UNIVERSAL
LIBRARY

OU_156369

UNIVERSAL
LIBRARY

OSMANIA UNIVERSITY LIBRARY

Call No. 517 /A51T Accession No. 38799

Author

Title Transactions of the symposium on fluid

This book should be returned on or before the date last marked below.
mechanics and Computers . 1957

TRANSACTIONS OF THE SYMPOSIUM

ON

FLUID MECHANICS AND COMPUTING

held at New York University
April 23-24, 1953

The First Symposium on Applied Mathematics

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY

and

OFFICE OF ORDNANCE RESEARCH, U. S. ARMY

Editorial Committee

GARRETT BIRKHOFF • K. O. FRIEDRICHS • T. E. STERNE

Reprinted from *Communications on Pure and Applied Mathematics*, a journal issued quarterly by the Institute of Mathematical Sciences of New York University: Volume VII, Number 1 (1954)

Copyright, 1954, Interscience Publishers, Inc.

ALL RIGHTS RESERVED

This book or any part thereof must not be reproduced in any form without permission of the publisher in writing. This applies specifically to photostatic and microfilm reproductions.

Reproduction in whole or in part is permitted for any purpose of the United States Government.

Library of Congress Catalog Card Number 54-7996

INTERSCIENCE PUBLISHERS, Inc., 250 Fifth Avenue, New York 1, N.Y.

For Great Britain or Northern Ireland:

Interscience Publishers Ltd., 88/90 Chancery Lane, London W.C.2

PRINTED IN THE UNITED STATES OF AMERICA

INTRODUCTION

This volume contains, with a single exception, all of the papers delivered at the first symposium in applied mathematics sponsored by the American Mathematical Society and the Office of Ordnance Research, U. S. Army. This symposium was held at New York University on 23 and 24 April, 1953. It is regretted that the many interesting discussions following the presentation of papers cannot be similarly reproduced.

The scientific fields explored at the symposium are many and diverse. They include broad areas in mathematics, fluid mechanics, mathematical physics and computer techniques. British, French, German and American scientists cooperated to make the symposium of significant international interest.

Each paper presented represents in a sense a critical point in the continuing frontal attack on the general problem of the understanding and control of the physical world. In addition to its general scientific interest to mathematicians in university and other laboratories, there is to the Ordnance Corps an intense user concern in each of the subjects treated.

It is hoped that these papers will stimulate further work in these areas in order not only to strengthen the fundamentals of Ordnance technology but also to advance our general research strength.

Office of Ordnance Research, U. S. Army
Durham, North Carolina

THOMAS J. KILLIAN
Chief Scientist

January 1954

CONTENTS

PAGE

Introduction iii

Mathematical methods in compressible flow theory.
By M. J. Lighthill 1

Boundary layer problems in applied mathematics.
By G. F. Carrier 11

Fourier synthesis of homogeneous turbulence.
By Garrett Birkhoff 19

One-dimensional anisentropic flows.
By G. S. S. Ludford and *M. H. Martin* 45

Approximate methods for computing flow fields.
By J. H. Giese 65

Results and conjectures in the mathematical theory of subsonic and transonic gas flow.
By Lipman Bers 79

The singular solutions and the Cauchy problem for generalized Tricomi equations.
By Alexander Weinstein 105

Remarks on the theory of partial differential equations of mixed type and applications to the study of transonic flow.
By P. Germain 117

Discussion on transonic flow.
By R. von Mises 145

Role of computing machines.* *By John von Neumann*

Some problems solvable on computing machines.
By Mark Lotkin 149

Weak solutions of nonlinear hyperbolic equations and their numerical computation.
By Peter D. Lax 159

Computation of one dimensional compressible flows including shocks.
By L. H. Thomas 195

The deformation of a thin material shell of non-uniform thickness by a detonation wave.
By G. E. Hudson 207

Measurement of spherical shock waves.
By Hubert Schardin 223

*This paper was presented at the Symposium. It is not published in these Transactions.

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. VII, 1–10 (1954)

Mathematical Methods in Compressible Flow Theory

By M. J. LIGHTHILL
University of Manchester, England

I am not going to give a catalogue of mathematical methods. I want rather to discuss the place of mathematical methods, their purpose and value, in scientific subjects, with my points illustrated by examples from compressible flow theory. Also I want to speak a little about what kinds of approach to problems lead to the most useful results.*

There are of course two special dangers—overvaluing mathematics and undervaluing it—but even when these extremes are recognized it is not easy to pick the right middle course. Overvaluing it is an *obvious* danger; mathematics is only one of many valuable ways of viewing the fantastic behaviour of things, and it cannot possibly replace the other essential and essentially different ways like physics, biology, statistics, economics, poetry, love and religion. And, even in the somewhat limited sphere of an engineering problem, the complications are notoriously always far outside the power of mathematics to grasp as a whole. It is necessary only to look at a dam, an automobile, or a guided missile to be assured of that. And certainly the airflows around the kind of objects which are chosen to move fast through the air are high on the list of the most complicated engineering phenomena.

In fact, of course, objects like these are designed on the basis of a great mass of experimental tests. The data in these tests *are* combined and used in a multitude of design calculations, and these can be called mathematics; but they are not at all like the kind of original mathematical research which I am trying to talk about, nor do they make use of any but a minute fraction of our mathematical knowledge of, say, compressible flow.

It is tempting then to go to the other extreme and say that mathematical research in subjects as complicated as this is no good. It is very easy to lose heart and give in to this proposition—two great men, each of whom is one of the top ten in the theory of compressible flow, have independently told me in confidence that they have at last come to that conclusion. Now humility is a virtue, but despair is a sin! I believe that if one analyses their views more closely

*The lecture includes, for illustration, brief surveys of three topics in compressible flow theory—the boundary layer, shock wave decay, and the noise of jets. The text is printed here in the original racy style, except that a sentence near the end, which caused some altercation when the lecture was given, has been altered.

one sees that intellectual humility is not with them duly balanced by the fruits of insight into the workings of the scientific process.

The real issue is the interaction between mathematical methods and physical ideas. Everyone agrees that basic physical ideas are necessary in organizing experiments on high-speed flow and in making proper use of the results. Dynamical similarity, boundary layer separation, sweepback are among the more elementary of these physical ideas. The question is whether mathematical methods are necessary to replenish our stock of physical ideas from time to time.

I think they are, and that this is the main purpose of mathematical methods. Take for example the boundary layer. If one is teaching this subject to students one does not laboriously describe all the mathematical methods which have been used to obtain solutions of the boundary layer equations; instead one tries to get across the essential physical picture that has emerged from them. One starts, for example, by pointing out that the shearing stresses will be of the order of the reciprocal of the thickness, and that for a constant pressure layer the thickness must grow in proportion as these shearing stresses slow down the fluid at the edge of the layer. Hence the product of thickness and rate of thickening remains constant, giving the parabolic law of growth. Next one observes that, where pressure gradient is present, the proportional velocity change it produces is greatest near the wall, because it tends to produce a constant change in kinetic energy, and because friction can only be partly effective in keeping down the huge proportional changes in velocity which must tend to result from this near the wall. The effect of these changes on the streamtube area near the wall is the cause of the thickening produced by positive pressure gradient and of the thinning resulting from negative pressure gradient. One points out also that a moderately large positive gradient must reduce the velocity near the wall to negative values, causing separation of the layer. Heating the wall causes the layer to respond in these ways more readily to pressure gradient, because the crucial fluid near the wall is then lighter. This is so whether the heating is accomplished artificially at low Mach number or aerodynamically at high Mach number, but there are slight differences for really high Mach number resulting from the fact that it is only the subsonic part of the layer which is thinned by a pressure drop, the supersonic part being slightly thickened.

From the point of view of leading to simple quantitative results, one important conclusion is that in any substantial change to the boundary layer extending over only a short distance viscous dissipation occurs appreciably only near the wall. This has recently led Gadd [1] and others to a method of calculating separations quickly by using energy conservation to determine the changed profile in the outer part of the layer, and fitting it with continuous curvature to a simple form of profile near the wall, which is only made to satisfy the balance of pressure gradient against the gradient of viscous stress at the wall, and conservation of mass for the inner sublayer as a whole. This simple calculation gives the separation point and other necessary information rapidly even in the case of high Mach number, and supersedes many laborious calcula-

tions of special cases which have been carried out. It also gives essential data for understanding the separations in supersonic flow which are produced upstream of compressive disturbances.

This illustrates how the physical ideas can lead to a simple mathematical method which is superior to those suggested in previous years by purely mathematical considerations. This is a typical situation. I shall cite later on, as another example from compressible flow, the Whitham theory [2] of the complete flow pattern of a supersonic projectile. It cannot be too much emphasized that a direct mathematical approach yields results uneconomically by comparison with an approach based on a sound physical idea.

However it is also essential not to forget that in most cases the mathematical approach has played an important part in laying bare what *is* the essential physical idea. Of course in the case of the boundary layer we cannot know what went on in Prandtl's mind, but the fact that his assumptions led to boundary layer equations and boundary conditions of a general type which one expects to give unique solutions was certainly an important step towards creating general confidence in the assumptions; and in later years each new solution of the equations underlined one of the ideas still further and added to that confidence. In addition the theory of boundary layer stability played a part in suggesting that for stationary disturbances of short length the viscous dissipation is effectively confined to an inner sublayer.

New significant physical ideas, then, *new* views about the mechanisms which operate in some fluid flow, do not always spring up out of nothing, or simply out of old physical ideas. A mathematical investigation is often an important intermediate step, even though the mathematics done at that stage is, essentially, the "wrong" mathematics, and the right mathematics will be done only later, when the correct physical picture has been found.

Some examples of the process are afforded by the theory of the upstream influence in supersonic boundary layers. It has been partly mathematical investigation and partly experimental investigation which have established the relative importance in different problems of the many different tentative physical pictures which were put forward originally, of which I shall mention three.

First, in 1941, Oswatitsch and Wieghardt [3] pointed out that in supersonic flow the wall pressure must be directly related to the slope of the displacement thickness contour of the boundary layer. On the other hand, according to our ideas of flow *in* the boundary layer, this slope, or rate of thickening, would vary with pressure *gradient*. When a rough, semi-empirical assumption about this variation was made, a relation between pressure and pressure gradient resulted, from which a definite "distance of upstream influence" could be inferred. This was of the order of 10 boundary layer thicknesses for laminar layers, but only 1 for turbulent layers, because in them the kinetic energy near the wall is being continually renewed by mixing, and therefore is not so readily changed by pressure gradient.

Several more physical pictures have been put forward in the past five

years. The most important one originated with Liepmann [4] and with Lees [5], who emphasized the important part played by separation and reversed flow near the surface, especially in the flows where the upstream influence was greatest. One could think of these flows being set up in such a way that the "dead air" region (so called) was finally just slender enough for the pressure gradients ahead of separation not to cause further separation. This condition effectively fixes the pressure coefficient *at* separation, and the pressure gradient beyond separation [1], [6] is a function of the quantity of circulatory flow in the "dead air" region, and of its thickness, and of whether the separated layer is laminar or turbulent. To complete the picture this pressure gradient has to adjust itself to the geometry of the dead air region, in accordance with supersonic flow theory.

From this point of view the differences between the behaviour of laminar and turbulent layers are explained by the resistance to separation on the part of turbulent layers, and by the large pressure gradients required to keep them separated. The theory is of course restricted to such compressive disturbances as produce separation, and it is broadly true that the Oswatitsch-Wieghardt explanation applies to all other disturbances, including all expansive ones.

A third picture was that originated by Howarth [7], and followed up by Tsien and Finston [8] and by me [9]. I have recently [10] done some rather elaborate investigations which show that this approach can be improved until it becomes in essence nothing but a precisely formulated version of that of Oswatitsch and Wieghardt. Tsien & Finston neglected viscosity, and replaced the boundary layer by a layer of uniform subsonic flow. Such a layer would of course thicken readily in response to the kinetic energy loss caused by pressure gradient. Accordingly the upstream influence was found to be fairly large, at least if the uniform velocity in the layer was taken to be low enough. But of course the theory gives no indication of what value for this velocity should be chosen, and obviously this must be influenced by viscosity, since it is only viscosity which prevents the energy change from taking its full value near the wall. It also neglects the fact that the velocity distribution is continuous from subsonic to supersonic, and that the change in stream tube area due to pressure has opposite signs in the subsonic and supersonic parts of the layer.

Now the theory is fairly easily extended [10] to take the continuous velocity distribution into account, but to get sensible results at this stage you still have to assume a non-vanishing velocity at the wall. Finally, however, taking viscosity into account, one finds, first, that it is important only in a thin sublayer near the wall, where compressibility can be neglected. This is what one would expect. And, next, the equations of motion in this sublayer are found to lead to quite a simple expression for the "effective Mach number at the wall," from which the degree of upstream influence can be determined. The resulting theory, which is really obtained by improving two theories until they become identical, gives some numerical results which are of value for flows without

separation, and also has been used to predict successfully the peculiar form of the reflected wave when a weak plane shock is incident on a turbulent layer.

But without going into further details, which would illustrate the point more conclusively, I want to suggest that the investigations I have just described constituted typical instances of what I have called the "wrong" mathematics (from a practical point of view) being used to straighten out the physical ideas. The checks with experiments, and the internal checks, show that the physical picture is valid under the conditions when it is expected to be. The various physical mechanisms which have been uncovered can then be used to design simpler mathematical theories which will get results as economically as possible. One such is that of Gadd [1] for predicting the least pressure rise to separation, which I mentioned before.

Another field where doing the wrong mathematics has proved extremely useful is the theory of the decay of shock waves. The valuable result has been found [11] that momentum methods which neglect energy degradation within shock waves give the same results as energy methods which have to take it into account, and that in many cases the easiest way to calculate the energy loss is from the momentum methods, even though they have neglected that loss in the first place. For example the flow pattern of a supersonic projectile was determined by Whitham [2] by assuming that the linear theory with its perfectly conical wavefronts could be turned into a good approximation if the values it predicted on a given wave front were taken to signify the values on the real non-conical wave front which originated in the same place on the projectile. This meant assuming that the most serious error in the linear theory at large distances from the body was its neglect of the gradual separation of the wave fronts; it follows from this hypothesis of Whitham's that the wave fronts are surfaces of revolution generated by parabolas. The shock waves are deduced fairly simply from this by applying the shock wave conditions to a first approximation. One then finds for the energy of the approximately conical blast wave emitted by the projectile, that it decays like the inverse fourth root of the distance travelled, the coefficient being known in terms of the size and speed of the projectile. This can then be checked quantitatively against the rate of degradation of energy in the shock waves, as calculated from the entropy change they produce.

Now, as some of you will be aware, several fierce mathematical papers had to be written by both Whitham [12] and myself [11], [13], and many careful checks applied, before we came to the relative simplicity of these ideas and had confidence in their essential soundness. Once this was achieved however, it was possible to use them in all sorts of steady and unsteady problems. To take only one instance, it was possible to show that the bangs which are heard as a result of aircraft accelerating and decelerating through the speed of sound, and which can be understood acoustically as due to the envelopes of the system of wavefronts, must in reality be lower in amplitude by factors of order 10 at

large distances than they would be in acoustic theory [14]. The reduction in wave energy is due to dissipation in the shock waves, which are what really correspond to the fictitious "envelopes of wavefronts." This energy decay probably follows fairly closely the inverse fourth root law which I mentioned before.

I hope I may have said enough to explain at least in part my contention that mathematical investigations have their greatest value when they lead to new physical ideas. But I could equally have spoken instead of the fundamental discoveries of R. T. Jones and his colleagues [15] on the yawed flow past projectiles, which used considerable mathematics on the splitting up of solutions of the Navier-Stokes equations, but which in their final form have a value which is almost purely qualitative; or (to take one more instance) the work of Lin and Lees on clarifying the physical processes governing the stability of the compressible boundary layer, which came only at the end of decades of mathematics.

In all these cases it was almost impossible to select the right idea in advance, because there were so many competing possibilities, and only the mathematics led to real confidence about what were the correct arguments, and to the picking of a single mechanism as the fundamental one. Later on, of course, in such a process, the idea in question becomes so well known that in the end everyone regards it as obvious, and designers will tell you that all this is quite clear to them on their engineer's intuition alone, and that they never make use of any high-flown mathematical stuff. This is simply part of the general phenomenon of the continuous rewriting of history.

Now I want to end my talk with a plea for doing it all in a different order. Put in another way, I think that a lot of us are publishing too much. We turn things out as soon as we have finished them, instead of putting them in a drawer and just thinking about them from to time. If I am one of the worst offenders in these respects that puts me in a good position to lecture the rest of you on the subject without causing offence!

If, when you have done one of these brilliant, involved bits of mathematics, you would just put it in a drawer, there would be so much the less for people to have to read! Then you could start gradually sorting out its meaning, in relation to the experimental data and to other bits of mathematics, until at last the basic physical ideas stood out quite clearly. Then, finally, you could write a *much* better paper! You would develop the subject in terms of these physical ideas and arguments right from the start, and include only the mathematics which was finally shown to be necessary in the light of them. The paper would be easier to read, and the reader would not have to change completely from one point of view to another at any stage. Of course you would not necessarily want to throw away all of the hard-won mathematics you had done originally, especially as some of it may still be valuable for increasing one's confidence in the physical arguments, but then what are appendices for?

To illustrate these principles I am going to describe briefly the work I have been doing on sound generated aerodynamically (and specially on jet

noise), in which I have been trying to follow them. I am using my own work as an illustration not because I doubt in the least that scientists have constantly adopted these methods, but rather because one cannot discuss other people's research methods unless one knows their "secrets of the study," which are not usually revealed. In any case you may probably wish to hear a little about how the work on jet noise is getting on.

When I started I thought it would be reasonable to look for an equation for the density, in which the fluctuating velocity field in the jet would appear but could be regarded as given. Outside the flow this equation for the density would be the equation of sound, and hence inside the flow it should be of the same form but with those quadratic terms, which the theory of sound neglects, retained on the right hand side. Actually various equations of this kind were possible, depending on the degree of approximation which was used on the right hand side. A solution in terms of an ordinary retarded potential then followed, with the integrand depending on the turbulent velocity field. The next question was how to take means; after a little thought it seemed reasonable to concentrate on the mean energy radiated as sound. This involved squaring the integral and taking mean values, and the result was a double integral involving mean products of quantities at two different points. This then had to be approximated to obtain the energy at large distances; and also a simplifying approximation for low Mach number was made, but this reduced the integral to zero. Next multiple applications of the divergence theorem were made to get the integral into a form where such a simplifying approximation could be applied without giving a zero answer. These showed that the sound output must depend on a high power of the Mach number, and they also showed just how little it was permissible to approximate in the equation of motion in the first place. Finally transformations were made indicating that a large mean shear should amplify the sound produced and cause it to be emitted at an angle of 45° to the shear layer. All this so far took about a fortnight, but at this stage it was all mathematics, and it did not seem to me to be very helpful in giving one ways in which one could really get a grasp of what was going on. So, before even writing up the *first half* of the work [17], I spent sixteen months thinking about how all the steps could be converted into physical arguments, starting at the end and working back to the beginning. And after this had been done of course the mathematics that was left was quite different. The second half of the work required another treatment. It was necessary to wait fourteen months longer, until a really reliable body of experimental information [18], [19], [20], [21] was available on the noise from a subsonic jet, before the relative importance of the various peripheral ideas of the subject could be established with confidence. I am happy to say however that the stage of submission of part II for publication is now very close at hand! I will spend my last five minutes trying to summarize briefly the physical ideas of which the theory as a whole is compounded.

Part I, which some of you may have seen, emphasized the distinction between three methods of sound generation. The most efficient one used fluctu-

ating sources of mass, which generate compression waves directly, as in the siren. The second, less efficient method operates by means of fluctuating sources of *momentum*, as when a bell vibrates. The *flux* of mass over fixed surfaces fluctuates, but not the net mass in a fixed region of space. If the bell is small compared with the wavelength, its motion is largely reciprocated by a local standing wave, and relatively little energy gets away as sound. Stokes showed that this was the explanation of the reduced sound output of a bell when hydrogen was added to the air in which it was sounded, so that the wavelength was augmented.

The third method of generation is by forcing the *flux of momentum* across fixed surfaces to vary, without any net fluctuation of momentum. This I call sound generated aerodynamically. The definition excludes any case where there is a fluctuating force between the fluid and a solid, as when sounding boards are present, or in phenomena like Aeolian tones. The fluctuating momentum flux in an airflow can be thought of in the manner of Reynolds as fluctuating external stresses applied to an acoustic medium at rest and causing it to execute forced oscillations. Acoustically these act like quadrupoles, because a fluctuating force acts like a dipole, and a stress field produces equal and opposite forces on both sides of a small element of fluid, and in consequence acts like a quadrupole distribution.

Naturally it is even more true of this third method of sound generation that its efficiency falls off rapidly at low frequencies, and in fact it follows from known properties of quadrupoles that the intensity at large distances varies as the square of the amplitude and the fourth power of the frequency. You can think of this as due to the fact that the four sources of which a quadrupole is composed cancel out in their effect at large distances, except in so far as the times when they emit sounds which arrive simultaneously are different, an effect obviously important only at high frequencies. It follows that for a jet the sound intensity varies approximately like the eighth power of the velocity, because the frequencies of the turbulence increase in proportion to the velocity of the jet.

Passing to the work of part II, it is necessary to make use of the fact that the values of the momentum flux at points with no eddy in common are uncorrelated, while its values at points with many eddies in common are well correlated. This leads to a result which one can express by saying that separate "average eddy volumes" give out their quadrupole radiation independently.

A second consequence of the statistical character of the turbulence is important for jet flows at high Mach number—and of course, as a result of the eighth power law, the noise is only really great at high Mach number. The eddies are carried along with the mean flow, and this distorts their sound field. In fact it greatly increases the sound radiated forwards at the expense of that radiated behind the jet, an effect which increases with Mach number. For the sound arriving simultaneously at a forward point from different parts of an

eddy is emitted from a set of points which fill a larger volume, and at a set of instants covering a longer interval of time, when the Mach number is large.

The last basic physical idea in the theory depends on the fact that the most important term contributing to the rate of fluctuation of momentum flux in the turbulence is the product of pressure and rate of strain. This is because high pressure at a point tends to make momentum flow outward in the direction of greatest expansion of the fluid element, and inwards in the direction of greatest contraction. One consequence is that in a heavily sheared turbulent layer the maximum sound radiated tends to be in the direction of the principal axes of rate of strain, that is to say at 45° to the shear layer. And in fact in a jet it is found that the high frequency sound emanating from the heavily sheared mixing region near the orifice has a maximum close to 45°, or slightly lower because of the distortion of the pattern in favour of downstream emission which I mentioned as being due to Mach number.

But rather more of the sound from a jet is at much lower frequencies, and comes from at least ten diameters downstream of the orifice, where the turbulence spreads over the whole jet, and in this region the *fluctuations* in the rate of strain exceed the mean shear. Accordingly there is much less special directional distribution in this low frequency sound, except the basic preference for forward emission, and calculations based on the assumption of isotropic turbulence [22] seem to be quite adequate.

I have had to rush through these details, but I hope I may have indicated the spirit of the investigation. A full account of the work will soon be available [23].

To sum up, one can make a marriage between mathematical and physical modes of argument, in which each successively aids the other; and only then is one attacking a compressible flow problem with the best chances of success. A mathematical result holds under precise conditions, so it is not sufficiently flexible to be applied in the complex circumstances of life. A physical idea may be too vague, and give in too many places when you probe it to find out what it really means. But what I wish to put to you is that a physical idea *founded on mathematics* is like good spring steel, which though you deflect it almost indefinitely from its original position will spring back unerringly into its correct shape.

Bibliography

[1] Gadd, G. E., *On the interaction with a completely laminar boundary layer of a shock wave generated in the mainstream,* Aeronautical Research Council, Report No. 15, 100 (unpublished).

[2] Whitham, G. B., *The flow pattern of a supersonic projectile,* Communications on Pure and Applied Mathematics, Volume 5, 1952, p. 301.

[3] Oswatitsch, K., and Wieghardt, K., *Theoretical analysis of stationary potential flows and boundary layers at high speed,* National Advisory Committee for Aeronautics, Technical Memorandum No. 1189, 1948, reprinted from German wartime report, 1941.

[4] Liepmann, H. W., Roshko, A., and Dhawan, S., *On the reflection of shock waves from boundary layers*, Guggenheim Aeronautical Laboratory, California Institute of Technology, Report, 1949, National Advisory Committee for Aeronautics, Technical Note No. 2334, 1951.

[5] Lees, L., *Interaction between the laminar boundary layer over a plane surface and an incident oblique shock wave*, Princeton University Aeronautical Engineering Laboratory, Report No. 143, 1949.

[6] Lees, L., and Crocco, L., *A mixing theory for the interaction between dissipative flows and nearly-isentropic streams*, Princeton University Aeronautical Engineering Laboratory, Report No. 187, 1952.

[7] Howarth, L., *The propagation of steady disturbances in a supersonic stream bounded on one side by a parallel subsonic stream*, Proceedings of the Cambridge Philosophical Society, Volume 44, 1948, p. 380.

[8] Tsien, H. S., and Finston, M., *Interaction between parallel streams of subsonic and supersonic velocities*, Journal of the Aeronautical Sciences, Volume 16, 1949, p. 515.

[9] Lighthill, M. J., *Reflection at a laminar boundary layer of a weak steady disturbance to a supersonic stream, neglecting viscosity and heat conduction*, Quarterly Journal of Mechanics and Applied Mathematics, Volume 3, 1950, p. 303.

[10] Lighthill, M. J., *On boundary layers and upstream influence*, Parts I and II, Proceedings of the Royal Society, Volume 217, pp. 344–357 and 378–407.

[11] Lighthill, M. J., *The energy distribution behind decaying shocks. I, Plane waves*, Philosophical Magazine, Volume 41, 1950, p. 1101.

[12] Whitham, G. B., *The behaviour of supersonic flow past a body of revolution, far from the axis*, Proceedings of the Royal Society, Volume 201, 1950, p. 89.

[13] Lighthill, M. J., *A technique for rendering approximate solutions to physical problems uniformly valid*, Philosophical Magazine, Volume 40, 1949, p. 1179.

[14] Warren, C. H. E., *Sonic bangs, a quantitative explanation*, Royal Aircraft Establishment, Technical Note No. 2192, Farnborough, 1952.

[15] Allen, H. J., and Perkins, E. W., *A study of effects of viscosity on flow over slender inclined bodies of revolution*, National Advisory Committee for Aeronautics, Report No. 1048, 1951.

[16] Lees, L., *The stability of the laminar boundary layer in a compressible fluid*, National Advisory Committee for Aeronautics, Technical Note No. 1360, July 1947.

[17] Lighthill, M. J., *On sound generated aerodynamically. I, General theory*, Proceedings of the Royal Society, Volume 211, 1952, p. 564.

[18] Westley, R., and Lilley, G. M., *An investigation into the noise field from a small jet and methods for its reduction*, College of Aeronautics, Report No. 53, Cranfield, 1952.

[19] Gerrard, J. H., *Experiments on jet noise*, to appear.

[20] Lassiter, L. W., and Hubbard, H. H., *Experimental studies of noise from subsonic jets in still air*, National Advisory Committee for Aeronautics, Technical Note No. 2757, 1952.

[21] Fitzpatrick, H. M., and Lee, R., *Measurements of noise radiated by subsonic air jets*, David Taylor Model Basin, Report No. 835, 1952.

[22] Proudman, I., *The generation of noise by isotropic turbulence*, Proceedings of the Royal Society, Volume 214, 1952, p. 119.

[23] Lighthill, M. J., *On sound generated aerodynamically. II. Turbulence as a source of sound*, to appear in Proceedings of the Royal Society.

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. VII, 11–17 (1954)

Boundary Layer Problems in Applied Mathematics

By G. F. CARRIER
Harvard University

1. *Introduction*

In two recent paper, M. J. Lighthill [1] and the author [2] have discussed techniques which may be used profitably on certain problems whose solution by conventional techniques may be extremely difficult. Here, the boundary layer technique is shown to be applicable to equations of the type treated by Lighthill and the advantages of the alternative representations to which these methods lead are compared. Since the methods under discussion here are outlined thoroughly in [1] and [2], we shall confine ourselves to a very concise description of the motivations and formalities.

2. *A Typical Problem*

We can most readily characterize the type of problem to which each method may apply by discussing a specific example. We consider, therefore,[1]

$$(x + \epsilon w)w' + \left(1 + \frac{2x^2}{1 + x^2}\right)w + 1 + 3x = 0 \tag{2.1}$$

where $\epsilon \ll 1$. The boundary condition is $w(1) = 0$ and we wish to find[2] $w(0)$.

If we now try a formal perturbation in ϵ (or an approximation where the ϵ term is omitted), we obtain as a first approximation

$$w_0(x) = \frac{c}{x(1 + x^2)} - \frac{1 + \frac{3x}{2} + \frac{x^2}{3} + \frac{3x^3}{4}}{1 + x^2} \tag{2.2}$$

c must take the value 43/12. It is clear that this approximation is useful only when x is not too close to the origin and that neither it nor the conventional perturbation series of which it is the first term will determine $w(0)$ for us. However, we may adopt either the boundary layer arguments and procedure or the Lighthill technique and obtain a satisfactory answer to this problem. The former proceeds as follows: We first note that the term in ϵ can become important only if w becomes large compared to unity. We also note that $w_0(x)$

[1] This is a specific example of the problem type treated in 1.

[2] It would be equally appropriate to ask for $w(x)$ with $x < 0$.

becomes large only when $x \ll 1$. In order to represent the terms in equation (2.1) in such a manner that their coefficients truly represent their order of magnitude we may make the following substitutions:

$$x = \epsilon^a z, \qquad w = \epsilon^b v. \tag{2.3}$$

We anticipate that $a > 0$, $b < 0$, in order that z and v will vary over a range of values of order unity in the region of interest. The substitution of the quantities defined in equation (2.3) into equation (2.1) leads to the following equation

$$(z + v)v' + v + O(\epsilon^{-b}) = 0 \tag{2.4}$$

provided $a - b = 1$. This choice is essential if the term in ϵ is to contribute equally (in order of magnitude) with the xw' and w terms. If we now ignore the third term of equation (2.4), the solution of this equation should provide an approximation which is valid in any region where the omitted term is small compared to the individual retained terms (v, for example). The solution of equation (2.4) can be written in the form

$$zv + \frac{v^2}{2} = c'. \tag{2.5}$$

However, for $x \ll 1$ (but not so small that w_0 is no longer a good approximation to the solution)

$$w_0 \simeq \frac{c}{x} - O(1)$$

and (if $w_0 = \epsilon^b v_0$)

$$v_0 = \left(\frac{c}{z}\right)\epsilon^{-a-b} + O(\epsilon^{-b}). \tag{2.6}$$

For those larger values of z for which equation (2.5) is a good approximation to a solution of equations (2.1), (2.6) and (2.5) should predict the same behavior if the method is to succeed. In this case, we choose $a + b = 0$, and $c' = c$ which implies that $a = 1/2$, $b = -1/2$. Thus, for $w(0)$, we have

$$w(0) = \left(\frac{2c}{\epsilon}\right)^{1/2}. \tag{2.7}$$

In principle, of course, one can continue the perturbations of which equations (2.2) and (2.5) provide the leading terms, but in practice this is sometimes very difficult.

The Lighthill method, as applied to the same problem, proceeds in the following manner: After noting the difficulties encountered in the conventional perturbation wherein we obtain equation (2.2), we arbitrarily introduce a new independent coordinate z and define the parametric representations

$$w = w_0(z) + \epsilon w_1(z) + \cdots$$

(2.8)

$$x = z + \epsilon x_1(z) + \cdots .$$

The essential object is to introduce functions $x_i(z)$ which are singular at $z = 0$ so that $x = 0$ corresponds to a positive value of z and so that, at this value of z, both series defined in equation (2.8) converge. Under these definitions, equation (2.1) takes the form

$$(zw_0') + \left[1 + \frac{2z^2}{(1+z^2)}\right]w_0 + 1 + 3z$$

$$+ \epsilon\left[w_0w_0' + x_1w_0' + zw_1' + (w_1 + w_0x_1')\left(1 + \frac{2z^2}{(1+z^2)}\right)\right. \tag{2.9}$$

$$\left. + \frac{4x_1zw_0}{(1+z^2)^2} + 3x_1 + (1+3z)x_1'\right] + O(\epsilon^2) = 0.$$

It is clear that $w_0(z)$ as defined by equation (2.2) annihilates the terms of order unity and we may choose x_1 to be a solution of

$$[(w_0 + 1)x_1]' + w_0w_0' = 0. \tag{2.10}$$

In particular, we may choose,

$$x_1 = -\frac{w_0^2}{2(w_0+1)} . \tag{2.11}$$

This choice (i.e., the choice of equation (2.10)), is made for two reasons: it is easy to solve and it puts equation (2.9) in such a form that w_1 is again of order $(1/z)$. The reader, after some manipulation can satisfy himself that $x_2(z)$ can be chosen so that $w_2(z)$ is again of order $(1/z)$ and so that, in the series,

$$x = z - \frac{\epsilon c^2}{2z(c+z)} + \epsilon^2 x_2(z) + \cdots , \tag{2.12}$$

the contribution of x_2 (and the later x_i) provide only a small correction to the root of $x(z) = 0$.

Since

$$w(x) = \frac{c}{z} + O\left(\frac{\epsilon}{z}\right), \tag{2.13}$$

Equations (2.12) and (2.13) can now provide us with a value for $w(0)$. We obtain, in fact,

$$w(0) = \left(\frac{2c}{\epsilon}\right)^{1/2} \tag{2.14}$$

which agrees with equation (2.7).

Before proceeding with a comparison of the advantages of the two methods,

we shall proceed to study two more examples which, superficially, are mild modifications of the foregoing problem.

3. *A "Higher Order" Problem*

In this section we shall consider the problem wherein the linearized coefficient of the term in w' has a second order zero. In particular we shall find the solution of

$$(x^2 + \epsilon w)w' + w - (2x^3 + x^2) = 0 \tag{3.1}$$

with $w(1) = 1 + e$. Again the approximate solution as generated by the linearized equation is singular at the origin. It is, in fact,

$$w_0 = e^{1/x} + x^2. \tag{3.2}$$

Proceeding by the Lighthill technique, precisely as in the preceding section, we obtain without difficulty

$$x = z - \epsilon[e^{1/z}(1 - 2z + z^2) + O(1)] + \epsilon^2[Oe^{2/z}] + \cdots, \tag{3.3}$$

$$w(x) = e^{1/z} + z^2 + \epsilon e^{1/z}[O(1)] + \cdots. \tag{3.4}$$

A careful analysis of the asymptotic behavior of $w(0)$ as $\epsilon \to 0$ provides the result

$$w(0) \sim \left[\epsilon \ln\left(\frac{1}{\epsilon}\right)\right]^{-1}. \tag{3.5}$$

The same problem, using the boundary layer technique, is not as straight forward and requires more detail. If we first were to proceed as in the foregoing section with the substitution $x = \epsilon^a z$, $w = \epsilon^b v$, we would find that the $v(z)$ obtained did not blend in with $w_0(x)$. There would, in fact, be no overlap in the regions of validity of the two solutions. However, a bit of thought about the nature of the solution of equation (3.1) indicates that one should expect three distinct regions: one in which w_0 is a good approximation to the solution and in which the $\epsilon ww'$ term is unimportant, one in which ϵw is of the same order as x^2 and one in which $x^2 \ll \epsilon w$. If we proceed to the negative x-axis, in fact, we should expect another region still in which x^2 and ϵw become comparable again.

The substitution we require in order to extablish the solution in the second zone is, then

$$x = \epsilon^\alpha + \epsilon^\beta z, \qquad w(x) = \epsilon^\gamma v(z) \tag{3.6}$$

We anticipate $\gamma < 0 < \alpha < \beta$. With this substitution, equation (3.1) takes the form

$$(\epsilon^{2\alpha+\gamma-\beta} + \epsilon^{1+2\gamma-\beta}v)v' + \epsilon^\gamma v + \cdots = 0. \tag{3.7}$$

If we choose $2\alpha = \beta = 1 + \gamma$, the three dots represent terms of higher order in ϵ. Omitting these terms, we have

$$(1 + v)v' + v = 0 \tag{3.8}$$

and

$$\ln v + v = c - z. \tag{3.9}$$

For $z \gg 1$ (but not so large that the omitted terms are important) v is essentially given by

$$v \simeq e^{c-z}. \tag{3.10}$$

In this region however, w_0 reduces to

$$w_0 = \epsilon^{\gamma} v_0 = e^{1/x} + x^2 \simeq \exp\{\epsilon^{-\alpha} - z\} + O(\epsilon^{2\alpha}).$$

Thus, with $\epsilon^{\gamma} = e^{\epsilon^{-\alpha}}$ and $c = 0$, w_0 and v are mutually consistent. Since $\gamma + 1 = 2\alpha$, the equation by which we determine α is

$$(1 - 2\alpha) \ln\left(\frac{1}{\epsilon}\right) = \left(\frac{1}{\epsilon}\right)^{\alpha}. \tag{3.11}$$

In particular, as $\epsilon \to 0$, $\alpha \sim ln\ ln\ (1/\epsilon)/ln\ 1/\epsilon$, and for $\epsilon = .001$, $\alpha = .2035$.

In the region where x^2 is to be unimportant we could again introduce formal substitutions but it is quite clear that the equation which determines w is

$$\epsilon w w' + w = 0 \tag{3.12}$$

so that

$$w_1 = k - \frac{x}{\epsilon}. \tag{3.13}$$

As z gets negative and moderately large, v, as given by equation (3.9), behaves like $-z$, or

$$w \sim -\epsilon^{\gamma-\beta}(x - \epsilon^{\alpha}). \tag{3.14}$$

Thus, in order that equations (3.13) and (3.14) be consistent, we choose $k = \epsilon^{\alpha-1}$. Thus, again, the asymptotic behavior of $w(0)$ as $\epsilon \to 0$ is given by $[\epsilon\ ln(1/\epsilon)]^{-1}$. For $\epsilon = .001$, the two methods give (respectively) the values 256 and 248 for $w(0)$. These values are clearly not obtained by the asymptotic formula $w_0(1) = 1/\epsilon ln(1/\epsilon)$ but by accurately evaluating α (via equation [3.11]) and finding the root of equation (3.3) for $x = 0$.

A similar problem, equation (3.1) with the final parenthesis omitted, admits a rigorous solution since the equation for $x(w)$ is a Riccati equation whose solution is

$$x = -y \frac{J_1(2y) + AN_1(2y)}{J_0(2y) + AN_0(2y)} \tag{3.15}$$

with $y = (\epsilon w)^{1/2}$. If we choose A so that $w(1) = 1$ and evaluate w at $x = 0$, we find, asymptotically, $w(0) = 1/\epsilon ln(1/\epsilon)$, and for $\epsilon = .001$, we find $w(0) =$

252. A treatment of this problem by the two methods under discussion here indicates that the representations obtained are essentially identical with those of equation (3.1) near $x = 0$. In other words, the boundary layer attack, the rigorous analysis, and the Lighthill method yield the same asymptotic result and, for $\epsilon = .001$, the respective values 248, 252, 256.

4. *Relaxation Oscillations of the van der Pol Equation*

The Van der Pol equation can be written in the form (where $v' \equiv v_u$)

$$\epsilon v v' - (1 - u^2)v + u = 0. \tag{4.1}$$

The substitution, $v = f(u)/\epsilon + w$, with $f(u) = u - u^3/3 - 2/3 - C\epsilon^{2/3}$, modifies this equation so that it takes the form

$$(f(u) + \epsilon w)w' + u = 0. \tag{4.2}$$

A boundary layer analysis of the problem of finding the solution of this equation corresponding to periodic $u(t)$, $v(t)$, is given in [2]. There are four representations corresponding to four domains, much as there were three in the problem of the preceding section. I had hoped, when I initially undertook this investigation that the Lighthill technique would provide a solution which could systematically be improved (this is extremely difficult in the boundary layer analysis) and, perhaps, provide a single representation for the entire result. Unfortunately, it so happens that equations like equation (4.1) in which: 1) $f(u)$ has either a double zero at some point in the domain or two simple zeros located (in the complex plane) near a point in the domain of integration, and 2) has a coefficient on the w term which vanishes, cannot be treated successfully by the Lighthill technique.

The positive contribution of this section, then, is the observation that, when problems of the type

$$(f(\alpha) + \epsilon w)w' + g(x) = 0 \tag{4.3}$$

(and, quite probably, other systems apparently falling within the scope of certain methods) are encountered, and when one of these techniques happens to fail, the substitutions $w = v - f/\epsilon$ will put the problem into the conventional boundary layer form and the conventional boundary layer ideas can be applied. It is to be noted that the problems of sections 2 and 3 were not of the type *conventionally* treated by that technique. In fact, most boundary layer problems are such that the powers of ϵ (a and b of equation (2.3)), by which the functions are stretched, are determined before the boundary layer equations (equation (2.4)) are solved. Examples of the more conventional type of boundary layer problem may be found in [2] and the papers of its bibliography.

5. *General Discussion*

It is clear now that the domains of applicability of these methods overlap considerably and that each method has its advantages. For those problems

where each method works the Lighthill technique has the following advantages: 1) The solution can be systematically improved in many cases merely by computing further terms in the series expansions. This is less easy to do in the boundary layer case and frequently does not improve the result. 2) A single representation is obtained.

The boundary layer method has the advantage that the individual representations in different domains allow an easier visualization of the functional behavior in these domains than does the parametric representation. It also seems to be more broadly applicable. For example, in the problem of section 3, the boundary layer method can be used to continue the solution into the large negative x-domain. The Lighthill series does not appear to give a useful representation in this region.

Perhaps the most useful notion, however, would be to use these two techniques as a common tool. Again, in the problem of section 3, the Lighthill technique could be used to get a highly accurate value for w near $x = 0$ and the boundary layer method would then be used (almost superficially) to continue the solution into the negative x-domain.

Although we have discussed only ordinary differential equations in this paper, there is also a variety of problems which give rise to partial differential equations whose solution may be effected by these methods. The reader is again referred to [1] and [2] for a more complete discussion of the methods, and examples of the treatment of partial differential equations.

Bibliography

[1] M. J. Lighthill, *A technique for rendering approximate solutions to physical problems uniformly valid*, Philosophic Magazine, Volume 7, XL, p. 1179, Dec. 1949.

[2] G. F. Carrier, *Boundary Layer Problems in Applied Mechanics, in Advances in Applied Mechanics*, Vol. III, Academic Press, 1953.

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. VII, 19–44 (1954)

Fourier Synthesis of Homogeneous Turbulence

By GARRETT BIRKHOFF
Harvard University

1. *Summary*

An attempt is made to clarify the foundations of the statistical theory of turbulence. For each instantaneous energy distribution $\mathcal{E}(K) = \int_0^K E(k)dk$, a specific *measure* is constructed in the space Ω of velocity-fields $\vec{u}(\vec{x})$. This measure is homogeneous, isotropic and "Gaussian" in the sense that quantities depending linearly on the velocity-field are normally distributed. The existence of such a measure will be called the First Consistency Principle (cf. §13).

In Part I, the preceding measure is constructed and shown to be finitely additive. The method is based on applying the Central Limit Theorem to a random superposition of Boussinesq flows, random in orientation and phase, so that the limiting measure will be isotropic and homogeneous. The technical difficulties are algebraic, and consist largely in extending the Central Limit Theorem from finite-dimensional to infinite-dimensional vector spaces.

In Part II, a σ-additive measure is constructed, which is essentially equivalent to that of Part I. The technical methods are not easy, but they have the added interest of being applicable to random solutions of general linear partial differential equations with constant coefficients.

In Part III, the statistics so derived are correlated with turbulence theory. It is shown that they are consistent with the accepted kinematic theory [2, Chaps. II-III], except that they indicate that, in general, $f(r) = O(r^{-2})$ instead of $f(r) = o(r^{-3})$, as suggested by Batchelor. Also, arguments are given which indicate that they are metrically transitive. The fact that the statistics are *normal,* and hence *reversible,* implies however that there is no interchange of energy between different "eddy sizes"; hence the statistics are inadequate for the dynamics of turbulence, except in the "final period" [1; 2, Chap. VII].

Part IV is concerned with the final period of decay. It is shown that the measures introduced are self-consistent in the final period (Theorem 10); moreover the assumption that $r^3 f(r) \to C \neq 0$ is shown to be consistent with the Navier-Stokes equations. As this possibility is excluded by Batchelor's theory of the final period, it seemed worth reviewing critically the experimental evidence supporting his theory; this is done in §§19–20.

I. MEASURE FOR VELOCITY FIELDS

2. *Turbulence as a Stochastic Process*

The idea that a complete statistical theory of turbulence should provide a measure in the "function space" of velocity fields is not new. It has been

stressed by Kampé de Fériet,[1] and is one with which every connoisseur of modern probability theory must agree.

In its broad outlines, such a measure is defined on an abstract "phase space" Ω, whose elements ω are velocity-fields $\vec{u}_\omega(\vec{x})$ in ordinary Euclidean space.[2] That is, it defines at each instant t a *measure* m_t, defined for suitable "measurable" subsets S of Ω. To be physically useful, it must satisfy the following postulates.

POSTULATE I: The measure must be "σ-additive" in the usual sense.[3]

POSTULATE II: Any quantity of physical interest[4] must be a "measurable" function on Ω.

Thus, since the velocity-component $u_1(\vec{x})$ at a point $\vec{x}$ may be of physical interest, the set $u_1(\vec{x}) < U$ must be measurable in Ω, for any real number U. Again, the second moment functions[5] $R_{ij}(\vec{h})$, defined by

$$R_{ij}(\vec{h}) = \int_\Omega u_i(\vec{x})u_j(\vec{x} + \vec{h})\, dm_\omega \,, \tag{1}$$

must exist in the sense of Lebesgue integration. So must the third moment functions

$$T_{ij,k}(\vec{h}) = \int_\Omega u_i(\vec{x})u_j(\vec{x})u_k(\vec{x} + \vec{h})\, dm_\omega \,. \tag{1'}$$

POSTULATE III: Each measure must be homogeneous and isotropic in the following sense.

DEFINITION: A measure m will be called *homogeneous* if it is invariant under translations of space; *isotropic* if it is invariant under rotations. Similarly, it will be called *reversible* in case it is invariant under $\vec{u}(\vec{x}) \to -\vec{u}(\vec{x})$.

(Here and below, when speaking of the invariance of measure, we mean that the class of measurable sets, as well as their measure, should be invariant.)

In the present paper, we shall consider only the case that Ω is the space of velocity fields $\vec{u}(\vec{x})$ on three-dimensional infinite "physical" space X (actually, fields for which Div $\vec{u} = 0$ will have measure one). The case that Ω consists of solutions $\vec{u}(\vec{x}, t)$ of the Navier-Stokes equations of motion in space-time is, of

---

[1] *Sur la mécanique statistique des mileux continus*, Actualité Sci. Ind. 1137, Paris, 1949, pp. 129–144. See also [2, §5.1 and §8.2], and E. Hopf, *Statistical hydrodynamics and functional calculus*, Journal of Rational Mechanics and Analysis, Volume I, 1952, pp. 87–124.

[2] The space Ω may be imagined as defined by some limiting process from an infinite sequence of laboratory experiments ω_1, ω_2, ω_3, $\cdots$, each producing a velocity-field; cf. [2, §2.1].

[3] This means that, if S_1, S_2, S_3, $\cdots$ are disjoint measurable sets $m(VS_i) = \Sigma\, m(S_i)$.

[4] Of course, this is a wish sometimes difficult to fulfill; even functions, introducing themselves quite naturally in probability theory, are not measurable, in some well known and simple problems; cf. J. L. Doob, *Stochastic Processes*, New York, Wiley, 1953, pp. 50–71.

[5] These are more convenient mathematically than the dimensionless correlation functions $R_{ij}(\vec{h})/u^2$. Our notation is thus that of [2].

course, also of great interest for turbulence. The former space involves the *kinematics*; the latter the *dynamics* of turbulence.[6]

3. *Random Fourier Synthesis*

All velocity fields of class L_2, and all triply periodic velocity fields, can be expressed as superpositions of *sinusoidal* velocity fields, obtained by translating and rotating velocity fields of the special form

$$u_2(\vec{x}) = \sin kx_1 , \qquad u_1(\vec{x}) = u_3(x) \equiv 0. \tag{2}$$

These facts follow immediately from the general theory of Fourier series and integrals, and the equation $\mathrm{Div}\ \vec{u} = 0$.

It is a short step from this to the idea that a turbulent velocity field is a *random* superposition of such "Boussinesq flows"; this idea underlies what may be called the *Fourier synthesis* of turbulence.[7] However, the giving of a precise mathematical content to this idea is not easy; I have been considering the problem for five years!

To develop Taylor's idea, note that for each wave-number $k > 0$, one obtains by rotating and translating (2) under the Euclidean group E_3, the compact 4-parameter set S_k of transforms ("Boussinesq flows")

$$u_\iota(x) = a_\iota \sin (\vec{k}\cdot\vec{x} + \delta). \tag{2'}$$

For each $\gamma \in E_3$, the wave vector $\vec{k} = (k_1 , k_2 , k_3)$ is the transform of $(k, 0, 0)$ under γ, $\vec{a} = (a_1 , a_2 , a_3)$ is the orthogonal velocity vector, and δ is a suitable phase constant. The four parameters are the *direction* of $\vec{k}$, the direction of $\vec{a}$, and the phase δ; clearly $k_1^2 + k_2^2 + k_3^2 = k^2$ and $a_1^2 + a_2^2 + a_3^2 = 1$. There is a unique measure[8] on S_k, which is invariant under the Euclidean group (i.e., homogeneous and isotropic), and has total measure one. In physical language, it assigns equal probabilities to all phases and to all directions of periodicity and velocity.

One should therefore, in synthesizing homogeneous isotropic turbulence, superpose the velocity-fields of Boussinesq flows (2′) picked *at random*, with respect to the preceding measure and the energy density $E(k)$. By an extension of the Central Limit Theorem, this should lead to a normal velocity distribution

---

[6]See [5, Chapter VIII], and Chapter XIV (by J. Kampé de Fériet) of A. Blanc-Lapierre and R. Fortet, *Théorie des fonctions aléatoires*. In these references, measures are defined for the space Ω of velocity-fields in space. The essential contribution of Parts I-II of this paper consists in constructing explicitly a Gaussian measure for each $\mathcal{E}(K)$, which has many of the essential characteristics of physical homogeneous, *isotropic* turbulence.

[7]It is due to Sir Geoffrey Taylor; see [2, §2.5]. The analogous concept for gravity waves was worked out, with applications, in [3].

[8]L. H. Loomis, Duke Mathematical Journal, Volume 16, 1949, pp. 193–208.

much like those observed in actual turbulence—i.e., to a Gaussian measure in the "space" of velocity fields.

We shall now give this idea a rigorous formulation.

4. *Gaussian Convolution*

Let Φ be any family of vector-valued functions $\vec{u}(\vec{x}, \phi)$ of a variable $\vec{x}$, and let μ be any non-negative, additive set-function (or "mass-function") on Φ, with respect to which $\vec{u}(\vec{x}, \phi)$ is quadratically integrable, for each fixed $\vec{x}$, as a function of ϕ. Consider the *double correlation function* defined by the Riemann integral

$$R_{ij}(\vec{x}, \vec{y}) = \int_{\Phi} u_i(\vec{x}, \phi)u_j(\vec{y}, \phi)\, d\mu_\phi . \tag{3}$$

This is *symmetric*, in the sense that

$$R_{ij}(\vec{x}, \vec{y}) = R_{ji}(\vec{y}, \vec{x}), \tag{4a}$$

and *non-negative definite*, in the sense that, for any $3n$-vector $C = (c_{11}, \cdots, c_{3n})$,

$$\sum c_{il}c_{jr}R_{ij}(\vec{x}_l, \vec{x}_r) = \int [\sum c_{il}u_i(\vec{x}_l, \phi)]^2\, d\mu_\phi \geqq 0. \tag{4b}$$

Hence[9] there is a unique Gaussian measure m (normal multivariate distribution) on the $3n$-dimensional vector space V of $u_i(\vec{x}_l)$, which has first moments

$$\overline{u_i(\vec{x}_l)} = \int_{\psi} u_i(\vec{x}_l, \psi)\, dm_\psi = 0 \qquad l = 1, \cdots, n, \tag{5}$$

and second moments given by[10]

$$\overline{u_i(\vec{x}_l)u_j(\vec{x}_r)} = \int u_i(\vec{x}_l, \psi)u_j(\vec{x}_r, \psi)\, dm_\psi = R_{ij}(\vec{x}_l, \vec{x}_r). \tag{5*}$$

These measures are all *consistent*, since they are all normal and defined by the same first and second moments (5)–(5*). Hence we may regard m as defining a *single* finitely additive Gaussian measure on the space ψ of all $\vec{u}(\vec{x})$. Equivalently, it defines a finitely additive Gaussian measure on the space of (almost periodic) finite linear combinations of Boussinesq flows (2′).

DEFINITION. The *Gaussian convolution* of the $\vec{u}(\vec{x}, \phi)$ over the *intensity function* μ, is the finitely additive Gaussian measure defined by (5)–(5*) on the space ψ of velocity fields.

---

[9] H. Cramer, *Random Variables and Probability Distributions*, Cambridge University Press, 1937, especially Chapter X.

[10] I have not tried to characterize $R_{ij}(\vec{x}, \vec{y})$ abstractly. This was done in a special case by H. Cramer, Annals of Mathematics, Volume 41, 1940, pp. 215–230.

5. *Motivation*

In order to motivate the preceding definition of Gaussian convolution, we now let Φ denote *any* subset of *any* finite-dimensional vector space V of elements $\vec{u}$, and let μ be any finitely additive mass-function on Φ. Also, we let α, α_i, α_j, $\cdots$ denote linear functions on V (elements of the "dual space" V^* of V).[11] Then clearly

$$\overline{\alpha_i\alpha_j(\mu)} = \int_\Phi \alpha_i(\vec{u}_\phi)\alpha_j(\vec{u}_\phi)\, d\mu_\phi \tag{6}$$

is a non-negative quadrative function on V^*. Hence, as before,[9] there is one and only one *Gaussian* measure m on V such that

$$\int_V \alpha_i(\vec{u})\alpha_j(\vec{u})\, dm_{\vec{u}} = \overline{\alpha_i\alpha_j} = \int_\Phi \alpha_i(\vec{u}_p)\alpha_j(\vec{u}_p)\, d\mu_p\,. \tag{6*}$$

For instance, in the usual positive definite case, if $(x_1, \cdots, x_n)$ is a coordinate system on V, and we write $\overline{x_ix_j} = b_{ij}$, the associated probability density is[9] proportional to

$$\rho = \exp\{-\sum a_{ij}x_ix_j\}, \tag{7}$$

where $A = \| a_{ij} \|$ is the matrix inverse of $B = \| b_{ij} \|$.

The use of the term "convolution" may now be motivated quite simply. Let m and m' denote the Gaussian measures associated by (6)–(6*) with the mass-functions μ and μ'. By (6),

$$\overline{\alpha_i\alpha_j}(\mu + \mu') = \overline{\alpha_i\alpha_j}(\mu) + \alpha_i\alpha_j(\mu').$$

On the other hand, the *convolution* $m*m'$ is Gaussian,[12] and also satisfies

$$\overline{\alpha_i\alpha_j}(m*m') = \overline{\alpha_i\alpha_j}(m) + \alpha_i\alpha_j(m'). \tag{8}$$

Hence we deduce $m(\mu + \mu') = m(\mu)*m(\mu')$.

A second motivation may also be given. In typical applications to turbulence, Φ will be compact, and μ will be invariant under the *reversal* transformation R: $\vec{u} \to -\vec{u}$. In any such case, we can suppose the total mass $\mu(\Phi)$ to be subdivided by a partition π into n parts $\mu_1, \cdots, \mu_n$, *each* invariant under R. We can form the convolution

$$m_\pi = \mu_1 * \cdots * \mu_n$$

of these partial measures. By symmetry, each $\int_\Phi \alpha(\vec{u})d\mu_k = 0$. Hence, by the Central Limit Theorem,[9] as $|\pi| = \sup \mu_i(\Phi)$ tends to zero, the m_π tends to a

[11] G. Birkhoff and S. MacLane, *Survey of Modern Algebra*, rev. ed., p. 195. For quotient-spaces, see p. 328; for lattices of subspaces, p. 352.

[12] See J. L. Doob, Annals of Mathematical Statistics, Volume 14, 1944, pp. 229–282.

Gaussian distribution m. Further, by the usual law (8) for the addition of second moments under convolution, and induction,

$$\int_V \alpha_i(\vec{u})\alpha_j(\vec{u})\, dm_\pi = \int_\pi \alpha_i(\vec{u})\alpha_j(\vec{u})\, d\mu_\phi\,, \qquad \text{for all } \pi.$$

Passing to the limit, we see that m is the Gaussian convolution of μ.

A third motivation is even more suggestive. For any measurable set Z, let $\mu(Z)/\mu(\Phi)$ denote the measure function associated with the mass-function μ. For each n, let velocity fields $\vec{u}_{\phi(1)}$, $\cdots$, $\vec{u}_{\phi(n)}$ be picked from Φ, at random with respect to the measure $\mu(Z)/\mu(\Phi)$ and completely independently. Then, by the Central Limit Theorem again, the *sum* of the random functions $\vec{u}_{\phi(k)}$, multiplied by the scale factor $\mu(\Phi)$, tends as $n \to \infty$ to the Gaussian convolution of the $\vec{u}_\phi$ over μ. In this sense, $m(\Omega)$ defines the effect of *superposition of velocity-fields of* Φ, *chosen at random from* Φ.

6. *Extension to Infinite-Dimensional V*

It is quite easy to extend the preceding results to infinite-dimensional V, on the algebraic level, of operations combining only finite sets of elements at a time. The essential point is to consider only finite-dimensional quotient spaces[11] V/S.

This might seem adequate to many physicists. Thus, it might even seem sufficient to restrict attention to values of $\vec{u}(\vec{x})$ at *finite* sets of points $\vec{x}_1$, $\cdots$, $\vec{x}_n$. This would give a measure in the usual sense of Kolmogoroff-Khinchine, that for any $\vec{x}_1$, $\cdots$, $\vec{x}_r$, the set

$$u_1(\vec{x}_1) < \gamma_{11}\,, \cdots, u_3(\vec{x}_r) < \gamma_{3r} \tag{9a}$$

is measurable.[13] Indeed, so far as correlation functions are concerned, this class of measurable sets is adequate.

More generally, let L be any lattice[11] of subspaces S, T, $\cdots$ of a vector space V, each having a finite-dimensional quotient-space V/S. Then the discussion of §§4–5 applies to each V/S. Moreover, comparing the measure on $V/S \cap T$ with those on V/S and V/T, we see that Gaussian convolution always gives a consistent, *finitely* additive measure.

In particular, we may let V be the space Ω of velocity-fields on physical space X, and let Φ be the family of Boussinesq flows (2′). For any energy distribution $\mathcal{E}(K)$ with finite $\mathcal{E}(+\infty)$, a unique measure μ is defined on Φ which is invariant under the Euclidean group (and hence under R), by §3.

On the subspace Ω_1 of bounded velocity fields, we consider linear functionals α associated with finite mass-functions ν_1 , ν_2 , ν_3 on physical space X, by the definition

$$\alpha(\vec{u}) = \int u_1(\vec{x})\, d\mu_1 + \int u_2(\vec{x})\, d\mu_2 + \int u_3(\vec{x})\, d\mu_3\,.$$

[13] See J. L. Doob, Annals of Mathematical Statistics, Volume 14, 1944, pp. 229–282.

Evidently, Ω_1 contains the vector space V_1 of finite linear combinations of Boussinesq flows.

By the usual Central Limit Theorem for (finite-dimensional) vectors with n components, if L consists of the subspaces $S(\alpha_1, \cdots, \alpha_n)$ of Ω_1 with $\alpha_1(\vec{u}) = \cdots = \alpha_n(\vec{u}) = 0$ for some finite set of n components, the random superposition of Boussinesq flows (2′) tends to a Gaussian limit in each $\Omega_1/S(\alpha_1, \cdots, \alpha_n)$. Hence it defines a finitely additive measure on Ω_1.

This can be extended to a finitely additive measure on Ω as follows. It is well-known, and easily shown by transfinite induction, that Ω contains a subspace W such that $W \cap \Omega_1 = 0$, $W + \Omega_1 = \Omega$. Hence we can let L^* be the lattice of subspaces $S + W$, and let the measure on $\Omega^*/(S + W)$ equal that on Ω_1/S, for all $S \in L$. This proves the following result.

THEOREM 1. Let μ be any finite mass-function defined on a subset Φ of a vector space Ω; let A be the set of linear functionals α bounded on Φ. Then the subspaces $S = S(\alpha_1, \cdots, \alpha_n)$ on which $\alpha_1(\vec{u}) = \cdots = \alpha_n(\vec{u}) = 0$ [$\alpha_1, \cdots, \alpha_n \in B$] form a lattice L; the Gaussian convolution of μ over Φ defines a finitely additive measure on the subspace $V(\Phi)$ spanned by Φ; this may be extended to a finitely additive measure on Ω.

Let now G be any group of transformations γ of the space of velocity-fields of the form

$$\gamma: \quad \vec{u}(\vec{x}) \to T_\gamma[\vec{u}(\vec{x}\gamma)], \tag{10}$$

where γ is point-to-point and T_γ is linear. Group composition $\gamma'' = \gamma\gamma'$ is here supposed defined by the natural formula[14]

$$\gamma'': \quad \vec{u}(\vec{x}) = (T_\gamma T_{\gamma'})[\vec{u}(\vec{x}\gamma\gamma')]. \tag{10′}$$

Since both sides of (5)–(5*) are transformed in the same way by γ, we see

THEOREM 2: If μ is invariant under a group G of transformation of Φ of the type (10), then so is the Gaussian convolution m defined on $V(\Phi)$ by (5)–(5*).

COROLLARY 1: If μ is homogeneous or isotropic, then so is m.

We will not give the extension to the infinite-dimensional case of the motivation discussion[15] of §5.

Part II. COUNTABLE ADDITIVITY

7. *Khinchine Measure*

It is well-known that, for a finitely additive measure $m(x)$ to be extensible to a σ-additive (i.e., countably additive) measure, each of the following dual conditions is necessary and sufficient:

[14]The conventions are: $(T_\gamma T_{\gamma'})\vec{u} = T_{\gamma'}(T_\gamma(\vec{u}))$, and $\vec{x}(\gamma\gamma') = (\vec{x}\gamma)\gamma'$.

[15]For convolutions on infinite-dimensional vector spaces like $V(\Phi)$, see L. Loomis, *An Introduction to Harmonic Analysis*, New York, 1953, pp. 120–122.

(i) $VX_\iota = I$ implies $\sum m(X_\iota) \geqq m(I)$, or

(ii) $X_1 \geqq X_2 \geqq X_3 \geqq \cdots$ and $m(X_\iota) \downarrow \epsilon > 0$ implies that ΛX_ι is non-void.

Thus the function space involved must contain *enough functions.* (In this respect, the set of finite linear combinations of Boussinesq flows is inadequate.)

Using condition (ii), it is easy to show that the Khinchine measure of §5 is extensible to a σ-additive measure. The argument is well-known: one establishes (ii) for those special (Jordan) measurable sets composed of finite sets of "rectangles"; each X_n includes all velocity-fields with velocities $\vec{u}_n(x_i)$, $i = 1, \cdots, r$, in a suitable *compact* set Y_n of $3r$-dimensional space, where $Y_n \leqq Y_{n-1}$ and $m(Y_n) > m(Y_{n-1}) - m(X_n - X_{n-1}) - \epsilon/2^n > 0$. One applies the Diagonal Process to get a limit $\vec{u}(\vec{x}) = \lim_{(n\to\infty)} \vec{u}_n(\vec{x})$ lying in all X_n.

Further, if a finitely additive measure is invariant under a group G (e.g., homogeneous or isotropic), then so is its minimal extension to a σ-additive measure. Hence, the Kolmogoroff-Khinchine type measure defined above from Boussinesq flows is homogeneous and isotropic.

This measure, although adequate for defining correlation functions, which involve only finite sets of points, must be extended before one can discuss the *continuity* of turbulent velocity fields, or of continuous *space averages* (cf. §17). For such purposes, a measure in Hilbert space will be defined.[16]

8. *Normal Distributions in Hilbert Space*

Let T be any bounded, non-negative definite symmetric operator on a Hilbert space H. Let $f_1, f_2, f_3, \cdots$ be any orthonormal basis for H, so that the elements $f \in H$ are just the linear combinations $\sum c_\iota f_\iota$ with $\sum c_\iota^2 < +\infty$. One can construct a normal distribution (or "Gaussian measure") on H, such that $\bar{c}_\iota = 0$ and

$$\overline{c_\iota c_\jmath} = (f_\iota T, f_\jmath) = (f_\iota, f_\jmath T) = T_{\iota\jmath} \tag{11}$$

in the usual Hilbert space notation [7, Chaps. I, II].

Indeed, if S is any closed subspace of H with finite deficiency-index s, take an orthonormal basis $f_1, f_2, f_3, \cdots$ of H such that $f_{s+1}, f_{s+2}, f_{s+3}, \cdots$ are a basis for S. Then, in the finite-dimensional quotient-space H/S, with orthonormal basis $f_1, \cdots, f_s$, there exists[7] as in (3) a unique Gaussian measure such that $\bar{f} = 0$ and $\overline{c_\iota c_\jmath} = t_{\iota\jmath}$, for $i, j = 1, \cdots, s$. As in §4, this defines a finitely additive measure on H.

Further, for any one-one orthogonal transformation P of H (i.e., change of orthonormal basis), $P' = P^{-1}$ by definition. Hence the quadratic form $\overline{c_\iota c_\jmath}$ transforms just like $(f_\iota T, f_\jmath)$ under P: $T \to PTP^{-1}$. It follows that reduction to canonical form is just the same as that for linear transformations, which is given by the standard "spectral theory" [7, Chap. VII].

---

[16] Cf. J. L. Doob, Bulletin of the American Mathematical Society. Vol. 53, 1947, pp. 15-30.

In many physical applications, including those of interest below, the squared norm in H corresponds to an *energy* integral, and the total energy is finite. Hence, in (11),

$$\overline{(f, f)} = \overline{\sum c_i^2} < +\infty . \tag{12a}$$

LEMMA: The inequality (12a) implies that T is of Hilbert-Schmidt type, so that

$$\sum_{i,j} (c_i c_j)^2 < +\infty . \tag{12b}$$

Proof: By the Schwarz inequality for the pseudo-norm $|| f_i || = (f_i T, f_i)^{1/2}$,

$$\overline{(c_i c_j)}^2 = (f_i T, f_j)^2 \leqq (f_i T, f_i)(f_j T, f_j),$$

for all i, j. But, squaring (12a),

$$+\infty > (\sum \overline{c_i^2})^2 = \sum_{i,j} (f_i T, f_j)(f_j T, f_j),$$

which completes the proof.

In more algebraic language, we have shown that non-negative definite infinite matrices of finite trace are of Hilbert-Schmidt type, and hence completely continuous. It is a corollary [7, p. 402] that they have a *discrete spectrum*. This proves a good deal more than

THEOREM 3: Any Gaussian distribution on Hilbert space with $f = 0$ and $f^2 < +\infty$ (mean finite "energy") has, relative to a suitable orthonormal basis, the special form

$$f = \sum c_i f_i , \tag{13}$$

where the c_i are independent, normally distributed constants with $\bar{c}_i = 0$ and $\sum \overline{c_i^2} < +\infty$.

A particular case of some interest is provided by Fourier series with independent random coefficients; a case of this type was first considered by Rayleigh[17].

More important is the following

COROLLARY 1: Any Gaussian distribution on H with $\overline{f^2} < +\infty$ can be extended so as to define a countably additive measure.

Proof: Neglecting the (degenerate) f_i with $c_i = 0$, we get from (13)

$$dm = \prod_i \exp(-c_i^2/2\overline{c_i^2})\, dc_i / \sqrt{\pi \overline{c_i^2}} . \tag{14}$$

If $y_i = \operatorname{sgn} c_i \operatorname{erf} | c_i |$ is introduced as a new variable, then $dm = dy_1 dy_2 dy_3 \cdots$, so that the measure may be regarded as measure in a compact infinite product space Y. It remains to show that H has measure one in Y; the well-known[18] countably additive measure in Y will then give the desired measure on H. But

[17]Philosophical Magazine. Vol. 10, 1880, pp. 73-78. See also R. E. A. C. Paley and N. Wiener, *Fourier Transforms*, New York, 1934, §37ff.

[18]A. Kolmogoroff, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Berlin, 1933, p. 27.

it is easy to show that the set H of infinite sequences $(c_1, c_2, c_3, \cdots)$ with $\sum c_i^2 < +\infty$ has measure one in Y (Lévy's Theorem: since $\sum \overline{c_i^2} < +\infty$, $\sum c_i^2$ is finite except on a set of measure zero.)

REMARK 1: With any distribution of the type described in §8 one can associate a diffusion process (Brownian movement) in Hilbert space, for which (relative to "principal axes") the coordinate variations $x_i(t)$ are independent; the curves $\vec{x}(t)$ are continuous in Hilbert space with probability one.

REMARK 2: The symmetric operators T form an interesting topological *Jordan algebra,* which would repay further study.

9. *Measure-Theoretic Properties*

The normal distribution defined by $\sum_1^\infty c_i f_i$, where the f_i are orthonormal, the c_i are independent, normally distributed random variables, $\bar{c}_i = 0$ and $\sum \overline{c_i^2} < +\infty$, has various convenient measure-theoretic properties. These seem to be most easily proved by using a countable generating family of measurable sets, $\alpha_i < c_i < \beta_i$ [α_i, β_i rational] with the obvious measure, and passing to the limit as $n \to \infty$. Corollary 1 of Theorem 3 shows that this countable family generates a countably additive measure in Hilbert space.

LEMMA 1: If $\sum \overline{c_i^2} = C$, then the sphere $\sum c_i^2 < R^2$ is measurable, with measure at most erf R/C if $R > C$.

Proof: Since the c_i are independent, the c_i^2 are "orthogonal random variables," each c_i^2 being distributed with appropriate negative exponential probability density.

LEMMA 2: For any bounded linear functional $\sum h_i c_i$ ($\sum h_i^2 < +\infty$) the set $\sum h_i c_i < \gamma$ is measurable, and its measure varies continuously with γ (and $\vec{h}$).

Proof: The $h_i c_i$ are orthogonal random variables, with $E\{|\ h_i c_i\ |^2\} = h_i^2 c_i^2$ satisfying $\sum h_i^2 \overline{c_i^2} \leqq (\sum h_i^2)(\sum \overline{c_i^2}) < +\infty$. Hence[19] $\sum h_i c_i$ converges with probability one. The rest of the proof is easy.

The preceding results show that our measure in Hilbert space need not be referred to any particular choice of principal axes; they also imply that the structure of the measurable sets is measure-isomorphic to those of the real line.[20]

10. *Central Limit Theorem*

An analog of Theorem 1 can easily be proved for vectors in Hilbert space, where convergence refers as before to convergence in measure. The application to velocity fields on *bounded* domains is obvious.

---

[19] J. L. Doob, *Stochastic Processes*, Wiley, New York, 1953, Thm. 4.1, p. 155.

[20] This is the Caratheodory-Halmos-von Neumann Theorem (G. Birkhoff, *Lattice Theory*, p. 169).

DEFINITION: Let Φ be any uniformly bounded family of vectors $h_\phi \in H$, and let μ be any non-negative, additive set-function on H, with respect to which h_ϕ is integrable.[21] Suppose further that

$$E = \int |h_\phi|^2 \, d\mu_\phi = \int E_\phi \, d\mu_\phi < +\infty . \tag{15}$$

Then the *Gaussian convolution* of the h_ϕ *with respect to* μ, is the normal distribution with $\bar{f} = 0$ and

$$\overline{c_i c_j} = \int_\Phi c_i c_j(\phi) \, d\mu_\phi . \tag{16}$$

Thus it is defined by (11) from

$$T: \; f \to \int (f, h_\phi) h_\phi \, d\mu_\phi . \tag{16'}$$

By (15), since $E_\phi = \overline{c_i c_i}(\phi)$, the hypotheses of Theorem 3 are fulfilled, proving

COROLLARY 2: For suitable "characteristic functions" f_i, the Gaussian convolution of the h_ϕ with respect to μ has the explicit form (13), and is countably additive.

Convergence in measure for the sets $\alpha_i < c_i < \beta_i$ is immediate, by the usual Central Limit Theorem; the details will be omitted.

11. *Extension to Infinite Domains*

To extend the measure to velocity fields in infinite space E, it seems easiest to form the *product space* $P = C_1 \times C_2 \times C_3 \times \cdots$, of the spaces C_q of functions of class L_2 on each cube of a countable family of disjoint cubes exhausting E. By Corollary 1 of Theorem 3, each C_q has measure one in a compact product $\bar{C}_q$ of infinitely many intervals $-\infty \leqq c_{iq} \leqq +\infty$; we may hope that our product space has measure one in the *compact* infinite product $\bar{P} = \bar{C}_1 \times \bar{C}_2 \times \bar{C}_3 \times \cdots$; this is however not obvious, since the measures on distinct C_q are *not* independent.

As in Part I, it is clear that a *finitely* additive measure is defined on P by mass-distributions $d\mu_i = \rho_i dx_1 \, dx_2 \, dx_3$ which have bounded ρ_i, vanishing outside some finite cube. We shall now show that this is extensible to a σ-additive measure, by proving (i) of §7.

By §9, we need only show that the *finitely* additive measure defined by the $\alpha_{iq} < c_{iq} < \rho_{iq}$ has the property (i) of §7, for this family generates all measurable sets. Hence we may suppose each X_i to be a subset of $\bar{C}_1 \times \bar{C}_2 \times \bar{C}_3 \times \cdots \times \bar{C}_{q(i)}$ for some finite $q(i)$. But, by §9, the measure in this finite product is perfectly normal and part of a countably additive measure; hence we can cover X_i by a measurable *open* set T_i, whose measure exceeds $m(X_i)$ by arbitrarily

[21] In any of the usual senses; cf. T. H. Hildebrandt, Bulletin of the American Mathematical Society. Vol. 59, 1953, pp. 111–139, especially, pp. 114–125. In our applications, Φ is finite-dimensional and h_ϕ continuous.

little. With a little skill, we can make $T_1 < T_2 < T_3 \cdots$, and sup $m(T_i) < 1$ also. It follows that the *compact complements* $T_1' > T_2' > T_3' > \cdots$ in $\overline{P}$ are non-void. But this implies that $\Lambda T_i' > 0$, q.e.d.

We are now able to show that the product ΠC_q has measure one in $\overline{P}$, because

$$\prod_{q=1}^{\infty} C_q = \underset{n}{\Lambda}\left(\prod_{q=1}^{n} C_q \times \prod_{n+1}^{\infty} \overline{C}_q\right),$$

and these all have measure one, by Corollary 1 of Theorem 3. Hence we have proved the following result.

THEOREM 4: There exists a σ-additive measure in the space of velocity fields having finite energy on bounded sets, which is also a limit of random superposition of Boussinesq flows, with energy distribution $\mathcal{E}(K)$.

12. *Khinchine and Hilbert Space Measures*

We have thus proved the Consistency Principle of §1 for two different types of measure: for Khinchine measure (cf. §7) and for Gaussian distributions in Hilbert space.

It would be desirable to show the equivalence of these two measures, in some sense. For instance, since functions differing only on a finite set are represented by the same point in Hilbert space, it is not clear how to *define* $R_{ij}(\vec{x}, \vec{y})$ from the measure of §9.

One procedure is the following; it is valid if $\int_0^\infty E(k)dk = \mathcal{E}(\infty) < +\infty$ and $\int_0^\infty k^2 E(k)dk < +\infty$. In this case, we can form a *new* Hilbert space on any bounded domain by the norm

$$||\vec{u}|| = \left\{\iiint [\sum u_i^2 + \sum (\partial u_i/\partial x_j)^2]\, dx_1\, dx_2\, dx_3\right\}^{1/2}.$$

The "points" of this Hilbert space H_1 correspond to *continuous* $\vec{u}(\vec{x})$; hence the countably additive measure on H_1 can be regarded as a countably additive measure for *continuous* functions, and the functional $u_i(\vec{x})$ is bounded. By Lemma 2 of §9, $\overline{u_i(\vec{x})\, u_j(\vec{y})}$ therefore has a meaning.

Presumably, in the general case $\int_0^\infty E(k)dk < +\infty$, velocity fields are continuous at a given point with probability one. It would be interesting to study this question further.

III. GENERAL PROPERTIES

13. *First Consistency Principle*

Since the preceding rigorous discussion of measure theory will hardly interest specialists in turbulence, it seems advisable to summarize the main result in less technical language.

For any energy distribution $\mathcal{E}(K)$, a *homogeneous, isotropic* measure (statistics) for volume-conserving velocity-fields exists, characterized by the following peculiarities:

(i) quantities depending *linearly* on the velocity field are normally distributed,
(ii) components coming from different wave-vectors are statistically independent.
The main conclusion is embodied in the following principle.

First Consistency Principle: Any spectral distribution of energy density is kinematically consistent with homogeneous isotropic turbulence.

In fact, it is consistent with a normal (i.e., Gaussian) statistics satisfying (i)-(ii).

This principle is plausible intuitively; it is the more remarkable that it should be explicitly contradicted in the literature.[22] Most of Part IV will be devoted to discussing this paradox. In Part III, we shall instead consider the more obvious consequences of our statistics, and compare them with experimental data. The first is well-known [2, p. 169].

COROLLARY 1: For any (homogeneous) linear functional ϕ of the velocity field,

$$\overline{\phi^4}/(\overline{\phi^2})^2 = 3, \qquad \overline{\phi^3}/(\overline{\phi^2})^{3/2} = 0, \tag{17}$$

for *any* admissible energy distribution.

This follows simply [23] from the more general hypothesis (i) that the statistics are Gaussian. This is known to be nearly true for u at one point, but deviations from normality for the joint probability distribution at two points are also known [2, p. 170]. Thus one finds reported[23] observed values of $\overline{\phi^4}/(\overline{\phi^2})^2$ equal to about 3.5 for $\phi = \partial u/\partial x$, and 3.0 for $\phi = \partial v/\partial x$; those of $\overline{\phi^3}/(\overline{\phi^2})^{3/2}$ are -0.39 for $\phi = \partial u/\partial x$. The prediction is thus pretty good, but the discrepancy is significant.

COROLLARY 2: If $E(k)$ decreases rapidly enough to zero as $k \to +\infty$, $\vec{u}(\vec{x})$ has n continuous space derivatives with probability one.

14. *Reversibility Principle*

We next come to a prediction which has not received attention in the literature.

COROLLARY 3: The normal statistics of velocity-fields are reversible: the transformation

$$\vec{x} \to \vec{x}, \qquad \vec{u} \to -\vec{u} \tag{18}$$

preserves the measure m.

[22] [9, p. 50] and [2, formula (3.4.11)].

[23] [4, formulas 861.4 −861.7]. For the experimental data, see G. Batchelor and A. A. Townsend, Proceedings of the Royal Society. Series A, Vol. 190, 1947, pp. 534-550; A. A. Townsend, Proceedings of the Cambridge Philosophical Society. Vol. 44, 1948, p. 560.

Proof: Like (17), this is the case for any Gaussian statistics. It is also a corollary of Theorem 2.

THEOREM 5: In any reversible statistics, all odd-order moments and correlation functions must vanish.

For the transformation (18) reverses the sign of all odd-order correlations; hence, if they are invariant under (18) they must vanish.

Thus, the observation $\overline{u_x^3}/(\overline{u_x^2})^{3/2} = -0.39$, if accurate, is inconsistent not only with Gaussian statistics, but even with any reversible statistics. The assumption of zero third-order correlations is however very plausible[24] *a priori*.

Theorem 5 has important dynamical implications, which show that our model is inadequate to describe the early stages of turbulence.

THEOREM 6: For any ensemble of velocity-fields having reversible statistics, the double correlation tensors satisfy the heat equation

$$\partial R_{\cdot,}/\partial t = 2\nu\nabla^2 R_{ij} \,. \tag{19}$$

Proof: In the Kármán-Howarth equation[25]

$$\frac{\partial}{\partial t} R_{\cdot,} = \frac{\partial}{\partial \xi_l} (T_{\cdot ij} + T_{ij\cdot}) = 2\nu\nabla^2 R_{\cdot,} \,, \tag{20}$$

all the third moments must vanish identically, by Theorem 4. Hence so do their spatial derivatives, which gives (19), q.e.d.

In the Fourier transform space, equation (19) reduces to

$$\partial \Phi_{\cdot,}/\partial t = -2\nu k^2 \Phi_{\cdot,} \,. \tag{21}$$

From this there follows immediately the

COROLLARY: If the velocity-fields of decay are statistically reversible at all times, then

$$\Phi_{i,}(\vec{k}, t) = \exp\{-2\nu k^2 t\}\Phi_{\cdot,}(\vec{k}, 0). \tag{21*}$$

The preceding formulas agree with generally accepted notions about turbulence in the "final stages of decay"; their inadequacy for the early stages is also well-known.

15. *Spectrum and Correlation*

In the usual theory [2, §3.1], the energy spectrum is defined as a linear transform of the double correlation. In our model, the correlation functions are constructed linearly from the energy spectrum, by (7)–(9*). We shall now show that our conventions regarding the relation between energy spectrum and

[24] See the footnote in [6, p. 208].

[25] [6, equation (50)]; our notation is that of [2]. Cf. [2, pp. 79–80], where a slightly different reduction to third moments of the other terms in (20) is given.

velocity correlation agree with those usually employed. Since both our model and the usual theory are linear, it suffices to prove the agreement in the *monochromatic* case. In this case, for the (scalar) wave-number k, we can compute $f(r) = R_{11}(r, 0, 0)$ as follows, ignoring universal constant factors.

The wave vector k makes an angle θ with the x_1-axis, with the probability $\sin\theta\, d\theta$. For each fixed k making such an angle θ, the velocity vector $\vec{a}$ makes all angles ϕ with the vector orthogonal to k in the x_2 , x_3-plane, with equal probability. Integrating, and ignoring universal constant factors,

$$R_{11}(r) = \int \sin\theta\, d\theta \int \sin^2\theta \cos^2\phi\, d\phi \cos(rk\cos\theta),$$

since $u_1(x) = \sin\theta\cos\phi$ times a phase-factor, which shifts by $rk\cos\theta$ as x_1 increases by h. Integrating with respect to ϕ,

$$R_{11}(r) = \int_0^\pi \sin^3\theta \cos(rk\cos\theta)\, d\theta.$$

Setting $\cos\theta = u$, and reversing the limits of integration, this gives

$$f(r) \propto \int_{-1}^{1} (1 - u^2)\cos rku\, du.$$

This can be easily evaluated, giving (since $f(0) = 1$)

$$f(r) = \frac{3}{r^2k^2}\left[\frac{\sin rk}{rk} - \cos rk\right] = 1 - \frac{r^2k^2}{10} + \cdots . \tag{22}$$

A graph of this function, for $k = 1$, is displayed in Figure 1. It agrees with that given by the usual theory [2, formulas (3.4.14)-(3.4.15)].

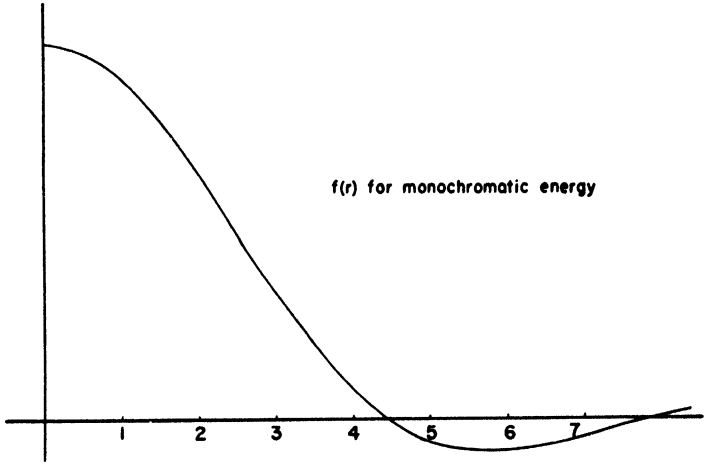


FIGURE 1

THEOREM 7: The longitudinal correlation function for the "normal statistics" of Part I is given by

$$f(r) = \int_0^\infty \frac{3}{r^2k^2}\left[\frac{\sin rk}{rk} - \cos rk\right] d\mathcal{E}(k), \tag{23}$$

just as in the usual theory.

For future reference, it will be convenient to determine the correspondence $E(k) \leftrightarrow f(r)$ in various specific cases. In doing this, it is convenient to use the spherical average $R(r)$ of $R_{ii}(\vec{h})$ over the sphere $|\vec{h}| = r$. We then have [2, p. 49] $2R(r) = 3f + rf'$, and hence

$$r^3 f(r) = 2\int_0^r R(r)r^2\, dr, \tag{24a}$$

where

$$R(r) = \int_0^\infty E(k)\frac{\sin kr}{kr}\, dr, \qquad E(k) = \frac{2}{\pi}\int_0^\infty R(r)kr \sin kr\, dr. \tag{24b}$$

We shall now list the examples; to avoid confusion between dimensionless and dimensional $f(r)$, we ignore constant factors.

EXAMPLE 1: The energy distribution

$$E(k) = d\mathcal{E}/dk = k^2e^{-k} \tag{25.1}$$

accords qualitatively with observation[26], and gives

$$\begin{aligned} R(r) &= \frac{1}{r}\int_0^\infty ke^{-k}\sin kr\, dr \\ &= \frac{1}{r+r^3}\int_0^\infty e^{-k}(\sin kr + r\cos kr)\, dr \\ &= \frac{2}{(1+r^2)^2} > 0. \end{aligned} \tag{26.1}$$

This and (24a) give by [4, formula (122.2)],

$$r^3 f(r) = \arctan r - r/(1+r^2). \tag{26.1a}$$

A plot of the corresponding $f(r)$ is shown in Figures 2a–2b; the fact that $R(r) > 0$ implies[27] $f(r) > 0$ by (24a).

EXAMPLE 2: Similarly, if $E(k) = ke^{-k}$, then

$$R(r) = 1/(1+r^2) > 0 \qquad \text{and} \qquad f(r) \propto \frac{1}{r^3}(r - \arctan r). \tag{26.2}$$

---

[26] [10, Fig. 10]; measurements of energy spectra seem to show a wide variation. The calculations involve formulas (863.1), (863.2) and (122.2) in [4].

[27] Cf. [2, p. 48], for the conjecture that $f(r) > 0$ in general.

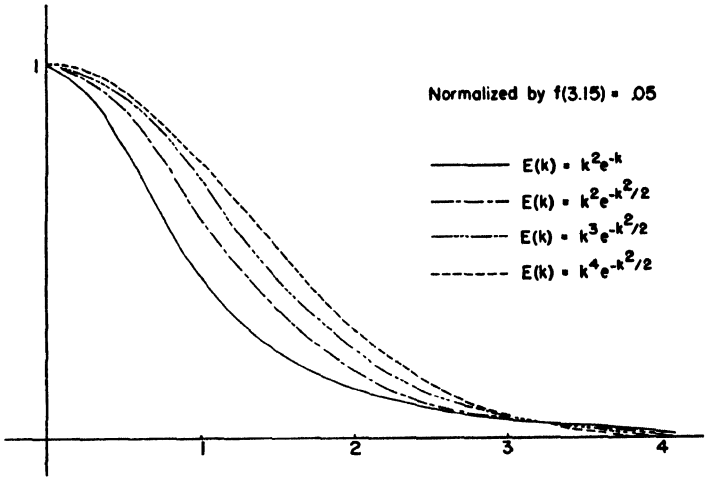


FIGURE 2a

Other interesting examples correspond to the presence of an exp $\{-r^2/2\}$ factor.

EXAMPLE 3: If $E(k) = k^2 \exp\{-k^2/2\}$, then $rR(r)$ is, by (24), the Fourier sine transform of $\psi_1(k) = k \exp\{-k^2/2\}$, whence $R(r) \exp\{-r^2/2\} > 0$. Hence, by (24a),

$$f(r) = \frac{3}{r^3}\left(\sqrt{\frac{\pi}{2}}\,\operatorname{erf} r - re^{-r^2/2}\right) \tag{26.3}$$

$$= 1 - 3\left(\frac{r^2}{10} - \frac{r^4}{56} + \cdots\right).$$

This is also shown in Figures 2a and 2b.

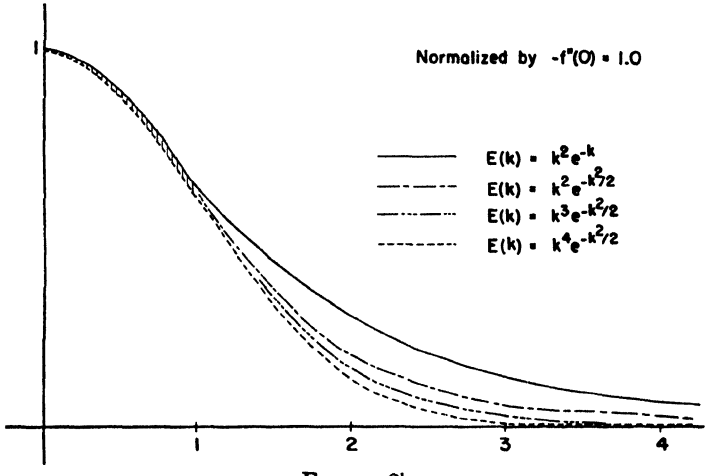


FIGURE 2b

EXAMPLE 4: If $E(k) = k^4 \exp\{-k^2/2\}$, then $rR(r)$ is by (24b) the Fourier sine transform of $k^3 \exp\{-k^2/2\} = \psi_3(k) + 3/2\ \psi_1(k)$, where $\psi_n(k) = H_n(k) \exp\{-k^2/2\}$ denotes the n-th Hermite function.[28] Hence $rR(r) = -\psi_3(r) + 3/2\,\psi_1(r) = (3r - r^3) \exp\{-r^2/2\}$; substituting in (24a), we get

$$f(r) \propto \frac{1}{r^3} \int (3r^2 - r^4) e^{-r^2/2}\, dr = e^{-r^2/2}. \tag{26.4}$$

The preceding calculations can easily be generalized. Thus, if $f(r) = p(r) \exp\{-r^2/2\}$, where $p(r)$ is any even polynomial of degree n, then $E(k) = q(k) \exp\{-k^2/2\}$, where $q(k)$ is an even polynomial of degree $n + 4$. Of course, only $E(k) > 0$ have any physical reality.

EXAMPLE 5: Let $E(k) = k^3 \exp\{-k^2/2\}$. Then

$$f(r) \propto \frac{3}{r^3} \int_0^\infty e^{-k^2/2} \sin rk\, dk - \frac{3}{r^2} \int_0^\infty k e^{-k^2/2} \cos rk\, dk$$

$$\propto \frac{1 + r^2}{r^2} \int_0^\infty e^{-k^2/2} \sin rk\, dk - \frac{1}{r^2}.$$

The definite integral involved has recently been computed at the Harvard Computation Laboratory, and Figures 2a–2b show the results.

The final example is chosen to have a high frequency cutoff.

EXAMPLE 6: Let $E(k) = k^3 \sin k$ on $0 \leqq k \leqq \pi$, and zero elsewhere. Then

$$f(r) = \frac{1}{r^3} \int_0^\pi (\sin rk - rk \cos rk) \sin k\, dk$$

$$= \frac{1}{2r^3} \int_0^\pi [\sin (r - 1)k - \cos (r + 1)k]\, dr$$

$$= \frac{1}{2r^2} \int_0^\pi [\sin (r + 1)k - \sin (r - 1)k]k\, dk.$$

These can easily be integrated in closed form.

16. *Approximation Principle*

Many of the deeper properties of the "normal statistics" defined in Parts I–II are hard to prove rigorously. They can however be made plausible by assuming the following heuristic

Approximation Principle: On bounded domains, the normal statistics of random incompressible flow is the *limit* of statistics involving discrete spectra and almost periodic velocity-fields.

---

[28] See N. Wiener, *The Fourier Integral*, Cambridge University Press, 1933, §6, for the relevant formulas. Formula (26.4) is due to Kármán-Howarth [6, (67)]; cf. [2, p. 94].

Specifically, any energy distribution $\mathcal{E}(k)$ can be apporoximated arbitrarily closely by one *truncated* so as to eliminate high-frequency components $k > k_s$, for sufficiently large finite cutoff frequencies k_s. The Boussinesq flows (2)–(2′) associated with the truncated spectrum form a compact set S (cf. §3), Hence a *finite* subset of couples $(k_i, i\vec{a})$ of wave-vectors k_i and orthogona. velocity-vectors $\vec{a}_i$ can be found, for any $\epsilon > 0$, which is ϵ-dense in S. The measure on Φ (cf. §3) can therefore be approximated arbitrarily closely follows from a measure μ_ϵ concentrated on the $(k_i, \vec{a}_i, \delta)$ obtained from this set by phase-shifts, in which all phases are equally probable.[29] Let m_ϵ be the measure on the space of velocity-fields, obtained by Gaussian convolution from the preceding $\mu_\epsilon(\Phi)$.

The measure m_ϵ will be homogeneous but not exactly isotropic. However, as $\epsilon \to 0$, it will be more and more nearly isotropic. For any finite set $\vec{x}_1, \cdots, \vec{x}_n$, the (Khinchine) measure defined by m_ϵ on the $3n$-dimensional space of $u_i(\vec{x}_r)$, tends to that defined by m as $\epsilon \to 0$. Hence in this sense (which would satisfy most physicists), it is demonstrable that $m_\epsilon \to m$. Since, for m_ϵ, $\vec{u}(\vec{x})$ is almost periodic with probability one, our Approximation Principle must be granted a certain sense.

17. *Metric Transitivity*

An important desideratum of any proposed measure for homogeneous turbulence is [2, p. 12] the equality of *space averages* and Gibbsian *ensemble averages*. Using the Approximation Principle of §16, one can make it very plausible that our "normal statistics" has this property.

As in §16, let Φ consist of the translates of a finite number n of the Boussinesq flows $\vec{a} \sin(\vec{k}\cdot\vec{x} + \delta)$ of (2′), all phases δ being equally likely. Then the Gaussian convolution of the Φ consists of linear combinations

$$u(\vec{x}; \vec{c}) = \sum_{l=1}^{n} c_l \vec{u}(\vec{x}, l) + \sum_{l=1}^{n} c_l \vec{u}(\vec{x}, n + l), \tag{27}$$

where $\vec{u}(\vec{x}, n + l)$ is obtained from $\vec{u}(\vec{x}, l)$ by translating through a quarter-phase. If the total probability of $(\vec{a}_l, \vec{k}_l)$ is μ_l, then the Gaussian convolution of Φ over μ can be shown to consist of the linear combinations

$$\vec{u}(\vec{x}, \vec{c}) = \sum_{l=1}^{2n} c_l \vec{u}(\vec{x}, l), \tag{28}$$

with probability density $(\vec{c})$ in $(c_1, \cdots, c_{2n})$-space given explicitly by

$$\rho(\vec{c}) = \frac{1}{(2\pi)^n \mu_1 \cdots \mu_n} \exp\left\{\sum_{k=1}^{2n} c_k^2/2\mu_k\right\}. \tag{28*}$$

---

[29] This selection corresponds to the Principle of Random Phase of [3]. Cf. also the theorem of P. Lévy, on p. 369 of Vol. 1 of California Publications in Statistics, 1953.

To prove (28*), note that the probabilities of $c_1, \cdots, c_{2n}$ are independent, so that the variables of integration are separable in

$$R_{ij}(\vec{x}, \vec{y}) = \int u_i(\vec{x}, \vec{c}) u_j(\vec{y}, \vec{c}) \rho(\vec{c})\, d\vec{c}$$

$$= \sum_{l,m} u_i(x, l) u_j(y, m) \left\{ \int c_l c_m \prod_{k=1}^{2n} \frac{\exp\{-c_k^2/2\mu_k\}}{(\sqrt{2\pi\mu_k}} \, dc_k \right\}.$$

If $l \neq m$, then the iterated integral over c_l, c_m vanishes since the integrand is odd. Hence

$$R_{ij}(\vec{x}, \vec{y}) = \sum u_i(\vec{x}, l) u_j(\vec{y}, l) \left[\prod_{k \neq l} \frac{1}{\sqrt{2\pi\mu_k}} \int_{-\infty}^{\infty} \exp\left\{-c_k^2/2\mu_k \, dc_k\right\} \right]$$

$$\cdot \frac{1}{\sqrt{2\pi\mu_l}} \int_{-\infty}^{\infty} c_l^2 \exp\{-c_l^2/2\mu_l\}\, dc_l .$$

The factors in curly brackets with $k \neq l$ are all unity; that involving c_l^2 is $\sqrt{2\pi\mu_l}^{3/2}$ [4, formula 861.7]. Hence

$$R_{ij}(\vec{x}, \vec{y}) = \sum \mu_l u_i(x, l) u_j(x, l), \tag{29}$$

which completes the proof of (28*), and proves further

THEOREM 8: If Φ is finite, then the Gaussian convolution of Φ over μ is given by (28*), with $dm = \rho(\vec{c}) dc_1 \cdots dc_n$.

The case of countable Φ can be treated similarly. In either case, the following result is evident.

LEMMA: If the $\vec{u}(\vec{x}, l)$ belong to the class of periodic velocity-fields (2′) and if the *wave-vectors* $\vec{k}_1, \cdots, \vec{k}_n$ are rationally independent,[30] then for every $\vec{c}$, the *space average* of the $u_i(\vec{x}, \vec{c})\, u_j(\vec{x} + \vec{h}, \vec{c})$ is

$$\sum c_l^2 R_{ij}(\vec{h}, l) = \sum c_l^2 u_i(\vec{x}, l) u_j(\vec{x} + \vec{h}, l). \tag{30}$$

Proof: Each $\vec{u}(\vec{x}, \vec{c})$ is almost periodic, and so if $k \neq l$, $u_i(\vec{x}, l)\, u_j(\vec{x} + h, m)$ has a zero space average.

THEOREM 9: Assuming the Approximation Principle, the space average of $R_{ij}(h)$ is equal to the ensemble average with probability one, for the normal statistics associated with $\mathcal{E}(k)$.

Proof: Since the k_l are rationally independent with probability one,[31] formula (30) holds with probability one for the ϵ-dense approximations m_ϵ of §16. Hence, by the law of large numbers, the space average (30) tends with probability one to the ensemble average (cf. (5*))

---

[30] In the usual sense, that $r_1\vec{k}_1 + r_n\vec{k}_n = \vec{0}$, and $r_1, \cdots, r_n$ *rational* scalars, imply $r_1 = \cdots = r_n = 0$.

[31] Simply because any countable set has Lebesgue measure zero.

$$R_{,,}(h) = \int u_{,}(\vec{x})u_{,}(\vec{x} + \vec{h})\, dm,$$

as $\epsilon \downarrow 0$. Assuming the Approximation Principle in a strong enough form, we conclude in the limit the *equality* of space averages and ensemble averages.

REMARK: Again, let $\mathcal{E}(k)$ be any "admissible energy distribution"—i.e., any non-decreasing function, with $\mathcal{E}(0) = 0$ and $\mathcal{E}(+\infty)$ finite. Following Khinchine,[32] one can easily construct a homogeneous, isotropic *statistical ensemble* of periodic velocity-fields of the class just described, and having the usual energy, second moment, and double correlation statistics associated with the energy density $E(k) = \mathcal{E}'(k)$. Namely, the phase-space Φ is the product of the line $0 < k < \infty$ with the set S_1, and *measure* on Φ is the product of the measure $\mathcal{E}(k)$ with the unique invariant measure on S_1 (extended to S_k by similarity), defined in §3.

IV. FINAL STAGES OF DECAY

18. *Exponential Decay Laws*

In view of (19), the dynamics of normal statistics are those which arise in the "final stages of decay", when the non-linear convection terms of the Navier-Stokes equations are negligible. Thus (cf. (21*)), the energy density as a function of time decays exponentially, according to the law

$$E(k, t) = e^{-2\nu k^2 t} E(k, 0), \tag{31}$$

or

$$\mathcal{E}(k, t) = \int_0^k e^{-2\nu k^2 t}\, d\mathcal{E}(k, 0). \tag{31'}$$

A sharper statement can also be proved.

THEOREM 10: Assuming the Approximation Principle, if convection terms are neglected, then initially normal statistics remain normal to all later times.

Proof: The Navier-Stokes equations then become linear. But Boussinesq flows (2)–(2') decay according to the similarity law $\vec{u}(\vec{x}; t) = \exp\{-\nu k^2 t\}\, \vec{u}(\vec{x}; 0)$; hence so do their transforms under the Euclidean group. This shows that their Gaussian convolution with respect to $\mathcal{E}(k, 0)$ decays into that defined by (31'), for the *almost periodic* case of a finite set of wave-vectors k_l. The proof is completed by the Approximation Principle.

Theorem 10 may be called the Second Consistency Principle for our normal statistics.

---

[32] Mathematische Annalen, Vol. 109, 1934, pp. 604–615.

Using (31), one is easily led to conjectures about energy decay in the "final period." Thus, from the formula

$$2\int_0^\infty k^n e^{-2\nu k^2 t}\,dk = \int_0^\infty u^{(n-1)/2} e^{-2\nu t u}\,du = \frac{\Gamma((n+1)/2)}{(2\nu t)^{(n+1)/2}},$$

we obtain Batchelor's first theorem,

THEOREM 11: The energy decay law in the "final period" is

$$\mathcal{E}(t) \propto t^{-(n+1)/2} \quad \text{if} \quad E(k) = O(k^n) \quad \text{near} \quad k = 0. \tag{32}$$

Batchelor's formula[33] $\mathcal{E}(t) \propto t^{-5/2}$ relies on a second mathematical theorem, of which a new proof will now be given.

THEOREM 12 (BATCHELOR): The three conditions $f(r) = o(1/r^3)$, $\iiint R_{ii}(\vec{h})dh_1dh_2dh_3 = 0$, and $E(\vec{0}) = 0$ are essentially equivalent. Hence, if $E(\vec{k})$ is continuous, so is $E(k) = o(k^2)$.

The new proof is based on the following

LEMMA: The average of $R_{ii}(\vec{h})$ over the sphere $|\vec{h}| = r$ is[34]

$$R(r) = S_{ii}(r) = \tfrac{1}{3}f(r) + \tfrac{2}{3}g(r); \tag{33}$$

that of $R_{ij}(\vec{h})$ is zero, if $i \neq j$.

Proof: If θ denotes colatitude with respect to the x_i-direction, and u, v denote velocity components parallel and perpendicular to $\vec{h}$ in the $(\vec{h}, x_i)$-plane, then the average of $R_{ii}(\vec{h})$ over a circle of latitude θ is

$$\overline{uu'}\cos^2\theta + \overline{vv'}\sin^2\theta + (\overline{uv'} + \overline{vu'})\cos\theta\sin\theta.$$

But $\overline{uv'} = \overline{vu'} = 0$, by reflection symmetry. Averaging with respect to the area factor $\sin\theta\,d\theta$, we get (33) by elementary integration. The fact that $S_{ij}(r) = 0$, if $i \neq j$, follows by reflection symmetry alone.

Since $E(\vec{k})$ is proportional to the Fourier transform of $R_{ii}(\vec{h})$, clearly

$$E(\vec{0}) \propto \iiint R_{ii}(\vec{h})\,dh_1\,dh_2\,dh_3 = \frac{1}{3}\int_0^\infty r^2 S_{ii}(r)\,dr. \tag{34}$$

But since $r^2S_{ii}(r)dr = d[r^3f(r)]$, by (33) and the Karman-Howarth identity $g = f + f'/2r$, we see [2, formula (3.4.9)]

$$E(\vec{0}) \propto \lim_{r\to\infty} r^3 f(r) = \frac{1}{3}\int_0^\infty r^2 S_{ii}(r)\,dr. \tag{35}$$

The truth of Theorem 12 is now evident.

Clearly, Theorem 12 is not incompatible with our Consistency Principle. It would seem *a priori* reasonable to suppose that a specific mechanism for creating turbulence might or might not have a zero density in the low-frequency

[33][1, p. 517]; see also the references of footnote 22 above.

[34]Cf. formula (3.4.14) of [2, p. 49].

end of the wave-vector spectrum. Batchelor's Theorem simply tells us that $E(\vec{0}) = 0$ if and only if the longitudinal correlation function $f(r)$ dies out faster than $1/r^3$.

However, it may be urged that, in the absence of any specific *cause* for a higher correlation at large distances, we should not expect a correlation larger than $o(1/r^3)$. The same intuitive argument would imply that since $f(r) = O(\exp\{-r^2/2\})$ is possible (Example 4 of §15), no higher correlation is to be expected. It would be interesting to know what $E(k)$ are compatible with this.

19. *Theoretical Discussion*

Consideration of the asymptotic behavior of solutions of the diffusion equation (19) for the most familiar types of initial conditions, would suggest[35] that $R_{\iota\iota}(r) \propto \exp\{-r^2/4\nu t\}$ for large t. Batchelor's Theorem 12 shows that this is kinematically impossible. Similarly, *dipole* solutions are impossible, since $R_{\iota\iota}(\vec{h}) = R_{ii}(-\vec{h})$. His Example 4 shows that, however, the quadrupole solution

$$R_{11}(r) = (1 - \tfrac{1}{2}(h_2^2 + h_3^2))e^{-r^2/2}$$

is kinematically possible, being associated with the energy distribution $E(k) = k^4 \exp\{-k^2/2\}$. It is doubtful whether any other solution will give as rapid a spatial decay of the correlation function.

Nevertheless, it seems clear that turbulence can be produced in which $E(k) \neq O(k^4)$. Mathematically, this is implied by our Consistency Principle, which would even permit $E(k) = k^{-.5}$ to become infinite as $k \downarrow 0$, or to have an essential singularity there. Although $E(\vec{k})$ is essentially the Fourier transform of $R_{\iota\iota}(\vec{h})$, regularity of $E(\vec{k})$ near $\vec{k} = 0$ corresponds to regularity of $R_{\iota\iota}(\vec{h})$ at infinity, which cannot generally be expected. Hence the claim simply amounts to the assertion that arbitrarily large eddies can be present, containing any preassigned proportion of the energy.

Secondly, I claim that the Navier-Stokes equations do *not* tend to make $E(k) = O(k^4)$, unless this condition is fulfilled initially. I shall in fact prove that $E(\vec{0})$ is an *invariant* (like Loitsyansky's invariant $\Lambda = \int_0^\infty r^4 f(r)dr$), under suitable conditions[36] of convergence. Consider

$$E(\vec{0}) = \iiint R_{\iota\iota}(\vec{h})\, dh_1\, dh_2\, dh_3 \,. \tag{36}$$

By formula (20),

$$\frac{\partial}{\partial t}[E(\vec{0})] = \iiint \frac{\partial}{\partial h_l}(T_{\iota\iota i} + T_{ii\iota} + 2\nu\, \partial R_{\iota\iota}/\partial h_l)\, dh_1\, dh_2\, dh_3 \,. \tag{37}$$

[35] See E. Reissner, Proceedings of the 5th International Congress on Applied Mechanics, p. 359.

[36] As is customary in turbulence theory, I am not being too fussy about mathematical niceties here.

Since the integrand on the right is a sum of derivatives, the integral clearly vanishes provided the T_{ijl} and ∇R_{ij} are $o(r^{-2})$ at infinity. Example 1 is a case in point; in this case, $E(\vec{k})$ happens to be an analytic function, and $E(k)$ to be even.

That is, under the conditions stated, $E(\vec{0}) = \text{const.}$, as is indeed suggested by (35). The significance of (36)–(37) is that the convection terms in the Navier-Stokes equations are *not* neglected. Thus we have proved the following result.

THEOREM 13: Under suitable convergence conditions, the quantity

$$E(\vec{0}) = \iiint R_{ii}(\vec{h})\, dh_1\, dh_2\, dh_3 = \lim_{r\to 0} r^3 f(r) \tag{38}$$

is a constant in time (i.e., an "invariant").

To obtain the last term of (38), note that

$$E(\vec{0}) \propto \int_0^\infty r^2 R(r)\, dr \propto \int_0^\infty d(r^3 f(r)) = \lim_{r\to\infty} r^3 f(r).$$

This is just a further refinement of Batchelor's Theorem 12.

The preceding considerations should make it clear that $E(\vec{0}) = 0$ is *not* necessary *a priori*.

If $E(\vec{0})$ is finite and non-zero, *and* if $E(\vec{k})$ is continuous near $\vec{k} = \vec{0}$, then by (31)–(32) we have the asymptotic formula $\overline{u^2} \propto t^{-1.5}$. Since I do not claim that $E(\vec{k})$ is continuous near $\vec{k} = 0$, I do *not* however assert the necessity of $\overline{u^2} \propto t^{-1.5}$; I shall amplify this remark below.

20. *Experimental Evidence*

The preceding discussion should make it clear that both $E(k) = O(k^4)$ and $E(k) = O(k^2)$ are compatible with the Navier-Stokes equations, but that $E(k)$ need not satisfy these or any other conditions of regularity near $k = 0$. This makes the question of whether or not $E(k) = O(k^4)$ and $\varepsilon(t) \propto t^{-2.5}$, in the final stages of decay of the homogeneous turbulence produced behind grids at the Cavendish Laboratory, depend on the experimental evidence. I should like to review this evidence briefly, in the light of the preceding theoretical discussion.

Two pieces of experimental evidence are known, which support the conjecture $E(k) = O(k^4)$. These are: (i) the approximate empirical formula $f(r) = \exp\{-r^2/\lambda^2\}$, and (ii) plots of $\lambda^2(t)$ and $\varepsilon^{-0.4}(t)$, which show a mean slope $d(\lambda^2)/dt$ of about 0.4. These will be reviewed in turn.

The property that $f(r)$ agrees well with $\exp\{-r^2/\lambda^2\}$ if suitably normalized[37] is shared by many non-negative energy density functions. In fact, in view of

[37]The strongest evidence supporting Batchelor's theory is perhaps the fact that, in Figure 2b, he determined the normalizing factor $f''(0)/f(0)$ independently, from measurements of $\overline{u^2}$ and $d(\overline{u^2})/dt$.

(22) (cf. Fig. 1), most reasonably smooth non-negative $E(k)$ will give a longitudinal correlation function $f(r)$ having the "bell-shape" typical of exp $\{-r^2/2\lambda^2\}$; this fact is illustrated in Figures 2a–2b. It would seem worth checking also $g(r) = (1 - r^2/2\lambda^2)$ exp $\{-r^2/2\lambda\}$, which would follow from Batchelor's conjecture.

More specifically, it should be noted that the question of whether or not $E(k) = O(k^4)$ depends on the values of $f(r)$ for large r (say, $r > 3\lambda$), which are especially uncertain and difficult to measure.[38] Hence the evidence from measurements of $f(r)$ can be described as compatible with Bathchelor's theory, or even favorable, but not conclusive—especially since the turbulence is non-isotropic [2, p. 97].

The second piece of evidence is the empirical approximate mean slope relation [2, p. 95]

$$d(\lambda^2)/dt = 4\nu, \qquad \text{or} \qquad \lambda^2 = \frac{10\nu}{n}(t - t_0) \qquad (n = 2.5). \tag{39}$$

Since $\varepsilon'/\varepsilon = -10\nu\lambda^2$, this is theoretically equivalent to

$$\varepsilon = \varepsilon_0 \exp\left\{\int (\varepsilon'/\varepsilon)\, dt\right\} = \varepsilon_0(t - t_0)^{-n}. \tag{39'}$$

Hence it is equivalent to a straight-line plot of $\varepsilon^{-0.4}$ against t, with intercept at $t = t_0$. The fact that this was also found [2, Fig. 5.1] shows essentially that the experimental technique used to measure (39) was correct; it does not constitute independent experimental confirmation.

This second piece of evidence depends mainly on *small* eddies, which largely determine λ^2. Hence it cannot be construed as proving the absence of larger eddies. Since absence of large eddies is what is asserted in the statement $E(k) = O(k^4)$, the second piece of evidence must also, in my opinion, be regarded as hopeful but not conclusive.

My view is that, in typical cases, there are energy-carrying eddies concentrated in various widely separated bands. As time passes, the major energy-dissipation occurs in one after another of these bands. During each interval when energy-dissipation is confined to a single band (several octaves long), a $t^{-5/2}$ decay law may well be observed. But ultimately, eddies from a band of longer wave-lengths become the dominant energy-containing eddies. During the transition period, the rate of decay is much slower (i.e., n is smaller), and the slope of λ^2 is correspondingly larger.

Thus as t increases by large factors (say 20), I predict that a much slower rate of energy decay than $t^{-5/2}$ will be observed. Thus it seems to me very

[38] For example, they would be seriously affected by propeller vibrations. Again, it is perhaps relevant that measurements of $f(r)$ by other means (though at earlier stages of decay) suggest $E(k) \propto k^2$ near $k = 0$ [10, Figs. 8–13]. Finally, a casual inspection of Figs. 3, 8 of Stewart and Townsend, loc. cit., suggests that $\int_0^\infty r^2[f(r) + 2g(r)]dr = \int_0^\infty r^3 R(r)dr$. Cf. also H. L. Dryden, Quarterly of Applied Mathematics, Vol. 1, 1943, p. 35, Fig. 10.

significant that, in the experiments cited [2, p. 95], the doubling of t corresponds to an increase in $\overline{u^2}^{-0.4}$ by a factor of 1.65, corresponding to a decrease in $\overline{u^2}$ by a factor of $2^{-1.5}$. This is considerably nearer to $2^{-1.5}$ than to $2^{-2.5}$.

In conclusion, it should be mentioned that, if my view is correct, then the Loitsyansky invariant $\Lambda = \int_0^\infty r^4 f(r)dr$ is a *divergent* integral.[39] This would explain the fact [10, Fig. 12] that rough experimental estimates of Λ differ by a factor of 10 or more.

I should like to express my appreciation to George Batchelor, J. Kampé de Fériet, C. C. Lin, Lynn Loomis, and Paul Lochak for various helpful comments.

I should also like to thank the Office of Naval Research, for providing technical assistance under Contract N5ori-07634.

Bibliography

[1] Batchelor, G. K., *The role of big eddies in homogeneous turbulence*, Proceedings of the Royal Society, Series A, Volume 195, 1948, pp. 513–532.

[2] Batchelor, G. K., *Homogeneous Turbulence*, Cambridge University Press, Cambridge, 1953.

[3] Birkhoff, G., and Kotik, J., *Fourier analysis of wave trains*, Gravity Wave Symposium, National Bureau of Standards Circular 521, 1951, pp. 221–234.

[4] Dwight, H. B., *Tables of Integrals and Other Mathematical Data*, Macmillan, New York, rev. ed., 1947.

[5] Kampé de Fériet, J., *Correlation and Spectrum*, University of Maryland Lecture Series, No. 8, 1951.

[6] von Kármán, T., and Howarth, L., *On the statistical theory of isotropic turbulence*, Proceedings of the Royal Society, Series A, Volume 164, 1938, pp. 192–215.

[7] Stone, M. H., *Linear Transformations in Hilbert Space*, American Mathematical Society, New York, 1932.

[8] Taylor, G. I., *Statistical theory of turbulence*, Proceedings of the Royal Society, Series A, Volume 151, 1935, p. 91.

[9] *Proceedings of the Seventh International Congress of Applied Mechanics*, London, 1948, Introduction.

[10] Stewart, R. W., and Townsend, A. A., *Similarity and preservation in isotropic turbulence*, Philosophical Transactions of the Royal Society, Series A, Volume 243, 1951, pp. 359–386.

[39] C. C. Lin has informed me that, with $E(k) = k^n e^{-ck^2}$, $\Lambda = 0$ if $n > 4$, Λ is finite if $n = 4$, and $\Lambda = \infty$ if $n < 4$.

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. VII, 45–63 (1954)

One-Dimensional Anisentropic Flows*

By G. S. S. LUDFORD and M. H. MARTIN
University of Maryland

1. *Introduction*

The adiabatic flow behind a decaying shock such as arises in one-dimensional flow when an expansion wave overtakes a shock, is anisentropic; that is, the specific entropy varies from one fluid particle to another. If we assume the gas is polytropic, with the caloric equation of state $p = a\rho^\gamma$, the quantity a, while remaining constant for any one fluid particle during the course of its motion,[1] will vary from particle to particle.

The mathematical study of such flows is the object of this paper. The methods employed, see Section 2, are based on previous work of one of the authors [1], [2], [3], in which the problem is reduced to the integration of a Monge-Ampère partial differential equation of special type [4].

For isentropic flows, there exist two intermediate integrals [5] of this equation and these represent simple waves, Section 3. Intermediate integrals also exist for anisentropic flows, whenever the distribution of specific entropy over the fluid particles is of a certain form; the flows corresponding to these integrals are therefore called anisentropic simple waves, and are studied in Section 4.

Whenever two intermediate integrals of the Monge-Ampère equation exist, its integration can be effected by Ampère's method [5]. Applied to isentropic flows it yields classical results [6] from a different viewpoint, Section 5; applied to anisentropic flows it yields new results, and in particular a concrete example of anisentropic flow, Section 6.

2. *Fundamental Principles*

The unsteady, one-dimensional, anisentropic, adiabatic flow of a gas with arbitrary equation of state is represented parametrically [3] in the x,t-plane by

$$x = \int [(\xi_\psi \xi_{\psi p} + \tau)\, d\psi + \xi_\psi \xi_{pp}\, dp], \qquad t = \xi_p\,, \qquad u = \xi_\psi\,, \tag{1}$$

where $\xi = \xi(\psi,p)$ is a solution of the Monge-Ampère equation

$$\xi_{\psi\psi}\xi_{pp} - \xi_{\psi p}^2 = \tau_p\,, \tag{2}$$

*This work was sponsored by the Office of Ordnance Research, U. S. Army, under Contract DA-36-034-ORD-966.

[1]The influence of viscosity, heat conduction and radiation being neglected.

in which the *specific volume* $\tau = \tau(\psi,p)$ is a function whose prescription characterizes the problem. The first two equations in (1) give, for fixed ψ, the trajectory of a gas particle parametrically in terms of the *pressure* p, while the last gives its *velocity* u at each point of its trajectory. For a polytropic gas, τ assumes the form [6, p. 10]

$$\tau = p^{-n} \exp\,[(S - S_0)/c_v], \qquad n = 1/\gamma, \qquad S_0 = \text{const.}, \tag{3}$$

where prescription of the *entropy distribution function* $S = S(\psi)$ determines the variation of specific entropy from trajectory to trajectory.

The characteristic strips of (2) satisfy one or other of the differential systems [5]

$$\begin{cases} d\xi - u\,d\psi - t\,dp = 0, \\ du + \lambda\,dp = 0, \\ dt - \lambda\,d\psi = 0, \end{cases} \qquad \begin{cases} d\xi - u\,d\psi - t\,dp = 0, \\ du - \lambda\,dp = 0, \\ dt + \lambda\,d\psi = 0, \end{cases} \tag{4}$$

where

$$\lambda^2 = -\tau_p \qquad \text{or} \qquad \lambda = \tau/c, \tag{5}$$

and $c = \sqrt{(\partial p/\partial \rho)_S}$ is the *local speed of sound*. Thus λ^{-1} is the *acoustic impedance* [see 6, p. 5]. It is worth noticing that τ does not appear explicitly in (4), and that there are only *two* families of characteristics instead of the expected *three* [6, pp. 91–92]. The effect of the third family (trajectories ψ = const.) is felt in λ, which may be prescribed to vary discontinuously with ψ. The last two pairs of equations in (4) are probably the simplest characteristic equations which have been given, and are well-suited for numerical computations.

A simple calculation shows that

$$J \equiv \frac{\partial(x,t)}{\partial(\psi,p)} = \tau t_p\,, \tag{6}$$

and consequently the mapping (1) of the ψ,p-plane upon the x,t-plane is locally one-to-one provided the specific volume and the time rate of change of pressure along a trajectory remain finite and different from zero.

As is to be expected, we find from (1) that $dx/dt = u$ holds along a trajectory. More generally, a curve $p = p(\psi)$ in the ψ,p-plane maps into a curve $x = x(t)$ in the x,t-plane upon which

$$\frac{dx}{dt} = u + \frac{\tau}{t_\psi + t_p p'}\,, \qquad p' = \frac{dp}{d\psi}\,. \tag{7}$$

Thus if some physical quantity q is constant along a curve $p = p(\psi)$, this constant value is propagated at a rate given by (7), or, relative to the gas itself, at a rate

$$R_q = \frac{\tau}{t_\psi + t_p p'}\,.$$

Taking in turn $q = p, u, \tau$, one finds that the rates at which constant pressure, velocity and specific volume are propagated with respect to the gas, are

$$R_p = \frac{\tau}{t_\psi}, \qquad R_u = -\frac{\tau t_\psi}{\tau_p}, \qquad R_\tau = \frac{\tau\tau_p}{\tau_p t_\psi - \tau_\psi t_p}. \tag{8}$$

For a polytropic gas the absolute temperature $T = p\tau/R$, so that the rate at which a given temperature is propagated is obtained by setting $q = p\tau$ to give

$$R_T = \frac{\tau(p\tau_p + \tau)}{p(\tau_p t_\psi - \tau_\psi t_p) + \tau t_\psi}.$$

From (5) it follows that $c^2 = R_p R_u$. Moreover when (3) is used to eliminate τ_p from R_T, and the formulas (8) for R_p and R_τ to eliminate t_ψ and $(\tau_p t_\psi - \tau_\psi t_p)$ respectively, we obtain

$$\frac{\gamma - 1}{R_T} = \frac{\gamma}{R_p} - \frac{1}{R_\tau}.$$

This proves the theorem:

THEOREM 1: *For any gas the local speed of sound is the geometric mean of the rates at which pressure and velocity are propagated with respect to the gas. For a polytropic gas the rate at which the absolute temperature is propagated with respect to the gas is a weighted harmonic mean of the rates at which density and pressure are propagated with respect to the gas.*

Along a characteristic base curve $\mu = \text{const.}$ in the ψ,p-plane

$$t_\psi + t_p p' = \pm\lambda, \tag{9}$$

according to the last pair of equations in (4), and hence, from (5), we see that $R_\mu = \pm c$. Thus, as is to be expected, the characteristic base curves are mapped by (1) into Mach lines.

3. *Isentropic Simple Waves*

Isentropic flows are characterized by $S = \text{const.}$, or $\tau = \tau(p)$. Consequently, from (5), $\lambda = \lambda(p)$ and the differential systems (4) possess the integrals [4]

$$u + f = 2r, \qquad u - f = 2s,$$

where r, s are constants, and the variable

$$f = \int \lambda(p)\, dp = f(p),$$

has been introduced by Riemann [7]. It follows that (2) has the *intermediate integrals* [5]

$$\xi_\psi + f = 2r, \qquad \xi_\psi - f = 2s,$$

the general solutions of which are

$$\xi = P(p) + (2r - f)\psi, \qquad \xi = P(p) + (2s + f)\psi, \tag{10}$$

respectively, where P is an arbitrary function. These are obviously solutions of (2), and (1) yields the formulas

$$x = \int uP''\,dp - (u - c)\lambda\psi, \qquad t = P' - \lambda\psi, \qquad u = 2r - f,$$

$$x = \int uP''\,dp + (u + c)\lambda\psi, \qquad t = P' + \lambda\psi, \qquad u = 2s + f. \tag{11}$$

From (11) it is clear that the isobars p = const. constitute one-parameter families of straight lines in the x,t-plane, and along each isobar u has a constant value. This identifies the flows represented by (11) as "simple waves" [see 6, pp. 92–95]. The Cartesian equations

$$x - (u - c)t = \int uP''\,dp - (u - c)P',$$

$$x - (u + c)t = \int uP''\,dp + (u + c)P',$$

of the isobars confirm this identification. By fixing ψ in (11) and allowing p to vary, the trajectories are given parametrically.

For simplicity consider just the first of the two kinds of simple waves in (11). Clearly from (6), $J = 0$ along the curve $t_p = P'' - \lambda'\psi = 0$ in the ψ,p-plane. The transform of this curve in the x,t-plane is given parametrically by

$$x = \int uP''\,dp - (u - c)\frac{\lambda}{\lambda'}P'', \qquad t = P' - \frac{\lambda}{\lambda'}P'',$$

which envelopes the isobars. To obtain the Mach lines, we observe that the differential equations (9) reduce to $(P'' - \lambda'\psi)p' = \pm\lambda + \lambda$. The lower sign yields p = const., that is, the isobars above. The upper sign gives

$$\frac{d\psi}{dp} + \frac{\lambda'}{2\lambda}\psi = \frac{P''}{2\lambda},$$

of which the general solution is

$$\psi = \frac{1}{\sqrt{\lambda}}\left(C + \frac{1}{2}\int \frac{P''}{\sqrt{\lambda}}\,dp\right),$$

and when the latter is inserted in (11), the parametric equations of the other family of Mach lines turn out to be

$$x = \int uP''\,dp - (u - c)\sqrt{\lambda}\left(C + \frac{1}{2}\int \frac{P''}{\sqrt{\lambda}}\,dp\right),$$

$$t = P' - \sqrt{\lambda}\left(C + \frac{1}{2}\int \frac{P''}{\sqrt{\lambda}}\,dp\right).$$

4. *Anisentropic Simple Waves*

We have seen in the previous section that for isentropic flows the intermediate integrals of (2) give rise to simple waves. This provides a natural extension of the idea of a simple wave to anisentropic flows.

DEFINITION: *The flow presented by* (1) *when* ξ *is a solution of an intermediate integral of* (2) *is called a simple wave.*

This immediately brings up the question of the existence of intermediate integrals of (2) and here we restrict ourselves to polytropic gases, so that (2) reads

$$\xi_{\psi\psi}\xi_{pp} - \xi_{\psi p}^2 + \lambda^2 = 0, \qquad \lambda = \Psi(\psi)P(p), \tag{12}$$

where

$$\Psi(\psi) = \sqrt{n}\exp\{(S - S_0)/2c_v\}, \qquad P(p) = p^{-(n+1)/2}. \tag{13}$$

To answer this question we recall that $u = \xi_\psi$, $t = \xi_p$ and use the following theorem, which is proved in [4].

THEOREM 2: *The Monge-Ampère equation* (12) *has intermediate integrals only in the following cases*:

Case	λ	Intermediate Integrals
1	0	$t = \phi(u)$
2	1	$t \pm \psi = \phi(u \mp p)$
3	$P(p) \not\equiv 0$	$u \pm f = \text{const.}; \quad f = \int P\,dp$
4	p^{-2}	$u\psi + tp - \xi \pm \psi/p = \phi(u \pm 1/p)$
5	$\psi^{m-1}/p^{m+1}(m \neq 0)$	$u\psi + tp - \xi \pm \frac{1}{m}\left(\frac{\psi}{p}\right)^m = \text{const.}$
6	ψ^{-1}/p	$u\psi + tp - \xi \pm \log\left(\frac{\psi}{p}\right) = \text{const.}$
7	$e^{\psi}e^{p}$	$u - t \pm e^{\psi}e^{p} = \text{const.}$

and in those which arise from these under translations, reflections in the line $p = \psi$, *and dilatations in the* ψ *and* p *directions, in the* ψ,p*-plane.*

In view of (13) we pass over case 1. Case 2 can only arise for isentropic flows of the Kàrmàn-Tsien gas ($\lambda = -1$). Case 3 occurs for the isentropic flows of an arbitrary gas, and has already been discussed. Its reflection is

obtained for general anisentropic flows of the Kàrmàn-Tsien gas. Case 4 corresponds to isentropic flows of a polytropic gas with $\lambda = 1/3$, and its reflection to special anisentropic flows of the Kàrmàn-Tsien gas. Case 6 offers anisentropic flows with $\gamma = 1$. Case 7 does not occur for a polytropic gas and does not appear to be of general interest.

We are left with case 5, to which we shall devote our attention from now on. Translations and reflections in the line $p = \psi$ provide intermediate integrals for $\lambda = (\psi - \psi_0)^{m-1}/(p - p_0)^{m+1}$ and $\lambda = (p - p_0)^{m-1}/(\psi - \psi_0)^{m+1}$, the second case being included in the first. For polytropic gases, see (13), $p_0 = 0$, and clearly there is no loss in generality if we take $\psi_0 = 0$, so as to return to case 5. However, the dilatations[2] in which ψ is replaced by $Ak^{1/2m}\psi$ and p by $Ak^{-1/2m}p$ provide a generalization, the intermediate integrals becoming

$$u\psi + tp - \xi - \frac{k}{m}\left(\frac{\psi}{p}\right)^m = r, \qquad u\psi + tp - \xi + \frac{k}{m}\left(\frac{\psi}{p}\right)^m = s, \tag{14}$$

for the Monge-Ampère equation

$$\xi_{\psi\psi}\xi_{pp} - \xi_{\psi p}^2 + \lambda^2 = 0, \qquad \lambda = k\frac{\psi^{m-1}}{p^{m+1}}\cdot \tag{15}$$

Here r and s are constants and clearly there is no loss in generality in taking $k > 0$. This corresponds, see (13), to

$$m = \frac{1}{2}(\gamma^{-1} - 1), \qquad S - S_0 = c_v \log\left\{\frac{k^2\psi^{2m-2}}{2m+1}\right\}. \tag{16}$$

Thus we have the theorem:

THEOREM 3: *The only anisentropic simple waves in a polytropic gas* ($\lambda \neq 1/3, \pm 1$) *arise from the solutions of the intermediate integrals* (14) *with* m *given by* (16). *They require the entropy distribution function* $S(\psi)$ *in* (16), *so that*

$$\tau = \left(\frac{k^2}{2m+1}\right)\frac{\psi^{2m-2}}{p^{2m+1}}\cdot \tag{17}$$

The general solutions of the first order partial differential equations (14) are (remembering $u = \xi_\psi$, $t = \xi_p$),

$$\xi = pK(\kappa) - \left(r + \frac{k}{m}\kappa^m\right), \qquad \xi = pK(\kappa) - \left(s - \frac{k}{m}\kappa^m\right), \qquad \kappa = \psi/p,$$

where $K(\kappa)$ is an arbitrary function. These correspond to (10) and when inserted into (1) yield

[2]These are clearly general dilatations, for $m \neq 0$.

$$x = -\int \kappa K'K''\,d\kappa + \frac{k\kappa^m K'}{p} - \frac{k^2(m+1)\kappa^{2m-1}}{(2m+1)\,p^2},$$

$$t = K - \kappa K' + \frac{k\kappa^m}{p}, \qquad u = K' - \frac{k\kappa^{m-1}}{p},$$

(18)

$$x = -\int \kappa K'K''\,d\kappa - \frac{k\kappa^m K'}{p} - \frac{k^2(m+1)\kappa^{2m-1}}{(2m+1)p^2},$$

$$t = K - \kappa K' - \frac{k\kappa^m}{p}, \qquad u = K' + \frac{k\kappa^{m-1}}{p},$$

which correspond to (11). The two rows of formulas in (18) represent two kinds of anisentropic simple waves in which the trajectories and isobars are presented parametrically when ψ and p are held fixed, respectively. Each kind may be derived from the other by replacing k by $-k$.

Again for simplicity, we consider the first simple wave in (18). Clearly the Jacobian $J = 0$ along the curve

$$t_p = \frac{\kappa^2}{p}K'' - \frac{k(m+1)\kappa^m}{p^2} = 0,$$

in the ψ,p-plane. The transform of this curve in the x,t-plane is presented parametrically by

(19) $$x = -\int \kappa K'K''\,d\kappa + \frac{\kappa^2 K'K''}{(m+1)} - \frac{\kappa^3 K''^2}{(m+1)(2m+1)},$$

$$t = K - \kappa K' + \frac{\kappa^2 K''}{(m+1)},$$

and is a "limiting line" for the simple wave. The differential equations (9) for the characteristics in the ψ,p-plane reduce to

$$\left(K'' - k(m \mp 1)\frac{\kappa^{m-2}}{p}\right)\frac{d\kappa}{dp} = (\mp 1 - 1)\frac{k\kappa^{m-1}}{p^2}.$$

The lower signs give $\kappa = \text{const.}$, corresponding to which the Mach lines are parabolas, see (18). These Mach lines are enveloped by the limiting line (19) as may easily be verified. With the upper signs

(20) $$\frac{1}{p} = \kappa^{(1-m)/2}\left[C + \frac{1}{2k}\int \kappa^{(1-m)/2}K''\,d\kappa\right], \qquad C = \text{const.},$$

and the other family of Mach lines is given parametrically when this equation is used to eliminate p from the first line in (18). Once again the family is enveloped by the limiting line.

As an example we consider the *centered simple wave*, that is, a simple wave such that the characteristics of one family all pass through a fixed point in the x,t-plane, say the origin, which is therefore their envelope. Since the envelope

of either family of characteristics of either type of simple wave is given by (19) parametrically, we must have $K - \kappa K' + \kappa^2 K''/(m+1) \equiv 0$ or

$$K = A\kappa^{m+1} + B\kappa, \tag{21}$$

and it is easily verified that this gives $x = 0$ for all κ, in (19). Such a wave is therefore represented by one or other of (18), where K has the form (21).

As a particular case we consider $A = B = 0$, that is $K \equiv 0$. The wave is given by

$$x = -\frac{k^2(m+1)\kappa^{2m-1}}{(2m+1)p^2}, \qquad t = \frac{k\kappa^m}{p}, \qquad u = -\frac{k\kappa^{m-1}}{p}, \tag{22}$$

and the Mach lines are obtained by setting $\kappa = $ const., or from (20), $p = C\kappa^{(m-1)/2}$ in these formulas. To obtain the trajectories and isobars we recall that $\kappa = \psi/p$, and in place of (22) write

$$x = -\frac{k^2(m+1)\psi^{2m-1}}{(2m+1)p^{2m+1}}, \qquad t = \frac{k\psi^m}{p^{m+1}}.$$

For air $\gamma = 7/5$ (approx.) or $m = -1/7$ (approx.), so that these formulas become

$$x = -\frac{6k^2}{5\psi^{9/7}p^{5/7}}, \qquad t = \frac{k}{\psi^{1/7}p^{6/7}}.$$

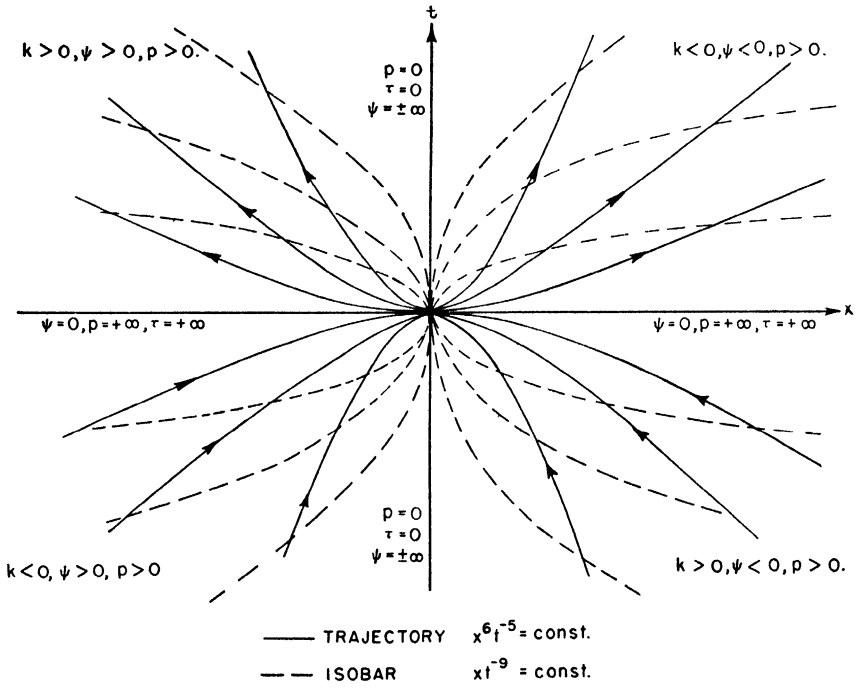


FIGURE 1

The trajectories ψ = const. and isobars p = const. are sketched in Figure 1. The wave occupies the second and fourth quadrants, the flows in these quadrants being symmetric with respect to the origin. A similar wave of the second kind is obtained in the first and third quadrants by changing k to $-k$; it is just the reflection of the original one in the x-axis.

The situation is essentially similar but much more complicated in the case of the general centered simple wave given by (21). Roughly speaking, for $\gamma = 7/5$ again, the flow field of Figure 1 is sheared parallel to the x-axis and parts of it become doubly-covered (due to the intrusion of a limit line $\tau = 0$, see (6)).

The simple waves for $K \equiv 0$ may be used as starting points for the discussion of general simple waves in (18). To see this replace x, t in (22) by x_0 , t_0 and notice that

$$\kappa = -\frac{(m+1)}{(2m+1)}\frac{t_0^2}{x_0}, \tag{23}$$

so that either line of (18) may be written as

$$x = x_0 + K't_0 - \int \kappa K'K''\, d\kappa, \qquad t = t_0 + K - \kappa K'.$$

With κ defined as in (23), this mapping of the x_0,t_0-plane upon the x,t-plane carries simple waves in the former plane into simple waves in the latter. Thus, for example, the centered simple waves for $A = 0$, $B = 1$, that is $K = \kappa$, are obtained from the simple waves for $K = 0$ by subjecting the x_0 , t_0-plane to the transformation

$$x = x_0 + t_0 , \qquad t = t_0 .$$

5. *General Isentropic Flows*

In this section we obtain the more important formulas used in the treatment of general isentropic flows, known since Riemann [7]. This will not only show the relation between our theory and the usual theory, but also serve as an introduction to the methods used in our treatment of general anisentropic flows in the following section. From our standpoint these amount to a linearization [4] of the Monge-Ampère equation (2).

Following Ampère [5] and using the results of Section 3, we write (4) in the form

$$\begin{cases} \xi_s - u\psi_s - tp_s = 0 , & \xi_r - u\psi_r - tp_r = 0 , \\ u + f = 2r, & u - f = 2s, \\ t_s - \lambda\psi_s = 0 , & t_r + \lambda\psi_r = 0 , \end{cases} \tag{24}$$

where r, s are now taken as independent (characteristic) variables. Clearly

$$u = r + s, \qquad f = r - s, \qquad \lambda = \lambda(r - s).$$

Elimination of ψ and t in turn from the last pair of equations in (24), by partial differentiation, gives[3] the conjugate equations [8]

$$(25) \qquad L(t) \equiv t_{rr} + \frac{\lambda'}{2\lambda}(t_r - t_s) = 0, \qquad M(\psi) \equiv \psi_{rr} - \frac{\lambda'}{2\lambda}(\psi_r - \psi_s) = 0.$$

These equations may be written in the more familiar form

$$t_{rs} - \frac{1 + c'}{2c}(t_r - t_s) = 0, \qquad \psi_{rs} + \frac{1 + c'}{2c}(\psi_r - \psi_s) = 0,$$

as may be seen from (5) and the formulas

$$r - s = -\int \frac{c}{\tau}\, d\tau, \qquad \frac{\tau'}{\tau} = -\frac{1}{c}.$$

If we note that (1) implies

$$(26) \qquad x = \int \{u\, dt + \tau\, d\psi\} = \int \{(u - c)t_r\, dr + (u + c)t_s\, ds\},$$

where (24) has been used to eliminate ψ_r, ψ_s and (5) to eliminate λ, we find

$$x_r = (u - c)t_r, \qquad x_s = (u + c)t_s.$$

These equations, first given by Riemann [7] are fundamental in the usual treatment of general isentropic flows, see [6] and [9].

A solution of either equation in (25) leads to a flow in the x,t-plane. To fix ideas suppose we have a solution $t(r,s)$ of the first equation in (25). Corresponding to $t(r,s)$ we have a flow in the x,t-plane given by

$$(27) \qquad x = \int \{(u - c)t_r\, dr + (u + c)t_s\, ds\},$$

$$t = t(r,s), \qquad u = r + s, \qquad c = c(r - s).$$

The first two of these equations map the r,s-plane, the *characteristic plane*, upon the x,t-plane; the last two show that the lines $(r + s) = \text{const.}$ and $(r - s) = \text{const.}$ in the characteristic plane are transformed into the *isovels* and *isobars* in the x,t-plane, respectively. Furthermore from (24)

$$(28) \qquad \psi = \int \left\{-\frac{t_r}{\lambda}\, dr + \frac{t_s}{\lambda}\, ds\right\} = \psi(r,s),$$

and the curves $\psi = \text{const.}$ in the characteristic plane transform into trajectories in the x,t-plane.

The solution of a Problem of Cauchy set for $L(t) = 0$ in (25) may always be reduced to quadratures, once the resolvent of $L(t) = 0$ is known [8]. This resolvent is a certain two-parameter family of solutions of the conjugate equation $M(\psi) = 0$.

[3]Primes indicate differentiation with respect to the argument $(r - s)$.

For polytropic gases

$$c = \frac{\gamma - 1}{2}(r - s), \quad \lambda = \alpha_0(r - s)^{-2N}, \quad N = \frac{\gamma + 1}{2(\gamma - 1)}, \quad \alpha_0 = \text{const.},$$

and (25) becomes

$$L_N(t) \equiv t_{rs} - \frac{N}{r - s}(t_r - t_s) = 0, \quad L_{-N}(\psi) \equiv \psi_{rs} + \frac{N}{r - s}(\psi_r - \psi_s) = 0,$$

the first of which appears in the literature [6] and [9]. For *positive integral* N, $\gamma = (2N + 1)/(2N - 1)$, the functions x, t, ψ in (27) and (28) may be given explicit representation in terms of two arbitrary functions $R_0 = R_0(r)$, $S_0 = S_0(s)$. Thus, from the Appendix, the general solution of the system in the third line of (24), with λ given as above, is

$$t = -N\alpha_0 \frac{\partial^{2N-2}}{\partial r^{N-1}\,\partial s^{N-1}}\left(\frac{R_0' + S_0'}{r - s}\right), \qquad \psi = (r - s)^{2N+1} \frac{\partial^{2N}}{\partial r^N\,\partial s^N}\left(\frac{R_0 + S_0}{r - s}\right).$$

Also, since

$$u - c = \frac{2N - 2}{2N - 1} r + \frac{2N}{2N - 1} s, \qquad u + c = \frac{2N}{2N - 1} r + \frac{2N - 2}{2N - 1} s,$$

one finds from (26) that

$$x = N\alpha_0\left[\frac{(N - 1)}{(2N - 1)} \frac{\partial^{2N-4}}{\partial r^{N-2}\,\partial s^{N-2}}\left(\frac{R_0'' + S_0''}{r - s}\right) - (r + s) \frac{\partial^{2N-2}}{\partial r^{N-1}\,\partial s^{N-1}}\left(\frac{R_0' + S_0'}{r - s}\right)\right].$$

6. *General Anisentropic Flows*

Referring to Section 4 and in particular to (15), we write (4) as

$$(29) \qquad \begin{cases} \xi_s - u\psi_s - tp_s = 0, & \xi_r - u\psi_r - tp_r = 0, \\ u_s + \lambda p_s = 0, & u_r - \lambda p_r = 0, \qquad \lambda = k\dfrac{\psi^{m-1}}{p^{m+1}}, \\ t_s - \lambda\psi_s = 0, & t_r + \lambda\psi_r = 0, \end{cases}$$

by introducing the characteristic variables (r,s) defined in (14) as independent variables, these equations taking the place of (24). From (14) we see that

$$(30) \qquad \frac{\psi}{p} = \left[\frac{m}{2k}(s - r)\right]^{1/m}, \qquad u\psi + tp - \xi = \tfrac{1}{2}(r + s),$$

so that the last two pairs of equations in (29) may be written as

$$(31) \qquad u_r = \beta(r - s)^{2M}\pi_r, \qquad u_s = -\beta(r - s)^{2M}\pi_s,$$

$$\pi = p^{-1}, \qquad \beta = -k[2k(2M - 1)]^{-2M},$$

$$(32) \qquad \omega_r = \alpha(r - s)^{2N} t_r, \qquad \omega_s = -\alpha(r - s)^{2N} t_s,$$

$$\omega = \psi^{-1}, \qquad \alpha = k^{-1}[2k(2N + 1)]^{-2N},$$

in which

$$M = \frac{1}{2}\left(1 - \frac{1}{m}\right) = \frac{3\gamma - 1}{2(\gamma - 1)}, \qquad N = -\frac{1}{2}\left(1 + \frac{1}{m}\right) = \frac{\gamma + 1}{2(\gamma - 1)} = M - 1.$$

Elimination by partial differentiation shows that π, u and t, ω are solutions[4] of the Euler-Poisson equations [4]

$$L_M(\pi) \equiv \pi_{rs} - \frac{M}{r - s}(\pi_r - \pi_s) = 0, \quad L_{-M}(u) \equiv u_{rs} + \frac{M}{r - s}(u_r - u_s) = 0,$$

(33)

$$L_N(t) \equiv t_{rs} - \frac{N}{r - s}(t_r - t_s) = 0, \quad L_{-N}(\omega) \equiv \omega_{rs} + \frac{N}{r - s}(\omega_r - \omega_s) = 0.$$

Starting with a solution $t = t(r,s)$ of $L_N(t) = 0$, for example, we obtain successively

$$\omega = \omega(r,s) = \alpha \int (r - s)^{2N}\{t_r\, dr - t_s\, ds\}, \qquad \text{from (32)},$$

(34)
$$\pi = \pi(r,s) = \left[\frac{2k(2M - 1)}{r - s}\right]^{2M-1}\omega, \qquad \text{from (30)},$$

$$u = u(r,s) = \beta \int (r - s)^{2M}\{\pi_r\, dr - \pi_s\, ds\}, \qquad \text{from (31)},$$

(35)
$$\tau = \tau(r,s) = \frac{k^2\pi^{2m+1}}{(2m + 1)\omega^{2m-2}}, \qquad \text{from (17)},$$

(36)
$$c = c(r,s) = \frac{k\pi^m}{(2m + 1)\omega^{m-1}}, \qquad \text{from (5) and (15)},$$

(37)
$$x = x(r,s) = \int \{(u - c)t_r\, dr + (u + c)t_s\, ds\}, \qquad \text{from (27)}.$$

The transformation $x = x(r,s)$, $t = t(r,s)$ maps the characteristic plane upon the x,t-plane and carries the curves $\psi(r,s) = \text{const.}$, $p(r,s) = \text{const.}$, $u(r,s) = \text{const.}$, $\tau(r,s) = \text{const.}$, $c(r,s) = \text{const.}$ in the characteristic plane into the trajectories, isobars, isovels, isopycnics, and isothermals, respectively, in the x,t-plane, the coordinate lines $r = \text{const.}$, $s = \text{const.}$, mapping into Mach lines.

Just as in the isentropic case, the solution of a Problem of Cauchy for any one of the partial differential equations (33), say $L_N(t) = 0$ reduces to quadratures, once the resolvent is known. For N a positive integer the resolvents of $L_N(t) = 0$ and $L_{-N}(\omega) = 0$ are given explicitly in the Appendix.

For positive integral N, corresponding to $\gamma = (2N + 1)/(2N - 1)$, the same values of γ as in the isentropic case, [5] the functions π, u; t, ω; τ, c may be

[4] The equation for t is exactly the same as in the isentropic case.

[5] The values $\gamma = 5/3$ and $\gamma = 7/5$ are thus covered, as in the isentropic case.

expressed explicitly in terms of two arbitrary functions $R(r)$ and $S(s)$. From Lemma 4 (Appendix) it follows that the general solution of (31) for positive integral M is

$$\pi = \frac{2(2M-1)}{\alpha} \frac{\partial^{2M-2}}{\partial r^{M-1}\,\partial s^{M-1}} \left(\frac{R'+S'}{r-s}\right),$$

$$u = -\frac{(r-s)^{2M+1}}{2M(2M-1)} \frac{\partial^{2M}}{\partial r^{M}\,\partial s^{M}} \left(\frac{R+S}{r-s}\right), \tag{38}$$

where $-R/2M(2M-1)$ and $-S/2M(2M-1)$ are taken as the arbitrary functions of r and s, respectively, and use has been made of the relation $\alpha/\beta = -4(2M-1)^2$. Likewise the general solution of (32) for positive integral N is

$$t = \frac{N}{\alpha} \frac{\partial^{2N-2}}{\partial r^{N-1}\,\partial s^{N-1}} \left(\frac{R_1'+S_1'}{r-s}\right), \qquad \omega = (r-s)^{2N+1} \frac{\partial^{2N}}{\partial r^{N}\,\partial s^{N}} \left(\frac{R_1+S_1}{r-s}\right), \tag{39}$$

with arbitrary functions $R_1 = R_1(r)$ and $S_1 = S_1(s)$. Moreover (34) will certainly hold if $R_1 = R'$, $S_1 = S'$, and further investigation shows that there is no loss in generality in making this identification. Hence (39) is replaced by

$$t = \frac{N}{\alpha} \frac{\partial^{2N-2}}{\partial r^{N-1}\,\partial s^{N-1}} \left(\frac{R''+S''}{r-s}\right), \quad \omega = (r-s)^{2N+1} \frac{\partial^{2N}}{\partial r^{N}\,\partial s^{N}} \left(\frac{R'+S'}{r-s}\right), \tag{40}$$

and from (35) and (36) we find

$$\tau = \frac{(r-s)^{4(N+1)}}{2\alpha(2N-1)} \left[\frac{\partial^{2N}}{\partial r^{N}\,\partial s^{N}} \left(\frac{R'+S'}{r-s}\right)\right]^3,$$

$$c = \frac{(r-s)^{2N+2}}{2(2N-1)} \frac{\partial^{2N}}{\partial r^{N}\,\partial s^{N}} \left(\frac{R'+S'}{r-s}\right). \tag{41}$$

The determination of $x(r,s)$ offers greater difficulties however. Direct substitution into (37) for u, t, c from (38), (40) and (41) leads to

$$x_r = -\frac{N}{\alpha} \left[\frac{(r-s)^{2N+3}}{(2N+1)(2N+2)} \frac{\partial^{2N+2}}{\partial r^{N+1}\,\partial s^{N+1}} \left(\frac{R+S}{r-s}\right) + \frac{(r-s)^{2N+2}}{2(2N-1)} \frac{\partial^{2N}}{\partial r^{N}\,\partial s^{N}} \left(\frac{R'+S'}{r-s}\right)\right] \frac{\partial^{2N-1}}{\partial r^{N}\,\partial s^{N-1}} \left(\frac{R''+S''}{r-s}\right),$$

$$x_s = -\frac{N}{\alpha} \left[\frac{(r-s)^{2N+3}}{(2N+1)(2N+2)} \frac{\partial^{2N+2}}{\partial r^{N+1}\,\partial s^{N+1}} \left(\frac{R+S}{r-s}\right) - \frac{(r-s)^{2N+2}}{2(2N-1)} \frac{\partial^{2N}}{\partial r^{N}\,\partial s^{N}} \left(\frac{R'+S'}{r-s}\right)\right] \frac{\partial^{2N-1}}{\partial r^{N-1}\,\partial s^{N}} \left(\frac{R''+S''}{r-s}\right), \tag{42}$$

and no simple integration of these seems to be possible in general.

For air (approx.)

$$\gamma = 7/5, \qquad m = -1/7, \qquad M = 4, \qquad N = 3,$$

and because of its simplicity and interest, we restrict the integration of (42) to this case. We shall only sketch the computations involved. Thus performing the operations indicated in (42), we find

$$x_r = a_{-1}(r - s) + a_0 + \frac{a_1}{r - s} + \cdots + \frac{a_6}{(r - s)^6},$$

$$x_s = b_{-1}(r - s) + b_0 + \frac{b_1}{r - s} + \cdots + \frac{b_6}{(r - s)^6},$$

where the a's and b's are polynomials in R, S and their derivatives, which do not contain r and s explicitly. These equations have the integral

$$x = -\frac{3(14k)^7}{490}\left[k_{-1}(r - s) + k_0 + \frac{k_1}{r - s} + \cdots + \frac{k_5}{(r - s)^5}\right], \tag{43}$$

with

$$k_{-1} = 2!3![3R^{(4)2} - R^{(4)}S^{(4)} + 3S^{(4)2}],$$

$$k_0 = -2!3!\Big[92R^{(3)}R^{(4)} - 35\int R^{(4)2}\,dr + 8(R^{(3)}S^{(4)} - R^{(4)}S^{(3)}) - 92S^{(3)}S^{(4)} + 35\int S^{(4)2}\,ds\Big],$$

$$k_1 = 2!4![5(33R^{(2)} + 12S^{(2)})R^{(4)} + 5(12R^{(2)} + 33S^{(2)})S^{(4)} + 6(8R^{(3)2} - 9R^{(3)}S^{(3)} + 8S^{(3)2})],$$

$$k_2 = -2!3!5![7(3R' - 2S')R^{(4)} + 7(2R' - 3S')S^{(4)} + (26R^{(2)} + 19S^{(2)})R^{(3)} - (19R^{(2)} + 26S^{(2)})S^{(3)}],$$

$$k_3 = 2!3!5![35(R + S)(R^{(4)} + S^{(4)}) + 14(8R' - 7S')R^{(3)} - 14(7R' - 8S')S^{(3)} + 40(R^{(2)} + S^{(2)})^2],$$

$$k_4 = -3!7!10[(R + S)(R^{(3)} - S^{(3)}) + (R^{(2)} + S^{(2)})(R' - S')],$$

$$k_5 = 5!7!(R + S)(R^{(2)} + S^{(2)}),$$

where the superscripts denote the number of differentiations of the functions R and S.

7. *An Example of General Anisentropic Flow*

For $\gamma = 7/5$, if we take

$$R = \frac{\alpha}{3\cdot 8\cdot 9\cdot 10}r^{10}, \qquad S = -\frac{\alpha}{3\cdot 8\cdot 9\cdot 10}s^{10}, \qquad \alpha = \frac{14}{(14k)^7},$$

then the following formulas are obtained

$$p = \frac{3}{14}(5r^2 + 8rs + 5s^2)^{-1}, \qquad u = -\frac{\alpha}{42}(r-s)^9(r+s),$$

$$t = (r+s)(5r^2 + 4rs + 5s^2), \qquad \psi = \frac{3}{\alpha}(r-s)^{-7}(5r^2 + 8rs + 5s^2)^{-1}$$

$$\tau = \frac{\alpha^2}{270}(r-s)^{16}(5r^2 + 8rs + 5s^2)^3, \qquad c = \frac{\alpha}{30}(r-s)^8(5r^2 + 8rs + 5s^2),$$

$$x = -\frac{\alpha}{455}(r-s)^9(100r^4 + 315r^3s + 444r^2s^2 + 315rs^3 + 100s^4).$$

The derivation of this last formula from (43) is very laborious, and it is more easily found directly from (37) knowing first the functions $u(r,s)$, $c(r,s)$ and $t(r,s)$ as above.

Elimination of r, s from the formulas for x, t by means of the formulas for ψ, p gives

$$x = \frac{18\delta}{65\psi p^2}\left[25\delta^2 - \frac{65}{84}\delta - \frac{13}{72}\right], \qquad \delta = \left(\frac{14k}{3}\right)^2 \frac{p^{9/7}}{\psi^{2/7}}. \tag{44}$$

$$t = \pm p^{-3/2}\left[\frac{1}{6} + 10\delta\right]\sqrt{\left(\frac{1}{21} - \delta\right)},$$

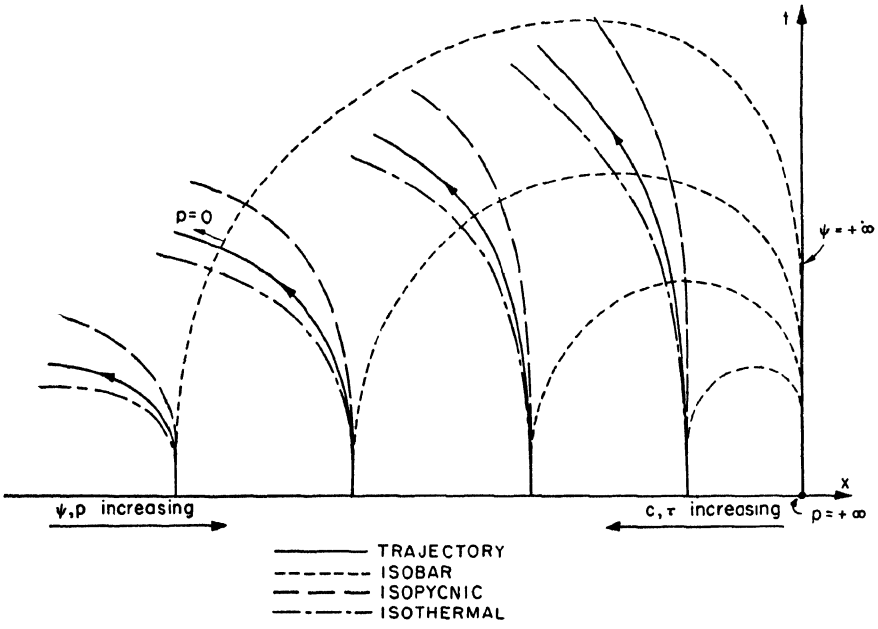


FIGURE 2

For fixed ψ, the pressure p is restricted to the range

$$0 \leq p \leq 21^{-7/9}\left(\frac{3}{14k}\right)^{14/9}\psi^{2/9},$$

and, with the positive sign on the radical in t, we have in this range $x < 0$, $t > 0$, $x_p > 0$, $t_p < 0$, so that the trajectories have the form indicated in Figure 2. As ψ varies from 0 to $+\infty$ the trajectories move from left to right.

For fixed p, the variable ψ is restricted to the range

$$21^{7/2}\left(\frac{14k}{3}\right)^{7}p^{9/2} \leq \psi < +\infty,$$

and within this range $x_\psi > 0$ and

$$t_\psi > 0 \quad \text{for} \quad 21^{7/2}\left(\frac{14k}{3}\right)^{7}p^{9/2} < \psi < \left(\frac{420}{11}\right)^{7/2}\left(\frac{14k}{3}\right)^{7}p^{9/2},$$

$$t_\psi < 0 \quad \text{for} \quad \left(\frac{420}{11}\right)^{7/2}\left(\frac{14k}{3}\right)^{7}p^{9/2} < \psi < +\infty.$$

Consequently, the isobars $p = \text{const.}$ are as indicated in Figure 2. As p varies from 0 to $+\infty$ the isobars move from left to right and shrink down on the origin.

From (35) and (36) we find in this case ($m = -1/7$)

$$\tau = \frac{7}{5}k^2\psi^{-16/7}p^{-5/7}, \qquad c = \frac{7}{5}k\psi^{-8/7}p^{1/7},$$

and when these equations are used in succession to eliminate p from (44) we obtain x, t in terms of ψ, τ and ψ, c, respectively. The isopycnics $\tau = \text{const.}$, and the isothermals $c = \text{const.}$, in Figure 2, were obtained by varying ψ with τ fixed in the first such pair of equations and c fixed in the second, respectively.

The failure of the isobars and isopycnics to coincide reflects the anisentropic character of the flow.

APPENDIX: *Remarks on the Euler-Poisson Equation*

We collect here a number of results concerning the Euler-Poisson equation

$$L_m(u) = u_{xy} - \frac{m}{x-y}(u_x - u_y) = 0,$$

which are used in the text.

LEMMA 1: *For positive integral m the general solution*[6] *of $L_m(u) = 0$ is*

$$u = \frac{\partial^{2m-2}}{\partial x^{m-1}\,\partial y^{m-1}}\left(\frac{X+Y}{x-y}\right),$$

where $X(x)$ and $Y(y)$ are arbitrary functions of their arguments.

[6] That is, one containing two arbitrary functions.

The proof of this lemma will be found in Darboux [10].

LEMMA 2: *If u is a solution of $L_m(u) = 0$, then $v = (x - y)^{2m-1}u$ is a solution of $L_{1-m}(v) = 0$, for all values of m.*

The proof of this lemma is given in Darboux [10], and merely involves calculating $L_{1-m}(v)$.

LEMMA 3: *For positive integral m the general solution of $L_{-m}(v) = 0$ is*

$$v = (x - y)^{2m+1} \frac{\partial^{2m}}{\partial x^m \, \partial y^m} \left(\frac{X + Y}{x - y} \right).$$

Lemma 2 sets up a (1,1) correspondence between solutions of the equation $L_{-m}(v) = 0$ and those of $L_{m+1}(u) = 0$. The result therefore follows by noticing that from Lemma 1 the general solution of the latter equation is

$$u = \frac{\partial^{2m}}{\partial x^m \, \partial y^m} \left(\frac{X + Y}{x - y} \right).$$

LEMMA 4: *For positive integral m the general solution of the system $v_x = \alpha(x - y)^{2m}u_x$, $v_y = -\alpha(x - y)^{2m}u_y$ ($\alpha = const.$), is*

$$u = \frac{m}{\alpha} \frac{\partial^{2m-2}}{\partial x^{m-1} \, \partial y^{m-1}} \left(\frac{X' + Y'}{x - y} \right), \qquad v = (x - y)^{2m+1} \frac{\partial^{2m}}{\partial x^m \, \partial y^m} \left(\frac{X + Y}{x - y} \right).$$

Any given pair of functions u, v which satisfy the system must also satisfy $L_m(u) = 0$ and $L_{-m}(v) = 0$, respectively. Thus, since every pair of solutions of these latter equations may, by Lemmas 1 and 3, be represented in the form

$$u = \frac{\partial^{2m-2}}{\partial x^{m-1} \, \partial y^{m-1}} \left(\frac{X_1 + Y_1}{x - y} \right), \qquad v = (x - y)^{2m+1} \frac{\partial^{2m}}{\partial x^m \, \partial y^m} \left(\frac{X_2 + Y_2}{x - y} \right),$$

the given pair of functions must allow of this representation. Furthermore, since

$$v_x - \alpha(x - y)^{2m}u_x = m(x - y)^{2m} \frac{\partial^{2m-1}}{\partial x^m \, \partial y^{m-1}} \left[\frac{\left(X_2' - \frac{\alpha}{m} X_1 \right) + \left(Y_2' - \frac{\alpha}{m} Y_1 \right)}{(x - y)} \right],$$

$$v_y + \alpha(x - y)^{2m}u_y = -m(x - y)^{2m} \frac{\partial^{2m-1}}{\partial x^{m-1} \, \partial y^m} \left[\frac{\left(X_2' - \frac{\alpha}{m} X_1 \right) + \left(Y_2' - \frac{\alpha}{m} Y_1 \right)}{(x - y)} \right],$$

must both vanish, it follows that for the given pair of functions, $X_1 = mX_2'/\alpha + \xi$, $Y_1 = mY_2'/\alpha + \eta$, where $\xi = \xi(x)$ and $\eta = \eta(y)$ are a pair of functions satisfying

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial y} = 0, \qquad Z = \frac{\partial^{2m-2}}{\partial x^{m-1} \, \partial y^{m-1}} \left(\frac{\xi + \eta}{x - y} \right).$$

Hence the given pair of functions allow of the representation

$$u = \frac{m}{\alpha}\frac{\partial^{2m-2}}{\partial x^{m-1}\,\partial y^{m-1}}\left(\frac{X_2' + Y_2'}{x-y}\right) + A$$

$$= \frac{m}{\alpha}\frac{\partial^{2m-2}}{\partial x^{m-1}\,\partial y^{m-1}}\left[\frac{(X_2 + Bx^{2m})' + (Y_2 - By^{2m})'}{(x-y)}\right],$$

$$v = (x-y)^{2m-1}\frac{\partial^{2m}}{\partial x^m\,\partial y^m}\left(\frac{X_2 + Y_2}{x-y}\right)$$

$$= (x-y)^{2m-1}\frac{\partial^{2m}}{\partial x^m\,\partial y^m}\left[\frac{(X_2 + Bx^{2m}) + (Y_2 - By^{2m})}{(x-y)}\right],$$

where A is some constant, and $B = A\alpha/2(m!)^2$. The lemma now follows on putting $X = X_2 + Bx^{2m}$, $Y = Y_2 - By^{2m}$, and verifying that conversely every pair of functions with the given representation satisfy the given system.

DEFINITION: *The resolvent of* $L_{-m}(u) = 0$ *is the solution* $v(x,y;\bar{x},\bar{y})$ *of* $L_m(v) = 0$ *which satisfies the characteristic conditions*

$$v = x - \bar{x} \quad on \quad y = \bar{y}, \qquad v = \bar{y} - y \quad on \quad x = \bar{x}.$$

Once the resolvent of $L_{-m}(u) = 0$ has been found, the solution of a Problem of Cauchy is reduced to quadratures [8].

LEMMA 5: *For positive integral* m *the resolvent of* $L_{-m}(u) = 0$ *is given by*

$$v_{-m} = \frac{\partial^{2m-2}}{\partial x^{m-1}\,\partial y^{m-1}}\left(\frac{X+Y}{x-y}\right), \qquad X = \frac{[(x-\bar{x})(x-\bar{y})]^m}{(m-1)!m!},$$

$$Y = \frac{[(y-\bar{x})(y-\bar{y})]^m}{(m-1)!m!},$$

and the resolvent of $L_m(u) = 0$ *by*

$$v_m = -\frac{\partial^{2m-2}}{\partial \bar{x}^{m-1}\,\partial \bar{y}^{m-1}}\left(\frac{\overline{X}+\overline{Y}}{\bar{x}-\bar{y}}\right), \qquad \overline{X} = \frac{[(\bar{x}-x)(\bar{x}-y)]^m}{(m-1)!m!},$$

$$\overline{Y} = \frac{[(\bar{y}-x)(\bar{y}-y)]^m}{(m-1)!m!}.$$

If m is a positive integer, v_{-m} is a solution of $L_m(v) = 0$ by Lemma 1. To verify that the required characteristic conditions are satisfied, we note that

$$v_{-m} = (m-1)!\frac{\partial^{m-1}}{\partial x^{m-1}}\left[\frac{X}{(x-y)^m}\right] + (-1)^{m-1}(m-1)!\frac{\partial^{m-1}}{\partial y^{m-1}}\left[\frac{Y}{(x-y)^m}\right],$$

so that when X, Y are replaced by their given values

$$v_{-m} = \frac{1}{m!}\frac{\partial^{m-1}}{\partial x^{m-1}}\left[\frac{(x-\bar{x})(x-\bar{y})}{(x-y)}\right]^m + \frac{(-1)^{m-1}}{m!}\frac{\partial^{m-1}}{\partial y^{m-1}}\left[\frac{(y-\bar{x})(y-\bar{y})}{(x-y)}\right]^m.$$

Along $x = \bar{x}$ the first term vanishes and the second reduces to $\bar{y} - y$; along $y = \bar{y}$ the second term vanishes and the first reduces to $x - \bar{x}$.

The second statement in the lemma follows from a general relation between the resolvents of conjugate equations [8]. Applied to this particular case, the result is that the resolvent of $L_m(u) = 0$ is obtained from that of $L_{-m}(u) = 0$ by interchanging the variables x, y with $\bar{x}$, $\bar{y}$ respectively, and changing the sign.

Bibliography

[1] Martin, M. H., *The rectilinear motion of a gas II*, American Journal of Mathematics, Volume 67, 1945, pp. 407–410.

[2] Martin. M. H., *A new approach to problems in two-dimensional flow*, Quarterly of Applied Mathematics, Volume 8, 1950, pp. 137–150.

[3] Martin, M. H., *The propagation of a plane shock into a quiet atmosphere*, Canadian Journal of Mathematics, Volume 5, 1953, pp. 37–39.

[4] Martin, M. H., *The Monge-Ampère partial differential equation* $rt - s^2 + \lambda^2 = 0$, Pacific Journal of Mathematics, Volume 3, 1953, pp. 165–187.

[5] Goursat, E., *Leçons sur l'intégration des équations aux dérivées partielles du second ordre*, Volume I, Hermann, Paris 1896, pp. 39–123.

[6] Courant, R., and Friedrichs, K. O., *Supersonic Flow and Shock Waves*, Interscience Publishers, New York, 1948, pp. 79–91.

[7] Riemann, B., *Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite*, Gesammelte Werke, Leipzig, 1892, 2nd edition, pp. 156–175.

[8] Martin, M. H., *The rectilinear motion of a gas*, American Journal of Mathematics, Volume 65, 1943, pp. 391–407.

[9] Ludford, G. S. S., *On an extension of Riemann's method of integration, with applications to one-dimensional gas dynamics*, Proceedings of the Cambridge Philosophical Society, Volume 48, 1952, pp. 499–510.

[10] Darboux, G., *Leçons sur la théorie générale des surfaces*, Deuxième partie, Gauthirr-Villars, Paris, 1915, pp. 54–65.

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. VII, 65–77 (1954)

Approximate Methods for Computing Flow Fields

By J. H. GIESE
Ballistic Research Laboratories, Aberdeen

1. *Introduction*

The announced title is rather misleading. We shall really discuss high speed computing machine calculations of steady non-viscous, non-linearized supersonic flows contemplated or actually performed on the ENIAC at the Ballistic Research Laboratories during the past five years by R. F. Clippinger, W. C. Carter, and the speaker, with the assistance of numerous other members of the laboratories. The physical motive for such work, of course, is to obtain pressure distributions on those parts of the body not adjacent to the wake. The mathematical motivation arises in the well known fact that the partial differential equations of compressible flow are non-linear. At present approximate solutions are usually found by linearizing or otherwise simplifying the equations. If simplifications are not acceptable, or if one wishes to measure their effects, numerical analysis with the aid of a high speed computer is the easiest general technique available for solving non-linear problems.

The remainder of the discussion will be devoted to a description of the method of computation, the influence of the computing machine's characteristics on the detailed computational procedure and on the nature and quantity of calculation performed, the rate of convergence of the computing method, a short catalog of computations completed, and speculative remarks about contemplated calculations.

2. *Outline of Computational Method*

Although characteristic methods for calculation of supersonic axisymmetric flows with vorticity were devised and applied to hand computation in Germany and Italy, and probably elsewhere, ten to fifteen or more years ago, and have been described and used in many places since then, it will still be desirable for use in the following sections to sketch the particular version of the method used on the ENIAC.

In terms of curvilinear coordinates based on the two Mach families of characteristic curves $\alpha(x,r) = \text{const.}$ and $\beta(x,r) = \text{const.}$, the equations for steady supersonic axisymmetric flow can be written as

$$\begin{aligned} (K - R)r_\alpha - Lx_\alpha &= 0, \\ Hr_\beta - (K - R)x_\beta &= 0, \\ Hu_\alpha + (K - R)v_\alpha + a^2vr^{-1}x_\alpha &= Rq^{-2}[(K - R)u - Hv]\pi_\alpha , \end{aligned} \tag{2.1}$$

$$(K - R)u_\beta + Lv_\beta + a^2vr^{-1}r_\beta = Rq^{-2}[(K - R)v - Lu]\pi_\beta ,$$

$$a^2q^2H\pi_\alpha x_\beta - (uR - va^2)^2\pi_\beta x_\alpha = 0. \tag{2.2}$$

Here u, v are cartesian velocity components at the point with cartesian coordinates (x,r), where the x-axis coincides with the axis of symmetry; $H = a^2 - u^2$, $K = -uv$, $L = a^2 - v^2$, $a^2 = \frac{1}{2}(\gamma - 1)(1 - q^2)$, $q^2 = u^2 + v^2$, $R^2 = a^2(q^2 - a^2)$, and

$$\gamma\pi = \log (p_0/p_0') \tag{2.3}$$

where $p_0(x,r)$ is the local stagnation pressure, and p_0' is some constant reference pressure.

On the body, defined by $r = f(x)$

$$\frac{dr}{dx} = f'(x) = \frac{v}{u}. \tag{2.4}$$

It is assumed that the body is pointed and that there is an attached shock defined by $r = g(x)$, not known in advance, at which

$$\frac{dr}{dx} = \frac{q_1 - u}{v} \equiv \tan \sigma,$$

$$v^2[(\gamma - 1)q_1^{-1} + 2q_1 - (\gamma + 1)u] = (q_1 - u)^2[(\gamma + 1)u - (\gamma - 1)q_1^{-1}], \tag{2.5}$$

$$\gamma(\gamma - 1)\pi = (\gamma + 1)\log(\gamma + 1) - \gamma \log [(\gamma - 1)M_1^2 \sin^2 \sigma + 2]$$
$$+ 2\gamma \log (M_1 \sin \sigma) - \log [2\gamma M_1^2 \sin^2 \sigma - (\gamma - 1)],$$

where $M_1^2 = 2q_1^2/[(\gamma - 1)(1 - q_1^2)]$, q_1 is the velocity of the undisturbed uniform flow ahead of the shock, and p_0' has been chosen to be the undisturbed stagnation pressure. At a corner on the body, where β, x, r, and π are constant, the first of equations (2.2) reduces to

$$\frac{dv}{du} = \frac{H}{K - R} \tag{2.6}$$

the integral curves of which are Prandtl-Busemann epicycloids. Finally, the pressure distribution is defined by

$$\frac{p}{p_1} = e^{\gamma\pi}\left[\frac{1 - q^2}{1 - q_1^2}\right]^{\gamma/(\gamma-1)} \tag{2.7}$$

Since equations (2.1) and (2.2) are homogeneous in partial derivatives with respect to α or β, the scales on the α and β axes can be adjusted so that segments of the shock and body with continuously turning tangents are mapped onto segments of slope unity. Corners, such as E in Figure 1(a), are nodes for one family of characteristics, and hence map onto segments parallel to one of the axes. Accordingly, the region $ABCDEF$ of the physical plane, where AB, BF, and CD are characteristics, can be mapped onto a polygon in the characteristic plane, as in Figure 1(b).

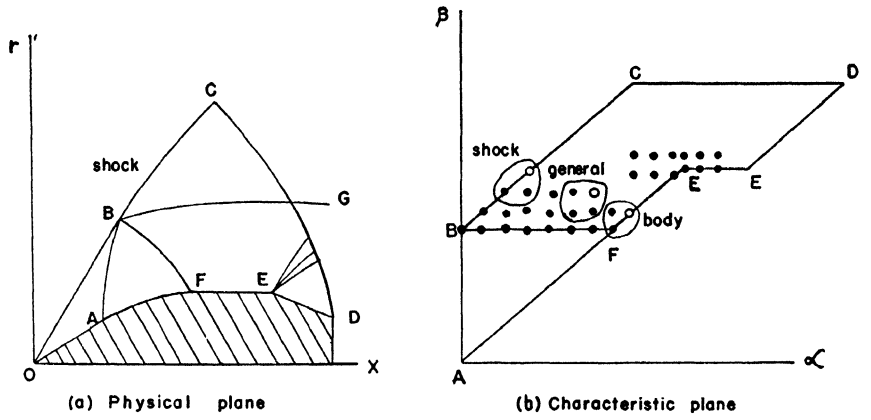


FIG. 1. Mapping onto characteristic plane.

To obtain a numerical solution the equations (2.1) and (2.2) and the various boundary or shock conditions (2.4) to (2.6) are approximated by linear difference equations which are solved step by step on a lattice in the polygon *ABCDEF*. If the solid dots in Figure 1(b) denote points at which values of x, r, u, v, and π have been previously determined, while these functions are to be determined next at the open dots, then the circled areas of Figure 1(b) indicate three situations to be considered, depending on whether boundary conditions are involved or not.

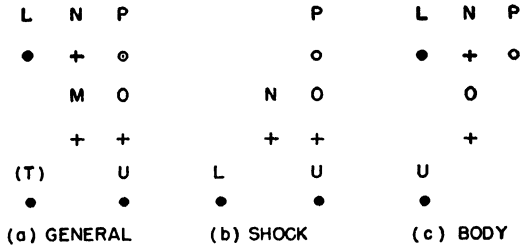


FIG. 2. Typical computing configurations.

To obtain information at a new general point, as in Figure 2(a), approximate equations (2.1) and (2.2) by

$$(K - R)_{M(N)}(r_P - r_L) - L_{M(N)}(x_P - x_L) = 0,$$

$$H_{M(O)}(r_P - r_U) - (K - R)_{M(O)})(x_P - x_U) = 0,$$

$$H_{M(N)}(u_P - u_L) + (K - R)_{M(N)}(v_P - v_L) + (a^2vr^{-1})_{M(N)}(x_P - x_L) = \{Rq^{-2}[(K - R)u - vH]\}_{M(N)}(\pi_P - \pi_L),$$

$$(K - R)_{M(O)}(u_P - u_U) + L_{M(O)}(v_P - v_U) + (a^2vr^{-1})_{M(O)}(r_P - r_U) = \{Rq^{-2}[(K - R)v - Lu]\}_{M(O)}(\pi_P - \pi_L),$$

$$(a^2q^2H)_M(\pi_P - \pi_L)(x_P - x_U) - (uR - va^2)^2_M(\pi_P - \pi_U)(x_P - x_L) = 0.$$

Subscripts P, L, U denote evaluation at the corresponding points of Figure 2(a), and M denotes evaluation with the averages of u, v, and r at U and L. This system, with coefficients with subscripts M, yields a first approximation to x, r, u, v, and π at P. Then the coefficients are evaluated a second time, subscripts N or O indicating that the arguments are taken to be the averages of first approximations at P and the values at L or U, respectively, and the system is solved again to obtain a second, final approximation at P.

To obtain information at a new shock point, as in Figure 2(b), first note that on the shock $v = v(u)$ and $\pi = \pi(u)$. Hence $dv/du = F(u,v,q_1)$, $d\pi/du = G(u,v,q_1)$, where the exact forms of F and G can be found from (2.5). Then the values of x, r, u, v, and π at P can be found iteratively from the system

$$H_{U(O)}(r_P - r_U) - (K - R)_{U(O)}(x_P - x_U) = 0,$$

$$v_{L(N)}(r_P - r_L) - (q_1 - u_{L(N)})(x_P - x_L) = 0,$$

$$(K - R)_{U(O)}(u_P - u_U) + L_{U(O)}(v_P - v_U) + (a^2vr^{-1})_{U(O)}(r_P - r_U)$$
$$= \{Rq^{-2}[(K - R)v - Lu]\}_{U(O)}(\pi_P - \pi_U),$$

$$v_P - v_L - F_{L(N)}(u_P - u_L) = 0,$$

$$\pi_P - \pi_L - G_{L(N)}(u_P - u_L) = 0,$$

where subscripts have the same significance as before. Similarly, to obtain information at a new point on the body, as in Figure 2(c), solve the system

$$(K - R)_{L(N)}(r_P - r_L) - L_{L(N)}(x_P - x_L) = 0,$$

$$r_P - r_U - f'(x)_{U(O)}(x_P - x_U) = 0,$$

$$H_{L(N)}(u_P - u_L) + (K - R)_{L(N)}(v_P - v_L) + (a^2vr^{-1})_{L(N)}(x_P - x_L)$$
$$= \{Rq^{-2}[(K - R)u - Hv]\}_{L(N)}(\pi_P - \pi_U),$$

$$v_P = f'(x)_{U(O)}u_P ,$$

$$\pi_P = \pi_U .$$

To start the calculation a short segment of the tip of the body is approximated by a conical section OA, as in Figure 1(a). Then between the shock wave OB, the body OA, and the characteristic AB is a region of Taylor-Maccoll, or axisymmetric conical flow in which u, v, and π are functions of a single parameter, e.g. r/x if the tip of the cone is at the origin. Only a boundary value problem for a system of ordinary differential equations is involved in determining the flow in this region. This is replaced by an initial value problem which is integrated numerically, appropriate initial values to produce flow about a given cone at given free stream Mach number being found by trial and error. Once initial data have been assigned on the characteristic AB, then by step by step applications of the three processes outlined above x, r, u, v, and π can be determined along successive characteristics $\beta = \text{const}$. In general, a corner E will

not fall on one of the characteristics used in the calculation. However, the body can be extended smoothly up to the first characteristic beyond E. Values for u and v at E can be found by interpolation between the values at the adjacent body points, and then velocities on EE can be found by numerical integration of (2.6). Since increments of α and β are not involved in the difference equations for the general process, EE can be considered a formal extension of the characteristic immediately preceding E, as suggested by the dots in Figure 1(b).

3. *Influence of the Computing Machine*

Although there have been numerous improvements in the design and performance of computing machines since the ENIAC was constructed, much of the experience gained with this machine carries over to more modern ones. For present purposes, important features of the ENIAC are a high speed memory of eighteen numbers, a switch set memory of 1200 orders, and punched card input and output. First, the size and nature of the memory restricts the complexity of the problems that can be handled by the machine. As a matter of fact, the axisymmetric flow calculations outlined above (Taylor-Maccoll flow excluded) come so close to the limits of the capacity of the ENIAC that the coders' ingenuity has been sorely taxed to code instructions efficiently enough to fit the machine. For lack of space it is also impossible to use floating decimal point routines. This makes it necessary to seek fairly sharp estimates of the sizes of numbers involved in a calculation part of whose interest stems from the fact that it is not known too definitely what sizes of numbers to expect. The alternative would be to calibrate the coding by watching the order by order operation of the machine for loss of the most significant parts of numbers. In any event, limited memory has made the process of coding the instructions for these computations a tedious, painfully slow undertaking. The effort required can only be justified, of course, by a large volume of calculation which involves many cases, for example studies of dependence on parameters, or by recurrent use of the computation in connection with design or experimental work. It should be remarked that to compensate for laborious coding the comparatively slow ENIAC computes values of x, r, u, v, π, and pressure and density wherever required at a rate of about twelve seconds per point of the characteristic net, as contrasted with upwards of two to four minutes which would be required by a relay type calculator, or perhaps three hours by hand.

Secondly, limited memory affects the formulation of a problem as well as the details of the numerical integration. For example, it is undesirable to use tabulated functions. Although numerous tables of Prandtl-Busemann characteristics are available, for machine purposes it is cheaper in memory requirements to solve the differential equation (2.7) than to store a table and a method for interpolation. Similarly, it is simpler to determine π at the shock from a differential equation than to evaluate the logarithms in (2.5). Likewise, the cartesian velocity components u, v have been used instead of q, $\theta = \arctan v/u$,

which appear in some versions of methods of characteristics, in order to avoid the need for determining tan $(\theta \pm \text{arc sin } a/q) = dr/dx$ in finding the coordinates. In a slightly different vein, because it is easier to code the solution of linear equations than to cope with the equation of the shock polar in (2.5) or with the equation $r = f(x)$ of the body, both have been replaced by differential equations.

Thirdly, the machine's input-output mechanism influences the organization of calculation. On the ENIAC results cannot be retained in the memory but can be recorded fairly rapidly on punched cards, and can be read back into the machine at about the same rate. Accordingly the rate of computation does not suffer intolerably from the fact that the very limited high speed memory permits only the results from the last computed point to be retained in the machine.

An effort to transfer the computation to a more modern machine appears to lead to very similar experiences. The ORDVAC, for example, can accommodate 1024 orders or numbers in its high speed memory. For normal operation its input and output are on teletype tape, or the output can be tabulated immediately. Because it can retain more numbers during the intermediate parts of a calculation, its coding is not as repetitive as the ENIAC's, and the number of orders can be decreased by about a third. Since tape output is quite slow, it is inadvisable to read great masses of results out of the ORDVAC and back in again, as on the ENIAC. Thus results must be kept in the high speed memory until they have been used to calculate the next characteristic. That is, there must be sufficient storage space for results from the longest characteristic; it seems possible to provide for about thirty-five points. Faster output, which would eliminate the need for storing an entire characteristic, could be achieved by use of punched cards, but is not worth while since the punched card input-output routine costs about a fifth of the memory. It should be remarked that the instructions plus thirty-five point storage completely fill the memory, which calls again for slow, tedious, efficient coding.

4. *Rate of Convergence*

H. Lewy [10] established existence and uniqueness theorems by showing that the solutions of initial value problems for hyperbolic partial differential equations are the limits approached by the solutions of corresponding partial difference equations as the grid size decreases to zero. F. Frankl and R. Aleksieva [6] extended the proofs to some of the types of boundary value problems encountered in supersonic flows with two independent variables. For these authors' difference quotient approximations of partial derivatives and evaluations of coefficients, the work of R. Courant, M. Rees, and E. Isaacson [5] suggests that the truncation errors are $O(h)$, where h is the length of a side of a characteristic mesh, taken to be square for the sake of definiteness. Truncation errors for such first order difference methods, for the iterative method described

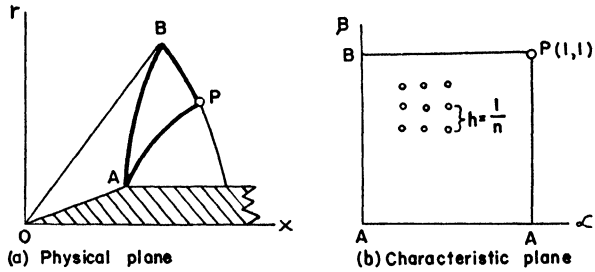


FIG. 3. Truncation error example.

in section 2, and in other methods have been studied empirically by R. F. Clippinger [3]. The example chosen was the expansion fan (Figure 3) in the flow at $M = 2.13$ about a 20°(half angle) cone cylinder, where the flow functions change fairly rapidly. This region was mapped onto a unit square in the plane of the characteristic variables, where the flow was calculated for square grids of n^2 points with mesh lengths $h = 1/n$ for fifteen values of n ranging from two to thirty-two. The variation of x, r, u, *and* v at $P(1,1)$ in the characteristic plane is shown in Figure 4 as a function of mesh size. Figure 4(a) shows the behavior of the first approximations in the method of Section 2. As one would expect, since the difference procedure resembles that used in [5], the truncation errors seem to be $O(h)$. For the iterated method of Section 2 truncation errors seem to be $O(h^2)$. For a method involving the use of results from three points L, U, T in Figure 2(a) the results oscillated between the two curves shown, but the truncation error still seems to be $O(h^2)$. Since it is inefficient to compute at extremely small grid sizes, the possibility of working at coarser grids and extrapolating to finer grids has been explored. The curves in Figure 4(b) suggest approximations at $\alpha = \beta = 1$ of the form $f(h) = A + Bh^2$ to the solution $f(h)$ of the difference equations at grid size h. This yields $f(0) = f(\frac{1}{2}h) + [f(\frac{1}{2}h) - f(h)]/3$ for extrapolation to zero grid size from results at two finite grids. It was found empirically that extrapolations at common points of 4×4 and 8×8 grids produced results comparable to those obtained, with *nine* times the effort, from a 27×27 grid. Figure 4(a) suggests approximations $f(h) = A + Bh$ which lead to $f(0) = f(\frac{1}{2}h) + [f(\frac{1}{2}h) - f(h)]$, an approximation used by Holt [10].

By a mixture of rigorous and heuristic arguments W. C. Carter and G. L. Spencer [2] have found rough bounds for truncation errors. Suppose that at a common point of three grids

$$\frac{[f(2^{-l-1}h) - f(2^{-l}h)]}{[f(2^{-l}h) - f(2^{-l+1})]} \sim 2^{-k}$$

for a computational method with local truncation errors $O(h^k)$. Then

$$f(0) - f(2^{-l}h) \sim \frac{[f(\frac{1}{2}h) - f(h)]}{(2^k - 1)2^{k(l-2)}}.$$

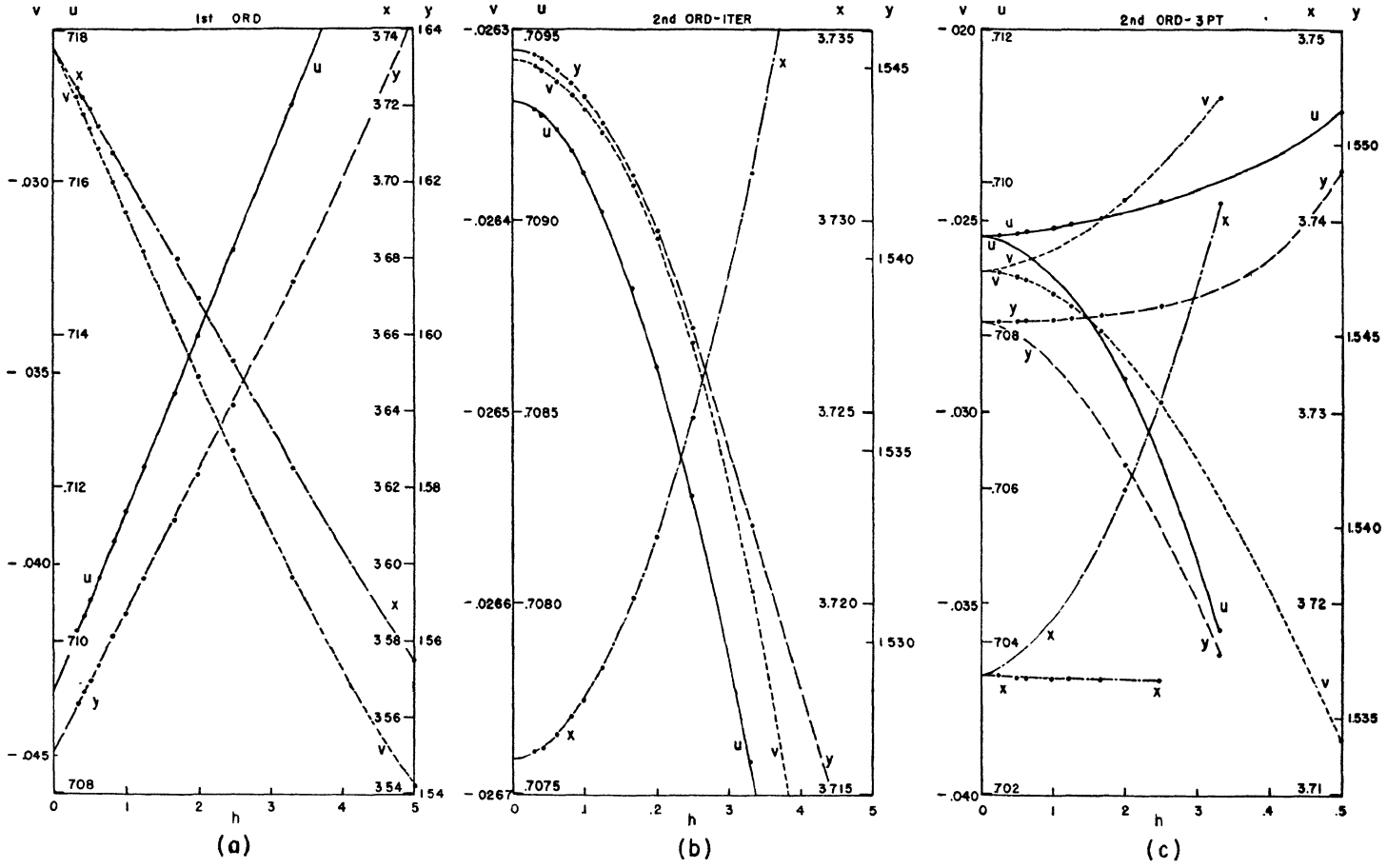


FIG. 4. Effect of grid size on calculations at point $\alpha = 1, \beta = 1$.

The table shows the results of an application of these formulae to the error study data mentioned above for the iterated method. The entries in the second column are quite close to the required value 0.25. The significance of the entries in the other columns for 16 × 16 and 32 × 32 calculations can be checked by reference to the limiting values in Figure 4(b).

		$k = 2$ $[f(0) - f(2^{-l})] \times 10^7$			
f	$\frac{f(2^{-4}) - f(2^{-3})}{f(2^{-3}) - f(2^{-2})}$	$l = 4$		$l = 5$	
		emp.	est.	emp.	est.
x	0.2597	−6336	−5484	−1584	−1371
r	0.2616	5133	4454	1283	1114
u	0.2775	698	631	175	158
v	0.2528	100	90	25	23

5. *Computed Results*

Large volume calculations include: (1) Tables of flows about cone cylinders with parameters restricted, for coding convenience, to the ranges $5° \leq \theta \leq 50°$

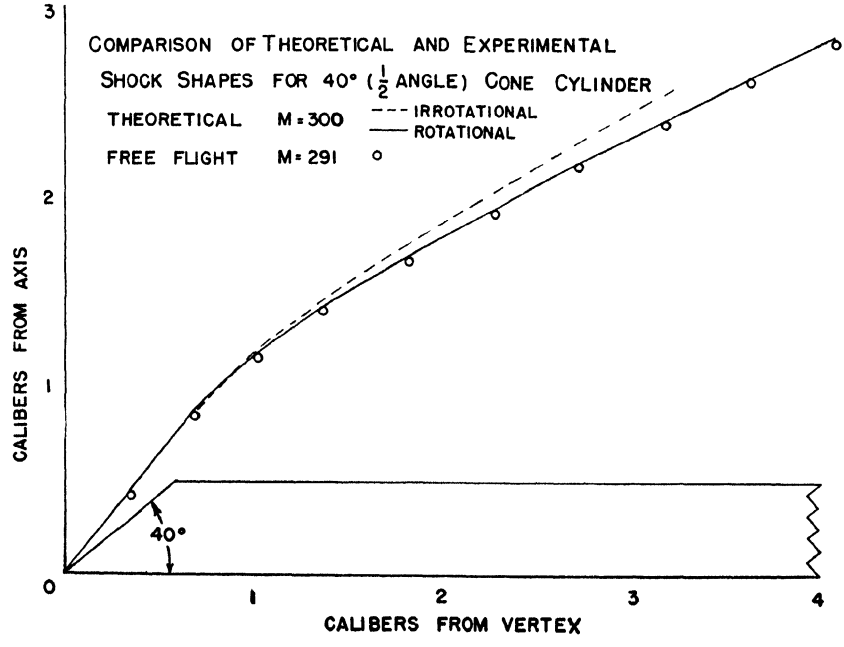


FIGURE 5

for cone half angles, and $1.3 \leq M_1 \leq 7.0$ for undisturbed Mach number. (2) Compilation of the supersonic part of a family of axisymmetric nozzles with plane sonic surfaces. (3) Computations of a large number of plane nozzle shapes with continuous wall curvatures, selected from a four parameter family, for use in designing cams for the control system of a flexible throat wind-tunnel.

Small volume computations consist of the determination of pressure distributions on particular models used in wind-tunnel or free flight range tests. The coding can handle convex bodies of revolution with profiles composed of straight and circular arc sections; it can easily be modified to handle segments of conic sections, or integrals of $dr/dx = (Ax + Br + C)/(Dx + Er + F)$. Incidentally, the importance of vorticity can be determined by computing flows once with variable π, and once suppressing the variation of π. Results for the rather extreme case of a 40° (half angle) cone cylinder at $M_1 = 3$ are shown in Figure 5.

6. *Contemplated Calculations*

Restriction to axisymmetric or plane supersonic flows about pointed objects with attached shocks is a grave handicap. Exterior ballistics is concerned with yawing bodies of revolution, which may also be blunt. The corresponding flows are certainly not axisymmetric, and the shock wave may be detached, with flow of mixed type. Preliminary efforts have been made at the Ballistic Research Laboratories to attack these more important types of flows with computing machines. Although the attempts have not yet been successful, some of the experiences may be worth mentioning. First, three-dimensional methods of characteristics for steady irrotational flow have been described in principle or detail by numerous authors [8], [7], [12], [14], [16], [17]. On a Mach characteristic surface in rotational flow the characteristic conditions are of the form [4]

$$(\mathbf{R}_\alpha \times \mathbf{R}_\beta)\cdot[(\mathbf{Q}\cdot\mathbf{Q}_\alpha - A^2\pi_\alpha)\mathbf{R}_\beta \times \mathbf{Q} + (\mathbf{Q}\cdot\mathbf{Q}_\beta - A^2\pi_\beta)\mathbf{Q} \times \mathbf{R}_\alpha] + (\mathbf{Q}\cdot\mathbf{R}_\alpha \times \mathbf{R}_\beta)[(\mathbf{R}_\beta \times \mathbf{Q}\cdot\mathbf{Q}_\alpha) + (\mathbf{Q} \times \mathbf{R}_\alpha\cdot\mathbf{Q}_\beta)] = 0,$$

$$(\mathbf{Q}\cdot\mathbf{R}_\alpha \times \mathbf{R}_\beta)^2 = A^2(\mathbf{R}_\alpha \times \mathbf{R}_\beta)^2$$

where $\mathbf{R}$ and $\mathbf{Q}$ are cartesian coordinate and velocity vectors, A is the speed of sound, and α and β are curvilinear coordinates on the surface. If α = const., β = const., and γ = const. are three families of Mach surfaces, then the six corresponding characteristic conditions and $\mathbf{Q}\cdot\nabla\pi = 0$ can be converted into difference form to yield a method of characteristics. While this may be suitable for wing-body combinations, it is not obviously applicable to bodies of revolution at large angles of attack since it is not known how to insert the vortices that stream off the leeward side. Moreover, the storage requirements are enormous. It was once planned to test the method by calculating Taylor-Maccoll flow and at the same time to test Moeckel's conjecture [12] that starting from fairly arbitrary wrong characteristic initial data and extending the flow downstream

to fit the cone, the computed results will converge to Taylor-Maccoll flow. Even after storage had been reduced by taking advantage of symmetry, at the time when the coding was temporarily halted, the ORDVAC's present memory was barely sufficient for this simplified problem. Any less symmetrical flow would have been out of the question.

Second, along slightly different lines, W. C. Carter [1] has perturbed non-linearized axisymmetric flow to develop a method for calculating flows about slightly yawing bodies of revolution, which is something of a hybrid between a method developed by Sauer [13] and A. H. Stone's [15] treatment of yawing cones. Since its memory requirements exceed those for axisymmetric flow, the ENIAC is unsuitable, but the ORDVAC may be adequate; the principal problem is to scale the computation. Results of a hand computation by this

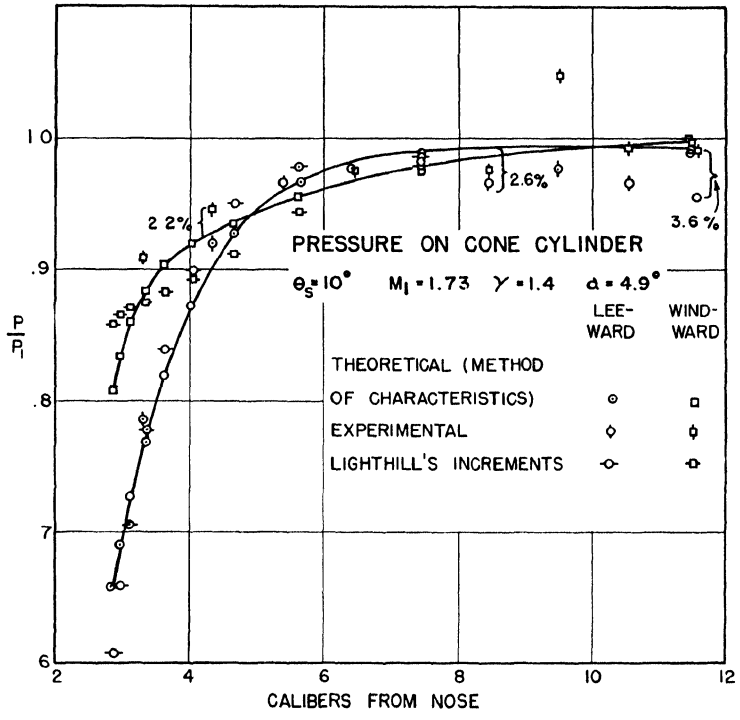


FIGURE 6

method are shown in Figure 6. Also shown are the results of superposing onto a non-linearized axisymmetric flow the velocity increments for linearized cross flow computed according to a method of M. J. Lighthill [11]. Preliminary coding of such a modification of van Dyke's [18] hybrid scheme for computing flows about bodies at an angle of attack, not using Lighthill's method for the linearized cross flow however, has been completed for the ENIAC.

As far as mixed flow is concerned, J. W. Fischbach has computed the plane flow about a finite wedge with detached shock on a relay computer with a net of forty-three points by the method of steepest descent. An outstanding feature of this work is its extremely slow convergence. It has been estimated that this same flow could be calculated on the ORDVAC by an N-step method for a net of $N = 50$ points.

Bibliography

[1] Carter, W. C., *Supersonic flow with vorticity about a slightly yawing body of revolution*, Ballistics Research Laboratory, Report No. 796, 1852.

[2] Carter, W. C., and Spencer, G. L., *On the numerical solution of hyperbolic systems of partial differential equations with two characteristic directions*, Ballistics Research Laboratory, Report No. 813, 1952.

[3] Clippinger, R. F., and Gerber, N., *Supersonic flow over bodies of revolution (with special reference to high speed computing)*, Ballistics Research Laboratory, Report No. 719, 1950.

[4] Clippinger, R. F., and Giese, J. H., *Characteristic conditions for three-dimensional flows with vorticity*, Ballistics Research Laboratory, Memorandum No. 615, 1952.

[5] Courant, R., Rees, M., and Isaacson, E., *On the solution of non-linear hyperbolic differential equations by finite differences*, Communications on Pure and Applied Mathematics, Volume 5, 1952, pp. 243–255.

[6] Frankl, F. I., and Aleksieva, R., *Zwei Randwertaufgaben aus der Theorie der hyperbolischen partiellen Differentialgleichungen zweiter Ordnung mit Anwendungen auf Gasströmungen mit Überschallgeschwindigkeit*, Receuil mathématique Moscou, Volume 41, 1934, pp. 483–502; English translation by R. F. Clippinger, Ballistics Research Laboratory, File X-123.

[7] Coburn, N., and Dolph, C. L., *The method of characteristics for three-dimensional supersonic flows*, Proceedings of the Symposium on Applied Mathematics, American Mathematical Society, 1949, pp. 55–66.

[8] Ferrari, C., *Interference between wing and body at supersonic speeds*, Journal of the Aeronautical Sciences, Volume 16, 1949, pp. 411–434.

[9] Holt, M., *The numerical method of characteristics for supersonic flows with axial symmetry*, Quarterly Journal of Mechanics and Applied Mathematics, Volume 2, 1949, pp. 473–478.

[10] Lewy, H., *Über das Anfangswertproblem einer hyperbolischen nichtlinearen partiellen Differentialgleichung zweiter Ordnung mit zwei unabhängigen Veränderlichen*, Mathematische Annalen, Volume 98, 1926, pp. 179–191.

[11] Lighthill, M. J., *Supersonic flow past slender bodies of revolution, the slope of whose meridian section is discontinuous*, Quarterly Journal of Mechanics and Applied Mathematics, Volume 1, 1948, pp. 90–103.

[12] Moeckel, W. E., *Use of characteristic surfaces for unsymmetrical flow problems*, National Advisory Committee for Aeronautics, Technical Note No. 1849, 1949.

[13] Sauer, R., *Supersonic flow about projectile heads of arbitrary shape at small incidence*, Luftfahrtforschung, Volume 19, 1942, pp. 148–152; R.T.P. Translation No. 1573, British Ministry of Aircraft Production.

[14] Sauer, R., *Dreidimensionale Probleme der Charakteristikentheorie partiellen Differentialgleichungen*, Zeitschrift für angewandte Mathematik und Mechanik, Volume 30, 1950, pp. 347–356.

[15] Stone, A. H., *On supersonic flow past a slightly yawing cone*, Journal of Mathematics and Physics, Volume 27, 1948, pp. 67–81.

[16] Teofilato, P., *Determinazione della corrente supersonica tridimensionale col metodo della carrateristiche*, Acta Pontificia Academia Scientarum, Volume 14, 1951, pp. 33–44.

[17] Thornhill, C. K., *The numerical method of charactertics for hyperbolic problems in three independent variables*, Aeronautical Research Council Reports and Memoranda, No. 2615 (11,767), 1952.

[18] Van Dyke, M., *First- and second-order theory of supersonic flow past bodies of revolution*, Journal of the Aeronautical Sciences, Volume 18, 1951, pp. 161–178.

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. VII, 79–104 (1954)

Results and Conjectures in the Mathematical Theory of Subsonic and Transonic Gas Flows

By LIPMAN BERS
New York University

1. *Introduction*

This paper is a progress report on a non-linear boundary value problem arising in gas dynamics. The problem has attracted a great deal of attention; a complete bibliography would have to include hundreds of titles. We do not, however, aim at completeness. The selection of material is purely subjective, and several important contributions are not even mentioned.

The flow of a gas around an obstacle can be described with a certain degree of accuracy by the classical theory of incompressible fluids as long as the velocities are sufficiently small. For higher values of the speed, compressibility must be taken into account. Experiments show, however, that as long as the speed of the gas nowhere exceeds that of sound the effects of compressibility are only quantitative. This is fully confirmed by the mathematical theory, as we shall show in a subsequent section. When supersonic velocities are reached the whole character of the flow may change and shocks may appear. We are concerned here with shockless flows only, and in particular, with the possibility of explaining the breakdown of potential shockless flow within the framework of a theory which does not take viscosity into account.

More precisely, the problem before us is that of describing fully the steady two-dimensional potential flow of a perfect compressible fluid around a given obstacle, the direction and (subsonic) speed of the flow at infinity being prescribed. Needless to say, the adjectives "steady", "potential", "two-dimensional" and "perfect" represent far-reaching idealizations of physical reality. Once an idealized model has been agreed upon, however, it is the mathematician's job to obtain from it as much information as possible. An applied mathematician ought to be constantly aware that he works with an idealized model; he should not use this awareness as an alibi for failure to practice his trade.

Experience shows that in most cases the value of an idealized model can not be assessed a priori. Only after the mathematics of the model has been sufficiently explored can one tell under which circumstances the model gives a satisfactory description of physical reality and in which cases it fails. The failure of an idealized model can be detected in two quite different ways. The model may furnish predictions which do not agree with, or which may even contradict, experimental data, or the model may turn out to be intrinsically inconsistent. This inconsistency will in general manifest itself in the failure of

the relevant problems to be *correctly posed.* The term "correctly posed" is used here in the technical sense formulated by Hadamard. A problem, say a boundary value problem for a partial differential equation, is called correctly posed if a solution exists, is unique, and depends continuously on the given data. In every concrete case one must, of course, specify in which class of functions one looks for a solution, and what precise meaning one gives to the requirement of continuous dependence on the data.

It is sometimes argued that existence and uniqueness proofs are of no interest to the "down to earth" applied mathematician, and in many cases such an attitude causes no trouble. In the problem under consideration, however, the main difficulties stem precisely from unsolved existence and uniqueness questions, and some investigators who have ignored these difficulties were led to quite erroneous conclusions.[1]

On the other hand, pure mathematicians have sometimes expressed a doubt as to whether mechanics still provides fruitful stimuli to mathematics. Again we believe that the problem under consideration can serve as a counter-example. The positive results achieved thus far have involved some very refined and modern techniques of analysis. The open questions have stimulated interest in the practically unexplored field of partial differential equations of mixed type.

It should be said from the outset that the problem on which we report is far from being fully solved. But significant progress has been achieved and the open questions can, perhaps, be formulated today with more precision than was possible some time ago. It might be worthwhile, therefore, to take stock.[2]

2. *Formulation of the Problem*

We are concerned with the non-linear partial differential equation[3]

$$(\rho\phi_x)_x + (\rho\phi_y)_y = 0 \tag{2.1}$$

where

$$\rho = \rho(q) \qquad (q^2 = \phi_x^2 + \phi_y^2) \tag{2.2}$$

is a given function. Here x, y are the coordinates in the plane of the flow (physical plane), $\phi(x, y)$ is the velocity potential, q the speed, and ρ the density. The density-speed relation (2.2) is obtained from Bernoulli's law and from the assumed equation of state. If the latter is

[1]This applies in particular to the alleged explanation of the breakdown of a potential flow past an airfoil by the appearance of a limiting line.

[2]The present report is written from the point of view of a mathematician. For recent surveys of the transonic problem in which the physicist's point of view is more emphasized, see Sears [59] and Guderley [40].

[3]Concerning the derivation of the differential equations and the underlying physical considerations see Sauer [58], Liepmann and Puckett [45], Courant and Friedrichs [20].

$$p\rho^{-\gamma} = \text{const.}$$

where p is the pressure and $\gamma > 1$ the constant ratio of specific heats, then the ρ–q relation is of the form

$$\rho = \left(1 - \frac{\gamma - 1}{2} q^2\right)^{1/(\gamma-1)} \tag{2.3}$$

provided that the units are chosen properly. In this case the speed of sound c is given by

$$c^2 = \frac{2}{\gamma - 1} - \phi_x^2 - \phi_y^2$$

and equation (2.1) can be written in the form

$$(c^2 - \phi_x^2)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + (c^2 - \phi_y^2)\phi_{yy} = 0. \tag{2.4}$$

The discriminant of this equation is

$$c^4(1 - M^2)$$

where $M = q/c$ is the so-called local Mach number. Equation (2.4) is elliptic for subsonic flow ($M < 1$), that is, as long as the speed q does not exceed the critical speed

$$q_{cr} = \sqrt{2/(\gamma + 1)}.$$

For supersonic flow ($M > 1$, $q > q_{cr}$) the equation is hyperbolic. For most of the following considerations the precise form of the density-speed relation is of little importance. Some authors prefer to use a slightly different ρ–q relation in order to facilitate computations. As long as the essential features of relation (2.3) are preserved there can hardly be any objections to this procedure. As a matter of fact, it can be justified rigorously for purely subsonic flows.

In the case of a general ρ–q relation the flow is subsonic or supersonic according to whether the derivative

$$d[q\rho(q)]/dq$$

is positive or negative.

In studying purely subsonic flows it is sometimes convenient to replace (2.3) by a density-speed relation which leads to a permanently elliptic differential equation and imposes no restrictions on the value of q. Thus Chaplygin [17] proposed the ρ–q relation.

$$\rho^2(1 + q^2) = 1.$$

This makes (2.1) into the classical equation of minimal surfaces which can be investigated by the tools of complex function theory. Chaplygin's approximation has been used extensively in recent aerodynamical literature. Another device, proposed by Vazsonyi [66] in 1944, but used for the first time successfully by Schiffman [60], consists in replacing (2.3) by a relation (2.2) which coincides

with (2.3) for values of q in an interval $0 \leq q \leq q_{cr} - \epsilon$, ϵ being a fixed arbitrarily small number, and makes (2.1) into the Euler-Lagrange equation of a regular variational problem. This "regularization" of the equation has an important advantage: a solution of the regularized equation for which $\phi_x^2 + \phi_y^2 < (q_{cr} - \epsilon)^2$ is at the same time a solution of the "exact" equation (2.4).

Now let P be a given simple closed curve in the z-plane ($z = x + iy$). We assume that P has a continuously turning tangent and a continuously changing curvature, except perhaps at one point, the trailing edge z_T, at which there may be a protruding corner or cusp.

A solution $\phi(x, y)$ of (2.1) defined in the domain exterior to P and having single-valued derivatives ϕ_x, ϕ_y will represent a flow past P if the derivatives are continuous on P and satisfy on it the condition $\partial\phi/\partial n = 0$, $\partial/\partial n$ denoting differentiation in the normal direction. In addition we demand that (i) the derivatives should satisfy a uniform Hölder condition on P, (ii) the derivatives should vanish at the trailing edge except if z_T is the vertex of a cusp (Kutta-Joukowski condition) and that (iii)

$$\phi_x \to q_\infty, \qquad \phi_y \to 0 \qquad \text{as} \qquad z \to \infty$$

where q_∞, $0 < q_\infty < q_{cr}$, is a given number. We do not require that ϕ itself be single-valued and do not prescribe the value of the circulation

$$\Gamma = \oint_P d\phi.$$

A function satisfying these conditions will be called a solution of Problem $\Pi(P, q_\infty)$ or a flow past P with free stream speed q_∞. For every such solution we set

$$q_{max} = \max \sqrt{\phi_x^2 + \phi_y^2}.$$

The flow is called subsonic if $q_{max} < q_{cr}$, transonic if $q_{max} > q_{cr}$.

Remark: Some of the conditions stated above are redundant. Thus the Kutta-Joukowski condition will be satisfied automatically whenever z_T is the vertex of a protruding corner of opening less than π. Condition (i) can be shown to be implied by the others. Also I proved [13] that whenever the flow is subsonic outside some large circle, the velocity components ϕ_x, ϕ_y necessarily approach subsonic limits as $z \to \infty$. This last statement implies, by the way, the following theorem: *a subsonic flow occupying the whole plane is necessarily uniform.*

3. *Alleged Approximate Solutions*

The boundary value problem formulated above presents three difficulties: The equation is non-linear. The value of the circulation is to be determined from the Kutta-Joukowski condition. Finally, and this is the most serious hurdle, the equation is of mixed type.

Confronted with a problem of this kind an applied mathematician usually

falls back on one of the three well-tried traditional methods for handling seemingly intractable problems: perturbation of known solutions, finite differences, construction of typical examples. All three methods have been applied to Problem II with apparent, I am tempted to say, with alarming success.

The solution of Problem II is well known in the limiting case $q_\infty = 0$ in which the problem reduces to the determination of an incompressible flow past P. The perturbation method applied to this solution consists in assuming for the solution an expansion of the form

$$\phi(x, y) = \phi_0(x, y) + \tau\phi_1(x, y) + \tau^2\phi_2(x, y) + \cdots . \tag{3.1}$$

Here ϕ_0 is the solution of the incompressible problem, τ a parameter characterizing the magnitude of q_∞ , say the square of the free stream Mach number. The coefficients $\phi_i(x, y)$ can be determined successively by solving linear elliptic boundary value problems. This method, known as the Rayleigh-Janzen method, has been the subject of innumerable papers.[4] While only a few terms of (3.1) have been computed in any given case, it is generally believed that the method fails to converge for values of τ which would lead to transonic flows. Whether convergence breaks down even for smaller values of τ is an open question. Frankl and Keldysh [30] proved, however, that the method coverges for sufficiently small values of τ; how small sufficiently small is depends, of course, on the profile.

Another known "solution" of II is the uniform flow which corresponds to the case when P degenerates into a straight segment. The perturbation method applied to this solution amounts to assuming an expansion of the form (3.1), ϕ_0 being of the form const. $\cdot$ x, and τ denoting the parameter characterizing the deviation of the flow from uniformity, say a typical thickness of the profile. (The first term of the expansion (3.1) is known as the Prandtl-Glauert approximation.) While I know of no convergence proof for this method,[5] numerical computations fail to indicate any trouble and several authors (in particular Görtler [35] and Kaplan [41]) produced flow patterns past thin profiles and along slightly curved walls which exhibit rather large but bounded supersonic regions within an otherwise subsonic flow.

Flows of this character were also obtained by G. I. Taylor [61] in the case of a symmetric nozzle. Taylor assumed for ϕ a power-series expansion in (x, y), of which only a few terms were retained in the computations.

The method of finite differences consists, as is well known, in replacing the partial derivatives occuring in a differential equation by corresponding partial difference quotients and in solving the resulting system of algebraic equations for a finite number of unknowns. In the case of equation (2.1) one obtains a system of non-linear algebraic equations. It seems that nobody investigated

---

[4]For a more precise description and references to the original papers, see Sauer [58] or Liepmann-Puckett [45]. A rather complete bibliography is given in Ehlers [21].

[5]For more details and references see the texts quoted in the preceding footnote.

whether this system has solutions, and whether these solutions, if they exist, approximate solutions of the differential equations. If, however, the difference equation is solved numerically by the so-called "relaxation method" no danger signals are observed. By this method also flow patterns with bounded supersonic regions have been obtained, for instance, by Emmons [22], [23].

We shall see later (see Section 7) that one may well doubt whether the exact solutions to which the methods just described are supposed to furnish approximations actually exist except in very special cases. If these doubts were justified one would have to investigate whether the alleged approximate solutions are simply "optical illusions" or whether they are valuable approximations to physically valid singular solutions, say solutions involving small discontinuities and small deviations from irrotationality.

4. *Rigorous Solutions. The Limiting Line*

The usefulness of indirect methods in applied mathematics has been demonstrated time and again.[6] If a boundary value problem can not be solved rigorously and sufficiently explicitly for a given shape (and few problems ever are), then one tries to find solutions for shapes which are determined a posteriori in the hope that these solutions will turn out to be typical. In the case of a correctly posed problem simple examples of solutions certainly furnish reliable information. The trouble is that examples of rigorous solutions can never be used in order to demonstrate that the problem is correctly posed.

In the case of problem II the construction of a single example already presents considerable difficulties, which can, however, be overcome in various ways. We shall discuss briefly only the methods which attach themselves directly to the pioneering work of Chaplygin [17]. Bergman's method is described rather fully in his papers [3], [4]. As far as I know, it has not been used for transonic flows. Another method, proposed independently by Christianovitch [19] and by myself [5] is in principle limited to purely subsonic flows.[7] All these methods for constructing solutions are based on the hodograph transformation.

Equation (2.1) implies the existence of a stream-function $\psi(x, y)$ such that

$$\rho\phi_x = \psi_y , \qquad \rho\phi_y = -\psi_x . \tag{4.1}$$

In the physical plane ψ does not satisfy a fixed differential equation. More precisely, one can not write down the second order equation satisfied by ψ in a domain of the z-plane, unless it is known that in this domain the flow is entirely subsonic or entirely supersonic. If, however, the velocity components ϕ_x and ϕ_y , or some fixed functions of the velocity components, are introduced as new independent variables, then ϕ and ψ become solutions of fixed linear partial

[6] Cf. Nemenyi [52] for a general discussion and many examples.

[7] Christianovitch's treatment of circulatory flows contains an error.

differential equations. It is customary to carry out this hodograph transformation[8] by introducing as independent variables the inclination of the velocity vector θ = arc tan (ϕ_y/ϕ_x) and a fixed function of the speed:

$$\sigma = \int_q^{q_{cr}} \frac{\rho\, dq}{q} . \tag{4.2}$$

As a function of (θ, σ) the stream-function satisfies the Chaplygin equation

$$K(\sigma)\psi_{\theta\theta} + \psi_{\sigma\sigma} = 0 \tag{4.3}$$

where

$$K(\sigma) = \frac{1 - M^2}{\rho^2} . \tag{4.4}$$

Note that

$$\sigma \to +\infty \text{ for } q \to 0, \qquad \sigma = 0 \text{ for } q = q_{cr}, \qquad K(0) = 0, \qquad K'(\sigma) > 0. \tag{4.5}$$

Chaplygin's equation is therefore elliptic in the subsonic half-plane $\sigma > 0$ and hyperbolic in the supersonic half-plane $\sigma < 0$.

Now let ψ be a solution of (4.3) defined in a domain Δ of the σ,θ-plane, and set

$$x + iy = \int \frac{e^{i\theta}}{q} \left\{ \left(\psi_\sigma + \frac{i}{\rho}\, \psi_\theta \right) d\theta + \left(-K\psi_\theta + \frac{i}{\rho}\, \psi_\sigma \right) d\sigma \right\}.$$

If this mappng takes Δ into a simply covered domain of the z-plane, then ψ is in this domain the stream function of a compressible potential flow, and the velocity potential of this flow satisfies (2.1).

The transformation from the hodograph to the physical plane is certainly impossible if there exists an arc Γ in Δ along which

$$x_\theta y_\sigma - x_\sigma y_\theta = 0. \tag{4.6}$$

This can never happen in the subsonic half-plane, but is very well possible in the supersonic region. The image C of Γ in the physical plane is a natural boundary of the flow, the locus of cusps of streamlines. For this reason C and Γ are called limiting lines. (The simplest example of a limiting line is furnished by the familiar flow with radial streamlines.)

There exist several ways of constructing complete systems of particular solutions of (4.3). The most useful system seems to be the one used by Chaplygin and consisting of the functions

$$\theta, \qquad f_\alpha(\sigma) \cos \alpha\theta, \qquad f_\alpha(\sigma) \sin \alpha\theta \tag{4.7}$$

[8] A detailed discussion of the hodograph transformation and of limiting lines will be found in Sauer [58], pp. 93–99 and in Courant-Friedrichs [20], pp. 248–256. Cf. also von Kármán [42].

where the f_α are solutions of appropriate ordinary differential equations. These solutions correspond to the harmonic functions

$$\theta, \qquad q^\alpha \cos \alpha\theta, \qquad q^\alpha \sin \alpha\theta \tag{4.8}$$

of the variables $q \cos \theta$, $q \sin \theta$. The important question is that of finding a combination of the solutions (4.7) which when transferred to the physical plane will yield a flow with desired properties, say a flow "similar" to a given incompressible flow. Chaplygin's prescription for making this choice is very simple: express the stream-function of the given incompressible flow $\tilde{\psi}$ which is a harmonic function of $(q \cos \theta, q \sin \theta)$, as a series of solutions (4.8), and then form with the same coefficients a series of corresponding solutions (4.7). This prescription works beautifully for a jet flow out of a vessel formed by two rectilinear walls and for a wake flow past a straight segment. But if $\tilde{\psi}$ is chosen as the stream-function of an incompressible flow past a profile, the series formed by Chaplygin's prescription can represent the desired solution only in a part of the domain. It becomes necessary to resort to analytic continuation, and it turns out that a solution with the desired properties can be obtained only by modifying the simple recipe used by Chaplygin. This is not the place to describe the ingenious methods by which the procedure sketched above was carried to a successful conclusion by Lighthill [46], [47], [48], Goldstein, Lighthill and Craggs [34], Cherry [18] and Tomatika and Tamada [63].[9] We state only the final result.

It is possible to find a family of solutions $\psi(\theta, \sigma; t)$ of Chaplygin's equation depending on a parameter t, $0 < t < t^*$, and having the following properties:

(i) For each t in the interval considered $\psi(t)$ is a (in general multiple-valued) solution of (4.3) defined in a domain Ω depending on t. In the closure of Ω the solution $\psi(t)$ and its partial derivatives with respect to θ, σ of order 2 are continuous and are continuous functions of t.

(ii) There exists a t_1, $0 < t_1 < t^*$, such that for each t, $0 < t < t_1$, the solution $\psi(t)$ when transferred to the physical plane represents a potential gas flow past a profile $P(t)$ depending on t. The velocity of this flow at infinity $q_\infty(t)$ is subsonic and is a monotone function of t. In other words: for $0 < t < t_1$, $\psi(t)$ is in the physical plane a solution of $\Pi(P(t), q_\infty(t))$. For $t \to 0$, the flow approaches an incompressible flow around a given profile $P(0)$.

(iii) There exists a t_2, $0 < t_2 < t_1$, such that for $t < t_2$ the flow represented by $\psi(t)$ is purely subsonic, whereas for $t_2 < t < t_1$, this flow is transonic, with bounded supersonic regions.

(iv) For $t > t_1$ the solution $\psi(t)$ can not be transferred to the physical plane in view of the appearance of limiting lines.

We remark that in all cases actually constructed the dependence of $\psi(t)$ on t is analytic.

Thus the results derived by the hodograph method seem to confirm the

[9] Tomatika and Tamada work with a slightly modified density-speed relation.

results obtained by the non-rigorous methods described in the preceding section. It was natural to assume that the computed flows were typical examples and that transonic flows with sufficiently small supersonic regions are possible for arbitrarily given profiles. In fact, several aerodynamicists concluded that for a given fixed profile there always exists a family of flows having properties (i)-(iv), and that the ultimate collapse of a potential transonic flow is due to the appearance of a limiting line in the hodograph.[10]

It turned out, however, that the attractive and elegant limiting line hypothesis was intrinsically wrong. In 1948 Friedrichs [31] proved that a family of solutions of the Chaplygin equation having properties (i)-(iii) and belonging to a *fixed* profile P can not have property (iv). The same is true if $P(t)$ is not constant but has a bounded curvature. A partial result in this direction was obtained previously by Nikolski and Taganov [53]. Friedrichs' proof was rather intricate; it made use of an algebraic lemma proved by Flanders, and it assumed the analyticity of ψ as a function of (σ, θ). Recently Kolodner and Cathleen Morawetz [44], using a variable-transformation proposed by Manwell [50], gave a new very short proof of Friedrichs' result, without assuming analyticity. Thus the limiting line hypothesis may be considered as *definitely disproved.*

Friedrichs' result applies, of course, only to the type of transonic flows in which the supersonic region is bounded. There are other cases in which the explanation of the breakdown of potential flow by limiting line considerations is perfectly legitimate.[11]

Once it is recognized that the examples computed by the hodograph method are not at all typical as far as the breakdown of potential flow is concerned, it is natural to ask whether they are not exceptional in other respects too. Indeed, quite independently of the limiting line considerations and at a relatively early time, several authors began to express doubts in the correctness of the boundary value problem for transonic flows. The starting point of this "critical" direction was the 1930 paper by G. I. Taylor [62]. Taylor applied the perturbation method to the explicitly known purely circulatory transonic flow around a circle, with vanishing velocity at infinity. He showed, not absolutely rigorously, but within the framework of the standard small perturbation technique, that the boundary value problem obtained by a slight change of the circular profile is not correctly posed. Later Nikolski and Taganov [53] discovered, by rather simple geometric considerations, that in a smooth transonic flow containing a supersonic region bounded by a fixed wall and by a sonic line on which $M = 1$ a definite inequality must hold along the wall connecting the rate of change of q with the curvature of the profile. In particular, a flow of this type is mathematically impossible if the profile contains an arbitrarily small straight segment. In an unpublished note Cathleen Morawetz showed that the straight segment condition can be replaced by the requirement that the curvature of the profile

[10]Cf. Tsien and Kuo [64].

[11]Cf. the references in footnote 8.

should vanish at a single point. The most general arguments against the possibility of solving the transonic boundary value problem in the general case, however, were given by Frankl [28], Busemann [15], [16] and Guderley [36], [37], [38], [39], [40]. A reformulation of the mathematical arguments advanced by these authors will be found in Section 7.

5. *Existence and Uniqueness Theorems for Subsonic Flow*

A critical examination of boundary value problem II must of course begin with the seemingly harmless case of a purely subsonic flow. Fortunately in this case a complete answer is available.

EXISTENCE AND UNIQUENESS THEOREM: *For a given profile P there exists a number $\hat{q} \leq q_{cr}$ such that for every value of q_∞, $0 < q_\infty < \hat{q}$ the boundary value problem $\Pi(P, q_\infty)$ has a unique solution $\phi = \phi^{(q_\infty)}(x, y)$. For this solution $q_{max} < q_{cr}$ and the derivatives ϕ_x, ϕ_y are continuous functions of q_∞. As q_∞ ranges over the interval $0 < q_\infty < \hat{q}$, q_{max} takes on all values in the interval $0 < q < q_{cr}$. (As $q_\infty \to 0$, $\phi^{(q_\infty)}$ converges to zero, but $\phi^{(q_\infty)}/q_\infty$ converges to the potential $\phi^{(0)}$ of an incompressible flow past P.)*

The first step toward the proof of this theorem was taken by Frankl and Keldysh [30] who proved the unique solvability of II for sufficiently small values of q_∞. This did not yet establish the existence of flows with arbitrarily high subsonic values of the maximal local speed. I proved the existence of such flows for the special case of the equation of minimal surfaces, by considering a modification of Problem II in which the value of q_{max} rather than that of q_∞ is prescribed [6], [8]. By an entirely different method Schiffman [60] treated a modified form of Problem II, assuming the profile to be smooth and prescribing the value of the circulation. Finally, I extended the method used for the minimal surface case to obtain the theorem stated above [12].[12]

The existence proofs are unfortunately too technical to be described here. Schiffman's proof uses the direct methods of the calculus of variation. He begins by observing that it is sufficient to prove the corresponding theorem for the regularized equation, cf. Section 2 (for which, of course, $\hat{q} = \hat{q}_{cr} = +\infty$). The existence of a "solution" follows from general theorems in calculus of variations, if one permits the solution to lie in a sufficiently wide class of functions.[13] The difficulty is thus shifted to proving that this generalized solution has the required smoothness properties, that the partial derivatives are continuous on the boundary, and that they depend continuously on q_∞. Schiffman overcomes these difficulties by a very ingenious method which is applicable also to other problems.

My proof (which also works with the regularized equation) is patterned

[12]The proof was presented in a series of seminar talks at New York University in 1952.

[13]Actually Schiffman works with the stream function rather than with the velocity potential.

after the conformal mapping method for solving the incompressible flow problem. Assume that the solution ϕ is known. Then one can map the domain exterior to P onto the domain $|\zeta| > 1$ in an auxiliary ζ-plane, conformally with respect to the Riemannian metric

$$(5.1) \qquad (c^2 - \phi_y^2)\, dx^2 + 2\phi_x\phi_y\, dx\, dy + (c^2 - \phi_x^2)\, dy^2$$

defined by the solutions. It turns out that the solution can be reconstructed once one knows the function f, a function of one variable, which describes the correspondence between points on $|\zeta| = 1$ and points on P. In fact, the solution ϕ can be computed from f in two different ways. This leads to a non-linear functional equation for f, which is then shown, by means of the Schauder-Leray method, to possess a solution. The most important step in the proof involves finding a priori bounds for f. Here the theory of quasi-conformal mappings and the theory of pseudo-analytic functions plays a decisive part.

In this proof the continuous dependence of the solution on q_∞ comes out almost automatically. It is also easy to conclude that the solution depends continuously on the shape of the profile and on the form of the density speed relation.

Remark (added September 15, 1953): I am now in possession of a considerably simpler existence proof. For details, see [12].

An important open question is the following: *is* $q_{\max}$ *a monotone function of* q_∞ ? I can answer this question *affirmatively* only under the assumption that P is symmetrical with respect to the x-axis.[13a]

It is natural to conjecture that

$$(5.2) \qquad \hat{q} < q_{cr}\,,$$

but the proof is not at all obvious. A proof has been given, however, by Schiffman.[14] Once inequality (5.2) is established it is not difficult to show that as $q_\infty \to \hat{q}$, $\phi^{(q_\infty)}$ approaches a solution of (2.1) defined in the domain exterior to P. Whether the derivatives of this solution are continuous on P is not known, and for this reason the existence of a critical flow (flow with $q_{\max} = q_{cr}$) past a given profile remains an open question.[15]

The smoothness conditions on the profile stated in Section 2 can be relaxed without impairing the validity of the existence proof. For instance, the profile may be permitted to have intruding corners.

Recently Berg [2] applied a similar method in order to prove the existence of a compressible subsonic Helmholtz flow with free boundaries for a wide class of curved obstacles.

The difference between the subsonic and transonic boundary value problems can be best explained by discussing the uniqueness proof.

[13a]This was also proved, under somewhat stronger hypotheses, by Gilbarg [67].

[14]Oral communication.

[15]It would be of interest to know very precisely the behaviour of the critical flow at the boundary since this might shed light on the conjectured collapse of potential flow.

Let $\phi(x, y)$ be a given subsonic solution of $\Pi(P, q_\infty)$, and let $\tilde{\phi}(x, y)$ be another solution of which we assume either that it is known to be subsonic or that its second derivatives are continuous up to the boundary. We must show that the function

$$\omega(x, y) = \phi(x, y) - \tilde{\phi}(x, y) \tag{5.3}$$

is constant. Now ω satisfies the linear equation

$$(c^2 - \phi_x^2)\omega_{xx} - 2\phi_x\phi_y\omega_{xy} + (c^2 - \phi_y^2)\omega_{yy} + A\omega_x + B\omega_y = 0 \tag{5.4}$$

where c is the speed of sound determined by the solution ϕ, and A, B are certain functions of (x, y). Also

$$\frac{\partial\omega}{\partial n} = 0 \text{ on } P, \qquad \omega_x^2 + \omega_y^2 = 0 \qquad \text{at} \qquad z_T\,, \tag{5.5}$$

and

$$\omega_x^2 + \omega_y^2 \to 0 \qquad \text{as} \qquad z \to \infty\,. \tag{5.6}$$

(We assume for the sake of simplicity that z_T is not a cusp.) Map the domain exterior to P onto $|\zeta| > 1$ by a transformation conformal with respect to the metric (5.1), and taking $z = \infty$ into $\zeta = \infty$, and $z = z_T$ into $\zeta = 1$. In the ζ-plane, ω satisfies an equation

$$\omega_{\xi\xi} + \omega_{\eta\eta} + \alpha(\xi, \eta)\omega_\xi + \beta(\xi, \eta)\omega_\eta = 0 \tag{5.7}$$

where $\zeta = \xi + i\eta$, and the boundary conditions are

$$\frac{\partial\omega}{\partial n} = 0 \text{ on } |\zeta| = 1, \qquad \omega_\xi - i\omega_\eta = 0 \text{ at } z = z_T\,, \qquad z = \infty\,. \tag{5.8}$$

The function $W = \omega_\xi - i\omega_\eta$ satisfies the equation

$$W_\xi + iW_\eta = -\tfrac{1}{2}(\alpha - i\beta)W - \tfrac{1}{2}(\alpha + i\beta)\overline{W} \tag{5.9}$$

and therefore is a so-called pseudo-analytic function.[16] From the similarity principle for pseudo-analytic functions[17] we conclude that W must be of the form

$$W = e^{s(\zeta)}f(\zeta) \tag{5.10}$$

where $s(\zeta)$ is uniformly bounded, continuous on $|\zeta| = 1$ and assumes on this circle real values, whereas $f(\zeta)$ is regular analytic. It is now an easy matter to show that $f \equiv 0$ and hence also $W \equiv 0$. (In order to be able to apply the similarity principle one must know that the coefficients α, β in (5.7) vanish of order greater than the first at infinity and are bounded by const. $\cdot\ |\zeta - 1|^{\epsilon-1}$,

[16]Concerning the theory of pseudo-analytic functions see Bers [9], [10].

[17]The proof of this principle in a form needed for the uniqueness proof will be found in Bers [11].

$\epsilon > 0$, near $|\zeta| = 1$. To verify these conditions here would take too much time; we merely state that this can be done.)

This argument, which shows that *the Kutta-Joukowski condition determines the value of the circulation*, was necessary since ω is not known to be a single-valued function. If we assume that the circulation is given, a more standard argument can be used. According to (5.4) ω is a solution of a linear homogeneous elliptic partial differential equation without zero-order term, whose normal derivative vanishes on the boundary. If the domain in question were bounded we could at once conclude that $\omega = \text{const}$. In our case the behaviour of ω at infinity must be investigated. One can either assume that $\omega(x, y)$ behaves at infinity in such a way that the standard uniqueness theorem holds (as is done by Schiffman who proves the uniqueness theorem for a class of functions for which a certain integral converges), or one can prove, say by using the theory of pseudo-analytic functions, that such is the case. At any rate, in this set-up the uniqueness theorem for subsonic flows turns out to be a direct consequence of the fact that for a linear *elliptic* equation of the form (5.4) the second boundary value problem possesses at most one solution (except for an additive constant).

As a matter of fact, as long as only purely subsonic flows are considered, the stream function could be used instead of the velocity potential. In this case the argument would have involved the unique solvability of the first boundary value problem.

6. *Uniqueness Theorems for Equations of Mixed Type*

In trying to extend the uniqueness proof sketched in the preceding section to the case of a transonic flow one is led to a boundary value problem for a linear equation of mixed elliptico-hyperbolic type. We must therefore interpolate here a brief discussion of such problems.[18]

The theory of equations of mixed type was originated by Tricomi in 1923. In a celebrated memoir Tricomi [65] studied the equation

$$y\psi_{xx} + \psi_{yy} = 0 \tag{6.1}$$

which is elliptic for $y > 0$, hyperbolic for $y < 0$. Note that Chaplygin's equation (4.3) is of a similar form. For values of q close to the critical speed q_{cr}, that is for values of σ close to zero, Chaplygin's equation (after a simple stretching in the σ-direction) may be approximated by Tricomi's equation. It is also possible to find a pressure-density relation (2.1) for which the Chaplygin equation will be precisely equation (6.1). Finally, we note that an approximate equation for the potential of a nearly uniform transonic flow proposed by von Kármán [43] can be transformed into Tricomi's equation.

[18]There are several classes of equations of mixed type, but we mention only those which are of interest in the present connection. For this reason no reference is made to the important work of Mme. Cinquini-Cibrario.

Equation (6.1) is a typical equation of mixed type in the following sense. Let

$$A_{11}(x, y)\psi_{xx} + 2A_{12}(x, y)\psi_{xy} + A_{22}(x, y)\psi_{yy} + A_1(x, y)\psi_x + A_2(x, y)\psi_y + A_0(x, y)\psi = 0 \tag{6.2}$$

be an equation of mixed type, that is such that the discriminant

$$A_{11}(x, y)A_{22}(x, y) - A_{12}(x, y)^2 \tag{6.3}$$

is positive on one side of a sufficiently smooth curve C and negative on the other side. If it is assumed that the discriminant (6.3) vanishes on C of the first order, and if sufficiently strong smoothness assumptions are made about the coefficients, then, as was observed by Tricomi, one can introduce new independent variables in such a way that equation (6.2) takes the form

$$y\psi_{xx} + \psi_{yy} + a(x, y)\psi_x + b(x, y)\psi_y + c(x, y)\psi = 0. \tag{6.4}$$

In particular, Chaplygin's equation (4.3) can be brought into this form.

It should be noted, though, that the fact that only the terms of highest order are present in equations (4.3) and (6.1) is of importance for the theory of these equations. This can be seen best by considering the Cauchy problem in which the values of the function and its derivative are given along the parabolic line $y = 0$, and the solution is required to exist in the hyperbolic half-plane. Frankl [25] showed that this problem is correctly posed for equation (6.4). This implies the same statement for an equation of the Chaplygin form

$$K(y)\psi_{xx} + \psi_{yy} = 0, \qquad yK(y) > 0 \text{ for } y \neq 0 \tag{6.5}$$

provided that one assumes that

$$K'(0) > 0. \tag{6.6}$$

On the other hand I showed [7] that the correctness of the Cauchy problem for equation (6.5) is independent of the behaviour of $K(y)$ near $y = 0$. That this result could be destroyed by the presence of first order terms follows from a theorem of Berezin [14] according to which the Cauchy problem for the equation

$$y^\alpha\psi_{xx} + \psi_{yy} + \psi_x = 0$$

is not correctly posed if $\alpha > 2$.

The possibility of solving the Cauchy problem for Chaplygin's equation implies, by the way, that *under certain rather mild restrictions a subsonic gas-flow which becomes sonic along an arc can always be continued across the sonic line.*[19]

In his paper Tricomi succeeded in stating a correctly posed boundary value problem for equation (6.1). He considers a domain D bounded by an

[19]See Bers [7] for a precise statement.

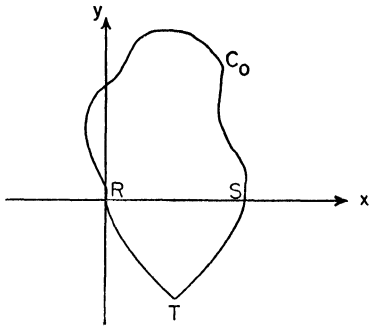


FIGURE 1

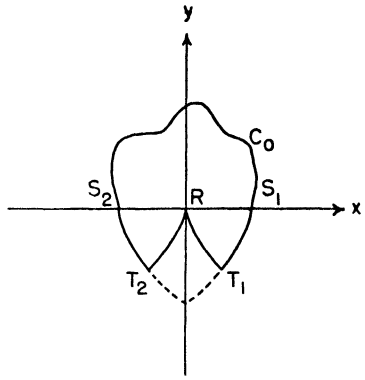


FIGURE 2

arc C_0 in the elliptic half-plane and by two characteristics RT and ST (Figure 1). The values of the function are prescribed along C_0 and along ST. Gellerstedt [32] extended this result to the case of equation (6.5) with $K(y) = y^n$. He also made the important observation that the *Tricomi problem* can be stated in a more general way. One can consider a domain D bounded by an elliptic arc and by four characteristics S_1T_1 , RT_2 , S_2T_2 (Figure 2) and prescribe[20] the values of the function along C_0 and along the characteristics S_1T_1 , S_2T_2 .

A generalization of the Tricomi problem was proposed by Frankl in connection with some transonic flow problems. Frankl replaces the characteristic ST of Figure 1 by an arc C_1 which may partly coincide with a characteristic, and which intersects every characteristic of the family containing RT at most once (Figure 3). A more general form of the *Frankl problem* is as follows: The domain D is bounded by an elliptic arc C_0 , by two characteristics RT_1 and RT_2 , and by two not necessarily characteristic curves C_1 and C_2 , subject to

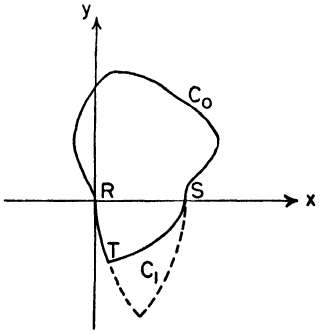


FIGURE 3

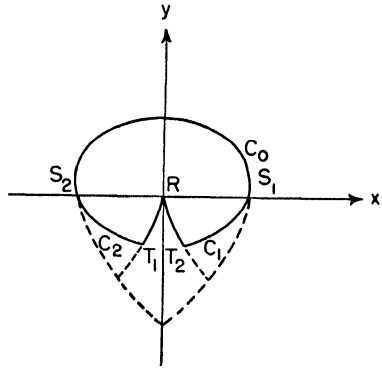


FIGURE 4

[20]There are other ramifications of the Tricomi problem which can be treated by the same methods. Cf. Gellerstedt [32], Protter [55].

the same restrictions as before (Figure 4). The values of the function are prescribed along the open arc $C_1 + C_0 + C_2$. The gas-dynamical importance of this problem was emphasized by Busemann and Guderley who also proposed to consider a limiting case of the Frankl problem: the values of the function are prescribed along the whole boundary consisting of an elliptic and a hyperbolic arc, but the function is permitted to possess an unspecified singularity at one intersection point R of the boundary and the x-axis.

It should be noted that in all cases the solution will be determined in the whole characteristic triangle adjacent to the "parabolic segment", even though the domain extends only into a part of this triangle. This follows from simple properties of hyperbolic equations.

Tricomi's uniqueness proof was rather intricate and made use of special properties of equation (6.1). It could be extended to the case considered by Gellerstedt (equation (6.5) with $K(y) = y^n$) but not to the more general equation (6.5). The first uniqueness proof for Tricomi's problem for an equation of the form (6.5) was given by Frankl. This proof is valid as long as

$$3K'^2 \geq 2KK'' \tag{6.7}$$

in the domain considered. This condition is always satisfied for $K = y$ (Tricomi's equation) or for $K = y^n$ (Gellerstedt's equation) but for a general equation (6.5), which we assume from now on to satisfy condition (6.6), it will hold only if the domain does not extend too far into the hyperbolic half-plane. In particular, Frankl's condition is not always satisfied by the actual Chaplygin equation (4.3).

Friedrichs observed that Frankl's proof can be recast as an energy-integral argument. Let $\psi(x, y)$ be a solution of (6.5) which vanishes along that part of the boundary of D along which one wants to prescribe data. For every choice of the functions $a(x, y)$, $b(x, y)$, $c(x, y)$ we have

$$\iint_D (a\psi + b\psi_x + c\psi_y)(K\psi_{xx} + \psi_{yy})\, dx\, dy = 0. \tag{6.8}$$

Now one tries to choose the functions a, b, c in such a way that the integral in (6.8), after it is transformed by Green's theorem, and after the boundary conditions have been taken into account, should become positive definite. If this is done, one may conclude that $\psi \equiv 0$.

Using this method Protter [55] succeeded in obtaining a new uniqueness proof for Tricomi's problem for equation (6.5). He assumes that either no restrictions are put onto the elliptic part of the domain, in which case it must be required that the hyperbolic part should not extend too far, or that no restrictions are put on the hyperbolic part, in which case the elliptic part is not permitted to extend too far in the x-direction.

A new approach to the uniqueness problem was given by Germain and Bader [33]. They proved the following *maximum-principle* for Tricomi's equa-

tion: if a solution of (6.1) defined in the domain of Figure 1 vanishes along the characteristic ST, then the maximum of $|\psi|$ is attained on the elliptic boundary C_0. Recently Agmon, Nirenberg, and Protter [1] showed that this result is a special case of a *general maximum principle* which holds for a wide class of equations of hyperbolic and mixed types, but by no means for all such equations. In the case of equation (6.5) the maximum-principle in the Germain-Bader form holds if (in the domain considered, for $y < 0$)

$$5K'^2 \geq 4KK''. \tag{6.9}$$

This condition is, unfortunately, stronger than Frankl's condition (6.7).

For the Frankl problem a uniqueness proof was given by Guderley for the special equation (6.1) under assumptions on the domain which are too complicated to be stated here. His proof is rather complicated and makes use of a special coordinate transformation. It is natural, however, to apply to the Frankl problem the energy-integral method. Thus Protter observed that by choosing a as 0 and b and c as appropriate constants one obtains a uniqueness proof for Frankl's problem in the case that the slope of the elliptic arc nowhere exceeds the minimum of the slopes of the non-characteristic boundary curves in the hyperbolic domain.

The most hopeful result thus far for the Frankl problem is due to Cathleen Morawetz [51]. The Frankl problem possesses at most one solution if the elliptic boundary curve is *starshaped* with respect to the point R. This result is achieved by the simple choice $a = 0$, $b = x$, $c = y$.

As far as the existence question is concerned, only partial results are known. A new treatment of the Tricomi problem for Tricomi's equation was given by Germain and Bader [33]. A not yet completed investigation by Agmon seems to indicate that one should be able to formulate a correctly posed problem for the domain complementary to the Tricomi domain of Figure 1. Protter [57] showed that by an extension of Gellerstedt's method the existence of a solution for Tricomi's problem as well as for Frankl's problem, in the case when the non-characteristic boundary arc initially coincides with the characteristic, can be obtained for the general equation (6.5) provided that the uniqueness theorem has been established within a sufficiently wide class of functions.[21] For the special equation (6.1) the same result was obtained by Frankl [29] by following the original procedure of Tricomi. Both authors impose the same conditions on the elliptic curve as Tricomi.

Summarizing, it must be admitted that the theory of the Tricomi and Frankl problems is as yet in a rather unsatisfactory state. For Tricomi's problem, for instance, there exist three different uniqueness results (Frankl, Protter, Morawetz). The three results are based on assumptions which are mutually independent. But no example of non-uniqueness is known, and it appears likely that an unrestricted uniqueness theorem holds. It seems that new methods will be needed to settle this problem, as well as the more complicated problems arising out of the boundary value problems for transonic flows past profiles.

[21] The degenerate cases of the Tricomi and Frankl problems in which the elliptic boundary curve collapses into the parabolic line can be treated very completely; cf. Protter [54], [56].

On the other hand, the existing proofs all point in the same direction and seem to confirm the conjectures to be formulated in the following section.

7. *Conjectured Non-existence Theorems for Transonic Flow*

We return now to the boundary value problem $\Pi(P, q_\infty)$ and try to carry out the uniqueness proof sketched at the end of Section 5 for the case when the flow described by the potential ϕ is transonic.

In order to simplify matters we assume from now on that the profile is smooth and symmetric with respect to both the x and the y-axis, and that all flows considered have the same symmetry properties (as indeed they must if an uniqueness theorem is to hold). In this case it will suffice to consider only the upper half of the flow domain which we denote by D.

We also assume that the flow is subsonic except in a region Σ bounded by an arc S_1S_2 of the profile and by a simple arc Λ along which $M = 1$. Within Σ the flow is supposed to be supersonic. All transonic flows obtained by the methods described in Sections 3, 4 are of this nature. It is not difficult to verify that every Mach line of the flow, that is every characteristic of equation (2.1), ϕ being the potential considered, which originates at the sonic line Λ intersects the arc S_1S_2 at exactly one point. It will be convenient for what follows to select a fixed but arbitrary point R on Λ. The two characteristics emanating from R meet P at the points T_1 , T_2 ; we call the arc T_1T_2 of P the critical arc (with respect to the transonic flow considered, and with respect to the point R chosen), the curvilinear triangle T_1RT_2 the critical triangle. By D_0 we shall denote the domain obtained from D removing the critical triangle [cf. Figure 5].

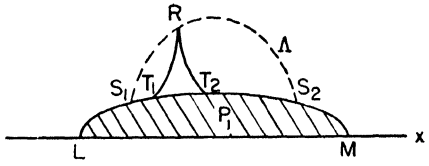


FIGURE 5

Now let $\tilde{\phi}$ be another solution of $\Pi(P, q_\infty)$ and set as before $\omega = \phi - \tilde{\phi}$. In D the function $\omega(x, y)$ satisfies the linear equation (5.4); it also satisfies the conditions

$$\omega_x^2 + \omega_y^2 \to 0 \qquad \text{as} \qquad z \to \infty, \tag{7.1}$$

$$\frac{\partial \omega}{\partial n} = 0 \qquad \text{on the boundary of } D. \tag{7.2}$$

In order to establish the uniqueness theorem we would have to conclude that $\omega = \text{const}$. But equation (5.4) is now of mixed type; it is hyperbolic in the domain Σ, elliptic outside of Σ. The discriminant of (5.4) vanishes along Λ.

Furthermore, the characteristics of this equation are precisely the Mach lines of the flow ϕ. Noting the uniqueness theorems discussed in the preceding section, we may expect that the constancy of ω could be established not using the full condition (7.2), but only the seemingly weaker requirement that

$$\frac{\partial \omega}{\partial n} = 0 \quad \text{on the boundary of } D \text{ except, perhaps, along the critical arc.} \tag{7.3}$$

Indeed, let us assume that the uniqueness theorem for Frankl's problem can be extended to the case when: (i) the equation is not of the special form (6.5) but the general linear equation (6.2) of mixed type, without a zero order term and with the discriminant vanishing along the transition line between the hyperbolic and elliptic regions, (ii) the domain in the elliptic part is permitted to extend to infinity in which case the derivatives of the solution must vanish at infinity, (iii) not the values of the function but the values of its normal derivative are prescribed along an arc of the boundary of the domain, chosen as in the case of the Frankl problem.

Applying this conjectured uniqueness theorem to the function ω we could conclude that it is constant in D_0 . But in the critical triangle equation (5.4) is hyperbolic, and since ω is now known to be constant along the two characteristics it must be constant everywhere. Hence $\tilde{\phi} = \phi + \text{const.}$, so that there exists only one transonic flow past P with free stream speed q_∞ .

Observe next, that in this (conditional) uniqueness proof the boundary condition imposed along the critical arc was not used at all. Hence if $\tilde{P}$ is a profile coinciding with P except along the critical arcs, and $\tilde{\phi}$ is a solution of $\Pi(\tilde{P}, q_\infty)$, we can repeat verbatim the argument showing that $\omega = \text{const.}$ But since the arc of $\tilde{P}$ is determined by the condition that along it $\partial\tilde{\phi}/\partial n = 0$, the profiles P and $\tilde{P}$ must coincide everywhere. Thus we are led to the following

CONJECTURED NON-EXISTENCE THEOREM A: *Let there be given a transonic flow past a profile P with speed q_∞ at infinity. Let $\tilde{P}$ be a profile coinciding with P except along the critical arcs. Then there exists no potential flow past $\tilde{P}$ with the same value of the speed at infinity.*

The significance of this conjecture lies in the fact that $\tilde{P}$ may be arbitrarily close to P, and have its slope, curvature, etc., as close to those of P as desired. Hence the conjecture implies that the boundary value problem $\Pi(P, q_\infty)$ is not correctly posed as far as transonic flows are concerned.

This formulation of the Frankl-Busemann-Guderly argument (which was given by C. S. Gardner[22]) has the advantage that it makes no use whatsoever of considerations involving infinitely close flows, and involves no estimates of orders of magnitude of unknown quantities. It depends, on the other hand, upon a very strong uniqueness theorem for linear equations, the proof of which seems to be very difficult at the present stage of the theory. Note that in this set-up there is no way of avoiding the difficulty due to the boundary condition

[22]In a seminar talk at New York University.

of the second kind by using the stream-function instead of the potential. In fact, the difference of two stream-functions will not, in general, satisfy a simple linear equation, for, the non-linear equation satisfied by each stream-function depends upon the location of the corresponding sonic line.

An even stronger indication for the incorrectness of Problem $P(\Pi, q_\infty)$ for transonic flows would be the following

CONJECTURED NON-EXISTENCE THEOREM B: *Under the hypotheses of* A *there exists no potential flow past* P *with a free stream speed* $\tilde{q}_\infty$ *distinct from but sufficiently close to* q_∞ .

As far as I know, however, it is not yet possible to formulate a reasonably sounding uniqueness theorem for linear equations of mixed type which would imply this non-existence theorem.

At present the proof of A looks like an exceedingly difficult task. It is therefore worthwhile to formulate a weaker non-existence conjecture.

We assume that the transonic flow described by ϕ is such that the upper half of the flow domain D has a one-to-one image in the hodograph plane. This assumption is not very restrictive since symmetric flows computed by the hodograph method will in general have this property, and since Nikolski and Taganov [53] proved that the supersonic region Σ always has a simply covered hodograph.

Let $\tilde{P}$ be again a profile coinciding with P except along the critical arc, and very close to P. One can try to find an approximation to the solution $\tilde{\phi}$ of $\Pi(\tilde{P}, q_\infty)$ by the classical method of small perturbations. It consists in setting $\tilde{\phi} = \phi - \omega$ where $\omega(x, y)$ will be considered (i) so small that the powers of ω and its derivatives may be neglected, and (ii) as a function defined in the domain exterior to $\tilde{P}$. In the usual way one obtains for ω the linear differential equation

$$(7.5)\qquad (c^2 - \phi_x^2)\omega_{xx} - 2\phi_x\phi_y\omega_{xy} + (c^2 - \phi_y^2)\omega_{yy} - 2[2\phi_x\phi_{xx} + \phi_y\phi_{xy} + \phi_x\phi_{yy}]\omega_x - 2[\phi_y\phi_{xx} + \phi_x\phi_{xy} + 2\phi_y\phi_{yy}]\omega_y = 0$$

and the conditions (7.1), (7.3) and

$$(7.6)\qquad \frac{\partial\omega}{\partial n} = \chi \qquad \text{on the critical arc,}$$

where χ is a function determined by the shapes of P and $\tilde{P}$ which will not be identically zero (the case $P = \tilde{P}$ being obviously excluded). We call the boundary value problem (7.5), (7.1), (7.3), (7.6) the *perturbation problem* belonging to ϕ. Such a problem can be formulated also in the case when the flow described by ϕ is purely subsonic. In this case the perturbation problem would be a correctly set linear elliptic problem.

CONJECTURED NON-EXISTENCE THEOREM C: *The perturbation problem belonging to a transonic flow is not correctly posed.*

This conjecture implies that even if problem $\Pi(P, q_\infty)$ can be solved for a

continuously varying family of profiles P, the dependence of the solution on the shape of the profile is not as smooth as one would expect it to be in the case of a correctly posed problem.

If statement C is correct, it should be possible to prove it in the hodograph plane. In fact, we may consider ω as a function of (ϕ_x, ϕ_y). If new independent variables θ and $\hat{\sigma}$,

$$\hat{\sigma} = \hat{\sigma}(q) = \int_q^{q_{cr}} \exp\left\{\int_{q_{cr}}^{q} M^2 \frac{dq}{q}\right\} \frac{dq}{q},$$

are introduced, the equation for ω becomes

$$\hat{K}(\hat{\sigma})\omega_{\theta\theta} + \omega_{\hat{\sigma}\hat{\sigma}} = 0$$

where

$$\hat{K}(\hat{\sigma}) = (1 - M^2) \exp\left\{2 \int_q^{q_{cr}} M^2 \frac{dq}{q}\right\}.$$

This equation is of the same form as the Chaplygin equation, and the characteristics of this equation are the images of the Mach lines of the flow ϕ. The image of the domain D_0 in the $\theta,\hat{\sigma}$-plane is shown in Figure 6, where cor-

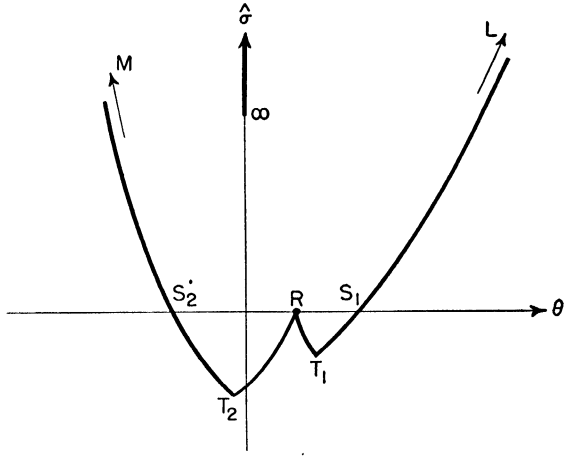


FIGURE 6

responding points are denoted by the same letters. Along the boundary lines corresponding to the boundaries of D_0 formed by the profile

$$\hat{K}\omega_\theta \, d\hat{\sigma} - \omega_{\hat{\sigma}} \, d\theta = 0$$

Along the vertical boundary line corresponding to the part of the boundary of D formed by the x-axis

$$\omega_\theta = 0.$$

The behaviour of ω at $\hat{\sigma} = +\infty$ (which corresponds to the stagnation point in the z-plane) and near the tip of the vertical slit (corresponding to the point $z = \infty$) could also be described without difficulties, though we do not do it here. The analogy of the present situation with that occurring in the Frankl problem is obvious. The conjectured theorem C would follow, for instance, if one could show that the conditions stated above imply that ω is constant or a linear combination of a finite number of "eigenfunctions."

The difference between the problem just considered and the Frankl problem consists primarily in the nature of the boundary conditions. One may try to regain the simpler conditions of the Frankl problem by considering the stream-function instead of the potential. But this leads to a singularity in the differential equation.

The considerations of this section, which indicate that there are good reasons to believe the existing examples of smooth transonic flows past profiles to be exceptional rather than typical, apply also to other types of flow, more precisely subsonic flows into which a bounded supersonic region is embedded. They definitely do not apply to cases when the supersonic region may extend to infinity, or be bounded by shocks like, for instance, the nozzle flows studied by Frankl [27], Falkovich [24] and Lighthill [49].

It should be noted that Busemann [68] and Guderley [40] describe their arguments as proofs of the non-existence of potential transonic flows in the general case. From the mathematical point of view, however, their arguments are only highly suggestive plausibility considerations. They pose but do not settle interesting (and far from simple!) mathematical problems.

8. *Concluding Remarks*

The theory of a subsonic flow past profiles may be considered as being put on a firm foundation, though some interesting questions are still open. (We mention, for instance, the influence of the angle of attack and of the free stream speed on the position of the front stagnation point, or on the value of the circulation. It would also be of interest to know whether the present theoretical understanding of such flows can be of use in devising a reasonably simple and reliable procedure for numerical computations.) The theory of transonic flows consists largely of conjectures.

This difference is, of course, not accidental. The theory of elliptic equations has been developed extensively over a long period and in dealing with such equations we have at our disposal some of the most powerful tools of analysis. A systematic developement in the theory of equations of mixed type which govern transonic flows is just beginning; this theory still depends on ad hoc devices and lucky guesses. Needless to say, this gives it a certain freshness and attractiveness which is sometimes missing in more firmly established disciplines.

Still, one may expect that the conjectures on transonic flows will be either

disproved or proved in the not too distant future. If these conjectures should be proved, the transonic flow problem will be solved only in a very limited sense. It would be known that one idealized model used to describe such flows fails, and it would be necessary to try to find a more workable model. In particular, even if the existing shockless transonic solutions should be proved to be exceptional and therefore without physical significance, it might still be possible to use them as a starting point for a rational description of the "normal" situation in which small discontinuities (shocks) are present. A tentative attempt at such a description was made by Frankl [28].

We shall not try to assess the value of the preceding considerations for aerodynamicists. Our main aim was rather to emphasize the intrinsic mathematical interest of problems arising in this chapter of gas dynamics.

BIBLIOGRAPHY

[1] Agmon, S., Nirenberg, L., and Protter, M. H., *A maximum principle for a class of equations of hyperbolic and mixed type*, Communications on Pure and Applied Mathematics, Volume 6, No. 4, 1953.

[2] Berg, P. W., *On the existence of Helmholtz flows of a compressible fluid*, Dissertation, New York University, 1953.

[3] Bergman, S., *Operator methods in the theory of compressible fluids*, Proceedings of the Symposium on Applied Mathematics, Volume 1, 1949, pp. 19–40.

[4] Bergman, S., *Operatorenmethoden in der Gasdynamik*, Zeitschrift für angewandte Mathematik und Mechanik, Volume 32, 1952, pp. 33–45.

[5] Bers, L., *On a method of constructing two-dimensional subsonic compressible flows around closed profiles*, National Advisory Committee for Aeronautics, Technical Note 969, March, 1945.

[6] Bers, L., *An existence theorem in two-dimensional gas dynamics*, Proceedings of the Symposium on Applied Mathematics, Volume 1, 1949, pp. 41–46.

[7] Bers, L., *On the continuation of a potential gas flow across the sonic line*, National Advisory Committee for Aeronautics, Technical Note No. 2058, 1950.

[8] Bers, L., *Boundary value problems for minimal surfaces with singularities at infinity*, Transactions, American Mathematical Society, Volume 70, 1951, pp. 465–491.

[9] Bers, L., *Partial differential equations and generalized analytic functions*, Proceedings of the National Academy of Sciences, U.S.A., Volume 36, 1950, pp. 130–136. *Second note*, ibid., Volume 37, 1951, pp. 42–47.

[10] Bers, L., *Theory of pseudo-analytic functions*, Lecture notes (mimeographed), New York University, 1953.

[11] Bers, L., *Function-theoretical properties of solutions of partial differential equations of elliptic type*, Annals of Mathematics Study XXXIII (to appear).

[12] Bers, L., *Existence and uniqueness of a subsonic compressible flow past a given profile*, Communications on Pure and Applied Mathematics (to appear).

[13] Bers, L., *Non-linear elliptic equations without non-linear entire solutions* (to appear).

[14] Berezin, I. S., *On the Cauchy problem for linear equations of second order with initial data on the parabolic line*, Matematiceskii Sbornik, Volume 24, 1949, pp. 301–320.

[15] Busemann, A., *The problem of drag at high subsonic speeds*, British Ministry of Aircraft Production, Translation 184, 1947.

[16] Busemann, A., *The drag problem at high subsonic speeds*, Journal of the Aeronautical Sciences, Volume 16, 1949, pp. 337–344.

[17] Chaplygin, S. A., *On gas jets*, Scientific Memoirs, Imperial University of Moscow, Mathe-

matical-Physical Section 21, 1902, pp. 1–121. (English Translation, National Advisory Committee for Aeronautics, Technical Memo No. 1063, 1944.)

[18] Cherry, T. M., *Flow of a compressible fluid about a cylinder*, Proceedings of the Royal Society, Series A, Volume 192, 1947, pp. 45–79; Volume 196, 1949, pp. 1–31.

[19] Christianovitch, S. A., *Gas flow over bodies at high subsonic velocities*, Trudy Central' nogo Aerogidrodinamičeskogo Instituta, No. 481, 1940.

[20] Courant, R. and Friedrichs, K. O., *Supersonic Flow and Shock Waves*, Interscience Publishers, New York-London, 1948.

[21] Ehlers, F. E., *Methods of linearization in compressible flow*, I, Janzen-Rayleigh method, Wright Field Report No. F-TR-1180A-ND.

[22] Emmons, H. W., *Flow of a compressible fluid past a symmetrical airfoil in a wind tunnel and in free air*, National Advisory Committee for Aeronautics, Technical Note, No. 1746, 1948.

[23] Emmons, H. W., *The theoretical flow of a frictionless, adiabatic, perfect gas inside of a two-dimensional hyperbolic nozzle*, National Advisory Committee for Aeronautics, Technical Note No. 1003, 1946.

[24] Falkovich, S. V., *On the theory of the Laval nozzle*, Prikladnaia Matematika i Mekhanika, Volume 10, 1946, pp. 503–512.

[25] Frankl, F. I., *On Cauchy's problem for equations of mixed elliptic-hyperbolic type with initial data on the transition line*, Izvestiya Akademiia Nauk SSSR, Volume 8 1944, pp. 195–224.

[26] Frankl, F. I., *On the problems of Chaplygin for mixed sub- and supersonic flows*, Izvestiya Akademiia Nauk SSSR, Volume 9, 1945, pp. 121–143.

[27] Frankl, F. I., *On the theory of the Laval nozzle*, Izvestiya Akademiia Nauk SSSR, Volume 9, 1945, pp. 387–422.

[28] Frankl, F. I., *On the appearance of compression shocks in subsonic flows with local supersonic speeds*, Prikladnaia Matematika i Mekhanika, Volume 11, 1947, pp. 192–202.

[29] Frankl, F. I., *On a new boundary value problem for the equation* $y(\partial^2 z/\partial x^2) + (\partial^2 z/\partial y^2) = 0$, Učenye Zapiski Moskovskogo Gosuearstvennogo Universiteta, Volume 152, 1951, pp. 99–116.

[30] Frankl, F. I., and Keldysh, M., *Die äussere Neumann'sche Aufgabe für nichtlineare elliptische Differentialgleichungen mit Anwendung auf die Theorie der Flügel im kompressiblien Gas*, Bulletin of the Academy of Sciences, URSS, Volume, 12 1934, pp. 561–607.

[31] Friedrichs, K. O., *On the non-occurrence of a limiting line in transonic flow*, Communications on Pure and Applied Mathematics, Volume 1, 1948, pp. 287–301.

[32] Gellerstedt, S., *Sur un problème aux limites pour une équation linéaire aux dérivées partielles du second ordre de type mixte*, Dissertation, Uppsala, 1935.

[33] Germain, P., and Bader, R., *Sur quelques problèmes relatifs a l'équation de type mixte de Tricomi*, Office Nationale d'Étude et de Recherche, Publication No. 54, 1952.

[34] Goldstein, S., Lighthill, M. J., and Graggs, J. W., *On the hodograph transformation for high speed flow, I. A flow without circulation*, Quarterly Journal of Mechanics and Applied Mathematics, Volume 1, 1948, pp. 344–357.

[35] Görtler, H., *Gasströmungen mit Übergang von Unterschall zu Überschallgeschwindigkeiten*, Zeitschrift für angewandte Mathematik und Mechanik, Volume 20, 1940, pp. 254–262.

[36] Guderley, G., *Considerations on the structure of transonic flow patterns*, Wright Field Report No. F-TR-2168-ND.

[37] Guderley, G., *Singularities at the sonic velocity*, Wright Field Report No. F-TR-1171-ND, June, 1948.

[38] Guderley, F., *On the transition from a transonic potential flow to a flow with shocks*, Wright Field Report No. F-TR-2160-ND.

[39] Guderley, G., *Two-dimensional flow patterns with a free stream Mach number close to 1*, Air Force Technical Report No. 6343.

[40] Guderley, G., *On the presence of shocks in mixed subsonic-supersonic flow patterns*, Advances in Applied Mechanics, Volume III, Academic Press, New York, 1953, pp. 145–184.

[41] Kaplan, C., *The flow of a compressible fluid past a curved surface*, National Advisory Committee for Aeronautics, Technical Report No. 768, 1943.

[42] Kármán, T. von, *Compressibility effects in aerodynamics*, Journal of the Aeronautical Sciences, Volume 8, 1941, pp. 337–356.

[43] Kármán, T. von, *The similarity law of transonic flow*, Journal of Mathematical Physics, Volume 26, 1947, pp. 182–190.

[44] Kolodner, I., and Morawetz, C. S., *On the non-existence of limiting lines in transonic flows*, Communications on Pure and Applied Mathematics, Volume 6, 1953, pp. 97–102.

[45] Liepmann, H. W., and Puckett, A. E., *Introduction to Aerodynamics of a Compressible Fluid*, Wiley, New York, 1947.

[46] Lighthill, M. J., *The hodograph transformation in transonic flow, II. Auxiliary theorems on the hypergeometric functions* $\psi_n(\tau)$, Proceedings of the Royal Society, Series A, Volume 191, 1947, pp. 341–351.

[47] Lighthill, M. J., *The hodograph transformation in transonic flow, III. Flow round a body*, Proceedings of the Royal Society, Series A, Volume 191, 1947, pp. 352–369.

[48] Lighthill, M. J., *On the hodograph transformation for high speed flow, II. A flow with circulation.* Quarterly Journal of Mechanics and Applied Mathematics, Volume I, 1948, pp. 442–450.

[49] Lighthill, M. J., *The hodograph transformation in transonic flow, I. Symmetric channels.* Proceedings of the Royal Society, Series A, Volume 191, 1947, pp. 323–341.

[50] Manwell, A. R., *A note on the hodograph transformation*, Quarterly of Applied Mathematics, Volume 10, 1952, pp. 177–184.

[51] Morawetz, C. S., *A uniqueness theorem for the Frankl problem*, Communications on Pure and Applied Mathematics (to appear).

[52] Neményi, P. F., *Recent developments in inverse and semi-inverse methods in the mechanics of continua*, Advances in Applied Mechanics, Volume II, Academic Press Inc., New York, 1951, pp. 123–151.

[53] Nikolsky, A. A., and Taganov, G. I., *Gas motion in a local supersonic region and conditions of breakdown of potential flow*, Prikladnaia Matematika o Mekhanika, Volume 10, 1946, pp. 481–502.

[54] Protter, M. H., *A boundary value problem for an equation of mixed type*, Transactions of The American Mathematical Society, Volume 71, 1951, p. 416.

[55] Protter, M. H., *Uniqueness theorems for the Tricomi equation*, Journal of Rational Mechanics and Analysis, Volume 2, 1953, pp. 107–114.

[56] Protter, M. H., *The two non-characteristics problems with data partly on the parabolic line*, Pacific Journal of Mathematics (to appear).

[57] Protter, M. H., *An existence theorem for the generalized Tricomi problem*, Duke Mathematical Journal (to appear).

[58] Sauer, R., *Theoretische Einführung in die Gasdynamik*, Springer, Berlin, 1943.

[59] Sears, W. R., *Transonic potential flow of a compressible fluid*, Journal of Applied Physics, Volume 21, 1950, pp. 771–778.

[60] Schiffman, M., *On the existence of subsonic flows of a compressible fluid*, Journal of Rational Mechanics and Analysis, Volume 1, 1952, pp. 605–652.

[61] Taylor, G. I., *The flow of air at high speed past curved surfaces*, Aeronautical Research Committee, Reports and Memoranda, No. 1381, London, 1930.

[62] Taylor, G. I., *Recent work on the flow of compressible fluids*, Journal of the London Mathematical Society, Volume 5, 1930, p. 224.

[63] Tomotika, S., and Tamada, K., *Studies on two-dimensional transonic flows of a compressible fluid*, Quarterly of Applied Mathematics, Volume 7, 1949, pp. 381–397; Volume 8, 1950, pp. 127–136; Volume 9, 1951, pp. 129–147.

[64] Tsien, H. S., and Kuo, Y. H., *Two-dimensional irrotational mixed subsonic and supersonic flow of a compressible fluid and the upper critical Mach number*, National Advisory Committee for Aeronautics, Technical Note No. 995, 1946.

[65] Tricomi, F., *Sulle equazioni lineari alle derivate parziali di 2° ordine, di tipo misto*, Atti Accademia Nazionale dei Lincei (5), Volume 14, 1923, p. 133.

[66] Vazsonyi, A., *An existence theorem in the theory of compressible fluids* (Abstract), Bulletin, American Mathematical Society, Volume 50, 1944, p. 673.

[67] Gilbarg, D., *Comparison methods in the theory of subsonic flows*, Journal of Rational Mechanics and Analysis, Volume 2, No. 2, 1953, pp. 233–251.

[68] Busemann, A., *The nonexistence of transonic flows*, Proceedings of Symposia in Applied Mathematics, Volume 4, 1953, pp. 29–39.

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. VII, 105–116 (1954)

The Singular Solutions and the Cauchy Problem for Generalized Tricomi Equations

By ALEXANDER WEINSTEIN
University of Maryland

1. *Introduction*

Recent developments in the theory of transonic flow have revived and stimulated interest in partial differential equations of mixed type. The best known equation of this kind is the Tricomi equation which occurs in the hodograph method for plane flows of compressible fluids. In its normal form, Tricomi's equation is related to the generalized axially symmetric potential theory (GASPT) and to the hyperbolic Euler-Poisson-Darboux (EPD) equation. An exposition of results in GASPT has been given recently [1] and will be used here. As to the EPD equation, the present paper will present some new previously unpublished results. In particular, since the EPD equation contains the wave equation as a special case, a new and probably the simplest solution of the Cauchy problem for the wave equation is obtained.

Using Chaplygin's notation, the Tricomi equation may be written in the form

$$\sigma\psi_{\theta\theta} + \psi_{\sigma\sigma} = 0, \tag{1.1}$$

where ψ denotes the stream function. For $\sigma > 0$ the equation is elliptic. For $\sigma < 0$, it is hyperbolic.

A problem of great importance in connection with Tricomi's equation is the determination of a fundamental solution in the large, that is, a solution defined in the entire θ,σ-plane and having a logarithmic singularity at one point in the elliptic half-plane ($\sigma > 0$), say $\theta = 0$, $\sigma = \beta > 0$. We shall determine such a solution by first obtaining a closed formula for $\sigma \geq 0$. Then we shall use the data given by our formula on $\sigma = 0$ as Cauchy data to determine the continuation of our solution in the entire hyperbolic half-plane ($\sigma < 0$). We shall see that this procedure will lead to more general problems of importance in various branches of pure and applied mathematics.

2. *The Normal Forms*

For $\sigma > 0$, we introduce the variables

$$x = \theta, \qquad y = \tfrac{2}{3}\sigma^{3/2}; \tag{2.1}$$

this yields the normal form

$$u_{xx} + u_{yy} + \frac{1}{3y} u_y = 0. \tag{2.2}$$

For $\sigma < 0$, introducing the variables

$$x = \theta, \qquad t = \tfrac{2}{3}(-\sigma)^{3/2}, \tag{2.3}$$

we obtain the normal form

$$u_{xx} - u_{tt} - \frac{1}{3t} u_t = 0. \tag{2.4}$$

As stated in the introduction, we shall consider in place of equations (2.2) and (2.4) the more general equations, containing an arbitrary real parameter, k,

$$\sum_{i=1}^{m} u_{x_i x_i} + u_{yy} + \frac{k}{y} u_y = 0, \tag{2.5}$$

$$\sum_{i=1}^{m} u_{x_i x_i} - u_{tt} - \frac{k}{t} u_t = 0 \tag{2.6}$$

for the functions $u(x_1, \cdots, x_m, y)$ and $u(x_1, \cdots, x_m, t)$ respectively.

For

$$k = (2n + 1)(2n + 3)^{-1}, \qquad n = 0, 1, 2, \cdots \tag{2.7}$$

the equations (2.5) and (2.6) are the normal forms of the equation

$$\sigma^{2n+1} \sum_{i=1}^{m} \psi_{\theta_i \theta_i} + \psi_{\sigma\sigma} = 0 \tag{2.8}$$

where we use the substitutions

$$x_i = \theta_i, \qquad y = 2(2n + 3)^{-1} \sigma^{(2n+3)/2}, \qquad \sigma \geq 0 \tag{2.9}$$

and

$$x_i = \theta_i, \qquad t = 2(2n + 3)^{-1}(-\sigma)^{(2n+3)/2}, \qquad \sigma \leq 0. \tag{2.10}$$

For values of k different from (2.7) the equations (2.5) and (2.6) are not connected with a common equation of mixed type such as (2.8). However, there is a formal link between equations (2.5) and (2.6), which can be obtained by putting $iy = t$ in any analytic solution of (2.5). In this way from each solution of (2.5) we obtain a solution of (2.6).

In order to emphasize the dependence of a solution of (2.5) and (2.6) on the index k, we denote such a solution by $u^{(k)}(x, y)$ and $u^{(k)}(x, t)$ respectively. A main tool of our investigation will be the elementary formulas for solutions of (2.6)

$$u^{(k)}(x, t) = t^{1-k} u^{(2-k)}(x, t) \tag{2.11}$$

$$u_t^{(k)}(x, t) = t^{-1} u^{(k+2)}(x, t), \tag{2.12}$$

which establish connections between solutions of different indices. By replacing t by y, we obtain the parallel formulas for (2.5). The relations (2.11) and (2.12) were given in a slightly different form by Darboux [2] for the case of one x variable ($m = 1$). They are valid for any m but have not been fully exploited until recently.

It is of importance to note that such relations are immediate consequences of the following two properties of the expression $v_{tt} + kt^{-1}v_t$ where v is a function of a single variable t. Put $w = t^{1-k}v$. Then

$$v_{tt} + kt^{-1}v_t = t^{1-k}(w_{tt} + (2 - k)t^{-1}w_t). \tag{2.13}$$

On the other hand, by putting $tz = v_t$ we obtain

$$t^{-1}(v_{tt} + kt^{-1}v_t)_t = z_{tt} + (k + 2)t^{-1}z_t\ . \tag{2.14}$$

Therefore (2.11) and (2.12) remain valid if we replace the EPD equation by the equation

$$u^{(k)}_{tt} + kt^{-1}u^{(k)}_t = Lu^{(k)} \tag{2.15}$$

where L is a linear differential operator which contains no derivatives with respect to the variable t.

3. *The Fundamental Solution in the Generalized Axially Symmetric Potential Theory*

The equation (2.5) may be regarded as a generalization of the classical Laplace equation in the axially symmetric case, a viewpoint which served as a guide in several investigations [1].

In this section we shall give *fundamental solutions in the large* of (2.5), namely, solutions which are continuous for all values $y \geq 0$ analytic for $y > 0$ and have an appropriate singularity at a single point, say $x = 0$, $y = b > 0$. It has been shown recently [3] that for $k > 0$ the formula

$$u^{(k)}_b = C_k \int_0^\pi \frac{\sin^{k-1}\alpha\, d\alpha}{\left(\sum_{i=1}^{m} x_i^2 + b^2 + y^2 - 2by\cos\alpha\right)^{(k+m-1)/2}} \tag{3.1}$$

where C_k is given by

$$C_k^{-1} = 2^{k-1}\Gamma^2\left(\frac{k}{2}\right)\Gamma^{-1}(k), \tag{3.2}$$

is a fundamental solution in the large, taking the values

$$1\Big/\left(\sum_{i=1}^{m} x_i^2 + b^2\right)^{(k+m-1)/2}$$

for $y = 0$. The derivative $\partial u^{(k)}_b/\partial y$ vanishes for $y = 0$. This formula is obviously a generalization of the Gauss formula for the potential of a circular ring, to which it reduces for $m = 1$, $k = 1$.

It is easy to see that $u_b^{(k)}$ is a solution of (2.5) because

$$y \sum_{i=1}^{m} \frac{\partial^2 u_b^{(k)}}{\partial x_i^2} + y \frac{\partial^2 u_b^{(k)}}{\partial y^2} + k \frac{\partial u_b^{(k)}}{\partial y} = C_k b(m - 1 + k)$$

$$\cdot \int_0^{\pi} \frac{d}{d\alpha} [(\sum x_i^2 + b^2 + y^2 - 2by \cos \alpha)^{-(m+1+k)/2} \sin^k \alpha] \, d\alpha = 0.$$

Replacing the expression in parenthesis in the denominator of (3.1) by

$$\sum_{i=1}^{m} x_i^2 + (y - b)^2 + 4by \sin^2 \frac{\alpha}{2},$$

we see that the only singular point of the function $u_b^{(k)}$ in the half space $y \geq 0$ is the point B with coordinates $x_1 = \cdots = x_m = 0$, $b > 0$. To determine the nature of the singularity, we make the substitution

$$\epsilon^2 = \frac{\sum_{i=1}^{m} x_i^2 + (y - b)^2}{4by}, \qquad \eta = \sin \frac{\alpha}{2}. \tag{3.3}$$

We then obtain

$$u_b^{(k)} = C_k 2^{1-m} (by)^{-(k+m-1)/2} \int_0^1 \frac{\eta^{k-1}(1 - \eta^2)^{(k-2)/2}}{(\eta^2 + \epsilon^2)^{(k+m-1)/2}} \, d\eta. \tag{3.4}$$

The integral in (3.4) is obviously a hypergeometric integral. It is simpler to investigate this integral directly for $\epsilon \to 0$ rather than the corresponding hypergeometric series, as the classical literature seemingly does not give explicit information for $m = 3, 5, 7, \cdots$ Omitting a regular part of the integral, we can replace the limits of integration by 0 and $\frac{1}{2}$. This allows us to develop $(1 - \eta^2)^{(k-2)/2}$ in a power series in η and integrate term by term. A short computation then shows that $u_b^{(k)}$ is a fundamental solution in the sense of Hadamard. For $m = 1$, $u_b^{(k)}$ has the form $u \log r +$ regular function, where

$$r^2 = \sum_{i=1}^{m} x_i^2 + (y - b)^2.$$

For m even, $u_k^{(b)}$ has the form $(U/r^{m-1}) +$ regular function, and for m odd, $u_b^{(k)} = (V/r^{m-1}) + W \log r$, where U, V and W are regular functions. According to a classical result of Hadamard, the singular parts are uniquely determined up to a constant factor (J. Hadamard, Lectures on Cauchy's Problem in Linear Partial Differential Equations, Dover, 1952, New York).

Since $k > 0$, it is easily seen from its definition, (3.1), that $u_b^{(k)}$ vanishes as $x^2 + y^2$ tends to infinity. For $k \geq 1$, this additional property insures that the fundamental solution (3.1) is unique up to a constant factor. To prove this, let us observe that a suitable linear combination, U, of two solutions is regular for $y > 0$, continuous for $y \geq 0$ and vanishes at infinity. A recent theorem of A. Huber [4] allows us to conclude that U is regular for $y \geq 0$. It

then follows from a mean value theorem [5] that U vanishes for $y = 0$ and therefore, by the maximum principle, U is identically zero. For $0 < k < 1$, the uniqueness property holds under the assumption that the solution is analytic for $y = 0$.

4. *A Second Fundamental Solution in the Large $k < 1$*

Using the correspondence principle

$$u^{(k)} = y^{1-k}u^{(2-k)}$$

which links solutions of (2.5) for the parameters k and $2 - k$, we easily obtain a second fundamental solution in the large for $k < 1$ with singularity at the point B, namely,

$$\begin{aligned} v_b^{(k)} &= y^{1-k}u_b^{(2-k)} \\ &= C_{2-k}y^{1-k}\int_0^{\pi} \frac{\sin^{1-k}\alpha\, d\alpha}{\left(\sum_{i=1}^{m} x_i^2 + b^2 + y^2 - 2by\cos\alpha\right)^{(1+m-k)/2}}. \end{aligned} \tag{4.1}$$

This solution v_b^k vanishes for $y = 0$ and at infinity but it is not analytic on $y = 0$ except for $k = 0, -1, -2, \cdots$. The solution $v_b^{(k)}$ was also given by Olevskii [6] in terms of hypergeometric series. Olevskii observes that his formula is a fundamental solution for $k \geq 1$ with the exception of $k = 2, 3, 4 \cdots$. However, since his solution is infinite for $y = 0$, it is not a solution in the large. It should also be noticed that Olevskii overlooks the solution (3.2) which is valid for all positive k.

5. *Continuation of the Fundamental Solutions of Tricomi's Equations into the Hyperbolic Half-Plane*

We turn back to the equations (2.8) in which we assume that n is a non-negative integer. Putting

$$b = 2(2n + 3)^{-1}\beta^{(2n+3)/2}$$

and using the transformation (2.9), we easily obtain two fundamental solutions for (2.8) corresponding to our formulas (3.2) and (4.1). The first fundamental solution, (3.1), considered as a function of θ_i and σ takes the value

$$\left[\sum_{i=1}^{m} \theta_i^2 + 4(2n + 3)^{-2}\beta^{2n+3}\right]^{-(k+m-1)/2} \tag{5.1}$$

on $\sigma = 0$, while its derivative with respect to σ is zero for $\sigma = 0$. The fundamental solution, (4.1), is of the type $\sigma u^{(2-k)}(\theta_i, \sigma)$, where $u^{(2-k)}$ assumes the values

$$\left[\sum_{i=1}^{m} \theta_i^2 + 4(2n + 3)^{-2}\beta^{2n+3}\right]^{-(m+1-k)/2} \tag{5.2}$$

on $\sigma = 0$ and also has a normal derivative which vanishes on $\sigma = 0$. To continue these solutions into the hyperbolic half-plane ($\sigma \leq 0$), we shall solve Cauchy problems using the values (5.1) and (5.2) as initial data, together with the vanishing normal derivatives, on $\sigma = 0$. As the normal form (2.6) shows, this is the singular Cauchy problem for the Euler-Poisson-Darboux equation. It will be treated in full generality in the next section.

It should be remarked that to obtain both fundamental solutions for a given k we have to consider not only the value k itself but, according to the correspondence principle used in (4.1), also the value $2 - k$. This is one of the several reasons for not confining ourselves to a single value of the parameter, say $k = 1/3$.

6. *The Singular Cauchy Problem for the Euler-Poisson-Darboux Equation*

We consider the equation

$$\sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2} = \frac{\partial^2 u}{\partial t^2} + \frac{k}{t}\frac{\partial u}{\partial t}, \qquad -\infty < k < \infty \tag{6.1}$$

with the initial conditions

$$u(x_1, \cdots, x_m, 0) = f(x_1, \cdots, x_m), \quad \text{or} \quad u(x, 0) = f(x) \tag{6.2}$$

$$u_t(x_1, \cdots, x_m, 0) = 0 \quad \text{or} \quad u_t(x, 0) = 0. \tag{6.3}$$

Since the coefficient of $\partial u/\partial t$ is infinite for $t = 0$, the ordinary existence and uniqueness theorems are not applicable except in the case $k = 0$, which is the wave equation. The solution is known to be unique for $k \geq 0$ (see section 10).

The following particular cases of (6.1) have been treated previously by special methods: $m = 1, 0 < k < 2$ [2]; $m = 3, k = 2$ [7]; for all positive integral values of m and $k = m - 1$ [8]; $m = 1, k = -1, -2, -3, \cdots$, [9]; $m = 2$, $0 < k < 1$ [10] this last case by a very complicated method involving finite parts of integrals.

Recently, in [11], we gave a unified elementary solution of the Cauchy problem for all values of k, in which the cases $k = 1, -3, \cdots$ were found to be exceptional. A second solution of the same problem was given by Diaz and Weinberger in [12], which includes an extensive discussion of the exceptional cases. The exceptional cases were also investigated independently by E. K. Blum who solved the problem under less restrictive conditions.

In this paper, we shall give a derivation of the solution of the Cauchy problem which is new and simpler in some points and leads to less cumbersome formulas.

The starting point of the solution of the Cauchy problem is the remark in [8] that for $k = m - 1$ the solution is given by Asgeirsson's mean value formula

$$u^{(m-1)}(x, t) = \frac{1}{\omega_m} \int_{\sum_{i=1}^{m} \alpha_i^2 = 1} f(x_1 + \alpha_1 t, \cdots, x_m + \alpha_m t)\, d\omega_m \equiv M(x, t; f) \tag{6.4}$$

where $\omega_m = 2\pi^{m/2}/\Gamma(m/2)$ is the area of the m dimensional unit sphere. Thus, M denotes the mean value of f. For $m = 3$, (6.4) reduces to Poisson's formula.

For $k > m - 1$, the solution is given by the formula

$$u^{(k)}(x, t) = \frac{\omega_{k+1-m}}{\omega_{k+1}} \cdot \int_{\sum_{i=1}^{m} \alpha_i^2 \leq 1} f(x_1 + \alpha_1 t, \cdots, x_m + \alpha_m t)\left(1 - \sum_{i=1}^{m} \alpha_i^2\right)^{(k-m-1)/2} d\alpha_1 \cdots d\alpha_m \tag{6.5}$$

which coincides with the formula obtained by Hadamard's method of descent for $k = m, m + 1, \cdots$. For all other $k > m - 1$, it can be verified by direct differentiation that (6.5) is actually a solution. This principle of obtaining a solution for integral values of $k > m - 1$ and extending it to non-integral values may be regarded as a *generalized method of descent.* It is obvious that (6.4) and (6.5) give even functions of t.

To solve the Cauchy problem $k < m - 1$, $k \neq -1, -3, -5, \cdots$, we shall make repeated use of formulas (2.11) and (2.12). Choose the smallest integer n such that $k + 2n \geq m - 1$. Let $u^{(k+2n)}$ denote the solution of the Cauchy problem for an EPD equation of index $k + 2n$ and initial values

$$u^{(k+2n)}(x, 0) = \frac{f(x)}{(k+1)(k+3)\cdots(k+2n-1)}, \qquad u_t^{k+2n}(x, 0) = 0. \tag{6.6}$$

Then $u^{(k+2n)}$ is determined by a formula of the type (6.4) or (6.5). By (2.11),

$$t^{k+2n-1}u^{(k+2n)} = u^{(2-k-2n)}. \tag{6.7}$$

Therefore, applying (2.12) n times, we have

$$\left(\frac{\partial}{t\,\partial t}\right)^n t^{k+2n-1}u^{(k+2n)} = u^{(2-k)}. \tag{6.8}$$

Finally, applying (2.11) again, we get $t^{1-k}u^{(2-k)} = u^{(k)}$, that is, from (6.8),

$$u^{(k)} = t^{1-k}\left(\frac{\partial}{t\,\partial t}\right)^n (t^{k+2n-1}u^{(k+2n)}). \tag{6.9}$$

It is clear that $u^{(k)}$ is a solution of the EPD equation. To verify that the initial conditions are satisfied, we simply carry out the differentiations and obtain an expression of the form

$$u^{(k)}(x, t) = (k+1)(k+3)\cdots(k+2n-1)u^{(k+2n)}(x, t) + \text{const. } tu_t^{(k+2n)} + O(t^2). \tag{6.10}$$

In view of (6.6), it follows immediately that $u^{(k)}(x, 0) = f(x)$ and $u_t^{(k)}(x, 0) = 0$.

Formula (6.9) is a solution of the Cauchy problem for all $k \neq -1, -3, \cdots$, but is of course mainly useful when $k < m - 1$. The only condition which has to be satisfied is that $k + 2n \geq m - 1$.

Let us emphasize that (6.9) reduces to the classical solution of the wave equation for $k = 0$. The method presented here provides a simpler derivation of this solution.

If $m - k$ is an odd positive integer, $k + 2n = m - 1$ and $u^{(k+2n)}$ in (6.9) is given by a mean value formula of the type (6.4). Therefore, we have a Huygens' principle as previously observed [12].

Finally, we remark that for (6.9) to be a solution it is sufficient to assume that the initial value $f(x_1, \cdots, x_m)$ has continuous derivatives of order not less than $\frac{1}{2}(m - k + 3)$.

7. *Representation of the Solution by the Riemann-Liouville Integral*

According to a remark of Diaz and Weinberger, by introducing polar coordinates, the solution (6.5), valid for $k > m - 1$, can be written in the form

$$(7.1)\quad u^{(k)}(x, t) = \frac{2\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k+1-m}{2}\right)\Gamma\left(\frac{m}{2}\right)} \int_0^1 M(x, \rho t; f)(1 - \rho^2)^{(k-m-1)/2}\rho^{m-1}\, d\rho,$$

where $M(x, \rho t; f)$ is again a mean value as in (6.4). Incidentally, the analytic continuation of (7.1) as a function of k is the basis of the method used by Diaz and Weinberger to obtain a solution for $k < m - 1$ [12].

Let us now introduce the Riemann-Liouville integral

$$(7.2)\qquad I^{\beta}\{F(\tau)\} = \frac{1}{F(\beta)} \int_0^{\tau} (\tau - \rho)^{\beta-1}F(\rho)\, d\rho, \qquad \beta > 0.$$

By making the substitution $\sigma = \rho^2 t^2$ in (7.1) and then putting $t^2 = \tau$, we easily obtain the formula

$$(7.3)\qquad u^{(k)}(x, \sqrt{\tau}) = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \tau^{(1-k)/2} I^{(k-m+1)/2}\{M(x, \sqrt{\tau}; f)\tau^{(m-2)/2}\},$$

which is valid for $k > m - 1$. For $k = m - 1$, (7.3) reduces to (6.4).

On the other hand, since $u^{(k+2n)}$ in (6.9) is given by a formula of the type (7.1), the same substitution allows us to write (6.9) in the form

$$(7.4)\quad u^{(k)}(x, \sqrt{\tau}) = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \tau^{(1-k)/2} \frac{d^n}{d\tau^n} I^{(k+2n-m+1)/2}\{M(x, \sqrt{\tau}; f)\tau^{(m-2)/2}\}$$

which is again valid for $k \geq m - 1 - 2n$, $k \neq -1, -3, \cdots$.

8. *Connections with the Asgeirsson Mean Value Theorem*

Let us now assume that $k = 0, 1, 2, 3, \cdots$. We shall show first that the EPD equation can be considered as a Coulon, or ultra-hyperbolic equation.

Suppose for the sake of definiteness that $k < m - 1$, we consider a function $u = u(x_1, \cdots, x_m ; y_1 \cdots y_m)$ which satisfies the Coulon equation

$$\Delta_x u = \Delta_y u. \tag{8.1}$$

Then we have the following mean value theorem [8],

$$\frac{1}{\omega_m}\int_{\omega m} u(x_i + \alpha_i t, 0)\, d\omega_m = \frac{1}{\omega_m}\int_{\omega m} u(x_i, \alpha_i t)\, d\omega_m . \tag{8.2}$$

Putting $u(x_i, 0) = f(x_i)$ and assuming that u depends only on $x_1, \cdots, x_m$, $y_1, \cdots, y_{k+1}$, we have by the method of descent (section 6)

$$M(x, t; f) = \frac{\omega_{m-k-1}}{\omega_m} \cdot \int_{\sum_{i=1}^{m} \alpha_i^2 \le 1} u(x_i, \alpha_1 t, \cdots, \alpha_{k+1} t)\left(1 - \sum_1^{k+1} \alpha_i^2\right)^{(m-k-3)/2} d\alpha_1 \cdots d\alpha_{k+1} . \tag{8.3}$$

If we further assume that u is radially symmetric in the space $y_1, \cdots, y_{k+1}$, and if we put $\rho^2 = \sum_1^{k+1} \alpha_i^2$, then (8.3) becomes

$$M(x_i, t; f) = \frac{\omega_{m-k-1}}{\omega_m}\,\omega_{k+1}\int_0^1 u(x_i, \rho t)(1 - \rho^2)^{(m-k-3)/2}\rho^k\, d\rho. \tag{8.4}$$

At the same time, equation (8.1) reduces to the EPD equation, $\Delta_x u = u_{tt} + kt^{-1}u_t$, $k = 0, 1, \cdots, m - 2$. The formula (8.4) has been considered for $k = 0$ by Asgeirsson. As in this special case, it is always an Abel integral equation the solution of which is given by formula (7.4). The difference between our consideration and Asgeirsson's is that we assume that u in Coulon's equation is radially symmetric in $y_1 \cdots y_{k+1}$, while Asgeirsson assumes that u depends only on y_1, which he identifies with t. Needless to say, a similar procedure would lead to formula (7.3) for $k = m - 1, m, m + 1, \cdots$. However, it should be realized that this procedure does not provide a proof for non-integer values of k and for negative values of k. Even for $k = 0, 1, 2, \cdots$, a verification of the formula would be required to make the application of Asgeirsson's theorem independent of the methods developed in Section 6. Moreover our method does not require the solution of an Abel integral equation or analytic continuation.

9. *The Exceptional Cases of the Odd Negative Integers*

In the previous sections, the solution has been obtained for $k \neq -1, -3, -5, \cdots$. The only assumption on $f(x_1, \cdots, x_m)$ was that $f(x_1, \cdots, x_m)$

should have continuous derivatives of order greater than or equal to $\frac{1}{2}(m + 3 - k)$. In that case, the solution also has continuous derivatives of the same order with respect to t, even for $t = 0$. The situation is quite different in the exceptional cases $k = -1, -3, -5, \cdots$. In these cases, a solution $u^{(k)}$ exists under the same differentiability conditions on f. However, the partial derivative $\partial^{1-k}u^{(k)}/\partial t^{1-k}$ of this solution becomes infinite like $\log t$ at $t = 0$ unless $f(x_1, \cdots, x_m)$ is polyharmonic of order $\frac{1}{2}(1 - k)$.

In this generality, a solution was found by Diaz and Weinberger [12] and Blum, in an unpublished paper, after the present author pointed out the exceptional role played by polyharmonic initial values. Blum shows that a solution can be obtained directly from (6.9). It should be noted that B. Friedman, in a communication to the author, gave a first example of a solution with non-polyharmonic initial values.

It is remarkable that for the exceptional values even a small deviation, no matter how regular, from polyharmonic initial values introduces logarithmic infinities in certain derivatives of the solution. In this sense, the solution does not depend continuously on the data.

10. *Uniqueness*

The uniqueness of the solution for $k > 0$ has been established by Asgeirsson by the method of Zaremba. However, for $k < 0$ there can be no uniqueness because any function of the type $t^{1-k}u^{(2-k)}(x, t)$ which vanishes together with its derivatives with respect to t may be added to a solution $u^{(k)}(x, t)$ of the Cauchy problem.

The situation has been recently clarified by E. K. Blum for the case $m = 1$ in a paper presented to the American Mathematical Society. He showed that for arbitrary $k < 0$, the difference of any two solutions is of the form $t^{1-k}u^{(2-k)}(x,t)$ where $u^{(2-k)}(x, t)$ is a solution such that

$$\lim_{t\to 0} t^{-k/2}u^{(2-k)}(x, t) = 0 \qquad \text{and} \qquad \lim_{t\to 0} t^{-(k-1)/2}u_t^{(2-k)}(x, t) = 0.$$

11. *The Formulas for the Fundamental Solutions of the Tricomi Equations in the Hyperbolic Half-Plane*

Let us return to the generalized Tricomi equations (2.8). As stated in section 5, the first fundamental solution assumes the values (5.1) on $\sigma = 0$. Using (2.10), we obtain the initial conditions

$$(11.1) \qquad u^{(k)}(x, 0) = f(x) = \left[\sum_{i=1}^{m} x_i^2 + b^2\right]^{-(k+m-1)/2}, \qquad u_t^{(k)}(x, 0) = 0$$

for our Cauchy problem with k given by (2.7) so that $0 < k < 1$. Hence, for $m = 1$, the solution is given by (6.5) with $f(x)$ as in (11.1), while for $m = 2, 3, \cdots$, the solution is given by (6.9).

The second fundamental solution is of the form $\sigma u^{(2-k)}(\theta, \sigma)$. The continuation into the hyperbolic half plane must have the same form, that is, using (2.10), it must be of the form $t^{2/(2n+3)}v(x, t)$. Since $t^{2/(2n+3)}v(x, t)$ must be a solution of (2.6) with $k = (2n + 1)/(2n + 3)^{-1}$, v must be a solution of (2.6) with k replaced by $2 - k = (2n + 5)(2n + 3)^{-1}$. That is $v = u^{(2-k)}$. The initial values of $u^{(2-k)}(x, t)$ are obtained from (5.2) by using (2.10) again. Thus, for our Cauchy problem

$$u^{(2-k)}(x, 0) = f(x) = \left[\sum_{i=1}^{m} x_i^2 + b^2\right]^{-(m+1-k)/2}, \qquad u_t^{(2-k)}(x, 0) = 0. \tag{11.2}$$

Since $1 < 2 - k < 2$, for $m = 1, 2$ the solution, $u^{(2-k)}$, is given by (6.5) with $f(x)$ as in (11.2), while for $m = 3, 4, \cdots$ the solution is given by (6.9).

In conclusion, we shall exemplify these results by giving explicitly the first fundamental solution of the generalized Tricomi equation (2.8) with $m = 1$. According to the rules given in this paragraph, the fundamental solution with singularity at $\theta_1 = 0$, $\sigma = \beta$ is given by

$$u^{(k)}(\theta_1, \sigma) = \frac{\omega_k}{\omega_{k+1}} \int_0^1 \left[\left(\theta_1 + \frac{2\cos\alpha}{2n+3}(-\sigma)^{(2n+3)/2}\right)^2 + \left(\frac{2}{2n+3}\right)^2 \beta^{2n+3}\right] d\alpha, \tag{11.3}$$

$$k = (2n + 1)(2n + 3)^{-1}, \qquad n = 0, 1, 2, \cdots$$

for $\sigma \leq 0$. For $\sigma \geq 0$ this fundamental solution is obtained from (3.1) and is given by

$$u^{(k)}(\theta_1, \sigma) = C_k \int_0^{\pi} \left[\theta_1^2 - 2\left(\frac{2}{2n+3}\right)^2 (\beta\sigma)^{(2n+3)/2} \cos\alpha + \left(\frac{2}{2n+3}\right)^2 \sigma^{2n+3} + \left(\frac{2}{2n+3}\right)^2 \beta^{2n+3}\right]^{-k/2} \sin^{k-1}\alpha \, d\alpha. \tag{11.4}$$

12. *Concluding Remarks*

By a process analogous to classical potential theory, namely, by integration and differentiation of the fundamental solution, other singular solutions can be obtained from our fundamental solutions as presented here. In the case $m = 1$, these results and others were given by the author in a series of papers going back to 1948, a resume of which is given in [17]. For further developments and applications to transonic flow see von Kármán and Fabri [13], Germain [14], Germain and Bader [15], [18], Cole [16] and Trilling and Walker [19].

Bibliography

[1] Weinstein, A., *Generalized axially symmetric potential theory*, Bulletin, American Mathematical Society, Volume 59, 1953, pp. 20–38.

[2] Darboux, G., *Leçons sur la théorie générale des surfaces*, 2nd edition, Volume II, Gauthier-Villars, Paris, 1915, Ch. III, IV.

[3] Diaz, J. B., and Weinstein, A., *On the fundamental solutions of a singular Beltrami operator*, in R. von Mises anniversary volume (in press).

[4] Huber, A., *On the uniqueness of generalized axially symmetric potentials*, Annals of Mathematics (in press).

[5] Weinstein, A., *On Tricomi's equation and generalized axially symmetric potential theory*, Bulletin, Académie royale de Belgique, Volume 37, 1951, pp. 348–358.

[6] Olevskii, M., *Solution of the Dirichlet problem for the equation* $\Delta u + (k/x_n)(\partial u/\partial x_n) = \rho$ *for a hemispherical region*, Doklady Akademii Nauk SSSR, (N.S.) Volume 64, 1949, pp. 767–770.

[7] Poisson, S. D., *Mémoire sur l'intégration des équations linéaires aux dérivées partielles*, Journal de l'école polytechnique, Volume 12, No. 19, 1823.

[8] Asgeirsson, L., *Über eine Mittelwerteigenschaft von Lösungen homogener linearer partiellen Differentialgleichungen zweiter Ordnung mit konstanten Koeffizienten*, Mathematische Annalen, Volume 113, 1937, pp. 321–346.

[9] Diaz, J. B., and Martin, M. H., *Riemann's method and the problem of Cauchy. II, The wave equation in n dimensions*, Proceedings of the American Mathematical Society, Volume 3, 1952, pp. 467–483.

[10] Kapilevic, M. B., *On an equation of mixed elliptic-hyperbolic type*, Matematiceskii Sbornik, (N.S.) Volume 30, No. 72, 1952, pp. 11–38.

[11] Weinstein, A., *Sur le problème de Cauchy pour l'équation de Poisson et l'équation des ondes*, Comptes rendus des séances de l'académie des sciences, Paris, Volume 234, 1952, pp. 2584–85.

[12] Diaz, J. B., and Weinberger, H. F., *A solution of the singular initial value problem for the Euler-Poisson-Darboux equation*, Proceedings of the American Mathematical Society, Volume 4, 1953, pp. 703–715.

[13] von Kármán, T., and Fabri, J., *Écoulement transsonique à deux dimensions le long d'une paroi ondulée*, Comptes rendus des séances de l'académie des sciences, Paris, Volume 231, 1950, pp. 1271–74.

[14] Germain, P., *Solutions élémentaires des équations regissant les écoulements des fluides compressibles*, Comptes rendus des séances de l'académie des sciences, Paris, Volume 234, 1952, pp. 1248–50.

[15] Germain, P., and Bader, R., *Sur quelques problèmes relatifs a l'équation de type mixte de Tricomi*, Office Nationale d'Étude et de Recherche Aeronautique, No. 54, 1952.

[16] Cole, J. D., *Drag of a finite wedge at high subsonic speeds*, Journal of Mathematics and Physics, Volume 30, 1951, pp. 79–83. *Note on the fundamental solution of* $wy_{vv} + y_{ww} = 0$, Zeitschrift für angewandte Mathematik und Physik, Volume 3, 1952, pp. 286–297.

[17] Weinstein, A., *The method of singularities in the physical and in the hodograph plane*, Proceedings of Symposia in Applied Mathematics, American Mathematical Society, Volume 4, 1953, pp. 167–178.

[18] Germain, P., and Bader, R., *Solutions élémentaires de certaines équations aux dérivées partielles du type mixte*, Bulletin de Société Mathématique de France, Volume 81, 1953, pp. 145–174.

[19] Trilling, L., and Walker, K., *On the transonic flow past a wedge*, Journal of Mathematics and Physics, Volume 32, 1953, pp. 72–79.

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. VII, 117–143 (1954)

Remarks on the Theory of Partial Differential Equations of Mixed Type and Applications to the Study of Transonic Flow

By P. GERMAIN
Office National d'Études et de Recherches Aeronautiques

Summary

I shall begin by recalling briefly some well known *statements* on problems arising in the theory of transonic flow. These, from the mathematical point of view, are related to a partial differential equation of the mixed type. But they are not the simplest basic problems of the theory of equations of this type. Although there are close similarities between the study of transonic flow and the mathematical study of mixed equations, one cannot say that the studies are strictly equivalent. The second part of this lecture is devoted to the formation and the study of elementary solutions of equations of this type; a few applications to certain boundary value problems are proposed.

Part I. TYPICAL PROBLEMS IN TRANSONIC FLOWS

I. Equations of Transonic Flow

It is well known that to every solution of the equation

$$L(f) = k(\sigma)\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial \sigma^2} = 0, \tag{1}$$

in which $k(\sigma)$ is a differentiable and increasing function of the variable σ, positive for $\sigma > 0$ and negative for $\sigma < 0$, there corresponds a steady two-dimensional flow for a compressible fluid which satisfies a certain equation of *state*. There are two simple ways of obtaining this correspondence (see for example [1]).

1. *The Hodograph Method*

One identifies f with a stream function $\psi(\theta, \sigma)$; the velocity potential $\phi(\theta, \sigma)$ is defined by

$$\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial \sigma}, \qquad \frac{\partial \phi}{\partial \sigma} = k(\sigma)\frac{\partial \psi}{\partial \theta}.$$

If one takes as the unit of the speed q and of the density ρ their values at the critical state ($\sigma = 0$), then q and $m = \rho q$ are functions of σ and solutions of

$$\frac{d}{d\sigma}\left(\frac{1}{q}\right) = \frac{1}{m}, \qquad \frac{d}{d\sigma}\left(\frac{1}{m}\right) = k(\sigma)\frac{1}{q}; \qquad q(0) = m(0) = 1.$$

This solution, together with Bernoulli's law determines the equation of the fluid under consideration. The cartesian coordinates (x, y) of the plane of motion are given by the integration of

$$dx + i\,dy = e^{i\theta}\left(\frac{d\phi}{q} + i\frac{d\psi}{m}\right)$$

(the variable θ defines the angle of the speed vector with Ox). In the case in which the fluid is a perfect polytropic gas, $k(\sigma)$ is defined only for $-\sigma_m \leq \sigma$ (σ_m positive constant); $k(\sigma)$ tends towards a positive limit when σ increases indefinitely and its absolute value $|\,k(\sigma)\,|$ increases indefinitely when σ tends towards $-\sigma_m$.

2. *The Method of Legendre's Potential*

One can also identify f with a function $\chi(\theta, \sigma)$, to which one associates a function $\omega(\theta, \sigma)$ by means of the system

$$\frac{\partial\omega}{\partial\theta} = \frac{\partial\chi}{\partial\sigma}, \qquad \frac{\partial\omega}{\partial\sigma} = -k(\sigma)\frac{\partial\chi}{\partial\theta}.$$

The coordinates of the plane of flow are given by

$$x + iy = e^{i\theta}\left(-\frac{1}{m}\frac{\partial\chi}{\partial\sigma} + \frac{i}{q}\frac{\partial\chi}{\partial\theta}\right)$$

(the variable θ still represents the angle of the vector with Ox) and the functions $q(\sigma)$ and $m(\sigma)$ are given by

$$\frac{dq}{d\sigma} = -m, \qquad \frac{dm}{d\sigma} = -k(\sigma)q; \qquad q(0) = m(0) = 1.$$

The first question for the aerodynamicist is the following: given one of these methods of correspondence (for the sake of uniformity we shall always take the first), what properties must a solution of (1) satisfy so that it shall correspond to a flow of given type? The second question is: what type of boundary value problem allows one to define the solution of a given problem? The following resume of results is meant simply to underline the complexity of these questions. Furthermore, from a practical point of view, one is often obliged to seek "approximations", i.e. functions $k(\sigma)$, for which these problems can be attacked. The simplest of these approximations corresponds to the case in which $k(\sigma)$ is proportional to σ; the corresponding equation (1) is the Tricomi equation [2].

II. The Problem of Nozzles

Numerous authors, Falkovitch [3], Lighthill [4], etc., have noted the essential property which a solution $\psi(\theta, \sigma)$ of (1) must satisfy if it is to represent the flow in a transonic nozzle.

(T): *In the neighbourhood of the origin O* ($\theta = 0$, $\sigma = 0$)—*i.e., the image in the* θ, σ-*plane of the point on the axis of the nozzle where the critical speed is reached—the characteristics issuing from O are the branch lines of the solution and* ψ *must take on a real value outside the angle made by these characteristics and three real values within this angle* (Figure 1).

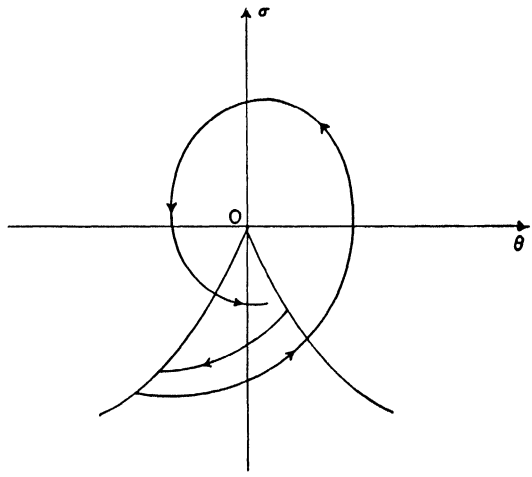


FIGURE 1

It is perhaps more satisfactory from a mathematical point of view to suppose that $\psi(\theta, \sigma)$ is an analytic function of the variables (θ, σ) and to state:

(T′): *In the neighbourhood of the origin of the complex* θ-*plane,* $\psi(\theta, \sigma)$ *has three branches but is uniform on the Riemann surface* $S(\sigma)$

$$\theta = a(\sigma)z + b(\sigma)z^3,$$

schematically represented in Figure 2, in which the branch points (of the second order) are given by

$$\theta = \pm l(\sigma), \qquad -il > 0 \text{ if } \sigma > 0;\ l > 0 \text{ if } \sigma < 0;$$

this equation represents the characteristics of equation (1) (real or imaginary) issuing from O. Furthermore, ψ is real for the following parts of the real axis in the θ-plane:

for $\sigma < 0$: $I(-l, +\infty)$, $II(-l, +l)$, $III(-\infty, l)$;

for $\sigma > 0$: $II(-\infty, +\infty)$.

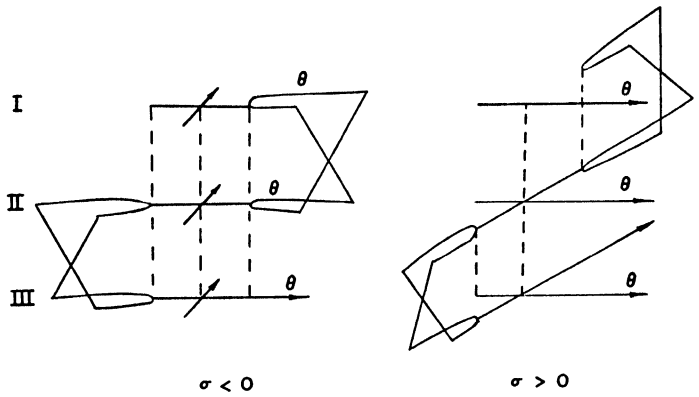


FIGURE 2

Cherry has given a complete solution to this problem in the case of a perfect polytropic gas [5]. But many authors have tried to define an equation (1) relating to a fluid which has an approximate equation of state and for which one can give a simple solution of (1) having the property (T). Recently M. Fenain [6] determined classes of functions $k(\sigma)$ which have the property that the equation (1), after transformation of the variable σ, admits a solution which remains constant on each tangent of the characteristics through O. Here is the statement of Fenain's theorem: The equation

$$f(s)\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial s^2} + H(s)\frac{\partial\psi}{\partial s} = 0 \tag{2}$$

admits a solution $\psi(\theta, s)$ defined by

$$\theta = sa(\psi) + B(\psi)$$

in the following cases:[1]

$$f'' = 0, \qquad H = \text{p};$$

$$\frac{f''}{f'} = -\frac{2}{s+a}, \qquad H = \frac{r}{s+a}; \qquad (a, p, r \text{ are arbitrary constants}).$$

$$\frac{f''}{f'} = -\frac{3}{s+a}, \qquad H = \frac{1}{s+a};$$

[1]The function $\psi(\theta, s)$ defined by $\theta = sa(\psi) + B(\psi)$ is a solution of (2) if and only if it is possible to satisfy an identity of the form

$$f(s) + a^2 = (sa' + B')(H(s)A(\psi) + C(s))$$

where $A(\psi)$ is a primitive of a and $C(s)$ an arbitrary function. By giving ψ three different values, it is obvious that $f(s)$ is the quotient of two polynomials of at most second degree. Using an advanced argument, it is possible to prove Fenain's theorem by determining the functions $f(s)$, $A(\psi)$, $B(\psi)$.

It is obvious that these solutions satisfy the property (T) or (T′). The corresponding equations or approximations cover a certain number of particular cases already mentioned by several authors, for instance, the equation of Tricomi (Falkovitch [2], H. Cabannes [7], Martin [8]) and the "homographic approximation" in which $k(\sigma)$ is a homographic function of σ (Germain [9]); only recently J. A. Tierney [10] independently found the "parabolic approximation" in which $k(\sigma)$ is a polynomial of the second degree in σ.

III. The Problem of Profiles

(a) *At first let us suppose that the flow admits a uniform velocity potential in the x, y-plane and that the speed at infinity is subsonic*; the image in the θ, σ-plane of this speed vector is denoted by A ($\theta = 0$, $\sigma = \sigma_A$; $\sigma_A > 0$). It is well known that: (P_1) *A is a second order branch point for $\psi(\theta, \sigma)$ (in other words, the image of the flow is in two sheets of the θ, σ-plane)*. Many theoretical studies have been devoted to such flows in the case of a perfect polytropic gas (see for example Tamada [11], M. J. Lighthill ([4] Part III), Cherry [12]).

For approximations which can lead to numerical calculations it is often useful to refer to a general remark made by Tomotika and Tamada in the particular case of Tricomi's equation ([13], Part II); if $\psi(\theta, \sigma)$ is a solution of (1) and satisfies the property (T′), then the real function $F(\theta, \sigma) = \psi(\theta - it, \sigma) + \psi(\theta + it, \sigma)$ for real θ and real σ is a solution of (1) satisfying (P_1).

This property seems almost evident when one uses the surface $S(\sigma)$ introduced earlier. The number t is chosen in such a way that $t = l\ (\sigma_A)$—σ_A can be assumed small so that the representation on $S(\sigma)$ is valid—the two values $\psi(\theta - it, \sigma)$ and $\psi(\theta + it, \sigma)$ which have to be associated to one another correspond to the points P and P' which are symmetric with respect to the real θ-axis of the plane II. When one draws in the real plane a path such as $XYZTX$

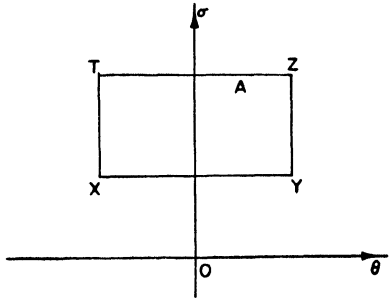


FIGURE 3

about A (Figure 3), it can be seen that points P and P' change sheets (when one draws YZ or TX the surfaces are continuously deformed but these points do not change sheets with the deformation since the θ corresponding to P and P' are constant.)

A solution with the desired singularity can be obtained by using the θ-derivative of the solution $F(\theta, \sigma)$ (see [13], Parts II and III, [14], and [15]).

(b) *If the speed at infinity remains subsonic and the flow includes a shock wave which limits the region in which the flow is supersonic, it is permissible to neglect the vorticity and to retain the same equations because the shock is generally of low intensity.* But the image of the shock wave in the θ, σ-plane consists of two arcs C and C'—one in the half-plane $\sigma > 0$, the other in the half-plane $\sigma < 0$—which correspond point for point and along which the following conditions must be satisfied:[2]

$$(3) \qquad (d\Phi)_C = (d\Phi)_{C'}, \qquad (d\psi)_C = (d\psi)_{C'}, \qquad \frac{d\psi}{d\Phi} = \pm\frac{m_1 m_2}{q_1 q_2}\sqrt{\frac{q_1^2 - q_2^2}{m_2^2 - m_1^2}},$$

$$\cos(\theta_2 - \theta_1) = \frac{q_2 m_2 + q_1 m_1}{q_2 m_1 + m_2 q_1};$$

this latter relationship is none other than that which defines the shock polar in the θ, σ-plane. This gives conditions which are exceedingly difficult to satisfy. As a result, authors who have worked on this difficult problem, particularly J. D. Cole [16] whose brilliant work has been so useful in aerodynamics, have been obliged to make important approximations in stating the boundary conditions in order to arrive at explicit expressions of $\psi(\theta, \sigma)$ (incidentally, Cole uses Tricomi's equation).

(c) *If the speed at infinity is sonic* (the point O being its image in the θ,σ-plane) then the function $\psi(\theta, \sigma)$, relative to the domain lying upstream from the transonic *frontier*, has at the point O a singularity of a well defined type (P_2) determined by F. J. Frankl [17]. The function must in fact increase indefinitely as one moves towards O but it must be finite at all points of the characteristics issuing from O (except for the point O). For instance, in the case of Tricomi's equation—investigated by Guderley and Yoshihara [18] for the case in which the obstacle is a "wedge"—it is easy to verify that the solution representing the singularity (P_2) can be written as

$$\psi = \rho^{-5/3}\left\{\left(1 - \frac{\theta}{\rho}\right)^{1/3}\left(1 + \frac{3\theta}{\rho}\right) - \left(1 + \frac{\theta}{\rho}\right)^{1/3}\left(1 - \frac{3\theta}{\rho}\right)\right\}, \qquad \rho^2 = \frac{4}{q}\sigma^3 + \theta^2.$$

(d) *Last, let us remember that if the speed at infinity is supersonic,* the shock wave being detached, the function $\psi(\theta, \sigma)$ in the transonic region can be defined as a solution of a boundary value problem which is a generalization of that of Tricomi, see F. J. Frankl [19], W. G. Vincenti and C. B. Wagoner [20]. In the case numerically solved by the last two authors the function $\psi(\theta, \sigma)$ vanishes along AA', $A'B$ and on the characteristic AC but satisfies along BD

[2]Equations (3) are obtained by assuming that the mechanical shock conditions hold; i.e. $d\Phi$ and $d\psi$ have the same value on both sides of the shock wave in the x, y-plane. On the other hand, $dx + i dy$ must also have the same value on both sides of the shock.

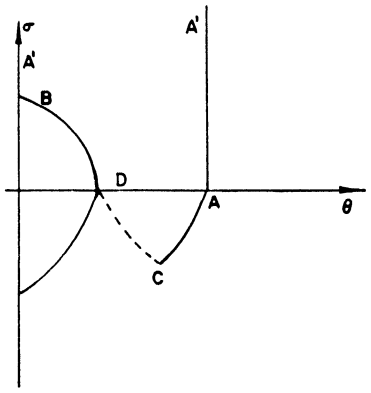


FIGURE 4

(which is an arc of the shock polar) more complex conditions (a linear relationship which joins the partial derivatives of ψ is known, see Figure 4).

It is enough to recall these few examples to show the variety of mathematical problems imposed by the theory of transonic flow; solutions presenting singularities of definite type, boundary value problems with complex data, all these difficulties result from the method of correspondence mentioned at the beginning. Unfortunately the mathematical investigation of partial differential equations of mixed type is not yet sufficiently developed to allow general study of such difficult problems.

The first problem (solutions with a given type of singularity) has been solved by L. Bers [21] for elliptic equations with the help of the Σ-monogenic functions. In the following, we give a method for writing explicit expressions (by means of Fourier integrals) for the solutions which are uniform in the neighbourhood of the singular point P (see [39], where R. Bader and the author have already studied this problem).

The study of boundary value problems is not yet very far developed for the domains which correspond to regions where the flow is partly subsonic and partly supersonic. We shall give later some references to recent works on this subject. But the existence of the solution of the Tricomi problem—which is the simplest—has not yet been proved. Nevertheless, many applications of the theory of mixed problems to transonic flows have been given by Guderley [23] (see also [22]).

Part II. ELEMENTARY SOLUTIONS FOR EQUATIONS OF MIXED TYPE

We shall now consider a precise problem of the theory of partial differential equations of mixed type: the derivation of "fundamental solutions" or, in French terminology (Hadamard) "elementary solutions." These are solutions which have a definite type of singularity and play an important part in the study

and the solution of *boundary value problems*. Since this notion has been used in work connected with elliptic as well as hyperbolic problems, I thought it would be interesting also to define such solutions for an equation of mixed type and investigate their behaviour when the singular point varies in a domain in which the equation can, eventually, change its type.

Several authors have given analytic solutions of Tricomi's equation in which the singular point is in the elliptic half-plane $\sigma \geq 0$. Holmgren and Gellersted [24] were apparently the first to approach the subject but their work has long remained unknown; independently, A. Weinstein [25], G. F. Carrier and F. E. Ehlers [26] and Cole [27] have given equivalent solutions. P. Germain and R. Bader [28] have studied the case where the singular point P may be in the hyperbolic region.

For simplicity, we use now the variables x and z, instead of θ and σ. Thus we consider the equation

$$\mathfrak{L}(u) \equiv k(z)\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} = 0, \tag{4}$$

in which u is an unknown function of x and z. The function $k(z)$ is still a continuously differentiable increasing function of z, positive for $z > 0$, negative for $z < 0$. We assume also that $k(z)$ has a bounded second derivative for $z = 0$.

IV. General Method

The method which we shall employ is a generalization of Weinstein's method; we use Fourier's transformation, defined by L. Schwartz in his theory of distributions [29], [30]. In this way we shall not only be able to simplify discussions on convergence but, above all, certain functions showing discontinuities or singularities within the domain in which they are defined can be considered globally as "distribution solutions."

This transformation is the extension to "temperate" distributions[3] of the transformation which for a set of two homologous functions $u(x)$, $U(\alpha)$, infinitely differentiable and decreasing rapidly, is written as

$$U(\alpha) = \mathfrak{F}(u) \rightarrow U(\alpha) = \int_{-\infty}^{\infty} u(x) \exp\{-2i\pi\alpha x\}\, dx,$$

$$u(x) = \mathfrak{F}^{-1}(U) \rightarrow u(x) = \int_{-\infty}^{\infty} U(\alpha) \exp\{2i\pi\alpha x\}\, d\alpha.$$

By means of this transformation equation (4) becomes the differential equation

$$l(U) = U_{zz} - 4\pi^2\alpha^2 k(z)U = 0. \tag{5}$$

Let $S(\alpha, z)$, $D(\alpha, z)$ be the solutions of this equation defined by the initial conditions

[3] For all references to distributions, see L. Schwartz [29], [30].

$$(6)\qquad \begin{aligned} S(\alpha, 0) &= 0, & D(\alpha, 0) &= 1, \\ \frac{\partial S(\alpha, 0)}{\partial z} &= 1, & \frac{\partial D(\alpha, 0)}{\partial z} &= 0. \end{aligned}$$

We set

$$(7)\qquad \mathfrak{D}(z, z_0) = S(\alpha, z)D(\alpha, z_0) - S(\alpha, z_0)D(\alpha, z);$$

this latter solution satisfies the conditions

$$(8)\qquad \mathfrak{D}(z_0, z_0) = 0, \qquad \frac{\partial \mathfrak{D}}{\partial z}(z_0, z_0) = 1.$$

One can, in fact, verify that the second equality is independent of z_0 since it is the Wronskian of two solutions of (5) and calculate it for $z_0 = 0$.

These functions S, D, $\mathfrak{D}$ are integral functions of the complex variable α by Poincaré's theorem; furthermore, thanks to the work done by R. E. Langer [31], one can write their asymptotic behaviour[4] for large values of α as follows:

$$(9)\qquad \left.\begin{aligned} S &\sim -a\,\frac{\Gamma(1/3)\cos(2\pi\alpha y - 5\pi/12)}{2^{1/6}\pi^{1/2}\,(2\pi\alpha)^{5/6}\mid k\mid^{1/4}} \\ D &\sim \frac{2^{1/6}\Gamma(2/3)\cos(2\pi\alpha y - \pi/12)}{a\pi^{1/2}\,(2\pi\alpha)^{5/6}\mid k\mid^{1/4}} \end{aligned}\right\} \quad \begin{aligned} &z < 0,\ dy = -\mid k\mid^{1/2} dz \\ &y(0) = 0 \\ &\mid \arg\alpha\mid \le \frac{\pi}{2} \\ &a = \lim_{z\to 0}\frac{\int_0^z \mid k\mid^{1/2} dz}{\mid k\mid^{1/4}} \end{aligned}$$

$$\left.\begin{aligned} S &\sim \frac{a\Gamma(1/3)}{2^{7/6}\pi^{1/2}}\,\frac{e^{2\pi\alpha s} + e^{-2\pi\alpha s + 5i\pi/6}}{(2\pi\alpha)^{5/6}k^{1/4}} \\ D &\sim \frac{\Gamma(2/3)}{2^{5/6}a\pi^{1/2}}\,\frac{e^{2\pi\alpha s} + e^{-2\pi\alpha s + i\pi/6}}{(2\pi\alpha)^{1/6}k^{1/4}} \end{aligned}\right\} \quad \begin{aligned} &z > 0,\ ds = k^{1/2} dz \\ &s(0) = 0,\ 0 \le \arg\alpha \le \pi. \end{aligned}$$

From this we have

$$(10)\qquad \begin{aligned} \mathfrak{D}(z, Z) &\sim \frac{\sinh 2\pi\alpha(s - S)}{2\pi\alpha(k(z)k(Z))^{1/4}} && z > 0,\ Z > 0, \\ \mathfrak{D}(z, Z) &\sim -\frac{\sin 2\pi\alpha(y - Y)}{2\pi\alpha\mid k(z)k(Z)\mid^{1/4}} && z < 0,\ Z < 0, \\ \mathfrak{D}(z, Z) &\sim -\frac{\cos 2\pi\alpha y \sinh 2\pi\alpha S + \cosh 2\pi\alpha S \sin 2\pi\alpha y}{2^{1/2}\pi\alpha\mid k(z)k(Z)\mid^{1/4}} && z < 0,\ Z > 0; \end{aligned}$$

these latter formulas are valid whatever the argument of α.

[4] The notation $f(\alpha, z) \sim g(\alpha, z)$ means: $f(\alpha, z) = g(\alpha, z)\,[1 + \epsilon(\alpha, z)/\alpha^p]$, where ϵ is uniformly bounded for large $\mid \alpha \mid$ and p is positive.

We now state the precise definition of an elementary solution at the point $P(x_0, z_0)$: this is a solution of the equation

$$\mathfrak{L}(u) = \delta_{(x_0, z_0)}, \tag{11}$$

where $\delta_{(x_0, z_0)}$ is Dirac's measure [29] for the x, z-plane, localized at the point $x = x_0$, $z = z_0$. Hereafter, unless the contrary is explicitly stated, we shall always assume $x_0 = 0$ without restriction of generality. One can write the Fourier transform of equation (11) if one knows that of the distribution $\delta_{(x_0, z_0)}$. Now, for every function $v(\alpha, z)$ infinitely differentiable and decreasing rapidly we have [30]

$$\begin{aligned}\mathfrak{F}(\delta_{(0, z_0)})v(\alpha, z) &= \delta_{(0, z_0)} \int_{-\infty}^{\infty} e^{-2i\pi\alpha x} v(\alpha, z)\, d\alpha \\ &= \int_{-\infty}^{\infty} v(\alpha, z_0)\, d\alpha = \Delta_{(\alpha, z_0)} v(\alpha, z_0),\end{aligned}$$

if $\Delta_{(\alpha, z_0)}$ is equal, in the α, z-plane, to the measure generated by the mass $+1$ placed at each point on the straight line $z = z_0$. On every straight line $\alpha = \text{const.}$ it is equal to Dirac's measure δ_{z_0} at the point $z = z_0$ on the z-axis; by taking α as a parameter, the transform of (11) can be written in the form

$$l(U) = \delta_{z_0}. \tag{12}$$

Thus the quest for elementary solutions of (11) becomes a search for solutions of (12). But this latter problem is a classical one [32]. It will be enough to effect the transformation $\mathfrak{F}^{-1}$, if possible, and to interpret the results.

It is known that a solution of (12), $E(\alpha, z)$, can be obtained from two solutions of (5), $E_1(\alpha, z)$, $E_2(\alpha, z)$, by setting

$$E(\alpha, z) = \begin{cases} E_1(\alpha, z) & \text{if} \quad z \geq z_0 \\ E_2(\alpha, z) & \text{if} \quad z \leq z_0 \end{cases} \tag{13}$$

if the following conditions are satisfied:

$$E_1(\alpha, z_0) = E_2(\alpha, z_0); \qquad \frac{\partial}{\partial z} E_1(\alpha_1, z_0) - \frac{\partial}{\partial z} E_2(\alpha, z_0) = 1. \tag{14}$$

We give at once a first simple application which will lead us to define the "Riemann function" of equation (1) in the hyperbolic domain.

The relationships (8) show that it is possible to take

$$E_1(\alpha, z) = 0, \qquad E_2(\alpha, z) = \mathfrak{D}(z, z_0),$$

but the function $e = \mathfrak{F}^{-1}(E)$ is defined only if $E(\alpha, z)$ is a temperate distribution of the variable α; this happens if and only if $z_0 \leq 0$. Let us consider this case.

Because of the Paley-Wiener theorem and the asymptotic behavior (10) of $\mathfrak{D}(z, z_0)$, e is identically zero outside the region bounded by the characteristics of P directed towards decreasing z; on these characteristics the function e is

equal to $-2^{-1} \mid k(z)k(z_0) \mid^{-1/4}$ and therefore in the region where it is not identically zero, it is proportional to the Riemann function for the point P.

Similarly one may set

$$E_1 = \mathfrak{D}(z, z_0), \qquad E_2 = 0;$$

the function E is a temperate distribution only if $z_0 < 0$, $z \leq 0$. The corresponding function $e = \mathfrak{F}^{-1}(E)$ is zero in the part of the strip $z_0 < z \leq 0$ outside the triangle PAB. (A and B are the intersections of characteristics from P with the x-axis.) On the arcs PA and PB, e takes the values $2^{-1} \mid k(z)\, k(z_0) \mid^{-1/4}$, and it is therefore proportional to the Riemann function for the point P defined in this triangle. Thus a Cauchy problem can be solved without difficulty with Cauchy data prescribed on a segment AB of the x-axis ($-x_0 \leq x \leq x_0$); direct calculation shows that if $T(\alpha)$ and $N(\alpha)$ are Fourier transforms of the given data

$$t(x) = u(x, 0), \qquad n(x) = \frac{\partial u}{\partial x}(x, 0), \qquad -x_0 \leq x \leq x_0$$

then the solution of the Cauchy problem within the triangle PAB is given by

$$\mathfrak{F}^{-1}(T(\alpha)\mathfrak{D}(\alpha, z) + N(\alpha)S(\alpha, z)).$$

Since these are classic problems in the theory of hyperbolic equations I do not stress them further.

V. Elementary Solutions in the Strip

There are obviously many ways of choosing the functions $E_1(\alpha, z)$ and $E_2(\alpha, z)$, solutions of (5) which satisfy conditions (14). It is essential for us to know the nature of these functions with respect to the complex variable α.

Since we know that the function $\mathfrak{D}(z, Z)$ is an integral function of α it is natural to try to obtain pairs E_1 , E_2 which can be expressed by means of the function $\mathfrak{D}$ alone; now we shall see that this is possible if we seek a function $e(x, z; 0, z_0)$ defined in the strip $z_2 \leq z \leq z_1$, e satisfying simple conditions along the straight line boundaries $z = z_1$; $z = z_2$. Later on, to be specific, we shall study the elementary solutions of (4) defined in this strip which vanish at the boundary.

The functions E_1 and E_2 must therefore be respectively proportional to $\mathfrak{D}(z, z_1)$ and $\mathfrak{D}(z, z_2)$, the factors of proportionality being independent of z.

Now, if one takes account of the identity

$$\mathfrak{D}(z, Z_2)\frac{\partial}{\partial z}\mathfrak{D}(z, Z_1) - \mathfrak{D}(z, Z_1)\frac{\partial}{\partial z}\mathfrak{D}(z, Z_2) = \mathfrak{D}(Z_1 , Z_2),$$

which can be established easily by noting that the first part is independent of z (since it is the Wronskian of two solutions of (5)) it is clear that we must assume

$$E_1 = \frac{\mathfrak{D}(z_0 , z_2)\mathfrak{D}(z, z_1)}{\mathfrak{D}(z_1 , z_2)}, \qquad E_2 = \frac{\mathfrak{D}(z_0 , z_1)\mathfrak{D}(z_0 , z_2)}{\mathfrak{D}(z_1 , z_2)}.$$

These functions are meromorphic functions of α and it is necessary to determine their poles. If α_0 is one pole, that is to say a zero of $\mathfrak{D}(z_1, z_2)$, then there exists a solution of (5) which vanishes for $z = z_1$ and $z = z_2$ for this value α_0 of the parameter α; in other words, α_0 is an eigenvalue for the problem which consists in defining a solution of (2) which takes on given values for $z = z_1$ and $z = z_2$. This is a classic problem [33] and we merely state the result: All the zeros of $\mathfrak{D}(z_1, z_2)$ are simple; they constitute a denumerable sequence of real numbers if $z_2 < z_1 \leq 0$, of pure imaginary numbers if $0 \leq z_2 < z_1$ and finally, a double sequence of real and pure imaginary numbers if $z_2 < 0 < z_1$. This result brings out three general cases which we shall examine; our discussion will depend on whether the strip belongs to a hyperbolic half-plane, an elliptic half-plane, or a mixed domain. We begin with the case:

(A) $0 \leq z_2 \leq z_1$ (*strip in the elliptic domain*).

Because of the preceding results $E(\alpha, z)$ is a continuous function on the real α-axis. For every z different from z_0, $e(x, z; 0, z_0)$ can be defined by the integral

$$e(x, z; 0, z_0) = \int_{-\infty}^{\infty} E(\alpha, z) e^{2i\pi\alpha x}\, d\alpha \tag{15}$$

because from (10) the modulus of the integrand behaves like the product of a bounded quantity multiplied by the exponential

$$\exp\{-2\pi \mid \alpha \mid (s - s_0)\}.$$

For $z = z_0$, e has a singularity at $P(0, z_0)$. But in the strip[5] $z_0 \leq z \leq z_1$, e is the sum of a continuous function and of

$$\mathfrak{F}^{-1}\left(\frac{\sinh(2\pi\alpha(s_0 - s_2))\sinh(2\pi\alpha(s - s_1))}{2\pi\alpha \sinh(2\pi\alpha(s_1 - s_2))}\right),$$

(which represents the solution of an analogous problem where (4) is a Laplace equation $k(z) = 1$); the singularity at P of this function is the same as that of

$$-\frac{1}{2\pi}\frac{1}{[k(z)k(z_0)]^{1/4}}\int_c^{\infty} e^{-2\pi\alpha(s-s_0)} \cos 2\pi\alpha x\, d\alpha, \qquad c > 0$$

and an easy calculation shows that finally e is the sum of a continuous function and

$$\frac{1}{2\pi \mid k(z_0) \mid^{1/2}} \log\sqrt{(s - s_0)^2 + x^2}. \tag{16}$$

This solution is of course even in x.

Finally let us note that instead of taking the integral on the real α-axis it can be taken on a straight line $\mathcal{I}m(\alpha) = c$ parallel to this axis without changing the final result, provided c is so small that $\mathfrak{D}(z_1, z_2)$ has no zeros in the strip $0 \leq \mathcal{I}m(\alpha) \leq c$ $(c > 0)$.

[5] A similar reasoning can be applied to the case: $z_2 \leq z \leq z_0$.

(B) $z_2 < 0 < z_1$ (*strip in the mixed domain*).

If z_2 is negative, $E(\alpha, z)$ has poles on the real α-axis and the integral (15) loses all meaning. Now we can use either integrals in the complex α-plane (this is what we shall do) or the theory of distributions (particularly division [29]).

Let us call L_1 the path $\mathcal{I}m(\alpha) = c$ $(c > 0)$ introduced earlier and let us define $e(x, z; 0, z_0)$ by the integral

$$e(x, z; 0, z_0) = \int_{L_1} E(\alpha, z) e^{2i\pi\alpha x}\, d\alpha. \tag{17}$$

If z_0 is positive and if z is different from z_0 the integrand decreases as an exponential and then $e(x, z; 0, z_0)$ is a continuous and even an analytic function at all points of the strip, except at the point P in the neighbourhood of which it behaves like the sum of a continuous function and of (16). If z_0 is negative the same argument shows that e is a continuous function for positive z but in the case of negative z a deeper study is needed. For this purpose we shall assume $z_0 \leq z < 0$ (the case $z_2 \leq z \leq z_0$ may be studied in the same way but we shall merely give the results). From (10) we see that the asymptotic behaviour of $E_1(\alpha, z)$ is

$$E_1 \sim \frac{e^{2i\pi\alpha(y_0 - y)}}{4i\pi\alpha \mid k(z_0)k(z) \mid^{1/4}} \tag{18}$$

in the region $\epsilon \leq \arg \alpha \leq -\epsilon + \pi$ $(\epsilon > 0)$ if for large values of $\mid \alpha \mid$ a denumerable sequence of circles with small radii centred at points $\beta_n = i(3/8s_1 + n/2s_1)$ is excluded. Each of these values is in fact the limit for large $\mid \alpha \mid$ of a zero of $\mathfrak{D}(z_1, z_2)$ and of a zero of $\mathfrak{D}(z, z_1)$. On the other hand we have

$$E_1 \sim \frac{\sin(2\pi\alpha(y_0 - y_2))\cos(2\pi\alpha y - \pi/4)}{2\pi\alpha \mid k(z)k(z_0) \mid^{1/4} \cos(2\pi\alpha y_2 - \pi/4)} = \tilde{E}_1 \tag{19}$$

in the region $-\pi/2 + \delta \leq \arg \alpha \leq \pi/2 + \delta$ $(\delta > 0)$ if for large values of $\mid \alpha \mid$ a denumerable sequence of circles of small radii centred at points $\alpha_n = (3/8y_2) + (n/2y_2)$ is excluded.

(a) *The function* $e(x, z; 0, z_0)$ *is continuous for* $x > y - y_0$. In fact, the integrals along L_1 and $\mathcal{L}_1$ (see Figure 5) are equal because of (19) since $x + y_0 - y$ is positive. Now the integral taken along $\mathcal{L}_1$ is a continuous function of x and z.

(b) *Singularities* of $e(x, z; 0, z_0)$ *in the region* $x < 0$. To study the behaviour of e in the neighbourhood of $x = y - y_0$ and for $0 > z > z_0$ we introduce the paths c_1 and c_2 (Figure 5), parts of L_1 and L_2 situated in the half-plane $\mathcal{R}e(\alpha) > 0$ and we set

$$\begin{aligned} \int_{c_1} E_1 e^{2i\pi\alpha x}\, d\alpha + \int_{c_2} E_1 e^{-2i\pi\alpha x}\, d\alpha &= \{E_1\} \\ \int_{c_1} E_1 e^{-2i\pi\alpha x}\, d\alpha + \int_{c_2} E_1 e^{2i\pi\alpha x}\, d\alpha &= [E_1]. \end{aligned} \tag{20}$$

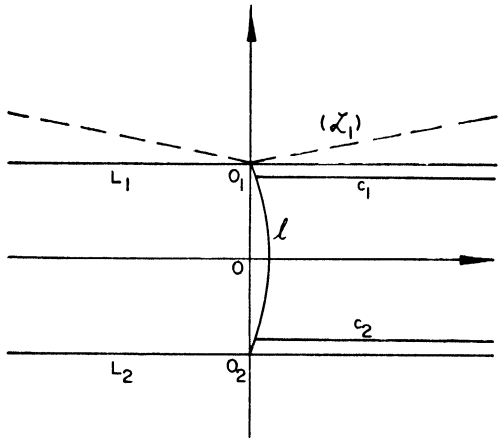


FIGURE 5

Because of the symmetry properties of E in the α-plane we have

$$e(x, z; 0, z_0) = \{E_1\}, \qquad e(-x, z; 0, z_0) = [E_1], \qquad z_0 < z < 0$$

Finally the symbols $\{\tilde{E}_1\}$ and $[\tilde{E}_1]$ are defined in the same way by replacing E_1 by $\tilde{E}_1$ in formulas (20).

To begin with, $\{E_1\} - \{\tilde{E}_1\}$ is a continuous function in the whole strip since by (19), the integrand with which this expression is calculated is uniformly bounded for large $|\alpha|$ by a power $|\alpha|^{-1+\rho}$ (ρ positive); on the other hand, if l is a path in $\mathcal{R}e(\alpha) > 0$ joining the points O_1 and O_2, every pole of $\tilde{E}_1$ in $\mathcal{R}e(\alpha) > 0$ being on the left of l, then by Cauchy's theorem we have

$$\{\tilde{E}_1\} = [E_1] - 2i\pi \sum_{\substack{\mathcal{I}m(\alpha)=0 \\ \mathcal{R}e(\alpha)>0}} \text{residues of } (2i\tilde{E}_1 \sin 2\pi\alpha x) + \int_l 2i \sin 2\pi\alpha x \tilde{E}_1 \, d\alpha.$$

Now, from the argument in (a), $[E_1]$ is a continuous function of x and z in the region studied; the same holds for the integral taken on l. Hence the singularities of $[\tilde{E}_1]$ and therefore those of e for $x < 0$, $z_0 < z < 0$ are those of the sum of the series.

$$2 \sum_{\substack{\mathcal{I}m(\alpha)=0 \\ \mathcal{R}e(\alpha)>0}} \text{residues of } \frac{\sin (2\pi\alpha(y_0 - y_2)) \cos (2\pi\alpha y - \pi/4) \sin 2\pi\alpha x}{\alpha \, |\, k(z)k(z_0) \,|^{1/4} \cos (2\pi\alpha y_2 - \pi/4)}.$$

Thus the study is reduced to an elementary calculation; after transforming the product of the trigonometric expressions in the numerator into a sum of cosines, we add series of the form

$$(21) \qquad \sum_{\substack{\mathcal{I}m(\alpha)=0 \\ \mathcal{R}e(\alpha)>0}} \text{residues of } \frac{\cos (2\pi\alpha X - \eta\, \pi/4)}{\alpha \cos (2\pi\alpha y_2 - \pi/4)}, \qquad \eta = \pm 1.$$

This sum is continuous for each value of X different from

$$X = (2N + 1)y_2 .$$

In the neighborhood $X = (2N + 1)y_2 + \epsilon y_2$ of such a value, the sum of the series (21) behaves like

$$-\frac{1}{\pi}\cos\left(3\left(N + \frac{1+\eta}{2}\right)\pi\right)\log|\epsilon| - \frac{\eta}{2}(\text{sign }\epsilon)\sin\left(\frac{3}{2}\left(N + \frac{\eta+1}{2}\right)\pi\right)$$

as can be seen if we use Euler's formulas and note that

$$\sum_{p=1}^{\infty}\frac{\zeta^p}{a+p} + \log(1-\zeta), \qquad a > 0, \tag{22}$$

is continuous in the domain $|\zeta| \leq 1$. All these results as well as those which

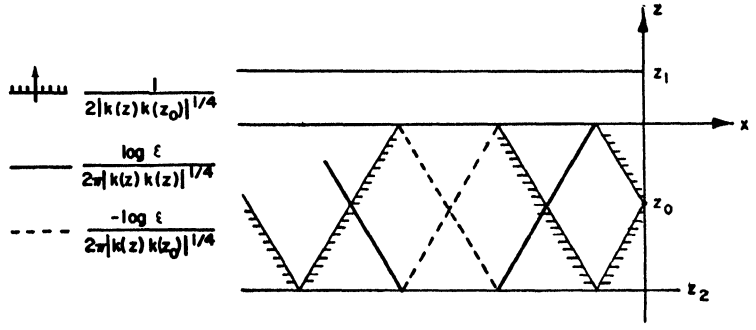


FIGURE 6

can be obtained by the same method when $z_2 < z < z_0$ are shown in Figure 6. In order not to interrupt the discussion certain necessary remarks will be postponed.

It remains to examine the limiting case in which $z_0 = 0$. If we consider a point P in the hyperbolic half-plane which tends towards the x-axis, we see that there exists a two-by-two superposition of the singular characteristics which we encountered earlier. A direct study can be made by noting that $\mathfrak{D}(z_0, z_2)$ and $\mathfrak{D}(z_0, z_1)$ are then reduced to $-S(z_2)$ and $-S(z_1)$; behaviours (9) must be taken into account and it can be seen that, in the strip $z_2 < z < 0$, the singularities of $e(x, z; 0, z_0)$ are those of the series

$$-\sum_{\substack{\Im m(\alpha)=0\\ \Re e(\alpha)>0}} 4\pi \text{ residues of } \frac{\Gamma(1/3)\sin(2\pi\alpha(y-y_2)\sin(2\pi\alpha x)}{\pi^{1/2}3^{1/6}(2\pi\alpha)^{5/6}\,|k(z)|^{1/4}\cos(2\pi\alpha y_2 - \pi/4)}.$$

This series can be studied as mentioned above provided one uses the continuity of

$$\sum_{p=1}^{\infty}\frac{\zeta^p}{p^s} - \frac{\Gamma(1-s)}{(1-\zeta)^{1-s}}, \qquad 0 < s < 1$$

in the domain $|\zeta| \leq 1$. The results of the calculation are given in Figure 7.

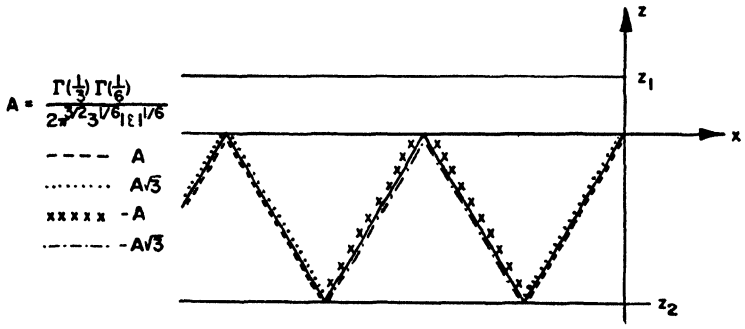


FIGURE 7

(C) $z_2 < z_1 < 0$ (*strip in the hyperbolic domain*).
The asymptotic behaviour of $E_1(\alpha, z)$ is given by

$$E_1 \sim -\frac{\sin(2\pi\alpha(y_0 - y_2))\sin(2\pi\alpha(y - y_1))}{2\pi\alpha \mid k(z)k(z_0) \mid^{1/4} \sin(2\pi\alpha(y_1 - y_2))} = \tilde{E}_1 .$$

If e is defined by formula (17), it can be seen that for $z_0 < z < z_1$

(a) the function e is identically zero for $x > y - y_0$ (to see this it is enough to apply Cauchy's theorem to the integrand in the domain of the half-plane $\mathcal{I}m(\alpha) > 0$ bounded by L_1 and a half-circle with very large radius and center at the origin),

(b) by using the argument developed in (B), the singularities of the solution e in the region $x < 0$ are shown to be those of the series

$$-2 \sum_{\substack{\mathcal{I}m(\alpha)=0 \\ \mathcal{R}e(\alpha)>0}} \text{residues of} \frac{\sin(2\pi\alpha(y_0 - y_2))\sin(2\pi(y - y_1))\sin 2\pi\alpha x}{\alpha \mid k(z_0)k(z) \mid^{1/4} \sin(2\pi\alpha(y_1 - y_2))}.$$

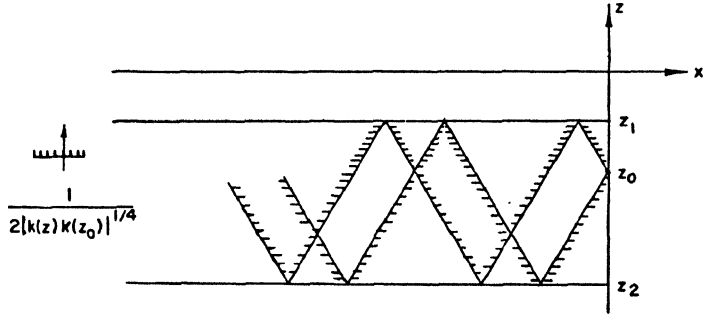


FIGURE 8

An easy calculation leads to the result shown in Figure 8 (the case $z_2 < z < z_0$ can be studied in the same way).

(D) *To complete the discussion we need only consider the case in which* $z_2 < 0$, $z_1 = 0$.

From the formulas (9) and (10) we deduce the behaviour

$$E_1 \sim \frac{\sin(2\pi\alpha(y_0 - y_2)) \mid \cos(2\pi\alpha y - 5\pi/12)}{2\pi\alpha \mid k(z_0)k(z) \mid^{1/4} \cos(2\pi\alpha y_2 - 5\pi/12)}$$

valid in the region $-\pi/2 + \delta \leq \arg \alpha \leq \pi/2 - \delta$, $(\delta > 0)$ if, for large values of $\mid \alpha \mid$, a sequence of circles of small radii centered at points $\alpha_n = 11/24y_2 + n/2y_2$ is excluded. On the other hand we have

$$E_1 \sim \frac{\exp\{2i\pi\alpha(y_0 - y)\}}{4i\pi \mid k(z_0)k(z) \mid^{1/4}}$$

in the region $\epsilon \leq \arg \alpha \leq \pi - \epsilon$ $(\epsilon > 0)$. With the help of these results it is possible as in (B), to determine the behaviour of $e(x, z; 0, z_0)$ (defined by (17)) when $z_0 < z < 0$. We find again that e is zero for $x > y - y_0$, and that the singularities of this solution are those shown in Figure 9.

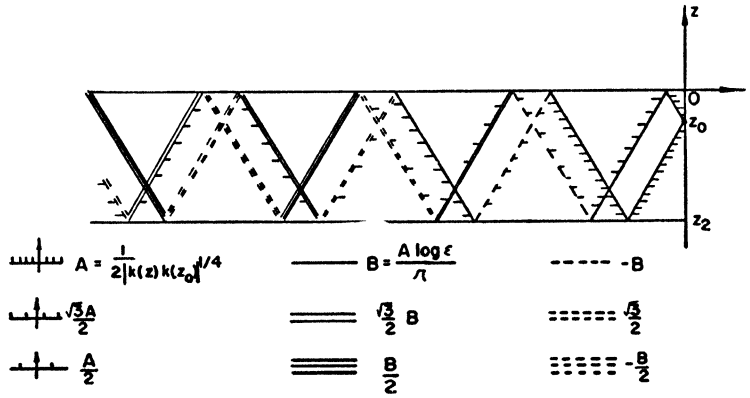


FIGURE 9

VI. Remarks on the Preceding Results

It is perhaps of interest to emphasize some aspects of the results obtained.

1. *Indeterminacy of the Elementary Solution in the Mixed and Hyperbolic Domains; "Directed Solutions"*

To define the elementary solution we have chosen as path of integration the straight line L_1 $(\Im m(\alpha) = c)$. We could just as easily have integrated along the path L_2 $(\Im m(\alpha) = -c)$. In case (A), we saw that the result was not changed; but for $z_2 < 0$, a new elementary solution, symmetric to the preceding one with respect to the z-axis is obtained. Hence for $z_2 < 0$, an indeterminacy appears which results from the existence of real poles for $E(\alpha, z)$. In fact, to each real

zero α_n of $\mathfrak{D}(z_1, z_2)$ there corresponds a real solution of (4) which vanishes along the straight lines $z = z_1$ and $z = z_2$:

$$[\mathfrak{D}(z, z_2)]_{\alpha=\alpha_n} \cos 2\pi\alpha_n(x - x_n). \tag{23}$$

Now, in a general way one can obtain an elementary solution by integrating along a path which lies along the real axis, leaving certain real poles on one side and other poles on the other, or even surrounding certain of these poles several times. All the solutions thus obtained differ from one another by a finite or infinite linear combination of expressions proportional to (23) with coefficients independent of z. Nevertheless, among all these solutions, it is easy to characterize the one defined in the previous paragraph: it is the only one which, for very large positive values of x, decreases like e^{-x} (this can be seen from the previous calculations); in particular when z_1 is negative or zero, it is the only one which is identically zero for $x > 0$. For these reasons we can say that this solution is "directed" towards decreasing x. If this supplementary condition of "direction" is introduced the indeterminacy mentioned previously disappears. In short, as soon as the domain in question contains a hyperbolic part, we have to introduce such a condition of "direction"; this is in agreement with the most classic notions for the elementary solutions of hyperbolic equations.

2. *Remarks on the Singularities of Elementary Solutions*

As one would expect, results concerning types of singularities are a generalization of those already obtained in the simple case of Tricomi [28].

These types were characterized in the elliptic case (A) by one logarithmic singularity at a point P, in the hyperbolic case (C) by discontinuities propagated towards decreasing x which are proportional to $| k(z) |^{-1/4}$, i.e. to the values taken by the Riemann function of a point Q along a characteristic through Q. Case (B) is the intermediate case since, along the x-axis, reflections of discontinuities (whose intensity increases indefinitely in the neighbourhood of $x = 0$) occur with logarithmic singularities, the coefficient of the logarithm being itself proportional to $| k(z) |^{-1/4}$. This kind of reflection, as Lighthill [34] has pointed out, enters into the interpretation of a first approximation of the reflection phenomenon of a shock wave on a boundary layer.

Note also the singular character of the particular case $z_0 = 0$ for which singularities in $\epsilon^{-1/6}$ are obtained; the coefficients of this infinitely large quantity have different values on each side of the singular characteristic. Finally, the case $z_1 = 0$, obtained when one of the boundaries of the strip coincides with the parabolic line (x-axis), gives rise to a configuration of reflections of a fairly complex kind.

3. *Other Conditions on the Boundary*

We have defined these elementary solutions by imposing the condition that they vanish along $z = z_1$ and $z = z_2$. It goes without saying that one could as

well have required these solutions to have a vanishing normal derivative along one of these straight lines, or to satisfy even more complicated conditions. If, for example, we impose on $z = z_2$ the conditions

$$E(\alpha, z_1) = 0, \qquad 2i\pi\alpha \mid k(z_2) \mid E(\alpha, z_2) + \frac{\partial}{\partial z} E(\alpha, z_2) = 0, \qquad z_2 < 0,$$

the singularities of the corresponding function e (defined by integration along the path L_1) do not reflect on the boundary $z = z_2$; (this is, of course, not true for the singularities of the derivatives).

4. *Case in which z_1 or z_2 Increases Indefinitely*

By restricting our study to a strip we have taken advantage of two facts; the interval being finite, we could always suppose on the one hand, that $k(z)$ did not show any singularity for the boundary values and on the other hand we knew exactly the analytical nature of the function $E(\alpha, z)$. Now, in the simple case of Tricomi's equation the elementary solutions, of which the explicit expression has been given, were defined over the whole x,z-plane; this has many advantages in certain cases.

The determination of such solutions[6] in the case of equation (1) requires a knowledge of solutions of (2) having in the α-plane a well determined asymptotic behaviour for large values of $\mid \alpha \mid$ ("subdominant" solutions according to an expression used by Langer [31]). His work proves the existence of such solutions in a quarter of the α-plane and for large values of $\mid \alpha \mid$. This domain may be extended to a half-plane, but we do not know if it is possible to have more information on the nature of these solutions as functions of α (we know, for example, that they are not uniform in the α-plane) without specifying the function $k(z)$.

Apart from the simple case of Tricomi, in which equation (5) can be reduced to a differential equation defining the cylindrical functions of order 1/3, one can also make a complete study of the case mentioned by Tomotika and Tamada[7] in which $k(z) = 1 - e^{-2z}$. Equation (5) admits as solutions the functions $J_{2\pi\alpha}(2\pi\alpha t)$, and $J_{-2\pi\alpha}(2\pi\alpha t)$ of the variable $t = e^{-z}$; the "subdominants" in the neighbourhood of the half-axes $\arg \alpha = 0$, $\arg \alpha = \pi/2$, $\arg \alpha = -\pi/2$ are, respectively,

$$J_{2\pi\alpha}(2\pi\alpha t), \qquad H^{(1)}_{2\pi\alpha}(2\pi\alpha t), \qquad H^{(2)}_{2\pi\alpha}(2\pi\alpha t)$$

the analytical nature of which is well known in terms of α.

In such cases, one can not only form the elementary solutions of equation

[6]We suppose $k(z)$ defined for all values of the variable z. The case of equation (4) is different for a perfect polytropic gas since $k(z)$ becomes infinite for the value $z = -\sigma_m$. The study of the behavior of these solutions along such a singular straight line does not seem to be essentially different from that in which z increases indefinitely, $k(z)$ remaining bounded.

[7]This function is one of those to which Fenain's theorem applies.

4), but also all solutions which have a prescribed type of singularity—for example, (T), (P_1), (P_2),—solutions generalizing those which Bader and the author called the "Darboux general solutions" in the case of Tricomi's equation. This is obvious from the Fourier transforms: the existence of solutions of (4) which satisfy certain conditions is equivalent to the existence of solutions of (2) having a certain behavior in α.

VII. Applications to Certain Boundary Value Problems

1. *Elementary Solutions and Green's Formula*

The classical use of elementary solutions for the study of boundary value problems depends on the application of Green's formula, which is written as

$$(24)\qquad \iint_D (u\mathfrak{L}(v) - v\mathfrak{L}(u))\,dx\,dz = \int_C u(k(z)v_x\,dz - v_z\,dx) - v(k(z)u_x\,dz - u_z\,dx),$$

to a solution u of (4) with bounded second derivatives and to a solution e with a singularity at a point P of the closed domain D with boundary C.

In a purely elliptic problem, if P is inside D, then by excluding from D a circle of small radius with center at P, one shows that the application of Green's formula to the remainder of the domain makes it possible to express the value of $u(P)$ in terms of the values of u and its first derivatives along C. In the case of a purely hyperbolic problem, the same result can be obtained by applying, if necessary, more complicated techniques, using for example, "Hadamard's finite part" or "Bureau's logarithmic part." But the theory of distributions permits these results to be described in a general way. In the present case one can thus show that, whatever the position of P in the domain D, $u(P)$ may be evaluated in terms of the values of u and its first derivatives along C by means of (24). In other words, the singularities of the function e which we have determined, "leave no trace" in the result obtained by applying Green's formula—since, as a distribution, the solution e satisfies equation (4) in all regions of the x, z-plane which do not contain the point P. Indeed, let us define the distribution T as equal to u in D and equal to 0 outside of D. As a distribution, T satisfies the equation

$$(25)\qquad \mathfrak{L}(T) = A,$$

when A is a distribution with support on the curve C. A makes the value

$$(26)\qquad A\cdot f = \int_C u(k(z)f_x\,dz - f_z\,dx) - f(k(z)u_x\,dz - u_z\,dx)$$

correspond to any function $f(x, z)$ which is continuously differentiable and zero outside a compact set. Since e satisfies

$$\mathfrak{L}(e) = \delta_P$$

where δ_P denotes the measure of Dirac at the point $P(x_0 , z_0)$ we have

$$T_P = \delta_P \cdot T = \mathcal{L}(e) \cdot T = e \cdot A = A \cdot e.$$

This formula which is rigorously valid when T is such that A is a function, remains true in the present case where A is a distribution and the last member has a meaning. One can describe this result without referring to the theory of distributions; let us indicate rapidly the principles of this demonstration (for further details, see [28]).

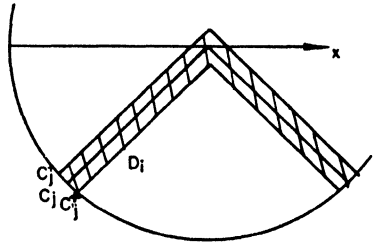


FIGURE 10

For this purpose (Figure 10) we exclude from the domain D strips of infinitely small width which include the characteristics along which the function e shows singularities and write Green's formula for each of the domains D_i thus determined. Let C_j^+ and C_j^- be two neighbouring boundaries of adjacent domains separated by a singular characteristic C_j . The result depends upon the following remarks:

If along C_j , e admits a discontinuity which is proportional to the value of the Riemann function, then the classical properties of this function show that in the limit, the difference in the values of the integrals along C_j^+ and C_j^- is independent of the values of u along this arc.

If along C_j , e admits a logarithmic singularity, then from the above remark the same conclusion follows because the coefficient of the logarithm remains proportional to $| k(z) |^{-1/4}$.

If along C_j , e admits a singularity in $\epsilon^{-1/6}$, the same result is obtained with the help of "Hadamard's finite part."

2. *Some Boundary Value Problems Related to Hyperbolic Domains*

In this section, we suppose that $z_1 \leq 0$ and we shall show that the elementary solutions defined previously permit us to solve certain problems even when certain data are given on the straight line $z_1 = 0$.

According to the preceding section, the following problem can be solved by the simple application of Green's formula: find the solution u of (1) with given values along $z = z_1$ and $z = z_2$ defined in that part of the strip $z_2 \leq z \leq z_1$ which is situated on the right of an arc AB having at each point a space-like direction (if the x-axis is taken as time-like axis) and having given values of u and its

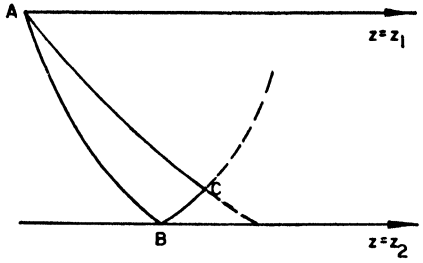


FIGURE 11

first derivatives along AB (see Figure 11). It will be noted that at a point P of the triangle ABC, the solution is that of a Cauchy problem obtained by Riemann's method because, in this triangle, the value of $e(M; P)$ is that of the Riemann function of P. The method can be compared to the completely analogous method used by Hadamard [35]; he defines the function e in each quadrilateral where it is regular whereas the theory of distributions gives an overall definition of this function. Moreover, Hadamard noted the difficulty encountered in the case in which $z_1 = 0$, a difficulty which is removed here.

It might be interesting to give explicitly the Fourier transform of the distribution equal to u on the right of AB, and equal to zero on the left; for simplicity, let us limit ourselves to the case in which u is zero on $z = z_1$ and $z = z_2$ and AB is a segment of the z-axis. The transform $U(\alpha, z)$ is the solution[8] of

$$U_{zz} - 4\pi^2\alpha^2 k(z)U = 2i\pi\alpha k(z)u(0, z) + k(z)u_z(0, z) = F(\alpha, z) \tag{27}$$

which vanishes for $z = z_1$ and $z = z_2$; then it follows that

$$U = \int_{z_2}^{z_1} F(\alpha, t)\frac{D(z, z_1)D(t, z_2)\,dt}{D(z_1\ , z_2)} + \int_{z}^{z_1} F(\alpha, t)\frac{D(t, z_1)D(z, z_2)\,dt}{D(z_1\ , z_2)}. \tag{28}$$

One also obtains a simple result when AB is an arc of the characteristic line $x = y$ (see Figure 12); the solution is the inverse transform of expression (28) if one assumes

$$F(\alpha, z) = \left(f_1(z)\frac{d}{dz} \mid k(z) \mid^{1/2} + 2 \mid k(z) \mid^{1/2}\frac{df_1}{dz}\right)e^{-2i\pi\alpha y}$$

---

[8] In order to obtain the left side, we have to find the Fourier transform of the distribution A, defined by (26), C being here the curve $AB(x = x(z))$. This distribution makes

$$\int_{-\infty}^{+\infty} \psi(z)\,dz[\phi(x)(u_x(k - x_z^2) - 2u_z x_z - ux_{zz}) - u\phi_x(k + x_z^2)]$$

correspond to any function $f = \phi(x)\,\psi(z)$, the functions $\phi(x)$, $\psi(z)$ being zero outside a closed interval. On each straight line z = constant, the expression between brackets defines the value $S \uparrow \phi$, where S is a distribution of x. For the transform of S we have

$$\mathfrak{F}\cdot(A) = (2i\pi\alpha u(k + x_z^2) + (k(z) - x_z^2)u_x - 2u_z x_z - ux_{zz})e^{-2i\pi\alpha x(z)}.$$

where $f_1(z)$ denotes the values of u given on the characteristic. Finally, we will consider another particular case, that in which u is zero on the characteristic line AB (see Figure 12) and on $z = z_2$ and the values of u are known on the

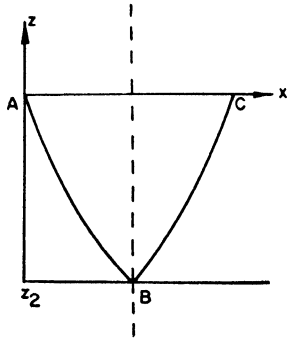


FIGURE 12

half-line Ax ($z_1 = 0$). In the triangle ABC, the values of the solution are independent of z_2 ; the solution is the inverse transform of

$$\frac{T(\alpha)\,\mathfrak{D}(z_2\,,\,z)}{S(z_2)} \tag{29}$$

if

$$t(x) = \mathfrak{F}^{-1}(T(\alpha))$$

is the expression of the values given along Ax; again u is calculated by integration along L_2 .

3. *Green's Function of Tricomi's Problem*

Tricomi's problem can be considered as a typical problem for equations of mixed type; given the values of the unknown function u along the arc (Γ) drawn in the elliptical half-plane and along the characteristic AC, we have to determine this function in the domain Δ (see Figure 13).

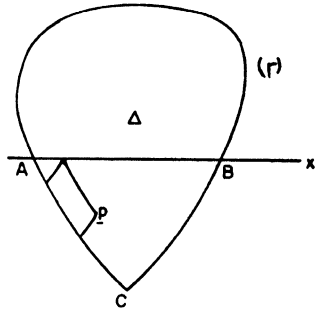


FIGURE 13

The existence and the uniqueness of the solution of this problem have been generally established only in the case in which $k(z) = z^m$, (see [2], [24], [28]); recently Protter [36] has established the uniqueness of the solution in more general cases than those treated by Frankl [19]. We shall now make some remarks which arise immediately from the preceding results. We shall call the problem which corresponds to data on (Γ) and on the characteristics BC the "conjugate problem of Tricomi." We shall say that the function $G(M; P)$ depending on points $M(x, z)$ and $P(x_0, z_0)$ is a Green's function of Tricomi's problem in Δ, if for any P in Δ

(G) (1) $G(M; P)$ *is equal to* 0 *when* M *is on* (Γ) *or on* BC,
(2) $G(M; P) - e(M; P)$ *is a continuous solution in* Δ *of equation* (1).

$e(M; P)$ denotes an elementary solution directed towards decreasing x; one can take for example the previous solution, and choose z_1 and z_2 in such a way that the domain Δ lies entirely within the strip $z_2 < z < z_1$.

The construction of Green's function $G(M; P)$ can be reduced to solving a conjugate problem of Tricomi for each position of the point P. Thus we know how to form a Green's function of the Tricomi problem provided the existence of the solution of the conjugate problem is established.

On the other hand, if one applies Green's formula to the function $G(M; P)$ and to a solution u of (4) in Δ, it can be seen that $u(P)$ is directly expressed in terms of the values of u on (Γ) and on AC. We can now state:

The existence of the solution of the conjugate problem implies the uniqueness of the solution of the Tricomi problem.

Let us now further suppose that the conjugate problem also admits a Green's function which we will denote by $\overline{G}(M; P)$. The application of Green's formula shows that these functions $G(M; P)$ and $\overline{G}(M; P)$ are connected by the symmetrical relation

$$G(M; P) = \overline{G}(P; M)$$

and hence that for fixed M, $G(M; P)$, considered as a function of the coordinates x_0, z_0 of P, is also a solution of equation (4). We also see that, if M is fixed within Δ and if P tends towards a point of (Γ) or of AC, the function tends towards zero. Conversely, if M is a point of AC, and if P also tends towards a point of AC, $G(M; P)$ tends towards $-2^{-1} \mid k(M)\, k(P) \mid^{-1/4}$ or towards 0 according to whether the limiting position of P belongs to the arc AM or to the arc MC. Let us denote by $g(M)$ the values of u on (Γ) and by $f(M)$ those of u on AC; then the function[9]

$$u(P) = \int_{(\Gamma)} g(kG_x\, dz - G_z\, dx) + \int_{AC} \mid k \mid^{1/2} (f\, dG - G\, df)$$

$$= \int_{(\Gamma)} g(kG_x\, dz - G_z\, dx) + \int_{AC} f(2 \mid k \mid^{1/2} dG + Gd \mid k^{1/2} \mid)$$

[9]The integral along AC is a Stieltjes integral; its principal value must also be taken.

is a solution of (4) in Δ; assuming G to have the properties mentioned above, we can verify that $u(P)$ tends towards $g(M_0)$ when P tends towards a point M_0 of(Γ) (thanks to the classic argument for elliptic problems), and that $u(P)$ tends towards $f(M_0)$ if P tends towards a point M_0 of AC since

$$\lim_{P \to M_0} \int_{AC} f(2 \mid k \mid^{1/2} dG + Gd \mid k^{1/2} \mid = f(M_0).$$

It is obvious that the function is the unique solution of the Tricomi problem thus set.

These remarks throw some light on the way in which the solution of a Tricomi problem is expressed linearly as a function of given data.

REMARK: Bader and the author have tried to express the solution of the Tricomi problem directly without first solving another problem as has just been done.

They have shown that a "semi-function" of Green $H(M; P)$ satisfying the conditions (G) can be constructed provided we replace (1) by the condition: $H(M; P)$ vanishes only when M is on (Γ).

The application of Green's formula to a solution u of (1) and to H permits one to express $u(P)$ as a function of data $g(M)$ on (Γ), $f(M)$ on AB, and of the unknown values $h(M)$ on BC; letting the point P tend towards a point of BC, we can write an integral equation of the Fredholm type which is satisfied by $h(M)$ and then derive the value from the data if this solution exists. It then remains to show that the function thus obtained is really a solution and is unique. It seems useless to dwell on this point since this attempt has been unsuccessful so far.

To establish the existence and uniqueness of the solution of the Tricomi problem, it seems necessary to use results about elementary solutions not stated above. In particular, the problem which seems to us the most urgent would be to seek "a priori" bounds of solutions which would allow one to establish the existence of the solution by using theorems of compactness. This method is classic to-day in dealing with elliptic problems. But recent work[10] of J. Leray also shows the importance of this method in the study of hyperbolic problems; we also think that for problems of the mixed type we have here a method of study which has not yet been sufficiently exploited. The work of L. Bers [37] and of Protter [38] who, using this method, have studied hyperbolic problems in which certain data are carried by the parabolic line, seems to be very important.

BIBLIOGRAPHY

[1] Crocco, L., *Transformations of the hodograph flow equation and the introduction of two generalized potential flows*, National Advisory Committee for Aeronautics, Technical Note No. 2432, August 1951.

[2] Tricomi, F., *Sulle equazioni lineari alle derivate parziali di 2° ordine di tipo misto*, Reale accademia dei Lincei, Memoria, Classe di scienze fisiche, mathematiche e naturali, Series 5, Volume 14, 1923, pp. 133–247.

[10] Lectures delivered at the Collège de France, 1953.

[3] Falkovitch, S. V., *Sur la théorie de la tuyère de Laval*, Prikladnaia matematika i mekhanika, Tome X, No. 4, 1946, p. 503. *Une classe de tuyères de Laval*, ibid., Tome XI, No. 2, 1947, p. 230.

[4] Lighthill, M. J., *The hodograph transformation in trans-sonic flow, Part I, Symmetrical channels*, Proceedings of the Royal Society, Series A, Volume 191, 1947, pp. 323–341. *Part III, Flow round a body*, ibid., pp. 352–369.

[5] Cherry, T. M., *Exact solutions for flow of a perfect gas in a two-dimensional Laval nozzle*, Proceedings of the Royal Society, Series A, Volume 203, 1950, pp. 551–571.

[6] Fenain, M., *Recherches d'approximations des équations régissant les écoulements des gaz. Applications aux tuyères planes*, La Recherche Aéronautique, No. 33, 1953, p. 11.

[7] Cabannes, H., *Écoulements transsoniques—ondes de choc*, Dissertation, Paris, 1950.

[8] Martin, M. M., and Thickston, W. R., *An example of transonic flow for the Tricomi gas*, Proceedings of the fourth Symposium in Applied Mathematics of the American Mathematical Society (to be published).

[9] Germain, P., *Sur l'approximation homographique dans l'étude des fluides compressibles*, La Recherche Aéronautique, No. 25, 1952, pp. 9–18.

[10] Tierney, J. A., *An approximation to transonic flow of a polytropic gas*, American Journal of Mathematics, Volume 75, 1953, p. 43.

[11] Tamada, K., *On the hodograph method and analytic continuation of solution in the theory of compressible fluid flow*, Memoirs of the College of Science of the University of Kyoto, Series A, Volume 26, Nos. 1 and 2, 1950, pp. 21–30.

[12] Cherry, T. M., *Flow of a compressible fluid about a cylinder. Part I*, Proceedings of the Royal Society, Series A, Volume 192, 1947, pp. 45–49. *Part II*, ibid., Volume 196, 1949, pp. 1–31.

[13] Tomotika, S., and Tamada, K., *Studies on two-dimensional transonic flows of compressible fluid, Part I*, Quarterly of Applied Mathematics, Volume 7, 1950, p. 381. *Part II*, ibid., Volume 8, 1950, p. 127. *Part III*, ibid., Volume 9, 1951, p. 129.

[14] Gilles, A., *Calcul d'une tuyère et d'un profil dans l'approximation homographique*, La Recherche Aéronautique, No. 32, 1953, p. 3.

[15] Liger, M., *Nouvelles équations approchées pour l'étude des écoulements subsoniques et transsoniques*, Office Nationale d'Étude et de Recherche Aéronautique, Pub. No. 64, 1953.

[16] Cole, J. D., *Drag of a finite wedge at high subsonic speeds*, Journal of Mathematics and Physics, Volume 30, 1951, pp. 79–93.

[17] Frankl, F., *An investigation of the theory of a wing of infinite span moving at the speed of sound*, Akademiia Nauk S. S. S. R., Doklady, Volume 57, No. 7, pp. 661–664.

[18] Guderley, G., and Yoshihara, H., *The flow over a wedge at Mach number one*, Journal of the Aeronautical Sciences, Volume 17, 1950, p. 723.

[19] Frankl, F., *On the problem of Chaplygin for mixed sub and supersonic flows*, Bulletin de l'académie des sciences U. S. S. R., Volume 9, 1945, p. 121.

[20] Vincenti, W. G., and Wagoner, C. B., *Transonic flow past a wedge with detached bow wave*, National Advisory Committee for Aeronautics, Technical Note 2339, 1951.

[21] Bers, L., *Partial differential equations and generalized analytic functions*, Proceedings of the National Academy of Sciences, Volume 36, 1950, pp. 190–196.

[22] Germain, P., *Introduction à l'étude mathématique des écoulements transsoniques*, La Recherche Aéronautique, No. 22, 1951, pp. 7–20.

[23] Guderley, G., *On the transition from a transonic potential flow with shocks*, G. S.—A. A. F., Wright Field, No. 13, 1947.

[24] Gellerstedt, S., *Sur un problème aux limites pour une équation aux dérivées partielles du second ordre de type mixte*, Dissertation, Uppsala, 1935.

[25] Weinstein, A., *Discontinuous integrals and generalized potential theory*, Transactions, American Mathematical Society, Volume 63, 1948, pp. 342–354.

[26] Carrier, G. F., and Ehlers, F. E., *On some singular solutions of the Tricomi equation*, Quarterly of Applied Mathematics, Volume 6, 1948, p. 331.

[27] Cole, J. D., *Note on the fundamental solution of* $wy_{vv} + y_{ww} = 0$, Zeitschrift für angewandte Mathematik und Physik, Volume 3, 1952, pp. 286–297.

[28] Germain, P., and Bader, R., *Sur quelques problèmes relatifs à l'équation de type mixte de Tricomi*, Office nationale d'étude et de recherche aéronautique, No. 54, 1952.

[29] Schwartz, L., *Théorie des distributions*, Volume 1, Hermann, Paris, 1950.

[30] Schwartz, L., *Théorie des distributions*, Volume 2, Hermann, Paris, 1951.

[31] Langer, R. E., *The asymptotic solutions of ordinary linear differential equations of the second order, with special reference to a turning point*, Transactions, American Mathematical Society, Volume 67, 1949, pp. 461–490.

[32] Courant, R., and Hilbert, D., *Methods of Mathematical Physics*, Interscience, New York-London, Vol. I, 1953.

[33] Goursat, E., *Cours d'analyse mathématique*, Volume III, Gauthier-Villars, Paris, 1942.

[34] Lighthill, M. J., *Reflection at a laminar boundary layer of a weak steady disturbance to a supersonic stream, neglecting viscosity and heat conduction*, Quarterly Journal of Mechanics and Applied Mathematics, Volume 3, 1950, pp. 303–325.

[35] Hadamard, J., *Sur un problème mixte aux dérivées partielles*, Bulletin de la Société Mathématique de France, Volume 31, 1903, pp. 208–224.

[36] Protter, M. H., *Uniqueness theorems for the Tricomi problem*, Journal of Rational Mechanics and Analysis, Volume 2, No. 1, 1953, pp. 107–114.

[37] Bers, L., *On the continuation of a potential gas flow across the sonic line*, National Advisory Committee for Aeronautics, Technical Note 2058, 1950.

[38] Protter, M. H., *A boundary value problem for an equation of mixed type*, Transactions, American Mathematical Society, Volume 71, 1951, pp. 416–429.

[39] Germain, P., and Bader, R., *Solutions élémentaires de certaines équations aux dérivées partielles du type mixte*, Bulletin de la Societe Mathématique de France, Volume 81, 1953, p. 145.

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. VII, 145–148 (1954)

Discussion on Transonic Flow

By R. von MISES*
Harvard University

Many and diversified questions can be raised and have been raised in connection with transonic or mixed flows, i.e., flow patterns including both subsonic and supersonic regions. Not many substantial answers could be found so far. The present discussion will be narrowed to one specific problem which seems fundamental from a theoretical point of view, namely: A steady uniform flow is disturbed by the presence of a simple (convex) body (Figure 1). An everywhere

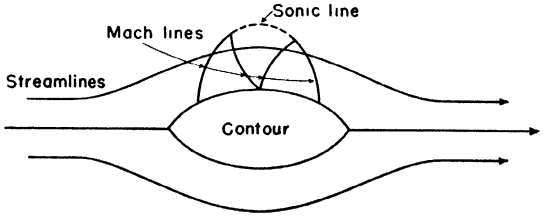


FIGURE 1

supersonic solution is excluded due to the presence of stagnation points. Provided the undisturbed velocity (afterwards the velocity at infinity) is subsonic, we ask whether a flow with a supersonic enclosure, as shown in the figure, exists under certain specifications.† For the sake of simplicity our computations will refer to two-dimensional motion only; in this case the enclosure will consist of two separate regions.

1. *Definition of the Problem*

In all discussions implying supersonic flow regions the term "shock" or "shock wave" emerges very soon. This is suggested by actual observations. Our present discussion, however, has to leave completely aside that notion. We try to investigate theoretical flow patterns that satisfy the conditions of motion of a fluid free of viscosity and heat conduction throughout. Shock flows are asymptotic or limiting solutions of equations in which viscosity and heat conduction are not both neglected; they correspond to the assumption that viscosity and/or heat conduction are effective on isolated surfaces only (lines, in the two-dimensional case). The exclusion of shocks does not imply

*To the great loss of the scientific community Professor von Mises died on July 14, 1953, before this article could appear.

that we restrict ourselves to continuous velocity distributions. As we know, the differential equations of the non-viscous and not-heat conducting fluid are, in a supersonic region, of hyperbolic character. Thus, certain types of discontinuities may occur without violating the equations.

We do suppose that the fluid is elastic (barotropic); i.e., at any two points with the same pressure value p also the density has the same value ρ. The one-to-one correspondence between p and ρ, which we call the p,ρ-relation, can without loss of generality, be considered as the condition that the entropy of a particle does not change through the motion and that its constant value is the same for all particles. (Neither of these conditions is fulfilled in shock flows).

In our problem, sketched in Figure 1, the vorticity is zero at infinity since here the flow is uniform and rectilinear. It follows from the Helmholtz-Kelvin vortex theory that the flow is vortexfree throughout. In a two-dimensional, steady, vortexfree motion the only characteristics along which discontinuities can occur are the Mach lines or lines crossing the streamlines under an angle α which depends on the local Mach number only. Across a Mach line, p, ρ and the velocity vector remain continuous, but the first order derivatives of these variables in the direction normal to the Mach lines may jump. In other words any two different analytic solutions with the same p, ρ, $\bar{q}$ along a Mach line can be patched together. We shall call a solution in which such jumps occur "non-analytic." If condensation of characteristics is excluded, no other type of non-analyticity can occur in the inner of a flow field. Thus, our problem can be formulated in the following way.

We ask for the existence or non-existence of an analytic or non-analytic solution of the differential equations of a two-dimensional, steady flow of an elastic, non-viscous and not-heat-conducting fluid, fulfilling the two boundary conditions: *a*) at infinity $\bar{q} = \overline{V} = \text{const.}$ (implying irrotationality); *b*) at the given contour $\bar{q}$ is tangent to it.

It has already been said that V will be supposed subsonic. Note also that discontinuities are restricted to the supersonic enclosures.

2. *The Equations*

Since a one-to-one relation between p and ρ is supposed to hold, we need three more equations for $\bar{q}$, p, ρ. We introduce the length q of $\bar{q}$ and the angle ϑ of q with the direction of the x-axis, furthermore the derivatives $\partial/\partial s$ and $\partial/\partial n$ in the direction of $\bar{q}$ and normal to it:

$$(1) \qquad \frac{\partial}{\partial s} = \cos\vartheta\,\frac{\partial}{\partial x} + \sin\vartheta\,\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial n} = -\sin\vartheta\,\frac{\partial}{\partial x} + \cos\vartheta\,\frac{\partial}{\partial y}.$$

Then

$$(2) \qquad \frac{\partial q}{\partial n} = q\,\frac{\partial\vartheta}{\partial s}, \qquad (M^2 - 1)\,\frac{\partial q}{\partial s} = q\,\frac{\partial\vartheta}{\partial n}.$$

Here, M^2 is the Mach number squared

$$M^2 = q^2 \frac{d\rho}{dp} \tag{3}$$

and q^2 is related to p, ρ by the Bernoulli equation

$$\frac{q^2}{2} + \int^p \frac{dp}{\rho} = \text{const.} \tag{4}$$

From (3) and (4) the variables p, ρ can be eliminated and thus M^2 computed as a function of q^2. This function monotonically increasing from the value zero at $q^2 = 0$ to ∞ at $q^2 = q_m^2$ (velocity maximum) where p and ρ vanish with $dp/d\rho = 0$. If M^2 is in this way expressed in terms of q^2, the two equations (2) control the variables q, ϑ while p, ρ can afterwards be found from equation (4).

In a supersonic region, $M > 1$, one sets

$$M = \frac{1}{\sin \alpha}, \qquad M^2 - 1 = \cot^2 \alpha, \qquad 0 \leq \alpha \leq \frac{\pi}{2} \tag{5}$$

and it is found that $\vartheta + \alpha$ and $\vartheta - \alpha$ are the two characteristic directions.

3. *The Local Approach*

Many efforts have been made to prove the non-existence of a solution with supersonic pockets, by studying the geometry of stream lines and Mach lines in the neighborhood. We call this the "local approach." This argument is based on two facts: (a) Along a Mach line pressure, density, velocity change monotonically. This is borne out by the fact that the hodograph transform of a Mach line is an epicycloid extending from the sonic circle to the circle of radius q_m. (b) If $\partial/\partial s_+$ and $\partial/\partial s_-$ signify differentiations in the directions $\vartheta + \alpha$, $\vartheta - \alpha$ respectively equations (2) supply immediately

$$\frac{1}{q \sin \alpha} \frac{\partial q}{\partial s_+} = k + \frac{1}{q} \frac{\partial q}{\partial s} \cot \alpha, \tag{6}$$

$$\frac{1}{q \sin \alpha} \frac{\partial q}{\partial s_-} = -k + \frac{1}{q} \frac{\partial q}{\partial s} \cot \alpha, \tag{7}$$

where k is the curvature of the streamline at the point of consideration. On two Mach lines emanating from one point of the contour the ds_+, ds_- towards the sonic line have different signs, q must be on both decreasing (from a supersonic to a sonic value). Therefore, the two expressions to the left in (6) and (7) must have opposite signs. It follows that any solution requires

$$|k| > 0, \qquad \frac{1}{q} \left| \frac{\partial q}{\partial s} \right| \cot \alpha < |k|. \tag{8}$$

The popular conclusions are: No solution of the type considered exists, if any small part of the contour is straight ($k = 0$) or even if an infinitesimal element of an otherwise convex contour is flattened. At least in the infinitesimal neighborhood of any contour for which a solution exists there must be sets of contours without solution.

These conclusions do not strike me as convincing. Take the simple problem of a Laplace potential. No solution is possible along a contour that has a corner, even an infinitesimal one; again one can say that in the infinitesimal neighborhood of any admissible contour there are sets of inadmissible (cornered) ones.

4. *The Complete Problem*

The complete boundary value problem that covers the question formulated at the beginning of this note was the essential subject matter of the papers of Messrs. Gelbart, Germain, and Weinstein. I would also like to refer to the detailed report presented just now by Mr. Guderley in Volume 3 of the Advances in Applied Mechanics. No definite, conclusive results can be drawn from all these most valuable sources. Two facts seem outstanding. First that exact solutions (with supersonic enclosures) exist for certain specific contours. Second, that "approximate" solutions in the usual sense can be found for "arbitrary" contours, e.g., by the numerical solution of a finite number of difference equations replacing the original differential equations. This is certainly not in disagreement with the above result of non-existence of solutions in the neighborhood of each one, since, at any rate the converse theorem is also valid: In the infinitesimal neighborhood of certain contours for which no solution exists there are sets of contours admitting solutions.

A general answer to the boundary value problem cannot be given at the present stage. At most a semblance of a general theory is available in the case of linear differential equations, which means transition to the hodograph variables. When intricate decisions about existence and non-existence have to be taken, the transition is not entirely without objection. On the other hand, in Mr. Weinstein's paper a way is indicated to use the theory of linear boundary value problems (Tricomi theory) directly in dealing with the physical plane equations. It seems that there is a strongly justified guess that, in certain circumstances, *less than the complete* boundary is sufficient to determine uniquely a solution. If this is correct it would be an indication that the problem that we took as point of departure is "incorrectly set" according to the expression of Hadamard. Whether or not this is in fact so, must, for the time being, be left undecided.

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. VII, 149–158 (1954)

Some Problems Solvable on Computing Machines

By MARK LOTKIN
Ballistic Research Laboratories, Aberdeen

1. *Introduction*

The general question of the role of computing machines is discussed elsewhere and so I shall not venture to review this very important question.

However, it is proper at least to emphasize a few special features of the impact of modern computing machines upon our general thinking. It is of course obvious that it is now possible to tackle problems whose size would otherwise be completely staggering. The mere existence of these facilities has tended to guide our research into new and appropriate directions. We are now explicitly concerned with questions of stability and convergence of difference procedures. We realize the importance of practical investigations into the effects of small disturbances. We face immediate and interesting problems related to the logical flow of information.

Indeed many problems each of which would have been called "spectacular" only a few years ago have already been attacked by modern large scale computing machines. This is true of weather prediction by the numerical integration of the equations of motion of the atmosphere. As a consequence, practical meteorology may soon acquire a new level of competence.

In remarking on the various problems from a computational point of view it is convenient to classify them roughly with regard to their magnitude. For example, we have first what might be called "small" problems. These, in some cases at least, are appropriately handled by hand computers. Such a problem may involve only a few thousand operations. The computation of the particle trajectory of a projectile is of this sort. An experienced computer should not require more than 2 to 3 days on one of these. But there are problems a thousand times as extensive, problems which I speak of as "medium" sized. These would seriously strain the facilities of a computing section operating only with desk computers. But we have been and shall continue to be faced in the future with such problems. During the war there were hundreds of computers working three shifts on problems of this type. It was exactly the need for more efficient methods that forced the Ordnance Department to consider more powerful computing tools. Now that we have such tools we can also, for the first time, deal effectively with problems that are of "large" size. A problem of this category may require billions of operations. Unsteady three-dimensional flows,

problems in atomic structure, and many others of real engineering interest are of this type.

The efficient and economic prosecution of computational problems of these magnitudes clearly demands suitably designed machines It is easy to list some criteria most of which a modern computer must satisfy in order to be capable of handling such problems satisfactorily. Let us explain first some of the terms which will be used in this connection. By "automatic sequencing" is meant the property of carrying out internally all the commands, decisions, and operations involved in the solution of the entire problem. By "primary" memory is meant the capacity to have the requisite information available at internal operational speeds. It is to be distinguished from "secondary" memory, which stores further information and makes it available at possibly lower speeds.

Some of the criteria referred to above, considered essential for a modern computer, are:

a. The machine must be automatically sequenced.

b. It must have a sufficiently large high speed primary memory.

c. Generally it must possess also a much larger secondary memory of sufficiently rapid accessibility.

d. The number of digits must be adequate for most accuracy requirements.

e. Internal operational speeds must be such as to permit the completion of a problem in a reasonable length of time.

f. The machine must be equipped to receive information in a speedy and effective manner.

g. It must be possible to record rapidly not only final but also selected intermediate results.

To be sure, the first of the modern computers, the ENIAC, which was installed in 1947 at the Ballistic Research Laboratories, did not have most of these features. But they are largely possessed by the more than 30 major general purpose computers that are already in or nearing operation by government agencies, educational institutions, and industrial organizations all over the world.

2. *Typical Problems Solvable on Machines*

Let us, however, turn away from wide generalities and come to more specific matters. Some of the essential features of the modern high speed computing machines at the Ballistic Research Laboratories will be discussed next, and brief outlines will be given of some major problems which have been solved on these machines. These problems deal with matrix inversion, ballistic computations, transonic air flows, and others.

The general purpose machines at the Ballistic Research Laboratories are the ENIAC, the EDVAC, the ORDVAC, the BELL machine, the IBM Relay Calculators, and the CPC. Some of the essential characteristics of these machines are shown in Table I.

As may readily be deduced from an inspection of this table, most of these machines satisfy many of the requirements mentioned above, deemed necessary for the efficient solution of medium and large scale problems. They are presently somewhat inadequate as to size of secondary storage as well as in the operational speeds of the input and output organs. However it is hoped that the incorporation into these machines of auxiliary memories of about 10,000 words each and the attachment of improved reading and printing devices will shortly remove these defects.

With such facilities on hand it is clearly quite possible to solve a large variety of problems. An average of 40 different problems is handled each week. More than 250 types of problems arising in many fields of applied science have already been completed, during the 30,000 hours of good operating time.

Now it clearly is neither advisable nor possible to give a complete listing of all of these problems. It seems better to point out that they may be grouped conveniently into certain mathematical categories. Such major categories are

(1) Matrix inversion.
(2) Finding characteristic values and vectors.
(3) Numerical solution of ordinary differential equations.
(4) Numerical solution of partial differential equations.
(5) Computation of statistical functions.
(6) Construction of tables.
(7) Reduction of observational data.

(1) *Solution of Linear Equations and Matrix Inversion.* Many problems require the solution of systems of linear algebraic equations or the inversion of matrices. From a theoretical point of view the solution of these problems is well known. From a computational point of view however, much remains to be done. The main difficulty arises from the accumulation of the unavoidable rounding error inherent in all digital calculations. For the solution of linear algebraic systems we are presently using a Gaussian elimination routine adapted for the ORDVAC by D. J. Wheeler of the University of Illinois. It permits the solution on the ORDVAC of such systems of orders not exceeding forty. A total of about 135 orders are involved in the entire routine, composed as follows: 96 orders are required for the basic routine itself, 21 orders are needed for the input conversion of the numerical data from the binary-decimal to the binary system, 12 orders are used for the reconversion, and there are finally 6 orders making up a printing sequence. Estimated times for systems of order n are as follows: about 6 seconds to read the routine from punched cards into the machine, $n^2/2$ seconds to read in the coefficient matrix, $n^3/400$ seconds to compute the solutions, and $2n$ seconds to print out the results. Thus the solution of a system of 40 equations takes approximately fifteen minutes.

This routine has lately been extended so as to include the computation of residuals, and the improvement of the solution by successive iterations.

For the inversion of matrices we are considering a number of methods such as elimination, square-rooting, and partitioning. While these and possibly

TABLE I. ABERDEEN COMPUTERS

	ENIAC	EDVAC	ORDVAC	BELL
Type	Synchronous Decimal Parallel	Synchronous Binary Serial	Asynchronous Binary Parallel	Asynchronous Bi-Quinary Parallel
Memory				
Ext. Type	Punched cards	5 hole tape	5 hole tape	6 hole tape
Int. Type	Vacuum tubes	Acoustic delay lines	Electrostat.	Relays
Capacity	20 Accumulators	1024 Cells	1024 Cells	16 Registers
Av. Access T. (msec.)	0.1	0.20	0.035	5
Av. Oper. Speed (msec.)	+ 0.60	0.85	0.06	300
	$\bar{x}$ 3.4	3	0.75	1000
	./. 24	2.9	1	2100
Input Rate				0 if comp. time more than 3000 msec. between reads
Tape (msec./word)		20	400	
Card (msec.)	300	300 (fut.)	1000 (8 words, dec. Sy.)	
Card (msec.)			300 (24 words, bi. Sy.)	
Output Rate				
Tape (msec/word)		2000	2000	as above
Card (msec)	200 if comp. time more than 600 msec. between prints	600 (fut.)	1000	
Code				
Type (Addr.)	1	4	1	3
Number Length (Dig.)	10 Dec.	44 Bin.	40 Bin.	7 Dec.
Order Length (Dig.)	2	44	19	3-12
Number of Orders	94	12 Types	53	9
Decimal Point	Fixed	Fixed	Fixed	Floating
Checking	Test runs Differencing Coding check	Self-checking Unused orders Exc. capac. indic.	Test runs Differencing Coding check	Self-checking Orders partially Self-checking

other methods are in general equally useful there are cases where the growth of the rounding error is so rapid as to make some inversion procedures altogether unstable. It is from this point of view that the partitioning method, when suitably modified by the inclusion of an iterative technique, seems to be capable of yielding more accurate results than some of the other methods. By stopping, at any desired step, to improve the intermediate inverses until satisfactory accuracy is attained it is possible to keep the growth of the rounding error in check.

The reciprocation is accomplished by the inversion of all successive principal submatrices as follows:

Let the matrix A of order n be symmetric and positive definite, and let it be assumed that the principal submatrix A_k, $k = 1, 2, \cdots, n - 1$, has already been inverted. Then the components S_k, V_k, q_k of A_{k+1}^{-1},

$$A_{k+1}^{-1} = \begin{bmatrix} S_k & V_k \\ V_k^* & q_k \end{bmatrix},$$

where S_k is a square matrix of order k, V_k a column matrix of k elements and V_k^* its transpose may be obtained in this manner:

$$\begin{aligned} X_k &= -A_k^{-1}C_k, \\ z_k &= a_{k+1,k+1} + C_k^* X_k, \\ q_k &= z_k^{-1}, \\ V_k &= X_k q_k, \\ S_k &= A_k^{-1} + X_k V_k^*. \end{aligned}$$

In these equations the asterisk denotes transposition and C_k is the $(k + 1)$-st column of the given matrix A.

A mechanization scheme using this approach has been devised and coded for high speed computing machines. The routine involves about 300 words and requires about $n^3/40$ seconds computing time.

A detailed description and error analysis may be found in [1].

It is worth while mentioning in this connection that ORDVAC computing times will be reduced, possibly by factors as high as ten, once the electrostatic memory tubes are replaced by improved ones of greatly increased read-around-ratio.

(2) *Ballistic Computations.* One of the principal missions of the Computing Laboratory of the BRL is the preparation of firing and bombing tables for use in the field.

The motion of a projectile, under certain simplifying assumptions, satisfies a system of ordinary differential equations of the form [2]

$$\ddot{x} = -E(\dot{x} - w_x) + 2\Omega\dot{y}\cos L\sin\alpha,$$

$$\ddot{y} = -E\dot{y} - g - 2\Omega\dot{x}\cos L\sin\alpha,$$

$$\dot{z} = -E(\dot{z} - w_z) + 2\Omega(\dot{x}\sin L + \dot{y}\cos L\cos\alpha),$$

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2,$$

$$E = a(y)\cdot H(y)G(v/a(y))C^{-1},$$

$$\dot{x}(0) = v_0\cos\theta_0\,, \qquad \dot{y}(0) = v_0\sin\theta_0\,, \qquad \dot{z}(0) = 0.$$

Here $O(x,y,z)$ is a system of Cartesian coordinates whose origin O is at the location of the gun, with x denoting distance down range, y altitude and z distance across the range, increasing to the right. The quantities w_x , w_z are wind components down and cross range, respectively. Further, α is the azimuth of fire and Ω, g are angular velocity of the earth and gravitational acceleration, respectively; L is the latitude. The function $H(y)$ is the relative air density, and $a(y)$ the relative speed of sound. Finally, $G(v/a)$ and C denote drag function and ballistic coefficient. They must be determined anew for each new missile, unless it bears close resemblance to one previously considered. Such determinations of drag function and ballistic coefficient utilize firings in free flight ranges, wind tunnel measurements, aerodynamical considerations, and the computation of "reduction" trajectories in order to fit the choice of C to observational data.

The bulk of a firing table consists of a large number of trajectories computed under "normal" conditions. Differential effects due to abnormal conditions such as may be caused by small variations in atmospheric temperature, range wind, muzzle velocity, projectile weight, etc. are obtained either by the integration of the system of adjoint differential equations or by interpolation in a suitably enlarged table of normal trajectories.

The numerical integrations are carried out on the high speed computing machines by procedures such as Euler's, Heun's, Iterated Simpson's, Runge-Kutta's, and others, depending on accuracy requirements and particular circumstances.

To indicate the magnitude of the computational labor involved let it be assumed that a table is required for a gun capable of firing a projectile at a number of different velocities. If one desires 25 reduction trajectories and 22 normals for each velocity and for each separate perturbation, in the positive as well as the negative direction, then about 1250 trajectories must be obtained.

Since the ORDVAC is capable of doing a trajectory of this type in about 25 seconds, or about half the real time of flight, the entire job would take about 9 hours. On the other hand, the same firing table would require the services of 25 computers working with desk machines for about 50 days.

(3) *Symmetrical Air Flows.* Various aspects concerning the computation of rotational flows of compressible fluids about symmetrical bodies moving at supersonic speeds have been discussed in the paper of J. H. Giese [3]. Extensive

calculations of such flows that are governed by partial differential equations of hyperbolic type have been carried out on the ENIAC. Many useful and interesting results are now readily obtainable.

Another problem which has recently been attacked with computing machinery, in particular the Bell Relay Computer, deals with the flow of air over a wedge with a detached shock wave. The problem was formulated and treated by J. W. Fischbach using relaxation techniques.

The flow, having separate regions of subsonic and supersonic behavior, must be computed by methods different from those applicable to wholly supersonic flows.

The situation is depicted in Figure 1(a). There BCF represents the contour of the upper half of the wedge whose symmetry axis is ABX, AE is the detached shock, CE the sonic line and DE the characteristic emanating from the corner D of the wedge and proceeding to the sonic point E on the shock. In the q,θ-plane, Figure 1(b), where q denotes magnitude of the velocity vector and θ its direction, AE represents part of the shock polar and DC, DE are characteristics.

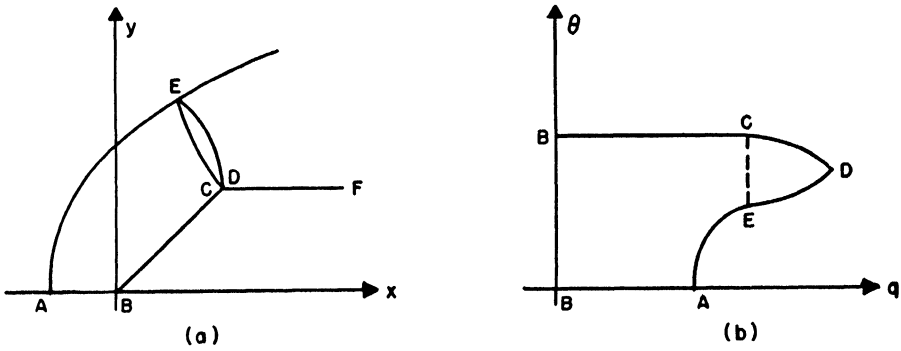


FIG. 1. Transonic flow over a finite wedge.

The pertinent equations to be solved are:

(*a*) *in the interior*

$$\psi_{qq} - (F_q/F)\psi_q - FG\psi_{\theta\theta} = 0,$$
$$Q = (1 - q^2)^{1/(\gamma-1)},$$
$$F = Q/q,$$
$$G = (M^2 - 1)/Qq,$$

(*b*) *at the shock*

$$\psi_q/\psi_\theta = F(H\mu - G)/(\mu - HF),$$
$$H = q_1 \sin\theta/Q(q_1 \cos\theta - q).$$

In these equations q, θ denote magnitude and direction of the velocity vector, $\psi(q,\theta)$ stands for the stream function, M the local Mach number, q_1 the free stream velocity, and μ the slope of the shock polar in the hodograph plane.

The nature of the main equation obviously depends on the behavior of $FG = (M^2 - 1)/q^2$; it is elliptic for $M < 1$, hyperbolic for $M > 1$. The problem is completely specified by prescribing the value of the stream function on the open boundary $EABCD$.

It is known [4] that if a solution of this problem exists at all, it must be unique.

A first attempt made at BRL at mechanical treatment of this problem showed quite promising results. A more complete mechanization, however, seems to require a larger secondary memory than has hitherto been available.

(4) *Tabulation of Cumulative Binomial Probabilities.* General purpose high speed machines obviously permit the rapid and economic evaluation and tabulation of mathematical functions. Thus a number of such functions whose knowledge was required in the course of a larger problem have been tabulated on the BRL machines.

One of these functions is

$$P(c,n,p) = \sum_{r=c}^{n} \binom{n}{r} p^r q^{n-r};$$

it is of major importance in the theory of probability as well as in its applications. This function P gives the probability of at least c successes in n trials when the chance of success in a single trial is p. It is related to the Incomplete Beta Function Ratio

$$I_p(s,t) = B_p(s,t)/B(s,t)$$

by means of

$$P(c,n,p) = I_p(c,n - c + 1).$$

Extensive tables of I_p were published in 1934 by K. Pearson [5]. However, when the prosecution of certain problems of interest to the BRL required many values of P not contained in Pearson's tables, it was decided to carry out an independent tabulation, at the same time extending the ranges of the various parameters involved.

This work was carried out on the IBM Relay Calculators of the BRL by the author and others, in collaboration with F. E. Grubbs. The basic formula employed was

$$I_p(s,t) = pI_p(s - 1, t) + qI_p(s, t - 1).$$

The ranges of the parameters are $n = 1(1)150$, $c = 0(1)n$, and $p = 0.01(0.01)0.50$; all entries are accurate to seven decimal places.

With the recent publication of these tables [6] another gap hitherto existing in this field has been closed.

(5) *Error Analysis for Integration Methods.* It has already been stated that modern computing machines may also be used in certain areas as substitutes for experiment and as tools in theoretical investigations. Thus, from the theoretical as well as practical point of view it is of importance to know the stability properties of numerical methods of integration and to have on hand procedures for estimating the growth of error in the large.

To be specific let us assume that we are concerned with the solution of the ordinary differential equation of n-th order

$$\bar{y}^{(n)} = \bar{f}_n(x, \bar{y}, \cdots, \bar{y}^{(n-1)})$$

subject to certain initial conditions. The bars above the various quantities indicate rounded values such as actually occur in numerical calculations.

The method of integration employed may be of the form

$$\bar{y}_{k+1}^{(i)} = \sum_{j=0}^{r} \bar{a}_{ij}^{(i)} \odot \bar{y}_{k-j}^{(i)} \oplus h \sum_{j=-1}^{r} \bar{a}_{i+1,j}^{(i)} \odot \bar{y}_{k-j}^{(i+1)} \oplus \cdots + h^{N-i} \sum_{j=-1}^{r} \bar{a}_{Nj}^{(i)} \odot \bar{y}_{k-j}^{(N)}$$

for $i = 0, 1, \cdots, n-1$, and

$$\bar{y}_{k+1}^{(i)} = \bar{f}_i(x_{k+1}, \bar{y}_{k+1}, \cdots, \bar{y}_{k+1}^{(i-1)})$$

for $i = n, n+1, \cdots, N$. Here $h = x_{k+1} - x_k$ is the step size of the independent variable, r a positive integer delimiting a set of points x_{k-j}, $j = 1, 2, \cdots, r$ to the left of x_k, $N \geq n$ indicates the order of the highest derivative $\bar{y}_{k-j}^{(i)}$ utilized in the integration process, and $a_{rj}^{(i)}$ denotes a set of constants characteristic of the method of integration. The circled symbols are used to represent digital operations such as pseudo-addition, pseudo-multiplication, etc., these being the mechanical counterparts of the corresponding arithmetical operations.

The stability and accuracy properties of the integration method employed are predicated upon the behavior of the errors

$$\eta_r^{(i)} = \bar{y}_r^{(i)} - y^{(i)}(x_r),$$

the latter function representing the i-th derivative of the supposedly existing exact solution evaluated at x_r. It seems that these errors are governed by the following:

PROPAGATION THEOREM: Let J, A be certain square matrices of order $(r+1)(N+1)$, I the unit matrix of the same order, and $\hat{J}$ the adjoint of J. Let, further, $\Gamma(\Lambda) = \Lambda I - \hat{J}A$, with Λ denoting a variable and

$$\Delta(\Lambda) = \det \Gamma(\Lambda), \qquad D = \det J.$$

If

$$\Omega_k^* = (\eta_k, 0, \cdots, \eta_k', 0 \cdots, \quad \cdots \eta_k^{(N)}, 0 \cdots 0)$$

is the transpose of the error vector Ω_k, and

$$c_i^* = (-T, 0, \cdots, -T_1, 0, \cdots; \phi_n, 0, \cdots \phi_N, 0 \cdots 0)$$

the transpose of a vector c_t composed of truncation errors T_μ and certain rounding errors ϕ_ν, then it may be shown that

$$\Omega_k = \sum_m [\hat{\Gamma}(\Lambda_m)\hat{J}/(D\Delta'(\Lambda_m))] \sum_{t=0}^{k-1} [\Lambda_m/D]^{k-t-1} c_t ,$$

the first summation to be extended over all distinct nonzero roots of $\Delta(\Lambda) = 0$.

The proof of this theorem will shortly be published elsewhere.

This theorem was used to advantage in obtaining estimates of the total error accumulated during the integration on machines of certain differential equations by means of a number of well-known methods.

3. *Conclusion*

Clearly it is hardly possible to make a few examples serve as an illustration of the variety and complexity of problems that have been—and may be—solved on computing machines. It is, however, safe to state that as computers become larger, faster, and cheaper, more and more problems of both theoretical and practical interest can be solved. More adequate mathematical formulations and more practical numerical methods will make possible the treatment of problems hitherto beyond mechanical reach. In fact, there are those who would intimate that we may yet live to see the day when even the creation of poetry will become a problem solvable on computing machines.

Bibliography

[1] Lotkin, M., and Remage, R., *Stability and error analysis of matrix inversion by partitioning*, Annals of Mathematical Statistics, Volume 24, 1953, pp. 428–439.

[2] Dederick, L. S., *The mathematics of exterior ballistic computations*, American Mathematical Monthly, Volume 47, 1940, pp. 628–634.

[3] Giese, J. H., *Approximate methods for computing flow fields*, Communications on Pure and Applied Mathematics, this issue (Volume 7, No. 1, 1954).

[4] Frankl, F., *On the problems of Chaplygin for mixed sub- and supersonic flows*, Translation in NACA TM 1155, 1947.

[5] Pearson, K., *Tables of the Incomplete Beta Function Ratio*, University College, London, 1934.

[6] Ordnance Corps, *Tables of the Cumulative Binomial Probabilities*, Department of Commerce, Washington, 1952.

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. VII, 159–193 (1954)

Weak Solutions of Nonlinear Hyperbolic Equations and Their Numerical Computation

By PETER D. LAX
New York University

Introduction

This paper describes a finite difference scheme for the calculation of *time dependent* one-dimensional compressible fluid flows containing *strong shocks*. This method is closely related to one proposed by J. von Neumann (see [12]) and modified more recently by him and R. D. Richtmyer (see [13]), inasmuch as the path of the shock is *not* regarded as an interior boundary.* The novel feature of the method described here is the use of the *conservation form* of the hydrodynamic equations and, to a lesser extent, the particular way of differencing the equations.

Although the method was designed to deal with hydrodynamic problems, it can be used to construct solutions of discontinuous initial value problems for any hyperbolic system of first order nonlinear conservation laws (to be defined below) in any number of space variables. The evidence for the convergence of the method is a number of calculations carried out on high speed computing machines that show every sign of convergence. Although the flows calculated so far all belong to a somewhat special class, I fully believe that the method will reproduce the most general type of flow. The question of *accuracy* of the approximate solution with a given mesh-size, specifically the detrimental effect of *contact discontinuities* on accuracy, is discussed at the end of Section 1.

In addition to the numerical evidence, I succeeded in proving the convergence of the scheme for arbitrary bounded measurable initial data, for the single conservation law

$$u_t - [\log (a + be^{-u})]_x = 0,$$

a and b being arbitrary positive constants (or even functions depending on x and t). The proof, modeled after a procedure of E. Hopf, see [8], will be published in a separate note.

For the discussion of the difference scheme presented here I found it useful to develop the theory of weak solutions of nonlinear conservation laws a little more systematically than customary. The theory and the numerical evidence supporting it is presented in Section 1. Section 2 contains some remarks, plus

**Added in proof*: See also Ludford, Polachek and Seeger, [18], who employ a linear viscosity term but one which is artificially large. The difference scheme employed in [18] is implicit and is solved by iteration. The difference equations are centered in time.

one numerical example, about mixed initial and boundary value problems for conservation laws. Section 3 discusses the manner in which the approximate solutions computed by our difference scheme approach steady state solutions. Section 4 discusses irreversibility and Section 5 the manner of dependence of weak solutions on the initial data. Section 6 describes how the finite difference scheme would be applied to problems with more space variables and in particular to the equations of compressible flow in Eulerian variables.

1. *Theory of Weak Solutions*

We start with the definition of weak solutions applicable to a certain class of nonlinear hyperbolic systems. We take the number of space variables to be one and consider hyperbolic systems of first order equations, i.e. systems of the form

$$U_t + AU_x + B = 0. \tag{1}$$

U here is a column vector of unknown functions, A and B are matrix and vector coefficients depending on x, t and U. We shall consider systems in which the x-derivatives are perfect derivatives, i.e. AU_x is equal to F_x (plus possibly a vector function of x, t, U), F being a (in general nonlinear) vector function of x, t, U. Such equations,

$$U_t + F_x + B = 0, \tag{2}$$

are called *conservation laws.*

U is called a *weak solution* of equation (2) with initial value Φ if the integral relation

$$\iint \{W_t U + W_x F - WB\}\, dx\, dt + \int W(x, 0)\Phi(x)\, dx = 0, \tag{3}$$

obtained by multiplying (1) by W on the left, integrating the resulting equation and integrating by parts, holds for every *test vector* W which has continuous first derivatives and which vanishes outside of some bounded set.

Clearly, a genuine solution is a weak solution, and conversely: a weak solution with continuous first derivatives is a genuine solution. Weak solutions need not be differentiable; if U_1 and U_2 are two genuine solutions of (2) whose domains of definition in the x,t-plane are separated by a smooth curve, the two taken together will constitute a weak solution if and only if the slope τ of the separating curve and the value of U_1 and U_2 along the curve satisfy the condition

$$\frac{1}{\tau}(U_1 - U_2) = F(U_1) - F(U_2).$$

For the conservation laws of mass, momentum and energy, this relation embodies precisely the Rankine-Hugoniot shock conditions.

These facts about weak solutions are basic:

(a) *The class of weak solutions associated with a given system of equations depends on the form in which the equations are written.*

By this we mean: Suppose that new unknowns V can be introduced as certain nonlinear functions of the old:

$$V = H(U), \tag{4}$$

so that when the differential equation (2) is rewritten in terms of V and solved for V_t, each of the resulting equations is again a conservation law

$$V_t + G_x + C = 0. \tag{2'}$$

Weak solutions of (2') can be defined the same way as before. But if U_0 is a weak solution of (2), $V_0 = H(U_0)$ is in general *not* a weak solution of (2'); in fact, it can be shown (at least in some special cases) that $V_0 = H(U_0)$ is a weak solution of (2') if and only if U_0 is a genuine solution of (2) ("genuine" here means Lipschitz continuous); this result holds of course only if the transformation (4) is genuinely nonlinear.

This noninvariance is not paradoxical. Whenever the concept of solution is generalized one has to sacrifice some properties of the original notion for the sake of saving others.

This dependence on the form is especially relevant for the equations of unsteady compressible flow which contain *four* conservation laws: conservation of mass, momentum, energy and entropy. Any one of these equations can be deduced from the other three, that is, a genuine solution of any three is necessarily a genuine solution of the fourth. But a weak solution of any three is in general, not a weak solution of the fourth. In physical problems one is looking for weak solutions of the first three equations.

(b) *Weak solutions cannot be obtained as limits of genuine solutions.*

(c) *The initial values do not in general determine a unique weak solution.*

There are many known examples of different weak solutions (even infinitely many) of the same equation with the same initial data.[1] This shows that the initial value problem for weak solutions is not a meaningful one—unless some additional principle is given which selects a unique weak solution for each initial value problem. However, if we believe that our mathematical model does describe an aspect of the physical world, then there is indeed assigned to each initial function a unique weak solution, namely the one that occurs in nature. The problem is to characterize mathematically this physically relevant solution.

---

[1]For example, the functions

$$u(x, t) = \begin{cases} 0 \text{ for } x < 0 \\ x/t \text{ for } 0 < x < t \\ 1 \text{ for } t < x \end{cases} \quad \text{and} \quad u(x, t) = \begin{cases} 0 \text{ for } 2x < t \\ 1 \text{ for } 2x > t \end{cases}$$

are both weak solution of the equation $u_t + (\frac{1}{2}u^2)_x = 0$ with initial value $\varphi(x) = 0$ for $x < 0$, $\varphi(x) = 1$ for $x > 0$.

First of all, we exclude all solutions where entropy of a particle has been decreased. It is not clear, however, whether this insures the uniqueness of the solution of the initial value problem especially if there are several space variables but even in the case of one space variable.** Some additional principle is needed to pick out a unique solution, such as:

(a) The weak solutions occurring in nature are limits of viscous flows.

(b) The weak solutions occurring in nature must be stable.

It is commonly believed that (a) does characterize uniquely the solutions occurring in nature. But whether the same is true of postulate (b) is seriously doubted by some.

We shall describe now in some detail the viscosity method, the notion of stability and the relation of the two.

Enlarge equation (2) by the additional term λU_{xx} on the right, obtaining a nonlinear parabolic system

$$U_t + F_x + B = \lambda U_{xx} . \tag{5}$$

The initial value problem $U(x, 0) = \Phi(x)$ can be solved for a fairly wide class of initial vectors Φ; it is commonly believed that the solution exists for a range of t which is independent of λ, and that if the initial vector is kept fixed and λ taken smaller and smaller, the corresponding solutions $U_\lambda(x, t)$ converge boundedly, almost everywhere in the strip $0 \leq t \leq T$ to a limit $U(x, t)$. This has been demonstrated so far only for a single equation,[2] $u_t + uu_x = 0$, by E. Hopf*** (see [8]), admitting all *bounded measurable* functions as initial values. For the hydrodynamic case only the convergence of *steady state* solutions[3] of the viscous flow equations to steady state weak solutions of the equations of ideal flow have been investigated.[4]

Granting the validity of the conjectures about the parabolic equation it is an easy matter to show, just as Hopf has shown for the equation considered by him, that the limit $U(x, t)$ of $U_\lambda(x, t)$ is a weak solution of equation (2). This follows from the fact that U, being a genuine solution of (5) is a weak solution of (5) as well; i.e. if we multiply (5) by any twice differentiable test vector W and integrate by parts we obtain the integral relation

---

***Added in proof*: Recently, Germain and Bader, see [16], have found an analogue of the entropy condition for the equation $u_t + (u^2/2)_x = 0$ and have succeeded in showing that this and the jump condition characterizes a unique solution of any initial value problem.

[2]The nonlinear parabolic equation $u_t + uu_x = \lambda u_{xx}$ was first considered by Burgers, see [8].

***See also J. H. Cole, [17].

[3]A steady state solution is one that depends only on a particular linear combination of x and t.

[4]See Becker, [1], Thomas [14], Gilbarg [6], Grad [7], and Courant and Friedrichs [2], pp. 134-138.

$$(6) \qquad -\iint \{W_t U_\lambda + W_x F(U_\lambda) - WB\}\, dx\, dt - \int W(x, 0)\Phi(x)\, dx = \lambda \iint W_{xx} U_\lambda\, dx\, dt.$$

Keep Φ and W fixed and let λ tend to zero. The left side of (6) approaches the left side of (3) and the right side of (6) tends to zero. This proves that U satisfies (3) for all twice differentiable test vectors W (and therefore à fortiore for all once differentiable ones too).

Observe that it was crucial for this argument that U be a *strong* limit of the sequence U_λ, i.e. that $\int\int \mid U_\lambda - U \mid$ over any bounded set of the x,t-plane tend to zero. For if U_λ converges to U in the *weak* sense only, the sequence $F(U_\lambda)$ will converge in the weak sense but *not* to $F(U)$. This phenomenon can be expressed concisely:

A weak limit of weak solutions is not a weak solution unless it is also a strong limit.

A precise statement and proof of this, in the case of a single conservation law, is given in Section 5.

There are a number of different ways of introducing a viscosity term; the way equation (5) does it is perhaps the simplest, although when applied to the equations of compressible flow it does not exactly correspond to the action of viscosity or heat conduction. All these methods are expected to produce the same weak solution in the limit. The aim of this paper is to describe a different type of limiting process, a finite difference scheme and to show—by mathematical reasoning, plausibility arguments and numerical evidence—that this scheme furnishes the experimentally observed flows.

The difference scheme is as follows:

Replace the space derivatives F_x by the *symmetric* difference quotients $[F(U(x + \Delta x, t)) - F(U(x - \Delta x, t))]/2\Delta x$, the time derivative U_t by the *forward* difference quotient $[U(x, t + \Delta t) - \overline{U}(x, t)]/\Delta t$, where $\overline{U}(x, t)$ is an abbreviation for the average[5] of U at $(x + \Delta x, t)$ and $(x - \Delta x, t)$. If $U(x,0)$ is known, we can determine $U(x, t)$ for all values of t which are integer multiples of Δt; in particular the value of $U(x, 0)$ at the lattice point $x = m\Delta x$, $m = 0, \pm 2, \cdots$ determines $U(x, t)$ at all points of the staggered lattice $t = n\Delta t$, $n = 0, 1, 2 \cdots$; $x = m\Delta x$, $m + (-1)^n = 0, \pm 2, \pm 4, \cdots$.

Denote by Δ a particular choice of mesh width, $\Delta = (\Delta t, \Delta x)$ and by U_Δ the corresponding solution of the finite difference scheme with initial value Φ. $U_\Delta(x, t)$ is defined for all values of t which are integer multiples of Δt, and for sake of convenience we might as well put $U_\Delta(x, t)$ equal to $U_\Delta(x, \nu\Delta t)$ for $\nu\Delta t \leq t < (\nu + 1)\Delta t$.

I conjecture that if Δt and Δx tend to zero so that the classical Courant-Friedrichs-Lewy stability criterion is observed; i.e., the domain of dependence of

[5] If we replace $\overline{U}(x, t)$ by $U(x, t)$, the resulting finite difference scheme is unconditionally unstable.

a point with respect to the differential equation always stays within the domain of dependence of this point with respect to the difference equation, then the functions $U_\Delta(x, t)$ remain uniformly bounded and converge almost everywhere to a limit $U(x, t)$ for a wide class of initial vectors. Should this conjecture be true, it is an easy matter to show that the limit $U(x, t)$ is a weak solution of the original system of equations. The proof goes the same way as in the viscosity method, summation by parts replacing integration by parts. And, just as in that case, it is crucial for the argument that U be a *strong* limit of U_Δ ; if U were a weak limit only, it would in general not be a weak solution.

Very likely this conjecture holds only for systems of conservation laws which, in addition to being hyperbolic, satisfy some additional condition, possibly in the large, see e.g. Weyl's paper [15].

I succeeded in proving this conjecture for the single equation

$$u_t - [\log(a + be^{-u})]_x = 0.$$

Details of the proof will be published in a separate note; I would like to mention however that in the course of proving the convergence, I obtained a fairly explicit formula relating $u(x, t)$ to its initial values $\varphi(x)$. This formula is strikingly similar to the formula Hopf obtained for the solution of the equation $u_t + (u^2/2)_x = 0$ by the linear viscosity method and suggests this

CONJECTURE: Both the linear viscosity method and the finite difference scheme described above, when applied to any single homogeneous first order conservation law

$$u_t + [f(u)]_x = 0, \qquad f'' < 0,$$

and arbitrary bounded measurable initial data $u(x, 0) = \varphi(x)$, converge to the same limit $u(x, t)$ given by the explicit formula

$$u(x, t) = g\left(\frac{x - y_0}{t}\right)$$

where $y_0 = y_0(x, t)$ is that value of y which maximizes

$$\int_0^y \varphi(s)\, ds + tG\left(\frac{x - y}{t}\right).$$

The function $g(s)$ is defined as the inverse of $f'(u) = df(u)/du$, i.e. $f'(g(s)) = s$; $G(s)$ is defined as the integral of $g(s)$: $dG/ds = g$. The maximum problem defining y_0 has a unique solution for almost all x and t so that $u(x, t)$ is well defined for almost all x and t. A similar result holds if "f" is positive.

Next we turn to the principle of stability formulated as a

CONJECTURE: *Among all functions $U = S(\Phi)$, which assign to each vector Φ a weak solution U with initial value Φ there exists one which is continuous in some reasonable topology and it is the only one which is continuous in any reasonable topology.*[6]

[6]It is understood that whenever an initial value problem has a genuine solution, $S(\Phi)$ has to coincide with it and that the semigroup property is satisfied by the solutions furnished by S.

This conjecture is a bit vague since it does not define reasonable topology, nor does it specify the domain of initial vectors Φ. Maybe further investigations will indicate what the proper choice for these undefined objects is.

The first part of the conjecture, asserting the existence of a continuous assignment, is the classical principle that in a physical problem the solution must depend continuously on the data.

That there is only *one* way of assigning $U = S(\Phi)$ continuously was proposed merely as a possible explanation of why various types of apparently different limiting procedures, such as the various viscosity methods and finite difference schemes, pick out the same weak solution. Because of the systematic nature of these procedures one would expect that the solutions picked out by each of them do depend continuously on the initial data. (This can be proved rigorously in simple cases; see Section 5.) If, therefore, there is only one way of assigning a weak solution continuously to each initial vector, it would follow that these various limiting procedures do lead to the same weak solution. On the other hand, the reason why these various limiting procedures lead to the same result could easily be some type of dissipative mechanism common to all of them. In this connection it should be pointed out that if the parabolic equation (5) is approximated by the standard finite difference scheme (centered space derivatives, forward time derivative) and if Δx, Δt and λ are let to approach zero *simultaneously*, keeping $\Delta t/\Delta x$ constant and λ equal to $(\Delta x)^2/2\Delta t$, we arrive at the finite difference scheme proposed in this paper. The von Neumann-Richtmyer method is explicitly based on such a simultaneous carrying out of two limiting procedures.

We present now a partial list of experimental calculations carried out for the equation $u_t + (u^2/2)_x = 0$, $u_t + (u^3/3)_x = 0$, and the hydrodynamic equations both in Eulerian and Lagrangean coordinates.

In the first group of calculations the initial data were picked to have a constant value Φ_l for x negative and another constant value Φ_r for x positive. This choice of the initial data leads to homogeneous problems; i.e., the solution depends only on x/t. This has the advantage that advancing in time in the finite difference scheme has the same effect as refining the mesh. The computational plan was to keep on grinding out time cycles until it becomes evident whether the method converges, diverges to infinity or oscillates. All calculations performed so far converged and quite rapidly at that. Unless stated otherwise, the difference scheme used was the one described before.

(1) *Equation*: $u_t + (u^2/2)_x = 0$.

Initial function: $\varphi_l = 1, \quad \varphi_r = 0$.

$$\Delta t/\Delta x = 1;$$

$$\text{Exact solution: } u(x, t) = \begin{cases} 1 \text{ for } x/t < 1/2 \\ 0 \text{ for } x/t > 1/2; \end{cases}$$

TABLE I

$n = 44$		$n = 48$	
k	u	k	u
17	1.00000	19	1.00000
19	.99548	21	.99548
21	.76818	23	.76817
23	.21061	25	.21061
25	.02343	27	.02344
27	.00210	29	.00210
29	.00018	31	.00018

TABLE II

Rarefraction wave, $n = 48$, $\Delta t/\Delta x = 1$

k	u
47	.92695
45	.88187
43	.83994
41	.79948
39	.7599
37	.7209
35	.6825
33	.6444
31	.6066
29	.5692
27	.5321
25	.4954
23	.4590
21	.4229
19	.3873
17	.3523
15	.3177
13	.2839
11	.2509
9	.2189
7	.1881
5	.1587
3	.1310
1	.1055
−1	.0823
−3	.0619
−5	.0447
−7	.0306
−9	.0198
−11	.0120

Values of the solution after 44 and 48 time cycles are listed in Table I. The values of u for k less than 17 are equal to one within five figures, those for k greater than 31 are zero within five figures. Notice that there is a very rapid transition from $u = 1$ to $u = 0$ around $k = 22$ for $n = 44$, and $k = 24$ for $n = 48$; this corresponds closely to the exact solution which has a sharp discontinuity along the line $x = 2t$.

(2) *Equation*: same.

Initial function: $\varphi_s = 0, \quad \varphi_f = 1.$

$$\Delta t/\Delta x = 1.$$

$$\textit{Exact solution: } u(x, t) = \begin{cases} 0 & \text{for } x < 0, \\ x/t & \text{for } 0 < x < t, \\ 1 & \text{for } t < x. \end{cases}$$

Values of the calculated solution after 48 time cycles are listed in Table II, and plotted in Figure 1. The dashed line in the graph is the exact solution.

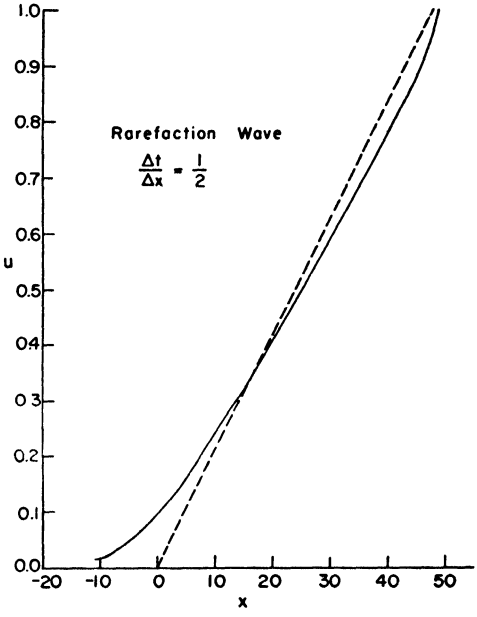


FIGURE 1

(3) *Equation*: same.

Initial function: same as in (2).

$$\Delta t/\Delta x = 1/2.$$

TABLE III

Rarefaction wave, $n = 63$, $\Delta t/\Delta x = 1/2$

k	u
64	.8553
60	.8170
56	.7758
52	.7322
48	.6869
44	.6405
40	.5933
36	.5457
32	.4980
28	.4506
24	.4039
20	.3580
16	.3134
12	.2704
8	.2295
4	.1911
0	.1555
−4	.1234
−8	.0949
−12	.0706
−16	.0505
−20	.0345
−24	.0225
−28	.0139

Values of calculated solution after 63 time cycle are listed in Table III and plotted in Figure 2. The dashed line in the graph is the exact solution.

(4) *Equation*: $u_t + (u^3/3)_x = 0$.

Initial function: $\varphi_l = 1, \quad \varphi_r = 0.$

$$\Delta t/\Delta x = 1.$$

Difference scheme: $u_k^{n+1} = u_k^n - 1/3[(u_k^n)^3 - (u_{k-1}^n)^3](\Delta t/\Delta x).$

Exact solution: $u(x, t) = \begin{cases} 1 & \text{for } x/t < 1/3, \\ 0 & \text{for } x/t > 1/3. \end{cases}$

Values of the calculated solution after 25, 26 and 27 time cycles are listed in Table IV. The values of u, for x to the left of the range listed, are equal to one within five figures, to the right of the range listed, equal to zero within five

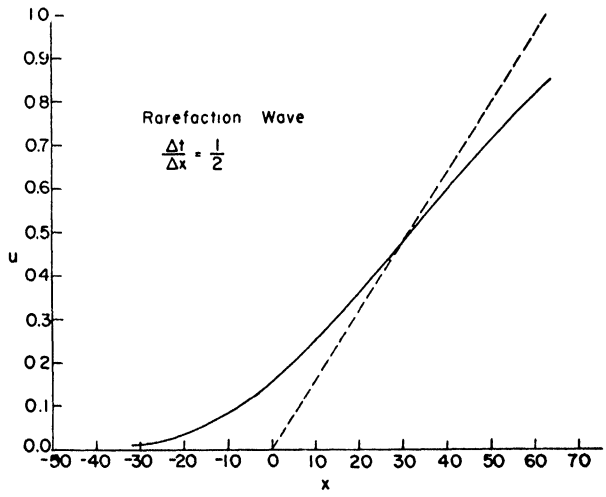


FIGURE 2

figures. The transition from $u = 1$ to $u = 0$ takes place very closely to where theory predicts it at $k = n/3$.

(5) *Equation*: Hydrodynamic equations in Eulerian form

$$\rho_t + (u\rho)_x = 0,$$

$$(u\rho)_t + (u^2\rho + p)_x = 0,$$

$$\left(\rho e + \frac{u^2\rho}{2}\right)_t + \left(\rho e u + \frac{u^3\rho}{2} + up\right)_x = 0.$$

Here ρ, u, p and e denote density, velocity, pressure and internal energy per unit mass. The equation of state expresses e as function of p and ρ; e.g. for an ideal gas e is $p/\rho(\gamma - 1)$.

In the computations the quantities ρ, $u\rho = m$ and $\rho(e + u^2/2) = E$, mass,

TABLE IV

$n = 25$		$n = 26$		$n = 27$	
k	u	k	u	k	u
6	1.00000				
7	.89330	7	.99989	7	1.00000
8	.10670	8	.65283	8	.98902
9	.00000	9	.01396	9	.34391
10		10	.00000	10	.00041

momentum and total energy per unit volume were used as dependent variables. In terms of these the equations are:

$$\rho_t + m_x = 0,$$

$$m_t + \left[(\gamma - 1)E + \frac{3 - \gamma}{2}\frac{m^2}{\rho}\right]_x = 0,$$

$$E_t + \left[\gamma \frac{m}{\rho} E - \frac{\gamma - 1}{2}\frac{m^3}{\rho^2}\right]_x = 0.$$

Initial Vector: $u_l = 2, \quad p_l = 50, \quad \rho_l = 50,$

$$u_f = 0, \quad p_f = 0, \quad \rho_f = 10,$$

$$\gamma = 1.5,$$

$$\frac{\Delta t}{\Delta x} = 0.25.$$

TABLE V

$n = 49$

ρ	$u/8$	p	k
4998	2500	4997	−6
4996	2501	4994	−4
4993	2501	4990	−2
4988	2503	4983	0
4981	2505	4972	2
4971	2508	4958	4
4959	2512	4941	6
4946	2515	4922	8
4931	2519	4904	10
4915	2522	4890	12
4898	2524	4882	14
4878	2524	4880	16
4852	2523	4887	18
4818	2520	4898	20
4777	2517	4913	22
4734	2513	4926	24
4695	2508	4928	26
4648	2492	4880	28
4497	2431	4620	30
3919	2196	3672	32
2622	1524	1753	34
1424	0490	0289	36
1047	0042	0012	38
1002	0001	0000	40
1000	0000	0000	42

$$\textit{Exact solution: } U = \begin{cases} U_i & \text{for } x/t > 2.5, \\ U_f & \text{for } x/t > 2.5. \end{cases}$$

Values of the calculated solution after 49 time cycles are given in Table V. There is a rapid transition from one state to the other around $k = 31$; this gives

TABLE VI

$n = 99$

ρ	$u/8$	p	k
4996	2501	4995	0
4994	2501	4992	2
4992	2502	4988	4
4988	2503	4983	6
4984	2504	4976	8
4979	2506	4969	10
4973	2508	4959	12
4966	2510	4949	14
4959	2512	4939	16
4952	2514	4929	18
4946	2516	4921	20
4942	2517	4914	22
4938	2518	4910	24
4937	2518	4909	26
4936	2518	4911	28
4937	2517	4915	30
4937	2515	4922	32
4935	2514	4931	34
4931	2512	4940	36
4923	2510	4949	38
4910	2508	4957	40
4893	2507	4965	42
4872	2505	4972	44
4850	2504	4977	46
4829	2503	4981	48
4814	2502	4984	50
4806	2502	4987	52
4809	2501	4987	54
4819	2499	4983	56
4824	2493	4956	58
4777	2469	4840	60
4508	2369	4388	62
3646	2027	3094	64
2208	1207	1150	66
1250	0278	0131	68
1024	0019	0005	70
1001	0000	0000	72
1000	0000	0000	74

$124/49 = 2.48$ for the speed of propagation of discontinuity, in pretty good agreement with the theoretically calculated value of 2.5.

Values of the calculated solution after 99 time cycles are given in Table VI. There is a rapid transition from one state to another around $k = 62$, giving as speed of propagation $248/99 = 2.50$.

(6) *Equation*: Hydrodynamic equations in Eulerian form.

$$\textit{Initial vector}: \quad u_l = 1, \quad p_l = 50, \quad \rho_l = 50,$$

$$u_r = 0, \quad p_r = 0, \quad \rho_r = 10,$$

$$\gamma = 1.5,$$

$$\frac{\Delta t}{\Delta x} = 0.25.$$

TABLE VII

$n = 49$

ρ	$u/8$	p	k
4995	1251	4992	−20
4989	1253	4983	−18
4977	1256	4966	−16
4957	1263	4936	−14
4924	1273	4887	−12
4874	1288	4814	−10
4804	1309	4713	−8
4712	1338	4581	−6
4598	1373	4421	−4
4464	1416	4237	−2
4314	1464	4034	0
4154	1517	3822	2
3990	1572	3609	4
3831	1628	3405	6
3685	1682	3218	8
3561	1731	3055	10
3469	1772	2921	12
3415	1804	2816	14
3405	1824	2733	16
3431	1828	2655	18
3461	1806	2539	20
3412	1731	2302	22
3138	1554	1835	24
2541	1212	1127	26
1795	0704	0444	28
1270	0244	0095	30
1062	0047	0012	32
1010	0006	0001	34
1001	0000	0000	36
1000	0000	0000	38

TABLE VIII

$n = 99$, $\gamma = 1.5$, $\Delta t/\Delta x = .25$

ρ	$u/8$	p	k
4990	1252	4986	−28
4984	1254	4977	−26
4975	1257	4963	−24
4961	1261	4943	−22
4942	1267	4914	−20
4917	1275	4876	−18
4883	1285	4827	−16
4841	1298	4765	−14
4790	1314	4691	−12
4730	1333	4604	−10
4660	1355	4504	−8
4582	1380	4393	−6
4496	1408	4272	−4
4403	1438	4144	−2
4306	1470	4011	0
4205	1504	3874	2
4102	1539	3737	4
3999	1575	3602	6
3898	1610	3471	8
3800	1645	3346	10
3708	1679	3229	12
3623	1711	3122	14
3546	1740	3026	16
3480	1766	2943	18
3426	1789	2873	20
3384	1808	2817	22
3357	1822	2772	24
3346	1834	2739	26
3352	1841	2716	28
3378	1847	2700	30
3425	1849	2691	32
3495	1851	2685	34
3586	1850	2680	36
3695	1847	2671	38
3809	1838	2646	40
3898	1817	2583	42
3907	1769	2437	44
3735	1666	2134	46
3273	1461	1613	48
2525	1103	0932	50
1747	0618	0350	52
1253	0211	0075	54
1061	0043	0010	56
1012	0006	0001	58
1002	0000	0000	60
1000	0000	0000	62

$$\textit{Exact solution: } U = \begin{cases} U_1 & \text{for } x/t < -0.225, \\ \text{rarefaction wave} & \text{for } -0.225 < x/t < 1.47, \\ U_2 & \text{for } 1.47 < x/t < 1.84, \\ U_r & \text{for } 1.84 < x/t. \end{cases}$$

The value of U_2 is: $u_2 = 1.47$, $p_2 = 27.1$, $\rho_2 = 50$.

Results of the calculation after 49 time cycles are given in Table VII. There is a rapid transition around $k = 22$, corresponding to shock speed $88/49 = 1.79$, in fairly good agreement with the theoretically calculated shock speed of 1.84.

TABLE IX

$n = 49$, $\gamma = 2$

ρ	$u/8$	p	k
4996	2501	9986	−16
4992	2503	9970	−14
4985	2507	9942	−12
4973	2513	9895	−10
4955	2522	9823	−8
4929	2535	9720	−6
4894	2552	9584	−4
4850	2574	9415	−2
4798	2601	9216	0
4740	2630	8997	2
4678	2661	8768	4
4615	2693	8541	6
4556	2723	8327	8
4501	2751	8138	10
4453	2774	7981	12
4411	2792	7860	14
4372	2805	7773	16
4328	2814	7717	18
4272	2819	7683	20
4193	2822	7667	22
4082	2823	7660	24
3936	2823	7659	26
3762	2823	7657	28
3573	2824	7655	30
3393	2824	7652	32
3241	2824	7654	34
3130	2823	7656	36
3064	2824	7679	38
3004	2805	7573	40
3064	2882	7971	42
1741	1376	2197	44
1007	0007	0002	46
1000	0000	0000	48

Results of the calculation after 99 time cycles are given in Table VIII; the transition here occurs around $k = 46$ which gives a shock speed $184/99 = 1.86$, in good agreement with the theoretically calculated value.

In Table VIII, u and p appear to be fairly constant for awhile behind the shock, the value of u being $(0.184 \pm 0.001) \times 8 = 1.47 \pm 0.01$, the value of p around 27 ± 0.3. These are fairly close to $u_s = 1.47$ and $p_s = 27.1$, in spite of the fact that the value of ρ in this range differs considerably from ρ_s.

(7) *Equation*: Hydrodynamical equations in Eulerian form.

$$\textit{Initial vector}: \quad u_i = 2, \quad p_i = 100, \quad \rho_i = 50,$$
$$u_f = 0, \quad p_f = 0, \quad \rho_f = 10,$$
$$\gamma = 2,$$
$$\frac{\Delta t}{\Delta x} = 0.25.$$

$$\textit{Exact solution}: \; U = \begin{cases} U_i & \text{for } x < 0, \\ \text{rarefaction wave} & \text{for } 0 < x/t < 2.26, \\ U_s & \text{for } 2.26 < x/t < 3.40, \\ U_f & \text{for } 3.40 < x/t. \end{cases}$$

The value of U_s is: $u_s = 2.26$, $p_s = 76.5$, $\rho_s = 30$.

Results of the calculation after 49, respectively 99 steps are listed in Tables IX and X; u and p appear to be fairly constant behind the shock, the value of u being $8 \times 0.2824 = 2.26$, p equal to 76.5. This is very close to $u_s = 2.26$, $p_s = 76.5$.

REMARK: The value of $\Delta t/\Delta x = 0.25$ is *larger* than its maximum permissible value according to the Courant-Fredrichs-Lewy theory at $p = 76.6$, $\rho = 30$. The indicated instability is indeed beginning to show around the shock front.

(8) *Equation*: Hydrodynamic equations in Lagrange mass variables

$$V_t - u_\xi = 0,$$
$$u_t + p_\xi = 0,$$
$$(e + \tfrac{1}{2}u^2)_t + (up)_\xi = 0.$$

V here denotes specific volume; if we introduce V, u and $E = e + \frac{1}{2}u^2$ as new unknowns (volume, momentum and energy per unit mass) the equations read

$$V_t - u_\xi = 0,$$
$$u_t + \left[(\gamma - 1)\frac{E - \frac{1}{2}u^2}{V}\right]_\xi = 0,$$
$$E_t + \left[(\gamma - 1)\frac{uE - \frac{1}{2}u^3}{V}\right]_\xi = 0.$$

TABLE X

$n = 99$, $\gamma = 2$, $\Delta t/\Delta x = .25$

ρ	$u/8$	p	k
4997	2501	9989	−24
4995	2502	9982	−22
4992	2503	9970	−20
4988	2505	9952	−18
4981	2509	9927	−16
4972	2513	9891	−14
4960	2519	9844	−12
4945	2527	9783	−10
4926	2536	9708	−8
4903	2548	9617	−6
4875	2562	9511	−4
4844	2577	9391	−2
4810	2595	9259	0
4772	2614	9117	2
4733	2634	8968	4
4692	2655	8816	6
4651	2676	8664	8
4611	2697	8515	10
4572	2717	8373	12
4535	2736	8242	14
4502	2753	8122	16
4472	2769	8017	18
4447	2782	7927	20
4425	2793	7853	22
4408	2802	7794	24
4395	2809	7749	26
4385	2814	7716	28
4377	2817	7692	30
4371	2820	7677	32
4365	2821	7667	34
4358	2822	7661	36
4349	2823	7657	38
4334	2823	7656	40
4312	2823	7655	42
4280	2823	7654	44
4234	2823	7654	46
4172	2823	7654	48
4091	2823	7654	50
3993	2823	7653	52
3878	2823	7653	54
3751	2824	7652	56
3619	2824	7652	58
3489	2824	7651	60
3368	2824	7651	62
3263	2824	7651	64
3178	2824	7650	66

TABLE X—*Continued*

$n = 99$, $\gamma = 2$, $\Delta t/\Delta x = .25$

ρ	$u/8$	p	k
3115	2825	7656	68
3066	2820	7628	70
3061	2841	7723	72
2981	2778	7342	74
3175	2949	8228	76
2734	2510	5891	78
3497	3137	9270	80
1993	1515	2139	82
3892	3514	2160	84
1853	1480	2426	86
1035	0045	0020	88
1000	0000	0000	90

Initial vector: $V_l = 1, \quad u_l = 4, \quad p_l = 8,$

$$V_r = 3, \quad u_r = 0, \quad p_r = 0,$$

$$\gamma = 2,$$

$$\frac{\Delta t}{\Delta x} = 0.25.$$

$$\text{Exact solution: } U = \begin{cases} U_l & \text{for } \xi/t < 2, \\ U_r & \text{for } 2 < \xi/t. \end{cases}$$

REMARK: The initial position $\xi = 0$ of the separation line of the two states was at $k = 100$. The tabulated results after 52, respectively 104 time cycles are listed in Tables XI and XII; the index k is listed in the leftmost column; the distance of the two consecutive lattice points is $2\Delta\xi$.

There is a rapid transition from the initial to the final state taking place around $k = 113$ for $n = 52$, and around $k = 126$ for $n = 104$, corresponding exactly to a shock speed of 2.

The experimental calculations presented have shown, I believe, the convergence of the method.

A large number of further shock calculations in Lagrange variables varying the initial state U_l and the value of $\Delta t/\Delta x$ were carried out by L. Baumhoff at Los Alamos. These calculations approximated the theoretically expected solutions very accurately with the exception noted further on and indicated that calculations carried out by this numerical scheme have these general features:

(i) *The width of the transition across a shock depends on the magnitude of* $\Delta t/\Delta x$; *it is narrowest if* $\Delta t/\Delta x$ *is taken as large as possible.*

TABLE XI

$n = 52$

k	V	E	u	p
076	1000	1600	4001	7993
077	1002	1602	4010	7959
078	1004	1603	4016	7935
079	1005	1604	4019	7920
080	1005	1604	4022	7911
081	1005	1604	4022	7908
082	1005	1604	4022	7909
083	1005	1604	4021	7913
084	1005	1604	4020	7918
085	1004	1603	4018	7925
086	1004	1603	4016	7932
087	1003	1603	4015	7939
088	1003	1602	4013	7946
089	1003	1602	4011	7953
090	1003	1602	4009	7960
091	1004	1603	4008	7966
092	1006	1605	4007	7971
093	1009	1607	4005	7976
094	1015	1612	4004	7980
095	1023	1618	4003	7984
096	1032	1625	4003	7987
097	1042	1634	4002	7989
098	1052	1642	4001	7992
099	1060	1648	4001	7994
100	1063	1650	4000	7995
101	1062	1650	4000	7996
102	1057	1645	4000	7997
103	1047	1638	4000	7998
104	1036	1629	3999	7998
105	1025	1620	3999	7999
106	1016	1612	3999	7999
107	1008	1606	3999	7999
108	1003	1602	3999	7999
109	1000	1600	3999	7999
110	0999	1599	3999	7999
111	1017	1556	3927	7717
112	1375	0931	2755	4011
113	2234	0190	0880	0680
114	2793	0020	0157	0068
115	2960	0001	0021	0006
116	2993	0000	0002	0000
117	3000	0000	0000	0000

TABLE XII
$n = 104$

k	V	E	u	p
47	1000	1600	4000	8000
56	1005	1604	4022	7912
77	1000	1600	4000	8000
100	1045	1636	4000	8000
114	1000	1600	4000	8000
123	1000	1600	4000	8000
124	1018	1557	3928	7716
125	1376	932	2755	4011
126	2235	191	881	680
127	2793	20	157	61
128	2961	2	21	5
129	2994	0	2	0
130	3000	0	0	0

The values of V, E, u, p vary monotonically between the lattice points $k = 47, 56, 77, 100, 114, 123$.

(ii) *The values of u and p converge faster than the values of ρ and V.*

The statement on the width of the shock is illustrated most strikingly in example 6, Table IX, where $\Delta t/\Delta x$ exceeds its maximum permissible value; here the shock transition takes place across one space interval. In fact, as J. Calkin pointed out, if U_l and U_r can be connected by a single shock, i.e. if $s[U_r - U_l]$ is equal to $F(U_r) - F(U_l)$, s being the shockspeed then, if we take $\Delta x/\Delta t$ to be exactly the shockspeed s, the numerically calculated solution is just $U_k^n = U_l$ for $k \leq sn$, $U_k^n = U_r$ for $k > sn$. Of course in hydrodynamics the character of the function F is such that the shock speed is always subsonic with respect to the state behind it, so that choosing $\Delta x/\Delta t$ to be s is necessarily an unstable mesh ratio. This is in sharp contrast to the linear case where s is the reciprocal of the slope of the characteristic curve involved and so $\Delta x/\Delta t = s$ is precisely the largest permissible mesh ratio.

Inaccuracies in the values of ρ and V are particularly noticeable if there are contact discontinuities present, i.e. lines across which u and p are continuous but not ρ; the reason for this is that contact discontinuities, in contrast to shocks, are very much like discontinuities of solutions of *linear*[7] hyperbolic equations and these, when calculated by the difference scheme described before, spread like $\sqrt{n}$. That is, after n time cycles the width of the contact discontinuity (measured, say, from the point where U is within 1% of U_l to the point where U is within 1% of U_r) is $O(\sqrt{n})$. One may check this by an explicit calculation

[7] In the same sense that solutions with contact discontinuities can be obtained as limits of continuous solutions; for proof of this fact, see a forthcoming paper of the author.

in a simple case, e.g. for the equation $u_t + u_x = 0$. In contrast, the width of shocks remains constant.

Of course whether or not there is any spreading of contact discontinuities depends on the difference scheme used and the magnitude of Δx and Δt. For instance, if we use the difference scheme described before for the equation $u_t + u_x = 0$ and choose $\Delta t/\Delta x = 1$, the solutions of the difference equations just propagate the initial data, unaltered in shape, at unit speed. There are other examples of equations and difference schemes where spreading does not occur. In all these examples the line of discontinuity passes through lattice points, and very likely this is a *necessary* condition on difference schemes for *linear* equations to avoid the spreading of initial discontinuities. This indicates that the spreading of contact discontinuities in difference calculations in *Eulerian* coordinates cannot be prevented unless one uses variable size space intervals which is tantamount to introducing Lagrange coordinates. In Lagrange coordinates a contact discontinuity is a straight line parallel to the t-axis so its path is known in advance. It is indeed possible to write down various schemes which, at least formally, maintain sharp contact discontinuities. The stability, effectiveness and accuracy of such schemes is being investigated by the author and L. Baumhoff, and by J. Calkin and N. Metropolis (who regard them from a somewhat different point of view, see a forthcoming report).

2. *Mixed Initial and Boundary Value Problems*

Let D be a domain bounded by an initial interval (say $-\infty < x \leq x_0$, $t = 0$) and a boundary curve (say $x = x_0$, $t \geq 0$) issuing from x_0 into the upper half-plane. In a mixed initial and boundary value problem for a system of equations we prescribe not only the values of all the unknowns (i.e. of all components of U) on the initial interval, but we also prescribe a certain number (say r) of relations between the unknowns along the boundary curve. The problem is to determine the solution in the domain D; this problem has a unique solution if the number r of relations prescribed is equal to the number of those characteristics issuing from x_0 which enter the domain D.

The theory of mixed initial and boundary value problems for *genuine solutions* of linear and nonlinear equations is in a fairly satisfactory state; the corresponding theory for *weak solutions* of nonlinear conservation equations is less well known. Here again one would expect to find, just as in the case of the pure initial value problem, new features which are not present in either the linear theory or the theory of genuine solutions.

Since most hydrodynamic problems arising in physics are mixed problems, there is a great need for effective finite difference schemes to compute their solutions. In this paper I would like to report on a method for handling a particularly simple case: reflection of a gas from a rigid wall (i.e. $u(x_0, t) = 0$). This boundary condition can be taken into account in the finite difference scheme simply by putting the value of u at the first lattice point *beyond* the

wall equal to the negative of its value at the last lattice point before the wall, while the values of p, V and E (Lagrange) or p, ρ and E (Euler) are transferred unchanged. I hope to be able to report in the near future on numerical schemes for handling more general boundary conditions.

Calculations were carried out in Lagrange coordinates using the following initial data: $U = U_i$ for all lattice points to the left of $k = 155$, $U = U_f$ for all lattice points between $k = 155$ and $k = 199$; the rigid wall is located at $k = 199$.

U_i and U_f were chosen as in example 8: $u_i = 4$, $V_i = 1$, $p_i = 8$, $u_f = 0$, $V_f = 3$, $p_f = 0$; γ was chosen as 2.

The exact solution is as follows: the shock propagates at the speed 2 until it reaches the wall, then it is reflected and propagated back at the speed of 8; the state of the gas U_r behind the reflected shock is $u_r = 0$, $V_r = 0.5$, $p_r = 40$. The original choice of $\Delta t/\Delta x$ was 0.25, which is the maximum permissible value for $p = 8$ and $V = 1$, and which is much too high at $p = 40$ and $V = 0.5$. Accordingly, $\Delta t/\Delta x$ was changed to 0.07 after 102 time cycles: this ratio is pretty close to the maximum permissible value of $\Delta t/\Delta x$ in the compressed state U_r . Calculations were carried on for an additional 650 time cycles. Taking Δx to be one, this corresponds to a total passage of time of $102 \times 0.25 + 65 \times 0.07 = 71$ units. During this time the shock will have reached the wall and will be reflected 216 units. Since meshpoints are $2\Delta x = 2$ units apart, this would locate the reflected shock at $k = 91$.

TABLE XIII

k	V	u	ϕ
88	0.9989	3.995	8.017
89	.9980	3.992	8.031
90	.9942	3.976	8.094
91	.9792	3.911	8.365
92	.9272	3.653	9.591
93	.7967	2.836	13.96
94	.6187	1.334	25.48
95	.5221	.292	36.41
96	.5039	.071	39.09
97	.5022	.050	39.36
98	.5019	.047	39.40

Table XIII lists the data around $k = 91$ after 752 time cycles. These calculations indicate that the reflected shock is located between $k = 91$ and 96, in pretty good agreement with theory.

The values of u beyond $k = 98$ decrease monotonically to $u = 0$ at $k = 199$; those of p increase monotonically to $p = 39.86$ at $k = 199$. The value of V decreases monotonically to $V = 0.5005$ which is reached at $k = 118$; from then

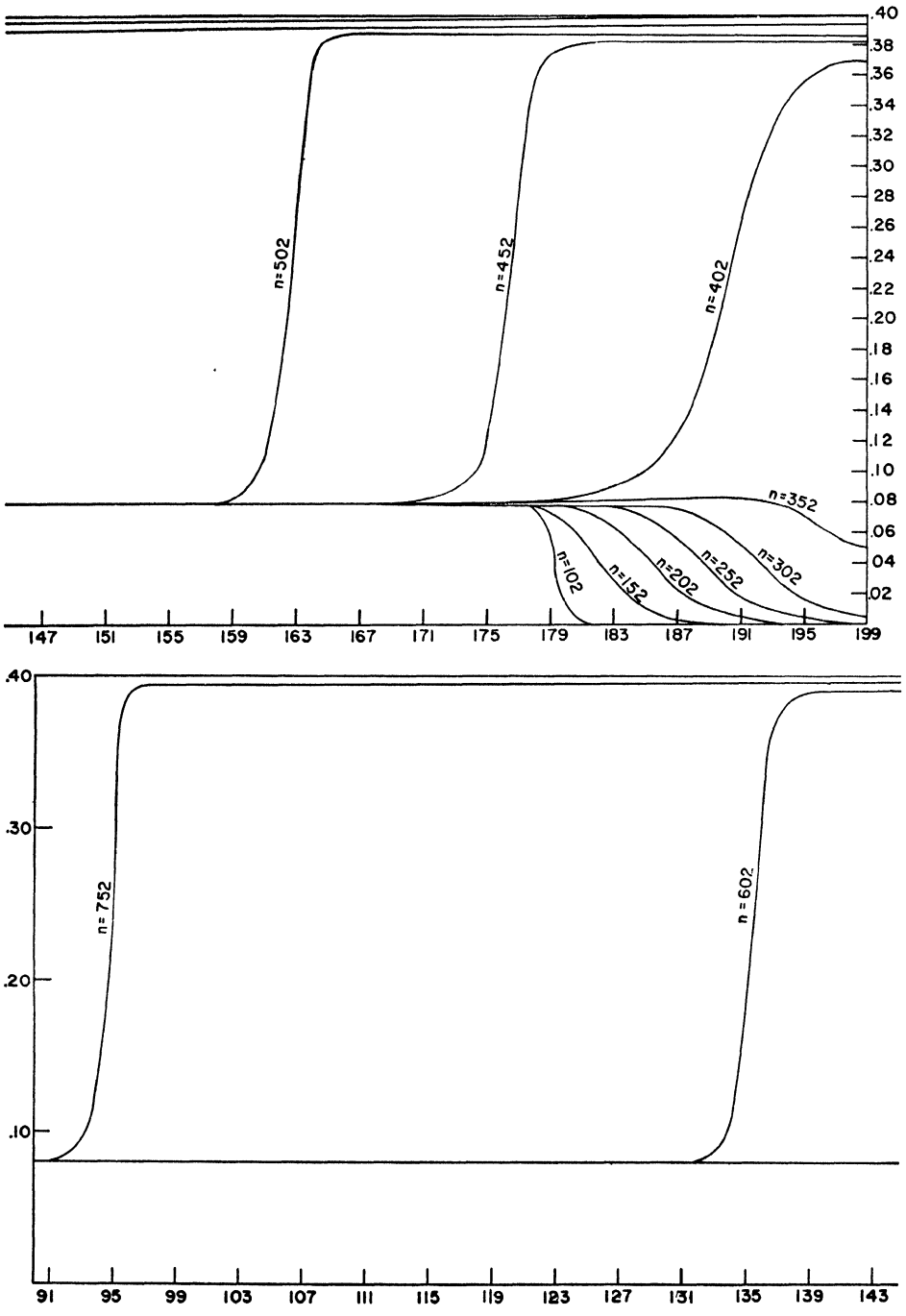

.40
.38
.36
.34
.32
.30
.28
.26
.24
.22
.20
.18
.16
.14
.12
.10
.08
.06
.04
.02
n=502
n=452
n=402
n=352
n=302
n=252
n=202
n=152
n=102
147 151 155 159 163 167 171 175 179 183 187 191 195 199
.40
.30
.20
.10
n=752
n=602
91 95 99 103 107 111 115 119 123 127 131 135 139 143

FIGURE 3

on it increases monotonically to $V = 0.5109$ which is reached at $k = 174$, and then again decreases monotonically until it reaches $V = 0.4950$ at $k = 199$.

Curves depicting the pressure distribution at various times are given in Figure 3.

REMARK: The time cycle $n = 102$ was a bit too early to switch from $\Delta t/\Delta x = 0.25$ to $\Delta t/\Delta x = 0.07$ since at that time the shock was still twenty meshpoints away from the wall. This resulted in a rather smeared out shock for a considerable portion of the calculations, as may be seen from the pressure curves. Nevertheless this did not affect the situation at the 752-nd time cycle.

An analogous calculation, starting with the same initial data, but keeping $\Delta t/\Delta x$ equal to 0.07 throughout the calculation led to substantially the same result.

3. *Finite Difference Approximations to Steady State Solutions*

The exact solutions in examples 1, 4, 5 and 8 were steady state solutions, i.e. they depended only on a linear combination of x and t. In this section we would like to analyze the manner in which the approximate solutions computed by our finite difference scheme approach steady state solutions.

The numerical evidence indicates fairly clearly that the width of the shock is constant; furthermore there is a strong indication, especially in examples 1, 4 and 8 that the shapes of the transition curves tend to a definite limiting shape. For instance in example 1 the profile of the transition curve changes only by one figure in the fifth place between the 44-th and the 48-th time cycle. The limiting shape can be characterized as the *steady state* solution of the finite difference equation. That is, into the difference equation

$$u_k^{n+1} = \tfrac{1}{2}(u_{k+1}^n + u_{k-1}^n) + \tfrac{1}{4}(u_{k-1}^n)^2 - (u_{k+1}^n)^2$$

we put $u_k^n = f(k + n/2)$, so that $f(x)$ satisfies the nonlinear difference equation

$$\frac{f(x-1) + f(x+1)}{2} + \frac{f^2(x-1) - f^2(x+1)}{4} = f(x + \tfrac{1}{2}). \tag{7}$$

and the boundary conditions

$$f(-\infty) = 1, \qquad f(\infty) = 0. \tag{8}$$

CONJECTURE: *The difference equation* (7) *has a continuous, monotonic solution* $f(x)$ *as function of the real variable* x, *subject to the boundary condition* (8); *this solution is unique except for an arbitrary phase shift.*

Furthermore: *Iterates of the transformation* $Tg = g'$ *defined by*

$$g'\left(x + \frac{1}{2}\right) = \frac{g(x-1) + g(x+1)}{2} + \frac{g^2(x-1) - g^2(x+1)}{4}$$

converge to a solution of the steady state equation.

TABLE XIV

$n = 44$		$n = 48$	
k	u	k	u
17	1.00000	19	1.00000
19	.99195	21	.99195
21	.71566	23	.71566
23	.17449	25	.17449
25	.01858	27	.01859
27	.00165	29	.00165
29	.00014	31	.00014

This last statement means: If $g_0(x)$ is any function defined over the odd integers and equal to 1 for x large enough negative, 0 for x large enough positive, and if we denote $T^n g_0$ by $g_n(x)$, then $g_n(x)$ tends uniformly to $f(x + \alpha)$, where $f(x)$ is the steady state solution of (7), (8) (made unique, say, by fixing $f(0)$ to be $\frac{1}{2}$). The phase shift α depends of course on the initial distribution $g_0(x)$.

Observe that $g_n(x)$ is defined only at points congruent $n/2$ modulo 2; consequently the only values of $f(x + \alpha)$ that enter this limiting statement are at points congruent 0 or $\frac{1}{2}$ modulo one. The somewhat exceptional situation in this example (and in examples 4 and 8 as well) arose because $\Delta t/\Delta x$ was chosen commensurable with the speed of propagation of the discontinuity.

The calculations in example 1 verify the conjecture. As further check,

TABLE XV

x	g	x	g
−3.0	0.995	−3.5	0.998
−2.5	.982	−3.0	.992
−2.0	.947	−2.5	.972
−1.5	.877	−2.0	.925
−1.0	.768	−1.5	.840
−0.5	.626	−1.0	.716
0.0	.471	−0.5	.566
0.5	.327	0.0	.412
1.0	.211	0.5	.277
1.5	.128	1.0	.174
2.0	.074	1.5	.104
2.5	.042	2.0	.060
3.0	.023	2.5	.034
3.5	.013	3.0	.019
4.0	.007	3.5	.010
		4.0	.006

calculations were carried out for this initial function: $g_0(x) = 1$ for odd negative x less than minus 1, $g_0(-1) = 0.9$, $g_0(x) = 0$ for positive odd x. The results after 44, respectively 48 time cycles, listed in Table XIV, support the conjecture.

Table XV lists g_{45}, g_{46}, g_{47} and g_{48}, both for original and the second choice of g_0.

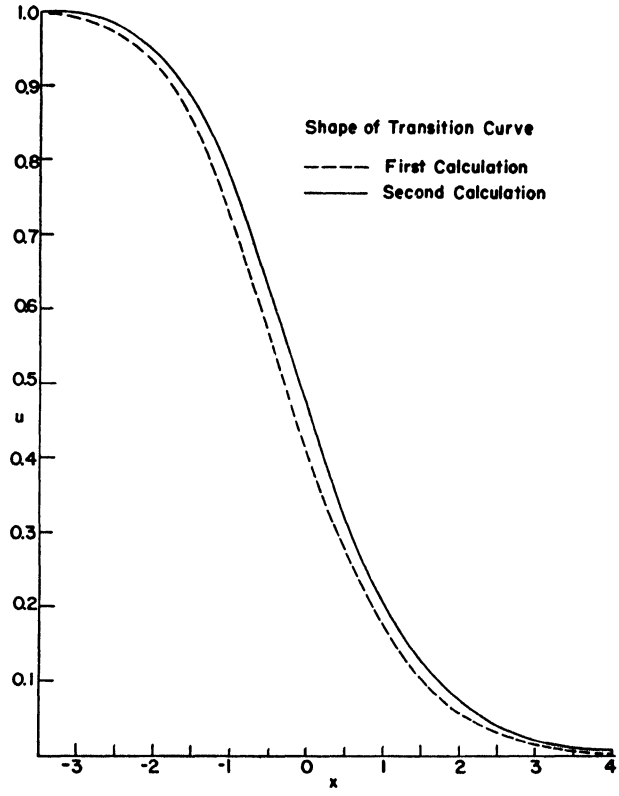


FIGURE 4

Figure 4 shows plots of these values; the two curves do indeed appear to be translates of each other, the phase shift being approximately 0.2.

Similar checks were carried out for examples 4 and 8; in each case the conjecture was verified.

It should be remarked that the nonlinear difference equation (7) has no solution if the boundary values in (8) are switched; i.e., $f(-\infty)$ is required to be zero, $f(\infty)$ one. This expresses the fact that the finite difference method furnishes compression shocks but no rarefaction shocks; I have no rigorous proof of this nonexistence at present. The result, if true, would be an analogue of a known result on viscous flows in steady state (See footnote 4). For completeness I shall present this result for the simplified equation $u_t + uu_x = \lambda u_{xx}$.

Let $u_0(x, t)$ be a steady state solution of this equation; that is, u_0 is a function of $x - ct$ only: $u_0(x, t) = f(x - ct)$. $f(\xi)$ satisfies the nonlinear ordinary differential equation

$$cf' + ff' = \lambda f''.$$

Integrating both sides with respect to ξ gives

$$\lambda f' = cf + \tfrac{1}{2}f^2 + k, \tag{9}$$

k being some constant.

We are interested in those solutions $f(\xi)$ of (9) which exist for all ξ and which approach constant values u_l and u_r as ξ tends to $-\infty$ and $+\infty$ respectively. Clearly if f approaches constant values, there must be two sequences of ξ tending to $+\infty$ and $-\infty$ respectively for which $f'(\xi)$ tends to zero, and so $u_l = f$ and $u_r = f$ are zeros of the quadratic function on the right of (9). But a quadratic function with two real roots is *negative* between its two roots, and so by (9) f' is negative for all ξ; this shows that u_r cannot exceed u_l. Conversely, if u_r is less than u_l, the two states can be connected by a solution of (9); the formula

$$\xi = \frac{2\lambda}{u_l - u_r} \log \frac{u_l - u}{u - u_r}$$

gives the shape of the connecting curve.

We emphasized at the beginning of this paper that the class of weak solutions depends on the form in which the equation is written. I would like to present here an example which shows that if in a conservation equation the exact space derivative is *not* replaced by an exact difference, the limit of the solutions of the difference equations is *not* a weak solution of the original conservation law.

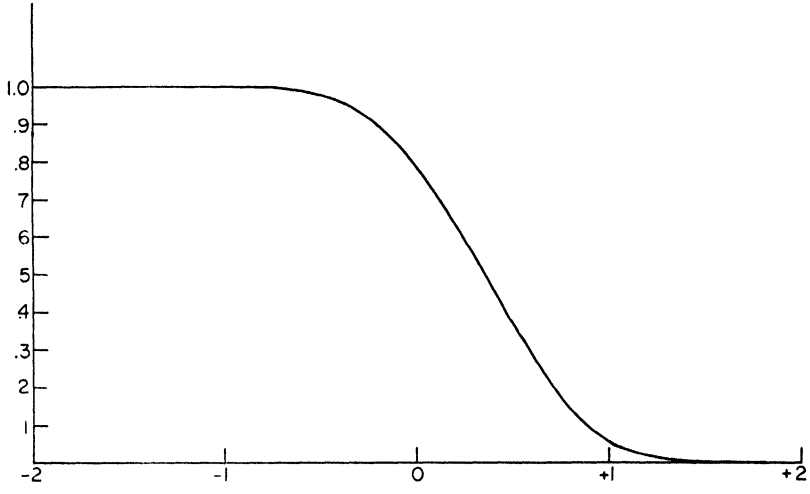


FIGURE 5

The equation in this example is the same as in example 4, $u_t + u^2 u_x = 0$; the following difference scheme was used:

$$u_k^{n+1} = u_k^n - \frac{(u_k^n)^2 + (u_{k-1}^n)^2}{2}(u_k^n - u_{k-1}^n)\frac{\Delta t}{\Delta x};$$

that is, instead of regarding the term $u^2 u_x$ as a perfect derivative, u^2 was regarded as coefficient of u_x. u_k^0 was chosen as 1 for $k < 0$, zero for $k > 0$; $\Delta t/\Delta x$ was chosen as one, and calculations were carried out for 48 time cycles. At all cycles the solution differed from 1 or zero only at two or three space points; the position of the interval of transition seemed to propagate at about the speed of 0.388; this differs appreciably from 0.333, the propagation speed of the exact weak solution. The shape of the transition curves seemed to settle down to a steady state shown in Figure 5.

The calculations were repeated with slightly altered initial data, taking u_1^0 to be 0.5 and u_k^0 for $k \neq 1$ the same as before. The solution again seemed to converge to a steady profile propagating at the speed of 0.388. The shape of

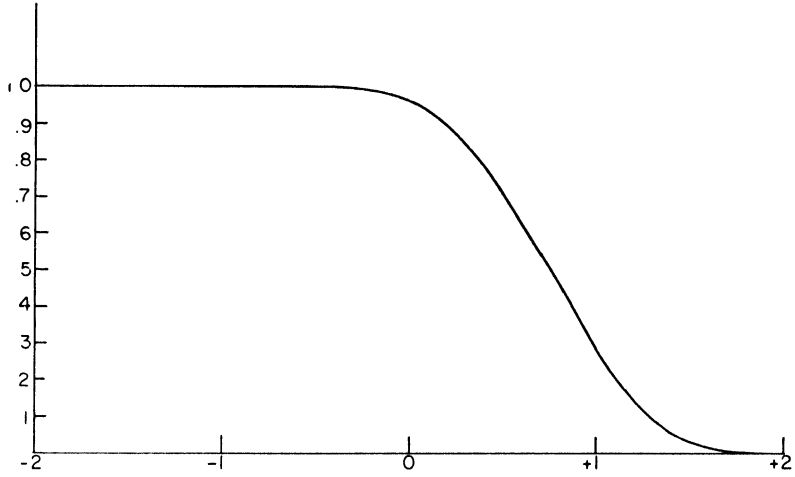


FIGURE 6

the profile, shown in Figure 6, seemed to be the same—except for a phase shift—as for the other set of initial values.

4. *Irreversibility*

The class of *genuine solutions* of differential equations is *reversible* in time; that is, if $U(x, t)$ is a genuine solution of the equation

$$U_t + AU_x + B = 0 \tag{1}$$

(in which we exclude, for sake of simplicity, explicit dependence of the coefficients

on time) in the strip $0 < t < T$, then $U'(x, t) = U(x, T - t)$, is, clearly, a genuine solution of

$$U'_t - AU'_x - B = 0.$$

Likewise, if $U(x, t)$ is a *weak solution* of the equation

$$U_t + F_x + B = 0, \tag{2}$$

$U'(x, t) = U(x, T - t)$ is a weak solution of

$$U'_t - F(U')_x - B = 0. \tag{2'}$$

But if $U(x, t)$ is a *physically relevant* solution of (2), U' need not be a physically relevant solution of (2'). This can be summarized thus: *the class of physically relevant weak solutions is irreversible.*

There is nothing paradoxical in this loss of reversibility; it is just another instance of that universal experience that when concepts are generalized, not all properties of the original can be retained. In our case we wanted to generalize the notion of solution of the initial value problem and retain the property of uniqueness and continuous dependence; this turned out to be at the expense of reversibility.

Note that both the viscosity method and the finite difference schemes described in this paper, as well as the von Neumann-Richtmyer scheme, described in [13], distinguish the positive t direstion from the negative one; thus it is not surprising that the class of solutions produced by these methods is irreversible. Conversely, any limiting procedure that can be expected to furnish the physically relevant solutions must be *unsymmetric* in time.

5. *Continuous Dependence of the Solution on the Initial Data*

In the section where the viscosity method and the finite difference scheme were described I asserted that on account of the systematic nature of these processes one would expect the solutions furnished by these methods to depend continuously on the initial data. This is of course merely a plausibility argument, no proof. However, if the number of unknowns is one and the equation is homogeneous in the first derivatives, it is possible to give a rigorous proof of this fact. The most convenient topology for this purpose is a sort of a *weak* topology. More precisely, we introduce the following norm for functions $\varphi(x)$:

$$|\varphi| = \max_{a,b} \left| \int_a^b \varphi(\xi)\, d\xi \right|.$$

We shall refer to this norm as the *weak norm.*

Let u and u' be two solutions of the equation $u_t + f(u)_x = \lambda u_{xx}$ with initial values φ and φ'. We shall show that their deviation, in the sense of the norm introduced, at any positive time t does not exceed twice their deviation at

$t = 0$. This would prove the continuous dependence of solutions of our parabolic equation on the initial data *uniformly* with respect to λ, and thus in the limit $\lambda \to 0$ as well.

Proof: Denote by U and U' integrals of u and u' with respect to x; that is, $U(x, t)$ and $U'(x, t)$ are two functions for which $\partial U/\partial x$ is u, $\partial U'/\partial x$ is u'. Clearly U and U' are determined only modulo a function of t. U and U' satisfy, modulo a function of t, the nonlinear parabolic equation

$$U_t + f(U_x) = \lambda U_{xx} \; ; \tag{10}$$

by adding to U and U' a suitable function of t we can achieve that these modified functions satisfy the equation (10) exactly. Imagine such terms already added on; there is still room for an arbitrary additive constant in U and U' which we choose so that $U(0, 0)$ and $U'(0, 0)$ are zero.

Subtract the equations satisfied by U and U' and apply the mean value theorem to the second term on the left; we obtain

$$(U - U')_t + f_u(\bar{u}) \cdot (U - U')_x = \lambda(U - U')_{xx} \; ,$$

where $\bar{u}$ denotes a value between u and u'. The resulting equation is a linear parabolic equation for $U - U'$; for this the *maximum principle* holds, therefore $| U(x, t) - U'(x, t) |$ is, for t positive,

$$\leq \max_x | U(x, 0) - U' (x, 0) | = \max_x \left| \int_0^x u(\xi, 0) - u'(\xi, 0) \, d\xi \right|.$$

This last quantity is clearly $\leq | u(x, 0) - u'(x, 0) | = | \varphi - \varphi' |$. Since $\int_a^b (u(x, t) - u'(x, t)) \, dx$ is equal to $U(b, t) - U'(b, t) - [U(a, t) - U'(a, t)]$, the maximum of this quantity with respect to a and b, $| u(t) - u'(t) |$, does not exceed $2 | \varphi - \varphi' |$.

The above result on solutions of the nonlinear parabolic equation has an analogue for the solutions of the finite difference scheme

$$u_k^{n+1} = u_{k+1}^n + \tfrac{1}{2} u_{k-1}^n - [f(u_{k+1}^n) - f(u_{k-1}^n)] \frac{\Delta t}{2\Delta x} \; . \tag{11}$$

If we introduce as new unknown

$$U_k^n = \sum_{j=0}^{k} u_k^n + c^n,$$

c^n being a function of n but not of k, U_k^n will satisfy the equation

$$U_k^{n+1} = U_{k+1}^n + \tfrac{1}{2} U_{k-1}^n - f(U_{k+1}^n - U_{k-1}^n) \frac{\Delta t}{2\Delta t} \tag{12}$$

modulo a function n; and if the constants c^n are chosen suitably (12) will be satisfied exactly.

Let u_k^n and $\tilde{u}_k^n$ be two solutions of (11) and U_k^n and $\tilde{U}_k^n$ the corresponding

solutions of (12). Subtract (12) and ($\tilde{12}$) and apply the mean value theorem to the second term on the right:

$$D_k^{n+1} = D_{k+1}^n + \tfrac{1}{2}D_{k-1}^n - f_u(\bar{u})[D_{k+1}^n - D_{k-1}^n]\frac{\Delta t}{2\Delta x}\,; \tag{13}$$

D_k^n is an abbreviation for the difference of U_k^n and $\tilde{U}_k^n$. The stability criterion demands that $f_u\ \Delta t/\Delta x$ should not exceed one; assume that this condition is satisfied. Then (13) is a recursion relation for D_k^n with *positive* coefficients whose sum is one, and for these the *maximum principle* applies: D_k^n never exceeds its largest value for $n = 0$. From this result one can deduce that the solutions constructed by the finite difference scheme are continuously dependent on the initial data in the sense of the weak norm.

The result we have just proved about the continuous dependence of the solutions constructed by the viscosity method on the initial data in the *weak* topology has this rather remarkable consequence:

The dependence of u on φ is completely continuous.

More precisely: Denote by $\{\varphi\}$ the collection of measurable functions bounded by some constants and vanishing outside some fixed interval[8], and denote by $\{u\}$ the corresponding weak solutions constructed by the viscosity method. We claim that the set $\{u\}$ is *compact in the L_1 sense.*

REMARK: The same result holds for the solutions constructed by the finite difference scheme.

Proof: Let $[u]$ be an infinite subset of the set of weak solutions considered; we wish to show that $[u]$ has a point of accumulation in the L_1 sense. Let $[\varphi]$ be the set of their initial data. The original collection $\{\varphi\}$ of initial functions is *compact* in the sense of the *weak* norm introduced before; therefore it is possible to select an infinite subsequence φ_ι in $[\varphi]$ which converges to a limit φ in the sense of the weak norm. According to the lemma proved before, the sequence of corresponding solutions u_ι converges in the sense of the weak norm to the solution u with initial value φ. However, according to a principle stated in one of the earlier sections, a weak limit of weak solutions is a weak solution if and only if it is a strong (i.e. L_1) limit. Therefore we can conclude that the sequence u_ι tends to u in the L_1 sense, and this proves the compactness of the set $\{u\}$ in the L_1 set.

For sake of completeness we shall state and prove rigorously the principle about weak limits of weak solutions. We have to assume to start with that our equation $u_t + f(u)_x = 0$ is genuinely nonlinear, meaning that f_u, the coefficient of u_x, genuinely depends on u, i.e. that its derivative with respect to u doesn't vanish: $f_{uu} \neq 0$, say f_{uu} is greater in the relevant range of u than some positive quantity.

LEMMA: If the sequence of functions u_n converges[9] in the *weak sense* to a

[8]These conditions imposed on the set of initial functions can be replaced by others.

[9]The range of the functions $u_n(x, t)$ is assumed to lie in a finite range: $a \leq u \leq b$.

limit u, then $f(u_n)$ converges in the weak sense to $f(u)$, if and only if u_n tends to u in the L_1 sense.

Proof: If $f(u_n)$ converges weakly to $f(u)$, $\int vf(u_n)$ must approach $\int vf(u)$ for every square integrable function v. Take for v any positive function and consider the integral

$$I(u) = \int vf(u)$$

as functional of u. It is an easy consequence of the assumption that f_{uu} is positive, that the second variation of $I(u)$ is positive definite, i.e.

$$I[u_n] - I[u] = \int vf_u(u) \cdot (u_n - u) + R,$$

where R is greater than const. $\int v(u_n - u)^2$. This implies the lower semicontinuity of the functional $I[u]$ in the *strict* sense:

$$\liminf I[u_n] \geq I[u]$$

for any sequence u_n weakly convergent to u; the sign of equality holding *if and only if* u is a strong limit of the sequence u_n .

The results derived can be summarized thus:

If the sequence of initial vectors φ_i converges in the weak norm, the corresponding solutions u_i converge in the L_1 sense in every bounded subset of the halfplane $t > 0$.

It follows from this result that for almost all t the functions $u_i(x, t)$ converge in the L_1 sense on every bounded interval of the x-axis. Most likely this is true for all values of t without *any* exception; this fact is expressed concisely by this statement:

The transformation mapping the initial state into the state at time t is completely continuous.

This is a particularly striking manifestation of irreversibility, in sharp contrast with the state of affairs in the theory of linear equations. There the transformations mapping the initial data into their value at time t is invertible.

The completely continuous dependence of the solution on the initial data can also be read off the explicit expression for the solution given in section 1. It would be interesting to know whether something like this is true for systems with more unknowns, in particular the equations of compressible flow.

6. *The Case of More Space Variables*

Much of what was said for the case of one space variable can be generalized if there are more space variables. In particular the same staggered difference scheme can be employed; the space derivatives are still replaced by centered difference quotients, and the time derivative U_t by a forward difference quotient

$$U_k^{n+1} - \overline{U}_k^n/\Delta t,$$

where $\overline{U}_k^n$ is the average of the value of U at the $2m$ neighbors of the lattice point k. The convergence of this scheme under a certain restriction on the ratio of Δt to the space increments has been proved for *linear* symmetric hyperbolic equations by K. O. Friedrichs.

The method can be applied to the Euler form of the flow equations in two (and three) dimensions. The conservation equations are:

$$\rho_t + (u\rho)_x + (v\rho)_y = 0,$$

$$(u\rho)_t + (u^2\rho + p)_x + (uv\rho)_y = 0,$$

$$(v\rho)_t + (uv\rho)_x + (v^2\rho + p)_y = 0,$$

$$E_t + (uE + up)_x + (vE + vp)_y = 0.$$

One would operate with ρ, $u\rho$, $v\rho$, and E as unknown functions.

A large portion of the work reported here was performed for the Los Alamos Scientific Laboratory. The calculations for examples 1, 2, 3, 4 were carried out on IBM calculators The calculations for examples 5, 6 and 7, as well as the calculations of Lester Baumhoff, were carried out on the Los Alamos MANIAC. The calculations for example 8 and for the reflection of a shock from a rigid wall were carried out on the UNIVAC of the Eckert-Mauchly Corporation. My thanks are due to Stewart Schlesinger for coding problems 1, 2, 4 and for programming problems 5, 6, 7, to Paul Stein and Paul Rosenthal for programming and coding the problems for the UNIVAC, to Lois Cook for coding problems 5, 6, 7, and to George Pimbley for coding problem 3. My thanks are also due to members of the staff of the Theoretical Physics Division of the Los Alamos Laboratory for their interest in this work; I had many fruitful conversations with them on finite difference methods.

Bibliography

[1] Becker, R., *Stosswelle und Detonation*, Zeitschrift für Physik, Volume 8, 1922, pp. 321–362.

[2] Courant, R., and Friedrichs, K. O., *Supersonic Flow and Shock Waves*, Interscience Publishers, New York-London, 1948.

[3] Courant, R., and Hilbert, D., *Methoden der mathematischen Physik*, Volume II, Springer, Berlin, 1937; reprinted by Interscience Publishers, New York, 1943.

[4] Courant, R., Isaacson, E., and Rees, M., *On the solution of nonlinear hyperbolic differential equations by finite difference methods*, Communications on Pure and Applied Mathematics, Volume V, 1952, pp. 243–256.

[5] Courant, R., Friedrichs, K. O., and Lewy, H., *Über die partielle Differentialgleichungen der mathematischen Physik*, Mathematische Annalen, Volume 100, 1928, pp. 32–74.

[6] Gilbarg, D., *The existence and limit behaviour of the one-dimensional shock layer*, American Journal of Mathematics, Volume 73, 1951, pp. 256–275.

[7] Grad, H., *The profile of a steady plane shock wave*, Communications on Pure and Applied Mathematics, Volume V, 1952, pp. 257–300.

[8] Hopf, E., *The partial differential equation* $u_t + u_{xx} = \mu u_{xx}$, Communications on Pure and Applied Mathematics, Volume III, 1950, pp. 201–230.

[9] Keller, J., and Lax, P. D., *Finite difference schemes for hyperbolic systems*, LAMS, 1205, 1950.

[10] Lax, P. D., *On discontinuous initial value problems for nonlinear equations and finite difference schemes*, Los Alamos Scientific Laboratory Report, 1332, 1952.

[11] Lax, P. D., *The initial value problem for nonlinear hyperbolic equations in two independent variables*, Princeton University Annals of Mathematics Studies, No. 33.

[12] von Neumann, J., *Proposal and analysis of a numerical method for the treatment of hydrodynamic shock problems*, National Defense and Research Committee Report AM551, 1944.

[13] von Neumann, J., and Richtmyer, R., *A method for numerical calculation of hydrodynamic shocks*, Journal of Applied Physics, Volume 21, 1950, pp. 232–237.

[14] Thomas, L. H., *Note on Becker's theory of the shock front*, Journal of Chemical Physics, Volume 12, 1944, pp. 449–453.

[15] Weyl, H., *Shock waves in arbitrary fluids*, Communications on Pure and Applied Mathematics, Volume 2, 1949, pp. 103–122.

[16] Germain, P. and Bader, R., *Unicité des écoulements avec chocs dans la mécanique de Burgers*, Office National d'Etudes et de Recherches Aeronautiques, Paris, 1953, pp. 1–13.

[17] Cole, J. D., *On a quasi-linear parabolic equation occurring in aerodynamics*, Quarterly of Applied Mathematics, 1951, pp. 226–236.

[18] Ludford, G. S. S., Polachek, R. J., and Seeger, R., *On unsteady flow of viscous fluids*, Journal of Applied Physics, Volume 24, 1953, pp. 490–495.

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. VII, 195–206 (1954)

Computation of One-Dimensional Compressible Flows Including Shocks

By L. H. THOMAS
Columbia University

Summary

The solutions of problems involving compressible flows have different functional forms in different regions, separated not only by discontinuities of the fluid itself but also by shock fronts and by characteristics starting from the origins of shock fronts and from points where such characteristics meet shocks. To obtain accuracy in numerical integration, small intervals and formulas using differences of low order are convenient immediately after crossing these lines, but much larger intervals, with differences of higher order, are sufficient elsewhere. The purpose of this paper is to explain a method of exploiting the economy made possible in this way.

If the equations for the one-dimensional case are transformed to characteristic variables, they form a 'canonical hyperbolic system'.[1,2] Adams' method for ordinary differential equations extends to a variable interval by using divided differences of the derivative, and it then becomes practical to use it for starting and across discontinuities. Further, it applies directly to canonical hyperbolic systems. Numerical solution is then very direct over regions determined by two characteristics and is convenient where the flow crosses already existing shocks. Finally, the origin of a new shock in any interval can be detected and arrangements can be made for it.

Introduction

Two features of fluid flow, turbulence and shock waves, make numerical representation difficult, because intervals of space and time small in proportion to the viscosity are required to represent their details. In time dependent one-dimensional flow, turbulence does not arise, because one-dimensional motion is not rotational, but, as the viscosity is reduced, shock waves simply approach shock fronts, surfaces of discontinuity for some of the variables describing the motion, and new shock fronts may start within the fluid.

The differential equations, with two dependent variables, have discrete

[1]See R. Courant and D. Hilbert, *Mathematische Physik*, Springer, Berlin, Volume 2, 1937, p. 324.

[2]See R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves*, Interscience, New York-London, 1948, p. 44 and p. 70.

characteristic directions at every point, which happen to be only two in number, so we may transform to the characteristic curves as parameter lines. The equations then reduce to a canonical hyperbolic system[3] of partial differential equations, and many of the numerical methods used for ordinary differential systems extend directly; in particular, large intervals may be used, since integration formulas using a number of backward differences may be found that do not result in numerical instability.

In order to provide for shock fronts and for their possible origin within an integration interval, we must divide the region of integration into domains across the boundaries of which the variables and their derivatives need not be continuous. We need, therefore, a method adapted to starting a new region of integration with a small interval and rapidly increasing it, and also to taking an irregular step at the end of a region.

The method described here applies to fluids with general equations of state, and to problems of plane, cylindrical, and spherical flow.

1. *The Integration of Ordinary Differential Equations*

The simplest method[4] of integrating an ordinary differential equation of the first order

$$dy/dx = f(x, y) \tag{1.1}$$

step by step, is to compute an increment, $\Delta y_r = y_{r+1} - y_r$, for the dependent variable y, for each step, $\delta x_r = x_{r+1} - x_r$, in the independent variable x, from the derivative computed from the differential equation for the point x_r, y_r, at the beginning of the step, by the simple approximate formula

$$\Delta y_r = \delta x_r f(x_r, y_r). \tag{1.2}$$

The calculation can be started at any point x, y; the interval δx can be varied as required, and an irregular step can be taken to a discontinuity in $f(x, y)$ at points given by $\varphi(x, y) = 0$, by using simply[5]

$$\varphi + \frac{\partial \varphi}{\partial x}\delta x + \frac{\partial \varphi}{\partial y}\Delta y = 0. \tag{1.3}$$

If ρ_r is the rounding error made in computing $f(x_r, y_r)$, the error made in the corresponding step is approximately

$$\{\delta x_r\, \rho_r - \tfrac{1}{2}(\delta x_r)^2 (d^2y/dx^2)_r\},$$

[3] Loc. cit., footnote 1.

[4] This method is given by Euler, *Institutiones Calculi Integralis*, Petropoli, 1768, p. 493, as a basis for an improved method using terms of the Taylor expansion at each point, but the simple method is commonly known as Euler's method.

[5] For the solution to exist and be unique the Lipschitz condition must continue to hold at points on the discontinuity. The statement made by E. L. Ince in *Ordinary Differential Equations*, Longmans, Green, London, 1927, p. 66, is not complete.

while a change η_r in the value of y at x_{r+1} produces a change in subsequent values of y given approximately by the appropriate solution of

$$\frac{d\eta}{dx} = \eta \frac{\partial f}{\partial y}, \tag{1.4}$$

and so at x_r , $\eta_r \exp \left(\int_{x_{r+1}}^{x_n} (\partial f/\partial y)\, dx\right.$. Hence the error after n steps is given approximately by[6,7]

$$\eta_n = \eta_0 \exp\left\{\int_{x_1}^{x_n} (\partial f/\partial y)\, dx\right\} + \sum_{r=0}^{n-1} \left\{\delta x_r\, \rho_r - \frac{1}{2}(\delta x_r)^2 \left(\frac{d^2y}{dx^2}\right)_r\right\} \exp\left\{\int_{x_{r+1}}^{x_n} (\partial f/\partial y)\, dx\right\}. \tag{1.5}$$

if the summation to form y is carried to so many digits that it does not increase the error.

The error falls off only as the first power of δx, and a great deal of work is saved if methods for which the error falls off as a higher power of δx can be used. In general, besides the inevitable term with the weighting factor, $\exp\{\int_{x_{r+1}}^{x_n} (\partial f/\partial y)\, dx\}$, proper to the differential equation, the error now contains terms with other weighting factors, and it is important that these should be small. This can be decided from a local linear approximation, a difference system with constant coefficients. Further, convenient provision must be possible for starting and for taking irregular steps.

Perhaps the best method is Adams' method[8] in which backward differences of dy/dx are used to correct (1.1) according to the formula

$$\Delta y_r = \delta x \left\{\frac{dy}{dx} + \frac{1}{2}\nabla \frac{dy}{dx} + \frac{5}{12}\nabla^2 \frac{dy}{dx} + \frac{3}{8}\nabla^3 \frac{dy}{dx} + \cdots\right\}_r, \tag{1.6}$$

now holding for uniform interval δx only. If we take the first three terms of the series and put the equation in Lagrangian form in terms of values instead of differences of dy/dx, we have

$$y_{r+1} = y_r + \delta x \left\{\frac{23}{12}\left(\frac{dy}{dx}\right)_r - \frac{16}{12}\left(\frac{dy}{dx}\right)_{r-1} + \frac{5}{12}\left(\frac{dy}{dx}\right)_{r-2}\right\} \tag{1.7}$$

and the local rates of growth of error are given by the homogeneous system

$$\eta_{r+1} = \eta_r + \delta x \frac{\partial f}{\partial y}\left\{\frac{23}{12}\eta_r - \frac{16}{12}\eta_{r-1} + \frac{5}{12}\eta_{r-2}\right\}, \tag{1.8}$$

---

[6] This is a special case of the formula given by Rademacher, Annals of the Computation Laboratory of Harvard University, Volume 16, 1948, p. 176, for a pair of simultaneous equations integrated by Heun's method.

[7] The special case is closely related to the Cauchy-Lipschitz method of treating the existence theorem for this equation. Cf. loc. cit., footnote 5, p. 75.

[8] F. Bashforth and J. C. Adams, *Capillary Action*, Cambridge, 1883, p. 17. Note that Adams actually used Gregory's formula to correct an approximate value of y_{r+1} obtained by extrapolation, although he also described the method repeated here, which is commonly known by his name.

the solution of which can be expressed as a sum of terms multiplied each step by the roots for λ of the equation

$$\lambda^3 = \lambda^2 + \delta x \frac{\partial f}{\partial y}\left\{\frac{23}{12}\lambda^2 - \frac{16}{12}\lambda + \frac{5}{12}\right\}, \tag{1.9}$$

which, besides the inevitable root proper to the differential equation, approximately $1 + \delta x(\partial f/\partial y)$, has two others of order $\delta x^{1/2}$, the terms corresponding to which can be neglected in the total error. The error can therefore be estimated from the first term neglected in (1.6), and is given approximately by (1.5), with $\frac{1}{2}(\delta x)^2(d^2y/dx^2)_r$ replaced by $\frac{3}{8}(\delta x)^4(d^4y/dx^4)_r$.

The formula (1.6) is in general only asymptotic, and the number of terms that may, and for economy should, be used, increases as δx diminishes. Indeed, if the function $y(x)$ is analytic with a radius of convergence R, and we have

$$\left|\frac{d^{s+1}y}{dx^{s+1}}\right| < KR^{-(s+1)}(s+1)!, \tag{1.10}$$

and since the coefficients of (1.6) approach $1/(\gamma + \log s)$, where γ is Euler's constant,[9] we get an approximate truncation error

$$\int_0^1 [\Gamma(t+s)/s!\Gamma(t)]\,dt \approx \int_0^1 \exp\{(t-1)(\log s + \gamma)\}\,ds.$$

$$\sum_{r=0}^{n-1} \frac{K}{R}\frac{\sqrt{2\pi s}}{\gamma + \log s}\left(\frac{s\,\delta x}{eR}\right)^s \delta x \exp\left\{\int_{x_{r+1}}^{x_n} (\partial f/\partial y)\,dx\right\}. \tag{1.11}$$

The value of s which makes this a minimum is given by

$$s\,\delta x = R, \tag{1.12}$$

and the minimum, which is the error, falls off like

$$e^{-R/\delta x} \tag{1.13}$$

as δx diminishes.[10]

Adams' method is easily modified to use unequal intervals, keeping a table of divided differences of dy/dx. By filling up a wedge of the table, repeating again and again the last argument, neglecting differences beyond some fixed order, we can build up a series of approximate differential coefficients at this argument, each already divided by a factorial. These can be used to give the next dependent variable by Taylor's series, and the new value of dy/dx obtained from the differential equation allows a new line of the original table of divided differences to be computed.[11]

---

[9] The coefficient (loc. cit., footnote 8) is

[10] The purpose of including this discussion is to emphasize the economy of using more accurate formulas.

[11] This method was explained in a paper given by the author at the September 1950 meeting of the Association for Computing Machinery in Washington, D. C. See Ohio State University Engineering Experiment Station News, 24, No. 3, June, 1952.

Alternatively we may keep a table of approximate derivatives of y up to some order and extrapolate this ahead regarding the next derivative as zero, obtaining values y'_{r+1} , $(dy/dx)'_{r+1}$, $(d^2y/dx^2)'_{r+1}$, $\cdots$. We then replace this value of dy/dx by that obtained from the differential equation; thus

$$\left(\frac{dy}{dx}\right)_{r+1} = f(x_{r+1}\,,\, y'_{r+1}), \tag{1.14}$$

and we correct the values of d^2y/dx^2, d^3y/dx^3, $\cdots$ by amounts proportional to $\{(dy/dx)_{r+1} - (dy/dx)'_{r+1}\}$, such as to make the method equivalent, for uniform interval, to Adams' method. This is always possible, and we have the same growth of error as for Adams' method[12]. Neglecting d^4y/dx^4 in the extrapolation, we must take

$$\delta x_{r+1}\left\{\left(\frac{d^2y}{dx^2}\right)_{r+1} - \left(\frac{d^2y}{dx^2}\right)'_{r+1}\right\} = \frac{3}{8}\left\{\left(\frac{dy}{dx}\right)_{r+1} - \left(\frac{dy}{dx}\right)'_{r+1}\right\}, \tag{1.15}$$

$$\delta x^2_{r+1}\left\{\left(\frac{d^3y}{dx^3}\right)_{r+1} - \left(\frac{d^3y}{dx^3}\right)'_{r+1}\right\} = \left\{\left(\frac{dy}{dx}\right)_{r+1} - \left(\frac{dy}{dx}\right)'_{r+1}\right\}, \tag{1.16}$$

while the truncation error per step is estimated from[13]

$$\frac{3}{8}\,(\delta x_{r+1})^4\,\frac{d^4y}{dx^4} \approx \frac{3}{8}\,\delta x_{r+1}\left\{\left(\frac{dy}{dx}\right)_{r+1} - \left(\frac{dy}{dx}\right)'_{r+1}\right\}. \tag{1.17}$$

Either of these methods, exactly equivalent for uniform interval to Adams' method, may be used to start an integration or to continue it from a discontinuity, by making the first step of length Re^{-s}, the second of length $\frac{1}{2}\,Re^{-s/2}$, and so on, one extra term being available and used at each new step. In practice the accuracy is controlled as we proceed. Further, a first approximation to an irregular step up to a discontinuity can be obtained from equation (1.3), then a second smaller step controlled in the same way, and s steps give a sufficiently accurate result.

For systems of ordinary differential equations exactly the same considerations hold. The rate of growth of error is now given by a linear system like (1.4), but of the order of the original differential system, provided that the extra terms, making the total number equal to the order of the difference system, can be neglected. For Adams' method, the extra terms can always be neglected.[14]

---

[12] This ensures that all terms in the error, other than the inevitable term growing at rate $1 + \delta\, x(\partial f/\partial y)$, are multiplied each step by a (fractional) power of δx.

[13] We may also correct y'_{r+1} using this term, making the error of one higher order in δx, but if we wish to estimate the error, we might just as well use the formulas for one more differential coefficient; cf. loc. cit. Footnote 8.

[14] Adam's method can be regarded as the adaptation to a first order equation of the methods actually used by astronomers for large systems. See, e.g., Astronomical papers prepared for the use of the American Ephemeris and Nautical Almanac, W. A. Eckert, Dirk Brouwer, and G. M. Clemence, *Coordinates of the Five Outer Planets*, Volume XII, 1951, p. x.

2. *The Integration of Canonical Hyperbolic Systems with Two Dependent Variables*

A canonical hyperbolic system[15] of the special form

$$\begin{aligned} \partial u/\partial\alpha &= f(u, v, \alpha, \beta), \\ \partial v/\partial\beta &= g(u, v, \alpha, \beta) \end{aligned} \tag{2.1}$$

can be integrated ahead directly by Adams' method using backward differences of $\partial u/\partial\alpha$ with respect to α, and of $\partial v/\partial\beta$ with respect to β, over a grid rectangular in α and β. The order in which new points are dealt with makes no difference to the computation and growth of error, although, of course, points α_r , β_{r-1} , and α_{r-1} , β_r , must precede α_r , β_r . The local rates of growth of error are given by a linear system, the solution of which can be expressed by a sum of terms multiplied by constants λ and μ respectively for steps in α and β, where λ and μ satisfy an equation analogous to (1.9), which, if we take three terms of each series of differences, is

$$\begin{vmatrix} \lambda^3 - \left\{\lambda^2 + \delta\alpha\dfrac{\partial f}{\partial u}\left(\dfrac{23}{12}\lambda^2 - \dfrac{16}{12}\lambda + \dfrac{5}{12}\right)\right\}, & \delta\alpha\dfrac{\partial f}{\partial v}\left(\dfrac{23}{12}\lambda^2 - \dfrac{16}{12}\lambda + \dfrac{5}{12}\right) \\ \delta\beta\dfrac{\partial g}{\partial u}\left(\dfrac{23}{12}\mu^2 - \dfrac{16}{12}\mu + \dfrac{5}{12}\right), & \mu^3 - \left\{\mu^2 + \delta\beta\dfrac{\partial g}{\partial v}\left(\dfrac{23}{12}\mu^2 - \dfrac{16}{12}\mu + \dfrac{5}{12}\right)\right\} \end{vmatrix} = 0. \tag{2.2}$$

The additional terms are still negligible, and the growth of error is given by the linear system in the errors ξ, η, respectively, in u and v,

$$\begin{aligned} \frac{\partial\xi}{\partial\alpha} &= \xi\frac{\partial f}{\partial u} + \eta\frac{\partial f}{\partial v} + \left\{\rho - \frac{3}{8}(\delta\alpha)^3\frac{\partial^4 u}{\partial\alpha^4}\right\}, \\ \frac{\partial\eta}{\partial\beta} &= \xi\frac{\partial g}{\partial u} + \eta\frac{\partial g}{\partial v} + \left\{\sigma - \frac{3}{8}(\delta\beta)^3\frac{\partial^4 v}{\partial\beta^4}\right\}, \end{aligned} \tag{2.3}$$

ρ and σ being rounding errors in finding f and g respectively.

The modified methods using divided differences or using derivatives make it convenient to start solutions and to deal with discontinuities. Starting a solution when values of u are known for $\alpha = \alpha_0$ and $\beta \geq \beta_0$, while values of v are known for $\beta = \beta_0$ and $\alpha \geq \alpha_0$, is done simply by taking $\alpha_1 - \alpha_0$ and $\beta_1 - \beta_0$ so small that the Euler approximation (1.1) is accurate enough; then $\alpha_2 - \alpha_1$ and $\beta_2 - \beta_1$ may be taken larger, but small enough for one difference of $\partial u/\partial\alpha$ or of $\partial v/\partial\beta$ respectively to be sufficient, Figure 1. Extra differences with correspondingly larger intervals are introduced till an optimum interval is reached.

[15] Loc. cit., footnote 1.

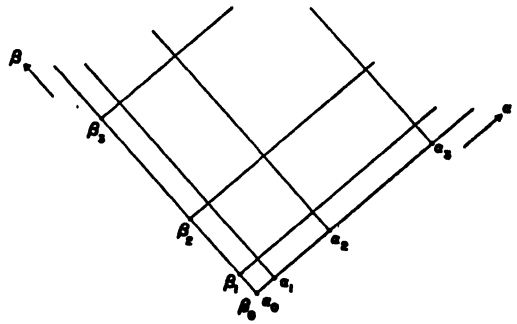


FIG. 1. Starting a solution between two characteristics.

Starting from an arbitrary initial situation is equally easy. If we compute at points arranged as in the diagram (Figure 2), we compute first points at a small interval, then at a larger interval with first differences of the derivative

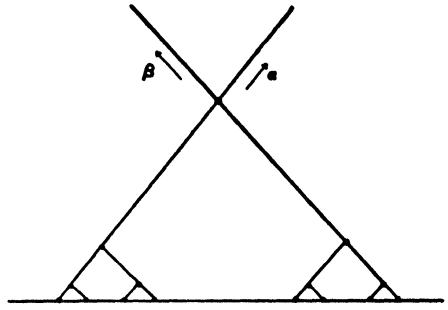


FIG. 2. Starting a solution from initial values.

available, then with second differences available, and so on. A line of discontinuity can be dealt with by computing extra sets of values at points where it cuts grid lines (Figure 3). By extrapolating ahead along the discontinuity,

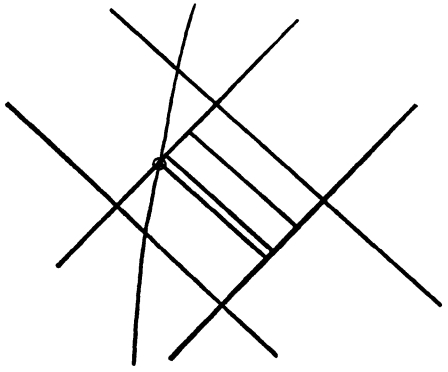


FIG. 3 Integrating across a line of discontinuity.

values are obtained for these points requiring only a single correction found from equations analogous to (1.3), and integration ahead from these points can be started with small steps.[16]

For the more general canonical hyperbolic system

$$A(u, v, \alpha, \beta)\frac{\partial u}{\partial \alpha} + B(u, v, \alpha, \beta)\frac{\partial v}{\partial \alpha} = f(u, v, \alpha, \beta), \tag{2.41}$$

$$C(u, v, \alpha, \beta)\frac{\partial u}{\partial \beta} + D(u, v, \alpha, \beta)\frac{\partial v}{\partial \beta} = g(u, v, \alpha, \beta). \tag{2.42}$$

after finding u and v at $\alpha - \delta\alpha$, β, and at α, $\beta - \delta\beta$, we have one relation, (2.41), between $\partial u/\partial\alpha$ and $\partial v/\partial\alpha$ at $\alpha - \delta\alpha$, β, and one relation, (2.42), between $\partial u/\partial\beta$ and $\partial v/\partial\beta$ at α, $\beta - \delta\beta$. The Adams equations for integrating ahead give u and v at α, β, in terms of $\partial u/\partial\alpha$, $\partial v/\partial\alpha$, at $\alpha - \delta\alpha$, β, and backward differences with α, and in terms of $\partial u/\partial\beta$, $\partial v/\partial\beta$, at α, $\beta - \delta\beta$, and backward differences with β. In all, there are six linear equations to determine u and v at α, β; $\partial u/\partial\alpha$ and $\partial v/\partial\alpha$, at $\alpha - \delta\alpha$, β; and $\partial u/\partial\beta$ and $\partial v/\partial\beta$ at α, $\beta - \delta\beta$. The coefficients of these equations have determinant proportional to

$$\delta\alpha\ \delta\beta(AD - BC). \tag{2.5}$$

When we investigate possible rates of increase of terms in the error for steps in α and β, we find, as before, besides the terms proper to the differential equation, only negligible additional terms.

The extension to further dependent variables is immediate, and we can deal with starting and with discontinuities just as before. In fact, for equations in canonical hyperbolic form, we can use Adams' method with larger intervals whenever we can use Euler's method, namely replace differential coefficients by differences at a small interval.

3. *The Equations of One-Dimensional Fluid Flow in Terms of Characteristic Variables*

We may express the equations of motion of an inviscid fluid in one dimension of space in terms of the position r at time t of a material particle with Lagrangian identifying variable s.

The material velocity u is given by

$$u = \partial r/\partial t \tag{3.11}$$

and s may be chosen so that the specific volume v is given by

$$v = \partial r/\partial s. \tag{3.12}$$

[16]This method has been used successfully to reduce greatly the work required to solve the Riemann equations for the propagation of a pulse along a non-uniform line. In more complicated cases where the discontinuities are given by functions involving the dependent variables, conditions of Lipschitz type must be satisfied.

The pressure p being regarded as a given function of v and s, we have the equation of motion[17]

$$\partial^2 r/\partial t^2 = -\partial p/\partial s. \tag{3.13}$$

These equations may be transformed to a canonical hyperbolic system by using the characteristic parameters α, β, as independent variables, r, t, s, u, and v, becoming dependent variables.[18,19] We arrive at the six equations

$$\frac{\partial t}{\partial \alpha} + \sqrt{\frac{-1}{p_v}}\frac{\partial s}{\partial \alpha} = 0, \tag{3.21}$$

$$\frac{\partial t}{\partial \beta} - \sqrt{\frac{-1}{p_v}}\frac{\partial s}{\partial \beta} = 0, \tag{3.22}$$

$$\frac{\partial r}{\partial \alpha} = u\frac{\partial t}{\partial \alpha} + v\frac{\partial s}{\partial \alpha}, \tag{3.31}$$

$$\frac{\partial r}{\partial \beta} = u\frac{\partial t}{\partial \beta} + v\frac{\partial s}{\partial \beta}, \tag{3.32}$$

and

$$\frac{\partial v}{\partial \alpha} + \sqrt{\frac{-1}{p_v}}\frac{\partial u}{\partial \alpha} = -\frac{p_s}{p_v}\frac{\partial s}{\partial \alpha}, \tag{3.41}$$

$$\frac{\partial v}{\partial \beta} + \sqrt{\frac{-1}{p_v}}\frac{\partial u}{\partial \beta} = -\frac{p_s}{p_v}\frac{\partial s}{\partial \beta}, \tag{3.42}$$

where p_v and p_s are the partial derivatives with respect to v and s of the pressure p supposed given as a function of these variables.

Five of these equations, say the first five, form a canonical hyperbolic system for the five dependent variables, and if the sixth holds initially, it will be automatically maintained.[20] Instead of α any function of α and instead of β any function of β may be used as convenient.

The function $p(v, s)$ may be a discontinuous function of s if the fluid is heterogeneous or if different shocks have passed over the different parts of the fluid. There is then a discontinuity in v at a value of s, while p must be continuous, and some simplicity is gained by using p as dependent variable in place of v.

---

[17] Loc. cit., footnote 2, p. 31.

[18] Cf. loc. cit., footnote 2, p. 70. The details for a general quasi-linear equation of the second order in one dependent and two independent variables are given in loc. cit., footnote 1, p. 326, by a method due to Hans Lewy.

[19] The straightforward derivation by this method of the form given here for the plane case, and of the slightly more complicated forms for cylindrical and spherical waves is given in L. H. Thomas, *The Numerical Integration of the Equations of Plane, Cylindrical, and Spherical Waves of Finite Amplitude*, Ballistic Research Laboratory Report 579.

[20] Loc. cit., footnote 1, p. 329.

4. *The Conditions at a Shock Front*

Suppose the internal energy of the material per unit mass, U, is known in terms of the specific volume v, and the entropy φ, as well, perhaps, as the identifying variable s; φ will then be given initially in terms of s, and will change only when the material crosses a shock front. The pressure is then given by

$$p = -U_v , \tag{4.1}$$

where U_v is the partial derivative of U with respect to v, φ and s kept constant.

If c is the speed of a shock front moving in the direction of increasing r, and if u_0 , v_0 , φ_0 , p_0 , U_0 , are the values of material velocity, specific volume, entropy, pressure, and internal energy just outside, and u, v, φ, p, U, their values just inside the front, the Rankine-Hugoniot conditions of conservation of mass, momentum, and energy,

$$\frac{c - u}{v} = \frac{c - u_0}{v_0}, \tag{4.2}$$

$$\frac{c - u}{v} u - p = \frac{c - u_0}{v_0} u_0 - p_0 , \tag{4.3}$$

$$\frac{c - u}{v}\left(\frac{1}{2}u^2 + U\right) - pu = \frac{c - u_0}{v_0}\left(\frac{1}{2}u_0^2 + U_0\right) - p_0 u_0 \tag{4.4}$$

which, of course, imply that the Bernouilli constant relative to the front does not change across the front, must be satisfied, while the entropy can only increase as the shock passes.[21]

Since U and p, for given s, are given functions of v and φ, these equations will give, for given u_0 , v_0 , φ_0 , and s, the values of c, v, φ, and u, and therefore of p and U, in terms of any one of these latter variables; in particular, the relation between p and v is the 'Hugoniot adiabatic'.

5. *Integration of the Compressible Flow Equations*

The equations of Sections 3 and 4 have been solved in various cases including shock waves by Euler's method,[22] the extra equation being useful as a check, and the arguments of Section 2 show that Adams' method can be extended to them. There is no special difficulty with discontinuity with the value of s, if the conditions for a solution of the differential equations to exist hold.

For integrating across an already established shock, it is only necessary to notice that the flow field in front of the shock may be integrated independently right under it to fictitious values, so that u_0 , v_0 , φ_0 , r, s, and t are known as

[21] Loc. cit., footnote 2, p. 124.

[22] Loc. cit., footnote 2, p. 202. Cf. loc. cit., footnote 19.

functions of two variables on this side. Behind the shock one set of new characteristics arises at the shock front and it is convenient to let them proceed from it at points where the other set meets the front, and in practice, at one or more points between also (Figure 4). In this case integration ahead to the shock

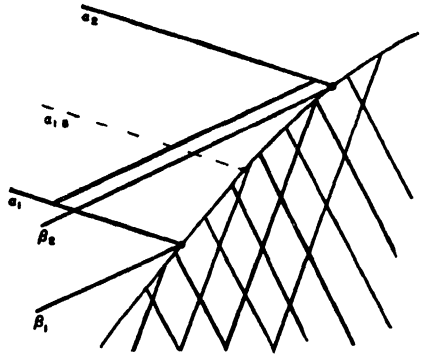


FIG. 4. Integration up to a shock front.

front is direct and we can integrate away from it starting with a small step.

The equations (3.21) to (3.42) can be integrated beyond the origin of a new shock, and the position of the origin of the shock, if it moves to the right, is fixed by the vanishing, necessarily simultaneous, of $\partial t/\partial\beta$, $\partial s/\partial\beta$, and $\partial r/\partial\beta$.[23] The old region extends on the left to the characteristic of constant α, $= \alpha_0$, through the shock origin, and on the right up to the shock front, while the new

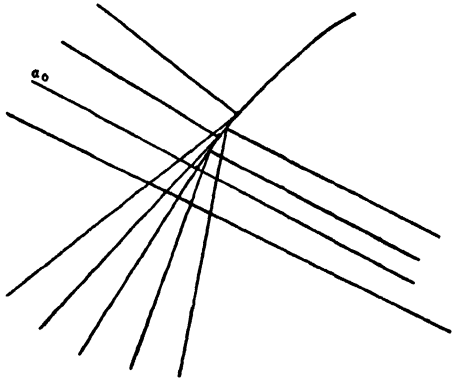


FIG. 5. The origin of a shock.

region is to be started between that characteristic and the new front (Figure 5); it is actually broken into two regions by the line of constant s through the shock origin.

[23] Loc. cit.. footnote 2, p. 168.

New regions also start where shocks, or discontinuities along characteristics, or discontinuities along lines of constant s, intersect each other.[24] In each case the suggested procedure is to take irregular steps up to the discontinuities, to restart with so small an interval that first order terms are sufficient, and then to expand the interval as quickly as is convenient.

[24] Loc. cit., footnote 2, p. 176.

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. VII, 207–222 (1954)

The Deformation of a Thin Material Shell of Non-uniform Thickness by a Detonation Wave

By G. E. HUDSON
New York University

Description of the Problem and the Theoretical Basis for its Solution

Consider a thin, semi-infinite shell of weak[1] material, initially coincident with the x,y-plane. Let the surface density, ρ_M , of this shell increase linearly, in the positive x-direction, from the value zero at x equal to zero. This uniform surface density gradient, denoted by δ, may be due, for example, to a non-uniform thickness which increases linearly in the x-direction. We suppose that the shell is in contact with an explosive material, for simplicity taken initially as semi-infinite, extending from the x,y-plane ($z = 0$) to infinity in the positive z-direction. We shall suppose further that the shell (or liner) is deformed by the pressures exerted on it by the burnt explosive material behind a detonation front which is sweeping through the unburnt explosive. Let us assume that the detonation front is plane and normal to the initial plane of the liner. The normal to the detonation front is, however, inclined at an angle θ_0 , measured positive in the counter-clockwise sense from the x-axis ($-\pi/2 < \theta_0 < \pi/2$). Our problem is to calculate the shape of the deformed shell.

The frame of reference we choose is one in which the detonation front is stationary and contains the origin, O, so that all the motions are reducible to steady state motions as in Figure 1. The shell particles in front of the detonation wave therefore move into it, in the y-direction. Behind the detonation front, the shell evidently curls up into a conically shaped surface, with apex at the origin. Because the only forces, which by assumption are effective in accelerating the shell particles, are due to the pressures exerted normally on the shell surface, the particle trajectories are evidently geodesics. These curves in general deviate from planes parallel to the y,z-plane, as well as from the x,y-plane. This deviation from the y,z-plane is of particular interest, and is due to two causes: (a) the surface density gradient, δ, and (b) the inclination angle, θ_0 . In the subsequent numerical calculations, we shall only consider the case when $\theta_0 = 0$, thus emphasizing the gradient effect, although in principle all cases can be

[1]A weak material is one whose yield stress in tension or shear, say, is small compared to the pressures in a detonation wave.

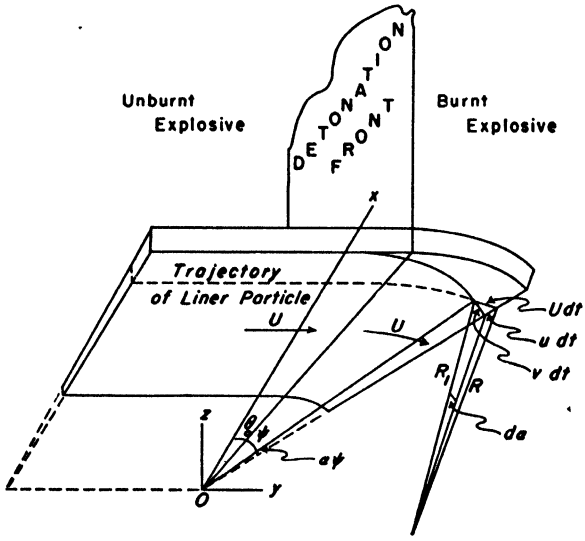


FIGURE 1

handled with essentially the same degree of difficulty. The analytical results which are derived are quite general, however.

In addition to the special conditions and assumptions described in the foregoing, we also assume that the flow of the gas from the burnt explosive is steady, isentropic and irrotational, and in fact, has the form of a simple wave; i.e., gas parameters are functions of a single function, σ, of position. In particular, let us assume that the gas obeys a "polytropic" equation of state, although this may easily be generalized to any functional dependence of pressure on density. The conditions imposed at the detonation front are those of Chapman-Jouguet—the normal velocity component equals the local sound speed.

Conical Surfaces

For orientation we first consider some geometry of conical surfaces. In this way we are able to formulate the problem in terms of finding first the "rate" of rotation, α', of the normal to the deformed shell surface as the surface's unit generating vector, $\mathbf{e}$, sweeps over it.

It is easy to see with the aid of Figure 2 that $\mathbf{e}$ obeys a linear third order differential equation. For, let the initial position of the generating vector be in the x,y-plane (intersection of the detonation front and the material shell) and make an angle θ_0 with the x-axis. Define the angular parameter, ψ, of the surface, by

$$\psi = \int_{\mathbf{e}_0}^{\mathbf{e}} |\, d\mathbf{e}\, |.$$

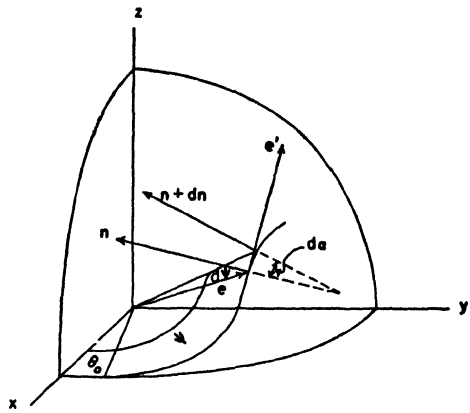


FIG. 2. A theorem on conical surfaces.

Let $\mathbf{e}'$ $(= d\mathbf{e}/d\psi)$ be the tangent vector to the curve traced on the surface of the unit sphere, centered at the origin, by the vector $\mathbf{e}$; let $\mathbf{n}$ be the unit normal to the conical surface generated by $\mathbf{e}$. As ψ changes by $d\psi$, let the direction of $\mathbf{n}$ change by $d\alpha$, so that in the limit $\alpha' = d\alpha/d\psi$ is the aforementioned rate of rotation of $\mathbf{n}$. It is clear from the diagram that

$$d\mathbf{e}' = -\mathbf{e}\, d\psi - \mathbf{n}\, d\alpha$$

and

$$d\mathbf{n} = \mathbf{e}'\, d\alpha.$$

Consequently

$$\mathbf{e}'' = -\mathbf{e} - \mathbf{n}\alpha'$$

and another differentiation yields

$$\mathbf{e}''' = -\mathbf{e}' - \mathbf{n}'\alpha' - \mathbf{n}\alpha''$$

so that, by eliminating $\mathbf{n}$ and $\mathbf{n}'$, we are led to

$$\mathbf{e}''' + \mathbf{e}'[1 + (\alpha')^2] = (\mathbf{e}'' + \mathbf{e})\frac{\alpha''}{\alpha'}, \tag{1}$$

as was to be shown. It is quite obvious that this can be rewritten in the more elegant and suggestive integro-differential form

$$\mathbf{e}'' + \mathbf{e} = -\alpha'\Big(\mathbf{k} + \int_0^{\psi} \mathbf{e}'\alpha'\, d\psi\Big). \tag{2}$$

Clearly, as far as formulation of the problem is concerned, we are left with the task of specifying equations leading to the determination of $\alpha'(\psi)$. For, if

$\alpha'(\psi)$ is once found, then the shape of the deformed surface is obtained from that solution, of equation (2), which has the initial values at $\psi = 0$,

$$\mathbf{e}_0 = (\cos\theta_0\ ,\ \sin\theta_0\ ,\ 0)$$

(2a)

$$\mathbf{e}_0' = (-\sin\theta_0\ ,\ \cos\theta_0\ ,\ 0).$$

Now, the shell particle trajectories are certain geodesics on this surface. If the surface were developed onto the x,y-plane, these geodesics would be straight lines parallel to the y-axis, with the polar equation

$$r\cos\theta = x_0\ .$$

In this equation x_0 is the distance of a rectified geodesic from the y-axis, r is the distance to the origin from a point on the geodesic, and $\theta(=\psi+\theta_0)$ is the polar angle to this point. Upon redeforming the surface, the equation of this geodesic obviously has the form

(3) $$\mathbf{r} = x_0\sec\theta\mathbf{e}$$

where $\mathbf{r} = r\,\mathbf{e}$ is the position vector to the point in question. If ds is the element of arc length along these geodesics, then their direction in space is given by

(4) $$\frac{d\mathbf{r}}{ds} = \mathbf{e}'\cos\theta + \mathbf{e}\sin\theta$$

or, if $\mathbf{e}'$ and $\mathbf{e}$ have limits as θ approaches $\pi/2$, the limiting direction of the geodesics is simply that of $\mathbf{e}$.

Under the assumption that α' and its derivatives exist at $\psi = 0$, one may derive the following series expansions for the equations of the geodesics, using equation (1)

$$\frac{x}{x_0} = 1 + \frac{1}{6}(\alpha_0')^2\tan\theta_0\psi^3 + \frac{1}{24}\{(\alpha_0')^2(1+4\tan^2\theta_0) + 3\alpha_0'\alpha_0''\tan\theta_0\}\psi^4 + \cdots$$

$$\frac{y}{x_0} = \tan(\psi+\theta_0) - \frac{1}{6}(\alpha_0')^2\psi^3 - \frac{1}{24}\{3(\alpha_0')^2\tan\theta_0 + 3\alpha_0'\alpha_0''\}\psi^4 + \cdots$$

(5a)

$$\frac{z}{x_0} = -\frac{1}{2}\alpha_0'\sec\theta_0\psi^2 - \frac{1}{6}\{\alpha_0'' + 3\alpha_0'\tan\theta_0\}\sec\theta_0\psi^3$$

$$+ \frac{1}{24}\{[\alpha_0'(7+(\alpha_0')^2) - \alpha'''] - 4\alpha_0''\tan\theta_0 - 12\alpha_0'\sec^2\theta_0\}\sec\theta_0\psi^4 + \cdots$$

or, if $\theta_0 = 0$

$$\frac{x}{x_0} = 1 + \frac{1}{24}(\alpha_0')^2\theta^4 + \cdots$$

$$\frac{y}{x_0} = \tan\theta - \frac{1}{6}(\alpha_0')^2\theta^3 - \frac{1}{8}\alpha_0'\alpha_0''\theta^4 + \cdots$$

(5b)

$$\frac{z}{x_0} = -\frac{1}{2}\alpha_0'\theta^2 - \frac{1}{6}\alpha_0''\theta^3 - \frac{1}{24}\{\alpha_0'(5-(\alpha_0')^2) + \alpha_0'''\}\theta^4 + \cdots.$$

It turns out, however, for our problem, that although α_0' exists, α_0'' may be infinite, and the expansion is not justified. However, it is interesting to observe that the gradient effect *by itself* (with $\theta_0 = 0$), affects (through α_0') the deviation from a plane parallel to the y,z-plane only through terms of the fourth order, whereas the inclination effect affects the deviation through terms of the third order.

Of more interest, perhaps, is the expression for the limiting value of $\mathbf{e}$ as θ approaches $\pi/2$, under the assumption of a constant value $\bar{\alpha}'$ for α', for all $\theta(\theta_0 \leq \theta \leq \pi/2)$. This is the case of a circular cone, and may be used to obtain approximate deviation estimates. From equations (2) and (2a), we find

$$\mathbf{e} = \mathbf{e}_0 \cos \omega\psi + \frac{\mathbf{e}_0'}{\omega} \sin \omega\psi + \frac{\mathbf{E}}{\omega^2} (1 - \cos \omega\psi), \tag{6}$$

where

$$\omega = \sqrt{1 + (\bar{\alpha}')^2}$$

$$\mathbf{E} = \bar{\alpha}'(\bar{\alpha}'\mathbf{e}_0 - \mathbf{k})$$

and the limiting value occurs for $\psi = \pi/2 - \theta_0$.

Liner Dynamics

Let the speed of the shell particles entering the detonation front be U. Because of the assumption of a weak material, this speed remains constant along the particle trajectories in the conical surface. The surface density distribution is, if r is the distance from the origin to a point on the deformed shell,

$$\rho_M = \delta r \cos \theta$$

and, since U is constant, the distribution is unperturbed by the deformation. Now the principal radii of curvature (Figure 1) of a cone are given by

$$R_\psi = r \frac{d\psi}{d\alpha}$$

$$R_r = \infty$$

so that the curvature of the particle trajectories is

$$\frac{1}{R} = \frac{\cos^2 \theta}{R_\psi} + \frac{\sin^2 \theta}{R_r} = \frac{\cos^2 \theta}{r} \alpha'.$$

Since the normal force per unit area is p, the gas pressure at the surface, we can equate it to the centripetal force/unit area $\rho_M U^2/R$. Hence we are led to the equation of motion

$$p = \delta U^2 \alpha' \cos^3 \theta. \tag{7}$$

"Initial" and Boundary Conditions

Just behind the detonation front, where $\theta = \theta_0$, or $\psi = 0$, we suppose the pressure, p, and density, ρ, of the burnt explosive to have the value p_0 and ρ_0, respectively. Hence, if the local sound speed is c_0, and γ is the polytropic parameter, then

$$c_0 = \sqrt{\gamma(p_0/\rho_0)}.$$

Let a and b be the radial and circumferential components of the burnt gas velocity $\mathbf{q}$ near the shell surface. Since the unburnt explosive material also moves into the detonation front with a speed U in the y-direction, its tangential speed at the front is $U \sin \theta_0$ while its normal speed, $U \cos \theta_0$, equals the detonation velocity V. The tangential speed is unchanged in passage through the front, therefore

$$a_0 = U \sin \theta_0 .$$

By the Chapman-Jouguet condition, we also have

$$b_0 = c_0$$

at the detonation front. Finally, from equation (7), the initial rate of rotation of the normal to the shell is

$$\alpha_0' = p_0/(\delta U^2 \cos^3 \theta_0).$$

At the limiting position specified by $\theta = \pi/2$, where the surface density of the shell is zero, that is, at the origin of coordinates, the burnt gas expands into a vacuum with the limit speed $\hat{q}$. From Bernoulli's law we have

$$\hat{q}^2 = \hat{a}^2 + \hat{b}^2 = U^2 \sin^2 \theta_0 + \left(\frac{\gamma + 1}{\gamma - 1}\right) c_0^2 .$$

Analysis of the Simple Expansion Wave Dynamics

The gas behind the detonation front is supposed to be moving as a simple expansion wave, that is, the gas particle velocity $\mathbf{q}$ is taken to be a function of a parameter $\sigma(x,y,z)$, so that $\sigma = K$ is a one-parameter family of surfaces of constant condition. Because the flow is assumed irrotational, $\nabla \times \mathbf{q} = 0$, from which we quickly derive

$$\nabla\sigma = \mu\mathbf{q}'(\sigma)$$

where μ is a factor of proportionality. Hence the direction of the gradient vector of each surface of constant condition is constant, so that these surfaces must be plane; because the gas velocity at the shell surface must be tangential to it, the surfaces of constant σ intersect the shell in its generating rays. Furthermore, changes $d\mathbf{q}$ in gas particle velocities must be perpendicular to the surfaces of constant condition. In addition, the relations

$$\nabla \cdot \mathbf{q} = \mu \mathbf{q}' \cdot \mathbf{q}'$$

and

$$\mathbf{q} \cdot (\mathbf{q} \cdot \nabla)\mathbf{q} = \mu(\mathbf{q} \cdot \mathbf{q}')^2$$

are easily derived. Multiplication of each side of the Eulerian equations of motion for steady compressible flow by $\mathbf{q}$ yields

$$\rho \mathbf{q} \cdot (\mathbf{q} \cdot \nabla)\mathbf{q} = -\mathbf{q} \cdot \nabla p$$

and the equation of continuity is of course

$$\rho \nabla \cdot \mathbf{q} = -\mathbf{q} \cdot \nabla \rho.$$

Division of each side of this form of the equation of motion by the corresponding sides of the continuity equation yields

$$c^2 \nabla \cdot \mathbf{q} = \mathbf{q} \cdot (\mathbf{q} \cdot \nabla)\mathbf{q}.$$

In this equation the sound speed c is defined by

$$c^2 = dp/d\rho. \tag{8}$$

Hence, from the relations derived above we obtain the simple result,

$$c \mid d\mathbf{q} \mid = \mathbf{q} \cdot d\mathbf{q}, \tag{S}$$

characteristic of simple waves. The Eulerian equation of motion may also be written in the form

$$c^2 \, d\rho + \tfrac{1}{2}\rho d(q^2) = 0 \tag{9}$$

where q is the gas particle speed, and to this must be appended the equation of state, which we take in the form

$$p = p_0(\rho/\rho_0)^\gamma. \tag{10}$$

Gas Kinematics

Figures 3 and 4 illustrate the state of the burnt gas behind the detonation front, and the kinematic relations which may exist at the interface between the burnt gas and the shell surface. Since $d\mathbf{q}$ is perpendicular to the generating rays, a reference to Figure 4 shows that

$$\frac{da}{d\psi} = b. \tag{11}$$

Since, furthermore,

$$q^2 = a^2 + b^2, \tag{12}$$

we have, by equation (11),

$$q \, dq = b(a \, d\psi + db).$$

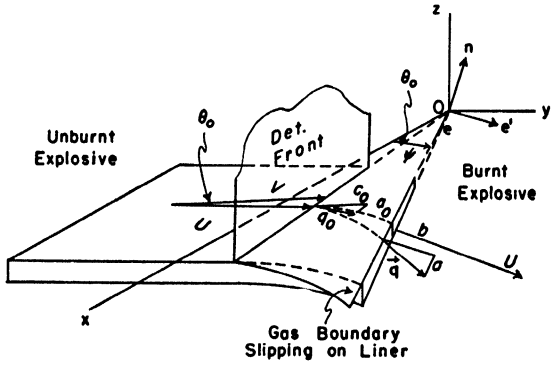


FIG. 3. State of explosive.

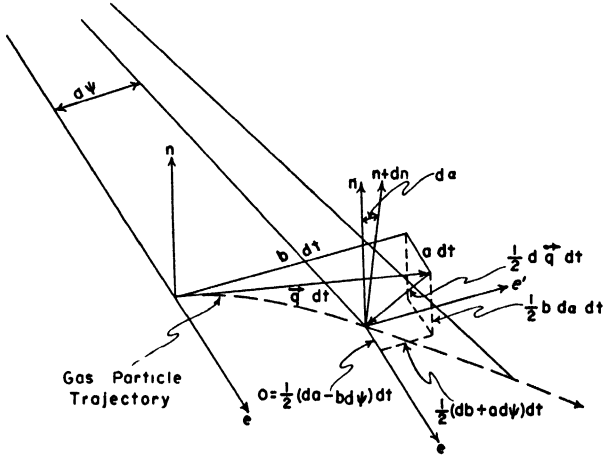


FIG. 4. Gas particle kinematics.

But from Figure 4, we have

$$| d\mathbf{q} |^2 = b^2 d\alpha^2 + (a\, d\psi + db)^2;$$

from this, and the foregoing, and from the simple wave relation (S) we finally derive

$$b^2\alpha'^2 = q^2q'^2\left(\frac{1}{c^2} - \frac{1}{b^2}\right). \tag{13}$$

Summary of the Formulation

The equations to be solved in order to determine the all-important deformation function $\alpha'(\psi)$, form a system of seven algebraic and differential relations (equations (7 to 13)). Equation (9) in conjunction with (8) and (10)

may be integrated at once to yield a Bernoulli relation. Then equation (13) may be rewritten entirely in terms of the variables a and ψ, and all other quantities may be defined in turn in terms of these variables, either algebraically, or by simple differentiation.

For calculational convenience, let us introduce certain dimensionless parameters and variables. Let

$$\lambda = \delta U^2/p_0 , \qquad \nu = (V/c_0) \tan \theta_0 \tag{14}$$

be the gradient and inclination parameters, respectively. Define the six auxiliary variables:

$$\begin{aligned} \bar{p} &= p/p_0 , & \bar{\rho} &= \rho/\rho_0 , \\ \bar{q} &= q/c_0 , & \bar{c} &= c/c_0 , \\ \bar{a} &= a/c_0 , & \bar{b} &= b/c_0 . \end{aligned} \tag{15}$$

Then the system of equations for α' may be summarized in the form

$$\bar{a}'' + \bar{a} = \frac{\bar{a}'}{\lambda \cos^3 \theta}\left(\frac{\gamma - 1}{2}\right)^{\frac{3\gamma-1}{2(\gamma-1)}} \frac{\left\{\frac{\gamma+1}{\gamma-1} + \nu^2 - (\bar{a})^2 - (\bar{a}')^2\right\}^{\frac{3\gamma-1}{2(\gamma-1)}}}{\sqrt{\frac{\gamma+1}{2}[(\bar{a}')^2 - 1] + \frac{\gamma-1}{2}[(\bar{a})^2 - \nu^2]}}, \tag{16a}$$

$$\bar{b} = \bar{a}', \tag{16b}$$

$$\bar{q}^2 = (\bar{a})^2 + (\bar{a}')^2, \tag{16c}$$

$$\bar{c}^2 = \frac{\gamma - 1}{2}\left\{\frac{\gamma+1}{\gamma-1} + \nu^2 - \bar{q}^2\right\}, \tag{16d}$$

$$\bar{\rho} = \bar{c}^{2/(\gamma-1)}, \tag{16e}$$

$$\bar{p} = \bar{\rho}^{\gamma}, \tag{16f}$$

$$\alpha' = \frac{\bar{p}}{\lambda \cos^3 \theta} \tag{16g}$$

where primes indicate differentiation with respect to either ψ or θ, since $\psi = \theta - \theta_0$.

Equation (16a) is to be solved subject to

$$\bar{a} = \nu, \qquad \bar{a}' = 1 \tag{17a}$$

at $\psi = 0$; furthermore, all solutions of physical interest, regardless of initial conditions, must satisfy

$$(\bar{a})^2 + (\bar{a}')^2 = \frac{\gamma+1}{\gamma-1} + \nu^2 \tag{17b}$$

at $\theta = \pi/2$, which appears as a kind of "natural boundary condition" on the solutions of (16a). There will be further discussion of this point later. After finding $\bar{a}$, the other parameters may be found in turn from equations (16b) to (16g). It is only then that $\alpha'(\psi)$ is determined, so that the shape of the deformed surface can be found by integrating equation (2).

The Investigation of the Singularities

Equation (16a) is in a suitable form for numerical integration provided it is known how to treat the singularities at $\psi = 0$, and $\theta = \pi/2$. The first singularity arises from the choice of the initial conditions, requiring the vanishing of the square root in the denominator of equation (16a). The second appears because of the inverse cube of $\cos\theta$ in this equation, and is due to the vanishing of the shell density at $\theta = \pi/2$.

The Initial Singularity

If one expands $\bar{a}$ about $\psi = 0$ in a power series in the cube root of ψ, it is found possible to satisfy the differential equation (16a) and the initial conditions (17a). Thus one obtains

$$\bar{a} = \nu + \psi + a_5\psi^{5/3} + a_6\psi^2 + a_7\psi^{7/3} + a_8\psi^{8/3} + \cdots \tag{18}$$

where

$$\begin{aligned} a_5 &= \frac{9}{10}G, \\ a_6 &= -\frac{2\gamma+1}{4}\left(\frac{\nu}{\gamma+1}\right), \\ a_7 &= \frac{27}{280G}\left\{\left(\frac{\nu}{\gamma+1}\right)^2 + (5-6\gamma)G^3\right\}, \\ a_8 &= \frac{3}{80G^2}\left\{\left(\frac{\nu}{\gamma+1}\right)^3 - \left(\frac{\nu}{\gamma+1}\right)\frac{20\gamma-3}{2}G^3 + 15G^3\tan\theta_0\right\}, \end{aligned} \tag{18a}$$

and

$$G = \sec^2\theta_0\left\{3\lambda^2\frac{\gamma+1}{2}\right\}^{-1/3}.$$

and one also finds

$$\bar{p} = \left[1 - \frac{\gamma}{3}\left\{5a_5\psi^{2/3} + (6a_6 + 3\nu)\psi + 7a_7\psi^{4/3} + (8a_8 - 2\nu a_5)\psi^{5/3} + \cdots\right\}\right] \tag{18b}$$

from which we may compute $\alpha' = \lambda^{-1}\sec^3\theta\bar{p}$. It is interesting to note that the pressure decreases with exceeding rapidity behind the detonation front, and that the derivatives of α' are infinite at the front, as claimed earlier.

It is of course now possible to begin the numerical solution of (16a) by using the series (18) for initial computation. Unfortunately the series converges very slowly, and so can be used only for very small angles ψ. The error can be estimated by comparing the value of $\bar{a}''$ calculated from the series approximation, with that from the differential equation (using the series for $\bar{a}$ and $\bar{a}'$).

The Final Singularity

Near $\theta = \pi/2$, all physically reasonable solutions would be expected to satisfy the condition

$$(\bar{a})^2 + (\bar{a}')^2 = \frac{\gamma + 1}{\gamma - 1} + \nu^2$$

approximately, since this simply states that the gas particle speed approaches the limit speed in a region where the shell yields quickly to the gas pressure because of its low surface density. Unfortunately, if the naive attempt is made to expand the solution in a series of fractional powers of ϵ $(= \pi/2 - \theta)$, the differential equation (16a) itself interposes the limit-speed condition on this type of solution, thereby cutting down the number of arbitrary constants at our disposal to one. In theory this is an insufficient number to join the series solution smoothly with one calculated numerically (i.e., computed by numerical integration up to a value of θ where the rounding-off errors and the singularity begin interacting so badly as to make further numerical integration impracticable). The situation is even worse than this, since a power series solution appears to be possible only for certain values of γ.[2] In spite of these restrictions, however, if the numerical integration can be carried close enough to $\theta = \pi/2$, so that extrapolation of the solution $\bar{a}$ to this limit is reasonable, then, at least for $\gamma = 3$, etc., the series method is useful in calculating the extrapolated values of the other quantities. This procedure is made somewhat more reasonable in what follows later. At this point, however, we simply state the result of the attempt at a series solution, for $\gamma = 3$. That is

$$\bar{a} = f_0\left(1 - \frac{\epsilon^2}{2} + \frac{\epsilon^4}{24} + \cdots\right) + \dot{f}_0\left(\epsilon - \frac{\epsilon^3}{6} + \cdots\right) + \epsilon^3 g_0\left(1 + 2\frac{f_0}{\dot{f}_0}\epsilon + \cdots\right),$$

(19)

$$\dot{\bar{a}} = \dot{f}_0\left(1 - \frac{\epsilon^2}{2} + \cdots\right) - f_0\left(\epsilon - \frac{\epsilon^3}{6} + \cdots\right) + g_0\epsilon^2\left(3 + 8\frac{f_0}{\dot{f}_0}\epsilon + \cdots\right)$$

[2] By a rather remarkable coincidence, the value of $\gamma = 3$ is among these—the value chosen for numerical calculation in the sequel. It is conjectured that the series approach is possible at least for γ-values of the form $(5 + j)/(3 - j)$, where j is 0, 1, or 2, for example.

where a dot means differentiation with respect to $\epsilon = \pi/2 - \theta$, and

$$f_0^2 + \dot{f}_0^2 = 2 + \nu^2,$$
$$g_0 = \frac{\lambda}{6}\frac{1}{\dot{f}_0^2}. \tag{19a}$$

With these values for $\bar{a}$ and $\dot{\bar{a}}$, we find that the dimensionless pressure is given by

$$\bar{p} = \{-6\dot{f}_0 g_0\}^{3/2}\epsilon^3\left\{1 + 3\frac{f_0}{\dot{f}_0}\epsilon + \cdots\right\} \tag{20a}$$

so that α' may be calculated from

$$\alpha' = \frac{1}{\lambda}\{-6\dot{f}_0 g_0\}^{3/2}\left\{1 + 3\frac{f_0}{\dot{f}_0}\epsilon + \cdots\right\}. \tag{20b}$$

Note that if f_0, say, is determined by extrapolation of $\bar{a}$ to $\epsilon = 0$, then all other relevant quantities are determined.

The Iteration Procedure

In order to investigate more fully the solution near $\theta = \pi/2$, we resort to an iteration procedure for calculating it. First let us put

$$n = \frac{3\gamma - 1}{2(\gamma - 1)} > \frac{3}{2}, \qquad H = 2\left(\frac{\gamma - 1}{2}\right)^n \lambda^{-1},$$
$$d = A^2 - \bar{a}^2 - \dot{\bar{a}}^2, \qquad A^2 = \frac{\gamma + 1}{\gamma - 1} + \nu^2 \tag{21}$$

and use ϵ as the independent variable. Now write equation (16a) in two ways:

$$\ddot{\bar{a}} + \bar{a} = -\frac{H}{2\sin^3\epsilon}\frac{\dot{a}d^n}{\sqrt{(\dot{\bar{a}})^2 - \frac{\gamma - 1}{2}d}} \equiv -G(\epsilon, \dot{\bar{a}}, d)\frac{d^n}{2},$$
$$\frac{\dot{d}}{d^n} = \frac{H}{\sin^3\epsilon}\frac{(\dot{\bar{a}})^2}{\sqrt{(\dot{\bar{a}})^2 - \frac{\gamma - 1}{2}d}} \equiv G(\epsilon, \dot{\bar{a}}, d)\dot{\bar{a}}. \tag{22}$$

Next consider two sequences of functions $\{\bar{a}_i\}$ and $\{d_i\}$ defined in part by:

$$d_0 = 0, \qquad \bar{a}_0 = A\cos(\epsilon + \epsilon_0), \tag{23a}$$

and

$$d_{i+1}/d_{i+1}^n = G(\epsilon, \dot{\bar{a}}_i, d_i)\dot{\bar{a}}_i \equiv G_i\dot{\bar{a}}_i; \qquad \dot{\bar{a}}_{i+1} + \bar{a}_{i+1} = -G_i(d_{i+1}^n/2). \tag{23b}$$

Here, ϵ_0 is to be chosen so that at $\epsilon = \beta$ ($\neq 0$), $\bar{a}_0$ has some specified value, k. The sequences will be completely defined if we require that, at $\epsilon = \beta$, $d_i = B^2$,

for $i \geq 1$, and $\bar{a}_i = k$, for $i \geq 0$, while $(\dot{\bar{a}}_i)^2 + (\bar{a}_i)^2 = A^2$ at $\epsilon = 0$. Now we may write, provided the integrals exist,

$$d_{i+1} = \left\{\left(\frac{1}{B^2}\right)^{n-1} - (n-1)\int_\beta^\epsilon G_i \dot{\bar{a}}_i \, d\epsilon\right\}^{-1/(n-1)} \qquad (i \geq 0),$$

(23c)

$$\bar{a}_{i+1} = -\frac{1}{2}\int_0^\epsilon G_i(\tau) d_{i+1}^n(\tau) \sin(\epsilon - \tau)\, d\tau + A \cos(\epsilon + \epsilon_0) \qquad (i \geq 0)$$

as formulas for the successive approximations to the solution of equation (22).

It would be highly desirable to prove that in the limit, as $i \to \infty$, $\bar{a}_i \to \bar{a}$ and $d_i \to d$, where $\bar{a}$ and d satisfy equation (22), with $d = B^2$ and $\bar{a} = k$ at $\epsilon = \beta$, while $d = A^2 - \bar{a}^2 - \dot{\bar{a}}^2$ everywhere. However, in this paper we shall not attempt this; instead we merely calculate d_1 and an approximation to $\bar{a}_1$, and as a check exhibit the agreement with the series "solution". But it should be expressly emphasized that the iteration procedure is more general, in the sense that now we have two parameters, B and ϵ_0 at our disposal.[3] We see that

$$d_1 = \left\{\left(\frac{1}{B^2}\right)^{n-1} - (n-1)\int_\beta^\epsilon \frac{HA \sin(\epsilon + \epsilon_0)}{\sin^3 \epsilon}\, d\epsilon\right\}^{-1/(n-1)}$$

(24a)

$$= \left\{\left(\frac{1}{B^2}\right)^{n-1} + (n-1)HA\left[\frac{\cos\epsilon_0 \cos(\beta - \epsilon)}{\sin\epsilon} + \frac{1}{2}\sin\epsilon_0\left(\frac{1}{\sin^2\epsilon} - \frac{1}{\sin^2\beta}\right)\right]\right\}^{-1/(n-1)}$$

so that, as

$$\epsilon \to 0, \qquad d_1 \simeq \frac{\epsilon^{2/(n-1)}}{\left(\frac{n-1}{2} HA \sin\epsilon_0\right)^{1/(n-1)}} \to 0$$

as it should. Moreover, for small ϵ,

(24b)

$$\bar{a}_1 \cong \int_0^\epsilon L\tau^{(3-n)/(n-1)} \sin(\epsilon - \tau)\, d\tau + A \cos(\epsilon + \epsilon_0)$$

where

$$L = \frac{H}{2\left[(n-1)HA \frac{\sin\epsilon_0}{2}\right]^{n/(n-1)}}.$$

As a check, for $\gamma = 3$ (i.e., $n = 2$), with $A \cos\epsilon_0 = f_0$ and $A \sin\epsilon_0 = -\dot{f}_0$,

$$\bar{a}_1 \cong f_0 \cos\epsilon + \dot{f}_0 \sin\epsilon + \frac{\lambda}{\dot{f}_0^2}(\epsilon - \sin\epsilon)$$

[3] This would have been the result, of course, if we had attempted a series expansion about a regular point—but then the tendency of the solution to the limit-speed condition might not have shown up as explicitly.

which agrees with the series expression (19) up to the third power of ϵ. It is gratifying to note, that for small ϵ values, the arbitrary constant B disappears from the expressions. This fact bears out the former conjecture that the limit speed condition is indeed a "natural boundary condition", in that the order of the second order differential equation (16a) is effectively reduced to unity in the neighborhood of the singular integral $A^2 = (\bar{a}')^2 + (\bar{a})^2$.

Numerical Calculations

We shall choose a typical numerical example to illustrate that calculation of the solution of (16a) is indeed feasible. Let us take $\gamma = 3$, which seems to be close to various estimates for the effective specific heat ratio of the products of detonation. Since we are primarily interested in the surface density gradient effect, we take $\theta_0 = 0$. The values of the remaining relevant constants are given in Table I, in which ρ_s denotes the density of the liner material.

TABLE I

Explosive Data	*Liner Data*
$p_0 = 1.08 \times 10^{11}$ dyne/cm^2	$\rho_s = 8.9$ gm/cm^3
$\rho_0 = 1.6$ gm/cm^3	
$V = 7.2 \times 10^5$ cm/sec	$\delta = .104$ gm/cm^2/cm
$c_0 = 4.5 \times 10^5$ cm/sec	

With these data, we find the value of the gradient parameter to be $\lambda = 1.50$, while the inclination parameter ν vanishes. We also quickly calculate the initial value of α': $\alpha_0' = 0.667 \cdots$.

To begin the solution of the equation (16a), the series approximations, equations (18), are used. Because of the slowness of convergence, these series are only accurate to 4 or 5 places for $\psi = 1^0$, even using the number of terms indicated in equations (18). To extend the solution farther, an analytical form of a graphical method due to E. Meissner[4] is useful, particularly because of the form of equation (16a). The method consists essentially of drawing a curve whose radii of curvature are given by the right side of (16a). Then the distances of the tangents to this curve from the origin of coordinates measure $\bar{a}$, while $\bar{a}'$ is measured by the lengths of the tangents from the points of tangency to the points on the tangents nearest the origin.

The numerical integration can be continued to $\psi = 84°$, where the effect of rounding-off errors and the final singularity make the extension of calculations by this method impractical. The trend of $\bar{a}$ is easily extrapolated to $\psi = 90°$ however, yielding the value $\bar{a} = f_0 = 1.38$. Then using equations (19) and (20),

[4]E. Meissner, *Graphische Analysis vermittelst des Linienbeltes einer Funktion*, Schweizerische Bauzeitung, Zürich, Volume 104, 1934.

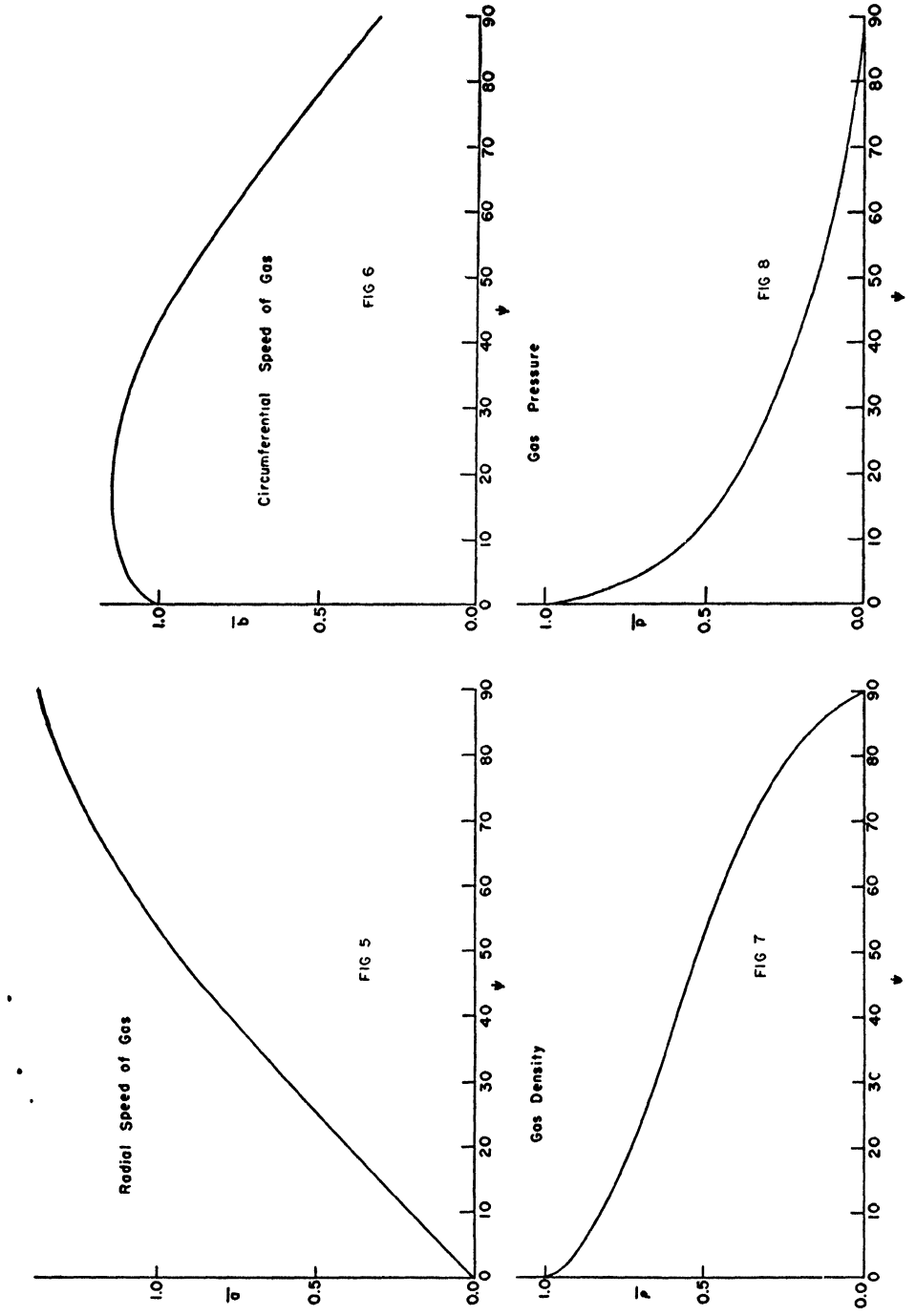


FIG 5

FIG 6

FIG 7

FIG 8

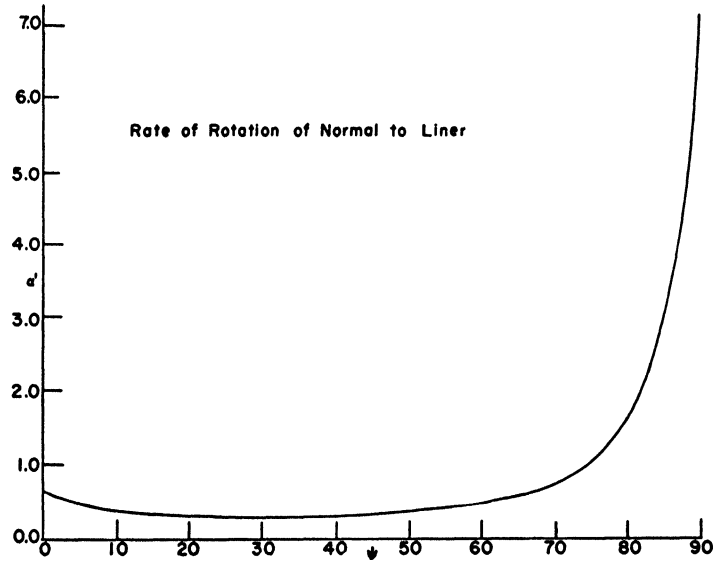


FIGURE 9

the extrapolated values for other quantities (and their slopes) at $\psi = 90°$ can be estimated with fair certainty.

Figures 5–9 summarize the results of these calculations. The rapid rise of α' in Figure 9, near $\psi = 90°$, is due to the small surface density in this region.

Because the example chosen is somewhat artificial, not corresponding exactly to any real explosive, we have not bothered to integrate equation (2) to determine the shape of the deformed plate, or the shape of the particle trajectories (equations (3) and (4)).

This can be done, of course, rather easily, using the Meissner integration method throughout the range of ψ. However, from Figure 9 we see that the average value of α' is of the order of unity. From equation (6), it is easy to show that the inclination effect, under such circumstances, can enhance, nullify, or even reverse the gradient effect, and this is of importance in applications.

I wish to acknowledge the considerable assistance of Dr. C. Gardner in the preliminary formulation of this problem, particularly as regards the simple wave analysis. Also I express my thanks to Dr. N. Rostoker, who pointed out certain errors in the original incomplete memo describing the formulation.

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. VII, 223–243 (1954)

Measurement of Spherical Shock Waves

By HUBERT SCHARDIN*

Introduction

This paper gives a description of measurements of spherical shock waves, produced by detonation, with the aim to verify the validity of theoretical solutions and to furnish the basis for practical solutions of the problems of blast effect.

Some introductory remarks are made on the one-dimensional plane case.

The theoretical basis for the spherical case is given by calculations which use the method of characteristics developed by F. Wecken in 1948–1952.

Experimental data result from different experimental methods e. g.; multiple-spark-camera, streak camera, condenser-microphone, Kerr-cell-photography, X-ray-photography.

Comparison shows that the mathematical calculations are in sufficient agreement with experimental results. They are the only means of getting a complete understanding of the whole phenomenon. The dependence of the peak pressure and impulse on distance, charge, chemical nature of the explosive, and on the physical state of the outside atmosphere gives the basis of the blast effect in general lines.

The theoretical and experimental work has been executed in the Research Laboratory in St. Louis, France.

1. *The One-Dimensional Plane Case*

Before starting the spherical problem, some introductory remarks on the one-dimensional plane case may be given. From the theoretical point of view this case is the simplest; moreover, quite a number of papers on the shock-tube have made the one-dimensional plane case well known.

In the shock-tube a membrane, closing a part of the tube, which is filled with compressed gas, is destroyed at the time zero, and starts the expansion of the gas.

This is equivalent to the case of a high explosive detonation in a tube, in which the compressed gas is replaced by the burnt gases of a cylindrical charge.

In this case, demonstrated in Figure 1, two shock waves of similar form will propagate to the right and left. In the interior two adiabatic rarefaction

*Under contract with the technical service of the French army.

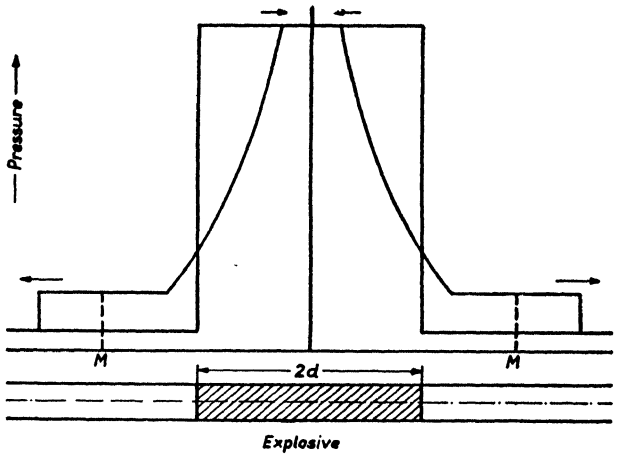


FIG. 1. The one-dimensional plane case: two shock waves propagate to the right and left, two adiabatic rarefaction waves to the interior. M is the boundary between burnt gas and compressed air.

waves arise. Between each compression shock and each zone of rarefaction there is a region of uniform pressure, the material of which originates partly from the air and partly from the gas.

This region was already found experimentally in 1899 by Vieille; in 1910 Kobes made the corresponding calculations up to the time, when the rarefaction waves reach the center.

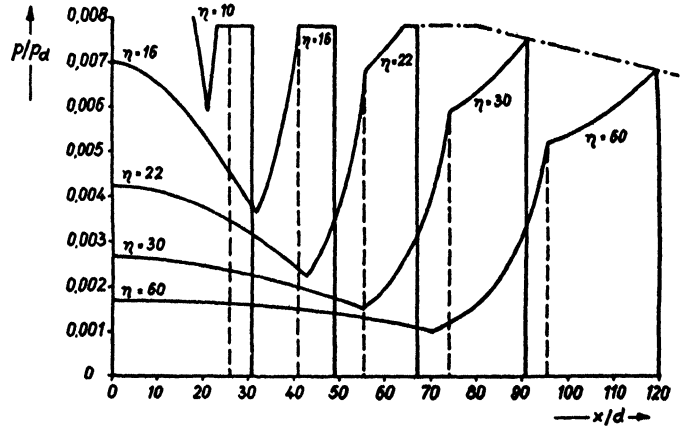


FIG. 2. The one-dimensional plane case at later states (only the right side). η = the time, p/p_d = the pressure over the detonation pressure, x/d = the distance from the center over half the length of the cylindrical charge of H. E. The dotted line is the material boundary (air-burnt gas).

In 1940 Döring continued the theoretical considerations for the case of the linear detonation also for the later states, treating the superpositions of the waves in both directions by means of the method of characteristics.

The results are represented in the diagram (see Figure 2).

As soon as the two rarefaction waves have crossed, the pressure decreases in the center also. Each rarefaction wave overtakes the opposite shock wave, reaching first the back of the region of constant pressure. It cuts down the pressure in this region in the manner shown in the diagram. The passing of the boundary between compressed air and burnt gas causes a new corner in the pressure-curve. Finally the rarefaction waves reach the shock front, the pressure of which now decreases.

In this state the central region has a rather low and uniform pressure; mass, and energy are concentrated near both ends.

In the later states, which have not been calculated, we may assume that in consequence of the supersonic expansion of the interior a second pair of shock waves start from points outside the pressure minimum in Figure 2 into the interior, compensating the pressure. These shocks cross again and follow the opposite first pair of shock waves.

In the shock-tube only the first phases with the uniform pressure region are of practical interest, while for the case of the detonation the later states are important. So much for the plane case.

2. *Calculation of the Spherical-Symmetric Case by the Method of Characteristics*

In practice we have always the case of a three-dimensional detonation. The general case cannot be attacked by today's mathematical means. But it is quite reasonable to restrict the calculation to the spherically symmetric case.

Some authors have tried to treat this problem by approximation methods. However, essential features are lost in these calculations. The adequate method seems to be a calculation step by step, particularly the method of characteristics.

The first attempts in this direction were made by Döring and K. H. Weise (1940). Results were obtained only for the first phases of the phenomenon.

We shall now discuss the calculation of F. Wecken which has been executed during the last years without high speed electronic-computors and has reached a state where an immediate comparison with practical measurements of the expanding shock wave is possible.

3. *The Initial Conditions*

The following initial data are either experimentally measured or derived from experiments and form the numerical basis of the calculation:

Explosive	50% TNT + 50% RDX
Charge	1 kg
Combustion heat	1210 kcal/kg
Detonation velocity	7600 m/sec
Detonation pressure	200.000 bar
Burnt gas velocity on the surface	7069 m/sec
(corresponding to initial pressure of the shock wave)	(654 bar)
Density of the explosive	1.65 g/cm³
Volume of the burnt gases at normal conditions	801 cm³/g

Heat conduction, friction and radiation are neglected as far as the former is not contained in the shock wave equations. Thermodynamic equilibrium is assumed, so that the Abelian equation of state is applicable with a constant exponent γ and a constant covolume.

Unfortunately, at the time of the start of these calculations the remarkable papers of Cook and Jones and Miller were not known.

4. *The Theoretical Results*

Figure 3 shows a radius-time-diagram of the phenomenon. The abscissa represents the distance from the center, and the ordinate the time, both in a log $(1 + x)$ scale.

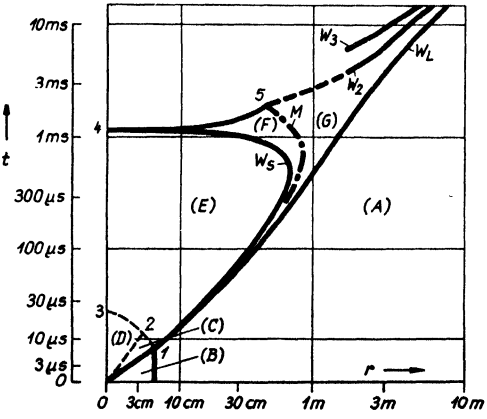


FIG. 3. Radius-time diagram of the formation of shock waves by the detonation of a spherical charge, log $(1 + x)$ scale: (A) undisturbed air, (B) H. E. before detonation, (C) expansion region behind the detonation wave, (D) domain of constant pressure, (E) domain of isentropical expansion of the burnt gas, (F) domain between second shock W_S and material boundary M, (G) domain between first shock and boundary (material-air), W_L first shock, W_2 second shock, W_3 third shock.

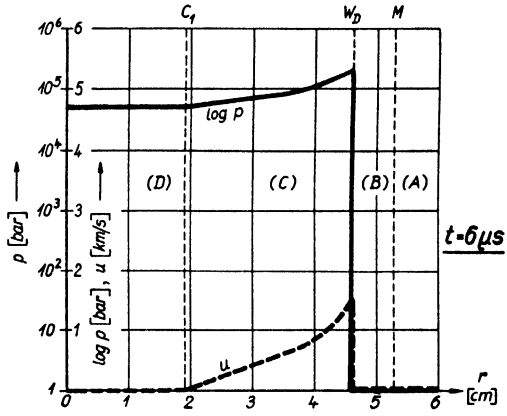


FIG. 4. Cross-section at the time 6 μsec showing pressure and velocity versus distance.

The detonation wave starts from the origin and travels at a constant velocity until it reaches the surface of the explosive (point 1 in Figure 3). Immediately behind the detonation front the pressure amounts to 200.000 atm. In the region (C) it decreases to 50.000 atm., this value being constant in the whole domain (D).

Figure 4 shows a cross-section at the time 6 μsec. after initiation of pressure and material-velocity as functions of radius. In the region (D) the velocity of the burnt gas is zero.

At the point (1) (see Figure 3) the detonation wave has just reached the surface. A shock wave is formed in the air with a pressure of 654 atm.

Contrary to the plane case the point (1) represents a special mathematical singularity. The development of the pressure p and the velocity u in series of $\sqrt{t-t_0}$ indicates already in the first approximation the existence of a second shock wave. The zero order term in this solution represents the shock-tube case.

This second shock wave propagates into the burnt gas with the initial amplitude zero, but later becomes quite strong. In consequence of the high expansion velocity this second shock wave, though facing into the burnt gas, is carried outward.

The expansion begins at the point (1) and spreads along the line (1) (2) (3) towards the center.

The undisturbed expansion region (E) is limited by the second shock wave W_2 and is not influenced from outside.

The material boundary M lies between the two shocks and is practically not far from the first shock during the first 100 μsec.

Figure 5 shows pressure and velocity at 50 μsec. The velocity of the material in the region (E) is about proportional to the distance from the center: the matter in this domain shows the same behavior as in an expanding universe.

Though at this time the pressure in the center is still about 100 atm., it

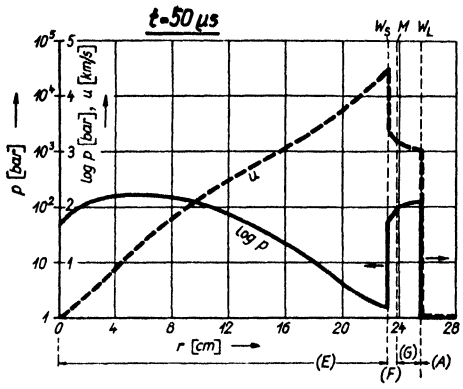


FIG. 5. Cross-section at the time 50 μsec showing pressure and velocity versus distance.

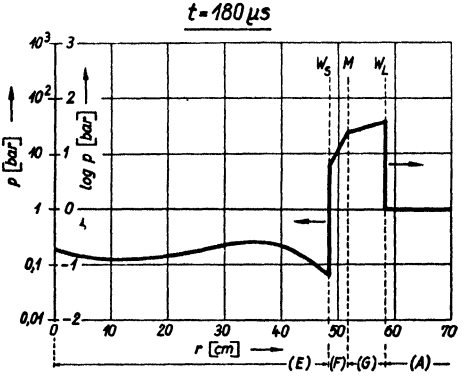


FIG. 6. Cross-section at the time 180 μsec showing pressure and velocity versus distance.

will decrease rapidly owing to the high expansion rate. It finally attains values of 10^{-5} atm., just before the second shock wave reaches the center.

As seen in Figure 6 for the time 180 μsec., nearly the whole explosive energy and most of the mass of the explosive is concentrated in the space between the two shocks, thus forming a hollow sphere at the center.

This phase can be varified experimentally. In this state of the detonation an X-ray photograph shows a ring representing the mass concentration (Figure 7).

Owing to the asymmetry of a real detonation and the instability of the material boundary, the photograph does not show a regular ring, but the essential distribution of material is in accordance with the calculations.

As indicated above, the second shock wave becomes stronger and stronger

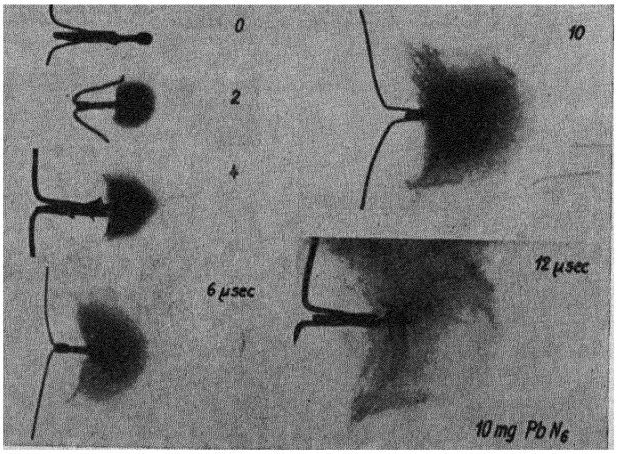


FIG. 7. X-ray photographs of the detonation of 10 mg lead azide at different times. At the front of the expansion of the burnt gas a ring is visible.

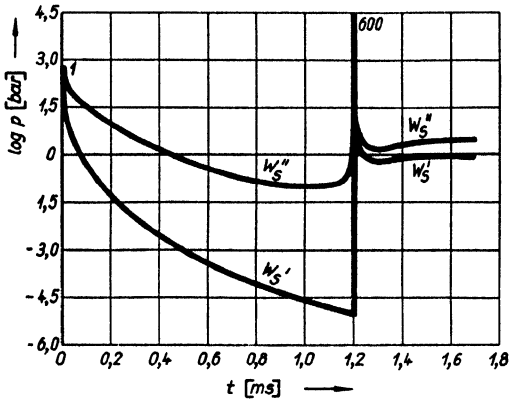


FIG. 8. The pressure in front of and behind the second shock versus time.

and increases its speed accordingly (Figure 8). At about 500 μsec. the velocity of the shock wave becomes higher than the material velocity of the burnt gas: therefore, considered by a resting observer, it turns back and travels concentrically towards the center (Figure 3).

This concentric shock wave is computed up to the neighborhood of the center (4) with the ordinary method of the characteristics. In 1943 Guderley

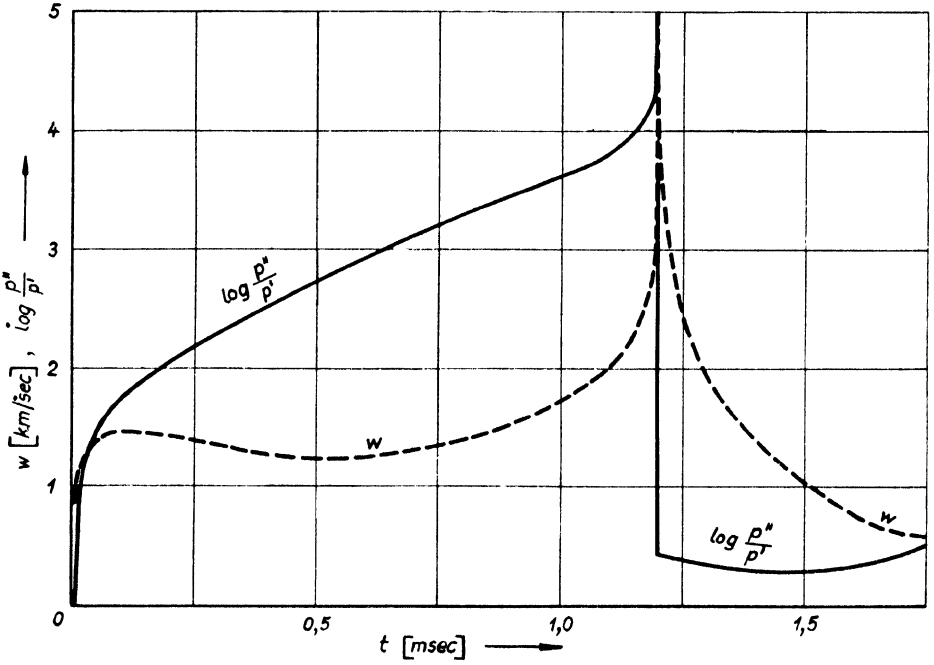


FIG. 9. The ratio of the pressure and the velocity of the second shock.

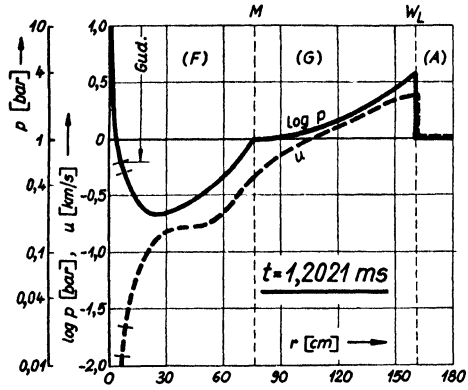


FIG. 10. Pressure and velocity versus distance at the moment when the second shock reaches the center.

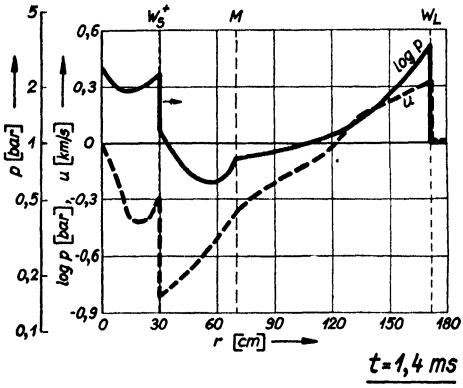


FIG. 11. Pressure and velocity versus distance at the time 1.4 *m* sec.

indicated a convenient method for treating the passage through the center (Figures 9 and 10). After this passage through the center the interior has about normal pressure (Figure 11).

The behavior of the second shock wave is treated by computation up to its intersection with the boundary of the material (5). From this point the second shock leaves the burnt gas and travels through the air behind the first shock wave.

The first shock wave could be computed up to much later times, because it is independent of the state in the interior region behind the second shock.

5. *Experiments on the Detonation Phenomena*

Before entering the comparison of theory and measurement of this shock wave, some figures may be completed which concern experimental facts about the detonation phenomena described hitherto.

In Figure 12 we see four Schlieren-photographs of the detonation of about 1.4 *g* lead azide.

In the first picture the detonation has already occured. The decomposition is not complete, for innumerable solid particles have been thrown through the wave front. Each of them draws a bow wave for itself. Consequently the outer limit of the entire process is irregular.

While in the first image the shock is still invisible, it has clearly separated itself from the burnt gas in the second picture. In the third picture we see the second shock wave. On the last picture this wave is very distinct.

Another representation of the same phenomenon by means of a streak camera is given in Figures 13, 14, and 15. The ordinate shows the direction of

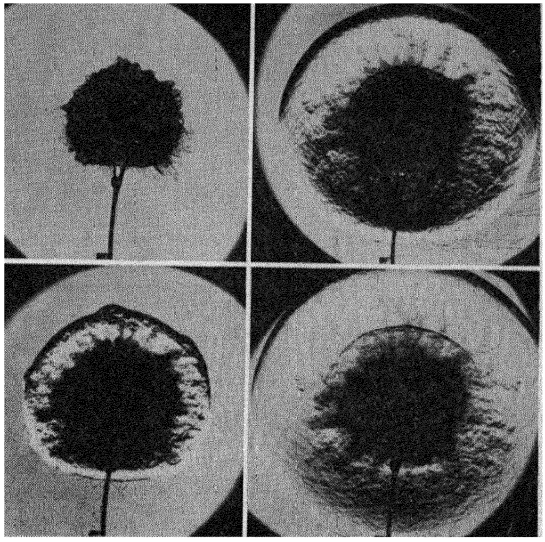


FIG. 12. Four schlieren pictures of the detonation of a small charge of lead azide showing the formation of the first and second shock wave.

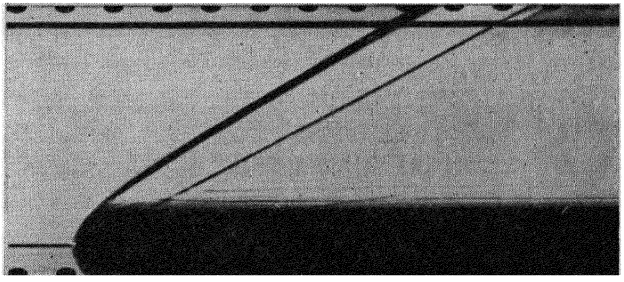


FIG. 13. Streak camera registration of the detonation of a small charge of lead azide. The shock waves are visible by means of a schlieren arrangement.

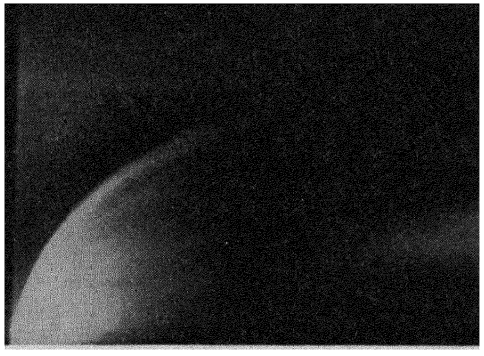


FIG. 14 Streak camera registration in the own light of the detonation of a sphere of H.E. The first shock wave is visible by means of the deviation of the light coming from a mirror.

the expansion; the abscissa gives the time: just as in the graphical representation shown before, we see the radius-time-curve of the first shock wave, of the burnt gas boundary, and of the second shock wave. We also recognize here that in the first instances shock wave and burnt gas-boundary are practically identical. The second shock wave leaves the burnt gas at a time when their expansion is nearly finished.

For higher charges of secondary explosives (as e.g. TNT and RDX) a recording of the detonation is possible by means of a Kerr-cell-arrangement in the detonation's own light.

Figures 16, 17 and 18 show some examples. We see that a difference between theory and practice exists in the assumption of the radial-symmetry of the material boundary. There is more reason for it than the non-ideal detonation: it is also an effect of a mechanical instability of the material-boundary, which does not exist for the shock waves.

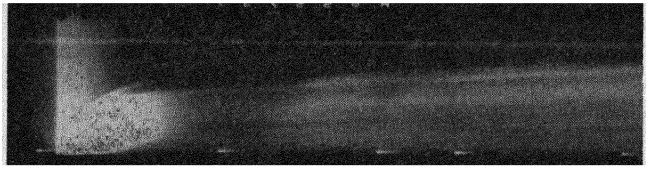


FIG. 15. The same as Figure 14 for later moments. The second shock is visible too.

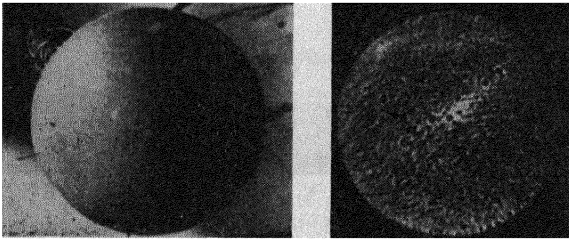


FIG. 16. On the left: normal photograph of a sphere of H.E., 1 kg. On the right: the same charge immediately when the shock wave leaves the surface. Kerr-cell photograph in the own light. Exposure time 10^{-6} sec.

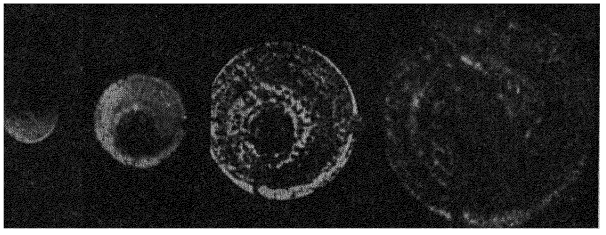


FIG. 17. First picture: normal photograph of the charge. Second, third and fourth pictures: different times after detonation (Kerr-cell photographs).

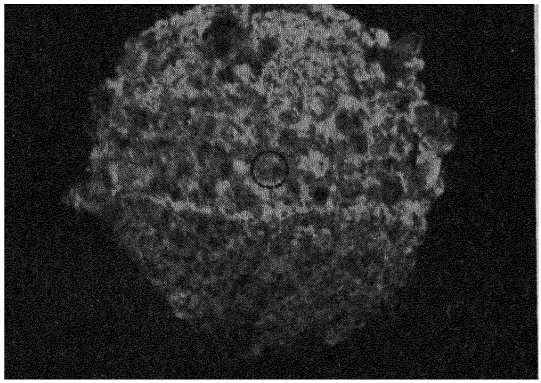


FIG. 18. Kerr-cell photograph of the detonation of a 750 g charge of H.E. at the time $\sim$ 150 μ sec after initiation. The diameter of the burnt gas: 100 cm, equal to 10 times the diameter of the original charge (dotted circle). Remarkable the irregularity of the burnt gas boundary.

Of course, these instabilities cannot be introduced into the calculation; but the following results show that they influence very little the average data for the shock waves.

6. *Time-Space-Curve of the Shock Wave*

The conclusive comparison between theory and experiment is based on the measurement of the time-space-curve or pressure-time-curve of the first shock wave.

The following methods were used:

(1) The determination of the time-space-curve of the shock wave for small charges of primary explosives by means of the multiple spark camera (Figure 19).

Every experiment furnishes 24 consecutive pictures. The time for each is determined with an error of less than 10^{-7} sec.

The accuracy in the space measurement is about 1/10 mm. The 24 frames of every experiment furnish 24 corresponding points in the time-space-curve.

In Figure 20 a sequence of experiments at different states is recorded. Two average curves join the experimental data for the first and the second shock wave respectively.

From this time-space-representation we can find out the local velocity and, by means of the Hugoniot-Rankine equation, the pressure. Thus all experimental results may be collected in one graphical representation, indicating peak-pressure versus distance.

(2) The second possibility to obtain experimental time-space curves is the streak-camera. With the aid of this method we get a record, as shown above.

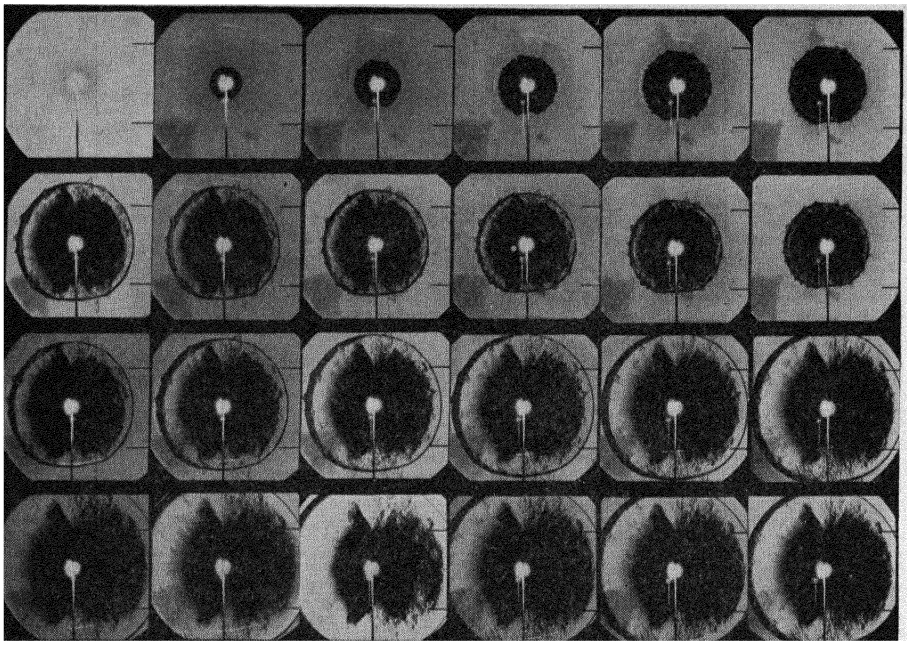


FIG. 19. Series of schlieren pictures of the detonation of a charge of lead azide by means of the multiple-spark camera. Frame rate 200,000/sec.

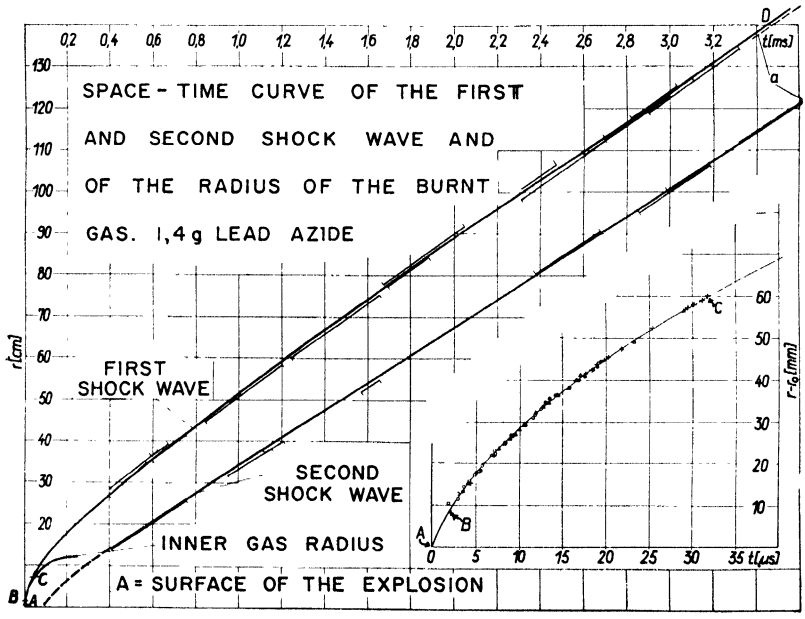


FIG. 20. Space-time curve of the first and second shock wave and of the radius of the burnt gas .01.4 *g* lead azide.

It covers a considerable region in time and space for small charges of initial explosive but a smaller one for bigger charges of secondary explosives.

(3) In order to record the time-space-curve of the shock wave for bigger charges and greater distances a direct measurement of the time interval between the moment of the detonation and the arrival of the shock wave can be applied. Any short-time recorder may be used.

In order to establish the whole curve, we need, however, either a great number of instruments or many repetitions of the experiment.

7. *Pressure-Registrations*

These three methods furnish only the velocity or the peak-pressure of the shock wave. The most important means for measuring the whole course of the pressure as a function of time for a given distance is the pressure registration with the aid of microphones (see Figure 21).

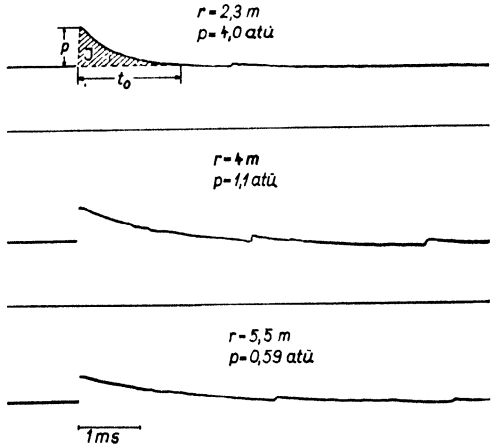


FIG. 21. Three curves of shock waves of a charge of 740 *g* for different distances, registered with the condenser microphone.

In the Research Laboratory in St. Louis, piezo-electric equipment was used occasionally. However, the condenser-microphone shown in Figure 22 is usually employed.

The main part of the condenser-microphone is a steel pot, the bottom of which is turned on a lathe to such a thickness, that it forms a membrane the natural frequency of which is 40.000 c/sec. This membrane forms one plate of a condenser, the capacity of which is a linear function of the pressure. The capacity can be measured either by direct current or by carrier frequency methods.

The direct current method is employed for short-time (small charges), the carrier frequency method is useful for large charges.

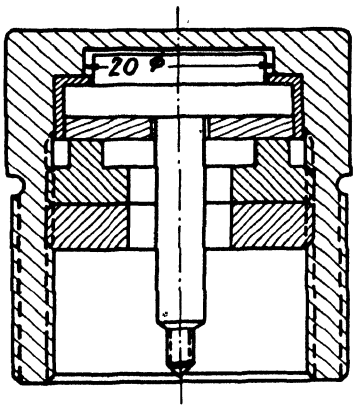


FIG. 22. Design of a condenser microphone, used for the registration of shock waves.

The mechanical natural frequency of the device is filtered by the amplifier circuit.

8. *Arrangement of the Detonation Experiments*

The microphone may be arranged in two different ways (Figure 23):

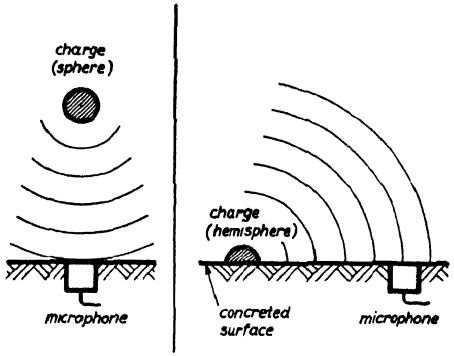


FIG. 23. Arrangements for the registration of shock waves with the microphone.

At the left the sphere of high explosive is suspended at a certain altitude over the ground, which is planed and concreted. Just under the charge is the microphone in the plane of the ground.

The pressure measured in this way is not the same as in the free propagation of the shock wave; it is increased by the reflection.

The increase ought to be computed by the characteristic method. But many examples have shown that the free pressure can be found sufficiently exactly from the reflected pressure by a constant reduction factor, which depends only on the peak pressure. The same factor is valit for the impulse.

The second arrangement uses a hemispherical charge on the ground. If the ground would form a perfect mirror, the microphone would record the free pressure. But the reflection is not perfect, and the ground attenuates the shock wave. The resulting mistake by employment of a charge factor 2 compared with a detonation in free space amounts to about 10%.

In general, the first method was employed; for big charges only technical complications caused the use of the second arrangement.

9. *Comparison of Theory and Experiment*

Now we consider the results of practical measurements and compare them with the theory.

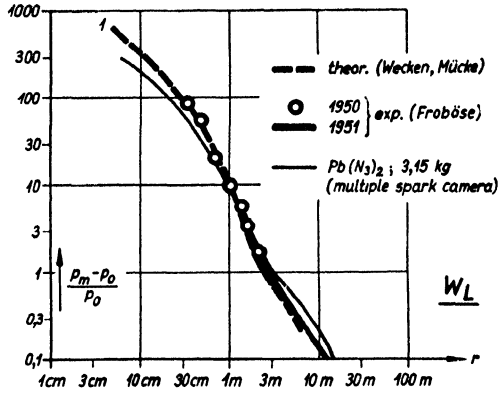


FIG. 24. Pressure versus time for the detonation of 1 kg TNT + RDX (1:1) and 3.15 kg $Pb(N_3)_2$.

Figure 24 shows for the detonations of 1 kg explosive (50% TNT + 50% RDX) the pressure-distance curve.

The dotted line shows the result of the calculation. The circles correspond to the direct time-space-measurements, and the thick line is the result of the pressure measurement by means of the condensor-microphone. As we see, there is a very good agreement. Further is drawn a thin line as the result of the measurements with a multiple spark camera found from the detonation of lead azide. But this curve does not correspond to a charge of 1 kg but of 3.15 kg. This charge gives the same energy as 1 kg of the high explosive. Thus this representation gives a good agreement of the pressure curves for both explosives. So much concerning the first shock wave.

For the second shock wave we again examine Figure 3; as formerly mentioned, the calculation ended at point 5. From the experiments we obtained the curve W_2, which reaches about 3 msec. The dotted line joins measurement and theory quite continuously.

The third shock wave, found experimentally, is represented by curve W_3.

10. *Influence of the Charge-Weight*

Until now we considered the dependence of the pressure on time and distance for the charge of 1 kg. The next question to be treated is the influence of the charge-weight. For this purpose we apply the law of similarity which was, as it seems, introduced into physics of detonation by Cranz. This law states, that detonation phenomenon may be derived from another, if for the same materials time and space have the same scale factor. Velocities and pressures in the corresponding time-space points are identical.

This law neglects heat conduction (except that contained in the shock waves), the Reynolds Number and all material properties which depend on time. It is quite important to establish experimentally the limits of validity of the Cranz law. The measurements have shown that the model-law is certainly valid within the precision of measurement itself for a charge domain from 10 *g* up to 10 *t*, that is for 6 orders of magnitude for the charge. As an example

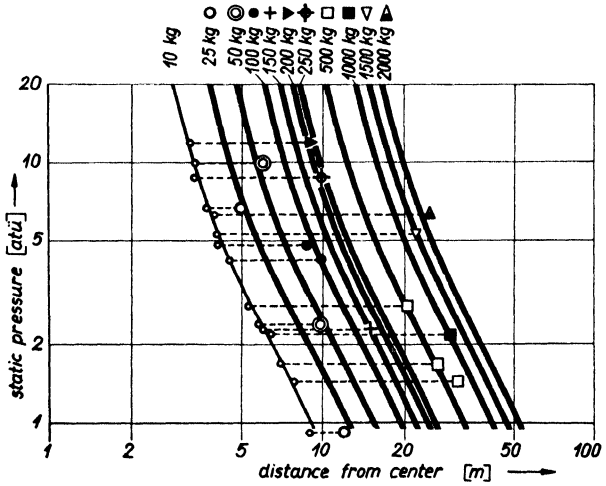


FIG. 25. Transformation of the peak shock wave pressure of different charges to a standard charge of 10 kg.

Figure 25 shows the transformation of the peak pressure of different charges to a standard charge of 10 kg. All transformed values fit well into one smooth curve. One can point out, that the experimental data are influenced

(1) by the imprecision of the measuring method,

(2) by the individual variation of each experiment, owing to the burnt gas front instability.

The first influence, the measuring precision, is not better than 3% for good measurements; the second influence, the dispersion of detonation itself, may reach about the same value.

11. *The Impulse*

For the consideration of the destruction effect of the shock wave, the so called blast effect, the impulse is as important as the peak pressure, a fact which has been neglected often. According to the Cranz law, the scale factor for the impulse is identical with the scale factor of the space, a behavior quite different from that of the pressure. A general destruction theory shows, how the destruction of a certain object depends on the two variables peak pressure and impulse.

12. *Influence of the Atmospheric Conditions*

The model law is only valid for identical atmospheric conditions, such as air-pressure, temperature, and composition. To find out the influence of the air-pressure, one made experiments at different altitudes. In particular measurements were executed on the Zugspitze in an altitude of 2600 m (8500 feet). Other experiments were arranged in a pressure tank. The results of these experiments regarding peak pressure and impulse are represented in Figure 26.

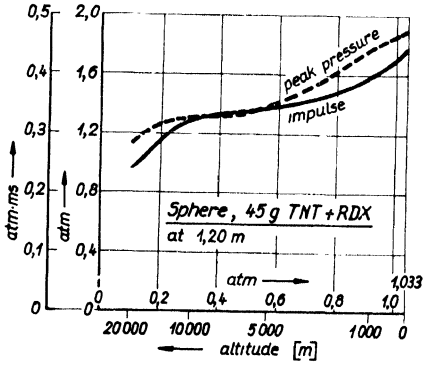


FIG. 26. Peak pressure and impulse versus outside air-pressure.

The influence of the temperature can be found out from the series of experiments which took place at different temperatures. Statistics had to be applied and yielded the following result: the coefficient of temperature-influence on the pressure for 1 kg of *HE* at 18° is negative for distances below 2 m, positive for greater distances and amounts to 1% per degree temperature variation at 4 m distance for 1:1 mixture of TNT + RDX.

13. *Influence of the Nature of the High Explosive*

If we consider a high explosive of different chemical nature the detonation phenomenon is changed. But owing to the similarity law we only have to establish the physical conditions for one standard charge. For this charge the

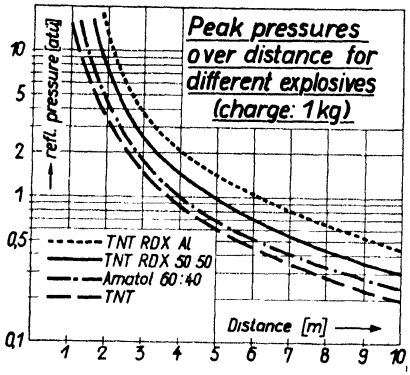


FIG. 27. Peak pressure versus distance for different explosives.

peak pressure and impulse must be measured anew for all distances (Figure 27).

Corresponding to the remarks made in Section 9 about lead azide and TNT and RDX, we can use in first approximation the equivalent of the pressure curve starting from the same energy. Thus, it is possible to find out first values for the pressure curve of a new explosive.

14. *Closing Remarks*

Now we know the theoretical and practical basis of the blast effect in general, that is, the dependence of peak pressure and impulse on distance, charge, chemical nature, air-pressure, and temperature. But there is a great number of details to be resolved.

Since the mathematical calculations are in sufficient agreement with practice, and since this agreement can still be improved, it is obvious that calculations with adequate computers may be very useful in the future.

At the end the question may be touched how far this knowledge may be applied to phenomena of a quite different *physical* origin, e.g. in the case of the atomic bomb. We compare the curve of peak pressures extrapolated from the above curves to a charge of 20.000 t of TNT with the pressure curve of a normal atomic bomb indicated in the book, *The Effect of Atomic Weapons*. The comparison is seen in Figure 28. The coincidence is astonishingly good. Thus it is possible to study some details of the effect of the atomic bombs with the aid of model experiments executed with normal high explosives.

BIBLIOGRAPHY

[1] Vieille, P., *Étude sur le rôle des discontinuités dans les phénomènes de propagation*, Mémoires des Poudres et Salpètres, Volume 10, 1899, pp. 117–260.

[2] Vieille, P., *Sur les discontinuités produites par la détente brusque de gaz comprimés*, Comptes Rendus de l'Académie des Sciences, Volume 129, 1899, pp. 1228–1230.

[3] Kobes, K., *Die Durchschlagsgeschwindigkeit bei den Luftsauge und Druckluftbremsen*,

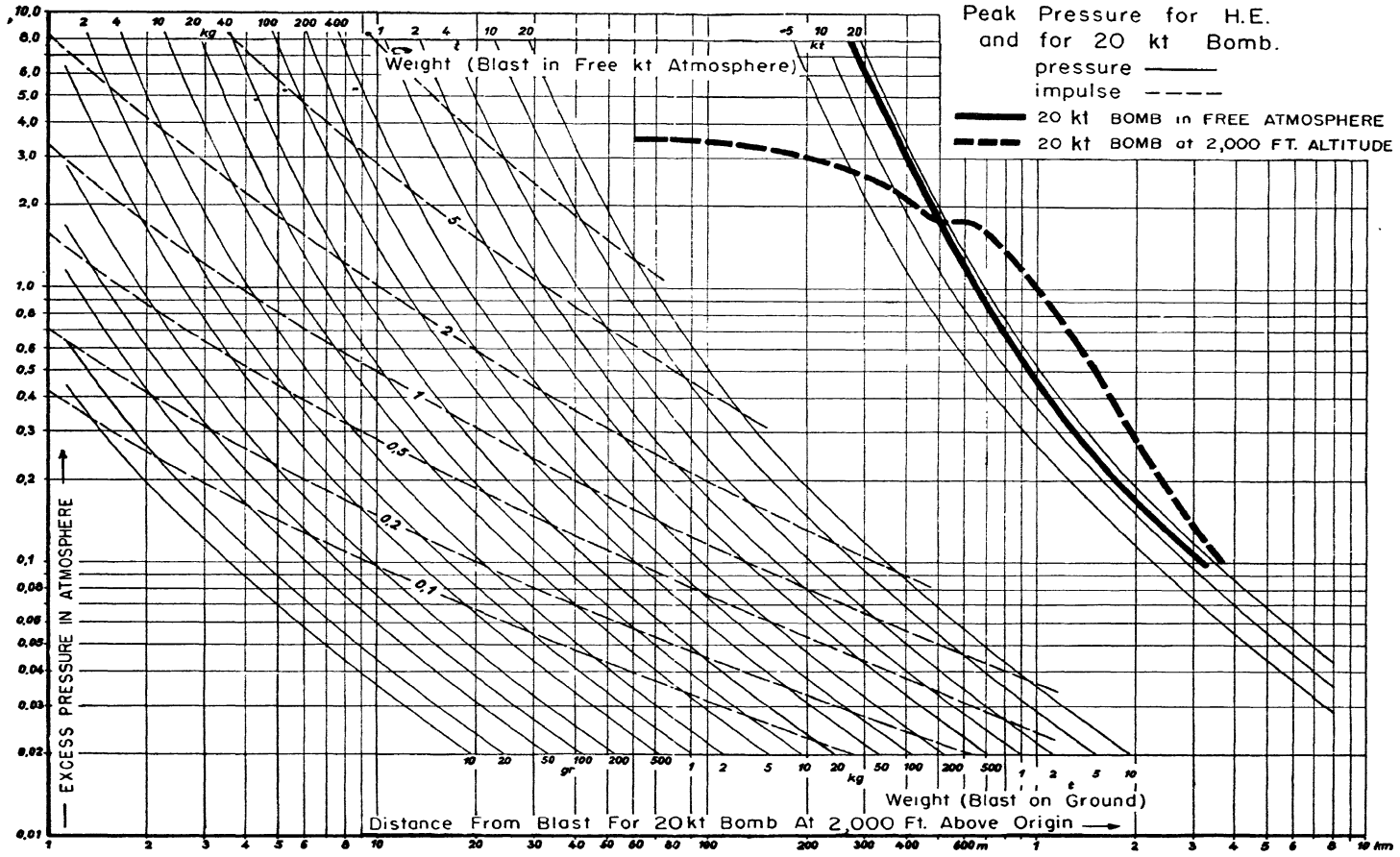


FIG. 28. Peak pressure and impulse for different charges of H.E. versus distance. Extrapolation of the pressure curves up to 5, 10, 20 kt; comparison with the normal atomic bomb.

Zeitschrift des Österreichischen Ingenieur- und Architektenvereins, Volume 62, 1910, p. 558.

[4] Schardin, H., *Bemerkungen zum Druckausgleichsvorgang in einer Rohrleitung*, Zeitschrift für Physik, Volume 33, 1932, pp. 60–64.

[5] Sauter, F., *Ansätze zur Behandlung der kugelförmigen Ausbreitung einer Knallwelle; Probleme der Detonation*, Deutschen Akademie der Luftfahrtforschung, Schrift 1032/41g, 1941.

[6] Guderley, G., *Starke kugelige und zylindrische Verdichtungsstösse in der Nähe des Kugelmittelpunktes beziehungsweise der Zylinderachse*, Luftfahrtforschung, Volume 19, 1942, pp. 302–312.

[7] Kirkwood, J. G., and Bethe, H. A., *The pressure wave produced by an underwater explosion, Part I, Progress report*, Office of Scientific Research and Development, Division B, Serial 252, No. 588, 1942.

[8] Landau, L. D., *On shock waves*, Journal of Physics, Moscow, Volume 6, No. 5, 1942, pp. 229–230.

[9] Schardin, H., *Die physikalischen Grundlagen der Wirkung der Detonation*, Deutsche Akademie der Luftfahrtforschung, 1943.

[10] Reynolds, G., *A preliminary study of plane shock-waves formed by bursting diaphragms in a tube*, Office of Scientific Research and Development, No. 1519, 1943.

[11] Döring, W., and Burkhardt, G., *Beiträge zur Theorie der Detonation*, Zentrale fur Wissenschaftliches Berichtswesen, No. 1939, 1943, pp. 203–207, 245–259.

[12] Kirkwood, J. G., and Brinkley, S. R., *Theory of the propagation of shock-waves from explosive sources in air and water*, Office of Scientific Research and Development, No. 4814, 1945.

[13] Du Mond, C., and Panofsky, D., *A determination of the wave forms and laws of propagation and dissipation of ballistic shock-waves*, Journal of the Acoustical Society of America, Volume 18, No. 1, 1946, pp. 97–118.

[14] Brinkley, S. R., and Kirkwood, J. G., *Theory of the propagation of shock-waves from infinite cylinders of explosives*, Physical Review, Volume 72, No. 11, 1947, pp. 1109–1113.

[15] Cassen, B., and Stanton, J., *Theory of non-steady shock-waves*, Physical Review, Volume 72, No. 2, 1947, p. 180.

[16] Richter, H., *Stoss- und Detonationsfronten bei Abelscher Zustandsgleichung*, Laboratoire de Recherche, St. Louis, France, 31/47, 1947.

[17] Sauer, R., *Mathematische Methoden für nichtstationäre Probleme der Gasdynamik*; see Sauer, R. and Sauter, F., *Nichtstationäre Gasdynamik*, Tagungsbericht, Laboratoire de Recherche, St. Louis, France, 21/47, Part I, 1947, p. 22.

[18] Stoner, R. G., and Bleakney, W., *The attenuation of spherical shock-waves in air*, Journal of Applied Physics, Volume 19, 1948, pp. 670–678.

[19] Mautz, C. W., Geiger, F. W., and Epstein, H. T., *On the investigation of supersonic flo patterns by means of a shock-tube*, Physical Review, Volume 74, 1948, p. 1872.

[20] Cole, R. H., *Underwater Explosions*, Princeton University Press, Princeton, 1948.

[21] Wecken, F., *Zur Energiebilanz bei unverdämmter Sprengung*, Laboratoire de Recherche, St. Louis, France, 5a/48, 1948.

[22] Cook, M. A., *Fungacity determinations of the products of detonation*, Journal of Chemical Physics, Volume 16, No. 11, 1948, pp. 1081–1086.

[23] Jones, H., and Miller, A. R., *The detonation of solid explosives, the equilibrium conditions in the detonation wave-front and the adiabatic expansion of the products of detonation*, Royal Society of London, Series A, Proceedings, Volume 194, 1948, p. 480.

[24] Mautz, C. W., *The use of the shock tube in the production of uniform fields of transonic and supersonic flow*, Dissertation, University of Michigan, 1949.

[25] Geiger, F. W., and Mautz, C. W., *The shock tube as an instrument for the investigation of*

transonic and supersonic flow patterns, Office of Naval Research, Report N6-ONR-232, University of Michigan, Engineering Research Institute, 1949.

[26] Taylor, G., *The dynamics of the combustion products behind plane and spherical detonation fronts in explosives*, Royal Society of London, Series A, Proceedings, Volume 200, 1950, pp. 235–247.

[27] Penny, W. G., and Pike, H. H., *Shock waves and the propagation of finite pulses in fluids*, Physical Society of London, Reports on Progress in Physics, Volume 13, 1950, pp. 46–82.

[28] Wecken, F., *Expansion einer Gaskugel hohen Druckes*, Zeitschrift für Angewandte Mathematik und Mechanik, Volume 30, No. 8/9, 1950, pp. 270 ff.

[29] Taylor, G., *The formation of a blast wave by a very intense explosion*, Royal Society of London, Series A, Proceedings, Volume 201, 1950, pp. 159–186.

[30] Los Alamos Scientific Laboratory, *The Effects of Atomic Weapons*, McGraw-Hill, New York, 1950.

[31] Wecken, F., *Die Detonation einer kugelformigen Sprengladung, Part I*, Laboratoire de Recherche, St. Louis, France, 8/50, 1950.

[32] Wecken, F., *Explosion kugelförmiger Ladungen*, Laboratoire de Recherche, St. Louis, France, 14m/51, 1951, pp. 141–162.

[33] McFadden, J. A., *Initial behavior of a spherical blast*, Journal of Applied Physics, Volume 23, No. 11, 1952, pp. 1269–1275.