature resistance is 1 ohm and the applied potential 100 volts. The following table gives the current and speed for various resisting couples:

| Couple in kilogram-metres | ·1 | ·5 | 1 | 2 |
|---|---|---|---|---|
| Armature current in amperes | 1·4 | 7 | 14 | 28 |
| Back E.M.F. in volts | 98·6 | 93 | 86 | 72 |
| Speed, revolutions per second | 22 | 21 | 19 | 16 |

For a given field and angular velocity, the direction of the armature current is different in a motor and a generator. Hence the effect of armature reaction is reversed; and the brushes of a motor have to be displaced *backwards* along the commutator in order to ensure sparkless commutation.

Motors may be either shunt or series-wound like generators. The two types, which have important and very distinct properties, are considered separately in the two following articles. Compound motors are seldom used.

**150. The shunt motor.** In the shunt-wound motor running normally, the armature $AB$ is joined in parallel with the field-magnet coil $CD$. Thus both armature and field-magnets are connected directly to the mains. But it would not be safe to start the motor by simply putting on the mains, because the armature resistance is so low that an excessive current would have passed

before the motor acquired enough speed to develop sufficient back E.M.F. The proper method of starting a shunt motor is by means of a starting switch, one form of which is shown diagrammatically in Fig. 207. The switch carries three terminals usually marked *L* (line),

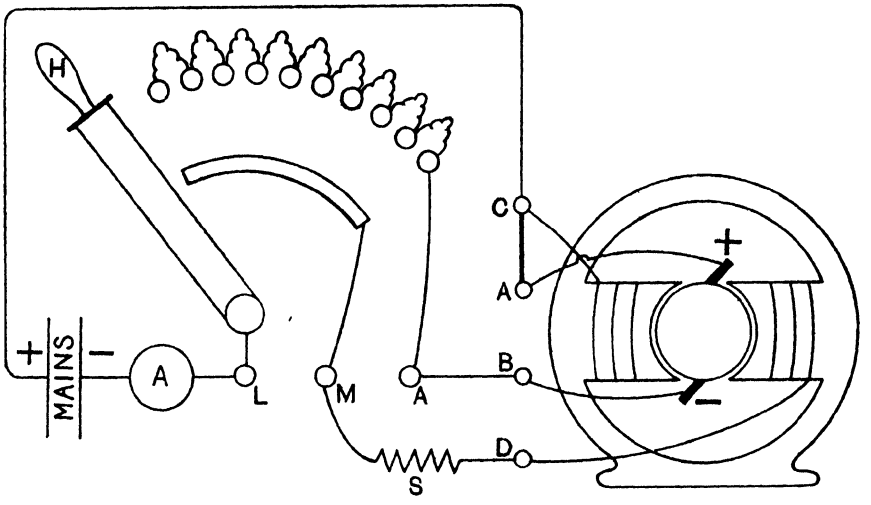



Fig. 207

*M* (magnets), *A* (armature) respectively. *L* is joined inside to a lever moved by a handle *H*, and carrying two metallic spring contacts. One of these contacts moves over a solid metal bar, in the form of a sector of a circle, connected to the field-magnets; the other over a row of metallic studs, between each of which is a resistance coil so that the whole forms a rheostat, the further end being connected to one end of the armature. The other ends of the armature and field-magnets are connected together and joined directly to the mains. On putting on the switch, connexion is first made with the metal sector, thus exciting the field-magnets fully while the motor is still at rest. Connexion with the first stud of the rheostat sends a weak current through the armature, sufficient to set it rotating. A back E.M.F. is thus developed tending to reduce the armature current. The motor can thus be started without excessive current at any stage by cutting out the resistance in the armature circuit gradually; and when the switch is pushed right over the mains are applied directly to both armature and field-magnets.

The shunt-wound motor, having constant excitation, has the properties mentioned in the last article: its speed decreases only slightly when the load is increased. The speed is usually regulated by means of a rheostat $S$ in the field-magnet circuit, being greater the greater the resistance. We shall consider this presently in detail, as it affords a good illustration of the principles underlying the action of motors.

When the motor is running unloaded a certain power $W_1$ will be required to overcome friction, hysteresis, and eddy currents. If $T_1$ is the corresponding torque that has to be exerted by the motor at speed $n$, $W_1 = 2\pi n T_1$. The power required to overcome friction is probably proportional to the speed: that required to overcome hysteresis varies as the speed and also depends on the excitation. The strength of the eddy currents at any point of the armature is proportional to $B$ and to the speed: hence the power lost in eddy currents is proportional to $n^2$ and also depends on the excitation. Thus $T_1$ is of the form $A + Bn$, where $A$ and $B$ depend on the excitation; from which it is not difficult to devise experiments to determine $T_1$ for all speeds and exciting currents.

In examining the shunt motor theoretically we neglect the variation of the relatively small torque $T_1$, and also neglect armature reaction. As in the theory of generators, we begin with the saturation curve, which can be found by driving the motor as a generator from an external source of power. Writing $N$ for $N_{\max}$, the brush voltage at constant speed $n_0$ is given by $V = 2mn_0N$; hence the saturation curve gives us the value of $mN$ for any value of the field-magnet current $j$. Consider the way in which the speed of a motor depends on the total resistance $S$ in the field-magnet circuit, when yielding a constant torque $T$ (inclusive of that required to overcome friction, hysteresis and eddy currents). If $V_0$ is the applied voltage, the field-magnet current $j = V_0/S$ is known in terms of $S$, and therefore also $mN$ is known. The equation $T = imN/\pi$ then gives the armature current $i$, and the back E.M.F. of the motor is $V = V_0 - Ri$. But if $n$ is the speed $V = 2mnN$; and thus $n$ can be calculated for all values of $S$. The following table gives some figures for the generator considered in Art. 148, when run as a motor off a 100 volt circuit. The saturation curve $V_0$ of Fig. 203, which is the mean of the ascending and descending curves

of Fig. 198, is taken as a basis, and the resisting torque is taken as 0·3 kilogram-metre.

| Total resistance $S$ in field-magnet circuit | Field-magnet current $j$ (amperes) | $mN$ in absolute units $10^8 \times$ | Armature current $i$ (amperes) | Back E.M.F. $V$ (volts) | Speed (revolutions per second) |
|---|---|---|---|---|---|
| 185 | 0·54 | 1·65 | 5·62 | 87·9 | 26·6 |
| 200 | 0·50 | 1·59 | 5·82 | 87·5 | 27·5 |
| 220 | 0·46 | 1·51 | 6·14 | 86·8 | 28·7 |
| 240 | 0·42 | 1·43 | 6·48 | 86·1 | 30·1 |
| 260 | 0·39 | 1·37 | 6·76 | 85·5 | 31·2 |
| 280 | 0·36 | 1·31 | 7·06 | 84·8 | 32·4 |
| 300 | 0·33 | 1·26 | 7·35 | 84·2 | 33·4 |
| 320 | 0·31 | 1·22 | 7·60 | 83·7 | 34·3 |
| 340 | 0·29 | 1·18 | 7·85 | 83·1 | 35·2 |

It is easy to see how the speed rises with decreasing excitation. The speed is proportional to $V/N$; but if $R$ is small $V$ does not vary very much, so that $n$ varies inversely as the flux. The armature current is accurately proportional to $1/N$ on the present theory, so that it rises even more rapidly than the speed. Hence the resistance of the field-magnet circuit cannot be altered much without causing a dangerous speed of the motor, or an excessive armature current, or both. It is important to make sure before starting a motor that the field-magnet circuit is not disconnected.

**151. The series motor.** In the series motor, as in the series generator, the same current flows through both armature and field-magnets. The connexions are shown in Fig. 208: the motor is started with a simple resistance cut out in stages.

The series motor differs from the shunt motor in the following respects:

(1) It has a high "initial torque"; that is, when started against a load it exerts a large couple to begin with, so that it comes quickly to its full speed.

(2) As the load increases the consumption of current increases, but not so rapidly. This, of course, necessitates a slowing down of the motor under heavy load.

These two properties render the series motor invaluable for electric traction. As regards (2), it is desirable that excessive

current should not be taken, for example, by an electric tram mounting an incline, even though it results in a decrease of speed. On the other hand, if the load of a series motor is suddenly removed the motor will run at a dangerously high speed; while its great variation of speed unfits it for laboratory work.

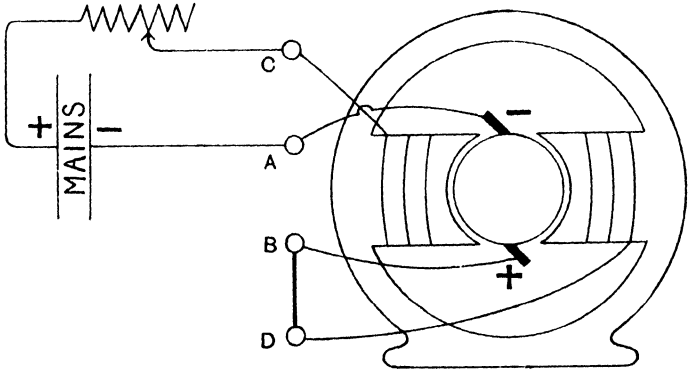



Fig. 208

In order to see how these effects are produced, let us assume that a certain constant torque $T_1$ is required to overcome friction, hysteresis and eddy currents. Let $T$ be the torque exerted by the motor at speed $n$ when carrying a current $i$, and let $R$ be the total resistance in the motor circuit, inclusive of any starting resistance that may be present. Then $T = imN/\pi$ and

$$Ri = V_0 - 2mnN.$$

As before, a knowledge of the saturation curve gives $mN$ as a function of the magnetising current, which in this case is $i$. Since $mN$ increases with $i$, the torque increases more rapidly than the current, which explains the above-mentioned property (2). The high initial torque has the same origin, since the large starting current is utilised to produce high saturation at the moment when it is needed.

In general, the speed is given by

$$n = \frac{V_0 - Ri}{2mN},$$

and in normal running, with the starting resistance short-circuited, $R$ is small. Hence the speed is not far from being inversely

proportional to the flux $N$. If the load is suddenly removed $T$ sinks immediately to $T_1$, the current falls to a low value and the speed becomes very high.

**152. Brake test of a motor.** We test a motor by applying known retarding couples to it. One method is with a *rope brake*, shown in Fig. 209. $A$ represents the end of a drum fixed on one end of the armature shaft. A rope $R$ is wound once or twice over the drum, and after passing over a pulley supports a mass $M$ whose magnitude can be varied. The other end of the rope is attached to a fixed beam, a spring balance $S$ being placed between the rope and the beam to register the pull on the "light" side of the rope, equivalent say to the weight of a mass $M_1$. If $M$ and $M_1$ are measured in grams and $a$ is the distance of the centre-line of the rope from the axis of the motor, measured in centimetres, the external couple is $(M - M_1)\, a/10^5$ kilogram-metres. If $n$ is the number of revolutions of the motor per second, the rate at which the motor is performing useful work is easily seen to be

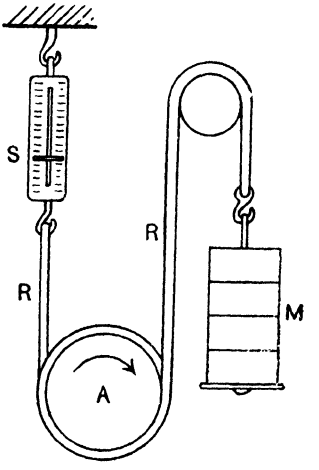


Fig. 209

$$2\pi n\,(M - M_1)\,ga \text{ erg/sec.} = \frac{6{\cdot}16}{10^4}\,(M - M_1)\,na \text{ watts.}$$

The test of a shunt-wound motor is carried out most instructively at constant speed and with a constant E.M.F. applied to the brushes. The voltage of the mains will tend to drop as more current is taken by the motor, so that it is necessary to have a resistance in the main circuit, which can be adjusted until a voltmeter attached to the brushes gives a certain specified reading $V_0$. The speed is kept constant by means of a regulating resistance in the field-magnet circuit.

Let $i$ be the main current, $j$ the current in the field-magnets at any stage. Then the electrical power supplied is $V_0 i$ watts. If $W$ is the useful power, measured as explained above, the fraction of

the power supplied which is really utilised is given by $\eta = W/V_0 i$. This fraction is called the *efficiency* of the motor.

To find where the energy which is not utilised goes to, we require to know the resistances of the armature and field-magnets of the motor. The field-magnet resistance can be measured on the Wheatstone's bridge. The armature resistance is often very low and can be measured by passing various currents through the armature and observing the corresponding difference of potential between the brushes, the motor of course remaining at rest during the process. The resistance is less for large than for small currents, because it partly consists in contact resistance between the brushes and the commutator. No great error is committed in taking the value corresponding to the normal working current, as has already been done in theoretical discussions. If $R$ is the armature resistance, $R(i-j)^2$ watts are lost in heating the armature conductors; and $V_0 j$ watts go to heat the field-magnet windings. The remainder, say $W_0$, is expended in friction, hysteresis and eddy currents. We have thus a means of estimating the relative magnitudes of the three losses; and methods have also been devised for further analysing the loss $W_0$. Fig. 210 shows the result of an experiment on a 4 horse-power shunt motor running at 1150 revolutions per minute and brush voltage 98.

The loss $W_0$ increases, sometimes very considerably, with the load. This is largely due to the increased friction of the bearings caused by the weight on the rope brake. Another brake method is sometimes used with small motors, which does away with this difficulty. A copper disc is fixed to the axle of the motor, and lies between the poles of two similar electromagnets placed symmetrically on either side of the axle. As the motor rotates eddy currents are developed in the disc, which react and produce a couple on the electromagnets. This couple can be altered by adjusting the exciting current, and can be counterbalanced by a sliding weight on the frame holding the electromagnets, which is made moveable about an axis. The retarding couple on the motor, which is equal and opposite to that on the electromagnets, is therefore known.

The testing of series motors is similar to that of shunt motors, except that care has to be taken not to reduce the load too much.

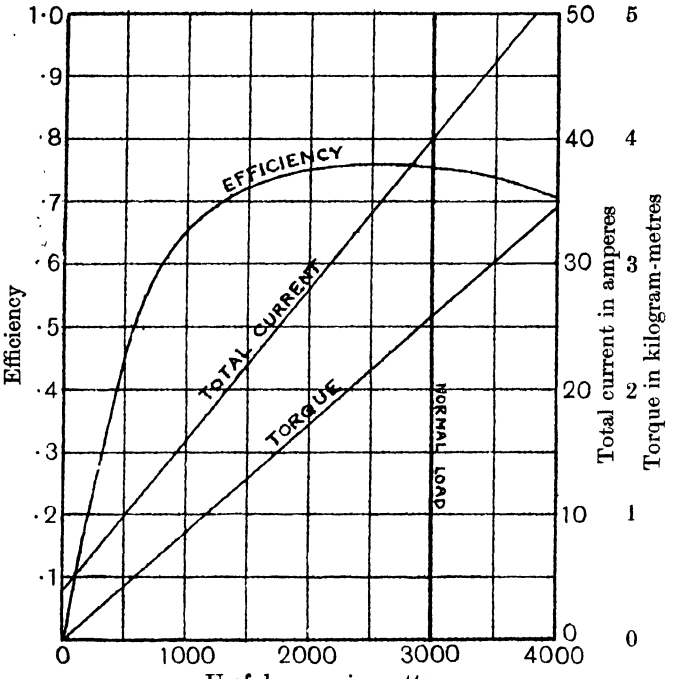



Fig. 210

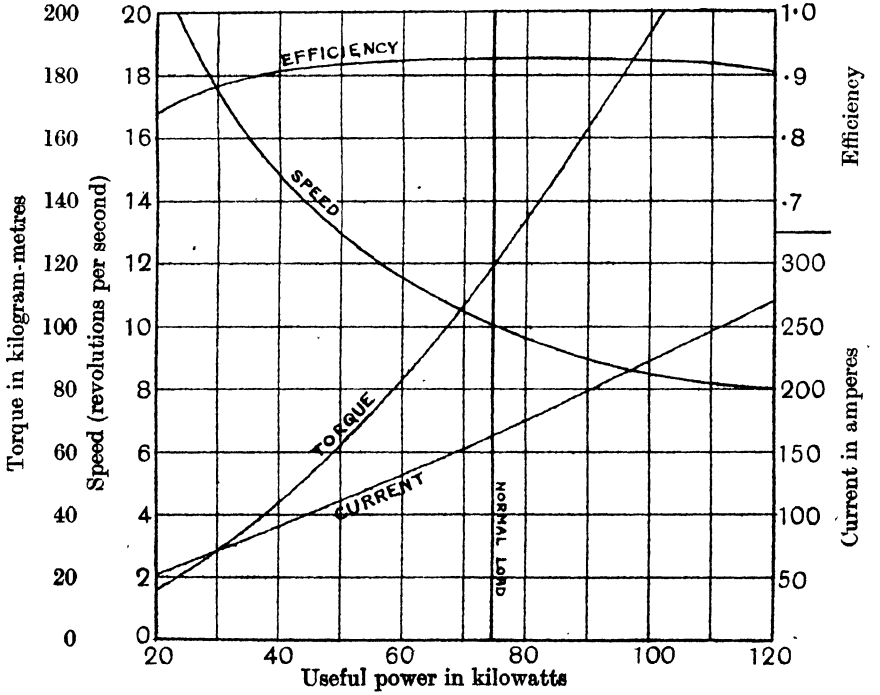



Fig. 211

Fig. 211 shows the result of a test on a 500 volt, 100 horse-power traction motor.

High efficiency is only attained by careful design. The two-pole construction is unusual except for small motors, as it does not advantageously utilise the space surrounding the armature.

**153. Testing and efficiency of generators.** If it requires $W$ watts to drive a generator which is only delivering $W'$ watts in electrical power, the ratio $\eta = W'/W$ is called the *efficiency* of the generator. Theoretically, the best way to find the efficiency of a generator for various currents is to drive it with an auxiliary motor which has already had a brake test performed on it, so that the power $W$ supplied to the generator is known in terms of the main motor current. In practice, however, there is always some uncertainty as to whether the friction of the bearings is the same in the two cases, and for laboratory purposes the method described below is more convenient, and of equal or greater accuracy.

Let $R$ be the resistance of the armature and $S$ that of the field-magnets of a shunt generator delivering a current $i$ at voltage $V$. The useful electrical power of the generator is $Vi$ watts; but it requires more power than this to drive the generator on account of the energy lost (1) in heating the armature coils, (2) in heating the field-magnet coils, (3) in hysteresis, eddy currents and friction. The loss under (1) is $R(i+j)^2$ watts, where $j = V/S$ is the field-magnet current, and the loss under (2) is $Sj^2$ watts. As regards (3), we notice that the drag of the generator on the motor, with direct coupling, does not tend to increase the friction of the bearings, and therefore the loss under (3) will certainly not increase to the same extent as under a rope brake test. This loss will be provisionally assumed to have the constant value $W_1$. Then the efficiency of the generator is given by

$$\eta = \frac{Vi}{Vi + R(i+j)^2 + Sj^2 + W_1}.$$

It follows that $\eta$ can be found for all currents if $W_1$ is known. To find it, first run the driving motor alone at the assigned speed. The electrical power consumed, less that lost in heating the armature and field-magnets, is the power $W_0$ required to overcome hysteresis, eddy currents and friction in the motor. Next couple

the generator to the motor and let it run with shunt winding on open circuit. Subtracting always the power lost in heating the various coils, we can find the power $W_0 + W_1$ absorbed in hysteresis, eddy currents and friction in motor and generator together, and hence $W_1$ is known.

There is a simple means of checking the assumption that $W_0$ and $W_1$ are constant, because the energy input, less the energy output and that lost in heating the conductors, gives the loss of energy in hysteresis, eddy currents and friction in motor and generator together. This does not usually differ very much from the $W_0 + W_1$ on open circuit; but if it does we can divide the excess equally between the two machines. This is all the more reasonable in that we have no means of judging exactly where the frictional losses arise.

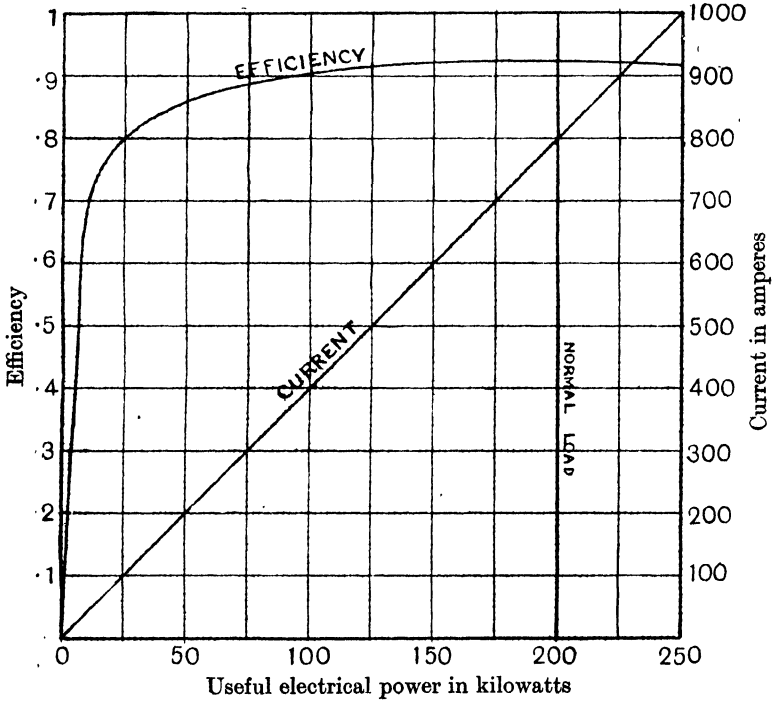



Fig. 212

Fig. 212, which refers to a 200 kilowatt compound generator delivering current at 250 volts, will give an idea of the efficiency attainable with large machines.

**154. Generation of alternating currents.** The principle of the generation of alternating currents has already been given (Art. 131). There are, however, practical reasons which make it inadvisable to construct large generators in that simple way. An alternator is designed to give high voltage, so that we can ultimately transform (Art. 158). It is, in consequence, best to dispense with slip-rings and collect the current from stationary windings, the field-magnets rotating instead of the armature, because the requisite magnetising current can always be obtained with a low voltage. The principle on which modern alternators are constructed will be understood from Fig. 213. The field-magnet frame, which carries an even number of coils, forms the rotating part of the machine. Current is furnished to the coils, which are not shown in the figure, by slip-rings attached to the axle, and the windings are arranged so as to produce positive and negative magnetic poles alternately along the circumference. The armature coils consist of one or more turns of stout wire or of copper strip, and are wound alternately in opposite directions, being finally connected to two fixed terminals $TT$ on the frame of the machine. In the position shown each coil is pierced by the maximum magnetic flux: the sign of the flux is the same for all the coils since their own winding alternates to keep pace with the alternations of the field-poles. When the axle has moved round so that a north pole is in the place originally occupied by a south pole, the sign of all the fluxes has changed, and this corresponds to one-half the period of alternation of the current. It is thus easy to see that if the speed of revolution is $n$ per second, and if there are $m$ poles on the machine, the number of complete periods of the alternating current is $\frac{1}{2}mn$ per second.

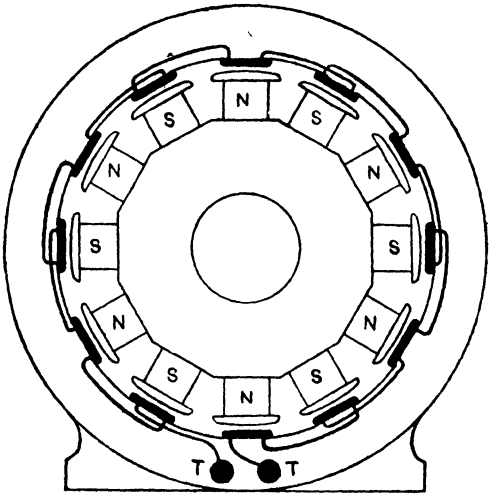


Fig. 213

The *rotary converter* (Fig. 214) exhibits the principles of the generation of alternating currents in another light, consisting

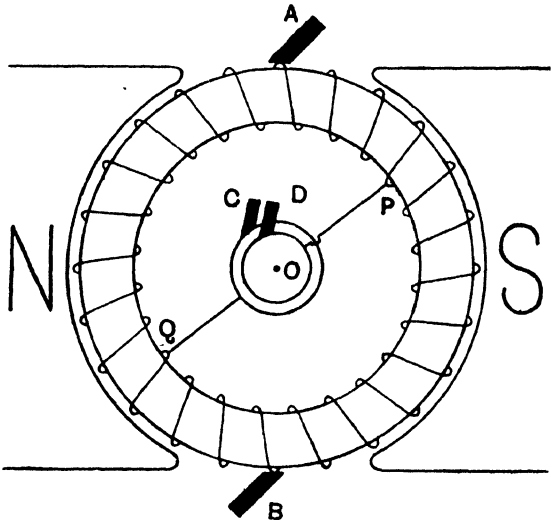



Fig. 214

essentially of a direct-current shunt motor with two tappings taken from the armature. The commutator has been omitted in order to simplify the figure, and the brushes $A$, $B$ shown as sliding directly over the armature windings. Two opposite points $P$, $Q$ on the armature are connected to two slip-rings sliding under two brushes $C$, $D$. Let us first calculate the difference of potential between $C$ and $D$ as a function of the time, supposing that no current is taken from $CD$ and that the lines of magnetic induction run straight across from left to right. If $OP$ makes an angle $\omega t$ with the initial line at time $t$, it follows from equation (176) that the potential of $P$ is $mnAB \sin \omega t$.

The potential of $Q$ is $mnAB \sin (\omega t + \pi) = - mnAB \sin \omega t$, so that the difference of potential between the brushes $C$, $D$ is $V = 2mnAB \sin \omega t$. Since the rotary converter is running as a motor with $A$, $B$ connected directly to the mains, this potential-difference must become equal to $V_0$, the E.M.F. of the mains, when $\omega t$ becomes equal to $\frac{1}{2}\pi$. Hence $2mnAB = V_0$, so that

$$V = V_0 \sin \omega t.$$

Thus alternating currents can be taken from the brushes $C$, $D$;

but the amplitude of the E.M.F. developed is not adjustable at will, being equal to the E.M.F. of the mains for all speeds of the converter. Smaller amplitudes can be obtained by connecting different points of the armature to slip-rings. Suppose for example that the angle $POQ$, instead of being two right angles, is equal to $\alpha$. Then the potentials at $P$ and $Q$ are $\frac{1}{2}V_0 \sin \omega t$ and $\frac{1}{2}V_0 \sin (\omega t + \alpha)$ respectively, so that

$$V = V_0 \sin \tfrac{1}{2}\alpha \cos (\omega t + \tfrac{1}{2}\alpha) \quad \ldots\ldots\ldots (184).$$

This is the most general case. The amplitude is proportional to $\sin \frac{1}{2}\alpha$, the effective value of the voltage being $\frac{V_0}{\sqrt{2}} \sin \frac{1}{2}\alpha$. We can thus find the angle $\alpha$ between any two tappings by measurements with an alternate-current voltmeter.

The frequency of alternating currents is found by calculation from the speed, or directly as follows. A number of steel springs are placed side by side, and near each one is placed a small electromagnet coil, all the coils being placed in parallel across the source of alternating potential. The alternating currents through the coils give rise to a forced vibration of the steel springs, which is most pronounced for that spring whose natural frequency of vibration most nearly coincides with the frequency of the alternating supply. This is the principle of the *resonance frequency-meter* of Hartmann-Kempf.

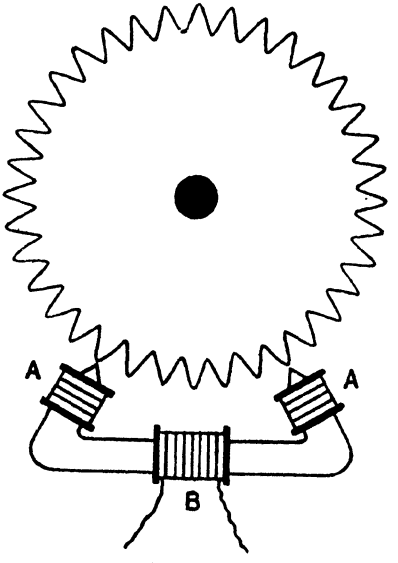


Fig. 215

High-frequency alternators (400 to 2000 complete periods per second) are constructed on the principle of Fig. 215. A small motor carries on the end of its axle a corrugated wheel made up of steel stampings. A small electromagnet $AA$ with wedge-shaped poles is placed near the wheel and arranged so that the poles are opposite projections on the corrugated wheel at the same time. When this occurs the lines of magnetic induction find an easy path through the wheel, and the flux of magnetic induction through

the core has its greatest value. It is smallest half-way between the teeth. The coils $AA$ are excited with direct current and the change of induction in the coil $B$ wound over the core gives rise to an alternating electromotive force. The frequency is $mn$, where $m$ is the number of teeth on the wheel and $n$ the number of revolutions per second.

**155. Delineation of alternating current curves. The oscillograph.** The curve connecting the E.M.F. of an alternator with the time can be determined experimentally by a method due to Joubert. On the axle $C$ of the alternator (Fig. 216) is fixed a thick disc of ebonite or insulating fibre, interrupted at one point of its circumference by a brass strip $EE$ let into it. Two copper brushes $FF$, carrying screw terminals, press on the disc, and are mounted on a holder so that they can be turned round into any position by the handle $G$. The angular position of the brushes $FF$ is indicated by means of a pointer fixed opposite a scale on a plate $K$ attached to the brush-holder. To determine the E.M.F. between the terminals $A$, $B$ of the alternator a circuit is set up containing the terminals $AB$, the brushes $FF$, a suitable resistance and a galvanometer as shown. The galvanometer should give readings on both sides of the zero.

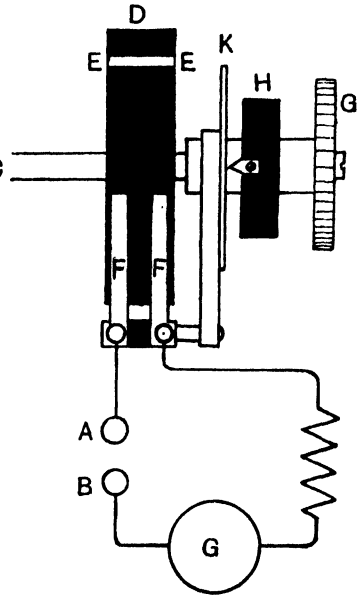


Fig. 216

Current can only flow in the galvanometer when the brass piece $EE$ completes the circuit, that is when the armature occupies a particular position in space. A momentary current will then be sent round the circuit. In consequence of these momentary currents, repeated once in every revolution, the galvanometer shows a steady deflexion proportional to the difference of potential between $A$ and $B$ at the moment of contact. The E.M.F. is found for

all positions of the armature by moving the brushes round. Absolute values are obtained by replacing $AB$ by an accumulator of known E.M.F. and observing the deflexion of the galvanometer.

This method can be used to find the time-variation of any current or potential having the same frequency as the alternator. Thus $AB$ may be replaced by the ends of a 1 ohm coil placed in any of the circuits used: the resulting curve is the curve of current in the circuit. In this way we can put in evidence the difference of phase between the current and E.M.F. in a circuit containing self-inductance (Art. 132).

Alternate current and potential curves can be exhibited to the eye, and also photographed, by means of the *oscillograph*. The principle of the instrument is shown in Fig. 217. A piece of phosphor-bronze is bent over a small pulley $P$ so as to form two thin parallel strips $ss$ lying between the wedge-shaped poles of a powerful electromagnet. The strips are kept tight by a spring attached to $P$, so that the restraining tension is high and the inertia low. The natural period of vibration of the strips, even with the attached mirror $M$, is very small (of the order 1/10,000 second). When a current flows through the strip the two sides are urged in opposite directions like the fibre of a string galvanometer, and the small mirror $M$ shows a deflexion. $KL$ is an oil-bath to damp out the natural vibrations of the strips. With alternating current the mirror oscillates backwards and forwards, but the oscillation is so much slower than natural vibration of the strips that it corresponds nearly to a succession of equilibrium positions.

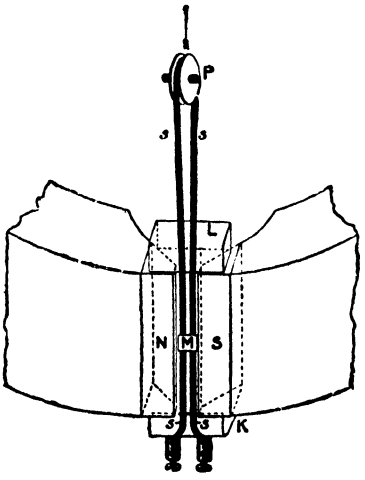


Fig. 217

The spot of light reflected from $M$ is received on a photographic plate moving at right angles to the direction of vibration. In this way the curve connecting the current with the time is recorded directly on the plate. Oscillographs are fitted with two separate

strips for simultaneous observation of the current and E.M.F. in a circuit, the current strip being shunted and the voltage strip in series with a high resistance.

The oscillograph is not essentially different from the string galvanometer (Art. 88), which is also used for recording purposes, e.g. of the passage of sound waves past selected points, in conjunction with a hot-wire microphone (Art. 76). A number of such microphones, with a central galvanometer of the same number of strings, enable the position of a distant gun or shell-burst to be located.

**156. Harmonic analysis of curves.** According to Fourier's theorem* an arbitrary function $f(x)$ can be expanded in the range $0 \leqq x \leqq 2\pi$, in a series of the form

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + \ldots + a_n \cos nx + b_n \sin nx + \ldots \quad \ldots.(185),$$

where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

and the other coefficients are given by

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\,dx.$$

In the case with which we are concerned $f(x)$ is a periodic function of $x$, of period $2\pi$, and the expansion (185) holds for all values of $x$.

The curve $y = f(x)$ is said to be harmonically analysed when the coefficients $a$ and $b$ are known. The constant term $a_0$ represents the mean value of $f(x)$ over a complete period, and is generally zero in electrical problems. The most important terms are the terms $a_1 \cos x + b_1 \sin x$ of the first order; and if the curve is sinoidal these are the only terms occurring. The higher terms are called *harmonics* by analogy with the modes of vibration of a stretched string, and the magnitude of the harmonics shows the extent to which the curve deviates from the sine-form.

It is interesting to note the effect of inductance on the current curve when harmonics are present in the alternating E.M.F. applied to a circuit. Let $p/2\pi$ be the frequency of the applied E.M.F. $V$, and

* The expansion holds for any function met with in electrical practice, since it does not require either $f(x)$ or its derivate to be continuous.

let $V$ be expanded in the Fourier's series

$$V = a_1 \cos pt + b_1 \sin pt + \ldots + a_n \cos npt + b_n \sin npt + \ldots,$$

the resistance in the circuit being neglected. The equation

$$L\,di/dt = V$$

gives on integration

$$i = \frac{a_1 \sin pt - b_1 \cos pt}{Lp} + \ldots + \frac{a_n \sin npt - b_n \cos npt}{nLp} + \ldots.$$

Since the denominators increase steadily with $n$, the effect of harmonics in the E.M.F. curve is diminished: self-inductance makes the current curve more nearly sinoidal. Capacity in the circuit, on the other hand, exaggerates harmonics, and may cause a considerable deviation from the sine-form even when the E.M.F. curve is nearly pure. For then

$$i = C\frac{dV}{dt} = C\sum_{n=1}^{\infty} np\,(b_n \cos npt - a_n \sin npt),$$

and the higher harmonics are multiplied by larger and larger factors.

The curves considered in electrical problems nearly always repeat themselves after a half-wave on the opposite side of the axis of $x$, i.e. they satisfy the condition $f(x + \pi) = -f(x)$. In this case only *odd* harmonics can occur. In practical harmonic analysis we take $2n$ ordinates on a half-wave corresponding to equidistant values of $x$ and determine the coefficients in the equation

$$\begin{aligned} y = {} & a_1 \cos x + a_3 \cos 3x + \ldots + a_{2n-1} \cos (2n-1)\,x \\ & + b_1 \sin x + b_3 \sin 3x + \ldots + b_{2n-1} \sin (2n-1)\,x \quad \ldots(186), \end{aligned}$$

so that $y$ may have assigned values for the values

$$0, \quad \frac{\pi}{2n}, \quad \frac{2\pi}{2n}, \ldots \frac{(2n-1)\,\pi}{2n}$$

of $x$. If these values are $y_0, y_1, \ldots y_{2n-1}$, we have then $2n$ equations to determine the coefficients, namely those of the type

$$\begin{aligned} y_r = {} & a_1 \cos \frac{r\pi}{2n} + a_3 \cos \frac{3r\pi}{2n} + \ldots + a_{2n-1} \cos \frac{(2n-1)\,r\pi}{2n} \\ & + b_1 \sin \frac{r\pi}{2n} + b_3 \sin \frac{3r\pi}{2n} + \ldots + b_{2n-1} \sin \frac{(2n-1)\,r\pi}{2n} \quad \ldots(187). \end{aligned}$$

To solve equations (187), multiply the $r$th equation by

$$\cos\frac{(2m-1)\,r\pi}{n}$$

and sum from $r=0$ to $r=2n-1$. The left-hand side becomes $\sum_{r=0}^{2n-1} y_r \cos\frac{(2m-1)\,r\pi}{n}$. On the right-hand side all the terms vanish except that containing $a_{2m-1}$. Hence

$$\sum_{r=0}^{2n-1} y_r \cos\frac{(2m-1)\,r\pi}{n} = a_{2m-1}\sum_{r=0}^{2n-1}\cos^2\frac{(2m-1)\,r\pi}{n},$$

so that $$na_{2m-1} = \sum_{r=0}^{2n-1} y_r \cos\frac{(2m-1)\,r\pi}{n}.$$

Similarly $$nb_{2m-1} = \sum_{r=0}^{2n-1} y_r \sin\frac{(2m-1)\,r\pi}{n}.$$ .....(188)

Thus the coefficients can be found. The following scheme, due to Runge, will be found convenient in calculating the first three harmonics from a set of 12 equidistant ordinates $y_0, y_1, \ldots y_{11}$ on the half-wave, corresponding to the values 0°, 15°, ... 165° of $x$. It is understood that the ordinates corresponding to $x = 180°$, 195°, ... 345° are equal and opposite to the above. The higher harmonics are not calculated.

*Scheme for twelve ordinates.*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ |
| | | $y_{11}$ | $y_{10}$ | $y_9$ | $y_8$ | $y_7$ | |
| Sums | | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ |
| Differences | $d_0$ | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | |

$$e_1 = d_1 - d_3 - d_5$$
$$e_2 = d_0 - d_4$$
$$e_3 = s_1 + s_3 - s_5$$
$$e_4 = s_2 - s_6$$

| | $6a_1$ | $6b_1$ | $6a_3$ | $6b_3$ | $6a_5$ | $6b_5$ |
|---|---|---|---|---|---|---|
| sin 15° = ·259 | $d_5$ | $s_1$ | — | — | $d_1$ | $s_5$ |
| sin 30° = ·500 | $d_4$ | $s_2$ | — | — | $d_4$ | $s_2$ |
| sin 45° = ·707 | $d_3$ | $s_3$ | $e_1$ | $e_3$ | $-d_3$ | $-s_3$ |
| sin 60° = ·866 | $d_2$ | $s_4$ | — | — | $-d_2$ | $-s_4$ |
| sin 75° = ·966 | $d_1$ | $s_5$ | — | — | $d_5$ | $s_1$ |
| sin 90° = 1 | $d_0$ | $s_6$ | $e_2$ | $e_4$ | $d_0$ | $s_6$ |

The meaning of the last table is that the rows are first to be multiplied by the sines on the left, and the columns are then to be added. Thus

$$6a_5 = d_1 \sin 15° + d_4 \sin 30° - d_3 \sin 45° - d_2 \sin 60° + d_5 \sin 75° + d_0 \sin 90° \quad \ldots(189).$$

The accuracy of this table can be verified from equation (188). Consider for example the last equation. Here $m = 3$ and $n = 6$, so that (188) gives

$$6a_5 = y_0 \cos 0° + y_1 \cos 75° + y_2 \cos 150° + y_3 \cos 225° + y_4 \cos 300° + y_5 \cos 375° + y_6 \cos 450° + y_7 \cos 525° + y_8 \cos 600° + y_9 \cos 675° + y_{10} \cos 750° + y_{11} \cos 825°.$$

The trigonometrical functions can clearly be expressed in terms of $\sin 15°$, $\sin 30°$, ... $\sin 90°$, and we find

$$6a_5 = (y_1 - y_{11}) \sin 15° + (y_4 - y_8) \sin 30° + (y_9 - y_3) \sin 45° + (y_{10} - y_2) \sin 60° + (y_5 - y_7) \sin 75° + y_0 \sin 90°,$$

which is identical with equation (189).

**157. Behaviour of a coil with an iron core.** When a sinoidal E.M.F. is applied to the terminals of a coil without iron, the current

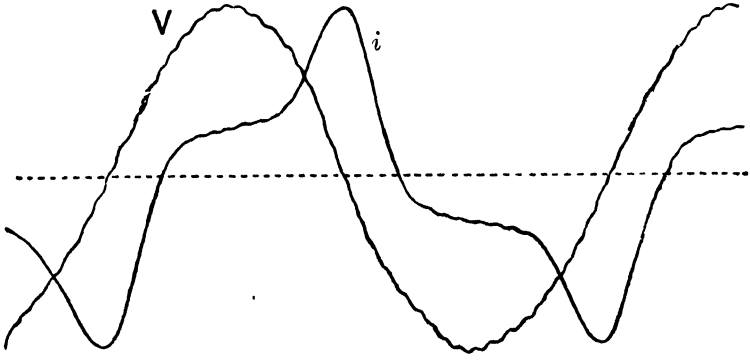



Fig. 218

curve is also sinoidal, but displaced from the E.M.F. curve by the amount $\tan^{-1}(Lp/R)$, reckoning a whole wave as $2\pi$. This ceases to be true when the coil has an iron core, the current curve being in general distorted as well as displaced. Fig. 218 is a copy of an oscillograph curve of current and potential for such a coil.

It is practically important to find what current a given coil takes for a given applied voltage (effective values), and the power absorbed by it. The connexions for this are shown in Fig. 219, where

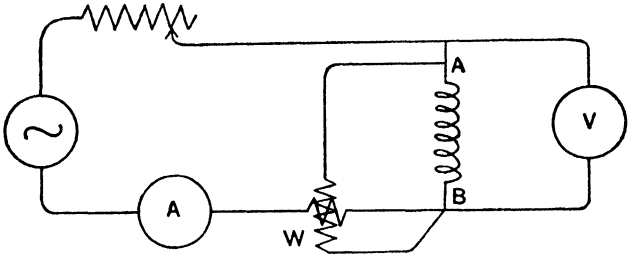



Fig. 219

the crossed coils represent symbolically the current and voltage coils of the wattmeter $W$. The current coil is placed in the main circuit and the voltage coil applied to the coil $AB$ as a shunt, so that the wattmeter measures the power $W$ directly. Strictly speaking, the coil cannot be said to have a self-inductance at all, since the flux through it is not proportional to the current. But there must be something which corresponds to inductance, and we should define it in such a way that the commonest electrical formulae still hold good.

Number of turns 32, cross-section of core 20 cm.$^2$,
actual resistance of coil 0·15 ohm

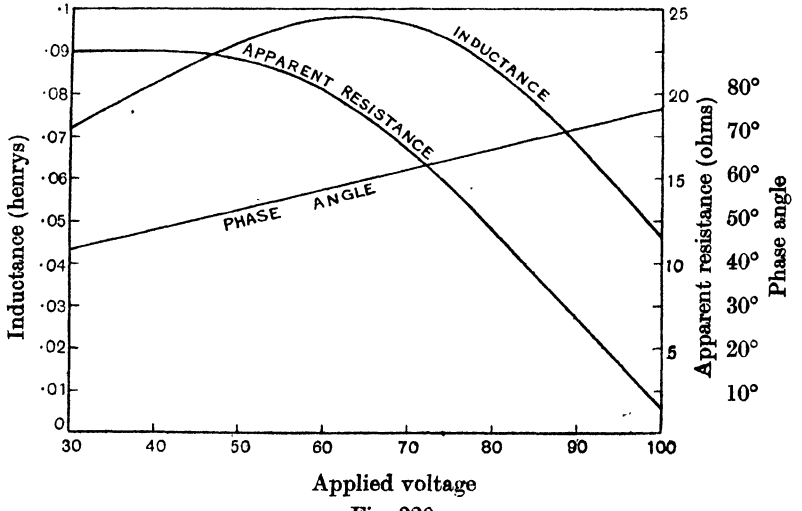



Fig. 220

The *apparent resistance* of the coil is defined so that the power $W$ is equal to $RI^2$. Here $R$ is no longer the true resistance of the coil, but considerably greater, since power is absorbed not only in heating the windings but also in hysteresis and eddy currents in the core. We next define a *phase angle* $\alpha$ so that $W = VI \cos \alpha$ (cf. Art. 132); and finally the *self-inductance* $L$ so that $\tan \alpha = Lp/R$, where $p/2\pi$ is the frequency of the E.M.F. applied.

The apparent resistance and the self-inductance so defined are not constants, but vary with the frequency and voltage. The extent to which they may vary in practice will be seen from Fig. 220.

**158. Transformers and the transmission of power.** The principle of the alternating current transformer has already been given (Art. 136). Its function is to change the E.M.F. of an alternating current circuit; and it is in this connexion that the special advantages of alternating currents begin to appear. In order to raise the E.M.F. of a direct-current circuit from 100 to 1000 volts it is necessary to employ a 100-volt motor to drive a 1000-volt generator; that is, to use rotating machinery. With alternating currents the transformation is performed by a stationary transformer which requires no attention whatever and also has a higher efficiency than any machine.

In transmitting power to long distances a change of voltage is sometimes necessary and generally advantageous. Suppose for example that 100 kilowatts have to be transmitted over a distance of 10 kilometres (6 miles). If this is done at 200 volts a current of 500 amperes would have to be sent with correspondingly thick cable. If, however, the voltage was raised to 2000 the current would be only 50 amperes. There is therefore a considerable saving in copper when high voltages are used, and 50,000 volts is not unusual for long-distance transmission. The high voltages used for transmission would be dangerous to the ordinary user, so that a second transformer must be placed at the end of the line to reduce the voltage again.

The circuits for testing a transformer are shown in Fig. 221. The primary circuit is set up with an ammeter and wattmeter as in Fig. 219, and various currents are taken off the secondary by glow-

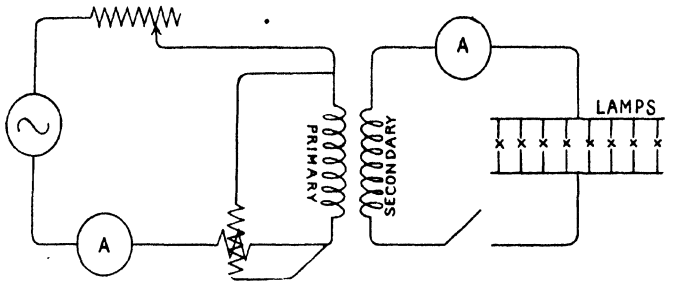



Fig. 221

lamps in parallel. We require a key for placing one and the same voltmeter across the primary and secondary terminals, which can be made by removing the cross-wires from a reversing-key.

Readings are taken of all the instruments when 1, 2, 3, . . . lamps are placed in parallel in the secondary circuit. It is best to adjust

Mean radius of ring 7·5 cm., cross-section of core 18·8 cm.$^2$ Primary 294 turns, resistance 1·76 ohms. Secondary 226 turns, resistance 1·61 ohms. Secondary voltage 53·5.

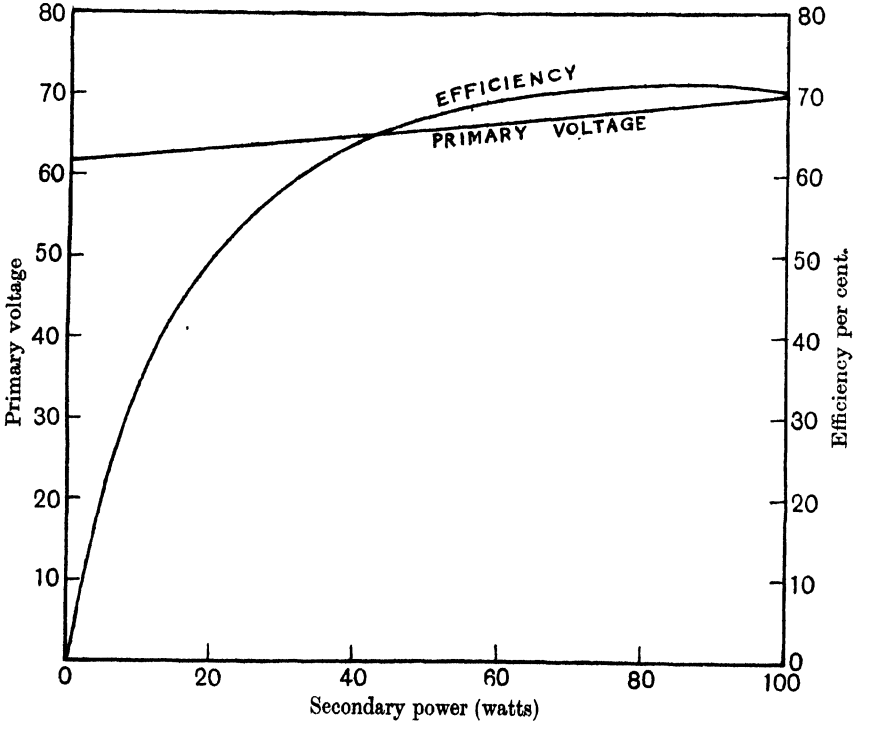



Fig. 222

the rheostat each time till the secondary voltage $V_2$ has an assigned value. The wattmeter gives the primary power $W_1$ directly. It is not necessary to have a wattmeter in the secondary, because the lamps are practically non-inductive* and the power is given by $W_2 = V_2 I_2$, where $I_2$ is the secondary current. The *efficiency* of the transformer is $\eta = W_2/W_1$.

Fig. 222 shows the result of a test on a small transformer, of which the primary voltage and efficiency are plotted in terms of the secondary power $W_2$. $V_1$ rises as more and more current is taken from the secondary; hence in a lighting system the primary voltage must increase with the load in order to maintain constant voltage across the lamps.

The efficiency of commercial transformers is very high, often exceeding 95 per cent. The losses in transformers are due to hysteresis, eddy currents, and heating of the copper windings. The core is of iron having a hysteresis loss, and the secondary coils are wound closely over or near to the primary coils, so that the lines of force passing through the primary shall also pass through the secondary. Small transformers are used in wireless telegraphy when a direct current is superposed on an alternating current, and it is undesirable to send the former through the telephones.

**159. Alternating current motors.** There is no alternating current machine having the properties of the direct-current shunt motor, because self-inductance in armature and field-magnets entirely alters the ordinary relationship of shunt circuits. The *alternating current series motor* is similar in principle to the corresponding direct-current machine. The torque is proportional at any instant to the product of the armature current $i$ and the flux $N$. These are always in phase with each other, so that if both of them are supposed to vary as $\cos pt$ the couple at any time is proportional to $\cos^2 pt$. There will therefore, on the average, be a couple tending to set the armature in rotation. The modifications required for converting a direct into an alternating current series motor are twofold. In the first place, the field-magnet core must be laminated

* For a single lamp $L$ is of probably the order 200 cm. $= 2 \times 10^{-7}$ henry (cf. Art. 124), and $R$ of the order 300 ohms. Hence when $p = 300$, $Lp/R$ is of the order $2 \times 10^{-7}$. The self-inductance of a laboratory circuit of ordinary dimensions is of the order of 2000 cm.

as well as the armature. Secondly, the effective self-inductance of the whole circuit must be diminished, for otherwise it would be unable to take up power from the mains. The method of overcoming this defect in the series motor is interesting. We cannot short-circuit either field-magnets or armature without preventing the motor from working: but the inductive effect of the armature can be neutralised by making it the primary of a transformer, the secondary winding being a fixed coil short-circuited on itself.

If $N$ is the flux of magnetic induction through the secondary, $R$ the resistance and $i$ the current in the secondary, $dN/dt + Ri = 0$. When $R$ is small, $Ri$ would remain finite if all the inductance of the primary was in the armature (Art. 136); but since this is not so $Ri$ becomes small with $R$. Hence the flux $N$ is small, which is the principle of compensation referred to. Since the magnetic field due to the armature currents is perpendicular to the main field, the additional windings on the field-magnet frame are placed at right angles to the ordinary coils.

Elihu Thomson's *repulsion motor* is another commutator motor. In this machine alternating currents are supplied to the field-magnets, and the armature, which is the same as that of a direct-current motor, is short-circuited across the brushes. The varying flux in the field-magnets first induces currents in the armature and then acts on them mechanically as in the experiments described in Art. 138. The armature is insulated from the field.

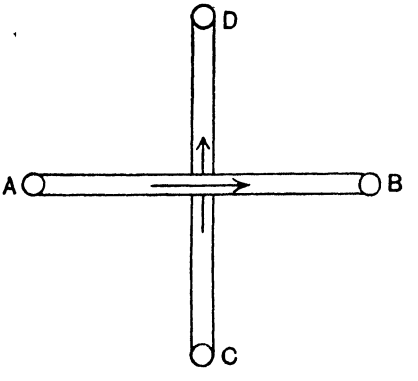


Fig. 223

**160. Polyphase currents and rotating magnetic fields.** Let $AB$, $CD$ (Fig. 223) represent a vertical view of two coils at right angles to one another, $AB$ being parallel to the axis of $x$ and $CD$ to the axis of $y$. The current in $AB$ is reckoned as positive when it flows upwards at $A$ and downwards at $B$, so that the current in the top half of the coil is in the direction of the arrow. Similarly the current in $CD$ is reckoned as positive when it flows upwards at $C$ and downwards at $D$. Let

alternating currents of equal amplitude be sent through the coils, the phases differing by the angle $\frac{1}{2}\pi$, so that the current in $AB$ is $\cos pt$ when that in $CD$ is $\sin pt$. The magnetic force at the centre of the coils is then compounded of a force $A \cos pt$ parallel to the $y$-axis and $-A \sin pt$ parallel to the $x$-axis, where $A$ is a constant. The resultant is of magnitude $A$ and makes an angle $\frac{1}{2}\pi + pt$ with the axis of $x$. Hence the effect of the two coils is to produce at the centre a magnetic field of constant intensity, which rotates with the angular velocity $p$. If iron is present the same is true of the induced magnetisation, and thus we have magnetic poles rotating in a mass of iron at rest, caused by two alternating currents in different phases.

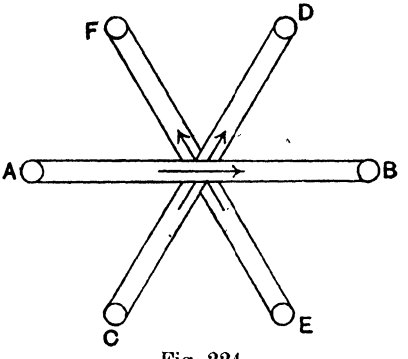


Fig. 224

The same effect can be produced by three (or more) alternating currents. Let $AB$, $CD$, $EF$ (Fig. 224) be traversed by the currents $\cos pt$, $\cos(pt + \frac{1}{3}\pi)$, $\cos(pt + \frac{2}{3}\pi)$ respectively. Then the components of magnetic force at the centre are

$$A\{-\tfrac{1}{2}\sqrt{3}\cos(pt + \tfrac{1}{3}\pi) - \tfrac{1}{2}\sqrt{3}\cos(pt + \tfrac{3}{2}\pi)\}$$

and

$$A\{\cos pt + \tfrac{1}{2}\cos(pt + \tfrac{1}{3}\pi) - \tfrac{1}{2}\cos(pt + \tfrac{2}{3}\pi)\},$$

or

$$\tfrac{3}{2}A \sin pt \text{ and } \tfrac{3}{2}A \cos pt.$$

The resultant field $\frac{3}{2}A$ therefore rotates with angular velocity $p$ as before.

The method of generating polyphase electromotive forces will be understood from Fig. 213. If an extra set of coils is inserted halfway between those shown, and brought out to two additional terminals, the phenomena of induction in them will take place a quarter of a period after those in the first set of coils. A machine so constructed would be a *two-phase generator*. Similarly a three-phase generator can be constructed by arranging three sets of coils between every two consecutive field-magnet coils.

Two-phase currents may also be taken from a rotary converter. For this purpose four points on the armature, separated from each

other by 90°, are connected to four slip-rings mounted on the axle. If the vectorial angles of $OP$ and $OQ$ in Fig. 214 are $\omega t$ and $\omega t + \pi$ at time $t$, the vectorial angles corresponding to the other phase (say $OR$, $OS$) are $\omega t + \frac{1}{2}\pi$ and $\omega t + \frac{3}{2}\pi$. The difference of potential between $P$ and $Q$ at time $t$ is $V_0 \sin \omega t$ (Art. 154), and hence the difference of potential between $R$ and $S$ is

$$V_0 \sin (\omega t + \tfrac{1}{2}\pi) = V_0 \cos \omega t.$$

The two potentials have therefore the proper phase-relation to give two-phase currents. A three-phase current can be taken off similarly with six slip-rings.

It is possible, with "star" and "delta" windings, to obtain three-phase currents with three, or at most four, wires instead of six. It is not proposed to describe these methods here.

**161. The induction motor.** One form of rotating field motor is the *squirrel-cage* motor, the principle of which is shown in Fig. 225. The currents, in this case two-phase, are led through two coils $AB$, $CD$ in a fixed laminated iron frame called the *stator*. The moving part, or *rotor*, is made up of thin discs threaded on a spindle like the drum armature of a direct-current machine. The slots in the discs hold a number of stout wires or strips of copper: these however are not wound according to any system, but simply connected together at both ends by a solid copper ring, so that they are all short-circuited.

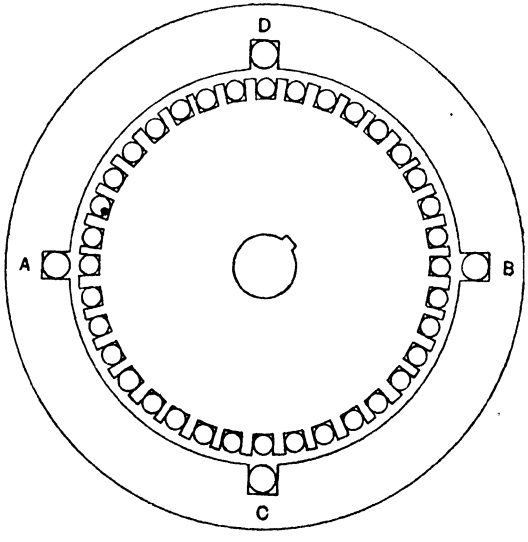


Fig. 225

In order to see how a squirrel-cage motor works, let us begin by calculating the currents in the bars under the assumption that the

inductive effect of the bars on one another can be neglected. Consider two opposite bars $P$, $Q$ (Fig. 226) situated in a uniform magnetic field $H$ which rotates with angular velocity $p$, the angular velocity of the rotor being $\omega$. Any current that may flow up the bar $P$ will enter the binding ring at the nearer end of the rotor, and the same current will flow down the opposite bar $Q$. Hence we may imagine the bars $P$, $Q$ joined at top and bottom so as to form an insulated circuit of resistance $R$, where $\frac{1}{2}R$ is the resistance of a single bar. Let $A$ be the area of this imaginary circuit, and write $N_0 = HA$ for the greatest possible flux of force through a coil. The actual flux through the coil $PQ$ is then $N = N_0 \sin\theta$. In calculating $dN/dt$ we must remember that the coil is rotating with angular velocity $\omega$, so that $\frac{d}{dt}(\theta + pt) = \omega$. Hence $\frac{d\theta}{dt} = -(p - \omega)$ and the current $i$ in the coil is given by

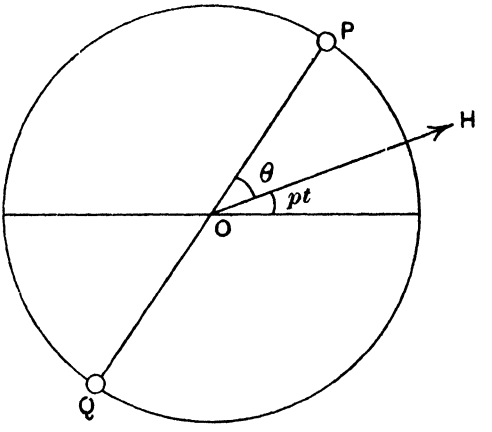


Fig. 226

$$Ri = -\frac{dN}{dt} = -N_0 \cos\theta \frac{d\theta}{dt} = N_0 (p - \omega) \cos\theta.$$

The induced current vanishes when $p = \omega$, since the coil would then rotate at the same rate as the field and the flux of force would never change. The speed of the motor must be less than that of the field, but if $R$ is small sufficient current will be developed for quite small values of $p - \omega$, so that the difference between the two speeds need not be great. This difference is called the *slip*, and in large motors does not amount to more than 2 or 3 per cent. at full load. The current in a bar depends on its position with respect to the field at the time being. It is greatest when $\theta = 0$ and zero when $\theta = \frac{1}{2}\pi$. The couple on the coil $PQ$ due to the field $H$ is

$$iHA \cos\theta = N_0^2 (p - \omega) \cos^2\theta / R,$$

which is positive for all values of $\theta$. Hence there is always a couple

on the rotor when $\omega$ is less than $p$, and this couple, *caeteris paribus*, is proportional to the slip.

In reality, the mutual inductive effects of the currents in the rotor are very great and have to be taken into account. It is however still possible to find the distribution of current, provided that two simplifying assumptions are made. Firstly, the number of rotor bars will be taken to be very great, so that the number between angles $\alpha$ and $\alpha + d\alpha$ is equal to $2n\,d\alpha/\pi$, where $4n$ is the total number of bars on the rotor. Secondly, we make a simplifying assumption concerning the mutual inductances. The flux of magnetic force through one coil due to unit current in another is greatest when the coils coincide and zero when their planes are at right angles. Hence we shall not be very far wrong in assuming that the flux is $L\cos\theta$, where $L$ is a constant and $\theta$ is the angle between the two coils. We shall also suppose that no iron is present in the field.

We have to calculate the flux of magnetic force through the coil $PQ$ (Fig. 226) due to the currents in the remaining coils. If $f(\theta)$ is the current in the bar $P$, the flux through $PQ$ due to the group of $2n\,d\alpha/\pi$ coils between $\theta = \alpha$ and $\theta = \alpha + d\alpha$ is

$$\frac{2n\,d\alpha}{\pi} f(\alpha)\, L\cos(\theta - \alpha).$$

Hence the required flux is

$$\frac{2nL}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(\alpha)\cos(\theta-\alpha)\,d\alpha,$$

and therefore

$$N = N_0\sin\theta + \frac{2nL}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(\alpha)\cos(\theta-\alpha)\,d\alpha.$$

As before, in calculating $dN/dt$ we put $d\theta/dt = -(p-\omega)$. Hence

$$\frac{dN}{dt} = -N_0(p-\omega)\cos\theta + \frac{2nL(p-\omega)}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(\alpha)\sin(\theta-\alpha)\,d\alpha,$$

and the law of electromagnetic induction gives

$$Rf(\theta) = N_0(p-\omega)\cos\theta - \frac{2nL(p-\omega)}{\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(\alpha)\sin(\theta-\alpha)\,d\alpha \quad (190).$$

Equation (190) has to be used to determine the distribution of current along the circumference. Assume a solution

$$f(\theta) = X\cos\theta + Y\sin\theta.$$

Substituting in (190),

$$R(X\cos\theta + Y\sin\theta) = N_0(p-\omega)\cos\theta - nL(p-\omega)(X\sin\theta - Y\cos\theta).$$

This holds for all values of $\theta$ if

$$XR - YnL(p-\omega) = N_0(p-\omega) \quad \text{and} \quad XnL(p-\omega) + YR = 0.$$

Solving for $X$ and $Y$ and substituting, we have

$$f(\theta) = N_0(p-\omega)\frac{R\cos\theta - nL(p-\omega)\sin\theta}{R^2 + n^2L^2(p-\omega)^2} \quad \ldots(191).$$

Thus the distribution of current changes according to a sine-law as we pass round the circumference, as before: but the maximum is displaced and the current is no longer zero at right angles to the field. On multiplying the expression (191) by $N_0\cos\theta \,.\, 2n\,d\theta/\pi$ and integrating we find that the torque exerted by the motor is

$$T = \frac{2n}{\pi}\frac{N_0^2(p-\omega)}{R^2 + n^2L^2(p-\omega)^2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\{R\cos^2\theta - nL(p-\omega)\cos\theta\sin\theta\}\,d\theta$$

or

$$T = \frac{nRN_0^2(p-\omega)}{R^2 + n^2L^2(p-\omega)^2} \quad \ldots\ldots\ldots\ldots(192).$$

$N_0$ is not a constant, because the rotor currents react on the stator and induce electromotive forces in it. Nevertheless, the performance of the motor is given in its main features by supposing that it is. Writing $x$ for $nL(p-\omega)/R$, $T$ is given as a function of the speed $\omega$ by the equation

$$T = A\frac{x}{1+x^2},$$

where $A$ is a constant. This expression has a maximum value $\frac{1}{2}A$ when $x = 1$: it vanishes when $x = 0$ and is small when $x$ is large. Hence we should expect the torque exerted by the motor to be greatest for some speed less than $p$, and to be comparatively small when the motor is starting ($\omega = 0$). This is what actually occurs, the relation between speed and torque being somewhat as shown in Fig. 227. In reality, the starting torque is smaller than we might expect, because the heavy currents induced in the rotor weaken the field just as the short-circuited secondary of a transformer weakens the flux, so that $N_0$ is smaller at low than at high speeds.

The combination of low starting torque with high starting current is a serious drawback to the squirrel-cage motor. For this reason

motors larger than about five horse-power require a starting device. The best plan is to give up the squirrel-cage construction and have regular windings on the rotor, which are brought out to slip-rings on the axle. The resistance of the rotor can then be made large to begin with, and cut out as the speed rises.

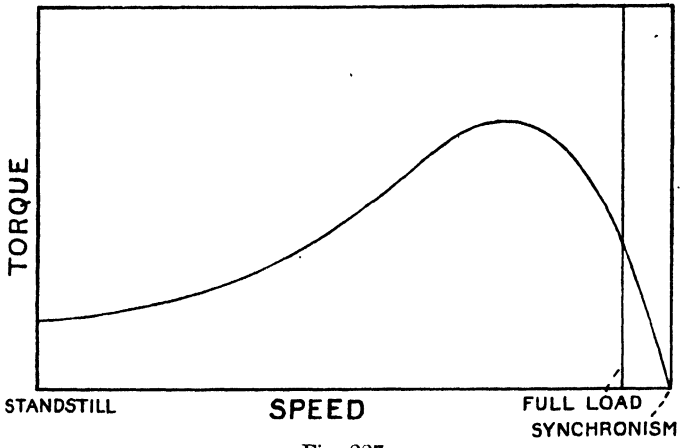



Fig. 227

**162. The electric arc.** The electric arc was discovered in 1806 by Davy, who observed that if two carbon rods, attached to the terminals of a battery of sufficiently high potential, are brought together and then separated, a brilliant light is produced so long as the rods are not too far from one another. Fig. 228 shows a carbon arc lamp arranged for lantern lighting, in which the screws are used for striking the arc and for adjusting the position of the carbons. The phenomena occurring on lighting are as follows. A large current flows through the circuit when the carbons are first brought into contact. Since the chief resistance

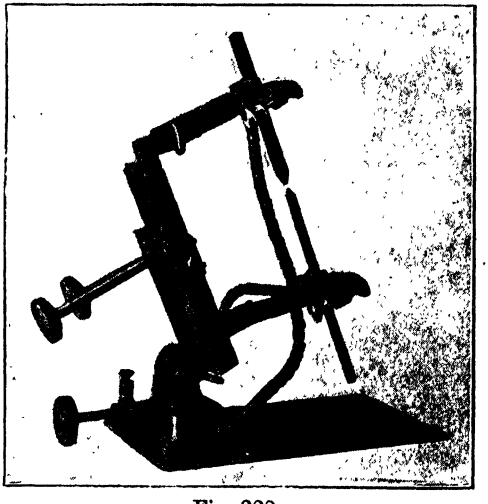


Fig. 228

is where the carbons touch, much heat is developed there, and the surfaces are raised to incandescence. On separating the electrodes the intervening space becomes a conductor, partly because air conducts at a sufficiently high temperature and partly because electrons are emitted from the negative carbon (cf. Art. 211), and the current continues to flow.

The arc is best observed by focusing its image on to a sheet of white paper. The positive carbon, which becomes the hotter of the two, is rounded off and has a white-hot depression in the centre, from which most of the light of the arc comes. The negative carbon is covered with small bubbles which constantly rise and evaporate and are probably due to impurities contained in the carbon. The air between the carbons emits comparatively little light, a yellow bow touching the negative carbon and a violet portion lying opposite the hottest part of the positive carbon.

Arcs can be struck with other materials besides carbon. Thus magnetite can be used for the negative electrode, and a copper rod for the positive, which increases the brightness of the arc itself. The same result is attained in the *flame arc*, where the positive carbon is impregnated with calcium or other salts, which evaporate and impart a brilliant luminosity to the arc stream. The high temperature of the electrodes of the carbon arc (positive 3430° C., negative about 3000° C.) causes them to be consumed in air at the rate of about 2 cm. per hour, and arc lamps are fitted with automatic devices for keeping the carbons at a constant distance apart. The rate of consumption is about 1 mm. per hour in *enclosed arcs*, contained in a globe which is nearly air-tight. The economy of different kinds of arc depends very much on the relative cost of current and carbons.

The characteristic of direct-current arcs is that, as the current rises, the difference of potential between the electrodes falls. Fig. 229 shows the potential-difference for magnetite arcs of various lengths and for currents up to 8 amperes. With the longer arcs more voltage is required for the same current, and Steinmetz found that the voltage can be represented over a large range by the formula

$$V = V_0 + \frac{a\,(l+b)}{\sqrt{i}},$$

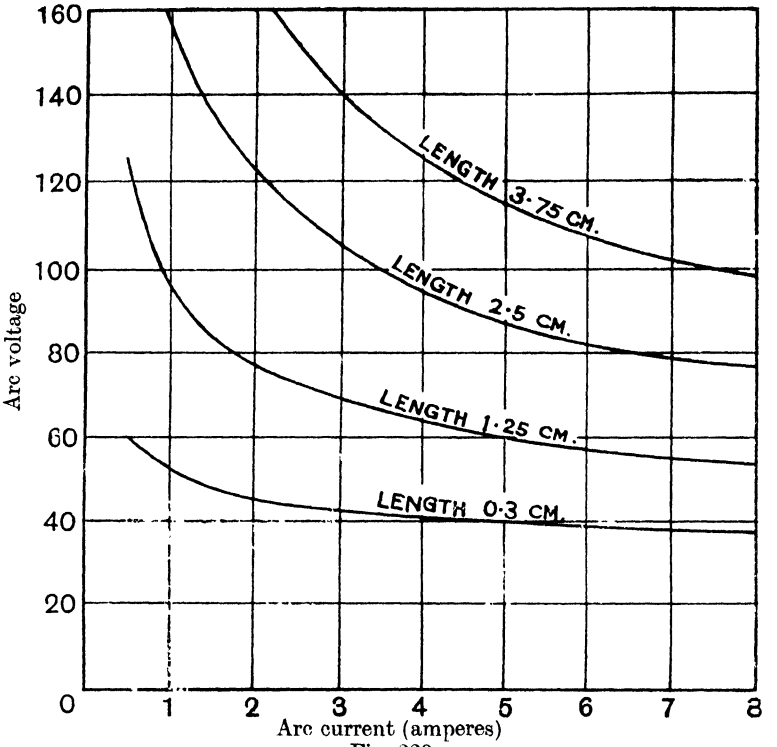


Fig. 229

where $l$ is the length of the arc and $V_0$, $a$, $b$ are constants. For the magnetite arc $V_0 = 30, \quad a = 48{\cdot}5, \quad b = 0{\cdot}125.$

The carbon arc has the constants

$$V_0 = 36, \quad a = 51, \quad b = 0{\cdot}8,$$

but the formula is less accurate. $V_0$ clearly represents the least possible voltage that will burn an arc at all.

Arcs can be struck between metals, but water cooling is necessary. The *mercury arc lamp*, which is sometimes used, burns between mercury electrodes in a vacuum. The arc can be struck by tilting the tube until the two streams of mercury run together.

The light from the carbon arc, when examined with the spectroscope, is found to be mainly continuous, being crossed by lines only slightly brighter than the background. Metal arcs, on the other hand, yield brilliant line-spectra. The absence of strong lines in the red unfits the mercury arc for certain purposes, as red objects appear black by its light.

**163. Electric lighting. Meters.** A description of the methods of determining the amount of light given out by a glow-lamp, or its candle-power, is beyond the scope of this book: but a relative determination may be made in the laboratory to illustrate the conditions regulating the economic emission of light. The comparison is conveniently carried out by means of the paraffin-block photometer, the most accurate of the simple photometers. It consists of two similar blocks of paraffin-wax pressed together after being slightly warmed, with a sheet of tinfoil between. Paraffin is translucent, and when illuminated from one side one-half of the compound block appears bright and the other half dark. When illuminated on opposite sides by lamps of candle-power $I$, $J$ at distances $r$, $s$ respectively, the two halves will appear equally bright when $I/r^2 = J/s^2$, so that the ratio of $I$ to $J$ can be determined by experiment. For our present purpose it is sufficient to take as a standard a new lamp, run it off the marked voltage, and assume that its marked candle-power is correct. The lamp to be tested is also run off the same mains in series with an ammeter and a resistance box, so that various currents can be sent through it. More than the normal current can be used by placing one or more accumulators in series with the lamp.

Suppose that at any particular stage the applied voltage is $V$ and the current $i$, and that the resistance in circuit, other than the lamp itself, is $R$. The difference of potential between the terminals of the lamp will be $V - Ri$, so that $Vi - Ri^2$ watts are consumed in the lamp. Dividing this by the candle-power of the lamp, as found with the photometer, we can find the watts required per candle-power for any current. The resistance of the filament will rise as its temperature increases, and is equal to $\frac{V - Ri}{i}$ for current $i$. Fig. 230 shows the result of such an experiment.

The efficiency of a lamp is measured by the number of watts consumed per candle-power. This is about 1·25 for the tungsten lamp. The efficiency falls very quickly when less than the normal current is sent through the lamp, because the filament then emits less of its energy in visible light, and more in long heat waves. Conversely, high efficiency results from the passage of a large current; but if this is carried too far the life of the filament, which

is limited by the gradual evaporation of the tungsten, is unduly shortened. The average life under normal conditions is roughly 1500 hours of actual running.

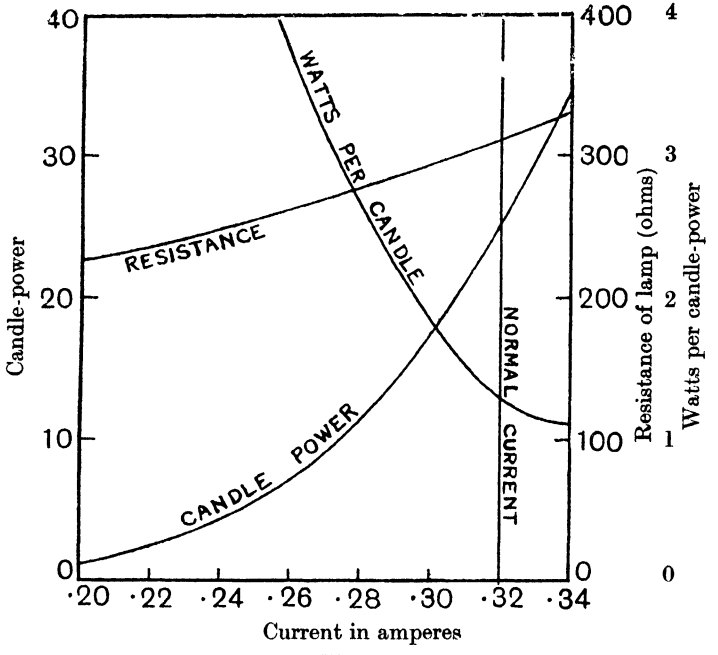



Fig. 230

An efficient glow-lamp must contain a filament of high melting-point. The melting-point of tungsten, which is about 3300° C., permits a normal working temperature of 2000° C. or thereabouts. An atmosphere of nitrogen allows us to go up to 2500° C., but the filament must be in a close spiral, otherwise too much heat escapes by convection.

The following table gives the approximate efficiency of various lamps:

| *Lamp* | *Watts per candle-power* |
|---|---|
| Mercury arc .. .. .. .. | 0·25 |
| Flame arc .. .. .. .. | 0·25 |
| Carbon arc .. .. .. .. | 0·5 |
| Tungsten filament lamp in nitrogen | 0·5–0·7 |
| Tungsten filament lamp *in vacuo* | 1·0–1·25 |

Direct-current supply in towns is at a voltage varying from 100 to 250. Hence the electrical energy used by a consumer can be ascertained by measuring the total quantity of electricity passing through the circuit. Ampere-hour meters on the electrolytic principle are extensively used.

Elihu Thomson's *watt-hour meter*, which registers the energy consumed on direct or alternating current circuits in any interval of time, consists of a small commutator motor without iron, of which the field is produced by the main current $i$ flowing in a coil. The voltage terminals are connected through the armature and also through a compensating coil assisting the field coil. The armature axle carries an aluminium disc which lies between the poles of a permanent magnet. When the impressed voltage is $V$, the driving couple on the motor is $(\lambda i + \mu V)\, V$, where $\lambda$ and $\mu$ are constants, and the retarding couple due to eddy currents is $\nu\, d\theta/dt$. Hence the equation of motion of the armature is

$$I\frac{d^2\theta}{dt^2} = \lambda i V + \mu V^2 - \nu\frac{d\theta}{dt} - R,$$

where $R$ is the retarding couple due to friction. On integrating over a long time $T$ the left-hand side becomes negligible, and we have

$$\lambda W + \mu\overline{V^2}T - \nu\alpha - RT = 0,$$

where $W = \int iV\,dt$ is the energy consumed and $\alpha = \int d\theta$ the angle turned through by the armature. The compensating coil is adjusted as nearly as possible so that $R = \mu\overline{V^2}$. Hence $\lambda W = \nu\alpha$. Thus $W$ can be measured by means of a clockwork attachment which registers $\alpha$.

# CHAPTER X

## ELECTROLYSIS

**164. Experimental evidence for Faraday's laws.** We now consider the laws of electrolytic decomposition, which were dismissed briefly in Art. 62. Faraday first examined the electrolysis of dilute sulphuric acid in order to see if consistent results could be obtained irrespective of the size of the electrodes, the absolute magnitude of the current, and the strength of the acid. The method was to place several voltameters in series, so that the same amount of electricity must pass through all. In one experiment three voltameters were placed in series, the first having small electrodes of platinum wire and the second having plates of the metal as electrodes, while in the third the products of electrolysis were collected separately, instead of together as in the first two. The sum of the volumes of hydrogen and oxygen in the three experiments were proportional to the numbers 74·30, 73·25 and 73·65. The small differences that occurred were such as might be explained by the solubility of the gases in the liquid, and Faraday concluded that the greatest accuracy was obtained with sulphuric acid of density 1·25, collecting the hydrogen only.

Faraday's second law is illustrated by the electrolysis of lead chloride ($PbCl_2$). Faraday found that the mass of lead deposited was to the mass of hydrogen liberated from a water voltameter in the same circuit in the ratio 100·85 to 1. The ratio of the atomic weights is 207, and the valency of lead is here 2, so that the agreement is fairly satisfactory. Similarly the amount of tin deposited from stannous chloride ($SnCl_2$) was about 58 times the amount of hydrogen liberated, the ratio of the atomic weights being 118.

It is not by any means every substance that lends itself to direct investigation, since disturbing reactions may mask the effect sought for. Faraday failed to obtain the expected amount of lead by electrolysis of fused lead iodide, and attributed the discrepancy to the formation of a periodide.

Lüpke recommends the following solutions for verifying Faraday's second law approximately:

(1) Solution of 3 gm. of silver nitrate and 5 gm. of potassium cyanide in 200 cm.$^3$ of water;

(2) Solution of 3 gm. of cuprous chloride in hydrochloric acid, diluted with 200 cm.$^3$ of water;

(3) 100 gm. of a saturated solution of copper sulphate added to 15 cm.$^3$ of nitric acid and diluted with 100 cm.$^3$ of water;

(4) 1 gm. of stannic chloride dissolved in 100 cm.$^3$ of water and added to 100 cm.$^3$ of a saturated solution of ammonium oxalate.

These are placed in voltameters with clean platinum cathodes, in series with a water voltameter. The following is the result of an experiment described by Lüpke:

| Cation | Valency | $\frac{\text{Atomic weight}}{\text{Valency}}$ | Electrolytic deposition in arbitrary units |
|---|---|---|---|
| Hydrogen | 1 | 1·008 | 1·008 |
| (1) Silver | 1 | 107·88 | 109·0 |
| (2) Copper | 1 | 63·57 | 64·1 |
| (3) Copper | 2 | 31·78 | 32·3 |
| (4) Tin | 4 | 29·75 | 28·6 |

The more accurate verifications have referred mainly to the ratio of the depositions of silver and copper. Shaw, working with solutions of silver nitrate and copper sulphate, tried to ascertain whether there was any error due to the solubility of the copper in the sulphate solution, by arranging his results in the order of increasing current-density. The results did not entirely agree in showing that this was the only cause, but when corrected in the light of this hypothesis they became more consistent. Finally only the results with the higher current-densities were taken, and no correction was applied. The ratio of the depositions of silver and copper was found to be 3·3998, while the value to be expected on the basis of the atomic weights is 3·3946.

Accurate experiments can also be made with cadmium (from cadmium sulphate) and iodine (from potassium iodide). Taking silver as a standard, the value 112·31 is obtained for the atomic weight of cadmium, the accepted value from chemical data being 112·40.

**165. The theory of electrolytic dissociation.** The view of electrolysis now generally adopted rests on a theory of the nature of solutions, which we proceed to expound. The physical molecule is the smallest amount of a substance capable of permanent independent existence in the gaseous form. The molecular weights of substances in the gaseous form are determined by a further appeal to *Avogadro's principle*, namely that the number of molecules per cubic centimetre of a gas at an assigned temperature and pressure is independent of the nature of the gas. This principle, which is not self-evident, is shown in the kinetic theory of gases to depend on a statistical theorem, that of the *equipartition of energy*, namely that two sets of molecules in presence of one another (as in a mixture of gases) ultimately reach a state in which the average translational kinetic energy of a molecule of either kind is the same.

If $p$ is the pressure in dyne/cm.$^2$ and $\rho$ the density of a substance in gaseous form at absolute temperature $\theta$, then the combined laws of Boyle and Gay-Lussac are expressed by

$$p = R\rho\theta/M \quad \ldots\ldots\ldots\ldots\ldots\ldots (193),$$

where $M$ is the molecular weight in terms of $O = 16$, and $R$ the *universal gas-constant* $= 8{\cdot}31 \times 10^7$.

Somewhat similar considerations apply to osmotic pressures, though the theory is not free from difficulty. The osmotic pressure of a substance in dilute solution is the same as would be produced by the same number of independent particles existing in the gaseous form. The term *independent particles* is used intentionally instead of molecules, because we have so far no means of knowing what happens to the molecules of a substance when it is dissolved. If the particles are molecules of the substance the osmotic pressure will be given by an equation analogous to (193), namely

$$P = Rc\theta/M \quad \ldots\ldots\ldots\ldots\ldots\ldots (194),$$

where $c$ is the concentration of the solution in gm./cm.$^3$

If this equation is found to hold, therefore, we may conclude that the particles in solution are molecules. Pfeffer found that a 1 per cent. solution of cane sugar in water had the osmotic pressure $6{\cdot}73 \times 10^5$, while the above formula ($c = 1/100$, $M = 342$) gives $6{\cdot}80 \times 10^5$.

Most of the experimental work has been done on the depression of the freezing-point of dilute solutions, which is easier to measure than osmotic pressure. This depression is connected with the osmotic pressure by the equation $\delta\theta = P\theta/J\lambda\rho$, where $J$ is the mechanical equivalent of heat, $\theta$ the absolute temperature of freezing of the solvent, $\lambda$ its latent heat of fusion and $\rho$ its density*. Using equation (194) we find for water solutions the equation

$$\delta\theta = 1850c/M.$$

This equation has been verified for cane sugar by Griffiths, who found that at great dilution $\delta\theta = 1858c/M$.

If each molecule of the dissolved substance gives rise on the average to $k$ independent particles from the point of view of osmotic pressure, instead of one, the last equation becomes $\delta\theta = 1850kc/M$. The factor $k$ can therefore be determined by experiment. It is found that, while the equation $\delta\theta = 1850c/M$ holds for non-electrolytes ($k = 1$), *it does not hold for electrolytic solutions*, for which $k$ exceeds unity. Thus with potassium chloride at extreme dilution Griffiths found that $\delta\theta = 3720c/M$, indicating that $k = 2$ very nearly. The following table gives some other results, derived chiefly from experiments by Loomis, Jahn and Bedford:

| Substance | $10^5\, c/M$ | $k$ | Substance | $10^5\, c/M$ | $k$ |
|---|---|---|---|---|---|
| $H_3PO_4$ | 1·0 | 1·52 | $K_2SO_4$ | 1·0 | 2·62 |
| | | | $BaCl_2$ | 1·0 | 2·70 |
| KOH | 1·0 | 1·86 | ,, | 0·02 | 3·02 |
| NaOH | 2·0 | 1·87 | $H_2SO_4$ | 0·1 | 2·70 |
| $KNO_3$ | 1·0 | 1·87 | ,, | 0·03 | 2·85 |
| $NH_4Cl$ | 1·0 | 1·93 | $K_2CO_3$ | 1·0 | 2·74 |
| NaCl | 2·5 | 1·93 | $Na_2CO_3$ | 1·0 | 2·75 |
| KBr | 2·5 | 1·93 | $MgCl_2$ | 1·0 | 2·78 |
| $NH_4NO_3$ | 1·0 | 1·94 | | | |
| HCl | 1·0 | 1·95 | $K_2Cr_2O_7$ | very dilute | >3·02 |
| $HNO_3$ | 0·1 | 1·97 | $Na_2SiO_3$ | 1·0 | 3·50 |
| $KMnO_4$ | 0·06 | 1·98 | $K_3Fe(CN)_6$ | 0·02 | 3·98 |
| $MgSO_4$ | 0·02 | 2·00 | $SnCl_4$ | 1·0 | 6·82 |

We observe the tendency of $k$ to be a whole number, which is more pronounced the more dilute the solution. In dilute hydro-

* See Whetham, *Theory of Solution*, pp. 122–6.

chloric acid ($k = 1{\cdot}95$) we conclude that practically all the molecules are split up into their constituent atoms. The weaker the solution the less chance there is of a hydrogen atom meeting a chlorine atom and becoming temporarily united with it, so that we should expect $k$ to increase with the dilution. Values of $k$ approaching 3 are ascribed to the presence of three dissociation products, complete dissociation being less easy to attain; and similarly the other figures receive a more or less satisfactory explanation. The chief difficulty occurs with stannic chloride, which has the factor $k = 6{\cdot}82$ while yielding at the most five atoms. Loomis attributes this to secondary reactions.

This theory of *electrolytic dissociation* just described, in which the molecules of electrolytic solutions are assumed to be split up before the electric force is applied, was first clearly enunciated by Arrhenius in 1887. Among known solvents, water is the one which most readily produces dissociation. This has been attributed by Nernst to its high dielectric constant, which would tend to weaken the attraction between the products of dissociation, assuming these to be charged.

**166. Ionic theory of electrolysis.** A neutral molecule contains a certain normal number of electrons, one or more of which are effective in holding the molecule together: if an element, in holding the component atoms together, and if a compound, in linking the constituents (Arts. 223, 276). We have seen that the molecules of a substance in electrolytic solution are wholly or partly dissociated, and we shall modify the phraseology of Faraday by giving to these parts the name of *ions*. Conduction of electricity can be explained if the separate ions are charged; that is, if one of the dissociation products contains one or more electrons in excess while others are in defect. The number of extra electrons in either case will be provisionally called the *valency* of the ion. On this view the primary function of the electrodes in electrolysis is to maintain a field of force in the liquid. A positively charged ion (or *cation*) is dragged along by the field and ultimately appears at the cathode, provided that no secondary reaction takes place. The negatively charged ions (or *anions*) move backwards along the lines of force and may appear at the anode.

We have now to show that the ionic theory accounts for Faraday's laws of electrolysis. Let $e$ be the charge on the electron, $M$ the atomic or molecular weight of an ion and $p$ its valency. Then we should expect the passage of a charge $pe$ to correspond to the transfer of an amount $\lambda M$ of the ion, where $\lambda$ is some constant. Hence the mass of the ion liberated by unit charge is $\mu M/p$, where $\mu = \lambda/e$ is also a constant. This, however, is precisely what would be given by Faraday's laws, provided that the number $p$ is identified with the chemical valency. Reviewing the theory as a whole, therefore, we find that it involves the following hypotheses:

(1) A chemically univalent ion carries a certain definite elementary charge $e$ independent of its nature;

(2) A $p$-valent ion carries a charge $pe$.

The first hypothesis, as was pointed out by Helmholtz in 1881, is sufficient of itself to lead to the theory of the atomic constitution of electricity. Faraday's laws of electrolysis thus yield an experimental confirmation of the electrical theory of matter, and an electrical theory of valency (cf. Arts. 223, 276).

This conclusion is not invalidated when secondary actions occur at the electrodes, for we may suppose that the ion, instead of being liberated, enters into combination and releases a chemically equivalent amount of some other substance. The amount of matter actually liberated has therefore the same relation to the charge as if the liberation had been caused by direct transfer of ions.

In the electrolysis of hydrochloric acid hydrogen appears at the cathode and therefore carries a positive charge, the chlorine ion carrying an equal and opposite charge. The separation of the molecules of the solute into cations consisting of hydrogen or metals, and anions consisting of halogens or basic radicles, is typical of those electrolytes for which $k$ approaches 2. It is natural on other grounds to suppose that the line of cleavage is drawn in this way, since the basic radicles ($NO_3$, $SO_4$, etc.) show great chemical stability in that they will bodily replace the halogens in compounds. Thus the simplest type of dissociation is exhibited by the equations

$$HCl = \overset{+}{H} + \overset{-}{Cl}, \quad KNO_3 = \overset{+}{K} + \overset{-}{NO_3},$$

and so on. In magnesium sulphate the $SO_4$ radicle is divalent and

therefore carries a double (negative) charge. Hence we can also have the type

$$MgSO_4 = \overset{++}{Mg} + \overset{--}{SO_4}.$$

Three ions must arise from the molecule when $k$ approaches 3. The type is that of sulphuric acid

$$H_2SO_4 = \overset{+}{H} + \overset{+}{H} + \overset{--}{SO_4},$$

agreeing with the charges assigned to the H and $SO_4$ ions in other cases. There is also the complementary type

$$BaCl_2 = \overset{++}{Ba} + \overset{-}{Cl} + \overset{-}{Cl}.$$

The evidence from osmotic pressure is less definite for the higher values of $k$.

A few ions can vary in valency. Thus the dissociation of ferrous chloride is represented by

$$FeCl_2 = \overset{++}{Fe} + \overset{-}{Cl} + \overset{-}{Cl},$$

| Cations ( + charge) | | | | Anions ( − charge) | | | |
|---|---|---|---|---|---|---|---|
| | Combining weight | Valency | Electrochemical equivalent $10^{-5}\times$ | | Combining weight | Valency | Electrochemical equivalent $10^{-5}\times$ |
| H | 1·008 | 1 | 1·045 | O | 16 | 2 | 8·291 |
| $NH_4$ | 18·04 | 1 | 18·7 | OH | 17·01 | 1 | 17·63 |
| Na | 23 | 1 | 23·83 | F | 19 | 1 | 19·69 |
| K | 39·1 | 1 | 40·52 | CN | 26·01 | 1 | 26·96 |
| Fe (ferrous salts) | 55·85 | 2 | 28·94 | S | 32·07 | 2 | 16·62 |
| (ferric salts) | | 3 | 19·29 | Cl | 35·46 | 1 | 36·75 |
| Cu (cuprous salts) | 63·57 | 1 | 65·88 | $C_2H_3O_2$ | 59·02 | 1 | 61·16 |
| (cupric salts) | | 2 | 32·94 | $CO_3$ | 60 | 2 | 31·09 |
| Zn | 65·37 | 2 | 33·87 | $NO_3$ | 62·01 | 1 | 64·26 |
| Ag | 107·88 | 1 | 111·8 | $SiO_3$ | 76·3 | 2 | 39·54 |
| Cd | 112·4 | 2 | 58·2 | Br | 79·92 | 1 | 82·82 |
| Sn (stannous salts) | 119 | 2 | 61·66 | $ClO_3$ | 83·46 | 1 | 86·5 |
| (stannic salts) | | 4 | 30·83 | $SO_4$ | 96·07 | 2 | 49·78 |
| Ba | 137·37 | 2 | 71·2 | $CrO_4$ | 116 | 2 | 60·11 |
| Hg (mercurous salts) | 200 | 1 | 207 | I | 126·92 | 1 | 131·5 |
| (mercuric salts) | | 2 | 103·5 | $BrO_3$ | 127·92 | 1 | 132·6 |
| Pb | 207·1 | 2 | 107·3 | | | | |

while that of ferric chloride is similarly

$$FeCl_3 = \overset{+++}{Fe} + \overset{-}{Cl} + \overset{-}{Cl} + Cl.$$

The electro-chemical equivalents of the more common ions are given in the table on p. 340, expressed in grams per coulomb. The values are not directly observed, but calculated in terms of silver as the standard.

**167. Physical constants derived from electrolysis.** There is an important relation connecting the elementary charge $e$ with the number of molecules $N$ of a gas at temperature 15° C. and at a pressure of 760 mm. of mercury. At this temperature and pressure 1 coulomb liberates 0·123 cm.$^3$ of hydrogen. Since each molecule corresponds to the passage of a charge $2e$, $0{\cdot}246\,Ne = 1$ coulomb $= 3 \times 10^9$ electrostatic units, so that

$$Ne = 1{\cdot}22 \times 10^{10} \quad \ldots\ldots\ldots\ldots (195),$$

where $e$ is expressed in electrostatic units.

Let $m_H$ be the mass of a hydrogen atom. Since each molecule contains two atoms, and the density of hydrogen at 15° C. and at a pressure of 760 mm. of mercury is $8{\cdot}517 \times 10^{-5}$,

$$2Nm_H = 8{\cdot}517 \times 10^{-5},$$

or

$$Nm_H = 4{\cdot}26 \times 10^{-5}.$$

Combining this with equation (195), we have

$$\frac{e}{m_H} = 2{\cdot}87 \times 10^{14} \ldots\ldots\ldots\ldots\ldots (196).$$

The value of $N$ can be found from a study of the Brownian movement in liquids*, but only approximately since the conditions contemplated in the kinetic theory of gases are not accurately fulfilled. Greater weight must be given to the electrical determinations of $e$ (Art. 219), from which we can also find $N$. We shall adopt the value

$$e = 4{\cdot}77 \times 10^{-10} \text{ E.S.U.} \quad \ldots\ldots\ldots\ldots (197).$$

From equation (196)

$$m_H = 1{\cdot}66 \times 10^{-24} \text{ grams} \quad \ldots\ldots\ldots (198),$$

and hence we can find the mass of any molecule of known molecular

* See J. Perrin, *Les Atomes*. Paris, F. Alcan.

weight. If $m$ is the mass of an ion or molecule of atomic or molecular weight $M$, it follows from (196) that

$$\frac{e}{m} = \frac{2{\cdot}89 \times 10^{14}}{M} \quad \ldots\ldots\ldots\ldots\ldots (199).$$

This equation holds whatever value of $e$ is adopted.

**168. Polarisation and secondary actions.** The E.M.F. of polarisation for various applied potentials may be measured by the method shown in Fig. 231. The source of applied potential must be one which can be varied by small stages from zero up to about 5 volts. A low-voltage direct-current generator, which can be excited separately by different currents, is convenient. The electrolyte is contained in a voltameter with electrodes $E$, $E$, one of which is earthed. The other is joined through a key $K$ to the insulated prong $B$ of a tuning-fork making contact alternately with the points $A$, $C$ connected to the generator and an electrometer respectively. When the key is up the electrometer quickly takes up the potential $V$ of the generator, the absolute magnitude of which is given by a voltmeter attached to its terminals. When the key is down the potential $V$ is applied to the electrolytic cell as long as $B$ is joined to $A$, and when $B$ is joined to $C$ it communicates to the electrometer a potential equal to the E.M.F. of polarisation in the cell. This method is applicable to small electrolytic cells, whose polarisation E.M.F. would disappear rapidly if they were joined directly to a voltmeter.

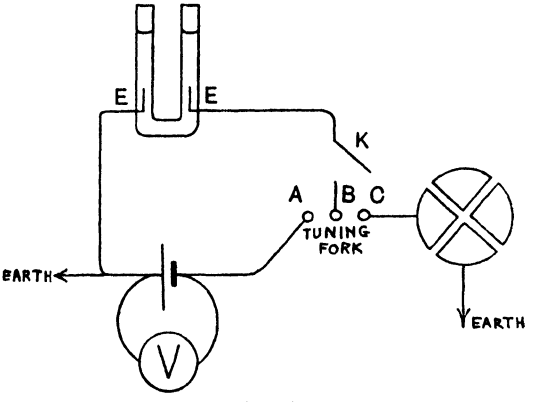


Fig. 231

Fig. 232 shows the relation of polarisation to applied E.M.F. for dilute sulphuric acid between electrodes of platinum foil. When $V$ is small the polarisation E.M.F. is nearly equal to $V$, and there is no appreciable electrolysis. But as $V$ increases the polarisation

E.M.F. tends to a limit which in this case is about 2 volts, but depends very much on the state of the electrodes.

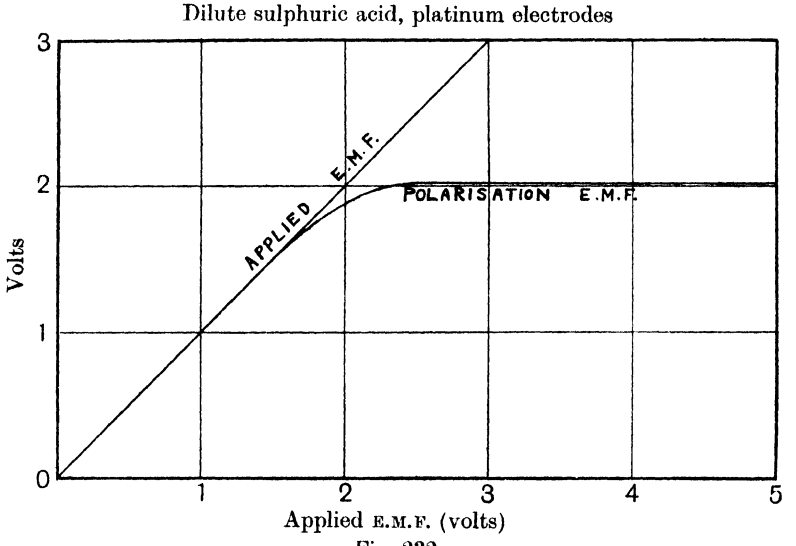


Fig. 232

Except for the statement that polarisation is due to the formation of an electrical "double layer" on or near the electrodes, i.e. a layer of positively charged particles in face of a layer of negatively charged ones, no physical theory of the phenomenon seems to have been offered.

The electrolysis of copper sulphate with copper electrodes results in a loss of copper by the anode and an equal gain by the cathode. Phenomena of this kind are called *secondary reactions*, and must occur when one of the ions in the liquid (here the $SO_4$ anion) is a radicle incapable of existence in the neutral state. In such cases the ion in question combines with the matter of the electrode or solvent and causes the liberation of some secondary product. In a copper voltameter the $SO_4$ ion attacks the anode with the formation of copper sulphate. With a platinum anode the radicle attacks the water in preference to the metal, and oxygen is liberated, the reaction being

$$2H_2O + 2SO_4 = 2H_2SO_4 + O_2.$$

The electrolyte therefore becomes more and more acid at the expense of the salt.

The liberation of the dissociation products of water from weak solutions of salts or acids depends on a similar reaction. The ions actually present in sulphuric acid are two hydrogen cations to every $SO_4$ anion. The hydrogen is liberated by electrolysis, and the $SO_4$ ion takes part in the foregoing reaction at the anode, whereby the strength of the acid solution is conserved.

**169. Accumulators.** Accumulators (see Art. 63) give an extremely constant electromotive force over the interval of time required for an ordinary experiment. As an example of a modern accumulator we may take the "Exide" accumulator. A grid on each electrode holds the active material, and prevents it from falling away when in use. The plates are separated by wooden partitions, rendered porous by special treatment, which keep the plates apart without adding seriously to the internal resistance of the cell. A medium-sized cell contains about a dozen positive and a like number of negative plates in a celluloid box, and two cells in series form a four-volt accumulator suitable for laboratory work. The cell should first be filled with the purest sulphuric acid diluted with distilled water till its density is 1·25. It is then charged at the rate of 2 amperes for at least 40 hours, when gas is evolved freely from all the plates. Subsequent chargings can be done at the rate of 4 amperes for 10–15 hours, or until each cell gives a potential-difference of 2·55 volts. Loss of liquid by evaporation is made up by the addition of distilled water.

The changes of the difference of potential between the terminals of a lead accumulator during charging and discharge are shown in Fig. 233, which gives the results of a typical experiment (Cohen and Donaldson). While practically all the electricity used in charging is recovered on discharge, the latter occurs throughout at a lower voltage, so that by no means all the *energy* is recoverable. The charging energy is represented by $\int V_1 de$ and the electrical work available on discharge by $\int V_2 de$, where $V_1$, $V_2$ are the ordinates of the upper and lower curve respectively for the abscissa $e$. We can calculate that about $5{\cdot}3 \times 10^{12}$ ergs are used in charging the accumulator, of which only $4{\cdot}4 \times 10^{12}$ ergs are recovered, giving an efficiency of 83 per cent. This is a very favourable value, not usually approached in practical working.

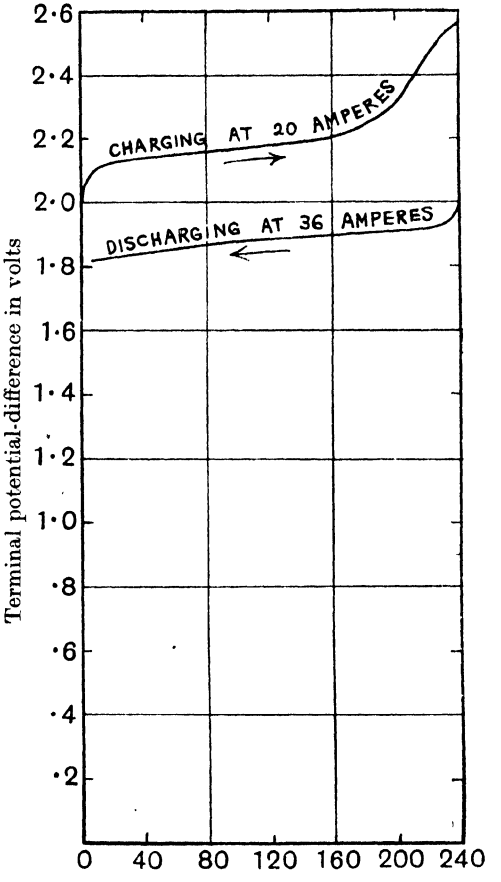



Fig. 233

It is generally maintained that the principal part of the reaction in the lead accumulator is as follows. At the end of a normal discharge both electrodes are covered with a layer of lead sulphate. When charging begins the hydrogen proceeding from the electrolysis of sulphuric acid reacts with the lead sulphate at the cathode and forms metallic lead according to the equation

$$H_2 + PbSO_4 = Pb + H_2SO_4.$$

At the anode the $SO_4$ ion reacts with the lead sulphate and the water of dilution to form lead peroxide and sulphuric acid, thus:

$$SO_4 + PbSO_4 + 2H_2O = PbO_2 + 2H_2SO_4.$$

Hence the passage of one coulomb round the circuit is accompanied by the disappearance of one equivalent of sulphuric acid; but three equivalents are formed by secondary action, so that the strength of the acid increases during charging. This process continues until the anode, which is to become the positive pole of the accumulator, is covered with a layer of lead peroxide and the cathode with a layer of pure lead; the potential-difference then being about 2·2 volts. Further charging results in the liberation of free hydrogen and oxygen from the electrodes, coupled with a sharp rise in the potential. Potentials above 2·3 volts disappear quickly on cessation of charging, even when no current is taken from the cell.

During normal discharge the current flows inside the cell from the negative to the positive pole, the latter forming the cathode in the electrolysis of the acid. The reaction at the positive pole is given by

$$H_2 + PbO_2 + H_2SO_4 = PbSO_4 + 2H_2O,$$

and that at the negative pole by

$$Pb + SO_4 = PbSO_4.$$

The visible reactions are therefore exactly reversed, the acid becoming weaker and lead sulphate appearing at both poles. It is inadvisable to run an accumulator lower than 1·8 volts.

**170. Measurement of electrolytic resistance.** Kohlrausch was the first to show that, when the effects of polarisation are eliminated or avoided, electrolytes obey Ohm's law. If small rapidly alternating currents are used, and electrodes of sufficient size, the products of polarisation form a very thin layer on the surface and are not actually liberated. It is reasonable to suppose, in these circumstances, that the polarisation E.M.F. is proportional to the charge $e$ that has passed up to the time considered. Assuming provisionally the existence of a resistance $R$ in the electrolyte, the current $i$ is given by $Ri = v - Pe$, where $v$ is the applied E.M.F. and $P$ a constant. Writing $v = V \cos pt$, and remembering that $i = de/dt$, we have

$$R\,di/dt + Pi = -Vp \sin pt.$$

Hence

$$i = \frac{\frac{V}{R}\left(\cos pt - \frac{P}{Rp}\sin pt\right)}{1 + \frac{P^2}{R^2p^2}}.$$

If $P$ is small in comparison with $Rp$, the current does not differ much, either in amplitude or phase, from its value $V \cos pt/R$ in the absence of polarisation. The coefficient $P$ is smaller the greater the surface of the electrode. A large effective surface can be obtained with quite small electrodes by depositing platinum on them in a finely divided form. These considerations enabled Kohlrausch to enunciate the conditions favourable to success with electrolytes: high frequency of alternation, moderately large resistance, and platinised electrodes.

The method of carrying out the measurement is as follows. The platinum electrodes of the electrolytic cell are first cleaned in a solution of sulphuric acid and potassium bichromate, washed well and placed in the platinising liquid. This consists of 3 grams of chloroplatinic acid and ·02 gram of lead acetate dissolved in 100 cm.$^3$ of water. A small current is passed between the electrodes so that there is a slight but steady evolution of gas, and is reversed at intervals until both electrodes are covered with a fine coating of platinum black. The electrodes are then placed in dilute sulphuric acid and a current passed in both directions in order to remove the last traces of the platinising solution. They are then washed well with distilled water and replaced in the electrolytic cell. This may be of the form shown in Fig. 234 if the electrolyte is a bad conductor: a long tube is used for a good conductor. The electrolytic cell must be of insoluble glass and the contents kept at a constant temperature, on account of the large variation of resistance with temperature.

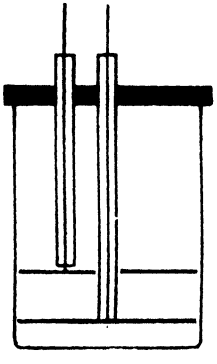


Fig. 234

The measurement of resistance is made on the Wheatstone's bridge. The alternating current is furnished by a high-frequency alternator or a small induction coil, and the galvanometer is replaced by a telephone receiver. The usual relation between the resistances in the arms of the bridge holds when no sound is heard in the telephone. Traces of inductance and capacity in the arms (cf. Art. 135) may prevent this condition from being actually realised, and all that is heard is a *minimum* sound in the telephone.

The existence of a definite resistance is proved by the agreement

of the results obtained under different experimental conditions. In one of his many experiments performed with this end in view, Kohlrausch measured the ratio of the resistances of various solutions in containing vessels $A$, $B$, with the following result:

| Solution | Method of measurement | $R_A/R_B$ |
|---|---|---|
| NaCl | Alternator and dynamometer | 22·676 |
| ,, | Coil and dynamometer | 22·693 |
| ,, | Coil and telephone | 22·697 |
| KCl | ,, ,, | 22·652 |
| HCl | ,, ,, | 22·683 |
| $H_2SO_4$ | ,, ,, | 22·686 |

Kohlrausch showed further that the error due to inductance and capacity in ordinary resistance boxes was negligible for coils up to 1000 ohms.

**171. Conductivity of electrolytes.** When the current flows in straight lines between the electrodes, as in Fig. 234, the absolute conductivity $\sigma$ of the electrolyte can be deduced from the measurements. If $A$ is the available area of the electrodes, $d$ their distance apart and $R$ the measured resistance, we have

$$R = \frac{l}{\sigma A} \quad \ldots\ldots\ldots\ldots\ldots\ldots (200).$$

Similar considerations apply to an electrolyte contained in a tube.

The water of solution has always an appreciable conductivity, which must be taken into account in weak solutions. The conductivity of ordinary distilled water is of the order $2 \times 10^{-6}$; for good specimens $\sigma$ may be as low as $6 \times 10^{-7}$. The existence of a limiting conductivity for very weak solutions is shown in Fig. 235, which represents the results of Whetham's experiments on sulphuric acid at extreme dilution.

Kohlrausch and Heydweiller found that when water was very carefully purified by distillation *in vacuo* its conductivity was about $4{\cdot}3 \times 10^{-8}$. Although every purification of water causes a reduction in its conductivity, yet there is reason to believe that absolutely pure water still conducts appreciably. A slight degree of natural dissociation in water is therefore indicated.

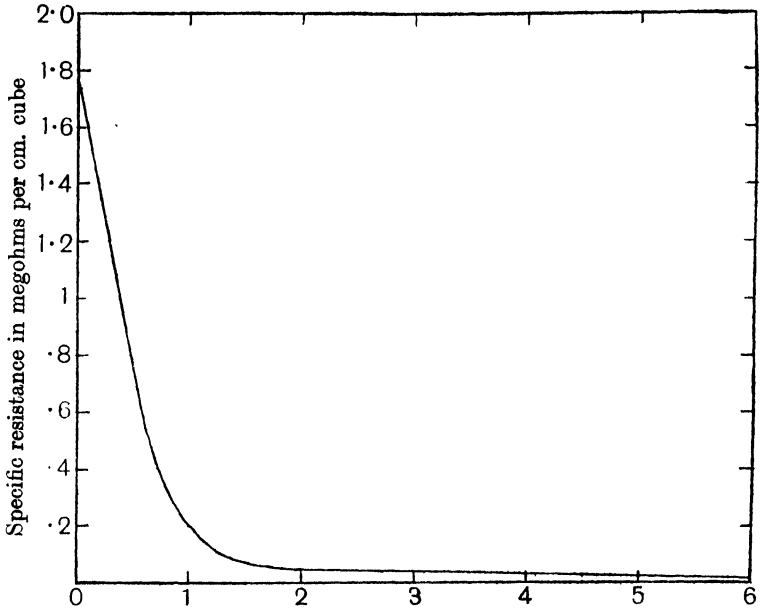



Fig. 235

The following table, due to Kohlrausch, gives the conductivities of some concentrated solutions at 18° C. in absolute measure (reciprocal ohms per centimetre cube). The strengths of the solutions are expressed in percentages of solute in the whole by weight.

| | 5 % | 10 % | 20 % | 30 % | 40 % | 50 % | 60 % | 70 % | 80 % |
|---|---|---|---|---|---|---|---|---|---|
| KCl | ·069 | ·136 | ·268 | — | — | — | — | — | — |
| $NH_4Cl$ | ·092 | ·178 | ·337 | — | — | — | — | — | — |
| NaCl | ·067 | ·121 | ·196 | — | — | — | — | — | — |
| $K_2SO_4$ | ·046 | ·086 | — | — | — | — | — | — | — |
| $MgSO_4$ | ·026 | ·041 | ·048 | — | — | — | — | — | — |
| $ZnSO_4$ | ·019 | ·032 | ·047 | ·044 | — | — | — | — | — |
| $CdSO_4$ | ·015 | ·025 | ·039 | ·044 | — | — | — | — | — |
| $CuSO_4$ | ·019 | ·032 | — | — | — | — | — | — | — |
| $AgNO_3$ | ·026 | ·048 | ·087 | ·124 | ·157 | ·186 | ·210 | — | — |
| KOH | ·172 | ·315 | ·499 | ·542 | ·450 | — | — | — | — |
| NaOH | ·195 | ·309 | ·328 | ·207 | ·121 | ·082 | — | — | — |
| HCl | ·395 | ·630 | ·762 | ·662 | ·515 | — | — | — | — |
| $HNO_3$ | ·258 | ·461 | ·711 | ·785 | ·733 | ·631 | ·513 | ·396 | ·267 |
| $H_2SO_4$ | ·209 | ·392 | ·653 | ·740 | ·680 | ·541 | ·373 | ·216 | ·111 |

Fig. 236 shows the relation of conductivity to temperature for

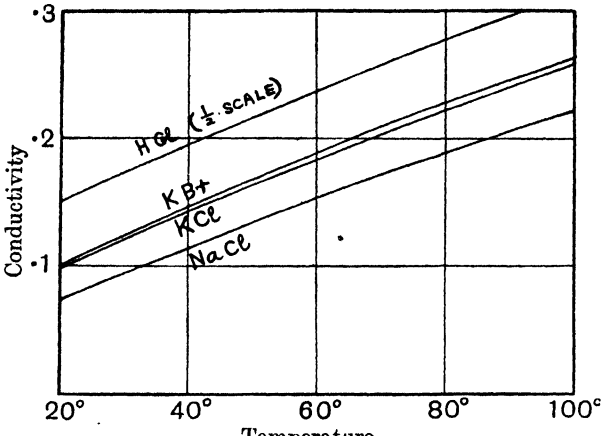


Fig. 236

normal solutions* of certain salts. At ordinary temperatures the rise of conductivity of salt solutions is of the order of 2 per cent. per degree, of acids somewhat less. Hence care is necessary in measuring the temperature, and in keeping it uniform, if accurate results are to be obtained.

**172. Voltaic cells.** Faraday's laws give the connexion between the amount of chemical action in a voltaic cell and the charge that has passed; but we require some further explanation of their mode of working. In a Daniell cell made of pure materials, very little zinc dissolves as long as the cell is on open circuit; but the zinc terminal begins to dissolve as soon as the circuit is completed, and copper is deposited on the copper terminal. Nernst gave a theory of these phenomena, which we shall now explain.

The theory is based on the fact that metallic ions are *cations*. Thus a metal dipped in pure water tends to go into solution positively charged, leaving the metal charged negatively. This can be verified immediately with an electrometer. The amount of the metal actually dissolved is very small, for the positive charge in the liquid creates a field of electric force tending to prevent any more ions leaving the metal with a positive charge, and thus a limit is soon reached. For the sake of illustration, suppose that the metal is in the form

* A normal solution is one which contains $M$ grams of the dissolved substance per litre, where $M$ is the molecular weight. It follows from Art. 167 that 1 cm.$^3$ of such a solution contains a definite number ($6{\cdot}15 \times 10^{20}$) of molecules, irrespective of the nature of the dissolved substance.

of a spherical shell entirely surrounding the water, say of 10 cm. radius. In order to produce at the surface an electric force of 1 volt per centimetre, or 1/300 electrostatic units, the charge $e$ in the liquid is given by $e/100 = 1/300$, or $e = \frac{1}{3}$. This corresponds to about $7 \times 10^8$ ions in solution. Large as this number seems, it represents the solution, relatively, of a very small amount of metal, since a layer of metal one molecule thick over the surface contains about $10^{18}$ molecules.

Another factor comes into play when metals are immersed in solutions of their salts. The metallic ions already present tend to separate out on the metal after giving up their positive charge to it. Thus copper becomes positively charged by contact with a concentrated solution of copper sulphate.

In a Daniell cell we suppose that the zinc tends to go readily into solution. Thus in a short time the electrolyte is charged positively and the zinc terminal negatively. The positive charge in the liquid, which resists the solution of the zinc, tends equally to deposit positive ions on the copper terminal, in this way assisting the action of the copper sulphate solution. Thus copper will dissolve less than it would normally, and may have ions deposited on it instead even in a weak solution. Under these conditions the state of the cell is as shown in Fig. 237. The actual amount of solution and deposition on making up the cell is very small, as already explained, the main result being that the potential of the copper is higher than that of zinc. On completing the circuit extra electrons from the zinc pass over into the copper: the negative charge on the zinc tends to diminish, so that more zinc is enabled to dissolve. Similar but opposite effects occur at the other terminal, and thus the process goes on continually. It follows from Nernst's theory that the E.M.F. of the cell will increase if anything is done to promote solution of zinc or deposition of copper, i.e. if the zinc sulphate solution is diluted or the copper sulphate solution made stronger. This is found to be the case.

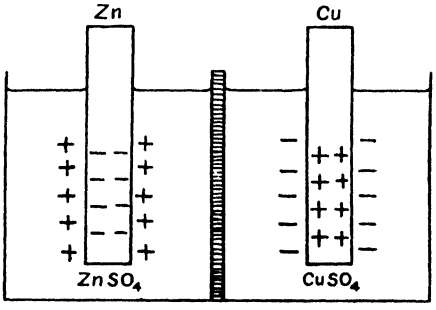



Fig. 237

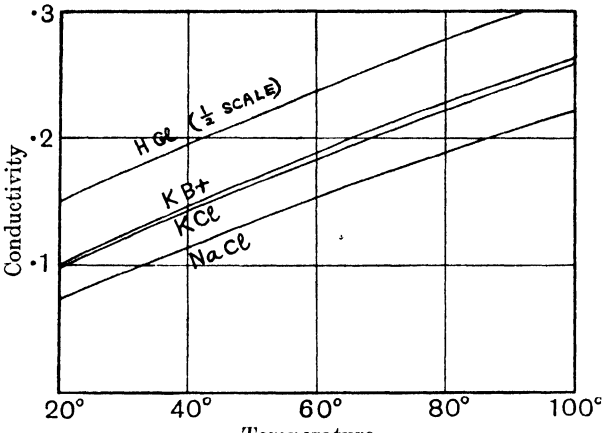



Fig. 236

normal solutions* of certain salts. At ordinary temperatures the rise of conductivity of salt solutions is of the order of 2 per cent. per degree, of acids somewhat less. Hence care is necessary in measuring the temperature, and in keeping it uniform, if accurate results are to be obtained.

**172. Voltaic cells.** Faraday's laws give the connexion between the amount of chemical action in a voltaic cell and the charge that has passed; but we require some further explanation of their mode of working. In a Daniell cell made of pure materials, very little zinc dissolves as long as the cell is on open circuit; but the zinc terminal begins to dissolve as soon as the circuit is completed, and copper is deposited on the copper terminal. Nernst gave a theory of these phenomena, which we shall now explain.

The theory is based on the fact that metallic ions are *cations*. Thus a metal dipped in pure water tends to go into solution positively charged, leaving the metal charged negatively. This can be verified immediately with an electrometer. The amount of the metal actually dissolved is very small, for the positive charge in the liquid creates a field of electric force tending to prevent any more ions leaving the metal with a positive charge, and thus a limit is soon reached. For the sake of illustration, suppose that the metal is in the form

* A normal solution is one which contains $M$ grams of the dissolved substance per litre, where $M$ is the molecular weight. It follows from Art. 167 that 1 cm.$^3$ of such a solution contains a definite number ($6{\cdot}15 \times 10^{20}$) of molecules, irrespective of the nature of the dissolved substance.

of a spherical shell entirely surrounding the water, say of 10 cm. radius. In order to produce at the surface an electric force of 1 volt per centimetre, or 1/300 electrostatic units, the charge $e$ in the liquid is given by $e/100 = 1/300$, or $e = \frac{1}{3}$. This corresponds to about $7 \times 10^8$ ions in solution. Large as this number seems, it represents the solution, relatively, of a very small amount of metal, since a layer of metal one molecule thick over the surface contains about $10^{18}$ molecules.

Another factor comes into play when metals are immersed in solutions of their salts. The metallic ions already present tend to separate out on the metal after giving up their positive charge to it. Thus copper becomes positively charged by contact with a concentrated solution of copper sulphate.

In a Daniell cell we suppose that the zinc tends to go readily into solution. Thus in a short time the electrolyte is charged positively and the zinc terminal negatively. The positive charge in the liquid, which resists the solution of the zinc, tends equally to deposit positive ions on the copper terminal, in this way assisting the action of the copper sulphate solution. Thus copper will dissolve less than it would normally, and may have ions deposited on it instead even in a weak solution. Under these conditions the state of the cell is as shown in Fig. 237. The actual amount of solution and deposition on making up the cell is very small, as already explained, the main result being that the potential of the copper is higher than that of zinc. On completing the circuit extra electrons from the zinc pass over into the copper: the negative charge on the zinc tends to diminish, so that more zinc is enabled to dissolve. Similar but opposite effects occur at the other terminal, and thus the process goes on continually. It follows from Nernst's theory that the E.M.F. of the cell will increase if anything is done to promote solution of zinc or deposition of copper, i.e. if the zinc sulphate solution is diluted or the copper sulphate solution made stronger. This is found to be the case.

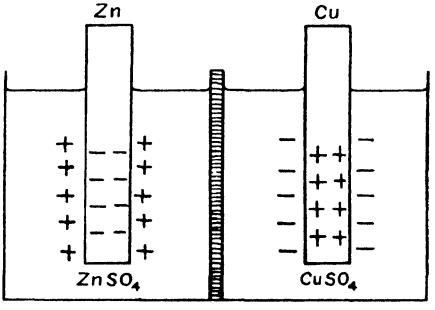



Fig. 237

Similar considerations hold for hydrogen, which tends to go into solution in the form of positive ions. It can therefore be driven out of the ionic and into the gaseous form at surfaces where there is sufficient electric force, directed away from the liquid. We can thus explain not only the evolution of hydrogen at a cathode in electrolysis, but also the spontaneous decomposition of water by the alkali metals. Potassium, for example, has such a great tendency to go into solution that the resulting electric force is sufficient to liberate hydrogen in this way, as there are always a number of hydrogen ions present in water.

Nernst in his presentation of the theory speaks of an *electrolytic solution pressure* tending to drive a metal into solution in ionic form, which does so until this is compensated by electrostatic action and by the osmotic pressure of the positive ion in the liquid. The existence of a definite magnitude characteristic of a metal in all circumstances does not, however, appear to have been placed beyond doubt.

Returning to the Daniell cell, there will in general be three places where the potential undergoes an abrupt change. These are (1) at the surface of separation of the zinc and its solution, (2) at the copper, (3) where the two liquids meet. In (1) the potential certainly rises in passing into the liquid; in (2) it rises or falls according to circumstances, but to a less degree; while the effect (3) is small and can be calculated (cf. Art. 176). If the external circuit is completed there is a fall of potential in the liquids due to their electrolytic resistance: this resistance is called the *internal resistance* of the cell.

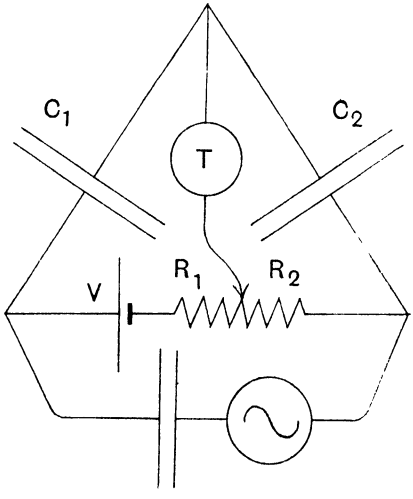


Fig. 238

Many methods have been devised for measuring the internal resistance of cells. Great accuracy is seldom required in the measurement, and the following method due to Nernst and Haagn will be found convenient, except in the case of accumulators, whose resist-

ance is too low. The cell is placed in a Wheatstone's bridge (Fig. 238) together with two adjustable resistances $R_1$, $R_2$, preferably resistance boxes containing fractions of an ohm. The other arms of the bridge are condensers $C_1$, $C_2$ of the order of 10 microfarads. Alternating current is supplied by a small induction coil, high-frequency alternator or oscillating triode, placed in series with a third large condenser. In this way the circulation of continuous currents in the bridge is prevented, and the bridge is balanced for no current in the telephone $T$. It is easy to show, as in Art. 135, that

$$\frac{V + R_1}{R_2} = \frac{C_2}{C_1},$$

and hence $V$ can be found. The resistance of an ordinary Leclanché cell is about 1 ohm, that of a small Weston cell 500 ohms.

The resistance of an accumulator can be found by connecting its terminals to a voltmeter while it is also delivering current to an external circuit. If $V_0$ is the voltmeter reading on open circuit, $V$ its value when a current $i$ is flowing, the resistance of the cell is given by $V_0 - V = Ri$. A medium-sized 8-volt battery has a resistance of about $\frac{1}{20}$ ohm.

**173. Thermodynamics of reversible cells.** The delivery of a current by a cell is reversible in the thermodynamical sense when the chemical and physical processes in the circuit can be made to go on in the opposite direction by an infinitesimal change in the circumstances under which we are operating. Thus we may imagine a cell whose E.M.F. on open circuit is $V$ sending a current round a circuit containing a direct-current generator of adjustable E.M.F. $V'$. The process may be reversible when $V$ exceeds $V'$ only infinitesimally, because then a slight increase of $V'$ raises it above $V$ and sends a current round the circuit in the opposite direction. In addition, the cell must be such that the chemical actions are exactly undone by the reverse transfer, so that the cell returns to the same state whenever the total charge that has passed any point of the circuit, measured algebraically, is zero. The two conditions necessary for a cell to be reversible are therefore (1) that it delivers only an infinitesimal current; (2) that the reactions which take place at the electrodes are reversed when the current is reversed. The second condition is satisfied very nearly in cells which do not evolve gas,

such as the Daniell and Weston cells and (practically) the accumulator. The primitive voltaic cell $Zn \mid H_2SO_4 \mid Cu$, on the other hand, is essentially irreversible since gas is evolved at the copper terminal. If current is sent through the cell in the wrong direction, gas is evolved at the zinc terminal, zinc not being deposited as it would be with the Daniell cell.

Let $\theta$ be the absolute temperature of the cell, $\eta$ its entropy, and let an infinitesimal charge of $de$ coulombs pass round the circuit in the ordinary direction. The work done *on* the cell in the process is $dW = -Vde$ joules or $-10^7 Vde$ ergs, where $V$ is the E.M.F. in volts, and the heat given to it is $dQ = \theta d\eta$ calories. Hence the gain of internal energy of the system, measured in calories, is

$$d\epsilon = \theta d\eta - \frac{10^7}{J} Vde.$$

Thus
$$d(\epsilon - \theta\eta) = -\eta d\theta - \frac{10^7}{J} Vde,$$

from which it follows that

$$\left(\frac{\partial V}{\partial \theta}\right)_e = \frac{J}{10^7}\left(\frac{\partial \eta}{\partial e}\right)_\theta.$$

This equation contains the thermodynamical theory of the reversible cell. The left-hand side is simply $dV/d\theta$, where $V$ is the E.M.F. on open circuit. Hence on multiplying by $\theta$ we have

$$\theta \frac{dV}{d\theta} = \frac{J}{10^7}\left(\frac{\theta d\eta}{de}\right)_\theta.$$

The right-hand side is the heat actually taken up by the cell during the isothermal passage of unit charge round the circuit, say $\lambda'$. If $\lambda$ is the heat *evolved* by the corresponding chemical change, *when conducted in a calorimeter without the passage of electric charge*, the difference between the heats $\lambda'$ and $-\lambda$ is due to the fact that in the former case an amount $10^7 V$ ergs of electrical work is performed, and in the latter case none. Hence

$$\lambda' = \frac{10^7 V}{J} - \lambda.$$

Substituting in the above equation and remembering that

$$J = 4{\cdot}18 \times 10^7,$$

we have Helmholtz's equation

$$V = 4{\cdot}18\lambda + \theta \frac{dV}{d\theta} \quad \ldots\ldots\ldots\ldots (201).$$

If the temperature-coefficient $dV/d\theta$ is sufficiently small $V = 4{\cdot}18\lambda$; the electrical work is "equivalent" to the heat of reaction. In the Daniell cell the effective reaction corresponding to the passage of one coulomb is the replacement of one electro-chemical equivalent of copper by zinc in sulphate solution, for which $\lambda = {\cdot}2592$. Thus $4{\cdot}18\lambda = 1{\cdot}09$, a number agreeing very well with the observed voltage. It is, however, clear both theoretically and experimentally that this simple relation does not generally hold. Bugarszky found a combination in which $\lambda$ and $V$ are of opposite signs, so that the cell at once absorbs heat (from the atmosphere) and delivers current.

An accurate verification of Helmholtz's equation was carried out by Jahn in 1886. The cell was enclosed in a Bunsen ice calorimeter, and closed through a resistance and also through a voltmeter circuit of high resistance. The heat actually developed in the cell during the passage of a known current at 0° C. was measured, and from it Jahn deduced the chemical heat $\lambda$. The following table shows how theory and observation agree:

| Cell | $V$ | $\lambda$ | $dV/d\theta$ | $V - 4{\cdot}18\lambda$ | $\theta dV/d\theta$ |
|---|---|---|---|---|---|
| Cu, $CuSO_4 + 100\ H_2O$, $ZnSO_4 + 100\ H_2O$, Zn | 1·0962 | ·26027 | +0·000034 | +·0083 | +·0093 |
| Cu, Cu $(C_2H_3O_2)_2$ aq, Pb $(C_2H_3O_2)_2 + 100\ H_2O$, Pb | 0·4764 | ·08582 | +0·000385 | +·1177 | +·1051 |
| Ag, $AgNO_3$, Cu $(NO_3)_2$, Cu | 0·4580 | ·15598 | −0·000708 | −·1940 | −·1930 |
| Ag, $AgNO_3$, Pb $(NO_3)_2$, Pb | 0·9320 | ·26413 | −0·000632 | −·1720 | −·1725 |
| Ag, AgCl, $ZnCl_2 + 100\ H_2O$, Zn | 1·0306 | ·27088 | −0·000409 | −·1017 | −·1117 |
| Ag, AgCl, $ZnCl_2 + 50\ H_2O$, Zn | 1·0171 | ·25485 | −0·000210 | −·0482 | −·0573 |
| Ag, AgCl, $ZnCl_2 + 25\ H_2O$, Zn | 0·9740 | ·24480 | −0·000202 | −·0493 | −·0551 |
| Ag, AgBr, $ZnBr_2 + 25\ H_2O$, Zn | 0·8409 | ·20736 | −0·000106 | −·0281 | −·0289 |

**174. Measurement of ionic velocity.** If the electric force in an electrolyte is doubled, the current is also; from which it follows that the ions move with velocities proportional to the electric force. The velocity of an ion under unit electric force (usually taken as

one volt per centimetre) is called its *mobility*, so that if $\beta$ is the mobility the velocity under an electric force $E$ is given by $v = \beta E$.

The mobility of ions can be found by the following method. It is applicable to dilute solutions of electrolytes in which only two ions are present.

The first part of the process, due to Hittorf, consists in finding the ratio of the mobilities by measuring the changes which take place in the concentration of the solution. In a vessel of moderate size there is always a region in the middle where the concentration is unaltered, even after the lapse of an hour or two, with an applied potential of a few volts. The thick line in Fig. 239 represents a plane

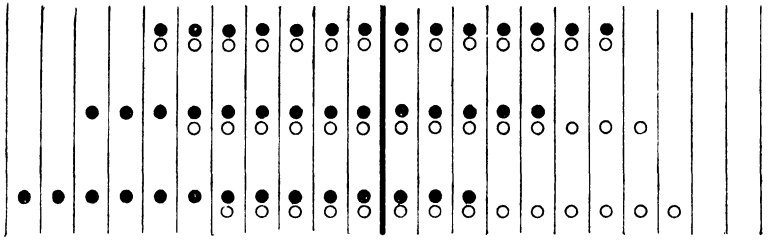


Fig. 239

near the centre of the electrolyte, and the black and white spheres stand respectively for negative and positive ions, so that the electric force is from left to right. At the beginning of the experiment the dissociated molecules near the plane are represented in the top diagram, and the other two diagrams show the position of the same ions at two subsequent epochs, the negative ion being supposed for simplicity to move twice as quickly as the positive. It is evident that the left-hand side of the plane has lost two paired ions while the right-hand side has lost four. Thus the strength of the solution to the left, as measured by the number of pairs of ions still remaining, has decreased half as much as the strength on the other side of the plane. The loss of salt in the anode region (up to where the solution is unaltered) in general is to that in the cathode region in the ratio $\beta_1$ to $\beta_2$, where $\beta_1$ is the mobility of the positive ion and $\beta_2$ that of the negative. These losses can be found by analysing the two portions of the liquid, and it is also easy to find whether the concentration in the central part of the liquid has altered during the experiment.

The theory of the experiment becomes a little perplexing if the regions near the electrodes are considered instead of the central portion of the electrolyte. Thus in the figure the negative ions are moving to the anode twice as fast as the positive ions are moving to the cathode. This cannot hold right up to the electrodes, for then two equivalents of the negative ion would be liberated for one of the positive, which would contradict Faraday's laws. The behaviour near the electrodes is therefore more complex, and controlled by electrostatic forces.

The sum of the ionic mobilities can be calculated from the conductivity of the electrolyte, as was shown by Kohlrausch. Let there be $n$ molecules of the salt per cm.$^3$ before dissociation, and let each produce $k_1$ positive ions of valency $p_1$ and $k_2$ negative ions of valency $p_2$, where

$$k_1 p_1 = k_2 p_2 = k \text{ (say)}.$$

Thus for ferric sulphate $k_1 = 2$, $p_1 = 3$, $k_2 = 3$, $p_2 = 2$, and $k = 6$. If $v_1$, $v_2$ are the actual velocities of the ions, the current-density in electrostatic units is

$$u = nk_1 p_1 e v_1 + nk_2 p_2 e v_2 = nke\,(v_1 + v_2),$$

or in amperes per square centimetre

$$u = nk\frac{e\,(v_1 + v_2)}{3 \times 10^9}.$$

If $\beta_1$ and $\beta_2$ are the mobilities and $E$ the electric force in volts per centimetre, $v_1 = \beta_1 E$ and $v_2 = \beta_2 E$. Also $u = \sigma E$, where $\sigma$ is the conductivity, so that

$$\sigma = nk\frac{e\,(\beta_1 + \beta_2)}{3 \times 10^9}.$$

Let $c$ be the concentration of the solution, $m$ the mass of a molecule of the solute. Then $c = mn$, and therefore

$$\frac{\sigma}{c} = k\frac{\beta_1 + \beta_2}{3 \times 10^9} \cdot \frac{e}{m}.$$

Hence from equation (199),

$$\frac{\sigma}{c} = k\frac{\beta_1 + \beta_2}{3 \times 10^9} \cdot \frac{2{\cdot}89 \times 10^{14}}{M},$$

where $M$ is the molecular weight of the solute. Thus

$$u_1 + u_2 = 1{\cdot}04 \times 10^{-5}\frac{M\sigma}{kc} \quad \ldots\ldots\ldots\ldots (202).$$

Since $\beta_1/\beta_2$ is given by Hittorf's method, and $\beta_1 + \beta_2$ by equation (202), both $\beta_1$ and $\beta_2$ can be found.

The theory is only strictly applicable when all the molecules of the electrolyte are dissociated, which is not easy to realise (Art. 165). It follows that the velocities in solutions of ordinary strengths cannot be determined with great accuracy. In dilute solutions the velocity of, say, a potassium ion may be expected to be independent of the nature of the other ion present, since the two ions are supposed to migrate independently. The mobility of the potassium ion in 0·01 normal solutions of KCl, KI and $KNO_3$ is found to be $6{\cdot}26 \times 10^{-4}$, $6{\cdot}34 \times 10^{-4}$ and $6{\cdot}08 \times 10^{-4}$ respectively. The following table gives the mobilities of various ions, as estimated by Kohlrausch for infinite dilution:

| Cations | $10^4\beta$ | Anions | $10^4\beta$ |
|---|---|---|---|
| H .. .. .. | 31·5 | OH | 17·4 |
| K .. .. .. | 6·46 | $CrO_4$ | 7·2 |
| $NH_4$ .. .. .. | 6·4 | $SO_4$ | 6·8 |
| Pb .. .. .. | 6·1 | Br | 6·7 |
| Ba .. .. .. | 5·5 | I | 6·65 |
| Ag .. .. .. | 5·43 | Cl | 6·55 |
| Cu (in cuprous salts) | 4·6 | $NO_3$ | 6·17 |
| Zn .. .. .. | 4·6 | $ClO_3$ | 5·50 |
| Cd .. .. .. | 4·6 | F | 4·66 |
| Mg .. .. .. | 4·5 | $C_2H_3O_2$ | 3·5 |
| Na .. .. .. | 4·35 | | |
| Li .. .. .. | 3·34 | | |

**175. Theory of the motion of ions in electrolytes.** Since the osmotic pressure of the dissolved particles in a liquid is the same as the pressure they would exert in the gaseous form, it is natural to regard both as due to the same cause, namely the motion of agitation of the particles. We shall, at least provisionally, regard the dissolved particles as moving in the solvent in the same way that the molecules of one gas move among those of another gas with which it is mixed. The motion of an ion in a liquid must be rather a continuously curved trajectory than a succession of zigzags; but there is one feature common to both, a certain average interval of time $T$ in which the velocity of an ion in any particular direction can be wiped out, so that all directions of motion afterwards are equally probable. In a gas this is the time of describing

a mean free path. Disregarding the differences between the two cases, therefore, we can speak of a *free path* of an ion in a liquid and of its *collisions* with molecules of the solvent.

We shall assume for simplicity that all the free paths are of length $l$ and all the velocities of agitation $V$. Let $e$ be the charge and $m$ the mass of an ion moving under an electric force of $E$ electrostatic units. The acceleration of the ion in the interval between collisions is $eE/m$, so that the ion has drifted along the lines of force a distance $eET^2/2m$ further than it would have done in the absence of the force. The number of collisions per second is $1/T$, and therefore the amount of drift per second, or the velocity of the ion under the electric force, is $eET/2m$. Since $T = l/V$, $v = eEl/2mV$. Hence if $\beta$ is the velocity *under unit electrostatic force*,

$$\beta = \frac{el}{2mV} \quad \ldots\ldots\ldots\ldots\ldots\ldots (203).$$

The mean free path $l$ in the liquid can be estimated from this equation. The velocity of agitation $V$ is given by the principle of equipartition of energy, which holds at ordinary temperatures. If $\Omega$ is the velocity of agitation of the molecules of hydrogen at pressure $p$ and density $\rho$ and at the same temperature as the liquid, $\frac{1}{2}mV^2 = \frac{1}{2} \cdot 2m_H \cdot \Omega^2$ and $\Omega^2 = 3p/\rho$. Hence

$$V^2 = \frac{6p}{\rho} \frac{m_H}{m} \quad \ldots\ldots\ldots\ldots\ldots\ldots (204).$$

For the hydrogen ion at 15° C., $V = 2{\cdot}67 \times 10^5$. The velocity under a force of 1 volt per cm. ($E = 1/300$) is $v = 3{\cdot}15 \times 10^{-3}$, so that $l = 1{\cdot}8 \times 10^{-9}$ cm. This is of the order of one-tenth the diameter of a hydrogen molecule, assuming the ion to consist of a single charged molecule. So intimate a contact being highly improbable, we conclude that the effective mass of the ion must be much greater than that of a hydrogen molecule, so that the ion is accompanied on its course by a number of molecules of the solvent which cluster round it. The simple kinetic representation is, therefore, merely illustrative, and the motion of an ion in a liquid is probably more analogous to the fall of a body in a viscous liquid according to Stokes' law (cf. Art. 219). We proceed to some further consequences of the theory, such as it is.

Diffusion comes into play when the ionic concentration is not

the same at all points, and there is on the whole a drift of ions even in the absence of electric force. The ordinary theory of diffusion is based on an analogy with the theory of the flow of heat. If $n$ is the number of ions per cm.$^3$ at a point $P$ in the electrolyte, the mean velocity of the ions is supposed to be parallel to the negative gradient $-\mathbf{D}n$, and the number of ions crossing unit area of a perpendicular plane $-K\mathbf{D}n$ per second, where $K$ is a constant called the *coefficient of diffusion*. Thus the number of ions crossing a vector area $d\mathbf{S}$ per second is $-K\mathbf{D}n\,d\mathbf{S}$.

If an electric field is acting as well as diffusion, the number of ions becomes

$$-K\mathbf{D}n\,d\mathbf{S} + n\mathbf{v}\,d\mathbf{S},$$

where $\mathbf{v}$ is the velocity of drift in the electric field alone. Integrate over a small closed surface $S$ surrounding $P$. If $\tau$ is the enclosed volume, the total number of ions leaving $S$ per second is

$$-K\int_S \mathbf{D}n\,d\mathbf{S} + \int_S n\mathbf{v}\,d\mathbf{S} = \tau\,[-K\Delta n + \operatorname{div}(n\mathbf{v})],$$

and this must be equal to $-\dfrac{\partial}{\partial t}(\tau n) = -\tau\dfrac{\partial n}{\partial t}$.

Hence

$$\frac{\partial n}{\partial t} + \operatorname{div}(n\mathbf{v}) = K\Delta n \quad \ldots\ldots\ldots\ldots (205).$$

This is the equation of continuity in an electrolyte in which diffusion is going on.

The physical meaning of the constant $K$ is illustrated by the following theorem:

*If $R^2$ is the mean value of the squares of the distances of a large number of ions in unlimited liquid from any fixed point in space, and if no electric force is acting, then $d(R^2)/dt = 6K$.*

Let $N$ be the total number of ions, $n$ the number per cm.$^3$ at $P$, $O$ the fixed point and $OP = r$. Then $N$ is equal to $\int n\,d\tau$, extended over the whole of space, and is a constant. By definition

$$NR^2 = \int r^2 n\,d\tau.$$

Hence

$$N\frac{d(R^2)}{dt} = \int r^2\frac{\partial n}{\partial t}\,d\tau.$$

If there is no electric force, $\mathbf{v} = 0$ and $\partial n/\partial t = K\Delta n$.

Thus

$$N\frac{d(R^2)}{dt} = K\int r^2\Delta n\,d\tau.$$

Now it follows from Green's reciprocal theorem applied to the whole of space that $\int(U\Delta V - V\Delta U)\,d\tau = 0$. Putting $U = r^2$ and $V = n$ in this equation, we have $\int r^2 \Delta n\, d\tau = \int n \Delta r^2 d\tau = 6\int n\, d\tau = 6N$, so that $Nd\,(R^2)/dt = 6NK$, which was to be proved.

It follows that ions tend to diffuse so that the mean value of the square of their distances from any point increases uniformly with the time; and hence they must ultimately diffuse to infinite distances. This theorem enables us to calculate $K$ in terms of molecular quantities. Consider the ions which have collisions in the neighbourhood of a point $P$ (Fig. 240). After a time $l/V$ they are found uniformly spread over a sphere of radius $l$ with $P$ as centre. If $Q$ is one of these ions, the square of its distance from $O$ has increased from $r^2$ to $r^2 + 2lr\cos\theta + l^2$. Hence on the average $R^2$ increases by an amount $l^2$ in time $l/V$, or by an amount $lV$ per second. Hence $d\,(R^2)/dt = lV$, and

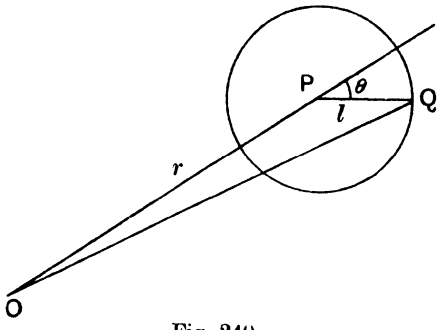


Fig. 240

$$K = \tfrac{1}{6}lV \quad \ldots\ldots\ldots\ldots\ldots\ldots (206).$$

From (203) and (206) we obtain by division

$$\frac{\beta}{K} = \frac{3e}{m\overline{V^2}}.$$

Let $N$ be the number of molecules per cm.$^3$ in a gas at the temperature of the liquid and pressure $\Pi$. Then $m\overline{V^2}$ is twice the mean kinetic energy of a molecule of the gas, so that $\Pi = \frac{1}{3}Nm\overline{V^2}$. Hence

$$\frac{\beta}{K} = \frac{Ne}{\Pi} \quad \ldots\ldots\ldots\ldots\ldots\ldots (207).$$

Although the present theory does not aim at mathematical rigour, equation (207) is exact.

**176. Concentration cells.** When two solutions of the same salt, but of different concentrations, are placed in contact, they will diffuse into one another and attain the same concentration. This however is a matter of days, and there is another effect which

takes place much sooner, the smoothing down of the surface of contact into a continuous transition layer.

Let the plane $x = 0$ originally separate two dilute solutions of the same salt, of concentrations $c_1$, $c_2$ respectively. At first more charge will cross the plane by diffusion in one direction than the other. This results in an accumulation of positive charge in one of the solutions which opposes further separation of the ions, and a steady state is ultimately attained in which as much charge passes in one direction as in the other. Let $n$ be the number of dissociated molecules per cm.$^3$ at distance $x$ from the original plane of separation in the steady state, $E$ the electric force caused by the separation of the ions. Very little separation will produce appreciable electric force (cf. Art. 172), so that we may put

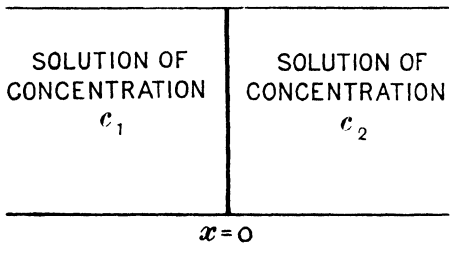


Fig. 241

$$n_1 p_1 = n_2 p_2 = nk \quad \ldots\ldots\ldots\ldots\ldots (208),$$

where $n_1$ is the number of positive ions per cm.$^3$ at any point, $n_2$ the number of negative ions, and $p_1$, $p_2$, $k$ are as in Art. 174. The number of positive ions crossing unit area of a plane parallel to $x = 0$ per second is $-K_1 dn_1/dx + n_1 v_1$, and each carries a charge $p_1 e$, so that the transfer of charge from left to right per second is $p_1 e\,(-K_1 dn_1/dx + n_1 v_1)$. Hence in the steady state

$$p_1\left(-K_1\frac{dn_1}{dx} + n_1 v_1\right) = p_2\left(-K_2\frac{dn_2}{dx} + n_2 v_2\right).$$

But $v_1 = \beta_1 E$ and $v_2 = -\beta_2 E$; and equation (207) now gives

$$K_1 = \frac{\Pi}{Ne}\frac{\beta_1}{p_1}, \quad K_2 = \frac{\Pi}{Ne}\frac{\beta_2}{p_2}.$$

Making these substitutions we find

$$-\frac{\Pi}{Ne}\beta_1\frac{dn_1}{dx} + n_1 p_1 \beta_1 E = -\frac{\Pi}{Ne}\beta_2\frac{dn_2}{dx} - n_2 p_2 \beta_2 E;$$

or, using (208),

$$n(\beta_1 + \beta_2)E = \frac{\Pi}{Ne}\left(\frac{\beta_1}{p_1} - \frac{\beta_2}{p_2}\right)\frac{dn}{dx}.$$

This last equation enables us to calculate the natural difference of potential that arises between the solutions. For writing it in the form

$$Edx = \frac{\beta_1 - \beta_2}{\beta_1 + \beta_2} \frac{p_1 - p_2}{Ne} \Pi \frac{dn}{n}$$

and integrating across the whole transition layer, we have

$$V = \frac{\beta_1 - \beta_2}{\beta_1 + \beta_2} \frac{p_1 - p_2}{Ne} \Pi \log_e \frac{\nu_1}{\nu_2},$$

where $n = \nu_1$ in the undisturbed parts of the left-hand solution and $n = \nu_2$ on the right. Since $\nu_1/\nu_2 = c_1/c_2$,

$$V = \frac{\beta_1 - \beta_2}{\beta_1 + \beta_2} \frac{p_1 - p_2}{Ne} \Pi \log_e \frac{c_1}{c_2} \quad \text{E.S.U.} \ldots\ldots (209).$$

The theory of the potential-differences arising at the boundary of any two electrolytes in contact is somewhat beyond the scope of this book; but there is one important general property, that $V$ depends only on the *ratio* of the concentrations. The condition of zero total flow of electricity, at a point where the electric force is $E$, is

$$\Sigma e_r \left( - K_r \frac{dn_r}{dx} + n_r \beta_r E \right) = 0,$$

where $e_r$ is the charge on an ion of the $r$th kind and summation is taken over all types of ion present. Hence $E$ would be unaltered if all the $n$'s were increased in the same ratio; that is, if the concentrations of the two solutions were so increased.

Cells of the kind just considered are called *concentration cells*. If we calculate the surface potential-differences at 15° C., and consider only univalent ions, equation (209) becomes

$$V = 0{\cdot}0574 \frac{\beta_1 - \beta_2}{\beta_1 + \beta_2} \log_{10} \frac{c_1}{c_2} \text{ volts} \ldots\ldots (210).$$

To verify the theory, Nernst set up a number of cells in which the effect of the electrodes is eliminated and the observed potential is due to change of concentration. One example is the cell

| KCl, 0·1 normal | KCl, 0·01 normal | HCl, 0·01 normal | HCl, 0·1 normal | KCl, 0·1 normal |
|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) |

with any similar electrodes. The potential arising at the interface

(2, 3) is equal and opposite to that arising at (4, 5), since the ratios of the concentrations are the same in both cases, and the effect of the electrodes also cancels out. Hence the only effective electromotive forces are those between (1, 2) and between (3, 4). The most general cell of the above type may be written symbolically

$$| c_1\beta_1\beta_2 | c_2\beta_1\beta_2 | c_2\beta_3\beta_4 | c_1\beta_3\beta_4 | c_1\beta_1\beta_2 |,$$

where $\beta_1$, $\beta_2$ are the ionic mobilities in the first electrolyte, $\beta_3$, $\beta_4$ those in the second, and $c_1$, $c_2$ the two concentrations occurring. The total E.M.F. of the cell is

$$V = 0{\cdot}0574 \left(\frac{\beta_1 - \beta_2}{\beta_1 + \beta_2} - \frac{\beta_3 - \beta_4}{\beta_3 + \beta_4}\right) \log_{10} \frac{c_1}{c_2} \text{ volts.}$$

The result of Nernst's experiments is shown in the following table:

| Electrolytes | E.M.F. | |
|---|---|---|
| | observed | calculated |
| KCl, HCl | -0·0357 | -0·0380 |
| KCl, $HNO_3$ | -0·0378 | -0·0390 |
| $NH_4Cl$, NaCl | 0·0098 | 0·0109 |
| KCl, NaCl | 0·0111 | 0·0112 |

# CHAPTER XI

## ELECTRIC OSCILLATIONS

**177. Theory of condenser discharges.** Let $C$ be a condenser whose plates are connected by a circuit of self-inductance $L$ and resistance $R$, all being measured in electromagnetic units, true or derived. If $e$ is the charge on the top plate at time $t$, the current is $i = -de/dt$ and the difference of potential between the plates $e/C$. Hence the equation $L\,di/dt + Ri = e/C$ gives

$$LC\frac{d^2e}{dt^2} + RC\frac{de}{dt} + e = 0 \quad \ldots\ldots\ldots\ldots (211).$$

The current $i$ and the potential-difference $V$ between the plates of the condenser both satisfy an equation of the same form. The equation (211) therefore determines the general type of electrical phenomena which can occur in a condenser circuit left to itself; and as we shall see that the quantities involved are usually oscillatory in nature, the equation may be said to represent the *electric oscillations* of the system.

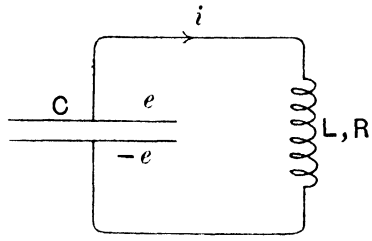


Fig. 242

When $R = 0$, equation (211) becomes $d^2e/dt^2 + n^2e = 0$, where

$$n^2 = 1/LC \quad \ldots\ldots\ldots\ldots\ldots\ldots (212).$$

Hence $e$ is then a sinoidal function of the time, of period

$$T = 2\pi/n = 2\pi\,(LC)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots (213).$$

It follows that, if a condenser is charged up to a certain potential and then discharged through a circuit of negligible resistance but appreciable self-inductance, the discharge is oscillatory, electricity surging backwards and forwards and charging the plates alternately positively and negatively. It had been observed by Savary in 1827 that when a needle was placed in a coil and magnetised by the discharge of a Leyden jar, the direction of magnetisation might vary according to circumstances. The present theory was given by Lord Kelvin in 1853, formula (213) being known by his name.

Writing $\mu$ for $R/2L$ in equation (211), we have

$$\frac{d^2e}{dt^2} + 2\mu\frac{de}{dt} + n^2e = 0.$$

The solution of this equation in real form differs according as $n \gtrless \mu$. If $n > \mu$ it is

$$e = \{A\cos(n^2 - \mu^2)^{\frac{1}{2}}t + B\sin(n^2 - \mu^2)^{\frac{1}{2}}t\}e^{-\mu t} \quad \ldots(214),$$

while when $n < \mu$ it becomes

$$e = \{Ae^{(\mu^2 - n^2)^{\frac{1}{2}}t} + Be^{-(\mu^2 - n^2)^{\frac{1}{2}}t}\}e^{-\mu t}.$$

We have thus to distinguish two types of discharge; the oscillatory type, which occurs when $R^2 < 4L/C$, and the non-oscillatory type, which occurs when $R^2 > 4L/C$. Oscillatory discharges, from equation (214), are represented in general by a sinoidal function with an exponential factor. The periodic time is defined as twice the interval between two consecutive epochs at which the charge or current vanishes, and is then given by $T = 2\pi(n^2 - \mu^2)^{-\frac{1}{2}}$. In general, therefore, the period exceeds that given by Kelvin's formula. The quantity

$$\delta = \frac{2\pi\mu}{n} = \pi R\left(\frac{C}{L}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots(215)$$

is called the *decrement*. It is such that the exponential factor sinks to $e^{-\delta}$ after one period given by Kelvin's formula. The discharge is only oscillatory when $\delta < 2\pi$; but even the value $\delta = 1$ represents a very highly damped discharge. For weakly damped oscillations in which $\delta$ is so small that its square is negligible, Kelvin's formula gives the periodic time with sufficient accuracy. It may be written

$$\text{Frequency} = \frac{159000}{(LC)^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots(216),$$

where $L$ is in millihenrys and $C$ in milli-microfarads.

Suppose that the condenser is first charged to potential $V_0$ and the circuit then closed. Thus at time $t = 0$, $e = CV_0$ and

$$i = -de/dt = 0.$$

If the damping is small, equation (214) becomes

$$e = (A\cos nt + B\sin nt)e^{-\mu t},$$

to the first order in $\mu$. The initial conditions give $A = CV_0$ and $B = \mu CV_0/n$, so that the solution is

$$e = CV_0\left(\cos nt + \frac{\mu}{n}\sin nt\right)e^{-\mu t} = CV_0 e^{-\frac{\delta t}{T}}\left(\cos\frac{2\pi t}{T} + \frac{\delta}{2\pi}\sin\frac{2\pi t}{T}\right).$$

By differentiation, $i = nCV_0 e^{-\mu t}\sin nt$ to the first order in $\mu$. Since the damping is small, the variation of the exponential factor from time $t = 0$ up to the time of the first maximum of $i$ is negligible, and we have with sufficient accuracy

$$i_{\max} = nCV_0.$$

If a condenser of 1 microfarad capacity is charged up to 100 volts and then discharged round a circuit of inductance 1/100 henry and resistance 1 ohm, $C = 10^{-6}$, $L = 10^{-2}$, $R = 1$ and $V_0 = 10^2$. Hence $n = 10^4$, $T = 1/1600$ sec., $i_{\max} = 1$ ampere, $\delta = 0{\cdot}03$. This is the type of comparatively slow oscillation with large inductance and capacity. The resistance necessary to make the discharge non-oscillatory would be 200 ohms, so that the discharge of a condenser in an experiment such as that of Art. 90 may very well be oscillatory. This does not impair the validity of equation (97), which assumes only that the charges $\pm CV_0$ originally on the plates are neutralised in a comparatively short time.

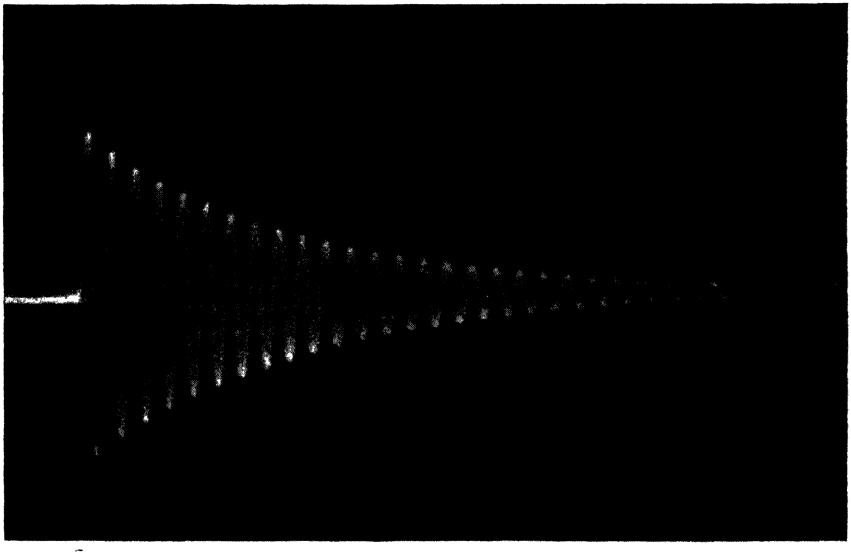


Fig. 243

As an example of more rapid oscillations consider a condenser of 1 milli-microfarad capacity charged to 5000 volts and then discharged round a coil of a few turns of wire, say of inductance 0·01 millihenry and resistance 1/10th ohm. The frequency is $1{\cdot}59 \times 10^6$ and the maximum current 50 amperes, while $\delta = 0{\cdot}003$. The exponential factor sinks to $1/e$ in 1/5000 second, during which time the system has carried out 300 complete oscillations. The resistance necessary to cause an aperiodic discharge would be 200 ohms.

Fig. 243 represents the current obtained by charging a condenser and then discharging it through an inductance*. The natural frequency was 250 per second and the decrement about 0·12. The record was made with a cathode ray oscillograph in a manner to be explained later (Art. 207).

**178. Electrical resonance.** The circuit of Fig. 242 can also have alternating currents induced in it from an external source. The theory has been indicated in Art. 132. If $V$ is the amplitude of the induced E.M.F., the amplitude of the current is

$$I = \frac{V}{\left\{\left(Lp - \frac{1}{Cp}\right)^2 + R^2\right\}^{\frac{1}{2}}} \quad \ldots\ldots\ldots(217).$$

Consider how $I$ varies with $p$. The current at low frequencies is small because it is blocked by the condenser. Very quick currents are small because they are choked by the inductance. The greatest amplitude $I = V/R$ occurs when $Lp = 1/Cp$, or

$$p^2 = 1/LC \quad \ldots\ldots\ldots\ldots\ldots(218).$$

Comparing (212) and (218) we see that the current is greatest when the frequency of the alternating current is equal to the natural frequency of the circuit, as calculated from Kelvin's formula. If $R$ is small the maximum current may be much greater than that corresponding to either larger or smaller frequencies. This effect is called *resonance* by analogy with the corresponding effect in acoustics. The phase-difference between the current and inducing E.M.F. vanishes when resonance occurs, and the circuit then behaves exactly like a coil of resistance $R$, with no inductance.

* I am indebted to Prof. J. Zenneck for this figure and for Fig. 250.

The condensers available in a laboratory do not usually exceed 10 microfarads, nor the inductances 1/10 henry (unless iron is used). This makes a circuit of frequency 160, so that a high-frequency alternator (Art. 154) is required to exhibit resonance. The alternating current is passed through a coil whose mutual inductance with the coil $L$ can be set at any convenient amount, and the main circuit contains a vacuum thermal junction with attached galvanometer. Curve I of Fig. 244 shows the result of an experiment with a circuit of inductance $2{\cdot}7 \times 10^{-2}$ henry and capacity 1 microfarad, the total resistance, including the junction, being about 3 ohms. Curve II shows the effect of inserting a non-inductive resistance of 10 ohms in the circuit. The reduction of current is, as we should expect, more marked near the position of resonance than elsewhere, so that resistance in a circuit reduces the sharpness of resonance.

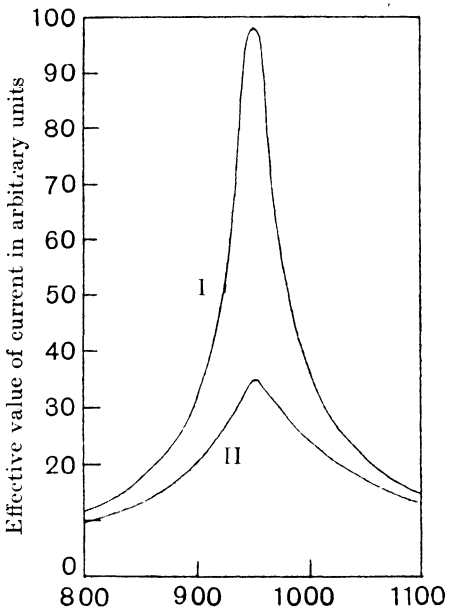


Fig. 244

**179. Production of oscillations with an induction coil and spark-gap.** A condenser can be charged to a high voltage and discharged automatically by utilising the properties of the electric spark. Let the circuit of Fig. 242 be interrupted anywhere by a spark-gap consisting of two zinc or brass balls brought to within a short distance (say 1 mm.) of one another and connected to the secondary of an induction coil. The "break" of the coil gives a rapidly increasing difference of potential between the plates of the condenser, which at first simply charges them up. When this potential-difference rises to a certain amount, in this case about 3000 volts, the spark-gap gives way and the condenser discharges somewhat as in Fig. 243; for when a current has once passed it

continues flowing below the potential necessary to start a spark in the first place. A limit is reached when the amplitude of voltage sinks to 200 volts or thereabouts, after which the spark-gap ceases to conduct. The condenser is then ready to be charged by the next break of the coil, and so indefinitely. Thus if the coil is interrupting regularly 50 times a second, 50 independent trains of damped oscillations are produced in every second. The oscillation considered in the first paragraph of p. 368 is almost certainly extinguished in 1/1000 sec., since the amplitude of potential is reduced to 200 volts in that time by resistance alone, without taking the spark-gap into account. It follows that the speed of interruption can be greatly increased without fear of cutting into the individual trains.

The oscillatory nature of condenser discharges in circuits of sufficiently low resistance was first shown by Feddersen (1861), who examined the image of the spark in a rotating mirror. The experiments have often been repeated, most carefully by Diesselhorst, who found the following results:

| $T$ observed | $T$ from Kelvin's formula |
|---|---|
| $1{\cdot}628 \times 10^{-6}$ | $1{\cdot}646 \times 10^{-6}$ |
| $2{\cdot}293 \times 10^{-6}$ | $2{\cdot}313 \times 10^{-6}$ |
| $4{\cdot}967 \times 10^{-6}$ | $4{\cdot}967 \times 10^{-6}$ |
| $5{\cdot}130 \times 10^{-6}$ | $5{\cdot}141 \times 10^{-6}$ |
| $7{\cdot}734 \times 10^{-6}$ | $7{\cdot}731 \times 10^{-6}$ |

The agreement between theory and observation is very close. A capacity of 10 milli-microfarads and an inductance of 1 millihenry give oscillations slow enough for a mirror fixed on the axle of a small motor, and the sparks can be shown widely separated by projecting the images on the ceiling of a room.

The effect of a spark-gap on the current curve was examined by Zenneck. If the electrodes are silver or copper the amplitude of current sinks almost linearly, and not exponentially, to zero. The mean decrement is greater with the spark-gap than without it, by an amount varying from 0·059 for magnesium to 0·116 for silver, in a 5 mm. gap. The frequency is also slightly reduced. Very short sparks give much greater damping, as was shown by Perot, and later by M. Wien. Such a highly damped circuit will act on a second

circuit as an impulse acts on a pendulum: the first discharge will induce a current in it which dies away in the natural period of the second circuit, after the first circuit has ceased to oscillate.

The current in an oscillatory circuit may be considerable, as we saw in Art. 177: the second of the circuits there mentioned would give $i_{max} = 150$ amperes with a 5 mm. spark-gap. Its inductive effect, at a frequency of $1{\cdot}6 \times 10^6$, is equal to that of an alternating current of no less than $5 \times 10^6$ amperes at a frequency of 50. The inductive effects are, therefore, extraordinarily great, even when reduced by the idle interval between the trains. A glow-lamp in a loop of one or two turns of wire glows brightly when brought near an oscillation circuit.

The inductive effect is seen to the greatest advantage after the oscillations have been *transformed*, as was first done by Tesla. His apparatus is shown in Fig. 245. The primary consists of a small number of turns of thick wire inserted in the oscillation circuit; the secondary coil is a solenoid of many turns well insulated from one another or immersed in oil. In this way voltages of the order of $10^6$ can be produced, and a variety of striking experiments performed. If the ends of the secondary are connected to two unequal metallic rings (diameters 15 cm. and 10 cm., about 10 cm. apart), a conical discharge takes place between them, the whole space being traversed by filmy blue streamers. Leads of bare and even insulated wire are surrounded by a halo. If the secondary terminals are joined to a pair of parallel wires about 10 cm. apart, a filmy discharge takes place along their entire length. Long crackling sparks are obtained from a pointed terminal attached to the top of the secondary, and glow-lamps and discharge tubes light up at a distance. A discharge tube held in one hand lights up when the secondary is touched with the other. The experiments are quite

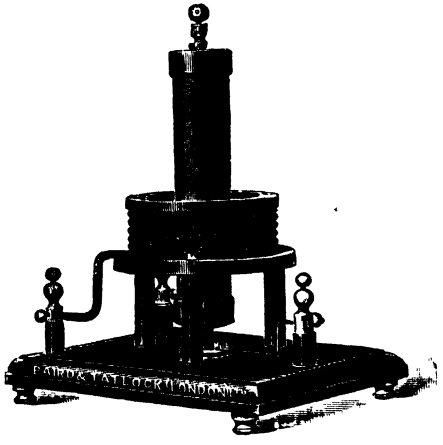


Fig. 245

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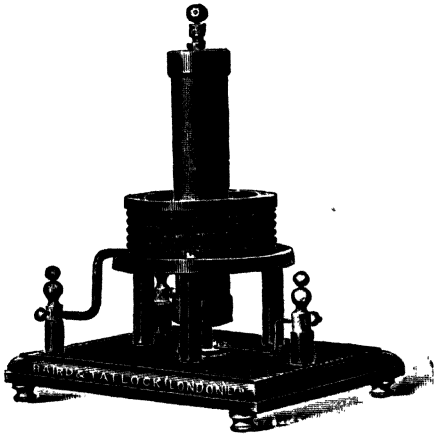


Fig. 245

safe, but touching the *primary* would give a dangerous low-frequency shock.

The secondary coil has one or more natural periods of its own (Art. 203), and we obtain the best effects by adjusting the frequency of the primary circuit to the fundamental.

**180. Buzzer and crystal method.** Weak trains of damped oscillations are produced very simply with a *buzzer*, which is a small interrupter like those used in electric bells, with a light spring vibrating several hundred times a second. The circuit is shown in the left-hand diagram of Fig. 246. $V$ is a dry cell or 2-volt accumulator, $B$ the buzzer, $C_1$ a variable condenser of the order of 1 millimicrofarad and $L_1$ an inductance of the order of 0·01 millihenry.

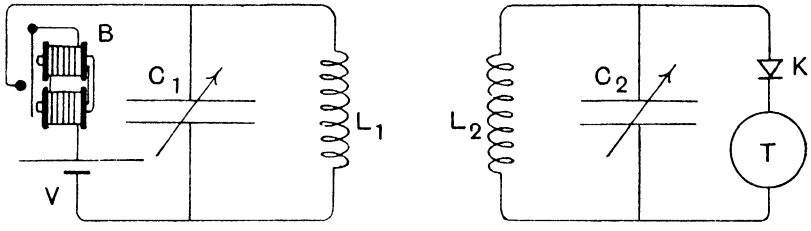



Fig. 246

The condenser charges up while the spring is completing the circuit, and discharges through $L_1$ on release, in the natural period of the circuit $L_1$, $C_1$.

The currents are detected in a second circuit containing a *crystal detector* $K$. One form is a piece of carborundum soldered into a cup and pressed on by a steel plate. Such a crystal lets through a current which is not only not proportional to the applied voltage, but is also different for positive and negative voltages (Fig. 247). Let us suppose (merely for illustration) that the relation of current to voltage is the same for high frequencies as for steady currents, and that the crystal is subjected to a voltage $A \cos pt$ at time $t$. If the relation between current and voltage is $i = f(V)$, the current at time $t$ is

$$f(A \cos pt) = f(0) + A \cos pt f'(0) + \tfrac{1}{2}A^2 \cos^2 pt f''(0)$$

to the second order in $A$, supposed small. The mean current is approximately $\frac{1}{4}A^2 f''(0)$, so that a galvanometer in series with the crystal will show a deflexion proportional to $A^2$ and to $f''(0)$.

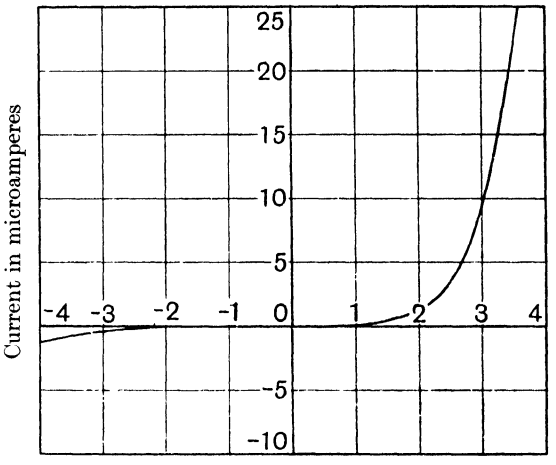



Fig. 247

The theory of the circuit of Fig. 246 is not quite so simple. The telephone would choke out high-frequency currents if it did not possess capacity, which it does in virtue of the amount of wire it contains. It behaves like a condenser in parallel with a self-inductance, and the condenser will let the (nearly periodic) high-frequency currents through freely (Art. 132). The whole train of oscillations will, from what has been said above, send a definite amount of electricity through the crystal in one direction, so that the condenser $C_2$ is charged when the oscillations cease. Part of the accumulated charge returns relatively slowly through the telephone, pulling the diaphragm aside and giving an audible click. Successive clicks combine into a musical note having the frequency of interruption of the buzzer. If a steady potential $V_0$ is applied from a separate battery, the strength of the impulse is proportional, by a simple extension of the preceding theory, to $f''(V_0)$, which is large if the characteristic is sharply curved at $V = V_0$. Most crystals are improved by applying potential from a cell and potentiometer, as shown in Fig. 248, the best point being found by trial.

In experimenting with the circuit of Fig. 246 we first take two identical inductances $L_1$, $L_2$ and identical variable condensers $C_1$, $C_2$. Equality of $C_1$ and $C_2$ is recognised by a greatly increased sound in the telephone at the resonance point, and resonance is

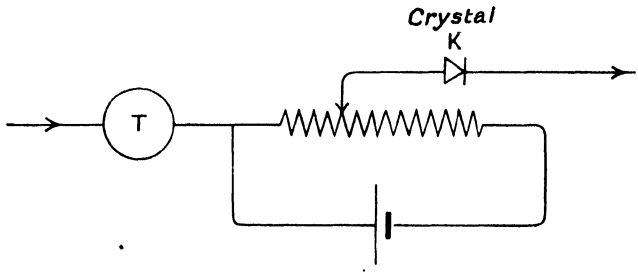



Fig. 248

sharp since the primary circuit contains no spark-gap and the crystal has a high mean resistance. We next remove the secondary circuit till only a faint sound is heard in the telephone. This sound is made stronger by adjusting the potentiometer, and the secondary can be removed still further till the loudest sound is only just audible. This is the adjustment for observing resonance with the greatest nicety. The application to electrical measurements is explained in the next article.

The cause of the rectifying action of crystals is obscure. Pierce found that the difference between the positive and negative currents, at the same voltage, became much less when the faces of the crystal were platinised so as to increase the area of effective contact with the electrodes. The action therefore appears to have its seat at or near the points where the crystal touches the electrodes. A crystal will certainly rectify alternating currents of frequency $4 \times 10^8$, and possibly higher.

**181. Frequency or wave-meters.** If the self-inductance of the secondary circuit in Fig. 246 is known, and also the capacity for different settings of the variable condenser, we have an instrument for determining the frequency of the oscillations in the primary. These instruments are made in compact form, with the self-inductance built into a box containing the variable condenser. The observer picks up the oscillations at short range and then moves away till he can only just hear the sound when the condenser is set for resonance. The frequency of the primary is then given by Kelvin's formula. This method is used with spark transmitters. The natural frequency of a *receiver* is found most conveniently by inducing oscillations in it from the meter with a

buzzer substituted for the crystal and telephone, adjusting the condenser for maximum sound in the receiving telephones.

Fig. 249 shows the essential parts of a meter suitable both for spark oscillations and for the continuous oscillations generated by triodes (Art. 185). There is a fixed condenser $C$ of the order of 1 milli-microfarad, made of mica and brass foil. The inductance $L$ is composed of two short solenoids in series, one of which can be rotated about an axis so as to vary the mutual inductance, and thus vary the self-inductance of the compound coil (Art. 134). $G$ is a small glow-lamp which lights up brightly in the position of resonance. A dry cell $D$ can be put across the lamp so as to raise it to a dull red heat, whereby the sensitiveness is increased and the oscillations are easier to pick up. The inductance $E$ prevents high-frequency current from going through the cell in preference to the lamp, while $B$ is a buzzer for generating oscillations in receivers, as already explained. The range of the instrument can be extended by switches for throwing in an extra condenser in parallel and for altering the inductance.

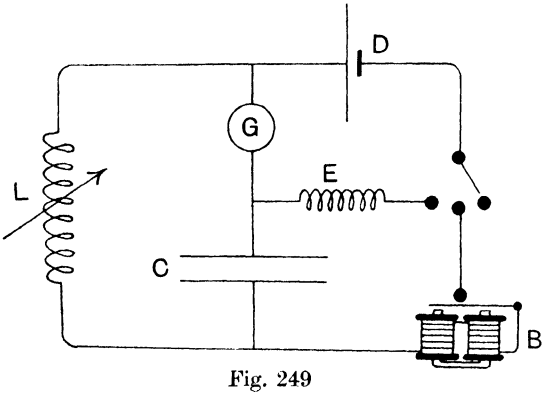


Fig. 249

Frequency-meters can be calibrated by comparison with one or more standard meters, whose inductance and capacity are found by calculation, checked by observations with a rotating mirror (Art. 179). Other absolute methods, one of which is explained in Art. 201, involve the use of harmonics.

It will appear shortly that electromagnetic effects are propagated through space with a finite velocity $c$, equal to the ratio of the electromagnetic to the electrostatic unit of charge. Thus electric *oscillations* are accompanied by electric *waves*, the wave-length $\lambda$ being connected with the period of oscillation by the relation $\lambda = cT$. For this reason frequency-meters are usually called *wave-meters*, as they measure the wave-length of the corre-

sponding electric waves. The formula

$$\lambda = 1885\,(LC)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots (219)$$

gives $\lambda$ in metres when $L$ is in millihenrys and $C$ in milli-microfarads. A wave-length of 300 metres corresponds to a frequency of one million per second.

A wave-meter is the most convenient instrument for measuring small inductances and capacities. We begin with an inductance whose magnitude can be calculated from its dimensions. The capacity of a condenser is found by making up an oscillation circuit of the two (taking care not to introduce more inductance than necessary in the leads) and applying equation (219) to the observed wave-length $\lambda$. Condensers which leak too much for electrostatic experiments can be measured in this way. Or we can begin with a known condenser and measure self-inductances; this is the most accurate method for small inductances. Knowing the self-inductances of two coils, their mutual inductance can be found as explained in Art. 134.

**182. Coupled oscillation circuits.** When the two circuits in Fig. 246 are brought so close together that their mutual inductance is appreciable, the oscillations in the circuits disturb each other. In order to examine this more particularly in the case in which the resistances in the circuits are negligible, suppose that $\pm e_1$ are the charges on the plates of $C_1$ at time $t$, $i_1 = -de_1/dt$ the primary current, and $e_2$, $i_2$ the corresponding quantities for the second circuit. Then

$$L_1C_1\frac{d^2e_1}{dt^2} + MC_1\frac{d^2e_2}{dt^2} + e_1 = 0, \quad MC_2\frac{d^2e_1}{dt^2} + L_2C_2\frac{d^2e_2}{dt^2} + e_2 = 0.$$

Write $n_1^2 = 1/L_1C_1$, $n_2^2 = 1/L_2C_2$, and try a solution $e_1 = A_1\cos pt$, $e_2 = A_2\cos pt$. Substituting in the above equations

$$(n_1^2 - p^2)\,A_1 = \frac{Mp^2}{L_1}A_2, \quad \frac{Mp^2}{L_2}A_1 = (n_2^2 - p^2)\,A_2.$$

Eliminating the ratio $A_1 : A_2$, the admissible values of $p$ are given by

$$(n_1^2 - p^2)\,(n_2^2 - p^2) = k^2p^4 \quad \ldots\ldots\ldots (220),$$

where

$$k = \frac{M}{(L_1L_2)^{\frac{1}{2}}} \quad \ldots\ldots\ldots\ldots\ldots (221).$$

The quantity $k$ is called the *coefficient of coupling*: the circuits

are said to be weakly coupled when $k$ is small, strongly coupled when $k$ approaches unity (cf. Art. 136). Equation (220) shows that there are two possible oscillation frequencies in a pair of coupled circuits, which in general are not the same as the natural frequencies of the two circuits by themselves.

In the case of resonance ($n_1 = n_2 = n$), equation (220) gives $n^2 - p^2 = \pm kp^2$. Hence if $T_0$ is the common period of the circuits separately, the combined periods are given by

$$T = T_0 (1 \pm k)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots (222).$$

The frequencies can be detected separately with a wave-meter, if not too close. Suppose for example that two coils of inductance 1·18 and 1·23 millihenrys are combined with condensers which are adjusted till each gives a wave-length of 1800 metres when oscillating alone, and then placed so as to have a mutual inductance of 0·42 millihenry. Wave-lengths of 1460 and 2100 metres are observed. Since $k = 0{\cdot}349$, the calculated wave-lengths are 1450 and 2090 metres, so that formula (222) is verified.

As in acoustics, the existence of two frequencies gives rise to *beats*, or periodic variations of the amplitude of the oscillations.

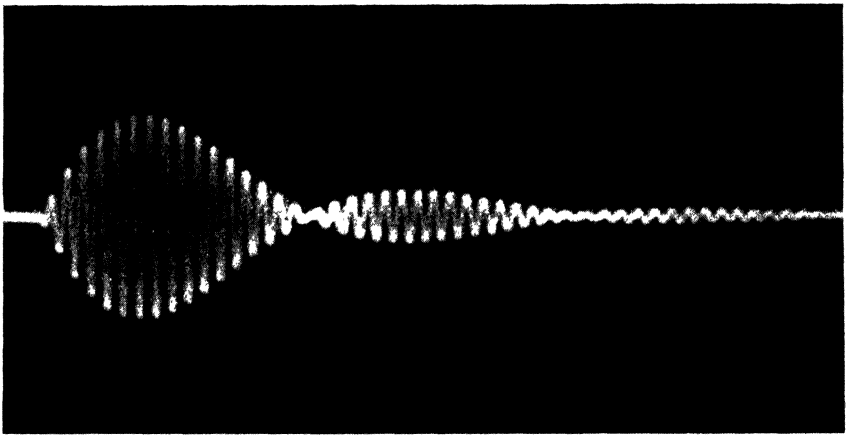


Fig. 250

Fig. 250 is one of Zenneck's records of the secondary current in a pair of coupled circuits. Beats occur whenever oscillations of two frequencies are superposed in the same circuit, whether by coupling or not (Art. 187).

A practical consequence of the theory is that a wave-meter cannot be brought too near an oscillation circuit without altering its frequency. A sensitive detector is therefore essential for accuracy. With weak oscillations, the glow-lamp in the wave-meter of Fig. 249 would be replaced by a vacuum thermal junction and galvanometer; or a crystal in series with a galvanometer can be placed across the condenser.

**183. Continuous oscillations with the electric arc.** The problem of exciting *continuous* oscillations in condenser circuits, as distinct from trains of damped oscillations, was first solved by Duddell in 1900. If a condenser $C$, of the order of 5 microfarads, and a self-inductance $L$, of the order of $\frac{1}{50}$ henry, are placed as a shunt across a carbon arc (Fig. 251), slow continuous oscillations are produced, which are accompanied by a musical note from the arc. The resistance $R$ is adjusted for a current of 2 to 3 amperes at 100 volts, though a higher voltage is required for good effects. The carbons are first brought together till the arc is struck, and then separated slowly until singing begins, which occurs just before the separation at which the arc goes out. The oscillations are detected with a secondary circuit containing self-inductance, capacity and a thermal detector; and resonance curves can be obtained showing that the primary frequency is given approximately by Kelvin's formula. The difference of potential between the plates of the condenser may rise to four or five times the potential of the mains, as may be seen by placing an electrostatic voltmeter across its terminals.

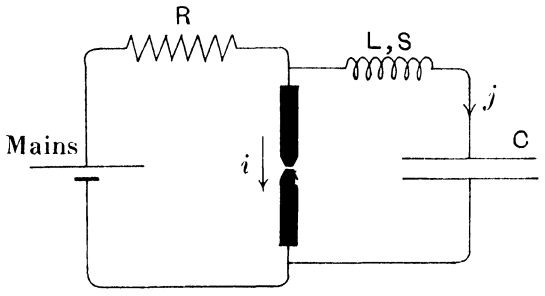


Fig. 251

A piece of iron placed near the coil $L$ raises the self-inductance and lowers the note emitted by the arc. The note may be altered continuously by making up the self-inductance of two coils in series with one another and moving one of them about so as to

change the mutual inductance. It is raised when a secondary circuit in the vicinity is short-circuited, on account of the diminution of the effective self-inductance of the primary (cf. Art. 136). The oscillations can be detected across a room by means of a receiving circuit with a telephone shunted across the condenser. When the capacity of this condenser is changed the intensity of the sound in the telephone changes also, but not the pitch: this illustrates the important principle of forced vibrations, namely that it is the frequency of the forcing agency which is established, and not that of the system acted on.

The following considerations show the possibility of continuous oscillations in the condenser circuit. The essential feature of the arc is that the voltage across the arc decreases as the current increases (Art. 162). Assume for simplicity that the current $i$ through the arc depends only on the difference of potential $V$ between the carbons; and further that the oscillatory current is small, so that we can write

$$i = a - bV,$$

where $a$ and $b$ are constants. If $j$ is the current in the condenser circuit and $V_0$ the voltage of the mains,

$$L\frac{d^2j}{dt^2} + S\frac{dj}{dt} + \frac{j}{C} = \frac{dV}{dt},$$

and $$V_0 = V + R(i + j) = V(1 - Rb) + Rj + Ra.$$

Hence $$(Rb - 1)\frac{dV}{dt} = R\frac{dj}{dt},$$

so that $$L\frac{d^2j}{dt^2} + \left(S - \frac{R}{Rb - 1}\right)\frac{dj}{dt} + \frac{j}{C} = 0.$$

The condenser therefore oscillates in its natural period, with an effective resistance $S - R/(Rb - 1)$. The effective resistance is greater than $S$ as long as $R < 1/b$, but less than $S$ if $R > 1/b$. It can vanish if

$$S = \frac{R}{Rb - 1}, \text{ or } R = \frac{S}{Sb - 1},$$

which is possible if $b$ is greater than either $1/R$ or $1/S$. Oscillations will then begin of their own accord, the energy of maintenance coming out of the mains. A rising characteristic, as of a glow-lamp, makes $b$ negative, and no oscillations are then possible.

The above theory is nothing more than an illustration, since the behaviour of the arc for frequencies as low as 50 to the second is very different from its behaviour for steady currents. Interesting results have been obtained by Blondel by means of the oscillograph. One of his curves is shown in Fig. 252, representing the most

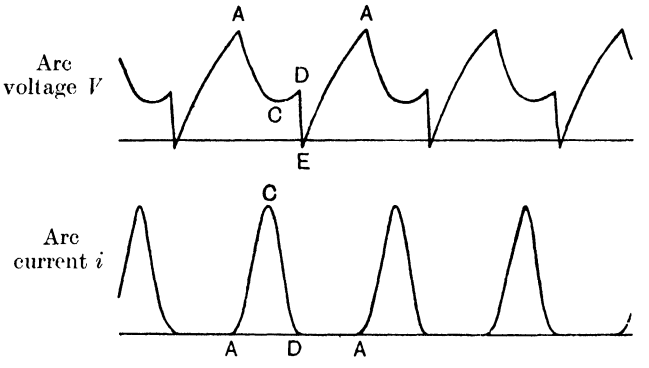



Fig. 252

common case. By taking corresponding values of $i$ and $V$ and plotting them against one another we obtain the "characteristic" curve (Fig. 253). The portion $ACD$ in the curves corresponds to the interval during which the arc is letting current through, $DEA$ to the interval of extinction. At $C$ the arc is fairly hot; but it cools slightly as the current diminishes, so that the potential at $D$ is slightly higher than at $C$. At $D$, when the arc ceases to conduct, the rapid removal of ions from the stream makes the potential fall quickly to $E$, and rise again as the mains charge up the condenser.

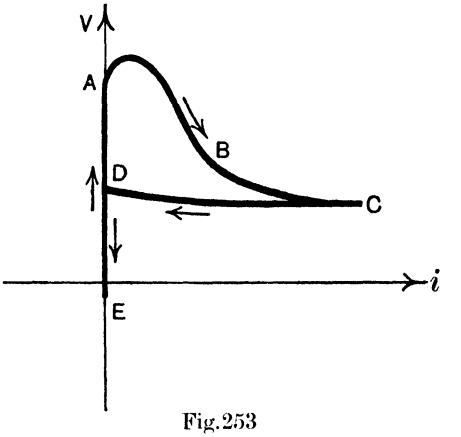



Fig. 253

Though the arc is cooling all this time it is still hot enough to re-light at $A$ for a certain potential higher than that at $D$. The cycle is then completed along a curve $ABC$ lying uniformly above the curve $CD$.

Duddell's arc fails to excite oscillations of frequency higher than about 10,000 per second. The difficulty was surmounted by Poulsen in 1902 by three main devices:

(1) a transverse magnetic field is applied;

(2) the arc burns in hydrogen, or in a gas containing hydrogen, such as alcohol vapour;

(3) the positive electrode is of copper, water-cooled.

The effect of (1), by blowing out the arc into a bow, is to raise the point $D$ in Fig. 252, and the effect of (2) and (3) together is to raise the point $A$ by thermal conduction during the period of extinction. Since the steady characteristic is also steepened, it is easy to see that these devices favour an increase both in the power and in the maximum working frequency.

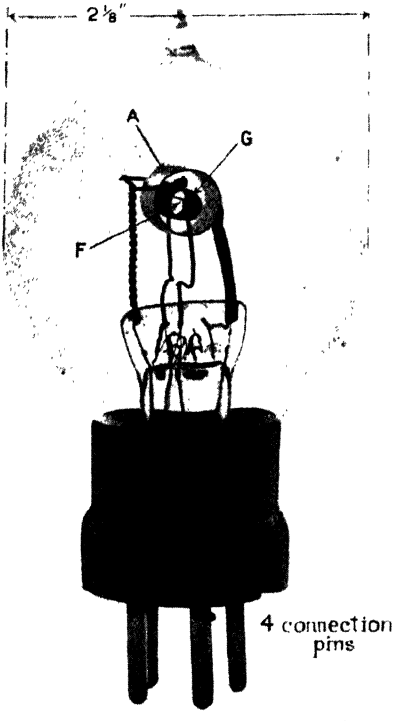



Fig. 254

**184. Triodes.** A *triode* (Fig. 254) is an exhausted glass vessel containing three electrodes, which, we shall see, can replace the crystal as a detector and the arc as a generator of continuous oscillations. One form is shown in Fig. 254. The *filament* $F$ is a tungsten wire, thicker than that of a glow-lamp, heated to a medium brightness by a 4 or 6 volt battery, with a current of about 1 ampere. The *grid* $G$ is a spiral of fine nickel wire, like an open solenoid, surrounding the filament, which it encloses from end to end. The *plate* $A$ is a nickel cylinder coaxial with the grid and filament, and about twice the diameter of the

grid. The two filament leads, the grid lead and the plate lead are brought out to four pins which fit into sockets in a holder.

The triode acts by emission of electrons from the filament, which are caught in different numbers by the grid and plate, according to their potentials (Arts. 211, 212). The electron streams constitute currents *towards* the grid and plate from their respective leads,

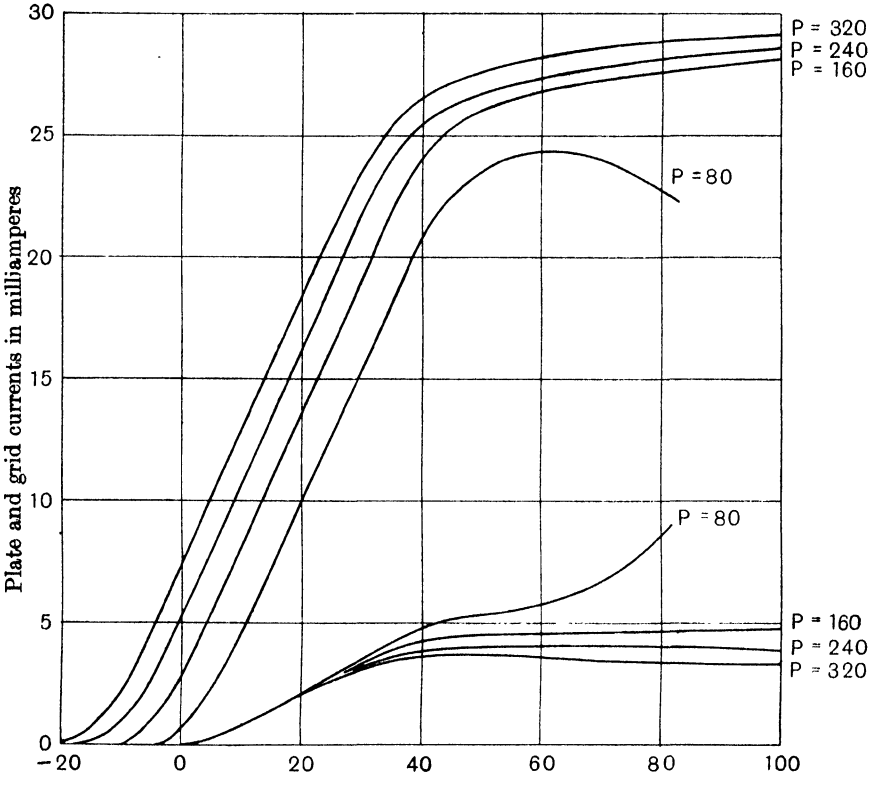



Fig. 255

called the grid and plate currents. The behaviour of a triode is determined by the curves of grid and plate current plotted against the grid potential. The grid potential $G$ and the plate potential $P$ are measured from the negative end of the filament. Fig. 255 shows some typical curves for a small triode, plate currents being above and grid currents below. The curves of plate current have a nearly straight part in the middle, the slope of which is nearly constant above $P = 80$; and any increase of $P$ moves this part back by an

amount roughly proportional to the increase. The plate current can therefore be represented by the formula

$$i = \text{const.} + aP + bG \quad \ldots\ldots\ldots\ldots (223)$$

over a considerable range in the neighbourhood of $G = 0$. The grid current is much smaller than the plate current in this region*. The plate current changes more with grid potential than with plate potential, in the ratio of $b$ to $a$. We have, near $P = 200$ and $G = 0$ in Fig. 255, $a = 3 \times 10^{-5}$ and $b = 6 \times 10^{-4}$ approximately, so that $b/a = 20$.

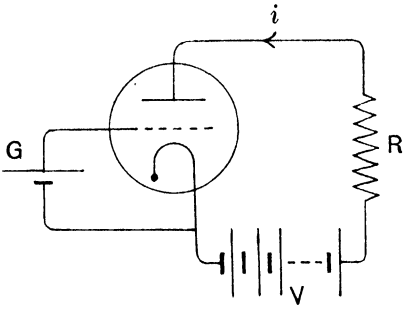


Fig. 256

The fact that $b/a$ may be considerable underlies the use of triodes to magnify small potentials (Art. 189). This phenomenon can be illustrated with the circuit of Fig. 256, in which the battery which heats the filament is omitted for simplicity. A battery of E.M.F. $G$ is applied to the grid, and one of E.M.F. $V$ to the plate, through a high resistance $R$. If $P$ is the plate potential,

$$i = \text{const.} + aP + bG, \quad P = V - Ri.$$

If $G$ is increased slightly to $G + dG$,

$$di = a\,dP + b\,dG \text{ and } dP = -R\,di,$$

so that

$$dP = -\frac{Rb\,dG}{1 + Ra}.$$

Suppose for example that $R = 100{,}000$ and that $G = 0$ and $P = 200$ initially, with the characteristics of Fig. 255. Then $dP = -15dG$, so that 2 volts applied to the grid make the plate potential fall by 30 volts, which is enough to deflect an electrostatic voltmeter. The maximum theoretical ratio, $b/a$, is called the *amplification ratio* of the triode. It lies usually between 5 and 20, and is greatest for thick grid wires close together (Art. 212).

**185. Continuous oscillations with the triode.** Triodes seem to have been first used to generate continuous electric oscillations

* These facts are useful in the elementary theory of triode oscillations (Art. 185).

by A. Meissner in 1914. A common circuit is shown in Fig. 257. One plate of the condenser $C$ is joined to the plate of the triode, the other to the positive pole of the battery, whose E.M.F. $V$ is of the order of 200 volts. The oscillating circuit induces into a coil placed between the grid and filament of the triode, with which it has a mutual inductance $M$.

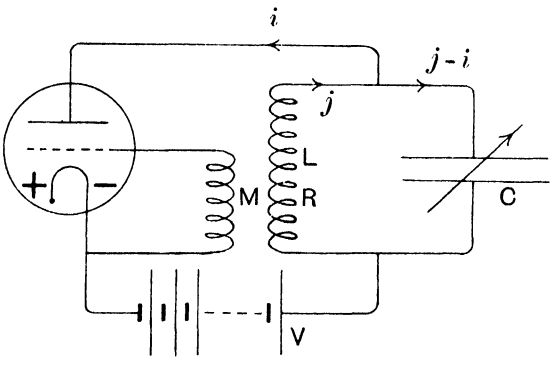


Fig. 257

Let $i$ be the plate current, $j$ the current through the inductance $L$. The plate current is given by equation (223), which would apply to a small oscillation in any case, but does actually hold over a considerable range. Grid currents are neglected in this simple theory, and also the internal resistance of the battery $V$. The circuit equations are

$$j - i = C\frac{dP}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots (224),$$

$$L\frac{dj}{dt} + Rj = V - P \quad \ldots\ldots\ldots\ldots\ldots (225),$$

$$M\frac{dj}{dt} = -G \quad \ldots\ldots\ldots\ldots\ldots\ldots (226).$$

From (223) and (224)

$$j = \text{const.} + C\frac{dP}{dt} + aP + bG,$$

and substituting for $P$ and $G$ from (225) and (226),

$$j = \text{const.} - LC\frac{d^2j}{dt^2} - RC\frac{dj}{dt} - La\frac{dj}{dt} - Raj - Mb\frac{dj}{dt}.$$

Hence the oscillatory part of $j$ (which is all that interests us) is given by

$$LC\frac{d^2j}{dt^2} + (La + Mb + RC)\frac{dj}{dt} + (1 + Ra)j = 0,$$

and is still of the damped sinoidal type. If $M$ is positive, increasing it only increases the damping. If $M$ is negative, increasing it

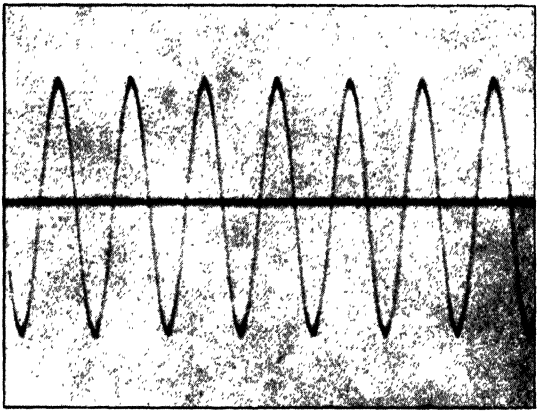
decreases the damping and can easily reduce it to zero, when oscillations will begin of their own accord. The effects can be distinguished experimentally with a grid coil moveable about an axis within the coil $L$, the two coils being placed first with their planes at right angles, so that the mutual inductance vanishes. Oscillations set in when the grid coil is turned through a certain finite angle in one direction, but none however much it is turned in the other.

The mutual inductance required to start an oscillation is of the order of 10 per cent. of $L$. There is no advantage in increasing $M$ beyond this point, and the disadvantage of introducing harmonics. Close coupling may also reduce the frequency by as much as 4 per cent., as can be shown with a wave-meter. The oscillatory E.M.F. increases with the steady E.M.F. applied to the plate, in such a way that the effective value of the one is usually roughly equal to the other.

A milliammeter in the grid circuit of the triode shows a sudden increase of deflexion when oscillations begin (cf. Art. 187), and is convenient as an indicator.

**186. Experiments at low frequency.** The circuit of Fig. 257 will generate alternating currents of acoustic frequency if $C$ is, for example, a mica condenser of 5 microfarads capacity and $L$ a coil of 30 millihenrys having a resistance of one or two ohms. The grid coil can be similar, but may have almost any resistance. Two small triodes in parallel, with 240 volts on the plate, will maintain an oscillatory current of an ampere or two in this circuit. With such an apparatus we can repeat the experiments with the singing arc, and also carry out measurements on the Wheatstone's bridge (Arts. 75, 172). The circuit itself emits a musical note, probably by internal vibration of the condenser. The oscillatory current is very constant both in frequency and amplitude, and the criticism levelled at alternating currents in Art. 134 does not apply here. It is also nearly sinoidal, as Appleton and van der Pol showed from oscillographic records (Fig. 258). This is natural since a triode has only to make good the losses in the condenser circuit, which it can do as a relatively high-resistance shunt, without having to carry the whole current like an arc. The plate current in general deviates considerably from the sine-form.

Frequencies as low as 8 or 10 per second are obtained with 20 microfarads and one or two telephone transformers with low resistance primaries, the secondaries being placed in the grid



Oscillatory current

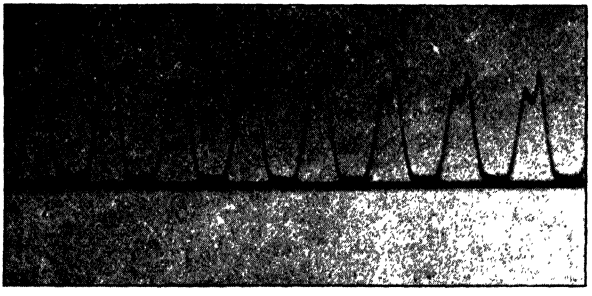


Plate current

Fig. 258

circuit. A milliammeter will show the fluctuations of current quite plainly, and it is interesting to find the effect of inductance and capacity on the frequency by direct observation. The effective range of frequency in triode oscillations is, therefore, very great, for only slight modification is required to generate oscillations as quick as 60 million to the second (Art. 202).

**187. The oscillating triode as detector.** High-frequency oscillations cannot be detected with a telephone direct, for the diaphragm will not respond to so quick a pull, nor are the resulting sound waves audible. What we detect are the *beats* between the

incoming oscillations and oscillations of slightly different frequency in the receiving circuit itself. Suppose that a circuit of frequency 1,000,000 is kept in continuous oscillation with a triode, and a second circuit of frequency 1,001,000 is also kept going, and acted on inductively by the first. Superposition of the two oscillations gives a compound oscillation of slowly varying amplitude, the cycle from least to greatest amplitude and back being performed with a frequency equal to the difference of the frequencies of the two circuits, that is 1000 per second. If the change of amplitude can be made to affect a telephone, we shall have an audible note of frequency 1000. This is the principle of *beat reception* due to Fessenden.

Fig. 259 shows a suitable circuit. $L$ is the coil into which the primary oscillations induce (in wireless telegraphy $L$ is coupled

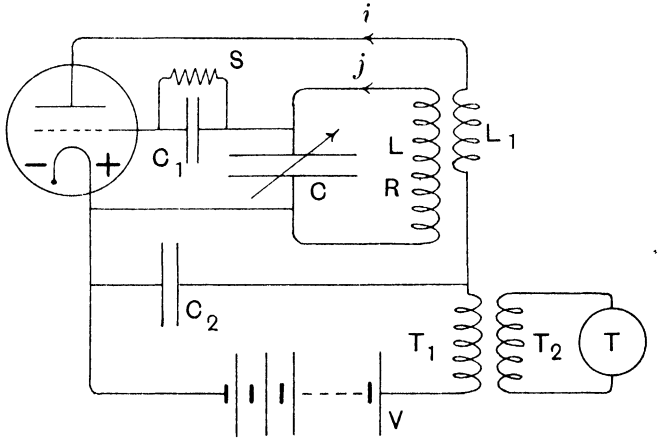



Fig. 259

inductively with the receiving aerial). The grid circuit contains, in addition to the variable condenser $C$, a condenser $C_1$ of about 0·1 milli-microfarad in parallel with a high resistance $S$. $L_1$ is a coil by which the plate currents can react on the grid circuit. The plate potential $V$ is of the order of 100 volts. Since the steady plate current is several milliamperes, it is sent, not through the telephones direct, but through the primary of a small transformer $T_1 T_2$, and the telephones are connected to the secondary. The condenser $C_2$, of the order of 1 milli-microfarad, conducts the high-frequency plate currents back to the filament. The large resistance

$S$ is designed to keep the grid normally at the potential where the grid current first begins to rise quickly (point $P$ in Fig. 260). If the positive end of the filament is, say, 4 volts above the negative end, and $k$ is the steady grid current, the resistance $S$ has to be chosen to make $Sk = 4 - G$. Since $G$ is about half a volt and $k$ a few microamperes, $S$ should be of the order of a megohm.

The circuit $LC$ is another self-oscillating circuit, different from that of Fig. 257.

There is one case in which we can easily see how oscillations are maintained in it. Let the condensers $C_1$ and $C_2$ be large, the mutual inductance between $L$ and $L_1$ small, and let the plate current $i$ be affected very little by variations of the plate potential. Then equation (223) gives $di/dt = b\,dG/dt$. If $j$ is the current in the condenser circuit at any time, and grid currents are neglected, the E.M.F. induced by the plate current in $L$ is

$$-M\frac{di}{dt} = -Mb\frac{dG}{dt} = -\frac{Mbj}{C},$$

since $j = C\,dG/dt$. The E.M.F. will assist the passage of current if $M$ is negative; if $Mb = -RC$ it will overcome the loss of potential $Rj$ which arises from the resistance of $L$, and oscillations will just be maintained.

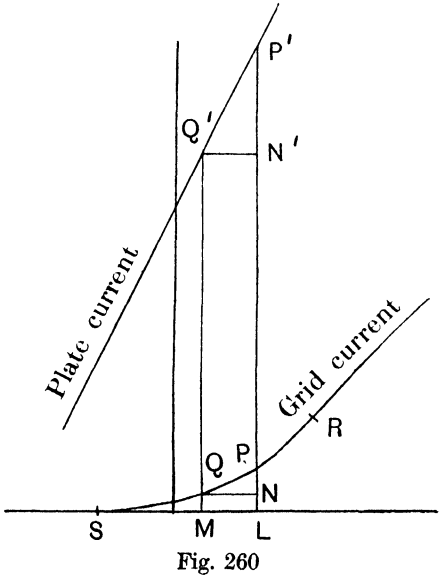


Fig. 260

In operation, the coil $L_1$ is brought near to $L$ until a click is heard in the telephones (indicating the beginning of continuous oscillation), and the capacity $C$ is then varied until an audible note is picked up. Towards explaining the origin of the note, suppose first that the circuit $LC$ is not oscillating. The primary oscillations, inducing into $L$, act on the grid through the condenser $C_1$. The space between the grid and filament conducts considerably while the potential is above the normal

(corresponding to the point $P$ in Fig. 260), but hardly at all below, acting in this respect like a crystal. The mean grid potential therefore falls while the oscillations are inducing into $L$, the actual grid potential oscillating between two points $R$, $S$, about a mean point $Q$ to the left of $P$. The mean plate current, which we take for simplicity to be the same for all plate potentials, is normally $LP'$, and falls to $MQ'$. On the whole, therefore, the plate current decreases by an amount $N'P'$, which is much greater than the decrease $NP$ of the grid current itself, and greater than could be obtained from a crystal. This is the first advantage of the triode.

Supposing for the moment that $T_1$ is a telephone, its diaphragm experiences a resultant pull while the primary oscillations are inducing into $L$. When the receiver oscillates as well, the amplitude varies slowly, as was explained on p. 387, and the pull with it. The diaphragm is therefore set into forced vibration with a frequency equal to the difference of the frequencies of the local and incoming oscillations. Similar considerations apply to the primary of the transformer, which is traversed by a low-frequency current superposed on a direct current. The condenser $C_2$ is chosen of such a size that it will pass high-frequency currents freely, while the low-frequency variations pass mainly through the coil $T_1$ (cf. Art. 132). The secondary currents act directly on the telephone $T$.

To resume, beat reception depends on a combination of three principles. The grid rectifies, the plate amplifies the rectified current, and the oscillation in the receiver makes this current vary slowly so that it can set a telephone diaphragm into vibration. For sensitiveness, the curve of plate current should be as steep as possible near $P'$, so that the plate potential should be neither too high nor too low, say 100 volts. There is also a best amplitude of the local oscillation. Let $RS$ in Fig. 260 represent the excursion of grid potential when the oscillation in the receiver is of amplitude $a$. Increase of amplitude from $a$ to $a + x$ displaces the point $Q$ to the left by an amount which is greatest when the extra current is being rectified best; that is, when the curve of grid current has its greatest slope at $R$ and no slope at $S$. The local oscillation should, therefore, make the grid potential vary between about $\pm$ 10 volts. The displacement of $Q$ being nearly proportional to $x$, it follows that the amplitude of the low-frequency telephone current is nearly

proportional to that of the incoming oscillations and not, like a crystal, to its square. This is the mathematical reason for the great sensitiveness of beat reception.

To hear a note we must have the receiver slightly out of resonance with the transmitter, as has been explained already. When we turn the handle of the condenser $C$ we hear first a note on the high limit of audibility, descending to the low limit of audibility, then an idle interval about the position of resonance, and finally a note rising again to the high limit. The note is controlled by the receiver, and not (as in the buzzer and crystal method) by the transmitter.

The triode is also a sensitive detector of spark or buzzer signals. For this purpose reaction is not pushed so far as to make the receiver oscillate, but only to reduce its damping. Each wave-train sends a definite extra amount of electricity through the plate circuit, charging up the condenser $C_2$, which discharges relatively slowly through the transformer. The action resembles that of a crystal, except that the plate circuit is used to increase the current rectified by the grid. A triode so used is more sensitive than the best crystal detector: further sensitiveness is obtained by amplification (Art. 189).

**188. Measurement of small changes of inductance and capacity.** Beat reception reveals small changes of frequency with great nicety. Suppose for example that the transmitter has a frequency of 1,000,000 and the receiver a frequency of 1,001,000. The audible note of frequency 1000 can itself be made to give beats with a low-frequency oscillation of frequency, say, 1002, whereby a change of frequency of transmitter or receiver by 1 part in $10^6$ is perceived immediately. The method seems to have been first used to detect small changes of capacity by Pungs and Preuner. Since $C$ is proportional to $1/n^2$, $dC/C = -2dn/n$ for the small correlated increases $dC$ and $dn$. Precautions must be taken against disturbance, by using filament and plate batteries of sufficient constancy, enclosing batteries and condensers in earthed metal cases, and waiting till the triodes have reached a steady state.

Whiddington used the method to measure very small displacements. If two parallel plates are set at, say, 0·1 mm. apart, an

increase of separation of $10^{-8}$ cm. makes $dC/C = -10^{-6}$, which is easily observed. Gill measured the dielectric constant of air from the change of capacity of a parallel-plate condenser inside a bell-jar when air was admitted. It was found to be independent of the frequency between $7 \times 10^4$ and $3 \times 10^5$, and equal to 1·00065, the static value being 1·00059 (Art. 50). The susceptibility of a weakly magnetic substance can be found from the change of self-inductance of a solenoid of which it is the core. The core is screened electrostatically from the solenoid by the thinnest of platinum deposits, and the absence of disturbing effect due to eddy currents is proved by experiments on highly conducting but non-magnetic solutions like sulphuric acid. Belz showed in this way that the susceptibility of iron, nickel and cobalt salts is the same at a frequency of $3 \times 10^5$ as for steady forces.

Wrong results may be obtained by this method if the incoming oscillations are strong. It is a common observation that beat notes below a certain frequency (several hundreds to the second) cannot then be heard. Appleton showed that the receiver is not stopped, but pulled into resonance with the incoming oscillations. This phenomenon is quite different from the ordinary reaction of coupled circuits (Art. 182), which tends to separate frequencies originally equal. Appleton showed that the beat frequency is less than $n_1 - n_2$ when the frequencies $n_1$, $n_2$ are nearly equal, but the error diminishes with increase of $n_1 - n_2$, so that moderate beat frequencies and weak coupling are required in the kind of experiment described earlier in this article.

**189. Amplification of low-frequency and high-frequency currents with the triode.** Lieben and Reiss discovered how to magnify small currents with a triode in 1913, before it was known that it could be used to generate oscillations. Their circuit makes use of the principle explained in Art. 184, in a slightly different form. Suppose that we wish to magnify a small current of audible frequency in the coil $T_1$ of Fig. 259. Instead of connecting the secondary coil $T_2$ to a telephone, we join it to the grid of a triode (Fig. 261). The plate circuit of the triode contains the primary $T_3$ of a second transformer, and it is the secondary $T_4$ of this transformer which goes to the telephone. Suppose for simplicity that

the telephone, like the grid circuit, is of high resistance. Writing $G$ for the E.M.F. induced in $T_2$ at any time by the current in $T_1$, $G$ is the grid potential, and the plate current is given by the equation $i = \text{const.} + aP + bG$ of Art. 184. The constant part of $i$ is not being considered here, and we can write simply $i = aP + bG$. If, therefore, either the self-inductance or the resistance of $T_3$ is large, we shall have approximately $aP = -bG$, or the potential at the end of $T_3$ is $b/a$ times that at the end of $T_1$ (cf. Art. 184). If the transformer $T_3T_4$ increases potentials further in the ratio of $n$ to 1 we shall, in theory, have $nb/a$ times the potential on the telephone $T$ that we should have had if it had been connected directly to $T_2$. The coil $T_4$ can itself be joined to the grid of a third triode, and so on.

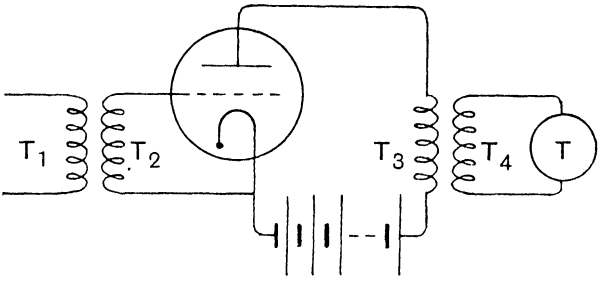


Fig. 261

Practical limits are set to the attainable magnification by the facts (1) that the coils have a certain effective capacity, so that the current through $T_3$, for example, is partly shunted without inducing into $T_4$, (2) we shall magnify not only the weak currents in $T_1$ but also those caused by variations in the emission from the filament of the first triode.

Modified design enables us to amplify the high-frequency current from the aerial, and not, as in Fig. 261, the low-frequency current whose origin has been explained in Art. 187. The coils $T_1$, $T_2$, $T_3$, $T_4$ are made as single-layer coils of the kind used in triode transmission, i.e. designed to have as little effective capacity as possible. Even so, the theoretical magnification ratio is not approached when the wave-length is short, say less than 1000 metres. High-frequency amplification has the advantage that spontaneous variations in the first triode, being of low frequency, are not transmitted to the same extent as in low-frequency amplification, so that a larger number of triodes can be used in series. The same filament and plate batteries can be used for all the triodes, in either kind of amplification.

High-frequency amplification becomes important when beat reception cannot be used, e.g. in wireless telephony at great distances. For the resulting low-frequency, or partially rectified, current (Arts. 180, 187) is proportional to the square of the amplitude of the incident oscillations, and may be too weak to amplify against the spontaneous variations in the first triode. High-frequency amplification may then bring it within the practicable range.

**190. Surface concentration of high-frequency currents.** High-frequency currents are not distributed uniformly over the cross-section of a wire, but tend to flow more and more on the outside as the frequency increases. The theory of this is given in the next article. While a good experimental proof is lacking, the phenomenon can be shown in two ways, one of which is of interest in the history of the science. In 1890 Lord Armstrong, happening to short-circuit a direct-current generator with a stout steel bar, was burnt severely in the hand and let the bar drop, when it was picked up immediately and found to be quite cold. This showed, as Lord Kelvin remarked, that the heat generated in a thick bar by a transient current, and therefore also by a high-frequency current, is superficial. The preference of the current for the outer layers can also be shown with a long hollow metal cylinder about 5 cm. in diameter, having metal end-plates perforated to hold a metal rod along the axis of the cylinder. Both rod and cylinder are divided near the centre by narrow gaps, and the gap in the rod can be viewed through a small hole in the cylinder. The gaps are adjusted with low-frequency current (from an induction coil) till sparks just pass across the inner gap in preference to the outer. High-frequency current from an oscillating spark circuit will now pass by preference across the *outer* gap.

The effect of this or any other inequality in the distribution of current is to increase the effective resistance of the wire. Imagine the cross-section to be cut into a large number $n$ of equal areas, carrying currents

$$i_1 \cos pt + j_1 \sin pt, \quad i_2 \cos pt + j_2 \sin pt, \ \ldots, \quad i_n \cos pt + j_n \sin pt.$$

The effective value of the current is given by

$$I^2 = (i_1 + i_2 + \ldots + i_n)^2 + (j_1 + j_2 + \ldots + j_n)^2,$$

and the heat developed per second is proportional to the quantity

$$H = i_1^2 + i_2^2 + \ldots + i_n^2 + j_1^2 + j_2^2 + \ldots + j_n^2.$$

If $H$ is to be a minimum subject to $I$ having a given value, then by known principles $I^2 + \lambda H$ must be an unrestricted minimum in the $2n$ variables $i_1, i_2, \ldots j_n$. This leads at once to the equations $i_1 = i_2 = \ldots = i_n$, $j_1 = j_2 = \ldots = j_n$. Thus the heating effect of a current of given effective value is least when it is distributed uniformly over the cross-section, in the same phase.

The heating effect of high-frequency currents has been measured by several observers. The apparatus used by Fleming for this purpose is shown in Fig. 262. Two wires $W_1$, $W_2$, as nearly as possible identical, are passed down two tubes $T_1$, $T_2$ connected by a cross-tube containing a thread of oil, so that the whole forms a differential air-thermometer. The left-hand wire is traversed by direct current and the right-hand wire by high-frequency currents from a condenser discharge, the effective values of the currents being measured by thermal ammeters $A_1$, $A_2$ with thin filaments. The direct current is adjusted by means of a rheostat until the oil thread remains in its ordinary position when both currents are flowing. Let $R_1$ be the normal resistance of the wire $W_1$ for steady currents, $kR_1$ its resistance for the high-frequency currents. Then the heat developed in the left-hand tube is represented by $R_1 I_1^2$, and in the right-hand tube by $kR_2 I_2^2$. Thus

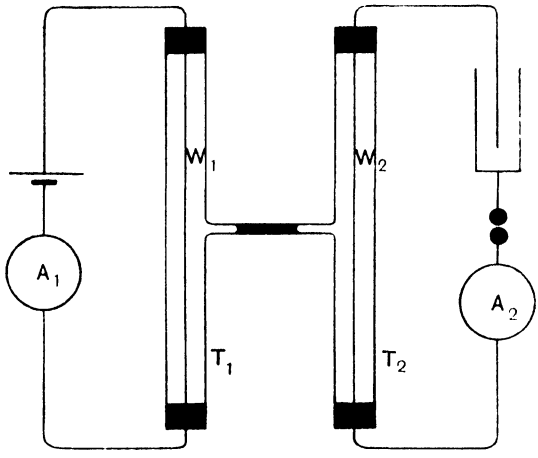


Fig. 262

$$R_1 I_1^2 = \lambda k R_2 I_2^2,$$

where $\lambda$ is a numerical constant which is equal to unity in a truly symmetrical apparatus, but in practice may have a slightly different value on account of differences in the sizes of the tubes and the

thermal emissivities of the wires. $\lambda$ may be eliminated by performing a second experiment in which high-frequency current flows through the left-hand wire and direct current through the right-hand wire. Thus

$$kR_1I_3^2 = \lambda R_2 I_4^2,$$

giving

$$k = \frac{I_1 I_4}{I_2 I_3}.$$

The fractional increase of resistance can thus be found experimentally.

The change in the distribution of current in a circuit also affects its self-inductance; but whereas the resistance increases indefinitely as the current flows more and more on the surface, the inductance tends to a new limiting value, called the *high-frequency inductance.* The theory of Art. 123 must be modified in this case, since the internal filaments of the wire no longer contribute to the magnetic flux. Instead of the result $L = M + \frac{1}{2}l$ we now find the simple result $L = M$. Thus *the high-frequency inductance of a circuit is equal to the mutual inductance of the line of centres of the cross-sections and a parallel curve drawn in the surface of the wire.* As an example of this rule we have the following formulae for high-frequency inductance:

(1) *Long solenoid of $n$ turns, length $l$, area $A$.*

$$L = 4\pi n^2 A/l,$$

the distribution of current being immaterial.

(2) *Rectangular circuit $a$ by $b$, diameter of wire $2r$.*

$$L = 9{\cdot}210\left\{c\log_{10}\frac{2ab}{r} - a\log_{10}(a+d) - b\log_{10}(b+d)\right\} - 8c + 8d,$$

where $c$ is the half perimeter, $d$ the diagonal of the rectangle.

(3) *Circular coil of one turn, radius $a$.*

$$L = a\left(28{\cdot}93\log_{10}\frac{a}{r} + 1{\cdot}00\right).$$

**191. Theory of the increase of resistance with frequency.** The following theory of the distribution of high-frequency currents over the cross-section of round wires was given by Lord Kelvin. Let $H$ be the magnetic force and $u$ the current-density at time $t$

and at a point in the wire distant $r$ from the centre. Since the line-integral of magnetic force round a circle of radius $r$, with its centre on the axis and its plane at right angles to it (Fig. 263), is $4\pi$ times the enclosed current,

$$2\pi rH = 4\pi \int_0^r 2\pi r u\, dr.$$

Differentiating with respect to $r$, we have

$$\frac{\partial H}{\partial r} + \frac{H}{r} = 4\pi u \quad \ldots\ldots\ldots\ldots\ldots (227).$$

Apply the law of electromagnetic induction to the rectangle bounded by the axis, a parallel line distant $r$ from it, and two

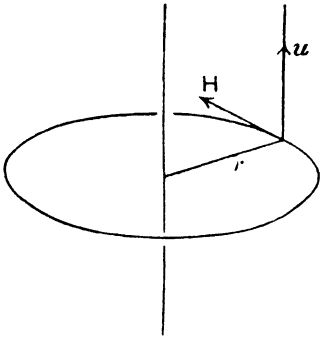



Fig. 263

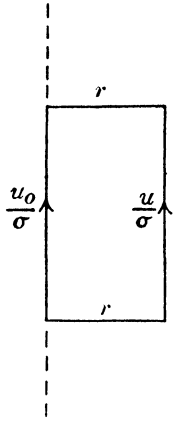



Fig. 264

radii vectores at unit distance apart (Fig. 264). Let $u_0$ be the current-density at a point on the axis. There are electric forces $u/\sigma$ and $u_0/\sigma$ acting along two sides of the rectangle, while the flux of magnetic force through it is $-\int_0^r H\, dr$. Hence

$$\frac{u - u_0}{\sigma} = \int_0^r \frac{\partial H}{\partial t}\, dr, \text{ or } \frac{\partial u}{\partial r} = \sigma \frac{\partial H}{\partial t} \quad \ldots\ldots\ldots (228).$$

From (227) and (228)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 4\pi\sigma \frac{\partial u}{\partial t}.$$

The solution corresponding to an alternating current of period $2\pi/p$ is got by writing $u = Re^{ipt}$, where $R$ is a function of $r$ only,

and it is understood that the real part is to be taken when the quantities come to be interpreted. Thus

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} = 4\pi i\sigma pR,$$

so that $R = AJ_0(kr)$ where $A$ is a constant and $k^2 = -4\pi i\sigma p$ (Art. 12). Lord Kelvin defined two new functions ber $x$ and bei $x$ by the equation

$$J_0(x\sqrt{-i}) = \text{ber } x + i \text{ bei } x,$$

so that $$\text{ber } x = 1 - \frac{x^4}{2^2 . 4^2} + \frac{x^8}{2^2 . 4^2 . 6^2 . 8^2} - \cdots$$

and $$\text{bei } x = \frac{x^2}{2^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \frac{x^{10}}{2^2 . 4^2 . 6^2 . 8^2 . 10^2} - \cdots .$$

Thus $u = A(\text{ber } x + i \text{ bei } x)\, e^{ipt}$, where $x = 2(\pi\sigma p)^{\frac{1}{2}} r$, or in real form $u = A(\text{ber } x \cos pt - \text{bei } x \sin pt)$. The currents vary across the cross-section in phase as well as amplitude. If $U$ is the effective value of the current-density at distance $r$ from the axis,

$$U^2 = \tfrac{1}{2}A^2(\text{ber}^2 x + \text{bei}^2 x) \quad \ldots\ldots\ldots (229).$$

The functions ber and bei have been tabulated and thus $U$ can be found numerically. Fig. 265 shows the distribution of $U$ over

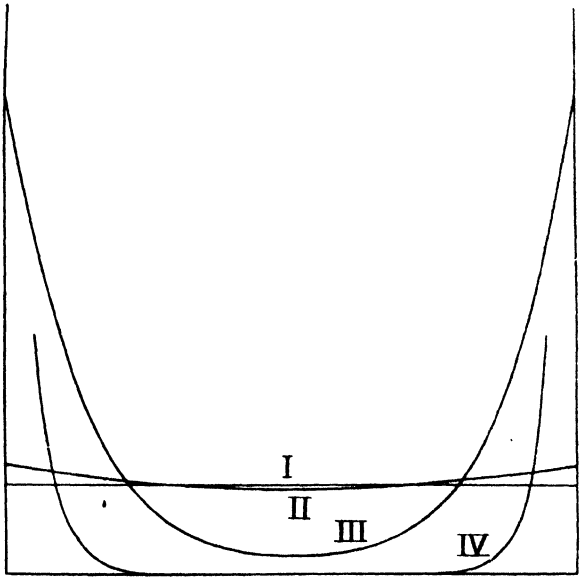



Fig. 265

the cross-section of a copper wire 2 mm. in diameter, curve I being for a frequency of $10^3$, curve II $10^4$, curve III $10^5$ and curve IV $10^6$ per second. The total current in the wire at time $t$ is given by $i = \int_0^a 2\pi r u\, dr$, where $a$ is the radius of the wire. Thus

$$i = 2\pi A e^{ipt} \int_0^a r J_0(kr)\, dr = -\frac{2\pi a A}{k^2} e^{ipt} \left[\frac{dJ_0(kr)}{dr}\right]_{r=a}$$

from equation (25) of Art. 12. Hence

$$i = \frac{A e^{ipt}}{2i\sigma p} \xi (\mathrm{ber}'\, \xi + i\, \mathrm{bei}'\, \xi),$$

where $\xi = 2(\pi\sigma p)^{\frac{1}{2}} a$ is the value of $x$ when $r = a$. If $I$ is the effective value of the current,

$$I^2 = \frac{A^2 \xi^2}{8\sigma^2 p^2} (\mathrm{ber}'^2\, \xi + \mathrm{bei}'^2\, \xi) \quad \ldots\ldots\ldots (230),$$

giving $A$ in terms of $I$.

The instantaneous rate of development of heat is $u^2/\sigma$ per cm.$^3$ per sec., and the mean rate $U^2/\sigma$. The element of volume is

$$2\pi l r\, dr = l x\, dx / 2\sigma p,$$

where $l$ is the length of the wire. The heat developed per second in the whole wire is therefore

$$\frac{l A^2}{4\sigma^2 p} \int_0^\xi x (\mathrm{ber}^2\, x + \mathrm{bei}^2\, x)\, dx.$$

Since $$\mathrm{ber}^2\, x + \mathrm{bei}^2\, x = J_0(x\sqrt{i})\, J_0(x\sqrt{-i}),$$

the integral can be found from equation (26) of Art. 12, and is equal to $\xi(\mathrm{ber}\, \xi\, \mathrm{bei}'\, \xi - \mathrm{ber}'\, \xi\, \mathrm{bei}\, \xi)$. Thus the heat developed per second is

$$\frac{l A^2 \xi}{4\sigma^2 p} (\mathrm{ber}\, \xi\, \mathrm{bei}'\, \xi - \mathrm{ber}'\, \xi\, \mathrm{bei}\, \xi) \quad \ldots\ldots\ldots (231).$$

The *apparent resistance* $R$ of the wire is defined so that the rate of heat-development is $RI^2$. Comparison with (230) gives

$$R = 2lp \frac{\mathrm{ber}\, \xi\, \mathrm{bei}'\, \xi - \mathrm{ber}'\, \xi\, \mathrm{bei}\, \xi}{\xi (\mathrm{ber}'^2\, \xi + \mathrm{bei}'^2\, \xi)}.$$

The resistance for low frequencies is $R_0 = l/\pi\sigma a^2$. Hence

$$\left.\begin{aligned} \frac{R}{R_0} &= \frac{\xi (\mathrm{ber}\, \xi\, \mathrm{bei}'\, \xi - \mathrm{ber}'\, \xi\, \mathrm{bei}\, \xi)}{2 (\mathrm{ber}'^2\, \xi + \mathrm{bei}'^2\, \xi)} \\ \text{where} \quad \xi^2 &= 4\pi\sigma p a^2 \end{aligned}\right\} \quad \ldots\ldots\ldots (232).$$

The quantity $R/R_0$ is that measured by calorimetric experiments (Art. 190). Some idea of its magnitude will be obtained from the following table, for copper wires.

| Frequency | $R/R_0$ | |
|---|---|---|
| | Diameter 2 mm. | Diameter 0·2 mm. |
| 10 | 1·000 | 1·000 |
| $10^2$ | 1·000 | 1·000 |
| $10^3$ | 1·000 | 1·000 |
| $10^4$ | 1·095 | 1·000 |
| $10^5$ | 2·626 | 1·000 |
| $10^6$ | 7·70 | 1·095 |
| $10^7$ | 23·8 | 2·626 |
| $10^8$ | 74·7 | 7·70 |

Equation (232) seems to be in good agreement with experiment. Thus for a certain copper wire Fleming found $R/R_0 = 6{\cdot}62$, while theory gave 6·64. The smaller the wire the less the fractional increase of resistance for a given frequency. Moreover, the increase is less for materials of low conductivity than for materials of high conductivity like copper. It is easily shown that $R/R_0$ is a function of $n/\rho$ only, where $n$ is the frequency and $\rho$ the resistance per unit length. The theory of this article is used in the design of thermal galvanometers for high-frequency measurements, where it is essential (for direct-current calibration) that $R/R_0$ should be nearly unity.

**192. Hertz's experiments.** In all our previous work we have postulated a direct influence of one object $A$ on another $B$ across the intervening space (action at a distance). No objection can be taken to this point of view as long as the phenomena are correctly explained. It would however be more satisfying if the action was traced all the way from $A$ to $B$ by postulating suitable properties in the separating medium. In 1888 Hertz published his famous experiments confirming the theory, propounded by Maxwell in 1863, that electric action is propagated in the form of waves having all the properties of light-waves.

Hertz's earlier experiments were performed with the apparatus

shown in Fig. 266. Two square brass plates $A$, $A'$, each of side 40 cm., were joined by a copper wire 60 cm. long. The wire was cut in the middle and a spark-gap inserted, the terminals of which were connected to a large induction coil. Although the apparatus

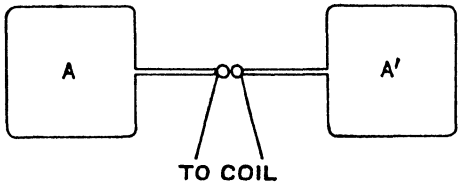



Fig. 266

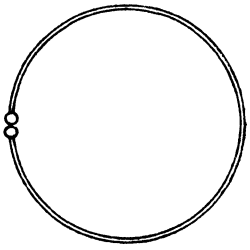


Fig. 267

does not form anything like a closed circuit, yet the plates $A$, $A'$ have capacity and the connecting wire something like inductance, so that the discharge is oscillatory and of very high frequency. For a receiver (Fig. 267) Hertz used a wire in the form of a circle of radius 35 cm., likewise interrupted by a spark-gap. When the oscillations fall on the receiver sparks are seen to pass across the gap, whose length can be adjusted by means of a micrometer screw.

The first observations were on reflexion, or rather on interference between the direct beam of rays and the beam reflected from a conducting surface. Working in a large lecture room, Hertz set up his oscillator in a horizontal position at a distance of 13 metres from a massive wall to which a sheet of zinc 4 metres high and 2 metres wide had been attached. Periodic variations were found in the intensity of sparking when the receiver was placed horizontally at increasing distances in front of the zinc sheet. This was attributed to the formation of standing waves, the wave-length of which was estimated at 9·6 metres.

Hertz worked afterwards with much shorter waves (65 cm.), using the apparatus shown in Fig. 268. The oscillator was now a brass cylinder 3 cm. in diameter and 26 cm. long, with spark-balls in the centre. It was placed vertically in the focal line of a parabolic mirror of height 2 metres and focal length 12·5 cm., the wires from the spark-gap to the induction coil being led through the mirror. The receiver was formed of two straight pieces of wire, each 50 cm. long and 5 mm. in diameter, placed with their ends 5 cm. apart in

the focal line of a second parabolic mirror. From the nearer ends two wires were led through the mirror to a small adjustable spark-gap at the back, in which the passage of sparks was observed.

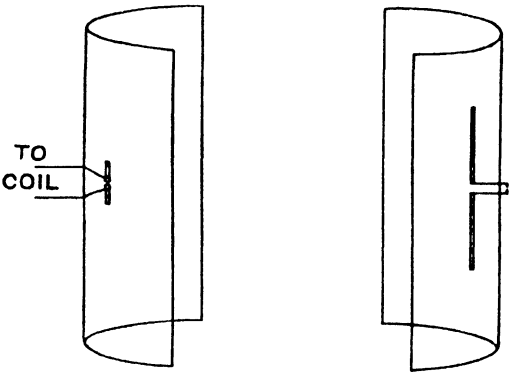



Fig. 268

In judging these experiments it should be borne in mind that the linear receiver is affected most powerfully when it is parallel to the lines of electric force, while the previous circular receiver responds best when its plane is perpendicular to the lines of magnetic force. The greatest distance at which the short waves could be detected was about 20 metres, but experiments were usually made at a distance of 6 to 10 metres.

(1) *Rectilinear propagation.* A sheet of zinc 2 metres high and 1 metre wide stopped all sparking when placed in the path between the mirrors, but did not when placed on one side. The body of the observer also had the same effect. Insulators such as wood did not stop the rays.

(2) *Reflexion.* When the mirrors were placed at such an angle that no direct action took place, vigorous sparks could be obtained by placing a sheet of zinc so that the normal to its plane bisected the angle between the axes of the mirrors. The effect was not very sharply marked, sparks being obtainable when the mirror was turned through 15° in either direction. But since the dimensions of the apparatus were of the same order of magnitude as the wave-length diffraction effects must come into play.

(3) *Refraction.* Hertz used a large prism of pitch. The height was 1·5 metres, the cross-section an isosceles triangle of base 60 centimetres and vertical angle 30°. Oscillator, prism and receiver were arranged like the collimator, prism and telescope of a spectrometer, the distance from oscillator to prism and from prism to receiver being 2·6 metres. With an angle of incidence of 25°,

sparks began to appear in the receiver at the deviation 11°, became strongest at 22° and vanished again at 34°. The angle of minimum deviation was estimated at about 22°, corresponding to the refractive index 1·69.

(4) *Polarisation.* A large grating was made of copper wires fixed parallel to one another at a distance of 3 centimetres apart. If the parabolic mirrors were vertical and the wires horizontal, practically no effect was produced by interposing the grating between the mirrors: but if the wires were vertical all traces of sparking ceased. This effect is analogous to that of a plate of tourmaline on plane-polarised light, the plate absorbing rays polarised in one direction much more than those polarised at right angles. It is however instructive to consider it purely electrically. It will appear shortly that the electric force in the wave as it falls on the grating is parallel to the direction of the primary oscillator. Such a wave induces no currents in the wires when they are perpendicular to the electric force: but currents and free charges appear if they are parallel to it, and their electric and magnetic effects are superposed on those of the original wave. The effects of the separate wires reinforce each other in certain directions (reflexion) and destroy the original wave in others (absorption). Hertz found, in fact, that the grating would *reflect* perfectly when the wires were vertical, but not when they were horizontal.

Hertz found that no sparks are produced if the oscillator and receiver are at right angles to one another, but they reappear when the grating is interposed in a direction which is not parallel to one of them. Let the lines of the grating make an angle $\theta$ with the oscillator. The charges induced in the wires are proportional to the component $E\cos\theta$ of electric force parallel to them, and their electric field makes an angle $\frac{1}{2}\pi - \theta$ with the receiver. Hence the electrical effect in the receiver is roughly proportional to $E\cos\theta\sin\theta$, which has a maximum value when $\theta = \frac{1}{4}\pi$.

Franklin used parabolic reflectors for signalling between London and Birmingham, with continuous waves 15 metres long. The signals, which could just be received without reflectors, were 200 times as strong when both transmitter and receiver were fitted with them. The received beam was about two miles wide.

**193. Experiments of Bjerknes, and of Sarasin and de la Rive.** We must now interrupt the main argument in order to describe two investigations which throw light on special points. Bjerknes examined the resonance and damping under different conditions. A set of circular receivers was constructed of various metals, but of geometrically identical form, and resonance curves taken off by altering the dimensions of the oscillator. (The quantity observed was potential, the spark-gap in the receiver being replaced by an electrometer of special design.) The sharpness of resonance was greatest with the copper receiver, and then followed brass, German silver, platinum, nickel, and iron in this order. The order of the non-magnetic metals is that of their conductivities, but iron and nickel fall out of the series and give proportionally greater damping, which shows that magnetic properties come into play even for the frequencies of Hertzian oscillations.

Choosing an oscillator of the same natural frequency as his receivers ($7{\cdot}1 \times 10^7$ per second), Bjerknes next tried the effect of depositing thin layers of various metals on the surface of the receivers. His results are shown in Fig. 269, in which the abscissa

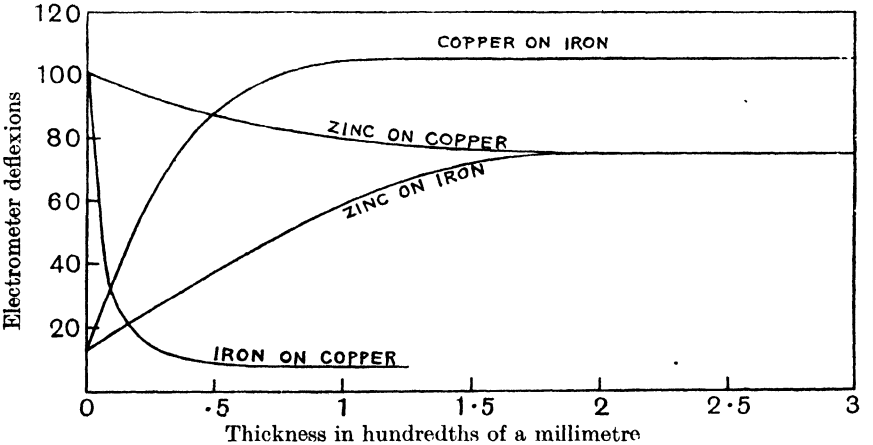



Fig. 269

represents the thickness of the added layer. Thus an iron wire covered with a layer of copper one-hundredth of a millimetre thick behaves like a solid copper wire*; from a certain thickness onwards

* The slight difference is probably due to the high conductivity of electrolytically deposited copper; while Bjerknes' iron deposit seems to have been of low conductivity.

the nature of the underlying metal is immaterial, the same effect being obtained with zinc on iron and with zinc on copper. Nothing could show more clearly that high-frequency currents flow only on the outsides of wires.

Bjerknes found that the decrement of his oscillator was 0·26, much higher than that of ordinary oscillation circuits. Some effect of this kind is to be expected if, as we believe, the Hertzian oscillator is continually emitting energy in the form of waves, so that its oscillations are more quickly damped out. This process is known as *damping by radiation*.

The circular receiver, on the other hand, approximates more to the ordinary re-entrant oscillation circuit. It hardly radiates, and its decrement is considerably smaller than that of the oscillator. This is brought out most clearly by the experiments of Sarasin and de la Rive. These physicists, in repeating Hertz's experiments on the interference between the direct rays and those reflected from a metal sheet, found that *the wave-length obtained depended only on the dimensions of the receiver, and not at all on the oscillator.* When a heavily-damped wave falls on a receiver it sets it into electrical vibration which continues long after the wave has passed over; and on the return of the wave similar free vibrations are set up. The two sets of vibrations will interfere destructively if the time $\tau$ required for the wave to move from the receiver to the mirror and back is an odd multiple of $\frac{1}{2}T$, where $T$ is the natural period of vibration of the receiver. Thus the distance $l$ of the first node from the mirror is given by $l = \frac{1}{4}cT$, $c$ being the velocity of propagation of the waves. The equation contains no quantity characteristic of the oscillator, which explains the observed facts.

Sarasin and de la Rive also examined the formation of stationary electric waves on wires, and found the same wave-length in air as on the wire, as shown in the following table:

| Diameter of receiver (cm.) .. .. | 100 | 75 | 50 | 35 (thick wire) | 35 (thin wire) | 25 | 20 (thick wire) | 20 (thin wire) | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $\lambda$ in air (cm.) .. | 812 | 564 | 444 | 304 | 320 | 240 | 172 | 204 | 76 |
| $\lambda$ on wire (cm.) .. | 768 | 592 | 392 | 292 | — | 224 | — | 180 | — |

It thus appears that the wave-length to which a circular receiver responds is nearly eight times its diameter. Experiments on waves along wires had previously been made by Hertz, but the results were less decisive.

**194. Laboratory experiments on electric waves.** The great size of Hertz's apparatus, and the consequent disturbing effect of the floor and walls of the room, render it unsuitable for regular laboratory use. Of the various arrangements that have been suggested for repeating Hertz's experiments on a smaller scale, that of Lebedew is the most reliable. The following simple apparatus, suitable for qualitative experiments, is on this plan. The oscillator (Fig. 270) is about 7 cm. long, the spark-balls being formed by fusing the end of a platinum wire in an electric arc. The oscillator is partly immersed in turpentine, in which the main sparks pass, and the length of the spark-gap is adjusted for maximum

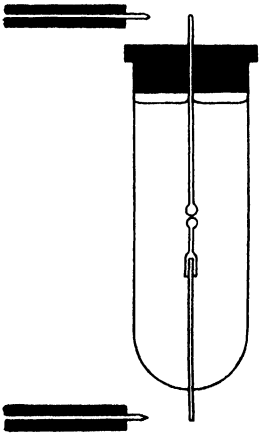


Fig. 270

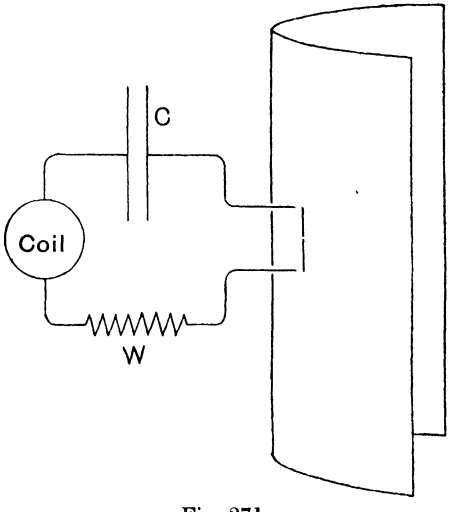



Fig. 271

effect by means of a screw, the best length being only a fraction of a millimetre. In order to avoid as far as possible the disturbing effects of the coil, a water-resistance $W$ and a Leyden jar $C$ are placed in the secondary circuit (Fig. 271), and the coil is separated from the oscillator by two subsidiary gaps about 1 cm. in length. The parabolic mirror, 40 cm. high and of focal length 8 cm., is

of tin plate, perforated with holes to take the leads to the subsidiary spark-gaps, sheathed in ebonite tubes.

The receiver, which is placed in the focal line of a second parabolic mirror, is a copper wire about 15 cm. long interrupted at the centre by a small thermal detector of the type shown in Fig. 138. It is mounted on a stand moveable like the arm of a spectrometer, and a prism table is also placed over the central pivot. A very sensitive galvanometer is required. The apparatus will be found very certain in action when first set up; but after a time the spark-balls become roughened and require to be remoulded in the electric arc*. The wave-length emitted is of the order of 15 cm., and the size of the subsidiary apparatus can be much reduced. Thus refraction can be demonstrated with a prism only 20 cm. high.

In this way ebonite, slate and other insulators can be shown to be more or less transparent, thin glass (even when coloured) almost completely so. Complete absorption is obtained with thin metal sheets, or with a grating of thin copper wires (No. 24 S.W.G.) stretched 1 cm. apart on a wooden frame 40 cm. square, provided that the wires are parallel to the lines of electric force. The same grating when turned through a right angle scarcely absorbs the rays at all. The grating reflects perfectly when the wires are parallel to the lines of electric force. An interesting result is obtained with a large block of wood whose edges are parallel and perpendicular to the grain: the absorption is greater when the grain is parallel to the electric force than when perpendicular. The block thus acts like a grating, on account of the greater electric conductivity along the grain.

Nichols and Tear used very small oscillators of Lebedew's type, 0·4 mm. long, giving wave-lengths down to 1·8 mm. The wave-lengths were measured by the interference of waves reflected from two parallel metallic sheets in different planes—a method first tried by Boltzmann. With a somewhat longer wave-length, Lebedew was able to demonstrate the double refraction of rhombic sulphur for electric waves. The greatest and least refractive indices were found to be 2·25 and 2·00, while Boltzmann found the square roots of the corresponding dielectric constants to be 2·18 and 1·95 (cf. Art. 198).

* The reason for this is that the electric discharge from a fine point, or from a roughened surface, takes place at a low potential, and with the small capacities involved it is impossible to get the potential of the oscillator high enough to produce the normal discharge. This is the main source of difficulty in work at very high frequencies.

**195. Equations of the free ether.** The experiments with the Hertzian oscillator cannot be explained on the ordinary lines by action at a distance due to the charges or currents present in the field: the far-reaching analogy with light-waves points rather to wave-motion in an imponderable medium (the *ether*), taking place with a finite velocity probably identical with that of light.

Any new theory propounded to cover these effects should keep as close as possible to the existing theories of electrostatics and electromagnetism, because those theories, within their limits, are confirmed most exactly by experiment. We notice that Hertzian and allied effects are best shown with rapid oscillations in rods and other bodies which differ considerably from the re-entrant electric circuit used in ordinary experiments. The fundamental electromagnetic laws however refer to closed circuits, and are otherwise devoid of meaning. What is wanted, therefore, is an adequate theory of *open* circuits. The line of least resistance would seem to be *to make all circuits closed, by counting as a current something which is not a flow of electricity*, and applying the original laws unaltered. This is the basis of the theory propounded by Maxwell in 1864, and his views have proved so successful in this particular domain that there is no need to consider alternative explanations. In order to see what is required in the simplest case, suppose that a current $i$ is flowing into a parallel plate condenser, $A$ being the area of each plate and $d$ their separation. If $V$ is the difference of potential at time $t$,

$$i = C\frac{dV}{dt} = \frac{A}{4\pi d}\frac{dV}{dt} = \frac{A}{4\pi}\frac{dE}{dt},$$

where $E$ is the electric force between the plates in electromagnetic units. The current can be completed by supposing that the changing electric force $E$ constitutes a current of current-density $\frac{1}{4\pi}\frac{dE}{dt}$ *between the plates*, or $\frac{1}{4\pi c}\frac{dE}{dt}$ if $E$ is in electrostatic units. We can reach the same result more generally as follows.

Imagine space to be filled with moving electricity, $\rho$ being the volume-density in electrostatic units and $\mathbf{v}$ the velocity at any point at time $t$. According to Art. 77 the current-density is $\mathbf{u} = \rho\mathbf{v}/c$. Application of electromagnetic laws fails since $\mathbf{u}$ is not solenoidal in general; for $\operatorname{div}\rho\mathbf{v} = -\partial\rho/\partial t$ (Art. 8), and vanishes only for

steady currents. To resolve the difficulty let us assume that Gauss' theorem on the normal component of electric force is always true, and not only in electrostatics. Hence from Art. 37

$$\operatorname{div} \mathbf{E} = 4\pi\rho \quad \ldots\ldots\ldots\ldots\ldots\ldots (233).$$

The equation of continuity now takes the form

$$\operatorname{div} \rho\mathbf{v} = -\frac{1}{4\pi} \operatorname{div} \frac{\partial \mathbf{E}}{\partial t}, \text{ or } \operatorname{div}\left(\rho\mathbf{v} + \frac{1}{4\pi}\frac{\partial \mathbf{E}}{\partial t}\right) = 0.$$

If, therefore, we write

$$\mathbf{u} = \frac{\rho\mathbf{v}}{c} + \frac{1}{4\pi c}\frac{\partial \mathbf{E}}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots (234),$$

$\mathbf{u}$ is always solenoidal, and we have the required generalisation. The vector $\frac{1}{4\pi c}\frac{\partial \mathbf{E}}{\partial t}$ is called the *ether-current-density*.

The fictitious electricity has served its turn and may now be dismissed. The general electromagnetic equation $4\pi\mathbf{u} = \operatorname{curl} \mathbf{H}$ gives, for the free ether

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \operatorname{curl} \mathbf{H}.$$

This is the first of Maxwell's equations. To obtain another we invoke the law of electromagnetic induction as a hypothesis in the following form:

The line-integral of electric force (in electromagnetic units) round any circuit $C$ is equal to the rate of decrease of the flux of magnetic force through $C$.

Since the electric force in electromagnetic units is $c\mathbf{E}$ this gives

$$c\int_C \mathbf{E}d\mathbf{s} = -\int_S \frac{\partial \mathbf{H}}{\partial t}\, d\mathbf{S}.$$

The left-hand side is $c\int \operatorname{curl} \mathbf{E}\, d\mathbf{S}$ by Stokes' theorem, and $S$ is arbitrary. Hence

$$-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = \operatorname{curl} \mathbf{E}.$$

It follows that $\frac{\partial}{\partial t}(\operatorname{div} \mathbf{H}) = 0$. It is usual to make this more precise by postulating that $\operatorname{div} \mathbf{H} = 0$, and equation (233), with $\rho = 0$, gives $\operatorname{div} \mathbf{E} = 0$. The equations of the free ether are therefore

$$\left.\begin{aligned} \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= \operatorname{curl} \mathbf{H}, & -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} &= \operatorname{curl} \mathbf{E} \\ \operatorname{div} \mathbf{E} &= 0, & \operatorname{div} \mathbf{H} &= 0 \end{aligned}\right\} \quad \ldots\ldots (235).$$

The new theory throws some light on the problem left unsolved in Art. 139, namely, how induction takes place in a fixed circuit when another circuit is moved. In Maxwell's view, a changing magnetic force is accompanied by an electric force, by the inherent constitution of the intervening medium. The contrast of the mechanisms is one of the least pleasing features of the ether theory, and a consequence of its non-relativistic origin.

**196. The electromagnetic theory of light.** The first and second of equations (235) give

$$\frac{1}{c}\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\operatorname{curl}\frac{\partial \mathbf{H}}{\partial t} = c \operatorname{curl}\operatorname{curl}\mathbf{E},$$

and the vector rule (20) gives

$$\operatorname{curl}\operatorname{curl}\mathbf{E} = -\Delta\mathbf{E}$$

since $\mathbf{E}$ is solenoidal. Hence

$$\Delta\mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Since $\mathbf{E} = \mathbf{i}E_x + \mathbf{j}E_y + \mathbf{k}E_z$, all the components of electric force satisfy the equation

$$\Delta\theta = \frac{1}{c^2}\frac{\partial^2 \theta}{\partial t^2},$$

which is the equation of wave-motion (Art. 11). The components of magnetic force satisfy an equation of the same form. It follows that electromagnetic actions are propagated through the ether with a velocity $c$ equal to the ratio of the electromagnetic to the electrostatic unit of charge.

The velocity of light *in vacuo*, according to the best measurements, is as follows:

| | | |
|---|---|---|
| Michelson (1885) | .. .. | $2{\cdot}9986 \times 10^{10}$, |
| Newcomb (1885) | .. .. | $2{\cdot}9986 \times 10^{10}$, |
| Perrotin (1900) | .. .. | $2{\cdot}9990 \times 10^{10}$. |

These numbers agree remarkably with the values of $c$ given in Art. 142. The natural consequence is to regard the electric and luminiferous media as the same, and to explain light as a form of electric wave (of very short wave-length). As Hertz remarked in connexion with one of his experiments, it is "a fascinating idea that the processes in air which we have been investigating represent

to us on a millionfold larger scale the same processes which go on in the neighbourhood of a Fresnel mirror or between the glass plates used for exhibiting Newton's rings."

The experimental evidence as to the velocity of propagation of electric waves in air is not quite direct. If we accept Sarasin and de la Rive's conclusion that this velocity is the same as that of waves on wires, we may utilise the measurements of Blondlot, and of Trowbridge and Duane (Art. 200), which show that the velocity is very nearly that of light. A direct method would consist in measuring the wave-length by an interference method with a resonator of known natural frequency: but the experiments made by MacLean in this direction are not conclusive.

We now proceed to lend definiteness to the idea of an electric wave by a simple example. Consider a plane wave travelling along the axis of $z$, and write

$$\mathbf{E} = (\mathbf{i}P + \mathbf{j}Q + \mathbf{k}R) \cos p\left(t - \frac{z}{c}\right),$$

$$\mathbf{H} = (\mathbf{i}S + \mathbf{j}T + \mathbf{k}U) \cos p\left(t - \frac{z}{c}\right).$$

Substituting in the equations of the ether we find

$$S = -Q, \quad T = P, \quad R = U = 0.$$

We may take $Q = 0$ without loss of generality, and thus we have the particular solutions

$$\mathbf{E} = \mathbf{i}P \cos p\left(t - \frac{z}{c}\right),$$

$$\mathbf{H} = \mathbf{j}P \cos p\left(t - \frac{z}{c}\right).$$

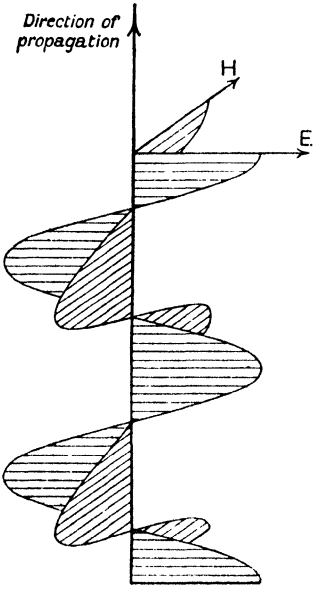


Fig. 272

There is no longitudinal component of either **E** or **H**, so that *the waves are transverse*. Moreover, **E** and **H** are in the same phase, but at right angles to one another. Fig. 272, which represents the electric and magnetic forces at a given time, at various points of the axis of $z$, shows the relation of the three vectors to one another.

**197. Theory of the ideal Hertzian oscillator.** Hertz also gave a theory of the waves in the ether in his experiments, in which the oscillator is reduced to its simplest theoretical form. The plate-oscillator and the rod-oscillator have this in common, that electricity surges backwards and forwards across the gap so as to have an electric moment which is sometimes positive, sometimes negative. Let us consider the field of a small electric doublet at the origin, with its axis along the axis of $z$ and its moment a prescribed function $f(t)$ of the time. Towards solving the equations of the ether in the space surrounding the doublet let us put

$$\mathbf{H} = \operatorname{curl} \mathbf{A} \quad \ldots\ldots\ldots\ldots\ldots\ldots (236)$$

as in Art. 95. Maxwell's second equation gives

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c} \operatorname{curl} \frac{\partial \mathbf{A}}{\partial t}.$$

Thus $\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$ is the gradient of some scalar function $-V$ which, like $\mathbf{A}$, also depends on $t$. Hence

$$\mathbf{E} = -\mathbf{D}V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots (237).$$

Maxwell's first equation now gives

$$-\frac{1}{c}\mathbf{D}\frac{\partial V}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \mathbf{D} \operatorname{div} \mathbf{A} - \Delta \mathbf{A},$$

or

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = \mathbf{D}\left(\operatorname{div} \mathbf{A} + \frac{1}{c}\frac{\partial V}{\partial t}\right),$$

and the third equation can be written

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V + \frac{1}{c}\frac{\partial}{\partial t}\left(\operatorname{div} \mathbf{A} + \frac{1}{c}\frac{\partial V}{\partial t}\right) = 0.$$

A simplification occurs if we put

$$\mathbf{A} = \frac{1}{c}\frac{\partial \mathbf{G}}{\partial t}, \quad V = -\operatorname{div} \mathbf{G} \quad \ldots\ldots\ldots (238),$$

where $\mathbf{G}$ is a vector to be determined; for then $\operatorname{div} \mathbf{A} + \frac{1}{c}\frac{\partial V}{\partial t}$ vanishes and we have only to make

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = 0, \quad \left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V = 0,$$

and both are satisfied if

$$\Delta \mathbf{G} = \frac{1}{c^2}\frac{\partial^2 \mathbf{G}}{\partial t^2} \quad \ldots\ldots\ldots\ldots\ldots (239).$$

Equations (236), (237) and (238) now give

$$\mathbf{E} = \mathbf{D}\,\mathrm{div}\,\mathbf{G} - \frac{1}{c^2}\frac{\partial^2 \mathbf{G}}{\partial t^2}, \quad \mathbf{H} = \frac{1}{c}\frac{\partial}{\partial t}\,\mathrm{curl}\,\mathbf{G} \quad \ldots\ldots(240).$$

This is Hertz's particular solution, $\mathbf{G}$ being any vector satisfying equation (239). We can specialise it further by writing $\mathbf{G} = \mathbf{k}\theta$, where

$$\theta = \frac{1}{r} f\left(t - \frac{r}{c}\right)$$

from Art. 11, and thus obtain

$$\mathbf{E} = \mathbf{i}\frac{\partial^2 \theta}{\partial x \partial z} + \mathbf{j}\frac{\partial^2 \theta}{\partial y \partial z} + \mathbf{k}\left(\frac{\partial^2 \theta}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \theta}{\partial t^2}\right),$$

$$\mathbf{H} = \frac{\mathbf{i}}{c}\frac{\partial^2 \theta}{\partial y \partial t} - \frac{\mathbf{j}}{c}\frac{\partial^2 \theta}{\partial x \partial t}.$$

We find after some reduction

$$\left.\begin{aligned}
\mathbf{E} = {} & \left\{\mathbf{i}\frac{3xz}{r^5} + \mathbf{j}\frac{3yz}{r^5} + \mathbf{k}\left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right)\right\} f\left(t - \frac{r}{c}\right) \\
& + \left\{\mathbf{i}\frac{3xz}{cr^4} + \mathbf{j}\frac{3yz}{cr^4} + \mathbf{k}\left(\frac{3z^2}{cr^4} - \frac{1}{cr^2}\right)\right\} f'\left(t - \frac{r}{c}\right) \\
& + \left\{\mathbf{i}\frac{xz}{c^2 r^3} + \mathbf{j}\frac{yz}{c^2 r^3} - \mathbf{k}\frac{x^2 + y^2}{c^2 r^3}\right\} f''\left(t - \frac{r}{c}\right) \\
\mathbf{H} = {} & \left\{-\mathbf{i}\frac{y}{cr^3} + \mathbf{j}\frac{x}{cr^3}\right\} f'\left(t - \frac{r}{c}\right) + \left\{-\mathbf{i}\frac{y}{c^2 r^2} + \mathbf{j}\frac{x}{c^2 r^2}\right\} f''\left(t - \frac{r}{c}\right)
\end{aligned}\right\} \quad \ldots\ldots(241).$$

So far all that has been found is a particular solution of the equations of the ether. The inverse powers of $1/r$ show that the source of disturbance is situated at the origin. Close up to the source the first term in the expression for the electric force predominates, and we have approximately

$$\mathbf{E} = \left\{\mathbf{i}\frac{3xz}{r^5} + \mathbf{j}\frac{3yz}{r^5} + \mathbf{k}\left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right)\right\} f(t).$$

This is the static force due to an electric doublet of moment $f(t)$, with its axis pointing along the axis of $z$ (cf. Art. 20). The finite velocity of propagation does not matter close up to the oscillator: hence we conclude that there actually is an electric doublet of moment $f(t)$ at the origin. The field at any place and time is then given by equations (241).

The magnetic force at any point is at right angles to the plane containing the point and the axis. The electric force is always in that plane. At points on the equatorial plane $z = 0$, $\mathbf{E}$ is perpendicular to the plane. These results hold good whatever the form of $f(t)$.

In order to find as nearly as possible the forces due to an actual Hertzian oscillator, we write $f(t) = A$ when $t < 0$, and $Ae^{-\frac{\delta t}{T}}\left(\cos\frac{2\pi t}{T} + \frac{\delta}{2\pi}\sin\frac{2\pi t}{T}\right)$ when $t > 0$ (cf. Art. 177). The form of the solution is then different according to whether $r$ is greater than or less than $ct$. Outside the sphere $r = ct$ we have simply the static field of the doublet of moment $A$. We have thus to look on a Hertzian wave as expanding continually outwards and pushing back the original electrostatic field as it goes. Fig. 273 shows the electric force at a point in the equatorial plane as a function of the distance from the oscillator after an interval of five complete periods, $\delta$ being taken as 0·2. The electric force outside the wave is, on this scale, too small to be shown. The phase of the electric force is not the same at all points, but changes sign at regular intervals. This is in most striking disagreement with the theory of action at a distance, according to which the phase, at a given time, should be the same everywhere. It is interesting to find what equations (241) would become on this latter hypothesis. The doublet may be regarded as consisting of a charge $-e$ at the origin and a moveable charge $e$ at distance $\zeta$, where $e\zeta = f(t)$; and the magnetic force can then be calculated from Art. 96. This would give

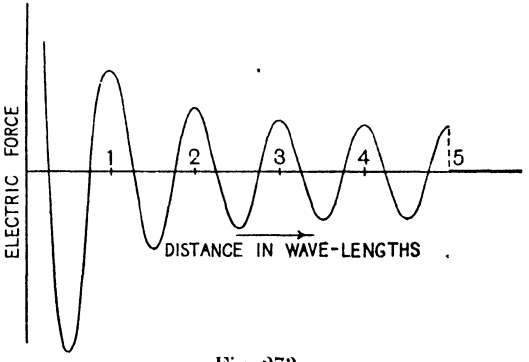



Fig. 273

$$\left.\begin{aligned}\mathbf{E} &= \left\{\mathbf{i}\frac{3xz}{r^5} + \mathbf{j}\frac{3yz}{r^5} + \mathbf{k}\left(\frac{3z^2}{r^5} - \frac{1}{r^3}\right)\right\} f(t),\\ \mathbf{H} &= \left\{-\mathbf{i}\frac{y}{cr^3} + \mathbf{j}\frac{x}{cr^3}\right\} f'(t)\end{aligned}\right\} \quad \ldots\ldots(242).$$

Hence the theory of action at a distance would give the first term of (241) in each case, with $t - r/c$ replaced by $t$. The discrepancy is most marked at great distances from the oscillator, where (241) becomes

$$\left.\begin{aligned}\mathbf{E} &= \left\{\mathbf{i}\frac{xz}{c^2r^3} + \mathbf{j}\frac{yz}{c^2r^3} - \mathbf{k}\frac{x^2+y^2}{c^2r^3}\right\} f''\left(t - \frac{r}{c}\right), \\ \mathbf{H} &= \left\{-\mathbf{i}\frac{y}{c^2r^2} + \mathbf{j}\frac{x}{c^2r^2}\right\} f''\left(t - \frac{r}{c}\right)\end{aligned}\right\} \quad \ldots(243).$$

While the terms in (242) are at most of the order $1/r^2$, those of (243) are of the order $1/r$, and therefore fall off less rapidly with increasing distance.

That the forces in a Hertzian wave are much greater than the corresponding static forces, even as near as one wave-length from the oscillator, is shown in Fig. 274, which shows the electric force in the equatorial plane at that distance as a function of the time. A slow falling off of intensity with distance is, of course, favourable to long-distance transmission.

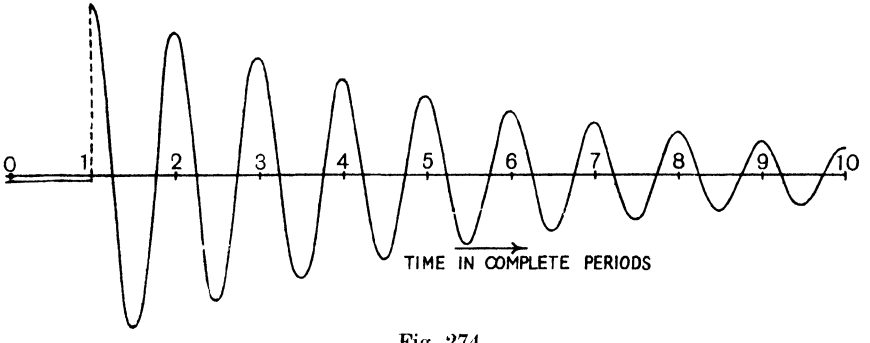



Fig. 274

**198. Electric waves in dielectrics.** In dielectrics there is, in addition to the ether current, a current due to changing polarisation. Consider a simple model dielectric with $n$ atoms per cm.$^3$, each containing a moveable electron of charge $e$, and a balancing nucleus. If $\mathbf{s}$ is the vector from the nucleus to the electron, the electric moment of the atom is $e\mathbf{s}$, and the polarisation $\mathbf{P} = ne\mathbf{s}$ (Art. 52). The motion of the electrons constitutes a current, and the current-density is, from Art. 77,

$$\frac{ne}{c}\frac{\partial \mathbf{s}}{\partial t} = \frac{1}{c}\frac{\partial \mathbf{P}}{\partial t}.$$

The ether current, more accurately the mean ether current in a small region of space, is $\frac{1}{4\pi c}\frac{\partial \mathbf{E}}{\partial t}$ as before. Hence the total current-density is

$$\mathbf{u} = \frac{1}{4\pi c}\frac{\partial}{\partial t}(\mathbf{E} + 4\pi\mathbf{P}) = \frac{1}{4\pi c}\frac{\partial \mathbf{D}}{\partial t},$$

where $\mathbf{D}$ is the electric induction (Art. 55). Maxwell's first equation for the dielectric is therefore

$$\frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \text{curl}\,\mathbf{H}.$$

The second is left unaltered (supposing the dielectric to be non-magnetic), while the equation $\text{div}\,\mathbf{D} = 0$ is introduced instead of $\text{div}\,\mathbf{E} = 0$. The fundamental equations for a dielectric are therefore

$$\left.\begin{aligned} &\frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \text{curl}\,\mathbf{H}, \quad -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = \text{curl}\,\mathbf{E} \\ &\text{div}\,\mathbf{D} = 0, \qquad \text{div}\,\mathbf{H} = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots(244).$$

We have now three vectors $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$ occurring instead of the original two, and the equations require supplementing by a relation between $\mathbf{D}$ and $\mathbf{E}$. This relation would depend on the structure of the atoms and their grouping, in a way that we could not expect to calculate in general. But when the oscillations are not too rapid we may expect an atom to respond to the electric force in the same way that it would to a steady force of the same magnitude, and write $\mathbf{D} = K\mathbf{E}$, where $K$ is the dielectric constant.

The above equations may now be treated in the same way as the equations of the free ether. The components of electric and magnetic force satisfy an equation of the form

$$\Delta\theta = \frac{K}{c^2}\frac{\partial^2\theta}{\partial t^2}.$$

It follows that electric waves of sufficient length are propagated in a dielectric with the velocity $c/K^{\frac{1}{2}}$. This explains the refraction of electric waves by insulators. The refractive index, being the ratio of the wave-velocity *in vacuo* to that in the dielectric, is equal to $K^{\frac{1}{2}}$. Hence we have the following remarkable consequence of Maxwell's theory:

*The refractive index of a dielectric for sufficiently long electric waves is equal to the square root of its dielectric constant.*

A strict test of this theory with very long waves is out of the question, since it would be necessary to use apparatus of enormous dimensions in order to get rid of diffraction effects. The only test, therefore, is to work with shorter waves of various lengths, and see whether the values of the refractive index thus obtained are the same. If this is so, these values may reasonably be compared with $K^{\frac{1}{2}}$; but if not, we shall have evidence of *electrical dispersion* analogous to that occurring in optics. The following table gives the experimental results for a few substances:

*Water.* $K^{\frac{1}{2}} = 9$.

| Wave-length (cm.) | $\mu$ | Observer |
|---|---|---|
| 644 | 8·96 | Weichmann |
| 273 | 8·97 | Weichmann |
| 65 | 8·88 | Ellinger |
| 8·8 | 8·89 | Eckert |
| 5·7 | 8·79 | Eckert |
| 5·0 | 8·80 | Cole |
| 3·7 | 8·10 | Eckert |
| 2·7 | 8·45 | Tear |
| 1·75 | 7·82 | Eckert |
| 1·8 | 6·65 | Tear |
| 0·84 | 5·68 | Tear |
| 0·42 | 5·33 | Tear |
| *1·26 × $10^{-4}$ | 1·32 | Rubens |
| †5·89 × $10^{-5}$ | 1·33 | — |

*Benzene.* $K^{\frac{1}{2}} = 1{\cdot}5$.

| Wave-length (cm.) | $\mu$ | Observer |
|---|---|---|
| 9·04 | 1·50 | Kossonogoff |
| 8·5 | 1·84 | v. Lang |
| 6·43 | 1·52 | Kossonogoff |
| 4·30 | 1·52 | Kossonogoff |
| 2·95 | 1·56 | Kossonogoff |
| 1·92 | 1·56 | Kossonogoff |
| 0·8 | 1·77 | Lampa |
| 0·6 | 1·76 | Lampa |
| 0·4 | 1·74 | Lampa |
| *1·85 × $10^{-4}$ | 1·48 | Rubens |
| †5·89 × $10^{-5}$ | 1·50 | — |

*Ethyl alcohol.* $K^{\frac{1}{2}} = 5{\cdot}1$.

| Wave-length (cm.) | $\mu$ | Observer |
|---|---|---|
| 65 | 4·89 | Ellinger |
| 5·7 | 3·4 | Eckert |
| 2·7 | 2·27 | Tear |
| 0·84 | 2·06 | Tear |
| 0·42 | 1·84 | Tear |
| †5·89 × $10^{-5}$ | 1·36 | — |

*Glycerine.* $K^{\frac{1}{2}} = 7{\cdot}5$ (?).

| Wave-length (cm.) | $\mu$ | Observer |
|---|---|---|
| 5·7 | 3·8 | Eckert |
| 4·5 | 4·1 | Merczyng |
| 2·7 | 2·38 | Tear |
| 0·84 | 2·16 | Tear |
| 0·42 | 1·98 | Tear |
| †5·89 × $10^{-5}$ | 1·47 | — |

*Paraffin wax.* $K^{\frac{1}{2}} = 1{\cdot}56$.

| Wave-length (cm.) | $\mu$ | Observer |
|---|---|---|
| 10 | 1·42 | Wiedeburg |
| 0·8 | 1·52 | Lampa |
| 0·6 | 1·41 | Lampa |
| 0·4 | 1·39 | Lampa |

* Heat waves.

† Sodium light.

The results for benzene and paraffin-wax lend strong support to Maxwell's theory, considering the difficulty of this kind of measurement. For these and some other substances the dispersion is small. The behaviour of water is remarkable. For the longer wave-lengths $\mu$ is nearly constant and equal to $K^{\frac{1}{2}}$, but falls rapidly near the wave-length of the shortest electric waves hitherto produced experimentally, the refractive index being quite low for the longest heat-waves.

**199. Waves along a pair of parallel wires.** A *perfect conductor* is one for which the lines of electric force leave the surface at right angles, even at the frequencies of electric waves. We shall assume, subject to verification, that all ordinary metals fulfil this condition approximately. Consider the propagation of electric waves along two wires of perfectly conducting material. The wires are taken as parallel to the axis of $z$, but not necessarily thin or round, the cross-sections by a plane parallel to $z = 0$ being bounded by any two curves $C_1$, $C_2$. The theory, which is due to Lord Rayleigh, is based on the following particular solution of the equations of the ether:

$$\left.\begin{aligned} \mathbf{E} = -\mathbf{i}\frac{\partial V}{\partial x} - \mathbf{j}\frac{\partial V}{\partial y}, \quad \mathbf{H} = \mathbf{i}\frac{\partial U}{\partial x} + \mathbf{j}\frac{\partial U}{\partial y} \\ \text{where} \qquad U + iV = f(x + iy,\ z - ct) \end{aligned}\right\} \quad \ldots\ldots(245).$$

$f$ being an arbitrary function of the two arguments $x + iy$, $z - ct$.

Confining attention to a particular time and a particular cross-section ($z$, $t$ constant), the condition of perfect conductivity is that $V$ is constant over $C_1$ and $C_2$. Thus $V$ is proportional to the potential in the corresponding electrostatical problem (Art. 46). The lines of electric force* coincide with those in the electrostatic problem, the lines of magnetic force with the equipotential curves.

We shall assume that the charge on the wires resides entirely on the surface, and is given by Coulomb's formula $E = 4\pi\sigma$. Suppose now that the wires have a length $l$ great in comparison with their distance apart, and that $C$ is the capacity in the electrostatic problem, which we suppose to have been already solved. In the actual wave-problem the line-integral $\int \mathbf{E}\,d\mathbf{s}$ is the same for all lines

* The electric force is not the gradient of $-V$, which is
$$-\mathbf{i}\partial V/\partial x - \mathbf{j}\partial V/\partial y - \mathbf{k}\partial V/\partial z.$$

of electric force passing from $C_1$ to $C_2$ in the particular cross-section considered. Denoting it by $V$, $V$ may be called the difference of potential between the wires *at that cross-section*, and is a function of $z - ct$ only. Let $e$ be the charge per unit length of $C_1$ at the cross-section $z$, $i$ the current in electromagnetic units at the same place, and $-e$, $-i$ the same quantities for the wire $C_2$. Then $e = CV/l$. The charge entering the part of $C_1$ between the planes $z$ and $z + dz$ per second is $-\frac{\partial i}{\partial z}dz$. This must be accounted for by the rate of increase of charge between the planes, which is $\frac{1}{c}\frac{\partial e}{\partial t}dz$. Hence $\frac{\partial i}{\partial z} = -\frac{1}{c}\frac{\partial e}{\partial t}$. Since $\frac{\partial}{\partial t} = -c\frac{\partial}{\partial z}$ applied to a function of $z - ct$, $i$ is simply equal to $e$.

A possible distribution of current and "potential" corresponding to the wave-length $\lambda$ in ether, is thus given by

$$V = A\cos\frac{2\pi}{\lambda}(z - ct), \quad e = \frac{CA}{l}\cos\frac{2\pi}{\lambda}(z - ct),$$
$$i = \frac{CA}{l}\cos\frac{2\pi}{\lambda}(z - ct) \ldots (246).$$

The effect of the wires appears only in the boundary conditions: in physical language, the wires serve only to guide the waves, which are really propagated through the ether. The velocity of propagation is $c$, so that the wave-length on the wire is the same as that of a wave of equal frequency in air.

For a wave travelling in the negative direction of the axis of $z$,

$$V = A\cos\frac{2\pi}{\lambda}(z + ct), \quad e = \frac{CA}{l}\cos\frac{2\pi}{\lambda}(z + ct),$$
$$i = -\frac{CA}{l}\cos\frac{2\pi}{\lambda}(z + ct) \ldots (247).$$

Adding equations (246) and (247), with a slight change of notation, we have the solution corresponding to a stationary wave, namely

$$V = A\cos\frac{2\pi z}{\lambda}\cos\frac{2\pi ct}{\lambda}, \quad e = \frac{CA}{l}\cos\frac{2\pi z}{\lambda}\cos\frac{2\pi ct}{\lambda},$$
$$i = \frac{CA}{l}\sin\frac{2\pi z}{\lambda}\sin\frac{2\pi ct}{\lambda} \ldots (248).$$

The amplitude of potential (or charge) at the section $z$ is here proportional to $\cos(2\pi z/\lambda)$, that of current to $\sin(2\pi z/\lambda)$.

Three cases have to be distinguished:

(1) *Free ends to both wires.* Here $i = 0$ when $z = 0$ and when $z = l$. The first condition is satisfied already in equation (248), and the second if $2\pi l/\lambda = n\pi$, where $n$ is an integer. Hence the possible wave-lengths of the free stationary waves are given by $\lambda = 2l$, $2l/2$, $2l/3$, etc.

The distribution of the amplitude of current and potential (or charge) along the wires is shown in Fig. 275 for the fundamental

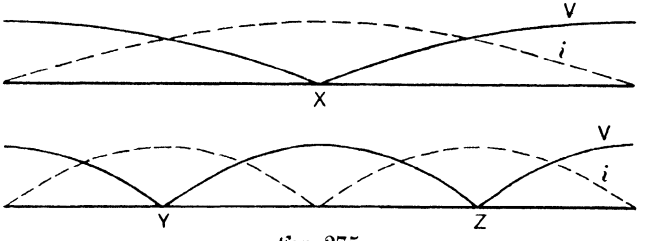



Fig. 275

and first harmonic. Points $X$, $Y$, $Z$ at which the amplitude of potential vanishes are called potential nodes. A wire can obviously be placed across at these points without stopping the oscillation (cf. Art. 201).

(2) *Free ends at $z = 0$, wires bridged across at $z = l$.* Here $i = 0$ when $z = 0$ and $V = 0$ when $z = l$, giving $2\pi l/\lambda = (n + \frac{1}{2})\pi$. Thus the admissible waves are given by $\lambda = 4l$, $4l/3$, $4l/5$, etc. The first two of these are represented in Fig. 276.

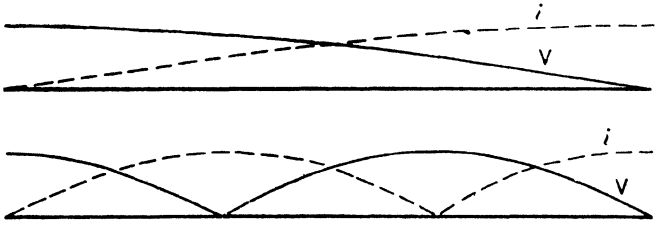



Fig. 276

(3) *Both ends bridged over.*

It is convenient to take another fundamental solution in place of (248), namely

$$V = A \sin\frac{2\pi z}{\lambda}\cos\frac{2\pi ct}{\lambda}, \quad e = \frac{CA}{l}\sin\frac{2\pi z}{\lambda}\cos\frac{2\pi ct}{\lambda},$$

$$i = -\frac{CA}{l}\cos\frac{2\pi z}{\lambda}\sin\frac{2\pi ct}{\lambda} \quad \ldots(249).$$

Here $V = 0$ when $z = 0$ and when $z = l$. Thus $2\pi l/\lambda = n\pi$, where $n$ is an integer; or $\lambda = 2l,\ 2l/2,\ 2l/3$, etc. The first two waves are shown in Fig. 277.

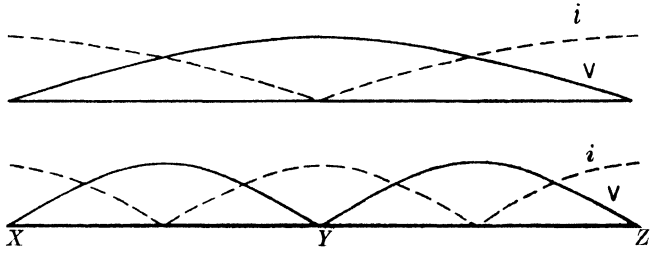



Fig. 277

Figs. 275–7 are not to scale as regards the amplitudes of $i$ and $V$ in the different harmonics, the higher ones being more and more difficult to excite.

**200. Measurement of the velocity of waves on wires.** The velocity of propagation of electric waves along wires was first measured by Blondlot. Two Leyden jars (Fig. 278) have their inner coatings continuous, but each of the outer coatings is cut into two parts $AB$, $A'B'$, separated from one another. The inner coatings are connected to the secondary of an induction coil, in parallel with the spark terminals $C$, $C'$. $A$ is joined to $A'$ by a wire interrupted by a small spark-gap $DD'$, and also by a piece of moistened cord $E$, the object of which will appear shortly. $BD$ and $B'D'$ are joined by wires $BFD$, $B'F'D'$, each 1000 metres long, doubled back on themselves, $BB'$ being also connected together by a moistened cord $E'$.

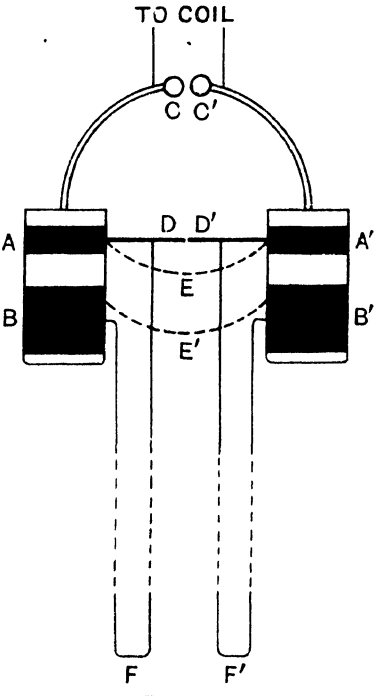



Fig. 278

The working of the apparatus is as follows. While the coil is charging up the inside coatings, and before

the spark-gap $CC'$ gives way, $A$ and $A'$ tend to become charged with electricity of opposite signs, and so do $B$ and $B'$. These charges are however equalised as they appear by the moist cords, which conduct sufficiently well for the purpose. An oscillatory discharge takes place as soon as the potential is high enough to spark across $CC'$, giving oscillatory potential-differences between $A$ and $A'$. Since the moist cords cannot neutralise these sufficiently rapidly, a spark passes between the terminals $DD'$. While this is occurring the corresponding disturbance due to $BB'$ is travelling along the wires $BFD$ and $B'F'D'$ with the velocity of propagation $v$, and after a time $l/v$ a second discharge occurs through $DD'$, $l$ being the length of either of the long wires. Blondlot examined the image of the spark in a rotating mirror, and found $v$ from the measured interval between the two discharges.

A set of five observations gave as the mean $v = 2{\cdot}964 \times 10^{10}$ cm./sec. and three similar observations on wires 1800 metres long gave 2·975, 2·985 and 2·980, mean $2{\cdot}980 \times 10^{10}$ cm./sec. The weakening of the waves in traversing such a great length of wire was distinctly noticeable, and Blondlot made the coatings $BB'$ larger than $AA'$ in order that the sparks might be of nearly equal intensity.

Another accurate determination of the velocity of waves on wires was made by Trowbridge and Duane. The primary condenser $ab$ (Fig. 279) consists of two metal plates 30 cm. square, separated by

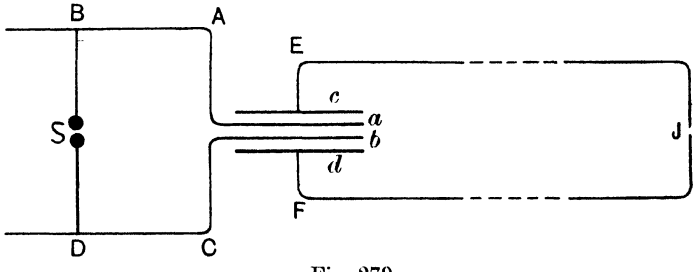



Fig. 279

a sheet of plate glass 2 cm. thick. Sheets of ebonite 1·8 cm. thick are laid on the other side of $ab$, and metal plates $cd$, each 26 cm. square, attached to them. These plates form one extremity of the parallel-wire system $EJF$, 58·6 metres in length, which is interrupted at $J$ by a spark-gap with pointed terminals of cadmium. The primary circuit is completed by a sliding piece $BD$ containing

a spark-gap $S$, the distance of the parallel wires $AB$, $CD$ from one another being 40 cm. Resonance can be obtained with one or more of the natural periods of oscillation of the wire-system by moving the sliding piece; this gives the greatest intensity of light in the spark-gap $J$. The most powerful effects were obtained when $AB = CD = 85$ cm., in which case resonance seems to have taken place with the first harmonic. Trowbridge and Duane observed the position of the nodes and also measured the frequency of the oscillatory spark $J$ with a very rapidly rotating mirror, with the following results:

| Wave-length $\lambda$ (cm.) | Periodic time $T$ (sec.) | Velocity of propagation $v = \lambda/T$ |
|---|---|---|
| 5670 | $1{\cdot}871 \times 10^{-7}$ | $3{\cdot}030 \times 10^{10}$ |
| 5670 | 1·876 | 3·022 |
| 5690 | 1·940 | 2·923 |
| 5690 | 1·897 | 3·000 |
| 5690 | 1·874 | 3·036 |
| 5690 | 1·899 | 2·996 |
| 5660 | 1·878 | 3·014 |
| | Mean | $3{\cdot}003 \times 10^{10}$ |

The velocity of propagation of waves on wires is therefore sensibly the same as that of light, a conclusion that has been utilised already. Since the waves are capable of traversing considerable distances before being extinguished, copper and other metals must fulfil approximately the conditions of perfect conductivity as used in Art. 199.

There is an important difference between the behaviour of a single wire and a pair of parallel wires, as carriers of electric waves. Waves on a single wire are damped heavily by radiation, but we have seen that parallel wires can carry waves damped only by resistance and a slight residual radiation from the ends.

**201. Experiments with stationary waves on wires.** The production of stationary waves on wires originated with Hertz. Lecher introduced the improvement of using two wires. The waves were excited by two small plates placed opposite the large plates of a Hertzian oscillator. These methods reveal the free oscillations in the wires, which persist after the Hertzian oscillations have been damped out. It is more convenient to examine the forced oscilla-

tions induced by a continuous wave generator. A suitable primary oscillator (Fig. 280) consists of two parallel rods $AB$, $A'B'$, 75 cm.

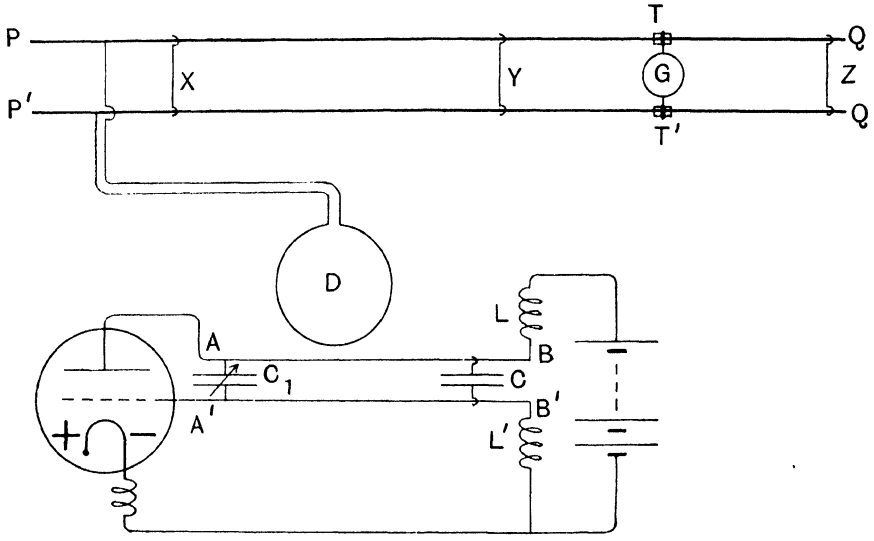



Fig. 280

long, closed at the far end by a mica condenser $C$ of about 1 millimicrofarad capacity, with two small inductances $L$, $L'$ to prevent high-frequency currents from passing through the plate battery. The grid and plate circuits act inductively on one another without the necessity for extra coils, as is evident from the figure. The condenser $C$ can be moved along the rods so as to vary their effective length, fine adjustment of frequency being made with a small variable condenser $C_1$ (of the order of 20 cm.) across $AA'$. The oscillator gives wave-lengths between 5 and 15 metres, with a plate potential of 240 volts. The parallel wires $PQ$, $P'Q'$ are of stout copper, 5 cm. apart and 10 metres long, excited inductively by a loop $D$ near the oscillator. A milliammeter in the plate circuit gives a greatly increased deflexion when oscillations are being generated, and is a convenient indicator.

It is instructive to detect the waves in the first instance electrostatically, as in some experiments of Rubens. Two short glass tubes $T$, $T'$ (Fig. 280) are slipped over the wires, and a single turn of wire wrapped round each, the loops being connected through a vacuum thermal junction $G$, with attached galvanometer. The charges on

the loops at any time are proportional to $\pm e$, so that the deflexion of the galvanometer is proportional to the square of the amplitude of $e$ or $V$, with the notation of Art. 199.

The oscillator having been set for a short wave-length, the wires are bridged first by a cross-wire $Z$ on the far side of $TT'$. The deflexion of the galvanometer is in general very small, but a large deflexion occurs at a well-defined position of the bridge, showing that the circuit from $D$ to $Z$ is in resonance with the oscillator. A second bridge $Y$ on the near side of $TT'$ in general extinguishes the oscillations completely, but the original deflexion can be reproduced for one particular position of the bridge. The segment $YZ$ is now in resonance with the oscillator, and $YZ$ is half the wave-length (cf. Fig. 277). Other equidistant bridges can be laid on so as not to impede the oscillations, if the wires are long enough. The one shown corresponds to $X$ in Fig. 277. We can remove the middle bridge $Y$ and verify that the deflexion of the galvanometer is very small near this point, in accordance with the theory.

A second and very simple method of demonstration is to pass along the wire a single bridge containing a glow-lamp. The lamp lights up in the positions $X$, $Y$, $Z$. A thermal junction is better for accurate measurements. Townsend and Morrell found that the wave-length so measured remained the same, to 1 part in 5000, when the distance between the wires was varied between 2 and 10 cm., and the resistance of the thermal junction between 0·3 and 8 ohms.

The Lecher wires thus give a means of measuring short wave-lengths with great accuracy. They can be used to calibrate wave-meters as follows. A condenser circuit, giving wave-lengths of the order of 1000 metres, is kept in oscillation by a triode, using close coupling to bring out the harmonics (Art. 185). A telephone transformer is inserted in the plate circuit of the long-wave generator, and a loop of wire in the grid circuit is brought near the short-wave generator of Fig. 280. The short-wave generator is first adjusted to resonance with some unknown harmonic, the $m$th, its wave-length $\lambda_1$ being found from the wires. Higher harmonics come into resonance successively as the short wave-length is diminished, until the $(m + n)$th harmonic (say) is in resonance for a wave-length $\lambda_2$. If $\lambda$ is the fundamental wave-length,

$$\lambda = m\lambda_1 = (m + n)\,\lambda_2,$$

whence $$\lambda = n\lambda_1\lambda_2/(\lambda_1 - \lambda_2).$$

In an experiment of Townsend and Morrell 30 harmonics came into resonance between 12·12 metres and 7·996 metres, giving $\lambda = 705$. A wave-meter gave 706 metres. As a check, $\lambda/\lambda_1$ is 58·17 instead of the nearest integer 58, the error being more probably in setting for resonance than in measuring on the wires.

The Lecher wire system was used by Drude to measure dielectric constants at high frequencies. Extension of the theory of Art. 199 shows that the velocity of waves on wires surrounded by a homogeneous dielectric is $c/K^{\frac{1}{2}}$. Thus if a Lecher wire-system is in forced oscillation and passes through a trough which is afterwards filled with a liquid of dielectric constant $K$, the wave-length is changed from $\lambda$ to $\lambda'$, where $\lambda/\lambda' = K^{\frac{1}{2}}$, or $K = \lambda^2/\lambda'^2$. $\lambda$ and $\lambda'$ are measured by methods already described. Drude used a small spark circuit to excite the oscillations. Greater accuracy can be obtained with continuous waves, as Holborn showed. The dielectric constants of substances like alcohol and glycerine depend markedly on the wave-length, but $K$ and $\mu^2$ agree fairly well with each other for a particular wave-length.

**202. Production of short waves with the triode.** The failure of triode oscillations below a wave-length of a few metres seems to be due to the fact that the periodic time of the oscillations is of the same order as the time of flight of the electrons from filament to plate. It is exactly this phenomenon which keeps oscillations going in another circuit, discovered by Barkhausen and Kurz, in which a Lecher wire system is connected across the grid and plate. The grid is at a positive potential of the order of 200 volts, the plate at a potential equal or near to that of the filament.

Barkhausen and Kurz described the phenomenon as a to-and-fro movement of electrons between the filament and plate, under the fixed field of the grid. Gill and Morrell explained the maintenance of the oscillations by calculating the work done on the electrons by the oscillatory E.M.F., supposed already developed: negative work means that the oscillatory circuit is abstracting energy from the electrons (i.e. ultimately from the grid battery) to maintain its own oscillations. To simplify the theory suppose that the filament

is removed and the grid represented by a plane emitting $n$ electrons per second with velocity $v$ normal to the surface, the plate being parallel to it at distance $a$. Suppose further that $n$ is constant and $v$ that velocity which the electrons must have in order to reach the grid plane under the steady E.M.F. $V$, so that $v^2 = 2eV/m$. An electron leaving the grid at time $t_0$ under the steady E.M.F. will be at distance $x$ from it at time $t$, where

$$\frac{dx}{dt} = v - \frac{eV}{ma}(t - t_0),$$

and will reach the plate at time $t_0 + T$, where $v = eVT/ma$. Now let an oscillation be superposed in which the potential of the grid becomes $V + A \sin pt$, where $A$ is small. The additional electric force between the grid and plate being $A \sin pt/a$, work

$$-eA \sin pt \,.\, dx/a$$

is done on an electron moving from $x$ to $x + dx$, which becomes

$$-\frac{e^2AV}{ma^2} \sin pt \,.\, (t_0 + T - t)\, dt$$

on substituting for $dx$ and $v$. Integrating from $t_0$ to $t_0 + T$, the work done on an electron leaving the grid at time $t_0$, up to the time at which it would normally reach the plate, is

$$\frac{2eA}{p^2T^2}\{pT \cos pt_0 + \sin pt_0 - \sin (pt_0 + pT)\}.$$

The electrons for which this expression is positive reach the plate. Those for which it is negative return to the grid, and the work done during the return is

$$\frac{2eA}{p^2T^2}\{pT \cos (pt_0 + 2pT) + \sin (pt_0 + pT) - \sin (pt_0 + 2pT)\}.$$

The mean work done during a cycle would be zero if all the electrons reached the plate, but differs from zero since half of them reach the plate and half return to the grid. It is not difficult to show that the additional E.M.F. is doing work altogether at a rate

$$-\frac{2neA}{\pi} \frac{p^2T^2 \cos 2pT - 2pT (\sin 2pT - \sin pT) - \cos 2pT + 2 \cos pT - 1}{p^2T^2\{(1 - \cos pT)^2 + (pT - \sin pT)^2\}^{\frac{1}{2}}}$$

on the $n$ electrons emitted by the idealised grid per second. This expression is negative for values of $pT$ between 1·2 and 3·4, and

has its greatest negative value $-0{\cdot}27neA$ when $pT = 2{\cdot}2$. The oscillations which are maintained most strongly, on this simple theory, are therefore those whose period is about $3T$.

Since $T^2 = 2ma^2/eV$, the wave-length of the oscillations which are maintained most strongly should be proportional to the linear dimensions of the triode, and to $V^{-\frac{1}{2}}$. These relations are confirmed approximately by experiment. Increased emission broadens the range of wave-lengths that can be maintained, as we should expect. The theory does not account for the maintenance of oscillations when the filament and anode differ much in potential, but makes it probable that only part of the stream of electrons falls on the anode when oscillations are taking place.

Barkhausen and Kurz produced waves 43 cm. long by applying 500 volts to the grid of a small triode, and were able to perform the simpler Hertzian experiments for the first time with continuous waves. Waves 30 cm. long are produced without difficulty with suitable triodes. Were the technique of the method developed further, so as to give continuous waves 10 cm. or so long, the experiments of Arts. 192, 194 might be carried out more accurately.

**203. Electric oscillations on rods and solenoids.** Consider first the simple Hertzian rod-shaped oscillator, represented in Fig. 270. If we leave the damping out of account we shall have a set of stationary waves, with the condition $i = 0$ at both ends of the rod; hence it is reasonable to suppose that the various types of vibration are given approximately by $i = A \sin(2\pi z/\lambda) \sin(2\pi ct/\lambda)$, where $\lambda$ is the wave-length. The general equation $\partial e/\partial t = -c\partial i/\partial z$ then shows that $e = A \cos(2\pi z/\lambda) \cos(2\pi ct/\lambda)$. The theory is therefore the same as that of Art. 199, and Fig. 275 shows the possible distributions of the amplitude of current and charge along the rod, $e$ taking the place of $V$.

These considerations make it very probable that the wave-length in air corresponding to the fundamental oscillation is approximately twice the length of the rod. Ives measured the wave-lengths of a number of small oscillators by an interference method, with results shown on p. 428.

The wave-length is therefore somewhat greater than $2l$, and in addition the damping is considerable ($\delta$ = about 0·2).

| Length of oscillator $l$ (cm.) | Fundamental wave-length $\lambda$ (cm.) | $\lambda/l$ |
|---|---|---|
| 4·93 | 10·42 | 2·10 |
| 7·49 | 15·24 | 2·04 |
| 9·85 | 19·86 | 2·03 |

The natural oscillations of solenoids, of length up to six times the diameter, were first examined by Drude. The solenoids were excited inductively by a small spark circuit with variable capacity, the wave-length corresponding to any particular position of which had been determined by a separate experiment with a pair of parallel wires. Resonance occurred in certain positions, and was recognised by the lighting up of a small vacuum tube attached to one end of the coil.

The constants necessary to specify a solenoid of one layer are

$n$ = number of turns of wire,
$g$ = distance between two consecutive turns,
$h$ = axial length,
$2r$ = diameter of cross-section, supposed circular,
$l$ = total length of wire,
$\delta$ = diameter of wire;

of which only four are independent, since $h = (n-1)g$ and $l = 2\pi rn$. The dielectric constant $K$ of the core will also have some effect. Drude set himself the problem of finding how the fundamental wave-length* $\lambda$ depends on these quantities. The wave-length of geometrically similar coils is proportional to the linear dimensions; for Maxwell's equations are unaltered when $x, y, z, t$ are replaced by $kx, ky, kz, kt$, $k$ being any constant. Thus a coil $k$ times as great has a natural period $k$ times as large as before, or

$$\lambda = lf\left(n, \quad \frac{h}{2r}, \quad \frac{g}{\delta}, \quad K\right),$$

where $f$ remains to be determined. Drude concluded that the variation with $n$ was unimportant, so that

$$\frac{\lambda}{l} = f\left(\frac{h}{2r}, \quad \frac{g}{\delta}, \quad K\right).$$

* By this we mean the wave-length in air which corresponds to the natural period $T$ of the coil, so that $\lambda = cT$. The distance between two consecutive nodes, measured along the axis of the solenoid, is of course much smaller than $\frac{1}{2}\lambda$.

The accompanying table gives the values of the factor $f$ for copper wires (1) wound on solid ebonite cores, (2) with no core. If the solenoid had been completely imbedded in ebonite, $\lambda$ would have been increased in the ratio $K^{\frac{1}{2}}$ to unity; the increase due to the core alone is naturally less.

| $h$ / $2r$ | Solid ebonite | | | Coreless | | |
|---|---|---|---|---|---|---|
| | $g/\delta$ = 1·09 | $g/\delta$ = 1·24 | $g/\delta$ = 2·4 | $g/\delta$ = 1·09 | $g/\delta$ = 1·24 | $g/\delta$ = 2·4 |
| 6·0 | 1·48 | 1·46 | 1·44 | 1·37 | 1·35 | 1·33 |
| 4·0 | 1·65 | 1·63 | 1·60 | 1·52 | 1·50 | 1·48 |
| 3·0 | 1·86 | 1·82 | 1·78 | 1·69 | 1·66 | 1·62 |
| 2·0 | 2·25 | 2·21 | 2·14 | 1·99 | 1·95 | 1·92 |
| 1·0 | 3·11 | 2·97 | 2·85 | 2·66 | 2·54 | 2·45 |
| 0·8 | 3·48 | 3·30 | 3·14 | 2·94 | 2·78 | 2·68 |
| 0·6 | 3·98 | 3·68 | 3·50 | 3·34 | 3·09 | 2·97 |
| 0·4 | 4·61 | 4·20 | 3·95 | 3·85 | 3·51 | 3·35 |
| 0·2 | 5·60 | 5·22 | 4·67 | 4·64 | 4·32 | 3·94 |
| 0·1 | 6·72 | 6·20 | 5·37 | 5·58 | 5·14 | 4 48 |
| 0·05 | 7·20 | 6·36 | 5·17 | 5·96 | 5·28 | 4·38 |

The fundamental wave-length does not, therefore, greatly exceed the total length of wire. Drude found that the higher oscillations are not harmonic, but have wave-lengths exceeding $\frac{1}{2}\lambda$, $\frac{1}{3}\lambda$, etc., the relative discrepancy increasing with the order of the oscillation. The nodes are closer together at the ends than at the centre. These results were confirmed by Townsend and Morrell, using continuous oscillations. In their experiments a long solenoid was excited by a circular coil of a few turns, in a central plane perpendicular to the axis, and the fundamental and higher oscillations were isolated successively by resonance. The following results were obtained for a solenoid 227 cm. long and 5 cm. in diameter, wound with 29 turns to the cm.:

| Order of oscillation | Wave-length (metres) |
|---|---|
| 1 (fundamental) | 870 |
| 3 | 338 |
| 5 | 231 |
| 7 | 184 |
| 9 | 158 |

A pair of coils near the ends of the solenoid will excite both odd and even oscillations, at the expense of a certain disturbance of wave-length. The apparatus is shown in Fig. 281. $S$ is the solenoid

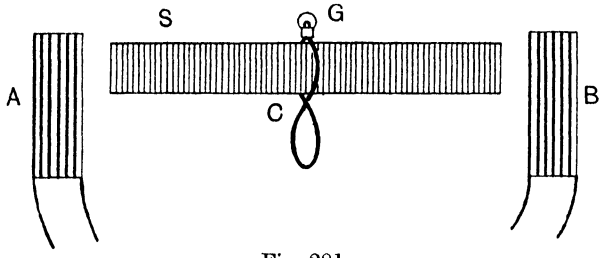



Fig. 281

and $A$, $B$ are the exciting coils, connected in parallel and forming part of an oscillatory circuit maintained by a large triode. Oscillations of odd order make the current flow in the same direction near the two ends of the solenoid and are, therefore, excited by currents in the same direction in $A$, $B$; even oscillations by currents flowing in opposite directions in $A$, $B$. The detector is a small glow-lamp $G$ in a figure-of-eight circuit formed by two equal loops crossing at $C$, with the object of neutralising the inductive effect of the end coils on the loop. This is attained since the point $C$ is on the line joining the centres of $A$ and $B$. The solenoid, on the other hand, acts almost entirely on the upper loop, and the induced current is nearly proportional to the current at points in its plane. Positions of maximum and zero current are found by moving the loop along the solenoid, and correspond generally to the $i$ curves of Fig. 275. Nodes of potential (cf. $X$, $Y$, $Z$ in Fig. 275) are located as the points at which the solenoid can be touched without extinguishing the lamp. Seven or eight higher oscillations can be observed in this way. Townsend and Morrell found that the distance between consecutive nodes of current, in the oscillation of order 5 on their long solenoid, diminished from 49·5 cm. at the centre to 37 cm. at the ends.

**204. Wireless telegraphy and telephony*.** It was recognised at once that Hertzian waves could be used for signalling without wires, if they could be produced with sufficient intensity or de-

* For information on telegraphy and telephony with wires see the *Dictionary of Applied Physics*, vol. 2, pp. 778–864.

tected with sufficient delicacy. By the end of 1891 Minchin had detected electric waves at a distance of 150 feet with his *impulsion cell*, a detector whose mode of action is obscure. Cardinal modifications to the oscillator were made by Marconi in 1895–6. One half was replaced by a simple earth-connexion, the other half by a long vertical wire, or *aerial*; the spark-gap being operated by an induction coil. The receiver was a similar aerial interrupted by a *coherer*, a tube of metallic filings, invented by Branly. Waves in the receiving aerial made the coherer conduct, giving a click in a telephone contained in a shunt circuit. The coherer had to be tapped back each time to a non-conducting state.

The coherer, though sensitive, was uncertain and also inefficient in that it only utilised the potential generated in the receiver up to the first maximum. Resonance was practically absent. The advent of integrating detectors, of which the crystal is a type, made it worth while to attend to resonance, in order to prevent dissipation of energy over a large range of wave-length. Marconi, while still using a coherer, had removed it from the aerial into a circuit having a mutual inductance with it. Braun used a coupled transmitter, producing spark discharges in a condenser circuit of relatively small decrement, which acted inductively on the aerial. All four circuits (spark circuit, transmitting aerial, receiving aerial, detector circuit) were tuned to resonance, or nearly so. Marconi's first transatlantic station used coupled circuits on this plan.

Braun's idea seems to have been to allow the aerial to radiate copiously (with a large natural decrement) while supplying energy to it from a weakly damped condenser circuit. This is theoretically possible with weak coupling; but the aerial current is then small and the efficiency low. Tight coupling gives two frequencies (Art. 182), of which only one is utilised in the receiver. The difficulty was obviated by M. Wien. The spark-gap was subdivided into 10 or more gaps of 0·1 mm. or so between discs, in order to make the damping in the condenser circuit very high (Art. 179). The damping of the aerial, on the other hand, was kept low by inserting self-inductances, so that it oscillates after the primary has ceased. The oscillating system is once more a single aerial, but it does not contain a spark-gap. This feature is essential. If $\delta_1$ is the part of the decrement arising from radiation and $\delta_2$ that arising from other

causes, efficiency is only high when $\delta_1$ is large compared with $\delta_2$. Damping is only low when $\delta_1 + \delta_2$ is small. Hence $\delta_2$ must be very small, which is impossible with a spark-gap. Efficient quenching can also be obtained (with long waves at least) by sparks between a fixed electrode and a studded disc rotating rapidly, as was done by Marconi.

The quenched spark system, embodying great progress in the use of damped waves, was much used in Germany before the European war. Large stations began to abandon it about 1914 in favour of high-frequency alternating currents. The oscillating triode appeared shortly after, so that there was a choice of three methods of generating continuous waves. The difficulty of constructing a suitable alternator (on the principle explained in Art. 154) is mitigated by the fact that the curvature of the earth favours long waves, and the best wave-length for transatlantic work seems to be about 8000 metres, corresponding to a frequency of 37,500. An alternator used at New Brunswick, U.S.A., generated 200 kilowatts at a frequency of 22,000. This is on the limit of mechanical possibility. The station at Nauen, near Berlin, generated current at a frequency of 8000 and passed it through two transformers nearly saturated in opposite directions by direct current in a third winding. The secondary windings, joined in opposition, give an approximately sinoidal current of twice the frequency. Repetition of the process gave a current of frequency 32,000, communicated direct to the aerial.

The Poulsen arc shares with the alternator the advantage that all the power can be concentrated in one unit, up to 500 kilowatts at least. Triodes of this power have not yet been constructed; but the triode is almost universal for small transmitting stations, and for reception.

The possibilities of the triode were not fully realised when it was invented by de Forest in 1907, the grid being used to improve rectification after the manner of a potentiometer. Experiments have proved it to be the most sensitive of known detectors for continuous waves or, when not quite oscillating, for damped wave-trains. It is reliable in that it does not depend on precarious contacts, like a crystal. Very small currents require preliminary amplification, as explained in Art. 189.

Only the briefest account can be given here of wireless telephony. The principle of the simplest method resembles that of the buzzer and crystal method explained in Art. 180. If a self-oscillating circuit contains a microphone, it will send out waves whose amplitude varies according to the intensity and frequency of the sound falling on it, and these produce a corresponding variation in the mean pull on the diaphragm of the receiving telephone (cf. Art. 187). The quality of the sound in the receiver depends largely on the transmitting microphone. One form used in broadcasting is a light coil of many turns of fine wire in a magnetic field, set in motion by the impinging waves of sound. The currents induced in the coil are amplified in six or eight stages, and pass finally by land wires to the transmitting station, if this is at a distance from the source of sound; the amplified currents being passed in the same way to distant stations, when it is desired to transmit the same message from them simultaneously. Messages to be relayed, for example, from America are received on a suitable aerial at a distance, and the received current passed by land wires to the amplifiers, for re-transmission as before. The oscillator at the transmitting station is usually a set of triodes, and the amplitude of the oscillations is changed by amplified low-frequency current applied directly to the plate, through a coil of sufficient inductance to choke out the high-frequency current and let through the low-frequency current. The receiver in wireless telephony is the same as in wireless telegraphy, except that reaction can be used only to reduce the effective resistance of the circuit (Art. 187), and not pushed so far as to make it oscillate.

# CHAPTER XII

## ELECTRICITY IN GASES

**205. Discharges in rarefied gases.** The electric discharge in air and other gases is most conveniently studied at a pressure of 5 mm. of mercury or less, as the potential required to produce the discharge is then much less than at atmospheric pressure, unless the electrodes are very close together. One of the simplest methods of manipulating air at low pressures is shown in Fig. 282. It is assumed that dry air has first been admitted and then pumped out to a pressure of about 1 cm. of mercury with an ordinary air-pump. Further exhaustion can be obtained with a Töpler mercury pump. Suppose that the mercury reservoir $A$, connected to the upright glass tube $B$ by flexible rubber tubing, is first of all lowered until the mercury in the tube is at the level of $C$. The chamber $D$ is then in connexion with the main apparatus, but on raising the reservoir $A$ this connexion is cut off as soon as the mercury reaches the level of $E$.

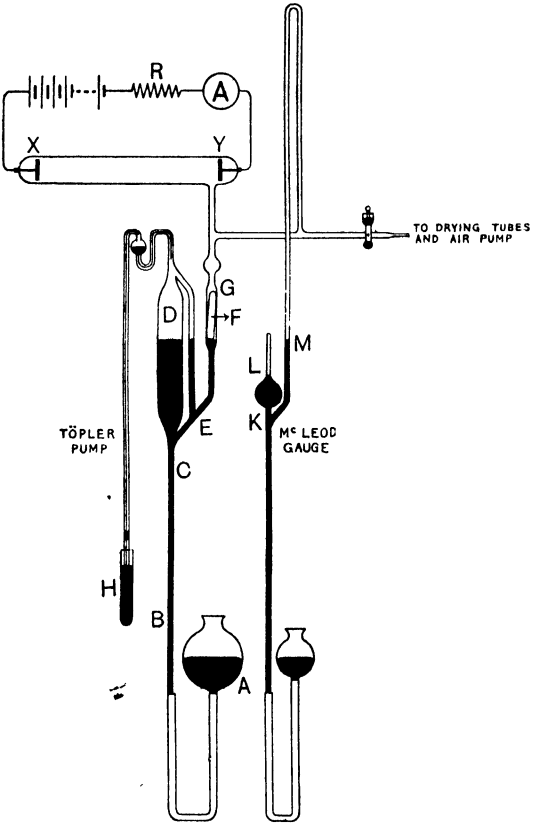


Fig. 282

A little higher up a glass float $F$ begins to be raised by the mercury, as shown in the figure, and is subsequently pressed against a constriction $G$ in the tube, which jams it tight. On further raising the reservoir the air in $D$ is forced down a long capillary tube

into a test tube $H$, where it escapes into the air. When the reservoir is lowered the space above the mercury in $D$ becomes a Torricellian vacuum, and by the time the mercury is again at $C$ air has diffused from the main apparatus into $D$, and the process can be repeated again and again. In this way it is an easy, though slow, process to reduce the pressure to $\frac{1}{100}$ mm. of mercury or less.

Pressures of the order of 1 mm. of mercury can be measured accurately with a McLeod gauge. The right-hand mercury reservoir in Fig. 282 is first kept low, and then raised until the mercury in the tube stands at the branching point $K$. At this point the pressure in both branches is the pressure $p$ in the main apparatus. As the mercury rises further, the air in the left-hand tube is compressed while the pressure on the right remains practically the same, so that the level in the two arms becomes different. Let $v$ be the volume above a fixed mark $L$ in the left-hand tube, $V$ the volume of bulb and tube down to the branching point $K$. Then when the mercury stands at $L$ the pressure in the left-hand tube is $pV/v$ and that in the right-hand tube $p$. Hence the difference of level of $L$ and $M$ is $p\left(\frac{V}{v} - 1\right)$ millimetres, where $p$ is the pressure in millimetres of mercury. The volumes $V$ and $v$ having been determined beforehand, a graduated scale can be attached to the tube $M$ giving the pressure in fractions of a millimetre. A ratio $V/v =$ about 100 is most generally convenient, giving a difference of level of about 10 cm. for every mm. pressure.

The discharge tube may be about 4 cm. in diameter and about 20 cm. long, containing two aluminium discs $X$, $Y$ (Fig. 282) to serve as electrodes. The necessary potential is furnished by about 500 small accumulators or a small high-voltage generator, in series with a resistance $R$ of the order of 50,000 ohms and a milliammeter $A$ reading up to about $\frac{1}{100}$ ampere. The discharge does not always begin when the battery is connected to the electrodes, since the potential required to start a discharge may be much greater than that required to maintain it. An induction coil is then used to start the discharge.

The appearance of the vacuum-tube discharge has several well-marked features, of which the most prominent are the following (cf. Fig. 284). The colours refer to air.

*A* a bright pink layer of gas at the surface of the anode, or positive electrode.

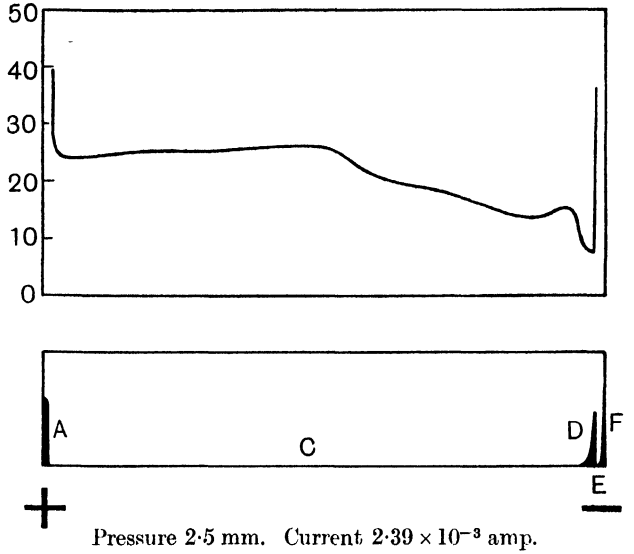
*B* the positive column, a column of luminous gas extending for a considerable distance *l* from the positive electrode and terminating abruptly at a certain distance *d* from the cathode. For a given current and pressure *d* is constant, and *l* increases with the distance between the electrodes.

*C* the Faraday dark space, merging into

*D* the negative glow (pale violet).

*E* the Hittorf (or Crookes) dark space.

*F* a red or orange glow adjacent to the cathode.

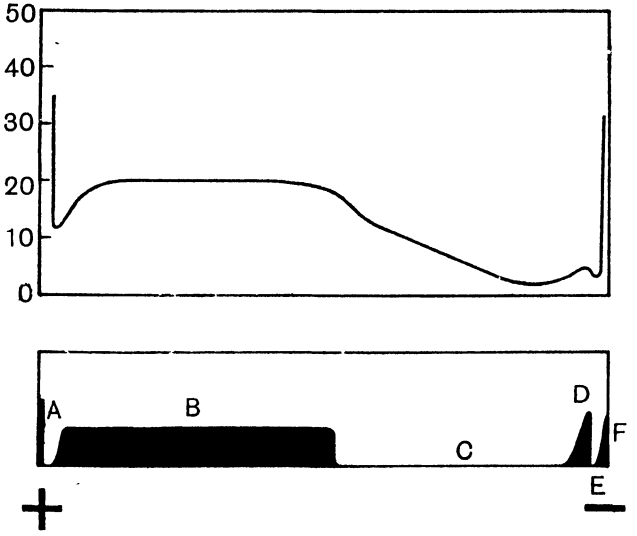



Pressure 2·5 mm. Current $2{\cdot}39 \times 10^{-3}$ amp.

Fig. 283

The four main stages of the discharge, as the pressure is reduced, are illustrated by Graham's experiments with nitrogen, which were made primarily in order to determine the electric force at different points of the gas. This was done by means of two wires placed close together along the axis of the tube. The upper diagrams in Figs. 283–6 give the electric force in volts per centimetre at any point, the lower showing the approximate distribution of the intensity of light along the tube. The tubes were about 20 cm. long.

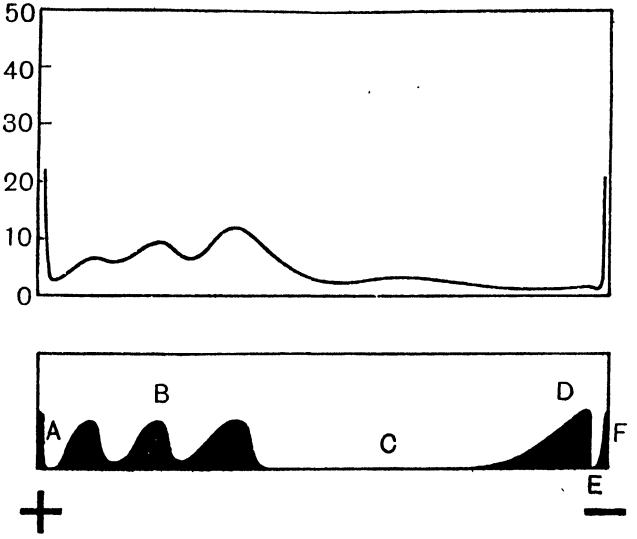
At the highest pressure (Fig. 283) the only light is near the electrodes, and the Hittorf dark space is very narrow. As the




Pressure 0·95 mm. Current $2{\cdot}46 \times 10^{-3}$ amp.

Fig. 284

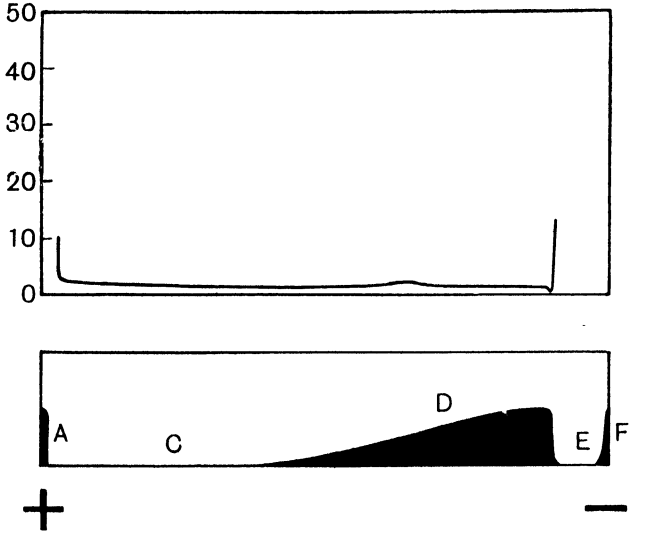
pressure is reduced (Fig. 284) the positive column appears, perfectly uniform except near the ends. Further rarefaction makes it break up into striations, or alternate bright and dark portions concave to the anode (Fig. 285), the process beginning at the end nearest




Pressure 0·63 mm. Current $1{\cdot}16 \times 10^{-3}$ amp.

Fig. 285

the cathode. The appearance in a long tube is very striking, as the additional length is all striated column and the rest of the tube is the same as before. The negative glow and Hittorf dark space expand continually as the pressure is reduced, until they push back the positive column and ultimately occupy the greater part of the tube (Fig. 286). The light in the tube becomes extremely faint and




Low pressure (probably about 0·02 mm.)
Current $3{\cdot}3 \times 10^{-4}$ amp.
Fig. 286

the electric force low, except near the electrodes. From this point the potential required to produce a discharge increases rapidly, and it becomes necessary to use an influence machine or induction coil instead of a battery of accumulators. As these low pressures are difficult to measure it is more generally convenient to specify the state of the tube by the potential required to produce the discharge.

Pressures of the order of $10^{-6}$ mm. of mercury, which are required for triodes and thermionic Röntgen ray bulbs (Art. 213) are obtained with Langmuir's mercury pump, after preliminary exhaustion to a few millimetres. In this pump the gas is carried away into an outlet chamber by a stream of vapour from boiling mercury, which is afterwards condensed and flows back into the heating chamber. The pump will exhaust a vessel of 10 litres capacity to $10^{-5}$ mm. pressure in about 5 minutes.

**206. Cathode rays.** The appearance of the discharge changes considerably when the pressure is reduced below the stage shown in Fig. 286. When the discharge potential exceeds 3000 volts, the cathode glow becomes concentrated more and more at the centre of the cathode, until finally a narrow pencil of light leaves the surface at right angles. Famous experiments on this subject were made by Hittorf in 1869, using as cathode a wire with a flat end,

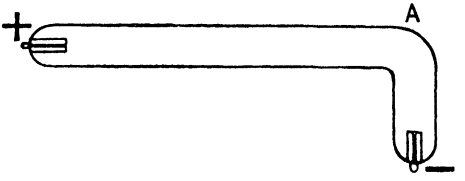



Fig. 287

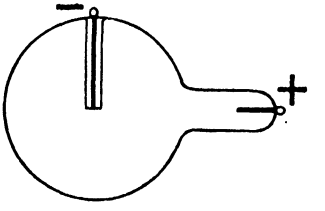


Fig. 288

enclosed in a tightly fitting glass tube, so that the available area is very small. With a tube of the form shown in Fig. 287 he found that the cathode rays, as they are called, did not bend round at the angle, but passed straight on and gave rise to a fluorescent spot on the glass near $A$. An obstacle in front of the cathode cast a shadow at $A$. In another experiment Hittorf used a tube like that shown in Fig. 288, and showed that the rays were deflected by a magnet, sometimes into circles but more generally into spirals.

Before explaining these results let us consider what would happen to a charged particle projected in a uniform magnetic field of strength $H$. We have shown (Art. 94) that it can reasonably be supposed to be acted on by a force equal to $\frac{e}{c} V\mathbf{vH}$, where $e$ is the charge in electrostatic units and $\mathbf{v}$ the velocity. Taking the direction of the field as axis of $z$, the equations of motion of the particle are

$$m\frac{d^2x}{dt^2} = \frac{eH}{c}\frac{dy}{dt}, \quad m\frac{d^2y}{dt^2} = -\frac{eH}{c}\frac{dx}{dt}, \quad m\frac{d^2z}{dt^2} = 0.$$

The last equation shows that the velocity along the axis of $z$ is constant, and the first two equations are satisfied by

$$\frac{dx}{dt} = A\sin(\omega t + \epsilon), \quad \frac{dy}{dt} = A\cos(\omega t + \epsilon),$$

where $\omega = He/mc$ and $A$, $\epsilon$ are constants. We may suppose, with-

out loss of generality, that the particle starts from the origin at time $t = 0$ with component velocities $v$, 0, $v'$. Thus $A = v$ and $\epsilon = \frac{1}{2}\pi$. Integrating, we have

$$x = \frac{v}{\omega}\sin\omega t, \quad y = -\frac{v}{\omega}(1 - \cos\omega t), \quad z = v't \ldots . (250).$$

The path in general is a helix, reducing to a circle if the initial velocity is at right angles to the magnetic field, and to a straight line if the two are parallel. The direction of description of the curves is right- or left-handed with reference to the axis of $z$ according as $\omega$ is negative or positive.

Hittorf's observations agree qualitatively with the theory, and he found that the spirals were always described in a right-handed direction with respect to the magnetic field. The explanation, which seems to have been first given by Varley in 1871, is that *the cathode rays are negatively charged particles emanating from the cathode, which, by the time they have travelled some distance from it, have acquired a considerable velocity.* Their origin and velocity of emission remain thus far unknown.

In 1895 Perrin proved independently that the cathode rays carry a negative charge. In his discharge tube the anode $A$ (Fig. 289),

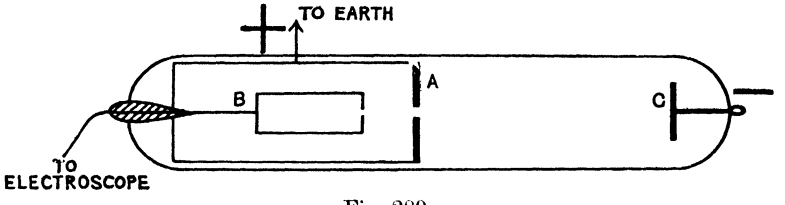



Fig. 289

which was connected to earth, was perforated so as to let the rays through, and nearly closed at the back in order to protect the space behind from electrostatic action. An insulated cylinder $B$ is placed behind the anode and connected to an electroscope. The cathode rays from $C$, after passing through $A$, enter $B$ through a small hole and charge the electroscope negatively. A single interruption of the coil gave enough charge to raise the potential of the insulated system to 300 volts. A magnet near the tube deflected the cathode stream, without stopping the discharge, until no electricity entered $B$.

**207. Measurement of e/m for the cathode particles. Electrons.** The path of a particle moving initially with velocity $v$ along the axis of $x$ is a circle of radius

$$\rho = \frac{v}{\omega} = \frac{mcv}{He}.$$

This follows at once from equation (250), or more simply as follows. Since the force on the moving charge is always at right angles to its path $Hev/c = mv^2/\rho$. Hence

$$\frac{e}{m} = \frac{cv}{\rho H} \quad \ldots\ldots\ldots\ldots\ldots\ldots (251).$$

It follows that if the velocity of the cathode particle is measured in any way the ratio $e/m$ of its charge to its mass is also known, since $\rho$ may be found from the magnetic deflexion produced by a known force $H$. This was first done by Wiechert in some experiments described in 1897. The cathode rays, after passing the ring-shaped anode $A$ (Fig. 290), fell on a sheet of glass $G$ placed behind

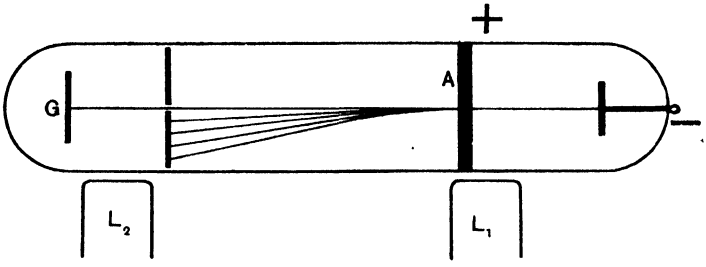



Fig. 290

a diaphragm, forming a fluorescent spot. They were deflected near the anode by a loop of wire $L_1$ in an electric oscillation circuit, so that they were broadened out into a fan-shaped beam. The beam could be deflected further by a magnet, and its edge brought precisely on the screen once more. A second loop of wire $L_2$ in the electric oscillation circuit was then brought near the beam between the screen and diaphragm, and the additional deflexion observed on the screen.

In order to see what is to be expected, let the current in the oscillatory circuit at time $t$ be $A \sin pt$, where $p/2\pi$ is the frequency of the circuit, and let $l$ be the distance between the loops $L_1$ and $L_2$. The only rays that get through the diaphragm are those which

are most deflected by the first loop; those, say, which passed it at the time $t$ given by $pt = \frac{1}{2}\pi$, etc. The first ray passes the loop $L_2$ at time $t = \frac{\pi}{2p} + \frac{l}{v}$, by which time the current in the loop has become $A \sin p\left(\frac{\pi}{2p} + \frac{l}{v}\right) = A \cos \frac{pl}{v}$. Similarly for all the other beams. It follows that the deflexion produced by the loop $L_2$ will be in the same direction as that produced by $L_1$, as long as $pl/v$ is less than $\frac{1}{2}\pi$. With $l = 20$ cm., Wiechert found that this was the case for frequencies up to $2 \times 10^7$, showing that $v$ was greater than $1{\cdot}6 \times 10^9$ cm./sec. For these rays $\rho H$ was equal to 150. Equation (251) then shows that

$$\frac{e}{m} > 3{\cdot}2 \times 10^{17}.$$

With the frequency $4 \times 10^7$ the deflexion seemed to be much reduced, and Wiechert concluded that the value of $e/m$ did not much exceed $5 \times 10^{17}$.

Wiechert's experiments lead to one extremely important conclusion. We have seen (Art. 167) that if $e$ is the charge on a univalent ion in electrolysis (the smallest charge known in electrolysis) and $m_H$ the mass of a hydrogen atom, then

$$\frac{e}{m_H} = 2{\cdot}89 \times 10^{14}.$$

The value of $e/m$ for cathode particles is about a thousand times as great. Among the possible ways of explaining this result, two limiting hypotheses are most probable:

(1) The charge on the cathode particle is the same as that on a univalent ion in electrolysis, and its mass very much smaller.

(2) The mass of the cathode particle is comparable with that of an atom, and its charge about a thousand times as large as that on an ion.

Wiechert adopted the first hypothesis, and was thus led to assert, for the first time, the existence of *particles smaller than the atom.* Evidence as to the equality of the charges in the two cases is given in Art. 218.

Shortly after Wiechert's discovery, Kaufmann made a more accurate determination of the value of $e/m$ for the cathode particles by a method first tried by Schuster in 1890. The cathode $C$ in his

discharge tube (Fig. 291) was a plate of aluminium or copper, the anode $A$ a platinum wire connected to earth. There was a large space behind the anode, screened electrostatically and enclosed

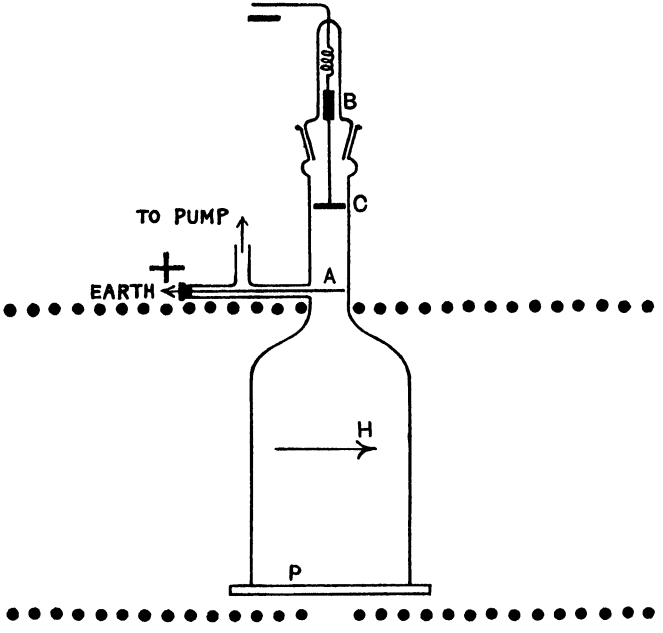



Fig. 291

between two solenoids giving an approximately uniform field $H$ over a considerable length. The tube was closed by a glass plate $P$ which fluoresced under the action of the rays, the anode $A$ casting a sharp shadow on it. The position of the cathode could be varied within certain limits by moving the iron cylinder $B$ up and down with a magnet. A Wimshurst machine was used to produce the discharge, and the difference of potential $V$ between $A$ and $C$ was measured with an electrostatic voltmeter.

Suppose for simplicity that the magnetic field is uniform and the deflexion small. On entering the field the particle has moved through a potential $V$, so that its velocity $v$ is given by

$$\tfrac{1}{2}m\,(v^2 - v_0^2) = eV,$$

where $v_0$ is the velocity on leaving the cathode. Kaufmann assumed provisionally that $v_0$ was negligible in comparison with $v$. Thus

$$mv^2 = 2eV \quad \ldots\ldots\ldots\ldots\ldots\ldots (252),$$

where $V$ is supposed to be measured in electrostatic units. If $l$ is the distance travelled in the magnetic field before reaching the plate and $\delta$ the deflexion, regardless of sign, equation (250) gives approximately

$$l = \frac{v}{\omega} \cdot \omega t, \quad \delta = \frac{v}{\omega} \cdot \frac{\omega^2 t^2}{2}.$$

Eliminating $t$ and remembering that $\omega = He/mc$, we have

$$\delta = \frac{He}{mc} \frac{l^2}{2v} = \frac{Hl^2}{2c} \left(\frac{e}{2mV}\right)^{\frac{1}{2}},$$

or

$$\frac{V^{\frac{1}{2}}\delta}{H} = \frac{l^2}{2c} \left(\frac{e}{2m}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots (253).$$

This equation may be tested by seeing whether $V^{\frac{1}{2}}\delta/i$ remains constant under various conditions, $i$ being the current in the solenoids. The following results were found by Kaufmann.

| Gases | Cathode | Voltage | Distance between electrodes | $\frac{V^{\frac{1}{2}}\delta}{i}$ (mean) |
|---|---|---|---|---|
| Air .. .. | Aluminium | 5920–11600 | 6 | 396 |
| Air .. .. | Copper | 4350–10630 | 6 | 393 |
| Air .. .. | Copper | — | 3·95–6 | 406 |
| Coal-gas .. | Copper | 6410–11850 | 5·8 | 401·5 |
| Carbon dioxide | Copper | 3870–11100 | 5·8 | 398 |
| Hydrogen .. | Copper | 4230–14400 | 5·8 | 404 |

The possibility of appreciable initial velocity is therefore excluded, and we also see that $e/m$ is *independent of the nature of the gas in the tube and of the material of the cathode.*

Kaufmann found it to be approximately $5 \times 10^{17}$, and Sir J. J. Thomson found a somewhat smaller value shortly afterwards. Kaufmann subsequently allowed for the fact that his field was not quite uniform, and found $e/m = 5{\cdot}31 \times 10^{17}$. In what follows we shall adopt the value

$$\frac{e}{m} = 5{\cdot}30 \times 10^{17} \quad \ldots\ldots\ldots\ldots\ldots (254),$$

which is the mean of the best measurements. Equation (252) now shows that the velocity acquired by the particle in moving through a potential difference of $V$ volts is

$$v = 5{\cdot}94 \times 10^7 V^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots (255).$$

The corrections which have to be made to formulae (253) and (255) for velocities near that of light are given in Art. 273.

Since the cathode particles are the same whatever their origin, we have evidence of a new fundamental unit or carrier of negative electricity, disproportionately light in comparison with its charge. These particles are called *electrons*. Schuster, in the experiments referred to above, might have discovered them, but concluded (partly from using too high a pressure) that the mass of the cathode particle was of the order of that of a molecule.

The part played by electrons in physical phenomena will become more and more clear as we go on, but as far as we know the ratio of their charge to their mass is always that of equation (254), from which we conclude that all electrons are the same. Every atom contains a number of electrons on which its physical and chemical properties depend (Arts. 223, 276). The absence of electrolysis in metals is good reason for supposing, as we have done in the elementary parts of the subject, that conduction in metals is due to a transport of electrons. Experiments described in Arts. 211, 212 and 258 confirm this view.

The small inertia of the cathode stream enables it to follow a change in the applied magnetic field almost instantaneously. This is utilised in the *cathode ray tube*, the quickest of oscillographs, in which cathode rays pass through a diaphragm and fall on a fluorescent screen of willemite or barium platinocyanide, producing a bright spot (cf. Fig. 297). An alternating current in a coil placed near the tube, as in Fig. 290, makes the spot of light move rapidly to and fro, and the form of the current curve can be traced on a plate moving at right angles to the direction of vibration.

**208. The photoelectric effect.** Hertz showed in 1887 that ultra-violet light facilitated the passage of a spark between metallic electrodes when it fell on the negative electrode. Following on this, Hallwachs was able to show that negatively charged metals lost their charge on exposure to ultra-violet light, but that positively charged bodies did not. The experiment can be made most simply with an electroscope and an arc lamp.

The discharge current is too small to affect a galvanometer, but can be measured with an electrometer connected as in Fig. 292.

The source of light is a zinc or aluminium spark-gap $S$ in a condenser circuit, worked from a fairly large induction coil. The whole is placed in an earthed metal box $L$ in order to guard against electrostatic influence. The rays emerge through a hole in the lid, and after passing through a wire grating in the plate $C_2$ fall on the zinc plate $C_1$. The plate $C_1$ is connected to one pair of quadrants of an electrometer, the other pair of which is earthed, and $C_2$ to the positive pole of a battery of small accumulators, of the type shown in Fig. 72. An experiment consists in insulating the electrometer, putting on the light for a measured time, and observing the difference between the initial and final steady positions of the spot of light. If the mean capacity of the insulated system is (say) 50 cm., and 200 mm. deflexion is obtained in 10 seconds, 80 mm. corresponding to 1 volt, the current is

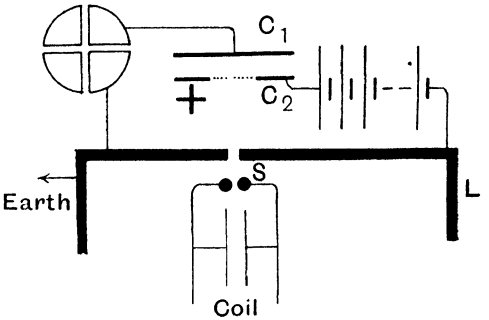


Fig. 292

$$\tfrac{1}{24} \text{ E.S.U.} = 1{\cdot}4 \times 10^{-11} \text{ amp.},$$

so that currents as low as $10^{-12}$ amp. can be measured without difficulty.

There are two distinct ways of showing that the leak is caused by ultra-violet light. The most direct is to mount the parallel-plate condenser $C_1C_2$ on the moveable arm of a spectrometer and to analyse the light from the spark-gap with a quartz prism. As the spectrometer arm is moved round in the direction of increasing deviation (so that red, blue, violet and ultra-violet light fall on the aperture in succession) no deflexion is observed until the visible spectrum is passed.

Another way is to observe the effect of placing various substances in the path of the light. Thus a glass plate laid over the aperture in Fig. 292 entirely suppresses the leak, while a quartz plate hardly reduces it at all; and generally all substances known to absorb ultra-violet light cut down the photoelectric effect, while substances which are transparent to it do not.

The order of sensitiveness of the common metals, for the ultra-violet light of medium wave-length transmitted by quartz, is somewhat as follows:

Aluminium, zinc, magnesium, lead, copper, silver, iron.

Much depends on the state of the surface. A zinc plate, for example, is most sensitive when freshly cleaned, but the deflexions fall off quickly at first, and finally become steady, though considerably smaller.

The leak from a given substance ceases at a well-defined wave-length, above which no intensity, however great, can discharge electricity. Barium, calcium, aluminium and magnesium are sensitive to yellow and sometimes to infra-red rays. Zinc has a long-wave limit of 3800 A.U., cadmium 3200, and mercury 2500 or thereabouts; but the limits, like the rate of leak, depend greatly on the state of the surface. Sodium also is sensitive to visible light, and can be used in photometry, since Elster and Geitel showed that the photoelectric current from it is proportional to the intensity of the light over a large range.

Nearly all substances are photoelectric: solids, liquids and, as Lenard first showed, gases. The volume effect in gases is very small in comparison with the effect on the metallic electrodes, if the light falls on them. It has, however, a special character in that an electron ejected from a gaseous molecule leaves it positively charged, so that charges of both signs are present in the gas. Lenard and Ramsauer found a considerable volume effect with short ultra-violet of wave-length $10^{-5}$ cm. Longer-waved light, such as is transmitted by quartz ($\lambda > 1{\cdot}8 \times 10^{-5}$ cm.), has no effect on pure gases, but will act on impurities such as condensible vapours or the emanations from rubber tubing. The volume effect is analogous to the ionisation of gases by Röntgen rays (Art. 215).

**209. Measurement of small currents in gases.** Fig. 293 shows an apparatus for measuring small currents in gases, whether produced by Röntgen rays, ultra-violet light or radioactive substances. It is here adapted for measuring photoelectric currents. A perforated brass plate $Q$ is fixed on a metal tripod, of which two legs are shown, which rests on the earthed metal box $L$. Three

vertical brass rods support a parallel plate $P$, the whole frame being thus kept automatically at zero potential. The method of supporting and insulating the plates $C_1$, $C_2$ will be evident from the figure.

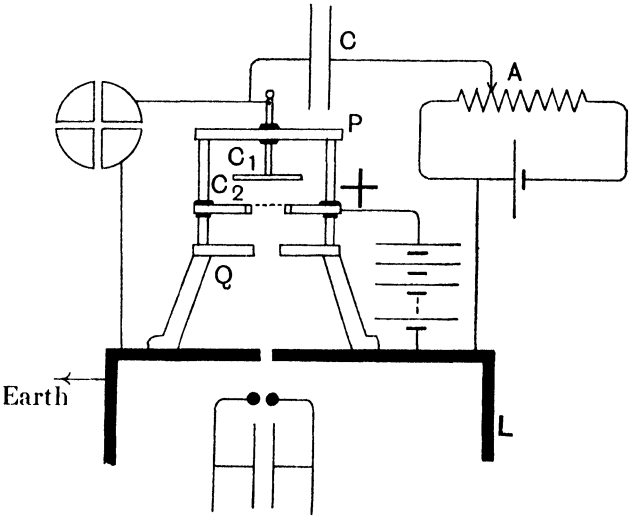



Fig. 293

The ultra-violet light, as in Fig. 292, passes through a wire grating in $C_2$ and falls on the zinc plate $C_1$. However great the difference of potential between $C_1$ and $C_2$ may be, the insulation of $C_1$ has only to withstand the potential to which $C_1$ may rise, which is of the order of 1 volt. This is the important principle of split insulation: direct support of $C_1$ from $C_2$ may introduce a leak sufficient to vitiate the experiment. One plate of a condenser $C$ is connected with the electrometer and the plate $C_1$, the other to a moveable point on a sliding potentiometer (Fig. 87), having 50 junctions and a total resistance of about 1000 ohms. To fix ideas, suppose that a positive charge on the electrometer sends the spot of light to the right. The photoelectric current then tends to send the spot to the right, and moving $A$ from left to right tends to send it to the left. The slider, which is first kept to the left, is moved along while the current is flowing so as to keep the spot permanently near zero, the position for a true zero being found by proportional parts between two positions giving small positive and negative deflexions, after the light has been cut off. Let $V$ be the corresponding

potential of $A$. Since neither the electrometer nor the zinc plate has received any access of charge, the whole of the electricity passing over from $C_1$ to $C_2$ has come out of the condenser and is of amount $CV$, where $C$ is its capacity. The current is therefore $CV/t$, where $t$ is the time during which the light is on. If $C$ is in cm. and $V$ in volts, $i = CV/9 \times 10^{11}\ t$ amperes. The method is independent of the variation of the effective capacity of the electrometer with deflexion, and of any change of capacity caused by altering the distance between $C_1$ and $C_2$.

Fig. 294 shows the way in which the photoelectric current, at atmospheric pressure, varies with the electric force between the

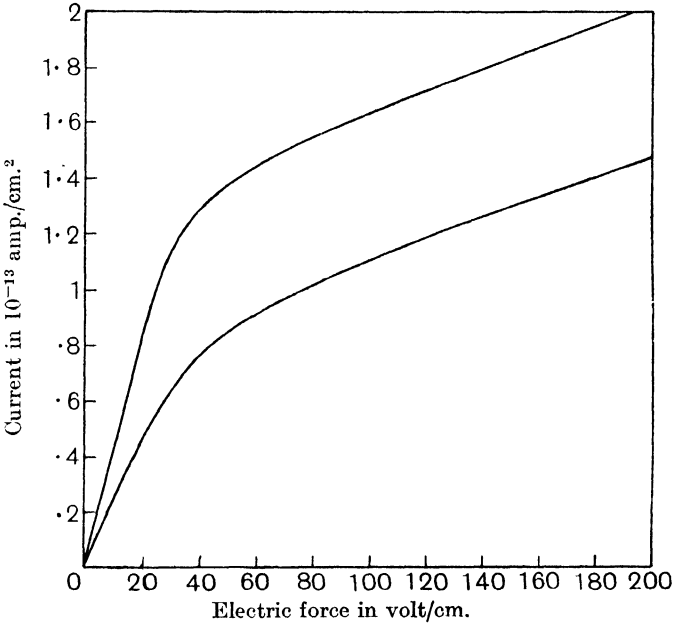



Fig. 294

plates. The two curves correspond to light of different intensity. These curves, like the corresponding curves obtained with Röntgen rays, are called *saturation curves* from their external resemblance to the magnetisation curve of iron. The current is at first nearly proportional to the electric force, then rises more slowly, and finally approaches saturation at forces of the order of 10,000 volt/cm., too high to show in the figure. The curves seem to owe their shape to two causes. We shall see that an electron, moving slowly through

a gas, attracts to itself a cluster of molecules and becomes a heavier charged particle, or *ion* (Art. 221). At low forces ions diffuse back to the negative electrode and are discharged by it, so that the current may be much less than the charge emitted by the electrode in one second. The action of electric force in overcoming diffusion would account for the bend in the curve: the subsequent rise seems to show that the emission itself may increase slowly with the force.

**210. Nature of the photoelectric effect.** The presence of air is not necessary to the photoelectric effect, which occurs in the highest attainable vacua. Lenard showed in 1899 that the photoelectric effect *in vacuo* consists in the emission of electrons from the negative electrode. His apparatus is shown in Fig. 295. Light

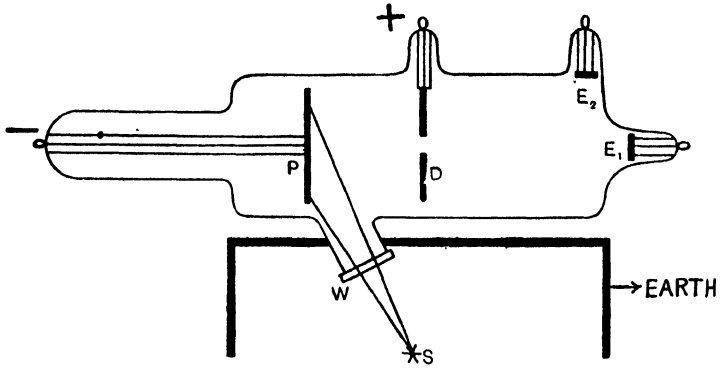



Fig. 295

from the spark-gap $S$ passes through the quartz plate $W$ and enters a highly exhausted glass vessel, falling on the aluminium plate $P$. The aluminium disc $D$ opposite to it is earthed, and the right-hand part of the tube is covered with tinfoil so as to form a chamber practically at zero potential. $E_1$, $E_2$ are two insulated electrodes which can be connected to an electrometer, and $P$ is kept at a negative potential of about 600 volts. A stream of negative electricity passes through the aperture in $D$ and falls mainly on the electrode $E_1$. A magnetic field is then applied perpendicular to the plane of the paper in the space $DE_1E_2$. Two curves are taken off showing the electrometer deflexions at $E_1$ and $E_2$ for various currents in the coils producing the magnetic field. The difference in the currents for maximum deflexion is that required to deflect the stream ions from

$E_1$ to $E_2$; and knowing also the potential of the plate $P$, $e/m$ and $v$ can be found as in Kaufmann's experiment.

Lenard found that $e/m = 3{\cdot}5 \times 10^{17}$. Accurate experiments by Alberti gave the value $e/m = 5{\cdot}28 \times 10^{17}$, so that there is no doubt the particles are electrons. The small velocity with which they leave the plate can be allowed for by working with two different voltages. The velocity of emission is, however, itself of great importance. Lenard estimated it by insulating the plate and observing the positive potential to which it rises when illuminated. If this is $V$, and we neglect surface effects for the moment, $\frac{1}{2}mv^2$ must be equal to $eV$; for as long as the kinetic energy exceeds $eV$ some electrons will arrive on $D$ and raise the potential of $P$ further. $V$ is about 2·1 volts for aluminium, corresponding to a velocity of emission of $8{\cdot}6 \times 10^7$ cm./sec. Lenard found that the velocity is *independent of the intensity of the light*, but *increases with its frequency*, being greatest for light of short wave-length.

The importance of these observations was first shown by Einstein in 1905. The maximum velocity is that of electrons which have taken up, in some way or other, the maximum amount of energy from the incident light, and lost the minimum amount in escaping from the metal. Lenard's experiments show that both these quantities are sharply defined. Einstein perceived in them a confirmation of Planck's views on radiation, and assumed that an electron can extract from a wave of light either no energy or an amount depending only on the frequency of the light, and increasing with it. Suppose that the plate $P$ in Fig. 295 is joined to the *positive* pole of a battery of E.M.F. $V$, the negative pole of which is earthed. If $e$ is the numerical value of the charge on an electron, energy $e\,(V + C)$ will be abstracted from an electron which passes from a point just inside $P$ to a point just inside $D$, where $C$ is a constant depending on the contact potentials. Let the energy acquired from the incident light be $eV_0$. Then the electron will just pass from $P$ to $D$ if $eV_0 = e\,(V + C)$, or $V = V_0 - C$. By measuring the potential $V$ at which the current just ceases, therefore, we measure $V_0 - C$; the change of $V_0$ with frequency is equal to the measured change of $V$.

Two main precautions are required in determining the critical potential $V$. The illuminated surface should be fresh, by distillation

or fresh cutting *in vacuo*, in order that the quantity $C$ should remain steady. The light should consist of a homogeneous beam, free from stray light of shorter wave-length; for such light tends to set up a positive potential corresponding to its own wave-length, and to do so it need only produce a current of (say) ten times the leak from the supports of the plate. In Millikan's experiments the source of light was a quartz mercury lamp, from which six homogeneous beams were isolated in succession. The pressure was as low as possible. The current was measured with a sensitive quadrant electrometer. With sodium (Fig. 296) increase of $V$ causes a sharp

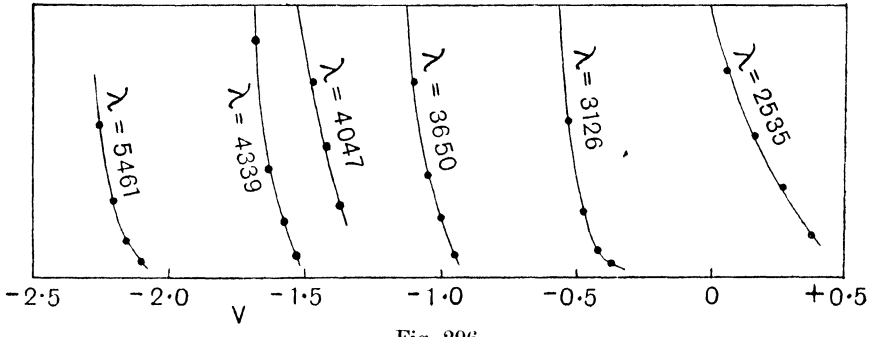



Fig. 296

fall of current towards zero, the currents shown being themselves only of the order of 1 per cent. of the saturation current, so that we are dealing already with only a small proportion of the fastest electrons. The points where the curves, if produced, meet the line of zero current can be estimated with fair accuracy, except at $\lambda = 3126$, where the effect is probably due to stray light. Millikan found that he could eliminate a residual bend of this kind by interposing a liquid filter; but obtained nearly the same critical potentials, with and without filter, by producing the steep parts of the curves downwards. The critical voltages, plotted against the frequency $\nu$ of the light, lie on a straight line, showing that $V_0$ is a linear function of $\nu$ over the range $\nu = 0{\cdot}55 \times 10^{15}$ to $1{\cdot}2 \times 10^{15}$. If we assume with Einstein that $V_0$ vanishes with $\nu$, Millikan's experiments show that $V_0 = 4{\cdot}12 \times 10^{-15}\,\nu$ over this range. The energy taken up by an electron from light of frequency $\nu$ is therefore $eV_0/300$ ergs $= h\nu$ ergs, where $h = 4{\cdot}12 \times 10^{-15}\,e/300$, or

$$h/e = 1{\cdot}37 \times 10^{-17} \quad \ldots\ldots\ldots\ldots (256),$$

$e$ being in electrostatic units. Experiments on lithium gave $1{\cdot}38 \times 10^{-17}$. We have here given part of the experimental evidence for the *theory of quanta*, according to which homogeneous light of frequency $\nu$ is associated with an indivisible amount of energy $h\nu$, where $h$ is a universal constant called *Planck's constant*. Further evidence will be given in Arts. 237, 239, 241 and the theory developed in Arts. 260 *et seq.*

To recapitulate, the long-wave limit of the photoelectric effect is, on Einstein's theory, a consequence of the quantum of energy being less than that necessary to expel an electron from the metal (more accurately to bring it on to a neighbouring electrode). When the surface of a metal changes, the energy lost in escaping, when measured in volts, changes by an amount to the charge of contact potential.

The photoelectric effect in gases is theoretically simpler than in solids, since there is no surface effect and the energy corresponding to the long-wave limit is that required to remove an electron from the molecule. Taking the limit as $1{\cdot}4 \times 10^{-5}$ cm. for air (Art. 208), the energy is that due to a fall of potential of 10 volts. Collision experiments (Art. 224) give results of the same order.

**211. Emission of electrons by hot metals.** The application to the study of electric oscillations has been described in Arts. 184–5 and elsewhere in Ch. XI. We are now concerned with its physical nature. Many old experiments show that both positive and negative electricity can escape from hot metals, according to the temperature, the pressure of the gas and the state of the surface. The pure effect is observed only in a high vacuum. The unilateral nature of the discharge in high vacua was first clearly proved by Elster and Geitel in 1885: negative electricity only escapes from metals with any persistence, the emission of positive electricity decaying in time so as to suggest that it is due to occluded salts or gases which are gradually driven off. Very little negative electricity escapes at temperatures below 500° C., but the current rises rapidly with increase of temperature, and at 1200° C. an ordinary wire will give currents of the order of $10^{-6}$ ampere. Occluded gases affect the negative leak so much that all parts of a triode or thermionic Röntgen ray bulb must be freed from them as far as possible. To

this end the whole tube is heated till the glass is on the point of softening, preferably in a vacuum furnace, with the interior connected constantly to a Langmuir pump (Art. 205). The metal parts are heated before assembly, and the filament kept hot while pumping is taking place, a high potential being applied to the grid and plate in succession in order that electronic bombardment (or the accompanying rise of temperature) may liberate gas from them. The pressure of the residual gas should not exceed $10^{-6}$ mm. of mercury.

The value of $e/m$ for the carriers of the current was first measured by Sir J. J. Thomson in 1899 and found to be $2{\cdot}6 \times 10^{17}$, which, though low, is sufficient to show that they are electrons. The method of magnetic deflexion and potential-difference was applied elegantly by Classen, who used as anode a box with a small hole near the filament, the electrons being subsequently turned through a complete semicircle by the magnetic field with a view to collecting them at one point (cf. Art. 241). He found the value $5{\cdot}32 \times 10^{17}$. A confirmation based on different principles is described in Art. 212, and a less accurate one in Art. 258.

Wehnelt found that the emission of electrons was much increased by coating the metal with calcium oxide. The tube shown in Fig. 297 is convenient for demonstrating the properties of moving

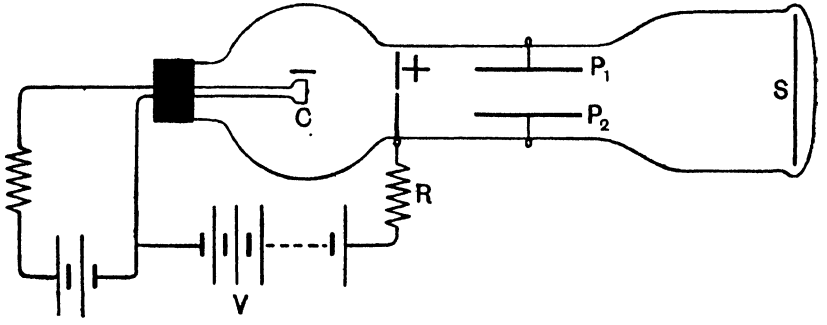



Fig. 297

electrons. There is an important difference between this and a cathode ray tube, in that the supply of electrons has no longer to be maintained by the impact of swift positive ions on the cathode. The high potential required for the discharge at low pressures is no longer necessary, and a battery of 250 small accumulators

suffices for most purposes. The Wehnelt cathode $C$ is a strip of platinum heated to redness by means of a subsidiary battery, and carrying a small spot of calcium oxide (better, a mixture of barium and strontium oxides). The stream of electrons passes through a small hole in the centre of the anode and produces a blue glow in the slight amount of gas remaining in the tube, which marks its path clearly. An electric deflexion may be obtained by means of the small condenser $P_1P_2$, a magnetic deflexion by a suitably placed coil, or pair of coils. Transparent squared paper or fabric is used as backing for the fluorescent screen $S$, so that the movement of the spot of light may be followed. In this way it is easy to make a rough measurement of $e/m$. The main circuit must contain a resistance $R$ of the order of 50,000 ohms, otherwise an excessive current may flow.

The facts just described give some insight into the mechanism of the electric arc. The essential part of a carbon arc is the incandescent negative electrode, emitting streams of electrons which fall on the positive electrode. The latter may be either hot or cold (Art. 162), but no arc can be formed with a cold cathode. The difference between an arc and a Wehnelt cathode discharge is that in the latter case the cathode is heated by an external agency, and in the former case by the impact of positive ions from the gas or positive electrode. The transition from one type of discharge to the other is well illustrated by an experiment described by Sir J. J. Thomson. If the resistance $R$ in Fig. 297 is not too large, and the electromotive force $V$ increased sufficiently, the cathode becomes hotter by the impact of the residual positive ions in the gas, and the current rises from a small value to an ampere or so. At this stage the current used to heat the cathode may be cut off and the discharge current will maintain itself and keep the cathode hot.

**212. Variation of thermionic current with potential.** Fig. 298, from an experiment by Langmuir, shows the current between a tungsten filament and a coaxial cylindrical sheath *in vacuo* for different applied potentials, measured from the negative end of the filament. Three main stages can be distinguished. No observable current (on the scale of the figure) flows while the

potential is negative. For positive potentials the current begins by being the same for all temperatures of the wire, and follows this common curve until it is nearly saturated. The final current is equal to the total charge emitted by the filament per second, and increases rapidly with the temperature, doubling itself for every increase of 80° or so in the neighbourhood of 2000° C.

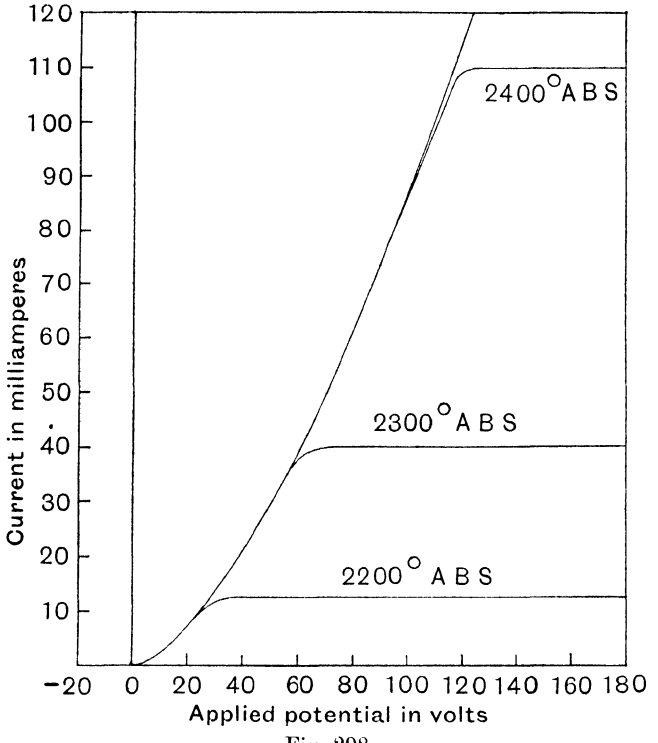



Fig. 298

In the first stage, current is first detected with a sensitive galvanometer at about − 1 volt, showing that few electrons are emitted with an energy exceeding one volt. In the second stage, when unsaturated currents of the order of $10^{-2}$ ampere are developed, an unexpected phenomenon appears, which Langmuir explained by the mutual repulsion of the electrons which accumulate between the filament and sheath. Let $V$ be the potential (in electrostatic units) at a point distant $r$ from the axis, $n$ the number of electrons per cm.$^3$ there, and $v$ the velocity of each. Then

$$\tfrac{1}{2}mv^2 = eV,$$

where $-e$ is the charge on an electron, $m$ its mass, and the small velocity of emission is neglected. The current radially outwards per cm. of the filament is

$$i = 2\pi rnev/c.$$

Poisson's equation becomes, from equation (19) of Art. 10,

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = 4\pi ne.$$

Eliminating $n$ and $v$ we have

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = \frac{ic}{r}\left(\frac{2m}{eV}\right)^{\frac{1}{2}}.$$

The last equation does not appear to be soluble in finite terms, so we seek a special solution and see whether it satisfies the conditions of the problem sufficiently accurately. Try $V = kr^a$, where $k$ and $a$ are constants. The differential equation is satisfied if $a = \frac{2}{3}$ and $k^{\frac{3}{2}} = \frac{9}{4}ic\,(2m/e)^{\frac{1}{2}}$. Hence a possible solution is contained in the equation

$$V^{\frac{3}{2}} = \tfrac{9}{4}\,irc\left(\frac{2m}{e}\right)^{\frac{1}{2}}.$$

The solution makes $V$ and $v$ small when $r$ is small, though neither vanishes for any finite value of $r$. Hence it gives nearly correct potentials everywhere in the case of a thin filament emitting electrons with a small velocity, as in Langmuir's experiments. Solving for $i$ and replacing $r$ by the radius $a$ of the cylindrical sheath, we have

$$i = \frac{2\sqrt{2}}{9c}\left(\frac{e}{m}\right)^{\frac{1}{2}}\frac{V^{\frac{3}{2}}}{a} \quad \ldots\ldots\ldots\ldots (257),$$

where $V$ is the (positive) potential applied to the sheath. This is Langmuir's formula, the theory for a hot plate and a parallel plane anode having been given previously by Child.

Langmuir's formula applies as long as $i < Ne/c$, where $N$ is the number of electrons emitted by unit length of the filament. Below this point some electrons (mainly those with the smallest velocity of emission) are forced back on the filament and others escape, giving the current (257), independent of $N$, as is found by experiment. Above this point all escape, and we have the saturation current $Ne/c$. The form of the $(i, V)$ curve will be influenced slightly by finite velocity of emission and more by cooling at the ends of the filament and by the fall of potential down it—effects which can

be reduced by adding guard-rings to the anode. With these precautions Langmuir's formula is confirmed by experiment, and gives a fresh proof that the current is carried by electrons. Dushman found $e/m = 5{\cdot}33$, $5{\cdot}30$ and $5{\cdot}26 \times 10^{17}$ from three experiments with a tungsten filament; Stoekle found $e/m = 5{\cdot}35$ and $5{\cdot}19 \times 10^{17}$ with a molybdenum filament.

The theory has a certain application to the characteristics of triodes (Fig. 255). Let the filament, grid and plate be at potentials 0, $G$, $P$ respectively. With the notation of Art. 45 the charge on the filament, when cold, is $q_{12}G + q_{13}P$. If the filament is thin, $q_{12}G + q_{13}P$ will also determine the whole distribution of potential round it, and hence the current leaving it when it is raised to an assigned high temperature. The filament current should therefore be a function of $q_{12}G + q_{13}P$ only. This relation holds very fairly for the filament current (the sum of the grid and plate currents) in Fig. 255, the derived curves for different values of $P$ being obtained from each other by simple translation parallel to the axis of $G$. The curves lie closer and closer together the smaller $q_{13}$ is, that is the better the electrostatic screening of the filament by the grid. Since the grid current is relatively small we have a theoretical explanation of the fact mentioned in Art. 184, that the amplification ratio (approximately $q_{12}/q_{13}$) is greatest for thick grid wires close together. The fall in the plate current when $P = 80$ and $G > 60$ is due to secondary electrons which reach the grid and increase the grid current at the expense of the plate current (cf. Art. 227). The grid emits secondary electrons also, but mostly on the side nearest the filament, so that they are dragged back on to it: few reach the plate and none reach the filament. The curve $P = 160$ is more normal since the secondary electrons emitted by the plate now return to it: it would fall in the same way for sufficiently high grid potentials.

**213. Röntgen rays.** We shall now give a brief account of the properties of the Röntgen rays, which, as is well known, were first produced in a discharge tube at very low pressures. The usual form of gas-filled Röntgen ray bulb is shown in Fig. 299. The cathode is of aluminium, concave in order to focus the cathode rays on the metal plate $T$, known as the *target*. This also acts as anode, but it

is found that the discharge is steadied by a second anode of aluminium on the axis of the tube. The target is the source of the Röntgen rays. It is made of a metal of high atomic weight,

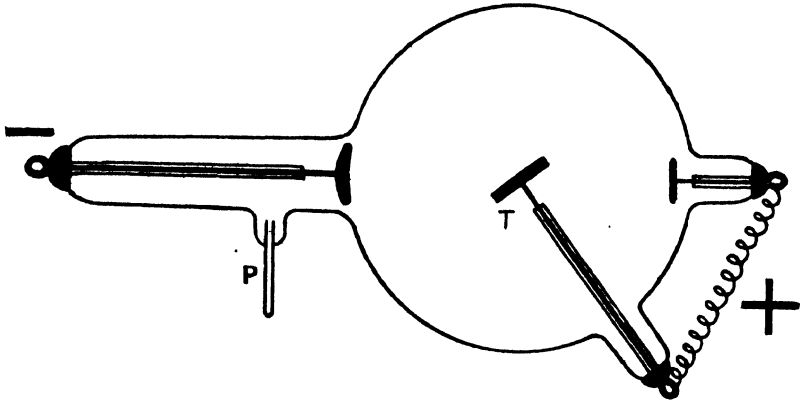



Fig. 299

such as platinum, rhodium or tungsten. Since cathode rays have a very considerable heating effect the melting point must be high, and bulbs designed for heavy discharges ($10^{-3}$ ampere or more) carry cooling vanes. The focussing and heating effects were observed by Crookes before the discovery of Röntgen rays. Röntgen ray bulbs become exhausted in time, from occlusion of gas by the walls of the bulb. More gas can be admitted at any time by means of a palladium tube $P$ closed at one end and sealed into the glass. Such a tube is gas-tight when cold, but lets hydrogen through freely when hot, so that, when a Bunsen flame plays gently on the tube, free hydrogen (which always exists in the flame) passes into the bulb. The pressure in the bulb determines the discharge potential and (indirectly) the quality of the rays. If it is high (discharge potential 20,000 volts or less) the rays are *soft*; that is, they are easily absorbed by matter. If it is low the rays are *hard* or penetrating; but below a certain point it becomes difficult to get the discharge to pass through the bulb at all.

Coolidge's Röntgen ray bulb (Fig. 300) contains no gas, the same precautions being taken to remove it as in a triode. Electrons are supplied by a flat tungsten spiral shown diagrammatically at $S$, heated with an 8-volt battery $B$. The spiral is surrounded by a short molybdenum tube $M$, which is found to focus the stream. The

target $T$, which is also the anode, is solid tungsten. This is a very powerful bulb. When run at full power, the impact of electrons makes the target red-hot in 10 or 20 seconds. It can be kept safely

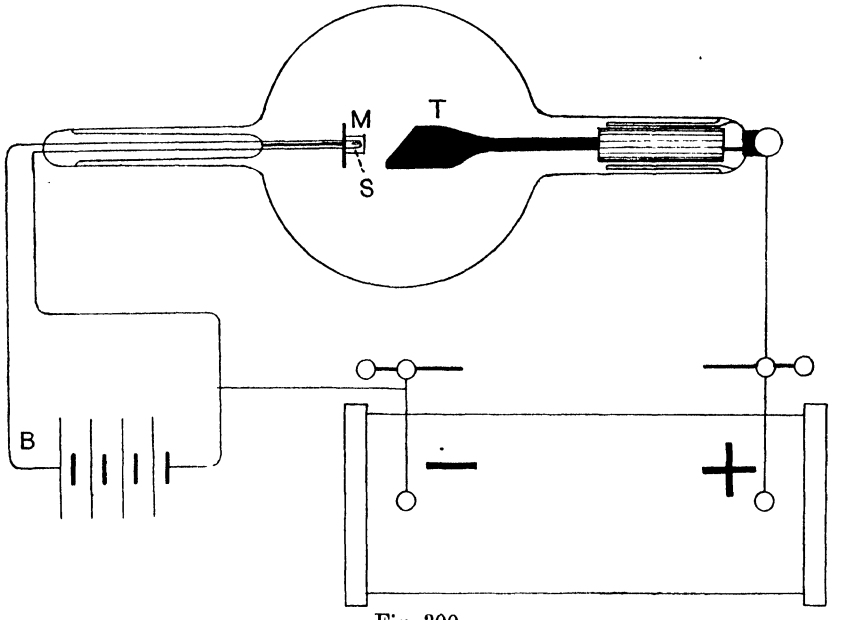



Fig. 300

at this temperature, but at higher temperatures the filament may be burnt out by reverse current. The strength and hardness of the rays are varied independently by varying the heating current and the potential on the anode, the latter up to about 50,000 volts.

The photographic action of Röntgen rays is well known. For demonstration purposes it is more convenient to make use of their power of exciting fluorescence in various substances. A screen of barium platinocyanide will light up brightly in the path of the rays, even when covered at the back with black paper. If the hand is held at the back of the screen the bones are seen clearly, and the absorption of various substances may be observed by the shadows that they cast on the screen. With a small gas-filled bulb of moderate hardness, $\frac{1}{2}$ mm. of lead is opaque to the rays, while 2 cm. of aluminium still lets through about one-third. A sheet of aluminium $\frac{1}{3}$ mm. thick is absolutely transparent, and invisible on the screen. Glass and quartz are about as opaque as aluminium, and ebonite is almost transparent in a sheet 1 mm. thick. It is

clear, therefore, that while the absorption of Röntgen rays in various substances is very different from that of light, it does not depend markedly on whether the substance is a conductor or not. It is much more of a mass-phenomenon, depending on whether the substance is light or heavy. A Coolidge bulb requires 6 mm. of lead to screen it at full power.

The sharpness of the shadow shows that the source of the rays is nearly a point, for a sheet of lead perforated with a small hole will give quite a small spot, even when the screen is some distance behind the hole. With two such sheets we can find the position of the source within the bulb, and with three we can prove the rectilinear propagation of the rays. We thus find that the rays emanate from the *centre of the target*, i.e. the point of impact of the electrons.

A fluorescent screen shielded from the direct action of the rays may be made to glow faintly when a sheet of metal or other substance is placed near it in the path of the beam. This may be due either to rays diffusely reflected from the surface or to some analogous radiation excited in the substance. For the present we may call them *secondary rays*, reserving further discussion for Arts. 215, 238 and 241.

The Röntgen rays show only the slightest trace of reflexion and refraction (Art. 235). Nevertheless, it has long been thought that they are waves in the ether, emitted by the target under the impact of the electrons. The virtual absence of reflexion and refraction is then to be ascribed to the shortness of the wave-length. This theory, which is due to Schuster, was confirmed in 1912 by Friedrich and Knipping, acting on a suggestion of v. Laue. Laue's idea was that the regular arrangement of atoms in a crystal might make it act like a diffraction grating, each atom scattering the rays and the resulting wavelets interfering so as to reinforce each other in certain directions and destroy each other in others. A narrow beam of rays, after passing through a plate of crystal, should therefore give rise to a diagram of discrete spots. Fig. 301 shows the position of the strongest spots on a photographic plate behind a crystal of zinc blende, cut parallel to one of the natural cube faces. The central spot $A$ arises from rays that have not suffered any diffraction. Other crystals, such as quartz, give different but very

beautiful patterns. The wave-lengths are of the order of $10^{-8}$ cm. The further development of this subject belongs to Ch. XIII: we are only concerned here with the fact that is proves the wave-like

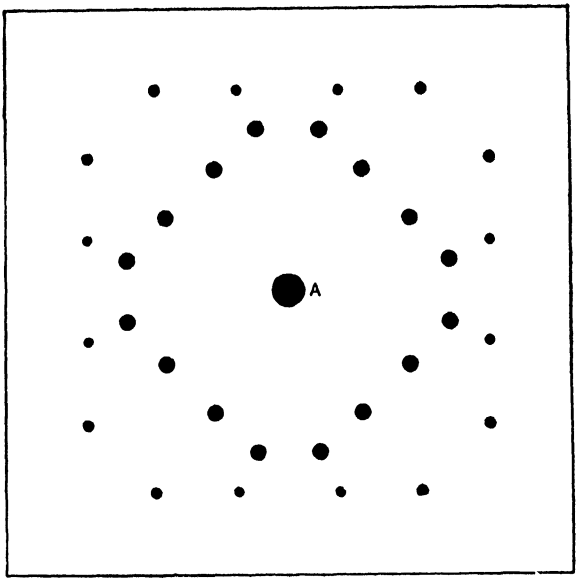



Fig. 301

nature of Röntgen rays. High discharge potential means predominance of short wave-lengths, so that *the penetrating power increases in general with decrease of wave-length* (see also Art. 238).

**214. Normal and abnormal conductivity of the air.** If an ordinary electroscope is left to itself for some days, the leaves sink gradually; but there is nothing to show whether this is due to passage of electricity through the air or leakage along the supports. After allowing for the latter it has been shown beyond doubt that atmospheric air conducts appreciably, but very slightly. Fig. 302 shows a method of split insulation used by C. T. R. Wilson for eliminating the leak over the supports. The gold leaf is attached to a metal rod $A$, insulated from the leading-in wire $B$ by a small sulphur bead $S$. The leaf is charged up to about 100 volts with a moveable rod $R$ connected to one end of a battery of accumulators, the other end of which, together with the case of the electroscope, is earthed. The leading-in wire $B$ is maintained throughout at the

same potential as $R$. If now the potential of $A$ falls on account of leakage through the air, electricity will tend to pass, if at all, from $B$ to $A$ and diminish the rate of fall of the leaves. The effect is initially zero, since $A$ is at the same potential as $B$.

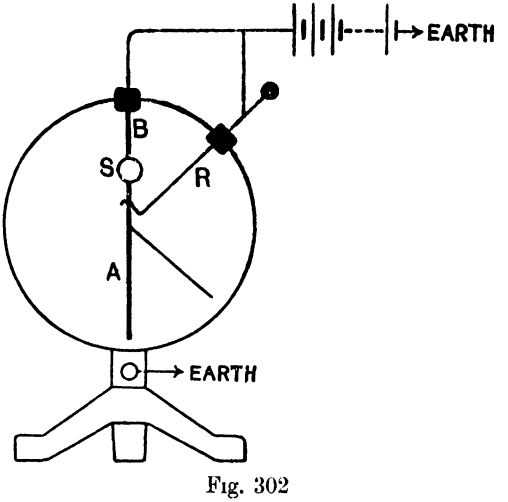


Fig. 302

With this apparatus the amount of electricity lost in a given time is the same whether the charge is positive or negative, and independent of the potential of the battery from about 50 volts upwards. We shall find this fact significant. The current is normally of the order of $10^{-8}$ electrostatic units per cubic centimetre of the vessel.

There are several ways of obtaining larger currents in air. The conductivity of air in the neighbourhood of flames is especially interesting, as it first led Giese, in 1882, to the modern theory of conduction in gases. It has long been known that the air which rises from a flame conducts electricity, so that a current may be made to pass between two electrodes on either side of the stream, whether the electrodes are warm or not. The main facts about this conduction are:

(1) The conductivity persists for some little time after the gas has left the flame, since currents may be detected in a stream of air which has risen to a considerable height above the flame.

(2) The current between a pair of electrodes is diminished by applying an electric force between two other electrodes lower down the stream.

(3) Ohm's law is not obeyed, the current being too small for the higher forces.

Giese tried to assimilate the conduction in gases as far as possible to that in electrolytes; that is, to explain it as a convection by charged particles, called *ions*, not necessarily identical with those

occurring in the electrolysis of liquids. The electrical theory of matter specifies these ions more exactly as aggregates of protons and electrons in unequal numbers. Giese supposed that ions are being constantly produced in a flame, either on account of the high temperature or as a result of chemical action. Only very few are present in air at ordinary temperatures, and give rise to the normal conductivity already described. When a mass of gas rises from a flame, it becomes cool and the formation of new ions practically ceases. At the same time those formed in the flame tend to *recombine*; that is, a positively charged ion can collide with a negatively charged ion and coalesce with it so as to form a neutral atom or molecule. The conductivity of the gas, if left alone, will therefore diminish more or less rapidly, but the last traces remain for some time, since the chance of a collision between the residual ions ultimately becomes very small. Any cause tending to remove ions from a stream of gas, such as a transverse electric force, will weaken the conductivity at further points in the stream. As regards (3), it is clear that there must ultimately be a limit to the rise of current with electric force, which occurs when all the ions which enter the space between the electrodes are taken away before they have time to recombine. The smaller the applied electric force the more time there is for recombination, and the less the current.

Lenard showed in 1897 that cathode rays, which can pass through thin sheets of aluminium into air outside the discharge tube, increase its conductivity; as long, at least, as they are capable of producing fluorescence. Electrons above a certain velocity can, therefore, expel electrons from the molecules of a gas. This effect, known as *ionisation by collision*, begins at velocities much lower than that of ordinary cathode rays (Arts. 224, 225).

**215. Ionisation by Röntgen rays.** Shortly after Röntgen's discovery Benoist and Hurmuzescu found that Röntgen rays would rapidly discharge positively or negatively charged bodies in their neighbourhood. This affords the most generally convenient method of ionising air or other gases. The discharge current is of the same order as a photoelectric leak, and is measured most simply with the apparatus of Fig. 303. The Röntgen ray bulb, worked off an induction coil, is placed inside a wooden box covered with lead

about 2 mm. thick in order to prevent rays from escaping in all directions. The box also acts as an electrostatic screen. A pencil of rays passes through a hole in the lid and afterwards traverses the space between two metal plates $C_1$, $C_2$, placed parallel to one another a centimetre or two apart. The plate $C_1$ is connected to an electrometer, and $C_2$ is maintained at a potential of the order of 40 volts.

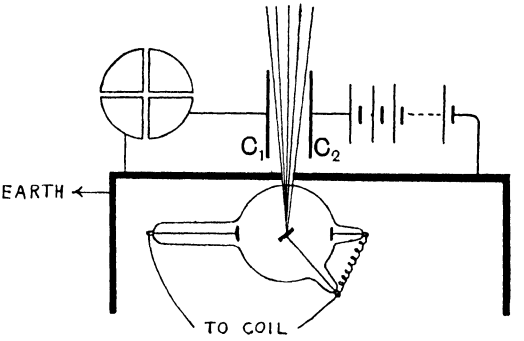
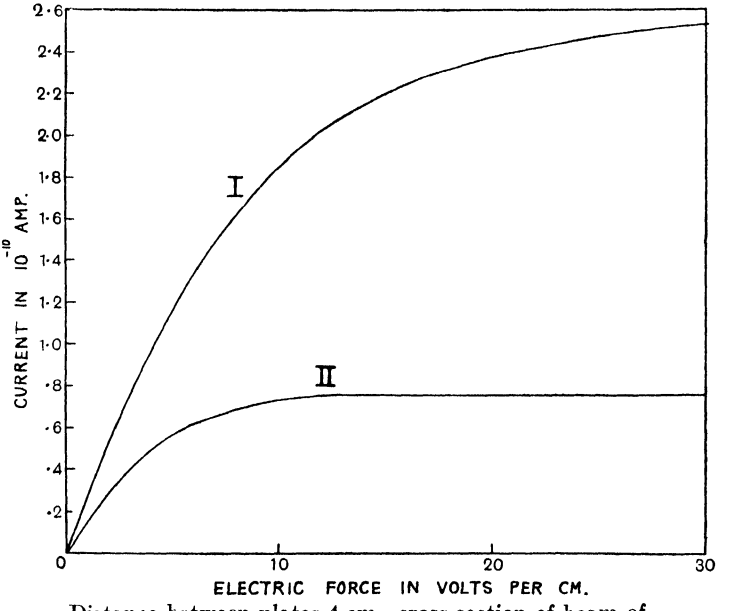


Fig. 303

In this simple method the current is estimated by deflexion as in Art. 208. Accurate measurements are made with the apparatus of Art. 209, replacing the wire grating in the plate $C_2$ by a thin sheet of aluminium.

Fig. 304 shows the relation of current to applied potential in a particular case. Curve I is obtained with an unscreened beam of



Distance between plates 4 cm., cross-section of beam of Röntgen rays 1 cm.²

Fig. 304

Röntgen rays, curve II with a sheet of aluminium $\frac{1}{2}$ mm. thick placed in the path of the rays. The stronger the ionisation, the greater the electric force required for saturation. This is in accordance with Giese's theory. The saturation current in curve I being about three times that in curve II, three times as many ions are present per cm.$^3$ and nine times as many recombine, so that a larger percentage is removed from the stream.

We have now to consider the secondary rays mentioned in Art. 213. Sagnac was the first to show that an electroscope, screened from the direct action of a beam of Röntgen rays, could be discharged by placing a plate of a solid substance in the path of the rays near it. Fig. 305 shows a method of measuring the ionisation produced. The primary rays pass upwards through a hole in the containing lead box, and fall on the plate *R*. *G* is a gauze cylinder containing a central rod connected to an electrometer, and sufficient potential *V* is applied to the cylinder to produce saturation between it and the rod, the wires leading to the electrometer being screened by passing them down the centre of metallic tubes connected to earth. In these circumstances the only ions reaching the electrometer are those generated inside the cylinder; for positive ions generated outside are sent from the gauze to the box, and negative ions, if attracted within the gauze, are collected on it before they can reach the electrometer. Elements of moderate atomic weight, like zinc or copper, give most ionisation inside the cylinder; light elements like aluminium and heavy elements like silver give less, for different reasons. The origin of the rays is discussed in Art. 238.

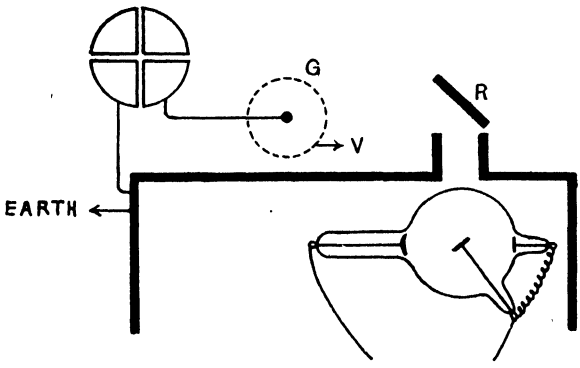


Fig. 305

Other experiments show that Röntgen rays, like ultra-violet light, expel negative electricity from metals on which they fall. Benoist and Hurmuzescu, who allowed the rays to fall on the body

in their experiments, found that it leaked more quickly when charged negatively than positively. Curie and Sagnac showed that a metal plate *in vacuo* becomes positively charged when exposed to Röntgen rays, and Dorn showed that rays are emitted which are deflected by a magnet, without however determining the value of $e/m$. Analogy with the photoelectric effect suggests that they are electrons, in which case their velocity is of the order $5 \times 10^9$ cm./sec. This view can be supported in two ways. Firstly, the velocity is that which would be communicated to an electron by light of wave-length $10^{-8}$ cm. or thereabouts, if Einstein's law continued to hold. Secondly, the velocity is that of cathode rays in a discharge tube at 12,000 volts, so that we should expect the particles to ionise the gas to a distance of a few millimetres from the electrode (Art. 214). Townsend found that when the primary rays fall on one or both electrodes, the saturation current is nearly proportional to the distance between the electrodes for the first few millimetres, but afterwards increases much more slowly. If $a$ represents the ionisation due to particles projected from one or both electrodes, and $b$ that due to the primary and secondary Röntgen radiations per unit length, the saturation current for a distance $x$ should be $i = a + bx$, provided that $x$ exceeds the limit of penetration of the particles. The experimental curve connecting $i$ and $x$ is straight when $x$ is greater than 5 mm., which is therefore the approximate limit of penetration. The velocities of emission are considered further in Art. 241.

**216. Condensation of water-vapour round gaseous ions.** Observations which would now be interpreted in this sense were made by Coulier in 1875. Commenting on the known fact that clouds appear when air saturated with water-vapour is suddenly expanded, Coulier stated that the explanation then current of precipitation of water-vapour by the lowering of temperature was not correct, because the formation of clouds ceased after the expansion had been performed several times with the same air. This he showed to be due to the settling of the dust in the air, which forms nuclei on which water-vapour can condense. But clouds could be formed in the absence of dust provided that the air was ozonised, or exposed for a short time to a red-hot platinum wire.

Richarz, using a steam jet, showed afterwards that Röntgen rays are equally effective.

The conditions under which clouds are formed were examined by C. T. R. Wilson, whose apparatus is shown in Fig. 306. The

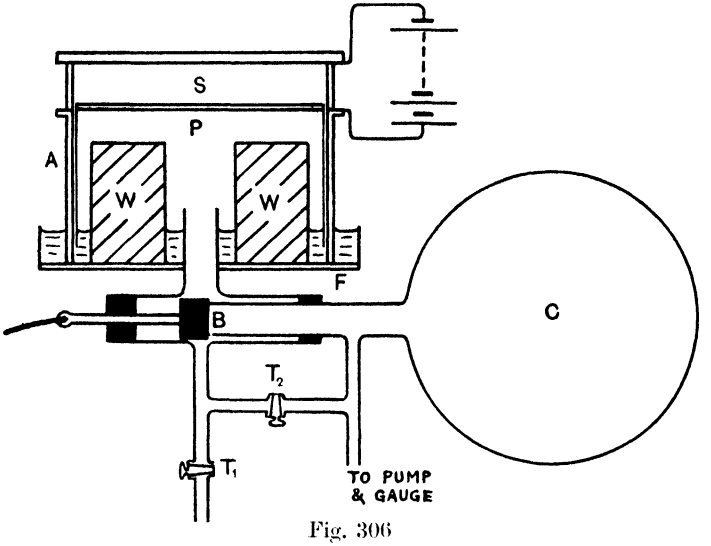



Fig. 306

expansion is produced in the space $S$ between two glass plates. The upper one forms the ceiling of the cylindrical glass vessel $A$; the lower one $P$ is fixed on the top of a thin-walled brass cylinder so as to form a moveable plunger. It dips into a little water resting on the floor $F$ of the outer vessel, which has a layer of indiarubber on the top. The pressure of the air below the plunger is first adjusted by means of the taps $T_1$, $T_2$, of which one goes to the air and the other to an air-pump, until the volume of $S$ has an assigned value $v_1$. $T_1$ and $T_2$ are then closed and the large vessel $C$ exhausted. On pulling off the valve $B$ the pressure below $P$ is suddenly reduced: the plunger falls and is held tight against the indiarubber floor. The volume $v_2$ of $S$ in the new position is known, and thus we can produce at will a sudden expansion of the air with a given expansion-ratio $v_2/v_1$. The hollow wooden cylinder shown in section at $WW$ is inserted to reduce the "dead space" below $P$.

The results of Wilson's experiments are as follows. No cloud appears on expansion in dust-free air as long as $v_2/v_1$ is less than

1·25. Between 1·25 and 1·38 there is a thin cloud, and dense clouds are always formed when $v_2/v_1$ exceeds 1·38. The latter seems a clear case of condensation without extraneous nuclei. Negative ions from Röntgen rays produce dense clouds when $v_2/v_1$ exceeds 1·28; with positive ions very few drops appear until $v_2/v_1 = 1·31$, and dense clouds are not obtained until $v_2/v_1 = 1·35$. It is obvious, therefore, that the method allows of considerable control, which is useful in what follows.

The apparatus of Fig. 306 was used by Wilson to photograph the paths of ionising particles in gases. The cylindrical form of the chamber avoids disturbing eddy currents after the expansion is complete. Condensation of water on the top plate is prevented by keeping the brass cylinder a little cooler than the rest of the apparatus. Wilson had a device in which the simultaneous release of three pendulums first created the expansion, then sent a flash through the Röntgen ray bulb, and finally illuminated the cloud for the short time necessary to take a photograph. The gas is supersaturated before the rays pass, so that the ions are immobilised almost immediately by condensation. An approximately vertical electric field, applied between $P$ and a tinfoil ring under the top plate, removes the charged drops when they are done with, without blurring the photographs appreciably. Electrical contact was made with $P$ by a layer of gelatine and copper sulphate, blackened with ink. The ratio of expansion was generally about 1·34.

In Shimizu's modification the plate $P$ is replaced by a piston moved up and down by a reciprocating movement, with commutators to perform the other operations in succession. In this way thousands of photographs can be taken, though with some sacrifice of quality.

Fig. 307 shows the cloud due to a narrow beam of Röntgen rays passing from left to right, in about twice natural size. The ionisation is concentrated along tracks originating in the beam, where a spot can usually be seen to mark the recoil of the parent molecule. The main part of the ionisation of a gas by Röntgen rays is, therefore, due to electrons ejected with a velocity sufficient for them to ionise by collision; a view first put forward by Sir W. H. Bragg. Larger photographs show that the total number of ions produced by one electron is of the order of 1000. More are produced near the end of a track than at the beginning, as the longer tracks of Fig. 307



Fig. 307

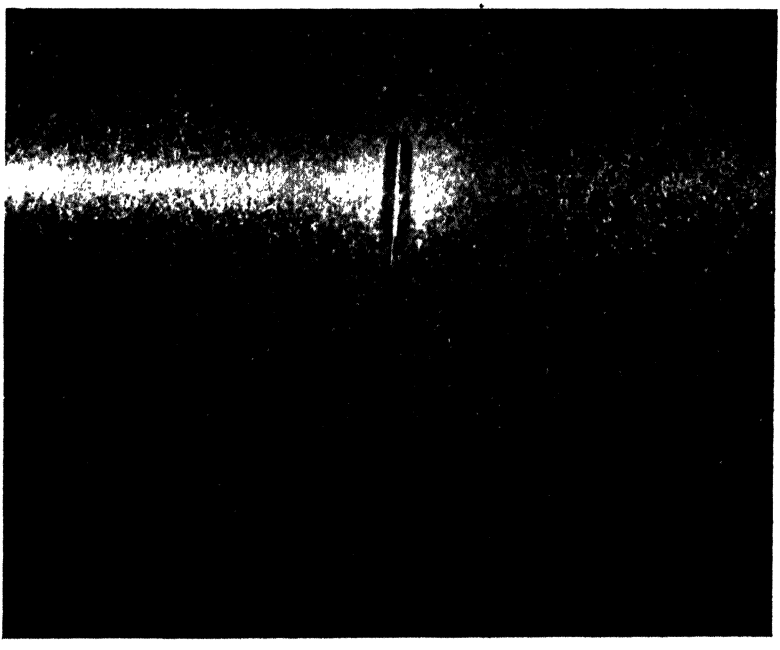


Fig. 308

show clearly. The end of a track is also more curled. An electron, as it loses velocity, loses its power of penetration (cf. Art. 222), but may nevertheless ionise more freely down to a certain point, since it remains longer within the atom.

Wilson found that Röntgen rays produced by a potential of 30,000 volts eject 60 per cent. of their electrons from air with a large component of velocity along the beam (most of them making an angle of nearly 45° with the beam), 20 per cent. making an angle of nearly 90° with the beam and 20 per cent. making a considerable angle backwards. The discrete distribution of angle is remarkable, and reminds one of the experiments of Gerlach and Stern (Art. 263).

Fig. 308 shows the passage of Röntgen rays through a thin sheet of copper. Sufficient supersaturation to produce a cloud is not obtained in the immediate neighbourhood of the sheet, as the temperature does not fall as rapidly as that of the air. The figure shows the weakening of the primary beam in passing through the sheet, the emission of large numbers of electrons from it, and the ionisation produced at a distance by the secondary rays. Further details are given in Art. 241.

**217. Theory of the motion of ions in gases.** The current between the plates $C_1$, $C_2$ in Fig. 303 consists of a flow of positive ions from right to left and negative ions from left to right. If the beam is narrow, positive ions are only found to the left of the beam and negative ions to the right. In either region, therefore, we have ions all of one sign, which move under the electric force in the interval between collisions with gas molecules. Similarly in the photoelectric effect, except that no separation is necessary. The nature of the ion must not be prejudged, since we cannot say whether an electron, for example, does or does not attach itself to a neutral molecule. The velocity of the ions is assumed to be so small that no new ions are generated by collision, and the mean free path also small. In these circumstances we can quote the results found for liquid electrolytes (Art. 175), remarking only that the kinetic theory is more applicable to this problem than to that of ions in liquids.

Taking all the free paths of the ions equal to $l$, and all the

velocities of agitation equal to $V$, the *mobility* of an ion, or its velocity under unit electric force, is

$$\beta = \frac{el}{2\pi V} \quad \ldots\ldots\ldots\ldots\ldots\ldots (258).$$

The *coefficient of diffusion* is

$$K = \tfrac{1}{6} lV \quad \ldots\ldots\ldots\ldots\ldots\ldots (259),$$

and

$$\frac{\beta}{K} = \frac{Ne}{\Pi} \quad \ldots\ldots\ldots\ldots\ldots\ldots (260),$$

$N$ being the number of molecules of a gas at 15° C. and pressure $\Pi$, $e$ the charge on the ion and $m$ its mass. The equation regulating the motion of the ions is

$$\frac{\partial n}{\partial t} + \operatorname{div}(n\mathbf{v}) = K\Delta n \quad \ldots\ldots\ldots\ldots (261),$$

$\mathbf{v}$ being the mean velocity of drift in the electric field and $n$ the number of ions per cm.$^3$ at any point.

These formulae only apply when the velocity of drift is small compared with the velocity of agitation. Equation (260) can be based on more general considerations, and is exact: other formulae derived from (258) and (259) may be in error by a numerical factor, which however nearly always lies between $\frac{1}{2}$ and 2. A method of finding the velocity experimentally is explained in Art. 221.

When slowly moving ions come into contact with a metal surface they give up their charge to it, and cease henceforth to carry current. This adds a boundary condition to the differential equation (261), which we proceed to find. Let an element $dS$ of the surface of the metal be at the origin of coordinates, and take the axis of $z$ normal to the surface and pointing inwards towards the metal. Suppose first of all that the metal above $z = 0$ is removed, allowing free motion to the ions. The number of ions crossing $dS$ per second from either side would be $\frac{1}{4} nV\,dS$ from the velocity of agitation alone. Diffusion and the electric force alter the equality and produce a flux $(\frac{1}{4} nV + \theta)\,dS$, say, from below to above and $(\frac{1}{4} nV - \theta)\,dS$ from above to below. The difference of the fluxes must be equal to $(-K\partial n/\partial z + nv)\,dS$, where $v$ is the mean velocity under the electric force: hence $\theta = \frac{1}{2}(-K\partial n/\partial z + nv)$. If the *régime* is such that the flux from above to below vanishes, the space above $z = 0$ can be replaced by metal with altering it. Hence the boundary condition

at a metal surface is $\frac{1}{4}nV = \theta$, or $\frac{1}{2}nV + K\partial n/\partial z - nv = 0$, and the actual number of ions falling on $dS$ per second is $(-K\partial n/\partial z + nv)\,dS$. These expressions are simplified by considering the relative magnitude of the terms. Since $v$ is small in comparison with $V$, $K\partial n/\partial z$ is comparable with $nV$, so that $nv$ is small in comparison with $K\partial n/\partial z$. The boundary condition becomes $n + \frac{1}{3}l\partial n/\partial z = 0$, or $n = 0$ since the second term is small in comparison with the first. Hence if no ions are generated near the surface of the metal or emitted from it*, the number per unit volume just there is negligible in comparison with that at some distance away. The flux across the boundary is approximately $-K\,(\partial n/\partial z)\,dS$.

**218. Measurement of Ne by the diffusion of ions.** Townsend showed how to determine the quantity $Ne$ by observing the lateral diffusion of a stream of ions moving in a uniform electric field, and hence to compare the charges on ions in liquids and gases (cf. Art. 167). The apparatus is shown in Fig. 309. The ions are generated by Röntgen rays in the space between the plates $A$ and $C$. An electric force $E$ drives positive ions (if it is desired to experiment with these) towards $C$, and some pass through a slit 2 mm. wide and 1·5 cm. long in a direction perpendicular to the plane of the paper. The field $E$ is maintained on the other side of the slit, and the ions, after travelling 4 cm. further, fall on a composite electrode $D$ shown in plan in the lower part of the figure. The segments 1, 2, 3 are insulated and separated from each other by narrow gaps, forming altogether

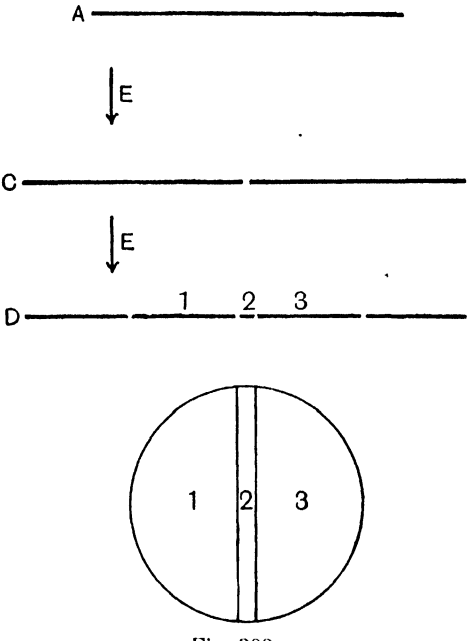


Fig. 309

* The latter condition is not satisfied when the velocity of the electrons is large (Art. 227).

a circle 7·6 cm. in diameter, the central segment being 5 mm. wide from centre to centre of the gap. The outer part of the disc $D$ forms a guard-ring connected to earth.

The ions have a long fall in a uniform electric field, in order that they may acquire a constant drifting motion before arriving at $C$. In the absence of diffusion all the ions would be caught on a strip 2 mm. wide in the centre of electrode 2. Lateral diffusion makes some spread out to 1 and 3, a certain charge $e_2$ reaching electrode 2 and $e_1 + e_2 + e_3$ reaching 1, 2 and 3 together in the same time. The ratio of these charges is measured by a compensation method, in which the three electrodes are kept as nearly as possible at zero potential.

The electric force is chosen so that very few ions reach the guard-ring. The value of $e_2/(e_1 + e_2 + e_3)$ is then the same as if the slit was infinitely long, and the electrodes 1 and 3 infinite. We may, therefore, treat the diffusion problem as a two-dimensional one. Take the centre of the slit as origin of coordinates, with an axis of $x$ perpendicular to it in the sheet $C$ and an axis of $y$ parallel to the electric force. Equation (261) becomes

$$K\left(\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2}\right) = \frac{\partial}{\partial y}(n\beta E) = \beta E \frac{\partial n}{\partial y};$$

or using (260),

$$\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2} = 2\gamma \frac{\partial n}{\partial y} \quad \ldots\ldots\ldots\ldots\ldots (262),$$

where

$$\gamma = NeE/2\Pi \quad \ldots\ldots\ldots\ldots\ldots (263).$$

Let $2a$ be the width of the slit, $2b$ the width of electrode 2, $2c$ the diameter of the guard-ring and $h$ the distance from $C$ to $D$. Since few ions reach the guard-ring, $n$ may be taken to be zero when $x = c$. We have then to find the solution of equation (262) subject to the following boundary conditions:

(i) when $y = h$, $n = 0$ for all values of $x$;

(ii) when $x = c$, $n = 0$ for all values of $y$;

(iii) when $y = 0$, $n = n_0$, a constant*, when $x < a = 0$,
$= 0$ when $a < x < c$,

only positive values of $x$ being considered. A special solution of equation (262) is

$$n = \cos kx e^{\theta y}, \quad \text{where} \quad k^2 = \theta^2 - 2\gamma\theta.$$

* This assumption is nearly, but not quite, accurate since some ions are lost by diffusion to $C$ near the edge of the slit.

Solving for $\theta$ in terms of $k$, and combining the two solutions into one, we have the special solution

$$n = \cos kx e^{\gamma y} (A \cosh \delta y + B \sinh \delta y),$$

where $$\delta = (k^2 + \gamma^2)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (264).$$

If the solution satisfies condition (i) it can clearly be put in the form

$$n = C \cos kx \, \frac{e^{\gamma y} \sinh \{\delta (h - y)\}}{\sinh \delta h},$$

where $C$ is a constant. Condition (ii) shows that $\cos kc = 0$, so that we can restrict $k$ to the values

$$k = (2r + 1)\, \pi/2c, \text{ where } r = 0, 1, 2, \ldots\ldots (265).$$

We now take a value of $C$ corresponding to each value of $k$, and sum over all those values. Such a solution makes $n = \Sigma C \cos kx$ when $y = 0$, and since this is a Fourier's series valid in the region $0 < x < c$, the coefficients can be adjusted to satisfy the last remaining condition (iii). Determining the constants $C$ by integration in the usual way (cf. Art. 156) and substituting, we find for the final solution

$$n = \frac{2n_0}{c} \Sigma \frac{\sin ka}{k} \cos kx \, e^{\gamma y} \frac{\sinh \{\delta (h - y)\}}{\sinh \delta h} \quad \ldots\ldots (266),$$

summation being over the values of $k$ in equation (265). The number of ions falling on electrode 2 per second is from Art. 217

$$e_2 = - K \left[ \int_0^b \frac{\partial n}{\partial y} dx \right]_{y=h},$$

and the number falling on the electrodes 1, 2 and 3 together is

$$e_1 + e_2 + e_3 = - K \left[ \int_0^c \frac{\partial n}{\partial y} dx \right]_{y=h}.$$

Evaluating the integrals, we find

$$\frac{e_2}{e_1 + e_2 + e_3} = \frac{\Sigma \dfrac{\sin ka \sin kb}{k^2} \dfrac{\delta}{\sinh \delta h}}{\Sigma \dfrac{\sin ka \sin kc}{k^2} \dfrac{\delta}{\sinh \delta h}} \quad \ldots\ldots (267).$$

It follows already from equation (262) that if the dimensions of the apparatus are given, the ratio $e_2/(e_1 + e_2 + e_3)$ depends only on $NeE/\Pi$. Its value can be computed from equation (267). The curve connecting the two quantities for the apparatus of Fig. 309 is shown

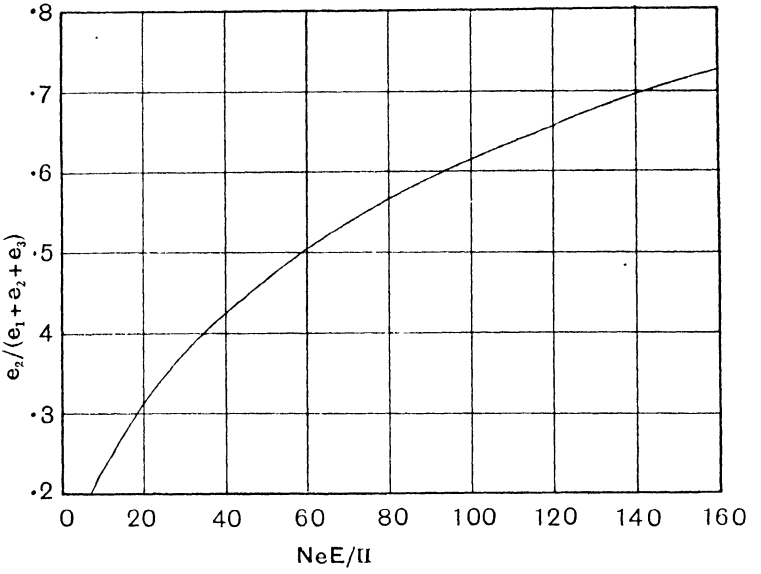



Fig. 310

in Fig. 310. A single observation of $e_2/(e_1 + e_2 + e_3)$ then gives $NeE/\Pi$ and hence $Ne$, since $\Pi = 1{\cdot}013 \times 10^6$. The following table gives the results of experiments on this principle with negative ions:

| Pressure of air in mm. | Electric force in volt/cm. | $Ne$ |
|---|---|---|
| 3 | 1·45 | $1{\cdot}23 \times 10^{10}$ |
| 6 | 0·98 | 1·23 |
| 6 | 1·47 | 1·25 |
| 6 | 1·96 | 1·15 |
| 6 | 2·96 | 1·27 |
| 12 | 1·00 | 1·20 |
| 12 | 0·98 | 1·20 |
| 12 | 1·46 | 1·25 |
| 12 | 2·75 | 1·26 |
| | Mean | $1{\cdot}23 \times 10^{10}$ |

The mean is identical, within the limits of experimental error, with the number $1{\cdot}22 \times 10^{10}$ found in Art. 167, $e$ there denoting the charge on a univalent ion in electrolysis. Hence the two charges are equal. Townsend's experiments with positive ions usually gave the same result, but freshly cleaned surfaces gave higher values (up to $2{\cdot}41 \times 10^{10}$), indicating that double charges were present. There is no evidence of charges less than that on a univalent ion. We conclude, as in Art. 207, that the quantity $e$ is the charge on the electron.

**219. The charge on a gaseous ion.** The elementary charge $e$ was first measured by Townsend, Sir J. J. Thomson, and H. A. Wilson. The method was improved by Millikan, who used it to give direct proof of the discontinuous or atomic structure of electricity. The principle of the experiment, as far as it need be stated at present, is as follows. Let $v_1$ be the rate of fall, under gravity, of a small oil-drop of mass $m$. If the air in the neighbourhood is ionised by Röntgen rays, one or more ions collide with the drop, and it acquires a charge $ne$, where $e$ is the elementary charge and $n$ an integer. To fix ideas suppose that the charge is positive. A vertical electric field $E$ of sufficient strength will stop the fall of the drop and convert it into a rise with uniform velocity $v_2$. Thus

$$mg = \lambda v_1, \quad neE - mg = \lambda v_2 \quad \ldots\ldots\ldots (268),$$

where $\lambda$ is a certain constant, which may however depend on the size of the drop and the pressure of the gas. Hence

$$\frac{v_1 + v_2}{v_1} = \frac{neE}{mg}.$$

If the ratio $(v_1 + v_2)/v_1$ is observed by allowing a particular drop to rise and fall many times through a measured distance, the values should, therefore, be small multiples of a common unit.

Millikan's apparatus for observing the velocities is shown in Fig. 311. The oil-drops are formed by a sprayer at $A$ in the figure, and fall slowly until one enters a small aperture $B$ in the plate $C_1$ of a parallel-plate condenser. This plate can be kept at a positive

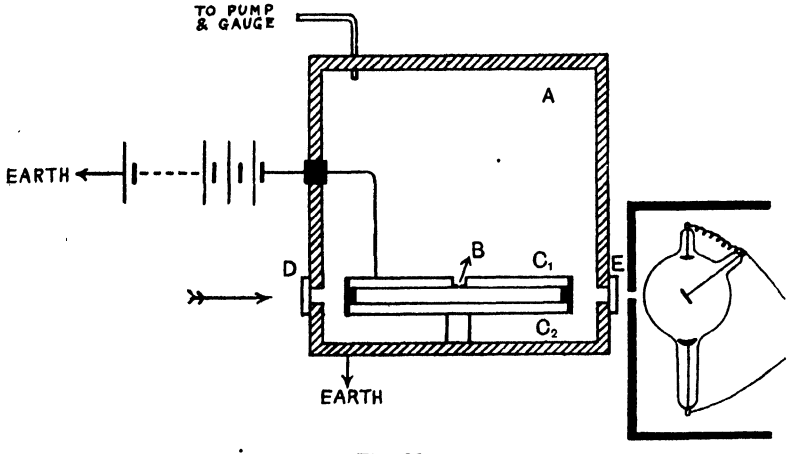



Fig. 311

or negative potential by means of a battery of about 2500 accumulators, the other plate $C_2$ being earthed. The condenser is illuminated by light entering through the window $D$, and ionised by Röntgen rays entering through $E$. A third window (not shown) allows the motion of the oil-drop to be observed through a short-focus telescope with a graduated eyepiece, so that the time of falling through a distance of 1·021 cm. can be measured accurately. The outer case is immersed in a constant-temperature bath in order to avoid convection currents, and the condenser $C_1C_2$, for the same reason, is edged with an ebonite strip containing three holes opposite the windows. The following table shows one set of observations:

$E = 6{\cdot}98$ E.S.U., $p = 169{\cdot}5$ mm., temp. 22·98° C.

| $v_1$ | $v_2$ | $\frac{v_1 + v_2}{v_1}$ |
|---|---|---|
| 0·020241 | 0·022972 | 2·1349 = 4 × 0·5337 |
| 0·020251 | 0·033748 | 2·6664 = 5 × 0·5333 |
| 0·020110 | 0·033668 | 2·6742 = 5 × 0·5348 |
| 0·020348 | 0·044577 | 3·1908 = 6 × 0·5318 |
| 0·020235 | 0·033590 | 2·6600 = 5 × 0·5320 |
| 0·020234 | 0·012196 | 1·6028 = 3 × 0·5343 |
| 0·020372 | 0·001284 | 1·0630 = 2 × 0·5315 |

The constancy of $v_1$ shows that the drop is not evaporating appreciably during the experiment. The submultiples in the last column are constant within the limits of experimental error, so that we have an accurate proof of the existence of the elementary charge.

The earlier experimenters found $e$ in absolute measure from Stokes' theorem* that the force required to drive a sphere of radius $a$ with velocity $v$ through a fluid of viscosity $\mu$ is $6\pi\mu av$. Thus $\lambda = 6\pi\mu a$, where $a$ is the radius of the drop. The weight $mg$ in equations (268) must be corrected for the buoyancy of the air, so that if $\rho$ is the density of the oil and $\sigma$ that of air, $mg = \frac{4}{3}\pi a^3(\rho - \sigma)g$. Hence for a drop carrying a single charge

$$\tfrac{4}{3}\pi a^3(\rho - \sigma)g = 6\pi\mu av_1, \quad eE - \tfrac{4}{3}\pi a^3(\rho - \sigma)g = 6\pi\mu av_2.$$

The first equation is solved for $a$, and the second equation then gives $e$, all the other quantities being known ($\mu = 1{\cdot}824 \times 10^{-4}$ at 23° C., density of oil 0·9199).

* Lamb's *Hydrodynamics*, 5th ed., p. 567.

Millikan, however, found that the values of $e$ thus obtained were not all the same, being larger for the smaller drops. As it is inconceivable that the charge on an ion should depend on the size of a drop with which it collides, the experiments show that Stokes' law ceases to be accurate for very small drops. The proof of the law, being purely hydrodynamical, fails when the mean free path $l$ of the air molecules becomes comparable with the radius of the drop. This is usually the case in Millikan's experiments. The most natural generalisation of Stokes' formula is to assume a retarding force due to viscosity of the form $6\pi\mu avf(l/a)$, where $f$ is unity for small values of $l/a$. Since $l$ is inversely proportional to the pressure $p$, the factor can be written $\phi(1/ap)$. Millikan found that the observations could be reconciled over a large range, and the same value of $e$ obtained in all cases, by taking

$$\phi\left(\frac{1}{ap}\right) = \left(1 + \frac{b}{ap}\right)^{-1},$$

the constant $b$ having the value $6{\cdot}254 \times 10^{-3}$ when $p$ is in mm. of mercury.

In this way Millikan found the value

$$e = 4{\cdot}774 \times 10^{-10} \text{ electrostatic units,}$$

estimated correct to 1 part in 500. The radius of the drops used was of the order $2 \times 10^{-4}$ cm.

**220. Fundamental physical constants.** We have now all the data for determining the quantities $N$, $e$ and $m$ separately. Adopting the values

$$Ne = 1{\cdot}22 \times 10^{10} \quad \ldots\ldots\ldots\ldots\ldots\ldots (269)$$

and

$$e = 4{\cdot}77 \times 10^{-10} \text{ E.S.U.} \quad \ldots\ldots\ldots\ldots (270),$$

we find, for the *number of molecules in a gas at* 15° C. *and* 760 *mm. pressure,*

$$N = 2{\cdot}56 \times 10^{19} \quad \ldots\ldots\ldots\ldots\ldots\ldots (271).$$

Again, since

$$e/m = 5{\cdot}30 \times 10^{17} \quad \ldots\ldots\ldots\ldots\ldots\ldots (272),$$

we find, for the *mass of the electron,*

$$m = 9{\cdot}00 \times 10^{-28} \text{ grams} \quad \ldots\ldots\ldots\ldots (273).$$

The mass of the hydrogen atom has already been estimated at $1{\cdot}66 \times 10^{-24}$ grams (Art. 167), so that

$$\frac{\text{mass of } H \text{ atom}}{\text{mass of electron}} = 1840 \quad \ldots\ldots\ldots\ldots (274).$$

The energy acquired by an electron, proton or any other body carrying the elementary charge, in falling through a potential of $V$ volts, is $eV/300 = 1{\cdot}59 \times 10^{-12}V$ ergs.

If $h$ is Planck's constant, $h/e = 1{\cdot}37 \times 10^{-17}$ from Art. 210, so that $h = 6{\cdot}53 \times 10^{-27}$. We shall adopt the value

$$h = 6{\cdot}55 \times 10^{-27} \quad \ldots\ldots\ldots\ldots\ldots (275)$$

on the basis of experiments described in Art. 237. Since the frequency of a wave of length $\lambda$ is $c/\lambda$, the energy acquired by an electron falling through $V$ volts is the quantum energy corresponding to a wave-length of $\lambda$ cm., where $eV/300 = hc/\lambda$, or

$$V = \frac{12300}{10^8\lambda} \quad \ldots\ldots\ldots\ldots\ldots (276).$$

Sodium light corresponds to a fall of potential of 2·05 volts, Schumann rays to about 10 volts, medium Röntgen rays to 10,000 volts, $\gamma$ rays to about 100,000 volts. These figures are significant since they represent the least voltage by which the radiation in question can possibly be excited (Art. 238). The energy quantum corresponding to sodium light is $3{\cdot}3 \times 10^{-12}$ ergs.

According to the principles of the kinetic theory, the kinetic energy of a molecule of a gas at absolute temperature $\theta$ (not too low) is $\alpha\theta$, where $\alpha$ is a constant which we shall call *Boltzmann's constant*. Its value can now be found. The pressure $p$ of a gas is given by $p = \frac{1}{3}mN\Omega^2$, where $\frac{1}{2}m\Omega^2 = \alpha\theta$. Hence $p = \frac{2}{3}N\alpha\theta$. Taking $\theta = 15°$ C. $= 288°$ absolute, $p = 760$ mm. of mercury $= 1{\cdot}013 \times 10^6$ dyne/cm.$^2$, $N = 2{\cdot}56 \times 10^{19}$, we have

$$\alpha = 2{\cdot}06 \times 10^{-16} \quad \ldots\ldots\ldots\ldots (277).$$

The kinetic energy of a gaseous molecule at ordinary temperatures is therefore about $5{\cdot}9 \times 10^{-14}$ ergs, which is the energy an electron would acquire in falling through $\frac{1}{27}$ volt. This is also the mean energy of an electron in a metal at ordinary temperatures.

**221. Measurement of ionic velocity.** The velocity of ions in gases was first measured by Rutherford, and afterwards by Zeleny and Langevin, by different methods. Fig. 312 shows the application of Rutherford's method to Röntgen rays.

Four plates $A$, $B$, $C$, $D$ are placed parallel to one another inside an earthed metal case. The plates $B$ and $C$ carry central gauzes,

and $D$ is connected to one pair of quadrants of a sensitive electrometer. Ions are generated in the space between $A$ and $B$. Suppose that it is desired to measure the velocity of the positive ions. An electric force $E$ is applied between $A$ and $B$, which drives the ions through the gauze. The electric force between $B$ and $C$ is made to alternate between $+E$ and $-E'$ by means of the rotating commutator shown in the figure, $E'$ being numerically greater than $E$. The gauze $C$ screens the plate $D$ from the alternating field.

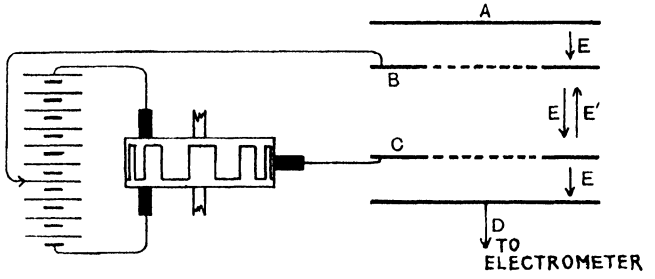



Fig. 312

The ions are thrown back upon the gauze by the reversed force, while it is acting; but at other times they proceed from $B$ towards $C$ in the same field as before. Let $t$ be the interval of time during which the direct force $E$ is acting, $l$ the distance between $B$ and $C$, and $v$ the velocity of the ions in the field $E$. If $t$ is less than $l/v$ the ions are overtaken by the reversed field before they reach $C$; but if $t$ exceeds $l/v$ some of them pass through $C$ and are received on $D$. Hence the electrometer will be unaffected when the commutator is rotating very rapidly, but a deflexion will begin to occur as it slows down. Diffusion prevents the transition from being absolutely sharp, but the critical speed of revolution can nevertheless be estimated to within a few per cent. The velocity of the ions under the electric force $E$ is then $v = l/t$.

The velocity is proportional to the electric force, for forces of a few volts per cm. at atmospheric pressure. The following table (p. 482) gives the *mobility*, or velocity of an ion under an electric force of 1 volt per cm. in this range.

The theoretical formula $v = eEl/2mV$ now gives a clue to the constitution of the ion. Let us calculate the mobility of a negative ion in hydrogen on the hypothesis that it is (1) a charged molecule,

| Gas | $\beta_+$ | $\beta_-$ | Observer |
|---|---|---|---|
| Air .. .. | 1·36 | 1·87 | Zeleny |
| | 1·40 | 1·70 | Langevin |
| | 1·37 | 1·80 | Franck and Pohl |
| Hydrogen .. | 6·70 | 7·95 | Zeleny |
| | 6·02 | 7·68 | Franck and Pohl |
| Carbon dioxide | 0·76 | 0·81 | Zeleny |
| | 0·86 | 0·90 | Langevin |
| Oxygen | 1·36 | 1·80 | Zeleny |

(2) an electron. The mean free path of a molecule at atmospheric pressure is about $2 \times 10^{-5}$ cm.; we assume that a slow electron penetrates the molecules to about the same extent as another molecule, so that its free path is about $10^{-4}$ cm. The velocity of agitation can be calculated from the equation $\frac{1}{2}mV^2 = a\theta$. Thus in case (1), $m = 3{\cdot}3 \times 10^{-24}$, $V = 1{\cdot}9 \times 10^5$ and $\beta = 25$. In case (2), $m = 9 \times 10^{-28}$, $V = 1{\cdot}1 \times 10^7$ and $\beta = 8000$. The results therefore show that in a weak field the ion is not a free electron, or even a charged molecule, but a group of molecules clustered round a positive or negative charge; the positive ion being somewhat greater than the negative.

The mean velocity of drift of a group of ions is in any case a function of the electric force $E$ and the pressure $p$ of the gas. The relation can be simplified by kinetic considerations. Consider a number of ions in a gas, having assigned velocities immediately after a collision. If the pressure is increased from $p$ to $\lambda p$ and the electric force from $E$ to $\lambda E$, the free paths of the ions are reduced in the ratio $\lambda$ to 1, but the same velocity is acquired during a free path. The velocities therefore continue to correspond indefinitely, so that the mean velocity of drift depends on $E/p$ only. The same is true of the average constitution of the ions; that is, the ratio of the numbers of electrons, charged atoms or molecules and molecular clusters at any time.

Fig. 313 shows the results of experiments by Lattey for values of $E/p$ up to 0·1, $E$ being in volts per cm. and $p$ in mm. of mercury. Air freshly admitted into the apparatus gives mobilities as above. A week's drying over phosphorus pentoxide does not affect the velocity of the positive ions, but the velocity of the negative ions becomes much greater, and is no longer proportional to the electric

force. The velocity for $E/p = 0{\cdot}08$ is intermediate between what would be expected for electrons and charged molecules, showing that the molecular cluster is breaking up and free electrons are becoming frequent. This process continues until only free electrons are present, as we shall see in the next article.

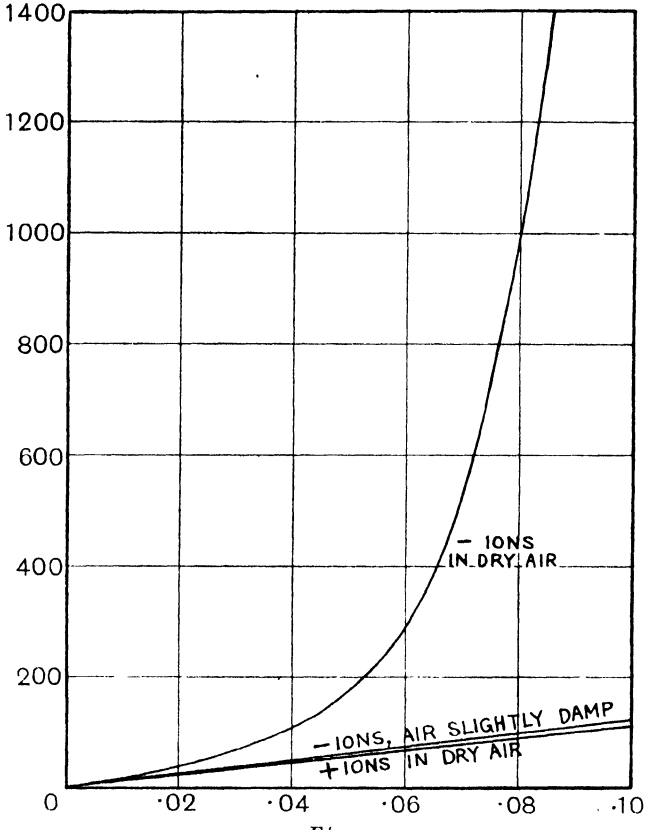



Fig. 313

The increase of velocity is accompanied, at slightly higher forces, by a more than proportional increase of lateral diffusion. This will be understood from Fig. 314, which is Fig. 310 redrawn on the hypothesis that $Ne$ is equal to $1{\cdot}22 \times 10^{10}$. Points corresponding to negative ions in slightly damp gas fall on the curve, as is evident from what has been said. Townsend found that negative ions in dry gas gave points on a new, dotted, curve to the right of the original curve. Fewer ions are then received on the central

electrode, notwithstanding that the velocity of drift is greater. It follows that the coefficient of diffusion of negative ions in dry gases, and therefore also their velocity of agitation, is abnormally great for high values of $E/p$.

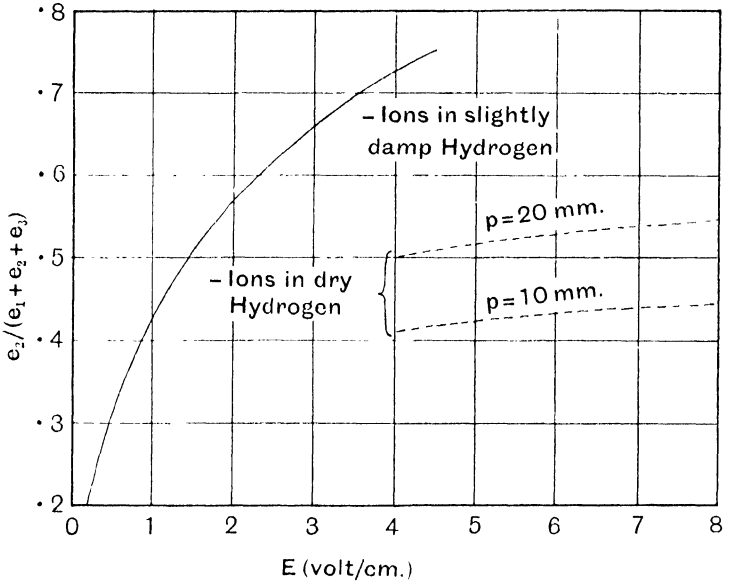



Fig. 314

Townsend assumed that the mean kinetic energy of agitation in dry gases is $k$ times that of a molecule of the gas, that is $k\alpha\theta$, where $k$ depends on the electric force and the pressure. The increase can be explained if the energy gained during a free path is not all lost on collision, but accumulates for some time: $k$ would then be a function of $E/p$ (cf. p. 482). Referring to equations (258) and (259), we have no longer $\frac{1}{2}mV^2 = \alpha\theta$, but $\frac{1}{2}mV^2 = k\alpha\theta$. Hence (260) becomes $\beta/K = Ne/k\Pi$, and (262) becomes

$$\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2} = \frac{NeE}{k\Pi}\frac{\partial n}{\partial y} \quad \ldots\ldots\ldots\ldots\ldots (278).$$

The equation having the same form as before, it follows that the same value of $e_2/(e_1 + e_2 + e_3)$ should be obtained with the electric force $E' = kE$ as was previously obtained with the force $E$. For example, in dry hydrogen at 20 mm. pressure, $e_2/(e_1 + e_2 + e_3) = 0{\cdot}507$ when $E' = 4{\cdot}35$ volt/cm., while a force $E = 1{\cdot}51$ volt/cm. would be required to produce the same ratio in damp hydrogen. Thus

$k = 2{\cdot}88$, and similarly $k$ can be determined for any other force and pressure. Townsend and his collaborators measured $k$ in this way, with the apparatus of Fig. 309, and also measured the velocity of drift $v$ by deflecting the ions in a magnetic field. A field $H$ is applied perpendicular to the plane of the paper, and adjusted till the charge received on 1 is equal to that received on 2 and 3 together, so that the centre of the stream falls on the middle of the gap between 1 and 2. The stream is thus deflected through a known small angle $\theta$. The force exerted by the magnetic field per unit charge is $Hv/c$ nearly at right angles to the electric force, so that $\theta = Hv/Ec$ approximately. Hence

$$v = Ec\theta/H \quad \ldots\ldots\ldots\ldots\ldots\ldots (279).$$

Fig. 315 shows the values of $k$ for air, hydrogen, argon and helium up to $E/p = 10$, Fig. 316 the velocities of drift. The air curve in

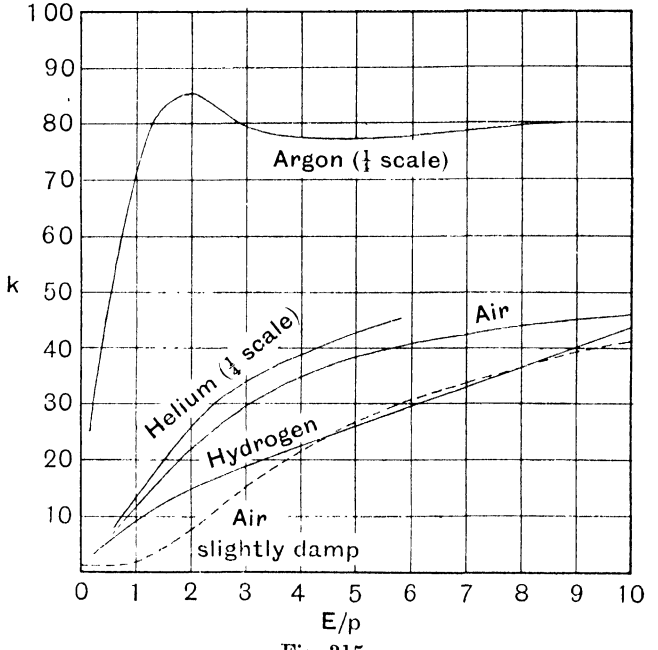



Fig. 315

Fig. 316 continues that of Fig. 313. The mobility (here the ratio of velocity to electric force) first increases as $m$ decreases, and then decreases with the rise of $k$, which increases $V$ in the ratio $k^{\frac{1}{2}}$ to 1. In order to see what the ions are, consider the lowest result in

hydrogen, $v = 6{\cdot}5 \times 10^5$ and $k = 3{\cdot}1$ for $E/p = 0{\cdot}25$, corresponding to 190 volt/cm. at atmospheric pressure. The theoretical velocity for a free electron, allowing for the increase in $V$, is then roughly

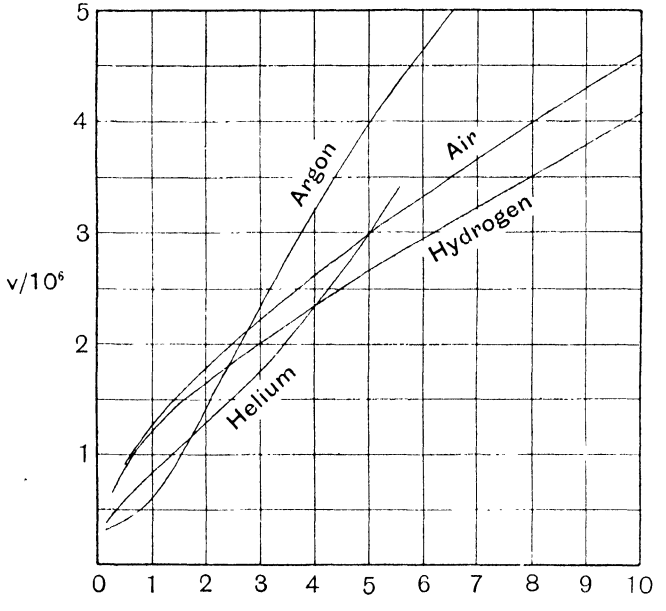



Fig. 316

$9 \times 10^5$, which is near the observed value. Excluding an improbable compensation, we conclude that the ions are electrons, and that an electron with a velocity of agitation of $2 \times 10^8$ cm./sec. penetrates the molecule to about the same extent as another molecule of hydrogen. The results for air are similar.

**222. Motion of electrons in gases.** The experiments described in the last article give the following information about negative ions in air or hydrogen:

(1) When $E/p$ is less than about 0·01 the ion consists of a group of molecules;

(2) A transition stage occurs between 0·01 and 0·25, in which the average mass of an ion continually diminishes;

(3) When $E/p$ exceeds 0·25 all the ions are free electrons.

This means that an electron will not attach itself to a molecule when its velocity exceeds $2 \times 10^7$ cm./sec. Less interest attaches

to positive ions since the most that can happen is the transition from a small group of molecules to a nucleus with or without attendant electrons, but always of atomic mass. The dotted curve in Fig. 315 refers to a sample of air not specially dried. The effect of a trace of water-vapour, which is considerable when $E/p = 1$, diminishes as the force increases, and the ions are electrons above $E/p = 20$ in moist gases, or even in water-vapour. It now becomes of interest to find the amount of energy lost when an electron collides with a molecule, and the extent to which it penetrates the molecule.

An electron must lose energy on collision, otherwise it would never dissipate what it gains in a free path, and a steady state would be impossible. Let it lose, on the average, a fraction $\lambda$ at each collision; that is, an amount $\lambda k\alpha\theta$. The number of collisions in one second is $V/l$ (Art. 175), during which time the electron drifts a distance $v$ back along the lines of electric force, and takes up energy $eEv$. Hence $\lambda k\alpha\theta V/l = eEv$, or

$$\lambda = \frac{eEvl}{k\alpha\theta V} = \frac{2mv^2}{k\alpha\theta} \quad \ldots\ldots\ldots\ldots\ldots (280).$$

Air and hydrogen give results of the order $\lambda = 0{\cdot}004$ at $E/p = 0{\cdot}25$ and $0{\cdot}1$ at $E/p = 20$. Such losses cannot be accounted for by merely elastic collision with a molecule of mass $M$, in which $\lambda$ would be $2m/M$, that is $0{\cdot}0006$ for the lightest molecule, hydrogen. The velocities of agitation at $E/p = 0{\cdot}25$ and $E/p = 20$ are $2 \times 10^7$ cm./sec. and $10^8$ cm./sec., corresponding to a fall of an electron through $0{\cdot}1$ volt and 3 volts respectively. Since an electron can ionise at an energy of about 10 volts* (Art. 224), it is not difficult to understand the rapid increase of $\lambda$ in this region. A similar increase takes place in argon, but $\lambda$ is very small for energies less than 5 volts, the loss of energy being probably all due to displacement of the argon atom. It is not, therefore, surprising that $k$ is large in argon (Fig. 315). Slow electrons in helium also lose but little of their energy. Franck and Hertz showed in 1913 that slow electrons could retain their energy over much greater distances in helium than in hydrogen or oxygen.

* This convenient expression will be used henceforth. The velocity corresponding to $V$ volts is $5{\cdot}94 \times 10^7 V^{\frac{1}{2}}$ cm./sec. (Art. 207).

The electrons in Wilson's photographs (Fig. 307) are deflected less at the beginning of their path than at the end. An atom must, therefore, have a yielding structure which allows electrons whose velocity is of the order of $5 \times 10^9$ cm./sec. to penetrate further into it than slower electrons (cf. also Art. 250). The effective diameter of the atom at any given velocity can be found by the following method, devised by Lenard.

If $l$ is the mean free path of an electron in a gas at pressure $p$, containing $n$ molecules per cm.$^3$, and the molecules are considered temporarily as elastic spheres of radius $\sigma$, we have from the kinetic theory of gases $\quad l = 1/n\pi\sigma^2, \quad p = \frac{2}{3}n\alpha\theta,$

$\alpha$ being Boltzmann's constant (Art. 220) and $\theta$ the absolute temperature. Hence $lp = 2\alpha\theta/3\pi\sigma^2$ or $l = 1/\omega p$, where

$$\omega = 3\pi\sigma^2/2\alpha\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (281).$$

A stream of electrons, moving with assigned velocity, is gradually weakened as it passes through a gas, since electrons are constantly being deflected from it by collision. The fraction which travel through a distance $x$ without appreciable deflexion is, from the kinetic theory of gases, $e^{-x/l} = e^{-\omega p x}$. Measurement of the weakening of the stream will therefore determine $\omega$, and hence $\sigma$. The apparatus is shown in Fig. 317. Electrons liberated photoelectric-

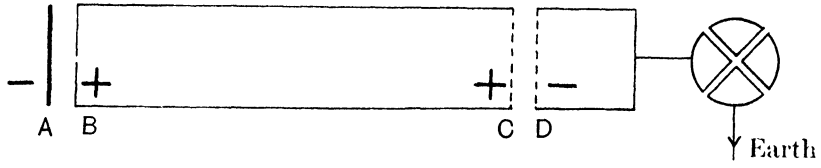



Fig. 317

ally from a metal plate $A$ (or emitted by a hot filament) are raised to a known velocity by a plate $B$ at a higher potential, a number of them passing through a hole in $B$ into a long metal tube closed at the far end by a grid $C$. The electrons are now in a region of no electric force, where they can collide with molecules of the gas. A fixed proportion of the undeflected electrons pass through $C$ and through a second grid $D$ into a metal box, connected to an electrometer, a retarding potential between $C$ and $D$ making it more difficult for deflected electrons to enter the box. If $x$ is the length

of the box and $i_1$, $i_2$ the electrometer currents for gas pressures $p_1$, $p_2$ respectively, $\omega$ is given by the equation

$$i_1/i_2 = e^{-\omega(p_1-p_2)x};$$

and then equation (281) gives $\sigma^2$. The quantity $\sigma$ so derived may be taken as the effective radius of the atom when it has a yielding structure.

Fig. 318, from experiments by Lenard and Mayer, shows how quickly the penetration of an electron into an atom or molecule

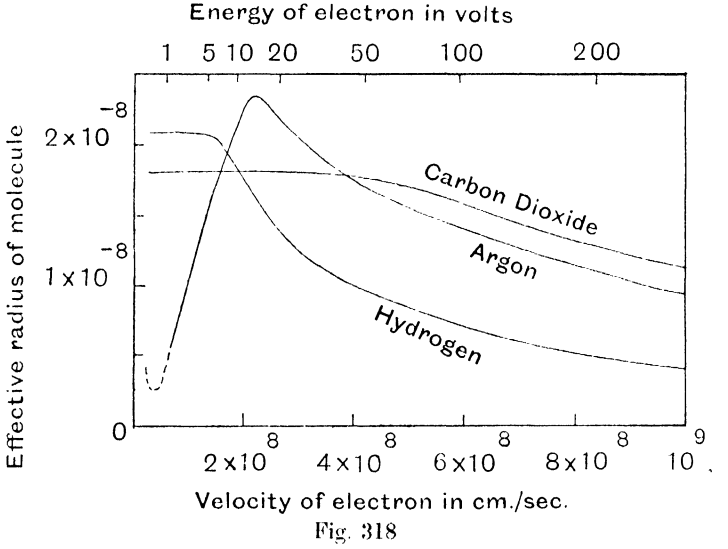



Fig. 318

increases when its velocity exceeds $3 \times 10^8$ cm./sec. It also shows a remarkable phenomenon discovered by Ramsauer, that argon (in common with krypton and xenon) presents a very small diameter to slow electrons. Townsend and Bailey showed that 20 mm. of argon can be added to 5 mm. of hydrogen without altering $k$ or $v$ very much, provided that $E/p$ is small, so that the argon atoms must be hit much less frequently by electrons, though four times as numerous as the molecules of hydrogen. They found a minimum effective radius of $2{\cdot}4 \times 10^{-9}$ cm., at a velocity of $4 \times 10^7$ cm./sec., indicated roughly by the dotted extension in Fig. 318. The effective radius of the hydrogen molecule also decreases slightly below $10^8$ cm./sec., that of helium and neon more so, though less than argon.

**223. Nuclear theory of the atom.** The suggestion of an atom with distributed electrons is due to Lord Kelvin, who showed that negatively charged particles can reside in stable configurations if they are imbedded in a sphere of diffused positive electricity. A set of $Z$ electrons requires a compensating sphere of charge $Ze$, if it is to be electrically neutral. Sir J. J. Thomson showed that, as $Z$ increases, concentric rings can be built up gradually from the inside, each ring containing electrons spaced equally round its circumference up to a certain maximum number, after which a new ring must begin to be formed outside it. A complete ring, or set of rings, may be supposed to constitute a particularly stable structure, to which an incomplete ring may tend by losing or gaining electrons; the former when it has only one or two electrons and the latter when it has nearly its full number. As $Z$ increases, therefore, we have atoms which tend to lose electrons, or become electropositive in solution, followed by atoms gradually less electropositive and finally electronegative; the cycle closing with an atom which is electrically, and probably also chemically, inert. We have thus a natural explanation of the recurrence of physical and chemical properties known as the periodic law. The assumed distribution of positive charge is difficult to reconcile with the observed deflexions of $\alpha$ particles (Art. 250), which Sir E. Rutherford explained as due to a concentrated charge, the *nuclear charge*, acting according to the inverse square law down to distances of the order of $10^{-11}$ cm. The hydrogen nucleus is a *proton*, having a charge of $+ 4{\cdot}77 \times 10^{-10}$ E.S.U. and a mass of $1{\cdot}65 \times 10^{-24}$ gm. Higher nuclei are in all probability clusters of protons and electrons, the excess of the former over the latter determining the nuclear charge (Art. 230). The idea that the nuclear charge determines the physical and chemical properties of the atom, latent in Thomson's theory, was put more definitely by van den Broek, who proposed to identify the number $Z$ (nuclear charge in terms of proton charge; number of circumnuclear electrons) with the *atomic number** of the element, or number specifying its order in the periodic classification.

On the nuclear theory, chemical and non-radioactive physical properties of the atom arise from the circumnuclear electrons, con-

* Shown in the table on pp. 492–3. The evidence for the existence of certain gaps is given in Art. 236.

trolled electrically by the nuclear charge. The absence of a direct effect of atomic weight (Art. 230) means that a proton and an electron can be added to the nucleus without altering its electrical attraction on the nearest circumnuclear electron—a further consequence of the smallness of the nuclear cluster. The circumnuclear electrons can no longer be at rest (cf. Art. 37), but must move in orbits round the nucleus. But we cannot admit all the orbits that Newtonian dynamics would allow as possible. The hydrogen atom ($Z = 1$) consists of a proton and one revolving electron, and the periodic time both of the orbit and of a small disturbance from it can take a continuous series of values. The spectrum of hydrogen would be continuous if either of these periodic times determined that of the radiation, whereas part of it consists of a line spectrum. Orbits must therefore be selected, and principles on which this should be done were laid down by Bohr and Wilson (Arts. 260, 261). Theory leads us to expect that electrons can move in different orbits and yet have the same, or nearly the same, total energy (that is the sum of the kinetic energy of the electron and its potential energy reckoned from infinity). Instead of speaking, therefore, of rings of electrons, we shall speak of the *energy levels* in the atom, leaving experiment to decide what energy is to be ascribed to each level, and how many electrons reside in each. The table on pp. 492–3 shows a scheme adopted by Bohr, simplified by neglecting such differences as exist between the electrons in the same level (Art. 240). The elements in the first group ($Z = 1, 2$) have in the normal state only one energy level, the $K$ level, corresponding to Thomson's innermost ring. This level has the largest negative energy, and may be visualised as the lowest of a series of levels under water, the surface of which corresponds to an electron at rest at a great distance from the nucleus (cf. Fig. 348). The $K$ level can contain two electrons, after which ($Z = 3$ to 10) a higher $L$ level (of smaller negative energy) is filled by electrons distant further on the average from the nucleus, and so on. The element of highest known atomic number (uranium, $Z = 92$) has seven levels $K$, $L$, $M$, $N$, $O$, $P$, $Q$. Electrons in the highest energy level of an atom can be detached with an energy of a few volts (Art. 224), and are, as Thomson observed, mainly instrumental in determining the chemical behaviour of the element.

*Energy levels of the elements.*

| Atomic number | | *K* | *L* | *M* | *N* | *O* |
|---|---|---|---|---|---|---|
| 1 | Hydrogen | 1 | | | | |
| 2 | Helium | 2 | | | | |
| 3 | Lithium | 2 | 1 | | | |
| 4 | Beryllium | | 2 | | | |
| 5 | Boron | | 3 | | | |
| 6 | Carbon | | 4 | | | |
| 7 | Nitrogen | | 5 | | | |
| 8 | Oxygen | | 6 | | | |
| 9 | Fluorine | | 7 | | | |
| 10 | Neon | 2 | 8 | | | |
| 11 | Sodium | 2 | 8 | 1 | | |
| 12 | Magnesium | | | 2 | | |
| 13 | Aluminium | | | 3 | | |
| 14 | Silicon | | | 4 | | |
| 15 | Phosphorus | | | 5 | | |
| 16 | Sulphur | | | 6 | | |
| 17 | Chlorine | | | 7 | | |
| 18 | Argon | 2 | 8 | 8 | | |
| 19 | Potassium | 2 | 8 | 8 | 1 | |
| 20 | Calcium | | | | 2 | |
| 21 | Scandium | | | (9) | (2) | |
| 22 | Titanium | | | (10) | (2) | |
| 23 | Vanadium | | | (11) | (2) | |
| 24 | Chromium | | | (12) | (2) | |
| 25 | Manganese | | | (13) | (2) | |
| 26 | Iron | | | (14) | (2) | |
| 27 | Cobalt | | | (15) | (2) | |
| 28 | Nickel | | | (16) | (2) | |
| 29 | Copper | | | 18 | 1 | |
| 30 | Zinc | | | | 2 | |
| 31 | Gallium | | | | 3 | |
| 32 | Germanium | | | | 4 | |
| 33 | Arsenic | | | | 5 | |
| 34 | Selenium | | | | 6 | |
| 35 | Bromine | | | | 7 | |
| 36 | Krypton | 2 | 8 | 18 | 8 | |
| 37 | Rudibium | 2 | 8 | 18 | 8 | 1 |
| 38 | Strontium | | | | | 2 |
| 39 | Yttrium | | | | (9) | (2) |
| 40 | Zirconium | | | | (10) | (2) |
| 41 | Niobium | | | | (11) | (2) |
| 42 | Molybdenum | | | | (12) | (2) |
| 43 | | | | | (13) | (2) |
| 44 | Ruthenium | | | | (14) | (2) |
| 45 | Rhodium | | | | (15) | (2) |
| 46 | Palladium | | | | (16) | (2) |
| 47 | Silver | | | | 18 | 1 |
| 48 | Cadmium | | | | | 2 |
| 49 | Indium | | | | | 3 |
| 50 | Tin | | | | | 4 |
| 51 | Antimony | | | | | 5 |
| 52 | Tellurium | | | | | 6 |
| 53 | Iodine | | | | | 7 |
| 54 | Xenon | 2 | 8 | 18 | 18 | 8 |

*Energy levels of the elements.*

| Atomic number | | *K* | *L* | *M* | *N* | *O* | *P* | *Q* |
|---|---|---|---|---|---|---|---|---|
| 55 | Caesium | 2 | 8 | 18 | 18 | 8 | 1 | |
| 56 | Barium | | | | | | 2 | |
| 57 | Lanthanum | | | | | (9) | (2) | |
| 58 | Cerium | | | | 19 | (9) | (2) | |
| 59 | Praseodymium | | | | 20 | (9) | (2) | |
| 60 | Neodymium | | | | 21 | (9) | (2) | |
| 61 | | | | | 22 | (9) | (2) | |
| 62 | Samarium | | | | 23 | (9) | (2) | |
| 63 | Europium | | | | 24 | (9) | (2) | |
| 64 | Gadolinium | | | | 25 | (9) | (2) | |
| 65 | Terbium | | | | 26 | (9) | (2) | |
| 66 | Dysprosium | | | | 27 | (9) | (2) | |
| 67 | Holmium | | | | 28 | (9) | (2) | |
| 68 | Erbium | | | | 29 | (9) | (2) | |
| 69 | Thulium | | | | 30 | (9) | (2) | |
| 70 | Ytterbium | | | | 31 | (9) | (2) | |
| 71 | Lutecium | | | | 32 | (9) | (2) | |
| 72 | Hafnium | | | | | (10) | (2) | |
| 73 | Tantalum | | | | | (11) | (2) | |
| 74 | Tungsten | | | | | (12) | (2) | |
| 75 | | | | | | (13) | (2) | |
| 76 | Osmium | | | | | (14) | (2) | |
| 77 | Iridium | | | | | (15) | (2) | |
| 78 | Platinum | | | | | (16) | (2) | |
| 79 | Gold | | | | | 18 | 1 | |
| 80 | Mercury | | | | | | 2 | |
| 81 | Thallium | | | | | | 3 | |
| 82 | Lead | | | | | | 4 | |
| 83 | Bismuth | | | | | | 5 | |
| 84 | Polonium | | | | | | 6 | |
| 85 | | | | | | | 7 | |
| 86 | Niton | 2 | 8 | 18 | 32 | 18 | 8 | |
| 87 | | 2 | 8 | 18 | 32 | 18 | 8 | 1 |
| 88 | Radium | | | | | | | 2 |
| 89 | Actinium | | | | | | (9) | (2) |
| 90 | Thorium | | | | | | (10) | (2) |
| 91 | Uranium $X_2$ | | | | | | (11) | (2) |
| 92 | Uranium | 2 | 8 | 18 | 32 | 18 | (12) | (2) |

NOTE.—*Numbers in brackets are uncertain.*

Elements with the same number of electrons in their highest energy level will resemble each other chemically: they will, so to speak, be partial isotopes (Art. 230). Elements with two identical upper levels ($Z = 58$ to 71) will resemble each other still more. In giving a plausible reason for the development of inner rings after an outer one has been begun, Bohr explained for the first time the occurrence of the series of rare earths, and their relative amenity to magnetic separation (Art. 276).

The account which the nuclear theory gives of electrolytic dissociation is of some interest. Consider the dissociation of HCl into $\overset{+}{\mathrm{H}}$ and $\overset{-}{\mathrm{Cl}}$. The former (except in so far as it may attract a cluster) is a hydrogen atom which has lost an electron, that is *a free proton*. The latter is a chlorine atom which has gained an electron and thereby completed its $M$ level to a semi-stable number of 8. It is the same as *an argon atom with an electron added to the nucleus*. Similarly the positive ion $\overset{++}{\mathrm{Ba}}$ of $BaCl_2$ is a xenon atom with two electrons taken from the nucleus. Metals have few electrons in their highest level, which they tend to lose and go into solution positively charged (Art. 172). But an atom does not in general lose all the electrons in its highest level when dissociated (cf. Art. 276).

We shall see that visible spectra arise from levels higher than those normally occupied by electrons (Arts. 259, 261): the visible spectrum of an atom, like its valency, is a *peripheral* property. Röntgen rays arise mainly from the $K$ and $L$ levels (Art. 239), while radioactivity is a property of the nucleus (Art. 244).

**224. Radiation and ionisation potentials.** The early stages of ionisation by collision are of interest in the nuclear theory, as we may expect to locate the highest energy level in an atom (or molecule) by finding the least energy required to detach an electron from it. The method was used by Lenard, and afterwards by Franck and Hertz. In their experiments electrons are emitted by the heated filament $A$ (Fig. 319), kept at a constant potential of, say, 10 volts above earth. A grid $B$, close to $A$, is put at potential $10 + V$, where $V$ is positive and variable. A plate or cylindrical sheath $D$ is connected to one pair of quadrants of an electrometer; or through a sensitive galvanometer to earth, as in Fig. 320. The

apparatus is thus a triode, with electrodes made of the same metal in order to minimise contact potentials, and filled with gas at a pressure of a few hundredths of a millimetre. When $V$ is increased from zero, no deflexion of the electrometer occurs up to a certain point, but there begins almost abruptly, so that the critical potential can be fixed to a fraction of a volt. Franck and Hertz found, for the critical potential $V$, 7·5 volts in nitrogen, 11 volts in hydrogen, 12 volts in argon, 16 volts in neon, 20·5 volts in helium. Since electrons emitted by the filament cannot reach $D$, the current must arise from ions (of either sign) produced in the space $BD$, or from electrons set free from $D$. Both are possible since an electron may either disintegrate an atom or molecule by collision, or stimulate it to emit light which acts photoelectrically on $D$, $B$ being already at a positive potential for drawing off the electrons so generated*. The potential required to produce light is less than that required to ionise, so that the first rise of current would be due to radiation.

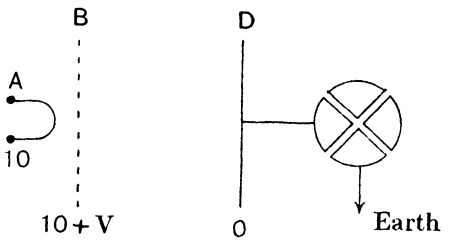


Fig. 319

Franck and Hertz, who fixed the critical potential of mercury vapour at 4·9 volts, found that the spectrum of mercury first appears at the same potential, the spectrum consisting in this case of the single line $\lambda = 2537$ A.U. Davis and Goucher proposed to separate the effects of radiation and ionisation by inserting a second grid $C$ between $B$ and $D$ (Fig. 320).

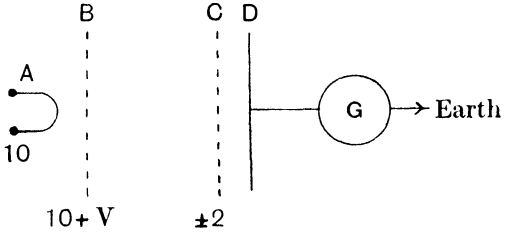


Fig. 320

Curve I of Fig. 321 shows the current for values of $V$ between 4 and 11 volts when $C$ is at + 2 volts, curve II when it is at − 2 volts. Deflexions begin at about 4·9 volts. Suppose for the moment that

* This possibility must be considered since an electron may, on Bohr's theory, be raised to an energy level higher than those normally occupied in the atom, and afterwards spring back with emission of light (Art. 259).

only radiation is produced, by collisions near $B$. The light falls partly on $C$ and partly on $D$. When $C$ is at $+2$ volts, the photoelectric emission from $D$ is drawn to $C$, giving a positive current from $C$ to $D$. When $C$ is at $-2$ volts, some of the emission from $C$ may reach $D$ in spite of its being on the unilluminated side, giving a negative current from $C$ to $D$. Had ionisation by collision occurred near $B$, the positive ions so generated would have moved towards $C$, and some would always have reached $D$, though more with $C$ at $+2$ volts than at $-2$ volts. The curves I and II would then have been on the same side. We conclude that the effect at 4·9 volts is due to radiation only. Curve I registers a further quick rise near 6·7 volts and curve II a corresponding fall (points $X$, $X'$), showing that more radiation is produced near 6·7 volts. The same thing occurs at $9{\cdot}8 = 2 \times 4{\cdot}9$ volts, for a reason given by Franck and Hertz. The electrons emitted by $A$ rise first through a potential of 4·9 volts, where some collide and excite radiation of wave-length 2537 A.U., losing energy in the process. Since the quantum voltage corresponding to this wave-length is $12300/2537 = 4{\cdot}85$ (Art. 220), all or nearly all the energy is lost, so that these electrons have to begin again and reach a velocity of 4·9 volts once more at the grid $B$. There will therefore be two layers of 2537 radiation, the sudden onset of the second causing the rise at $Y$ and the fall at $Y'$. The rise at $Z$ (10·3 volts) is accompanied by a *rise* at $Z'$, pointing now to ionisation by collision. Davis and Goucher confirmed their results by putting $C$ at a high negative potential, $-20$ volts. All the positive ions generated between $B$ and $C$ are now caught on $C$, so that the pure radiation effect is isolated*.

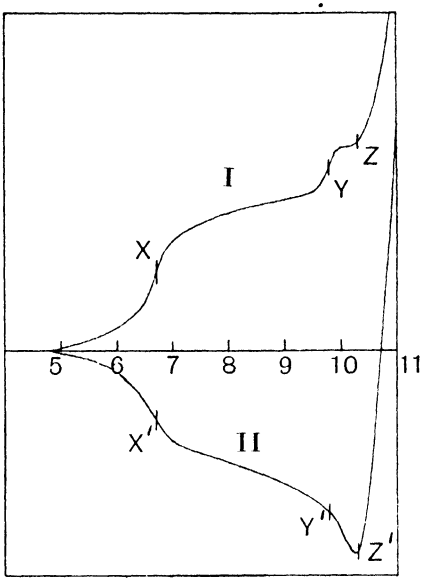



Fig. 321

* The chance of ionisation by collision between $C$ and $D$ is minimised by having the two conductors close together.

The curve corresponding to II then falls continually, with acceleration at $X'$ and $Y'$ but nothing conspicuous at $Z'$. The evidence, therefore, is that radiation is produced in mercury vapour at 4·9 and 6·7 volts, and ionisation at 10·3 volts.

Critical potentials are located most sharply with two grids $B$, $C$ kept at the *same* potential, say $10 + V$. This is a modification of Lenard's original method, introduced by Horton and Miss Davies. It gives the electrons which have acquired any critical velocity a greater chance of colliding at that velocity, whereas a single grid gives only a small region of maximum velocity near the actual grid wires, the electrons entering this space being also few in number and deflected somewhat from their path by the grid wires. Fig. 322

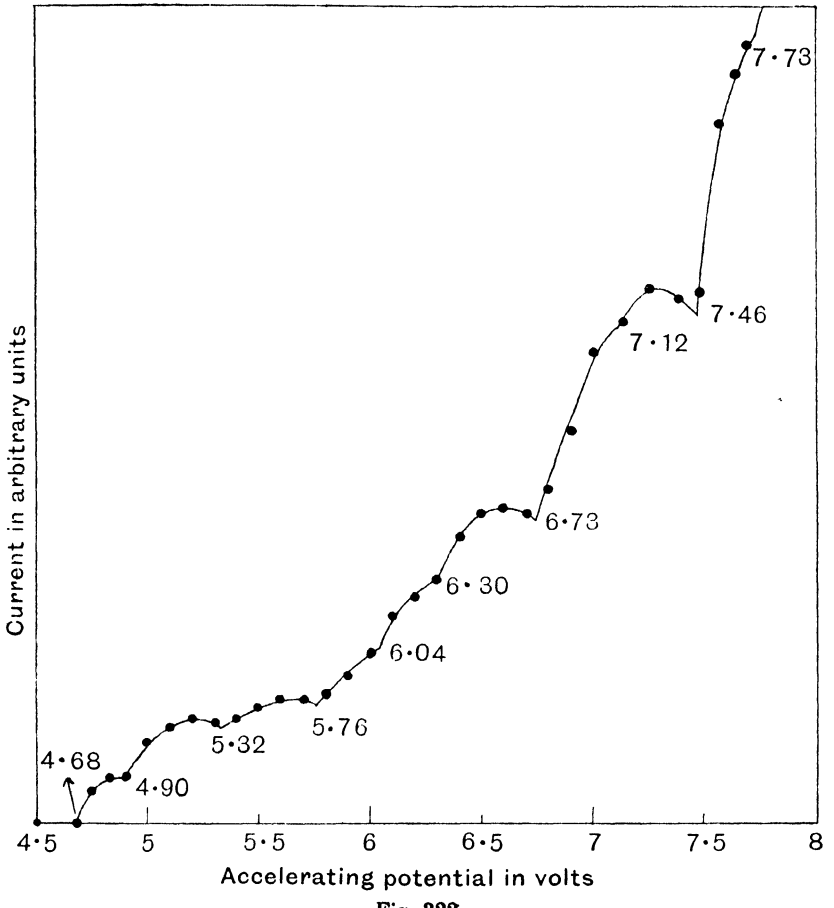



Fig. 322

shows part of a curve obtained by Franck and Einsporn with mercury vapour: the weaker breaks are confirmed by other curves, which also indicate one at 5·47 volts. Franck and Einsporn found sixteen breaks between 4·68 volts and the ionising potential of 10·38 volts. Confining attention to the eight strongest, six of these correspond to observed lines in the spectrum of mercury by the quantum relation, as shown in the following table, and one is a double voltage. The ionising potential corresponds theoretically not to a line, but to the limit of a series (Art. 261). The experiments therefore confirm the quantum theory, whatever model may be adopted to explain the properties of particular atoms; though one might wish that the breaks were sometimes sharper. The highest

| Critical potential $V$ in volts | Observed wave-length $\lambda$ in A.U. | $\lambda V$ | Remarks |
|---|---|---|---|
| 4·68 | 2656 | 12430 | |
| 4·90 | 2537 | 12420 | |
| 5·76 | 2150 | 12370 | |
| 6·73 | 1850 | 12450 | |
| 7·46 | — | — | |
| 7·73 | 1604 | 12390 | |
| 8·86 | 1403, 1400 | 12430, 12400 | |
| 9·79 | — | — | $2 \times 4{\cdot}90 = 9{\cdot}80$ |
| 10·38 | 1188 | 12340 | Series limit |

energy level of the mercury atom ($Z = 80$) is thus fixed with some certainty at $-$ 10·38 volts $= -$ 1·65 $\times$ $10^{-11}$ ergs.

There is greater theoretical complication about the simplest element, hydrogen, since we begin with a molecule instead of an atom, and the stages which must be passed through before ionisation are not obvious in advance. The evidence is improved by Olmstead's device of heating the gas on its way to the grid, whereby the proportion of free atoms is increased. The ionising potential of the hydrogen atom seems to be 13·5 volts. Helium has at least two critical points, 20·45 and 21·25, below the ionising potential of 25·3 volts. Franck and Knipping found indications of a break at 79·5 volts, due apparently to the expulsion of both electrons from the atom.

Ionisation, when once produced, gives much more current than radiation. Experiments on critical potentials aim primarily at

fixing the velocity at which an electron can just produce a certain effect in a favourable collision. The rarity or otherwise of the phenomenon is not taken into account.

**225. Ionisation by collision in a uniform field.** Ionisation by collision at low velocities was discovered by Townsend, with the apparatus of Fig. 293. Let the plate $C_1$ emit $n_0$ electrons per second in a given time, and let $\alpha$ be the average number of electrons which a single electron will detach from molecules in moving through 1 cm. in the direction of the electric force. The number of electrons crossing a plane distant $x$ from $C_1$ will increase with $x$. If $n$ is that number, $\alpha n dx$ new electrons are produced in a succeeding short distance $dx$, so that $dn = \alpha n dx$. Integrating, $n = n_0 e^{\alpha x}$, and the number of electrons reaching $C_2$ is $n_0 e^{\alpha d}$, where $d$ is the distance between $C_1$ and $C_2$. Experiments are made by giving $d$ various values and increasing the voltage in proportion, so that $E$ remains constant. A typical result is the following:

*Air*, $p = 4$ *mm.*, $E = 700$ *volt/cm.*, $\alpha = 8{\cdot}16$, $\beta = 0{\cdot}0067$.

| Distance $d$ between plates in cm. | 0·2 | 0·3 | 0·4 | 0·5 | 0·6 | 0·7 | 0·8 |
|---|---|---|---|---|---|---|---|
| Current in arbitrary units | 5·12 | 11·4 | 26·7 | 61 | 148 | 401 | 1500 |
| $e^{\alpha d}$ | 5·11 | 11·6 | 26·1 | 59 | 133 | 301 | 680 |
| $\frac{(\alpha-\beta)e^{(\alpha-\beta)d}}{\alpha-\beta e^{(\alpha-\beta)d}}$ | 5·11 | 11·6 | 26·5 | 62 | 149 | 399 | 1544 |

The current is accurately exponential over a 12 to 1 ratio, after which the law ceases to hold. The discrepancy will be explained shortly; meanwhile it is clear that $\alpha$ can be determined from observations at the shorter distances. The values of $\alpha/p$ for a given gas, when plotted against $E/p$, all lie on a single curve (Fig. 323). This relation would be expected from the considerations of Art. 221; for the velocity of agitation of an electron is unchanged by changing $E$ and $p$ to $\lambda E$ and $\lambda p$, but the number of collisions is $\lambda$ times as great as before, so that $\alpha$ becomes $\lambda\alpha$.

An extension of the theory shows that the failure of the exponential law at large distances can be ascribed to ionisation by

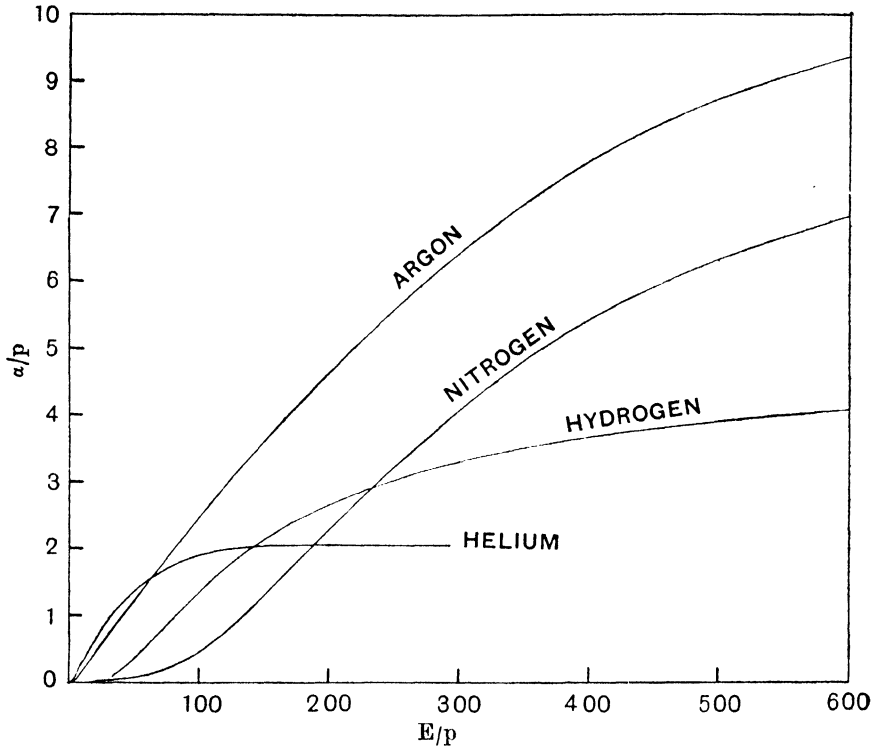



Fig. 323

positive ions. Let $n_0$ be the number of electrons emitted by the negative electrode per second, $n$ the number arriving at the positive electrode, and $p$, $q$ the numbers produced by collision on the two sides of the plane $x$, as shown. Then $n = n_0 + p + q$. In one second $n_0 + p$ electrons cross the plane $x$ from left to right, and $q$ positive ions from right to left. If $\beta$ is the number of electrons produced by a positive ion in moving through 1 cm. in the direction of the electric force $(n_0 + p)\,\alpha dx + q\beta dx$ electrons are generated in the strip $dx$, so that $dp/dx = (n_0 + p)\,\alpha + q\beta$, or

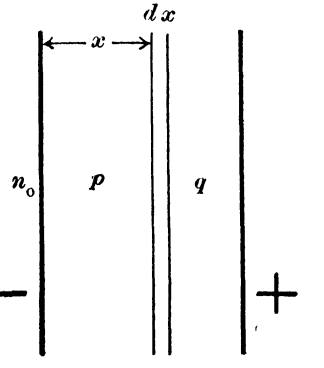



Fig. 324

$$\frac{d}{dx}(n_0 + p) - (\alpha - \beta)(n_0 + p) = n\beta.$$

Since $p$ vanishes with $x$,

$$n_0 + p = \left(n_0 + \frac{n\beta}{\alpha - \beta}\right) e^{(\alpha-\beta)x} - \frac{n\beta}{\alpha - \beta}.$$

When $x = d$, $n_0 + p = n$. Substituting and solving for $n$, we have

$$n = n_0 \frac{(\alpha - \beta)\, e^{(\alpha-\beta)d}}{\alpha - \beta\, e^{(\alpha-\beta)d}} \quad \ldots\ldots\ldots\ldots (282).$$

The last line of the table shows that the higher currents can be explained accurately by a suitable choice of $\beta$. Though few electrons are detached by a positive ion in moving through a centimetre, the effect on the current is very great at the larger distances.

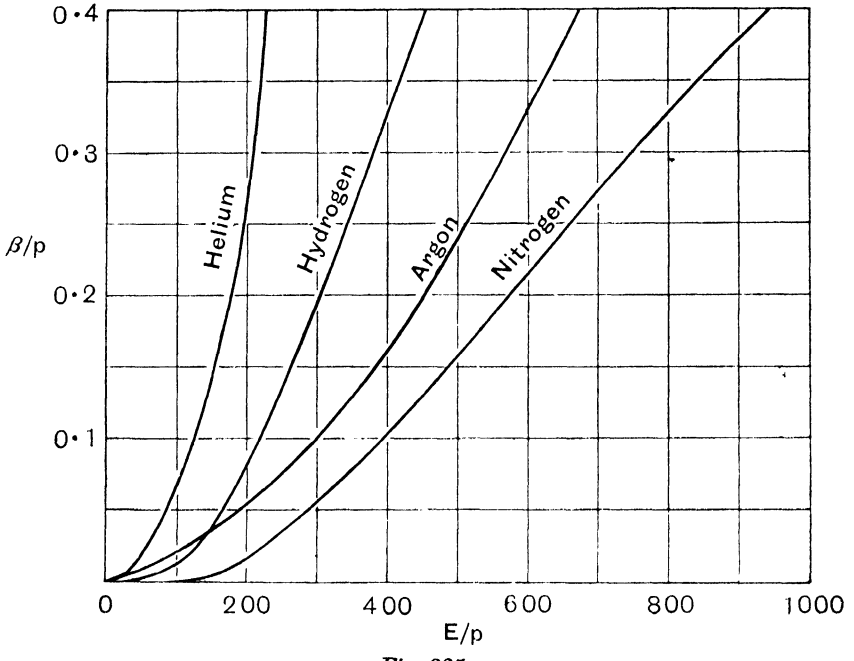



Fig. 325

Fig. 325 shows the curves connecting $\beta/p$ and $E/p$ in various gases. The lower ionising power of the positive ion in a given field is a consequence of its lower velocity: we shall see that an $\alpha$ particle, which is a helium nucleus moving with a velocity of about $2 \times 10^9$ cm./sec., produces more ions per centimetre than an electron moving at any speed (Art. 249).

**226. Theory of the sparking potential.** Ionisation by collision begins to modify the Röntgen ray saturation curve at atmospheric pressure, when the electrodes are a centimetre or two apart, at electric forces of the order of 20,000 volt/cm. The complete curve is shown diagrammatically in Fig. 326. The part $AB$ corresponds to Fig. 304. At some point $C$, depending on the amount of

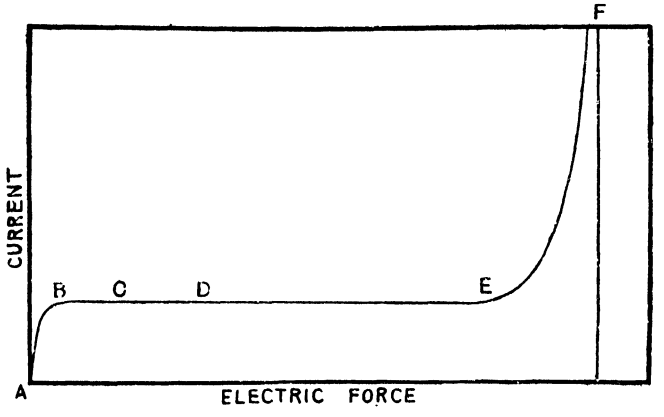



Fig. 326

moisture present (Art. 221), the ions begin to decrease in average size, all of them becoming electrons at $D$ (10,000 volt/cm.). There is nothing to show so far on the curve, since the charge transported remains the same as that produced by the rays per second. Ionisation by collision, which begins at $E$, gives a gradually increasing current until a spark passes at $F$: the gas becomes luminous and the current continues to flow even after the Röntgen rays are cut off.

The explanation of the sparking potential follows naturally from Townsend's theory. If $d$ is such that

$$\alpha - \beta e^{(\alpha-\beta)d} = 0,$$

the denominator of equation (282) vanishes, and $n/n_0$ becomes infinite. In physical language, a finite current can be generated from a very small number of ions present originally, which is what happens when a spark passes. The sparking distance $d$, for a field $E$ between two parallel plates, should therefore be

$$d = \frac{\log_e \alpha - \log_e \beta}{\alpha - \beta} \quad \ldots\ldots\ldots\ldots\ldots (283).$$

To test the theory a force $E$ is selected, $\alpha$ and $\beta$ calculated in terms of $E$ and the pressure, and the plates set at a distance apart given by (283). The potential $V$ required to produce a spark is then compared with $Ed$. Experiments on air give the following results:

| $E$ in volts per cm. | $p$ in mm. of mercury | $d$ in cm. | $Ed$ | $V$ in volts | $pd$ |
|---|---|---|---|---|---|
| 1050 | 8 | ·765 | 803 | 803 | 6·12 |
| 1400 | 8 | ·431 | 601 | 603 | 3·45 |
| 1050 | 6 | ·572 | 601 | 604 | 3·43 |
| 700 | 4 | ·871 | 610 | 615 | 3·48 |
| 1050 | 4 | ·454 | 477 | 480 | 1·82 |
| 525 | 2 | ·910 | 481 | 488 | 1·82 |
| 700 | 2 | ·575 | 403 | 407 | 1·15 |
| 350 | 1 | 1·13 | 395 | 398 | 1·13 |
| 437 | 1 | ·832 | 364 | 365 | ·83 |
| 350 | ·66 | ·965 | 338 | 340 | ·64 |
| 437 | ·66 | ·766 | 335 | 336 | ·505 |

These numbers exemplify a law first enunciated by de la Rue and Müller, that the sparking potential is a function of the product $pd$ of pressure and sparking distance only. Thus $V$ is practically the same when $p = 4$ and $d = 0{\cdot}454$ as it is when $p = 2$ and $d = 0{\cdot}910$. In order to prove this theoretically, put $\alpha = pf(E/p)$ and $\beta = p\phi(E/p)$ in equation (283). Since $E = V/d$, the equation becomes

$$pd\left[f\left(\frac{V}{pd}\right) - \phi\left(\frac{V}{pd}\right)\right] = \log_e f\left(\frac{V}{pd}\right) - \log_e \phi\left(\frac{V}{pd}\right),$$

which involves $V$ and $pd$ only.

The curves connecting $V$ and $pd$ for certain gases are given in Fig. 327. They have much the same shape: $V$ is large when $pd$ is either small or large, and has a minimum value between. The following are the minimum sparking potentials for certain gases, in volts:

| | | | |
|---|---|---|---|
| Carbon dioxide | 470 | Nitrogen | 290 |
| Sulphur dioxide | 455 | Hydrogen | 275 |
| Oxygen | 455 | Argon | 194 |
| Air | 340 | Helium | 154 |

Neon also has a low sparking potential, and gives a bright glow, which has been utilised in low-power lamps. The least potential for air corresponds to $pd = 0{\cdot}5$, or a sparking distance at atmospheric

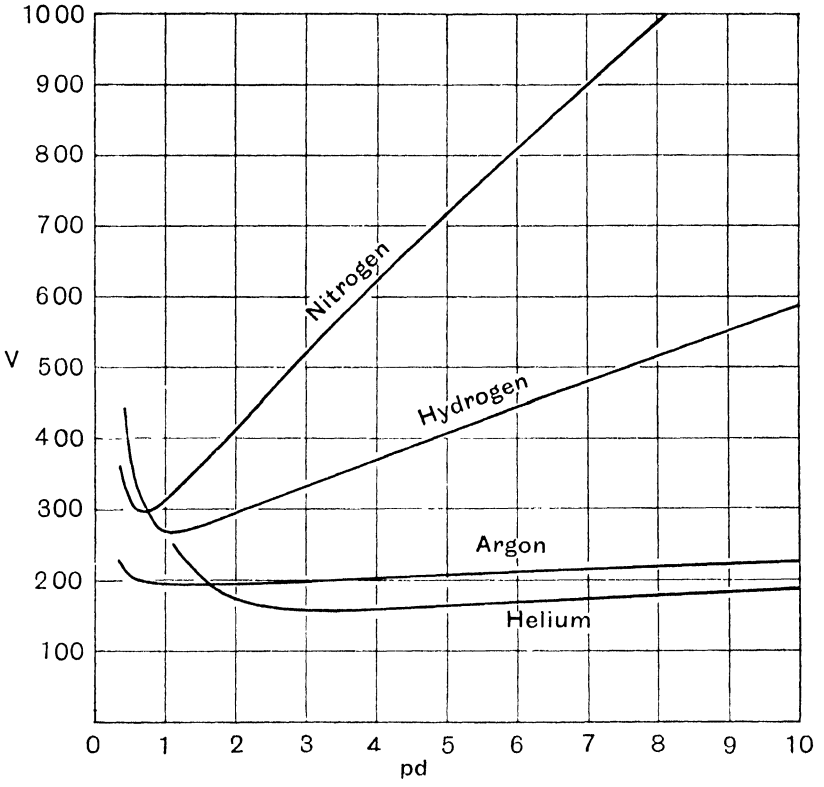



Fig. 327

pressure of 1/150 mm. Sparking potentials at atmospheric pressure are, therefore, usually much above the minimum. A rough rule is to allow 30,000 volts to the centimetre for all distances above 1 mm. There is a tendency, below $pd = 0{\cdot}5$, for the discharge to choose a path round the back of the electrodes in preference to the straight path between them, since the potential in the latter case exceeds that in the former. This can be prevented by embedding the electrodes in ebonite.

Another precaution is necessary in measuring sparking potentials. Though a discharge takes place when the applied potential is equal to the sparking potential, it does not necessarily do so at once, because the few ions present may not be favourably placed for multiplying themselves by collision. As much as a quarter of an hour may elapse if the applied potential is only a volt or so above the sparking potential. The "lag" is much reduced by allowing

ultra-violet light to fall on the negative electrode. This is no doubt the explanation of Hertz's observation (Art. 208), which led to the discovery of the photoelectric effect.

An electric discharge, when once started, goes on by itself as long as a sufficiently high potential is applied to the electrodes. The potential required to maintain a current is not in general the same as that required to start it, being greater, for example, in brush discharges from fine points and less in the discharges between parallel plates for the higher values of $pd$. The current itself generates the supply of ions necessary for its own continuance. It is very difficult to give a comprehensive theory, because there are several processes of ionisation at work, and their relative efficacy is uncertain. Cathode rays are produced by the impact of positive ions which have acquired a high velocity in the Hittorf dark space. Positive ions may therefore liberate electrons from the negative electrode as well as from the molecules of the gas. This effect is negligible at the higher pressures, because the sparking potential has been almost universally found to be independent of the material of the electrodes; but it certainly comes into play at very low pressures.

The larger the linear dimensions of an apparatus, the lower the pressure required to exhibit a particular phenomenon. These results are covered by a general theorem, due to Holm, on discharges in geometrically similar systems. Consider two discharge tubes $D_1$, $D_2$ (Fig. 328), made of the same materials and filled with the same gas. Let the linear dimensions be in the ratio 1 to $k$, the pressures in the inverse ratio $k$ to 1.

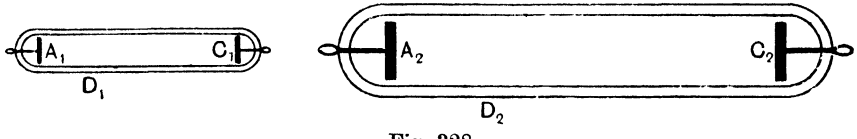



Fig. 328

Suppose that at a particular time the distribution and movement of the ions in the tubes are exactly similar, but that the total numbers are in the ratio 1 to $k$. By this we mean that to every group of $n$ ions in an assigned small region of $D_1$, and with velocities in an assigned small range, corresponds a group of $kn$ ions in the corresponding region of $D_2$ and with velocities in the same range.

To an electric force $E_1 = e_1/r_1^2$ in $D_1$ corresponds the force $E_2 = ke_1/k^2r_1^2 = E_1/k$ in $D_2$. These correlations, with others easily deduced from them, are given here for reference:

| | $D_1$ | $D_2$ |
|---|---|---|
| Linear dimensions .. .. | $d$ | $kd$ |
| Pressure .. .. .. .. | $p$ | $p/k$ |
| Current .. .. .. .. | $i$ | $i$ |
| Potential at corresponding points | $V$ | $V$ |
| Electric force .. .. .. | $E$ | $E/k$ |
| Free path of an ion .. .. | $l$ | $kl$ |

Now specialise the group still further by assigning to it a given range of free paths in the neighbourhood of a value $l$. The velocity acquired in a free path is proportional to $El$, and is therefore the same in both tubes. Hence the equality of the distribution of velocities subsists till the next collision, and the effect of that collision is the same; the same fraction of each group is deflected through a given angle and the same fraction of collisions results in ionisation. It follows that the correlation between the groups persists indefinitely. The argument is independent of particular assumptions as to equality of free paths, velocities of agitation, and so forth, and covers diffusion, velocity under the electric force and ionisation by collision of both kinds. The conclusion still holds in the presence of the electrodes and walls of the tube. Consider two groups of ions in $D_1$, one of which is colliding in the gas and the other impinging on an electrode. In $D_2$ each group has $k$ times as far to go, so that there is a constant ratio between the number of collisions. The argument of permanence proceeds as before, whatever the effect of a collision with an electrode, whether in absorbing ions or in generating cathode rays. Similarly for the walls. The theory will include the effect of Röntgen rays if their ionising power varies nearly inversely as the square of the distance, as it seems to do. It does not include recombination, but we may expect this to be relatively small at high forces.

These considerations make it highly probable that the potential $V$ required to maintain a current $i$ in a discharge tube depends only on the product $pd$, and is therefore given by an equation of the form

$$V = f(i, pd),$$

where $f$ depends on the shape only, and $d$ is a linear dimension.

When $i = 0$, $V$ is the sparking potential, and we recover de la Rue and Müller's law. In discharge tubes at pressures inversely proportional to the linear dimensions we should have similar distributions of potential and luminosity.

**227. Secondary emission of electrons from metals.** If the anode in a tube like that of Fig. 287 is placed obliquely at the corner $A$, fluorescence is produced in the left-hand branch of the tube, as if cathode rays had been reflected diffusely by the anode. The secondary rays cast shadows, and are deflected electrically and magnetically like negatively charged particles. These observations are due to Silvanus Thompson. Swinton applied Perrin's method to prove that the rays carry a negative charge, and found (what some observers do not confirm) that the secondary emission may be greater when electrons fall obliquely on the obstacle than when they fall normally. We shall assume, in the light of these observations, that the secondary rays are electrons.

Lenard worked with electrons liberated photoelectrically and accelerated by known voltages. He also applied retarding potentials to the secondary electrons (cf. Art. 212) and concluded that most of them had energies of less than 11 volts, when that of the primary electrons was 1000 volts or more. A further experiment showed that some of the secondary electrons, at least, must come from the atoms of the obstacle and thus constitute ionisation of the solid metal; for *the secondary electrons may exceed the primary electrons in number*. This effect can be shown with a triode having a galvanometer in its plate circuit (Fig. 329). If $P$ is kept at (say) 200 volts above the filament, the plate current $i$ is usually positive when the grid is at 100 volts and negative when it is at 400 volts, showing that in the second case more electrons leave the plate than are caught on it. An electron may therefore, on the average, produce more than one secondary electron; but we have no means of knowing whether it remains in the metal itself or not.

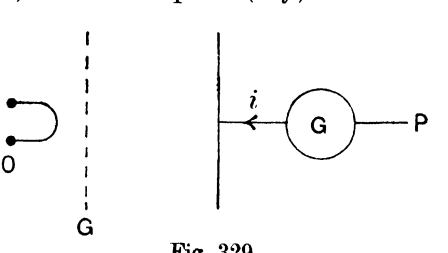


Fig. 329

The phenomena in a triode are extremely complicated, since

electrons may be emitted from the grid wires and an electron may pass either between the grid wires (when it does not acquire the velocity corresponding to the grid potential) or so near them as to be deflected from its path. Let us, however, in order to gain a rough notion of the phenomena, suppose that the grid acts like an equipotential plane, perfectly transparent to electrons. The general type of curve obtained when the plate potential is raised from 0 to 500 volts, with the grid at about 400 volts, is shown diagrammatically in curve 1 of Fig. 330. The plate current rises

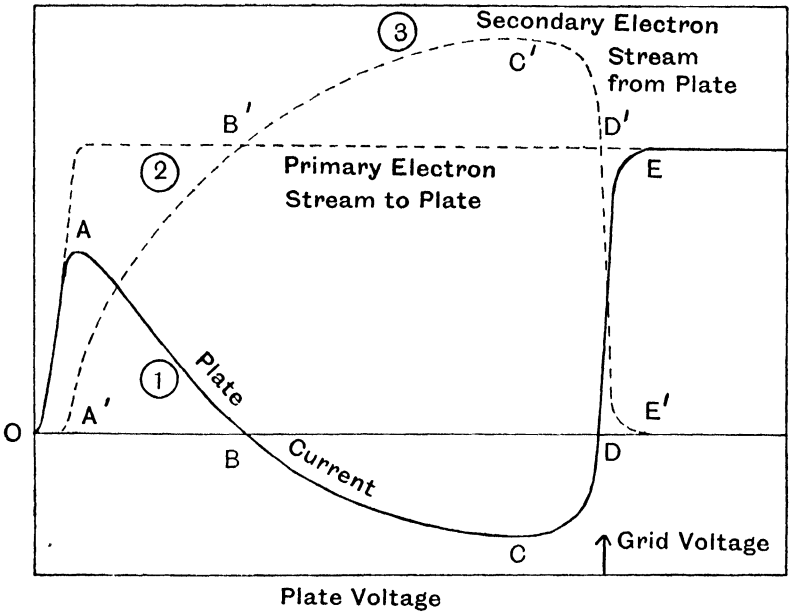



Fig. 330

at first to a point $A$, falls and is negative over a range $BCD$, and finally tends to a constant positive value at $E$. The final current, on our simple assumptions, is equal to the charge sent off by the filament per second; for the plate will, at high potentials, attract back the secondary electrons emitted from it, Lenard's experiments having made it probable that few are emitted with a velocity comparable with that of the incident primary electrons. In the absence of secondary emission we should expect a curve of the form 2, reaching saturation (for currents of the order of $10^{-7}$ ampere) at a positive potential of a volt or two at the most. The

difference between the two curves represents the secondary emission, and is plotted as curve 3. Its form (uncertain below $A'$) is in general what we should expect, knowing that the potential of the plate determines the velocity of the electrons which fall on it. The secondary stream rises at first, becoming equal to the primary stream at $B'$, and would doubtless continue to increase if all the secondary electrons could be removed from the plate. But at $C'$, where the potential of the plate approaches that of the grid, some begin to fall back on the plate, and beyond $E'$ all (or nearly all) fall back, as we have explained. A fair reason has therefore been given for each of the stages of curve 1.

As the velocity of the primary electrons increases, the average velocity of the secondary electrons decreases, being less than 5 volts at 1000 volts. There are, however, always a few secondary electrons with the velocity of the primary electrons, which may be due to scattering. Triode experiments indicate that secondaries may be produced at a velocity of 1·5 volts, and also that a corrugated plate emits more than a plain plate, confirming Swinton's observation.

**228. Positive rays.** The glow nearest to the cathode in a discharge tube spreads out at low pressures (Art. 205), and extends to a distance of 2 cm. or so from the cathode at a discharge potential of 10,000 volts. If the cathode is perforated, the glow is seen also on the far side, taking the form of parallel streams of red or orange light. These rays were first observed by Goldstein in 1886, and are called *positive rays*, since, as we shall see, they contain positively charged particles moving with a high velocity.

The magnetic deflexion of the rays is very small compared with that of the cathode rays; but in 1898 W. Wien succeeded in de-

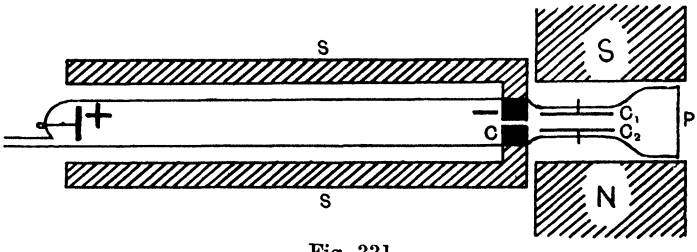



Fig. 331

flecting them by both electric and magnetic forces. His apparatus, which has been used with slight modifications by all subsequent observers, is shown in Fig. 331. The rays pass through the perforated iron cathode $C$ and fall on a glass plate $P$ which fluoresces at the point of impact, so that the deflexions are easily observed. An electric field is applied between the pair of plates $C_1$, $C_2$, and a magnetic field in the same direction from a large electromagnet. The discharge tube is screened from the direct action of the electromagnet by an iron cylinder $S$.

Suppose for simplicity that the electric and magnetic fields are uniform over the whole distance $l$ from the cathode to the plate $P$, and consider the effect on a charged particle moving with velocity $v$. The electric deflexion is $eEt^2/2m$, and the magnetic deflexion, supposed small, is $evHt^2/2mc$ in the perpendicular direction, where $t = l/v$ is the time occupied by the free flight of the particle. Calling these deflexions $x$ and $y$ respectively, we have

$$x = \frac{eEl^2}{2mv^2}, \quad y = \frac{eHl^2}{2mcv} \quad \ldots\ldots\ldots\ldots (284).$$

Hence

$$y^2 = \frac{H^2l^2}{2Ec^2} \cdot \frac{e}{m} \cdot x \quad \ldots\ldots\ldots\ldots\ldots (285),$$

so that all particles having the same value of $e/m$ lie on a parabola. The electric deflexion is, *caeteris paribus*, inversely proportional to the kinetic energy of the particle. It is natural to suppose that the positive rays are the positive ions generated in the discharge tube, which move past the cathode with a high velocity. In this case no particle can have a kinetic energy greater than $eV$, where $V$ is the discharge potential; hence there is a definite inferior limit to the electric deflexion, and the points corresponding to the different velocities should be spread over a parabolic arc, as shown in Fig. 332.

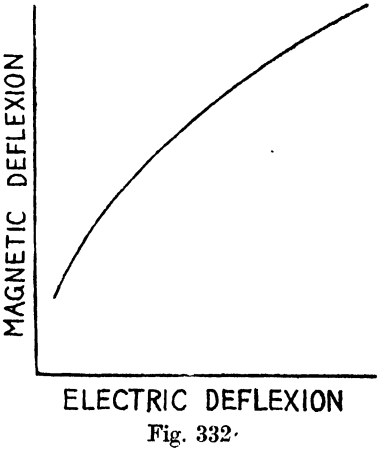


Fig. 332.

All experiments on electric and magnetic deflexion agree in

giving values of $e/m$ of the order $10^{14}$, corresponding to that of a charged atom or molecule, and velocities of the order $10^8$ cm./sec.; but the phenomenon is not so simple as we have supposed it to be. There is always an undeflected central spot, and no gap between it and the deflected curve, nor do the curves usually much resemble parabolas. This may be explained by supposing that some of the particles become discharged between the cathode and the fluorescent screen, and thus experience less than the normal deflexion. Wien showed that the opposite phenomenon also occurs: a stream from which the charged particles have been removed becomes charged as it moves through the gas, and the proportion of charged to uncharged particles is the same as if all the particles had been charged originally. This is undoubtedly an effect of collision with circumnuclear electrons from the atoms of the gas, whereby they gain or lose electrons without great loss of velocity (cf. Art. 249). The extent to which this interchange takes place when slow positive ions move through a gas is not clear, but the theory of Art. 225 is not disturbed by such interchange, since it makes no assumptions as to the mechanism by which new electrons are made to appear.

The positive rays described above are not strictly analogous to cathode rays, since they are generated in the gas and not at the anode. True *anode rays* were first observed by Gehrcke and Reichenheim in 1906. They are produced by anodes containing salts, either cold under high discharge potentials or hot under low discharge potentials, the latter phenomenon having been observed by other experimenters also. Anode rays of metals are produced from an anode consisting of their salts mixed with graphite. The rays appear to shoot from the anode, as cathode rays shoot from the cathode. Gehrcke and Reichenheim detected the positive charge on the rays by Perrin's method (Art. 206), and deflected them electrically and magnetically to an extent suggesting that they are positively charged atoms (see also Art. 229 below).

It is clear that the normal parabolas can only be obtained when the pressure in the space behind the cathode is so low that very few collisions occur there. The pressure in the main tube must at the same time be high enough to allow the discharge to pass with ease; but no separation of the chambers is possible since the rays cannot pass through even the thinnest membranes. Wien solved

the difficulty by having a long capillary tube between the discharge tube and the tube in which the properties of the rays are examined. The latter is kept very highly exhausted, and gas is continually supplied to the discharge tube by a second fine capillary to make up the loss of gas by diffusion. Using these methods v. Dechend and Hammer were able to obtain parabolic bands like that shown in Fig. 332. In hydrogen the strongest band corresponds to a hydrogen atom with a single positive charge ($e/m = 2{\cdot}9 \times 10^{14}$), and the kinetic energy of the least deflected particles is that which would be acquired in falling through the whole discharge potential. Evidence was also obtained of charged hydrogen molecules ($H_2+$) and of singly and doubly charged carbon atoms ($C+$, $C++$), the latter arising from inpurities present in the tube. These experiments gave the first experimental proof that the atoms and molecules of a particular gas are all of approximately the same mass. Similar experiments were made shortly afterwards by Sir J. J. Thomson, who devised a method of deciding whether the charge is single or multiple, and thus determining the atomic weight of the corresponding particle quite unequivocally. This will be understood from Fig. 333 which shows (diagrammatically) two curves obtained with oxygen in the discharge tube. Here $O$ is the undeflected spot, 1 a parabolic arc corresponding to $m/e = 16$ in terms of the hydrogen atom. The thick part of the curve is no doubt caused by singly charged oxygen atoms: it has, however, a thinner prolongation extending to exactly half its distance from the vertical through $O$. This must be due to particles with kinetic energy up to twice the normal amount; that is, to oxygen atoms which have lost two electrons in the discharge tube, and regained one of them after passing the cathode. This view is confirmed by the appearance, on all plates showing the beaded parabola 1, of a curve 2 corresponding to $m/e = 8$, caused by atoms which here remained short of two electrons throughout.

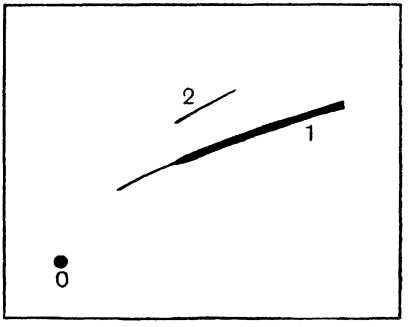


Fig. 333

The experiments of this article, together with those of Arts. 207 and 250, constitute the basis of the nuclear theory of matter. The singly charged hydrogen atom is, on that theory, a free nucleus. It is the atom of positive charge, or *proton* (cf. Art. 229). The absence of a doubly charged particle of the same mass, first proved by Hammer, shows that two electrons cannot be detached from a neutral hydrogen atom. Similarly three electrons cannot be detached from a helium atom, though the $\alpha$ particle of radioactive disintegration is a helium nucleus without attendant electrons, carrying a charge $2e$ (Art. 248). Helium only loses one electron in a positive ray tube, krypton three, mercury as many as eight: no high element loses more than a small part of its circumnuclear electrons. Hence the particle remains an effective atom, with a spectrum. Stark found that positive rays show a Doppler effect, spectral lines changing in position according as the rays approach the spectroscope or recede from it.

**229. The mass-spectrograph.** Positively charged particles whose masses differ by less than 10 per cent. give parabolas so close that they can hardly be separated. Greater resolution is attained in Aston's *mass-spectrograph*, in which particles of varying velocity, but the same value of $e/m$, are brought to a focus at the same point of a photographic plate. The principle of the instrument is shown in Fig. 334. Positively charged particles from a discharge tube pass

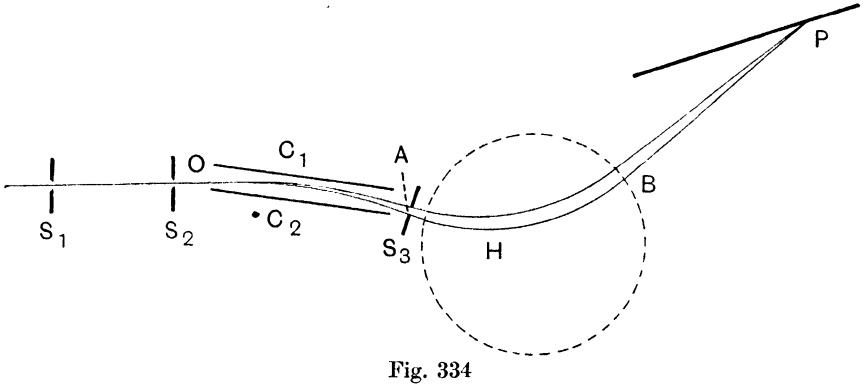



Fig. 334

through two slits $S_1$, $S_2$, each about $\frac{1}{20}$ mm. wide, into a parallel-plate condenser $C_1C_2$, in which a field of the order of 2000 volt/cm.

is maintained over a length of 5 cm. The particles are spread out by the field, and a wider slit $S_3$ selects those whose deflexions lie within a certain small range. These enter a magnetic field $H$, of the order of 10,000 gausses, perpendicular to the plane of the paper and practically confined to the space inside the dotted circle. The field is produced by a du Bois electromagnet (Fig. 150) with flat pole pieces 8 cm. in diameter and 3 mm. apart. The particles, after leaving the field, strike a photographic plate, 12 cm. long and 2 cm. wide, at a point $P$. Exposures range from 20 sec. to half an hour. The slits $S_1$, $S_2$, in addition to limiting the beam, play the part of the fine tube in Wien's apparatus, the space to the right of $S_1$ being kept highly exhausted with a pump and charcoal tubes immersed in liquid air.

To explain the focussing take an origin at $O$, where the particles enter the condenser, and an axis of $x$ along their initial path. Let $A$ be the centre of the slit $S_3$, and suppose that the particles are subjected to a uniform transverse electric field $E$ for a distance $l$ from $O$ to $A$, and then to a uniform magnetic field $H$ for a distance $L$ from $A$ to $B$, all deflexions being assumed to be small. The electric field turns the beam through an angle $\theta$ and the magnetic field through an angle $\phi$, where

$$\theta = \frac{eEl}{mv^2}, \quad \phi = \frac{eHL}{mcv} \quad \ldots\ldots\ldots\ldots (286).$$

$A$ is the point $(l, -\frac{1}{2}l\theta)$ to the first order of small quantities, and $B$ the point $(l + L, -\frac{1}{2}l\theta - L\theta + \frac{1}{2}L\phi)$. Hence the equation of $BP$ is

$$y + \tfrac{1}{2}l\theta + L\theta - \tfrac{1}{2}L\phi = (\phi - \theta)(x - l - L),$$

or

$$\theta(x - \tfrac{1}{2}l) - \phi(x - l - \tfrac{1}{2}L) + y = 0 \quad \ldots\ldots (287).$$

A neighbouring particle will pass through the same point $(x, y)$ if

$$d\theta(x - \tfrac{1}{2}l) - d\phi(x - l - \tfrac{1}{2}L) = 0.$$

If the two particles have the same $e/m$, $d\theta/\theta = 2d\phi/\phi$ from (286), so that

$$2\theta(x - \tfrac{1}{2}l) - \phi(x - l - \tfrac{1}{2}L) = 0 \quad \ldots\ldots\ldots (288).$$

From (287) and (288)

$$\theta = \frac{y}{x - \frac{1}{2}l}, \quad \phi = \frac{2y}{x - l - \frac{1}{2}L} \quad \ldots\ldots\ldots\ldots (289).$$

The slit picks out those particles for which $\theta$ is near an assigned

small value. Hence $P$ lies on the straight line $y = \theta\,(x - \frac{1}{2}l)$, which is *the same for all values of e/m.* A single plate in a fixed position will therefore collect all focussed beams. Elimination of $v$, $\phi$ and $y$ from equations (286) and (289) gives

$$\left(\frac{x - l - \frac{1}{2}L}{x - \frac{1}{2}l}\right)^2 = \frac{4mc^2El\theta}{eH^2L^2} \quad \ldots\ldots\ldots (290),$$

showing how the position of the focus changes with $m/e$. The curve connecting $x$ and $m$ has an inflexion at $x = \frac{5}{4}l + \frac{3}{4}L$, and $m$ in this region is nearly proportional to the distance of $P$ from the point $x = \frac{17}{16}l + \frac{9}{16}L$, i.e. a point on the plate near the foot of the perpendicular let fall from the centre of the magnetic field.

Accurate comparison of masses is carried out as follows. Carbon dioxide in the discharge tube gives seven lines ascribed provisionally to C + + (6), O + + (8), C (12), O (16), CO (28), $O_2$ (32), $CO_2$ (44), where the numbers are the values of $m/e$ in terms of O as 16, taken roughly from the mass scale, which is at first only assumed to be linear enough for purposes of identification. If the electric field is doubled the O + + line should, from (290), occupy the position previously occupied by O. Similarly O should occupy the position previously occupied by $O_2$, and C + + that occupied by C. Aston found these coincidences to hold within the limit of experimental error, about 1 part in 1000. These experiments confirm on the one hand the exact relation of multiple charges, and on the other the mass relation of atom and molecule. Increase of $E$ in the ratio of 4 to 3 brings the C line on to the previous O line, so that C is accurately 12·00 in terms of O = 16·00; and similarly for the others. The seven lines thus measured are used to calibrate the mass scale over the range 5 to 50. They are obtained on all plates by adding $CO_2$ to the gas in the discharge tube; hydrocarbons being also present as impurities.

Typical mass spectra are shown in Fig. 335. The plates are measured from a fixed mark on the plate to the left-hand edge of the line. The numbers marked are values of $m/e$ on the oxygen scale, all being *exact* (to within 1 or 2 parts in 1000). The lines 20 and 22 in spectrum I belong to neon; 15 belongs to $CH_3$, 17 to HO, 18 to $H_2O$, 24 to $C_2$, 25 to $C_2H$ and so on, 40 probably to a trace of argon.

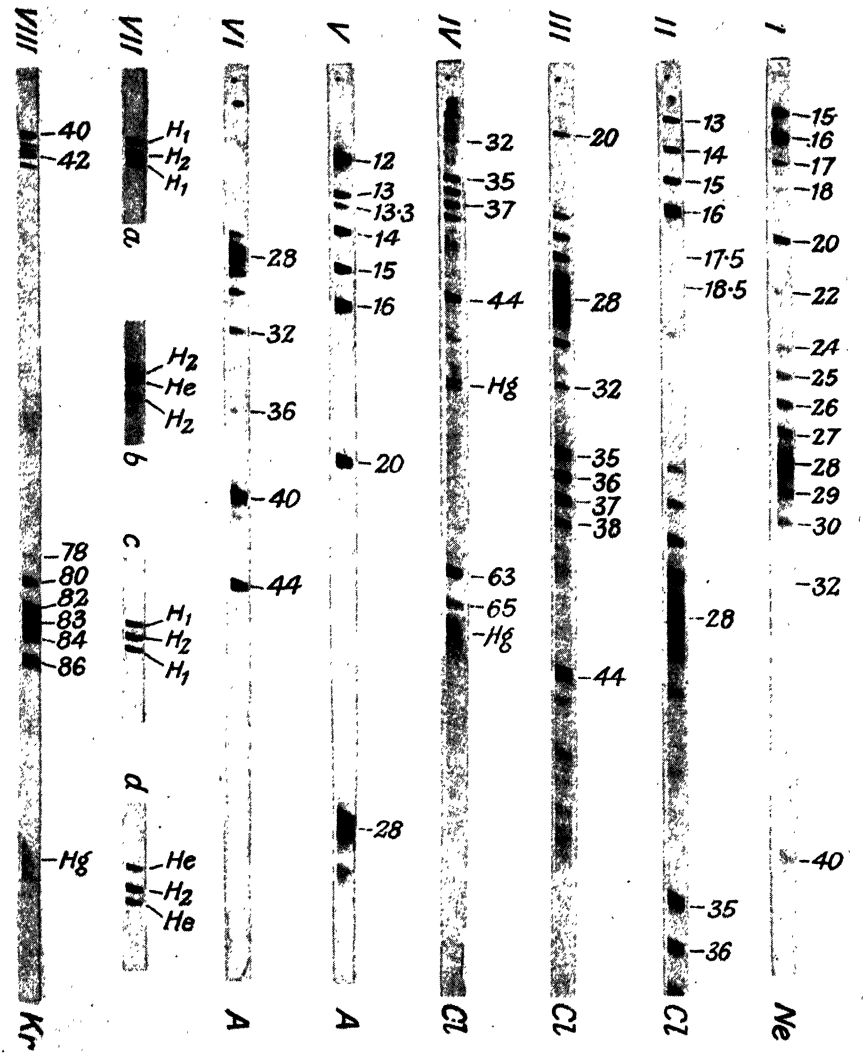



Fig. 335

Spectrum VII *a*, *c* show how nearly the 2 to 1 ratio holds for the hydrogen molecule and hydrogen atom*. The $H_2$ line is here photographed with 250 volts on the condenser and the H line twice, with $500 \pm 12$ volts, so that it should bracket the $H_2$ line symmetrically

* One extra electron, of mass 0·00055 on the oxygen scale, hardly affects the comparison.

(plate *c* is the best; *a* should be judged by the left-hand edges). A similar comparison of He with O + + shows that the mass of the helium atom is accurately 4·00. But $H_2$ is bracketed unsymmetrically by He, and *vice versâ* (VII, *b* and *d*). The mass of the hydrogen molecule is therefore not exactly 2, but 2·015 from Aston's measurements. This makes the mass of a neutral hydrogen atom 1·008 on the oxygen scale, as it should be from determinations of the atomic weight. Other deviations of atomic weights from whole numbers are accounted for quite differently, as is shown in the next article.

**230. Composite nature of the chemical elements.** The composite nature of neon was discovered by Sir J. J. Thomson in 1912. A strong parabola was found at $m/e = 20$ and a weaker one at $m/e = 22$. Spectrum I of Fig. 335, already referred to, gives more exact information of the same kind. There are two lines at 20·00 and 22·00, but none to correspond with the atomic weight at 20·20. Also, Sir J. J. Thomson's parabola at 20 can be reasonably estimated at 9 times the strength of the parabola at 22, and the estimate is confirmed by the mass spectrograph. The conclusion is that neon is a mixture of two substances in this ratio, more elementary than itself in that their atoms are all of the same mass—a conclusion that cannot be drawn from a measurement of density, since the constituents may be mixed in constant ratios.

The experiments suggest that other elements may be heterogeneous, the constituents being nearly or quite inseparable by chemical means. Such constituents, if they exist, are called *isotopes*, since they occupy the same place in the periodic table. There is also the suggestion that isotopes are best sought in elements of fractional atomic weight. Aston examined a large number of elements with his mass spectrograph, with results that will now be discussed.

Spectra II, III and IV (Fig. 335) are obtained from phosgene, $COCl_2$. There is no line corresponding to the atomic weight of chlorine at 35·46, but four lines at 35, 36, 37 and 38, integral numbers to one or two parts in 1000. The lines at 35 and 37 appear doubly charged at 17·5 and 18·5, but the weak line at 18 is probably due to water-vapour, and there is no line at 19. The difference

seems to show that 35 and 37 are isotopes of chlorine, 36 and 38 the corresponding hydrochloric acids.

Of the other lines requiring explanation, 20 appears to be neon left over from a previous experiment, and the broader mercury lines (see below) Hg + + + + and Hg + + +. Argon (V and VI) gives a strong line at 40 and a weak line at 36, the former doubly charged at 20 and triply charged at 13·33, by accurate measurement. Krypton (VIII) gives singly charged lines at 78, 80, 82, 83, 84, 86. In the doubly charged set 39 would hardly be seen, and 40 is covered by an argon line in the sample; otherwise the ratio of intensities is well preserved. The broad mercury bands in IV and VIII show that isotopes are present, covering a range of five or six units in the neighbourhood of $m/e = 200$, but not resolved.

These examples will perhaps suffice to show Aston's mode of procedure. The alkali metals can be generated at a hot anode, and $m/e$ measured by magnetic deflexion and potential difference, as in Classen's experiment (Art. 211). This method was used by Dempster. Aston excited the mass-spectra of other metals by the method of anode rays (Art. 228).

Isotopes have, on the nuclear theory, the same nuclear charge and the same arrangement of circumnuclear electrons. The difference between, say, $Cl^{35}$ and $Cl^{37}$ is that the latter has two more protons and two more electrons *in the nucleus*. Equality of nuclear charge is vouched for not only by chemical properties, but by the absence of more than one type of Röntgen ray spectrum (Art. 236). Aston's experiments throw great light on the constitution of the nucleus. Had the mass of the proton been 1 instead of 1·007 on the oxygen scale, we could say with confidence that atomic nuclei are composed of protons and electrons. As it is, most of the nucleus must be of "bricks" having nearly integral atomic weight, with a few extra protons. Experiments on the disintegration of elements by $\alpha$ particles (Art. 252) show that nuclei of mass $4n$ are especially stable, and the $\alpha$ particle is itself a helium nucleus expelled with high velocity (Art. 248). The evidence, therefore, is that higher nuclei are probably built up of helium nuclei and protons. A purely unitary theory is possible if four protons can combine to form a helium nucleus, with *loss of mass*: the theory is less speculative than might appear at first sight (Art. 273).

| Atomic number | Element | Atomic weight | Minimum number of isotopes | Masses of isotopes in order of intensity |
|---|---|---|---|---|
| 1 | Hydrogen | 1·008 | 1 | 1·008 |
| 2 | Helium | 4·00 | 1 | 4 |
| 3 | Lithium | 6·94 | 2 | 7, 6 |
| 4 | Beryllium | 9·1 | 1 | 9 |
| 5 | Boron | 10·9 | 2 | 11, 10 |
| 6 | Carbon | 12·00 | 1 | 12 |
| 7 | Nitrogen | 14·01 | 1 | 14 |
| 8 | Oxygen | 16·00 | 1 | 16 |
| 9 | Fluorine | 19·00 | 1 | 19 |
| 10 | Neon | 20·20 | 2 | 20, 22 |
| 11 | Sodium | 23·00 | 1 | 23 |
| 12 | Magnesium | 24·32 | 3 | 24, 25, 26 |
| 13 | Aluminium | | | |
| 14 | Silicon | 28·3 | ? 3 | 28, 29, ? 30 |
| 15 | Phosphorus | 31·04 | 1 | 31 |
| 16 | Sulphur | 32·06 | 1 | 32 |
| 17 | Chlorine | 35·46 | 2 | 35, 37 |
| 18 | Argon | 39·88 | 2 | 40, 36 |
| 19 | Potassium | 39·10 | 2 | 39, 41 |
| 20 | Calcium | 40·07 | 2 | 40, 44 |
| 21 | Scandium | 45·1 | 1 | 45 |
| 22 | Titanium | 48·1 | 1 | 48 |
| 23 | Vanadium | 51·0 | 1 | 51 |
| 24 | Chromium | 52·0 | 1 | 52 |
| 25 | Manganese | 54·93 | 1 | 55 |
| 26 | Iron | 55·84 | 2 | 56, 54 |
| 27 | Cobalt | 58·97 | 1 | 59 |
| 28 | Nickel | 58·68 | 2 | 58, 60 |
| 29 | Copper | 63·57 | 2 | 63, 65 |
| 30 | Zinc | 65·37 | ? 4 | 64, 66, 68, 70 |
| 31 | Gallium | 69·72 | 2 | 69, 71 |
| 32 | Germanium | 72·5 | 3 | 74, 72, 70 |
| 33 | Arsenic | 74·96 | 1 | 75 |
| 34 | Selenium | 79·2 | 6 | 80, 78, 76, 82, 77, 74 |
| 35 | Bromine | 79·92 | 2 | 79, 81 |
| 36 | Krypton | 82·92 | 6 | 84, 86, 82, 83, 80, 78 |
| 37 | Rubidium | 85·45 | 2 | 85, 87 |
| 38 | Strontium | 87·63 | 2 | 88, 86 |
| 39 | Yttrium | 88·9 | 1 | 89 |
| 40 | Zirconium | | | |
| 41 | Niobium | | | |
| 42 | Molybdenum | | | |
| 43 | | | | |
| 44 | Ruthenium | | | |
| 45 | Rhodium | | | |
| 46 | Palladium | | | |
| 47 | Silver | 107·88 | 2 | 107, 109 |
| 48 | Cadmium | | | |
| 49 | Indium | 114·8 | 1 | 115 |

| Atomic number | Element | Atomic weight | Minimum number of isotopes | Masses of isotopes in order of intensity |
|---|---|---|---|---|
| 50 | Tin | | | |
| 51 | Antimony | 121·77 | 2 | 121, 123 |
| 52 | Tellurium | | | |
| 53 | Iodine | 126·92 | 1 | 127 |
| 54 | Xenon | 130·2 | 9 | 129, 132, 131, 134, 136, 128, 130, 124, 126 |
| 55 | Caesium | 132·81 | 1 | 133 |
| 56 | Barium | 137·37 | 1 | 138 |
| 57 | Lanthanum | 138·91 | 1 | 139 |
| 58 | Cerium | | | |
| 59 | Praseodymium | 140·92 | 1 | 141 |
| 60 | Neodymium | | | |
| 61 | | | | |
| 62 | Samarium | | | |
| 63 | Europium | | | |
| 64 | Gadolinium | | | |
| 65 | Terbium | | | |
| 66 | Dysprosium | | | |
| 67 | Holmium | | | |
| 68 | Erbium | | | |
| 69 | Thulium | | | |
| 70 | Ytterbium | | | |
| 71 | Lutecium | | | |
| 72 | Hafnium | | | |
| 73 | Tantalum | | | |
| 74 | Tungsten | | | |
| 75 | | | | |
| 76 | Osmium | | | |
| 77 | Iridium | | | |
| 78 | Platinum | | | |
| 79 | Gold | | | |
| 80 | Mercury | 200·6 | ? 6 | 197–200, 202, 204 |
| 81 | Thallium | | | |
| 82 | Lead | | | |
| 83 | Bismuth | | | |
| 84 | Polonium | | | |
| 85 | | | | |
| 86 | Niton | | | |
| 87 | | | | |
| 88 | Radium | | | |
| 89 | Actinium | | | |
| 90 | Thorium | | | |
| 91 | Uranium $X_2$ | | | |
| 92 | Uranium | | | |

# CHAPTER XIII

## RÖNTGEN RAYS

**231. Diffraction of Röntgen rays by crystals.** We have now to consider in greater detail the diffraction phenomena mentioned in Art. 213. They are analogous to the effects produced when light falls on a diffraction grating: but whereas the dimensions of an optical grating are known beforehand, here both the structure of the crystal and the nature of the incident "light" have to be found from the observations. It is essential to bear in mind the two problems involved.

Consider a simple cubical array of identical atoms; that is, one which is such that axes can be chosen so that there is an atomic nucleus at each of the points $(pd, qd, rd)$, where $d$ is a constant length and $p$, $q$, $r$ are positive or negative integers. Let a plane wave of Röntgen rays of wave-length $\lambda$ pass through the crystal in the direction $(l, m, n)$ referred to these axes. The rays are scattered in all directions by the circumnuclear electrons, and subsequently interfere. Consider the effect produced in the direction $(l', m', n')$. Let $O$, $A$, $B$, $C$ be the mean centres of the circumnuclear electrons of four consecutive atoms, which we identify for simplicity with their nuclei, and let $O$ be taken as origin. Since the projection of $OA$ on the normal to the wave-front is $ld$, the wave reaches $A$ at a time $ld/c$ after it reaches $O$, $c$ being the velocity of light. By similar reasoning it follows that a ray scattered from $A$ in the direction $(l', m', n')$ is in advance of one from $O$ by the amount $l'd/c$; hence the total time-lag between the two rays after scattering is $(l' - l)\, d/c$. The directions of appreciable intensity of the scattered beam are those for which the time-lags between rays scattered by $O$, $A$, $B$, $C$ differ by integral multiples of the periodic time $\lambda/c$ of

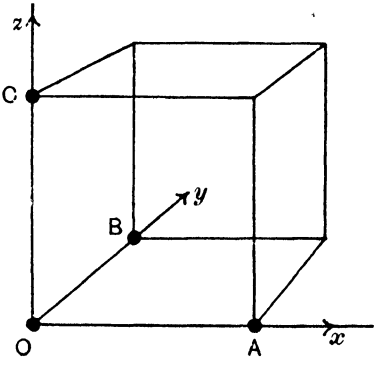


Fig. 336

the waves. Then $(l' - l)\, d/c = h_1\lambda/c$, ...., or

$$l' - l = \frac{h_1\lambda}{d}, \quad m' - m = \frac{h_2\lambda}{d}, \quad n' - n = \frac{h_3\lambda}{d} \quad ..(291),$$

where $h_1$, $h_2$, $h_3$ are integers. Solving for $l'$, $m'$, $n'$, squaring and adding, we have

$$\frac{\lambda}{d}(h_1^2 + h_2^2 + h_3^2) + 2\,(h_1 l + h_2 m + h_3 n) = 0 \ldots .(292),$$

since $l'^2 + m'^2 + n'^2 = l^2 + m^2 + n^2 = 1$. The last equation shows that $\lambda$ can only have one of a certain set of values. If $\lambda$ has not one of these values no spot is formed, and if it has the direction $(l', m', n')$ is given uniquely by equations (291). Hence a heterogeneous beam of Röntgen rays will give a diagram of discrete spots, while homogeneous rays in general give the central spot only.

If the rays fall normally on a crystal plate, $l = m = 0$ and $n = 1$, so that

$$l' = \frac{h_1\lambda}{d}, \quad m' = \frac{h_2\lambda}{d}, \quad n' = 1 + \frac{h_3\lambda}{d}.$$

Hence

$$\frac{l'}{h_1} = \frac{m'}{h_2} = -\frac{1 - n'}{h_3},$$

showing that there are simple numerical relations between the direction-cosines corresponding to the different spots.

For visual demonstration of spots like those shown in Fig. 301 we require a Coolidge bulb working at full power and apertures about 5 mm. wide. If a large fluorescent screen is placed 20 cm. behind the apertures, the spots are seen clearly after resting the eye for a quarter of an hour or so in the dark, and change rapidly as the crystal is turned about an axis.

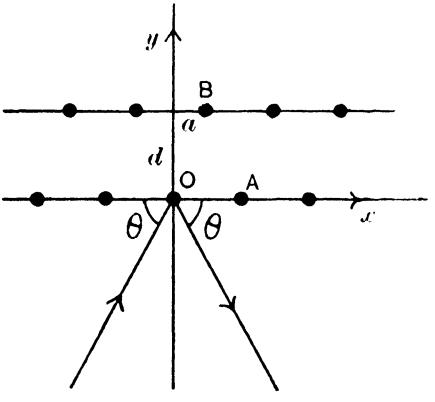


Fig. 337

The following theorem applies to any crystal. From the regular arrangement of atoms in a crystal, it is clear that many planes can be found containing large numbers of atoms. A cleavage plane is one such. If $OA$ (Fig. 337) is one of these planes, another will be found parallel to it at a perpendicular distance $d$ (say).

The atoms in the second plane are not in general precisely behind those in the first; they may be sheared over through a distance $a$ as shown in the figure. Let Röntgen rays fall obliquely on the plane $OA$, and consider only the waves diffracted at an angle equal to the angle of incidence $i$*. The condition of equality of phase from $O$ and $A$ is then satisfied identically, as in the reflexion of light. The condition of reinforcement of waves from $O$ and $B$ is

$$\frac{d\sin\theta + a\cos\theta}{c} + \frac{d\sin\theta - a\cos\theta}{c} = n\frac{\lambda}{c},$$

or

$$2d\sin\theta = n\lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots (293),$$

$n$ being an integer. Since $a$ has disappeared, we see that the shearing over of the atoms in the second plane is immaterial, as is obvious otherwise. $d$ is called the *grating constant* of the plane.

**232. The Röntgen ray spectrometer.** The observation of waves diffracted at an angle equal to the angle of incidence, as in

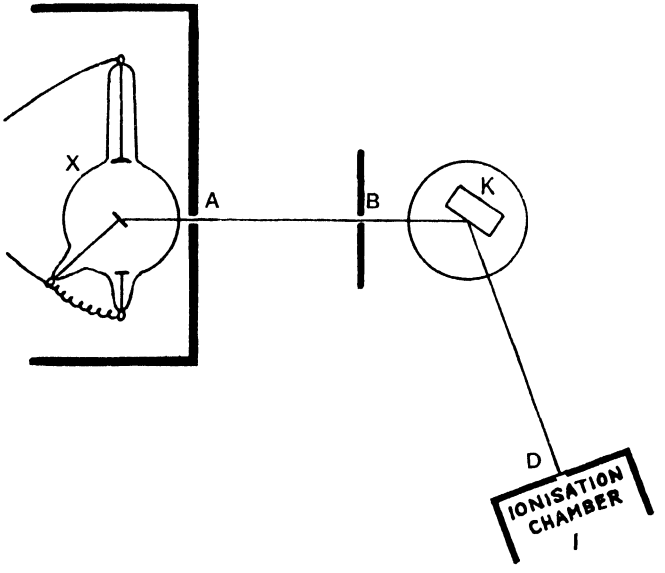



Fig. 338

Fig. 337, has one important advantage over Laue's diagram. Each wave-length is examined in turn by varying $\theta$ (with some overlapping as in grating spectrometry), so that we can form an idea

* As the angle of incidence is generally nearer 90° than 0° it is more convenient to use instead the *glancing angle* $\theta = \frac{1}{2}\pi - i$.

of the distribution of the intensity of the original beam among the different wave-lengths. The method was introduced by Sir W. H. Bragg, and utilised in his *Röntgen ray spectrometer*, shown in outline in Fig. 338. The beam of Röntgen rays from the bulb $X$ is narrowed and made nearly parallel by two slits $A$, $B$ in thick lead sheets. The crystal $K$ is mounted on the central table of a spectrometer, and the diffracted beam passes through a third slit $D$ into an ionisation chamber $I$ occupying the position of the telescope in an optical spectrometer. A sensitive electroscope is required to measure the weak ionisation current.

A completed instrument is shown in Fig. 339. Here $C$ is the crystal holder, $E$ the electroscope read with a microscope $M$, and $W$ a cylinder screening the wire which goes to the electroscope.

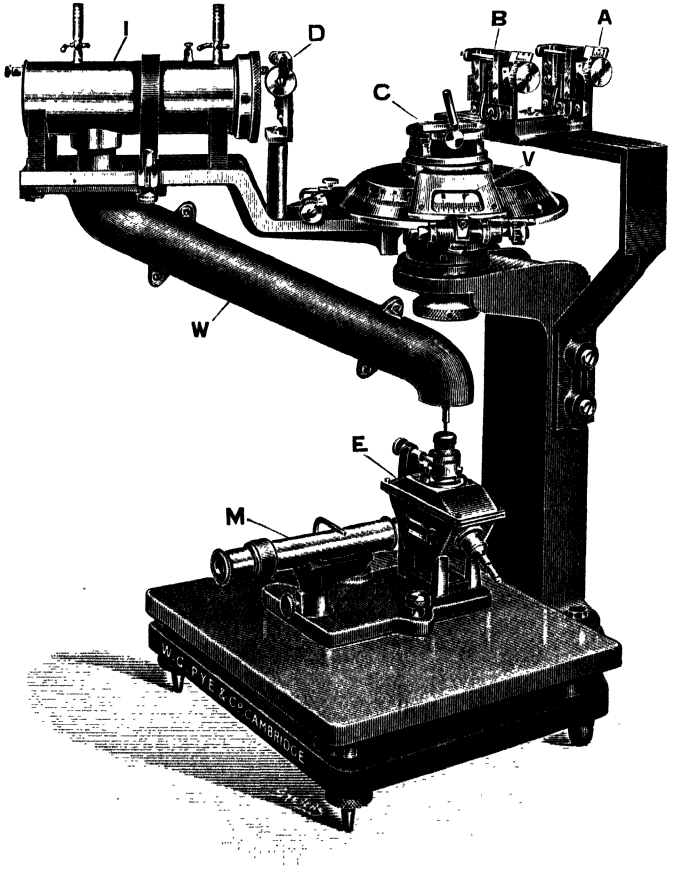



Fig. 339

Figs. 341 and 346 below show the type of curve obtained by varying the glancing angle, the former being early curves of Sir W. H. and W. L. Bragg. In both we have the continuous radiation which we should expect from Laue's diagrams, but there are certain angles at which the intensity of the diffracted pencil is abnormally great. The peaks in Fig. 341 correspond undoubtedly to three rays of definite wave-length emitted by the platinum target, $A_1$, $B_1$, $C_1$ forming the first order spectrum ($n = 1$ in equation (293)) and $A_2$, $B_2$, $C_2$ the second ($n = 2$); for after applying certain small angular corrections the peaks $B_1$, $B_2$ were found to occur at angles 11° 33′ and 23° 39′, and a third peak $B_3$ was found at 36° 39′. The sines of these angles are 0·200, 0·401 and 0·597, which are nearly in the ratio 1 : 2 : 3.

Fig. 346, in which the abscissae can be considered for the present to be glancing angles, shows that there is no radiation corresponding to small angles, when the bulb is excited by a constant voltage (Art. 237; cf. however Art. 235).

The homogeneous rays change in position when the material of the target is changed. These rays must, in view of the perfect analogy between Röntgen rays and light, be part of the spectrum of the material of the target. Both the continuous and homogeneous rays are waves set up by the impact of electrons on the target. The former arise somehow from the retardation of an electron in close collision with an atom, the latter from the atom itself, after it has been put in an abnormal state by the impinging electron. They yield, in different ways, some of the best evidence in support of the quantum theory (Arts. 237, 239).

**233. Analysis of crystal structure.** Laue's diagrams were first used to determine the structure of crystals by W. L. Bragg. He found the deductions of Art. 231 entirely verified for potassium chloride, and concluded that the atomic nuclei were in simple cubical array. The behaviour of sodium chloride and potassium bromide shows that the agreement is accidental, depending on the proximity of the atomic numbers of potassium and chlorine. The diagram of sodium chloride agrees with the structure shown in Fig. 340, where the black spots represent sodium nuclei and the white spots chlorine nuclei. The complete set of spots form a simple

cubical lattice: the black spots only, or the white spots only, a "face-centred lattice" of cubes with a point at the centre of each face. The deduction from the Laue diagram need not be followed out, since the model can be checked more simply with a Röntgen ray spectrometer. Take coordinate axes as shown in Fig. 336, and let a plane of atoms parallel to the plane $lx + my + nz = 0$ be called an $(lmn)$ plane. The top curve of Fig. 341 shows an early observation of Sir W. H. and W. L. Bragg on the (100) plane of rock-salt, $A_1, B_1, C_1$ being three homogeneous rays of platinum and $A_2$, $B_2$, $C_2$ the same in the second order. Identical curves are found for the other principal planes (010), (001),

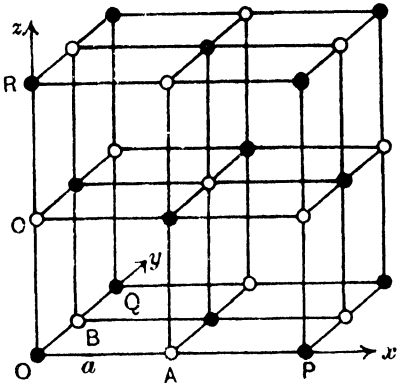


Fig. 340

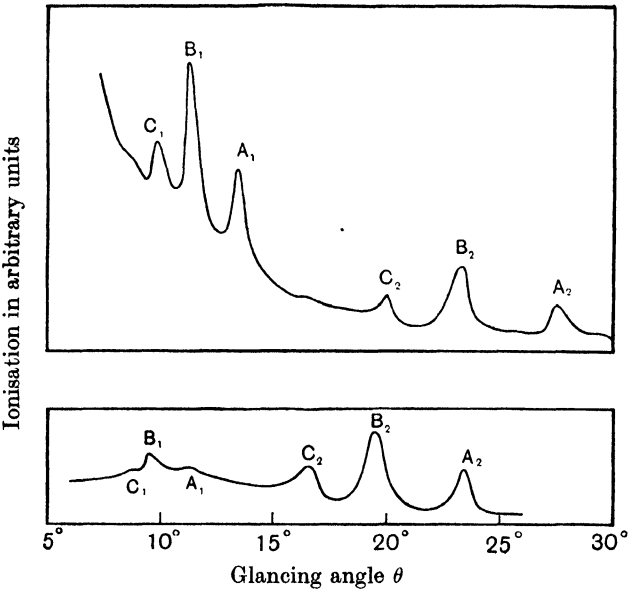


Fig. 341

which is to be expected since the distance between successive planes of nuclei in each case is equal to the side $d$ of the elementary cube. The lower curve is taken from the (111) plane, of which the plane

$ABC$ in Fig. 340 is an example. The parallel plane $PQR$, at a distance of $d/\sqrt{3}$ from it, contains only black nuclei, and we have to move through a distance $2d/\sqrt{3}$ before coming to a similar plane of nuclei. Hence the relation of corresponding peaks in the curves (100) and (111) should be given by $d \sin \theta_{100} = 2d \sin \theta_{111}/\sqrt{3}$, or $\sin \theta_{100} = 1{\cdot}15 \sin \theta_{111}$, if the model is correct. Experiments on the $B$ peak give $\sin \theta_{100} = 1{\cdot}16 \sin \theta_{111}$. The result is confirmed by observation of the (110) plane, which gives $\sin \theta_{100} = 0{\cdot}718 \sin \theta_{110}$, the theoretical factor being $1/\sqrt{2} = 0{\cdot}707$. Better agreement has since been obtained between theory and experiment. It is clear that we have a method of finding the structure of simple crystals, and of checking any proposed model of a more complicated crystal.

When the structure of a crystal is known, the absolute dimensions of its lattice follow from its density. Consider a crystal of rock-salt. The mass of a sodium atom is $3{\cdot}795 \times 10^{-23}$ gm., and the mean mass of the chlorine atoms $5{\cdot}851 \times 10^{-23}$ gm. If $d$ is the side of the elementary cube (the whole cube in Fig. 340), the number of atoms of either kind in 1 cm.$^3$ of crystal is $1/2d^3$. Hence the mass of 1 cm.$^3$ of the crystal is $4{\cdot}823/10^{23}d^3$. Equating this to the known density 2·164 we have $d = 2{\cdot}814 \times 10^{-8}$ cm. Sodium chloride is not a very regular crystal, and apt to give curved cleavage planes, so that it has been largely superseded for accurate work by calcite and other crystals.

Debye and Scherrer invented a method for crystals which are too small for the method just described. The crystal is powdered and placed in a small tube at the centre of the spectrometer table. There will, on account of the irregular distribution of the grains, be some which will diffract rays of any given wave-length (cf. equation (292)), and these give a series of concentric rings or halos on a photographic plate placed in the direct path of the rays. Consider the simple cubical array of Art. 231. If $\theta$ is the angle between the incident ray $(l, m, n)$ and the diffracted ray $(l', m', n')$,

$$\cos \theta = ll' + mm' + nn',$$

and equations (291) give, on multiplying by $l$, $m$, $n$ and adding,

$$\cos \theta - 1 = \frac{\lambda}{d}(h_1 l + h_2 m + h_3 n),$$

or

$$1 - \cos \theta = \frac{\lambda^2}{2d^2}(h_1^2 + h_2^2 + h_3^2) \ldots\ldots\ldots (294)$$

from equation (292). Debye and Scherrer found that nearly all the rings observed with LiF were given by (294), where $h_1$, $h_2$, $h_3$ were either all even or all odd. This means that the structure consists of face-centred lattices like NaCl, which gives exactly this selection of the possible integral values of $h_1$, $h_2$, $h_3$.

Consideration of the evidence for the structure of other crystals would lead us too far. Perhaps the most interesting are:

(1) *Calcite.* The calcium nuclei are on a face-centred lattice like that of Fig. 340, except that the figure is strained till the three angles which meet at $O$ are each 101° 54′. If the calcium nuclei are taken to be the black spots, the carbon nuclei are the white spots. The oxygen nuclei (three for each one of the others) lie on lines joining the carbon nuclei in the deformed (111) planes. The crystal is of great importance for exact measurement of wave-length, and has a grating constant of $3{\cdot}028 \times 10^{-8}$ cm.

(2) *Diamond.* The structure is of carbon atoms, whose nuclei lie on two face-centred cubic lattices, which penetrate each other in such a way that each nucleus of either lattice is surrounded by four nuclei of the other forming a regular tetrahedron with the first nucleus at the centre.

(3) *Quartz.* The silicon nuclei are arranged in a set of approximate spirals, which are linked to each other by oxygen nuclei. A quasi-spiral structure is also possessed by tartaric acid, the structure changing enantiomorphically in both crystals in the change from right-handed to left-handed optical activity.

The forces which keep the nuclei of crystals in such definite positions are unknown, or at least not reduced to any simple laws. Each nucleus has as a rule a number of nuclei all at the same distance from it, and the outermost electrons are probably shared.

**234. Measurement of the wave-length of Röntgen rays.** A crystal of known grating constant $d$ will measure wave-lengths between 0 and $2d$, from equation (293). The best results are obtained by a focussing method devised by Sir W. H. and W. L. Bragg. There is, in principle, only one slit $S$ (Fig. 342). The photographic plate is placed to touch a circle with centre $O$ and radius $OS$ at a point $P$ where a line is expected, $O$ being the centre of the

spectrometer table. If the angle $AOP = 2\theta$, and the crystal face bisects the angle, the ray $SO$ is diffracted to $P$. If the crystal is in a slightly different position, let the circle through $SOP$ meet it at $B$. Then the angle $SBO = SPO = \theta$, so that a ray $SB$ of the same wave-length as before, but from a slightly different part of the target, is also diffracted to $P$. A large crystal need not therefore be set perfectly. An improvement is to rotate the crystal to and fro through a small angle, so that all parts are utilised, instead of a particular part which may be less perfect than the rest. This method was introduced by de Broglie, who was also the first to register line-spectra photographically. Line-spectra were observed visually by de Broglie and Lindemann, who used a bulb with a window of lithium glass in order to diminish absorption. A line so observed remains stationary when the crystal is rotated, instead of moving like Laue's spots.

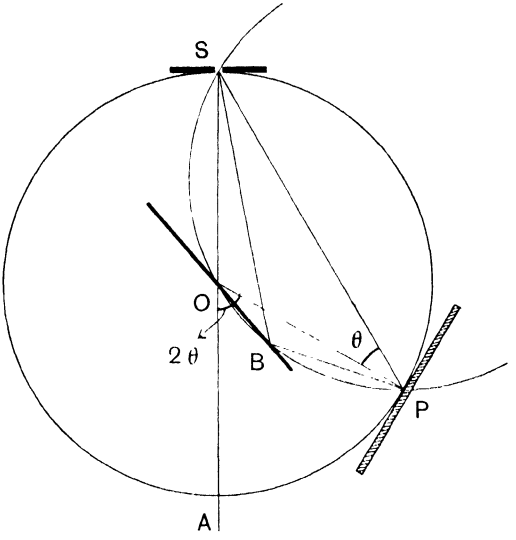


Fig. 342

Fig. 343 shows some spectra taken by Siegbahn with a rotating crystal: they belong to the $K$ series of the elements, described in Art. 236. Many types of spectrometer have been used to measure wave-lengths, of which those of Siegbahn are designed for the greatest attainable accuracy (of the order of 1 part in 200,000), the same line being photographed twice on opposite sides of the direct beam. The glancing angle is then one-quarter of the angle between the diffracted beams on the two sides. The whole spectrometer must be exhausted for wave-lengths greater than $3 \times 10^{-8}$ cm. or thereabouts, to avoid absorption by the air. Since the mass of an atom enters into the calculation of wave-length (Art. 233), relative wave-lengths are known more accurately than absolute wave-

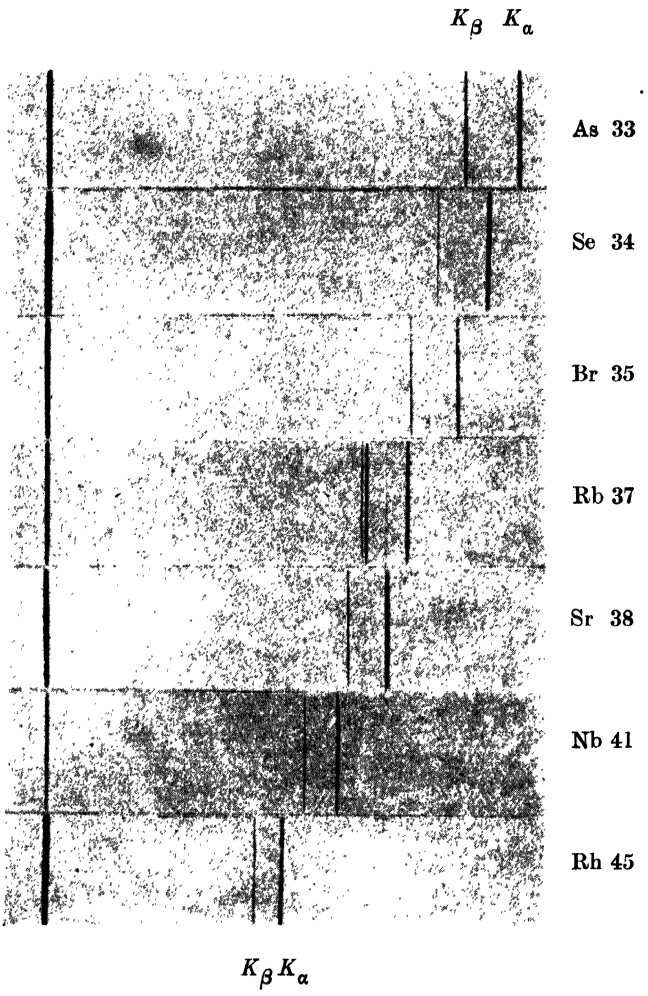



Fig. 343

lengths. Those now in use are based on the figure $2{\cdot}814 \times 10^{-8}$ cm. for the grating constant of rock-salt, or on that of calcite derived from it by comparing the glancing angles of the two crystals for the same rays.

**235. Refraction and total reflexion of Röntgen rays.** Schuster and other writers pointed out, on the discovery of Röntgen rays, that the apparent absence of refraction would be explained by their high frequency. To obtain a rough idea of the refractive

index, on the classical wave theory, let us suppose that the electrons are subject only to the electric force in the wave (washed about, as it were, like a cork upon water), and that there are $n$ of them per cm.$^3$, with an equal number of fixed protons to make the whole substance neutral. If $E$ is the electric force, supposed to be sinoidal and of frequency $\nu$, the displacement of an electron at any time is $-eE/4\pi^2m\nu^2$ parallel to the electric force, constituting an effective polarisation of magnitude $-ne^2E/4\pi^2m\nu^2$ in the same direction. The refractive index of the medium is therefore (Art. 198) given by

$$\frac{\mu^2-1}{4\pi}=\frac{P}{E}=-\frac{ne^2}{4\pi^2m\nu^2},$$

or

$$\mu-1=-\frac{ne^2}{2\pi m\nu^2}=-\delta\text{ (say)}\ldots\ldots\ldots(295)$$

approximately, $\mu-1$ being small. The medium might therefore behave like one of refractive index slightly less than 1. For example, if $n=8\times10^{23}$ (the number of circumnuclear electrons per cm.$^3$ in calcite) and $\nu=2\times10^{18}$ (corresponding to a wave-length of $1{\cdot}5\times10^{-8}$ cm.), $\delta=8\times10^{-6}$.

To apply the theory to crystals we should want to be sure (1) that the classical theory is adequate in this field of inquiry, (2) that the calculations hold for regular distributions of electrons separated by distances comparable with the wave-length, (3) that there is a sufficiently long train of waves inside the crystal for the normal refractive index to be developed. C. G. Darwin observed that if the theory is applicable it involves a correction to equation (293). If $\lambda$, $\theta$ are the wave-length and glancing angle, the waves which enter the crystal have wave-length $\lambda'$ and a glancing angle $\theta'$ inside, where $\lambda=\mu\lambda'$, $\cos\theta=\mu\cos\theta'$.

Equation (293) applies to $\lambda'$ and $\theta'$, i.e. $n\lambda'=2a\sin\theta'$ for spectra of order $n$. When $\delta$ is small, we find the approximate equation

$$n\lambda=2a\sin\theta\,(1-\delta\operatorname{cosec}^2\theta),$$

so that the glancing angle is greater than that given by equation (293) by an amount $\delta/\cos\theta\sin\theta$ .....(296);

or if the glancing angle is taken as correct, wave-lengths are over-estimated by an amount $2a\delta/n\sin\theta$ .....(297).

Deviations from equation (293) were first observed by Stenström and explained by him on the principles just described. Siegbahn obtained the following figures for the glancing angle of rays of wave-length $1{\cdot}537 \times 10^{-8}$ cm. on calcite:

First order, $\sin\theta = 0{\cdot}2537608.$
Second order, $\sin\theta = 2 \times 0{\cdot}2537345.$
Third order, $\sin\theta = 3 \times 0{\cdot}2537307.$

These observations can be reconciled by taking $\delta = 8{\cdot}6 \times 10^{-6}$, which is of the order calculated above. The deviations are too small to affect the tables of Art. 236.

A. H. Compton observed a total reflexion of Röntgen rays by glass, which should accompany a refractive index less than unity. Rays of wave-length $1{\cdot}279 \times 10^{-8}$ cm. were reflected almost totally at glancing angles less than 10′, above which the intensity fell rapidly, becoming inappreciable at 13′. The theoretical limit of total reflexion is given by $\cos\theta = \mu$, or $\theta^2 = 2\delta$ approximately. The observations therefore indicate a value of $\delta$ of about $4 \times 10^{-6}$ for glass at this wave-length, while equation (295) makes $\delta = 5 \times 10^{-6}$. There is a fair agreement between the values of $\delta$ obtained for a calcite crystal face by Stenström's method and by total reflexion.

**236. Röntgen ray spectra of the elements.** The earliest wave-length measurements are those of Sir W. H. and W. L. Bragg, who found the three platinum lines shown in Fig. 341, two of which increased resolution showed to be double. They also examined nickel, copper, rhodium, palladium, osmium and iridium. A systematic study led Moseley in 1913 to results of far-reaching importance in atomic theory. Moseley's results, as completed by Siegbahn and others, are shown in part in the tables on pp. 533–4. The spectra hitherto observed form four series, called the $K$, $L$, $M$ and $N$ series respectively*. The $K$ series has the shortest wave-length, and is given out freely by light elements. It consists of two strong lines $K_\alpha$, $K_{\alpha'}$ close together, with a line $K_\beta$ and some weak satellites. Fourteen lines of the $L$ series have been dis-

* The existence of homogeneous rays in general, and of $K$ and $L$ radiation in particular, was proved by Barkla before the discovery of the diffraction phenomena, by considering the absorption of the rays in metals. The method is not suitable for separating rays of nearly equal wave-length.

tinguished, of which the three strongest $L_\alpha$, $L_\beta$, $L_\gamma$ are shown in the table. They are of medium wave-length in heavy elements. Sir W. H. Bragg's rhodium lines are $K$ lines, his platinum lines $L$ lines. The $M$ and $N$ series, of still greater wave-length, were discovered by Siegbahn and Dolejšek respectively, the last series having been observed so far in bismuth, thorium and uranium only.

*Wave-lengths in* $10^{-8}$ *cm.; K and L series.*

| Atomic number | Element | $K_\alpha$ | $K_{\alpha'}$ | $K_\beta$ | $L_\alpha$ | $L_\beta$ | $L_\gamma$ |
|---|---|---|---|---|---|---|---|
| 11 | Sodium | 11·884 | | 11·591 | | | |
| 12 | Magnesium | 9·868 | | 9·534 | | | |
| 13 | Aluminium | 8·319 | | 7·940 | | | |
| 14 | Silicon | 7·109 | | 6·739 | | | |
| 15 | Phosphorus | 6·142 | | 5·785 | | | |
| 16 | Sulphur | 5·361 | 5·364 | 5·021 | | | |
| 17 | Chlorine | 4·719 | 4·722 | 4·394 | | | |
| 18 | Argon | | | | | | |
| 19 | Potassium | 3·734 | 3·737 | 3·446 | | | |
| 20 | Calcium | 3·352 | 3·355 | 3·083 | | | |
| 21 | Scandium | 3·025 | 3·029 | 2·774 | | | |
| 22 | Titanium | 2·742 | 2·746 | 2·509 | | | |
| 23 | Vanadium | 2·498 | 2·502 | 2·280 | | | |
| 24 | Chromium | 2·285 | 2·290 | 2·081 | | | |
| 25 | Manganese | 2·098 | | 1·902 | | | |
| 26 | Iron | 1·932 | 1·937 | 1·753 | | | |
| 27 | Cobalt | 1·786 | 1·790 | 1·617 | | | |
| 28 | Nickel | 1·655 | 1·659 | 1·497 | | | |
| 29 | Copper | 1·537 | 1·541 | 1·389 | 13·309 | | |
| 30 | Zinc | 1·432 | 1·436 | 1·293 | 12·222 | 11·951 | |
| 31 | Gallium | 1·338 | 1·342 | 1·206 | | | |
| 32 | Germanium | 1·257 | 1·261 | 1·131 | 10·414 | | |
| 33 | Arsenic | 1·173 | 1·177 | 1·052 | 9·650 | 9·394 | |
| 34 | Selenium | 1·104 | 1·109 | 0·993 | 8·971 | 8·717 | |
| 35 | Bromine | 1·035 | 1·040 | 0·929 | 8·357 | 8·108 | |
| 36 | Krypton | | | | | | |
| 37 | Rubidium | 0·922 | 0·926 | 0·825 | 7·303 | 7·060 | |
| 38 | Strontium | 0·871 | 0·876 | 0·779 | 6·848 | 6·609 | |
| 39 | Yttrium | 0·835 | 0·840 | 0·746 | 6·435 | 6·198 | |
| 40 | Zirconium | 0·788 | 0·793 | 0·705 | 6·056 | 5·823 | 5·573 |
| 41 | Niobium | 0·749 | 0·754 | 0·669 | 5·711 | 5·480 | 5·295 |
| 42 | Molybdenum | 0·708 | 0·712 | 0·631 | 5·394 | 5·166 | 4·909 |
| 43 | | | | | | | |
| 44 | Ruthenium | 0·645 | | 0·574 | 4·836 | 4·611 | 4·362 |
| 45 | Rhodium | 0·612 | 0·616 | 0·545 | 4·588 | 4·365 | 4·122 |
| 46 | Palladium | 0·584 | 0·588 | 0·521 | 4·358 | 4·137 | 3·899 |
| 47 | Silver | 0·562 | 0·567 | 0·501 | 4·146 | 3·927 | 3·694 |
| 48 | Cadmium | 0·534 | 0·538 | 0·479 | 3·948 | 3·730 | 3·507 |
| 49 | Indium | 0·510 | 0·515 | 0·453 | 3·764 | 3·548 | 3·332 |

*Wave-lengths in* $10^{-8}$ *cm.; K, L and M series.*

| Atomic number | Element | $K_\alpha$ | $K_{\alpha'}$ | $K\beta$ | $L_\alpha$ | $L\beta$ | $L_\gamma$ | $M_\alpha$ | $M\beta$ |
|---|---|---|---|---|---|---|---|---|---|
| 50 | Tin | 0·489 | 0·494 | 0·432 | 3·592 | 3·378 | 3·169 | | |
| 51 | Antimony | 0·468 | 0·472 | 0·416 | 3·432 | 3·218 | 3·017 | | |
| 52 | Tellurium | 0·456 | | 0·404 | 3·282 | 3·070 | 2·877 | | |
| 53 | Iodine | 0·437 | | 0·388 | 3·142 | 2·931 | 2·746 | | |
| 54 | Xenon | | | | | | | | |
| 55 | Caesium | 0·398 | 0·402 | 0·352 | 2·886 | 2·678 | 2·507 | | |
| 56 | Barium | 0·388 | 0·393 | 0·343 | 2·770 | 2·562 | 2·399 | | |
| 57 | Lanthanum | 0·372 | 0·376 | 0·329 | 2·660 | 2·453 | 2·298 | | |
| 58 | Cerium | 0·355 | 0·360 | 0·314 | 2·556 | 2·351 | 2·204 | | |
| 59 | Praseodymium | 0·342 | 0·347 | 0·301 | 2·458 | 2·254 | 2·115 | | |
| 60 | Neodymium | 0·330 | 0·335 | 0·292 | 2·365 | 2·162 | 2·031 | | |
| 61 | | | | | | | | | |
| 62 | Samarium | | | | 2·195 | 1·994 | 1·877 | | |
| 63 | Europium | | | | 2·116 | 1·916 | 1·807 | | |
| 64 | Gadolinium | | | | 2·042 | 1·842 | 1·741 | | |
| 65 | Terbium | | | | 1·971 | 1·773 | 1·679 | | |
| 66 | Dysprosium | | | | 1·905 | 1·707 | 1·619 | 9·509 | 9·323 |
| 67 | Holmium | | | | 1·841 | 1·644 | 1·564 | 9·150 | 8·943 |
| 68 | Erbium | | | | 1·780 | 1·583 | 1·511 | 8·783 | 8·573 |
| 69 | Thulium | | | | 1·723 | 1·527 | 1·460 | | |
| 70 | Ytterbium | | | | 1·668 | 1·472 | 1·412 | 8·125 | 7·891 |
| 71 | Lutecium | | | | 1·616 | 1·417 | 1·366 | 7·820 | 7·582 |
| 72 | Hafnium | | | | 1·566 | 1·371 | 1·324 | 7·521 | 7·286 |
| 73 | Tantalum | | | | 1·518 | 1·324 | 1·280 | 7·238 | 7·001 |
| 74 | Tungsten | 0·209 | 0·213 | 0·184 | 1·473 | 1·279 | 1·242 | 6·973 | 6·745 |
| 75 | | | | | | | | | |
| 76 | Osmium | | | | 1·388 | 1·194 | 1·168 | 6·481 | 6·256 |
| 77 | Iridium | | 0·196 | 0·168 | 1·348 | 1·155 | 1·133 | 6·250 | 6·030 |
| 78 | Platinum | 0·185 | 0·190 | 0·163 | 1·310 | 1·117 | 1·099 | 6·041 | 5·820 |
| 79 | Gold | | | | 1·274 | 1·081 | 1·068 | 5·831 | 5·619 |
| 80 | Mercury | | | | 1·240 | 1·049 | 1·042 | 5·649 | 5·439 |
| 81 | Thallium | | | | 1·205 | 1·013 | 1·008 | 5·443 | 5·233 |
| 82 | Lead | | | | 1·172 | 0·980 | 0·980 | 5·273 | 5·065 |
| 83 | Bismuth | | | | 1·141 | 0·949 | 0·953 | 5·107 | 4·894 |
| 84 | Polonium | | | | 1·101 | 0·920 | | | |
| 85 | | | | | | | | | |
| 86 | Niton | | | | | | | | |
| 87 | | | | | | | | | |
| 88 | Radium | | | | 1·010 | | | | |
| 89 | Actinium | | | | | | | | |
| 90 | Thorium | | | | 0·953 | 0·763 | 0·791 | 4·129 | 3·931 |
| 91 | Uranium $X_2$ | | | | | | | | |
| 92 | Uranium | | | | 0·908 | 0·718 | 0·753 | 3·901 | 3·709 |

Moseley's first important result is connected with *atomic number* (Art. 223). He showed that atomic number, not atomic weight, determines the Röntgen ray spectrum. Thus cobalt comes between

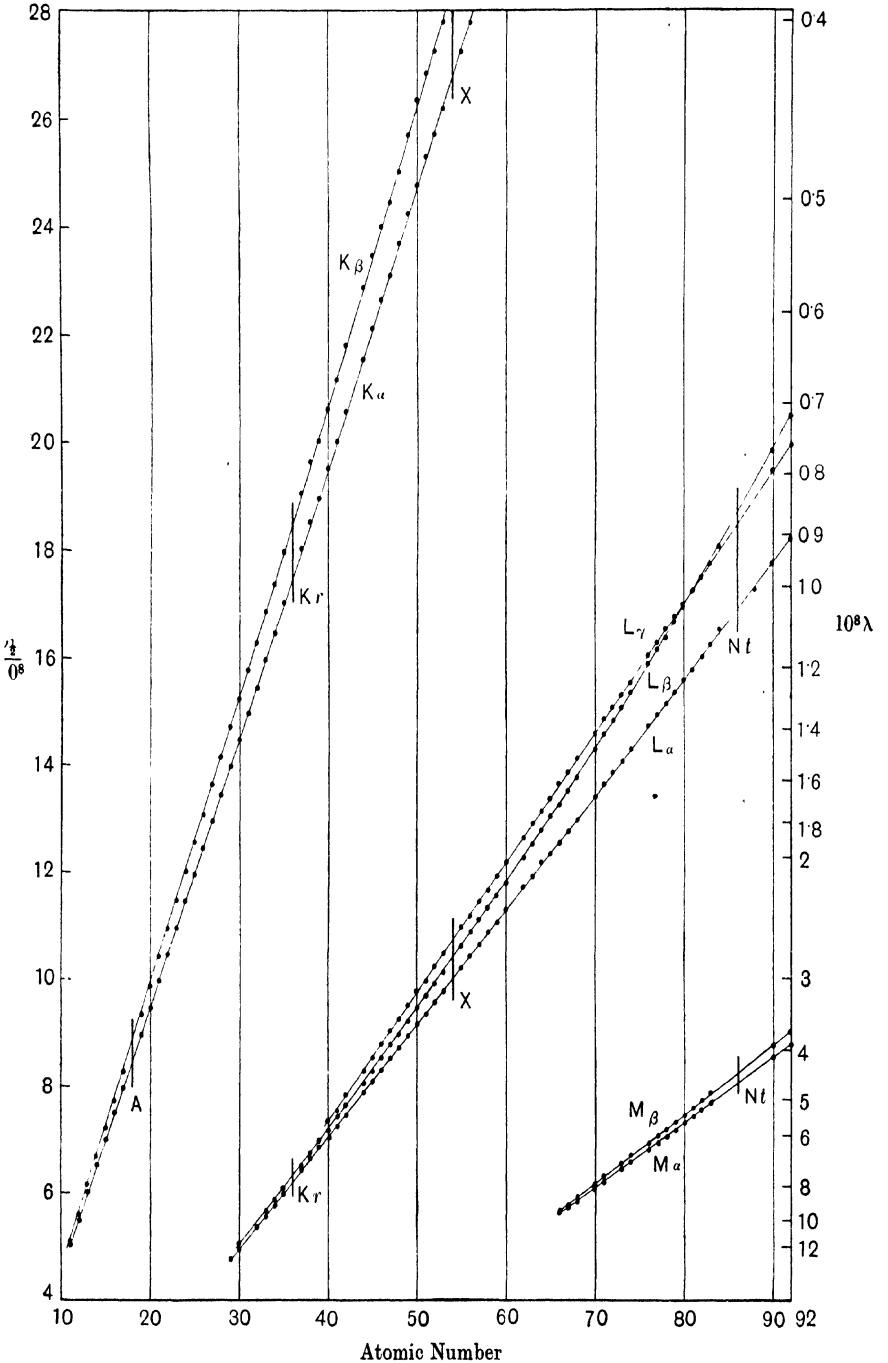

28
26
24
22
20
18
16
14
12
10
8
6
4
$\frac{\nu}{10^8}$
0·4
0·5
0·6
0·7
0·8
0·9
1·0
1·2
1·4
1·6
1·8
2
3
4
5
6
8
10
12
$10^8\lambda$
10
20
30
40
50
60
70
80
90
92
Atomic Number
K$_\beta$
K$_\alpha$
K$_r$
X
A
L$_\gamma$
L$_\beta$
L$_\alpha$
N$_t$
M$_\beta$
M$_\alpha$

Fig. 344

iron and nickel both chemically and in its Röntgen ray spectrum, though its atomic weight exceeds that of nickel. Moseley plotted the square root of the frequency against atomic number (cf. Art. 276), and found curves which are straight in the main, though not exactly so (Fig. 344)*. Gaps indicate missing elements (if such exist) with atomic numbers 43, 61, 75, and *there are no other gaps between sodium and polonium.* Chlorine, with two isotopes, has a spectrum exactly like those of the preceding elements phosphorus and sulphur; from which we conclude that *isotopes have the same Röntgen ray spectra.* Moseley's experiments, therefore, provide a new method of putting the elements in order, and give the strongest support to van den Broek's hypothesis (Art. 223); for it is difficult to ascribe the regular progression of frequency to anything but a change of nuclear charge by one unit at a time. The regularity is not however absolute (see Art. 240).

Röntgen ray spectra show the constituents of mixtures and compounds, as we should expect from their atomic origin. Thus brass gives the spectra of zinc and copper, chlorine can be examined in sodium chloride spread on the target, the rare earths in impure preparations. The inert gases, which cannot be so examined, are represented by the marked gaps in Fig. 344. No effect of temperature has yet been discovered, though many accurate experiments have been made with hot and cold targets.

The element of atomic number 72 was discovered in 1922 by Coster and Hevesy, by Röntgen ray analysis, and given the name of hafnium. Its chemical properties are similar to those of zirconium.

Extension of the $K$ series below $Z = 11$, and of the $L$ series below $Z = 29$, is hindered by the lack of crystals of large grating constant; for a crystal of grating constant $d$ will only measure wave-lengths up to $2d$. Further information is obtained by photo-electric methods (Art. 241).

**237. The continuous spectrum.** The sharp beginning of the continuous spectrum, referred to in Art. 232, was discovered by Duane and Hunt with an ionisation spectrometer. The experiment

* The regular decrease of wave-length with increasing atomic number is shown in Siegbahn's photograph of the $K$ series of elements between arsenic and rhodium (Fig. 343).

can be carried out in two ways. We may either keep the high potential constant and vary the glancing angle (Figs. 341, 346) or set the spectrometer for one wave-length and vary the voltage.

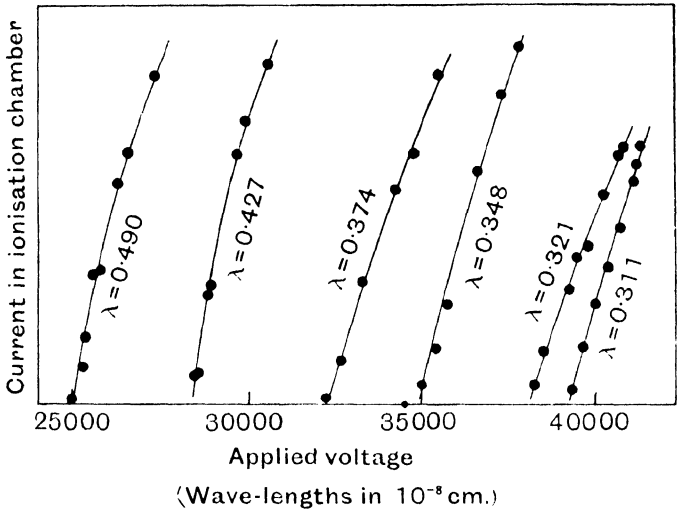



Fig. 345

In the second method no rays of that wave-length are produced below a well-marked potential, which we call the starting potential. Duane and Hunt used a Coolidge bulb in conjunction with a battery of 20,000 accumulators, and a calcite crystal. Their curves are shown in Fig. 345, and the starting potentials in the following table.

| Wave-length λ in A.U. ($10^{-8}$ cm.) | Frequency $\nu$ | Starting potential $V$ in volts | Energy $E$, in ergs, of an electron falling through potential $V$ | $E/\nu$ |
|---|---|---|---|---|
| 0·490 | $6{\cdot}12 \times 10^{18}$ | 25000 | $3{\cdot}98 \times 10^{-8}$ | $6{\cdot}50 \times 10^{-27}$ |
| 0·427 | 7·03 | 28400 | 4·52 | 6·43 |
| 0·374 | 8·02 | 32250 | 5·13 | 6·40 |
| 0·348 | 8·62 | 34900 | 5·55 | 6·44 |
| 0·321 | 9·35 | 37950 | 6·03 | 6·45 |
| 0·311 | 9·65 | 39150 | 6·22 | 6·45 |

The energies in the fourth column are practically those which the electrons have on arrival at the target, since the energy of emission corresponds only to a fraction of a volt. To produce rays of frequency

$\nu$, therefore, we require electrons of energy $h\nu$, where $h$ is a constant which is about $6{\cdot}45 \times 10^{-27}$, that is *the same constant which was found previously from the photoelectric effect* (Art. 210). The quantum theory, there confirmed for frequencies between $0{\cdot}55 \times 10^{15}$ and $1{\cdot}2 \times 10^{15}$, is shown to hold for frequencies of the order of $10^{19}$. It would seem, therefore, to hold over the whole range of the visible and Röntgen ray spectra, and the evidence for the linear law of Art. 210 is much strengthened. Duane and Hunt's experiment is the converse of the photoelectric effect: the direct analogue is described in Art. 241.

Accurate experiments have to take into account the width of the primary beam of Röntgen rays. Duane, Palmer and Yeh gave the result $6{\cdot}56 \times 10^{-27}$, with a tungsten target. Perhaps the main source of error, the measurement of the high potential, was avoided by potentiometric comparison with a standard cell. Webster found $6{\cdot}53 \times 10^{-27}$ with a rhodium target, and Wagner $6{\cdot}53 \times 10^{-27}$ with copper. Wagner worked with low potentials, down to 4500 volts, in order to obtain large glancing angles, thereby confirming Einstein's law at a lower frequency. The value

$$h = 6{\cdot}55 \times 10^{-27} \quad ..(298)$$

is adopted in this book. Photographic experiments are less accurate, as it is not easy to see exactly where the darkening at the short-wave limit begins.

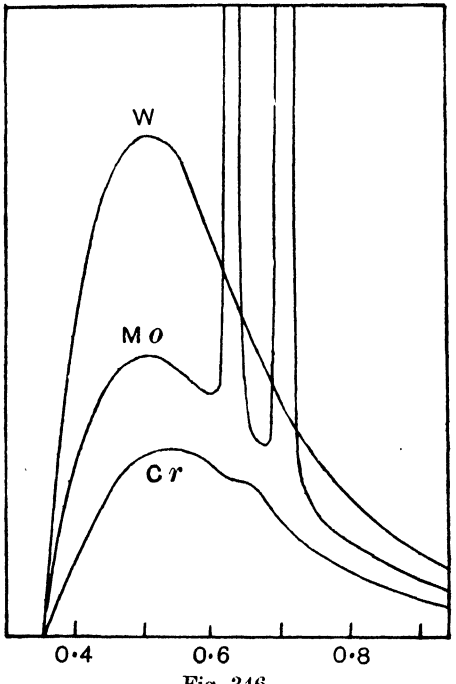


Fig. 346

The universal nature of the constant $h$ has been shown directly in two ways. Duane and Hunt found the same short-wave limit for unobstructed rays and rays that had passed through an aluminium screen. Ulrey mounted plates of six heavy elements on a six-faced target which could be turned by a magnet outside the bulb. His

Fig. 346 shows the equality of the short-wave limits of tungsten, molybdenum and chromium at 35,000 volts, and shows also that the $K$ lines of molybdenum are excited by a potential which does not suffice to excite the $K$ lines of tungsten. The voltage necessary to excite homogeneous rays of various kinds is considered in Art. 238, where it will appear that more is wanted than that corresponding to the quantum $h\nu$. A wave-length of $\lambda$ cm. cannot therefore be obtained in any way with a discharge potential of less than $V$ volts, where $V = 12300/10^8\lambda$ (cf. Art. 220).

**238. Absorption of Röntgen rays.** Homogeneous Röntgen rays are absorbed according to an exponential law; that is, if the ionising power of a beam is $I$, it sinks to $Ie^{-Ax}$ (apart from scattering) after the rays have passed through a sheet of matter of thickness $x$. The *absorption coefficients* $A$ are much greater for heavy than light substances: they become more comparable when divided by the density $\rho$. Fig. 347 gives the values of $A/\rho$ for aluminium,

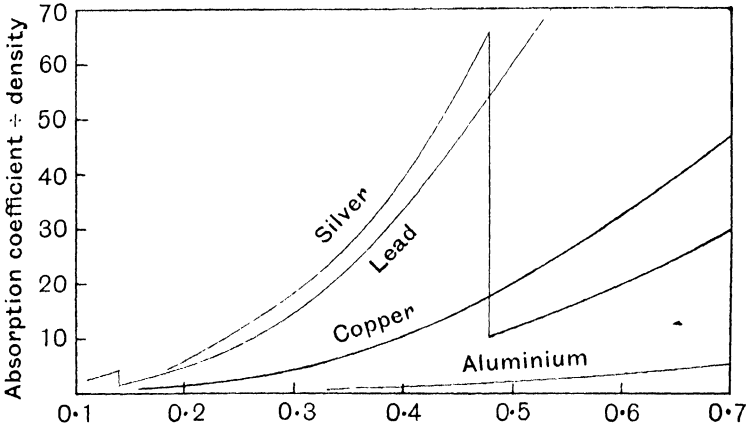



Fig. 347

copper, silver and lead over a certain range of wave-lengths, from experiments by Hull, Rice and Richtmyer. The absorption is in general greatest for long wave-lengths, but is set back sharply at one or more wave-lengths characteristic of the substance: thus the absorption coefficient of lead diminishes discontinuously at $\lambda = 0{\cdot}14 \times 10^{-8}$ cm., that of silver at $\lambda = 0{\cdot}485 \times 10^{-8}$ cm. These

*absorption limits* are seen clearly in photographs of continuous spectra taken with a rotating crystal, when absorbing screens are placed in the path of the rays. As the glancing angle increases, the blackening of the negative decreases up to a certain point, then increases suddenly to form a dark edge, and shades off continuously above. There is always an absorption limit at a wave-length *slightly less than that of the shortest K line of the absorbing medium.*

Similar effects are observed in the *L* and *M* series. There are three absorption limits near the *L* lines and five near the *M* lines, one having in each case a wave-length slightly shorter than any line of the series. High resolution shows each absorption limit to be accompanied by faint absorption *lines*, sometimes on one side and sometimes on the other. Kossel explained those on the long-wave side by assuming that an electron is removed, not to infinity, but to an "optical" level above those normally occupied in the atom (Art. 259): the origin of the others is obscure. The table on p. 541 gives the wave-lengths of the absorption limits observed hitherto in the *K* and *L* series.

The general effect of Röntgen rays on matter has been described in Art. 215. Clark and Duane found that when tungsten rays fall on potassium iodide and other substances, the secondary beam contains the wave-lengths of the primary radiation, as required by Art. 231. A. H. Compton found also a change of wave-length on scattering, the full consequences of which have yet to be perceived. When Röntgen rays are absorbed the energy goes, as far as can be ascertained, in exciting the characteristic rays of the absorber and in expelling electrons from its atoms—two effects which the theory of Art. 239 shows to be closely related to each other. Barkla, who discovered characteristic rays, found them in a secondary beam, and de Broglie and others confirmed the discovery by analysing the secondary radiation with a Röntgen ray spectrometer. An element can, therefore, be made to yield its spectrum either by the impact of swift electrons or under the action of Röntgen rays; and we should expect the two to be equivalent when the energy of one is related to the wave-length of the other by equation (276). This is quite a different thing from saying that voltage *V* will excite *characteristic* radiation given by equation (276). The theory of

*Absorption limits in $10^{-8}$ cm.; K and L series.*

| Atomic number | Element | $K$ | Atomic number | Element | $K$ | $L_1$ | $L_2$ | $L_3$ |
|---|---|---|---|---|---|---|---|---|
| | | | 50 | Tin | 0·424 | | | |
| | | | 51 | Antimony | 0·406 | 2·633 | 2·831 | 2·994 |
| | | | 52 | Tellurium | 0·390 | 2·503 | 2·684 | 2·847 |
| | | | 53 | Iodine | 0·374 | 2·382 | 2·548 | 2·712 |
| 11 | Sodium | | 54 | Xenon | 0·359 | | | |
| 12 | Magnesium | 9·511 | 55 | Caesium | 0·344 | 2·160 | 2·307 | 2·468 |
| 13 | Aluminium | 7·947 | 56 | Barium | 0·331 | 2·060 | 2·199 | 2·358 |
| 14 | Silicon | | 57 | Lanthanum | 0·319 | 1·971 | 2·098 | 2·250 |
| 15 | Phosphorus | 5·758 | 58 | Cerium | 0·307 | 1·887 | 2·005 | 2·160 |
| 16 | Sulphur | 5·012 | 59 | Praseodymium | | 1·808 | 1·920 | 2·073 |
| 17 | Chlorine | 4·384 | 60 | Neodymium | 0·286 | 1·736 | 1·838 | 1·990 |
| 18 | Argon | 3·866 | 61 | | | | | |
| 19 | Potassium | 3·434 | 62 | Samarium | | 1·608 | 1·701 | 1·841 |
| 20 | Calcium | 3·063 | 63 | Europium | | | | 1·773 |
| 21 | Scandium | 2·752 | 64 | Gadolinium | | 1·470 | 1·550 | 1·699 |
| 22 | Titanium | 2·494 | 65 | Terbium | 0·240 | | | |
| 23 | Vanadium | 2·265 | 66 | Dysprosium | 0·231 | 1·362 | 1·435 | 1·576 |
| 24 | Chromium | 2·067 | 67 | Holmium | 0·221 | | | |
| 25 | Manganese | 1·889 | 68 | Erbium | | 1·265 | 1·336 | 1·478 |
| 26 | Iron | 1·740 | 69 | Thulium | 0·208 | 1·220 | 1·285 | 1·430 |
| 27 | Cobalt | 1·602 | 70 | Ytterbium | 0·201 | 1·171 | 1·242 | 1·386 |
| 28 | Nickel | 1·489 | 71 | Lutecium | 0·195 | 1·136 | 1·194 | 1·338 |
| 29 | Copper | 1·378 | 72 | Hafnium | 0·190 | 1·097 | 1·151 | 1·293 |
| 30 | Zinc | 1·296 | 73 | Tantalum | | 1·058 | 1·112 | 1·253 |
| 31 | Gallium | 1·190 | 74 | Tungsten | 0·178 | 1·024 | 1·072 | 1·213 |
| 32 | Germanium | 1·115 | 75 | | | | | |
| 33 | Arsenic | 1·043 | 76 | Osmium | 0·168 | 0·951 | 0·998 | 1·138 |
| 34 | Selenium | 0·979 | 77 | Iridium | | 0·919 | 0·965 | 1·104 |
| 35 | Bromine | 0·918 | 78 | Platinum | 0·158 | 0·888 | 0·932 | 1·070 |
| 36 | Krypton | 0·865 | 79 | Gold | 0·153 | 0·861 | 0·899 | 1·038 |
| 37 | Rubidium | 0·814 | 80 | Mercury | 0·149 | 0·833 | 0·870 | 1·007 |
| 38 | Strontium | 0·770 | 81 | Thallium | 0·144 | 0·805 | 0·841 | 0·978 |
| 39 | Yttrium | 0·725 | 82 | Lead | 0·140 | 0·780 | 0·813 | 0·950 |
| 40 | Zirconium | 0·687 | 83 | Bismuth | 0·136 | 0·753 | 0·787 | 0·922 |
| 41 | Niobium | 0·650 | 84 | Polonium | | | | |
| 42 | Molybdenum | 0·618 | 85 | | | | | |
| 43 | | | 86 | Niton | | | | |
| 44 | Ruthenium | 0·558 | 87 | | | | | |
| 45 | Rhodium | 0·533 | 88 | Radium | | | 0·668 | 0·802 |
| 46 | Palladium | 0·507 | 89 | Actinium | | | | |
| 47 | Silver | 0·485 | 90 | Thorium | 0·113 | 0·604 | 0·629 | 0·760 |
| 48 | Cadmium | 0·463 | 91 | Uranium $X_2$ | | | | |
| 49 | Indium | 0·443 | 92 | Uranium | 0·106 | 0·568 | 0·592 | 0·721 |

Art. 239, and the experiments described below, show that more voltage is wanted.

Webster observed the effect of applying increasing potentials to

a Coolidge bulb with a rhodium target. Continuous radiation of wave-length $0{\cdot}614 \times 10^{-8}$ cm. (the mean wave-length of the $K_\alpha$ doublet of rhodium) begins at a potential of 20,000 volts, in approximate agreement with Duane and Hunt, but the spectrum does not yet contain the $K_\alpha$ doublet, which rises first from the continuous background at 23,200 volts, the $K_\beta$ line appearing at the same potential. The "quantum wave-length" corresponding to the potential is $0{\cdot}532 \times 10^{-8}$ cm., which is *the wave-length of the K absorption limit.* The suggestion, therefore, is that characteristic rays are not excited by Röntgen rays of their own wave-length, but the $K$ series remains unexcited till the wave-length is reduced to that of the $K$ absorption limit, when all the lines appear at once (see Art. 239). De Broglie observed absorption limits first while working with a photographic spectrometer in a range containing the $K$ limits of silver and bromine. This is an inverse effect. Consider a wave-length just below the absorption limit of bromine. The rays are absorbed more than they are just above it, but the energy goes in secondary rays (ethereal or electronic), all of which affect the photographic plate, since they are emitted from the bromine atoms of the emulsion itself. Had the photographic plate been used as an absorbing screen in front of an ionisation spectrometer, few secondary rays would have entered, and the ionisation would have been less below the absorption limit than above it (Fig. 347). The bands are therefore shaded in opposite directions in the two experiments.

The approximate absorption data given in Art. 213 can now be made more precise. A bulb worked at 100,000 volts, for example, gives out a continuous spectrum whose short-wave limit is at $0{\cdot}12 \times 10^{-8}$ cm. There will therefore be a considerable amount of radiation for which $A/\rho$ in aluminium is about 0·18, requiring a thickness of 1·5 cm. (or 0·2 mm. of lead) to cut it down to half strength. If the target is of tungsten, the spectrum will contain the characteristic lines, of which $K_\alpha$ is reduced to half strength by 1 cm. of aluminium, and $L$ by a thickness of 0·6 mm. Most $L$ radiation is absorbed considerably by the walls of a Röntgen ray bulb. The contrast obtainable in Röntgen ray photographs depends on the ratio of the absorption coefficients of the substances traversed by the rays, and must therefore vary with the wave-length; a silver

disc being, for example, more opaque to a wave-length of $0{\cdot}4 \times 10^{-8}$ cm. than a similar lead disc, and less opaque to a wave-length of $0{\cdot}5 \times 10^{-8}$ cm.

**239. Origin of the Röntgen ray spectra.** The nuclear theory of the atom (Art. 223) was first connected with the quantum theory by Bohr. He assumed that an electron does not radiate as long as it remains in the same energy level, but does when it falls from a level of energy $W_1$ to one of lower energy $W_2$, the frequency of the radiation emitted being connected with the loss of energy by the relation

$$h\nu = W_1 - W_2.$$

Conversely, an electron can be raised from $W_2$ to $W_1$ by absorbing light of frequency $\nu$, given by the same equation. An atom must, therefore, be put first into an abnormal state, or *excited*, before it can radiate. These views were first applied by Bohr to optical spectra (Art. 259): important details of their application to Röntgen ray spectra, such as the *rôle* of the absorption limits (see below), are due to Kossel.

Consider for simplicity an atom with three energy levels $K$, $L$, $M$ (Fig. 348). An electron striking it with sufficient velocity may remove one of its $K$ electrons to a great distance from the nucleus, represented in the figure by the level $S$ of zero energy. The $K$ level is now incomplete and tends to attract electrons from higher levels. Most of these will come from the $L$ level, emitting radiation of frequency $(W_L - W_K)/h$ in the process. This is the $K_\alpha$ radiation of the element. The $K_\beta$ radiation, weaker and of higher frequency, is produced by electrons falling from the $M$ level to the $K$ level. We have therefore a natural explanation of the multiplicity of the $K$ lines, and of the general diminution of their intensity with wave-length. The origin of the $L_\alpha$ line is shown in the right-hand part of Fig. 348. We see now why

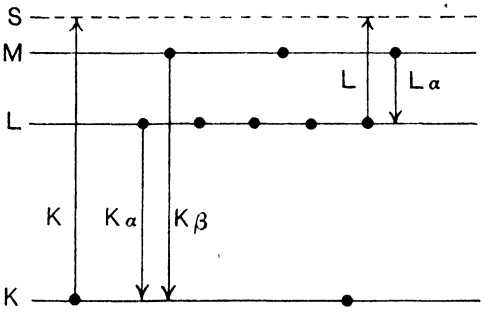


Fig. 348

the same notation has been chosen for energy levels and spectral series: all the $K$ lines are produced by electrons falling into the $K$ level, the $L$ lines by those falling into the $L$ level. Kossel's theory indicates the presence of three lines in the spectrum of an element having three energy levels, six lines in one of four levels, and so on. We shall see most clearly how to extend it, so as to explain the spectra actually observed, by considering the *rôle* of the absorption limits.

Kossel's theory explains at once why the lines of the $K$ series are all excited together by incident energy exceeding a certain amount (Art. 238), which is obviously $-W_K$, the energy required to move a $K$ electron to infinity. An atom with three energy levels $L, M, N$ has three absorption limits, of frequency $-W_K/h$, $-W_L/h$ and $-W_M/h$. This interpretation of the absorption limits can be subjected to a crucial test as follows. The frequency of the $K_\alpha$ line being $(W_L - W_K)/h$, we should have $K_\alpha = K - L$, where $K_\alpha$ denotes the frequency of the $K_\alpha$ line, $K$ that of the $K$ absorption limit and $L$ that of the $L$ absorption limit. The multiplicity of the $L$ absorption limits requires us to assume a corresponding number of energy levels, and we should naturally begin by trying to associate the strongest line $K_\alpha$ with the limit $L_3$, which is strongest on all absorption photographs. The relation

$$K_\alpha = K - L_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (299)$$

is, in fact, found to hold for all elements for which it can be tested; thus for caesium (omitting a factor of $10^{18}$ in the frequency) $K = 8{\cdot}72$, $L_3 = 1{\cdot}22$, $K_\alpha = 7{\cdot}54$, and for platinum $K = 18{\cdot}98$, $L_3 = 2{\cdot}80$, $K_\alpha = 16{\cdot}21$. It is clear, therefore, that the $K_\alpha$ line is emitted by an electron which begins in the $L_3$ level and ends in the $K$ level. The relations

$$K_{\alpha'} = K - L_2, \quad K_\beta = K - M \quad \ldots\ldots\ldots (300)$$

indicate similarly the origin of the next strongest lines of the $K$ series, the relative smallness of the $M$ frequencies making it unnecessary to inquire (at this stage) which of the $M$ levels is concerned.

The multiplicity of the $L$ levels, to which we have been led by consideration both of emission lines and absorption limits, is vouched for most strongly by observations of excitation voltage. Hoyt found that the $L_\alpha$ and $L_\gamma$ lines of tungsten are excited by

10,200 volts, $L_\beta$ by 11,600 volts and some other lines by 12,000 volts. The quantum wave-lengths are 1·206, 1·060 and 1·026 A.U., practically those of the $L_3$, $L_2$ and $L_1$ limits. The lines $L_\alpha$ and $L_\gamma$ have, therefore, $L_3$ as terminal level, and $L_\beta$ has $L_2$.

Regarding the relations (299) and (300) as a general confirmation of Kossel's theory, we see (1) that an absorption limit is evidence of an energy level, (2) that the corresponding energy is $-h$ times the absorption frequency. The lines of the Röntgen ray spectra, as such, play a secondary part in atomic theory. The energy levels express the real structure, the lines their distance apart on the scale of energy. It is difficult to imagine a greater contrast between the processes of emission of (relatively long) electric waves by a Hertzian oscillator and of Röntgen rays on the Bohr-Kossel theory. The quantum theory strikes at the base of those theories which would regard an atom as a Hertzian oscillator of very high frequency, its own point of view being that a Hertzian oscillator represents a limiting case of large quantum numbers (cf. Art. 265).

**240. The energy levels of the elements.** Smekal and Coster showed that most lines of the Röntgen ray spectra can be explained by transitions between one $K$ level, three $L$ levels, five $M$ levels, seven $N$ levels and so on, corresponding to the known absorption limits, so far as they have been observed. Not every transition is allowed; thus no electron can move from the $L_1$ level to the $K$ level, or from one $L$ level to another (cf. Art. 265). We note (without justifying the classification in detail) that lines with the same excitation voltage have the same final level (Art. 239), and that if the frequencies of four lines $X$, $Y$, $Z$, $W$ are connected by the equation $X - Y = Z - W$, $X$ and $Y$ must have the same final (or initial) level, $Z$ and $W$ the same final (or initial) level, $X$ and $Z$ the same initial (or final) level, and $Y$ and $W$ the same initial (or final) level. The lines tabulated in Art. 236 are accounted for by the scheme

$$\left.\begin{array}{lll} K_\alpha = K - L_3, & K_{\alpha'} = K - L_2, & K_\beta = K - M_3 \\ L_\alpha = L_3 - M_5, & L_\beta = L_2 - M_4, & L_\gamma = L_3 - N_5 \\ M_\alpha = M_5 - N_7, & M_\beta = M_4 - N_6 & \end{array}\right\} \quad \ldots(301)$$

and shown in Fig. 349, a generalisation of Fig. 348.

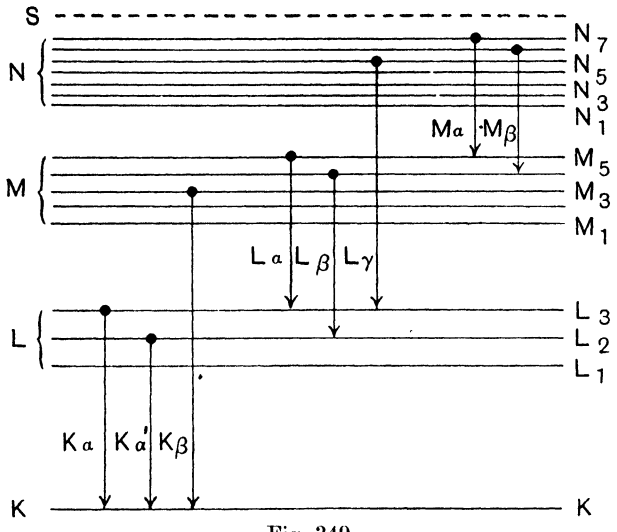



Fig. 349

These equations can be used to calculate the energy of levels whose absorption wave-length is too long to measure, e.g. the $L$ levels of light elements. The calculation is of great interest, for reasons first given by Kossel. Since the frequency of the absorption limits increases with atomic number, the levels as drawn in Fig. 349 gradually sink, allowing others to form above them. Their fall need not however be regular. To use a rough analogy, the effect of pouring water into a glass changes suddenly when the glass is full. We might expect a level to fall according to one law when it is being filled up and to another when full, so that the curves of energy against atomic number might show irregularities of slope. Also a level which is already full might be affected by the completion of one above it.

Fig. 350, adapted from Bohr and Coster, shows the square root of the theoretical excitation voltage $V^*$ of different levels, plotted against the atomic number $Z$. The figure should be compared with Bohr's scheme of distribution of circumnuclear electrons (Art. 223). We notice first an increase in the slope of $M_2$ and $M_3$ at $Z = 29$, where the $M$ levels are filled up to their final number of 18. The slope decreases at $Z = 39$, where the $N$ level resumes its interrupted

* That is, the energy of the level (with sign reversed) expressed in volts. The ionising potential (Art. 224) is the least excitation voltage of any level in the atom.

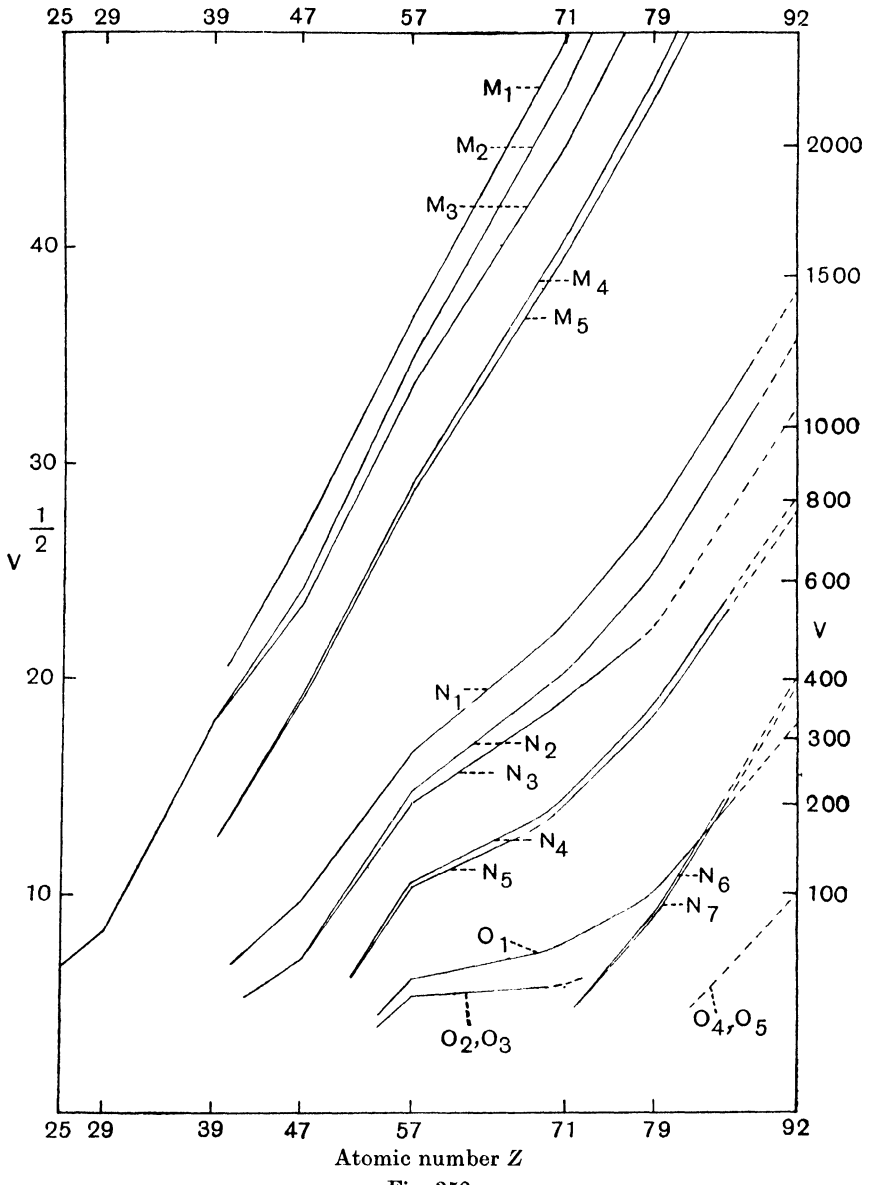



Fig. 350

development, increasing after it has stopped again at $Z = 47$. At $Z = 57$ we have a marked change in all the levels, the $O$ levels becoming almost flat. While the resumption of the $N$ levels at $Z = 58$ does not show on the curves, its final completion at $Z = 71$

is shown in the $M$ levels, and less clearly in the $N$ and $O$ levels, where the experimental error is greater. Development of the $O$ level to 18, at $Z = 79$, gives a further change of slope of the same character. Thus the Röntgen ray measurements, while not yet accurate enough for all purposes, give a general support to Bohr's theory, though leaving open distinct possibilities of modification. They do not at present show the distribution of electrons amongst the different sub-levels, nor (with certainty) where new lines (i.e. new levels) first appear: obviously a point of great importance to the theory.

The $L$ levels, which are complete at $Z = 10$, show slight changes of slope thereafter. To show the corresponding irregularities in the emission lines (Fig. 344) would require greater accuracy of tabulation and plotting than was attempted in Art. 236. The interest of Moseley's curves is however undiminished, for they show that the inner orbits of an atom are of fixed type, controlled mainly by the nucleus and relatively immune from disturbance by outer layers (cf. Art. 276). The possibility of slight disturbance, even of the $L$ levels, was shown independently by Bergengren and Lindh, who found that chemical combination affects the $L$ absorption limits of phosphorus and chlorine by amounts of the order of 0·010 A.U. This observation is unwelcome in so far as it may prevent the finer details of atomic structure from being exhibited in diagrams like Fig. 350.

To resume, the regularity of Röntgen ray spectra decreases as we go from the inside of the atom to the outside, where the electrons are held with an energy of a few volts only. Röntgen ray spectra are not periodic, but show characteristic changes of a different kind. Real periodicity is a property of the outermost electrons of all, and chemical evidence shows that the top level should contain only 0, 1, 2, ... 8 electrons, corresponding to the nine columns of the periodic table (Art. 276).

Low excitation potentials are bound up closely with the *transition from Röntgen rays to ultra-violet light.* Medium Röntgen rays have excitation voltages of the order of 10,000, and all those which can be measured with a crystal require 1000 volts at least. On the optical side, the quantum voltage of visible light varies from 1·7 (red) to 3·1 (violet). Millikan measured the ultra-violet spectrum of

aluminium down to 144 A.U., with a quantum voltage of 85. We are thus left with a gap of about four octaves, which was explored photoelectrically by Richardson and Bazzoni, and others. The substance whose radiation is to be excited is made the target of a Coolidge bulb, inside which is placed a metal plate (connected to an electrometer) and a grid, as in Fig. 292. The photoelectric current increases with the potential applied to the bulb, but irregularly after the manner of Fig. 321, the points of rapid change being taken to indicate voltages at which new radiations first appear, i.e. excitation voltages of characteristic Röntgen rays. While the experimental results are not very consistent, there is evidence for Kossel's view that the curves connecting $Z$ and $V^{\frac{1}{2}}$ for the $L$ levels increase in slope at $Z = 10$, where the $L$ levels are completed.

**241. Velocity of the electrons ejected from matter by Röntgen rays.** The earliest experiments have been referred to in Art. 215. The velocity of ejection was measured by Robinson and Rawlinson, and by de Broglie, with the apparatus shown in Fig. 351.

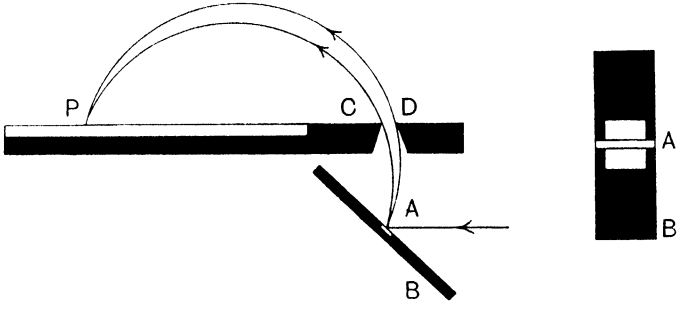



Fig. 351

A narrow beam of Röntgen rays falls on a strip $A$ of the substance under investigation, which is supported on an aluminium plate $B$, cut away in the neighbourhood of $A$, as shown on the right of the figure. Metals are examined in thin sheets; iodine, barium and other substances as salts on a strip of filter paper. The apparatus is enclosed in a flat box which can be evacuated and placed between the poles of a large electromagnet, giving a field of the order of 2000 gausses perpendicular to the plane of the paper. The line joining

$A$ to the centre of the slit is perpendicular to the sensitive face $CP$ of a photographic plate. The radii of the circles $ACP$ and $ADP$ are equal to the first order of small quantities, if $CD$ is taken to be of that order. Hence electrons of equal velocity are united very approximately at $P$, even with a wide slit, and each point $P$ corresponds to a known velocity. This method of focussing was first used by Classen in measuring $e/m$ for the electrons from hot metals (Art. 211). The radius of curvature is deduced from measurements along the plate, and then $v$ is found approximately from equation (251); more accurately from the table of Art. 253, since the variation of mass has to be taken into account. Exposures of one to three hours are required, with a powerful primary bulb.

The theory of energy levels leads us to expect several groups of electrons. Suppose, for example, that the $K_\alpha$ rays of rhodium fall on copper, and let $RhK_\alpha$ denote the quantum voltage of the incident radiation. A certain number of electrons will take up energy $RhK_\alpha$ and leave their parent atoms, but will lose the energy corresponding to the levels from which they start. Thus if $CuK$ is the excitation voltage of the $K$ level of copper, electrons expelled from that level will leave the atom with energy $RhK_\alpha - CuK$. All electrons lose energy in clearing the surface (Art. 210), but those which come from underneath lose larger and varying amounts by collision with atoms. We should therefore expect a band on the photographic plate, ending sharply on the side of high velocity, for each line of the incident radiation and for each energy level in the atoms of the obstacle. If $W$ is the observed energy corresponding to one of the bands, we should have approximately

$$W = RhK_\alpha - CuK,$$

since the surface loss is a volt or two at most. De Broglie found four bands, with edges shown in the table on p. 551. The energy assigned to $CuL$ in the table has been calculated from the formulae

$$L_2 = K - K_{\alpha'}, \quad L_3 = K - K_\alpha,$$

which apply equally to quantum voltages. The quantum voltages of $CuK$, $CuK_{\alpha'}$ and $CuK_\alpha$ being 8930, 7990 and 8000, $CuL_2$ and $CuL_3$ must both be near 900. This is the value taken for $CuL$: similarly $RhL$ is taken as 3000. De Broglie was not able to distinguish between the different $L$ limits, nor (usually) between

| Rhodium rays falling on copper | | Tungsten rays falling on rhodium | |
|---|---|---|---|
| Observed energy in volts | Energy differences in volts | Observed energy in volts | Energy differences in volts |
| 11300 | $RhK_{\alpha'} - CuK = 11100$<br>$RhK_{\alpha} - CuK = 11200$ | 16800 | $RhK_{\alpha'} - RhL = 17000$<br>$RhK_{\alpha} - RhL = 17100$ |
| 13600 | $RhK_{\beta} - CuK = 13700$ | 19400 | $RhK_{\beta} - RhL = 19600$ |
| 19200 | $RhK_{\alpha'} - CuL = 19100$<br>$RhK_{\alpha} - CuL = 19200$ | 35800 | $WK_{\alpha'} - RhK = 34700$<br>$WK_{\alpha} - RhK = 35800$ |
| 21600 | $RhK_{\beta} - CuL = 21700$ | | |

$K_\alpha$ and $K_{\alpha'}$. The experiments leave little doubt that the four bands are due respectively to electrons ejected by the $K_\alpha$ doublet from the $CuK$ level, the $K_\beta$ line from the $CuK$ level, the $K_\alpha$ doublet from the $CuL$ level and the $K_\beta$ line from the $CuL$ level. The $K$ radiation of copper is excited and would no doubt give bands of energy $CuK_\alpha - CuL$, ..., lower than those observed. De Broglie found a similar effect when tungsten rays fall on rhodium (right-hand table). We have not only the band $WK_\alpha - RhK$, but also $RhK_\alpha - RhL$ and $RhK_\beta - RhL$. Other substances gave results agreeing generally with the theory, but less accurate as the edges of the bands were often badly defined.

Robinson showed that it is possible, by more accurate measurement, to distinguish the $M$ and $N$ levels of heavy elements. The energies of these levels generally agree well with those found by the method of Art. 240.

The varying velocities of electrons ejected from copper were also observed by C. T. R. Wilson by the method of condensation (Art. 216). Most of the tracks are between 2 and 3 mm. long, as if arising from the $K$ radiations of copper. Others are small spherical clouds, not more than 0·1 mm. in diameter, indicating electrons of still smaller velocity ejected apparently by the $L$ radiations of copper. This interpretation is confirmed by experiments with aluminium, where the 2 mm. tracks are absent, the $K$ radiation of aluminium having a frequency comparable with that of the $L$ radiation of copper. We conclude that electrons whose energy is 1500 volts or less cannot travel more than 0·1 mm. in air

at normal temperature and pressure. This small penetration was already known to Hertz and Lenard: still slower electrons require maintaining by electric force, or they would lose their velocity in a few collisions.

Wilson found, by using very weak rays, that *for every long track which leaves the obstacle, a second short track starts in the air near it.* This agrees with the quantum theory of emission; for if the first electron is ejected, say, from the $K$ level of copper by radiation of quantum voltage $V$, it has energy $V - CuK$, and the atom is now ready to emit $K$ radiation, which may itself eject an electron from the air.

Reviewing the whole subject, it is interesting to note how photo-electric phenomena change as the wave-length decreases. The effect is, so to speak, latent above the long-wave limit. Electrons are no doubt ejected from atoms, but they remain in the body of the metal and their energy is dissipated as heat. Ultra-violet light expels electrons only from the upper levels. Röntgen rays attack the lower levels; and since the quanta of energy greatly exceed that necessary to clear the surface of the metal, the effect is seen in its purest form. Further reduction of wave-length confirms the observations made with Röntgen rays, without altering their character (Art. 255).

# CHAPTER XIV

## RADIOACTIVITY

**242. Discovery of radioactivity.** H. Becquerel observed in 1896 that crystals of the double sulphate of uranium and potassium, $K(UO)SO_4 . H_2O$, gave a well-marked impression on a photographic plate after a day's exposure, when the plate was screened from the light by wrapping it in two thicknesses of black paper. This was attributed at first to the fluorescence of the crystals; but the effect is produced equally well by crystals kept for a long time in the dark, and also by non-fluorescent uranium salts; while no effect is produced by non-uranium salts, however fluorescent. Becquerel showed that the determining factor was the amount of uranium present, and that pure uranium was more active, weight for weight, than any of its compounds. The effect is still produced when a thin sheet of glass is interposed between the metal and the photographic plate to cut off possible vapours rising from the uranium. Thick objects, such as coins, cast shadows on the plate, but the radiations pass easily through thin sheets of aluminium or mica.

Becquerel also found that uranium and its compounds would discharge an electroscope when brought near it. This electrical

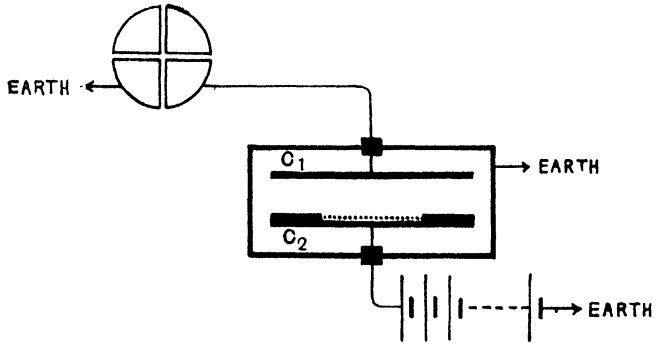



Fig. 352

effect can be examined by any of the methods used in the conduction of electricity through gases. Fig. 352 shows the simplest form of apparatus. The active matter is spread in a thin layer on

the lower plate $C_2$ of a parallel-plate condenser, giving rise to a steady current through the gas, which rises with the applied electromotive force and tends towards a saturation value. This saturation current, as determined by the rate at which the spot of light moves over the electrometer scale, is taken as a measure of the activity of the substance. It is important to notice that the current is not unidirectional, as in the photoelectric effect, but similar to that produced by weak Röntgen rays.

In accurate work it is best to measure the charge acquired by the insulated system in a given time (Art. 209). Since the rays cannot be conveniently put on and off, a special *electrometer key* has to be used, as shown in Fig. 353. $P_1$ and $P_2$ are two insulated brass pieces contained in the earthed cylindrical case $C$, so that by moving the handle $P_2$ can be connected either to earth or to the electrometer at will, and without disturbing the charges on the insulated system. Sensitive electroscopes have also been used extensively for measuring weak ionisations.

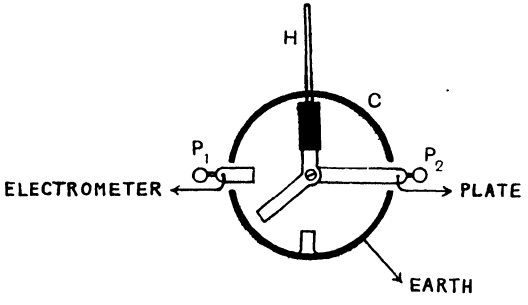


Fig. 353

The electric action, like the photographic, can be produced across thin sheets of matter, but is much reduced by thick ones. The absorption of the radiations by the active substance itself may also become serious. Using very thin layers, however, McCoy found that the activity of uranium salts is accurately proportional to the amount of uranium present, as shown in the following table:

| Salt | Activity of unit mass | Percentage of uranium present | Ratio |
|---|---|---|---|
| Uranium oxide $U_3O_8$ | 16·69 | 84·9 | ·197 |
| Uranium oxalate | 11·59 | 57·6 | ·201 |
| Ammonium uranate | 14·95 | 74·4 | ·201 |

This atomic property is of the utmost importance. The activity of a thin layer depends only on the number of uranium atoms present, and is entirely independent of their state of chemical combination.

The ordinary physical influences, change of temperature, light and darkness, are without effect. Radioactivity cannot, therefore, be a peripheral property of the atom, but pertains to the nucleus (Art. 244). Mme Curie measured the activity of uranium electrically over an interval of three years, and found it constant within the limits of experimental error, and certainly within 1 per cent.

G. C. Schmidt found in 1898 that thorium and its compounds had properties similar to those of uranium; Mme Curie was engaged at the same time in a more comprehensive search. Her apparatus would detect an activity of one-hundredth of that of uranium; but no effect was found with substances not containing uranium or thorium. Certain uranium *minerals* were found to be abnormally active, in apparent disagreement with Becquerel's law. Thus while the activity of a layer of uranium was represented by 2·3, that of a similar layer of chalcolite, a crystallised phosphate of copper and uranium, was 5·2. An artificial chalcolite, prepared from the pure components, had an activity of about 1, corresponding to the amount of uranium present; from which Mme Curie concluded that the most active minerals (pitchblende, carnotite and chalcolite) owe their activity to an unknown but potent element present in very small quantities, probably so small as to escape detection by ordinary means. By subjecting the mineral to a regular chemical analysis it should be possible to separate out the unknown substance together with some element to which it is chemically allied, and thus obtain preparations of much greater activity than uranium. The conclusion was justified, since Pierre and Mme Curie were able to extract from pitchblende a highly active preparation of bismuth, which they regarded as containing a new element called *polonium*. Afterwards, in collaboration with Bémont, they obtained a second element called *radium*, chemically allied to barium. This time it was found possible to separate the radium from the barium, on account of a slight difference in solubility of the chlorides, and obtain the pure salts of the new element. Mme Curie and Debierne isolated the pure element in 1910: polonium has not been separated in a weighable quantity.

Radium has an atomic weight of about 226 and a visible spectrum not belonging to any other element. Part of its $L$ radiation has also been measured. Its activity is very great, and its physical

properties correspondingly striking (see following article). About 5 kilograms of pitchblende are required to yield 1 milligram of radium: a little over 1 gram of it was used by Hönigschmid in his determination of the atomic weight. The term "radium" is often used to denote the amount of radium present in a radium salt*; for it is not convenient to use the pure metal, which is difficult to prepare and is affected chemically by the atmosphere. There is no doubt that the radioactivity of a radium salt is due entirely to the radium contained in it.

**243. The three types of radiation.** It was recognised very early, in connexion with uranium, that the radiation could not all be of one quality. Thus while the strength of the photographic image is hardly altered when a sheet of aluminium $\frac{1}{20}$ mm. thick is placed over an uncovered uranium compound, its ionising power is much reduced. We shall now show that radioactive substances probably emit three kinds of rays.

The simplest type of activity is that of polonium. It gives brilliant fluorescence on a screen of zinc sulphide, which is resolved by a magnifying glass into countless scintillations, or bright flashes of short duration: but the luminosity ceases abruptly at a distance of 4 cm. from the active surface. This, as Mme Curie first pointed out, suggests an emission of particles projected with a high velocity, but only able to pass through 4 cm. of air. They will be called $\alpha$ rays for the present. The $\alpha$ rays do not penetrate far into solid substances. Thus when a sheet of aluminium 0·02 mm. thick is placed over polonium the fluorescence ceases, as abruptly as before, at a distance of 1 cm. from the source. A sheet of thickness 0·03 mm. cuts off the rays altogether. With a zinc sulphide screen on glass it is curious to see brilliant fluorescence when the zinc sulphide side is presented to the polonium, and none when the glass side is presented.

The behaviour of radium is more complicated†. There is always a residual fluorescence which is visible at 10 cm. from the source,

* One gram of radium chloride contains 0·761 gram of radium, one gram of radium bromide 0·586 gram.

† Most of the following experiments can be carried out with $\frac{1}{4}$ mg. of radium bromide, contained in a capsule with a thin mica cover, to let through the $\alpha$ rays. The greater the quantity of radium, the easier it is to demonstrate its effects.

even when $\frac{1}{2}$ mm. of aluminium is interposed. Observations are best made on a screen of barium platinocyanide mounted on cardboard, which cuts off the $\alpha$ rays. The residual radiation is profoundly affected by a magnetic field of the order of 1000 gausses. To show this place the radium between the poles of a small electromagnet, and fix the screen horizontally about 1 cm. above it. On exciting the electromagnet part of the fluorescence is spread out into a bright band at right angles to the magnetic field, as shown in Fig. 354, and its direction changes with that of the magnetic field. These magnetically deviable rays are called $\beta$ rays. The figure shows that they behave like negatively charged particles emitted by the radium, with velocities extending over a large range. They are shown to be electrons in Art. 253.

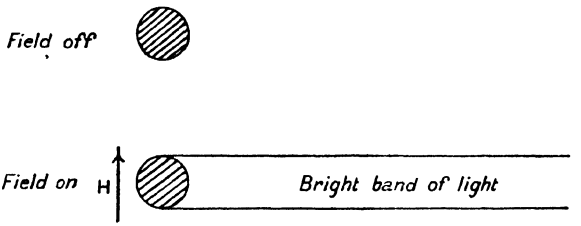


Fig. 354

Most of the $\beta$ rays will pass through $\frac{1}{2}$ mm. of aluminium. With increasing thickness they are cut off more and more, beginning with the most deviable rays, until only a faint undeflected spot remains. The rays coming through 5 mm. of aluminium or 2 mm. of lead are no longer appreciably affected by a magnetic field. It is probable, therefore, that there is a third type of radiation, very penetrating and not deflected by a magnetic field. These rays are called $\gamma$ rays. They produce visible fluorescence through 4 mm. of lead; their electrical effect can be detected, with care, through 8 or 10 cm.

The $\alpha$ rays produce intense ionisation near the active body. This is illustrated by Rutherford's experiments, showing the effect of placing thin sheets of aluminium over a layer of uranium oxide (Fig. 355). The ionisation falls rapidly until a thickness of 0·02 mm. is reached, and afterwards very much more slowly. The complexity of the radiation was first shown in this way, before the discovery of the magnetic deflexion of $\beta$ rays. The difference between $\beta$ and $\gamma$ rays can be shown similarly with sheets of lead, about 2 millimetres being required to cut off the former.

We cannot say exactly how much ionisation is due to each kind of ray, since that depends on the size of the ionisation chamber and the thickness of the layer of active matter. With a thin layer of radium, and plates 5 cm. or so apart, the ionisation due to the $\alpha$ rays may be 100 times that due to the $\beta$ rays, and that of the $\gamma$ rays 100 times as small again. A thick layer tends to absorb its own $\alpha$ rays and give greater prominence to the $\beta$ and $\gamma$ rays. The $\gamma$ rays produce comparatively little ionisation per unit volume; but they will travel almost any distance, and are measured with advantage in a large ionisation chamber. The photographic effect of a uranium compound seems to be largely due to its $\beta$ rays.

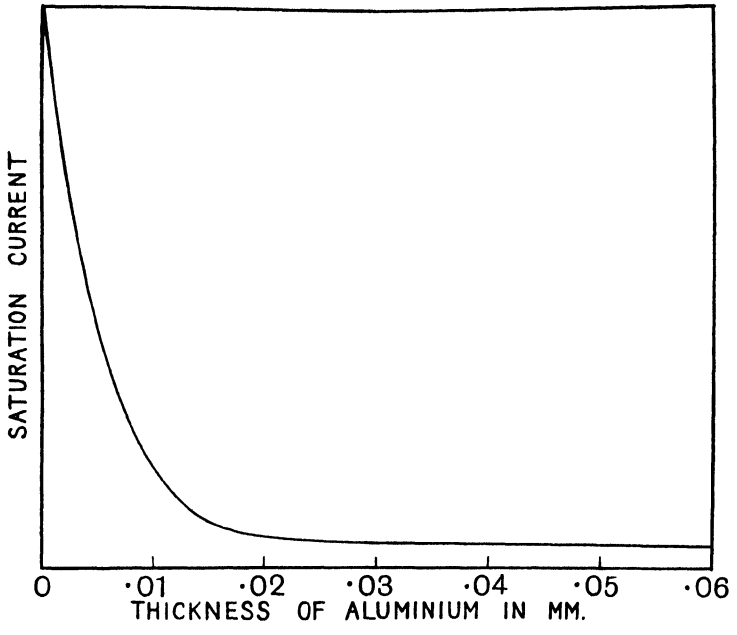



Fig. 355

It must be admitted that the evidence of complexity furnished by absorption measurements is not as clear as it might be. In particular the $\gamma$ rays might well be very fast $\beta$ rays, which would be but little affected by a magnet and would also have high penetrating power. Nevertheless, the correctness of the classification has been proved by other experiments (Arts. 248, 253, 255), which have settled definitely the nature of the three kinds of rays.

In the following pages we shall give a short account of the radioactivity of radium. For the properties of uranium and thorium, and of actinium, a radioactive element separated from pitchblende by Debierne, reference may be made to Mme Curie, *Traité de Radioactivité*, or Rutherford, *Radioactive Substances and their Radiations*.

**244. Radioactive transformations.** It is generally agreed that radioactivity consists in a disintegration of the atomic nuclei of the radioactive substance itself. These nuclei are not absolutely stable, but some of them are always breaking up, with the emission of $\alpha$ or $\beta$ rays. (The $\gamma$ rays, being as we shall see waves in the ether, have not here the same importance.) The result is an internal redistribution of the nucleus, which alters its nuclear charge, giving rise to a new element of different physical and chemical properties. The new element thus formed may itself be unstable, and break up with the formation of a third element, and so on.

The theory was stated qualitatively by Rutherford and Soddy in 1902. They assumed that if there are $P$ atoms of a radioactive substance present at any time, the number whose nuclei break up per second is $\lambda P$, where $\lambda$ is a constant called the *decay constant* of the substance. Thus for a single substance decaying by itself (i.e. being transformed into an inactive product), $dP = -\lambda P dt$, which gives on integration

$$P = P_0 e^{-\lambda t} \quad \ldots\ldots\ldots\ldots\ldots\ldots (302),$$

$P_0$ being the number of atoms present at time $t = 0$. This law is satisfied accurately by polonium, for which $\lambda = 5{\cdot}9 \times 10^{-8}$. The *half-value period*, or time $T$ required for the activity to fall to half its original value, is connected with $\lambda$ by the equation

$$T = \frac{\log_e 2}{\lambda} = \frac{0{\cdot}693}{\lambda}.$$

For polonium $T = 136$ days.

On the above theory the amount of a radioactive substance, when left to itself, is constantly diminishing, but may appear constant if the loss is inappreciable in the interval of time over which observations extend. This is the case with uranium, thorium and radium. Polonium, whose activity largely disappears in the course of a year, is typical, and long-lived elements are exceptional.

We next consider the theory of more than one transformation. The simplest case is that of a radioactive substance $A$ generated at a constant rate, say $\lambda_0 P_0$, by a substance of very slow decay. Let $P$ be the number of atoms of $A$ present at time $t$, $\lambda_1$ its decay constant. In time $dt$, $\lambda_0 P_0 dt$ atoms are produced by the primary substance, and $\lambda_1 P dt$ disappear by decay. Hence

$$dP/dt = \lambda_0 P_0 - \lambda_1 P \quad \ldots\ldots\ldots\ldots (303).$$

Integrating, we have $P = Ce^{-\lambda_1 t} + \lambda_0 P_0/\lambda_1$, where $C$ is a constant. If no atoms of $A$ are present at time $t = 0$, $C = -\lambda_0 P_0/\lambda_1$, and

$$P = \frac{\lambda_0 P_0}{\lambda_1}(1 - e^{-\lambda_1 t}) \ldots\ldots\ldots\ldots\ldots (304).$$

After a sufficient length of time the term $e^{-\lambda_1 t}$ becomes negligible, and $P = \lambda_0 P_0/\lambda_1$. Thus a state is ultimately reached in which the loss of the substance $A$ by decay is counterbalanced by the gain from the source. This is known as *radioactive equilibrium.* The quicker the decay of the substance $A$, the less the amount of it present in the equilibrium state, and the less also the time required to attain it, as equation (304) shows. The theory can be extended to any number of successive products. If $P$, $Q$, $R$, ... are the numbers of atoms of the various products present at time $t$, and $\lambda_1, \lambda_2, \lambda_3, \ldots$ their decay constants,

$$dP/dt = \lambda_0 P_0 - \lambda_1 P, \quad dQ/dt = \lambda_1 P - \lambda_2 Q, \quad dR/dt = \lambda_2 Q - \lambda_3 R, \ldots.$$

The differential coefficients vanish in the equilibrium state, and then

$$\lambda_1 P = \lambda_2 Q = \lambda_3 R = \ldots = \lambda_0 P_0 \quad \ldots\ldots (305).$$

The numbers of atoms of the different substances are therefore proportional to their half-value periods (cf. Art. 256).

The following theory will be useful in discussing the "active deposit of radium." Let there be three substances $A$, $B$, $C$ of decay constants $\lambda_1$, $\lambda_2$, $\lambda_3$, $A$ producing $B$ and $B$ producing $C$; and let all the matter be initially of one kind $A$. It is required to find the numbers of atoms of $A$, $B$, $C$ present at all subsequent times. We have

$$dP/dt = -\lambda_1 P, \quad dQ/dt = \lambda_1 P - \lambda_2 Q, \quad dR/dt = \lambda_2 Q - \lambda_3 R.$$

These equations can be integrated in succession, the first giving $P$, the second then giving $Q$, and the third $R$. Adjusting the con-

stants of integration so that $P = P_0$, $Q = R = 0$ at time $t = 0$, we find

$$\left.\begin{aligned} P &= P_0 e^{-\lambda_1 t}, \quad Q = -\frac{\lambda_1 P_0}{\lambda_1 - \lambda_2}(e^{-\lambda_1 t} - e^{-\lambda_2 t}), \\ R &= \frac{\lambda_1 \lambda_2 P_0}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \\ &\quad \times \{(\lambda_2 - \lambda_3) e^{-\lambda_1 t} - (\lambda_1 - \lambda_3) e^{-\lambda_2 t} + (\lambda_1 - \lambda_2) e^{-\lambda_3 t}\} \end{aligned}\right\} \ldots(306).$$

The numbers of atoms of $A$, $B$, $C$ breaking up per second are $\lambda_1 P$, $\lambda_2 Q$, $\lambda_3 R$ respectively; hence if each substance in decaying emitted rays of the same quality the activity would be represented by $\lambda_1 P + \lambda_2 Q + \lambda_3 R$. This, however, is not so, nor need all the products emit rays of the kind considered. The activity at any time is represented by $k_1 P + k_2 Q + k_3 R$, where $k_1$, $k_2$, $k_3$ are constants depending on the size and shape of the apparatus and the kind of rays ($\alpha$, $\beta$ or $\gamma$) that are being observed. The activity varies as $e^{-\lambda t}$ when $t$ is large, $\lambda$ being the least of the quantities $\lambda_1$, $\lambda_2$, $\lambda_3$; hence the law of decay of the activity is ultimately that of the slowest of the separate constituents.

**245. Generation of niton by radium.** Although radium itself is a substance of slow decay, it can be seen from its behaviour that it has short-lived products. For example, pure radium on preparation emits only $\alpha$ rays, together with weak $\beta$ rays. At the end of a month the $\alpha$ ray activity has quadrupled, the $\beta$ ray activity increased fifty-fold, and there is a strong $\gamma$ ray activity. Henceforth the activity remains nearly constant, increasing slightly for years before its final decrease. The problem of finding how many products there are, and their relation to one another, will occupy us here and in Arts. 246, 247.

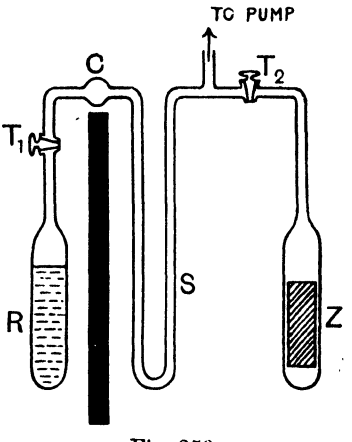


Fig. 356

The following experiment shows that radium is constantly giving off a radioactive gas, called the radium emanation, or *niton*. A vessel $R$ (Fig. 356), closed by a tap $T_1$, contains a solution of a radium salt. Another vessel, communicating with it through a

U-tube $S$, contains a zinc sulphide screen $Z$, which fluoresces whenever $\alpha$ rays fall on it. The tap $T_1$ is first closed and the remainder of the apparatus exhausted with a pump, and the radium solution is then boiled. On opening the tap $T_1$ a bright fluorescence immediately appears on the screen $Z$. Since this takes place equally well when the vessel $R$ is surrounded by thick lead screens, it cannot be due to any direct action of the rays emitted by the radium. Nor is it due to the passage of ionised gases from the radium chamber to that containing the fluorescent screen: for it is not stopped by filling the space $C$ with cotton-wool, which would remove all the ions from the stream. But the luminosity is stopped by immersing the tube $S$ in liquid air, either before or after opening the tap $T_1$. These facts point to the emission of a radioactive gas, which is liquefied at the temperature of liquid air. A great increase in the luminosity occurs when the right-hand tube itself is immersed in liquid air, as the liquid niton then collects at the bottom of the tube.

The gas given off from radium solutions does not consist of pure niton, but chiefly of hydrogen and oxygen, arising from the action of radium on water. Helium and carbon dioxide are also present. After removal of these impurities a small amount of pure niton is left, which may be condensed by pumping it above mercury into a fine capillary tube. In this way enough has been collected to determine its physical and chemical properties. The volume obtained is never more than a fraction of a cubic millimetre at atmospheric pressure; but most of its properties can be examined without isolating a weighable quantity of it, since its presence is everywhere indicated by its intense radioactive effect. Niton is a true, if transitory, element like radium, with its own spectrum. It belongs to the group of inert gases, and is the heaviest gas known. Its atomic weight has been found by density measurements to be about 223, on the assumption that the molecule is monatomic.

Pure niton, contained in a sealed tube, emits at first only $\alpha$ rays, the $\beta$ and $\gamma$ ray activity appearing in the course of a few hours. There is thus evidence of the existence of further transformation products, which, as we shall see, are solid and deposited on the walls of the tube. The rate of decay of niton itself is found by

extracting a known fraction of the gas at intervals, and measuring its activity either immediately or at a definite time after removal. The activity is found to diminish in time according to an exponential law, with a half-value period of 3·85 days; a number which is probably correct to about 1 part in 1000. The corresponding decay constant is $2{\cdot}085 \times 10^{-6}$.

The volume of niton in equilibrium with one gram of radium is about 0·60 mm.$^3$ at 0° C. and 760 mm. pressure, from which we can find the half-value period of radium itself. For the above volume contains $1{\cdot}63 \times 10^{16}$ atoms of niton, which are in equilibrium with $2{\cdot}71 \times 10^{21}$ atoms of radium. The equation $\lambda_0 P_0 = \lambda_1 P_1$, with $\lambda_1 = 2{\cdot}085 \times 10^{-6}$, gives $\lambda_0 = 1{\cdot}25 \times 10^{-11}$. The half-value period of radium is therefore about 1750 years.

It is important to decide whether niton is produced directly by radium, or whether there is another transformation product in between. The method of solving such problems is as follows. It is known that the equilibrium amount of niton can be collected from radium regularly at intervals of one month, which, as equation (304) shows, is to be expected if radium changes directly into niton: the rate of growth would be much slower if a product of long life intervened. The same method of reasoning has been applied in other cases to prove that one substance could not be the direct parent of another.

In 1903 Ramsay and Soddy made the important discovery that helium is continually being produced by niton. When purified niton is pumped into a small vacuum tube and allowed to decay *in situ*, its spectrum grows gradually fainter, while the helium spectrum appears and becomes gradually more and more distinct. This fact has an important bearing on the nature of the $\alpha$ rays (see Art. 248).

**246. Radium A, radium B, radium C.** It was discovered by P. and Mme Curie in 1899 that substances left for some time in the presence of uncovered radium salts acquire a temporary activity, which largely disappears in the course of a few hours. The same is true of the walls of a vessel that has contained niton. This was at first called "induced radioactivity"; but since the activity is, in similar circumstances, the same for all non-porous

substances, it can hardly be due to any effect on the substance itself. Further, the activity is confined to a very thin layer on the surface, which can be removed by friction. The most natural explanation is that there is a radioactive substance deposited on the surface of the body. The *active deposit*, as it is called, is only produced in places accessible to niton: thus its formation can be prevented by blowing away the gas with a current of air as it rises, or by enclosing the radium in an air-tight case, even if the walls are thin enough to let through all the three kinds of rays. On the other hand, active deposits can be formed at great distances from the source by blowing a stream of niton down a long tube. The view adopted by Rutherford, to whom the explanation of these effects is chiefly due, is that the products of decay of niton are solid at ordinary temperatures, and are therefore deposited on the walls of the tube.

It is found that while the activity of exposed wires is very much reduced in the course of 12 hours, it does not actually vanish, but there is always a residue which persists for many years. It is thus convenient to distinguish between the *active deposit of quick decay* and the *active deposit of slow decay*. The activity of the latter, which never constitutes more than one-thousandth part of the whole, is considered separately in the next article, and what follows refers to the active deposit of quick decay.

If a wire is exposed to niton for a few seconds and then removed, its activity falls off as shown in Fig. 357, curve I representing the $\alpha$ ray activity and curve II that due to the $\beta$ and $\gamma$ rays together. It is clear from this that the active deposit of quick decay is itself complex. A full account of the experiments necessary to disentangle the various products, and to assign to each its proper radiation, would occupy too much space: but we can form some idea of these as follows.

The deposit, just after a short exposure, consists almost entirely of the first transformation product of niton, which we may call radium A. The $\alpha$ ray activity falls at first according to an exponential law, with a half-value period of three minutes. It is therefore probable that radium A emits $\alpha$ rays and has a period of about three minutes; and curve II shows that it has at most a very weak $\beta$ ray activity. For large values of $t$ all the activity curves tend

to the exponential form, with a half-value period of 26·7 minutes; which gives the period of the product of slowest decay (not necessarily the last product of the series). It is obviously useless to attempt to separate any of the intermediate products by chemical means, for even if once prepared they will not remain pure, but change constantly into other substances. If any pure product is obtained it must be the last product of the series. By dipping a copper or nickel plate into a solution of the active deposit v. Lerch obtained a deposit which decayed according to an exponential law throughout, with a half-value period of 19·5 minutes.

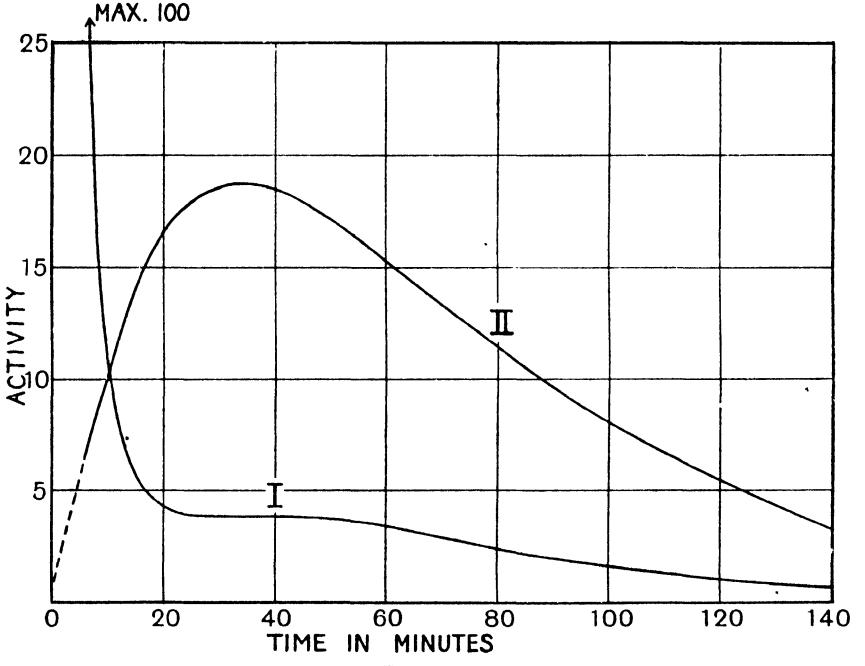



Fig. 357

These facts are most simply explained by supposing that there are three successive products in the active deposit of quick decay —radium A with half-value period 3 minutes, radium B with period 26·7 minutes, and radium C with period 19·5 minutes. The corresponding decay constants are ·00385, ·000433 and ·000593 respectively. The evidence from the activity curves can be used to test this assumption and to find what rays are given off by the different products in decaying. For example, curve I of Fig. 357

can be represented fairly accurately by the equation

$$y = \frac{100P + 12{\cdot}3R}{P_0},$$

where $P$, $R$ are taken from equation (306), and $\lambda_1$, $\lambda_2$, $\lambda_3$ have the values given above. Since good agreement can be obtained by putting $k_2 = 0$ in the formula $k_1P + k_2Q + k_3R$, it is probable that radium B emits no $\alpha$ rays. Indirect evidence of this kind must usually be accepted with caution; its cogency depends very much on the number of constants available for adjustment. There are additional checks in the present case, since the values of $k_1$, $k_3$, when once chosen, have to serve for all times of exposure to the emanation, from a few seconds to several hours. Information as to the rays emitted by the different products is afforded by experiments with various thicknesses of lead or aluminium over the active layer. In this way it has been shown, beyond reasonable doubt, that radium A emits $\alpha$ rays only, radium B emits $\beta$ rays and weak $\gamma$ rays, and radium C emits $\alpha$, $\beta$ and $\gamma$ rays. The transformations occurring between radium and radium C are shown in the following scheme:

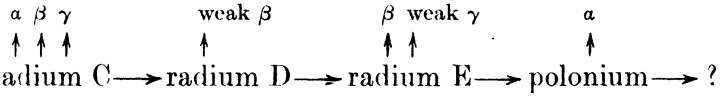
$$\begin{array}{ccccccccc} & \alpha\ \text{weak}\ \beta & & \alpha & & & \beta\ \text{weak}\ \gamma & & \alpha\ \beta\ \gamma \\ & \uparrow\ \uparrow & & \uparrow & & & \uparrow\ \uparrow & & \uparrow\ \uparrow\ \uparrow \\ \text{Radium} & \longrightarrow & \text{niton} & \longrightarrow & \text{radium A} & \longrightarrow & \text{radium B} & \longrightarrow & \text{radium C.} \end{array}$$

We can now give a more precise account than before of the origin of the different rays from radium, say one month after preparation. By this time radium is in equilibrium with niton and with the products A, B, C. The $\alpha$ rays are given off by radium, niton, radium A and radium C. The $\beta$ rays arise mainly from radium B and radium C, and the $\gamma$ rays from radium C. It is simplest to suppose that a disintegrating nucleus emits either $\alpha$ rays or $\beta$ rays, but not both, so that radium C consists most probably of two successive products, one of which is of very short life. There is also evidence to show that the atoms of radium C can break up in two distinct ways. Consideration of these would lead us too far, but we give the results in Art. 256.

Radium C, in decaying, gives rise to the active deposit of slow decay, which we now consider.

**247. Radium D, radium E and polonium.** The behaviour of the active deposit of slow decay somewhat resembles that of radium immediately after preparation. At first there is only a weak $\beta$ ray activity, which rises gradually for about two months, and afterwards remains nearly constant. An $\alpha$ ray activity is also developed, but more slowly, requiring two years to reach its maximum value. It is probable, therefore, that the first product of radium C is a substance of slow decay, which we may call radium D, and which itself gives rise to further products.

The separation of these products is comparatively simple. If a polished plate of bismuth or copper is immersed in a solution of the active deposit of slow decay, a deposit is obtained which emits only $\alpha$ rays, and decays exponentially with a period of about 140 days. This exact coincidence with the properties of polonium cannot be accidental; hence we conclude that polonium is a transformation product of radium D, and it is the *last* active product. If a nickel plate is used instead of bismuth or copper, the deposit at first emits no $\alpha$ rays, but only $\beta$ rays and weak $\gamma$ rays, and both activities decay exponentially with a period of 5 days. These facts are exactly accounted for by the following scheme of transformation:




As before, we can test for intermediate products by means of the curves giving the rise of $\alpha$ and $\beta$ ray activity of radium D with time. No evidence of any such product has been obtained, and there is certainly none of long life.

Equilibrium between radium D, radium E and polonium is practically attained in two years, after which the product shows $\alpha$, $\beta$ and $\gamma$ ray activity, all of which fall off exponentially with the period of radium D. Mme Curie, by direct observations over a period of five years, estimated the period of radium D at 17 years. The chemical properties of radium D closely resemble those of lead, from which it cannot be separated by any known means. It was discovered at an early period that the lead extracted from pitchblende had a very considerable activity, which was ascribed to the

presence of a radioactive constituent known as radio-lead. This is now known to be identical with radium D.

The main transformation series of radium, as far as it is necessary to consider it here, is shown in the following table:

| | Radiation | Period |
|---|---|---|
| Radium .. .. | $\alpha$ + weak $\beta$ | 1750 years |
| Niton .. .. | $\alpha$ | 3·85 days |
| Radium A .. | $\alpha$ | 3 minutes |
| Radium B .. | $\beta$ + weak $\gamma$ | 26·7 minutes |
| Radium C .. | $\alpha + \beta + \gamma$ | 19·5 minutes |
| Radium D .. | weak $\beta$ | 17 years |
| Radium E .. | $\beta$ + weak $\gamma$ | 5 days |
| Polonium .. | $\alpha$ | 136 days |
| An isotope of lead (see Art. 256) | — | — |

Over 100 years must elapse before radium is in equilibrium with *all* its products, the delay arising from the long period of radium D. Neither radium D nor its products emit any very penetrating rays, the $\gamma$ ray activity when measured through 3 mm. of lead being due entirely to radium B and radium C, and most of it to the latter. Hence the $\gamma$ ray activity is proportional to the amount of radium present, provided that at least one month has elapsed after preparation. This is the basis of an accurate method of comparing quantities of radium, which has the further advantage that, since only very penetrating rays are used, the absorption by the walls of the containing vessel and the radium preparation itself is small and need not generally be allowed for.

In handling radium great care must be taken to avoid radioactive contamination. Any considerable or prolonged escape of niton in a laboratory will deposit enough solid products on the walls to disturb electrometer work. No chemical operations on radium should be performed in a physical laboratory.

**248. The $\alpha$ particles.** The $\alpha$ rays were at first thought to be unaffected by magnetic fields; and certainly they are but little deflected in fields strong enough to produce a very great deflexion of the $\beta$ particles. Rutherford was the first to show that the $\alpha$ rays behave like positively charged particles projected with a great velocity. The exact measurement of $e/m$ and $v$ is made more difficult

by the fact that radium sends out several sets of rays with different initial velocity, and if the active layer is at all thick there is an additional loss of velocity by transmission through the layer. In one of Rutherford's experiments (Fig. 358) the source of the $\alpha$ rays was a deposit of radium C on a thin wire, laid in a groove at $A$ in the figure. Above it was a metal plate $B$ pierced with a slit parallel to the wire, and higher still a photographic plate $P$. A strong and uniform magnetic field was applied parallel to the wire, and the whole space inside the case was highly exhausted. If $a$ is the distance from $A$ to $B$, $b$ the distance from $B$ to $P$, and $\rho$ the radius of the circle into which the rays are bent, the deflexion $\delta$ (supposed small) is given by $2\rho\delta = b\,(a + b)$. But from Art. 207, $\rho = mcv/He$. Hence

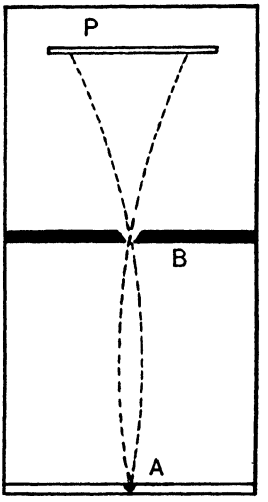


Fig. 358

$$\frac{e}{mv} = \frac{2c\delta}{Hb\,(a+b)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (307).$$

The double deflexion caused by reversing the magnetic field was measured on the plate. It was found that the width of the deflected band was the same as that of the undeflected band: hence all the rays from a homogeneous product like radium C are sent off with the same velocity. For these rays $\rho$ is about 40 cm. in a field of 10,000 gausses.

Strong fields are required for the electric deflexion. Rutherford used two plates $C_1$, $C_2$ (Fig. 359), $\frac{1}{4}$ mm. apart, placed immediately over the active matter. The ray $ABCD$ which experienced the greatest deflexion usually left the active matter at $A$, grazed the left-hand plate at $B$, and passed over the edge $C$ of the right-hand plate, falling on the photographic plate $P$ at $D$. On reversing the field a similar extreme ray $A'B'C'D'$ is obtained. The path of either ray is parabolic between the plates, and straight outside. If $d$ is the distance between the plates $C_1$, $C_2$, $g$ the distance from $C$ to the photographic plate, and $2\epsilon$ the double deflexion $DD'$, supposed small, it is not difficult to show that

$$\frac{e}{mv^2} = \frac{(\epsilon - \frac{1}{2}d)^2}{2g^2Ed} \quad \ldots\ldots\ldots\ldots\ldots\ldots (308).$$

Equations (307) and (308) can be solved for $e/m$ and $v$ separately. Rutherford found that the values of $e/m$ for the $\alpha$ particles from niton, radium A and radium C agree to within 1 part in 400, and are equal to $1{\cdot}446 \times 10^{14}$; which shows that the $\alpha$ particle is the same from whatever source it is derived. The velocity of projection of the particles from an uncovered preparation of radium C is $1{\cdot}92 \times 10^{9}$ cm./sec.

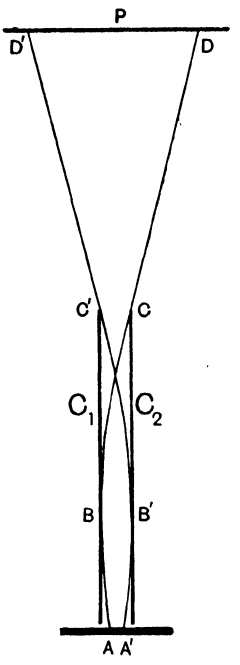


Fig. 359

The charge on an $\alpha$ particle, if at all comparable with that of an electron, is too small to be measured directly; but it can be found indirectly as follows. It is practically certain that scintillations (Art. 243) are produced by the separate impacts of $\alpha$ particles on the fluorescent screen. By counting scintillations we can find the total number of $\alpha$ particles emitted by a radioactive layer per second; after which it is only necessary to measure the charge carried away by the $\alpha$ particles in a given time. Scintillations were first counted systematically by Regener. Rutherford and Geiger magnified the ionising effect of the $\alpha$ particles by collision (Art. 225) until the entry of a single particle into the ionisation chamber deflected an electrometer. The number of particles, as so counted, was nearly the same as that found from observations on a zinc sulphide screen.

It is not easy to measure the total charge. Sir J. J. Thomson found that a metallic plate covered with polonium, and placed in a high vacuum, emits a negative and not a positive charge. The negative leak is reversed when the whole apparatus is placed between the poles of a powerful electromagnet, so that it is probably due to emission of slow-moving electrons from the plate. It is now believed that slow electrons are emitted whenever $\alpha$ rays fall on matter. But even in high vacua, and strong magnetic fields, the current in an apparatus like that of Fig. 352 differs according as the active plate is positive or negative. Rutherford assumed that the difference arises from ionisation of the residual gas between the plates, and that the charge carried by the $\alpha$ particles per second is equal to the mean of the two currents.

In this way Rutherford and Geiger obtained values of $e$ ranging from $8{\cdot}3 \times 10^{-10}$ to $10 \times 10^{-10}$. Shortly afterwards Regener completed his counting experiments by measuring the total charge, and found that $e = 9{\cdot}58 \times 10^{-10}$ E.S.U. Comparing this number with Millikan's value $4{\cdot}77 \times 10^{-10}$, we see that the $\alpha$ particle carries a charge numerically double that of the electron. Conversely, Regener's experiments give $4{\cdot}79 \times 10^{-10}$ for the charge of the electron. The value $4{\cdot}77 \times 10^{-10}$ has been adopted in this book, giving $9{\cdot}54 \times 10^{-10}$ for the charge on the $\alpha$ particle.

The mass of an $\alpha$ particle can now be found. Putting $e = 9{\cdot}54 \times 10^{-10}$ and $e/m = 1{\cdot}446 \times 10^{14}$, we find $m = 6{\cdot}60 \times 10^{-24}$ gm. Since the mass of a helium atom is also $6{\cdot}60 \times 10^{-24}$ gm., it is simplest to assume an $\alpha$ particle is an atom of helium that has lost two electrons, that is *a helium nucleus*. It is unlikely on general grounds that a helium atom can retain any of its circumnuclear electrons when travelling at the highest speeds. As the nucleus slows down it picks up first one and then another electron and becomes a normal helium atom (see below).

These conclusions receive independent support as follows. One gram of radium in equilibrium with niton and with radium A, B, C emits $1{\cdot}43 \times 10^{11}$ $\alpha$ particles per second; and since there are *four* $\alpha$ ray products the number emitted by radium itself is $3{\cdot}57 \times 10^{10}$. If, as is probable, each corresponds to the disintegration of one radium atom, the last number is equal to $\lambda P$, where $\lambda$ is the decay constant of radium and $P = 2{\cdot}71 \times 10^{21}$ the number of atoms in one gram. Thus $\lambda = 1{\cdot}32 \times 10^{-11}$, which agrees well with the value $1{\cdot}25 \times 10^{-11}$ found previously. If each neutralised $\alpha$ particle is a helium atom, 0·175 cm.$^3$ of helium should be given off in one year by one gram of radium in equilibrium. The observed amount is about 0·170 cm.$^3$

The scintillations from an $\alpha$ particle cease abruptly after it has traversed a certain distance through matter, called its *range*. Rutherford demonstrated the loss of velocity in transit by magnetic deflexion. The following table shows the velocity of the $\alpha$ particles from radium C after traversing various distances, measured (as is most convenient) by the mass of a cylinder whose height is the distance traversed, and whose base has an area of 1 cm.$^2$:

| Velocity in cm./sec. | Mass of obstacle per cm.² of area, in milligrams | | | | |
|---|---|---|---|---|---|
| | Gold | Copper | Aluminium | Mica | Air |
| $1{\cdot}92 \times 10^9$ | 0 | 0 | 0 | 0 | 0 |
| 1·73 | 7·05 | 3·90 | 2·79 | 2·75 | 2·32 |
| 1·54 | 12·27 | 5·69 | 5·01 | 4·86 | 4·08 |
| 1·34 | 17·04 | 9·20 | 7·03 | 6·40 | 5·46 |
| 1·15 | 20·71 | 11·40 | 8·50 | 7·50 | 6·48 |
| 0·96 | 23·89 | 13·13 | 9·64 | 8·47 | 7·29 |
| 0·80 | 26·65 | 14·60 | 10·46 | 9·35 | 7·96 |
| End of range | 29·50 | 16·00 | 11·40 | 10·15 | 8·50 |

The law of diminution of velocity is not quite the same in all substances, but we should not go far wrong in assuming that it was, nor in assuming proportional diminution for other $\alpha$ particles. Hence, since the ranges of the $\alpha$ particles from radium C are known accurately in many substances, we can estimate the loss of velocity of any $\alpha$ particle in passing through any thin foil, a correction necessary in accurate experiments (Art. 250). A rough empirical rule, which has proved useful (Art. 252), is that the range of an $\alpha$ particle is proportional to the cube of its velocity.

The fastest $\alpha$ particles move unchanged through matter, but slower particles are constantly getting charged and discharged, like positive rays (Art. 228). Henderson showed this by placing different thicknesses of mica over a source of radium C in the apparatus of Fig. 358. While only one beam is observed for thicknesses less than 0·005 mm. (the range being 0·035 mm.), a layer 0·007 mm. thick gives a second beam deflected half as much as the first, and therefore having a value of $e/mv$ half as great as before. Electrostatic deflexion shows that $e/mv^2$ is also half as great as before, so that $e/m$ is half as great as before and $v$ the same. This is what we should expect if some of the $\alpha$ particles had picked up an electron. Neutral particles begin to appear at 0·03 mm. of mica (or 6 cm. of air), and the principal interchange at low velocities is between these and singly charged particles. In other words, a helium nucleus can keep one electron while moving through matter at a moderate speed, and two electrons when moving still more slowly (cf. Art. 224).

**249. Ionisation produced by $\alpha$ particles.** The $\alpha$ particles produce intense ionisation close up to the active substance, which

ceases near the point at which scintillations cease. This phenomenon, first observed by Mme Curie for polonium, shows that $\alpha$ particles move straight on through a gas, losing energy as they go, until finally their velocity is insufficient to enable them to ionise by collision. The deviation from the straight line is slight, except near the end of the path (Fig. 365): hence if a nearly parallel beam of $\alpha$ rays is obtained, the ionisation per unit length will tell us approximately how the ionising power of a single $\alpha$ particle varies along its path.

The preceding considerations are due to Sir W. H. Bragg, who first measured the ionisation at various points of a beam of $\alpha$ rays. The source of rays is a thin layer of polonium deposited on a disc of copper, attached to a rod $R$ passing through a hole in a fixed brass plate (Fig. 360). A cap $C$, consisting of a thick piece of brass perforated with holes about 1 mm. in diameter, is placed over the disc, so that only vertical rays escape. The ionisation is measured in the space between a wire grating $G$ and a brass plate $P$, and the distance $x$ from the polonium to the centre of the gap can be varied by sliding the rod $R$ up and down. The effect of electrons emitted from the plate $P$ is relatively small at atmospheric pressure, as we can show by reversing the field. Hence with a narrow gap the saturation current is nearly proportional to the ionisation per millimetre of the path of an $\alpha$ particle at distance $x$ from the source. The results of this experiment are shown in Fig. 361. With an uncovered layer (curve I) the ionisation per unit length increases at first, attains a maximum at 2·5 cm. from the source, and then sinks rapidly to zero. It would probably cease at about 3·9 cm. from the source, if diffusion could be prevented. The distance so defined is nearly that at which scintillations cease (Art. 248), and it hardly matters which we call the range. Bragg explained the maximum in the ionisation curve by observing that the fastest particles might

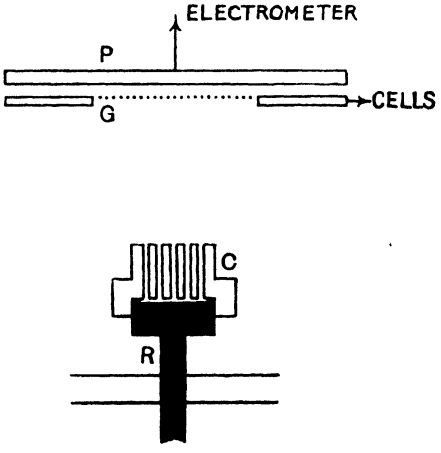


Fig. 360

pass over an atom too quickly to remove an electron from it, while the slowest would not disturb it enough. The behaviour of $\alpha$ particles is thus similar to that of electrons (Art. 216).

Curve II is obtained with a sheet of aluminium 0·006 mm. thick placed over the polonium. The effect is to move curve I back as a whole 1 cm. nearer to the active layer; which confirms the view that the $\alpha$ particles, though retarded, are but little deflected. The ionisation is *increased* at certain distances, where we get into a more favourable range of velocity.

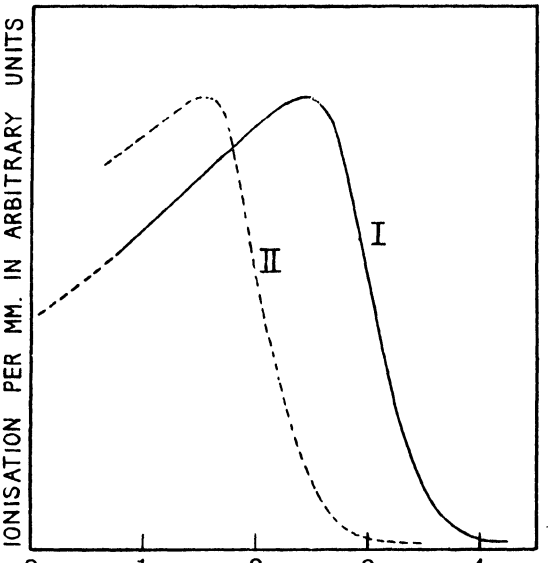


Fig. 361

Experiments with radium give a more complicated curve (Fig. 362) which can be analysed into four simple curves, as shown. It is known that radium in equilibrium with its products of quick decay contains four substances which emit $\alpha$ rays—radium, niton, radium A and radium C. The numbers of the different atoms breaking up per second are equal (Art. 244), and presumably also the numbers of $\alpha$ particles emitted by the four products. The curve in Fig. 362 can be explained exactly on this hypothesis, as Bragg showed. The rays are assigned to their respective products as follows. A freshly prepared layer gives a single curve corresponding to portion 1 of the figure: a deposit of radium C, prepared by v. Lerch's method, gives curve 4 alone. As regards the remaining two, it is known that the $\alpha$ rays from niton are more easily absorbed than those from radium A: hence curve 1 is to be ascribed to radium, curve 2 to niton, curve 3 to radium A and curve 4 to radium C. An old preparation of radium would show a disturbance between 1 and 2 due to the $\alpha$ rays from polonium; but about 100 years would be required for its full development.

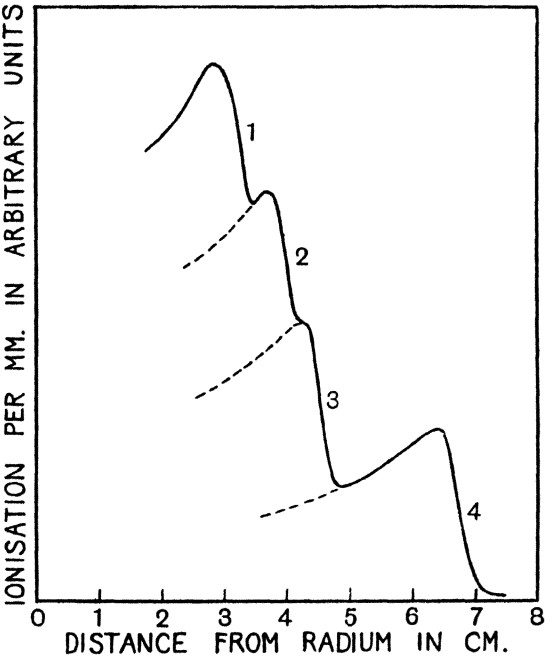



Fig. 362

The ranges of the different $\alpha$ particles are shown on p. 576. Radium C also gives out a few particles of range 9·3 cm., and possibly others of 11·2 cm.

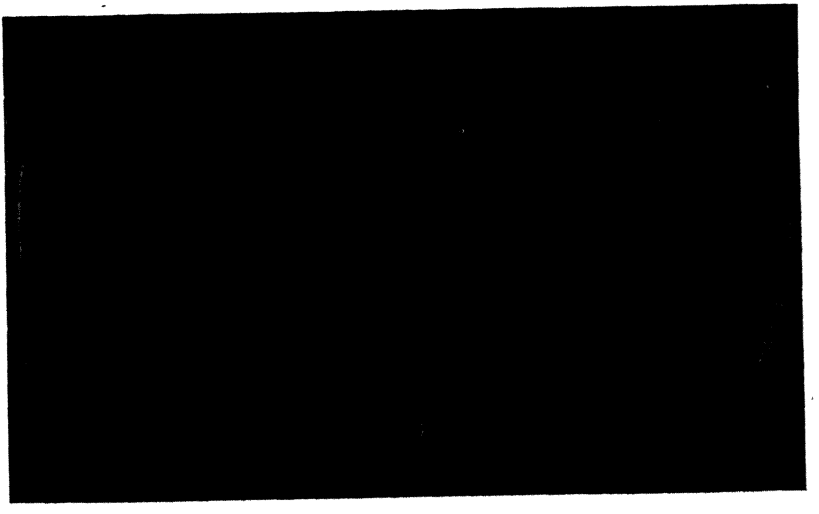


Fig. 363

The paths of $\alpha$ particles were first made visible by C. T. R. Wilson by the method of Art. 216. Fig. 363 shows a photograph obtained with a small quantity of radium on the end of a wire. The tracks are straight in the main and quite continuous, the cloud being too dense to resolve into drops. The rays are often deflected through finite angles near the end, as in the enlarged Fig. 364.

*Ranges in cm. in air at* 15° C. *and* 760 *mm. pressure.*

| | |
|---|---|
| Radium | 3·39 |
| Niton | 4·12 |
| Radium A | 4·72 |
| Radium C | 6·97 |
| Polonium | 3·92 |

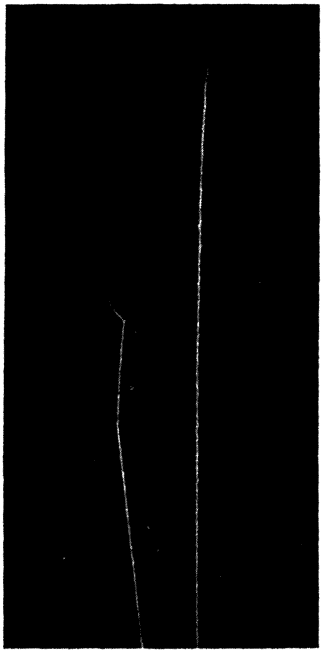


Fig. 364

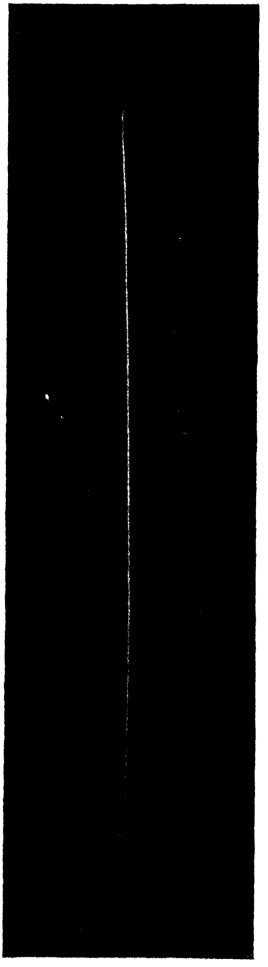


Fig. 365

Wilson's experiments, like Bragg's, confirm in a striking way the theory of successive transformations. At first all the $\alpha$ rays radiate from the radium at the end of the wire; but as niton diffuses

into the expansion chamber more and more tracks originate *in the gas itself*, and others, due to the active deposit, start from various points on the walls of the chamber. Fig. 365 shows a complete track from niton. We notice the characteristic hook at the end, the spot at the beginning due to the recoil of the atom of the emanation, and the gradual increase of the density of ionisation as the ray slows down. One of these tracks measured by Wilson had a length of 4·3 cm., reduced to 15° C. and 760 mm. pressure; which agrees well with the range given on p. 576.

The average number of ions produced by an $\alpha$ particle from radium C is about 3000 per millimetre of its path, or about 200,000 in all. It is obvious, from the concentration of the ions in channels, that there must be a great deal of recombination, especially when the electric force is parallel to the direction of the $\alpha$ particles, since an ion is then constantly meeting other ions produced by the same particle. Moulin showed that the electric force required for saturation does, in fact, vary with the direction of the rays, as much as 1000 volts per cm. being necessary when the rays are parallel to the electric force.

**250. Deflexion of $\alpha$ particles by nuclei. The nuclear charge.** Wilson's experiments, like Bragg's, throw light on the nuclear theory of matter. The dense cloud along the track is due, without doubt, to electrons ejected from the atoms of nitrogen and oxygen. Most are ejected slowly, but a few have tracks of visible length, seen as short twigs leaving the earlier part of the $\alpha$ ray track. This is one type of collision of an $\alpha$ particle with matter. The other is a single deflexion through a finite angle, shown in the left-hand track of Fig. 364. If the nucleus was a solid structure occupying any considerable part of the atom, large deflexions would be frequent, and equally so at the beginning and end of the track. Experiment shows them to be rare, and more frequent at the end. Hence the nucleus acts yieldingly, like a centre of force, and the force it exerts on an $\alpha$ particle at a distance of, say, $10^{-9}$ cm. does not deflect it appreciably.

The frequency of deflexion can be found from a large number of photographs, or more expeditiously by passing $\alpha$ particles through metal foils. Geiger and Marsden found that a certain thin platinum

foil turned one $\alpha$ particle in 8000 through a right angle or more. Elements of high atomic number deflect more, atom for atom, than those of low atomic number. Rutherford explained their experiments by the hypothesis that a nucleus repels an $\alpha$ particle according to the inverse square law. Let $E$ be its charge, $2e$ the charge and $m$ the mass of the $\alpha$ particle, and let the nucleus (on account of its greater mass) be assumed to be at rest. An $\alpha$ particle will be said to be *aimed so as to pass a nucleus at distance* $\delta$ when it would do so in the absence of a deflecting force. Suppose that an $\alpha$ particle so aimed has velocity $u$. The force on it at distance $r$ from the nucleus is $2Ee/r^2$ by hypothesis, and the theory of central forces shows that it is turned altogether through an angle $\phi$ given by

$$\delta = \frac{2Ee}{mu^2} \cot \tfrac{1}{2}\phi.$$

Now let a narrow beam of $\alpha$ particles fall normally at $P$ on a metal foil of small thickness $t$, containing $n$ atoms per cm.$^3$ The chance of an $\alpha$ particle being aimed so as to pass some nucleus within a distance $\delta$ is $\pi nt\delta^2$, since 1 cm.$^2$ of the foil contains $nt$ atomic targets, each of area $\pi\delta^2$. A particle so aimed will be turned through an angle equal to or greater than $\phi$. It follows that the chance of an $\alpha$ particle being deflected through an angle between $\phi$ and $\phi + d\phi$ is

$$2\pi nt\delta\, d\delta = 4\pi nt \left(\frac{Ee}{mu^2}\right)^2 \frac{\cos \tfrac{1}{2}\phi}{\sin^3 \tfrac{1}{2}\phi}\, d\phi.$$

Suppose for the moment that a unit sphere is described with $P$ as centre. The particles deflected through angles between $\phi$ and $\phi + d\phi$ fall on an area $2\pi \sin\phi d\phi = 4\pi \cos \tfrac{1}{2}\phi \sin \tfrac{1}{2}\phi d\phi$ of the sphere, the chance of a particle falling on *unit* area in this region being

$$\left(\frac{Ee}{mu^2}\right)^2 \frac{nt}{\sin^4 \tfrac{1}{2}\phi}.$$

If, therefore, a small screen of area $A$ is set up at distance $r$ from $P$, so as to receive particles scattered at an angle $\phi$, the number received per second is

$$\left(\frac{Ee}{mu^2}\right)^2 \frac{AQnt}{r^2 \sin^4 \tfrac{1}{2}\phi} \quad \ldots\ldots\ldots\ldots\ldots\ldots (309),$$

where $Q$ is the number falling on the foil per second. Geiger and Marsden verified that the number was proportional to $\operatorname{cosec}^4 \tfrac{1}{2}\phi$ for angles between 45° and 150°.

The nuclear charge $E$ was determined approximately by Geiger and Marsden from equation (309), and more accurately by Chadwick as follows. The beam from a radium source $R$ (Fig. 366) is limited by diaphragms so as to strike the metal foil on a circular annulus shown in section at $AA'$, $BB'$. A zinc sulphide screen $S$ is placed on the other side of the annulus, and at the same distance from it as $R$. Source and foil are enclosed in an exhausted box,

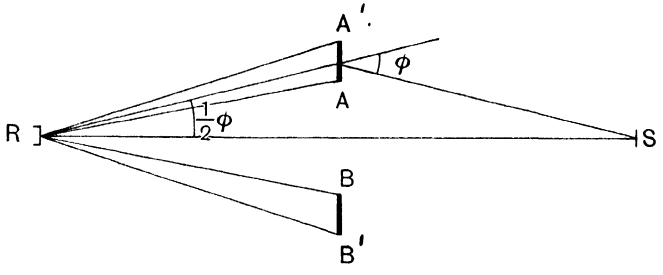



Fig. 366

with a window of thin mica to let the particles through to $S$, which is outside the box. Another diaphragm allows particles from $R$ to strike $S$ direct when required. Discrepancy in numbers between the direct and scattered particles, the chief source of error in these experiments, is overcome by counting the direct particles through a rotating sector which reduces them in the ratio of about 500 to 1. Particles leaving $R$ at an angle $\frac{1}{2}\phi$ with $RS$ reach $S$ after being deflected through an angle $\phi$, so that it is not difficult to calculate the ratio of the number of scattered particles reaching $S$ to the number of direct particles reaching it in the same time. The ratio is

$$4\pi nt\left(\frac{Ee}{mu^2}\right)^2\left[\frac{\cos\theta}{\sin^2\theta}-\frac{\cos\theta'}{\sin^2\theta'}+\log\tan\tfrac{1}{2}\theta-\log\tan\tfrac{1}{2}\theta'\right],$$

where $\theta$ is the angle $SRA$ and $\theta'$ the angle $SRA'$. $nt$ is found by weighing the foil, and the average velocity of the $\alpha$ particles inside the foil can be estimated from the table on p. 572. Chadwick found that $E = 77{\cdot}4e$, $46{\cdot}3e$ and $29{\cdot}3e$ for platinum, silver and copper respectively, with a probable error of about $1\frac{1}{2}$ per cent. The atomic numbers are 78, 47 and 29. We have thus an accurate confirmation of van den Broek's hypothesis that the nuclear charge is $Z$ times the charge on a proton (Arts. 223, 236), the charge $2e$ of the

$\alpha$ particle corresponding to its atomic number 2. The experiments of Geiger, Marsden and Chadwick also prove that the inverse square law holds for distances of the order of $10^{-11}$ cm., so that the diameter of the largest nuclei cannot greatly exceed $10^{-12}$ cm.

**251. Motion of the nucleus in a close collision.** The left-hand track of Fig. 364 has a spur at the point where the $\alpha$ particle is last deflected, due no doubt to the nucleus of nitrogen or oxygen struck by it. Shimizu and others observed branched tracks in which the lengths of the branches are comparable with each other. The angle between the branches can be found from photographs taken with two lenses at right angles, and it has been shown that energy and momentum are conserved in close collisions. Let $m$ be the mass of an $\alpha$ particle (Fig. 367), $u$ its velocity before collision, $v$ its velocity after, and $M$, $V$ the mass and final velocity of the nucleus. Then if the final tracks make angles $\alpha$, $\beta$ with the initial track, conservation of energy and momentum makes the three tracks lie in one plane, and

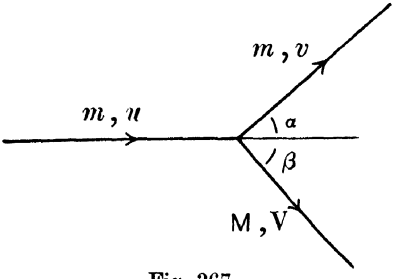


Fig. 367

$$mu = mv\cos\alpha + MV\cos\beta, \quad mv\sin\alpha = MV\sin\beta,$$
$$mu^2 = mv^2 + MV^2.$$

Eliminating the velocities we should have

$$\frac{m}{M} = \frac{\sin(\alpha + 2\beta)}{\sin\alpha}.$$

Auger and F. Perrin found 1·00 and 1·14 for the mass $M$ of a hydrogen nucleus in two rough experiments, taking $m = 4$ in this formula; Blackett found 1·02 for hydrogen, 4·03 for helium and 16·7 for argon. In helium we should have $\alpha + \beta = \frac{1}{2}\pi$, so that the final tracks should be at right angles to each other. Blackett found tracks inclined at an angle of 89° 45′, agreeing well with the theoretical angle.

Close collisions in hydrogen are specially interesting, since the nucleus moves faster than the $\alpha$ particle, and might have a longer range. This phenomenon was discovered by Marsden. The range of

$\alpha$ particles from radium C is 24 cm. (that is, there is a great drop in the number of scintillations just before this point), but a few scintillations are observed up to 80 cm., which must be due to hydrogen nuclei; for they can be restored by admitting hydrogen, after they have been cut out by absorbing screens: also the value of $e/m$ is also that of a hydrogen nucleus.

Since the charge on a hydrogen nucleus is but one unit, an $\alpha$ particle will come nearer to it than to a nucleus of copper or platinum, and hence we can test the inverse square law below the limit of the last article. Considerable discrepancies were found by Rutherford. Chadwick and Bieler measured the angular distribution of the projected hydrogen nuclei, and found failure only when the velocity of the $\alpha$ particle exceeds $1{\cdot}4 \times 10^9$ cm./sec. Particles of this velocity can get to a distance of $5 \times 10^{-13}$ cm. from the nucleus, below which the law of action changes so rapidly as to suggest that the $\alpha$ particle has a solid structure of about these dimensions.

**252. Disintegration of nuclei by $\alpha$ particles.** To resume, while a heavy nucleus keeps an $\alpha$ particle at a distance in a field of electrostatic force, a hydrogen nucleus suffers an intimate collision with it. The intermediate stage is of still greater interest, since the nuclei of certain light elements are broken up by impact. This phenomenon was discovered in 1919 by Rutherford, who detected long-range particles from nitrogen by the method which Marsden had used for hydrogen. These particles are deflected magnetically to the same extent as hydrogen nuclei of the same range, so that they must be protons. It appeared also that protons could be ejected from boron, since boron nitride gave more fast particles than would be expected from its nitrogen content.

Rutherford and Chadwick afterwards made the decisive discovery that the particles from nitrogen are faster than those from hydrogen, so that they cannot be due to contamination by hydrogen. Similar particles are ejected from boron, fluorine, sodium, aluminium and phosphorus. There is no doubt that the nuclei of all these elements are disintegrated by the close approach of an $\alpha$ particle. An atom so modified must, on the analogy of radioactive emission, rearrange itself and become one of lower atomic

weight, so that we have the solution, on a minute scale, of the long-standing problem of the artificial transmutation of elements. The apparatus for detecting long-range particles is shown in Fig. 368. The source of $\alpha$ particles is a brass disc $R$ coated on one side with radium $B+C$, and contained in a tube $T$. The zinc sulphide screen $S$ is placed outside the tube, opposite a hole covered with a silver foil $F$ of stopping power equal to 6·3 cm. of air. Scintillations are observed through a low power microscope $M$, the distance from $R$ to $S$ being normally 3·5 cm. The element $E$ to be tested is either a metal foil or a powder deposited on a gold foil, placed directly in front of $R$; while a slot between $F$ and $S$ holds mica sheets of different thickness, for estimating the penetration of the ejected particles. The sheets will be specified in what follows by the thicknesses of air which have the same stopping power for $\alpha$ particles (cf. Art. 248).

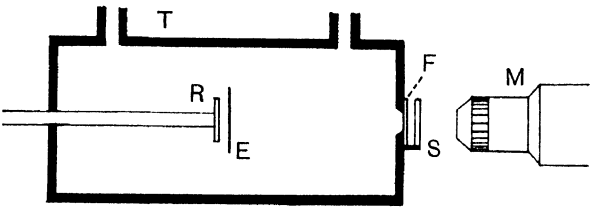


Fig. 368

Suppose for the moment that a mica sheet of 1 cm. equivalent is in the slot. The total stopping power being 7·3 cm. and the range of the $\alpha$ particles 7 cm., all such particles are cut off from the screen $S$, so that its scintillations are due to ejected protons. A few such scintillations are always present. Many more appear when hydrogen is admitted to the apparatus, but both kinds are stopped by mica sheets equivalent to 29 cm. of air. If we allow a little margin, and use sheets equivalent to 32 cm. of air, the observed scintillations are due to particles with a range greater than 32 cm., which cannot come from any of the materials used in making up the apparatus, or from hydrogen occluded anywhere. No such particles are emitted by gold or oxygen, so that the use of the gold foil is unexceptionable, and an element can be deposited on it in the form of an oxide if necessary.

The ejected particles have not all the same velocity. Air, from its nitrogen content, yields particles with a range up to 40 cm. Rutherford and Chadwick examined all the elements between

lithium and chlorine, except neon, with results shown in the following table.

| Element | Atomic weight | Maximum range of ejected particle in equivalent cm. of air | |
|---|---|---|---|
| | | Forwards | Backwards |
| Lithium | 7, 6 | — | — |
| Beryllium | 9 | — | — |
| Boron | 11, 10 | 58 | 38 |
| Carbon | 12 | — | — |
| Nitrogen | 14 | 40 | 18 |
| Oxygen | 16 | — | — |
| Fluorine | 19 | 65 | 48 |
| Sodium | 23 | 58 | 36 |
| Magnesium | 24, 25, 26 | — | — |
| Aluminium | 27 | 90 | 67 |
| Silicon | 28, 29, ? 30 | — | — |
| Phosphorus | 31 | 65 | 49 |
| Sulphur | 32 | — | — |
| Chlorine | 35, 37 | — | — |

The backward particles were observed by coating a gold foil with active deposit on the side remote from $S$, and placing $E$ on the same side: particles are almost certainly expelled in all directions, but more slowly backwards than forwards. No fast particles were found with heavier elements, though many were tried.

Magnetic deflexion can be observed by tilting $R$ backwards (making it, in effect, a line-source) and interposing a knife-edge between $R$ and $S$, so that scintillations fall on one-half of the screen. A magnetic field then increases or decreases their number according to its direction. Particles from aluminium, having an average range of 45 cm., are deflected about four-fifths as much as hydrogen nuclei with an average range of 22 cm., as we should expect if the range varies as the cube of the velocity (cf. Art. 248). The values of $e/m$ must, therefore, be approximately the same. Bearing in mind the limited number of plausible hypotheses, it is hardly possible that the ejected particles are other than protons.

The number of protons increases rapidly with the velocity of the impinging $\alpha$ particle, though not more than one collision in 100,000 is effective at the best. Harkins and Ryan photographed the tracks of 80,000 particles and found only one triple track, ascribed to

disintegration. An $\alpha$ particle must have a certain minimum energy in order to disintegrate a given element—3 million volts for aluminium. This explains the failure of innumerable attempts to disintegrate elements by weak stimuli such as temperature or chemical action, the energy of which can hardly exceed 100 volts.

All the elements known to emit long-range particles have atomic weights $4n + 2$ or $4n + 3$, where $n$ is an integer. This suggests that their nuclei are built up mainly of helium nuclei, or specially stable aggregates of four protons; one further proton being attached firmly, a second more loosely and a third more loosely still.

**253. The $\beta$ particles.** The magnetic deflexion of the $\beta$ rays was discovered by Giesel in 1899. Early experiments showed that the rays were complex, and that the slowest rays were most easily absorbed by matter. The $\beta$ particles were also shown to have a negative charge by receiving them on a plate completely imbedded in ebonite.

The charge on the $\beta$ particles has never been measured, but many experiments have been made to determine the velocity and the ratio of the charge to the mass. Perhaps the most accurate are those of Bucherer, repeated by G. Neumann. The principle of the experiment is shown in Fig. 369. A cylindrical box $C$ contains two parallel plates $C_1$, $C_2$, about 20 cm. in diameter and $\frac{1}{4}$ mm. apart. A strong and uniform electric field is applied by connecting the plates to the terminals of a battery of small accumulators. The box is enclosed in a solenoid long enough to give a uniform magnetic field parallel to the plates, and the pressure inside the box is kept as low as possible. A small cavity in the lower plate contains

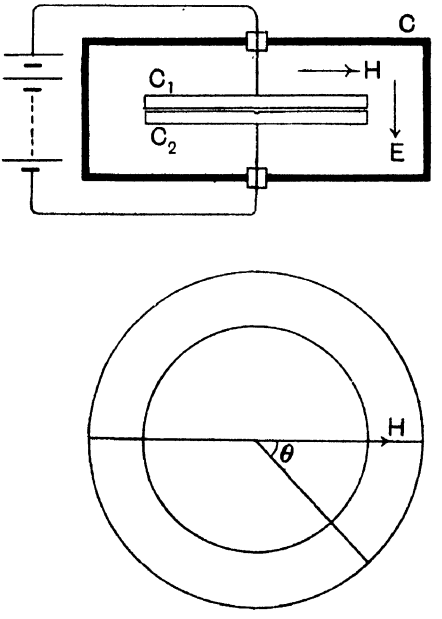



Fig. 369

radium fluoride, and the inner curved wall of the box is covered with a photographic film pressed against it by springs. $\beta$ rays spread straight out from the radium when neither field is applied, and those which are projected horizontally escape, giving a straight band on the photographic film. To find what rays escape when both the electric and magnetic fields are applied, consider those projected horizontally in a direction making an angle $\theta$ with the magnetic field (Fig. 369). The electric field exerts a force $eE$ downwards, and the magnetic field a force $eHv \sin\theta/c$ upwards, where $v$ is the velocity. If these forces are not equal the particle is drawn to one or other of the plates; hence only those particles escape for which $E = Hv \sin\theta/c$. The effect of the combined fields is therefore to sort out the particles of different velocity, so that each corresponds to a particular value of $\theta$. After leaving the condenser the rays are acted on by the magnetic field only, and are bent into a circle of radius $\rho$, where $\rho = mcv/He$. Bucherer only considered the rays experiencing the maximum deflexion, for which $\theta = \frac{1}{2}\pi$. For these

$$\frac{v}{c} = \frac{E}{H} \text{ and } \frac{e}{m} = \frac{cv}{\rho H} = \frac{c^2 E}{\rho H^2}.$$

Since $\rho$ is known from the deflexion, $e/m$ and $v$ are known.

No rays are observed to emerge when $E > H$. Hence *the velocity of the $\beta$ particles is always less than that of light.* Also, as Kaufmann first observed, *$e/m$ is not constant, but decreases as $v$ increases.* The smaller the velocity the more $e/m$ approximates to the value $5{\cdot}30 \times 10^{17}$ which it has for cathode rays and other slow electrons. Thus, though there is every reason for believing that the $\beta$ particles are electrons, the variability of $e/m$ suggests a new problem. The experimental facts might be explained in several ways; for example, that the mass of an electron depends on its velocity, that its charge does, or that the law of action of a magnetic field on a moving charge (Art. 94) has to be modified for high velocities of the particle. It has been found most convenient to assume that the *mass* alone varies: the theoretical aspects of this are discussed in Art. 273.

The experiments of Bucherer and Neumann agree accurately with the formula

$$\frac{e}{m} = \frac{e}{m_0}\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (310)$$

over a range $v/c = 0{\cdot}3$ to $0{\cdot}8$, $e/m_0$ having the normal value $5{\cdot}30 \times 10^{17}$. This formula has a theoretical foundation (Art. 273), but may at present be regarded as empirical. It follows that the mass $m$ of an electron at velocity $v$ is given by

$$m = m_0\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (311),$$

where $m_0 = 9{\cdot}00 \times 10^{-28}$ is the mass of an electron with a velocity small compared with that of light. The values of $m/m_0$ and of $\rho H = mcv/e$ for various values of $v$ are shown in the table on p. 587, which can be used to find the velocity of a $\beta$ particle by magnetic deflexion alone.

Fig. 370 shows one of Wilson's photographs of the tracks of $\beta$ particles in gases. The complete tracks are too long to be seen on a single photograph.

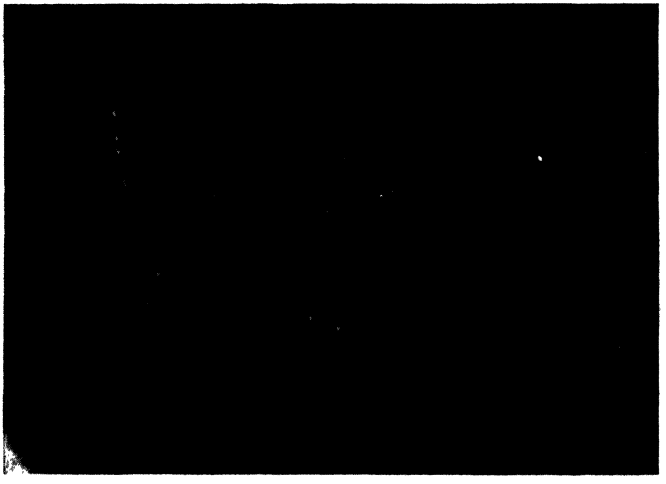


Fig. 370

The tracks of the fastest $\beta$ particles are quite straight for many centimetres. Wilson observed a few sudden deflexions of $\beta$ particles through finite angles, and some branched tracks arising from the expulsion of a fast electron from the atom. As in the collision of $\alpha$ particles with helium nuclei (Art. 251), the branches are (with fast particles) nearly at right angles; but not with slow particles,

no doubt since the velocity of the impinging electron is then comparable with that of a circumnuclear electron in its orbit.

| Velocity in cm./sec. | $\frac{m}{m_0}$ | Energy in volts (Art. 273) | Quantum wave-length in $10^{-8}$ cm. (from equation (276)) | $\rho H$ |
|---|---|---|---|---|
| $10^9$ | 1·000 | 290 | 42·4 | 56·8 |
| $2 \times 10^9$ | 1·002 | 1130 | 9·18 | 113 |
| 3 | 1·005 | 2570 | 4·79 | 171 |
| 4 | 1·009 | 4600 | 2·68 | 229 |
| 5 | 1·014 | 7240 | 1·70 | 287 |
| 6 | 1·021 | 10500 | 1·18 | 346 |
| 7 | 1·028 | 14500 | 0·849 | 408 |
| 8 | 1·038 | 19100 | 0·644 | 470 |
| 9 | 1·048 | 24600 | 0·500 | 534 |
| $10^{10}$ | 1·061 | 31000 | 0·397 | 600 |
| $1·1 \times 10^{10}$ | 1·075 | 38200 | 0·322 | 669 |
| 1·2 | 1·091 | 46500 | 0·265 | 741 |
| 1·3 | 1·110 | 55900 | 0·220 | 817 |
| 1·4 | 1·131 | 67200 | 0·183 | 896 |
| 1·5 | 1·155 | 78900 | 0·156 | 980 |
| 1·6 | 1·182 | 92900 | 0·132 | 1071 |
| 1·7 | 1·214 | 109000 | 0·113 | 1168 |
| 1·8 | 1·250 | 127000 | 0·0969 | 1273 |
| 1·9 | 1·292 | 149000 | 0·0826 | 1390 |
| 2·0 | 1·342 | 174000 | 0·0707 | 1519 |
| 2·05 | 1·370 | 188000 | 0·0655 | 1585 |
| 2·10 | 1·400 | 204000 | 0·0603 | 1665 |
| 2·15 | 1·434 | 221000 | 0·0557 | 1745 |
| 2·20 | 1·471 | 240000 | 0·0513 | 1832 |
| 2·25 | 1·512 | 261000 | 0·0471 | 1925 |
| 2·30 | 1·557 | 284000 | 0·0433 | 2027 |
| 2·35 | 1·609 | 310000 | 0·0397 | 2140 |
| 2·40 | 1·667 | 340000 | 0 0362 | 2264 |
| 2 45 | 1·733 | 374000 | 0·0329 | 2403 |
| 2·50 | 1·809 | 412000 | 0·0299 | 2561 |
| 2·55 | 1·899 | 458000 | 0·0269 | 2740 |
| 2·60 | 2·00 | 512000 | 0·0240 | 2949 |
| 2·65 | 2·13 | 578000 | 0·0213 | 3200 |
| 2·70 | 2·29 | 660000 | 0·0186 | 3507 |
| 2·75 | 2·50 | 767000 | 0·0160 | 3891 |
| 2·80 | 2·79 | 911000 | 0·0135 | 4416 |
| 2·85 | 3·20 | 1124000 | 0·0109 | 5165 |
| 2·90 | 3·90 | 1480000 | 0·0083 | 6412 |
| 2·95 | 5·50 | 2291000 | 0·0054 | 9177 |
| 2·96 | 6·14 | 2622000 | 0·0047 | 10290 |
| 2·97 | 7·08 | 3102000 | 0·0040 | 11910 |
| 2·98 | 8·68 | 3913000 | 0·0031 | 14640 |
| 2·99 | 12·2 | 5736000 | 0·0021 | 20730 |
| 3 | $\infty$ | $\infty$ | 0 | $\infty$ |

**254. Homogeneous groups of $\beta$ particles.** The $\beta$ particles from a thick layer of a radioactive substance have all velocities over a considerable range; with a thin layer a number of the particles, at least, form homogeneous groups of definite velocity. This was first shown by v. Baeyer, Hahn and Frl. Meitner, by magnetic deflexion in a vacuum. Their apparatus resembled that used for the magnetic deflexion of the $\alpha$ particles (Fig. 358). A thin film of radium deprived of its transformation products gave two weak bands on the photographic plate; a deposit of radium C gave four strong bands, and so on. The velocity of the rays can be found by interpolation from the table on p. 587.

The main results of these experiments are shown in Fig. 371. Some of the rays from radium C have velocities closely approaching that of light. The rays from radium D are both weak and feebly penetrating, and were, in fact, discovered by the photographic method. So far radium E, which gives a continuous spectrum of $\beta$ rays with velocities between 2·1 and 2·8 $\times$ $10^{10}$ cm./sec., has not been found to emit any homogeneous groups.

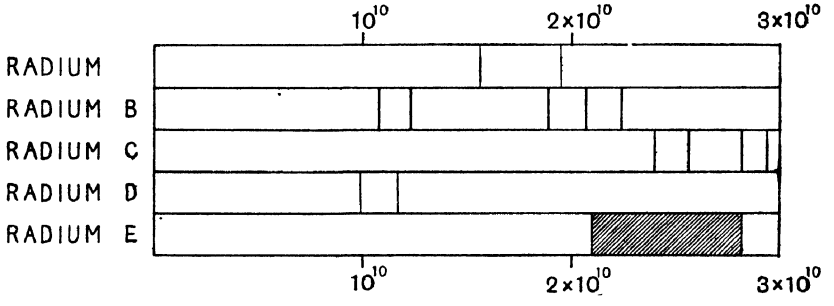



Fig. 371

More detailed observations were made by Danysz and others. The following table gives the energies, in volts, of some of the $\beta$ particles from radium B and radium C, from experiments by Ellis and Skinner. The energy is calculated from the equations of Art. 273, to which reference should be made: it would no longer be even approximately correct to calculate it from elementary formulae, as in Art. 241. Radium B and C also give out $\beta$ particles varying continuously in velocity like those of radium E.

*Energies of β particles in volts.*

| Radium B | | Radium C |
|---|---|---|
| 37250 | 106800 | 42000 |
| 37920 | 116000 | |
| 40160 | 152900 | 60100 |
| 49830 | 169700 | 63200 |
| 52880 | 186000 | 66700 |
| 53650 | 206700 | 72500 |
| | 227500 | 76400 |
| 58060 | 244200 | |
| 58720 | 263800 | 147000 |
| 61060 | 282400 | 158200 |
| 61720 | 292600 | 181900 |
| 64120 | 303300 | 188200 |
| 66670 | 337900 | 240600 |
| 70940 | 350200 | 258000 |
| 74260 | 353600 | 297500 |
| | 380900 | 314200 |
| | 392500 | 337900 |
| | | 370400 |

This table will be useful in discussing the relation of β particles and γ rays in the next article.

**255. The γ rays.** The γ rays were discovered by Villard in 1900. They are remarkable for their great penetrating power. After the γ rays from radium have passed through 3 mm. of lead, every additional thickness of 1·5 cm. of lead, or 6 cm. of aluminium, cuts them down to about one-half of their original intensity. The electrical effect of a strong radium preparation can be measured through 20 cm. of lead. One metre of air at atmospheric pressure cuts the rays down by about one per cent.

From the fact that γ rays are not deflected in magnetic fields, and that they are always found together with fairly swift β rays, it was concluded almost at once that they are similar to Röntgen rays, only harder. Imperfect diffraction effects were observed by Shaw in 1913, and subsequently Rutherford and Andrade measured the wave-lengths of the γ rays from radium B and C together, using as a source of the rays some niton in a small glass tube. It was known that both radium B and radium C emit hard γ rays, and the former also emits some rays of about the same penetrating power as ordinary Röntgen rays. This suggested a search for wave-lengths of the order of $10^{-8}$ cm., and also for shorter wave-lengths,

probably of the order $10^{-9}$ cm. Those actually found are shown in Fig. 372, and in the following table.

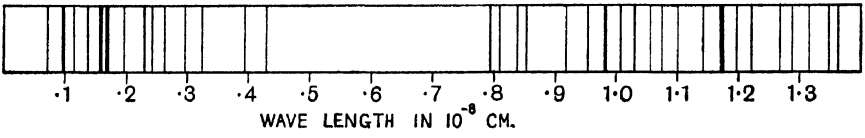



Fig. 372

| Radium B+C | | Radium B only | |
|---|---|---|---|
| Wave-length in $10^{-8}$ cm. | Quantum energy in volts | Wave-length in $10^{-8}$ cm. | Quantum energy in volts |
| 0·071 | 173000 | 0·793 | 15500 |
| 0·099 | 124000 | 0·809 | 15200 |
| 0·115 | 107000 | 0·838 | 14700 |
| 0·137 | 89800 | 0·853 | 14400 |
| 0·159 | 77400 | 0·917 | 13400 |
| 0·169 | 72800 | 0·953 | 12900 |
| 0·196 | 62800 | 0·982 | 12500 |
| 0·229 | 53700 | 1·006 | 12200 |
| 0·242 | 50800 | 1·029 | 12000 |
| 0·262 | 47000 | 1·055 | 11800 |
| 0·296 | 41600 | 1·074 | 11500 |
| 0·324 | 38000 | 1·100 | 11200 |
| 0·393 | 31300 | 1·141 | 10800 |
| 0·428 | 28700 | 1·175 | 10500 |
| | | 1·196 | 10300 |
| | | 1·219 | 10100 |
| | | 1·266 | 9700 |
| | | 1·286 | 9500 |
| | | 1·315 | 9300 |
| | | 1·349 | 9100 |
| | | 1·365 | 9000 |

Only an imperfect account of the origin of these lines can be given in the present state of our knowledge. Assuming from Art. 256 that the circumnuclear electrons of radium B can behave like those of an element of atomic number 82, and radium C like one of atomic number 83, we can calculate the wave-lengths of their $K$ and $L$ lines by extrapolation from the tables of Art. 236. Four of the strongest lines, 0·159, 0·169, 0·982 and 1·175, then seem to be the $K_\alpha$ doublet of radium C, the $K_\alpha$ doublet of radium B, the $L_\beta$ line of radium B and the $L_\alpha$ line of radium B respectively. The $\gamma$ radiation certainly contains lines much shorter than this, which must

be ascribed to the nucleus. We shall be better prepared to discuss this problem after considering the effect of $\gamma$ rays on a normal atom.

The $\gamma$ rays ionise gases like Röntgen rays of short wave-length; that is, they eject electrons which move through the gas and produce more ions by collision. These electrons can penetrate many centimetres of air at atmospheric pressure, and C. T. R. Wilson's experiments show that their tracks are indistinguishable from those of $\beta$ particles. Fast electrons are also ejected when $\gamma$ rays fall on solids. This effect, first measured by Rutherford, Robinson and Rawlinson, was shown by Ellis to be analogous to that of Röntgen rays (Art. 241). When, for example, the $\gamma$ rays from radium B + C fall on tungsten, platinum, lead and uranium, three main groups of electrons are ejected in each case, as shown in the following table.

| | Energy of $K$ level in volts | Energy of ejected particle | Sum |
|---|---|---|---|
| Tungsten | 69000 | 166000 | 235000 |
| | | 220000 | 289000 |
| | | 276000 | 345000 |
| Platinum | 78000 | 158000 | 236000 |
| | | 212000 | 290000 |
| | | 269000 | 347000 |
| Lead | 88000 | 149000 | 237000 |
| | | 203000 | 291000 |
| | | 260000 | 348000 |
| Uranium | 116000 | 122000 | 238000 |
| | | 174000 | 290000 |
| | | 231000 | 347000 |

These electrons are undoubtedly produced by three distinct $\gamma$ rays, whose energies are 237000, 290000 and 347000 volts, corresponding to wave-lengths of 0·0519, 0·0425 and 0·0355 $\times 10^{-8}$ cm., shorter than any measured by Rutherford and Andrade and too short, possibly, to measure with a crystal.

The origin of the $\beta$ particles from radium B was discussed in detail by Ellis. Since radium B is isotopic with lead, we should expect electrons of the same energy as those tabulated above, and these are found approximately at 152900, 206700 and 263800

volts (see table on p. 589). Three more groups, at 227500, 282400 and 337900 volts, are ejected by the same rays from the $L_3$ level of radium B. The six slowest groups of electrons from radium B can be ascribed to the action on the $L$ and lower levels of a $\gamma$ ray of wave-length $0{\cdot}229 \times 10^{-8}$ cm., one of those measured by Rutherford and Andrade. Notwithstanding certain difficulties, Ellis found evidence of six energy levels in the nucleus of radium B, with energies of 53700, 62500, 257200, 294200, 404800 and 531000 volts; with a similar system of levels for radium C. The fact that the quantum mechanism applies to the nucleus is very important, and opens up a large field of inquiry. The difficulties centre mainly round the continuous spectrum of $\beta$ particles, the lack of complete agreement between the natural $\beta$ particles of radium B and those ejected from lead by the same $\gamma$ rays, and the uncertainty as to the distribution of circumnuclear electrons round a nucleus whose atomic number is changing. The primary cause of all is, presumably, ejection of an electron from the nucleus; but $\gamma$ rays play an important intermediary part.

**256. Radioactive elements in general.** Radium is found only in minerals which contain considerable quantities of uranium, which suggests that radium is a product of uranium. On this view, sufficiently old uranium minerals should contain a definite percentage of radium, corresponding to radioactive equilibrium (Art. 244). Boltwood found, in confirmation, that the masses of radium and uranium in many minerals are in a constant ratio, about $3{\cdot}3 \times 10^{-7}$. Since the atomic weight of uranium is 238·2 and that of radium 226, the ratio of the half-value periods is $3{\cdot}5 \times 10^{-7}$, so that the half-value period of uranium is about $5 \times 10^9$ years. A period of this order would be expected from the observed emission of $\alpha$ particles from uranium. There is reason to believe that all the stages of the transformation from uranium to radium are known, and that there are two intermediate products emitting $\alpha$ particles. The transformation from uranium to the inactive product of polonium is shown in the following table, which includes the element radium C′ formed from radium C by its $\beta$ ray change*.

* Radium C can also break up with emission of an $\alpha$ particle into radium $C_2$ and then into an inactive substance, but only one atom in 3000 breaks up in this way.

*Transformations of uranium (main branch).*

| | Particles emitted by the nucleus | Atomic number | Atomic weight |
|---|---|---|---|
| Uranium .. .. | $\alpha$ | 92 | 238 |
| Uranium $X_1$ .. | $\beta$ | 90 | 234 |
| Uranium $X_2$ .. | $\beta$ | 91 | 234 |
| Uranium 2 .. | $\alpha$ | 92 | 234 |
| Ionium .. .. | $\alpha$ | 90 | 230 |
| Radium .. .. | $\alpha$ | 88 | 226 |
| Niton .. .. | $\alpha$ | 86 | 222 |
| Radium A .. | $\alpha$ | 84 | 218 |
| Radium B .. | $\beta$ | 82 | 214 |
| Radium C .. | $\beta$ | 83 | 214 |
| Radium C′ .. | $\alpha$ | 84 | 214 |
| Radium D .. | $\beta$ | 82 | 210 |
| Radium E .. | $\beta$ | 83 | 210 |
| Polonium .. | $\alpha$ | 84 | 210 |
| An isotope of lead | — | 82 | 206 |

In discussing this table we assume that the slow $\beta$ particles from radium are circumnuclear electrons ejected by its $\alpha$ particles; or at least do not come from the nucleus. They have, therefore, been omitted. The atomic number of uranium is known to be 92 from the wave-length of its $L$ radiation (Art. 236). Since the loss of an $\alpha$ particle reduces the atomic number by two, and the loss of a $\beta$ particle increases it by one, we can calculate the atomic number of all the elements, which is done in the third column. The atomic numbers of radium (88) and polonium (84) are also known independently from their Röntgen ray spectra, so that we can feel confident that the intermediate numbers are correct. Since the loss of an $\alpha$ particle reduces atomic weight by 4, and the loss of a $\beta$ particle leaves it practically unchanged, we can also calculate the atomic weights, with uranium at 238. The agreement is again good (better if we assume an isotope of mass 238 exactly); for the atomic weight of radium is 226, and that of niton 223 from the (difficult) determination of density by Gray and Ramsay.

Let us now consider the evidence for van den Broek's hypothesis (Art. 223). Radium B and radium D are chemically indistinguishable from lead ($Z = 82$), radium C and radium E from bismuth ($Z = 83$). Ionium, which Boltwood discovered in 1906, is chemically

indistinguishable from thorium ($Z = 90$). Russell and Rossi found that the arc spectrum of thorium is identical with that of a mixture of thorium and ionium, containing a large percentage of the latter. The possibility of isotopes, or elements with identical physical and chemical properties but different atomic weights, was first stated by Russell and Rossi.

In 1911 Soddy showed that emission of an $\alpha$ particle shifts a radioactive element two places in the periodic table. The pure dependence on nuclear charge is, however, obscured by the simultaneous loss of atomic weight, and is seen first in Fajans' supplementary law that emission of a $\beta$ particle shifts the element one place in the opposite direction. These laws account without exception for the chemical properties of the elements between radium and lead, as we should expect from Bohr's scheme (Art. 223).

The theory of isotopes received much support, in its early stages, from a study of the final products of radioactive disintegration. Lead (or a substance chemically identical with it) occurs in all uranium minerals, and the Pb—Ur ratio is often fairly constant. Since a polonium atom which has lost an $\alpha$ particle has atomic number 82, the final product of the uranium series should be an isotope of lead of atomic weight 206. Richards found 206·08 for the atomic weight of lead from Norwegian cleveite, Hönigschmid 206·05 for lead from pitchblende. The probability of descent from radium is very great in both cases. Thorium also yields an isotope of lead, but since six $\alpha$ particles are emitted in the change, the atomic weight should be $232 - 24 = 208$. Soddy, who first discovered divergences from accepted atomic weights, found 207·69 for the atomic weight of lead from a thorium mineral: Hönigschmid found 207·9 for another sample. The atomic weight of ordinary lead, 207·2, agrees with the view that it is a mixture of (mainly) these two isotopes, in nearly equal amounts.

# CHAPTER XV

## THE ELECTRICAL THEORY OF MATTER

**257. The metallic state.** It has been assumed, in Arts. 28, 94 and elsewhere, that electrical conduction in metals is by electrons which can pass away from the atoms to which they were originally attached. Difficulties arise when we try to form a detailed picture of the motion of the free electrons. Drude assumed that the free electrons are in a state of thermal agitation like the molecules of a gas, colliding at intervals with the atoms of the metal. If all move with velocity $V$ and have a free path $l$, their mean velocity of drift under an electric force of $E$ electrostatic units is $v = eEl/2mV$ (Art. 175). If $n$ is the number of free electrons per cm.$^3$ of the metal, the current-density in electromagnetic units is

$$u = nev/c = ne^2El/2mcV.$$

If $\sigma$ is the conductivity in electromagnetic units, $u = \sigma cE$, so that on this theory $\sigma = ne^2l/2mc^2V$.

A plausible hypothesis would be that $\frac{1}{2}mV^2 = \alpha\theta$ (Art. 220), which makes $V = 1{\cdot}1 \times 10^7$ cm./sec. at 15° C. The energy required to raise the temperature of 1 cm.$^3$ of copper by 1° is $3{\cdot}5 \times 10^7$ ergs, and the number of atoms is $8{\cdot}6 \times 10^{22}$. We should expect at least $8{\cdot}6 \times 10^{22}\alpha = 1{\cdot}8 \times 10^7$ ergs to go to increase the energy of vibration of the atoms; and if the rest is taken up by the electrons, their number must be comparable with the number of atoms. To explain the observed conductivity of copper ($\sigma = 6 \times 10^{-4}$) we have to make $l$ at least $6 \times 10^{-7}$ cm., which is about 30 times the average distance from nucleus to nucleus in the solid state. A long free path is not peculiar to metals, for slow electrons penetrate freely through inert gases (Art. 222), and it is conceivable that their free path would be very long at velocities of the order of $10^7$ cm./sec., if we could be sure (as we can in metals) that an electron would remain free, and not carry atoms with it. So long a free path is, however, very difficult to explain.

The hypothesis that $\frac{1}{2}mV^2 = \alpha\theta$ for moderate absolute temperatures is supported by measurements of thermal conductivity. Since metals conduct heat much better than non-metals, we can assume that most of the conduction is by electrons, and the thermal conductivity is then, on kinetic principles, $nl\alpha V/6J$, where $J$ is the mechanical equivalent of heat. Writing $k$ for the thermal conductivity, we should have $k/\sigma = 2\alpha^2c^2\theta/3e^2J$, which is the same for all metals and proportional to the absolute temperature. Jäger, Diesselhorst and Lees found that $k/\sigma\theta$ only varies from 4·4 to 6 for copper, silver, lead and zinc between $-170°$ C. and 100° C., the value of $2\alpha^2c^2/3e^2J$ being 2·7. This approximate agreement constitutes the principal success of Drude's theory.

Drude's theory gives a qualitative explanation of, amongst other things, thermoelectric effects. Let two metals be placed in contact, and let there be more electrons per cm.$^3$ in one than the other. Electrons will diffuse at first from one metal to the other, but their accumulation in the second metal gives an electrical repulsion which tends to prevent further electrons from crossing the boundary. A steady state will ultimately occur in which the boundary is a seat of electric force, which does not penetrate far into either metal (cf. Art. 176). The first metal will then be at a higher potential than the second, and we have a natural (or Peltier) difference of potential which is physically distinct from the so-called contact potential difference of the two metals (Art. 210). In the same way we should expect differences of potential in a single unequally heated conductor (Art. 103).

Consideration of the fact that the "atomic heat" (or product of atomic weight and specific heat at constant volume) is nearly the same for metals and non-metals leads to a second view of the metallic state. Haber first suggested that the electrons in a metal might form a lattice, like the nuclei of a crystal, rather than a gas. The specific heat of a solid becomes small at a temperature which is highest for substances whose atomic weight is lowest, so that if the electrons form a solid their contribution to the specific heat at ordinary temperatures may be negligible. This theory is due to Keesom. Lindemann pointed out that the thermal conductivity of such a lattice might be very great, so that we have the possibility of explaining the high thermal conductivity of metals on lines quite

distinct from Drude's theory. These theories have not so far explained the approximate constancy of $k/\sigma\theta$, or its magnitude.

The phenomenon of super-conduction (Art. 67) is a challenge to theories of the metallic state, from the practically complete absence of collisions. Super-conduction ceases in a magnetic field above a certain critical strength. It is not likely that any theory can be made very definite, for the interaction of electron and atom is much more intimate, and less capable of representation by simplifying assumptions, than that of molecules in the kinetic theory of gases, and we must wait for more information about the nuclear atom than we have at present.

**258. The Hall effect. Currents produced by acceleration of a conductor.** Hall found in 1877 that the lines of current-flow in a metal are altered in a magnetic field. Let a current enter a thin sheet of metal at $A$ (Fig. 373) and leave it at $B$, the points $C$, $D$ being chosen so that no deflexion occurs when they are joined to a galvanometer. If a magnetic field is applied normal to the sheet, and towards the observer, a current will usually flow through the galvanometer in the direction shown. This is the case with bismuth, copper, nickel, platinum and silver; with tellurium, antimony, cobalt, iron and zinc the current is in the opposite direction. Very thin sheets and a sensitive galvanometer are required to detect the current, except in tellurium, bismuth and antimony, where the effect is considerable. The simplest theoretical assumptions make the Hall effect have in all metals the direction shown in the figure. The carriers of the current, whatever their sign, experience in the magnetic field a deflecting force in the general direction of $CD$ (Art. 94), and are deflected from $C$ towards $D$. As electrons they give a current from $D$ to $C$ inside the sheet, as shown. Since the nuclear theory leaves no room for conduction by positively charged particles in metals, an explanation of the negative Hall effects must be sought in the nature of the particular

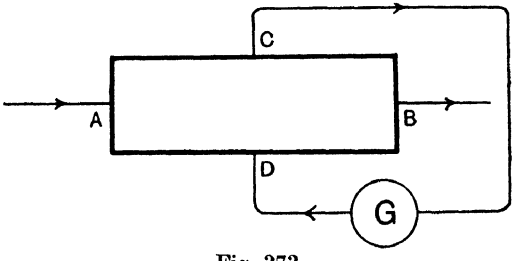


Fig. 373

atoms concerned, on which no light has hitherto been thrown. The relation of ordinary electromagnetic actions to the Hall effect has been noticed in Art. 94.

Maxwell observed, in his *Treatise on Electricity and Magnetism*, that if the carriers of the electric current possess inertia, currents will flow in a conductor while it is being accelerated; in particular, in a coil of wire which is rotating about an axis perpendicular to its plane, and suddenly stopped. The carriers of the current will move with the velocity of the wire while it remains constant, but will outstrip it during the retardation, and a current will flow for a short time. The sign of the current will depend on whether positive or negative electricity is free to move. An effect of this kind, sought for by many experimenters, was measured by Tolman and Stewart. In their experiments the ends of the coil were brought to small rings on the axis, from which sliding contacts were taken to a ballistic galvanometer. The chief difficulty arose from the smallness of the ballistic throw. The coil had about 600 turns and a mean diameter of 25 cm., and could be stopped in a fraction of a second from a speed of 5000 revolutions per minute. The galvanometer was read at a distance of 10 metres, the steady current for a deflexion of one millimetre being about $10^{-5}$ ampere and the time of swing 10 seconds.

Let $e$ be the charge on the carrier in electrostatic units, $m$ its mass. Consider for the moment a steady current $i$ driven round the coil, when at rest, by a battery. If $R$ is the resistance of the coil and $a$ its total length of wire, there is a fall of potential $Ri$ down the wire, and a mean electric force of $Ri/a$ electromagnetic units in it, so that the mean driving force on a carrier is $eRi/ac$. We take $e$ as positive for the moment in order to fix ideas. Since the current is steady, the mean dragging force on a carrier, due to its collisions with the atoms of the metal, must also be $eRi/ac$. This formula may be assumed to hold for a current produced by other means, if it does not vary very rapidly. Let each element of the coil in Tolman's experiment be moving, at time $t = 0$, with velocity $v$ in the direction of its length. Let $i$ be the current due to the relative motion of the carriers $t$ seconds afterwards, and $-V$ the fall of potential down the coil. A carrier is acted on by two retarding forces, $eV/ac$ due to the electric field and $eRi/ac$ due to

the metal. But $V = Ldi/dt + Si$, where $L$ is the self-inductance and $S$ the resistance of the galvanometer circuit. The mean retarding force on a carrier inside the coil at time $t$ is therefore

$$F = \frac{eV}{ac} + \frac{eRi}{ac} = \frac{e}{ac}\left(L\frac{di}{dt} + Ri + Si\right).$$

This force ultimately reduces the velocity of the carrier from $v$ to 0, so that $\int_0^\infty F\,dt$ is equal to $mv$, the momentum destroyed. Hence

$$mv = \frac{e}{ac}\int_0^\infty \left(L\frac{di}{dt} + Ri + Si\right)dt.$$

The first term contributes nothing since $i$ vanishes both at the beginning and at the end. Writing $Q$ for the total charge sent round the galvanometer, we have therefore $mv = (R + S)\,eQ/ac$, or

$$\frac{e}{m} = \frac{acv}{(R+S)\,Q}.$$

Tolman and Stewart had an extra coil in the circuit to compensate for variation of the earth's magnetic field, and also neutralised the steady horizontal and vertical components so that slight tilting of the coil during and after rotation should not produce irregular throws. The error in a single experiment was about 50 per cent., but 100 experiments on a coil of copper wire showed that the carrier is negative, and gave as a mean numerical value $e/m = 4{\cdot}8 \times 10^{17}$. Experiments on aluminium gave $4{\cdot}6 \times 10^{17}$, and on silver $4{\cdot}4 \times 10^{17}$. These experiments, like those of Arts. 211 and 212, support the theory of electronic conduction.

**259. The origin of spectra*.** We cannot always say whether any given radiation is emitted by an atom or molecule. It is clear that band spectra are emitted only by molecules; for the spectrum of iodine vapour changes from a band spectrum to an ordinary line spectrum when the temperature changes from 500° C. to 1000° C., in accordance with the known change from diatomic to monatomic iodine. The spectrum of hydrogen, which we select for its theoretical interest (Arts. 261, 262, 270, 275), consists of

* For further information see Bohr, *The Theory of Spectra and Atomic Constitution*, and Sommerfeld, *Atombau und Spektrallinien*, 4th ed.

(1) the primary spectrum (including Balmer's series) in which all the frequencies are connected by simple relations;

(2) the secondary spectrum, a large number of lines of which only a few have hitherto been found to be connected;

(3) continuous radiation, of which part extends from the limit of Balmer's series to 2000 A.U. or less.

While a hydrogen tube, worked directly from an induction coil, shows both Balmer's series and the secondary spectrum, the former is much stronger when a condenser is placed in parallel with it. Currents of the order of 100 amperes then flow in the tube (Art. 177), and may be expected to dissociate the gas into its atoms, so that Balmer's series must be emitted by atomic hydrogen. It is observed in great perfection in stellar spectra. Wood found that when a sufficient current is sent through a very long tube, Balmer's series is emitted from the centre and the secondary spectrum from the ends or from isolated spots in the tube, where foreign matter might help hydrogen atoms, which would otherwise remain free, to recombine into molecules.

Bohr's theory of optical spectra is a generalisation of that described in Art. 239. By a normal atom we mean a nucleus surrounded by electrons in various energy levels (Art. 223), the total charge of the system being zero. Excitation of the atom consists in the removal of one or more electrons from their levels. Suppose for simplicity that one is removed. It will, as we know, return ultimately to its old level; but Bohr assumed that there are other levels, in each of which it can remain for a time, though not indefinitely. These levels, which are called optical levels, are represented by horizontal lines above the highest shown in Fig. 349. The rest of the theory of emission remains as in Art. 239: an electron can fall from an optical level of energy $W_1$ to one of energy $W_2$ and emit light of frequency $\nu$, where $h\nu = W_1 - W_2$. The lines of optical spectra, like those of Röntgen ray spectra, play a secondary part in atomic theory, the energy levels expressing the real structure. The more abstract principle, that $\nu$ is a difference of two numbers, and that frequencies of spectral lines have additive relations amongst themselves (cf. Art. 240), was foreshadowed by Balmer, and first stated generally by Ritz.

The distinction between permanent and temporary levels need

not be absolute, but nearly all the circumnuclear electrons in a normal monatomic gas will be in the lowest energy levels, the other levels having a metastability like that of a supercooled liquid. A rational and not unimportant distinction between Röntgen ray spectra and optical spectra now emerges. If the radiating electron both begins and ends in a permanent level, the spectral line is said to belong to a Röntgen ray spectrum; if not, to an optical spectrum. We also require a more general theory of excitation than that given in Art. 239. An electron need not be removed to infinity, but may go into an optical level, and may then spring back with emission of light. The distinction is that between absorption *limits* and absorption *lines*; for while all light above a certain frequency can take an electron from a level $W_2$ to infinity, only light for which $h\nu = W_1 - W_2$ will take it from $W_2$ to a higher level $W_1$. The corresponding absorption line will coincide with the bright line emitted by electrons which fall from $W_1$ to $W_2$. This theory explains the "resonance" spectra observed by Wood: sodium vapour, for example, can itself emit a $D$ line when illuminated by the light of a $D$ line. The absorption spectrum of a monatomic gas in the normal state will not contain all possible lines, but only those whose electrons start from permanent levels. Since Balmer's lines (as distinct from Lyman's lines) arise wholly from optical levels (Art. 261), they are not absorption lines of a normal hydrogen atom (to say nothing of a molecule). The great transparency of hydrogen to visible light is, therefore, in no way paradoxical.

The origin of the secondary spectrum of hydrogen is obscure, though it is emitted by molecules and has some affinity with band spectra.

According to Bohr, continuous spectra can be produced by electrons which approach an atom with finite velocity and end in an optical level. In this way they lose energy $W + T$, where $-W$ is the energy of the level and $T$ the kinetic energy of approach, and emit light of frequency $(W + T)/h$. Since $T$ may have any positive value, we have a continuous range of frequencies from $W/h$ upwards, the latter corresponding to a series limit (Art. 261). The experimental facts are thus explained. Similarly, as we have just explained, we should expect continuous absorption above the series limit, by electrons which take up more energy than is necessary

to escape from the atom, and leave it with finite velocity. Such absorption has been observed in the spectra of certain stars.

**260. Wilson's rule for the selection of atomic orbits.** We have contented ourselves hitherto with the idea of an energy level as an orbit whose total energy has to be found by experiment (Arts. 223, 240). Bohr, to whom the theory is due, defined them as specially favoured orbits (whose existence he postulated) which do not radiate energy into the ether. The ideal of atomic theory is that each orbit should be specified by a simple rule, towards which progress has been made by a compromise between Newtonian dynamics and the quantum theory. Planck, in his theory of radiation, considered an electron which can oscillate about a fixed point with frequency $\nu$, but can gain or lose energy only by amounts $h\nu$ at a time, so that its energy at any time (apart from an absolute constant) is $nh\nu$, where $n$ is a positive integer. Its amplitude can, therefore, have only one of a number of discrete values. Planck's theory has since taken a double form. On one side it has crystallised into the law that light of frequency $\nu$ is associated with an indivisible unit of energy $h\nu$, so that an electron falling from energy $W_1$ to energy $W_2$ emits light whose frequency is given by $h\nu = W_1 - W_2$ (Arts. 210, 239, 259). On the other side it has led to a more general law for specifying the favoured orbits of dynamical systems, which we now explain.

A system consisting of a nucleus and $n$ electrons is specified completely by their $3(n+1)$ coordinates; but if there are internal constraints a smaller number of quantities $q_1, q_2, \ldots q_k$ will suffice to determine their positions. These are the *generalised coordinates* of the system. The kinetic energy $T$ is a homogeneous function of the second degree in $\dot{q}_1, \dot{q}_2, \ldots \dot{q}_k$ (involving also $q_1, q_2, \ldots q_k$), and the *generalised momenta* corresponding to the coordinates $q_1, q_2, \ldots q_k$ are defined to be $p_1, p_2, \ldots p_k$, where

$$p_1 = \frac{\partial T}{\partial \dot{q}_1}, \quad p_2 = \frac{\partial T}{\partial \dot{q}_2}, \ldots, \quad p_k = \frac{\partial T}{\partial \dot{q}_k}.$$

Planck's linear oscillator, just referred to, has one coordinate $q$, its distance from the fixed point, and $p = m\dot{q}$. The mean kinetic energy of the oscillator is equal to its mean potential energy, so

that the mean kinetic energy is $\frac{1}{2}nh\nu$. Hence

$$\frac{\int T\,dt}{\int dt} = \tfrac{1}{2}nh\nu,$$

where the integrals are taken over a single period. Since $\int dt = 1/\nu$ we have simply $\int T\,dt = \frac{1}{2}nh$. Also $T = \frac{1}{2}m\dot{q}^2 = \frac{1}{2}p\dot{q}$, so that

$$\int p\,dq = nh \quad \ldots\ldots\ldots\ldots\ldots\ldots (312),$$

integration being taken over a complete cycle of values of $q$. W. Wilson generalised equation (312) into the hypothesis that for some coordinate systems

$$\int p_1 dq_1 = n_1 h, \quad \int p_2 dq_2 = n_2 h, \ \ldots, \quad \int p_k dq_k = n_k h \ldots (313),$$

where $n_1, n_2, \ldots n_k$ are positive integers. This rule is by no means general, but will suffice for the problems we have to consider.

Take the problem by which Bohr first explained Balmer's series (Art. 261). What circular orbits can an electron describe round a fixed nucleus? Let the nucleus have a charge $Ze$, and let the charge and mass of the electron be $-e$, $m$ respectively. If $a$ is the radius of the orbit and $\omega$ the angular velocity of the electron in it, we have from Newtonian mechanics $ma\omega^2 = Ze^2/a^2$, or

$$ma^3\omega^2 = Ze^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (314).$$

The generalised momentum corresponding to the angular co-ordinate $\theta$ is $ma^2\dot{\theta}$, since $T = \frac{1}{2}ma^2\dot{\theta}^2$. Equation (312) now gives $\int ma^2\dot{\theta}\,d\theta = nh$, or

$$2\pi ma^2\omega = nh \quad \ldots\ldots\ldots\ldots\ldots\ldots (315),$$

an equation first used by Nicholson and a particular case of the theorem that the moment of momentum of the electron round the nucleus is a multiple of $h/2\pi$ (see below).

From (314) and (315) we have

$$a = \frac{n^2h^2}{4\pi^2me^2Z}, \quad \omega = \frac{8\pi^3me^4Z^2}{n^3h^3} \quad \ldots\ldots\ldots\ldots (316).$$

The admissible orbits, therefore, have radii proportional to the squares of the natural numbers. The kinetic energy of an electron on the $n$th orbit is $\frac{1}{2}ma^2\omega^2 = 2\pi^2me^4Z^2/n^2h^2$ and its potential energy, measured from infinity, is $-Ze^2/a = -4\pi^2me^4Z^2/n^2h^2$. The total energy of the $n$th orbit is therefore

$$W = -\frac{2\pi^2me^4Z^2}{n^2h^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (317).$$

Wilson considered, more generally, what elliptic orbits are allowed by equations (313). Taking polar coordinates $q_1 = r$ and $q_2 = \theta$, $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$, so that $p_1 = m\dot{r}$ and $p_2 = mr^2\dot{\theta}$. The latter quantity is the moment of momentum of the electron round the nucleus, which is constant. The second of equations (313) then shows that the moment of momentum is $n_2h/2\pi$, since $\int d\theta = 2\pi$. We call $n_2$ the *azimuthal quantum number*. The first of equations (313) gives $\int m\dot{r}\,dr = n_1h$, and $n_1$ is called the *radial quantum number*. We evaluate $\int m\dot{r}\,dr$ by the theory of central orbits*. If $2l$ is the latus rectum, $\epsilon$ the eccentricity of the orbit and $u = 1/r$,

$$lu = 1 + \epsilon\cos\theta.$$

Hence
$$\int m\dot{r}\,dr = \int_0^{2\pi} m\left(\frac{dr}{d\theta}\right)^2 \dot{\theta}\,d\theta,$$

and $m\dot{\theta} = n_2hu^2/2\pi$, so that

$$n_1h = \frac{n_2h}{2\pi}\int_0^{2\pi}\left(\frac{1}{u}\frac{du}{d\theta}\right)^2 d\theta = \frac{n_2h\epsilon^2}{2\pi}\int_0^{2\pi}\frac{\sin^2\theta\,d\theta}{(1+\epsilon\cos\theta)^2}.$$

We leave it to the reader to prove that

$$\int_0^{2\pi}\frac{\sin^2\theta\,d\theta}{(1+\epsilon\cos\theta)^2} = \frac{2\pi}{\epsilon^2}\{(1-\epsilon^2)^{-\frac{1}{2}} - 1\}.$$

Hence we have finally $n_1 = n_2\{(1-\epsilon^2)^{-\frac{1}{2}} - 1\}$, or

$$1 - \epsilon^2 = \left(\frac{n_2}{n_1 + n_2}\right)^2.$$

The eccentricities of the possible orbits have, therefore, discrete values. The moment of momentum round the nucleus is

$$m(\mu l)^{\frac{1}{2}} = m\{\mu a(1-\epsilon^2)\}^{\frac{1}{2}},$$

where $2a$ is the major axis and $\mu = Ze^2/m$ the acceleration of the electron at unit distance. Hence $m^2\mu a(1-\epsilon^2) = (n_2h/2\pi)^2$, or

$$a = \frac{(n_1+n_2)^2h^2}{4\pi^2me^2Z}.$$

The total energy of the orbit is $W = -m\mu/2a$. Writing

$$n = n_1 + n_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (318),$$

we have therefore

$$W = -\frac{2\pi^2me^4Z^2}{n^2h^2} \quad \ldots\ldots\ldots\ldots\ldots (319).$$

* See Love's *Theoretical Mechanics*, 3rd Ed., pp. 41–46.

The energy depends only on $n$, which is called the *principal quantum number*, and not on $n_1$ or $n_2$ separately. Further consequences of these equations are considered in the next article.

**261. The hydrogen atom and its spectrum.** The primary spectrum of hydrogen in the visible region consists of Balmer's series, a series of lines $H_\alpha, H_\beta, H_\gamma, \ldots$ of decreasing wave-length which are heaped together more and more till they converge in a *series limit*. Balmer showed in 1885 that the wave-lengths of these lines in Ångström units could be represented, with the measurements of his day, by the formula

$$\lambda = 3645{\cdot}6\,\frac{k^2}{k^2 - n^2}, \text{ with } n = 2,\ k = 3, 4, 5, \ldots.$$

He conjectured that other series might be found for which $n = 1$ and $k = 2, 3, 4, \ldots$; $n = 3$ and $k = 4, 5, 6, \ldots$; and so on. Three lines of the first series were subsequently found by Lyman in the extreme ultra-violet, and two lines of the second series by Paschen in the infra-red.

The importance of Balmer's formula appears first in connexion with Ritz's principle, to the discovery of which it no doubt contributed. Writing $\lambda$ for the wave-length *in vacuo*, expressed in centimetres, Balmer's formula is now written

$$\frac{1}{\lambda} = R\left(\frac{1}{n^2} - \frac{1}{k^2}\right) \quad \ldots\ldots(320),$$

where

$$R = 109678{\cdot}8 \quad \ldots\ldots(321).$$

Fig. 374 shows the primary spectrum of hydrogen on the scale of $1/\lambda$, and the following table the degree of accuracy of the formula.

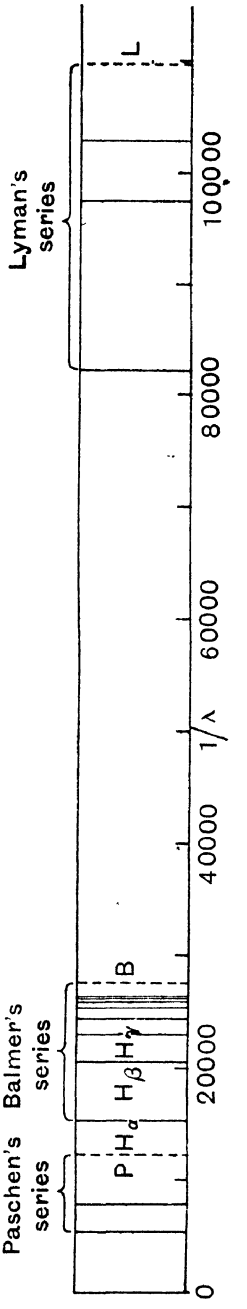



Fig. 374

*Primary spectrum of hydrogen* ($Z = 1$).

| | $k$ | Wave-lengths in Ångström units | |
|---|---|---|---|
| | | Observed | Calculated |
| Lyman's series $n = 1$ | 2 | 1216·0 | 1215·7 |
| | 3 | 1026·0 | 1025·7 |
| | 4 | 972·7 | 972·5 |
| | $\infty$ ($L$) | — | 911·8 |
| Balmer's series $n = 2$ | 3 ($H_\alpha$) | 6564·602 | 6564·623 |
| | 4 ($H_\beta$) | 4862·680 | 4862·684 |
| | 5 ($H_\gamma$) | 4341·683 | 4341·682 |
| | 6 | 4102·891 | 4102·889 |
| | 7 | 3971·194 | 3971·192 |
| | 8 | 3890·149 | 3890·147 |
| | $\infty$ ($B$) | — | 3647·062 |
| Paschen's series $n = 3$ | 4 | 18756·5 | 18756·0 |
| | 5 | 12821·1 | 12821·5 |
| | $\infty$ ($P$) | — | 8205·8 |

The formula makes Lyman's series converge to a limit $L$ for which $\lambda = 1/R = 911{\cdot}8$ A.U., Balmer's series to a limit $B$ for which $\lambda = 2^2/R = 3647{\cdot}062$ A.U., and Paschen's series to a limit $P$ for which $\lambda = 3^2/R = 8205{\cdot}8$ A.U. Enough lines have been measured in Balmer's series, both in the laboratory and in stellar spectra, to make the convergence evident.

The primary spectrum of hydrogen was first explained by Bohr in 1913. We put $Z = 1$ in equation (317) or (319), since the hydrogen atom after excitation consists of a single proton. Thus when an electron falls from a $k$-quantum orbit to an $n$-quantum orbit, whether circular or elliptic, it emits light for which

$$h\nu = \frac{2\pi^2 me^4}{h^2}\left(\frac{1}{n^2} - \frac{1}{k^2}\right), \text{ or } \frac{1}{\lambda} = \frac{2\pi^2 me^4}{h^3 c}\left(\frac{1}{n^2} - \frac{1}{k^2}\right),$$

since $\nu = c/\lambda$. This agrees with (320) if

$$R = \frac{2\pi^2 me^4}{h^3 c} \quad \ldots\ldots\ldots\ldots\ldots\ldots (322).$$

From the known values of $m$, $e$ and $h$ we find $R = 1{\cdot}09 \times 10^5$, which agrees with (321) within the limits of experimental error. Thus Bohr's theory gives the law and absolute wave-lengths of

all the lines, and proves that the hydrogen atom (and by implication every atom) is subject to a quantum mechanism.

It is now clear that Lyman's series is emitted by electrons which end in a 1-quantum orbit, and its series limit by electrons which come from infinity and have no energy there (cf. Art. 259). The electrons in Balmer's series end in 2-quantum orbits, and so on. Since the 1-quantum orbit is the only one occupied in the normal atom, the limit of Lyman's series plays the part of an absorption limit in Kossel's theory (Art. 239). Its quantum energy is the least energy necessary to remove an electron from a normal atom. In other words, the ionising potential of a hydrogen atom is *the quantum voltage corresponding to the limit of Lyman's series*, which is $12300R/10^8$ volts $= 13{\cdot}5$ volts from equation (276). A critical potential of 13·5 volts has been found for hydrogen in which free

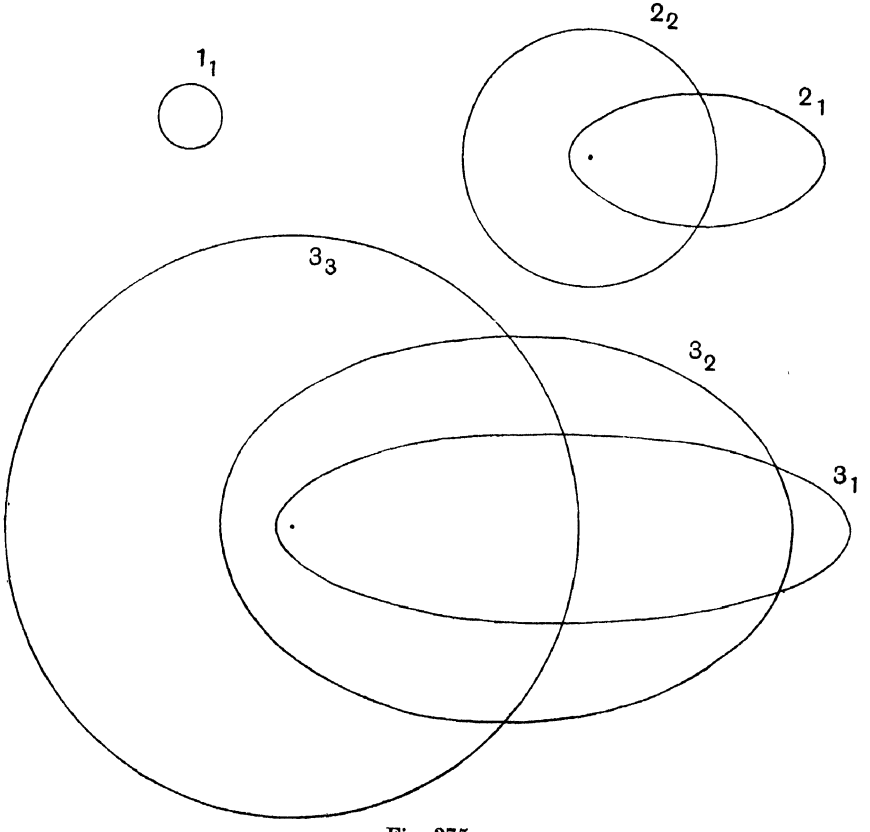



Fig. 375

atoms are present (Art. 224), and this is an independent confirmation of Bohr's theory. The diameter $2a$ of the 1-quantum orbit is calculated to be $1{\cdot}06 \times 10^{-8}$ cm., which is of the same order as that of the hydrogen molecule in the kinetic theory of gases.

An orbit which has $n$ as principal quantum number and $q$ as azimuthal quantum number is called an $n_q$-orbit. We shall see that the equality of energy of all $n$-quantum orbits is disturbed by an electric field, and also when the variation of mass with velocity is taken into account (Arts. 270, 275), two phenomena which give the strongest support to the theory of multiple orbits. The first few orbits are drawn to scale in Fig. 375, orbits for which $q = 0$ being excluded since the electron would collide with the nucleus. These orbits play a part in Bohr's general theory of atomic structure (Art. 276).

**262. Effect of the finite mass of the nucleus.** The nucleus was, in effect, taken to have an infinite mass in the theory of Art. 260. Suppose now that it has a finite mass $M$, and let $r$, $\theta$ be the polar coordinates of the electron with respect to the nucleus. The common centroid of the system is at distance $Mr/(M+m)$ from the electron and $mr/(M+m)$ from the nucleus. The kinetic energy of the system, when the centroid is at rest, is

$$T = \tfrac{1}{2}\left\{\left(\frac{M}{M+m}\right)^2 m + \left(\frac{m}{M+m}\right)^2 M\right\}(\dot{r}^2 + r^2\dot{\theta}^2)$$
$$= \tfrac{1}{2}\frac{Mm}{M+m}(\dot{r}^2 + r^2\dot{\theta}^2).$$

The potential energy being $-Ze^2/r$ as before, it follows that the energies of the stationary orbits are given by equation (319), with the substitution of $Mm/(M+m)$ for $m$. The same is true of the constant $R$ in equation (322). Writing now more definitely

$$R_\infty = \frac{2\pi^2 me^4}{h^3 c} \quad \ldots\ldots\ldots\ldots\ldots\ldots (323)$$

we see that an atom which consists after excitation of a nucleus of charge $Ze$ and mass $M$ gives out a Balmer's series for which

$$R = \frac{M}{M+m} Z^2 R_\infty \quad \ldots\ldots\ldots\ldots\ldots\ldots (324).$$

The term "orbit" is here taken in a generalised sense, since both

nucleus and electron move, and any gain or loss of kinetic energy is shared between them in a fixed ratio.

Any atom can, in theory, emit a series of Balmer's type, but must be excited by having *all* its circumnuclear electrons removed from it. The absolute wave-lengths change when we turn from a fixed nucleus to a moveable nucleus of mass $M$, and from a nuclear charge $e$ to $Ze$; the former increasing all wave-lengths in the ratio of $1 + m/M$ to 1, and the latter reducing them in the ratio of $Z^2$ to 1*. Since both electrons can be removed from a helium atom with an expenditure of energy of about 80 volts (Art. 224), a strong discharge in helium should yield Balmer's series, with a constant $R$ rather more than four times that of hydrogen. These series have been found. They correspond to those of hydrogen, with a constant

$$R_{He} = 438892{\cdot}9 = 4 \times 109723{\cdot}2 \quad \ldots\ldots\ldots(325),$$

as shown in the following table:

*Spectrum of ionised helium* ($Z = 2$).

| | $k$ | Wave-lengths in Ångström units | |
|---|---|---|---|
| | | Observed | Calculated |
| Lyman's series $n=1$ | 2 | 303·6 | 303·8 |
| | 3 | 256·3 | 256·3 |
| Lyman's series $n=2$ | 3 | 1640·2 | 1640·5 |
| | 4 | 1215·1 | 1215·2 |
| Fowler's series $n=3$ | 4 | 4687·115 | 4687·118 |
| | 5 | 3204·087 | 3204·085 |
| | 6 | 2734·153 | 2734·153 |
| | 7 | 2512·005 | 2512·003 |
| | 8 | 2386·167 | 2386·169 |
| | 9 | 2306·926 | 2306·941 |

We now write $$R_H = 109678{\cdot}8 \quad \ldots\ldots\ldots\ldots\ldots(326)$$

for the Balmer constant of hydrogen, and $m_H$ for the mass of a hydrogen nucleus. Thus

$$\frac{R_\infty}{1 + \dfrac{m}{m_H}} = 109678{\cdot}8.$$

* The $K$ series of a heavy element is, practically, its Lyman series (Art. 276).

The mass of a helium nucleus being 3·97 times that of a hydrogen nucleus (Art. 229), we have from (324) and (325)

$$\frac{R_\infty}{1+\frac{m}{3{\cdot}97m_H}} = 109723{\cdot}2.$$

Solving these equations, we find

$$R_\infty = 109738{\cdot}1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (327),$$

and

$$m/m_H = 1/1850 \quad \ldots\ldots\ldots\ldots\ldots\ldots (328).$$

The last ratio is practically the ratio of the mass of an electron to that of a hydrogen atom, and agrees well with equation (274) on p. 479. By our present theory, which is due to Bohr, it is found from measurements of wave-length only. The absolute agreement between equations (321) and (322), referred to in Art. 261, is really one between equations (323) and (327), which still holds within the errors of experiment.

**263. Experiments of Gerlach and Stern.** That atomic orbits have definite orientation in a magnetic field was shown by Gerlach and Stern as follows. A stream of silver atoms from a small electric furnace $F$ (Fig. 376) passes through a circular aperture 0·06 mm. in diameter in the platinum sheet $S_1$, and a horizontal slit 0·8 mm. long and 0·03—0·04 mm. wide in the sheet $S_2$. Beyond $S_2$ is an electromagnet, of which the lower pole-piece, marked $N$ in Figs. 376 and 377, is wedge-shaped, with its edge just below the stream. The silver atoms follow the edge for a distance of 3·5 cm., and are then deposited on a glass plate $G$. The apparatus is enclosed in a thick brass case, the pressure inside which is kept below $10^{-5}$ mm. of mercury, so that the silver atoms move outside the furnace with a constant velocity, corresponding on the average to the temperature of the furnace. Faint deposits are observed after 4 to 8 hours' exposure.

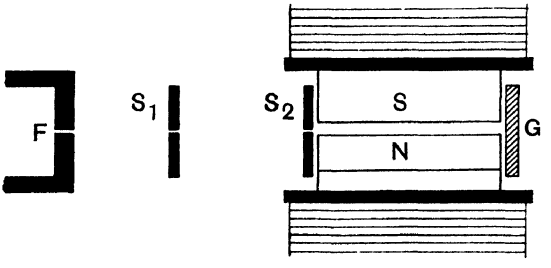


Fig. 376

The idea of Gerlach and Stern's experiment is that if the silver atoms have a magnetic moment, those whose axes are parallel to the field will tend to the strongest parts of it (Art. 113) and those, if any, whose magnetic axes are opposed to the field will tend to the weakest parts. A continuous distribution of axes would give a broadened band. The observed deposits have the general form shown in Fig. 377. In the absence of the magnetic field we have a band $AB$, slightly irregular in width from irregularity in the slit. When the field is applied the central band disappears, and instead we have *two* bands $CED$, $CFD$, one repelled and the other attracted by the edge of the magnet.

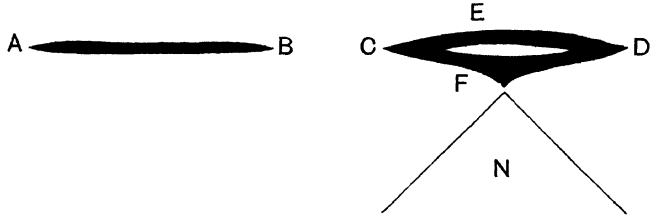



Fig. 377

The simplest explanation of these experiments is that silver atoms can take up two positions in a magnetic field, in which the magnetic axis is either parallel to the field or directly opposed to it. If $M$ is the magnetic moment of a silver atom and $F$ the magnitude of $\mathbf{D}H$, the deflecting force is $MF$ (Art. 113) and the deflexion $MFl^2/2mv^2$, where $l$ is the length of the edge of the electromagnet, $m$ the mass of a silver atom and $v$ its velocity. The deflected bands are broader than the undeflected band, since $F$ varies a little from point to point, and $v$ also varies about the mean value corresponding to the temperature of the furnace. Writing $\theta$ for this latter quantity and $\delta$ for the mean deflexion, $\delta = MFl^2/4\alpha\theta$, from which $M$ can be found. Gerlach and Stern found $F$ at the centre of the repelled stream by magnetic measurements, and estimated $M$ at $10^{-20}$ (cf. p. 613). The attracted beam is less suitable for measurement since $F$ varies greatly near the edge of the magnet, and part of the stream is pulled out into a point.

**264. Theory of the orientation of orbits in space.** Gerlach and Stern's experiments give a striking illustration of Wilson's

rules in circumstances in which they might appear uncertain. Let us assume that the magnetic field is not strong enough to deform the atomic orbits appreciably, but only points out a specially favoured direction in space which we should not be able to select otherwise, and which we take as the axis of a set of polar co-ordinates in space. Wilson's rules now show that the plane of an orbit can only make one of a number of discrete angles with the field. Let $R$, $\Theta$, $\Phi$ be the polar coordinates in space, and let $r$, $\theta$ be polar coordinates in the plane of the orbit, $\alpha$ the acute angle between the normal and the magnetic field.

The quantisation of the $r$, $\theta$ coordinates was carried out in Art. 260. We write $Q_1$, $Q_2$, $Q_3$ respectively for $R$, $\Theta$, $\Phi$, so that

$$P_1 = m\dot{R}, \quad P_2 = mR^2\dot{\Theta}, \quad P_3 = mR^2 \sin^2 \Theta\dot{\Phi}.$$

If $N_1$, $N_2$, $N_3$ are the quantum numbers corresponding to $Q_1$, $Q_2$, $Q_3$,

$$\int P_1 dQ_1 = N_1 h, \quad \int P_2 dQ_2 = N_2 h, \quad \int P_3 dQ_3 = N_3 h.$$

Obviously $R = r$, so that, by comparison with the theory of Art. 260,

$$n_1 = N_1.$$

Since $T$ must have the same value in both sets of coordinates, and since it is a quadratic function of both sets of generalised co-ordinates,

$$2T = p_1\dot{q}_1 + p_2\dot{q}_2 = P_1\dot{Q}_1 + P_2\dot{Q}_2 + P_3\dot{Q}_3.$$

Hence
$$\int p_2 dq_2 = \int P_2 dQ_2 + \int P_3 dQ_3,$$

or
$$n_2 = N_2 + N_3.$$

$P_3$ is the (constant) moment of momentum of the electron about the axis of $z$, and $p_2$ the total moment of momentum of the electron. Hence $P_3 = p_2 \cos\alpha$, so that the third quantum condition becomes $p_2 \cos\alpha \int d\Phi = N_3 h$, or $2\pi p_2 \cos\alpha = N_3 h$. Since $2\pi p_2 = n_2 h = (N_2 + N_3)\, h$, we have

$$\cos\alpha = \frac{N_3}{N_2 + N_3}.$$

The principal quantum number is now

$$n = n_1 + n_2 = N_1 + N_2 + N_3,$$

and all orbits with the same principal quantum number have the same energy, as before. $N_3$ is called the *equatorial quantum number*

to distinguish it from the azimuthal quantum number $n_2$. The inclination of the orbit varies with the way in which the azimuthal quantum number is subdivided into integers; thus if $n_2 = 1$ we can have either

$$N_2 = 1, \quad N_3 = 0, \quad \cos\alpha = 0,$$

or

$$N_2 = 0, \quad N_3 = 1, \quad \cos\alpha = 1.$$

Hence $\alpha = 0$ or $\frac{1}{2}\pi$, so that the plane of the orbit either contains the magnetic field or is normal to it, with either direction of circulation. Gerlach and Stern's observations show that there is no orbit for which $\alpha = \frac{1}{2}\pi$, and we shall take this as an indication that the equatorial quantum number cannot be zero, either in an electric or magnetic field (Arts. 267, 270). It remains to show that the magnetic moment, as measured by Gerlach and Stern, does actually correspond to $n_2 = 1$.

Let $A$ be the area of the orbit. Since $mr^2\dot{\theta} = n_2 h/2\pi$ (Art. 260), areas are described at a rate $n_2 h/4\pi m$, so that the orbit is described $n_2 h/4\pi mA$ times a second. An electron therefore passes any particular point of the orbit $n_2 h/4\pi mA$ times a second, and constitutes on the average a current $i = en_2 h/4\pi mcA$ electromagnetic units. The equivalent magnetic moment (Art. 81) is

$$iA = en_2 h/4\pi mc = 9{\cdot}2 \times 10^{-21}\, n_2.$$

Gerlach and Stern's experiments could be explained by a single effective orbit, of azimuthal quantum number unity, in each silver atom, the other orbits together having no magnetic moment. This is no doubt the orbit of the single electron which a silver atom has in its highest energy level (p. 492). Gerlach and Cillier found similar patterns with copper and gold, but no magnetic effect on the pattern of lead, the atoms of which seem to be non-magnetic. Bismuth gave two beams, an undeflected and an attracted beam; while tin and thallium gave broadened bands which could not be resolved. These experiments introduce a new method of studying the structure of the atom, from which more definite information will be gleaned in time.

**265. Bohr's correspondence principle.** We have hitherto considered only the frequency of the waves emitted when an electron falls from one orbit to another. Bohr showed that an idea

of the intensity and polarisation can be got by considering orbits of large quantum numbers. Let us see how far the quantum theory of radiation can be reconciled with that of a Hertzian oscillator (Art. 197), the distinguishing feature of which is that the waves have the frequency of the oscillator and any harmonics it may possess. The frequency emitted when an electron falls from an $n$-quantum orbit to an $(n-p)$-quantum orbit is

$$\nu = \frac{2\pi^2 me^4 Z^2}{h^3}\left\{\frac{1}{(n-p)^2} - \frac{1}{n^2}\right\},$$

and if $p$ is finite and $n$ large $\nu = 4\pi^2 pme^4Z^2/n^3h^3$ approximately. An easy calculation shows this to be $p$ times the frequency of revolution of the electron in its orbit. In the region of large quantum numbers (where the quantum of energy is small) we have, therefore, Hertzian emission, the $p$th harmonic being emitted when the quantum number changes by $p$. The Hertzian theory goes further and states, firstly that the $p$th harmonic will not be emitted unless it is present in the orbital motion, and secondly, that the waves have a well-defined polarisation. Bohr sought to make as much use as possible of these results, applying them to finite as well as large quantum numbers. Though the Hertzian theory does not give the correct frequency of the radiation in general, it is still (in the absence of a better theory) allowed to decide on the presence or absence of harmonics and on the polarisation. Since the co-ordinates of an electron moving in a *circular* $n$-quantum orbit are of the form $A\cos(\omega t + \epsilon)$, the radiation has no harmonics, and $p = 1$. Such an electron can, therefore, fall only into an $(n-1)$-quantum orbit, possibly only into a circular orbit. Hence Bohr's theory limits the possible transitions from one orbit into another (cf. Art. 240), a limitation which is essential to a proper explanation of the Zeeman and Stark effects and of the fine structure of spectral lines (Arts. 267, 270, 275). The example just given is not, however, typical since the principal and subordinate quantum numbers are not distinguished. We proceed more generally as follows.

We shall show in Art. 267 that a uniform magnetic field parallel to the axis of $z$ makes any central orbit rotate with small angular velocity $\omega$ about that axis. Let us assume that the admissible orbits are those of Art. 264 with the rotation superposed; in

particular, that $mR^2 \sin^2 \Theta\dot{\Phi} = N_3 h/2\pi$, where $R$, $\Theta$, $\Phi$ are polar coordinates referred to axes rotating with the angular velocity $\omega$. The kinetic energy of a rotating orbit is

$$T = \tfrac{1}{2}m \{\dot{R}^2 + R^2\dot{\Theta}^2 + R^2 \sin^2 \Theta (\dot{\Phi} + \omega)^2\},$$

or to the first order in $\omega$,

$$T = T_0 + mR^2 \sin^2 \Theta\dot{\Phi}\omega = T_0 + N_3 h\omega/2\pi,$$

where $T_0$ is the kinetic energy of the same orbit without rotation. If $W$, $W_0$ are the corresponding total energies,

$$W = W_0 + N_3 h\omega/2\pi.$$

An electron falling from a rotating $(K_1, K_2, K_3)$ orbit to a rotating $(N_1, N_2, N_3)$ orbit emits radiation of frequency

$$\nu = \nu_0 + (K_3 - N_3)\,\omega/2\pi \quad \ldots\ldots\ldots\ldots (329),$$

where $\nu_0$ is the frequency emitted in the absence of rotation. Values of $K_3 - N_3$ greater than unity are assumed to correspond, as before, with harmonics in the orbital motion. The effect of rotation is, however, to introduce a new frequency $\omega/2\pi$ into all the orbital motions, *without any harmonics*. We have, therefore, the suggestion that $K_3 - N_3 = \pm 1$ or 0, the latter corresponding (for large quantum numbers at least) with an orbit which would not be affected by rotation, that is a motion along the axis of $z$. In other words, transitions for which $K_3 = N_3$ give out waves like those of an ideal Hertzian oscillator (Art. 197), invisible along the axis and polarised at right angles to the equatorial plane at all points on it, "polarisation" referring as usual to the electric force in the wave.

The transition $K_3 = N_3 + 1$ has been noticed already in connexion with circular orbits, which suggest (more doubtfully) that the waves may be such as an electron would emit, according to Hertz's theory, while describing a circle in the equatorial plane. Such an electron can be replaced by two Hertzian oscillators parallel to the axes of $x$ and $y$, just as a uniform circular motion can be replaced by two sinoidal vibrations. The formulae of Art. 197 then show that the radiation is circularly polarised in a right-handed sense at (distant) points on the axis of $z$, and is polarised in the equatorial plane at points in that plane. We shall assume that the transition $K_3 = N_3 - 1$ gives left-handed circular polarisation at points on the axis, and plane polarisation as before at points

in the equatorial plane. The vagueness of the argument need not trouble us since the correspondence principle itself has not yet reached a final form. Rules for limiting the possible transitions between orbits, and for the polarisation of the waves, were first enunciated by Rubinowicz on quite different grounds. They are supported by the Zeeman effect, the theory of which (Art. 267) should be read in connexion with this article.

The preceding rules will be assumed to hold also in an electric field (cf. Art. 270). In an atom subject to no external force, it is reasonable to assign to the azimuthal quantum number $n_2$ the part played previously by the equatorial quantum number $N_3$, with one difference. Since the original orbit contains nothing to correspond to a Hertzian oscillator *perpendicular* to its plane, transitions for which $k_2 = n_2$ should be absent. Thus the azimuthal quantum number can change only by one unit (cf. Art. 275).

Assuming for the moment that the selection principle is rigorously true in a magnetic field, we should expect it to be relaxed when an electric field acts on the atom as well; for the rotation theorem will then break down (Art. 267). The equatorial quantum number may change by more than one unit, giving (in a weak field) weak extra lines. Such lines have been observed in the Zeeman effect and in Fowler's series under high resolution (Art. 275).

**266. The Zeeman effect.** Zeeman observed in 1896 that the yellow lines of sodium were broadened slightly when the flame was placed between the poles of a strong electromagnet. He suspected that it was due to the action of the field on moving charges within the atom, and at Lorentz's suggestion examined the edges of the broadened line for traces of circular or plane polarisation, according to whether the light was observed along or at right angles to the lines of force. Zeeman found afterwards that three separate lines could be produced from a single line, each having a definite polarisation. The normal Zeeman effect, as it is called, is as follows.

Let $A$ be the original spectral line, shown diagrammatically in Fig. 378. The light emitted along the lines of magnetic force contains two spectral lines $B$, $C$ as shown. Of these $B$ is circularly polarised in a right-handed direction with respect to the magnetic force, and $C$ in a left-handed direction. The light emitted at right

angles to the lines of force contains three equidistant lines $D$, $E$, $F$, of which $D$ and $F$ are in the same position as $B$ and $C$, and $E$ is in the position of the original line $A$. All three lines are plane-polarised, $E$ parallel to the lines of magnetic force and $D$, $F$ at right angles to them. The term "polarisation," here and elsewhere, has reference to the direction of vibration of the electric force. These effects are called the longitudinal and transverse effects respectively. If the separation is insufficient we get a broadened line polarised partially at the edges, as in Zeeman's original observations.

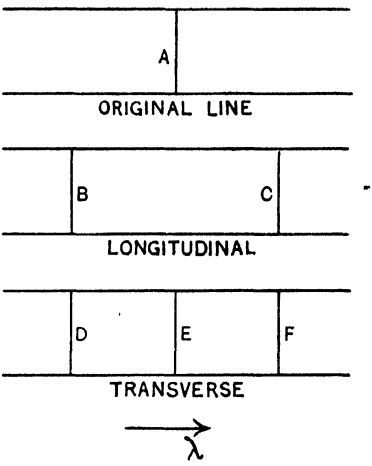


Fig. 378

The Zeeman effect is a small effect, 10,000 gausses giving a separation of the order of 1 A.U. It is measured with a concave grating and an electromagnet of the type shown in Fig. 150. For demonstration on a small scale we can use a Lummer-Gehrcke plate* in conjunction with an ordinary spectroscope (Fig. 379).

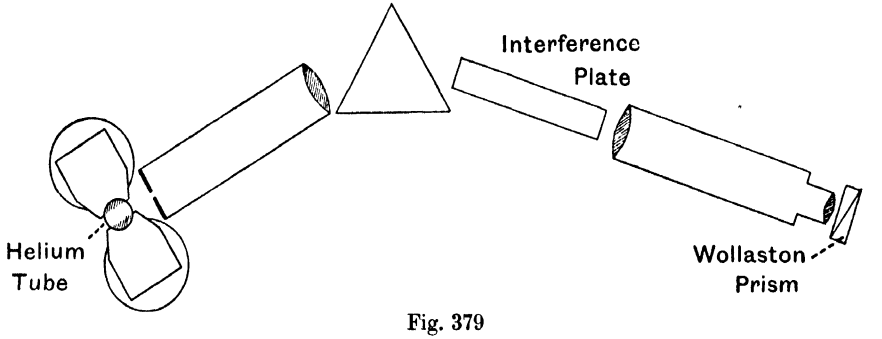


Fig. 379

The spectrum tube lies with its constriction between the poles of a small electromagnet, and the slit of the collimator is wide. After passing through the prism, the light falls on the interference plate, whose reflecting planes are horizontal, emerges at nearly grazing angle and passes into the telescope. The usual spectrum is seen on

* See Wood's *Physical Optics*, 2nd Ed., p. 282.

looking into the eyepiece, but each line is crossed by a series of horizontal fringes produced by the Lummer-Gehrcke plate. A suitable line to observe is the yellow line of helium, $\lambda = 5876$ A.U. The fringes are tripled in a magnetic field. To find the polarisations we place a Wollaston prism behind the eyepiece and turn it until there are two sets of fringes in register with each other side by side, one of light polarised parallel to the lines of force and the other of light polarised at right angles to the lines of force. The first set of fringes are unaffected by the magnetic field and the second set are doubled, the original line disappearing. This agrees with what has been said of the normal effect.

The simple triplet is exhibited by lines of Balmer's type (Art. 267) but is not the only, or usual, type of resolution. Spectroscopists distinguish between single lines, doublets and triplets, the latter being pairs and triads of lines in numerical relation to each other, sometimes close together and sometimes not. Doublets and triplets have complex resolutions: the $D_1$ line of sodium, for example, is resolved into four lines, none of which coincides with the original line, and the $D_2$ line into six. The complex Zeeman effects are subject to the following rules:

(1) *Preston's rule.* Lines of the same series show the same pattern, and if $\nu$ is the frequency of the original line and $\nu + \delta\nu$ that of a line in the pattern, $\delta\nu$ is the same for all lines of the series. Like other important scientific rules, Preston's rule can be inverted and used as a test for sorting unknown lines into series.

(2) *Runge's rule.* In a given field the value of $\delta\nu$ is either that of a Balmer line or $p/q$ times as large, where $p$ and $q$ are *small* integers. Thus $q = 2$ for certain lines of mercury, cadmium and zinc, $q = 3$ for either $D$ line of sodium. The rule seems to be weakened by the necessity of taking $q = 15$ for certain lines, but the factorisation $15 = 3 \times 5$ is significant (Art. 267).

The preceding rules apply to magnetic fields so weak that the magnetic separation is small compared with the natural separation of the doublet, triplet, etc. Paschen and Back found that all simplicity is lost when the two separations are comparable, the lines influencing each other; finally, when the magnetic separation is large compared with the natural separation, we have the normal Zeeman effect, both in pattern and magnitude. Obviously, there-

fore, the same electron takes a part in the emission of all the lines of a doublet or triplet. On the other hand, the patterns of independent lines which are accidentally close together are merely superposed.

**267. Theory of the normal Zeeman effect.** Let an electron, whose charge is $-e$, be moving under a central force, on which a uniform magnetic field $\mathbf{H}$ is superposed. If $\mathbf{r}$ is the vector from the centre to the electron and $V$ the potential energy, the equation of motion is

$$m\ddot{\mathbf{r}} = -\mathbf{D}V - \frac{e}{c}V\dot{\mathbf{r}}\mathbf{H},$$

or

$$m\left(\ddot{\mathbf{r}} + \frac{e}{mc}V\dot{\mathbf{r}}\mathbf{H}\right) = -\mathbf{D}V \quad \ldots\ldots\ldots\ldots (330).$$

Let $\mathbf{r}$ now be specified by reference to axes rotating with angular velocity $\mathbf{a}$. The velocity of the electron is found by compounding $\dot{\mathbf{r}}$ with the velocity it would have in relative rest, which is easily seen to be $V\mathbf{ar}$. The total velocity is therefore $\dot{\mathbf{r}} + V\mathbf{ar}$. Writing $\mathbf{v}$ for this velocity, the acceleration is $\dot{\mathbf{v}} + V\mathbf{av}$. Hence the acceleration is $\ddot{\mathbf{r}} + V\mathbf{a}\dot{\mathbf{r}} + V\mathbf{a}(\dot{\mathbf{r}} + V\mathbf{ar})$. If the terms in $\mathbf{H}$ and $\mathbf{a}$ are relatively small, we can leave $\dot{\mathbf{r}}$ unaltered in the second term of (330), and replace $\ddot{\mathbf{r}}$ by $\ddot{\mathbf{r}} + 2V\mathbf{a}\dot{\mathbf{r}}$. Equation (330) then becomes

$$m\left(\ddot{\mathbf{r}} + 2V\mathbf{a}\dot{\mathbf{r}} + \frac{e}{mc}V\dot{\mathbf{r}}\mathbf{H}\right) = -\mathbf{D}V,$$

or $m\ddot{\mathbf{r}} = -\mathbf{D}V$ if we take $\mathbf{a}$ equal to $e\mathbf{H}/2mc$. If, therefore, we take axes rotating round the direction of $\mathbf{H}$ with angular velocity

$$\omega = eH/2mc \quad \ldots\ldots\ldots\ldots\ldots (331),$$

the relative motion is a central orbit under the original potential energy $V$. This interpretation, due to Larmor, has been used in Art. 265. Since $e/mc = 1{\cdot}8 \times 10^7$, $\omega$ is of the order $10^{12}$ at most, while the orbital angular velocities given by equation (316) are of the order $10^{17}$, so that the impressed angular velocity is relatively small.

Equations (329) and (331) contain the theory of the normal Zeeman effect, $\nu_0$ being the frequency of the original spectral line. The appearance of new components is due to the fact that the total energy now depends on more than one quantum number (cf. Arts. 270, 275). If $\lambda$ is the original wave-length and $\lambda + \delta\lambda$ that of one of the components, $\nu_0 = c/\lambda$ and $\nu = c/(\lambda + \delta\lambda)$, so that

$$\frac{\delta\lambda}{H\lambda^2} = -\frac{(K_3 - N_3)\,e}{4\pi mc^2} \quad \ldots\ldots\ldots\ldots (332).$$

The transition $K_3 = N_3$ gives a line of wave-length $\lambda$ polarised parallel to the lines of force when seen transversely, and invisible when seen longitudinally. This is the line $E$ in Fig. 378. The transition $K_3 = N_3 + 1$ gives right-handed circular polarisation when seen longitudinally, and polarisation perpendicular to the lines of force when seen transversely (lines $B$, $D$), while the transition $K_3 = N_3 - 1$ gives left-handed circular polarisation when seen longitudinally, and plane polarisation as before when seen transversely (lines $C$, $F$). The normal Zeeman effect is therefore explained in full; or if we prefer, the frequencies are explained and the number and polarisation of the lines strengthen the rules arrived at in Art. 265. The theoretical wave-length of the lines $C$, $F$ is given by

$$\frac{\delta\lambda}{H\lambda^2} = \frac{e}{4\pi mc^2} = 4{\cdot}73 \times 10^{-5} \ldots\ldots\ldots\ldots (333).$$

Croze measured the patterns of $H_\alpha$, $H_\beta$, $H_\gamma$ and $H_\delta$ in a field of 24,000 gausses, and found $\delta\lambda/H\lambda^2 = 4{\cdot}73$, $4{\cdot}65$, $4{\cdot}74$ and $4{\cdot}73 \times 10^{-5}$ respectively. Lohmann found $\delta\lambda/H\lambda^2 = 4{\cdot}70 \times 10^{-5}$ for a helium line: the theoretical separation is independent of the nuclear charge.

The normal Zeeman effect should be exhibited not only by lines of Balmer's type, but also by those lines (if any) which are due to one electron in the field of a set of nuclei on the same line of magnetic force. An oblique electric field would invalidate Larmor's theorem, and give harmonics in the disturbed motion. Bohr and Sommerfeld observed weak components of two and three times the normal separation in a photograph of the Zeeman effect in hydrogen, taken by Paschen and Back. These components correspond to transitions in which $K_3 - N_3 = \pm 2$ and $\pm 3$. A similar failure will be noticed in Art. 275—a failure not of the correspondence principle but of the conditions assumed in applying it.

In the theory of complex Zeeman effects we notice only a general remark of Sommerfeld's. The energy of an $(N_1, N_2, N_3)$ orbit is increased by $N_3 ehH/4\pi mc$ in the magnetic field, according to the simple theory. Runge's rule suggests that it is in general increased by $\alpha/\beta$ times this amount, where $\alpha$ and $\beta$ are integers. The energy of a $(K_1, K_2, K_3)$ orbit is increased similarly by $\gamma/\delta$ times the normal amount. Hence with the notation of Art. 266,

$$\frac{p}{q} = \frac{\gamma}{\delta} - \frac{\alpha}{\beta},$$

with the suggestion that the denominator $q = 15$ corresponds to $\beta = 3$ and $\delta = 5$, or *vice versâ*, neither being as high as 15. The essential validity of Runge's rule is therefore strengthened.

**268. The Stark effect.** The effect of electric fields on the hydrogen spectrum was observed by Stark in 1913. The cathode $C$ of a discharge tube (Fig. 380) is perforated with holes about 1 mm.

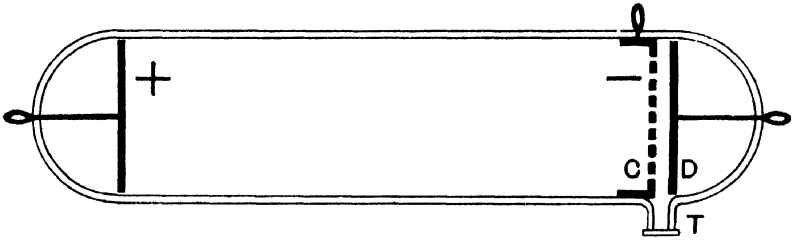



Fig. 380

in diameter, through which positive rays pass. A parallel plate $D$ is placed 1·5 mm. behind $C$ and kept at 15,000 volts above or below it, so that the atoms are subjected, while radiating, to a field of the order of 100,000 volt/cm. The rays are observed through a side tube $T$ covered with a plate of optical glass, whereby any effect on the spectrum is separated from the Doppler effect, which is observed only when the rays approach or recede from the spectroscope (Art. 228). The Stark effect as so observed is analogous to the transverse Zeeman effect. Stark also observed a longitudinal effect with an apparatus resembling that of Fig. 331, in which the magnetic parts are omitted and the plate $C_2$ perforated for observation: a high discharge potential is used so that the rays do not acquire much lateral velocity in passing between the plates.

Fig. 381 shows the transverse effect in hydrogen. As in the transverse Zeeman effect, there are components polarised parallel to the field and components polarised at right angles to it, which are shown separately in the figure. The pattern expands and contracts in the ratio of the applied electric force, but is different for each line of Balmer's series, increasing steadily in complexity as we go on. Notwithstanding this complexity, the position and polarisation of all the main lines are accounted for by the theory of the next article, according to which each component arises from

a different pair of elliptic orbits in space, which have no longer the same energy in an electric field.

In the longitudinal effect the upper set of lines are absent and the lower set are unpolarised.

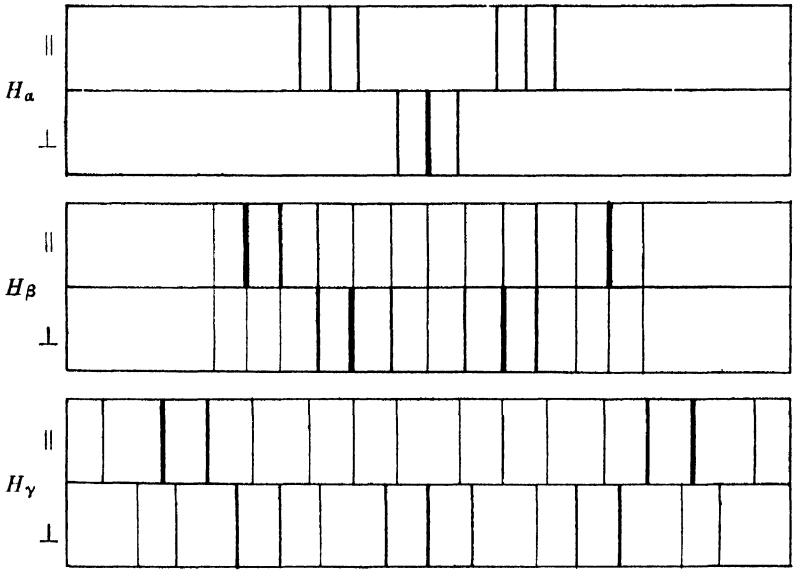



Fig. 381

**269. Quantisation of orbits in an electric field.** Schwarzschild first showed how to quantise the orbit of an electron in the field of a charge $Ze$ with a uniform electric field superposed on it, by using parabolic coordinates. Let the field be of strength $E$ parallel to the axis of $z$. Take polar coordinates $R$, $\Theta$, $\Phi$ in space, and then write $R\cos\Theta = \frac{1}{2}(\xi^2 - \eta^2)$, $R\sin\Theta = \xi\eta$,

where $\xi$ and $\eta$ are positive. The position of the electron is specified completely by $\xi$, $\eta$, $\Phi$, and $R = \frac{1}{2}(\xi^2 + \eta^2)$. Since $\frac{1}{2}(\xi^2 - \eta^2)$ and $\xi\eta$ are rectangular coordinates in the plane $\Phi = \text{const.}$, and the velocity perpendicular to the plane is $R\sin\Theta\dot{\Phi}$, the kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}m\{(\xi\dot{\xi} - \eta\dot{\eta})^2 + (\eta\dot{\xi} + \xi\dot{\eta})^2 + \xi^2\eta^2\dot{\Phi}^2\} \\ &= \tfrac{1}{2}m\{(\xi^2 + \eta^2)(\dot{\xi}^2 + \dot{\eta}^2) + \xi^2\eta^2\dot{\Phi}^2\}. \end{aligned}$$

The potential energy is

$$V = -\frac{Ze^2}{R} + eER\cos\Theta = -\frac{2Ze^2}{\xi^2 + \eta^2} + \tfrac{1}{2}eE(\xi^2 - \eta^2),$$

and the total energy of the orbit is

$$W = \tfrac{1}{2}m(\xi^2+\eta^2)(\dot\xi^2+\dot\eta^2) + \tfrac{1}{2}m\xi^2\eta^2\dot\Phi^2 - \frac{2Ze^2}{\xi^2+\eta^2} + \tfrac{1}{2}eE(\xi^2-\eta^2) \quad \ldots(334).$$

The generalised momenta corresponding to the coordinates $q_1 = \xi$, $q_2 = \eta$, $q_3 = \Phi$ are

$$p_1 = m(\xi^2+\eta^2)\dot\xi,\ p_2 = m(\xi^2+\eta^2)\dot\eta,\ p_3 = m\xi^2\eta^2\dot\Phi \quad \ldots(335).$$

We write Lagrange's equations of motion in the form

$$\frac{dp_1}{dt} = \frac{\partial T}{\partial \xi} - \frac{\partial V}{\partial \xi},\quad \frac{dp_2}{dt} = \frac{\partial T}{\partial \eta} - \frac{\partial V}{\partial \eta},\quad \frac{dp_3}{dt} = \frac{\partial T}{\partial \Phi} - \frac{\partial V}{\partial \Phi}.$$

The last equation shows that $p_3$ is constant, a known result since $p_3$ is the moment of momentum round the axis of $z$. The first equation gives

$$\frac{dp_1}{dt} = m\xi(\dot\xi^2+\dot\eta^2) + m\xi\eta^2\dot\Phi^2 - \frac{4Ze^2\xi}{(\xi^2+\eta^2)^2} - eE\xi.$$

Writing equation (334) in the form

$$\frac{2W\xi}{\xi^2+\eta^2} = m\xi(\dot\xi^2+\dot\eta^2) + \frac{m\xi^3\eta^2\dot\Phi^2}{\xi^2+\eta^2} - \frac{4Ze^2\xi}{(\xi^2+\eta^2)^2} + \frac{2E(\xi^2-\eta^2)\xi}{\xi^2+\eta^2}$$

and subtracting we have, with the aid of (335),

$$\frac{dp_1}{dt} = \frac{2W\xi}{\xi^2+\eta^2} + \frac{{p_3}^2}{m\xi^3(\xi^2+\eta^2)} - \frac{2eE\xi^3}{\xi^2+\eta^2}.$$

Since

$$\frac{dt}{d\xi} = \frac{m(\xi^2+\eta^2)}{p_1},$$

$$\frac{p_1 dp_1}{d\xi} = 2mW\xi + \frac{{p_3}^2}{\xi^3} - 2meE\xi^3.$$

Hence $p_1$ is a function of $\xi$ only, given by

$${p_1}^2 = 2mW\xi^2 - \frac{{p_3}^2}{\xi^2} - meE\xi^4 + A \quad \ldots\ldots(336),$$

and similarly

$${p_2}^2 = 2mW\eta^2 - \frac{{p_3}^2}{\eta^2} + meE\eta^4 + B \quad \ldots\ldots(337).$$

The constants $A$ and $B$ are not independent, for if we add the last two equations and substitute from (335) we have

$$m^2(\xi^2+\eta^2)^2(\dot\xi^2+\dot\eta^2) = 2mW(\xi^2+\eta^2) - m^2(\xi^2+\eta^2)\xi^2\eta^2\dot\Phi^2 - meE(\xi^4-\eta^4) + A + B,$$

which from (334) reduces to

$$A + B = 4me^2Z \quad \ldots\ldots\ldots\ldots (338).$$

The quantum conditions are

$$\int p_1 d\xi = N_1 h, \quad \int p_2 d\eta = N_3 h, \quad 2\pi p_3 = N_3 h.$$

We shall evaluate the first integral on the assumption that $E$ is small. When $E = 0$ the orbits are ellipses, so that $\xi$ oscillates between two finite values. This must also hold when $E$ is small, and the turning values are, from (335), roots of the equation $p_1 = 0$. Hence the equation $p_1 = 0$, considered as a cubic in $\xi^2$, has two positive roots $\alpha$ and $\beta$, in addition to a root which is infinite when $E = 0$. If $\alpha < \beta$ we integrate from $\xi^2 = \alpha$ to $\xi^2 = \beta$ taking the positive value of $p_1$ from equation (336), and from $\beta$ back to $\alpha$ taking the negative value. The two halves of the integral are equal, so that the first quantum condition becomes

$$\int_\alpha^\beta \frac{(-p_3^2 + Ax + 2mWx^2 - meEx^3)^{\frac{1}{2}}\, dx}{x} = N_1 h,$$

where the variable of integration has been changed from $\xi$ to $\xi^2$. Keeping only the first power of $E$,

$$\int_\alpha^\beta \frac{(2mWx^2 + Ax - p_3^2)^{\frac{1}{2}}\, dx}{x} - \frac{meE}{2} \int_\alpha^\beta \frac{x^2 dx}{(2mWx^2 + Ax - p_3^2)^{\frac{1}{2}}} = N_1 h.$$

Since the first integrand is small at both limits, and the second integral is finite over any finite range, we can take $\alpha$ and $\beta$ to be the roots of the equation $2mWx^2 + Ax - p_3^2 = 0$ without incurring any error of order $E$. Thus

$$(-2mW)^{\frac{1}{2}} \int_\alpha^\beta \frac{(\beta - x)^{\frac{1}{2}} (x - \alpha)^{\frac{1}{2}}\, dx}{x} - \frac{meE}{2(-2mW)^{\frac{1}{2}}} \int_\alpha^\beta \frac{x^2 dx}{(\beta - x)^{\frac{1}{2}} (x - \alpha)^{\frac{1}{2}}} = N_1 h.$$

The integrals are respectively

$$\tfrac{1}{2}\pi(\alpha + \beta) - \pi(\alpha\beta)^{\frac{1}{2}} \text{ and } \tfrac{1}{8}\pi(3\alpha^2 + 2\alpha\beta + 3\beta^2),$$

and the first quantum condition is therefore

$$\frac{\pi A}{2(-2mW)^{\frac{1}{2}}} - \frac{N_3 h}{2} - \frac{\pi meE}{16(-2mW)^{\frac{3}{2}}} \left( 3A^2 - 2mW - \frac{N_3^2 h^2}{\pi^2} \right) = N_1 h.$$

$A$ can be replaced by its approximate value

$$(-2mW)^{\frac{1}{2}} (2N_1 + N_3)\, h/\pi$$

in the term involving $E$, and thus we have, correct to the first power of $E$,

$$A - \frac{(-2mW)^{\frac{1}{2}}(2N_1+N_3)h}{\pi} - \frac{meh^2(12N_1^2+12N_1N_3+N_3^2)E}{8\pi^2(-2mW)} = 0.$$

Similarly

$$B - \frac{(-2mW)^{\frac{1}{2}}(2N_2+N_3)h}{\pi} + \frac{meh^2(12N_2^2+12N_2N_3+N_3^2)E}{8\pi^2(-2mW)} = 0,$$

so that equation (338) gives

$$4me^2Z - \frac{2(-2mW)^{\frac{1}{2}}nh}{\pi} + \frac{3meh^2nn'E}{2\pi^2(-2mW)} = 0,$$

where $$n = N_1 + N_2 + N_3, \quad n' = N_2 - N_1 \quad \ldots\ldots(339).$$

This is the equation for the total energy, which we treat as before, using the first approximation for $(-2mW)^{\frac{1}{2}}$ in the last term. We have finally

$$W = -\frac{2\pi^2me^4Z^2}{n^2h^2} - \frac{3h^2nn'E}{8\pi^2meZ} \quad \ldots\ldots\ldots(340).$$

Here $N_3$ is still the equatorial quantum number, but $N_1$ and $N_2$ have not the same meaning as in Art. 264. We now show how equation (340) accounts for the Stark effect.

**270. Theory of the Stark effect.** When an electron falls into the orbit (340) from one in which $k$ and $k'$ take the place of $n$ and $n'$, the frequency emitted is

$$\nu = \frac{2\pi^2me^4Z^2}{h^3}\left(\frac{1}{n^2} - \frac{1}{k^2}\right) - \frac{3hE}{8\pi^2meZ}(kk' - nn').$$

The second term, as in Art. 267, determines the separation of the lines, the wave-lengths of the components into which a line of wave-length $\lambda$ is split up being $\lambda + \delta\lambda$, where

$$\frac{\delta\lambda}{\lambda^2} = \frac{3hE}{8\pi^2meZc}(kk' - nn') \ldots\ldots\ldots(341).$$

A first success of the formula is to explain why the number of components change as we pass from one line of a series to another, since the number of possible values of $k'$ increases with $k$. The correspondence principle only allows transitions for which $N_3 = K_3$ or $K_3 \pm 1$, the former giving lines polarised parallel to the field (with transverse observation) and the latter lines polarised at right angles to the field (Art. 265). Neglecting states for which $K_3$ or $N_3$

vanishes (see Art. 264), we find that sixteen transitions correspond to the $H_\alpha$ line of hydrogen, as shown in the following table:

$H_\alpha$ ($k = 3$, $n = 2$).

| | $K_1K_2K_3$ | $N_1N_2N_3$ | $k'$ | $n'$ | $kk' - nn'$ |
|---|---|---|---|---|---|
| ∥ component $K_3 = N_3$ | 201 | 011 | −2 | 1 | −8 |
| | 201 | 101 | −2 | −1 | −4 |
| | 102 | 002 | −1 | 0 | −3 |
| | 111 | 011 | 0 | 1 | −2 |
| | 111 | 101 | 0 | −1 | 2 |
| | 012 | 002 | 1 | 0 | 3 |
| | 021 | 011 | 2 | 1 | 4 |
| | 021 | 101 | 2 | −1 | 8 |
| ⊥ component $K_3 = N_3 + 1$ | 102 | 011 | −1 | 1 | −5 |
| | 102 | 101 | −1 | −1 | −1 |
| | 003 | 002 | 0 | 0 | 0 |
| | 012 | 011 | 1 | 1 | 1 |
| | 012 | 101 | 1 | −1 | 5 |
| ⊥ component $K_3 = N_3 - 1$ | 201 | 002 | −2 | 0 | −6 |
| | 111 | 002 | 0 | 0 | 0 |
| | 021 | 002 | 2 | 0 | 6 |

Similar tables can be constructed for $H_\beta$ and $H\gamma$. We next calculate the separations in absolute measure, for the field of 104000 volt/cm. used by Stark. Putting $Z = 1$ and remembering that $E$ is 104000/300 electrostatic units, equation (341) gives

$$\delta\lambda = 6{\cdot}7 \times 10^{-8}\,\lambda^2\,(kk' - nn'),$$

or

$$\begin{aligned}\delta\lambda &= 2{\cdot}89\,(kk' - nn') \text{ A.U. for } H_\alpha,\\ &= 1{\cdot}58\,(kk' - nn') \quad ,, \quad H_\beta,\\ &= 1{\cdot}26\,(kk' - nn') \quad ,, \quad H_\gamma.\end{aligned}$$

The following table gives the observed separations in Ångström units for these three lines, and the calculated values for those lines admitted by the principles laid down above.

Of the eight ∥ components of $H_\alpha$ indicated by the theory six are observed, and of the seven ⊥ components three are observed. Of the eleven components of $H_\beta$, of either kind, nine are observed: six observed components of $H_\beta$ are not allowed for by the theory, but Stark regarded four of these as doubtful. All the fourteen ∥ components and the thirteen ⊥ components of $H_\gamma$, and no more, are indicated by the theory; and a similar agreement holds for

*Transverse Stark effect in a field of* 104000 *volt/cm.*

| | Observed | Calculated | | Observed | Calculated | | Observed | Calculated |
|---|---|---|---|---|---|---|---|---|
| | — | −23·1 | | — | −22·2 | | −29·4 | −27·7 |
| | −11·5 | −11·6 | | −19·4? | — | | −23·9 | −22·7 |
| | − 8·8 | − 8·7 | | −16·4 | −15·8 | | −19·9 | −18·9 |
| $H_\alpha \parallel$ | − 6·2 | − 5·8 | | −13·2 | −12·7 | | −15·9 | −15·1 |
| | 6·2 | 5·8 | | −10·0 | − 9·5 | | −10·6 | −10·1 |
| | 8·8 | 8·7 | | − 6·7 | — | | − 6·6 | − 6·3 |
| | 11·5 | 11·6 | | − 3·3 | − 3·2 | $H_\gamma \parallel$ | − 2·7 | − 2·5 |
| | — | 23·1 | $H_\beta \parallel$ | 0 | 0 | | 2·7 | 2·5 |
| | | | | 3·3 | 3·2 | | 6·6 | 6·3 |
| | | | | 6·7 | — | | 10·6 | 10·1 |
| | | | | 10·0 | 9·5 | | 15·9 | 15·1 |
| | | | | 13·2 | 12·7 | | 19·9 | 18·9 |
| | | | | 16·4 | 15·8 | | 23·9 | 22·7 |
| | | | | 19·4? | — | | 29·4 | 27·7 |
| | | | | — | 22·2 | | | |
| | — | −17·3 | | −19·3 | −19·0 | | −26·3 | −25·2 |
| | — | −14·5 | | −16·4 | −15·8 | | −22·8 | −21·4 |
| | − 2·6 | − 2·9 | | −13·2? | — | | −17·3 | −16·4 |
| $H_\alpha \perp$ | 0 | 0 | | − 9·7 | − 9·5 | | −13·3 | −12·6 |
| | 2·6 | 2·9 | | − 6·6 | − 6·3 | | − 9·7 | − 8·8 |
| | — | 14·5 | | − 3·4 | − 3·2 | | − 3·9 | − 3·8 |
| | — | 17·3 | $H_\beta \perp$ | 0 | 0 | $H_\gamma \perp$ | 0 | 0 |
| | | | | 3·4 | 3·2 | | 3·9 | 3·8 |
| | | | | 6·6 | 6·3 | | 9·7 | 8·8 |
| | | | | 9·7 | 9·5 | | 13·3 | 12·6 |
| | | | | 13·2? | — | | 17·3 | 16·4 |
| | | | | 16·4 | 15·8 | | 22·8 | 21·4 |
| | | | | 19·3 | 19·0 | | 26·3 | 25·2 |

$H_\delta$, whose resolution is not considered here. The theory of the Stark effect constitutes, perhaps, the greatest success of the quantum theory as hitherto developed. There is a systematic discrepancy of about 4 per cent. between the observed and calculated separations, due most likely to uncertainty in the measured voltage.

One point has been passed over in the table on p. 626. Since $K_3$ and $N_3$ are the absolute values of the moment of momentum, the actual moments round the axis of $z$ can also be $-K_3$ and $-N_3$ (cf. Art. 264), and there is no reason why one occurrence should be more frequent than the other. Thus the equation $K_3 = N_3 + 1$ does not here mean right-handed circular polarisation when viewed in the direction of the field, but a mixture of right-handed and left-handed polarisation in equal intensities, that is unpolarised light. This agrees with Stark's observations of the longitudinal effect (Art. 268).

**271. The special theory of relativity.** Although it is not clear how matter interacts with ether so as to emit energy in quanta, we may provisionally retain Maxwell's equations for the free ether and consider whether they fall in with the general principle that an observer cannot detect a uniform motion of himself "in space" (that is, with respect to the mean of those stars which have no observable annual parallax) by mechanical, electrical or optical experiments. This principle rests on several experiments, of which the best known is that of Michelson and Morley*, proving that light appears to travel with the same velocity in all directions when both source and observer move with the same velocity in space.

In mechanics, if we select any particular set of axes $A$ as "fixed," the equations of motion are not altered by taking instead a set $B$, provided that the relative motion of $B$ and $A$ is uniform motion in a straight line. The equations of motion are not changed when the space-coordinates $x, y, z$ and the time $t$ are changed to $x', y', z', t'$, where

$$x' = x, \quad y' = y, \quad z' = z - vt, \quad t' = t \quad \ldots\ldots(342),$$

$v$ being constant. In finding whether Maxwell's equations are altered by this transformation we have to remember that electric and magnetic forces are not the same in the fixed and moving systems. Charges which the moving observer would consider at rest have a velocity $\mathbf{v}$ with respect to the fixed observer, and the new electric force $\mathbf{E}'$ is the mechanical force divided by the charge. Hence

$$\mathbf{E}' = \mathbf{E} + \frac{1}{c} V\mathbf{v}\mathbf{H},$$

or in Cartesian coordinates

$$E_x' = E_x - \frac{vH_y}{c}, \quad E_y' = E_y + \frac{vH_x}{c}, \quad E_z' = E_z \ldots(343).$$

In estimating the magnetic field (e.g. of an electric current) the moving observer would not allow for the fact that the moving charges have an additional motion of translation with velocity $\mathbf{v}$. Confining our attention, as we shall at first, to small velocities of translation, we should expect one of these charges $e$ to produce at any point $P$ a magnetic force

$$\frac{e}{cr^3} V\mathbf{v}\mathbf{r},$$

* Wood's *Physical Optics*, 2nd Ed., p. 672; Michelson, *Light Waves and their Uses*, p. 158.

where $\mathbf{r}$ is the vector from $e$ to $P$ (Art. 96). The electric force due to $e$ can only differ from the static force $e\mathbf{r}/r^3$ by quantities of the order $v$, so that we can write with some degree of plausibility

$$\mathbf{H} = \mathbf{H}' + \frac{1}{c} V\mathbf{v}\mathbf{E},$$

or
$$H_x' = H_x + \frac{vE_y}{c}, \quad H_y' = H_y - \frac{vE_x}{c}, \quad H_z' = H_z \ldots (344).$$

Making these substitutions, we see easily that the form of Maxwell's equations is altered, even to the first order in $v$.

Towards resolving the difficulty let us inquire more closely what is implied in the definition of time. Suppose that two observers, each moving with velocity $v$ in the direction of the axis of $z$, are setting their clocks by optical or electrical signals, the first observer being at $A$ and the second at $B$ (Fig. 382). The signal which arrives at $B$ at time $t$ came from the position $A'$ occupied by $A$ at some previous time $t-\tau$,

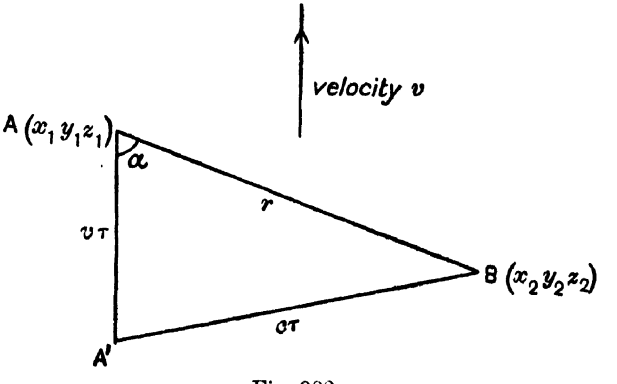



Fig. 382

where $A'A = v\tau$ and $A'B = c\tau$. If $AB = r$ and the angle $A'AB = \alpha$, we have from the triangle $AA'B$

$$c^2\tau^2 = v^2\tau^2 + r^2 - 2vr\tau\cos\alpha.$$

If $v$ is so small that $v^2/c^2$ is negligible, we have approximately

$$\tau = \frac{r}{c} - \frac{vr\cos\alpha}{c^2},$$

or
$$\tau = \frac{r}{c} - \frac{v(z_1 - z_2)}{c^2}.$$

A signal from $B$ to $A$ similarly requires time

$$\tau' = \frac{r}{c} + \frac{v(z_1 - z_2)}{c^2}.$$

Hence to this order $\tau + \tau' = 2r/c$, as if both observers were at rest. Supposing temporarily that $B$ is a reflecting mirror, an observer at

$A$ will conclude that the velocity of light is $c$. If $A$'s clock is started and a signal sent at the same time, it is received at $B$ at a time $\tau$ by $A$'s clock. The observer at $B$, concluding that the velocity of propagation is $c$, will set his clock to read time $r/c$ on receipt of the signal. Thus when $A$'s clock reads $\tau$, $B$'s clock reads $r/c$. Adding $t - \frac{r}{c} - \frac{vz_2}{c^2}$ to each, we see that when $A$'s clock reads $t - vz_1/c$, $B$'s clock reads $t - vz_2/c^2$. The time $t' = t - vz/c^2$ is called the *local time* at the point $(x, y, z)$. Since time at different places can only be correlated by signalling, a moving observer has good reason to regard local time as true time. We should, therefore, write

$$x' = x, \quad y' = y, \quad z' = z - vt, \quad t' = t - vz/c^2 \quad \ldots(345)$$

instead of (342).

We have now all the data for finding what Maxwell's equations become when transformed by equations (343), (344) and (345). If $v^2/c^2$ is neglected throughout, we can write

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial y'}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'} - \frac{v}{c^2}\frac{\partial}{\partial t'}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - v\frac{\partial}{\partial z'},$$

$$E_x = E_x' + \frac{vH_y'}{c}, \quad E_y = E_y' - \frac{vH_x'}{c}, \quad E_z = E_z',$$

$$H_x = H_x' - \frac{vE_y'}{c}, \quad H_y = H_y' + \frac{vE_x'}{c}, \quad H_z = H_z',$$

and then Maxwell's equations, in their Cartesian form, are unaltered. The principle of relativity has been *proved*, to the first order in $v/c$, by means of the idea of local time. This theory is due jointly to Lorentz and Einstein: the idea of local time to Lorentz and the idea that it is, to all intents and purposes, true time for a moving observer to Einstein. The extension to velocities of any order is considered in Art. 273.

**272. Theory of Fizeau's experiment.** The merit of the preceding theory does not lie merely in the fact that it explains the negative result of certain experiments (which might conceivably be done in other ways): it explains at the same time an experiment of Fizeau's. Fizeau observed a shift in a set of interference fringes caused by the motion of a stream of water in the path of a ray of light, and was thus able to estimate the speed at which light travelled

within the moving water itself. If $c$ is the velocity of light *in vacuo*, $\mu$ the refractive index and $v$ the speed of the water, the observed velocity was not $c/\mu + v$, as we might expect, but $c/\mu + kv$, where $k$ was certainly less than unity.

We shall assume that the forces which bind electrons to their atoms in dielectrics are not affected by a uniform motion of translation of the whole system, to the first order of small quantities. The velocity of propagation of the waves, as registered by an observer moving with the system, is then $c/K^{\frac{1}{2}}$, where $K$ is the dielectric constant when at rest. Let us assume at first that $K$ is independent of the frequency, and write $K = \mu^2$.

Let a sinoidal wave of absolute frequency $\nu$ be travelling in the direction of the axis of $z$. To an observer moving with the velocity of the water it will appear as a wave of velocity $c/\mu$ and (possibly) different frequency $\nu'$. Let $V$ be the velocity of the wave as perceived by an observer at rest. Then since each vector associated with the wave is proportional to

$$\cos 2\pi\nu\left(t - \frac{z}{V}\right) \text{ or } \cos 2\pi\nu'\left(t' - \frac{\mu z'}{c}\right),$$

we have

$$\nu\left(t - \frac{z}{V}\right) = \nu'\left(t' - \frac{\mu z'}{c}\right) = \nu'\left\{\left(1 + \frac{\mu v}{c}\right)t - \left(\frac{\mu}{c} + \frac{v}{c^2}\right)z\right\}.$$

Hence $$\nu = \nu'\left(1 + \frac{\mu v}{c}\right), \quad V = \left(1 + \frac{\mu v}{c}\right)\Big/\left(\frac{\mu}{c} + \frac{v}{c^2}\right),$$

or to the first order in $v$,

$$\nu' = \nu\left(1 - \frac{\mu v}{c}\right), \quad V = \frac{c}{\mu} + \left(1 - \frac{1}{\mu^2}\right)v.$$

Let $\lambda = c/\nu$ be the wave-length *in vacuo* corresponding to the frequency $\nu$. If $\mu$ varies with $\lambda$, we have to correct the last equation by taking the refractive index corresponding to $\nu'$ instead of that corresponding to $\nu$, that is we must replace $\mu$ by

$$\mu - \frac{\mu v \nu}{c}\frac{d\mu}{d\nu} = \mu + \frac{\mu v \lambda}{c}\frac{d\mu}{d\lambda}.$$

Hence we have finally

$$V = \frac{c}{\mu} + \left(1 - \frac{1}{\mu^2} - \frac{\lambda}{\mu}\frac{d\mu}{d\lambda}\right)v \ldots\ldots\ldots (346).$$

Zeeman measured Fizeau's factor $k$ for light of four colours,

using a stream of water of mean velocity 465 cm./sec. The results were as follows:

| Wave-length in A.U. | Fizeau's factor | |
|---|---|---|
| | Observed | Calculated |
| 4500 | 0·465 | 0·464 |
| 4580 | 0·463 | 0·463 |
| 5460 | 0·451 | 0·454 |
| 6870 | 0·445 | 0·447 |

Experiments like Fizeau's, in which the velocity of light in a moving medium is measured by a fixed observer, are, of course, to be distinguished from the "null" experiments in which the observer and all his apparatus move together.

**273. Theory of the increase of mass with velocity.** We shall account most quickly for Michelson and Morley's experiment by showing that equations (343), (344) and (345) can be generalised so as to leave Maxwell's equations unaltered for any velocity less than that of light. In the first place we cannot retain equations (345), since they make

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + (1 - v^2/c^2)(z^2 - c^2t^2),$$

whereas $x'^2 + y'^2 + z'^2 - c^2t'^2$ and $x^2 + y^2 + z^2 - c^2t^2$ must vanish together if the moving observer is to make the velocity of light $c$ in all directions. Einstein, following Lorentz, proposed the transformation

$$\left.\begin{aligned} &x' = x, \quad y' = y, \quad z' = \beta(z - vt), \quad t' = \beta(t - vz/c^2) \\ &E_x' = \beta(E_x - vH_y/c), \quad E_y' = \beta(E_y + vH_x/c), \quad E_z' = E_z \\ &H_x' = \beta(H_x + vE_y/c), \quad H_y' = \beta(H_y - vE_x/c), \quad H_z' = H_z \end{aligned}\right\} \quad (347),$$

where $$\beta = (1 - v^2/c^2)^{-\frac{1}{2}}$$

which leaves Maxwell's equations unaltered for all values of $v$, and the transformations are real if $v$ is numerically less than $c$. They reduce to those of Art. 271 when $v^2/c^2$ is neglected.

When equations (347) are solved for $x, y, z, \ldots$ in terms of $x', y', z', \ldots$, we have

$$\left.\begin{aligned} &x = x', \quad y = y', \quad z = \beta(z' + vt'), \quad t = \beta(t' + vz'/c^2) \\ &E_x = \beta(E_x' + vH_y'/c), \quad E_y = \beta(E_y' - vH_x'/c), \quad E_z = E_z' \\ &H_x = \beta(H_x' - vE_y'/c), \quad H_y = \beta(H_y' + vE_x'/c), \quad H_z = H_z' \end{aligned}\right\} \quad (348),$$

equations of the same form as (347), with $-v$ written for $v$. The relation of the two sets of axes to each other is therefore perfectly reciprocal, so that either set can be regarded as fixed and the other moveable, or neither fixed, at will.

The transformation (347) will only be satisfactory in relation to material media if we use it in the mechanics of moving charges. Suppose that a proton or electron is moving with any velocity less than that of light, and take axes so that it has velocity $v$ along the axes of $z$ at the time $t$ considered. To an observer moving with constant velocity $v$ it will appear to be instantaneously at rest, and will therefore behave like a particle of constant mass $m_0$. The equations of motion will be

$$m_0 \frac{d^2x'}{dt'^2} = eE_x' = e\beta\left(E_x - \frac{vH_y}{c}\right),$$

$$m_0 \frac{d^2y'}{dt'^2} = eE_y' = e\beta\left(E_y + \frac{vH_x}{c}\right),$$

$$m_0 \frac{d^2z'}{dt'^2} = eE_z' = eE_z,$$

where $\mathbf{E}$ is the impressed electric force. Since

$$\frac{dx'}{dt'} = \frac{dx}{\beta\left(dt - \frac{vdz}{c^2}\right)} = \frac{\dot{x}}{\beta\left(1 - \frac{v\dot{z}}{c^2}\right)}, \quad \frac{dt'}{dt} = \beta\left(1 - \frac{v\dot{z}}{c^2}\right),$$

we have

$$\frac{d^2x'}{dt'^2} = \frac{d}{dt}\left(\frac{dx'}{dt'}\right)\frac{dt'}{dt} = \frac{\left(1 - \frac{v\dot{z}}{c^2}\right)\ddot{x} + \dot{x}\frac{v\ddot{z}}{c^2}}{\beta^3\left(1 - \frac{v\dot{z}}{c^2}\right)^3}.$$

At the time we are considering $\dot{x} = \dot{y} = 0, \dot{z} = v$, so that $d^2x'/dt'^2 = \beta^2\ddot{x}$. Similarly $d^2y'/dt'^2 = \beta^2\ddot{y}$, $d^2z'/dt'^2 = \beta^3\ddot{z}$. The equations of motion, referred to "fixed" axes, are therefore

$$m_0\beta\ddot{x} = e\left(E_x - \frac{vH_y}{c}\right), \quad m_0\beta\ddot{y} = e\left(E_y + \frac{vH_x}{c}\right), \quad m_0\beta^3\ddot{z} = eE_z.$$

The terms on the right are the components of external force acting on the particle. Writing $X$, $Y$, $Z$ for them we have

$$m_0\beta\ddot{x} = X, \quad m_0\beta\ddot{y} = Y, \quad m_0\beta^3\ddot{z} = Z \quad \ldots\ldots(349).$$

It follows that a particle which has mass $m_0$ at slow velocities

has in general a mass $m_0\beta$ for transverse accelerations and $m_0\beta^3$ for accelerations in the direction of motion. It is more instructive to re-write (349) so as to give the particle a single mass. If we regard $\beta$ as a variable we have at the moment considered

$$\beta = \left(1 - \frac{\dot{z}^2}{c^2}\right)^{-\frac{1}{2}}, \quad \frac{d\beta}{dt} = \beta^3 \frac{\dot{z}\ddot{z}}{c^2} \ldots\ldots\ldots (350).$$

Since $\dot{x} = \dot{y} = 0$ and $\dot{z} = v$ at the moment considered, it is easy to verify that equations (349) can be put in the symmetrical form

$$\frac{d}{dt}(m_0\beta\dot{x}) = X, \quad \frac{d}{dt}(m_0\beta\dot{y}) = Y, \quad \frac{d}{dt}(m_0\beta\dot{z}) = Z \ldots (351).$$

Hence the particle has a single mass

$$m = m_0\beta = m_0 (1 - v^2/c^2)^{-\frac{1}{2}} \ldots\ldots\ldots (352)$$

in the sense that its momentum is $m_0\beta v$ when its velocity is $v$. This agrees with the mass found experimentally in Art. 253, the indefinite increase of $m$ as $v$ approaches $c$ preventing any particle from acquiring a velocity greater than that of light.

If $T$ is the kinetic energy of the particle at the moment considered, $\dot{T}$ is the rate at which the forces are doing work, or

$$\dot{T} = Z\dot{z} = m_0\beta^3\dot{z}\ddot{z} = m_0c^2\dot{\beta}$$

from (349) and (350). Hence $T = \text{const.} + m_0c^2\beta$, or since $T$ vanishes when $v = 0$,

$$T = m_0c^2(\beta - 1) = (m - m_0)c^2 \ldots\ldots (353).$$

If the particle is an electron and $T$ the kinetic energy acquired in falling through $V$ volts, $T = eV/300$, from which we find

$$V = 510000\left(\frac{m}{m_0} - 1\right) = 510000\left\{\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} - 1\right\} \ldots (354).$$

This formula has been used to calculate the kinetic energy of $\beta$ particles in Arts. 253 and 254. Working over the formulae in Art. 207, we find that Kaufmann's formula for $V^{\frac{1}{2}}\delta$ only requires correcting by a factor $1 - v^2/8c^2$ (approximately), which is 1 in 170 at 12,000 volts. Equation (255) is correct to 1 per cent. up to 3400 volts.

Einstein observed that equation (353) takes a still more simple form if we assume that a proton or electron has potential energy $m_0c^2$ when at rest, from the mutual attraction of its parts. The

total energy when in motion is then $mc^2$, leading to the doctrine that energy $E$ in any form is associated with a mass $E/c^2$. This might explain the disappearance of mass when four protons combine to form a helium nucleus (Art. 230).

**274. Central orbits in the special theory of relativity.** Let an electron move round a fixed nucleus of charge $Ze$, and let the variation of its mass with velocity be taken into account to terms of order $v^2/c^2$. The momentum of the electron is

$$m_0 v\,(1 - v^2/c^2)^{-\frac{1}{2}} = m_0 v\,(1 + v^2/2c^2)$$

and its moment of momentum round the nucleus in polar coordinates $m_0 r^2\dot{\theta}\,(1 + v^2/2c^2)$. Writing $H$ for this constant moment of momentum, we have approximately

$$r^2\dot{\theta} = \frac{H}{m_0}\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right) \ldots\ldots\ldots\ldots\ldots (355).$$

The kinetic energy of the electron is

$$m_0 c^2\,\{(1 - v^2/c^2)^{-\frac{1}{2}} - 1\} = \tfrac{1}{2} m_0 v^2\,(1 + 3v^2/4c^2)$$

and its potential energy $-\,m_0\mu/r$, where $\mu = Ze^2/m_0$ as in Art. 260. Thus

$$\tfrac{1}{2} m_0\,(\dot{r}^2 + r^2\dot{\theta}^2)\left(1 + \frac{3}{4}\frac{v^2}{c^2}\right) - \frac{m_0\mu}{r} = \text{const.} = -\frac{m_0\mu}{2a}\ \text{(say)},$$

where $a$ is a constant; or

$$\dot{r}^2 + r^2\dot{\theta}^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)\left(1 - \frac{3}{4}\frac{v^2}{c^2}\right) \ldots\ldots\ldots (356).$$

Using equation (355) to get rid of the time, as in the theory of Newtonian orbits, we have

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{\mu m_0^{\,2}}{H^2}\left(2u - \frac{1}{a}\right)\left(1 + \frac{1}{4}\frac{v^2}{c^2}\right).$$

We can further replace $v^2$ by its approximate value $\mu\,(2u - 1/a)$, so that

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{\mu m_0^{\,2}}{H^2}\left\{2u - \frac{1}{a} + \frac{\mu}{4c^2}\left(2u - \frac{1}{a}\right)^2\right\}.$$

The equation is of the form

$$\left(\frac{du}{d\theta}\right)^2 + Pu^2 + 2Qu + R = 0,$$

and the solution is

$$u = \frac{-Q + (Q^2 - PR)^{\frac{1}{2}}\cos\,(P^{\frac{1}{2}}\theta + C)}{P},$$

where $C$ is arbitrary. Putting in the known values of $P$, $Q$, $R$ we find

$$lu = 1 + \epsilon \cos(\gamma\theta + C) \quad \ldots\ldots\ldots\ldots (357),$$

where we shall not want the value of $l$, but

$$1 - \epsilon^2 = \frac{H^2}{\mu m_0{}^2 a}\left(1 - \frac{\mu^2 m_0{}^2}{H^2 c^2} + \frac{3\mu}{4c^2 a}\right) \quad \ldots\ldots\ldots (358),$$

and

$$\gamma = 1 - \frac{\mu^2 m_0{}^2}{2H^2 c^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (359).$$

Since $\gamma$ is nearly 1, the interpretation of (357) is that the orbit is nearly an ellipse, but the radius vector resumes its original value when $\theta$ increases by an angle $2\pi/\gamma$, slightly greater than $2\pi$, so that the line of apses is constantly moving round in the same direction as the electron in its orbit.

In applying the quantum conditions we shall (following Sommerfeld) define $p_1$ and $p_2$ as the radial momentum and moment of momentum respectively, which are not the same as $\partial T/\partial \dot{q}_1$ and $\partial T/\partial \dot{q}_2$. Thus

$$p_1 = m\dot{r},\ p_2 = mr^2\dot{\theta},$$

where $m$ is the (variable) mass of the electron. Since $p_2 = H$ the second quantum condition gives

$$2\pi H = n_2 h \quad \ldots\ldots\ldots\ldots\ldots\ldots (360),$$

where $n_2$ is the azimuthal quantum number. The calculation of $\int m\dot{r}\,dr$ proceeds as in Art. 260, except that the limits of integration are from $\phi = 0$ to $\phi = 2\pi$, where $\phi = \gamma\theta + C$. In this way the first quantum condition becomes

$$n_1 h = \frac{n_2 h \epsilon^2 \gamma}{2\pi} \int_0^{2\pi} \frac{\sin^2\phi\, d\phi}{(1 + \epsilon\cos\phi)^2},$$

or

$$1 - \epsilon^2 = \left(\frac{n_2\gamma}{n_1 + n_2\gamma}\right)^2 \quad \ldots\ldots\ldots\ldots (361).$$

Substituting from (360) in (359) and writing $Ze^2$ for $m_0\mu$, we have

$$\gamma = 1 - \frac{2\pi^2 e^4 Z^2}{n_2{}^2 h^2 c^2} = 1 - \frac{Z^2 F}{2n_2{}^2},$$

where

$$F = \left(\frac{2\pi e^2}{hc}\right)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (362),$$

a constant of which more use will be made in the next article. Since $F = 5{\cdot}30 \times 10^{-5}$ and $Z = 1$ or 2, the assumption that we can

stop at terms of order $F$ is justified, and to this order equation (361) gives

$$1-\epsilon^2=\left(\frac{n_2}{n_1+n_2}\right)^2\left(1-\frac{n_1Z^2F}{(n_1+n_2)\,n_2^2}\right)\ldots\ldots(363).$$

Comparing (358) and (363) we have

$$\frac{n_2^2h^2}{4\pi^2m_0eZ^2a}\left(1-\frac{Z^2F}{n_2^2}+\frac{3\mu}{4c^2a}\right)=\left(\frac{n_2}{n_1+n_2}\right)^2\left(1-\frac{n_1Z^2F}{(n_1+n_2)\,n_2^2}\right),$$

or

$$a=\frac{(n_1+n_2)^2\,h^2}{4\pi^2m_0e^2Z}\left(1-\frac{Z^2F}{(n_1+n_2)\,n_2}+\frac{3\mu}{4c^2a}\right).$$

The quantity $a$ still remains on the right, but only in terms of order $F$, where it can be replaced by its approximate value

$$(n_1+n_2)^2\,h^2/4\pi m_0e^2Z.$$

To this order

$$\frac{3\mu}{4c^2a}=\frac{3Z^2F}{4\,(n_1+n_2)^2}.$$

Writing $n=n_1+n_2$ for the principal quantum number, as in Art. 260, we have finally

$$a=\frac{n^2h^2}{4\pi^2m_0e^2Z}\left\{1-\left(\frac{1}{nn_2}-\frac{3}{4n^2}\right)Z^2F\right\}.$$

The total energy of the orbit is $W=-Ze^2/2a$, or

$$W=-\frac{2\pi^2m_0e^4Z^2}{n^2h^2}\left\{1+\left(\frac{1}{nn_2}-\frac{3}{4n^2}\right)Z^2F\right\}\ldots.(364).$$

Comparing (364) with (319) we see that the energy is no longer the same for all $n$-quantum orbits, and this affords a partial explanation of the subdivision of levels in Art. 240. We now consider whether its consequences are confirmed in the hydrogen and helium spectra.

**275. Fine structure of spectral lines.** Let an electron fall from one orbit of the type (364) to another, the principal quantum number changing from $k$ to $n$ and the azimuthal quantum number from $k_2$ to $n_2$. The wave-length emitted is given by

$$\frac{1}{\lambda}=RZ^2\left(\frac{1}{n^2}-\frac{1}{k^2}\right)+RZ^4F\left(\frac{1}{n^3n_2}-\frac{1}{k^3k_2}\right)-\frac{3RZ^4F}{4}\left(\frac{1}{n^4}-\frac{1}{k^4}\right)$$
$$\ldots.(365),$$

where $R$ is Balmer's constant for a fixed nucleus. It follows that Balmer's lines are not strictly single, but each consists of a number of components, corresponding to different azimuthal quantum

numbers. Since $F$ is small, the wave-lengths corresponding to a given $n$ and $k$ are approximately

$$\lambda + RZ^4F\lambda^2\left(\frac{1}{k^3k_2} - \frac{1}{n^3n_2}\right) \quad \ldots\ldots\ldots(366),$$

where $\lambda$ is a line near to the theoretical Balmer line. In the second term, which determines the separation of the components, we can take an approximate value of $\lambda$; and we need not allow for the motion of the nucleus, any of the values hitherto used for $R$ being sufficiently accurate. The number of components is limited by the correspondence principle, according to which $k_2 = n_2 \pm 1$ (Art. 265). Since neither $n_2$ nor $k_2$ can vanish (Art. 261) the possible combinations of $n_2$ and $k_2$ are given by

$$\begin{aligned} &n_2 = 1, \quad k_2 = 2; \\ &n_2 = 2, \quad k_2 = 1 \text{ or } 3; \\ &\ldots\ldots\ldots\ldots\ldots \\ &n_2 = n, \quad k_2 = n - 1 \text{ or } n + 1, \end{aligned}$$

and are therefore $2n - 1$ in number. Hence the number of components is the same for all lines of the same series, and equal to $2n - 1$.

Fine structure of hydrogen lines has been observed, but ionised helium (Art. 262) is better for measurement, since the relativity separation is increased in the ratio of 16 to 1 (equation (366)), the Stark effect of any chance electric fields is reduced to one-half (equation (341)), and the lines are sharper. Paschen studied the lines $k = 4$ and $k = 5$ of Fowler's series, of which the components 4687·115 and 3204·087 have been selected for tabulation on p. 639. The coefficient $RZ^4F\lambda^2$ is 20·43 for the first group and 9·55 for the second group, adapted to Ångström units. We see at once, in fitting the theory to Paschen's measurements, that the tabulated component corresponds to $n_2 = 3$, $k_2 = 4$ in each case. The following table gives the observed wave-length, and the observed and calculated differences of wave-length with respect to the tabulated component.

Paschen observed all but one of the lines allowed by the correspondence principle. Asterisks denote transitions not so allowed, two of which were observed as faint lines in each group. Their emergence would be ascribed to the electric field of the discharge

*Fine structure of lines of ionised helium* ($Z = 2$).

| 4687 group ($n=3$, $k=4$) | | | | | 3204 group ($n=3$, $k=5$) | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Observed wave-length in A.U. | $n_2$ | $k_2$ | Differences of wave-length in A.U. | | Observed wave-length in A.U. | $n_2$ | $k_2$ | Differences of wave-length in A.U. | |
| | | | Calculated | Observed | | | | Calculated | Observed |
| 4686·695 | 1 | 2 | −0·425 | −0·420 | 3203·874 | 1 | 2 | −0·216 | −0·203 |
| 4686·857 | 1* | 1* | −0·266* | −0·258 | 3203·918 | 1* | 1* | −0·178* | −0·169 |
| 4687·010 | 2 | 3 | −0·098 | −0·105 | 3204·033 | 2 | 3 | −0·051 | −0·054 |
| **4687·115** | 3 | 4 | **0** | **0** | **3204·087** | 2 | 1 | 0·000 | **0** |
| 4687·212 | 2 | 1 | +0·114 | +0·097 | | 3 | 4 | **0** | |
| | 3 | 2 | +0·080 | | — | 3 | 2 | +0·020 | — |
| 4687·335 | 3* | 1* | +0·239* | +0·220 | 3204·145 | 3* | 1* | +0·058 | +0·058 |

(Art. 267), which Paschen kept low by using accumulators instead of an induction coil. On the whole we have a good confirmation of Sommerfeld's theory, and also by implication of the theory of Art. 273.

Since no line of the group obeys Balmer's law, the value of $R_{He}$ given in equation (325) is only approximate. A closer approximation is found by applying equation (365) to the lines 4687·115 and 3204·087. The values so obtained for $\frac{1}{4}R_{He}$ are 109724·5 and 109723·9, the mean of which makes $m/m_H = 1/1810$ instead of 1/1850.

**276. Physical and chemical properties of the elements.** This last article will be in the main a continuation of Art. 223, in the light of information that we have gained in the meantime.

Moseley observed in 1913 that the $K$ series of higher elements corresponds to Lyman's series of hydrogen, the $L$ series to Balmer's series—an electron falling into a 1-quantum orbit emits a $K$ line, into a 2-quantum orbit an $L$ line, and so on. The analogy would be strict if each circumnuclear electron moved in the field of the nucleus only, which is approximately true of the innermost electrons, since the nucleus overpowers the few electrons which are as near as itself. The spectrum of such an atom is given by the equation

$$\frac{1}{\lambda} = RZ^2\left(\frac{1}{n^2} - \frac{1}{k^2}\right).$$

It was this theory which led Moseley to plot $\nu^{\frac{1}{2}}$ against the atomic

number (Art. 236). For the $K_\alpha$ line we find empirically

$$\frac{1}{\lambda} = R(Z-1)^2\left(\frac{1}{1^2}-\frac{1}{2^2}\right),$$

where the reduction from $Z$ to $Z-1$ represents, in general terms, a weakening of the nuclear attraction by other electrons near it. The theoretical excitation voltages are $13{\cdot}5Z^2/n^2$. Fig. 350 shows how these simple considerations fail more and more as we go outwards from the nucleus. The diameter of the ideal 1-quantum orbit becomes smaller and smaller as $Z$ increases (equation (316)). Atomic "diameters" do not, therefore, increase after the manner of a building to which bricks are added. They vary between relatively small limits, the greatest being those of sodium, potassium, rubidium and caesium, where the outermost electron is by itself in an orbit of a new type.

The tendency of an atom to lose its outer electrons, or augment them to a more stable number, was mentioned in Art. 223. Abegg

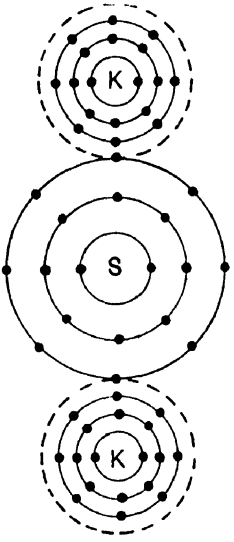



Fig. 383

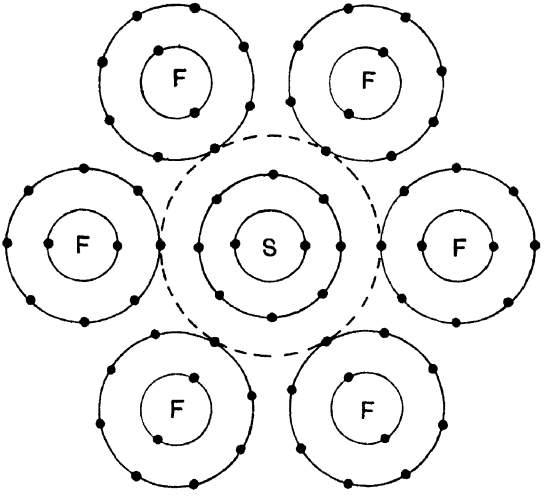



Fig. 384

showed that some atoms can do either. Consider the compounds $K_2S$ and $SF_6$, whose probable structure is shown diagrammatically in Figs. 383 and 384. The normal sulphur atom has its $K$ and $L$ levels complete, and 6 electrons in its $M$ level. In $K_2S$ it completes its $M$ level to the semi-stable number of 8 by taking an $N$ electron

from each potassium atom. The sulphur atom has (in a modified sense) gained two electrons, while the potassium atoms have lost one each; for though two electrons are shared in the molecule, its dissociation into $\overset{+}{\mathrm{K}} + \overset{+}{\mathrm{K}} + \overset{--}{\mathrm{S}}$ shows that sulphur holds them hardest. Both atoms tend to the electronic configuration of argon — 2 *K* electrons, 8 *L* electrons and 8 *M* electrons. In assigning the valencies + 1 and — 2 to potassium and sulphur we introduce an idea closely allied to that of Art. 166, but more general since the full power of an atom to gain or lose electrons is not usually called forth by solution. Abegg showed that sulphur can also have valency + 6, in the compound $SF_6$ (Fig. 384). Fluorine has a great affinity for one electron (but no more), so that the 6 *M* electrons of sulphur can be taken away by 6 fluorine atoms, and sulphur tends to the electronic configuration of neon. Abegg's law is that some atoms can have both positive and negative valency, the sum of which is 8. The suggestion, therefore, is that an atom can have eight electrons in its highest energy level—as Abegg first expressed it, an electron can have eight "strategical positions," the positive valency being the number which it occupies in the neutral atom. Hydrogen, which can have valencies + 1 and — 1, is an exception to Abegg's law, just as helium is to the rule that the highest level contains eight electrons.

Similar considerations, emphasised especially by Kossel and Lewis, apply to many compounds. Detailed application of the rule of Art. 260, and even such plausible arrangement of the nuclei as will allow two of them to share an electronic orbit, are entirely lacking. But however loosely we interpret them, the sharing of electrons and the tendency to a stable form are important principles.

Fig. 385, from Kossel and Langmuir, shows this and other tendencies. Sets of 28 and 46 electrons have stability scarcely less than that of the inert gases, but the corresponding elements are not inert gases, and those just below them have more than one valency. Ladenburg assumed that the development of the *N* level ceases at $Z = 21$, further electrons going into the *M* level until it is filled up to its final number of 18. Variable valency means that extraction of electrons does not always cease when the two outermost electrons (see table, p. 492) have been taken away; an electron

may, indeed, be taken from an inner level first. Ladenburg observed, in confirmation, that the elements in question exhibit pronounced colour, corresponding to relatively small change of energy in transitions from one orbit to another.

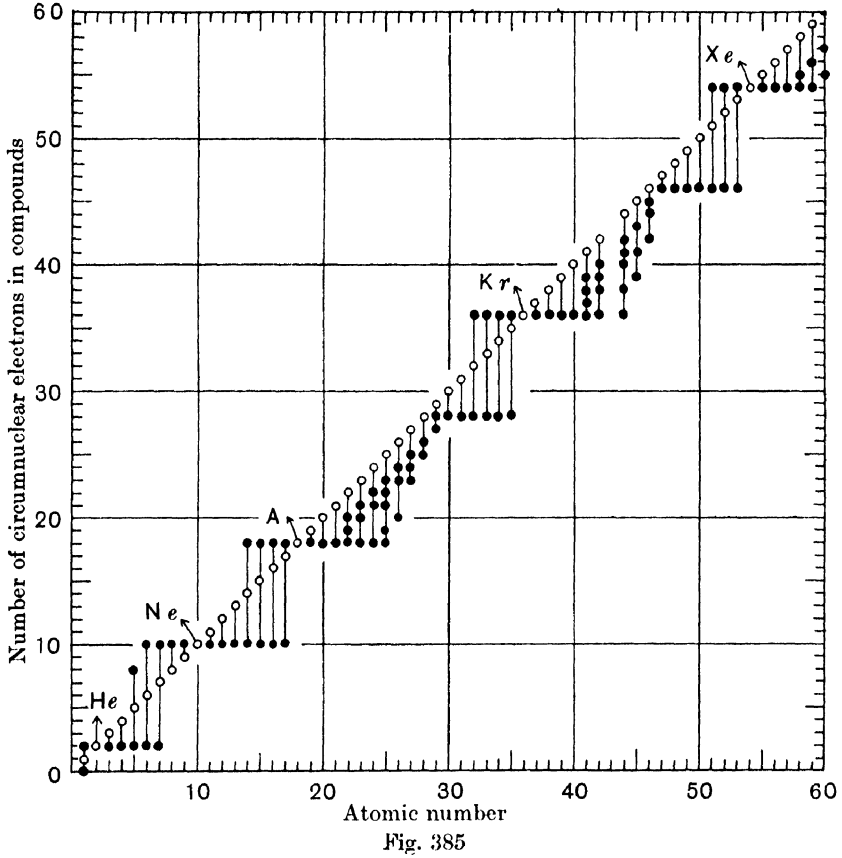



Fig. 385

The development of inner levels was first suggested by Vegard in connexion with the rare earths. It is, as we have hinted, a consequence of the excitation potential of some inner level being greater than that of some outer level, for which evidence is already available in Fig. 350. The spectroscopic evidence adduced by Bohr to show that it occurs between $Z = 21$ and $Z = 28$ would take too long to explain. Fig. 350 does, however, show that some $N$ levels in the region of the rare earths should be filled up before some $O$ levels.

Development of an inner level implies, according to Bohr, an

asymmetry associated with a resultant magnetic moment, so that the occurrence of paramagnetism at this stage is made plausible (cf. Art. 114). Such development is, in general terms, a consequence of the very different shapes of orbit which even the simple theory allows, whereby a $5_1$ orbit, for example, comes nearer the nucleus than a $4_4$ orbit. The disturbing presence of other circumnuclear electrons is assumed by Bohr, on spectroscopic evidence, to be so great that a type of orbit which one might expect to have less energy than another may actually have greater energy.

# APPENDIX

## UNITS AND NOTATION

The *electrostatic system* is based on the unit charge of electricity, which repels a similar charge placed 1 cm. away with a force of 1 dyne.

The *electromagnetic system* is based on the unit magnetic pole, which repels a similar pole placed 1 cm. away with a force of 1 dyne.

The ratio of the electromagnetic to the electrostatic unit of charge is denoted by $c$. Its value is very nearly $3 \times 10^{10}$ C.G.S. units.

The *derived electromagnetic system* is obtained by taking multiples or submultiples of the true electromagnetic units as new units, as shown in the following scheme:

| | | | |
|---|---|---|---|
| 1 ampere | $= \frac{1}{10}$ | E.M.U. of | current |
| 1 coulomb | $= \frac{1}{10}$ | ,, | charge |
| 1 volt | $= 10^8$ | ,, | potential |
| 1 ohm | $= 10^9$ | ,, | resistance |
| 1 farad | $= 10^{-9}$ | ,, | capacity |
| 1 henry | $= 10^9$ | ,, | inductance |

The next table shows the notation used in this book for the commoner electrical quantities. The use of a single letter indicates the system of units in which any given quantity is usually expressed. Thus, unless otherwise stated, *charge* is expressed in electrostatic units, *current* in electromagnetic units, and so on. The usage for potential and capacity varies. The table shows how the *measure of a given quantity* changes when the system of units is changed; thus 1 E.S.U. of capacity $= 1/9 \times 10^{11}$ farads, 1 E.S.U. of potential $=$ 300 volts, and so on.

| Quantity | Measure in E.S.U. | Measure in E.M.U. | Measure in derived E.M.U. | Name of derived unit |
|---|---|---|---|---|
| Charge .. .. | $e$ | $e/c$ | $10e/c = e/3 \times 10^9$ | Coulomb |
| Volume- and surface-density | $\rho$, $\sigma$ | — | — | — |
| Electric force .. | $E$ | $cE$ | — | — |
| Potential (E.S.U.) .. | $V$ | $cV$ | $cV/10^8 = 300\ V$ | Volt |
| Capacity (E.S.U.) .. | $C$ | $C/c^2$ | $10^9 C/c^2 = C/9 \times 10^{11}$ | Farad |
| Electric polarisation | $P$ | — | — | — |
| Electric induction | $D$ | — | — | — |
| Dielectric constant | $K$ | — | — | — |
| Magnetic pole .. | — | $m$ | — | — |
| Magnetic force .. | — | $H$ | — | — |
| Magnetic potential | — | $\Omega$ | — | — |
| Intensity of magnetisation | — | $I$ | — | — |
| Magnetic induction | — | $B$ | — | — |
| Current .. .. | $ci$ | $i$ | $10i$ | Ampere |
| Current-density .. | — | $u$ | — | — |
| Electromotive force (potential) | $V/c$ | $V$ | $V/10^8$ | Volt |
| Capacity (E.M.U.) .. | $c^2C$ | $C$ | $10^9\ C$ | Farad |
| Resistance .. .. | — | $R$ | $R/10^9$ | Ohm |
| Specific resistance | — | $k$ | $k/10^9$ | — |
| Conductivity .. | — | $\sigma$ | $10^9\ \sigma$ | — |
| Flux of magnetic induction | — | $N$ | — | — |
| Self-inductance .. | — | $L$ | $L/10^9$ | Henry |
| Mutual inductance | — | $M$ | $M/10^9$ | Henry |

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