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GEOMETRY
OF
FOUR DIMENSIONS

BY

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IN THE UNIVERSITY OF CAMBRIDGE

VOLUME II

CAMBRIDGE

AT THE UNIVERSITY PRESS

1930

PRINTED IN GREAT BRITAIN

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CHAPTER XV.

REGIONS: FUNDAMENTAL MAGNITUDES: LINEAR CURVATURE.

Analytical definition of a region: the fundamental primary magnitudes.

262. We now proceed to consider the geometry of a region in homaloidal quadruple space, a region constituting the most extended sub-amplitude in that space. The properties discussed will be those of a descriptive and organic character, intrinsic and not topographical. Moreover, flats have already been discussed: and therefore it will be sufficient to discuss non-homaloidal triple regions. For the purpose, we require invariantive forms: that is, combinations of magnitudes which remain unchanged in value when the independent variables are changed. The last requirement will be met, initially, merely by the construction of analytical expressions for the geometrical quantities concerned: the occurrence of these particular forms leads to the general theory of the associated invariantive forms. The development of this theory will be deferred for the present.

A curved region in the quadruple space can be defined analytically by the postulation of a relation between the variables x, y, z, v, the relation not being linear. Such a definition, like the Cartesian definition of a surface in triple space, is readily convenient for the discussion of topographical properties: it is less suited for the discussion of intrinsic properties, if only because the initial axes of reference tend to occur too obtrusively in their irrelevance as regards these intrinsic properties.

An alternative definition is provided by the full parametric expression of the four coordinates of any point included in the region. As these coordinates are connected by only a single equation, three independent parameters are needed for the expression: they are denoted by p, q, r. Consequently, the region will be represented parametrically by means of equations

$$x = x(p, q, r), \quad y = y(p, q, r), \quad z = z(p, q, r), \quad v = v(p, q, r);$$

and these functions are to be such that not more than one relation, free from parameters, subsists among the four variables x, y, z, v.

We shall be concerned with derivatives of x, y, z, v, and of cognate quantities, with respect to p, q, r; and these derivatives will be of various orders. In accordance with one customary notation, we take

$$\frac{\partial\omega}{\partial p}, \frac{\partial\omega}{\partial q}, \frac{\partial\omega}{\partial r}; \frac{\partial^2\omega}{\partial p^2}, \frac{\partial^2\omega}{\partial p\partial q}, \frac{\partial^2\omega}{\partial p\partial r}, \frac{\partial^2\omega}{\partial q^2}, \frac{\partial^2\omega}{\partial q\partial r}, \frac{\partial^2\omega}{\partial r^2};$$

$$= \omega_1, \omega_2, \omega_3; \omega_{11}, \omega_{12}, \omega_{13}, \omega_{22}, \omega_{23}, \omega_{33};$$

respectively: and so on, in the various orders of differentiation. Thus the assumption concerning the functional expressions of x, y, z, v, in terms of p, q, r, is analytically represented by an assumption that no two of the determinants

$$\left\| \begin{matrix} x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \\ z_1, & z_2, & z_3, & z_4 \end{matrix} \right\|$$

can vanish identically.

As the fundamental element for a region, we take the elementary distance between two neighbouring points x, y, z, v; and $x+dx, y+dy, z+dz, v+dv$; in the homaloidal quadruple space. This elementary distance is denoted by ds: thus

$$\begin{aligned} ds^2 &= \Sigma \{(x+dx)-x\}^2 \\ &= dx^2+dy^2+dz^2+dv^2, \end{aligned}$$

where Σ denotes summation over the four coordinates. Now, to the first order of small quantities, we have

$$\left.\begin{aligned} dx &= x_1 dp + x_2 dq + x_3 dr \\ dy &= y_1 dp + y_2 dq + y_3 dr \\ dz &= z_1 dp + z_2 dq + z_3 dr \\ dv &= v_1 dp + v_2 dq + v_3 dr \end{aligned}\right\}.$$

For the expression of ds^2 in terms of dp, dq, dr, we take

$$\left.\begin{aligned} A &= x_1^2+y_1^2+z_1^2+v_1^2, & F &= x_2x_3+y_2y_3+z_2z_3+v_2v_3 \\ B &= x_2^2+y_2^2+z_2^2+v_2^2, & G &= x_3x_1+y_3y_1+z_3z_1+v_3v_1 \\ C &= x_3^2+y_3^2+z_3^2+v_3^2, & H &= x_1x_2+y_1y_2+z_1z_2+v_1v_2 \end{aligned}\right\};$$

and then

$$\begin{aligned} ds^2 &= A\,dp^2+2H\,dp\,dq+2G\,dp\,dr+B\,dq^2+2F\,dq\,dr+C\,dr^2 \\ &= (A, B, C, F, G, H \mathbin{\text{⟩⟨}} dp, dq, dr)^2, \end{aligned}$$

the central expression for the arc-element of the region. We shall also require derivatives with regard to s: again the customary notation

$$p' = \frac{dp}{ds}, \quad p'' = \frac{d^2p}{ds^2}, \ \ldots$$

and similarly for q and r, is used. The foregoing expression for ds^2 thus yields a permanent relation

$$(A, B, C, F, G, H \mathbin{\text{⟩⟨}} p', q', r')^2 = 1.$$

The quantities A, B, C, F, G, H, are called the *fundamental magnitudes of the first order* (or *primary magnitudes*) belonging to the region.

We at once infer that, if ω be the angle between two directions p_1', q_1', r_1', and p_2', q_2', r_2',

$$\cos\omega = (A, B, C, F, G, H \mathbin{\text{⟩⟨}} p_1', q_1', r_1' \mathbin{\text{⟩⟨}} p_2', q_2', r_2');$$

and thus the directions are perpendicular if

$$Ap_1'p_2' + H(p_1'q_2' + q_1'p_2') + G(p_1'r_2' + r_1'p_2') + Bq_1'q_2' + F(q_1'r_2' + r_1'q_2') + Cr_1'r_2' = 0.$$

Properties, corresponding to those for directions on surfaces (§ 206) in free space, can easily be established—they will be obtained hereafter, when they are required—for curves and for surfaces in a region.

Within the region, we shall have families of surfaces and families of curves. In particular, there are parametric surfaces,

the surface P, over which p is constant, while q and r are variable,

..... Q, ... q ..... r ... p ... variable,

..... R, ... r ..... p ... q ... variable.

In particular, there are parametric curves,

the curve p, along which q and r are constant, while p is variable,

..... q, ... r ... p ..... q ... variable,

..... r, ... p ... q ..... r ... variable.

Thus the curves q and r are parametric on the surface P: the curves r and p are parametric on the surface Q: the curves p and q are parametric on the surface R.

Ex. Shew that, if p_1', q_1', r_1'; p_2', q_2', r_2'; p_3', q_3', r_3'; are three directions in the region which are perpendicular to one another in pairs,

$$p_l' = \begin{vmatrix} a, & h, & g \\ p_m', & q_m', & r_m' \\ p_n', & q_n', & r_n' \end{vmatrix}, \quad q_l' = \begin{vmatrix} h, & b, & f \\ p_m', & q_m', & r_m' \\ p_n', & q_n', & r_n' \end{vmatrix}, \quad r_l' = \begin{vmatrix} g, & f, & c \\ p_m', & q_m', & r_m' \\ p_n', & q_n', & r_n' \end{vmatrix},$$

where $l, m, n, = 1, 2, 3$, in cyclical rotation, and

$$a = BC - F^2, \quad b = CA - G^2, \quad c = AB - H^2,$$
$$f = GH - AF, \quad g = HF - BG, \quad h = FG - CH.$$

Supplementary magnitudes: relations.

263. Arising out of combinations of the first-order derivatives of the coordinates, there are certain supplementary magnitudes: and these magnitudes, as will be seen later, initially are associated—one set, with the flat that is tangential to the region at any point: the other set, with the orientation-coordinates of the tangent planes to the parametric surfaces P, Q, R.

The first set of supplementary magnitudes is constituted by

$$J_x = \begin{vmatrix} y_1, & z_1, & v_1 \\ y_2, & z_2, & v_2 \\ y_3, & z_3, & v_3 \end{vmatrix}, \quad J_y = \begin{vmatrix} z_1, & v_1, & x_1 \\ z_2, & v_2, & x_2 \\ z_3, & v_3, & x_3 \end{vmatrix},$$

$$J_z = \begin{vmatrix} v_1, & x_1, & y_1 \\ v_2, & x_2, & y_2 \\ v_3, & x_3, & y_3 \end{vmatrix}, \quad J_v = \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}.$$

By the lemma (§ 16), we have

$$J_x^2+J_y^2+J_z^2+J_v^2=\begin{vmatrix} \Sigma x_1^2 , & \Sigma x_1x_2, & \Sigma x_1x_3 \\ \Sigma x_1x_2, & \Sigma x_2^2 , & \Sigma x_2x_3 \\ \Sigma x_1x_3, & \Sigma x_2x_3, & \Sigma x_3^2 \end{vmatrix}$$

$$=\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix}=\Omega^2,$$

where Ω^2 will be used, throughout, to denote this determinant involving A, B, C, F, G, H. We have already noted that no two of the quantities J_x, J_y, J_z, J_v, can vanish identically; hence the quantity Ω does not vanish for real regions. It will appear that

$$J_x/\Omega, \quad -J_y/\Omega, \quad J_z/\Omega, \quad -J_v/\Omega,$$

are the direction-cosines of the normal to the tangent flat to the region, that is, of the normal to the region.

The second set of supplementary magnitudes is constituted by

$$a_{mn}=y_mz_n-z_my_n, \quad f_{mn}=v_mx_n-x_nv_n,$$
$$b_{mn}=z_mx_n-x_mz_n, \quad g_{mn}=v_my_n-y_mv_n,$$
$$c_{mn}=x_my_n-y_mx_n, \quad h_{mn}=v_mz_n-z_mv_n,$$

for $m, n, =1, 2, 3$. Thus $a_{23}, \ldots, h_{23}$, are orientation-coordinates (proportional to the orientation-cosines) of the tangent plane to the parametric surface P: the quantities $a_{31}, \ldots, h_{31}$, stand in the same relation to the parametric surface Q, and the quantities $a_{12}, \ldots, h_{12}$, stand in the same relation to the parametric surface R.

Among these supplementary magnitudes, there are various identical relations, which are noted for possible reference and use.

The only relation among the orientation-coordinates, not being the actual orientation-cosines, is

$$a_{mn}f_{mn}+b_{mn}g_{mn}+c_{mn}h_{mn}=0,$$

for $m, n, =1, 2, 3$.

There are several types of relations connecting these magnitudes, all of which involve only the first derivatives of the coordinates. The principal types appear to be:

I. The relations linear in all the three sets:

$$\left.\begin{aligned} J_x= & -y_lh_{mn}+z_lg_{mn}+v_la_{mn} \\ J_y= & -x_lh_{mn}+z_lf_{mn}-v_lb_{mn} \\ J_z= & -x_lg_{mn}+y_lf_{mn}+v_lc_{mn} \\ J_v= & \ x_la_{mn}+y_lb_{mn}+z_lc_{mn} \end{aligned}\right\},$$

for $l, m, n, =1, 2, 3$, in cyclical order: and the particular relation

$$x_rJ_x-y_rJ_y+z_rJ_z-v_rJ_v=0,$$

for $r=1, 2, 3$.

II. There are relations, quadratic in the orientation-coordinates:

$$\left.\begin{aligned} b_{lm}c_{mn} - c_{lm}b_{mn} &= x_m J_v \\ c_{lm}a_{mn} - a_{lm}c_{mn} &= y_m J_v \\ a_{lm}b_{mn} - b_{lm}a_{mn} &= z_m J_v \end{aligned}\right\}, \quad \left.\begin{aligned} g_{lm}h_{mn} - h_{lm}g_{mn} &= v_m J_x \\ f_{lm}h_{mn} - h_{lm}f_{mn} &= v_m J_y \\ f_{lm}g_{mn} - g_{lm}f_{mn} &= v_m J_z \end{aligned}\right\},$$

$$\left.\begin{aligned} a_{lm}f_{mn} - f_{lm}a_{mn} &= -x_m J_x + v_m J_v \\ a_{lm}g_{mn} - g_{lm}a_{mn} &= -y_m J_x \\ a_{lm}h_{mn} - h_{lm}a_{mn} &= -z_m J_x \end{aligned}\right\},$$

$$\left.\begin{aligned} b_{lm}f_{mn} - f_{lm}b_{mn} &= x_m J_y \\ b_{lm}g_{mn} - g_{lm}b_{mn} &= -y_m J_y + v_m J_v \\ b_{lm}h_{mn} - h_{lm}b_{mn} &= z_m J_y \end{aligned}\right\},$$

$$\left.\begin{aligned} c_{lm}f_{mn} - f_{lm}c_{mn} &= -x_m J_z \\ c_{lm}g_{mn} - g_{lm}c_{mn} &= -y_m J_z \\ c_{lm}h_{mn} - h_{lm}c_{mn} &= -z_m J_z + v_m J_v \end{aligned}\right\};$$

$$\begin{vmatrix} a_{lm}, & h_{lm}, & g_{lm} \\ a_{mn}, & h_{mn}, & g_{mn} \\ a_{nl}, & h_{nl}, & g_{nl} \end{vmatrix} = -J_x^2, \quad \begin{vmatrix} h_{lm}, & b_{lm}, & f_{lm} \\ h_{mn}, & b_{mn}, & f_{mn} \\ h_{nl}, & b_{nl}, & f_{nl} \end{vmatrix} = -J_y^2,$$

$$\begin{vmatrix} g_{lm}, & f_{lm}, & c_{lm} \\ g_{mn}, & f_{mn}, & c_{mn} \\ g_{nl}, & f_{nl}, & c_{nl} \end{vmatrix} = -J_z^2, \quad \begin{vmatrix} a_{lm}, & b_{lm}, & c_{lm} \\ a_{mn}, & b_{mn}, & c_{mn} \\ a_{nl}, & b_{nl}, & c_{nl} \end{vmatrix} = J_v^2.$$

III. There are combinations of the orientation-coordinates of the parametric surfaces P, Q, R, expressible by means of the fundamental magnitudes:

$$\left.\begin{aligned} a_{23}^2 + b_{23}^2 + c_{23}^2 + f_{23}^2 + g_{23}^2 + h_{23}^2 &= BC - F^2 \\ a_{31}^2 + b_{31}^2 + c_{31}^2 + f_{31}^2 + g_{31}^2 + h_{31}^2 &= CA - G^2 \\ a_{12}^2 + b_{12}^2 + c_{12}^2 + f_{12}^2 + g_{12}^2 + h_{12}^2 &= AB - H^2 \end{aligned}\right\},$$

$$\left.\begin{aligned} a_{12}a_{13} + b_{12}b_{13} + c_{12}c_{13} + f_{12}f_{13} + g_{12}g_{13} + h_{12}h_{13} &= AF - GH \\ a_{23}a_{21} + b_{23}b_{21} + c_{23}c_{21} + f_{23}f_{21} + g_{23}g_{21} + h_{23}h_{21} &= BG - HF \\ a_{31}a_{32} + b_{31}b_{32} + c_{31}c_{32} + f_{31}f_{32} + g_{31}g_{32} + h_{31}h_{32} &= CH - FG \end{aligned}\right\}.$$

Spherical trigonometry in any tangent flat of a region.

264. Consider the configuration of the region in the immediate vicinity of any point O, with the coordinates x, y, z, v. Passing through that point, there are three parametric curves p, q, r; and there are three directions, determined by the tangent lines to these respective curves at O. These three directions determine a flat which, as will be established (§ 269), is the flat tangential to the region at O. The tangent lines to the curves p and q

determine a plane which is tangential to the parametric surface R at O: the tangent lines to the curves q and r determine a plane which is tangential to the parametric surface P at O; and the tangent lines to the curves r and p determine a plane which is tangential to the parametric surface Q at O.

Accordingly in the flat, which at O is tangential to the region, we take a spherical surface having O for its centre. Let the tangent lines to the parametric curves p, q, r, meet this spherical surface in p, q, r; the arcs of great circles qr, rp, pq, are the respective intersections of the spherical surface by the tangent planes at O to the parametric surfaces P, Q, R. We thus have a spherical triangle on the spherical surface: the whole of this configuration lies in the tangent flat at O to the region: and we can use the customary spherical trigonometry of three-dimensional homaloidal space, to obtain the inclinations of the various associated lines and planes at O lying in the flat. Thus the angle qpr, the angle between the great-circle arcs pq and pr, is the angle between the planes pq and pr, that is, is equal to the angle at which the parametric surfaces Q and R intersect at the point O; and so for the other angles of that spherical triangle pqr.

Along the line Op, the direction-cosines are proportional to the values of dx, dy, dz, dv, at O measured along that tangent line: that is, they are proportional to $x_1 dp$, $y_1 dp$, $z_1 dp$, $v_1 dp$: consequently, they are

$$\frac{x_1}{A^{\frac{1}{2}}}, \quad \frac{y_1}{A^{\frac{1}{2}}}, \quad \frac{z_1}{A^{\frac{1}{2}}}, \quad \frac{v_1}{A^{\frac{1}{2}}}.$$

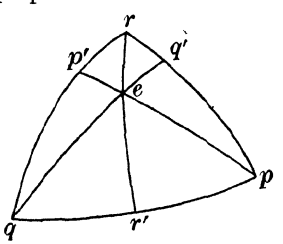


Fig. 22.

Similarly for the direction-cosines of the tangents Oq and Or to the other parametric curves through O. Thus

$$\left.\begin{array}{l} \text{the direction-cosines of } Op \text{ are } x_1 A^{-\frac{1}{2}},\ y_1 A^{-\frac{1}{2}},\ z_1 A^{-\frac{1}{2}},\ v_1 A^{-\frac{1}{2}} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ Oq \ \ldots\ x_2 B^{-\frac{1}{2}},\ y_2 B^{-\frac{1}{2}},\ z_2 B^{-\frac{1}{2}},\ v_2 B^{-\frac{1}{2}} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ Or \ \ldots\ x_3 C^{-\frac{1}{2}},\ y_3 C^{-\frac{1}{2}},\ z_3 C^{-\frac{1}{2}},\ v_3 C^{-\frac{1}{2}} \end{array}\right\},$$

the positive sign being associated with the radicals.

We denote by α, β, γ, respectively the sides qr, rp, pq, of the spherical triangle, that is, the respective inclinations of the parametric tangents Oq and Or, Or and Op, Op and Oq. Then

$$\cos\alpha = \Sigma\left(x_2 B^{-\frac{1}{2}} . x_3 C^{-\frac{1}{2}}\right) = (BC)^{-\frac{1}{2}} \Sigma x_2 x_3 = \frac{F}{(BC)^{\frac{1}{2}}},$$

$$\cos\beta = \Sigma\left(x_3 C^{-\frac{1}{2}} . x_1 A^{-\frac{1}{2}}\right) = (CA)^{-\frac{1}{2}} \Sigma x_3 x_1 = \frac{G}{(CA)^{\frac{1}{2}}},$$

$$\cos\gamma = \Sigma\left(x_1 A^{-\frac{1}{2}} . x_2 B^{-\frac{1}{2}}\right) = (AB)^{-\frac{1}{2}} \Sigma x_1 x_2 = \frac{H}{(AB)^{\frac{1}{2}}};$$

and therefore

$$\sin\alpha=\frac{(BC-F^2)^{\frac{1}{2}}}{(BC)^{\frac{1}{2}}},\quad \sin\beta=\frac{(CA-G^2)^{\frac{1}{2}}}{(CA)^{\frac{1}{2}}},\quad \sin\gamma=\frac{(AB-H^2)^{\frac{1}{2}}}{(AB)^{\frac{1}{2}}},$$

the positive signs again being associated with the new radicals, because α, β, γ, are (each of them) less than π. Manifestly the conditions, necessary and sufficient to secure that the lines of reference are perpendicular to one another, are

$$F=0,\quad G=0,\quad H=0.$$

For the angles p, q, r, of the spherical triangle, the customary formulæ lead to the expressions

$$\left.\begin{aligned}\cos p&=\frac{AF-GH}{\{(AB-H^2)(AC-G^2)\}^{\frac{1}{2}}},&\sin p&=\frac{(A\Omega^2)^{\frac{1}{2}}}{\{(AB-H^2)(AC-G^2)\}^{\frac{1}{2}}}\\ \cos q&=\frac{BG-HF}{\{(BC-F^2)(BA-H^2)\}^{\frac{1}{2}}},&\sin q&=\frac{(B\Omega^2)^{\frac{1}{2}}}{\{(BC-F^2)(BA-H^2)\}^{\frac{1}{2}}}\\ \cos r&=\frac{CH-FG}{\{(CA-G^2)(CB-F^2)\}^{\frac{1}{2}}},&\sin r&=\frac{(C\Omega^2)^{\frac{1}{2}}}{\{(CA-G^2)(CB-F^2)\}^{\frac{1}{2}}}\end{aligned}\right\}.$$

Hence we have the relations

$$\frac{\sin p}{\sin\alpha}=\frac{\sin q}{\sin\beta}=\frac{\sin r}{\sin\gamma}=\left\{\frac{ABC\Omega^2}{(BC-F^2)(CA-G^2)(AB-H^2)}\right\}^{\frac{1}{2}},$$

$$1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma=\frac{\Omega^2}{ABC},$$

$$1-\cos^2 p-\cos^2 q-\cos^2 r+2\cos p\cos q\cos r=\frac{\Omega^4}{(BC-F^2)(CA-G^2)(AB-H^2)}.$$

In the spherical triangle, let perpendiculars pp', qq', rr', be drawn from the points p, q, r, upon the opposite sides: then the planes Opp' and Oqr are perpendicular (not orthogonal), likewise the planes Oqq' and Orp, and likewise the planes Orr' and Opq. The following results, among others, hold:

$$\left.\begin{aligned}\frac{\cos qp'}{\{H^2(BC-F^2)\}^{\frac{1}{2}}}&=\frac{\sin qp'}{BG-FH}=\frac{1}{\{B(BG^2+CH^2-2FGH)\}^{\frac{1}{2}}}\\ \frac{\cos rp'}{\{G^2(BC-F^2)\}^{\frac{1}{2}}}&=\frac{\sin rp'}{CH-FG}=\frac{1}{\{C(BG^2+CH^2-2FGH)\}^{\frac{1}{2}}}\end{aligned}\right\},$$

$$\left.\begin{aligned}\frac{\cos rq'}{\{F^2(CA-G^2)\}^{\frac{1}{2}}}&=\frac{\sin rq'}{CH-FG}=\frac{1}{\{C(CH^2+AF^2-2FGH)\}^{\frac{1}{2}}}\\ \frac{\cos pq'}{\{H^2(CA-G^2)\}^{\frac{1}{2}}}&=\frac{\sin pq'}{AF-GH}=\frac{1}{\{A(CH^2+AF^2-2FGH)\}^{\frac{1}{2}}}\end{aligned}\right\},$$

$$\left.\begin{aligned}\frac{\cos pr'}{\{G^2(AB-H^2)\}^{\frac{1}{2}}} &= \frac{\sin pr'}{AF-GH} = \frac{1}{\{A(AF^2+BG^2-2FGH)\}^{\frac{1}{2}}}\\ \frac{\cos qr'}{\{F^2(AB-H^2)\}^{\frac{1}{2}}} &= \frac{\sin qr'}{BG-FH} = \frac{1}{\{B(AF^2+BG^2-2FGH)\}^{\frac{1}{2}}}\end{aligned}\right\};$$

$$\left.\begin{aligned}\frac{\sin pp'}{\Omega^{\frac{1}{2}}} &= \frac{\cos pp'}{(BG^2+CH^2-2FGH)^{\frac{1}{2}}} = \frac{1}{(ABC-AF^2)^{\frac{1}{2}}}\\ \frac{\sin qq'}{\Omega^{\frac{1}{2}}} &= \frac{\cos qq'}{(CH^2+AF^2-2FGH)^{\frac{1}{2}}} = \frac{1}{(ABC-BG^2)^{\frac{1}{2}}}\\ \frac{\sin rr'}{\Omega^{\frac{1}{2}}} &= \frac{\cos rr'}{(AF^2+BG^2-2FGH)^{\frac{1}{2}}} = \frac{1}{(ABC-CH^2)^{\frac{1}{2}}}\end{aligned}\right\};$$

$$\left.\begin{aligned}\tan ep &= \frac{AF-GH}{GH}\left(\frac{BG^2+CH^2-2FGH}{\Omega^2}\right)^{\frac{1}{2}}\\ \tan eq &= \frac{BG-HF}{HF}\left(\frac{CH^2+AF^2-2FGH}{\Omega^2}\right)^{\frac{1}{2}}\\ \tan er &= \frac{CH-FG}{FG}\left(\frac{AF^2+BG^2-2FGH}{\Omega^2}\right)^{\frac{1}{2}}\end{aligned}\right\}.$$

Parametric elements of area: parametric element of volume.

265. We take short distances along the parametric curves p, q, r, through a point O; these distances are respectively $A^{\frac{1}{2}}dp$, $B^{\frac{1}{2}}dq$, $C^{\frac{1}{2}}dr$.

These distances define elementary areas upon the parametric surfaces, the areas being bounded by parametric lines. The areas of the elementary parallelograms on the respective surfaces are easily obtained. Thus the area of the elementary parallelogram on the parametric surface P is

$$= B^{\frac{1}{2}}dq \,.\, C^{\frac{1}{2}}dr \,.\, \sin\alpha$$
$$= (BC-F^2)^{\frac{1}{2}}\, dq\, dr;$$

and the similar elements of area on the parametric surfaces Q and R respectively are

$$(CA-G^2)^{\frac{1}{2}}\, dr\, dp, \quad (AB-H^2)^{\frac{1}{2}}\, dp\, dq.$$

Similarly the three distances define a trihedral volume, the concurrent edges of which are the elements of the three parametric lines. When the parallelepiped having three elementary arcs for its determining current edges is completed, the element of volume is

$$= (\text{elementary area in plane } R) \,.\, Or \,.\, \sin rr'$$
$$= \{(AB-H^2)^{\frac{1}{2}}\, dp\, dq\} \,.\, C^{\frac{1}{2}}\, dr \,.\, \sin rr'$$
$$= \Omega\, dp\, dq\, dr.$$

Corresponding expressions will be found immediately, for similar elementary areas and similar elementary volumes, when these are bounded by any two assigned arcs, or by any three assigned arcs, respectively, lying within the region and concurrent at a point in the region: these arcs not being portions of the parametric curves.

Ex. 1. *The volume of the globular region*

$$x^2+y^2+z^2+v^2=c^2$$

is $2\pi^2c^3$.

Anticipating the parametric expression (p. 84) of the content of the region, we have

$$x=c\sin p\sin q,\quad y=c\sin p\cos q,\quad z=c\cos p\cos r,\quad v=c\cos p\sin r,$$
$$A=c^2,\quad B=c^2\sin^2 p,\quad C=c^2\cos^2 p,\quad F=0,\quad G=0,\quad H=0,$$

so that

$$\Omega^2=c^6\sin^2 p\cos^2 p.$$

For the full range of the whole content, we take the one-sixteenth part, one of the sixteen symmetric portions bounded by semi-axes; and, for that part, the range is

$$p=0 \text{ to } \tfrac{1}{2}\pi,\quad q=0 \text{ to } \tfrac{1}{2}\pi,\quad r=0 \text{ to } \tfrac{1}{2}\pi.$$

Hence the volume

$$=16\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} c^3\sin p\cos p\,dp\,dq\,dr=2\pi^2c^3.$$

Ex. 2. Prove that the volume of the dicyclic region

$$\frac{x^2+y^2}{a^2}+\frac{z^2+v^2}{c^2}=1$$

is equal to

$$\frac{4}{3}\pi^2 ac\,\frac{a^2+ac+c^2}{a+c}.$$

Ex. 3. Prove that the volume of the region

$$\frac{x^2+y^2+z^2}{a^2}+\frac{v^2}{c^2}=1$$

is equal to

$$\frac{8}{3}\pi a^3\left\{E+\frac{c^2}{a^2-c^2}(K-E)\right\},$$

where K and E are the first and the second complete elliptic integrals with modulus k given by $k^2a^2=a^2-c^2$.

General elements of area: parametric surface coordinates: inclinations: normals.

266. It is necessary to consider not merely lines and their elements of length but also surfaces and their elements of area. An initial indication has been given in § 265.

We have taken a direction dp, dq, dr, and have associated an elementary length ds with that direction. Now take two different directions, d_1p, d_1q, d_1r, with an element of length ds_1, and d_2p, d_2q, d_2r, with an element of length ds_2; and let ω be the angle between the two directions. Then

$$ds_1^2=\Sigma A\,d_1p^2,\quad ds_2^2=\Sigma A\,d_2p^2,\quad ds_1\,ds_2\cos\omega=\Sigma A\,d_1p\,d_2p.$$

Let dS denote the element of area, constituted by the rudimentary parallelogram having ds_1 and ds_2 for adjacent sides: then

$$dS = ds_1\, ds_2 \sin\omega,$$

and therefore

$$dS^2 = (\Sigma A\, d_1p^2)(\Sigma A\, d_2p^2) - (\Sigma A\, d_1p\, d_2p)^2.$$

There are certain combinations which occur in the evaluated expression for the right-hand side. We write

$$\xi = \begin{vmatrix} d_1q, & d_1r \\ d_2q, & d_2r \end{vmatrix}, \quad \eta = \begin{vmatrix} d_1r, & d_1p \\ d_2r, & d_2p \end{vmatrix}, \quad \zeta = \begin{vmatrix} d_1p, & d_1q \\ d_2p, & d_2q \end{vmatrix};$$

and it will appear later (§ 273) that we need the combinations

$$\left.\begin{aligned} BC - F^2 &= a, & GH - AF &= f \\ CA - G^2 &= b, & HF - BG &= g \\ AB - H^2 &= c, & FG - CH &= h \end{aligned}\right\}.$$

Then we have

$$\begin{aligned} dS^2 &= a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + 2g\zeta\xi + 2h\xi\eta \\ &= (a, b, c, f, g, h)(\xi, \eta, \zeta)^2, \end{aligned}$$

a fundamental expression for the general element of area within the region.

The spatial orientation-cosines of this elementary area are required. The spatial direction-coordinates determined by d_1p, d_1q, d_1r, are proportional to d_1x, d_1y, d_1z, d_1v, where

$$\begin{aligned} d_1x &= x_1\, d_1p + x_2\, d_1q + x_3\, d_1r, \\ d_1y &= y_1\, d_1p + y_2\, d_1q + y_3\, d_1r, \\ d_1z &= z_1\, d_1p + z_2\, d_1q + z_3\, d_1r, \\ d_1v &= v_1\, d_1p + v_2\, d_1q + v_3\, d_1r; \end{aligned}$$

and similarly d_2p, d_2q, d_2r, determine elements d_2x, d_2y, d_2z, d_2v, where

$$\begin{aligned} d_2x &= x_1\, d_2p + x_2\, d_2q + x_3\, d_2r, \\ d_2y &= y_1\, d_2p + y_2\, d_2q + y_3\, d_2r, \\ d_2z &= z_1\, d_2p + z_2\, d_2q + z_3\, d_2r, \\ d_2v &= v_1\, d_2p + v_2\, d_2q + v_3\, d_2r. \end{aligned}$$

The orientation-cosines in question, **a**, **b**, **c**, **f**, **g**, **h**, are (§ 85)

$$\mathbf{a}\,dS = \begin{vmatrix} d_1y, & d_1z \\ d_2y, & d_2z \end{vmatrix}, \quad \mathbf{b}\,dS = \begin{vmatrix} d_1z, & d_1x \\ d_2z, & d_2x \end{vmatrix}, \quad \mathbf{c}\,dS = \begin{vmatrix} d_1x, & d_1y \\ d_2x, & d_2y \end{vmatrix},$$

$$\mathbf{f}\,dS = \begin{vmatrix} d_1x, & d_1v \\ d_2x, & d_2v \end{vmatrix}, \quad \mathbf{g}\,dS = \begin{vmatrix} d_1y, & d_1v \\ d_2y, & d_2v \end{vmatrix}, \quad \mathbf{h}\,dS = \begin{vmatrix} d_1z, & d_1v \\ d_2z, & d_2v \end{vmatrix}.$$

Now

$$\begin{vmatrix} d_1y, & d_1z \\ d_2y, & d_2z \end{vmatrix}$$

$$= \begin{vmatrix} y_2, & y_3 \\ z_2, & z_3 \end{vmatrix} \begin{vmatrix} d_1q, & d_1r \\ d_2q, & d_2r \end{vmatrix} + \begin{vmatrix} y_3, & y_1 \\ z_3, & z_1 \end{vmatrix} \begin{vmatrix} d_1r, & d_1p \\ d_2r, & d_2p \end{vmatrix} + \begin{vmatrix} y_1, & y_2 \\ z_1, & z_2 \end{vmatrix} \begin{vmatrix} d_1p, & d_1q \\ d_2p, & d_2q \end{vmatrix}$$

$$= (y_2z_3 - z_2y_3)\,\xi + (y_3z_1 - z_3y_1)\,\eta + (y_1z_2 - z_1y_2)\,\zeta,$$

so that

$$\mathbf{a}\,dS = a_{23}\xi + a_{31}\eta + a_{12}\zeta,$$

in the notation of § 263; and, similarly,

$$\mathbf{b}\,dS = b_{23}\xi + b_{31}\eta + b_{12}\zeta, \quad \mathbf{c}\,dS = c_{23}\xi + c_{31}\eta + c_{12}\zeta,$$

with like expressions for $\mathbf{f}\,dS$, $\mathbf{g}\,dS$, $\mathbf{h}\,dS$. It is easy to verify that these expressions satisfy the identical relation

$$\mathbf{af} + \mathbf{bg} + \mathbf{ch} = 0;$$

and they satisfy the permanent relation

$$\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 + \mathbf{f}^2 + \mathbf{g}^2 + \mathbf{h}^2 = 1,$$

in virtue of the foregoing value for dS^2.

267. Now let there be, within a region, a surface

$$\theta\,(p,\ q,\ r) = \text{constant},$$

belonging possibly (though not necessarily) to a family. On that surface, let there be two directions $d_3p,\ d_3q,\ d_3r$, and $d_4p,\ d_4q,\ d_4r$; and let $d\Sigma$ denote the element of area on the surface, given by the parallelogram having the two arcs for adjacent sides. Let the superficial coordinates, which are $\xi,\ \eta,\ \zeta$, for the elementary area dS, be $\bar{\xi},\ \bar{\eta},\ \bar{\zeta}$, for the elementary area $d\Sigma$. The orientation-coordinates $\bar{\mathbf{a}},\ \bar{\mathbf{b}},\ \bar{\mathbf{c}},\ \bar{\mathbf{f}},\ \bar{\mathbf{g}},\ \bar{\mathbf{h}}$, for the surface are given by

$$\bar{\mathbf{a}}\,d\Sigma = a_{23}\bar{\xi} + a_{31}\bar{\eta} + a_{12}\bar{\zeta}, \quad \bar{\mathbf{b}}\,d\Sigma = b_{23}\bar{\xi} + b_{31}\bar{\eta} + b_{12}\bar{\zeta}, \quad \bar{\mathbf{c}}\,d\Sigma = c_{23}\bar{\xi} + c_{31}\bar{\eta} + c_{12}\bar{\zeta},$$

and so for $\bar{\mathbf{f}}\,d\Sigma$, $\bar{\mathbf{g}}\,d\Sigma$, $\bar{\mathbf{h}}\,d\Sigma$.

We denote by ϵ the inclination of the two elements of area $d\Sigma$ and dS, that is, the inclination of the two tangent planes to the surfaces determined by the guiding directions d_1 and d_2, d_3 and d_4, respectively: then (§ 94)

$$\cos\epsilon = \mathbf{a}\bar{\mathbf{a}} + \mathbf{b}\bar{\mathbf{b}} + \mathbf{c}\bar{\mathbf{c}} + \mathbf{f}\bar{\mathbf{f}} + \mathbf{g}\bar{\mathbf{g}} + \mathbf{h}\bar{\mathbf{h}}.$$

When substitution is made for the two sets of orientation-coordinates, we find

$$dS\,d\Sigma\cos\epsilon = (a,\ b,\ c,\ f,\ g,\ h\,|\,\xi,\ \eta,\ \zeta\,|\,\bar{\xi},\ \bar{\eta},\ \bar{\zeta}),$$

with an expression for $d\Sigma^2$ similar to the expression for dS^2, given by

$$d\Sigma^2 = (a,\ b,\ c,\ f,\ g,\ h\,|\,\bar{\xi},\ \bar{\eta},\ \bar{\zeta})^2.$$

268. As the two directions $d_3p,\ d_3q,\ d_3r$, and $d_4p,\ d_4q,\ d_4r$, lie in the surface $\theta(p,\ q,\ r) = \text{constant}$, we have

$$\theta_1 d_3p + \theta_2 d_3q + \theta_3 d_3r = 0,$$
$$\theta_1 d_4p + \theta_2 d_4q + \theta_3 d_4r = 0,$$

and therefore

$$\frac{\theta_1}{\bar{\xi}} = \frac{\theta_2}{\bar{\eta}} = \frac{\theta_3}{\bar{\zeta}}.$$

We write

$$D = a\theta_1^2 + b\theta_2^2 + c\theta_3^2 + 2f\theta_2\theta_3 + 2g\theta_3\theta_1 + 2h\theta_1\theta_2$$
$$= (a,\ b,\ c,\ f,\ g,\ h \mathbin{\text{\textbackslash\!\!\textbackslash}} \theta_1,\ \theta_2,\ \theta_3)^2,$$

and we have

$$d\Sigma^2 = (a,\ b,\ c,\ f,\ g,\ h \mathbin{\text{\textbackslash\!\!\textbackslash}} \bar{\xi},\ \bar{\eta},\ \bar{\zeta})^2;$$

hence

$$\frac{\bar{\xi}}{\theta_1} = \frac{\bar{\eta}}{\theta_2} = \frac{\bar{\zeta}}{\theta_3} = \frac{d\Sigma}{D^{\frac{1}{2}}}.$$

Again, let $\delta p,\ \delta q,\ \delta r$, be a direction within the region normal to the surface $\theta = \text{constant}$, and therefore perpendicular to the two directions $d_3p,\ d_3q,\ d_3r$, and $d_4p,\ d_4q,\ d_4r$: hence

$$(A,\ B,\ C,\ F,\ G,\ H \mathbin{\text{\textbackslash\!\!\textbackslash}} d_3p,\ d_3q,\ d_3r \mathbin{\text{\textbackslash\!\!\textbackslash}} \delta p,\ \delta q,\ \delta r) = 0,$$
$$(A,\ B,\ C,\ F,\ G,\ H \mathbin{\text{\textbackslash\!\!\textbackslash}} d_4p,\ d_4q,\ d_4r \mathbin{\text{\textbackslash\!\!\textbackslash}} \delta p,\ \delta q,\ \delta r) = 0.$$

Now

$$\begin{vmatrix} A\,d_3p + H\,d_3q + G\,d_3r, & H\,d_3p + B\,d_3q + F\,d_3r \\ A\,d_4p + H\,d_4q + G\,d_4r, & H\,d_4p + B\,d_4q + F\,d_4r \end{vmatrix}$$
$$= \begin{vmatrix} A, & H \\ H, & B \end{vmatrix} \begin{vmatrix} d_3p, & d_3q \\ d_4p, & d_4q \end{vmatrix} + \begin{vmatrix} H, & G \\ B, & F \end{vmatrix} \begin{vmatrix} d_3q, & d_3r \\ d_4q, & d_4r \end{vmatrix} + \begin{vmatrix} G, & A \\ F, & H \end{vmatrix} \begin{vmatrix} d_3r, & d_3p \\ d_4r, & d_4p \end{vmatrix}$$
$$= g\bar{\xi} + f\bar{\eta} + c\bar{\zeta},$$

and so for the other combinations; hence

$$\frac{\delta p}{a\bar{\xi} + h\bar{\eta} + g\bar{\zeta}} = \frac{\delta q}{h\bar{\xi} + b\bar{\eta} + f\bar{\zeta}} = \frac{\delta r}{g\bar{\xi} + f\bar{\eta} + c\bar{\zeta}}.$$

Let $d\nu$ denote a length along this direction normal to the elementary area $d\Sigma$, the length being determined by the increments $\delta p,\ \delta q,\ \delta r$. If, for the moment, we denote the common value of these fractions by $d\nu/M$, and note that

$$d\nu^2 = \Sigma A\,\delta p^2,$$

we have

$$M^2 = \Sigma A\,(a\bar{\xi} + h\bar{\eta} + g\bar{\zeta})^2.$$

Also

$$A\,(a\bar{\xi} + h\bar{\eta} + g\bar{\zeta}) + H\,(h\bar{\xi} + b\bar{\eta} + f\bar{\zeta}) + G\,(g\bar{\xi} + f\bar{\eta} + c\bar{\zeta}) = \bar{\xi}\Omega^2,$$

and so for the others; hence

$$M^2 = \Omega^2 \Sigma \bar{\xi}\,(a\bar{\xi} + h\bar{\eta} + g\bar{\zeta})$$
$$= \Omega^2 \Sigma a\bar{\xi}^2 = \Omega^2 d\Sigma^2,$$

so that

$$M = \Omega\, d\Sigma.$$

Consequently

$$a\bar{\xi} + h\bar{\eta} + g\bar{\zeta} = \frac{dp}{d\nu} M = \Omega \frac{dp}{d\nu} d\Sigma,$$

$$h\bar{\xi} + b\bar{\eta} + f\bar{\zeta} = \frac{dq}{d\nu} M = \Omega \frac{dq}{d\nu} d\Sigma,$$

$$g\bar{\xi} + f\bar{\eta} + c\bar{\zeta} = \frac{dr}{d\nu} M = \Omega \frac{dr}{d\nu} d\Sigma;$$

and we have had

$$\bar{\xi} D^{\frac{1}{2}} = \theta_1 d\Sigma, \quad \bar{\eta} D^{\frac{1}{2}} = \theta_2 d\Sigma, \quad \bar{\zeta} D^{\frac{1}{2}} = \theta_3 d\Sigma.$$

Hence, as

$$\frac{d\theta}{d\nu} = \theta_1 \frac{dp}{d\nu} + \theta_2 \frac{dq}{d\nu} + \theta_3 \frac{dr}{d\nu},$$

it follows that

$$\begin{aligned}
\Omega \frac{d\theta}{d\nu} d\Sigma &= \theta_1 (a\bar{\xi} + h\bar{\eta} + g\bar{\zeta}) + \theta_2 (h\bar{\xi} + b\bar{\eta} + f\bar{\zeta}) + \theta_3 (g\bar{\xi} + f\bar{\eta} + c\bar{\zeta}) \\
&= \frac{d\Sigma}{D^{\frac{1}{2}}} (a\theta_1^2 + 2h\theta_1\theta_2 + 2g\theta_1\theta_3 + b\theta_2^2 + 2f\theta_2\theta_3 + c\theta_3^2) \\
&= d\Sigma \,.\, D^{\frac{1}{2}},
\end{aligned}$$

so that

$$\Omega \frac{d\theta}{d\nu} = D^{\frac{1}{2}},$$

or

$$D = \Omega^2 \left(\frac{d\theta}{d\nu}\right)^2.$$

Further, returning to the expression for $\cos\epsilon$, we had

$$\begin{aligned}
dS\, d\Sigma \cos\epsilon &= (a, b, c, f, g, h\mathbin{\text{\textbigcircle}}\xi, \eta, \zeta\mathbin{\text{\textbigcircle}}\bar{\xi}, \bar{\eta}, \bar{\zeta}) \\
&= \frac{d\Sigma}{D^{\frac{1}{2}}} (a, b, c, f, g, h\mathbin{\text{\textbigcircle}}\xi, \eta, \zeta\mathbin{\text{\textbigcircle}}\theta_1, \theta_2, \theta_3);
\end{aligned}$$

hence

$$D^{\frac{1}{2}} \,.\, dS \cos\epsilon = (a, b, c, f, g, h\mathbin{\text{\textbigcircle}}\xi, \eta, \zeta\mathbin{\text{\textbigcircle}}\theta_1, \theta_2, \theta_3),$$

or

$$\Omega \frac{d\theta}{d\nu} dS \cos\epsilon = (a, b, c, f, g, h\mathbin{\text{\textbigcircle}}\xi, \eta, \zeta\mathbin{\text{\textbigcircle}}\theta_1, \theta_2, \theta_3).$$

With these, we associate the two cognate forms

$$\Omega^2 \left(\frac{d\theta}{d\nu}\right)^2 = (a, b, c, f, g, h\mathbin{\text{\textbigcircle}}\theta_1, \theta_2, \theta_3)^2,$$

$$dS^2 = (a, b, c, f, g, h\mathbin{\text{\textbigcircle}}\xi, \eta, \zeta)^2.$$

Ex. 1. Let dn denote a direction which lies within the region and is perpendicular to the elementary area dS within the region; prove that ϵ is the angle between the two directions dn and $d\nu$.

Ex. 2. Through any point O draw directions OP, OQ, OR, ON, respectively parallel to the lines

(i) $q=$constant, $r=$constant:

(ii) $r=$constant, $p=$constant;

(iii) $p=$constant, $q=$constant; and

(iv) the normal to the surface $\theta(p, q, r)=$constant.

Let α', b', c', denote the angles NOP, NOQ, NOR, respectively; and let A', B', C', denote the angles between the planes NOQ and NOR, NOR and NOP, NOP and NOQ, respectively. Prove that

$$\cos\alpha'=\frac{\Omega\theta_1}{(AD)^{\frac{1}{2}}},\quad \cos b'=\frac{\Omega\theta_2}{(BD)^{\frac{1}{2}}},\quad \cos c'=\frac{\Omega\theta_3}{(CD)^{\frac{1}{2}}};$$

$$\left.\begin{aligned}\{(BD-\Omega^2\theta_2^2)(CD-\Omega^2\theta_3^2)\}^{\frac{1}{2}}\cos A'&=DF-\Omega^2\theta_2\theta_3\\ \{(CD-\Omega^2\theta_3^2)(AD-\Omega^2\theta_1^2)\}^{\frac{1}{2}}\cos B'&=DG-\Omega^2\theta_3\theta_1\\ \{(AD-\Omega^2\theta_1^2)(BD-\Omega^2\theta_2^2)\}^{\frac{1}{2}}\cos C'&=DH-\Omega^2\theta_1\theta_2\end{aligned}\right\};$$

and verify the property $A'+B'+C'=2\pi$.

A corresponding calculation for a general *element of volume*, in a parallelepiped with three non-complanar arcs ds_1, ds_2, ds_3, for conterminous arcs, leads to the expression

$$\Omega\begin{vmatrix} p_1', & q_1', & r_1' \\ p_2', & q_2', & r_2' \\ p_3', & q_3', & r_3' \end{vmatrix} ds_1 ds_2 ds_3,$$

where p_i' denotes dp/ds_i, and so for the others. When the arcs are taken along the parametric curves, the expression assumes the earlier form (§ 265)

$$\Omega\, dp\, dq\, dr.$$

Equation of the tangent flat.

269. The equation of any flat through the point x, y, z, v, in quadruple space can be taken in the form

$$\Sigma X(\bar{x}-x)=X(\bar{x}-x)+Y(\bar{y}-y)+Z(\bar{z}-z)+V(\bar{v}-v)=0.$$

The line through that point and through any neighbouring point $x+dx$, $y+dy$, $z+dz$, $v+dv$, is represented by the equations

$$\frac{\bar{x}-x}{dx}=\frac{\bar{y}-y}{dy}=\frac{\bar{z}-z}{dz}=\frac{\bar{v}-v}{dv}.$$

If this line lies wholly in the foregoing flat, the condition

$$X\,dx+Y\,dy+Z\,dz+V\,dv=0$$

must be satisfied.

Let that neighbouring point be a point, still contiguous to x, y, z, v, and now restricted to lie within the specified parametric region of § 262, so that

$$dx=x_1\,dp+x_2\,dq+x_3\,dr,$$

with corresponding expressions for dy, dz, dv. Then the condition is

$$(\Sigma Xx_1)\,dp+(\Sigma Xx_2)\,dq+(\Sigma Xx_3)\,dr=0.$$

This condition can be interpreted in two ways. When it is not required to be evanescent, it specifies a relation between dp, dq, dr, in order that the hypothetical line may lie in the arbitrarily assigned flat: in other words, the direction of the line must satisfy the equation

$$(\Sigma X x_1)\, p' + (\Sigma X x_2)\, q' + (\Sigma X x_3)\, r' = 0,$$

as well as the permanent equation

$$Ap'^2 + Bq'^2 + Cr'^2 + 2Fq'r' + 2Gr'p' + 2Hp'q' = 1.$$

Consequently there is, in general, a single infinitude of directions which, passing through any point of a region, also lie within any arbitrarily assigned flat through the point.

If however every direction, through the point of the region, should lie within the flat, then the foregoing condition must no longer impose any limitation of relation between dp, dq, dr: it must be satisfied for all values of dp, dq, dr, and consequently must become evanescent as a relation between those quantities. To satisfy this requirement, we must have

$$\Sigma X x_1 = 0, \quad \Sigma X x_2 = 0, \quad \Sigma X x_3 = 0.$$

The first of these relations ensures that a neighbouring point on the parametric curve p lies in the flat: that is, the tangent at O to that curve lies in the flat. The second relation ensures that the tangent at O to the parametric curve q lies in the flat; and the third ensures that the tangent at O to the parametric curve r lies in the flat.

Conversely, when these three relations are satisfied, the relation

$$\Sigma X\, (x_1 dp + x_2 dq + x_3 dr) = 0$$

is satisfied; that is, every direction, through the point x, y, z, v, and lying in the region, is contained in the flat, or every point in the region contiguous to O provides a tangent line through O lying in the flat. We therefore declare the flat to be tangential to the region at the point O: or more briefly, we call it the *tangent flat* at O. Accordingly, the equation of the tangent flat at a point x, y, z, v, in the region is

$$\Sigma X\, (\bar{x} - x) = 0,$$

where the coefficients X, Y, Z, V, satisfy the relations

$$\Sigma X x_1 = 0, \quad \Sigma X x_2 = 0, \quad \Sigma X x_3 = 0.$$

Manifestly, the equation of the tangent flat at O can be written

$$\begin{vmatrix} \bar{x} - x, & \bar{y} - y, & \bar{z} - z, & \bar{v} - v \\ x_1 , & y_1 , & z_1 , & v_1 \\ x_2 , & y_2 , & z_2 , & v_2 \\ x_3 , & y_3 , & z_3 , & v_3 \end{vmatrix} = 0.$$

In the first form of equation, the quantities X, Y, Z, V, are proportional to the direction-cosines of the normal to the flat. We shall now define X, Y, Z, V, to be the actual direction-cosines of the normal to the flat; thus we are to have

$$X^2 + Y^2 + Z^2 + V^2 = 1,$$
$$\Sigma X x_1 = 0, \quad \Sigma X x_2 = 0, \quad \Sigma X x_3 = 0.$$

The last three equations give, with the notation of § 263,

$$\frac{X}{J_x} = \frac{Y}{-J_y} = \frac{Z}{J_z} = \frac{V}{-J_v},$$

while

$$J_x^2 + J_y^2 + J_z^2 + J_v^2 = \begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix} = \Omega^2.$$

We take the positive root of Ω^2 as the value of Ω; and we define the positive direction of the normal as that which is given by direction-cosines

$$J_x/\Omega, \quad -J_y/\Omega, \quad J_z/\Omega, \quad -J_v/\Omega.$$

To compare the definition with the earlier convention (§ 49) concerning the positive direction of a perpendicular to a flat, we consider the particular instance of a region which is represented by the flat $v = 0$. For that instance, we can take $x = p$, $y = q$, $z = r$: then $A = 1$, $B = 1$, $C = 1$, $F = G = H = 0$, so that $\Omega = 1$: and $J_x = 0$, $J_y = 0$, $J_z = 0$, $J_v = 1$. Thus, under the definition, we should have $X = 0$, $Y = 0$, $Z = 0$, $V = -1$, for the positive direction of the perpendicular to the flat: that is, the definition for the positive direction of the normal accords with the earlier convention.

Thus *the direction-cosines of the normal to the tangent flat are* X, Y, Z, V, *where*

$$\Omega X = \begin{vmatrix} y_1, & z_1, & v_1 \\ y_2, & z_2, & v_2 \\ y_3, & z_3, & v_3 \end{vmatrix}, \quad \Omega Y = -\begin{vmatrix} z_1, & v_1, & x_1 \\ z_2, & v_2, & x_2 \\ z_3, & v_3, & x_3 \end{vmatrix},$$

$$\Omega Z = \begin{vmatrix} v_1, & x_1, & y_1 \\ v_2, & x_2, & y_2 \\ v_3, & x_3, & y_3 \end{vmatrix}, \quad \Omega V = -\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}.$$

When we have to specify directions inwards to a region, we shall take them as directions making an acute angle with the positive direction of the normal.

270. Should the region be defined by an equation

$$R(x, y, z, v) = 0$$

between the coordinates, the direction-cosines of the normal are obtained at once from the equation of the tangent flat.

When a point, contiguous to x, y, z, v, and having coordinates

$$x + dx, \quad y + dy, \quad z + dz, \quad v + dv,$$

also lies in the region, so that the line through x, y, z, v, in the direction dx, dy, dz, dv, lies in the region, the equation

$$\frac{\partial R}{\partial x}dx+\frac{\partial R}{\partial y}dy+\frac{\partial R}{\partial z}dz+\frac{\partial R}{\partial v}dv=0$$

is satisfied. Now the specified line has

$$\frac{\bar{x}-x}{dx}=\frac{\bar{y}-y}{dy}=\frac{\bar{z}-z}{dz}=\frac{\bar{v}-v}{dv}$$

for its equations; and therefore the coordinates $\bar{x}, \bar{y}, \bar{z}, \bar{v}$, of any point on the line satisfy the equation

$$(\bar{x}-x)\frac{\partial R}{\partial x}+(\bar{y}-y)\frac{\partial R}{\partial y}+(\bar{z}-z)\frac{\partial R}{\partial z}+(\bar{v}-v)\frac{\partial R}{\partial v}=0.$$

This relation is the equation of the tangent flat to the region at the point x, y, z, v. Accordingly, the direction-cosines of the normal to the flat at that point, that is, of the normal to the region, are proportional to

$$\frac{\partial R}{\partial x}, \quad \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial z}, \quad \frac{\partial R}{\partial v}.$$

The same result follows at once from the use of the parametric expressions of the coordinates. When substituted in $R=0$, they make the equation an identity; hence

$$\Sigma x_1\frac{\partial R}{\partial x}=0, \quad \Sigma x_2\frac{\partial R}{\partial x}=0, \quad \Sigma x_3\frac{\partial R}{\partial x}=0,$$

and therefore

$$\frac{\begin{vmatrix} y_1, & z_1, & v_1 \\ y_2, & z_2, & v_2 \\ y_3, & z_3, & v_3 \end{vmatrix}}{\frac{\partial R}{\partial x}}=\frac{-\begin{vmatrix} z_1, & v_1, & x_1 \\ z_2, & v_2, & x_2 \\ z_3, & v_3, & x_3 \end{vmatrix}}{\frac{\partial R}{\partial y}}=\frac{\begin{vmatrix} v_1, & x_1, & y_1 \\ v_2, & x_2, & y_2 \\ v_3, & x_3, & y_3 \end{vmatrix}}{\frac{\partial R}{\partial z}}=\frac{-\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}}{\frac{\partial R}{\partial v}},$$

that is,

$$\frac{1}{X}\frac{\partial R}{\partial x}=\frac{1}{Y}\frac{\partial R}{\partial y}=\frac{1}{Z}\frac{\partial R}{\partial z}=\frac{1}{V}\frac{\partial R}{\partial v}.$$

Fundamental magnitudes of the second order.

271. The perpendicular from any point $\xi, \eta, \zeta, \upsilon$, on the tangent flat at x, y, z, v, is equal to P, where

$$P=X(\xi-x)+Y(\eta-y)+Z(\zeta-z)+V(\upsilon-v),$$

and X, Y, Z, V, are the direction-cosines of the normal to the flat. Now let $\xi, \eta, \zeta, \upsilon$, be the coordinates of a point of the region, in the immediate

vicinity of the point x, y, z, v, and determined by values $p+dp$, $q+dq$, $r+dr$, of the parameters: then

$$\xi - x = (x_1, x_2, x_3 \mathbin{\text{\textbackslash\!\!\textbackslash}} dp, dq, dr) + \tfrac{1}{2}(x_{11}, x_{22}, x_{33}, x_{23}, x_{31}, x_{12} \mathbin{\text{\textbackslash\!\!\textbackslash}} dp, dq, dr)^2 + \ldots,$$

$$\eta - y = (y_1, y_2, y_3 \mathbin{\text{\textbackslash\!\!\textbackslash}} dp, dq, dr) + \tfrac{1}{2}(y_{11}, y_{22}, y_{33}, y_{23}, y_{31}, y_{12} \mathbin{\text{\textbackslash\!\!\textbackslash}} dp, dq, dr)^2 + \ldots,$$

$$\zeta - z = (z_1, z_2, z_3 \mathbin{\text{\textbackslash\!\!\textbackslash}} dp, dq, dr) + \tfrac{1}{2}(z_{11}, z_{22}, z_{33}, z_{23}, z_{31}, z_{12} \mathbin{\text{\textbackslash\!\!\textbackslash}} dp, dq, dr)^2 + \ldots,$$

$$v - v = (v_1, v_2, v_3 \mathbin{\text{\textbackslash\!\!\textbackslash}} dp, dq, dr) + \tfrac{1}{2}(v_{11}, v_{22}, v_{33}, v_{23}, v_{31}, v_{12} \mathbin{\text{\textbackslash\!\!\textbackslash}} dp, dq, dr)^2 + \ldots.$$

Now

$$\Sigma X x_1 = 0, \quad \Sigma X x_2 = 0, \quad \Sigma X x_3 = 0,$$

so that the terms of the first order in dp, dq, dr, disappear from the expression for P. For the terms of the second order in dp, dq, dr, we have to introduce new quantities L, M, N, K, I, J, according to the definitions

$$\left.\begin{aligned} L &= Xx_{11} + Yy_{11} + Zz_{11} + Vv_{11} \\ M &= Xx_{12} + Yy_{12} + Zz_{12} + Vv_{12} \\ N &= Xx_{22} + Yy_{22} + Zz_{22} + Vv_{22} \\ K &= Xx_{13} + Yy_{13} + Zz_{13} + Vv_{13} \\ I &= Xx_{23} + Yy_{23} + Zz_{23} + Vv_{23} \\ J &= Xx_{33} + Yy_{33} + Zz_{33} + Vv_{33} \end{aligned}\right\}.$$

These six quantities, involving the second derivatives of the coordinates, are called the *fundamental magnitudes of the second order*, or the *secondary magnitudes*, characteristic of the region. We thus have

$$\begin{aligned} P &= \Sigma X (\xi - x) \\ &= \tfrac{1}{2}(L, N, J, I, K, M \mathbin{\text{\textbackslash\!\!\textbackslash}} dp, dq, dr)^2 + \ldots. \end{aligned}$$

Curvature of a normal section of the region: linear measure of curvature: indicatrix quadric: asymptotic cone.

272. Consider the plane through the point x, y, z, v, having as its guiding lines the direction x', y', z', v', (or p', q', r') and the normal X, Y, Z, V, to the tangent flat of the region: it is represented by the two Cartesian equations

$$\begin{Vmatrix} \bar{x} - x, & \bar{y} - y, & \bar{z} - z, & \bar{v} - v \\ x', & y', & z', & v' \\ X, & Y, & Z, & V \end{Vmatrix} = 0.$$

The amplitude in which this plane intersects the region, represented by its single Cartesian equation, is represented by combining the single equation of the region with the two equations of the plane. Thus there are three equations in $\bar{x}, \bar{y}, \bar{z}, \bar{v}$; the represented amplitude is therefore a curve: that is, the intersection of the region by the plane in question (or, indeed, by any plane) is a curve.

That this intersection is a curve can also be seen thus. Any point lying in the region can be expressed in terms of the three parameters which describe the region; let the current values of these parameters be λ, μ, ν. Then for every point $\bar{x}, \bar{y}, \bar{z}, \bar{v}$, which is common to the region and the plane, we substitute the values of $\bar{x}, \bar{y}, \bar{z}, \bar{v}$, in terms of λ, μ, ν; and the two equations of the plane will then limit the admissible values of λ, μ, ν. Consequently λ, μ, ν, satisfy two independent equations; and therefore the amplitude, thus represented in the region by two equations between the three current parameters, is a plane curve.

Moreover, this plane contains the normal to the flat that touches the region, that is, contains the normal to the region: thus the plane is perpendicular (§ 75) to the flat and therefore, at the point O, is normal to the region. We thus have a plane curve, which is the intersection of the region by a normal plane drawn through a tangential direction determined by p', q', r'.

The curvature of this normal (plane) section of the region can be at once obtained. The perpendicular, in the plane of the section, drawn from the point Q contiguous to O upon the tangent to the curve is the preceding perpendicular P from Q upon the tangent flat; and the length of the arc QO has been denoted by ds. The radius of curvature of the curve is the limiting value of the quantity

$$\frac{ds^2}{2P},$$

as Q approaches O; and therefore

$$\begin{aligned}\frac{1}{\rho} &= \frac{2P}{ds^2} \\ &= \frac{(L, N, J, I, K, M)(dp, dq, dr)^2}{(A, B, C, F, G, H)(dp, dq, dr)^2} \\ &= Lp'^2 + 2Mp'q' + 2Kp'r' + Nq'^2 + 2Iq'r' + Jr'^2,\end{aligned}$$

a result that will occasionally be quoted in the form

$$\frac{ds^2}{\rho} = L\,dp^2 + 2M\,dp\,dq + 2K\,dp\,dr + N\,dq^2 + 2I\,dq\,dr + J\,dr^2.$$

The quantity $1/\rho$ is called *the curvature of a normal section* of the region.

Later, it will be found convenient (as, in the case of a surface, it has proved convenient) to substitute, for the normal section, the regional geodesic having the same tangent: but the elementary properties of geodesics must first be established (§§ 297, 298).

Further, as (§ 323) we shall have superficial measures and volumetric measures of curvature, we shall sometimes call this the *linear* measure of curvature of the region.

An obvious interpretation of this expression for the curvature can be derived from the customary geometry of homaloidal triple space with oblique axes of reference; here, it needs no more than a passing mention. We have

$$\frac{1}{\rho} = Lp'^2 + 2Mp'q' + 2Kp'r' + Nq'^2 + 2Iq'r' + Jr'^2,$$

with the relation

$$1 = Ap'^2 + 2Hp'q' + 2Gp'r' + Bq'^2 + 2Fq'r' + Cr'^2.$$

Let α, β, γ, denote the angles between the three curves of reference given by the pairs of parametric equations from $p = \text{constant}$, $q = \text{constant}$, $r = \text{constant}$, so that

$$(BC)^{-\frac{1}{2}} F = \cos\alpha, \quad (CA)^{-\frac{1}{2}} G = \cos\beta, \quad (AB)^{-\frac{1}{2}} H = \cos\gamma.$$

Also, let

$$p'A^{\frac{1}{2}} = \lambda, \quad q'B^{\frac{1}{2}} = \mu, \quad r'C^{\frac{1}{2}} = \nu,$$

so that the permanent relation becomes

$$\lambda^2 + \mu^2 + \nu^2 + 2\mu\nu\cos\alpha + 2\nu\lambda\cos\beta + 2\lambda\mu\cos\gamma = 1.$$

Finally, let there be quantities $\bar{x}$, $\bar{y}$, $\bar{z}$, R, such that

$$\bar{x} = \lambda R, \quad \bar{y} = \mu R, \quad \bar{z} = \nu R;$$

then

$$R^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2 + 2\bar{y}\bar{z}\cos\alpha + 2\bar{z}\bar{x}\cos\beta + 2\bar{x}\bar{y}\cos\gamma,$$

and

$$\frac{R^2}{\rho} = \frac{L}{A}\bar{x}^2 + \frac{N}{B}\bar{y}^2 + \frac{J}{C}\bar{z}^2 + 2\frac{I}{F}\bar{y}\bar{z}\cos\alpha + 2\frac{K}{G}\bar{z}\bar{x}\cos\beta + 2\frac{M}{H}\bar{x}\bar{y}\cos\gamma = U.$$

If then we take a quadric

$$U = c,$$

where c is a constant, we at once have

$$\rho = \frac{R^2}{c};$$

that is, the radius of curvature of the geodesic is proportional to the square of the radius vector of the quadric. It is an *indicatrix quadric*; and the equation

$$U = 0$$

is the *asymptotic cone of the quadric.*

Derived relations among the primary magnitudes.

273. In preparation for the analytical development of curvature properties of the region, it is convenient to obtain certain algebraical and derived relations among the quantities which have been introduced, viz. the direction-cosines X, Y, Z, V, of the normal to the flat; the primary magnitudes A, B, C, F, G, H; and the secondary magnitudes L, M, N, K, I, J.

I. We have a central determinant Ω^2, composed of the primary magnitudes according to the definition

$$\Omega^2 = \begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix}.$$

The first minors of the constituents of Ω^2 will be needed: we define

$$\left.\begin{aligned} a &= BC - F^2, & f &= GH - AF \\ b &= CA - G^2, & g &= HF - BG \\ c &= AB - H^2, & h &= FG - CH \end{aligned}\right\}.$$

II. The first derivatives, and certain combinations of the first derivatives, of the primary magnitudes repeatedly occur: and there are some specially useful combinations. We write

$$\left.\begin{aligned} \alpha_1 &= \Sigma x_1 x_{11} = \tfrac{1}{2} A_1 \\ \alpha_2 &= \Sigma x_2 x_{11} = -\tfrac{1}{2} A_2 + H_1 \\ \alpha_3 &= \Sigma x_3 x_{11} = -\tfrac{1}{2} A_3 + G_1 \end{aligned}\right\}, \qquad \left.\begin{aligned} \epsilon_1 &= \Sigma x_1 x_{23} = \tfrac{1}{2}(G_2 + H_3 - F_1) \\ \epsilon_2 &= \Sigma x_2 x_{23} = \tfrac{1}{2} B_3 \\ \epsilon_3 &= \Sigma x_3 x_{23} = \tfrac{1}{2} C_2 \end{aligned}\right\},$$

$$\left.\begin{aligned} \delta_1 &= \Sigma x_1 x_{22} = -\tfrac{1}{2} B_1 + H_2 \\ \delta_2 &= \Sigma x_2 x_{22} = \tfrac{1}{2} B_2 \\ \delta_3 &= \Sigma x_3 x_{22} = -\tfrac{1}{2} B_3 + F_2 \end{aligned}\right\}, \qquad \left.\begin{aligned} \gamma_1 &= \Sigma x_1 x_{31} = \tfrac{1}{2} A_3 \\ \gamma_2 &= \Sigma x_2 x_{31} = \tfrac{1}{2}(H_3 + F_1 - G_2) \\ \gamma_3 &= \Sigma x_3 x_{31} = \tfrac{1}{2} C_1 \end{aligned}\right\},$$

$$\left.\begin{aligned} \omega_1 &= \Sigma x_1 x_{33} = -\tfrac{1}{2} C_1 + G_3 \\ \omega_2 &= \Sigma x_2 x_{33} = -\tfrac{1}{2} C_2 + F_3 \\ \omega_3 &= \Sigma x_3 x_{33} = \tfrac{1}{2} C_3 \end{aligned}\right\}, \qquad \left.\begin{aligned} \beta_1 &= \Sigma x_1 x_{12} = \tfrac{1}{2} A_2 \\ \beta_2 &= \Sigma x_2 x_{12} = \tfrac{1}{2} B_1 \\ \beta_3 &= \Sigma x_3 x_{12} = \tfrac{1}{2}(F_1 + G_2 - H_3) \end{aligned}\right\}.$$

In connection with these quantities, we introduce eighteen new quantities

$$\Gamma_{lm}, \ \Delta_{lm}, \ \Theta_{lm}, \quad (l, m, = 1, 2, 3),$$

where $\Gamma_{lm} = \Gamma_{ml}$, $\Delta_{lm} = \Delta_{ml}$, $\Theta_{lm} = \Theta_{ml}$, their values being assigned by the following definitions*:

* It may be convenient, for purposes of reference, to state here the equivalence (for $n = 4$) of these symbols α, β, γ, δ, ϵ, ω, Γ, Δ, Θ, with the Christoffel three-index symbols for the Riemann combinations as used in n-fold geometry: see § 423, *post*. It is to be remembered that, in the usual exposition of that theory, any external homaloidal m-fold space ($m > n$) enclosing the n-fold amplitude is left out of account. Its foundation is the postulated expression for ds^2, in terms of the parameters; and the coefficients, in that expression for the metric element, are taken as functions of the parameters, without any assumption as to their genesis from combinations of first derivatives of point-coordinates in some homaloidal m-fold space.

(i) For the symbols α, β, γ, δ, ϵ, ω, the equivalence is

$$\Sigma x_k x_{ij} = [ij, k].$$

(ii) For the symbols Γ, Δ, Θ, the equivalence is

$$\Gamma_{ij} = \{ij, 1\}, \quad \Delta_{ij} = \{ij, 2\}, \quad \Theta_{ij} = \{ij, 3\},$$

taken from the equations such as $\Omega^2 \Gamma_{11} = a\alpha_1 + h\alpha_2 + g\alpha_3$.

The Riemann four-index symbols (ij, kl) are used hereafter when they occur (§ 423); their significance, for quadruple space, is contained implicitly in the relations in § 276.

$$\left.\begin{aligned}\alpha_1 &= A\Gamma_{11} + H\Delta_{11} + G\Theta_{11}\\ \alpha_2 &= H\Gamma_{11} + B\Delta_{11} + F\Theta_{11}\\ \alpha_3 &= G\Gamma_{11} + F\Delta_{11} + C\Theta_{11}\end{aligned}\right\},\quad \left.\begin{aligned}\epsilon_1 &= A\Gamma_{23} + H\Delta_{23} + G\Theta_{23}\\ \epsilon_2 &= H\Gamma_{23} + B\Delta_{23} + F\Theta_{23}\\ \epsilon_3 &= G\Gamma_{23} + F\Delta_{23} + C\Theta_{23}\end{aligned}\right\},$$

$$\left.\begin{aligned}\delta_1 &= A\Gamma_{22} + H\Delta_{22} + G\Theta_{22}\\ \delta_2 &= H\Gamma_{22} + B\Delta_{22} + F\Theta_{22}\\ \delta_3 &= G\Gamma_{22} + F\Delta_{22} + C\Theta_{22}\end{aligned}\right\},\quad \left.\begin{aligned}\gamma_1 &= A\Gamma_{31} + H\Delta_{31} + G\Theta_{31}\\ \gamma_2 &= H\Gamma_{31} + B\Delta_{31} + F\Theta_{31}\\ \gamma_3 &= G\Gamma_{31} + F\Delta_{31} + C\Theta_{31}\end{aligned}\right\},$$

$$\left.\begin{aligned}\omega_1 &= A\Gamma_{33} + H\Delta_{33} + G\Theta_{33}\\ \omega_2 &= H\Gamma_{33} + B\Delta_{33} + F\Theta_{33}\\ \omega_3 &= G\Gamma_{33} + F\Delta_{33} + C\Theta_{33}\end{aligned}\right\},\quad \left.\begin{aligned}\beta_1 &= A\Gamma_{12} + H\Delta_{12} + G\Theta_{12}\\ \beta_2 &= H\Gamma_{12} + B\Delta_{12} + F\Theta_{12}\\ \beta_3 &= G\Gamma_{12} + F\Delta_{12} + C\Theta_{12}\end{aligned}\right\}.$$

These relations of definition for the quantities Γ, Δ, Θ, when resolved, yield the following values:

$$\left.\begin{aligned}\Omega^2\Gamma_{11} &= a\alpha_1 + h\alpha_2 + g\alpha_3\\ \Omega^2\Delta_{11} &= h\alpha_1 + b\alpha_2 + f\alpha_3\\ \Omega^2\Theta_{11} &= g\alpha_1 + f\alpha_2 + c\alpha_3\end{aligned}\right\},\quad \left.\begin{aligned}\Omega^2\Gamma_{23} &= a\epsilon_1 + h\epsilon_2 + g\epsilon_3\\ \Omega^2\Delta_{23} &= h\epsilon_1 + b\epsilon_2 + f\epsilon_3\\ \Omega^2\Theta_{23} &= g\epsilon_1 + f\epsilon_2 + c\epsilon_3\end{aligned}\right\},$$

$$\left.\begin{aligned}\Omega^2\Gamma_{22} &= a\delta_1 + h\delta_2 + g\delta_3\\ \Omega^2\Delta_{22} &= h\delta_1 + b\delta_2 + f\delta_3\\ \Omega^2\Theta_{22} &= g\delta_1 + f\delta_2 + c\delta_3\end{aligned}\right\},\quad \left.\begin{aligned}\Omega^2\Gamma_{31} &= a\gamma_1 + h\gamma_2 + g\gamma_3\\ \Omega^2\Delta_{31} &= h\gamma_1 + b\gamma_2 + f\gamma_3\\ \Omega^2\Theta_{31} &= g\gamma_1 + f\gamma_2 + c\gamma_3\end{aligned}\right\},$$

$$\left.\begin{aligned}\Omega^2\Gamma_{33} &= a\omega_1 + h\omega_2 + g\omega_3\\ \Omega^2\Delta_{33} &= h\omega_1 + b\omega_2 + f\omega_3\\ \Omega^2\Theta_{33} &= g\omega_1 + f\omega_2 + c\omega_3\end{aligned}\right\},\quad \left.\begin{aligned}\Omega^2\Gamma_{12} &= a\beta_1 + h\beta_2 + g\beta_3\\ \Omega^2\Delta_{12} &= h\beta_1 + b\beta_2 + f\beta_3\\ \Omega^2\Theta_{12} &= g\beta_1 + f\beta_2 + c\beta_3\end{aligned}\right\}.$$

The actual first derivatives of the primary magnitudes are expressible in terms of these quantities Γ, Δ, Θ, in the forms

$$\left.\begin{aligned}\tfrac{1}{2}A_1 &= A\Gamma_{11} + H\Delta_{11} + G\Theta_{11}\\ \tfrac{1}{2}A_2 &= A\Gamma_{12} + H\Delta_{12} + G\Theta_{12}\\ \tfrac{1}{2}A_3 &= A\Gamma_{13} + H\Delta_{13} + G\Theta_{13}\end{aligned}\right\},\quad \left.\begin{aligned}\tfrac{1}{2}B_1 &= H\Gamma_{21} + B\Delta_{21} + F\Theta_{21}\\ \tfrac{1}{2}B_2 &= H\Gamma_{22} + B\Delta_{22} + F\Theta_{22}\\ \tfrac{1}{2}B_3 &= H\Gamma_{23} + B\Delta_{23} + F\Theta_{23}\end{aligned}\right\},\quad \left.\begin{aligned}\tfrac{1}{2}C_1 &= G\Gamma_{31} + F\Delta_{31} + C\Theta_{31}\\ \tfrac{1}{2}C_2 &= G\Gamma_{32} + F\Delta_{32} + C\Theta_{32}\\ \tfrac{1}{2}C_3 &= G\Gamma_{33} + F\Delta_{33} + C\Theta_{33}\end{aligned}\right\}$$

$$\left.\begin{aligned}F_1 &= (H\Gamma_{31} + B\Delta_{31} + F\Theta_{31}) + (G\Gamma_{21} + F\Delta_{21} + C\Theta_{21})\\ F_2 &= (H\Gamma_{32} + B\Delta_{32} + F\Theta_{32}) + (G\Gamma_{22} + F\Delta_{22} + C\Theta_{22})\\ F_3 &= (H\Gamma_{33} + B\Delta_{33} + F\Theta_{33}) + (G\Gamma_{23} + F\Delta_{23} + C\Theta_{23})\end{aligned}\right\}$$

$$\left.\begin{aligned}G_1 &= (A\Gamma_{31} + H\Delta_{31} + G\Theta_{31}) + (G\Gamma_{11} + F\Delta_{11} + C\Theta_{11})\\ G_2 &= (A\Gamma_{32} + H\Delta_{32} + G\Theta_{32}) + (G\Gamma_{12} + F\Delta_{12} + C\Theta_{12})\\ G_3 &= (A\Gamma_{33} + H\Delta_{33} + G\Theta_{33}) + (G\Gamma_{13} + F\Delta_{13} + C\Theta_{13})\end{aligned}\right\},$$

$$\left.\begin{aligned}H_1 &= (A\Gamma_{21} + H\Delta_{21} + G\Theta_{21}) + (H\Gamma_{11} + B\Delta_{11} + F\Theta_{11})\\ H_2 &= (A\Gamma_{22} + H\Delta_{22} + G\Theta_{22}) + (H\Gamma_{12} + B\Delta_{12} + F\Theta_{12})\\ H_3 &= (A\Gamma_{23} + H\Delta_{23} + G\Theta_{23}) + (H\Gamma_{13} + B\Delta_{13} + F\Theta_{13})\end{aligned}\right\}$$

III. We deduce the values of Ω_1, Ω_2, Ω_3. Immediate differentiation gives

$$2\Omega\Omega_1 = aA_1 + 2hH_1 + 2gG_1 + bB_1 + 2fF_1 + cC_1.$$

When substitution takes place on the right-hand side for the derivatives of A, B, C, F, G, H, the aggregate of terms in Γ_{11}, Δ_{11}, Θ_{11}, is

$$\begin{aligned}&= 2a(A\Gamma_{11} + H\Delta_{11} + G\Theta_{11}) + 2h(H\Gamma_{11} + B\Delta_{11} + F\Theta_{11}) \\ &\qquad + 2g(G\Gamma_{11} + F\Delta_{11} + C\Theta_{11}) \\ &= 2\Gamma_{11}(aA + hH + gG) + 2\Delta_{11}(aH + hB + gF) + 2\Theta_{11}(aG + hF + gC) \\ &= 2\Gamma_{11}\Omega^2;\end{aligned}$$

the aggregate of terms in Γ_{12}, Δ_{12}, Θ_{12}, is equal to

$$2\Delta_{12}\Omega^2,$$

and the aggregate of terms in Γ_{13}, Δ_{13}, Θ_{13}, is equal to

$$2\Theta_{13}\Omega^2.$$

Hence

$$\Omega\Omega_1 = (\Gamma_{11} + \Delta_{21} + \Theta_{31})\Omega^2.$$

Similarly for Ω_2 and for Ω_3. The complete set of values is

$$\left.\begin{aligned}\frac{\Omega_1}{\Omega} &= \Gamma_{11} + \Delta_{21} + \Theta_{31} \\ \frac{\Omega_2}{\Omega} &= \Gamma_{12} + \Delta_{22} + \Theta_{32} \\ \frac{\Omega_3}{\Omega} &= \Gamma_{13} + \Delta_{23} + \Theta_{33}\end{aligned}\right\}.$$

IV. It is convenient, at this stage, to put on record the expressions for the first derivatives of the minors a, b, c, f, g, h, of the constituents of the determinant Ω^2. They are as follows:

$$\left.\begin{aligned}\tfrac{1}{2}a_1 &= a\frac{\Omega_1}{\Omega} - (a\Gamma_{11} + h\Gamma_{21} + g\Gamma_{31}) \\ \tfrac{1}{2}a_2 &= a\frac{\Omega_2}{\Omega} - (a\Gamma_{12} + h\Gamma_{22} + g\Gamma_{32}) \\ \tfrac{1}{2}a_3 &= a\frac{\Omega_3}{\Omega} - (a\Gamma_{13} + h\Gamma_{23} + g\Gamma_{33})\end{aligned}\right\},$$

$$\left.\begin{aligned}\tfrac{1}{2}b_1 &= b\frac{\Omega_1}{\Omega} - (h\Delta_{11} + b\Delta_{21} + f\Delta_{31}) \\ \tfrac{1}{2}b_2 &= b\frac{\Omega_2}{\Omega} - (h\Delta_{12} + b\Delta_{22} + f\Delta_{32}) \\ \tfrac{1}{2}b_3 &= b\frac{\Omega_3}{\Omega} - (h\Delta_{13} + b\Delta_{23} + f\Delta_{33})\end{aligned}\right\},$$

$$\left.\begin{aligned}
\tfrac{1}{2}c_1 &= c\,\frac{\Omega_1}{\Omega} - (g\Theta_{11} + f\Theta_{21} + c\Theta_{31}) \\
\tfrac{1}{2}c_2 &= c\,\frac{\Omega_2}{\Omega} - (g\Theta_{12} + f\Theta_{22} + c\Theta_{32}) \\
\tfrac{1}{2}c_3 &= c\,\frac{\Omega_3}{\Omega} - (g\Theta_{13} + f\Theta_{23} + c\Theta_{33})
\end{aligned}\right\},$$

$$\left.\begin{aligned}
f_1 &= 2f\,\frac{\Omega_1}{\Omega} - (g\Delta_{11} + f\Delta_{21} + c\Delta_{31}) - (h\Theta_{11} + b\Theta_{21} + f\Theta_{31}) \\
f_2 &= 2f\,\frac{\Omega_2}{\Omega} - (g\Delta_{12} + f\Delta_{22} + c\Delta_{32}) - (h\Theta_{12} + b\Theta_{22} + f\Theta_{32}) \\
f_3 &= 2f\,\frac{\Omega_3}{\Omega} - (g\Delta_{13} + f\Delta_{23} + c\Delta_{33}) - (h\Theta_{13} + b\Theta_{23} + f\Theta_{33})
\end{aligned}\right\},$$

$$\left.\begin{aligned}
g_1 &= 2g\,\frac{\Omega_1}{\Omega} - (a\Theta_{11} + h\Theta_{21} + g\Theta_{31}) - (g\Gamma_{11} + f\Gamma_{21} + c\Gamma_{31}) \\
g_2 &= 2g\,\frac{\Omega_2}{\Omega} - (a\Theta_{12} + h\Theta_{22} + g\Theta_{32}) - (g\Gamma_{12} + f\Gamma_{22} + c\Gamma_{32}) \\
g_3 &= 2g\,\frac{\Omega_3}{\Omega} - (a\Theta_{13} + h\Theta_{23} + g\Theta_{33}) - (g\Gamma_{13} + f\Gamma_{23} + c\Gamma_{33})
\end{aligned}\right\},$$

$$\left.\begin{aligned}
h_1 &= 2h\,\frac{\Omega_1}{\Omega} - (h\Gamma_{11} + b\Gamma_{21} + f\Gamma_{31}) - (a\Delta_{11} + h\Delta_{21} + g\Delta_{31}) \\
h_2 &= 2h\,\frac{\Omega_2}{\Omega} - (h\Gamma_{12} + b\Gamma_{22} + f\Gamma_{32}) - (a\Delta_{12} + h\Delta_{22} + g\Delta_{32}) \\
h_3 &= 2h\,\frac{\Omega_3}{\Omega} - (h\Gamma_{13} + b\Gamma_{23} + f\Gamma_{33}) - (a\Delta_{13} + h\Delta_{23} + g\Delta_{33})
\end{aligned}\right\}.$$

Derivatives of X, Y, Z, V, the direction-cosines of the normal to the tangent flat.

274. All the preceding relations ultimately involve only the primary magnitudes and their first derivatives.

We now pass to the expressions for the first derivatives of X, Y, Z, V: at once it appears that these require the use of the secondary magnitudes. By differentiating the three relations

$$\Sigma X x_1 = 0, \quad \Sigma X x_2 = 0, \quad \Sigma X x_3 = 0,$$

with respect to p, we have

$$\Sigma X_1 x_1 + \Sigma X x_{11} = 0, \quad \Sigma X_1 x_2 + \Sigma X x_{12} = 0, \quad \Sigma X_1 x_3 + \Sigma X x_{13} = 0,$$

that is,

$$\begin{aligned}
X_1 x_1 + Y_1 y_1 + Z_1 z_1 + V_1 v_1 &= -L, \\
X_1 x_2 + Y_1 y_2 + Z_1 z_2 + V_1 v_2 &= -M, \\
X_1 x_3 + Y_1 y_3 + Z_1 z_3 + V_1 v_3 &= -K;
\end{aligned}$$

and there is the relation

$$X_1X + Y_1Y + Z_1Z + V_1V = 0,$$

because $X^2 + Y^2 + Z^2 + V^2 = 1$. To resolve these equations, introduce four subsidiary magnitudes θ_p, ϕ_p, ψ_p, χ_p, such that

$$\begin{aligned} X_1 &= x_1\theta_p + x_2\phi_p + x_3\psi_p + X\chi_p, \\ Y_1 &= y_1\theta_p + y_2\phi_p + y_3\psi_p + Y\chi_p, \\ Z_1 &= z_1\theta_p + z_2\phi_p + z_3\psi_p + Z\chi_p, \\ V_1 &= v_1\theta_p + v_2\phi_p + v_3\psi_p + V\chi_p. \end{aligned}$$

When these values are substituted in the four relations, we find

$$\begin{aligned} A\theta_p + H\phi_p + G\psi_p &= -L, \\ H\theta_p + B\phi_p + F\psi_p &= -M, \\ G\theta_p + F\phi_p + C\psi_p &= -K, \\ \chi_p &= 0 \quad , \end{aligned}$$

so that

$$\begin{aligned} \Omega^2\theta_p &= -(aL + hM + gK), \\ \Omega^2\phi_p &= -(hL + bM + fK), \\ \Omega^2\psi_p &= -(gL + fM + cK), \\ \chi_p &= 0. \end{aligned}$$

Thus

$$\left.\begin{aligned} -\Omega^2X_1 &= x_1\lambda_1 + x_2\mu_1 + x_3\nu_1 \\ -\Omega^2Y_1 &= y_1\lambda_1 + y_2\mu_1 + y_3\nu_1 \\ -\Omega^2Z_1 &= z_1\lambda_1 + z_2\mu_1 + z_3\nu_1 \\ -\Omega^2V_1 &= v_1\lambda_1 + v_2\mu_1 + v_3\nu_1 \end{aligned}\right\},$$

where

$$\lambda_1 = aL + hM + gK, \quad \mu_1 = hL + bM + fK, \quad \nu_1 = gL + fM + cK.$$

Proceeding similarly after differentiating the same relations with respect to q and with respect to r separately, we find respectively

$$\left.\begin{aligned} -\Omega^2X_2 &= x_1\lambda_2 + x_2\mu_2 + x_3\nu_2 \\ -\Omega^2Y_2 &= y_1\lambda_2 + y_2\mu_2 + y_3\nu_2 \\ -\Omega^2Z_2 &= z_1\lambda_2 + z_2\mu_2 + z_3\nu_2 \\ -\Omega^2V_2 &= v_1\lambda_2 + v_2\mu_2 + v_3\nu_2 \end{aligned}\right\},$$

where

$$\lambda_2 = aM + hN + gI, \quad \mu_2 = hM + bN + fI, \quad \nu_2 = gM + fN + cI;$$

and

$$\left.\begin{aligned} -\Omega^2X_3 &= x_1\lambda_3 + x_2\mu_3 + x_3\nu_3 \\ -\Omega^2Y_3 &= y_1\lambda_3 + y_2\mu_3 + y_3\nu_3 \\ -\Omega^2Z_3 &= z_1\lambda_3 + z_2\mu_3 + z_3\nu_3 \\ -\Omega^2V_3 &= v_1\lambda_3 + v_2\mu_3 + v_3\nu_3 \end{aligned}\right\},$$

where

$$\lambda_3 = aK + hI + gJ, \quad \mu_3 = hK + bI + fJ, \quad \nu_3 = gK + fI + cJ.$$

Ex. Later, in connection with the discussion of minimal regions in quadruple space (Chapter XXIII), we shall require the values of the quantities

$$\frac{\partial}{\partial p}\left(\frac{\partial \Omega}{\partial x_1}\right)+\frac{\partial}{\partial q}\left(\frac{\partial \Omega}{\partial x_2}\right)+\frac{\partial}{\partial r}\left(\frac{\partial \Omega}{\partial x_3}\right),$$

$$\frac{\partial}{\partial p}\left(\frac{\partial \Omega}{\partial y_1}\right)+\frac{\partial}{\partial q}\left(\frac{\partial \Omega}{\partial y_2}\right)+\frac{\partial}{\partial r}\left(\frac{\partial \Omega}{\partial y_3}\right),$$

and the remaining two expressions with derivatives in $z_1, z_2, z_3; v_1, v_2, v_3$.

We have

$$\Omega^2 = J_x^2 + J_y^2 + J_z^2 + J_v^2,$$

and therefore

$$\Omega \frac{\partial \Omega}{\partial x_1} = J_x \frac{\partial J_x}{\partial x_1} + J_y \frac{\partial J_y}{\partial x_1} + J_z \frac{\partial J_z}{\partial x_1} + J_v \frac{\partial J_v}{\partial x_1}.$$

Now

$$\frac{\partial J_x}{\partial x_1}=0, \quad \frac{\partial J_y}{\partial x_1}=\begin{vmatrix} z_2, & v_2 \\ z_3, & v_3 \end{vmatrix}, \quad \frac{\partial J_z}{\partial x_1}=\begin{vmatrix} y_2, & v_2 \\ y_3, & v_3 \end{vmatrix}, \quad \frac{\partial J_v}{\partial x_1}=\begin{vmatrix} y_2, & z_2 \\ y_3, & z_3 \end{vmatrix},$$

and

$$J_x = \Omega X, \quad J_y = -\Omega Y, \quad J_z = \Omega Z, \quad J_v = -\Omega V;$$

hence

$$\frac{\partial \Omega}{\partial x_1} = -\begin{vmatrix} Y, & Z, & V \\ y_2, & z_2, & v_2 \\ y_3, & z_3, & v_3 \end{vmatrix}.$$

Similarly we find

$$\frac{\partial \Omega}{\partial x_2} = -\begin{vmatrix} Y, & Z, & V \\ y_3, & z_3, & v_3 \\ y_1, & z_1, & v_1 \end{vmatrix}, \quad \frac{\partial \Omega}{\partial x_3} = -\begin{vmatrix} Y, & Z, & V \\ y_1, & z_1, & v_1 \\ y_2, & z_2, & v_2 \end{vmatrix}.$$

When these values are substituted in the quantity

$$-\left\{\frac{\partial}{\partial p}\left(\frac{\partial \Omega}{\partial x_1}\right)+\frac{\partial}{\partial q}\left(\frac{\partial \Omega}{\partial x_2}\right)+\frac{\partial}{\partial r}\left(\frac{\partial \Omega}{\partial x_3}\right)\right\},$$

there are the three terms

$$\begin{vmatrix} Y_1, & Z_1, & V_1 \\ y_2, & z_2, & v_2 \\ y_3, & z_3, & v_3 \end{vmatrix} + \begin{vmatrix} Y_2, & Z_2, & V_2 \\ y_3, & z_3, & v_3 \\ y_1, & z_1, & v_1 \end{vmatrix} + \begin{vmatrix} Y_3, & Z_3, & V_3 \\ y_1, & z_1, & v_1 \\ y_2, & z_2, & v_2 \end{vmatrix};$$

and there are three pairs of terms, one such pair which comes from

$$-\frac{\partial}{\partial p}\left(\frac{\partial \Omega}{\partial x_1}\right)-\frac{\partial}{\partial q}\left(\frac{\partial \Omega}{\partial x_2}\right),$$

being

$$\begin{vmatrix} Y, & Z, & V \\ y_{12}, & z_{12}, & v_{12} \\ y_3, & z_3, & v_3 \end{vmatrix} + \begin{vmatrix} Y, & Z, & V \\ y_3, & z_3, & v_3 \\ y_{12}, & z_{12}, & v_{12} \end{vmatrix}.$$

The sum of this pair, taken together, vanishes: the sum of each of the other two pairs vanishes: the sum of all the six terms vanishes.

There remain the three terms. Using the values of Y_1, Z_1, V_1, we have

$$\begin{vmatrix} Y_1, & Z_1, & V_1 \\ y_2, & z_2, & v_2 \\ y_3, & z_3, & v_3 \end{vmatrix} = -\frac{1}{\Omega^2}\left\{\lambda_1 \begin{vmatrix} y_1, & z_1, & v_1 \\ y_2, & z_2, & v_2 \\ y_3, & z_3, & v_3 \end{vmatrix} + \mu_1 \begin{vmatrix} y_2, & z_2, & v_2 \\ y_2, & z_2, & v_2 \\ y_3, & z_3, & v_3 \end{vmatrix} + \nu_1 \begin{vmatrix} y_3, & z_3, & v_3 \\ y_2, & z_2, & v_2 \\ y_3, & z_3, & v_3 \end{vmatrix}\right\}$$

$$= -\frac{1}{\Omega^2}\lambda_1 J_x$$

$$= \frac{X}{\Omega}(aL + hM + gK).$$

Similarly we find

$$\begin{vmatrix} Y_2, & Z_2, & V_2 \\ y_3, & z_3, & v_3 \\ y_1, & z_1, & v_1 \end{vmatrix} = \frac{X}{\Omega}(hM+bN+fI),$$

$$\begin{vmatrix} Y_3, & Z_3, & V_3 \\ y_1, & z_1, & v_1 \\ y_2, & z_2, & v_2 \end{vmatrix} = \frac{X}{\Omega}(gK+fI+cJ).$$

Consequently

$$\frac{\partial}{\partial p}\left(\frac{\partial\Omega}{\partial x_1}\right)+\frac{\partial}{\partial q}\left(\frac{\partial\Omega}{\partial x_2}\right)+\frac{\partial}{\partial r}\left(\frac{\partial\Omega}{\partial x_3}\right)$$
$$= -\frac{X}{\Omega}(aL+2hM+2gK+bN+2fI+cJ)$$
$$= -\frac{X}{\Omega}\mathfrak{G}_1;$$

and similarly

$$\frac{\partial}{\partial p}\left(\frac{\partial\Omega}{\partial y_1}\right)+\frac{\partial}{\partial q}\left(\frac{\partial\Omega}{\partial y_2}\right)+\frac{\partial}{\partial r}\left(\frac{\partial\Omega}{\partial y_3}\right)=-\frac{Y}{\Omega}\mathfrak{G}_1,$$

$$\frac{\partial}{\partial p}\left(\frac{\partial\Omega}{\partial z_1}\right)+\frac{\partial}{\partial q}\left(\frac{\partial\Omega}{\partial z_2}\right)+\frac{\partial}{\partial r}\left(\frac{\partial\Omega}{\partial z_3}\right)=-\frac{Z}{\Omega}\mathfrak{G}_1,$$

$$\frac{\partial}{\partial p}\left(\frac{\partial\Omega}{\partial v_1}\right)+\frac{\partial}{\partial q}\left(\frac{\partial\Omega}{\partial v_2}\right)+\frac{\partial}{\partial r}\left(\frac{\partial\Omega}{\partial v_3}\right)=-\frac{V}{\Omega}\mathfrak{G}_1.$$

The required values are thus obtained. The quantity $\mathfrak{G}_1$ will be found (§ 281) to be connected with one of the principal measures of linear curvature of the region.

Equations of the second order satisfied by point-coordinates.

275. By means of the secondary magnitudes L, N, J, I, K, M, and of the quantities Γ, Δ, Θ, constructed from the first derivatives of the primary magnitudes A, B, C, F, G, H, it is possible to express the second derivatives of the point-coordinates x, y, z, v. The relations, thus to be obtained, are fundamental in connection with the region.

Among the equations established between the quantities, there are four which involve x_{11}, y_{11}, z_{11}, v_{11}, linearly, viz.

$$Xx_{11}+Yy_{11}+Zz_{11}+Vv_{11}=L,$$
$$x_1x_{11}+y_1y_{11}+z_1z_{11}+v_1v_{11}=\alpha_1=A\Gamma_{11}+H\Delta_{11}+G\Theta_{11},$$
$$x_2x_{11}+y_2y_{11}+z_2z_{11}+v_2v_{11}=\alpha_2=H\Gamma_{11}+B\Delta_{11}+F\Theta_{11},$$
$$x_3x_{11}+y_3y_{11}+z_3z_{11}+v_3v_{11}=\alpha_3=G\Gamma_{11}+F\Delta_{11}+C\Theta_{11};$$

and therefore

$$x_{11}=XL+x_1\Gamma_{11}+x_2\Delta_{11}+x_3\Theta_{11},$$
$$y_{11}=YL+y_1\Gamma_{11}+y_2\Delta_{11}+y_3\Theta_{11},$$
$$z_{11}=ZL+z_1\Gamma_{11}+z_2\Delta_{11}+z_3\Theta_{11},$$
$$v_{11}=VL+v_1\Gamma_{11}+v_2\Delta_{11}+v_3\Theta_{11}.$$

These resolved results are easily obtained by assuming four quantities ξ, η, ζ, θ, such that

$$\begin{aligned} x_{11} &= X\theta + x_1\xi + x_2\eta + x_3\zeta, \\ y_{11} &= Y\theta + y_1\xi + y_2\eta + y_3\zeta, \\ z_{11} &= Z\theta + z_1\xi + z_2\eta + z_3\zeta, \\ v_{11} &= V\theta + v_1\xi + v_2\eta + v_3\zeta. \end{aligned}$$

When these are multiplied by X, Y, Z, V, and added, we have

$$L = \theta.$$

When they are multiplied by x_1, y_1, z_1, v_1, and added: by x_2, y_2, z_2, v_2, added: and by x_3, y_3, z_3, v_3, and added: we find, respectively,

$$\alpha_1 = A\xi + H\eta + G\zeta, \quad \alpha_2 = H\xi + B\eta + F\zeta, \quad \alpha_3 = G\xi + F\eta + C\zeta,$$

so that

$$\xi = \Gamma_{11}, \quad \eta = \Delta_{11}, \quad \zeta = \Theta_{11}.$$

The same process can be applied to the other second derivatives of the coordinates, and leads to the following aggregate of results:

$$\left.\begin{aligned} x_{11} &= XL + x_1\Gamma_{11} + x_2\Delta_{11} + x_3\Theta_{11} \\ y_{11} &= YL + y_1\Gamma_{11} + y_2\Delta_{11} + y_3\Theta_{11} \\ z_{11} &= ZL + z_1\Gamma_{11} + z_2\Delta_{11} + z_3\Theta_{11} \\ v_{11} &= VL + v_1\Gamma_{11} + v_2\Delta_{11} + v_3\Theta_{11} \end{aligned}\right\},$$

$$\left.\begin{aligned} x_{12} &= XM + x_1\Gamma_{12} + x_2\Delta_{12} + x_3\Theta_{12} \\ y_{12} &= YM + y_1\Gamma_{12} + y_2\Delta_{12} + y_3\Theta_{12} \\ z_{12} &= ZM + z_1\Gamma_{12} + z_2\Delta_{12} + z_3\Theta_{12} \\ v_{12} &= VM + v_1\Gamma_{12} + v_2\Delta_{12} + v_3\Theta_{12} \end{aligned}\right\},$$

$$\left.\begin{aligned} x_{13} &= XK + x_1\Gamma_{13} + x_2\Delta_{13} + x_3\Theta_{13} \\ y_{13} &= YK + y_1\Gamma_{13} + y_2\Delta_{13} + y_3\Theta_{13} \\ z_{13} &= ZK + z_1\Gamma_{13} + z_2\Delta_{13} + z_3\Theta_{13} \\ v_{13} &= VK + v_1\Gamma_{13} + v_2\Delta_{13} + v_3\Theta_{13} \end{aligned}\right\},$$

$$\left.\begin{aligned} x_{22} &= XN + x_1\Gamma_{22} + x_2\Delta_{22} + x_3\Theta_{22} \\ y_{22} &= YN + y_1\Gamma_{22} + y_2\Delta_{22} + y_3\Theta_{22} \\ z_{22} &= ZN + z_1\Gamma_{22} + z_2\Delta_{22} + z_3\Theta_{22} \\ v_{22} &= VN + v_1\Gamma_{22} + v_2\Delta_{22} + v_3\Theta_{22} \end{aligned}\right\},$$

$$\left.\begin{aligned} x_{23} &= XI + x_1\Gamma_{23} + x_2\Delta_{23} + x_3\Theta_{23} \\ y_{23} &= YI + y_1\Gamma_{23} + y_2\Delta_{23} + y_3\Theta_{23} \\ z_{23} &= ZI + z_1\Gamma_{23} + z_2\Delta_{23} + z_3\Theta_{23} \\ v_{23} &= VI + v_1\Gamma_{23} + v_2\Delta_{23} + v_3\Theta_{23} \end{aligned}\right\},$$

$$\left.\begin{aligned} x_{33} &= XJ + x_1\Gamma_{33} + x_2\Delta_{33} + x_3\Theta_{33} \\ y_{33} &= YJ + y_1\Gamma_{33} + y_2\Delta_{33} + y_3\Theta_{33} \\ z_{33} &= ZJ + z_1\Gamma_{33} + z_2\Delta_{33} + z_3\Theta_{33} \\ v_{33} &= VJ + v_1\Gamma_{33} + v_2\Delta_{33} + v_3\Theta_{33} \end{aligned}\right\}.$$

All of these can be represented by a single set

$$\left.\begin{aligned} x_{mn} &= X\eta_{mn} + x_1\Gamma_{mn} + x_2\Delta_{mn} + x_3\Theta_{mn} \\ y_{mn} &= Y\eta_{mn} + y_1\Gamma_{mn} + y_2\Delta_{mn} + y_3\Theta_{mn} \\ z_{mn} &= Z\eta_{mn} + z_1\Gamma_{mn} + z_2\Delta_{mn} + z_3\Theta_{mn} \\ v_{mn} &= V\eta_{mn} + v_1\Gamma_{mn} + v_2\Delta_{mn} + v_3\Theta_{mn} \end{aligned}\right\},$$

with the conventions

$$\eta_{11} = L, \quad \eta_{22} = N, \quad \eta_{33} = J, \quad \eta_{23} = I, \quad \eta_{31} = K, \quad \eta_{12} = M.$$

Values of the quantities $\Sigma x_{ij}x_{kl}$.

276. Many relations arise from the quadratic combinations of the preceding equations; among them, there occur the relations associated with the extension of the Gauss fundamental equation and the Codazzi-Mainardi conditions subsisting for a surface in homaloidal triple space.

By squaring each of the equations in the first of the foregoing sets, and adding, we have

$$\begin{aligned} \Sigma x_{11}^2 &= L^2 + 2L\{\Gamma_{11}\Sigma Xx_1 + \Delta_{11}\Sigma Xx_2 + \Theta_{11}\Sigma Xx_3\} \\ &\quad + \Gamma_{11}^2\Sigma x_1^2 + \Delta_{11}^2\Sigma x_2^2 + \Theta_{11}^2\Sigma x_3^2 \\ &\quad + 2\Delta_{11}\Theta_{11}\Sigma x_2x_3 + 2\Theta_{11}\Gamma_{11}\Sigma x_3x_1 + 2\Gamma_{11}\Delta_{11}\Sigma x_1x_2 \\ &= L^2 + (A, B, C, F, G, H\,\mathrm{X}\,\Gamma_{11}, \Delta_{11}, \Theta_{11})^2 = L^2 + (\ast\,\mathrm{X}\,\Gamma_{11}, \Delta_{11}, \Theta_{11})^2. \end{aligned}$$

Similarly, with this notation for the coefficients A, B, C, F, G, H,

$$\begin{aligned} \Sigma x_{11}x_{12} &= LM + (\ast\,\mathrm{X}\,\Gamma_{11}, \Delta_{11}, \Theta_{11}\mathrm{X}\,\Gamma_{12}, \Delta_{12}, \Theta_{12}), \\ \Sigma x_{11}x_{13} &= LK + (\ast\,\mathrm{X}\,\Gamma_{11}, \Delta_{11}, \Theta_{11}\mathrm{X}\,\Gamma_{13}, \Delta_{13}, \Theta_{13}), \\ \Sigma x_{11}x_{22} &= LN + (\ast\,\mathrm{X}\,\Gamma_{11}, \Delta_{11}, \Theta_{11}\mathrm{X}\,\Gamma_{22}, \Delta_{22}, \Theta_{22}), \\ \Sigma x_{11}x_{23} &= LI + (\ast\,\mathrm{X}\,\Gamma_{11}, \Delta_{11}, \Theta_{11}\mathrm{X}\,\Gamma_{23}, \Delta_{23}, \Theta_{23}), \\ \Sigma x_{11}x_{33} &= LJ + (\ast\,\mathrm{X}\,\Gamma_{11}, \Delta_{11}, \Theta_{11}\mathrm{X}\,\Gamma_{33}, \Delta_{33}, \Theta_{33}), \\ \Sigma x_{12}^2 &= M^2 + (\ast\,\mathrm{X}\,\Gamma_{12}, \Delta_{12}, \Theta_{12})^2, \\ \Sigma x_{12}x_{13} &= MK + (\ast\,\mathrm{X}\,\Gamma_{12}, \Delta_{12}, \Theta_{12}\mathrm{X}\,\Gamma_{13}, \Delta_{13}, \Theta_{13}), \\ \Sigma x_{12}x_{22} &= MN + (\ast\,\mathrm{X}\,\Gamma_{12}, \Delta_{12}, \Theta_{12}\mathrm{X}\,\Gamma_{22}, \Delta_{22}, \Theta_{22}), \\ \Sigma x_{12}x_{23} &= MI + (\ast\,\mathrm{X}\,\Gamma_{12}, \Delta_{12}, \Theta_{12}\mathrm{X}\,\Gamma_{23}, \Delta_{23}, \Theta_{23}), \\ \Sigma x_{12}x_{33} &= MJ + (\ast\,\mathrm{X}\,\Gamma_{12}, \Delta_{12}, \Theta_{12}\mathrm{X}\,\Gamma_{33}, \Delta_{33}, \Theta_{33}), \\ \Sigma x_{13}^2 &= K^2 + (\ast\,\mathrm{X}\,\Gamma_{13}, \Delta_{13}, \Theta_{13})^2, \\ \Sigma x_{13}x_{22} &= KN + (\ast\,\mathrm{X}\,\Gamma_{13}, \Delta_{13}, \Theta_{13}\mathrm{X}\,\Gamma_{22}, \Delta_{22}, \Theta_{22}), \\ \Sigma x_{13}x_{23} &= KI + (\ast\,\mathrm{X}\,\Gamma_{13}, \Delta_{13}, \Theta_{13}\mathrm{X}\,\Gamma_{23}, \Delta_{23}, \Theta_{23}), \\ \Sigma x_{13}x_{33} &= KJ + (\ast\,\mathrm{X}\,\Gamma_{13}, \Delta_{13}, \Theta_{13}\mathrm{X}\,\Gamma_{33}, \Delta_{33}, \Theta_{33}), \end{aligned}$$

$$\Sigma x_{22}^2 = N^2 + (\divideontimes \Gamma_{22}, \Delta_{22}, \Theta_{22})^2,$$
$$\Sigma x_{22} x_{23} = NI + (\divideontimes \Gamma_{22}, \Delta_{22}, \Theta_{22} \Gamma_{23}, \Delta_{23}, \Theta_{23}),$$
$$\Sigma x_{22} x_{33} = NJ + (\divideontimes \Gamma_{22}, \Delta_{22}, \Theta_{22} \Gamma_{33}, \Delta_{33}, \Theta_{33}),$$
$$\Sigma x_{23}^2 = I^2 + (\divideontimes \Gamma_{23}, \Delta_{23}, \Theta_{23})^2,$$
$$\Sigma x_{23} x_{33} = IJ + (\divideontimes \Gamma_{23}, \Delta_{23}, \Theta_{23} \Gamma_{33}, \Delta_{33}, \Theta_{33}),$$
$$\Sigma x_{33}^2 = J^2 + (\divideontimes \Gamma_{33}, \Delta_{33}, \Theta_{33})^2.$$

All these relations can be included in the single form

$$\Sigma x_{mn} x_{rs} = \eta_{mn} \eta_{rs} + (\divideontimes \Gamma_{mn}, \Delta_{mn}, \Theta_{mn} \Gamma_{rs}, \Delta_{rs}, \Theta_{rs}),$$

where $m, n, = 1, 2, 3$, in any combination; and $r, s, = 1, 2, 3$, in any combination, the two combinations being independent of one another; and where there are the conventions, as before,

$$\eta_{11} = L, \quad \eta_{22} = N, \quad \eta_{33} = J, \quad \eta_{23} = I, \quad \eta_{31} = K, \quad \eta_{12} = M.$$

Moreover, to abbreviate the notation, we write

$$(\divideontimes \Gamma_{mn}, \Delta_{mn}, \Theta_{mn} \Gamma_{rs}, \Delta_{rs}, \Theta_{rs}) = (\divideontimes mn\, rs),$$

for all the combinations of m, n, and of r, s; and thus the comprehensive form becomes

$$\Sigma x_{mn} x_{rs} = \eta_{mn} \eta_{rs} + (\divideontimes mn\, rs).$$

First derivatives of Γ, Δ, Θ.

277. For the later establishment of the extended Gauss relations and the extended Codazzi-Mainardi relations, as well as for the construction of magnitudes of the third order, the first derivatives of the magnitudes Γ, Δ, Θ, are required. By direct differentiation of the relation

$$\Gamma_{11} = \frac{a}{\Omega^2} \alpha_1 + \frac{h}{\Omega^2} \alpha_2 + \frac{g}{\Omega^2} \alpha_3,$$

we have

$$\Omega^2 \frac{\partial \Gamma_{11}}{\partial p} = a \frac{\partial \alpha_1}{\partial p} + h \frac{\partial \alpha_2}{\partial p} + g \frac{\partial \alpha_3}{\partial p}$$
$$+ \alpha_1 \left(a_1 - 2 \frac{a}{\Omega} \Omega_1 \right) + \alpha_2 \left(h_1 - 2 \frac{h}{\Omega} \Omega_1 \right) + \alpha_3 \left(g_1 - 2 \frac{g}{\Omega} \Omega_1 \right)$$
$$= a \frac{\partial \alpha_1}{\partial p} + h \frac{\partial \alpha_2}{\partial p} + g \frac{\partial \alpha_3}{\partial p}$$
$$- 2 (A \Gamma_{11} + H \Delta_{11} + G \Theta_{11}) (a \Gamma_{11} + h \Gamma_{12} + g \Gamma_{13})$$
$$- (H \Gamma_{11} + B \Delta_{11} + F \Theta_{11}) \{ (a \Delta_{11} + h \Delta_{12} + g \Delta_{13}) + (h \Gamma_{11} + b \Gamma_{12} + f \Gamma_{13}) \}$$
$$- (G \Gamma_{11} + F \Delta_{11} + C \Theta_{11}) \{ (a \Theta_{11} + h \Theta_{12} + g \Theta_{13}) + (g \Gamma_{11} + f \Gamma_{12} + c \Gamma_{13}) \}.$$

The last three lines, together, are equal to

$$
\begin{aligned}
&-a(*\,\Gamma_{11},\ \Delta_{11},\ \Theta_{11})^2-h(*\,\Gamma_{11},\ \Delta_{11},\ \Theta_{11}\Gamma_{12},\ \Delta_{12},\ \Theta_{12})\\
&\qquad\qquad -g(*\,\Gamma_{11},\ \Delta_{11},\ \Theta_{11}\Gamma_{13},\ \Delta_{13},\ \Theta_{13})\\
&-(a\Gamma_{11}+h\Gamma_{12}+g\Gamma_{13})(A\Gamma_{11}+H\Delta_{11}+G\Theta_{11})\\
&-(h\Gamma_{11}+b\Gamma_{12}+f\Gamma_{13})(H\Gamma_{11}+B\Delta_{11}+F\Theta_{11})\\
&-(g\Gamma_{11}+f\Gamma_{12}+c\Gamma_{13})(G\Gamma_{11}+F\Delta_{11}+C\Theta_{11})\\
=&-a(*\,11\,11)-h(*\,11\,12)-g(*\,11\,13)\\
&\qquad\qquad -\Omega^2(\Gamma_{11}\Gamma_{11}+\Delta_{11}\Gamma_{12}+\Theta_{11}\Gamma_{13}),
\end{aligned}
$$

in conformity with the abbreviated notation at the end of § 276. Thus

$$
\begin{aligned}
&\Omega^2\frac{\partial\Gamma_{11}}{\partial p}=\tfrac{1}{2}aA_{11}+h(H_{11}-\tfrac{1}{2}A_{12})+g(G_{11}-\tfrac{1}{2}A_{13})\\
&-a(*\,11\,11)-h(*\,11\,12)-g(*\,11\,13)-\Omega^2(\Gamma_{11}\Gamma_{11}+\Delta_{11}\Gamma_{12}+\Theta_{11}\Gamma_{13}),
\end{aligned}
$$

on substituting for the first derivatives of α_1, α_2, α_3, in terms of the second derivatives of the primary magnitudes.

Similarly for the other derivatives of Γ_{11} and for the derivatives of all the quantities Γ, Δ, Θ. The full tale of results is:

$$
\left.
\begin{aligned}
\Omega^2\frac{\partial\Gamma_{11}}{\partial p}&=\tfrac{1}{2}aA_{11}+h(H_{11}-\tfrac{1}{2}A_{12})+g(G_{11}-\tfrac{1}{2}A_{13})-\Omega^2(\Gamma_{11}\Gamma_{11}+\Delta_{11}\Gamma_{12}+\Theta_{11}\Gamma_{13})\\
&\quad -a(*\,11\,11)-h(*\,11\,12)-g(*\,11\,13)\\
\Omega^2\frac{\partial\Gamma_{11}}{\partial q}&=\tfrac{1}{2}aA_{12}+h(H_{12}-\tfrac{1}{2}A_{22})+g(G_{12}-\tfrac{1}{2}A_{23})-\Omega^2(\Gamma_{11}\Gamma_{12}+\Delta_{11}\Gamma_{22}+\Theta_{11}\Gamma_{23})\\
&\quad -a(*\,11\,12)-h(*\,11\,22)-g(*\,11\,23)\\
\Omega^2\frac{\partial\Gamma_{11}}{\partial r}&=\tfrac{1}{2}aA_{13}+h(H_{13}-\tfrac{1}{2}A_{23})+g(G_{13}-\tfrac{1}{2}A_{33})-\Omega^2(\Gamma_{11}\Gamma_{13}+\Delta_{11}\Gamma_{23}+\Theta_{11}\Gamma_{33})\\
&\quad -a(*\,11\,13)-h(*\,11\,23)-g(*\,11\,33)
\end{aligned}
\right\},
$$

$$
\left.
\begin{aligned}
\Omega^2\frac{\partial\Delta_{11}}{\partial p}&=\tfrac{1}{2}hA_{11}+b(H_{11}-\tfrac{1}{2}A_{12})+f(G_{11}-\tfrac{1}{2}A_{13})-\Omega^2(\Gamma_{11}\Delta_{11}+\Delta_{11}\Delta_{12}+\Theta_{11}\Delta_{13})\\
&\quad -h(*\,11\,11)-b(*\,11\,12)-f(*\,11\,13)\\
\Omega^2\frac{\partial\Delta_{11}}{\partial q}&=\tfrac{1}{2}hA_{12}+b(H_{12}-\tfrac{1}{2}A_{22})+f(G_{12}-\tfrac{1}{2}A_{23})-\Omega^2(\Gamma_{11}\Delta_{12}+\Delta_{11}\Delta_{22}+\Theta_{11}\Delta_{23})\\
&\quad -h(*\,11\,12)-b(*\,11\,22)-f(*\,11\,23)\\
\Omega^2\frac{\partial\Delta_{11}}{\partial r}&=\tfrac{1}{2}hA_{13}+b(H_{13}-\tfrac{1}{2}A_{23})+f(G_{13}-\tfrac{1}{2}A_{33})-\Omega^2(\Gamma_{11}\Delta_{13}+\Delta_{11}\Delta_{23}+\Theta_{11}\Delta_{33})\\
&\quad -h(*\,11\,13)-b(*\,11\,23)-f(*\,11\,33)
\end{aligned}
\right\},
$$

$$
\left.
\begin{aligned}
\Omega^2\frac{\partial\Theta_{11}}{\partial p}&=\tfrac{1}{2}gA_{11}+f(H_{11}-\tfrac{1}{2}A_{12})+c(G_{11}-\tfrac{1}{2}A_{13})-\Omega^2(\Gamma_{11}\Theta_{11}+\Delta_{11}\Theta_{12}+\Theta_{11}\Theta_{13})\\
&\quad -g(*\,11\,11)-f(*\,11\,12)-c(*\,11\,13)\\
\Omega^2\frac{\partial\Theta_{11}}{\partial q}&=\tfrac{1}{2}gA_{12}+f(H_{12}-\tfrac{1}{2}A_{22})+c(G_{12}-\tfrac{1}{2}A_{23})-\Omega^2(\Gamma_{11}\Theta_{12}+\Delta_{11}\Theta_{22}+\Theta_{11}\Theta_{23})\\
&\quad -g(*\,11\,12)-f(*\,11\,22)-c(*\,11\,23)\\
\Omega^2\frac{\partial\Theta_{11}}{\partial r}&=\tfrac{1}{2}gA_{13}+f(H_{13}-\tfrac{1}{2}A_{23})+c(G_{13}-\tfrac{1}{2}A_{33})-\Omega^2(\Gamma_{11}\Theta_{13}+\Delta_{11}\Theta_{23}+\Theta_{11}\Theta_{33})\\
&\quad -g(*\,11\,13)-f(*\,11\,23)-c(*\,11\,33)
\end{aligned}
\right\},
$$

$$\left.\begin{aligned}
\Omega^2\frac{\partial\Gamma_{12}}{\partial p} &= \tfrac{1}{2}aA_{12}+\tfrac{1}{2}hB_{11}+\tfrac{1}{2}g\,(F_{11}+G_{12}-H_{13})-\Omega^2\,(\Gamma_{12}\,\Gamma_{11}+\Delta_{12}\,\Gamma_{12}+\Theta_{12}\,\Gamma_{13})\\
&\quad -a\,(\divideontimes\,12\,|\,11)-h\,(\divideontimes\,12\,|\,12)-g\,(\divideontimes\,12\,|\,13)\\
\Omega^2\frac{\partial\Gamma_{12}}{\partial q} &= \tfrac{1}{2}aA_{22}+\tfrac{1}{2}hB_{12}+\tfrac{1}{2}g\,(F_{12}+G_{22}-H_{23})-\Omega^2\,(\Gamma_{12}\,\Gamma_{12}+\Delta_{12}\,\Gamma_{22}+\Theta_{12}\,\Gamma_{23})\\
&\quad -a\,(\divideontimes\,12\,|\,12)-h\,(\divideontimes\,12\,|\,22)-g\,(\divideontimes\,12\,|\,23)\\
\Omega^2\frac{\partial\Gamma_{12}}{\partial r} &= \tfrac{1}{2}aA_{23}+\tfrac{1}{2}hB_{13}+\tfrac{1}{2}g\,(F_{13}+G_{23}-H_{33})-\Omega^2\,(\Gamma_{12}\,\Gamma_{13}+\Delta_{12}\,\Gamma_{23}+\Theta_{12}\,\Gamma_{33})\\
&\quad -a\,(\divideontimes\,12\,|\,13)-h\,(\divideontimes\,12\,|\,23)-g\,(\divideontimes\,12\,|\,33)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\Omega^2\frac{\partial\Delta_{12}}{\partial p} &= \tfrac{1}{2}hA_{12}+\tfrac{1}{2}bB_{11}+\tfrac{1}{2}f\,(F_{11}+G_{12}-H_{13})-\Omega^2\,(\Gamma_{12}\,\Delta_{11}+\Delta_{12}\,\Delta_{12}+\Theta_{12}\,\Delta_{13})\\
&\quad -h\,(\divideontimes\,12\,|\,11)-b\,(\divideontimes\,12\,|\,12)-f\,(\divideontimes\,12\,|\,13)\\
\Omega^2\frac{\partial\Delta_{12}}{\partial q} &= \tfrac{1}{2}hA_{22}+\tfrac{1}{2}bB_{12}+\tfrac{1}{2}f\,(F_{12}+G_{22}-H_{23})-\Omega^2\,(\Gamma_{12}\,\Delta_{12}+\Delta_{12}\,\Delta_{22}+\Theta_{12}\,\Delta_{23})\\
&\quad -h\,(\divideontimes\,12\,|\,12)-b\,(\divideontimes\,12\,|\,22)-f\,(\divideontimes\,12\,|\,23)\\
\Omega^2\frac{\partial\Delta_{12}}{\partial r} &= \tfrac{1}{2}hA_{23}+\tfrac{1}{2}bB_{13}+\tfrac{1}{2}f\,(F_{13}+G_{23}-H_{33})-\Omega^2\,(\Gamma_{12}\,\Delta_{13}+\Delta_{12}\,\Delta_{23}+\Theta_{12}\,\Delta_{33})\\
&\quad -h\,(\divideontimes\,12\,|\,13)-b\,(\divideontimes\,12\,|\,23)-f\,(\divideontimes\,12\,|\,33)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\Omega^2\frac{\partial\Theta_{12}}{\partial p} &= \tfrac{1}{2}gA_{12}+\tfrac{1}{2}fB_{11}+\tfrac{1}{2}c\,(F_{11}+G_{12}-H_{13})-\Omega^2\,(\Gamma_{12}\,\Theta_{11}+\Delta_{12}\,\Theta_{12}+\Theta_{12}\,\Theta_{13})\\
&\quad -g\,(\divideontimes\,12\,|\,11)-f\,(\divideontimes\,12\,|\,12)-c\,(\divideontimes\,12\,|\,13)\\
\Omega^2\frac{\partial\Theta_{12}}{\partial q} &= \tfrac{1}{2}gA_{22}+\tfrac{1}{2}fB_{12}+\tfrac{1}{2}c\,(F_{12}+G_{22}-H_{23})-\Omega^2\,(\Gamma_{12}\,\Theta_{12}+\Delta_{12}\,\Theta_{22}+\Theta_{12}\,\Theta_{23})\\
&\quad -g\,(\divideontimes\,12\,|\,12)-f\,(\divideontimes\,12\,|\,22)-c\,(\divideontimes\,12\,|\,23)\\
\Omega^2\frac{\partial\Theta_{12}}{\partial r} &= \tfrac{1}{2}gA_{23}+\tfrac{1}{2}fB_{13}+\tfrac{1}{2}c\,(F_{13}+G_{23}-H_{33})-\Omega^2\,(\Gamma_{12}\,\Theta_{13}+\Delta_{12}\,\Theta_{23}+\Theta_{12}\,\Theta_{33})\\
&\quad -g\,(\divideontimes\,12\,|\,13)-f\,(\divideontimes\,12\,|\,23)-c\,(\divideontimes\,12\,|\,33)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\Omega^2\frac{\partial\Gamma_{13}}{\partial p} &= \tfrac{1}{2}aA_{13}+\tfrac{1}{2}h\,(F_{11}-G_{12}+H_{13})+\tfrac{1}{2}gC_{11}-\Omega^2\,(\Gamma_{13}\,\Gamma_{11}+\Delta_{13}\,\Gamma_{12}+\Theta_{13}\,\Gamma_{13})\\
&\quad -a\,(\divideontimes\,13\,|\,11)-h\,(\divideontimes\,13\,|\,12)-g\,(\divideontimes\,13\,|\,13)\\
\Omega^2\frac{\partial\Gamma_{13}}{\partial q} &= \tfrac{1}{2}aA_{23}+\tfrac{1}{2}h\,(F_{12}-G_{22}+H_{23})+\tfrac{1}{2}gC_{12}-\Omega^2\,(\Gamma_{13}\,\Gamma_{12}+\Delta_{13}\,\Gamma_{22}+\Theta_{13}\,\Gamma_{23})\\
&\quad -a\,(\divideontimes\,13\,|\,12)-h\,(\divideontimes\,13\,|\,22)-g\,(\divideontimes\,13\,|\,23)\\
\Omega^2\frac{\partial\Gamma_{13}}{\partial r} &= \tfrac{1}{2}aA_{33}+\tfrac{1}{2}h\,(F_{13}-G_{23}+H_{33})+\tfrac{1}{2}gC_{13}-\Omega^2\,(\Gamma_{13}\,\Gamma_{13}+\Delta_{13}\,\Gamma_{23}+\Theta_{13}\,\Gamma_{33})\\
&\quad -a\,(\divideontimes\,13\,|\,13)-h\,(\divideontimes\,13\,|\,23)-g\,(\divideontimes\,13\,|\,33)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\Omega^2\frac{\partial\Delta_{13}}{\partial p} &= \tfrac{1}{2}hA_{13}+\tfrac{1}{2}b\,(F_{11}-G_{12}+H_{13})+\tfrac{1}{2}fC_{11}-\Omega^2\,(\Gamma_{13}\,\Delta_{11}+\Delta_{13}\,\Delta_{12}+\Theta_{13}\,\Delta_{13})\\
&\quad -h\,(\divideontimes\,13\,|\,11)-b\,(\divideontimes\,13\,|\,12)-f\,(\divideontimes\,13\,|\,13)\\
\Omega^2\frac{\partial\Delta_{13}}{\partial q} &= \tfrac{1}{2}hA_{23}+\tfrac{1}{2}b\,(F_{12}-G_{22}+H_{23})+\tfrac{1}{2}fC_{12}-\Omega^2\,(\Gamma_{13}\,\Delta_{12}+\Delta_{13}\,\Delta_{22}+\Theta_{13}\,\Delta_{23})\\
&\quad -h\,(\divideontimes\,13\,|\,12)-b\,(\divideontimes\,13\,|\,22)-f\,(\divideontimes\,13\,|\,23)\\
\Omega^2\frac{\partial\Delta_{13}}{\partial r} &= \tfrac{1}{2}hA_{33}+\tfrac{1}{2}b\,(F_{13}-G_{23}+H_{33})+\tfrac{1}{2}fC_{13}-\Omega^2\,(\Gamma_{13}\,\Delta_{13}+\Delta_{13}\,\Delta_{23}+\Theta_{13}\,\Delta_{33})\\
&\quad -h\,(\divideontimes\,13\,|\,13)-b\,(\divideontimes\,13\,|\,23)-f\,(\divideontimes\,13\,|\,33)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\Omega^2\frac{\partial\Theta_{13}}{\partial p} &= \tfrac{1}{2}gA_{13}+\tfrac{1}{2}f(F_{11}-G_{12}+H_{13})+\tfrac{1}{2}cC_{11}-\Omega^2(\Gamma_{13}\Theta_{11}+\Delta_{13}\Theta_{12}+\Theta_{13}\Theta_{13})\\
&\quad -g(*\,13\,11)-f(*\,13\,12)-c(*\,13\,13)\\
\Omega^2\frac{\partial\Theta_{13}}{\partial q} &= \tfrac{1}{2}gA_{23}+\tfrac{1}{2}f(F_{12}-G_{22}+H_{23})+\tfrac{1}{2}cC_{12}-\Omega^2(\Gamma_{13}\Theta_{12}+\Delta_{13}\Theta_{22}+\Theta_{13}\Theta_{23})\\
&\quad -g(*\,13\,12)-f(*\,13\,22)-c(*\,13\,23)\\
\Omega^2\frac{\partial\Theta_{13}}{\partial r} &= \tfrac{1}{2}gA_{33}+\tfrac{1}{2}f(F_{13}-G_{23}+H_{33})+\tfrac{1}{2}cC_{13}-\Omega^2(\Gamma_{13}\Theta_{13}+\Delta_{13}\Theta_{23}+\Theta_{13}\Theta_{33})\\
&\quad -g(*\,13\,13)-f(*\,13\,23)-c(*\,13\,33)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\Omega^2\frac{\partial\Gamma_{22}}{\partial p} &= a(H_{12}-\tfrac{1}{2}B_{11})+\tfrac{1}{2}hB_{12}+g(F_{12}-\tfrac{1}{2}B_{13})-\Omega^2(\Gamma_{22}\Gamma_{11}+\Delta_{22}\Gamma_{12}+\Theta_{22}\Gamma_{13})\\
&\quad -a(*\,22\,11)-h(*\,22\,12)-g(*\,22\,13)\\
\Omega^2\frac{\partial\Gamma_{22}}{\partial q} &= a(H_{22}-\tfrac{1}{2}B_{12})+\tfrac{1}{2}hB_{22}+g(F_{22}-\tfrac{1}{2}B_{23})-\Omega^2(\Gamma_{22}\Gamma_{12}+\Delta_{22}\Gamma_{22}+\Theta_{22}\Gamma_{23})\\
&\quad -a(*\,22\,12)-h(*\,22\,22)-g(*\,22\,23)\\
\Omega^2\frac{\partial\Gamma_{22}}{\partial r} &= a(H_{23}-\tfrac{1}{2}B_{13})+\tfrac{1}{2}hB_{23}+g(F_{23}-\tfrac{1}{2}B_{33})-\Omega^2(\Gamma_{22}\Gamma_{13}+\Delta_{22}\Gamma_{23}+\Theta_{22}\Gamma_{33})\\
&\quad -a(*\,22\,13)-h(*\,22\,23)-g(*\,22\,33)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\Omega^2\frac{\partial\Delta_{22}}{\partial p} &= h(H_{12}-\tfrac{1}{2}B_{11})+\tfrac{1}{2}bB_{12}+f(F_{12}-\tfrac{1}{2}B_{13})-\Omega^2(\Gamma_{22}\Delta_{11}+\Delta_{22}\Delta_{12}+\Theta_{22}\Delta_{13})\\
&\quad -h(*\,22\,11)-b(*\,22\,12)-f(*\,22\,13)\\
\Omega^2\frac{\partial\Delta_{22}}{\partial q} &= h(H_{22}-\tfrac{1}{2}B_{12})+\tfrac{1}{2}bB_{22}+f(F_{22}-\tfrac{1}{2}B_{23})-\Omega^2(\Gamma_{22}\Delta_{12}+\Delta_{22}\Delta_{22}+\Theta_{22}\Delta_{23})\\
&\quad -h(*\,22\,12)-b(*\,22\,22)-f(*\,22\,23)\\
\Omega^2\frac{\partial\Delta_{22}}{\partial r} &= h(H_{23}-\tfrac{1}{2}B_{13})+\tfrac{1}{2}bB_{23}+f(F_{23}-\tfrac{1}{2}B_{33})-\Omega^2(\Gamma_{22}\Delta_{13}+\Delta_{22}\Delta_{23}+\Theta_{22}\Delta_{33})\\
&\quad -h(*\,22\,13)-b(*\,22\,23)-f(*\,22\,33)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\Omega^2\frac{\partial\Theta_{22}}{\partial p} &= g(H_{12}-\tfrac{1}{2}B_{11})+\tfrac{1}{2}fB_{12}+c(F_{12}-\tfrac{1}{2}B_{13})-\Omega^2(\Gamma_{22}\Theta_{11}+\Delta_{22}\Theta_{12}+\Theta_{22}\Theta_{13})\\
&\quad -g(*\,22\,11)-f(*\,22\,12)-c(*\,22\,13)\\
\Omega^2\frac{\partial\Theta_{22}}{\partial q} &= g(H_{22}-\tfrac{1}{2}B_{12})+\tfrac{1}{2}fB_{22}+c(F_{22}-\tfrac{1}{2}B_{23})-\Omega^2(\Gamma_{22}\Theta_{12}+\Delta_{22}\Theta_{22}+\Theta_{22}\Theta_{23})\\
&\quad -g(*\,22\,12)-f(*\,22\,22)-c(*\,22\,23)\\
\Omega^2\frac{\partial\Theta_{22}}{\partial r} &= g(H_{23}-\tfrac{1}{2}B_{13})+\tfrac{1}{2}fB_{23}+c(F_{23}-\tfrac{1}{2}B_{33})-\Omega^2(\Gamma_{22}\Theta_{13}+\Delta_{22}\Theta_{23}+\Theta_{22}\Theta_{33})\\
&\quad -g(*\,22\,13)-f(*\,22\,23)-c(*\,22\,33)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\Omega^2\frac{\partial\Gamma_{23}}{\partial p} &= \tfrac{1}{2}a(-F_{11}+G_{12}+H_{13})+\tfrac{1}{2}hB_{13}+\tfrac{1}{2}gC_{12}-\Omega^2(\Gamma_{23}\Gamma_{11}+\Delta_{23}\Gamma_{12}+\Theta_{23}\Gamma_{13})\\
&\quad -a(*\,23\,11)-h(*\,23\,12)-g(*\,23\,13)\\
\Omega^2\frac{\partial\Gamma_{23}}{\partial q} &= \tfrac{1}{2}a(-F_{12}+G_{22}+H_{23})+\tfrac{1}{2}hB_{23}+\tfrac{1}{2}gC_{22}-\Omega^2(\Gamma_{23}\Gamma_{12}+\Delta_{23}\Gamma_{22}+\Theta_{23}\Gamma_{23})\\
&\quad -a(*\,23\,12)-h(*\,23\,22)-g(*\,23\,23)\\
\Omega^2\frac{\partial\Gamma_{23}}{\partial r} &= \tfrac{1}{2}a(-F_{13}+G_{23}+H_{33})+\tfrac{1}{2}hB_{33}+\tfrac{1}{2}gC_{23}-\Omega^2(\Gamma_{23}\Gamma_{13}+\Delta_{23}\Gamma_{23}+\Theta_{23}\Gamma_{33})\\
&\quad -a(*\,23\,13)-h(*\,23\,23)-g(*\,23\,33)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\Omega^2\frac{\partial\Delta_{23}}{\partial p} &= \tfrac{1}{2}h(-F_{11}+G_{12}+H_{13})+\tfrac{1}{2}bB_{13}+\tfrac{1}{2}fC_{12}-\Omega^2(\Gamma_{23}\Delta_{11}+\Delta_{23}\Delta_{12}+\Theta_{23}\Delta_{13})\\
&\quad -h(\divideontimes\,23\,11)-b(\divideontimes\,23\,12)-f(\divideontimes\,23\,13)\\
\Omega^2\frac{\partial\Delta_{23}}{\partial q} &= \tfrac{1}{2}h(-F_{12}+G_{22}+H_{23})+\tfrac{1}{2}bB_{23}+\tfrac{1}{2}fC_{22}-\Omega^2(\Gamma_{23}\Delta_{12}+\Delta_{23}\Delta_{22}+\Theta_{23}\Delta_{23})\\
&\quad -h(\divideontimes\,23\,12)-b(\divideontimes\,23\,22)-f(\divideontimes\,23\,23)\\
\Omega^2\frac{\partial\Delta_{23}}{\partial r} &= \tfrac{1}{2}h(-F_{13}+G_{23}+H_{33})+\tfrac{1}{2}bB_{33}+\tfrac{1}{2}fC_{23}-\Omega^2(\Gamma_{23}\Delta_{13}+\Delta_{23}\Delta_{23}+\Theta_{23}\Delta_{33})\\
&\quad -h(\divideontimes\,23\,13)-b(\divideontimes\,23\,23)-f(\divideontimes\,23\,33)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\Omega^2\frac{\partial\Theta_{23}}{\partial p} &= \tfrac{1}{2}g(-F_{11}+G_{12}+H_{13})+\tfrac{1}{2}fB_{13}+\tfrac{1}{2}cC_{12}-\Omega^2(\Gamma_{23}\Theta_{11}+\Delta_{23}\Theta_{12}+\Theta_{23}\Theta_{13})\\
&\quad -g(\divideontimes\,23\,11)-f(\divideontimes\,23\,12)-c(\divideontimes\,23\,13)\\
\Omega^2\frac{\partial\Theta_{23}}{\partial q} &= \tfrac{1}{2}g(-F_{12}+G_{22}+H_{23})+\tfrac{1}{2}fB_{23}+\tfrac{1}{2}cC_{22}-\Omega^2(\Gamma_{23}\Theta_{12}+\Delta_{23}\Theta_{22}+\Theta_{23}\Theta_{23})\\
&\quad -g(\divideontimes\,23\,12)-f(\divideontimes\,23\,22)-c(\divideontimes\,23\,23)\\
\Omega^2\frac{\partial\Theta_{23}}{\partial r} &= \tfrac{1}{2}g(-F_{13}+G_{23}+H_{33})+\tfrac{1}{2}fB_{33}+\tfrac{1}{2}cC_{23}-\Omega^2(\Gamma_{23}\Theta_{13}+\Delta_{23}\Theta_{23}+\Theta_{23}\Theta_{33})\\
&\quad -g(\divideontimes\,23\,13)-f(\divideontimes\,23\,23)-c(\divideontimes\,23\,33)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\Omega^2\frac{\partial\Gamma_{33}}{\partial p} &= a(G_{13}-\tfrac{1}{2}C_{11})+h(F_{13}-\tfrac{1}{2}C_{12})+\tfrac{1}{2}gC_{13}-\Omega^2(\Gamma_{33}\Gamma_{11}+\Delta_{33}\Gamma_{12}+\Theta_{33}\Gamma_{13})\\
&\quad -a(\divideontimes\,33\,11)-h(\divideontimes\,33\,12)-g(\divideontimes\,33\,13)\\
\Omega^2\frac{\partial\Gamma_{33}}{\partial q} &= a(G_{23}-\tfrac{1}{2}C_{12})+h(F_{23}-\tfrac{1}{2}C_{22})+\tfrac{1}{2}gC_{23}-\Omega^2(\Gamma_{33}\Gamma_{12}+\Delta_{33}\Gamma_{22}+\Theta_{33}\Gamma_{23})\\
&\quad -a(\divideontimes\,33\,12)-h(\divideontimes\,33\,22)-g(\divideontimes\,33\,23)\\
\Omega^2\frac{\partial\Gamma_{33}}{\partial r} &= a(G_{33}-\tfrac{1}{2}C_{13})+h(F_{33}-\tfrac{1}{2}C_{23})+\tfrac{1}{2}gC_{33}-\Omega^2(\Gamma_{33}\Gamma_{13}+\Delta_{33}\Gamma_{23}+\Theta_{33}\Gamma_{33})\\
&\quad -a(\divideontimes\,33\,13)-h(\divideontimes\,33\,23)-g(\divideontimes\,33\,33)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\Omega^2\frac{\partial\Delta_{33}}{\partial p} &= h(G_{13}-\tfrac{1}{2}C_{11})+b(F_{13}-\tfrac{1}{2}C_{12})+\tfrac{1}{2}fC_{13}-\Omega^2(\Gamma_{33}\Delta_{11}+\Delta_{33}\Delta_{12}+\Theta_{33}\Delta_{13})\\
&\quad -h(\divideontimes\,33\,11)-b(\divideontimes\,33\,12)-g(\divideontimes\,33\,13)\\
\Omega^2\frac{\partial\Delta_{33}}{\partial q} &= h(G_{23}-\tfrac{1}{2}C_{12})+b(F_{23}-\tfrac{1}{2}C_{22})+\tfrac{1}{2}fC_{23}-\Omega^2(\Gamma_{33}\Delta_{12}+\Delta_{33}\Delta_{22}+\Theta_{33}\Delta_{23})\\
&\quad -h(\divideontimes\,33\,12)-b(\divideontimes\,33\,22)-g(\divideontimes\,33\,23)\\
\Omega^2\frac{\partial\Delta_{33}}{\partial r} &= h(G_{33}-\tfrac{1}{2}C_{13})+b(F_{33}-\tfrac{1}{2}C_{23})+\tfrac{1}{2}fC_{33}-\Omega^2(\Gamma_{33}\Delta_{13}+\Delta_{33}\Delta_{23}+\Theta_{33}\Delta_{33})\\
&\quad -h(\divideontimes\,33\,13)-b(\divideontimes\,33\,23)-g(\divideontimes\,33\,33)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\Omega^2\frac{\partial\Theta_{33}}{\partial p} &= g(G_{13}-\tfrac{1}{2}C_{11})+f(F_{13}-\tfrac{1}{2}C_{12})+\tfrac{1}{2}cC_{13}-\Omega^2(\Gamma_{33}\Theta_{11}+\Delta_{33}\Theta_{12}+\Theta_{33}\Theta_{13})\\
&\quad -g(\divideontimes\,33\,11)-f(\divideontimes\,33\,12)-c(\divideontimes\,33\,13)\\
\Omega^2\frac{\partial\Theta_{33}}{\partial q} &= g(G_{23}-\tfrac{1}{2}C_{12})+f(F_{23}-\tfrac{1}{2}C_{22})+\tfrac{1}{2}cC_{23}-\Omega^2(\Gamma_{33}\Theta_{12}+\Delta_{33}\Theta_{22}+\Theta_{33}\Theta_{23})\\
&\quad -g(\divideontimes\,33\,12)-f(\divideontimes\,33\,22)-c(\divideontimes\,33\,23)\\
\Omega^2\frac{\partial\Theta_{33}}{\partial r} &= g(G_{33}-\tfrac{1}{2}C_{13})+f(F_{33}-\tfrac{1}{2}C_{23})+\tfrac{1}{2}cC_{33}-\Omega^2(\Gamma_{33}\Theta_{13}+\Delta_{33}\Theta_{23}+\Theta_{33}\Theta_{33})\\
&\quad -g(\divideontimes\,33\,13)-f(\divideontimes\,33\,23)-c(\divideontimes\,33\,33)
\end{aligned}\right\}.$$

CHAPTER XVI.

LINEAR CURVATURE OF A REGION.

Extended Gauss relations.

278. By differentiating the relations (§ 273)

$$\Sigma x_1 x_{12} = \tfrac{1}{2} A_2, \quad \Sigma x_1 x_{22} = H_2 - \tfrac{1}{2} B_1,$$

with regard to q and p respectively, we have

$$\Sigma x_{12}^2 + \Sigma x_1 x_{122} = \tfrac{1}{2} A_{22}, \quad \Sigma x_{11} x_{22} + \Sigma x_1 x_{122} = H_{12} - \tfrac{1}{2} B_{11};$$

and therefore

$$\Sigma x_{12}^2 - \Sigma x_{11} x_{22} = \tfrac{1}{2} (A_{22} - 2H_{12} + B_{11}).$$

There are other five (and not more than five) such relations, which express particular combinations of second derivatives of the primary magnitudes in terms solely of second derivatives of the variables: the five relations are

$$\begin{aligned}
\Sigma x_{23}^2 - \Sigma x_{22} x_{33} &= \tfrac{1}{2} (B_{33} - 2F_{23} + C_{22}), \\
\Sigma x_{31}^2 - \Sigma x_{33} x_{11} &= \tfrac{1}{2} (C_{11} - 2G_{31} + A_{33}), \\
\Sigma x_{12} x_{13} - \Sigma x_{11} x_{23} &= \tfrac{1}{2} (A_{23} - G_{12} - H_{13} + F_{11}), \\
\Sigma x_{23} x_{22} - \Sigma x_{22} x_{31} &= \tfrac{1}{2} (B_{31} - H_{23} - F_{12} + G_{22}), \\
\Sigma x_{31} x_{32} - \Sigma x_{33} x_{12} &= \tfrac{1}{2} (C_{12} - F_{13} - G_{23} + H_{33}).
\end{aligned}$$

All these relations can be exchanged for equivalent relations which involve the secondary magnitudes of the region instead of the derivatives of x, y, z, v. By the results of § 276, we have

$$\Sigma x_{12}^2 = M^2 + (\divideontimes \chi 12 \chi 12), \quad \Sigma x_{11} x_{22} = LN + (\divideontimes \chi 11 \chi 22);$$

and therefore

$$LN - M^2 = -\tfrac{1}{2} (A_{22} - 2H_{12} + B_{11}) - (\divideontimes \chi 11 \chi 22) + (\divideontimes \chi 12 \chi 12).$$

The other five similarly changed relations are

$$\begin{aligned}
NJ - I^2 &= -\tfrac{1}{2} (B_{33} - 2F_{23} + C_{22}) - (\divideontimes \chi 22 \chi 33) + (\divideontimes \chi 23 \chi 23), \\
JL - K^2 &= -\tfrac{1}{2} (C_{11} - 2G_{31} + A_{33}) - (\divideontimes \chi 33 \chi 11) + (\divideontimes \chi 31 \chi 31), \\
MK - LI &= \tfrac{1}{2} (A_{23} - G_{12} - H_{13} + F_{11}) - (\divideontimes \chi 12 \chi 13) + (\divideontimes \chi 11 \chi 23), \\
IM - NK &= \tfrac{1}{2} (B_{31} - H_{23} - F_{12} + G_{22}) - (\divideontimes \chi 23 \chi 12) + (\divideontimes \chi 22 \chi 31), \\
KI - JM &= \tfrac{1}{2} (C_{12} - F_{13} - G_{23} + H_{33}) - (\divideontimes \chi 13 \chi 23) + (\divideontimes \chi 33 \chi 12).
\end{aligned}$$

These are the extensions, to regions in homaloidal quadruple space, of the single fundamental Gauss relation for surfaces in homaloidal triple space. Accordingly, they will be called the *extended Gauss relations* for such regions. The six quantities on the left-hand sides of the relations are denoted by j, l, n, i, k, m, respectively (§ 286).

Extended Codazzi-Mainardi relations.

279. Again, we have the equations

$$x_{11} = XL + x_1 \Gamma_{11} + x_2 \Delta_{11} + x_3 \Theta_{11}, \quad x_{12} = XM + x_1 \Gamma_{12} + x_2 \Delta_{12} + x_3 \Theta_{12};$$

when these values of x_{11} and x_{12} are inserted in the condition

$$\frac{\partial}{\partial q} (x_{11}) = \frac{\partial}{\partial p} (x_{12}),$$

and terms are collected, we find

$$\begin{aligned}
X(L_2-M_1)+x_1\left(\frac{\partial\Gamma_{11}}{\partial q}-\frac{\partial\Gamma_{12}}{\partial p}\right)+x_2\left(\frac{\partial\Delta_{11}}{\partial q}-\frac{\partial\Delta_{12}}{\partial p}\right)+x_3\left(\frac{\partial\Theta_{11}}{\partial q}-\frac{\partial\Theta_{12}}{\partial p}\right)\\
=MX_1-LX_2+\Gamma_{12}x_{11}+\Delta_{12}x_{12}+\Theta_{12}x_{13}-\Gamma_{11}x_{12}-\Delta_{11}x_{22}-\Theta_{11}x_{23}\\
=MX_1-LX_2\\
+X\{\Gamma_{12}L+\Delta_{12}M+\Theta_{12}K-(\Gamma_{11}M+\Delta_{11}N+\Theta_{11}I)\}\\
+x_1\{\Gamma_{12}\Gamma_{11}+\Delta_{12}\Gamma_{12}+\Theta_{12}\Gamma_{13}-(\Gamma_{11}\Gamma_{12}+\Delta_{11}\Gamma_{22}+\Theta_{11}\Gamma_{23})\}\\
+x_2\{\Gamma_{12}\Delta_{11}+\Delta_{12}\Delta_{12}+\Theta_{12}\Delta_{13}-(\Gamma_{11}\Delta_{12}+\Delta_{11}\Delta_{22}+\Theta_{11}\Delta_{23})\}\\
+x_3\{\Gamma_{12}\Theta_{11}+\Delta_{12}\Theta_{12}+\Theta_{12}\Theta_{13}-(\Gamma_{11}\Theta_{12}+\Delta_{11}\Theta_{22}+\Theta_{11}\Theta_{23})\}.
\end{aligned}$$

When we proceed similarly from the expressions for y_{11} and y_{12}, those for z_{11} and z_{12}, and those for v_{11} and v_{12}, we find equations in Y, Y_1, Y_2, y_1, y_2, y_3; Z, Z_1, Z_2, z_1, z_2, z_3; V, V_1, V_2, v_1, v_2, v_3; respectively, exactly similar to the foregoing equation and having the same coefficients.

(i) In the first place, multiply the four equations by X, Y, Z, V, and add: then

$$L_2-M_1=\Gamma_{12}L+\Delta_{12}M+\Theta_{12}K-(\Gamma_{11}M+\Delta_{11}N+\Theta_{11}I).$$

Thus the terms involving X alone on the two sides of the foregoing equation cancel: and similarly for the terms involving Y alone, Z alone, V alone, in the other equations—always because of the same condition. This condition is a new relation.

Next, we have (§ 274)

$$\begin{aligned}
MX_1-LX_2&=-\frac{M}{\Omega^2}(x_1\lambda_1+x_2\mu_1+x_3\nu_1)+\frac{L}{\Omega^2}(x_1\lambda_2+x_2\mu_2+x_3\nu_2)\\
&=\frac{x_1}{\Omega^2}\{h(LN-M^2)+g(LI-MK)\}\\
&\quad+\frac{x_2}{\Omega^2}\{b(LN-M^2)+f(LI-MK)\}\\
&\quad+\frac{x_3}{\Omega^2}\{f(LN-M^2)+c(LI-MK)\}.
\end{aligned}$$

But, by the results in § 277,

$$\begin{aligned}
\Omega^2\left\{\frac{\partial\Gamma_{11}}{\partial q}-\frac{\partial\Gamma_{12}}{\partial p}-(\Gamma_{12}\Gamma_{11}+\Delta_{12}\Gamma_{12}+\Theta_{12}\Gamma_{13})+(\Gamma_{11}\Gamma_{12}+\Delta_{11}\Gamma_{22}+\Theta_{11}\Gamma_{23})\right\}\\
=h(-\tfrac{1}{2}B_{11}+H_{12}-\tfrac{1}{2}A_{22})+g(-\tfrac{1}{2}A_{23}+\tfrac{1}{2}H_{13}+\tfrac{1}{2}G_{12}-\tfrac{1}{2}F_{11})\\
+h\{(*\,12\,12)-(*\,11\,22)\}+g\{(*\,12\,13)-(*\,11\,23)\}\\
=h(LN-M^2)+g(LI-MK),
\end{aligned}$$

by the result in § 278; hence the terms in x_1 in the two sides cancel. Similarly for the terms in x_2, and for the terms in x_3. Thus the required condition is satisfied for the first of the equations, because of the relations between the coefficients. On the same ground, the other three similar conditions are satisfied. It therefore appears that one, and only one, additional new relation is obtained. It may be written in the form

$$L_2+\Gamma_{11}M+\Delta_{11}N+\Theta_{11}I=M_1+\Gamma_{12}L+\Delta_{12}M+\Theta_{12}K;$$

and it follows from the identically satisfied equations of the type

$$\frac{\partial \theta_{11}}{\partial q} = \frac{\partial \theta_{12}}{\partial p},$$

for $\theta = x, y, z, v$.

Similarly, one relation between L_3 and K_1—it is the only such relation—survives from the condition

$$\frac{\partial \theta_{11}}{\partial r} = \frac{\partial \theta_{13}}{\partial p},$$

for $\theta = x, y, z, v$; one relation, between N_1 and M_2, survives from the condition

$$\frac{\partial \theta_{22}}{\partial p} = \frac{\partial \theta_{12}}{\partial q};$$

one relation, between N_3 and I_2, survives from the condition

$$\frac{\partial \theta_{22}}{\partial r} = \frac{\partial \theta_{23}}{\partial q};$$

one relation, between J_1 and K_3, survives from the condition

$$\frac{\partial \theta_{33}}{\partial p} = \frac{\partial \theta_{13}}{\partial r};$$

one relation, between J_2 and I_3, survives from the condition

$$\frac{\partial \theta_{33}}{\partial q} = \frac{\partial \theta_{23}}{\partial r};$$

and two relations, between I_1, K_2, M_3, survive from the two conditions

$$\frac{\partial \theta_{23}}{\partial p} = \frac{\partial \theta_{31}}{\partial q} = \frac{\partial \theta_{12}}{\partial r}.$$

These are all the identical conditions of the specified type. Thus we find eight necessary relations, in the following form

$$\left.\begin{aligned}
L_2 + \Gamma_{11} M + \Delta_{11} N + \Theta_{11} I &= M_1 + \Gamma_{12} L + \Delta_{12} M + \Theta_{12} K \\
L_3 + \Gamma_{11} K + \Delta_{11} I + \Theta_{11} J &= K_1 + \Gamma_{13} L + \Delta_{13} M + \Theta_{13} K \\
N_1 + \Gamma_{22} L + \Delta_{22} M + \Theta_{22} K &= M_2 + \Gamma_{12} M + \Delta_{12} N + \Theta_{12} I \\
N_3 + \Gamma_{22} K + \Delta_{22} I + \Theta_{22} J &= I_2 + \Gamma_{23} M + \Delta_{23} N + \Theta_{23} I \\
J_1 + \Gamma_{33} L + \Delta_{33} M + \Theta_{33} K &= K_3 + \Gamma_{13} K + \Delta_{13} I + \Theta_{13} J \\
J_2 + \Gamma_{33} M + \Delta_{33} N + \Theta_{33} I &= I_3 + \Gamma_{23} K + \Delta_{23} I + \Theta_{23} J
\end{aligned}\right\},$$

together with

$$\begin{aligned}
I_1 + L\Gamma_{23} + M\Delta_{23} + K\Theta_{23} \\
= K_2 + M\Gamma_{13} + N\Delta_{13} + I\Theta_{13} \\
= M_3 + K\Gamma_{12} + I\Delta_{12} + J\Theta_{12}.
\end{aligned}$$

These are the extension, to a region in homaloidal quadruple space, of the two Codazzi-Mainardi relations for a surface in homaloidal triple space. They may therefore be called the *extended Codazzi-Mainardi relations* for the region.

The application of these relations, in connection with the construction of the third derivatives of x, y, z, v, and of the associated magnitudes of the third kind, will be deferred until after the consideration of some organic curves belonging to the region.

(ii) There apparently are other relations, which arise out of the conditions on p. 36 and which must be satisfied.

When we use, again, the values of X_1 and X_2, given in § 274, we have

$$MX_1 - LX_2 = \frac{1}{\Omega^2}\{L(x_1\lambda_2 + x_2\mu_2 + x_3\nu_2) - M(x_1\lambda_1 + x_2\mu_1 + x_3\nu_1)\}$$

$$= \frac{1}{\Omega^2}\{x_1(lj - gi) + x_2(bj - fi) + x_3(fj - ci)\},$$

with the notation of § 278. Hence we infer that the conditions

$$\frac{\partial\Gamma_{11}}{\partial q} - \frac{\partial\Gamma_{12}}{\partial p} = \frac{1}{\Omega^2}(hj - gi) + (\Gamma_{12}\Gamma_{11} + \Delta_{12}\Gamma_{12} + \Theta_{12}\Gamma_{13}) - (\Gamma_{11}\Gamma_{12} + \Delta_{11}\Gamma_{22} + \Theta_{11}\Gamma_{23}),$$

$$\frac{\partial\Delta_{11}}{\partial q} - \frac{\partial\Delta_{12}}{\partial p} = \frac{1}{\Omega^2}(bj - fi) + (\Gamma_{12}\Delta_{11} + \Delta_{12}\Delta_{12} + \Theta_{12}\Delta_{13}) - (\Gamma_{11}\Delta_{12} + \Delta_{11}\Delta_{22} + \Theta_{11}\Delta_{23}),$$

$$\frac{\partial\Theta_{11}}{\partial q} - \frac{\partial\Theta_{12}}{\partial p} = \frac{1}{\Omega^2}(fj - ci) + (\Gamma_{12}\Theta_{11} + \Delta_{12}\Theta_{12} + \Theta_{12}\Theta_{13}) - (\Gamma_{11}\Theta_{12} + \Delta_{11}\Theta_{22} + \Theta_{11}\Theta_{23}),$$

must be satisfied. But from the values of the derivatives of Γ, Δ, Θ, (§ 277), we have

$$\begin{aligned}\Omega^2\left(\frac{\partial\Gamma_{11}}{\partial q} - \frac{\partial\Gamma_{12}}{\partial p}\right) &= \tfrac{1}{2}h(-A_{22} + 2H_{12} - B_{11}) + \tfrac{1}{2}g(G_{12} + H_{13} - F_{11} - A_{23}) \\ &\quad + h\{-(*\,11\,22) + (*\,12\,12)\} + g\{(*\,12\,13) - (*\,11\,23)\} \\ &\quad - \Omega^2(\Gamma_{11}\Gamma_{12} + \Delta_{11}\Gamma_{22} + \Theta_{11}\Gamma_{23}) + \Omega^2(\Gamma_{12}\Gamma_{11} + \Delta_{12}\Gamma_{12} + \Theta_{12}\Gamma_{13}) \\ &= hj - gi - \Omega^2(\Gamma_{11}\Gamma_{12} + \Delta_{11}\Gamma_{22} + \Theta_{11}\Gamma_{23}) + \Omega^2(\Gamma_{12}\Gamma_{11} + \Delta_{12}\Gamma_{12} + \Theta_{12}\Gamma_{13}),\end{aligned}$$

on account of the values of l, n, j, i, k, m, given in § 278. Thus the first of the foregoing conditions is satisfied in virtue of the relations which already have been retained.

Similarly, the other conditions are satisfied in virtue of those same relations: and so for all the other conditions which similarly arise.

It therefore appears that all the conditions, which thus can be derived in connection with the various equations, are satisfied in virtue of the fourteen relations which have been retained, these relations being constituted by

(α) the six extended Gauss relations,

(β) the eight extended Codazzi-Mainardi relations.

Curves of curvature.

280. Among organic curves characteristic of a region, curves of curvature naturally claim the earliest consideration. As usual for other amplitudes in other spaces, the curves arise in two different ways. In one mode, they are the loci which, at every point, give the directions of maximum or minimum curvature of the normal section of the region (or, as will be seen, the directions of maximum or minimum circular curvature among geodesic curves through the point). In the other mode, they are the curve-loci along which normals to the consecutive tangent flats of the region intersect. We take these definitions in turn, so far as to deduce the critical equations.

The curvature of a normal section of the region through the point x, y, z, v, in the direction p', q', r', is given (§ 272) by the expression

$$\frac{1}{\rho} = (L, N, J, I, K, M \mathbin{\text{\textbigcircle}} p', q', r')^2.$$

Adopting the first of the two modes of definition, we require the directions p', q', r', which give the maximum or the minimum values of ρ; and we require the equation or equations for these values of ρ. Thus it is necessary to find the maximum or the minimum values of ρ for the range of variation of the quantities p', q', r', which are subject to the permanent relation

$$(A, B, C, F, G, H \mathbin{\text{\textbigcircle}} p', q', r')^2 = 1.$$

The critical equations accordingly are

$$\left.\begin{aligned} Lp' + Mq' + Kr' &= \epsilon(Ap' + Hq' + Gr') \\ Mp' + Nq' + Ir' &= \epsilon(Hp' + Bq' + Fr') \\ Kp' + Iq' + Jr' &= \epsilon(Gp' + Fq' + Cr') \end{aligned}\right\},$$

where, so far as concerns the customary formation of the equations, the quantity ϵ is an undetermined multiplier.

Principal measures of curvature.

281. We multiply the equations by p', q', r', and add: then $1/\rho = \epsilon$. When this value of ϵ is inserted in the three critical equations, they become homogeneous and linear in p', q', r'. Elimination of these three quantities leads to the equation

$$E\left(\frac{1}{\rho}\right) = \begin{vmatrix} L - \frac{A}{\rho}, & M - \frac{H}{\rho}, & K - \frac{G}{\rho} \\ M - \frac{H}{\rho}, & N - \frac{B}{\rho}, & I - \frac{F}{\rho} \\ K - \frac{G}{\rho}, & I - \frac{F}{\rho}, & J - \frac{C}{\rho} \end{vmatrix} = 0.$$

With the notation (§§ 273, 278, 286) adopted for the two determinants

$$\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix} = \Omega^2, \qquad \begin{vmatrix} L, & M, & K \\ M, & N, & I \\ K, & I, & J \end{vmatrix} = \Upsilon,$$

and for the first minors of Ω^2 and Υ, the equation for the determination of ρ becomes

$$E\left(\frac{1}{\rho}\right) = \Upsilon - \mathfrak{G}_2 \frac{1}{\rho} + \mathfrak{G}_1 \frac{1}{\rho^2} - \Omega^2 \frac{1}{\rho^3} = 0,$$

where

$$\left.\begin{aligned} \mathfrak{G}_2 &= Al + Bn + Cj + 2Fi + 2Gk + 2Hm \\ \mathfrak{G}_1 &= La + Nb + Jc + 2If + 2Kg + 2Mh \end{aligned}\right\}.$$

Now the roots of this cubic equation are real. By the properties of determinants, we have

$$\left(L-\frac{A}{\rho}\right)E\left(\frac{1}{\rho}\right)$$

$$= \begin{vmatrix} \left(L-\frac{A}{\rho}\right)\left(N-\frac{B}{\rho}\right)-\left(M-\frac{H}{\rho}\right)^2 & , & \left(L-\frac{A}{\rho}\right)\left(I-\frac{F}{\rho}\right)-\left(M-\frac{H}{\rho}\right)\left(K-\frac{G}{\rho}\right) \\ \left(L-\frac{A}{\rho}\right)\left(I-\frac{F}{\rho}\right)-\left(M-\frac{H}{\rho}\right)\left(K-\frac{G}{\rho}\right), & & \left(L-\frac{A}{\rho}\right)\left(J-\frac{C}{\rho}\right)-\left(K-\frac{G}{\rho}\right)^2 \end{vmatrix}.$$

Let $1/\rho_1$ and $1/\rho_2$, where $1/\rho_1 > 1/\rho_2$, be the roots of the quadratic equation

$$f\left(\frac{1}{\rho}\right)=\left(L-\frac{A}{\rho}\right)\left(N-\frac{B}{\rho}\right)-\left(M-\frac{H}{\rho}\right)^2=0;$$

and as regards the quantities L/A, and N/B, let $L/A \geqslant N/B$. Then, as A, B, $AB-H^2$, are positive, we have

$$f(+\infty)>0, \quad f\left(\frac{L}{A}\right)<0, \quad f\left(\frac{N}{B}\right)<0, \quad f(-\infty)>0.$$

Thus one root of the quadratic lies between $+\infty$ and L/A, and the other between L/A and $-\infty$: that is,

$$\frac{1}{\rho_1}>\frac{L}{A}, \quad \frac{1}{\rho_2}<\frac{L}{A},$$

and therefore

$$L-\frac{A}{\rho_1}<0, \quad L-\frac{A}{\rho_2}>0.$$

But

$$\left(L-\frac{A}{\rho_1}\right)E\left(\frac{1}{\rho_1}\right)=-\left\{\left(L-\frac{A}{\rho_1}\right)\left(I-\frac{F}{\rho_1}\right)-\left(M-\frac{H}{\rho_1}\right)\left(K-\frac{G}{\rho_1}\right)\right\}^2,$$

a negative quantity;

$$\left(L-\frac{A}{\rho_2}\right)E\left(\frac{1}{\rho_2}\right)=-\left\{\left(L-\frac{A}{\rho_2}\right)\left(I-\frac{F}{\rho_2}\right)-\left(M-\frac{H}{\rho_2}\right)\left(K-\frac{G}{\rho_2}\right)\right\}^2,$$

a negative quantity;

hence

$$E\left(\frac{1}{\rho_1}\right)>0, \quad E\left(\frac{1}{\rho_2}\right)<0.$$

Obviously

$$E(+\infty)<0, \quad E(-\infty)>0,$$

for Ω^2 is a positive quantity. Hence the cubic equation in $1/\rho$ has one root between $+\infty$ and $1/\rho_1$, one root between $1/\rho_1$ and $1/\rho_2$, and one root between $1/\rho_2$ and $-\infty$: that is, all the roots are real.

Let $1/\alpha$, $1/\beta$, $1/\gamma$ be these three real roots, which accordingly are the principal values of $1/\rho$; and let

$$\mathfrak{L}=\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}, \quad \mathfrak{F}=\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}+\frac{1}{\alpha\beta}, \quad \mathfrak{R}=\frac{1}{\alpha\beta\gamma},$$

so that $\mathfrak{L}$ may be called the *linear measure of curvature* of the region at the point, $\mathfrak{F}$ the *superficial measure of curvature* of the region, and $\mathfrak{R}$ the *spatial measure of curvature* (or the *volumetric curvature*) of the region at the point.

These three measures of curvature are expressible in terms of the fundamental magnitudes of the first order and the fundamental magnitudes of the second order, by the equations

$$\Omega^2\mathfrak{L} = \mathfrak{C}_1, \quad \Omega^2\mathfrak{F} = \mathfrak{C}_2, \quad \Omega^2\mathfrak{N} = \Upsilon.$$

We shall assume that Υ is not zero: also that the three roots α, β, γ, are unequal to one another in general, because an equality between the roots would imply a relation among the primary and the secondary magnitudes alone—a relation entirely different from the relations (§ 278) established between the secondary magnitudes and the primary magnitudes with the derivatives of the latter. When the unequal roots, arranged in order of magnitude, are such that $\alpha > \beta > \gamma$, we infer that α provides a maximum value for ρ, that γ provides a minimum value, and that β provides a merely stationary value that is neither a true maximum nor a true minimum.

Alternative derivation of the measures: their values for a Cartesian region.

282. When the alternative definition is adopted, whereby a line of curvature in the region is a locus along which consecutive normals to the region intersect, the modified analysis leads to the same result. The normal to the tangent flat at x, y, z, v, is given by

$$\frac{\bar{x}-x}{X} = \frac{\bar{y}-y}{Y} = \frac{\bar{z}-z}{Z} = \frac{\bar{v}-v}{V}.$$

If this normal is intersected by the normal to the tangent flat at a consecutive point $x+dx,\ y+dy,\ z+dz,\ v+dv$, let $\bar{x}, \bar{y}, \bar{z}, \bar{v}$, now denote the coordinates of the point of intersection, and let its distance from x, y, z, v, be ρ. Then, for the first normal, we have

$$\bar{x} = x+\rho X, \quad \bar{y} = y+\rho Y, \quad \bar{z} = z+\rho Z, \quad \bar{v} = v+\rho V;$$

and, for the second normal, we have

$$\bar{x} = x+dx+\rho(X+dX), \quad \bar{y} = y+dy+\rho(Y+dY),$$
$$\bar{z} = z+dz+\rho(Z+dZ), \quad \bar{v} = v+dv+\rho(V+dV):$$

hence

$$dx+\rho dX = 0, \quad dy+\rho dY = 0, \quad dz+\rho dZ = 0, \quad dv+\rho dV = 0.$$

Now these four equations are not independent of one another, because

$$\Sigma X(dx+\rho dX) = \Sigma X\,dx + \rho\,\Sigma X\,dX = 0;$$

hence we need reserve only three relations. When the due values are substituted for dx and dX, for dy and dY, for dz and dZ, and for dv and dV, the four equations become

$$(x_1+\rho X_1)\,dp + (x_2+\rho X_2)\,dq + (x_3+\rho X_3)\,dr = 0,$$
$$(y_1+\rho Y_1)\,dp + (y_2+\rho Y_2)\,dq + (y_3+\rho Y_3)\,dr = 0,$$
$$(z_1+\rho Z_1)\,dp + (z_2+\rho Z_2)\,dq + (z_3+\rho Z_3)\,dr = 0,$$
$$(v_1+\rho V_1)\,dp + (v_2+\rho V_2)\,dq + (v_3+\rho V_3)\,dr = 0.$$

Multiply the equations by x_1, y_1, z_1, v_1, and add: using the relations

$$\Sigma x_1 X_1 = -L, \quad \Sigma x_1 X_2 = -M, \quad \Sigma x_1 X_3 = -K,$$

we have the equation

$$(A-\rho L)\,dp+(H-\rho M)\,dq+(G-\rho K)\,dr=0.$$

Multiply by x_2, y_2, z_2, v_2, and add: using the relations

$$\Sigma x_2 X_1=-M,\quad \Sigma x_2 X_2=-N,\quad \Sigma x_2 X_3=-I,$$

we have the equation

$$(H-\rho M)\,dp+(B-\rho N)\,dq+(F-\rho I)\,dr=0.$$

Multiply by x_3, y_3, z_3, v_3, and add: using the relations

$$\Sigma x_3 X_1=-K,\quad \Sigma x_3 X_2=-I,\quad \Sigma x_3 X_3=-J,$$

we have a third equation

$$(G-\rho K)\,dp+(F-\rho I)\,dq+(C-\rho J)\,dr=0.$$

These are the three independent equations: effectively they are equivalent to the three equations in the former investigation.

It thus appears that the two modes of proceeding yield the same results analytically. Further, stationary values of the curvature require the intersection of the normals to the tangent flats at consecutive points along the directions.

Ex. 1. When a single equation of the form

$$v-\chi(x, y, z)=0$$

is used to specify a region in the homaloidal quadruple space, the coordinates x, y, z, can be taken as parameters for the region. We then have

$$ds^2=dx^2+dy^2+dz^2+(v_x dx+v_y dy+v_z dz)^2,$$

so that

$$\left.\begin{aligned} A&=1+v_x^2, & F&=v_y v_z\\ B&=1+v_y^2, & G&=v_z v_x\\ C&=1+v_z^2, & H&=v_x v_y \end{aligned}\right\},$$

and

$$\Omega^2=1+v_x^2+v_y^2+v_z^2.$$

We then have, for the outward-drawn normal,

$$\Omega X=-J_x=-v_x,\quad \Omega Y=J_y=-v_y,\quad \Omega Z=-J_z=-v_z,\quad \Omega V=J_v=1.$$

Again,

$$\left.\begin{aligned} \Omega L&=v_{xx}, & \Omega I&=v_{yz}\\ \Omega N&=v_{yy}, & \Omega K&=v_{zx}\\ \Omega J&=v_{zz}, & \Omega M&=v_{xy} \end{aligned}\right\}.$$

The measures of curvature of the region, as derived through the circular curvature of the principal normal sections, are given by the equation

$$\begin{aligned} &\frac{1}{\rho^3}-\frac{1}{\Omega^3\rho^2}\{(1+v_y^2+v_z^2)\,v_{xx}+(1+v_z^2+v_x^2)\,v_{yy}+(1+v_x^2+v_y^2)\,v_{zz}\\ &\qquad -2v_x v_y v_{xy}-2v_y v_z v_{yz}-2v_z v_x v_{zx}\}\\ &+\frac{1}{\Omega^4\rho}\{(1+v_x^2)\,(v_{yy}v_{zz}-v_{yz}^2)+(1+v_y^2)\,(v_{zz}v_{xx}-v_{zx}^2)+(1+v_z^2)\,(v_{xx}v_{yy}-v_{xy}^2)\\ &\qquad +2v_y v_z\,(v_{zx}v_{xy}-v_{xx}v_{yz})+2v_z v_x\,(v_{xy}v_{yz}-v_{yy}v_{zx})+2v_x v_y\,(v_{yz}v_{zx}-v_{zz}v_{xy})\}\\ &-\frac{1}{\Omega^5}\begin{vmatrix} v_{xx}, & v_{xy}, & v_{xz}\\ v_{xy}, & v_{yy}, & v_{yz}\\ v_{xz}, & v_{yz}, & v_{zz} \end{vmatrix}=0. \end{aligned}$$

This result may be obtained directly without quoting the general formula: it can be derived by using the property that, along a line of curvature in the region, consecutive normals to the region intersect. The direction-cosines of the outward-drawn normal are

$$X=-\frac{v_x}{\Omega},\quad Y=-\frac{v_y}{\Omega},\quad Z=-\frac{v_z}{\Omega},\quad V=\frac{1}{\Omega},$$

where $\Omega^2=1+v_x^2+v_y^2+v_z^2$. When two consecutive normals intersect, the distance of their intersection from the point x, y, z, v, is ρ, and the coordinates of the intersection are

$$\bar{x}=x+\rho X=x+dx+\rho(X+dX),$$
$$\bar{y}=y+\rho Y=y+dy+\rho(Y+dY),$$
$$\bar{z}=z+\rho Z=z+dz+\rho(Z+dZ),$$
$$\bar{v}=v+\rho V=v+dv+\rho(V+dV).$$

From the last equality, we have

$$dv+\rho dV=0,$$

and therefore

$$\frac{1}{\rho}(v_x dx+v_y dy+v_z dz)-\frac{d\Omega}{\Omega^2}=0.$$

The first equality gives a relation

$$dx+\rho dX=0,$$

that is,

$$\frac{dx}{\rho}-\frac{1}{\Omega}(v_{xx}dx+v_{xy}dy+v_{xz}dz)+\frac{v_x}{\Omega^2}d\Omega=0.$$

When we substitute for $\frac{d\Omega}{\Omega^2}$, and rearrange the terms, this relation becomes

$$\left(\frac{1+v_x^2}{\rho}-\frac{v_{xx}}{\Omega}\right)dx+\left(\frac{v_xv_y}{\rho}-\frac{v_{xy}}{\Omega}\right)dy+\left(\frac{v_xv_z}{\rho}-\frac{v_{xz}}{\Omega}\right)dz=0.$$

The other two equalities similarly yield the relations

$$\left(\frac{v_xv_y}{\rho}-\frac{v_{xy}}{\Omega}\right)dx+\left(\frac{1+v_y^2}{\rho}-\frac{v_{yy}}{\Omega}\right)dy+\left(\frac{v_yv_z}{\rho}-\frac{v_{yz}}{\Omega}\right)dz=0,$$

$$\left(\frac{v_xv_z}{\rho}-\frac{v_{xz}}{\Omega}\right)dx+\left(\frac{v_yv_z}{\rho}-\frac{v_{yz}}{\Omega}\right)dy+\left(\frac{1+v_z^2}{\rho}-\frac{v_{zz}}{\Omega}\right)dz=0.$$

The magnitudes of the principal radii of curvature are given by means of the roots of the cubic

$$\begin{vmatrix} \frac{1+v_x^2}{\rho}-\frac{v_{xx}}{\Omega}, & \frac{v_xv_y}{\rho}-\frac{v_{xy}}{\Omega}, & \frac{v_xv_z}{\rho}-\frac{v_{xz}}{\Omega} \\ \frac{v_xv_y}{\rho}-\frac{v_{xy}}{\Omega}, & \frac{1+v_y^2}{\rho}-\frac{v_{yy}}{\Omega}, & \frac{v_yv_z}{\rho}-\frac{v_{yz}}{\Omega} \\ \frac{v_xv_z}{\rho}-\frac{v_{xz}}{\Omega}, & \frac{v_yv_z}{\rho}-\frac{v_{yz}}{\Omega}, & \frac{1+v_z^2}{\rho}-\frac{v_{zz}}{\Omega} \end{vmatrix}=0.$$

which, when expanded, is the former equation.

For the directions of the lines of curvature, we have

$$dx-\rho d\frac{v_x}{\Omega}=0,\quad dy-\rho d\frac{v_y}{\Omega}=0,\quad dz-\rho d\frac{v_z}{\Omega}=0,\quad dv+\rho d\frac{1}{\Omega}=0.$$

The last equation gives

$$\frac{dv}{\rho}=\frac{d\Omega}{\Omega^2};$$

and then the first equation becomes

$$\frac{dx}{\rho} - \frac{1}{\Omega}\,dv_x + \frac{v_x}{\rho}\,dv = 0,$$

with similar forms for the other two equations. Thus the elements dx, dy, dz, dv, satisfy the three equations

$$\left.\begin{aligned} \left(\frac{1}{\rho} - \frac{v_{xx}}{\Omega}\right)dx \quad & -\frac{v_{xy}}{\Omega}\,dy \quad & -\frac{v_{xz}}{\Omega}\,dz + \frac{v_x}{\rho}\,dv = 0 \\ -\frac{v_{xy}}{\Omega}\,dx + \left(\frac{1}{\rho} - \frac{v_{yy}}{\Omega}\right)dy \quad & & -\frac{v_{yz}}{\Omega}\,dz + \frac{v_y}{\rho}\,dv = 0 \\ -\frac{v_{xz}}{\Omega}\,dx \quad & -\frac{v_{yz}}{\Omega}\,dy + \left(\frac{1}{\rho} - \frac{v_{zz}}{\Omega}\right)dz + \frac{v_z}{\rho}\,dv = 0 \end{aligned}\right\},$$

which accordingly are the differential equations of a line of curvature, the three lines being given by the respective values of ρ.

Ex. 2. Let dp_1, dq_1, dr_1; dp_2, dq_2, dr_2; dp_3, dq_3, dr_3; be three directions in a region, not lying in one plane; and let $1/\rho_1$, $1/\rho_2$, $1/\rho_3$, be the circular curvatures of normal sections of the region through these directions. Also, denoting rudimentary arcs in these directions by ds_1, ds_2, ds_3, respectively, let

$$L_{12}\,ds_1 ds_2 = \Sigma L\,dp_1 dp_2, \quad L_{23}\,ds_2 ds_3 = \Sigma L\,dp_2 dp_3, \quad L_{31}\,ds_3 ds_1 = \Sigma L\,dp_3 dp_1;$$

and let $\widehat{12}$, $\widehat{23}$, $\widehat{31}$ be the angles between the respective pairs of directions. Shew that the curvature of any normal section is given by

$$\frac{1}{\rho} = \frac{\xi^2}{\rho_1} + \frac{\eta^2}{\rho_2} + \frac{\zeta^2}{\rho_3} + 2L_{12}\,\xi\eta + 2L_{23}\,\eta\zeta + 2L_{31}\,\zeta\xi,$$

where the parameters ξ, η, ζ, are connected by a relation

$$\xi^2 + \eta^2 + \zeta^2 + 2\xi\eta\cos\widehat{12} + 2\eta\zeta\cos\widehat{23} + 2\zeta\xi\cos\widehat{31} = 1;$$

and that the principal curvatures of the region are the roots of the cubic equation

$$\begin{vmatrix} \frac{1}{\rho} - \frac{1}{\rho_1}, & \frac{\cos\widehat{12}}{\rho} - L_{12}, & \frac{\cos\widehat{13}}{\rho} - L_{13} \\ \frac{\cos\widehat{12}}{\rho} - L_{12}, & \frac{1}{\rho} - \frac{1}{\rho_2}, & \frac{\cos\widehat{23}}{\rho} - L_{23} \\ \frac{\cos\widehat{13}}{\rho} - L_{13}, & \frac{\cos\widehat{23}}{\rho} - L_{23}, & \frac{1}{\rho} - \frac{1}{\rho_3} \end{vmatrix} = 0.$$

Hence (or otherwise) obtain geometrical interpretations for the magnitudes L_{23}, L_{31}, L_{12}.

Ex. 3. Consider the region enveloping a flat, when the equation of the flat contains only a single parameter. (Such a region has already arisen for consideration, under the discussion of twisted curves; the osculating flat of a twisted curve is a particular instance and, in §§ 180, 181, the reconstruction of the curve as the edge of regression of the enveloping region has been effected.)

We take, for our immediate purpose, the equation of the flat in the form

$$v = xp + yq + zr - u,$$

where p, q, r, are functions solely of the single parameter u. The enveloping region, touching the flats for all values of u, has its equation given as the eliminant of the equations

$$\left.\begin{aligned} v &= xp + yq + zr - u \\ 0 &= xp' + yq' + zr' - 1 \end{aligned}\right\},$$

where p', q', r', are the derivatives of p, q, r, with respect to u. Along the enveloping region, we have

$$v_x = p + (xp' + yq' + zr' - 1)\frac{\partial u}{\partial x} = p;$$

and similarly

$$v_y = q, \quad v_z = r.$$

We write

$$\square^2 = 1 + v_x^2 + v_y^2 + v_z^2 = 1 + p^2 + q^2 + r^2;$$

then the direction-cosines of the outward-drawn normal are

$$X = -\frac{p}{\square}, \quad Y = -\frac{q}{\square}, \quad Z = -\frac{r}{\square}, \quad V = \frac{1}{\square}.$$

Again, from the second of the two equations for the enveloping region, we have

$$(xp'' + yq'' + zr'')\frac{\partial u}{\partial x} + p' = 0.$$

Let

$$xp'' + yq'' + zr'' = -\theta;$$

then

$$\frac{\partial u}{\partial x} = \frac{p'}{\theta}, \quad \frac{\partial u}{\partial y} = \frac{q'}{\theta}, \quad \frac{\partial u}{\partial z} = \frac{r'}{\theta}.$$

Hence

$$v_{xx} = \frac{p'^2}{\theta}, \quad v_{yy} = \frac{q'^2}{\theta}, \quad v_{zz} = \frac{r'^2}{\theta},$$

$$v_{yz} = \frac{q'r'}{\theta}, \quad v_{zx} = \frac{r'p'}{\theta}, \quad v_{xy} = \frac{p'q'}{\theta}.$$

Consequently

$$v_{yy}v_{zz} - v_{yz}^2 = 0, \quad v_{zz}v_{xx} - v_{zx}^2 = 0, \quad v_{xx}v_{yy} - v_{xy}^2 = 0,$$

$$v_{zx}v_{xy} - v_{xx}v_{yz} = 0, \quad v_{xy}v_{yz} - v_{yy}v_{zx} = 0, \quad v_{yz}v_{zx} - v_{zz}v_{xy} = 0,$$

$$\begin{vmatrix} v_{xx}, & v_{xy}, & v_{xz} \\ v_{xy}, & v_{yy}, & v_{yz} \\ v_{xz}, & v_{yz}, & v_{zz} \end{vmatrix} = 0,$$

$$(1 + v_y^2 + v_z^2)v_{xx} + (1 + v_z^2 + v_x^2)v_{yy} + (1 + v_x^2 + v_y^2)v_{zz} - 2v_yv_zv_{yz} - 2v_zv_xv_{zx} - 2v_xv_yv_{xy}$$

$$= \frac{1}{\theta}\{(1 + p^2 + q^2 + r^2)(p'^2 + q'^2 + r'^2) - (pp' + qq' + rr')^2\}$$

$$= \frac{1}{\theta}\{p'^2 + q'^2 + r'^2 + (qr' - rq')^2 + (rp' - pr')^2 + (pq' - qp')^2\}$$

$$= \frac{U}{\theta},$$

with an obvious significance for U. Thus the equation, which gives the principal line-curvatures for the enveloping region, is

$$\frac{1}{\rho^3} - \frac{1}{\rho^2\square^3}\frac{U}{\theta} = 0.$$

Consequently, two of the lines of curvature are straight lines; and the only non-zero measure of linear curvature is

$$\frac{p'^2 + q'^2 + r'^2 + (qr' - rq')^2 + (rp' - pr')^2 + (pq' - qp')^2}{(1 + p^2 + q^2 + r^2)^{\frac{3}{2}}(xp'' + yq'' + zr'')},$$

measured inwards.

The region is a developable region: the process of development into a flat has already been discussed (§ 178). The region is planar: for its equation is satisfied by all the values of x, y, z, v, that simultaneously satisfy

$$v = xp + yq + zr - u, \quad 0 = xp' + yq' + zr' - 1,$$

that is, for all points in this plane which accordingly is contained in the region.

When these two equations are taken simultaneously, they represent a plane. The envelope of the plane is constituted by the surface having, as its equations, the two equations resulting from the elimination of u between

$$\left.\begin{aligned} v &= xp + yq + zr - u \\ 0 &= xp' + yq' + zr' - 1 \\ 0 &= xp'' + yq'' + zr'' \end{aligned}\right\}.$$

As the three equations represent a line, this envelope surface is ruled: and the original flat is its osculating flat.

Moreover, the consecutive generators of this envelope surface intersect: such a point of intersection is given by

$$\left.\begin{aligned} v &= xp + yq + zr - u \\ 0 &= xp' + yq' + zr' - 1 \\ 0 &= xp'' + yq'' + zr'' \\ 0 &= xp''' + yq''' + zr''' \end{aligned}\right\};$$

and the locus of this point is the edge of regression of the developable region.

When the region is developed into a flat, the osculating surface becomes a developable surface in that flat: and further development within the flat can change the developable surface into a plane. Reference may be made to the former discussion (§§ 178, 180) of the whole configuration associated with a twisted curve.

Directions of the lines of curvature.

283. The directions of the lines of curvature are derivable from the same three equations

$$\left(L - \frac{A}{\rho}\right)p' + \left(M - \frac{H}{\rho}\right)q' + \left(K - \frac{G}{\rho}\right)r' = 0,$$

$$\left(M - \frac{H}{\rho}\right)p' + \left(N - \frac{B}{\rho}\right)q' + \left(I - \frac{F}{\rho}\right)r' = 0,$$

$$\left(K - \frac{G}{\rho}\right)p' + \left(I - \frac{F}{\rho}\right)q' + \left(J - \frac{C}{\rho}\right)r' = 0,$$

in which ρ must be one of the three principal radii. When the first two equations are combined, they give

$$\frac{p'}{\left(M - \frac{H}{\rho}\right)\left(I - \frac{F}{\rho}\right) - \left(N - \frac{B}{\rho}\right)\left(K - \frac{G}{\rho}\right)} = \frac{q'}{\left(K - \frac{G}{\rho}\right)\left(M - \frac{H}{\rho}\right) - \left(L - \frac{A}{\rho}\right)\left(I - \frac{F}{\rho}\right)};$$

and when the second and the third are combined, they give

$$\frac{q'}{\left(I - \frac{F}{\rho}\right)\left(K - \frac{G}{\rho}\right) - \left(J - \frac{C}{\rho}\right)\left(M - \frac{H}{\rho}\right)} = \frac{r'}{\left(M - \frac{H}{\rho}\right)\left(I - \frac{F}{\rho}\right) - \left(N - \frac{B}{\rho}\right)\left(K - \frac{G}{\rho}\right)}.$$

Hence the direction p', q', r', for a principal radius of curvature is given by the equations

$$\left\{\left(K-\frac{G}{\rho}\right)\left(M-\frac{H}{\rho}\right)-\left(L-\frac{A}{\rho}\right)\left(I-\frac{F}{\rho}\right)\right\}p'$$
$$=\left\{\left(M-\frac{H}{\rho}\right)\left(I-\frac{F}{\rho}\right)-\left(N-\frac{B}{\rho}\right)\left(K-\frac{G}{\rho}\right)\right\}q'$$
$$=\left\{\left(I-\frac{F}{\rho}\right)\left(K-\frac{G}{\rho}\right)-\left(J-\frac{C}{\rho}\right)\left(M-\frac{H}{\rho}\right)\right\}r',$$

together with the equation

$$Ap'^2+Bq'^2+Cr'^2+2Fq'r'+2Gr'p'+2Hp'q'=1.$$

Let

$$\left(M-\frac{H}{\rho}\right)\left(K-\frac{G}{\rho}\right)-\left(L-\frac{A}{\rho}\right)\left(I-\frac{F}{\rho}\right)=\bar{\eta}\bar{\zeta},$$
$$\left(I-\frac{F}{\rho}\right)\left(M-\frac{H}{\rho}\right)-\left(N-\frac{B}{\rho}\right)\left(K-\frac{G}{\rho}\right)=\bar{\zeta}\bar{\xi},$$
$$\left(K-\frac{G}{\rho}\right)\left(I-\frac{F}{\rho}\right)-\left(J-\frac{C}{\rho}\right)\left(M-\frac{H}{\rho}\right)=\bar{\xi}\bar{\eta},$$

then from the equation satisfied by the principal radii

$$\left(N-\frac{B}{\rho}\right)\left(J-\frac{C}{\rho}\right)-\left(I-\frac{F}{\rho}\right)^2=\bar{\xi}^2,$$
$$\left(J-\frac{C}{\rho}\right)\left(L-\frac{A}{\rho}\right)-\left(K-\frac{G}{\rho}\right)^2=\bar{\eta}^2,$$
$$\left(L-\frac{A}{\rho}\right)\left(N-\frac{B}{\rho}\right)-\left(M-\frac{H}{\rho}\right)^2=\bar{\zeta}^2.$$

The equations for p', q', r', can then be written in the form

$$\bar{\eta}\bar{\zeta}p'=\bar{\zeta}\bar{\xi}q'=\bar{\xi}\bar{\eta}r';$$

or

$$\frac{p'}{\bar{\xi}}=\frac{q'}{\bar{\eta}}=\frac{r'}{\bar{\zeta}},$$

but the earlier form is simpler because it requires no radical. In any of the forms, we have the directions of the principal radii of curvature of normal sections of the region.

There is the further proposition that *each two of these principal directions are perpendicular to one another*. Let two of the directions be p_1', q_1', r_1', for a principal radius ρ_1, and p_2', q_2', r_2', for another principal radius ρ_2, these radii being unequal (§ 281). Then

$$Ap_1'p_2'+H(p_1'q_2'+q_1'p_2')+G(p_1'r_2'+r_1'p_2')+Bq_1'q_2'+F(q_1'r_2'+r_1'q_2')+Cr_1'r_2'$$
$$=p_2'(Ap_1'+Hq_1'+Gr_1')+q_2'(Hp_1'+Bq_1'+Fr_1')+r_2'(Gp_1'+Fq_1'+Cr_1')$$
$$=\rho_1\{p_2'(Lp_1'+Mq_1'+Kr_1')+q_2'(Mp_1'+Nq_1'+Ir_1')+r_2'(Kp_1'+Iq_1'+Jr_1')\}$$
$$=\rho_1\{Lp_1'p_2'+M(p_1'q_2'+q_1'p_2')+K(p_1'r_2'+r_1'p_2')$$
$$+Nq_1'q_2'+I(q_1'r_2'+r_1'q_2')+Jr_1'r_2'\};$$

and the same expression

$$p_1'(Ap_2' + Hq_2' + Gr_2') + q_1'(Hp_2' + Bq_2' + Fr_2') + r_1'(Gp_2' + Fq_2' + Cr_2')$$
$$= \rho_2\{p_1'(Lp_2' + Mq_2' + Kr_2') + q_1'(Mp_2' + Nq_2' + Ir_2') + r_1'(Kp_2' + Iq_2' + Jr_2')\}$$
$$= \rho_2\{Lp_1'p_2' + M(p_1'q_2' + q_1'p_2') + K(p_1'r_2' + r_1'p_2')$$
$$+ Nq_1'q_2' + I(q_1'r_2' + r_1'q_2') + Jr_1'r_2'\}.$$

Now ρ_1 and ρ_2 are assumed (§ 281) to be unequal; hence

$$Ap_1'p_2' + H(p_1'q_2' + q_1'p_2') + G(p_1'r_2' + r_1'p_2') + Bq_1'q_2' + F(q_1'r_2' + r_1'q_2') + Cr_1'r_2' = 0,$$

and therefore (§ 262) the two directions are perpendicular.

Any two of the three principal directions have been chosen; hence we infer that, in each pair, the two directions are perpendicular. Further, each direction is perpendicular to the plane through the other two directions.

Parametric curves as lines of curvature.

284. We deduce at once the conditions that the parametric curves of reference in the region shall be lines of curvature.

The condition, that two directions p_1', q_1', r_1', and p_2', q_2', r_2', shall be perpendicular, is that the relation

$$Ap_1'p_2' + H(p_1'q_2' + q_1'p_2') + G(p_1'r_2' + r_1'p_2') + Bq_1'q_2' + F(q_1'r_2' + r_1'q_2') + Cr_1'r_2' = 0$$

shall be satisfied. In the present case, this relation must be satisfied for

$$p_1' = \text{variable}, \quad q_1' = \text{zero}, \quad r_1' = \text{zero},$$
$$p_2' = \text{zero}, \quad q_2' = \text{variable}, \quad r_2' = \text{zero}:$$

that is, $H = 0$. It must also be satisfied for

$$p_1' = \text{zero}, \quad q_1' = \text{variable}, \quad r_1' = \text{zero},$$
$$p_2' = \text{zero}, \quad q_2' = \text{zero}, \quad r_2' = \text{variable}:$$

that is, $F = 0$. It must also be satisfied for

$$p_1' = \text{variable}, \quad q_1' = \text{zero}, \quad r_1' = \text{zero},$$
$$p_2' = \text{zero}, \quad q_2' = \text{zero}, \quad r_2' = \text{variable}:$$

that is, $G = 0$.

We thus have $F = 0$, $G = 0$, $H = 0$, as necessary conditions. But these conditions merely secure the perpendicularity of the three directions of the parametric curves through a point: the intrinsic property, that they are to be lines of curvature, has not been used.

Accordingly, those same three lines—it is no longer necessary to take them in pairs—must satisfy the equations

$$Lp' + Mq' + Kr' = \frac{1}{\rho}(Ap' + Hq' + Gr') = \frac{A}{\rho}p',$$

$$Mp' + Nq' + Ir' = \frac{1}{\rho}(Hp' + Bq' + Fr') = \frac{B}{\rho}q',$$

$$Kp' + Iq' + Jr' = \frac{1}{\rho}(Gp' + Fq' + Cr') = \frac{C}{\rho}r'.$$

In order that these equations may be satisfied by

$$p' = \text{variable}, \quad q' = \text{zero}, \quad r' = \text{zero},$$

we must have

$$L = \frac{A}{\rho}, \quad M = 0, \quad K = 0:$$

thus giving A/L as the corresponding radius of curvature of a principal normal section. In order that they may be satisfied by

$$p' = \text{zero}, \quad q' = \text{variable}, \quad r' = \text{zero},$$

we must have

$$M = 0, \quad N = \frac{B}{\rho}, \quad I = 0:$$

thus giving B/N as the corresponding radius of curvature. In order that they may be satisfied by

$$p' = \text{zero}, \quad q' = \text{zero}, \quad r' = \text{variable},$$

we must have

$$K = 0, \quad I = 0, \quad J = \frac{C}{\rho}:$$

thus giving C/J as the corresponding radius of curvature.

Hence the aggregate of conditions, necessary and sufficient to secure that the parametric curves of reference for the region are its lines of curvature, is

$$F = 0, \quad G = 0, \quad H = 0;$$
$$K = 0, \quad I = 0, \quad M = 0.$$

Ex. We can verify, directly, the expectation that the lines of curvature (other than the linear generators) of a cylindrical region are the same as those of the surface (in ordinary triple space) which is the section of the region by a flat perpendicular to its generators; and likewise for its measures of curvature (other than the zero curvature of the linear generators).

We take a columnar region

$$z = f(x, y),$$

the generators of which are perpendicular to the flat $v = 0$. For the analytical representation of the region, we take x, y, v, as the three parameters: then

$$x_1 = 1, \quad y_1 = 0, \quad z_1 = p, \quad v_1 = 0,$$
$$x_2 = 0, \quad y_2 = 1, \quad z_2 = q, \quad v_2 = 0,$$
$$x_3 = 0, \quad y_3 = 0, \quad z_3 = 0, \quad v_3 = 1.$$

Thus

$$A=1+p^2, \quad B=1+q^2, \quad C=1,$$
$$F=0, \quad G=0, \quad H=pq,$$
$$\Omega^2=1+p^2+q^2,$$
$$\Omega X=-p, \quad \Omega Y=-q, \quad \Omega Z=1, \quad \Omega V=0.$$

Also all the second derivatives of x, y, z, v, with respect to the parameters, are zero except z_{11}, z_{12}, z_{22}; and

$$z_{11}=r, \quad z_{12}=s, \quad z_{22}=t.$$

Hence

$$\Omega L=r, \quad \Omega N=t, \quad \Omega J=0, \quad \Omega I=0, \quad \Omega K=0, \quad \Omega M=s;$$

and therefore

$$\Omega^2 j=rt-s^2, \quad l=0, \quad n=0, \quad i=0, \quad k=0, \quad m=0.$$

Consequently

$$\Omega\mathfrak{S}_1=r(1+q^2)-2pqs+(1+p^2)t,$$
$$\Omega^2\mathfrak{S}_2=rt-s^2,$$
$$\Upsilon=0;$$

and therefore the equation for the principal curvatures of the region becomes

$$\frac{1}{\rho}\left[\frac{\Omega^2}{\rho^2}-\frac{1}{\rho\Omega}\{r(1+q^2)-2pqs+(1+p^2)t\}+\frac{rt-s^2}{\Omega^2}\right]=0.$$

Thus one measure is zero; the other two are given by the equation

$$\frac{1}{\rho^2}-\frac{1}{\rho}\frac{r(1+q^2)-2pqs+(1+p^2)t}{(1+p^2+q^2)^{\frac{3}{2}}}+\frac{rt-s^2}{(1+p^2+q^2)^2}=0,$$

which is the equation for the principal curvatures of the surface

$$z=f(x, y)$$

existing in the homaloidal triple space $v=0$.

We assume that this surface is not a developable surface, so that $rt-s^2$ does not vanish. By § 283, the direction of the line of curvature, with $1/\rho$ for its measure of curvature, is given by the equations

$$\frac{1}{\Omega}(r\,dx+s\,dy)=\frac{1}{\rho}\{(1+p^2)\,dx+pq\,dy\},$$
$$\frac{1}{\Omega}(s\,dx+t\,dy)=\frac{1}{\rho}\{pq\,dx+(1+q^2)\,dy\},$$
$$0=-\frac{1}{\rho}\,dv.$$

First, when $1/\rho$ is zero, then

$$dx=0, \quad dy=0,$$

because $rt-s^2$ is not zero: and then dv is not zero, for otherwise every line would be a nul line. Hence one set of lines of curvature is constituted by the generators.

Next, for non-zero curvatures, we have

$$\frac{r\,dx+s\,dy}{s\,dx+t\,dy}=\frac{(1+p^2)\,dx+pq\,dy}{pq\,dx+(1+q^2)\,dy},$$

which is the equation for the directions of the lines of curvature of the surface $z=f(x, y)$ in homaloidal triple space.

Hence the proposition.

Magnitudes and relations for parametric lines of curvature.

285. It is convenient to place on record the values of the various magnitudes Γ, Δ, Θ: the forms of the extended Codazzi-Mainardi relations: and the forms of the extended Gauss relations: when the parametric curves of the region are its lines of curvature.

The special values of the quantities Γ, Δ, Θ, in this event are

$$\left.\begin{aligned} \Gamma_{11} &= \frac{1}{2}\frac{A_1}{A} \\ \Delta_{11} &= -\frac{1}{2}\frac{A_2}{B} \\ \Theta_{11} &= -\frac{1}{2}\frac{A_3}{C} \end{aligned}\right\}, \quad \left.\begin{aligned} \Gamma_{22} &= -\frac{1}{2}\frac{B_1}{A} \\ \Delta_{22} &= \frac{1}{2}\frac{B_2}{B} \\ \Theta_{22} &= -\frac{1}{2}\frac{B_3}{C} \end{aligned}\right\}, \quad \left.\begin{aligned} \Gamma_{33} &= -\frac{1}{2}\frac{C_1}{A} \\ \Delta_{33} &= -\frac{1}{2}\frac{C_2}{B} \\ \Theta_{33} &= \frac{1}{2}\frac{C_3}{C} \end{aligned}\right\},$$

$$\left.\begin{aligned} \Gamma_{23} &= 0 \\ \Delta_{23} &= \frac{1}{2}\frac{B_3}{B} \\ \Theta_{23} &= \frac{1}{2}\frac{C_2}{C} \end{aligned}\right\}, \quad \left.\begin{aligned} \Gamma_{31} &= \frac{1}{2}\frac{A_3}{A} \\ \Delta_{31} &= 0 \\ \Theta_{31} &= \frac{1}{2}\frac{C_1}{C} \end{aligned}\right\}, \quad \left.\begin{aligned} \Gamma_{12} &= \frac{1}{2}\frac{A_2}{A} \\ \Delta_{12} &= \frac{1}{2}\frac{B_1}{B} \\ \Theta_{12} &= 0 \end{aligned}\right\}.$$

The special forms of six of the extended Codazzi-Mainardi relations are

$$\left.\begin{aligned} L_2 - \frac{1}{2}\frac{A_2}{A}L &= \frac{1}{2}\frac{A_2}{B}N \\ L_3 - \frac{1}{2}\frac{A_3}{A}L &= \frac{1}{2}\frac{A_3}{C}J \\ N_3 - \frac{1}{2}\frac{B_3}{B}N &= \frac{1}{2}\frac{B_3}{C}J \\ N_1 - \frac{1}{2}\frac{B_1}{B}N &= \frac{1}{2}\frac{B_1}{A}L \\ J_1 - \frac{1}{2}\frac{C_1}{C}J &= \frac{1}{2}\frac{C_1}{A}L \\ J_2 - \frac{1}{2}\frac{C_2}{C}J &= \frac{1}{2}\frac{C_2}{B}N \end{aligned}\right\};$$

the other two are evanescent.

The special forms of the six extended Gauss relations are

$$\left.\begin{aligned} 4NJ + 2(B_{33} + C_{22}) &= -\frac{1}{A}B_1C_1 + \frac{1}{B}(B_2C_2 + B_3^2) + \frac{1}{C}(B_3C_3 + C_2^2) \\ 4JL + 2(C_{11} + A_{33}) &= \frac{1}{A}(C_1A_1 + A_3^2) - \frac{1}{B}C_2A_2 + \frac{1}{C}(C_3A_3 + C_1^2) \\ 4LN + 2(A_{22} + B_{11}) &= \frac{1}{A}(A_1B_1 + A_2^2) + \frac{1}{B}(A_2B_2 + B_1^2) - \frac{1}{C}A_3B_3 \end{aligned}\right\},$$

$$\left.\begin{aligned} 2A_{23} &= \frac{A_2A_3}{A} + \frac{A_2B_3}{B} + \frac{A_3C_2}{C} \\ 2B_{31} &= \frac{B_1A_3}{A} + \frac{B_3B_1}{B} + \frac{B_3C_1}{C} \\ 2C_{12} &= \frac{C_1A_2}{A} + \frac{C_2B_1}{B} + \frac{C_1C_2}{C} \end{aligned}\right\}.$$

The first three of these extended Gauss relations serve to express L, N, J, in terms of the primary magnitudes A, B, C, and of their first and second derivatives. Substitution of these inferred expressions in the various Codazzi-Mainardi relations leads to six partial differential equations, each of the third order, connecting the primary magnitudes; and these six equations, at least in their initial forms, are distinct from the three partial equations constituted by the last three of the extended Gauss relations.

Theorem concerning the fundamental secondary magnitudes of a non-developable region.

286. The relations in § 278, between the secondary magnitudes on the one hand and the primary magnitudes (with their derivatives) on the other hand, lead to one comprehensive inference which has no analogue in the Gauss theory of surfaces in homaloidal triple space.

Let Υ denote the determinant

$$\Upsilon = \begin{vmatrix} L, & M, & K \\ M, & N, & I \\ K, & I, & J \end{vmatrix}$$

of § 281, being a quantity concerned with the third (the volumetric) measure of curvature of the region; and suppose that Υ is not zero. Let the first minors of Υ be

$$\left.\begin{aligned} l &= NJ - I^2, & i &= MK - LI \\ n &= JL - K^2, & k &= IM - NK \\ j &= LN - M^2, & m &= KI - JM \end{aligned}\right\};$$

then

$$\begin{vmatrix} l, & m, & k \\ m, & n, & i \\ k, & i, & j \end{vmatrix} = \Upsilon^2.$$

Now each constituent in this determinant Υ^2 has been expressed (§ 278) solely in terms of the primary magnitudes and their derivatives: and therefore Υ^2 is similarly expressible. But

$$\left.\begin{aligned} L\Upsilon &= nj - i^2, & I\Upsilon &= km - li \\ N\Upsilon &= jl - k^2, & K\Upsilon &= mi - nk \\ J\Upsilon &= ln - m^2, & M\Upsilon &= ik - jm \end{aligned}\right\};$$

and all the right-hand sides in these relations are expressible, as Υ is expressible, in terms of the fundamental primary magnitudes of the region and the derivatives of these magnitudes. Hence, unless Υ vanishes—that is, unless the measure of volumetric curvature of the region, (as estimated, § 281), vanishes—the secondary quantities L, N, J, I, K, M, are expressible in terms of the primary magnitudes and their derivatives.

Now when Υ vanishes, one (at least) of the principal curvatures is zero. Consequently, along the corresponding curve of curvature the meeting normals (§ 282) to the consecutive tangential flats meet only at infinity: that is, they are parallel. The curve of curvature is a straight line, because (§ 146) its circular curvature is zero; and along that line of curvature, there is one and the same tangential flat to the region.

Consider four successive points P, Q, R, S, in the region, no two of them lying on the rectilinear lines of curvature through P, Q, R, S, respectively, which will be denoted by PP', QQ', RR', SS'. In general, four flats meet in a point; the four flats at P, Q, Q', R, actually meet in the line QQ', because the flats at Q and Q' are the same; and this line of intersection QQ' lies in the tangent plane which is the ultimate intersection of the tangential flats at Q and R. Similarly, the four flats at Q, R, R', S, actually meet in the line RR', because the flats at R and R' are the same; and this line of intersection RR' lies in the tangent plane which is the ultimate intersection of the tangential flats at Q and R. Thus QQ' and RR' lie in one and the same tangent plane; therefore they meet: that is, these successive rectilinear lines of curvature of the region meet.

Take a single flat through the four points $PQQ'R$; it can be turned, about a plane orthogonal to the plane which is the intersection of the tangential flats at Q and R, until it coincides with $QRR'S$: that is, the tangential flat at P can be turned, owing to the complanarity of PP' and QQ', until it coincides with the tangential flat at Q.

This amplified tangential flat at Q can similarly be turned, owing to the complanarity of QQ' and RR', until it coincides with the tangential flat at R.

The process can be continued until all the tangential flats to the region with the rectilinear lines of curvature are merged into one single flat. The region, still a curved configuration now lying wholly in one flat, has become the aggregate of points lying in the tangent planes to a surface in the flat*. In such circumstances, we say that the region is *developable*; the term implies that all its tangential flats can be brought (without stretching or tearing) into geometrical coincidence in one flat and that, then, the region has become a portion of the flat bounded by a surface in the flat.

* The change is analogous to the change by which, in a flat, a developable surface becomes the aggregate of points lying in the tangent lines to a curve.

This possibility has arisen through the assumption that $\Upsilon = 0$, implying a zero measure of volumetric curvature.

In general, Υ does not vanish, and the secondary magnitudes are expressible in terms of the primary magnitudes. The arc-element, in and by itself, thus becomes of sovereign importance in the region.

We therefore are in a position to infer the theorem:

All the fundamental secondary magnitudes of a region in homaloidal quadruple space are expressible in terms of the fundamental primary magnitudes of the region and of derivatives of these primary magnitudes, provided the measure of volumetric curvature of the region is not zero: or, what is the geometrical equivalent, provided the region cannot be developed into a portion of a flat bounded by a surface.

But the inference is not valid if the measure of volumetric curvature is zero, that is, if the region be developable.

Ex. As an example of the kind of development, consider the sphero-conical region *

$$v^2 = (x^2 + y^2 + z^2) \tan^2 \alpha,$$

where α is constant.

For the parametric representation, we use the spherical coordinates in triple space, and assume

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta, \quad v = r \tan\alpha.$$

We find

$$\left.\begin{aligned} X &= \sin\theta \cos\phi \sin\alpha \\ Y &= \sin\theta \sin\phi \sin\alpha \\ Z &= \cos\theta \sin\alpha \\ V &= -\cos\alpha \end{aligned}\right\};$$

$$A = \sec^2\alpha, \quad B = r^2, \quad C = r^2 \sin^2\theta, \quad F = G = H = 0;$$

$$L = 0, \quad N = -r \sin\alpha, \quad J = -r \sin^2\theta \sin\alpha, \quad I = K = M = 0.$$

Thus the three principal curvatures are

$$0, \quad -\frac{\sin\alpha}{r}, \quad -\frac{\sin\alpha}{r},$$

the negative signs being due to the fact that the region is convex to the positive directions of the coordinate axes of x, y, z. Thus one of the measures is zero: the region is developable, though not in the same sense as is the region which is the envelope of the osculating flat of a skew curve (§ 178).

The linear curve of curvature of the region is that along which, in this representation, the distance r is variable while θ and ϕ are constant: that is, a generator. When the region is developed by the rotation of the tangent flat, all these flats ultimately coincide, coalescing into the one flat which contains the developed region. A tangent flat to the region is

$$(\bar{x} \sin\theta \cos\phi + \bar{y} \sin\theta \sin\phi + \bar{z} \cos\theta) \tan\alpha = \bar{v}.$$

When the region is developed, the portion of it corresponding to a range

$$x^2 + y^2 + z^2 \leqslant c^2, \qquad v \leqslant c \tan\alpha,$$

corresponds to the solid content

$$x^2 + y^2 + z^2 \leqslant c^2.$$

* The geodesics in this region are obtained later: § 302, Ex. 3.

But now the solid content is relatively not of significance as compared with its superficial boundary; so, ignoring the content as compared with the boundary, we consider the surface

$$x^2+y^2+z^2=c^2.$$

The particular tangent flat, with the foregoing equation, now gives rise to the tangent plane

$$\bar{x}\sin\theta\cos\phi+\bar{y}\sin\theta\sin\phi+\bar{z}\cos\theta=\frac{\bar{v}}{\tan\alpha},$$

when, in the right-hand side, the appropriate value $\bar{v}=c\tan\alpha$ for the boundary is taken: that is, we have the tangent plane

$$\bar{x}\sin\theta\cos\phi+\bar{y}\sin\theta\sin\phi+\bar{z}\cos\theta=c.$$

The sphero-conical region is developable.

Conjugate directions.

287. The notion of conjugate directions in a plane arises in the discussion of the properties of conic sections: and the corresponding notion, extended so as to include the conjugate relation between duly associated lines and planes, occurs in the discussion of the properties of quadric surfaces in homaloidal triple space. It is extended so as to lead to the investigation of conjugate directions in the Gauss geometry of surfaces in that same space: and its extension may naturally be expected in amplitudes belonging to space of any number of dimensions.

For the purpose of determining conjugate properties in a region occurring in quadruple space, we take a point P in the region and any point Q adjacent to P; and we draw the flats, tangential to the region at P and at Q respectively. These two flats intersect in a plane, which may be denoted by S'_{PQ}. Now let Q approach P, so that in the limit there is a direction PQ, arbitrarily drawn through P; then the plane S'_{PQ} also approaches a definite limiting position, which may be denoted by S_{PQ}. The limiting direction PQ and the limiting plane S_{PQ} are said to be *conjugate* to one another; and any direction in the limiting plane S_{PQ} is said to be *conjugate* to the limiting direction PQ.

The analytical expression of conjugate direction is easily derived from the definition. Let P be the point x, y, z, v; and Q, a neighbouring point arbitrarily chosen in the region, be $x+dx, y+dy, z+dz, v+dv$. When the parameters at P are p, q, r, and those at Q are $p+dp, q+dq, r+dr$, the limiting direction PQ is determined by means of dp, dq, dr. The flat, tangential at P to the region, is given by the equation

$$\Sigma(\bar{x}-x)X=0.$$

The tangential flat to the region at Q is given by the equation

$$\Sigma\{\bar{x}-(x+dx)\}(X+dX)=0.$$

Now Q is adjacent to P so that the limiting direction PQ lies in the tangential flat at Q; thus

$$\Sigma X\,dx=0,$$

and the equation of the tangential flat at Q tends to the form

$$\Sigma(\bar{x}-x)(X+dX)=0,$$

when quantities of the second order are neglected in comparison with those of the first. Accordingly, the equations of the plane S_{PQ} can be taken

$$\Sigma(\bar{x}-x)X=0, \quad \Sigma(\bar{x}-x)(X+dX)=0;$$

and when these are combined, we can take them in the form

$$\Sigma(\bar{x}-x)X=0, \quad \Sigma(\bar{x}-x)dX=0.$$

Now

$$dX=X_1dp+X_2dq+X_3dr=0;$$

and therefore *the equations of a plane through the point P, conjugate to the direction dp, dq, dr*, are

$$\left.\begin{aligned}\Sigma(\bar{x}-x)X=0\\ \Sigma(\bar{x}-x)(X_1dp+X_2dq+X_3dr)=0\end{aligned}\right\}.$$

Let δp, δq, δr, determine any direction which, passing through P, lies in this plane, so that the line

$$\frac{\bar{x}-x}{x_1\delta p+x_2\delta q+x_3\delta r}=\frac{\bar{y}-y}{y_1\delta p+y_2\delta q+y_3\delta r}=\frac{\bar{z}-z}{z_1\delta p+z_2\delta q+z_3\delta r}=\frac{\bar{v}-v}{v_1\delta p+v_2\delta q+v_3\delta r}$$

lies in the plane: then the equations

$$\Sigma X(x_1\delta p+x_2\delta q+x_3\delta r)=0,$$

$$\Sigma(x_1\delta p+x_2\delta q+x_3\delta r)(X_1dp+X_2dq+X_3dr)=0,$$

must be satisfied. The former is satisfied unconditionally, as is to be expected because the new direction necessarily lies in the tangential flat at P. The second equation is a homogeneous lineo-linear relation in dp, dq, dr; δp, δq, δr. For the coefficients in this relation, we have

$$\Sigma x_1X_1=-L, \quad \Sigma x_2X_2=-N, \quad \Sigma x_3X_3=-J,$$

$$\Sigma x_1X_2=\Sigma x_2X_1=-M, \quad \Sigma x_1X_3=\Sigma x_3X_1=-K, \quad \Sigma x_2X_3=\Sigma x_3X_2=-I;$$

and therefore the relation becomes

$$\begin{aligned}L\,dp\,\delta p+M(dp\,\delta q+dq\,\delta p)+K(dp\,\delta r+dr\,\delta p)\\ +N\,dq\,\delta q+I(dq\,\delta r+dr\,\delta q)+J\,dr\,\delta r=0,\end{aligned}$$

which accordingly is the condition that the direction δp, δq, δr, should be conjugate to dp, dq, dr. As the condition is unaltered by the interchange of the two sets of symbols specifying the two directions, the relation is reciprocal; it is the condition that the two directions dp, dq, dr; δp, δq, δr; shall be conjugate to one another.

We note, at once, that the lines of curvature through a point P in the region are conjugate to one another in pairs. For, in § 283, where the perpendicularity of these lines was established, it was proved that two such lines with directions p_1', q_1', r_1', and p_2', q_2', r_2', are such that

$$\begin{aligned}Lp_1'p_2'+M(p_1'q_2'+q_1'p_2')+K(p_1'r_2'+r_1'p_2')\\ +Nq_1'q_2'+I(q_1'r_2'+r_1'q_2')+Jr_1'r_2'=0;\end{aligned}$$

and this condition shews that the two lines are conjugate. Moreover, the relation is satisfied for each pair of the lines of curvature.

Orientation of a plane conjugate to a direction.

288. Any direction $\delta p, \delta q, \delta r$, conjugate to a given direction dp, dq, dr, must satisfy the relation

$$\Sigma L\, dp\, \delta p = 0.$$

Let $\delta' p, \delta' q, \delta' r$, be another direction, conjugate to the same direction dp, dq, dr, so that the relation

$$\Sigma L\, dp\, \delta' p = 0$$

is satisfied; and let the two directions $\delta p, \delta q, \delta r$, and $\delta' p, \delta' q, \delta' r$, be conjugate to one another, so that the relation

$$\Sigma L\, \delta p\, \delta' p = 0$$

is satisfied. Then we have

$$\frac{L\,dp + M\,dq + K\,dr}{\delta q\,\delta' r - \delta r\,\delta' q} = \frac{M\,dp + N\,dq + I\,dr}{\delta r\,\delta' p - \delta p\,\delta' r} = \frac{K\,dp + I\,dq + J\,dr}{\delta p\,\delta' q - \delta q\,\delta' p}.$$

Now the two directions $\delta p, \delta q, \delta r$, and $\delta' p, \delta' q, \delta' r$, lie in the plane S_{PQ} conjugate to the direction dp, dq, dr; so, writing

$$\xi_1 = \delta q\,\delta' r - \delta r\,\delta' q, \quad \eta_1 = \delta r\,\delta' p - \delta p\,\delta' r, \quad \zeta_1 = \delta p\,\delta' q - \delta q\,\delta' p,$$

we call ξ_1, η_1, ζ_1, the *regional orientation-coordinates** of the plane. Hence

$$\frac{L\,dp + M\,dq + K\,dr}{\xi_1} = \frac{M\,dp + N\,dq + I\,dr}{\eta_1} = \frac{K\,dp + I\,dq + J\,dr}{\zeta_1},$$

and therefore also

$$\frac{dp}{l\xi_1 + m\eta_1 + k\zeta_1} = \frac{dq}{m\xi_1 + n\eta_1 + i\zeta_1} = \frac{dr}{k\xi_1 + i\eta_1 + j\zeta_1}.$$

We thus have the regional orientation-coordinates of the plane, conjugate to a direction dp, dq, dr. The magnitudes dp, dq, dr, determine the direction conjugate to a plane which has ξ_1, η_1, ζ_1, for its regional orientation-coordinates.

It follows at once that, if the three directions

$$dp, dq, dr; \quad \delta p, \delta q, \delta r; \quad \delta' p, \delta' q, \delta' r,$$

be conjugate to one another in pairs, and if

$$\xi_1, \eta_1, \zeta_1; \quad \xi_2, \eta_2, \zeta_2; \quad \xi_3, \eta_3, \zeta_3,$$

are the regional orientation-coordinates of the planes respectively conjugate to these directions,

$$\frac{\delta p}{l\xi_2 + m\eta_2 + k\zeta_2} = \frac{\delta q}{m\xi_2 + n\eta_2 + i\zeta_2} = \frac{\delta r}{k\xi_2 + i\eta_2 + j\zeta_2},$$

$$\frac{\delta' p}{l\xi_3 + m\eta_3 + k\zeta_3} = \frac{\delta' q}{m\xi_3 + n\eta_3 + i\zeta_3} = \frac{\delta' r}{k\xi_3 + i\eta_3 + j\zeta_3}.$$

* The association of ξ_1, η_1, ζ_1, with the direction-cosines of the normal to a surface in the region will appear later (§ 340).

If, then, three directions

$$p_\alpha, q_\alpha, r_\alpha; \quad p_\beta, q_\beta, r_\beta; \quad p_\gamma, q_\gamma, r_\gamma;$$

(where $\alpha, \beta, \gamma, = 1, 2, 3$, in any cyclical order) are conjugate to one another,

$$\frac{p_\alpha}{\begin{vmatrix} l, & m, & k \\ p_\beta, & q_\beta, & r_\beta \\ p_\gamma, & q_\gamma, & r_\gamma \end{vmatrix}} = \frac{q_\alpha}{\begin{vmatrix} m, & n, & \imath \\ p_\beta, & q_\beta, & r_\beta \\ p_\gamma, & q_\gamma, & r_\gamma \end{vmatrix}} = \frac{r_\alpha}{\begin{vmatrix} k, & i, & j \\ p_\beta, & q_\beta, & r_\beta \\ p_\gamma, & q_\gamma, & r_\gamma \end{vmatrix}}.$$

Similarly, for the regional orientation-coordinates of three planes determined by the pairs from three conjugate lines, we have

$$\frac{\xi_\alpha}{\begin{vmatrix} L, & M, & K \\ \xi_\beta, & \eta_\beta, & \zeta_\beta \\ \xi_\gamma, & \eta_\gamma, & \zeta_\gamma \end{vmatrix}} = \frac{\eta_\alpha}{\begin{vmatrix} M, & N, & I \\ \xi_\beta, & \eta_\beta, & \zeta_\beta \\ \xi_\gamma, & \eta_\gamma, & \zeta_\gamma \end{vmatrix}} = \frac{\zeta_\alpha}{\begin{vmatrix} K, & I, & J \\ \xi_\beta, & \eta_\beta, & \zeta_\beta \\ \xi_\gamma, & \eta_\gamma, & \zeta_\gamma \end{vmatrix}}.$$

Three such planes, determined in turns by the pairs from three conjugate lines, are said to be conjugate to one another; but the formal use of three conjugate planes is less frequent than the use of three conjugate lines.

Conditions that the parametric curves are conjugate.

289. The conditions that the parametric curves of reference shall be conjugate in pairs (or, more simply, be conjugate lines) are immediate. In order that the curves

$$dp = \text{variable}, \quad dq = 0, \quad dr = 0, \quad \text{and} \quad \delta p = 0, \quad \delta q = \text{variable}, \quad \delta r = 0,$$

may be conjugate, we must have

$$M\,dp\,\delta q = 0,$$

that is, $M = 0$: in order that the curves

$$\delta p = 0, \quad \delta q = \text{variable}, \quad \delta r = 0, \quad \text{and} \quad \delta' p = 0, \quad \delta' q = 0, \quad \delta' r = \text{variable},$$

may be conjugate, we must have

$$I\,\delta q\,\delta' r = 0,$$

that is, $I = 0$: and in order that the curves

$$dp = \text{variable}, \quad dq = 0, \quad dr = 0, \quad \text{and} \quad \delta' p = 0, \quad \delta' q = 0, \quad \delta' r = \text{variable},$$

may be conjugate, we must have

$$K\,dp\,\delta' r = 0,$$

that is, $K = 0$. Thus the conditions are

$$I = 0, \quad K = 0, \quad M = 0.$$

COROLLARY. We have seen that the parametric curves are perpendicular to one another if

$$F = 0, \quad G = 0, \quad H = 0,$$

and that the lines of curvature are perpendicular to one another and are also conjugate to one another. Thus, as before (§ 284), *the conditions that the parametric curves shall be lines of curvature* are

$$F=0, \quad G=0, \quad H=0, \quad I=0, \quad K=0, \quad M=0.$$

Consequently, when these conditions are satisfied, the three principal measures of linear curvature of the region are

$$\frac{1}{\alpha}=\frac{L}{A}, \quad \frac{1}{\beta}=\frac{N}{B}, \quad \frac{1}{\gamma}=\frac{J}{C}.$$

Ex. It is easy to verify that the parametric curves of reference are conjugate in a region

$$\left.\begin{aligned} x&=A\,(p-a)^l\,(q-a)^m\,(r-a)^n\\ y&=B\,(p-b)^l\,(q-b)^m\,(r-b)^n\\ z&=C\,(p-c)^l\,(q-c)^m\,(r-c)^n\\ v&=D\,(p-d)^l\,(q-d)^m\,(r-d)^n \end{aligned}\right\},$$

where A, B, C, D; a, b, c, d; and l, m, n; are constants. For we easily find

$$\begin{aligned} (q-p)\,\theta_{12}&=m\theta_1-l\theta_2,\\ (r-q)\,\theta_{23}&=n\theta_2-m\theta_3,\\ (p-r)\,\theta_{31}&=l\theta_3-n\theta_1, \end{aligned}$$

for $\theta=x, y, z, v$, in turn; and therefore

$$I=\Sigma Xx_{23}=0, \quad K=\Sigma Xx_{31}=0, \quad M=\Sigma Xx_{12}=0,$$

shewing the property as stated.

When $l=m=n$, we have the desmic region

$$\left(\frac{x}{A}\right)^{\frac{1}{n}}(b-c)\,(c-d)\,(d-b)$$
$$-\left(\frac{y}{B}\right)^{\frac{1}{n}}(c-d)\,(d-a)\,(a-c)$$
$$+\left(\frac{z}{C}\right)^{\frac{1}{n}}(d-a)\,(a-b)\,(b-d)$$
$$-\left(\frac{v}{D}\right)^{\frac{1}{n}}(a-b)\,(b-c)\,(c-a)=(a-b)\,(b-c)\,(c-a)\,(a-d)\,(b-d)\,(c-d).$$

When $l, m, n=\frac{3}{2}, \frac{1}{2}, \frac{1}{2}$, in successive arrangements, we have the three separate volumes of the region of centres belonging to an ovoidal region (§ 394). In this instance, the parametric lines are lines of curvature for the original ovoid: and thus there is a verification of the property (*l.c.*) that the lines, in the region of centres corresponding to the lines of curvature in the ovoid, are conjugate lines in that central region.

Asymptotic directions: inflexional surface.

290. A direction, which is self-conjugate in a region, is called an *asymptotic* direction. Thus the parameters of an asymptotic direction satisfy the equation

$$\Sigma L\,dp^2=0.$$

Consequently the curvature of a normal section along an asymptotic direction is zero.

As there is only a single homogeneous relation to be satisfied between the parameters in order that a direction may be asymptotic, it follows that these asymptotic directions at a point generate a surface. Now, as the tangential flat at any point x, y, z, v, in the region has the equation

$$\Sigma (\bar{x} - x) X = 0,$$

the perpendicular from a contiguous point $x + dx,\ y + dy,\ z + dz,\ v + dv$, drawn upon the flat, is

$$\Sigma X\, dx,$$

$$= \tfrac{1}{2} \Sigma L\, dp^2.$$

Thus any tangent line to the region, drawn in any direction, meets the region in two consecutive points; a tangent line to the region, drawn in an asymptotic direction, meets the region in three consecutive points, and therefore behaves as an inflexional tangent. But every such direction lies in the surface having the equation

$$\Sigma L\, dp^2 = 0;$$

consequently, such a surface may be called an *inflexional surface*, and it is, at any point, the locus of asymptotic directions through that point of the region.

We see at once that *the conditions, to be satisfied if the parametric curves are asymptotic*, are

$$L = 0, \quad N = 0, \quad J = 0.$$

The curvature indicatrix.

291. Now suppose that the region has been referred to parametric curves which are perpendicular to one another: that is, we may take $F = 0$, $G = 0$, $H = 0$, so that for every direction in the region we have

$$Ap'^2 + Bq'^2 + Cr'^2 = 1.$$

Now let $p'A^{\frac{1}{2}} = \lambda$, $q'B^{\frac{1}{2}} = \mu$, $r'C^{\frac{1}{2}} = \nu$, so that the permanent relation becomes

$$\lambda^2 + \mu^2 + \nu^2 = 1,$$

and λ, μ, ν, may be taken as direction-cosines of a direction in a region referred to three axes at right angles to one another. The curvature of any normal section of the region is given by

$$\frac{1}{\rho} = Lp'^2 + 2Mp'q' + 2Kp'r' + Nq'^2 + 2Iq'r' + Jr'^2$$

$$= \frac{L}{A}\lambda^2 + 2\frac{M}{(AB)^{\frac{1}{2}}}\lambda\mu + 2\frac{K}{(AC)^{\frac{1}{2}}}\lambda\nu + \frac{N}{B}\mu^2 + 2\frac{I}{(BC)^{\frac{1}{2}}}\mu\nu + \frac{J}{C}\nu^2.$$

Let a length R be measured in the direction λ, μ, ν, and write

$$R\lambda = \xi, \quad R\mu = \eta, \quad R\nu = \zeta;$$

and take the quadric surface

$$\frac{L}{A}\xi^2+2\frac{M}{(AB)^{\frac{1}{2}}}\xi\eta+2\frac{K}{(AC)^{\frac{1}{2}}}\xi\zeta+\frac{N}{B}\eta^2+2\frac{I}{(BC)^{\frac{1}{2}}}\eta\zeta+\frac{J}{C}\zeta^2=\theta,$$

referred to the three perpendicular axes. Then we have

$$\frac{1}{\rho}=\frac{\theta}{R^2},$$

so that the surface can be regarded as an *indicatrix* for the curvature of normal sections of the region. Associated with the inflexional surface, there is an inflexional quadric cone; referred to these axes, its equation is

$$\frac{L}{A}\xi^2+2\frac{M}{(AB)^{\frac{1}{2}}}\xi\eta+2\frac{K}{(AC)^{\frac{1}{2}}}\xi\zeta+\frac{N}{B}\eta^2+2\frac{I}{(BC)^{\frac{1}{2}}}\eta\zeta+\frac{J}{C}\zeta^2=0:$$

that is, the inflexional cone through any point is the asymptotic cone of the curvature indicatrix at the point.

The customary properties of a central quadric

$$\frac{L}{\theta A}\xi^2+2\frac{M}{\theta(AB)^{\frac{1}{2}}}\xi\eta+2\frac{K}{\theta(AC)^{\frac{1}{2}}}\xi\zeta+\frac{N}{\theta B}\eta^2+2\frac{I}{\theta(BC)^{\frac{1}{2}}}\eta\zeta+\frac{J}{\theta C}\zeta^2=1,$$

as regards principal axes in direction and magnitude, are given in text-books on solid geometry. The directions of these principal axes give the directions of the lines of curvature in the region. The magnitudes of the principal axes of the indicatrix, taken in connection with the equation

$$\frac{\theta}{R^2}=\frac{1}{\rho},$$

give the principal linear curvatures of the region at the point. The three customary 'invariants' of the quadric, that is, those combinations of the coefficients in the equation which remain invariantive under change of the perpendicular coordinate axes to any other similar set of axes, give the three measures of curvature of the region at the point. Similarly for relations of conjugacy between directions of lines, and between a line and a plane: indeed, all the properties of the central quadric, which are of this descriptive (non-topographical) type, find an interpretation in corresponding properties of the region, as regards curvature, and certain intrinsic directions and orientations in the region.

Ex. 1. *Required the principal radii of curvature at any point of the region*

$$\frac{x^2+y^2}{a}+\frac{z^2+v^2}{c}=1.$$

Any point in the region can be represented parametrically by the coordinate-values

$$x=a^{\frac{1}{2}}\cos p\cos r,\quad y=a^{\frac{1}{2}}\sin p\cos r,\quad z=c^{\frac{1}{2}}\cos q\sin r,\quad v=c^{\frac{1}{2}}\sin q\sin r.$$

The direction-cosines X, Y, Z, V of the inward-drawn normal to the region at the point are

$$\Theta X=-\frac{x}{a},\quad \Theta Y=-\frac{y}{a},\quad \Theta Z=-\frac{z}{c},\quad \Theta V=-\frac{v}{c},$$

where

$$\Theta^2 = \frac{x^2+y^2}{a^2} + \frac{z^2+v^2}{c^2} = \frac{1}{ab}(a\sin^2 r + c\cos^2 r).$$

The primary magnitudes are

$$A = \Sigma x_1^2 = a\cos^2 r, \quad B = \Sigma x_2^2 = c\sin^2 r, \quad C = \Sigma x_3^2 = a\sin^2 r + b\cos^2 r,$$
$$F = 0, \quad G = 0, \quad H = 0.$$

For the secondary magnitudes, we have

$$\Theta L = -\Sigma \frac{x}{a} x_{11} = \cos^2 r, \quad \Theta N = -\Sigma \frac{x}{a} x_{22} = \sin^2 r, \quad \Theta J = -\Sigma \frac{x}{a} x_{33} = 1,$$
$$l = 0, \quad K = 0, \quad M = 0.$$

Thus the parametric curves are the lines of curvature of the region. Consequently, the principal radii of curvature at the point are

$$\frac{1}{\alpha} = \frac{E}{A} = \left\{\frac{b}{a}\frac{1}{a\sin^2 r + b\cos^2 r}\right\}^{\frac{1}{2}},$$
$$\frac{1}{\beta} = \frac{N}{B} = \left\{\frac{a}{b}\frac{1}{a\sin^2 r + b\cos^2 r}\right\}^{\frac{1}{2}},$$
$$\frac{1}{\gamma} = \frac{J}{C} = \left\{\frac{ab}{(a\sin^2 r + b\cos^2 r)^3}\right\}^{\frac{1}{2}}.$$

Ex. 2. Prove that, in the sphero-cylindrical region

$$x^2+y^2+z^2 = a^2,$$

two of the principal curvatures are equal to $1/a$ and the remaining principal curvature is zero: and that, in the sphero-conical region

$$x^2+y^2+z^2 = v^2\cot^2\alpha,$$

two of the principal curvatures are equal to $\dfrac{\sin^2\alpha}{v\cos\alpha}$ and the remaining principal curvature is zero.

Ex. 3. Find the principal linear curvatures at any point of the region

$$x^2+y^2+z^2 = 2f(v),$$

where f is a given function of v; and obtain the measure of solid curvature at any point of the sphero-paraboloidal region

$$x^2+y^2+z^2 = 4cv.$$

Ex. 4. A dicyclic conical region is given by the equation

$$z^2+v^2 = (x^2+y^2)\tan^2\alpha.$$

Shew that its three principal curvatures are

$$\frac{\sin\alpha}{r}, \quad -\frac{\cos^2\alpha}{r\sin\alpha}, \quad 0,$$

where $r^2 = x^2+y^2$.

Ex. 5. At any point P of the plane curve $y=f(x)$, the radius of plane curvature is ρ; the normal at P meets the axis of x in G; and a line, drawn through the origin O parallel to the normal at P, meets the ordinate of P in H. A region is generated by the complete revolution of the curve round the orthogonal planes of x and v, and of y and z. Prove that the principal radii of curvature of the generated region are

$$\rho, \quad PG, \quad OH.$$

Magnitudes of the third order.

292. It proves desirable to have fundamental quantities of the third order belonging to the region, that is to say, ultimately involving third-order derivatives of its point-coordinates in the quadruple space taken with respect to the parameters of the region. Initially, however, such quantities arise in connection with curvature properties of a geodesic; and therefore, for the immediate purpose, we must anticipate the discussion of geodesics as given in the succeeding chapter. One set of magnitudes of the third order is derived from a consideration of the variation of the circular curvature along the geodesic, as given by the customary formula

$$\frac{1}{\rho} = (L, N, J, I, K, M \mathbin{\text{\textbigcircle}} p', q', r')^2.$$

We have, along any curve,

$$\begin{aligned}\frac{d}{ds}\left(\frac{1}{\rho}\right) = {} & (L_1, N_1, J_1, I_1, K_1, M_1 \mathbin{\text{\textbigcircle}} p', q', r')^2 p' \\ & + (L_2, N_2, J_2, I_2, K_2, M_2 \mathbin{\text{\textbigcircle}} p', q', r')^2 q' \\ & + (L_3, N_3, J_3, I_3, K_3, M_3 \mathbin{\text{\textbigcircle}} p', q', r')^2 r' \\ & + 2(Lp' + Mq' + Kr')p'' + 2(Mp' + Nq' + Ir')q'' + 2(Kp' + Iq' + Jr')r''.\end{aligned}$$

But, along the geodesic through p', q', r', we have (§ 298, *post*)

$$p'' = -(\Gamma \mathbin{\text{\textbigcircle}} p', q', r')^2, \quad q'' = -(\Delta \mathbin{\text{\textbigcircle}} p', q', r')^2, \quad r'' = -(\Theta \mathbin{\text{\textbigcircle}} p', q', r')^2,$$

and the first derivatives of L, N, J, I, K, M, are connected by the eight Codazzi-Mainardi relations of § 279

$$\left.\begin{aligned} L_2 + \Gamma_{11}M + \Delta_{11}N + \Theta_{11}I &= M_1 + \Gamma_{12}L + \Delta_{12}M + \Theta_{12}K \\ L_3 + \Gamma_{11}K + \Delta_{11}I + \Theta_{11}J &= K_1 + \Gamma_{13}L + \Delta_{13}M + \Theta_{13}K \end{aligned}\right\},$$

$$\left.\begin{aligned} N_3 + \Gamma_{22}K + \Delta_{22}I + \Theta_{22}J &= I_2 + \Gamma_{23}M + \Delta_{23}N + \Theta_{23}I \\ N_1 + \Gamma_{22}L + \Delta_{22}M + \Theta_{22}K &= M_2 + \Gamma_{12}M + \Delta_{12}N + \Theta_{12}I \end{aligned}\right\},$$

$$\left.\begin{aligned} J_1 + \Gamma_{33}L + \Delta_{33}M + \Theta_{33}K &= K_3 + \Gamma_{13}K + \Delta_{13}I + \Theta_{13}J \\ J_2 + \Gamma_{33}M + \Delta_{33}N + \Theta_{33}I &= I_3 + \Gamma_{23}K + \Delta_{23}I + \Theta_{23}J \end{aligned}\right\},$$

$$\left.\begin{aligned} I_1 + \Gamma_{23}L + \Delta_{23}M + \Theta_{23}K &= K_2 + \Gamma_{13}M + \Delta_{13}N + \Theta_{13}I \\ &= M_3 + \Gamma_{12}K + \Delta_{12}I + \Theta_{12}J \end{aligned}\right\}.$$

When the values of p'', q'', r'', are substituted in the foregoing expression for the variation of the curvature, it becomes a homogeneous cubic ternary form in p', q', r', which may be written in the forms

$$(e \mathbin{\text{\textbigcircle}} p', q', r')^3, = \Sigma \frac{3!}{t_1!\,t_2!\,t_3!} e_{t_1 t_2 t_3} p'^{t_1} q'^{t_2} r'^{t_3},$$

for all the positive integer values of t_1, t_2, t_3, (zero included), such that $t_1 + t_2 + t_3 = 3$. Accordingly, we have

$$\frac{d}{ds}\left(\frac{1}{\rho}\right) = (e \mathbin{\text{\textbigcircle}} p', q', r')^3.$$

The ten literal coefficients in the form, when regard is paid to the Codazzi-Mainardi relations, are *

$$
\begin{aligned}
e_{300} &= L_1 - 2(L\Gamma_{11} + M\Delta_{11} + K\Theta_{11}),\\
e_{030} &= N_2 - 2(M\Gamma_{22} + N\Delta_{22} + I\Theta_{22}),\\
e_{003} &= J_3 - 2(K\Gamma_{33} + I\Delta_{33} + J\Theta_{33}),\\
e_{210} &= L_2 - 2(L\Gamma_{12} + M\Delta_{12} + K\Theta_{12})\\
&= M_1 - (M\Gamma_{11} + N\Delta_{11} + I\Theta_{11}) - (L\Gamma_{12} + M\Delta_{12} + K\Theta_{12}),\\
e_{201} &= L_3 - 2(L\Gamma_{13} + M\Delta_{13} + K\Theta_{13})\\
&= K_1 - (K\Gamma_{11} + I\Delta_{11} + J\Theta_{11}) - (L\Gamma_{13} + M\Delta_{13} + K\Theta_{13}),\\
e_{120} &= N_1 - 2(M\Gamma_{12} + N\Delta_{12} + I\Theta_{12})\\
&= M_2 - (L\Gamma_{22} + M\Delta_{22} + K\Theta_{22}) - (M\Gamma_{12} + N\Delta_{12} + I\Theta_{12}),\\
e_{021} &= N_3 - 2(M\Gamma_{23} + N\Delta_{23} + I\Theta_{23})\\
&= I_2 - (K\Gamma_{22} + I\Delta_{22} + J\Theta_{22}) - (M\Gamma_{23} + N\Delta_{23} + I\Theta_{23}),\\
e_{102} &= J_1 - 2(K\Gamma_{13} + I\Delta_{13} + J\Theta_{13})\\
&= K_3 - (L\Gamma_{33} + M\Delta_{33} + K\Theta_{33}) - (K\Gamma_{13} + I\Delta_{13} + J\Theta_{13}),\\
e_{012} &= J_2 - 2(K\Gamma_{23} + I\Delta_{23} + J\Theta_{23})\\
&= I_3 - (K\Gamma_{22} + I\Delta_{22} + J\Theta_{22}) - (K\Gamma_{23} + I\Delta_{23} + J\Theta_{23}),\\
e_{111} &= I_1 - (M\Gamma_{13} + N\Delta_{13} + I\Theta_{13}) - (K\Gamma_{12} + I\Delta_{12} + J\Theta_{12})\\
&= K_2 - (K\Gamma_{12} + I\Delta_{12} + J\Theta_{12}) - (L\Gamma_{23} + M\Delta_{23} + K\Theta_{23})\\
&= M_3 - (L\Gamma_{23} + M\Delta_{23} + K\Theta_{23}) - (M\Gamma_{13} + N\Delta_{13} + I\Theta_{13}).
\end{aligned}
$$

293. The first derivatives of L, N, J, I, K, M, are thus expressible in terms of the quantities e of the third order. It is convenient to have also the expressions for the derivatives of the first minors l, n, j, i, k, m, of the determinant Υ; they are as follows:

$$
\left.
\begin{aligned}
l_1 - 2l\frac{\Omega_1}{\Omega} &= Je_{120} - 2Ie_{111} + Ne_{102} - 2(l\Gamma_{11} + m\Gamma_{12} + k\Gamma_{13})\\
l_2 - 2l\frac{\Omega_2}{\Omega} &= Je_{030} - 2Ie_{021} + Ne_{012} - 2(l\Gamma_{12} + m\Gamma_{22} + k\Gamma_{23})\\
l_3 - 2l\frac{\Omega_3}{\Omega} &= Je_{021} - 2Ie_{012} + Ne_{003} - 2(l\Gamma_{13} + m\Gamma_{23} + k\Gamma_{33})
\end{aligned}
\right\},
$$

$$
\left.
\begin{aligned}
n_1 - 2n\frac{\Omega_1}{\Omega} &= Je_{300} - 2Ke_{201} + Le_{102} - 2(m\Delta_{11} + n\Delta_{12} + i\Delta_{13})\\
n_2 - 2n\frac{\Omega_2}{\Omega} &= Je_{210} - 2Ke_{111} + Le_{012} - 2(m\Delta_{12} + n\Delta_{22} + i\Delta_{23})\\
n_3 - 2n\frac{\Omega_3}{\Omega} &= Je_{201} - 2Ke_{102} + Le_{003} - 2(m\Delta_{13} + n\Delta_{23} + i\Delta_{33})
\end{aligned}
\right\},
$$

* Professor Neville points out that, with the notation on p. 30, each expression for a coefficient e is of the form

$$(\eta_{rs})_t - (\eta_{r1}\Gamma_{st} + \eta_{r2}\Delta_{st} + \eta_{r3}\Theta_{st}) - (\eta_{s1}\Gamma_{rt} + \eta_{s2}\Delta_{rt} + \eta_{s3}\Theta_{rt}),$$

where $(\eta_{rs})_t$ denotes the derivative of η_{rs} with respect to p, q, r, according as $t = 1, 2, 3$: also that, in virtue of the Codazzi-Mainardi relations, these last forms are symmetrical in r, s, t.

$$\left.\begin{aligned} j_1 - 2j\,\frac{\Omega_1}{\Omega} &= Ne_{300} - 2Me_{210} + Le_{120} - 2(k\Theta_{11} + i\Theta_{12} + j\Theta_{13}) \\ j_2 - 2j\,\frac{\Omega_2}{\Omega} &= Ne_{210} - 2Me_{120} + Le_{030} - 2(k\Theta_{12} + i\Theta_{22} + j\Theta_{23}) \\ j_3 - 2j\,\frac{\Omega_3}{\Omega} &= Ne_{201} - 2Me_{111} + Le_{021} - 2(k\Theta_{13} + i\Theta_{23} + j\Theta_{33}) \end{aligned}\right\},$$

$$\left.\begin{aligned} i_1 - 2i\,\frac{\Omega_1}{\Omega} &= -Le_{111} + Ke_{210} + Me_{201} - Ie_{300} - (k\Delta_{11} + i\Delta_{12} + j\Delta_{13}) - (m\Theta_{11} + n\Theta_{12} + i\Theta_{13}) \\ i_2 - 2i\,\frac{\Omega_2}{\Omega} &= -Le_{021} + Ke_{120} + Me_{111} - Ie_{210} - (k\Delta_{12} + i\Delta_{22} + j\Delta_{23}) - (m\Theta_{12} + n\Theta_{22} + i\Theta_{23}) \\ i_3 - 2i\,\frac{\Omega_3}{\Omega} &= -Le_{012} + Ke_{111} + Me_{102} - Ie_{201} - (k\Delta_{13} + i\Delta_{23} + j\Delta_{33}) - (m\Theta_{13} + n\Theta_{23} + i\Theta_{33}) \end{aligned}\right\},$$

$$\left.\begin{aligned} k_1 - 2k\,\frac{\Omega_1}{\Omega} &= -Ne_{201} + Me_{111} + Ie_{210} - Ke_{120} - (l\Theta_{11} + m\Theta_{12} + k\Theta_{13}) - (k\Gamma_{11} + i\Gamma_{12} + j\Gamma_{13}) \\ k_2 - 2k\,\frac{\Omega_2}{\Omega} &= -Ne_{111} + Me_{021} + Ie_{120} - Ke_{030} - (l\Theta_{12} + m\Theta_{22} + k\Theta_{23}) - (k\Gamma_{12} + i\Gamma_{22} + j\Gamma_{23}) \\ k_3 - 2k\,\frac{\Omega_3}{\Omega} &= -Ne_{102} + Me_{012} + Ie_{111} - Ke_{021} - (l\Theta_{13} + m\Theta_{23} + k\Theta_{33}) - (k\Gamma_{13} + i\Gamma_{23} + j\Gamma_{33}) \end{aligned}\right\},$$

$$\left.\begin{aligned} m_1 - 2m\,\frac{\Omega_1}{\Omega} &= -Je_{210} + Ie_{201} + Ke_{111} - Me_{102} - (m\Gamma_{11} + n\Gamma_{12} + i\Gamma_{13}) - (l\Delta_{11} + m\Delta_{12} + k\Delta_{13}) \\ m_2 - 2m\,\frac{\Omega_2}{\Omega} &= -Je_{120} + Ie_{111} + Ke_{021} - Me_{012} - (m\Gamma_{12} + n\Gamma_{22} + i\Gamma_{23}) - (l\Delta_{12} + m\Delta_{22} + k\Delta_{23}) \\ m_3 - 2m\,\frac{\Omega_3}{\Omega} &= -Je_{111} + Ie_{102} + Ke_{012} - Me_{003} - (m\Gamma_{13} + n\Gamma_{23} + i\Gamma_{33}) - (l\Delta_{13} + m\Delta_{23} + k\Delta_{33}) \end{aligned}\right\}.$$

Third derivatives of the point-coordinates.

294. It is possible to express, in terms of these quantities of the third order, the third derivatives of the coordinates x, y, z, v, with respect to the parameters. We write

$$\frac{\partial^{\alpha+\beta+\gamma}\theta}{\partial p^\alpha \partial q^\beta \partial r^\gamma} = \theta_{\alpha\beta\gamma},$$

for $\theta = x, y, z, v$, and for positive integer values of α, β, γ, such that $\alpha+\beta+\gamma > 2$; the briefer notation for the cases $\alpha+\beta+\gamma \leqslant 2$ is given in § 262.

For convenience of reference, we repeat the formulæ (§ 274)

$$\left.\begin{aligned} -\Omega^2 X_1 &= x_1(aL + hM + gK) + x_2(hL + bM + fK) + x_3(gL + fM + cK) \\ &= x_1\lambda_1 + x_2\mu_1 + x_3\nu_1 \\ -\Omega^2 X_2 &= x_1(aM + hN + gI) + x_2(hM + bN + fI) + x_3(gM + fN + cI) \\ &= x_1\lambda_2 + x_2\mu_2 + x_3\nu_2 \\ -\Omega^2 X_3 &= x_1(aK + hI + gJ) + x_2(hK + bI + fJ) + x_3(gK + fI + cJ) \\ &= x_1\lambda_3 + x_2\mu_3 + x_3\nu_3 \end{aligned}\right\},$$

and so for the derivatives of Y, Z, V; and we proceed from the formulæ (§ 275) expressing the second derivatives of x, y, z, v. We have

$$x_{300} = \frac{\partial}{\partial p}(x_{11}) = \frac{\partial}{\partial p}(XL + x_1\Gamma_{11} + x_2\Delta_{11} + x_3\Theta_{11})$$
$$= XL_1 + LX_1 + x_{11}\Gamma_{11} + x_{12}\Delta_{11} + x_{13}\Theta_{11} + x_1\frac{\partial\Gamma_{11}}{\partial p} + x_2\frac{\partial\Delta_{11}}{\partial p} + x_3\frac{\partial\Theta_{11}}{\partial p}.$$

When substitution takes place for L_1, X_1, x_{11}, x_{12}, x_{13}, we find

$$x_{300} = XE_{300} + x_1R_{300} + x_2S_{300} + x_3T_{300},$$

where

$$E_{300} = e_{300} + 3(L\Gamma_{11} + M\Delta_{11} + K\Theta_{11}),$$
$$\left.\begin{aligned} R_{300} &= \frac{\partial\Gamma_{11}}{\partial p} + (\Gamma_{11}, \Delta_{11}, \Theta_{11}\text{X}\Gamma_{11}, \Gamma_{12}, \Gamma_{13}) - \frac{1}{\Omega^2}L\lambda_1 \\ S_{300} &= \frac{\partial\Delta_{11}}{\partial p} + (\Gamma_{11}, \Delta_{11}, \Theta_{11}\text{X}\Delta_{11}, \Delta_{12}, \Delta_{13}) - \frac{1}{\Omega^2}L\mu_1 \\ T_{300} &= \frac{\partial\Theta_{11}}{\partial p} + (\Gamma_{11}, \Delta_{11}, \Theta_{11}\text{X}\Theta_{11}, \Theta_{12}, \Theta_{13}) - \frac{1}{\Omega^2}L\nu_1 \end{aligned}\right\}.$$

Alternative forms, in terms of the second derivatives of A, B, C, F, G, H, can be given to R_{300}, S_{300}, T_{300}, and to all the succeeding quantities R, S, T, in these expressions, by means of the formulæ in § 277. The values of y_{300}, z_{300}, v_{300}, are

$$y_{300} = YE_{300} + y_1R_{300} + y_2S_{300} + y_3T_{300},$$
$$z_{300} = ZE_{300} + z_1R_{300} + z_2S_{300} + z_3T_{300},$$
$$v_{300} = VE_{300} + v_1R_{300} + v_2S_{300} + v_3T_{300}.$$

Similarly for all the other derivatives of x, y, z, v, of the third order with respect to p, q, r; the results can be embodied in the general formulæ

$$\left.\begin{aligned} x_{stu} &= XE_{stu} + x_1R_{stu} + x_2S_{stu} + x_3T_{stu} \\ y_{stu} &= YE_{stu} + y_1R_{stu} + y_2S_{stu} + y_3T_{stu} \\ z_{stu} &= ZE_{stu} + z_1R_{stu} + z_2S_{stu} + z_3T_{stu} \\ v_{stu} &= VE_{stu} + v_1R_{stu} + v_2S_{stu} + v_3T_{stu} \end{aligned}\right\},$$

for all the ten partitions $s+t+u=3$ in positive integers or zero, while the groups of the coefficients for the respective partitions are as follow, the values for the partition $s=3$, $t=0$, $u=0$, having already been given. The values are

$$E_{210} = e_{210} + 2(L\Gamma_{12} + M\Delta_{12} + K\Theta_{12}) + (M\Gamma_{11} + N\Delta_{11} + I\Theta_{11}),$$
$$\left.\begin{aligned} R_{210} &= \frac{\partial\Gamma_{11}}{\partial q} + (\Gamma_{11}, \Delta_{11}, \Theta_{11}\text{X}\Gamma_{12}, \Gamma_{22}, \Gamma_{23}) - \frac{1}{\Omega^2}L\lambda_2 \\ &= \frac{\partial\Gamma_{12}}{\partial p} + (\Gamma_{12}, \Delta_{12}, \Theta_{12}\text{X}\Gamma_{11}, \Gamma_{12}, \Gamma_{13}) - \frac{1}{\Omega^2}M\lambda_1 \\ S_{210} &= \frac{\partial\Delta_{11}}{\partial q} + (\Gamma_{11}, \Delta_{11}, \Theta_{11}\text{X}\Delta_{12}, \Delta_{22}, \Delta_{23}) - \frac{1}{\Omega^2}L\mu_2 \\ &= \frac{\partial\Delta_{12}}{\partial p} + (\Gamma_{12}, \Delta_{12}, \Theta_{12}\text{X}\Delta_{11}, \Delta_{12}, \Delta_{13}) - \frac{1}{\Omega^2}M\mu_1 \\ T_{210} &= \frac{\partial\Theta_{11}}{\partial q} + (\Gamma_{11}, \Delta_{11}, \Theta_{11}\text{X}\Theta_{12}, \Theta_{22}, \Theta_{23}) - \frac{1}{\Omega^2}L\nu_2 \\ &= \frac{\partial\Theta_{12}}{\partial p} + (\Gamma_{12}, \Delta_{12}, \Theta_{12}\text{X}\Theta_{11}, \Theta_{12}, \Theta_{13}) - \frac{1}{\Omega^2}M\nu_1 \end{aligned}\right\},$$

$$E_{201}=e_{201}+2(L\Gamma_{13}+M\Delta_{13}+N\Theta_{13})+(K\Gamma_{11}+I\Delta_{11}+J\Theta_{11}),$$

$$\left.\begin{aligned}
R_{201}&=\frac{\partial\Gamma_{11}}{\partial r}+(\Gamma_{11},\ \Delta_{11},\ \Theta_{11}\text{X}\Gamma_{13},\ \Gamma_{23},\ \Gamma_{33})-\frac{1}{\Omega^2}L\lambda_3\\
&=\frac{\partial\Gamma_{13}}{\partial p}+(\Gamma_{13},\ \Delta_{13},\ \Theta_{13}\text{X}\Gamma_{11},\ \Gamma_{12},\ \Gamma_{13})-\frac{1}{\Omega^2}K\lambda_1\\
S_{201}&=\frac{\partial\Delta_{11}}{\partial r}+(\Gamma_{11},\ \Delta_{11},\ \Theta_{11}\text{X}\Delta_{13},\ \Delta_{23},\ \Delta_{33})-\frac{1}{\Omega^2}L\mu_3\\
&=\frac{\partial\Delta_{13}}{\partial p}+(\Gamma_{13},\ \Delta_{13},\ \Theta_{13}\text{X}\Delta_{11},\ \Delta_{12},\ \Delta_{13})-\frac{1}{\Omega^2}K\mu_1\\
T_{201}&=\frac{\partial\Theta_{11}}{\partial r}+(\Gamma_{11},\ \Delta_{11},\ \Theta_{11}\text{X}\Theta_{13},\ \Theta_{23},\ \Theta_{33})-\frac{1}{\Omega^2}L\nu_3\\
&=\frac{\partial\Theta_{13}}{\partial p}+(\Gamma_{13},\ \Delta_{13},\ \Theta_{13}\text{X}\Theta_{11},\ \Theta_{12},\ \Theta_{13})-\frac{1}{\Omega^2}K\nu_1
\end{aligned}\right\};$$

$$E_{120}=e_{120}+2(M\Gamma_{12}+N\Delta_{12}+I\Theta_{12})+(L\Gamma_{22}+M\Delta_{22}+K\Theta_{22}),$$

$$\left.\begin{aligned}
R_{120}&=\frac{\partial\Gamma_{12}}{\partial q}+(\Gamma_{12},\ \Delta_{12},\ \Theta_{12}\text{X}\Gamma_{12},\ \Gamma_{22},\ \Gamma_{23})-\frac{1}{\Omega^2}M\lambda_2\\
&=\frac{\partial\Gamma_{22}}{\partial p}+(\Gamma_{22},\ \Delta_{22},\ \Theta_{22}\text{X}\Gamma_{11},\ \Gamma_{12},\ \Gamma_{13})-\frac{1}{\Omega^2}N\lambda_1\\
S_{120}&=\frac{\partial\Delta_{12}}{\partial q}+(\Gamma_{12},\ \Delta_{12},\ \Theta_{12}\text{X}\Delta_{12},\ \Delta_{22},\ \Delta_{23})-\frac{1}{\Omega^2}M\mu_2\\
&=\frac{\partial\Delta_{22}}{\partial p}+(\Gamma_{22},\ \Delta_{22},\ \Theta_{22}\text{X}\Delta_{11},\ \Delta_{12},\ \Delta_{13})-\frac{1}{\Omega^2}N\mu_1\\
T_{120}&=\frac{\partial\Theta_{12}}{\partial q}+(\Gamma_{12},\ \Delta_{12},\ \Theta_{12}\text{X}\Theta_{12},\ \Theta_{22},\ \Theta_{23})-\frac{1}{\Omega^2}M\nu_2\\
&=\frac{\partial\Theta_{22}}{\partial p}+(\Gamma_{22},\ \Delta_{22},\ \Theta_{22}\text{X}\Theta_{11},\ \Theta_{12},\ \Theta_{13})-\frac{1}{\Omega^2}N\nu_1
\end{aligned}\right\};$$

$$E_{111}=e_{111}+(L\Gamma_{23}+M\Delta_{23}+K\Theta_{23})+(M\Gamma_{31}+N\Delta_{31}+I\Theta_{31})+(K\Gamma_{12}+I\Delta_{12}+J\Theta_{12}),$$

$$\left.\begin{aligned}
R_{111}&=\frac{\partial\Gamma_{12}}{\partial r}+(\Gamma_{12},\ \Delta_{12},\ \Theta_{12}\text{X}\Gamma_{13},\ \Gamma_{23},\ \Gamma_{33})-\frac{1}{\Omega^2}M\lambda_3\\
&=\frac{\partial\Gamma_{23}}{\partial p}+(\Gamma_{23},\ \Delta_{23},\ \Theta_{23}\text{X}\Gamma_{11},\ \Gamma_{12},\ \Gamma_{13})-\frac{1}{\Omega^2}I\lambda_1\\
&=\frac{\partial\Gamma_{31}}{\partial q}+(\Gamma_{31},\ \Delta_{31},\ \Theta_{31}\text{X}\Gamma_{12},\ \Gamma_{22},\ \Gamma_{23})-\frac{1}{\Omega^2}K\lambda_2\\
S_{111}&=\frac{\partial\Delta_{12}}{\partial r}+(\Gamma_{12},\ \Delta_{12},\ \Theta_{12}\text{X}\Delta_{13},\ \Delta_{23},\ \Delta_{33})-\frac{1}{\Omega^2}M\mu_3\\
&=\frac{\partial\Delta_{23}}{\partial p}+(\Gamma_{23},\ \Delta_{23},\ \Theta_{23}\text{X}\Delta_{11},\ \Delta_{12},\ \Delta_{13})-\frac{1}{\Omega^2}I\mu_1\\
&=\frac{\partial\Delta_{31}}{\partial q}+(\Gamma_{31},\ \Delta_{31},\ \Theta_{31}\text{X}\Delta_{12},\ \Delta_{22},\ \Delta_{23})-\frac{1}{\Omega^2}K\mu_2\\
T_{111}&=\frac{\partial\Theta_{12}}{\partial r}+(\Gamma_{12},\ \Delta_{12},\ \Theta_{12}\text{X}\Theta_{13},\ \Theta_{23},\ \Theta_{33})-\frac{1}{\Omega^2}M\nu_3\\
&=\frac{\partial\Theta_{23}}{\partial p}+(\Gamma_{23},\ \Delta_{23},\ \Theta_{23}\text{X}\Theta_{11},\ \Theta_{12},\ \Theta_{13})-\frac{1}{\Omega^2}I\nu_1\\
&=\frac{\partial\Theta_{31}}{\partial q}+(\Gamma_{31},\ \Delta_{31},\ \Theta_{31}\text{X}\Theta_{12},\ \Theta_{22},\ \Theta_{23})-\frac{1}{\Omega^2}K\nu_2
\end{aligned}\right\};$$

$$E_{102}=e_{102}+2\,(K\Gamma_{13}+I\Delta_{13}+J\Theta_{13})+(L\Gamma_{33}+M\Delta_{33}+K\Theta_{33}),$$

$$\left.\begin{aligned}
R_{102}&=\frac{\partial\Gamma_{33}}{\partial p}+(\Gamma_{33},\,\Delta_{33},\,\Theta_{33}\text{X}\Gamma_{11},\,\Gamma_{12},\,\Gamma_{13})-\frac{1}{\Omega^2}J\lambda_1\\
&=\frac{\partial\Gamma_{13}}{\partial r}+(\Gamma_{13},\,\Delta_{13},\,\Theta_{13}\text{X}\Gamma_{13},\,\Gamma_{23},\,\Gamma_{33})-\frac{1}{\Omega^2}K\lambda_3\\
S_{102}&=\frac{\partial\Delta_{33}}{\partial p}+(\Gamma_{33},\,\Delta_{33},\,\Theta_{33}\text{X}\Delta_{11},\,\Delta_{12},\,\Delta_{13})-\frac{1}{\Omega^2}J\mu_1\\
&=\frac{\partial\Delta_{13}}{\partial r}+(\Gamma_{13},\,\Delta_{13},\,\Theta_{13}\text{X}\Delta_{13},\,\Delta_{23},\,\Delta_{33})-\frac{1}{\Omega^2}K\mu_3\\
T_{102}&=\frac{\partial\Theta_{33}}{\partial p}+(\Gamma_{33},\,\Delta_{33},\,\Theta_{33}\text{X}\Theta_{11},\,\Theta_{12},\,\Theta_{13})-\frac{1}{\Omega^2}J\nu_1\\
&=\frac{\partial\Theta_{13}}{\partial r}+(\Gamma_{13},\,\Delta_{13},\,\Theta_{13}\text{X}\Theta_{13},\,\Theta_{23},\,\Theta_{33})-\frac{1}{\Omega^2}K\nu_3
\end{aligned}\right\};$$

$$E_{030}=e_{030}+3\,(M\Gamma_{22}+N\Delta_{22}+I\Theta_{22}),$$

$$\left.\begin{aligned}
R_{030}&=\frac{\partial\Gamma_{22}}{\partial q}+(\Gamma_{22},\,\Delta_{22},\,\Theta_{22}\text{X}\Gamma_{12},\,\Gamma_{22},\,\Gamma_{23})-\frac{1}{\Omega^2}N\lambda_2\\
S_{030}&=\frac{\partial\Delta_{22}}{\partial q}+(\Gamma_{22},\,\Delta_{22},\,\Theta_{22}\text{X}\Delta_{12},\,\Delta_{22},\,\Delta_{23})-\frac{1}{\Omega^2}N\mu_2\\
T_{030}&=\frac{\partial\Theta_{22}}{\partial q}+(\Gamma_{22},\,\Delta_{22},\,\Theta_{22}\text{X}\Theta_{12},\,\Theta_{22},\,\Theta_{23})-\frac{1}{\Omega^2}N\nu_2
\end{aligned}\right\};$$

$$E_{021}=e_{021}+2\,(M\Gamma_{23}+N\Delta_{23}+I\Theta_{23})+(K\Gamma_{22}+I\Delta_{22}+J\Theta_{22}),$$

$$\left.\begin{aligned}
R_{021}&=\frac{\partial\Gamma_{23}}{\partial q}+(\Gamma_{23},\,\Delta_{23},\,\Theta_{23}\text{X}\Gamma_{12},\,\Gamma_{22},\,\Gamma_{23})-\frac{1}{\Omega^2}I\lambda_2\\
&=\frac{\partial\Gamma_{22}}{\partial r}+(\Gamma_{22},\,\Delta_{22},\,\Theta_{22}\text{X}\Gamma_{13},\,\Gamma_{23},\,\Gamma_{33})-\frac{1}{\Omega^2}N\lambda_3\\
S_{021}&=\frac{\partial\Delta_{23}}{\partial q}+(\Gamma_{23},\,\Delta_{23},\,\Theta_{23}\text{X}\Delta_{12},\,\Delta_{22},\,\Delta_{23})-\frac{1}{\Omega^2}I\mu_2\\
&=\frac{\partial\Delta_{22}}{\partial r}+(\Gamma_{22},\,\Delta_{22},\,\Theta_{22}\text{X}\Delta_{13},\,\Delta_{23},\,\Delta_{33})-\frac{1}{\Omega^2}N\mu_3\\
T_{021}&=\frac{\partial\Theta_{23}}{\partial q}+(\Gamma_{23},\,\Delta_{23},\,\Theta_{23}\text{X}\Theta_{12},\,\Theta_{22},\,\Theta_{23})-\frac{1}{\Omega^2}I\nu_2\\
&=\frac{\partial\Theta_{22}}{\partial r}+(\Gamma_{22},\,\Delta_{22},\,\Theta_{22}\text{X}\Theta_{13},\,\Theta_{23},\,\Theta_{33})-\frac{1}{\Omega^2}N\nu_3
\end{aligned}\right\};$$

$$E_{012}=e_{012}+2\,(K\Gamma_{23}+I\Delta_{23}+J\Theta_{23})+(M\Gamma_{33}+N\Delta_{33}+I\Theta_{33}),$$

$$\left.\begin{aligned}
R_{012}&=\frac{\partial\Gamma_{33}}{\partial q}+(\Gamma_{33},\,\Delta_{33},\,\Theta_{33}\text{X}\Gamma_{12},\,\Gamma_{22},\,\Gamma_{23})-\frac{1}{\Omega^2}J\lambda_2\\
&=\frac{\partial\Gamma_{23}}{\partial r}+(\Gamma_{23},\,\Delta_{23},\,\Theta_{23}\text{X}\Gamma_{13},\,\Gamma_{23},\,\Gamma_{33})-\frac{1}{\Omega^2}I\lambda_3\\
S_{012}&=\frac{\partial\Delta_{33}}{\partial q}+(\Gamma_{33},\,\Delta_{33},\,\Theta_{33}\text{X}\Delta_{12},\,\Delta_{22},\,\Delta_{23})-\frac{1}{\Omega^2}J\mu_2\\
&=\frac{\partial\Delta_{23}}{\partial r}+(\Gamma_{23},\,\Delta_{23},\,\Theta_{23}\text{X}\Delta_{13},\,\Delta_{23},\,\Delta_{33})-\frac{1}{\Omega^2}I\mu_3\\
T_{012}&=\frac{\partial\Theta_{33}}{\partial q}+(\Gamma_{33},\,\Delta_{33},\,\Theta_{33}\text{X}\Theta_{12},\,\Theta_{22},\,\Theta_{23})-\frac{1}{\Omega^2}J\nu_2\\
&=\frac{\partial\Theta_{23}}{\partial r}+(\Gamma_{23},\,\Delta_{23},\,\Theta_{23}\text{X}\Theta_{13},\,\Theta_{23},\,\Theta_{33})-\frac{1}{\Omega^2}I\nu_3
\end{aligned}\right\};$$

and

$$\left.\begin{aligned} E_{003} &= e_{003} + 3(K\Gamma_{33} + I\Delta_{33} + J\Theta_{33}), \\ R_{003} &= \frac{\partial \Gamma_{33}}{\partial r} + (\Gamma_{33}, \Delta_{33}, \Theta_{33}\}\Gamma_{13}, \Gamma_{23}, \Gamma_{33}) - \frac{1}{\Omega^2} J\lambda_3 \\ S_{003} &= \frac{\partial \Delta_{33}}{\partial r} + (\Gamma_{33}, \Delta_{33}, \Theta_{33}\}\Delta_{13}, \Delta_{23}, \Delta_{33}) - \frac{1}{\Omega^2} J\mu_3 \\ T_{003} &= \frac{\partial \Theta_{33}}{\partial r} + (\Gamma_{33}, \Delta_{33}, \Theta_{33}\}\Theta_{13}, \Theta_{23}, \Theta_{33}) - \frac{1}{\Omega^2} J\nu_3 \end{aligned}\right\}.$$

Many results, as regards the bilinear symmetric combinations of derivatives of x, y, z, v, of the first three orders, follow at once from these expressions. Thus

$$\left.\begin{aligned} \Sigma X x_{stu} &= E_{stu}; \\ \Sigma x_1 x_{stu} &= AR_{stu} + HS_{stu} + GT_{stu} \\ \Sigma x_2 x_{stu} &= HR_{stu} + BS_{stu} + FT_{stu} \\ \Sigma x_3 x_{stu} &= GR_{stu} + FS_{stu} + CT_{stu} \end{aligned}\right\};$$

$$\Sigma x_{11} x_{stu} = LE_{stu} + (A\Gamma_{11} + H\Delta_{11} + G\Theta_{11}) R_{stu} + (H\Gamma_{11} + B\Delta_{11} + F\Theta_{11}) S_{stu} + (G\Gamma_{11} + F\Delta_{11} + C\Theta_{11}) T_{stu},$$

and so on; and

$$\Sigma x_{stu} x_{s't'u'} = E_{stu} E_{s't'u'} + (A, B, C, F, G, H\}R_{stu}, S_{stu}, T_{stu}\}R_{s't'u'}, S_{s't'u'}, T_{s't'u'}),$$

for all values of s, t, u, and s', t', u', such that $s+t+u=3$, $s'+t'+u'=3$.

Ex. 1. Establish the relations among the derivatives of the quantities E, R, S, T:

(i) $$\frac{\partial E_{l-1,m,n}}{\partial p} + LR_{l-1,m,n} + MS_{l-1,m,n} + KT_{l-1,m,n} = \frac{\partial E_{l,m-1,n}}{\partial q} + MR_{l,m-1,n} + NS_{l,m-1,n} + IT_{l,m-1,n} = \frac{\partial E_{l,m,n-1}}{\partial r} + KR_{l,m,n-1} + IS_{l,m,n-1} + JT_{l,m,n-1},$$

for $l+m+n=3$, provided $l \geqslant 1$ for the first expression, $m \geqslant 1$ for the second expression, and $n \geqslant 1$ for the third expression, being a further extension of the Codazzi-Mainardi relations (§ 279);

(ii) $$\frac{\partial R_{l-1,m,n}}{\partial p} + \Gamma_{11} R_{l-1,m,n} + \Gamma_{12} S_{l-1,m,n} + \Gamma_{13} T_{l-1,m,n} - \frac{1}{\Omega^2}(aL+hM+gK) E_{l-1,m,n} = \frac{\partial R_{l,m-1,n}}{\partial q} + \Gamma_{12} R_{l,m-1,n} + \Gamma_{22} S_{l,m-1,n} + \Gamma_{23} T_{l,m-1,n} - \frac{1}{\Omega^2}(hL+bM+fK) E_{l,m-1,n} = \frac{\partial R_{l,m,n-1}}{\partial r} + \Gamma_{13} R_{l,m,n-1} + \Gamma_{23} S_{l,m,n-1} + \Gamma_{33} T_{l,m,n-1} - \frac{1}{\Omega^2}(gL+fM+cK) E_{l,m,n-1};$$

(iii) $$\frac{\partial S_{l-1,m,n}}{\partial p} + \Delta_{11} R_{l-1,m,n} + \Delta_{12} S_{l-1,m,n} + \Delta_{13} T_{l-1,m,n} - \frac{1}{\Omega^2}(aM+hN+gI) E_{l-1,m,n} = \frac{\partial S_{l,m-1,n}}{\partial q} + \Delta_{12} R_{l,m-1,n} + \Delta_{22} S_{l,m-1,n} + \Delta_{23} T_{l,m-1,n} - \frac{1}{\Omega^2}(hM+bN+fI) E_{l,m-1,n} = \frac{\partial S_{l,m,n-1}}{\partial r} + \Delta_{13} R_{l,m,n-1} + \Delta_{23} S_{l,m,n-1} + \Delta_{33} T_{l,m,n-1} - \frac{1}{\Omega^2}(gM+fN+cI) E_{l,m,n-1};$$

(iv)
$$\frac{\partial T_{l-1,m,n}}{\partial p}+\Theta_{11}R_{l-1,m,n}+\Theta_{12}S_{l-1,m,n}+\Theta_{13}T_{l-1,m,n}-\frac{1}{\Omega^2}(aK+hI+gJ)E_{l-1,m,n}$$
$$=\frac{\partial T_{l,m-1,n}}{\partial q}+\Theta_{12}R_{l,m-1,n}+\Theta_{22}S_{l,m-1,n}+\Theta_{23}T_{l,m-1,n}-\frac{1}{\Omega^2}(hK+bI+fJ)E_{l,m-1,n}$$
$$=\frac{\partial T_{l,m,n-1}}{\partial r}+\Theta_{13}R_{l,m,n-1}+\Theta_{23}S_{l,m,n-1}+\Theta_{33}T_{l,m,n-1}-\frac{1}{\Omega^2}(gK+fI+cJ)E_{l,m,n-1}.$$

Each of the equal expressions in (ii) is denoted later (§ 501) by U_{lmn}, in (iii) by V_{lmn}, and in (iv) by W_{lmn}; and each of the equal expressions in (i) is denoted by η_{lmn}.

Ex. 2. Establish the aggregate of relations of the type

$$\frac{\partial E_{300}}{\partial p}=\frac{\partial e_{300}}{\partial p}+3(\Gamma_{11}e_{300}+\Delta_{11}e_{210}+\Theta_{11}e_{201})+3(LR_{300}+MS_{300}+KT_{300})$$
$$+3(L\{\Gamma_{11},\ \Delta_{11},\ \Theta_{11}\}^2+3\frac{L}{\Omega^2}(a\{L,\ M,\ K\})^2,$$

$$\frac{\partial E_{300}}{\partial q}=\frac{\partial e_{300}}{\partial q}+3(\Gamma_{11}e_{210}+\Delta_{11}e_{120}+\Theta_{11}e_{111})+3(LR_{210}+MS_{210}+KT_{210})$$
$$+3(L\{\Gamma_{11},\ \Delta_{11},\ \Theta_{11}\}\{\Gamma_{12},\ \Delta_{12},\ \Theta_{12}\})+3\frac{L}{\Omega^2}(a\{L,\ M,\ K\}\{M,\ N,\ I\}),$$

$$\frac{\partial E_{300}}{\partial r}=\frac{\partial e_{300}}{\partial r}+3(\Gamma_{11}e_{201}+\Delta_{11}e_{111}+\Theta_{11}e_{102})+3(LR_{201}+MS_{201}+KT_{201})$$
$$+3(L\{\Gamma_{11},\ \Delta_{11},\ \Theta_{11}\}\{\Gamma_{13},\ \Delta_{13},\ \Theta_{13}\})+3\frac{L}{\Omega^2}(a\{L,\ M,\ K\}\{K,\ I,\ J\}).$$

These results, and the results in the preceding example, are used in the analysis for the construction of the magnitudes of grade four (§ 501), appertaining to the region.

Second derivatives of X, Y, Z, V.

295. Later (§ 309), in order to obtain an expression for the curvature of tilt of a regional geodesic, we shall require the values of the quantities

$$X_{lm}-X_1\Gamma_{lm}-X_2\Delta_{lm}-X_3\Theta_{lm},$$

for all the combinations l, m, $=1, 2, 3$, and the like combinations of the derivatives of Y, Z, V.

In the first place, as regards the quantities λ_1, μ_1, ν_1; λ_2, μ_2, ν_2; λ_3, μ_3, ν_3; let

$$D_{11}=L\lambda_1+M\mu_1+K\nu_1,$$
$$D_{22}=M\lambda_2+N\mu_2+I\nu_2,$$
$$D_{33}=K\lambda_3+I\mu_3+J\nu_3,$$
$$D_{23}=K\lambda_2+I\mu_2+J\nu_2=M\lambda_3+N\mu_3+I\nu_3,$$
$$D_{31}=L\lambda_3+M\mu_3+K\nu_3=K\lambda_1+I\mu_1+J\nu_1,$$
$$D_{12}=M\lambda_1+N\mu_1+I\nu_1=L\lambda_2+M\mu_2+K\nu_2,$$

the last equalities being due to the values of the quantities λ, μ, ν.

By direct substitution, we find

$$\left.\begin{aligned}
\Omega^2\frac{\partial}{\partial p}\left(\frac{\lambda_1}{\Omega^2}\right) &= ae_{300}+he_{210}+ge_{201}+(\Gamma_{11}\lambda_1+\Delta_{11}\lambda_2+\Theta_{11}\lambda_3)-(\Gamma_{11}\lambda_1+\Gamma_{12}\mu_1+\Gamma_{13}\nu_1)\\
\Omega^2\frac{\partial}{\partial p}\left(\frac{\mu_1}{\Omega^2}\right) &= he_{300}+be_{210}+fe_{201}+(\Gamma_{11}\mu_1+\Delta_{11}\mu_2+\Theta_{11}\mu_3)-(\Delta_{11}\lambda_1+\Delta_{12}\mu_1+\Delta_{13}\nu_1)\\
\Omega^2\frac{\partial}{\partial p}\left(\frac{\nu_1}{\Omega^2}\right) &= ge_{300}+fe_{210}+ce_{201}+(\Gamma_{11}\nu_1+\Delta_{11}\nu_2+\Theta_{11}\nu_3)-(\Theta_{11}\lambda_1+\Theta_{12}\mu_1+\Theta_{13}\nu_1)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\Omega^2\frac{\partial}{\partial q}\left(\frac{\lambda_1}{\Omega^2}\right) &= ae_{210}+he_{120}+ge_{111}+(\Gamma_{12}\lambda_1+\Delta_{12}\lambda_2+\Theta_{12}\lambda_3)-(\Gamma_{12}\lambda_1+\Gamma_{22}\mu_1+\Gamma_{23}\nu_1)\\
\Omega^2\frac{\partial}{\partial q}\left(\frac{\mu_1}{\Omega^2}\right) &= he_{210}+be_{120}+fe_{111}+(\Gamma_{12}\mu_1+\Delta_{12}\mu_2+\Theta_{12}\mu_3)-(\Delta_{12}\lambda_1+\Delta_{22}\mu_1+\Delta_{23}\nu_1)\\
\Omega^2\frac{\partial}{\partial q}\left(\frac{\nu_1}{\Omega^2}\right) &= ge_{210}+fe_{120}+ce_{111}+(\Gamma_{12}\nu_1+\Delta_{12}\nu_2+\Theta_{12}\nu_3)-(\Theta_{12}\lambda_1+\Theta_{22}\mu_1+\Theta_{23}\nu_1)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\Omega^2\frac{\partial}{\partial r}\left(\frac{\lambda_1}{\Omega^2}\right) &= ae_{201}+he_{111}+ge_{102}+(\Gamma_{13}\lambda_1+\Delta_{13}\lambda_2+\Theta_{13}\lambda_3)-(\Gamma_{13}\lambda_1+\Gamma_{23}\mu_1+\Gamma_{33}\nu_1)\\
\Omega^2\frac{\partial}{\partial r}\left(\frac{\mu_1}{\Omega^2}\right) &= he_{201}+be_{111}+fe_{102}+(\Gamma_{13}\mu_1+\Delta_{13}\mu_2+\Theta_{13}\mu_3)-(\Delta_{13}\lambda_1+\Delta_{23}\mu_1+\Delta_{33}\nu_1)\\
\Omega^2\frac{\partial}{\partial r}\left(\frac{\nu_1}{\Omega^2}\right) &= ge_{201}+fe_{111}+ce_{102}+(\Gamma_{13}\nu_1+\Delta_{13}\nu_2+\Theta_{13}\nu_3)-(\Theta_{13}\lambda_1+\Theta_{23}\mu_1+\Theta_{33}\nu_1)
\end{aligned}\right\},$$

with corresponding expressions for each of the three derivatives of λ_2, μ_2, ν_2; λ_3, μ_3, ν_3. The expressions can be deduced, from those given for the derivatives of λ_1, μ_1, ν_1, by appropriate interchange of the magnitudes concerned.

Again, we have

$$-X_1=x_1\frac{\lambda_1}{\Omega^2}+x_2\frac{\mu_1}{\Omega^2}+x_3\frac{\nu_1}{\Omega^2};$$

and therefore

$$\begin{aligned}
-\Omega^2X_{11} &= x_{11}\lambda_1+x_{12}\mu_1+x_{13}\nu_1+x_1\Omega^2\frac{\partial}{\partial p}\left(\frac{\lambda_1}{\Omega^2}\right)+x_2\Omega^2\frac{\partial}{\partial p}\left(\frac{\mu_1}{\Omega^2}\right)+x_3\Omega^2\frac{\partial}{\partial p}\left(\frac{\nu_1}{\Omega^2}\right)\\
&= X(L\lambda_1+M\mu_1+N\nu_1)\\
&\quad+x_1(ae_{300}+he_{210}+ge_{201})+x_2(he_{300}+be_{210}+fe_{201})\\
&\qquad+x_3(ge_{300}+fe_{210}+ce_{201})\\
&\quad+x_1(\Gamma_{11}\lambda_1+\Delta_{11}\lambda_2+\Theta_{11}\lambda_3)+x_2(\Gamma_{11}\mu_1+\Delta_{11}\mu_2+\Theta_{11}\mu_3)\\
&\qquad+x_3(\Gamma_{11}\nu_1+\Delta_{11}\nu_2+\Theta_{11}\nu_3).
\end{aligned}$$

We introduce symbols ξ_1, ξ_2, ξ_3, according to the definitions

$$\left.\begin{aligned}
ax_1+hx_2+gx_3 &= \xi_1\\
hx_1+bx_2+fx_3 &= \xi_2\\
gx_1+fx_2+cx_3 &= \xi_3
\end{aligned}\right\},$$

so that the former values for X_1, X_2, X_3, become

$$\left.\begin{aligned} -\Omega^2 X_1 &= L\xi_1 + M\xi_2 + K\xi_3 \\ -\Omega^2 X_2 &= M\xi_1 + N\xi_2 + I\xi_3 \\ -\Omega^2 X_3 &= K\xi_1 + I\xi_2 + J\xi_3 \end{aligned}\right\};$$

and we find

$$\begin{aligned} -\Omega^2 X_{11} = {} & X(L\lambda_1 + M\mu_1 + K\nu_1) \\ & + \xi_1(e_{300} + \Gamma_{11}L + \Delta_{11}M + \Theta_{11}K) \\ & + \xi_2(e_{210} + \Gamma_{11}M + \Delta_{11}N + \Theta_{11}I) \\ & + \xi_3(e_{201} + \Gamma_{11}K + \Delta_{11}I + \Theta_{11}J). \end{aligned}$$

The quantities, which will be found directly useful in connection with the tilt of geodesics, are of the type

$$X_{11} - X_1\Gamma_{11} - X_2\Delta_{11} - X_3\Theta_{11},$$

being six in number so far as concerns combinations of derivatives of X, with of course the corresponding combinations of derivatives of Y, Z, V. Then we have at once, with the foregoing notation for the quantities D and the quantities ξ,

$$-\Omega^2(X_{11} - X_1\Gamma_{11} - X_2\Delta_{11} - X_3\Theta_{11}) = XD_{11} + \xi_1 e_{300} + \xi_2 e_{210} + \xi_3 e_{201}.$$

There are symbols η_1, η_2, η_3, derived from ξ_1, ξ_2, ξ_3, by changing x_1, x_2, x_3, into y_1, y_2, y_3; symbols ζ_1, ζ_2, ζ_3, derived from ξ_1, ξ_2, ξ_3, by changing x_1, x_2, x_3, into z_1, z_2, z_3; and symbols υ_1, υ_2, υ_3, derived from ξ_1, ξ_2, ξ_3, by changing x_1, x_2, x_3, into v_1, v_2, v_3; and then we have the set of results

$$\left.\begin{aligned} -\Omega^2(X_{11} - X_1\Gamma_{11} - X_2\Delta_{11} - X_3\Theta_{11}) &= XD_{11} + \xi_1 e_{300} + \xi_2 e_{210} + \xi_3 e_{201} \\ -\Omega^2(Y_{11} - Y_1\Gamma_{11} - Y_2\Delta_{11} - Y_3\Theta_{11}) &= YD_{11} + \eta_1 e_{300} + \eta_2 e_{210} + \eta_3 e_{201} \\ -\Omega^2(Z_{11} - Z_1\Gamma_{11} - Z_2\Delta_{11} - Z_3\Theta_{11}) &= ZD_{11} + \zeta_1 e_{300} + \zeta_2 e_{210} + \zeta_3 e_{201} \\ -\Omega^2(V_{11} - V_1\Gamma_{11} - V_2\Delta_{11} - V_3\Theta_{11}) &= VD_{11} + \upsilon_1 e_{300} + \upsilon_2 e_{210} + \upsilon_3 e_{201} \end{aligned}\right\}.$$

The corresponding results for expressing the other second derivatives of X are similarly found to be

$$\left.\begin{aligned} -\Omega^2(X_{12} - X_1\Gamma_{12} - X_2\Delta_{12} - X_3\Theta_{12}) &= XD_{12} + \xi_1 e_{210} + \xi_2 e_{120} + \xi_3 e_{111} \\ -\Omega^2(X_{13} - X_1\Gamma_{13} - X_2\Delta_{13} - X_3\Theta_{13}) &= XD_{13} + \xi_1 e_{201} + \xi_2 e_{111} + \xi_3 e_{102} \\ -\Omega^2(X_{22} - X_1\Gamma_{22} - X_2\Delta_{22} - X_3\Theta_{22}) &= XD_{22} + \xi_1 e_{120} + \xi_2 e_{030} + \xi_3 e_{021} \\ -\Omega^2(X_{23} - X_1\Gamma_{23} - X_2\Delta_{23} - X_3\Theta_{23}) &= XD_{23} + \xi_1 e_{111} + \xi_2 e_{021} + \xi_3 e_{012} \\ -\Omega^2(X_{33} - X_1\Gamma_{33} - X_2\Delta_{33} - X_3\Theta_{33}) &= XD_{33} + \xi_1 e_{102} + \xi_2 e_{012} + \xi_3 e_{003} \end{aligned}\right\},$$

with the analogous expressions for the corresponding second derivatives of Y, Z, V.

First derivatives of the principal measures of curvature of a region.

296. The derivatives of the three principal measures of curvature (§ 281) of a region with respect to the regional parameters are immediately expressible in terms of these quantities e of the third order.

The linear measure of curvature of the region is (§ 281) given by

$$\Omega^2 \mathfrak{L} = aL + 2hM + 2gK + bN + 2fI + cJ;$$

and therefore

$$\begin{aligned}
\mathfrak{L}_1 = {} & \frac{1}{\Omega^2}(aL_1 + 2hM_1 + 2gK_1 + bN_1 + 2fI_1 + cJ_1) \\
& + L\frac{\partial}{\partial p}\left(\frac{a}{\Omega^2}\right) + 2M\frac{\partial}{\partial p}\left(\frac{h}{\Omega^2}\right) + 2K\frac{\partial}{\partial p}\left(\frac{g}{\Omega^2}\right) \\
& \qquad + N\frac{\partial}{\partial p}\left(\frac{b}{\Omega^2}\right) + 2I\frac{\partial}{\partial p}\left(\frac{f}{\Omega^2}\right) + J\frac{\partial}{\partial p}\left(\frac{c}{\Omega^2}\right) \\
= {} & \frac{a}{\Omega^2}\{e_{300} + 2(L\Gamma_{11} + M\Delta_{11} + K\Theta_{11})\} \\
& + 2\frac{h}{\Omega^2}\{e_{210} + (M\Gamma_{11} + N\Delta_{11} + I\Theta_{11}) + (L\Gamma_{12} + M\Delta_{12} + K\Theta_{12})\} \\
& + 2\frac{g}{\Omega^2}\{e_{201} + (K\Gamma_{11} + I\Delta_{11} + J\Theta_{11}) + (L\Gamma_{13} + M\Delta_{13} + K\Theta_{13})\} \\
& + \frac{b}{\Omega^2}\{e_{120} + 2(M\Gamma_{12} + N\Delta_{12} + I\Theta_{12})\} \\
& + 2\frac{f}{\Omega^2}\{e_{111} + (K\Gamma_{12} + I\Delta_{12} + J\Theta_{12}) + (M\Gamma_{13} + N\Delta_{13} + I\Theta_{13})\} \\
& + \frac{c}{\Omega^2}\{e_{102} + 2(K\Gamma_{13} + I\Delta_{13} + J\Theta_{13})\} \\
& - 2\frac{L}{\Omega^2}(a\Gamma_{11} + h\Gamma_{12} + g\Gamma_{13}) \\
& - 2\frac{M}{\Omega^2}\{(h\Gamma_{11} + b\Gamma_{12} + f\Gamma_{13}) + (a\Delta_{11} + h\Delta_{12} + g\Delta_{13})\} \\
& - 2\frac{K}{\Omega^2}\{(g\Gamma_{11} + f\Gamma_{12} + c\Gamma_{13}) + (a\Theta_{11} + h\Theta_{12} + g\Theta_{13})\} \\
& - 2\frac{N}{\Omega^2}\{(h\Delta_{11} + b\Delta_{12} + f\Delta_{13})\} \\
& - 2\frac{I}{\Omega^2}\{(g\Delta_{11} + f\Delta_{12} + c\Delta_{13}) + (h\Theta_{11} + b\Theta_{12} + f\Theta_{13})\} \\
& - 2\frac{J}{\Omega^2}\{(g\Theta_{11} + f\Theta_{12} + c\Theta_{13})\} \\
& = \frac{1}{\Omega^2}(ae_{300} + 2he_{210} + 2ge_{201} + be_{120} + 2fe_{111} + ce_{102}).
\end{aligned}$$

Similarly

$$\mathfrak{L}_2 = \frac{1}{\Omega^2}(ae_{210} + 2he_{120} + 2ge_{111} + be_{030} + 2fe_{021} + ce_{012}),$$

$$\mathfrak{L}_3 = \frac{1}{\Omega^2}(ae_{201} + 2he_{111} + 2ge_{102} + be_{021} + 2fe_{012} + ce_{003}).$$

Proceeding similarly to the construction of the derivatives of $\mathfrak{F}$, the additive measure of superficial curvature of a region at a point, we find

$$\left.\begin{aligned}\mathfrak{F}_1 &= \frac{1}{\Omega^2}(\bar{l}e_{300} + 2\bar{m}e_{210} + 2\bar{k}e_{201} + \bar{n}e_{120} + 2\bar{i}e_{111} + \bar{j}e_{102})\\ \mathfrak{F}_2 &= \frac{1}{\Omega^2}(\bar{l}e_{210} + 2\bar{m}e_{120} + 2\bar{k}e_{111} + \bar{n}e_{030} + 2\bar{i}e_{021} + \bar{j}e_{012})\\ \mathfrak{F}_3 &= \frac{1}{\Omega^2}(\bar{l}e_{201} + 2\bar{m}e_{111} + 2\bar{k}e_{102} + \bar{n}e_{021} + 2\bar{i}e_{012} + \bar{j}e_{003})\end{aligned}\right\},$$

where

$$\begin{aligned}\bar{l} &= BJ + CN - 2FK, & \bar{i} &= -AI - FL + GM + HK,\\ \bar{n} &= CL + AJ - 2GI, & \bar{k} &= -BK + FM - GN + HI,\\ \bar{j} &= AN + BL - 2HM, & \bar{m} &= -CM + FK + GI - HJ.\end{aligned}$$

In the same way, the first derivatives of $\mathfrak{R}$, the measure of spatial (or volumetric) curvature of a region at a point, are

$$\left.\begin{aligned}\mathfrak{R}_1 &= \frac{1}{\Omega^2}(le_{300} + 2me_{210} + 2ke_{201} + ne_{120} + 2ie_{111} + je_{102})\\ \mathfrak{R}_2 &= \frac{1}{\Omega^2}(le_{210} + 2me_{120} + 2ke_{111} + ne_{030} + 2ie_{021} + je_{012})\\ \mathfrak{R}_3 &= \frac{1}{\Omega^2}(le_{201} + 2me_{111} + 2ke_{102} + ne_{021} + 2ie_{012} + je_{003})\end{aligned}\right\}.$$

We thus have expressions for the variation, along any direction in the region, of the three measures in the forms

$$\left.\begin{aligned}\frac{d\mathfrak{L}}{ds} &= \mathfrak{L}_1 p' + \mathfrak{L}_2 q' + \mathfrak{L}_3 r'\\ \frac{d\mathfrak{F}}{ds} &= \mathfrak{F}_1 p' + \mathfrak{F}_2 q' + \mathfrak{F}_3 r'\\ \frac{d\mathfrak{R}}{ds} &= \mathfrak{R}_1 p' + \mathfrak{R}_2 q' + \mathfrak{R}_3 r'\end{aligned}\right\},$$

quantities to which we shall return in Chapter XXVII, because of their covariantive quality.

CHAPTER XVII.

GEODESICS IN A REGION.

Geodesics in a region.

297. Measurements of regional curvatures have been obtained from the curvature of a normal section of the region, by a consideration of the region in the immediate vicinity of the point of contact with a tangent flat, perpendicular to which the plane section is taken. The march of the curve in any such section, beyond the immediate vicinity of the point, is of no significance and thenceforward is ignored. But the same circular curvature of a curve in the region, touching the same direction in the region through the point x, y, z, v, is provided by the geodesic of the region drawn through that point in that direction; and, owing to the organic character of the geodesic, the march of the geodesic beyond the immediate vicinity of the point is of essential significance. We therefore proceed to consider geodesics.

A geodesic in a region is defined as the shortest curve joining two points in the region, the joining curve being required to lie wholly within the region. Accordingly, when a current variable t is adopted initially, instead of the variable s the total value of which is restricted to provide a minimum regional length, we have to make

$$\int \frac{ds}{dt}\,dt = \int \left\{ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2 \right\}^{\frac{1}{2}} dt$$

a minimum, the dependent variables x, y, z, v, being subject to the relation

$$R(x, y, z, v) = 0,$$

taken to be the Cartesian equation of the region. The characteristic equations for the determination of the curve are the four equations of the type

$$\frac{d}{dt}\left[\frac{\dfrac{dx}{dt}}{\left\{ \left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 + \left(\dfrac{dz}{dt}\right)^2 + \left(\dfrac{dv}{dt}\right)^2 \right\}^{\frac{1}{2}}} \right] = \lambda \frac{\partial R}{\partial x},$$

that is,

$$\frac{d}{dt}\left(\frac{dx}{ds}\right) = \lambda \frac{\partial R}{\partial x},$$

or

$$\frac{d^2x}{ds^2} = \frac{\partial R}{\partial x}\lambda \frac{ds}{dt}.$$

Now $\frac{\partial R}{\partial x}, \frac{\partial R}{\partial y}, \frac{\partial R}{\partial z}, \frac{\partial R}{\partial v}$, are proportional (§ 270) to X, Y, Z, V, the direction-cosines of the normal to the region at a point; the required curve therefore satisfies the equations

$$\frac{1}{X}\frac{d^2x}{ds^2} = \frac{1}{Y}\frac{d^2y}{ds^2} = \frac{1}{Z}\frac{d^2z}{ds^2} = \frac{1}{V}\frac{d^2v}{ds^2}.$$

Hence *the radius of circular curvature of the geodesic curve coincides, in direction, with the normal to the region*—a fundamental property of the geodesic.

Moreover, this normal to the region is the direction of the radius of curvature of the normal section of the region through the same direction: thus the geodesic and the normal section touch one another, and the two curves have the same circular curvature. Consequently, we can take the circular curvature of the normal section of a region through any direction as the circular curvature of the tangent geodesic, that is, of the geodesic through that initial direction: or *as regards the circular curvature of a normal section of a region we can substitute, for it, the circular curvature of a geodesic which touches the direction* (or passes initially through the direction) *determining the normal section.*

Parametric equations of geodesics.

298. When the region is represented parametrically, the determination of geodesics is initiated by making the integral

$$\int \frac{ds}{dt}\,dt, = \int S^{\frac{1}{2}}\,dt,$$

a minimum, where

$$S = A\left(\frac{dp}{dt}\right)^2 + 2H\frac{dp}{dt}\frac{dq}{dt} + 2G\frac{dp}{dt}\frac{dr}{dt} + B\left(\frac{dq}{dt}\right)^2 + 2F\frac{dq}{dt}\frac{dr}{dt} + C\left(\frac{dr}{dt}\right)^2.$$

There are three critical equations, corresponding to variations of p, of q, and of r, respectively. The first of these three equations is

$$\frac{d}{dt}\left\{S^{-\frac{1}{2}}\left(A\frac{dp}{dt} + H\frac{dq}{dt} + G\frac{dr}{dt}\right)\right\}$$

$$= \tfrac{1}{2}S^{-\frac{1}{2}}\left\{A_1\left(\frac{dp}{dt}\right)^2 + 2H_1\frac{dp}{dt}\frac{dq}{dt} + 2G_1\frac{dp}{dt}\frac{dr}{dt} + B_1\left(\frac{dq}{dt}\right)^2 + 2F_1\frac{dq}{dt}\frac{dr}{dt} + C_1\left(\frac{dr}{dt}\right)^2\right\}.$$

Now

$$\frac{ds}{dt} = S^{\frac{1}{2}},$$

and therefore this equation is

$$\frac{d}{ds}(Ap' + Hq' + Gr') = \tfrac{1}{2}(A_1p'^2 + 2H_1p'q' + 2G_1p'r' + B_1q'^2 + 2F_1q'r' + C_1r'^2).$$

The left-hand side is equal to

$$Ap''+Hq''+Gr''$$
$$+A_1p'^2+(A_2+H_1)\,p'q'+(A_3+G_1)\,p'r'+H_2q'^2+(H_3+G_2)\,q'r'+G_3r'^2;$$

and so the first critical equation becomes

$$\begin{aligned}Ap''+Hq''+Gr''&\\ &=-\tfrac{1}{2}A_1p'^2-A_2p'q'-A_3p'r'-(H_2-\tfrac{1}{2}B_1)\,q'^2\\ &\qquad -(H_3+G_2-F_1)\,q'r'-(G_3-\tfrac{1}{2}C_1)\,r'^2\\ &=-(\alpha_1p'^2+2\beta_1p'q'+2\gamma_1p'r'+\delta_1q'^2+2\epsilon_1q'r'+\omega_1r'^2),\end{aligned}$$

with the notation of § 273. Similarly, the second critical equation becomes

$$\begin{aligned}Hp''+Bq''+Fr''&\\ &=-(\alpha_2p'^2+2\beta_2p'q'+2\gamma_2p'r'+\delta_2q'^2+2\epsilon_2q'r'+\omega_2r'^2);\end{aligned}$$

and the third critical equation becomes

$$\begin{aligned}Gp''+Fq''+Cr''&\\ &=-(\alpha_3p'^2+2\beta_3p'q'+2\gamma_3p'r'+\delta_3q'^2+2\epsilon_3q'r'+\omega_3r'^2).\end{aligned}$$

Resolving these three equations, linear in p'', q'', r'', and using the magnitudes Γ, Δ, Θ, introduced in § 273, we have

$$\left.\begin{aligned}p''&=-(\Gamma_{11}p'^2+2\Gamma_{12}p'q'+2\Gamma_{13}p'r'+\Gamma_{22}q'^2+2\Gamma_{23}q'r'+\Gamma_{33}r'^2)=-(\Gamma\,|\,p',\,q',\,r')^2\\ q''&=-(\Delta_{11}p'^2+2\Delta_{12}p'q'+2\Delta_{13}p'r'+\Delta_{22}q'^2+2\Delta_{23}q'r'+\Delta_{33}r'^2)=-(\Delta\,|\,p',\,q',\,r')^2\\ r''&=-(\Theta_{11}p'^2+2\Theta_{12}p'q'+2\Theta_{13}p'r'+\Theta_{22}q'^2+2\Theta_{23}q'r'+\Theta_{33}r'^2)=-(\Theta\,|\,p',\,q',\,r')^2\end{aligned}\right\},$$

which accordingly are the characteristic equations of geodesics in the region.

Ex. The geodesics in a flat are lines.

Let the flat pass through a point f, g, h, e; and let three guiding lines be chosen perpendicular to one another, with direction-cosines l_1, m_1, n_1, k_1; l_2, m_2, n_2, k_2; l_3, m_3, n_3, k_3. The flat can be represented parametrically by the equations

$$\begin{aligned}x&=f+l_1p+l_2q+l_3r,\\ y&=g+m_1p+m_2q+m_3r,\\ z&=h+n_1p+n_2q+n_3r,\\ v&=e+k_1p+k_2q+k_3r.\end{aligned}$$

Then

$$ds^2=dx^2+dy^2+dz^2+dv^2=dp^2+dq^2+dr^2,$$

so that, for the flat,

$$A=1,\quad B=1,\quad C=1,\quad F=0,\quad G=0,\quad H=0.$$

Thus each of the quantities $\Gamma_{\lambda\mu}$, $\Delta_{\lambda\mu}$, $\Theta_{\lambda\mu}$, (for $\lambda, \mu = 1, 2, 3$) is zero; and so the equations characteristic of the geodesics in the flat are

$$p''=0, \quad q''=0, \quad r''=0.$$

Consequently

$$p=a_1 s+b_1, \quad q=a_2 s+b_2, \quad r=a_3 s+b_3,$$

where the six quantities a and b are constants. When these values of p, q, r, are inserted in the expressions for x, y, z, v, we find relations of the form

$$\frac{x-A}{\lambda}=\frac{y-B}{\mu}=\frac{z-C}{\nu}=\frac{v-D}{\kappa}, \quad =s:$$

that is, the geodesic is a line.

This result can be obtained immediately, without recourse to the characteristic equations. For the flat is a homaloidal triple space; and therefore geodesics in the flat are geodesics in the ordinary Euclidean triple space, that is, they are lines.

Critical geodesic tests, due to Jacobi, Legendre, Weierstrass.

299. In order to be assured that the geodesics provide, not merely a stationary value of the regional distance between two points, but an actual minimum, it is necessary to secure that the customary tests for weak variations, viz. the Jacobi test and the Legendre test, and the customary Weierstrass test for strong variations, shall be satisfied.

(A) The Jacobi test can be applied only after the geodesic curve is specified. It is quantitative; and it assigns a limit, beginning with any point, within which the distance from that point measured along the curve is actually a minimum. Thus an integral of the characteristic equations is required: the integral may be very special or it may be the primitive; there are corresponding methods of applying the Jacobi test. But the details of the test are particular to each instance: they must be settled for each instance as it arises.

(B) The Legendre test is of general application. The integral has been taken in the form $\int S^{\frac{1}{2}} dt$, where

$$S=(A, B, C, F, G, H)(p_0, q_0, r_0)^2,$$

where

$$\frac{dp}{dt}=p_0, \quad \frac{dq}{dt}=q_0, \quad \frac{dr}{dt}=r_0.$$

We write

$$S_{11}=\frac{\partial^2 S^{\frac{1}{2}}}{\partial p_0^2}, \quad S_{22}=\frac{\partial^2 S^{\frac{1}{2}}}{\partial q_0^2}, \quad S_{33}=\frac{\partial^2 S^{\frac{1}{2}}}{\partial r_0^2},$$

$$S_{23}=\frac{\partial^2 S^{\frac{1}{2}}}{\partial q_0 \partial r_0}, \quad S_{31}=\frac{\partial^2 S^{\frac{1}{2}}}{\partial r_0 \partial p_0}, \quad S_{12}=\frac{\partial^2 S^{\frac{1}{2}}}{\partial p_0 \partial q_0}.$$

Then the Legendre tests are the requirement* that, if the integral is to be a minimum, the magnitude $\mathbf{U}$, where

$$\begin{aligned}\mathbf{U} &= S_{11}\xi^2 + 2S_{12}\xi\eta + 2S_{13}\xi\zeta + S_{22}\eta^2 + 2S_{23}\eta\zeta + S_{33}\zeta^2\\ &= (S_{11},\ S_{22},\ S_{33},\ S_{23},\ S_{31},\ S_{12}\mathbin{\text{\textbackslash\!\!\textbackslash}}\xi,\ \eta,\ \zeta)^2\end{aligned}$$

shall be positive for all real quantities ξ, η, ζ, which do not simultaneously vanish. We have

$$\begin{aligned}S_{11} &= S^{-\frac{3}{2}}(cq_0^2 - 2fq_0r_0 + br_0^2),\\ S_{22} &= S^{-\frac{3}{2}}(ar_0^2 - 2gr_0p_0 + cp_0^2),\\ S_{33} &= S^{-\frac{3}{2}}(bp_0^2 - 2hp_0q_0 + aq_0^2),\\ S_{23} &= S^{-\frac{3}{2}}(-aq_0r_0 + gq_0p_0 + hr_0p_0 - fp_0^2),\\ S_{31} &= S^{-\frac{3}{2}}(-br_0p_0 + hr_0q_0 + fp_0q_0 - gq_0^2),\\ S_{12} &= S^{-\frac{3}{2}}(-cp_0q_0 + fp_0r_0 + gq_0r_0 - hr_0^2).\end{aligned}$$

Substituting these values of the coefficients in the quadratic form $\mathbf{U}$, and collecting terms, we have

$$\begin{aligned}\mathbf{U}S^{\frac{3}{2}} &= (a, b, c, f, g, h\mathbin{\text{\textbackslash\!\!\textbackslash}}\eta r_0 - \zeta q_0,\ \zeta p_0 - \xi r_0,\ \xi q_0 - \eta p_0)^2\\ &= (\Sigma A p_0^2)(\Sigma A\xi^2) - (\Sigma A p_0\xi)^2.\end{aligned}$$

Now

$$\begin{aligned}\Sigma A p_0^2 &= \Sigma\ (x_1p_0 + x_2q_0 + x_3r_0)^2,\\ \Sigma A\xi^2 &= \Sigma\ (x_1\xi + x_2\eta + x_3\zeta)^2,\\ \Sigma A p_0\xi &= \Sigma\ \{(x_1p_0 + x_2q_0 + x_3r_0)(x_1\xi + x_2\eta + x_3\zeta)\};\end{aligned}$$

and therefore

$$\mathbf{U}S^{\frac{3}{2}} = \Sigma \begin{vmatrix} x_1p_0 + x_2q_0 + x_3r_0, & x_1\xi + x_2\eta + x_3\zeta \\ y_1p_0 + y_2q_0 + y_3r_0, & y_1\xi + y_2\eta + y_3\zeta \end{vmatrix}^2,$$

where the last summation of quantities, each of which is positive, extends over the six terms arising from the pair-combinations of x, y, z, v.

The right-hand side is a positive magnitude; and the positive sign is associated, throughout, with the radical $S^{\frac{1}{2}}$. Consequently $\mathbf{U}$ is always positive for quantities ξ, η, ζ, that do not simultaneously vanish.

Hence the Legendre test is satisfied: it admits the possibility of a minimum.

(C) As regards the Weierstrass test† through the Excess-function $\mathfrak{E}$, we take a direction P', Q', R', distinct from the tangential direction p', q', r', along the characteristic curve: we still denote by $S^{\frac{1}{2}}$ its value when p', q', r',

* See my *Calculus of Variations*, §§ 172, 178.

† *Calculus of Variations*, § 225.

are substituted for p_0, q_0, r_0; and we denote by $\bar{S}^{\frac{1}{2}}$ its value when P', Q', R', are substituted for p_0, q_0, r_0. Then the Excess-function is given by

$$\mathfrak{E} = \bar{S}^{\frac{1}{2}} - \left\{P' \frac{\partial S^{\frac{1}{2}}}{\partial p'} + Q' \frac{\partial S^{\frac{1}{2}}}{\partial q'} + R' \frac{\partial S^{\frac{1}{2}}}{\partial r'}\right\}$$

$$= \bar{S}^{\frac{1}{2}} - \frac{1}{S^{\frac{1}{2}}} \{P'(Ap' + Hq' + Gr') + Q'(Hp' + Bq' + Fr') + R'(Gp' + Fq' + Cr')\}$$

$$= \bar{S}^{\frac{1}{2}} - \frac{\bar{S}^{\frac{1}{2}}}{S^{\frac{1}{2}}} S^{\frac{1}{2}} \cos\theta$$

$$= \bar{S}^{\frac{1}{2}} (1 - \cos\theta),$$

where θ is the inclination between the differing directions P', Q', R', and p', q', r', and where $S^{\frac{1}{2}}$ (with p', q', r') actually is equal to unity. Now $\bar{S}^{\frac{1}{2}}$ is positive, and θ differs from 0 and from 2π: the Excess-function is always positive. The Weierstrass test for strong variations is satisfied: it admits the possibility of a minimum.

Hence, subject to the Jacobi test which may limit the length of the range of the arc and which must be specially (not generally) considered for the particular data in each instance, the geodesic provides a true minimum length within the region.

There are two (not three) independent characteristic equations.

300. The general investigation leads to the symmetrical set of three characteristic equations for a geodesic; but it appears that, owing to the equation

$$(A, B, C, F, G, H \mathcal{X} p', q', r')^2 = 1$$

satisfied by all curves in the region, the three characteristic equations are equivalent to only two independent equations.

Differentiation of this universal equation leads to an equation

$$2(Ap' + Hq' + Gr')p'' + 2(Hp' + Bq' + Fr')q'' + 2(Gp' + Fq' + Cr')r''$$
$$+ \left(\frac{dA}{ds}, \frac{dB}{ds}, \frac{dC}{ds}, \frac{dF}{ds}, \frac{dG}{ds}, \frac{dH}{ds}\right)(p', q', r')^2 = 0.$$

Now

$$\frac{dA}{ds} = A_1 p' + A_2 q' + A_3 r',$$

and similarly for the derivatives of B, C, F, G, H, with respect to s. When the values of A_1, A_2, A_3, and the corresponding derivatives of the other primary magnitudes are substituted from § 273, we have an aggregate of

terms in the quantities Γ, Δ, Θ. Now the whole coefficient of Γ_{11} is

$$\begin{aligned} & p'^2 . 2Ap' \text{ from the term } \frac{dA}{ds} p'^2 \\ + & 2p'q' . Hp' \quad \ldots\ldots\ldots\ldots\ldots\ldots 2\frac{dH}{ds} p'q' \\ + & 2p'r' . Gp' \quad \ldots\ldots\ldots\ldots\ldots\ldots 2\frac{dG}{ds} p'r' : \end{aligned}$$

that is, the whole term in Γ_{11} is $2(Ap' + Hq' + Gr') . \Gamma_{11} p'^2$. Similarly, the whole coefficient of Γ_{12} is

$$\begin{aligned} & p'^2 . 2Aq' \quad \text{from the term } \frac{dA}{ds} p'^2 \\ + & \; q'^2 . 2Hp' \quad \ldots\ldots\ldots\ldots\ldots\ldots \frac{dB}{ds} q'^2 \\ + & 2q'r' . Gp' \quad \ldots\ldots\ldots\ldots\ldots\ldots 2\frac{dF}{ds} q'r' \\ + & 2p'q' (Ap' + Hq') \quad \ldots\ldots\ldots\ldots\ldots\ldots 2\frac{dH}{ds} p'q' \\ + & 2p'r' . Gq' \quad \ldots\ldots\ldots\ldots\ldots\ldots 2\frac{dG}{ds} p'r' : \end{aligned}$$

that is, the whole term in Γ_{12} is $2(Ap' + Hq' + Gr') . 2\Gamma_{12} p'q'$. Similarly for the others: the aggregate of the terms in the quantities Γ resulting from the substitutions is

$$2(Ap' + Hq' + Gr') . (\Gamma_{11}, \Gamma_{22}, \Gamma_{33}, \Gamma_{23}, \Gamma_{31}, \Gamma_{12} \{ p', q', r')^2,$$

that is, it is

$$2(Ap' + Hq' + Gr') . (\Gamma \{ p', q', r')^2.$$

Similarly the aggregate of the terms in the quantities Δ resulting from the substitutions is

$$2(Hp' + Bq' + Fr') . (\Delta \{ p', q', r')^2 ;$$

and the aggregate of the corresponding terms in the quantities Θ is

$$2(Gp' + Fq' + Cr') . (\Theta \{ p', q', r')^2.$$

Thus, when the universal equation is differentiated along any curve, it yields the relation

$$\begin{aligned} & 2(Ap' + Hq' + Gr') \{ p'' + (\Gamma \{ p', q', r')^2 \} \\ & \quad + 2(Hp' + Bq' + Fr') \{ q'' + (\Delta \{ p', q', r')^2 \} \\ & \quad\quad + 2(Gp' + Fq' + Cr') \{ r'' + (\Theta \{ p', q', r')^2 \} = 0 \end{aligned}$$

as an equation identically satisfied along the curve. Moreover, the three quantities

$$Ap' + Hq' + Gr', \quad Hp' + Bq' + Fr', \quad Gp' + Fq' + Cr',$$

cannot vanish together. Consequently, the three characteristic equations

$$p'' + (\Gamma \nabla p', q', r')^2 = 0, \quad q'' + (\Delta \nabla p', q', r')^2 = 0, \quad r'' + (\Theta \nabla p', q', r')^2 = 0,$$

being connected by this identical linear relation, are equivalent to only two independent equations.

If, then, two first integrals of the characteristic geodesic equations can be obtained, they can be associated with the equation

$$(A, B, C, F, G, H \nabla p', q', r')^2 = 1;$$

and thus, in passing from the stage of second derivatives of the parameters p, q, r, with respect to s, to the stage of first derivatives of those parameters, two arbitrary constants are introduced. In the further integration, which is preliminary to the complete expression of p, q, r, as functions of s without the occurrence of any derivatives, three arbitrary constants will be introduced. One of these new constants can be taken as additive to the independent variable s, which does not occur explicitly in the original equations: there is no need, and there is no datum, to specify any origin for the measurement of s: and, since the arc can be considered as eliminable for the final relations between p, q, r, this additive constant will disappear with s, and consequently is an unessential constant. It follows that the primitive of the geodesic equations contains four (and not more than four) essential arbitrary constants.

301. This result can also be seen as follows. As the variable s is ultimately not to occur in the primitive of the equations, let one of the surviving variables, say p, be chosen as the independent variable in the integration. We have

$$\frac{dq}{dp} = \frac{q'}{p'}, \quad \frac{d^2q}{dp^2} = \frac{p'q'' - q'p''}{p'^3},$$
$$\frac{dr}{dp} = \frac{r'}{p'}, \quad \frac{d^2r}{dp^2} = \frac{p'r'' - r'p''}{p'^3};$$

and therefore

$$\begin{aligned}\frac{d^2q}{dp^2} = {} & \Delta_{11} + (2\Delta_{12} - \Gamma_{11})\frac{dq}{dp} + 2\Delta_{13}\frac{dr}{dp} \\ & + (\Delta_{22} - 2\Gamma_{12})\left(\frac{dq}{dp}\right)^2 + 2(\Delta_{23} - \Gamma_{13})\frac{dq}{dp}\frac{dr}{dp} + \Delta_{33}\left(\frac{dr}{dp}\right)^2 \\ & - \Gamma_{22}\left(\frac{dq}{dp}\right)^3 - 2\Gamma_{23}\left(\frac{dq}{dp}\right)^2\frac{dr}{dp} - \Gamma_{33}\frac{dq}{dp}\left(\frac{dr}{dp}\right)^2, \\ \frac{d^2r}{dp^2} = {} & \Theta_{11} + 2\Theta_{12}\frac{dq}{dp} + (2\Theta_{13} - \Gamma_{11})\frac{dr}{dp} \\ & + \Theta_{22}\left(\frac{dq}{dp}\right)^2 + 2(\Theta_{23} - \Gamma_{12})\frac{dq}{dp}\frac{dr}{dp} + (\Theta_{33} - 2\Gamma_{13})\left(\frac{dr}{dp}\right)^2 \\ & - \Gamma_{22}\left(\frac{dq}{dp}\right)^2\frac{dr}{dp} - 2\Gamma_{23}\frac{dq}{dp}\left(\frac{dr}{dp}\right)^2 - \Gamma_{33}\left(\frac{dr}{dp}\right)^3.\end{aligned}$$

These are two equations, ordinary and of the second order, for the determination of q and r as functions of p. Their form shews* that the primitive contains four independent arbitrary constants; and these arbitrary constants can be determined by (i), the assignment of values of q and r for any initially assumed value of p, that is, by the assignment of any initial position in the region: and (ii), the assignment of values of dq/dp and dr/dp for that initial value of p, that is, as $dq/dp = q'/p'$ and $dr/dp = r'/p'$, and as $(A, B, C, F, G, H \mathbin{\rangle\!\langle} p', q', r')^2 = 1$, by the assignment of the direction of the curve through this initial position. Moreover, owing to the form of the equation, the primitive, which thus is determinate, is unique; and therefore we infer that *a geodesic of the region is uniquely defined by the assignment of any point on its course and the simultaneous assignment of its direction at that point.*

Other conditions may be assigned for the specification of a geodesic: thus it might be required to be a curve in the region drawn so as to join two assigned positions. But a question would then arise as to any limitations upon the relative positions of the points, e.g., possible limitations associated with the determination of the restricted region within which the conjugate of the initial position lies; and the necessary investigation can be undertaken only after the Jacobi criteria for the conjugate have been obtained, usually for the particular instance.

Examples of geodesics, in selected quadric regions.

302. We proceed to some examples† of geodesics, to indicate their course. After the first two examples, we leave the application of the Jacobi test for the geodesic range as exercises in analysis.

Ex. 1. We begin with geodesics in the globular region

$$x^2+y^2+z^2+v^2=a^2.$$

I. When the intrinsic property of a geodesic is used, whereby its radius of circular curvature coincides with the normal to the region, the differential equations of the geodesic are

$$\frac{x''}{x}=\frac{y''}{y}=\frac{z''}{z}=\frac{v''}{v}.$$

Each fraction is equal to

$$\frac{xx''+yy''+zz''+vv''}{x^2+y^2+z^2+v^2}$$

$$=-\frac{x'^2+y'^2+z'^2+v'^2}{a^2}=-\frac{1}{a^2},$$

so that the equations are

$$a^2x''+x=0, \quad a^2y''+y=0, \quad a^2z''+z=0, \quad a^2v''+v=0.$$

The primitive of the system is

$$x=A_1\cos\frac{s}{a}+A_2\sin\frac{s}{a}, \quad y=B_1\cos\frac{s}{a}+B_2\sin\frac{s}{a},$$

$$z=C_1\cos\frac{s}{a}+C_2\sin\frac{s}{a}, \quad v=D_1\cos\frac{s}{a}+D_2\sin\frac{s}{a},$$

* See my *Theory of Differential Equations*, vol. ii, chap. ii.

† Several of these occur in a paper by the author, *Bulletin of the Calcutta Mathematical Society*, vol. xx (1928), pp. 75—100.

where the arbitrary constants A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, satisfy the relations

$$A_1^2 + B_1^2 + C_1^2 + D_1^2 = a^2,$$
$$A_2^2 + B_2^2 + C_2^2 + D_2^2 = a^2,$$
$$A_1A_2 + B_1B_2 + C_1C_2 + D_1D_2 = 0.$$

Thus there appear to be five independent arbitrary constants: but one of these is unessential, as being additive to s in another mode of expression. Consequently there remain the four essential independent arbitrary constants.

From the primitive, we have

$$\begin{vmatrix} x, & y, & z, & v \\ A_1, & B_1, & C_1, & D_1 \\ A_2, & B_2, & C_2, & D_2 \end{vmatrix} = 0:$$

that is, the curve lies in a plane passing through the centre of the globe.

It is unnecessary to consider the specific application of the Legendre test and of the Weierstrass test: these have been proved to be satisfied by every geodesic in a region.

For the particular instance under consideration, the Jacobi test can be used in the geometric form of requirement, that the range shall not extend so far as the intersection of the circular curve with a consecutive circular curve continued until the two meet. To find this range, let s_0 be the value of s for the initial point and s_1 its value at the conjugate point: let $A_1 + \kappa a_1$, $A_2 + \kappa a_2$, be the changed values of A_1, A_2: let $B_1 + \kappa b_1$, $B_2 + \kappa b_2$, be the changed values of B_1, B_2; and so for the other constants. At the initial point, the circle and its consecutive intersect, so that the variations of x, of y, of z, and of v, vanish there: that is,

$$\kappa(a_1 \cos s_0 + a_2 \sin s_0) = 0, \quad \kappa(b_1 \cos s_0 + b_2 \sin s_0) = 0,$$
$$\kappa(c_1 \cos s_0 + c_2 \sin s_0) = 0, \quad \kappa(d_1 \cos s_0 + d_2 \sin s_0) = 0.$$

At the conjugate point, if any, the curves again meet for the first time after the initial point: the variations of x, of y, of z, and of v, must vanish there: that is,

$$\kappa(a_1 \cos s_1 + a_2 \sin s_1) = 0, \quad \kappa(b_1 \cos s_1 + b_2 \sin s_1) = 0,$$
$$\kappa(c_1 \cos s_1 + c_2 \sin s_1) = 0, \quad \kappa(d_1 \cos s_1 + d_2 \sin s_1) = 0.$$

There also are the limitations

$$\Sigma A_1 a_1 = 0, \quad \Sigma A_2 a_2 = 0, \quad \Sigma(A_1 a_2 + A_2 a_1) = 0;$$

but these have no effect on the relation between s_0 and s_1. From the earlier equations, because not all the quantities a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, could vanish (for then there would be no variation), we have

$$\sin\left(\frac{s_1}{a} - \frac{s_0}{a}\right) = 0:$$

and therefore, as we are to select the first point of intersection,

$$s_1 = s_0 + a\pi:$$

or, for a true unique minimum, the range of the geodesic along the great circle in the globe must not be greater than a half of the circle.

II. When the parametric representation of the globular region is adopted, we can take the three parameters p, q, r, so that

$$x = a \sin p \sin q, \quad y = a \sin p \cos q, \quad z = a \cos p \cos r, \quad v = a \cos p \sin r.$$

With this representation, we find

$$A = a^2, \quad B = a^2 \sin^2 p, \quad C = a^2 \cos^2 p, \quad F = 0, \quad G = 0, \quad H = 0,$$
$$\Omega^2 = a^6 \sin^2 p \cos^2 p;$$
$$X = -\sin p \sin q, \quad Y = -\sin p \cos q, \quad Z = -\cos p \cos r, \quad V = -\cos p \sin r,$$

it being remembered that (under the convention, p. 16) the positive direction for the normal is taken inwards into the region.

The secondary magnitudes are

$$L=a, \quad N=a\sin^2 p, \quad J=a\cos^2 p, \quad I=0, \quad K=0, \quad M=0.$$

Thus the radius of curvature of every normal section is a.

The quantities Γ, Δ, Θ, are

$$\begin{array}{lll} \Gamma_{11}=0 & , \ \Gamma_{12}=0 & , \ \Gamma_{13}=0 \ , \\ \Delta_{11}=0 & , \ \Delta_{12}=\cot p, & \Delta_{13}=0 \ , \\ \Theta_{11}=0 & , \ \Theta_{12}=0 & , \ \Theta_{13}=-\tan p \ , \\ \Gamma_{22}=-\sin p\cos p, & \Gamma_{23}=0 & , \ \Gamma_{33}=\sin p\cos p, \\ \Delta_{22}=0 & , \ \Delta_{23}=0 & , \ \Delta_{33}=0 \ , \\ \Theta_{22}=0 & , \ \Theta_{23}=0 & , \ \Theta_{33}=0 \ . \end{array}$$

Thus the differential equations of the geodesics are

$$\left.\begin{array}{l} -p''=-q'^2\sin p\cos p+r'^2\sin p\cos p \\ -q''=\ 2p'q'\cot p \\ -r''=-2p'r'\tan p \end{array}\right\},$$

equations which also can be derived immediately by applying the conditions characteristic of the minimum value of an integral

$$\int\left\{\left(\frac{dp}{dt}\right)^2+\left(\frac{dq}{dt}\right)^2\sin^2 p+\left(\frac{dr}{dt}\right)^2\cos^2 p\right\}^{\frac{1}{2}}dt.$$

From the second of these equations, we have

$$q'\sin^2 p=c;$$

and from the third, we have

$$r'\cos^2 p=k;$$

where c and k are arbitrary constants. When q' and r' are eliminated from the general arc-equation of the region

$$p'^2+q'^2\sin^2 p+r'^2\cos^2 p=1,$$

the equation in p alone is

$$p'^2+\frac{c^2}{\sin^2 p}+\frac{k^2}{\cos^2 p}=\frac{1}{a^2}.$$

An integral of this equation is

$$\sin^2 p=\frac{c}{\mu}\sin^2 t+c\mu\cos^2 t,$$

where now $at=s$, the (unessential) constant of integration is absorbed into the variable s, and μ is a constant dependent upon c and k by the relation

$$\left(1-\frac{c}{\mu}\right)(1-c\mu)=k^2.$$

The equation

$$q'\sin^2 p=c$$

now gives

$$\mu\tan(q+\alpha)=\tan t,$$

where α is an arbitrary constant; and the equation

$$r'\cos^2 p=k$$

now gives

$$\left(\frac{1-c\mu}{1-\frac{c}{\mu}}\right)^{\frac{1}{2}} \tan(r+\beta)=\tan t,$$

where β is an arbitrary constant.

Regarding k as expressible in terms of μ and c, we now have four arbitrary constants c, μ, α, β; or writing

$$\left(\frac{c}{\mu}\right)^{\frac{1}{2}}=\cos\gamma, \quad \left(1-\frac{c}{\mu}\right)^{\frac{1}{2}}=\sin\gamma,$$

$$(c\mu)^{\frac{1}{2}}=\cos\delta, \quad (1-c\mu)^{\frac{1}{2}}=\sin\delta,$$

we have α, β, γ, δ, as the four essential arbitrary constants for the primitive. With these constants, we have

$$\frac{\cos\gamma\sin t}{\sin(q+\alpha)}=\frac{\cos\delta\cos t}{\cos(q+\alpha)}=\sin p,$$

$$\frac{\sin\gamma\sin t}{\sin(r+\beta)}=\frac{\sin\delta\cos t}{\cos(r+\beta)}=\cos p.$$

Thus

$$\begin{aligned}
a\cos\gamma\sin t &= a\sin p\sin(q+\alpha) = \quad x\cos\alpha+y\sin\alpha,\\
a\cos\delta\cos t &= a\sin p\cos(q+\alpha) = -x\sin\alpha+y\cos\alpha,\\
a\sin\gamma\sin t &= a\cos p\sin(r+\beta) = \quad z\sin\beta+v\cos\beta,\\
a\sin\delta\cos t &= a\cos p\cos(r+\beta) = \quad z\cos\beta-v\sin\beta;
\end{aligned}$$

and therefore

$$\left.\begin{aligned}
x/a &= \cos\alpha\cos\gamma\sin t-\sin\alpha\cos\delta\cos t\\
y/a &= \sin\alpha\cos\gamma\sin t+\cos\alpha\cos\delta\cos t\\
z/a &= \sin\beta\sin\gamma\sin t+\cos\beta\sin\delta\cos t\\
v/a &= \cos\beta\sin\gamma\sin t-\sin\beta\sin\delta\cos t
\end{aligned}\right\},$$

which accordingly are the equations of a geodesic in the globe, t denoting s/a.

As before, the geodesic is a section of the globe by a plane through the centre, and therefore it is a great circle in the region.

The Jacobi test is concerned with the range between any point on the geodesic curve and the conjugate of that point; and the conjugate is the next intersection of a consecutive geodesic drawn through the initial point. Proceeding as in the earlier method, by now taking a curve with constants $\alpha+d\alpha$, $\beta+d\beta$, $\gamma+d\gamma$, $\delta+d\delta$, where $d\alpha$, $d\beta$, $d\gamma$, $d\delta$, are small, we find that the test gives the first value of t_1, greater than t_0, by the condition

$$\sin(t_1-t_0)=0,$$

that is,

$$s_1-s_0=a\pi:$$

or the curve is a unique geodesic shortest distance in the region if its length be less than half of the circle.

The curvatures of the geodesic follow at once from the general formulæ for the curvatures of skew curves (Chapter VII). In the present case, we have, by direct substitution,

$$\Sigma x'^2=1, \quad \Sigma x''^2=\frac{1}{a^2}, \quad \Sigma x'''^2=\frac{1}{a^4},$$

$$\begin{vmatrix} x', & y', & z', & v'\\ x'', & y'', & z'', & v''\\ x''', & y''', & z''', & v'''\\ x^{\mathrm{iv}}, & y^{\mathrm{iv}}, & z^{\mathrm{iv}}, & v^{\mathrm{iv}} \end{vmatrix}=0.$$

The last result shews that there is no curvature of tilt. The second result shews that the radius of circular curvature is equal to a, the constant radius of the globular region. If σ be the radius of torsion, the third result gives

$$\frac{1}{\sigma^2\rho^2}+\frac{1}{\rho^4}+\frac{\rho'^2}{\rho^4}=\frac{1}{a^4},$$

so that, as $\rho=a$, we have $1/\sigma=0$, or there is no curvature of torsion. These results are in agreement with the fact that the curve is a plane curve, so that there is neither torsion nor tilt.

The radii of spherical curvature and of globular curvature are, each of them, equal to a.

Ex. 2. As a second example, consider the geodesics in the sphero-cylindrical region

$$x^2+y^2+z^2=a^2.$$

A parametric representation is given by

$$x=a\sin\theta\cos\phi, \quad y=a\sin\theta\sin\phi, \quad z=a\cos\theta, \quad v=v,$$

so that θ, ϕ, v, are the parameters. Then

$$A=a^2, \quad B=a^2\sin^2\theta, \quad C=1, \quad F=0, \quad G=0, \quad H=0,$$
$$\Omega^2=a^4\sin^2\theta.$$

All the quantities Γ, Δ, Θ, vanish except Γ_{22} and Δ_{12}; while

$$\Gamma_{22}=-\sin\theta\cos\theta, \quad \Delta_{12}=\cot\theta.$$

Thus the characteristic geodesic equations are

$$\left.\begin{aligned} -\theta'' &= -\sin\theta\cos\theta\,.\,\phi'^2 \\ -\phi'' &= \quad 2\cot\theta\,.\,\theta'\phi' \\ -v'' &=0 \end{aligned}\right\}.$$

The last equation yields a first integral

$$v'=\text{constant}=\sin\alpha.$$

The second equation yields another first integral

$$\phi'\,.\,\sin^2\theta=\text{constant}=k\sin\beta.$$

Both of these equations can be derived at once from the critical equations for the minimisation of the arc-integral

$$a\int\left\{\left(\frac{d\theta}{dt}\right)^2+\left(\frac{d\phi}{dt}\right)^2\sin^2\theta+\left(\frac{dv}{dt}\right)^2\right\}^{\frac{1}{2}}dt.$$

The former integral arises from the critical equation for the parameter v: the latter integral arises from the critical equation for the parameter ϕ.

The permanent arc-equation becomes, in this instance,

$$\theta'^2+\phi'^2\sin^2\theta+\frac{1}{a^2}v'^2=\frac{1}{a^2}.$$

When substitution is made for v', we find

$$\theta'^2+\phi'^2\sin^2\theta=\frac{\cos^2\alpha}{a^2}=k^2,$$

where k is a constant, alternative to α; and then, on substitution for ϕ',

$$\theta'^2=k^2\left(1-\frac{\sin^2\beta}{\sin^2\theta}\right),$$

so that

$$\cos\theta=\cos\beta\cos ks,$$

where an (unessential) arbitrary constant of integration has been absorbed into the variable s.

We now have

$$\frac{d\phi}{ds}=\frac{k\sin\beta}{\sin^2\theta}=\frac{k\sin\beta}{1-\cos^2\beta\cos^2 ks},$$

and therefore

$$\sin\beta\tan(\phi-\delta)=\tan ks,$$

where δ is a constant of integration. Thus

$$\frac{\sin(\phi-\delta)}{\sin ks}=\frac{\cos(\phi-\delta)}{\sin\beta\cos ks}=\frac{1}{(\sin^2 ks+\sin^2\beta\cos^2 ks)^{\frac{1}{2}}}=\frac{1}{\sin\theta},$$

so that

$$\sin\theta\sin(\phi-\delta)=\sin ks,\quad \sin\theta\cos(\phi-\delta)=\sin\beta\cos ks.$$

The former of these two relations is

$$-x\sin\delta+y\cos\delta=a\sin ks:$$

the latter is

$$x\cos\delta+y\sin\delta=a\sin\beta\cos ks.$$

Thus

$$\left.\begin{aligned} x&=a(\sin\beta\cos\delta\cos ks-\sin\delta\sin ks)\\ y&=a(\sin\beta\sin\delta\cos ks+\cos\delta\sin ks)\\ z&=a\cos\beta\cos ks\\ v&=v_0+s\sin\alpha \end{aligned}\right\},$$

where v_0, δ, β, k (or α), are the four essential arbitrary constants.

For the curvatures of this geodesic (which manifestly is an extension, to quadruple space, of the cylindrical circular helix in triple space), we can proceed from the integral equations. By direct substitution, we find

$$\Sigma x'^2=1,\quad \Sigma x''^2=\frac{\cos^4\alpha}{a^2},\quad \Sigma x'''^2=\frac{\cos^6\alpha}{a^4},$$

$$\begin{vmatrix} x', & y', & z', & v'\\ x'', & y'', & z'', & v''\\ x''', & y''', & z''', & v'''\\ x^{\mathrm{iv}}, & y^{\mathrm{iv}}, & z^{\mathrm{iv}}, & v^{\mathrm{iv}} \end{vmatrix}=0.$$

(The last result comes at once from the fact that $x^{\mathrm{iv}}=-k^2x''$, $y^{\mathrm{iv}}=-k^2y''$, $z^{\mathrm{iv}}=-k^2z''$, $v''=0$, $v^{\mathrm{iv}}=0$.)

From the second of these equations, we have

$$\frac{1}{\rho^2}=\frac{\cos^4\alpha}{a^2},$$

that is, ρ is constant and equal to $a\sec^2\alpha$. The third gives

$$\frac{1}{\sigma^2\rho^2}+\frac{1}{\rho^4}+\frac{\rho'^2}{\rho^4}=\frac{\cos^6\alpha}{a^4}:$$

and therefore, on the insertion of the constant value of ρ,

$$\sigma=a\sec\alpha\operatorname{cosec}\alpha.$$

The third equation gives zero as the curvature of tilt.

Thus, along the geodesic, there is no tilt: the circular curvature and the curvature of torsion are constant.

The radius of spherical curvature and the radius of globular curvature are equal, each of them, to $a\sec^2\alpha$.

The Legendre test and the Weierstrass test are known to be satisfied. The Jacobi test limits the range so that it shall not pass beyond the conjugate of an initial point: and this conjugate is obtained as the next intersection of a consecutive geodesic through the initial point. We can take two of the integral equations in the form

$$x\cos\delta+y\sin\delta=z\tan\beta,\quad x^2+y^2+z^2=a^2.$$

When a consecutive curve is drawn through an initial point, the constants change: let β become $\beta+\Delta\beta$, and δ become $\delta+\Delta\delta$. The second equation provides no limitation. When the first equation is taken in the form

$$z=(x\cos\delta+y\sin\delta)\cot\beta,$$

the coincidence of the initial points for the consecutive curve requires that

$$\frac{\partial z}{\partial\delta}\Delta\delta+\frac{\partial z}{\partial\beta}\Delta\beta=0,$$

and therefore, at that point,

$$\frac{\Delta\beta}{\Delta\delta}=\frac{x\sin\delta-y\cos\delta}{x\cos\delta+y\sin\delta}\cos\beta\sin\beta$$
$$=\cos\beta\tan ks_0.$$

Similarly the coincidence of the curves at the conjugate point, defined by a value s_1 of s, requires the relation

$$\frac{\Delta\beta}{\Delta\delta}=\cos\beta\tan ks_1.$$

Thus we must have

$$\tan ks_1=\tan ks_0,$$

as defining the conjugate point: that is,

$$\tan(\phi_1-\delta)=\tan(\phi_0-\delta),$$

or the conjugate is given by the relation

$$\phi_1-\phi_0=\pi.$$

In order to have a geodesic minimum, the range of the curve must not extend more than half-way round the sphero-cylindric region.

Ex. 3. In the sphero-conical region

$$v^2=(x^2+y^2+z^2)\tan^2\alpha,$$

we take

$$x=r\sin\theta\cos\phi,\quad y=r\sin\theta\sin\phi,\quad z=r\cos\theta,\quad v=r\tan\alpha.$$

We have

$$A=\sec^2\alpha,\quad B=r^2,\quad C=r^2\sin^2\theta,\quad F=0,\quad G=0,\quad H=0,$$
$$\Omega^2=r^4\sin^2\theta\sec^2\alpha.$$

The normal direction inwards at the point is given by

$$X=-\sin\alpha\sin\theta\cos\phi,\quad Y=-\sin\alpha\sin\theta\sin\phi,\quad Z=-\sin\alpha\cos\theta,\quad V=\cos\alpha;$$

and the secondary magnitudes are

$$L=0,\quad N=r\sin\alpha,\quad J=r\sin\alpha\sin^2\theta,\quad I=0,\quad K=0,\quad M=0.$$

The quantities Γ, Δ, Θ are

$$\begin{array}{lll}
\Gamma_{11}=0, & \Gamma_{12}=0, & \Gamma_{13}=0,\\
\Delta_{11}=0, & \Delta_{12}=1/r, & \Delta_{13}=0,\\
\Theta_{11}=0, & \Theta_{12}=0, & \Theta_{13}=1/r,\\
\Gamma_{22}=-r\cos^2\alpha, & \Gamma_{23}=0, & \Gamma_{33}=-r\cos^2\alpha\sin^2\theta,\\
\Delta_{22}=0, & \Delta_{23}=0, & \Delta_{33}=-\sin\theta\cos\theta,\\
\Theta_{22}=0, & \Theta_{23}=\cot\theta, & \Theta_{33}=0
\end{array}$$

The characteristic equations of the geodesics are

$$\left.\begin{aligned} -r'' &= -r\theta'^2\cos^2\alpha - r\phi'^2\cos^2\alpha\sin^2\theta \\ -\theta'' &= \frac{2}{r}r'\theta' - \phi'^2\sin\theta\cos\theta \\ -\phi'' &= \frac{2}{r}r'\phi' + 2\theta'\phi'\cot\theta \end{aligned}\right\};$$

they can be derived at once, by minimising the integral

$$\int\left\{\left(\frac{dr}{dt}\right)^2\sec^2\alpha + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\sin^2\theta\right\}^{\frac{1}{2}}dt.$$

Integrals of the characteristic equations are

$$r^2\phi'\sin^2\theta = k\cos\alpha\sin\beta,$$

$$r'^2 = \left(1 - \frac{k^2}{r^2}\cos^2\alpha\right)\cos^2\alpha,$$

where k and β are constants of integration.

The march of the geodesic is given by

$$\left.\begin{aligned} \cos\theta &= \cos\beta\cos u \\ \sin\beta\tan(\phi-\gamma) &= \tan u \\ s &= k\tan(u\cos\alpha+\delta) \\ r &= k\sec(u\cos\alpha+\delta)\cos\alpha \end{aligned}\right\};$$

and the Cartesian coordinates of any point on the geodesic become

$$\left.\begin{aligned} y\cos\gamma - x\sin\gamma &= r\sin u \\ y\sin\gamma + x\cos\gamma &= r\sin\beta\cos u \\ z &= r\cos\beta\cos u \\ v &= r\tan\alpha \end{aligned}\right\},$$

where r is expressible in terms of u, and u is expressible in terms of s, by the preceding equations. The arbitrary constants are k, β, γ, δ.

The circular curvature of the geodesic at any point is

$$\frac{k^2}{r^3}\sin\alpha\cos^2\alpha.$$

The determination of the curvature of torsion and of the curvature of tilt is left as an exercise.

Ex. 4. Shew that, when the dicyclic-conical region

$$z^2 + v^2 = (x^2+y^2)\tan^2\alpha$$

is represented in the form

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = r\cos\theta\tan\alpha, \quad v = r\sin\theta\tan\alpha,$$

the geodesics are given by the equations

$$\phi - \gamma = u\cos\beta\sec\alpha, \quad \theta - \delta = u\sin\beta\operatorname{cosec}\alpha, \quad r = k\sec u,$$

where k is a constant, and u is a function of s given by

$$s\cos\alpha = k\tan u.$$

Ex. 5. Consider the geodesics in the dicyclic-elliptic region

$$\frac{x^2+y^2}{a} + \frac{z^2+v^2}{b} = 1.$$

For a parametric representation, we can take

$$x=p\cos\theta,\quad y=p\sin\theta,\quad z=q\cos\phi,\quad v=q\sin\phi,$$

so that

$$\frac{p^2}{a}+\frac{q^2}{b}=1.$$

We can express ρ and σ in terms of a single parameter λ in the forms

$$p^2=\frac{a(a+\lambda)}{a-b},\quad q^2=\frac{b(b+\lambda)}{b-a};$$

and now we take θ, ϕ, λ, as the parameters for the region. Then

$$\begin{aligned}ds^2&=dp^2+p^2d\theta^2+dq^2+q^2d\phi^2\\&=\frac{a(a+\lambda)}{a-b}d\theta^2+\frac{b(b+\lambda)}{b-a}d\phi^2+\frac{1}{4}\frac{\lambda}{(a+\lambda)(b+\lambda)}d\lambda^2.\end{aligned}$$

The characteristic equations are obtained by minimising the integral

$$\int\left\{\frac{a(a+\lambda)}{a-b}\left(\frac{d\theta}{dt}\right)^2+\frac{b(b+\lambda)}{b-a}\left(\frac{d\phi}{dt}\right)^2+\frac{1}{4}\frac{\lambda}{(a+\lambda)(b+\lambda)}\left(\frac{d\lambda}{dt}\right)^2\right\}^{\frac{1}{2}}dt.$$

As the subject of integration does not involve θ explicitly or ϕ explicitly, the two simplest of the three equations—and only two out of the three need be retained (§ 300)—are

$$\frac{d}{ds}\left\{\frac{a(a+\lambda)}{a-b}\frac{d\theta}{ds}\right\}=0,\quad \frac{d}{ds}\left\{\frac{b(b+\lambda)}{b-a}\frac{d\phi}{ds}\right\}=0.$$

Thus two first integrals are

$$\frac{a(a+\lambda)}{a-b}\theta'=\text{constant}=\gamma a^{\frac{1}{2}},$$

$$\frac{b(b+\lambda)}{b-a}\phi'=\text{constant}=\kappa b^{\frac{1}{2}},$$

where γ and κ are arbitrary constants. We also have the permanent arc-equation, which here is

$$\frac{a(a+\lambda)}{a-b}\theta'^2+\frac{b(b+\lambda)}{b-a}\phi'^2+\frac{1}{4}\frac{\lambda}{(a+\lambda)(b+\lambda)}\lambda'^2=1.$$

Let the values of θ' and ϕ' from the two first integrals be used: then

$$\begin{aligned}\frac{1}{4}\lambda\left(\frac{d\lambda}{ds}\right)^2&=(a+\lambda)(b+\lambda)-(a-b)\{\gamma^2(b+\lambda)-\kappa^2(a+\lambda)\}\\&=(\xi-\lambda)(\eta-\lambda),\end{aligned}$$

where ξ and η are two new arbitrary constants, substituted for γ and κ according to the relations

$$(a+\xi)(a+\eta)=\gamma^2(a-b)^2,\quad (b+\xi)(b+\eta)=\kappa^2(b-a)^2.$$

Consequently,

$$\left.\begin{aligned}ds&=\frac{1}{2}\left\{\frac{\lambda}{(\xi-\lambda)(\eta-\lambda)}\right\}^{\frac{1}{2}}d\lambda\\\left\{\frac{a}{(a+\xi)(a+\eta)}\right\}^{\frac{1}{2}}d\theta&=\frac{1}{2}\left\{\frac{\lambda}{(\xi-\lambda)(\eta-\lambda)}\right\}^{\frac{1}{2}}\frac{d\lambda}{a+\lambda}\\\left\{\frac{b}{(b+\xi)(b+\eta)}\right\}^{\frac{1}{2}}d\phi&=\frac{1}{2}\left\{\frac{\lambda}{(\xi-\lambda)(\eta-\lambda)}\right\}^{\frac{1}{2}}\frac{d\lambda}{b+\lambda}\end{aligned}\right\},$$

which can be regarded as equivalent to the equations of the geodesics. Manifestly, elliptic functions are required to obtain the explicit primitive of the equations.

One arbitrary constant is associated with θ additively, and another with ϕ additively. These results are to be expected. For the region can be rotated through any angle round the plane ZOV without changing the geometry of the region, and such rotation implies an arbitrary addition to the value of θ; and similarly for an independent arbitrary rotation round the plane XOY, with an arbitrary addition to the value of ϕ.

Ex. 6. Geodesics in the sphero-elliptic region

$$\frac{x^2+y^2+z^2}{a}+\frac{v^2}{c}=1$$

are similarly represented by expressions which involve elliptic functions.

We take

$$x=r\sin\theta\cos\phi,\quad y=r\sin\theta\sin\phi,\quad z=r\cos\theta,$$

$$r=a^{\frac{1}{2}}\sin\psi,\quad v=c^{\frac{1}{2}}\cos\psi;$$

and then

$$\begin{aligned}ds^2&=dx^2+dy^2+dz^2+dv^2\\&=r^2d\theta^2+r^2\sin^2\theta\,d\phi^2+dr^2+dv^2\\&=a\sin^2\psi\,d\theta^2+a\sin^2\psi\sin^2\theta\,d\phi^2+a\,(1-k^2\sin^2\psi)\,d\psi^2,\end{aligned}$$

where k^2 is a definite constant such that

$$k^2a=a-c.$$

The characteristic equations can be obtained by minimising the integral

$$\int\left\{\left(\frac{d\theta}{dt}\right)^2\sin^2\psi+\left(\frac{d\phi}{dt}\right)^2\sin^2\psi\sin^2\theta+\left(\frac{d\psi}{dt}\right)^2(1-k^2\sin^2\psi)\right\}^{\frac{1}{2}}dt.$$

As the permanent arc-equation

$$\theta'^2\sin^2\psi+\phi'^2\sin^2\psi\sin^2\theta+\psi'^2(1-k^2\sin^2\psi)=\frac{1}{a}$$

is an integral, we retain only the two simplest of the three characteristic equations.

The simplest of all these equations arises out of the variable ϕ which does not occur explicitly in the subject of integration: it is

$$\frac{d}{ds}(\phi'\sin^2\psi\sin^2\theta)=0,$$

leading to a first integral

$$\phi'\sin^2\psi\sin^2\theta=\text{constant}=\frac{1}{l},$$

where l is an arbitrary constant.

The characteristic equation, which arises out of the variation of the variable θ, is

$$\frac{d}{ds}\left(\frac{d\theta}{ds}\sin^2\psi\right)-\left(\frac{d\phi}{ds}\right)^2\sin\theta\cos\theta\sin^2\psi=0.$$

When the preceding first integral is used to eliminate $\frac{d\phi}{ds}$, this equation becomes

$$\sin^2\psi\frac{d}{ds}\left(\frac{d\theta}{ds}\sin^2\psi\right)-\frac{1}{l^2}\frac{\cos\theta}{\sin^3\theta}=0;$$

and an integral of this equation is

$$\left(\frac{d\theta}{ds}\right)^2\sin^4\psi=\frac{1}{l^2}\left(\frac{1}{\sin^2\alpha}-\frac{1}{\sin^2\theta}\right),$$

where α is an arbitrary constant of integration.

When these two integrals are used to remove θ' and ϕ' from the permanent arc-equation, the latter becomes

$$\begin{aligned}\frac{1}{a}&=\frac{1}{l^2}\left(\frac{1}{\sin^2\alpha}-\frac{1}{\sin^2\theta}\right)\frac{1}{\sin^2\psi}+\frac{1}{l^2}\frac{1}{\sin^2\theta\sin^2\psi}+\psi'^2(1-k^2\sin^2\psi)\\&=\frac{1}{l^2\sin^2\alpha\sin^2\psi}+\psi'^2(1-k^2\sin^2\psi),\end{aligned}$$

an equation which leads to an elliptic function relation between s and ψ.

The equation

$$\theta'^2 \sin^2\psi = \frac{1}{l^2}\left(\frac{1}{\sin^2\alpha} - \frac{1}{\sin^2\theta}\right),$$

on substitution for ψ in terms of s, then leads to an expression for θ in terms of elliptic functions. Finally, the equation

$$\phi' \sin^2\psi \sin^2\theta = \frac{1}{l},$$

then leads to an expression for ϕ in terms of elliptic functions.

Ex. 7. Consider the geodesics in the cyclo-ellipsoidal region

$$\frac{x^2+y^2}{a} + \frac{z^2}{b} + \frac{v^2}{c} = 1.$$

We take

$$x = t\cos r, \quad y = t\sin r$$

where

$$\frac{t^2}{a} + \frac{z^2}{b} + \frac{v^2}{c} = 1.$$

With the customary ellipsoidal coordinates in homaloidal triple space, we can take

$$t^2 = \frac{a}{(a-b)(a-c)}(a+p)(a+q),$$

$$z^2 = \frac{b}{(b-a)(b-c)}(b+p)(b+q),$$

$$v^2 = \frac{c}{(c-a)(c-b)}(c+p)(c+q);$$

and then*, as usual, we have

$$dt^2+dz^2+dv^2 = \frac{1}{4}\left\{\frac{p(p-q)}{(a+p)(b+p)(c+p)}dp^2 + \frac{q(q-p)}{(a+q)(b+q)(c+q)}dq^2\right\}.$$

Then, for the arc in this region,

$$ds^2 = dt^2 + t^2dr^2 + dz^2 + dv^2$$
$$= A\,dp^2 + B\,dq^2 + C\,dr^2,$$

where

$$C = t^2 = \frac{a}{(a-b)(a-c)}(a+p)(a+q),$$

$$A = (p-q)P, \quad B = (q-p)Q, \quad F = 0, \quad G = 0, \quad H = 0,$$

while

$$P = \frac{1}{4}\frac{p}{(a+p)(b+p)(c+p)}, \quad Q = \frac{1}{4}\frac{q}{(a+q)(b+q)(c+q)},$$

so that A, B, C, do not involve the parameter r.

As the region is cyclic round the plane ZOV, an equation of the geodesic belonging specially to the orthogonal plane XOY is immediately integrable: it is

$$\frac{x''}{x} = \frac{y''}{y},$$

(the other equations of the geodesic are not so directly useful), and so

$$xy' - yx' = \text{constant},$$

that is,

$$Cr' = \mu,$$

where μ is an arbitrary constant of integration.

* *Lectures on Differential Geometry*, § 78.

Proceeding to the formal set of three equations, either by framing the values of the quantities Γ, Δ, Θ, or by minimising the integral

$$\int \left\{ A\left(\frac{dp}{d\theta}\right)^2 + B\left(\frac{dq}{d\theta}\right)^2 + C\left(\frac{dr}{d\theta}\right)^2 \right\}^{\frac{1}{2}} d\theta,$$

we obtain them in the form

$$\left.\begin{aligned} -2Ap'' &= A_1 p'^2 + 2A_2 p'q' - B_1 q'^2 - C_1 r'^2 \\ -2Bq'' &= -A_2 p'^2 + 2B_1 p'q' + B_2 q'^2 - C_2 r'^2 \\ -2Cr'' &= 2C_1 p'r' + 2C_2 q'r' \end{aligned}\right\},$$

the third of which immediately leads to the foregoing first integral

$$Cr' = \mu.$$

With the preceding values of A, B, C, we have

$$\left.\begin{aligned} \frac{A_1}{A} &= \frac{1}{p-q} + \frac{P_1}{P}, & \frac{A_2}{A} &= \frac{1}{q-p} \\ \frac{B_1}{B} &= \frac{1}{p-q}, & \frac{B_2}{B} &= \frac{1}{q-p} + \frac{Q_2}{Q} \\ \frac{C_1}{C} &= \frac{1}{a+p}, & \frac{C_2}{C} &= \frac{1}{a+q} \end{aligned}\right\}.$$

To integrate these equations, we introduce two new variables ξ and η, by the defining relations

$$\frac{Pp'^2}{\xi+p} = \frac{Qq'^2}{\xi+q} = \frac{Cr'^2}{\eta}.$$

When these are used to remove p' and q' from the arc-relation, it becomes

$$\begin{aligned} 1 &= (p-q)\,Pp'^2 + (q-p)\,Qq'^2 + Cr'^2 \\ &= \left\{ 1 + \frac{(p-q)^2}{\eta} \right\} Cr'^2, \end{aligned}$$

and, therefore, as $Cr' = \mu$,

$$\eta = \frac{\mu^2}{C-\mu^2}(p-q)^2:$$

or, writing

$$\mu^2 = \frac{a}{(a-b)(a-c)}\kappa, \quad (a+p)(a+q) = \zeta,$$

where κ is a new constant and ζ is a new variable, we have

$$\eta = \frac{\kappa}{\zeta-\kappa}(p-q)^2;$$

and the postulated defining relations become

$$\frac{Pp'^2}{\xi+p} = \frac{Qq'^2}{\xi+q} = \frac{\zeta-\kappa}{\zeta}\frac{1}{(p-q)^2}.$$

Taking logarithmic derivatives of the first and third fractions with regard to s, we have

$$\frac{2p''}{p'} + \frac{P_1}{P}p' - \frac{\xi'+p'}{\xi+p} = \frac{\zeta'\kappa}{\zeta(\zeta-\kappa)} - 2\frac{p'-q'}{p-q}.$$

When we use the first of the three characteristic equations, substitution of the values of A, A_1, A_2, B_1, C_1, leads to the relation

$$-2\frac{p''}{p'} = \left(\frac{1}{p-q} + \frac{P_1}{P}\right)p' + \frac{2}{q-p}q' + \frac{1}{p-q}\frac{\xi+q}{\xi+p}p' - \frac{\kappa}{\zeta-\kappa}\frac{p-q}{(\xi+p)(a+p)}p';$$

and therefore the foregoing derived equation is equivalent to

$$\frac{\zeta'\kappa}{\zeta(\zeta-\kappa)} = -\frac{\xi'}{\xi+p} + \frac{\kappa}{\zeta-\kappa}\frac{p-q}{(\xi+p)(a+p)}p'.$$

Similarly, logarithmic derivation of the second and third fractions leads to the equation

$$\frac{\zeta'\kappa}{\zeta(\zeta-\kappa)} = -\frac{\xi'}{\xi+q} + \frac{\kappa}{\zeta-\kappa}\frac{q-p}{(\xi+q)(a+q)}q'.$$

Now ζ is an explicit function of p and q; and ξ is a desired function of p and q. These two equations are satisfied by making the coefficients of p' and of q' respectively equal to one another on the two sides of each equation. The first equation thus gives

$$-\frac{1}{\xi+p}\xi_2 = \frac{\kappa}{\zeta-\kappa}\frac{1}{a+q},$$
$$-\frac{1}{\xi+p}\xi_1 = \frac{\kappa}{\zeta-\kappa}\frac{1}{a+p} - \frac{\kappa}{\zeta-\kappa}\frac{p-q}{(\xi+p)(a+p)};$$

and the second equation thus gives

$$-\frac{1}{\xi+q}\xi_1 = \frac{\kappa}{\zeta-\kappa}\frac{1}{a+p},$$
$$-\frac{1}{\xi+q}\xi_2 = \frac{\kappa}{\zeta-\kappa}\frac{1}{a+q} - \frac{\kappa}{\zeta-\kappa}\frac{q-p}{(\xi+q)(a+q)}.$$

All these four equations are satisfied by, and are algebraically equivalent to, the two equations

$$\xi_1 + \frac{\kappa}{\zeta-\kappa}\frac{\xi+q}{a+p} = 0, \quad \xi_2 + \frac{\kappa}{\zeta-\kappa}\frac{\xi+p}{a+q} = 0.$$

These two equations satisfy the condition

$$\frac{\partial\xi_1}{\partial q} = \frac{\partial\xi_2}{\partial p}$$

of coexistence, account being taken of the value of ζ. Thus the differential equation

$$d\xi + \frac{\kappa}{\zeta-\kappa}\left(\frac{\xi+q}{a+p}dp + \frac{\xi+p}{a+q}dq\right) = 0$$

is exact; and its integral is

$$\xi = \frac{\kappa}{\zeta-\kappa}(p+q+a) + E\frac{\zeta}{\zeta-\kappa},$$

where E is an arbitrary constant of integration. As κ is an arbitrary constant (in place of μ), we thus have two arbitrary constants of integration in the first order equations of the geodesic.

To complete the integration leading to a primitive, we have

$$\begin{aligned}(\xi+p)(\zeta-\kappa) &= \kappa(p+q+a) + E\zeta + p\zeta - p\kappa \\ &= (E+p)\zeta + \kappa(a+q) \\ &= \zeta\left(E+p+\frac{\kappa}{a+p}\right) \\ &= \frac{\zeta}{a+p}(p+\alpha)(p+\beta),\end{aligned}$$

where α and β are two constants, taken in place of the arbitrary constants E and κ (and therefore also arbitrary), such that

$$(E+p)(a+p) + \kappa = (p+\alpha)(p+\beta)$$

for all values of p. Manifestly

$$\kappa = (a-\alpha)(a-\beta),$$

and thus

$$\frac{(a-\alpha)(a-\beta)}{(a-b)(a-c)}.$$

The preceding defining relations, connecting p'^2 and q'^2 with ξ, now give

$$Pp'^2(p-q)^2=(\xi+p)\frac{\zeta-\kappa}{\zeta}$$
$$=\frac{(p+\alpha)(p+\beta)}{a+p};$$

and therefore, when we write

$$f(p)=p(p+b)(p+c)(p+\alpha)(p+\beta),$$

we have

$$\tfrac{1}{2}p\frac{dp}{ds}(p-q)=\{f(p)\}^{\frac{1}{2}}.$$

Similarly, we have

$$\tfrac{1}{2}q\frac{dq}{ds}(q-p)=\{f(q)\}^{\frac{1}{2}}.$$

Consequently, we have

$$\frac{pdp}{2\{f(p)\}^{\frac{1}{2}}}+\frac{qdq}{2\{f(q)\}^{\frac{1}{2}}}=0.$$

Let a new variable ω be introduced by the definition

$$\frac{dp}{2\{f(p)\}^{\frac{1}{2}}}+\frac{dq}{2\{f(q)\}^{\frac{1}{2}}}=d\omega,$$

in association with the preceding equation; then

$$ds=-pq\,d\omega,$$

giving the relation between ds and $d\omega$.

Finally, for the expression of r along the geodesic, we have

$$a\frac{(a+p)(a+q)}{(a-b)(a-c)}dr=\mu ds$$
$$=\left\{a\frac{(a-\alpha)(a-\beta)}{(a-b)(a-c)}\right\}^{\frac{1}{2}}ds,$$

so that

$$a(a+p)(a+q)\,dr=\{a(a-b)(a-c)(a-\alpha)(a-\beta)\}^{\frac{1}{2}}ds,$$

that is,

$$dr=-\frac{pq}{(a+p)(a+q)}\{-f(-a)\}^{\frac{1}{2}}\frac{d\omega}{a}.$$

Obviously, as the function f is a quintic polynomial, p and q are hyperelliptic functions (of the simplest rank, two) of the argument ω and a constant argument. Hence also r is expressible by means of hyperelliptic functions, integrals of the second kind being required; and therefore the coordinates of any point on a geodesic in the cyclo-ellipsoidal region are also expressible by means of hyperelliptic functions of the simplest rank.

Ex. 8. The full analytical integral expression of geodesics in a paraboloidal region

$$\frac{y^2}{a}+\frac{z^2}{b}+\frac{v^2}{c}=4x$$

also requires the use of hyperelliptic functions.

(i) To represent the region parametrically, we associate with it the confocal paraboloidal regions

$$\frac{y^2}{a+\mu}+\frac{z^2}{b+\mu}+\frac{v^2}{c+\mu}=4(x+\mu),$$

being the regions the sections of which by the planes YOZ, ZOV, VOY, are parabolas respectively confocal with the same sections of the given region. There manifestly are four such regions, distinct from one another, passing through any given point; and they are

determined by the roots of the last equation, regarded as a quartic in μ. One root is zero: it determines the given region. Let the other three roots be p, q, r, which are unequal to one another and are distinct from zero. Now

$$\frac{y^2}{a+\omega}+\frac{z^2}{b+\omega}+\frac{v^2}{c+\omega}-4(x+\omega)=-4\omega\frac{(\omega-p)(\omega-q)(\omega-r)}{(\omega+a)(\omega+b)(\omega+c)},$$

for all values of ω; hence

$$\left.\begin{aligned} y^2&=-4a\frac{(a+p)(a+q)(a+r)}{(a-b)(a-c)}\\ z^2&=-4b\frac{(b+p)(b+q)(b+r)}{(b-c)(b-a)}\\ v^2&=-4c\frac{(c+p)(c+q)(c+r)}{(c-a)(c-b)}\\ x&=-a-b-c-p-q-r \end{aligned}\right\},$$

equations which accordingly represent the region in terms of three parameters p, q, r.

The primary magnitudes for the region are

$$\left.\begin{aligned} A&=\frac{p(p-q)(p-r)}{(a+p)(b+p)(c+p)}=(p-q)(p-r)P, & F&=0\\ B&=\frac{q(q-p)(q-r)}{(a+q)(b+q)(c+q)}=(q-p)(q-r)Q, & G&=0\\ C&=\frac{r(r-p)(r-q)}{(a+r)(b+r)(c+r)}=(r-p)(r-q)R, & H&=0 \end{aligned}\right\},$$

where

$$P=\frac{p}{(a+p)(b+p)(c+p)},\quad Q=\frac{q}{(a+q)(b+q)(c+q)},\quad R=\frac{r}{(a+r)(b+r)(c+r)}.$$

By constructing the quantities Γ, Δ, Θ, or immediately by minimising the integral

$$\int\left\{A\left(\frac{dp}{dt}\right)^2+B\left(\frac{dq}{dt}\right)^2+C\left(\frac{dr}{dt}\right)^2\right\}^{\frac{1}{2}}dt,$$

the three characteristic equations of the geodesics are

$$\left.\begin{aligned} -2Ap''=&\ A_1p'^2+2A_2p'q'+2A_3p'r'-B_1q'^2 &&-C_1r'^2\\ -2Bq''=&-A_2p'^2+2B_1p'q' &&+B_2q'^2+2B_3q'r'-C_2r'^2\\ -2Cr''=&-A_3p'^2 &&+2C_1p'r'-B_3q'^2+2C_2q'r'+C_3r'^2 \end{aligned}\right\}.$$

(ii) Three new variables ξ, η, ζ, are introduced, initially as to their ratios alone, by the defining relations

$$\frac{Pp'^2}{\xi}=\frac{Qq'^2}{\eta}=\frac{Rr'^2}{\zeta}.$$

(iii) We proceed to modify the characteristic equations. We have

$$\frac{A_1}{A}=\frac{P_1}{P}+\frac{1}{p-q}+\frac{1}{p-r},\quad \frac{A_2}{A}=\frac{1}{q-p},\quad \frac{A_3}{A}=\frac{1}{r-p};$$

$$\frac{B_2}{B}=\frac{Q_2}{Q}+\frac{1}{q-r}+\frac{1}{q-p},\quad \frac{B_3}{B}=\frac{1}{r-q},\quad \frac{B_1}{B}=\frac{1}{p-q};$$

$$\frac{C_3}{C}=\frac{R_3}{R}+\frac{1}{r-p}+\frac{1}{r-q},\quad \frac{C_1}{C}=\frac{1}{p-r},\quad \frac{C_2}{C}=\frac{1}{q-r}.$$

With these values, the first of the three characteristic equations of the geodesics becomes

$$-2p''=\left(\frac{P_1}{P}+\frac{1}{p-q}+\frac{1}{p-r}\right)p'^2+\frac{2p'q'}{q-p}+\frac{2p'r'}{r-p}+\frac{Bq'^2}{A(q-p)}+\frac{Cr'^2}{A(r-p)}.$$

But

$$\frac{Bq'^2}{A(q-p)} = \frac{(q-r)p'^2}{(p-q)(p-r)}\frac{\eta}{\xi}, \qquad \frac{Cr'^2}{A(r-p)} = \frac{(r-q)p'^2}{(p-r)(p-q)}\frac{\zeta}{\xi};$$

and therefore the first characteristic equation becomes

$$-\frac{2p''}{p'} - \frac{P_1 p'}{P} = \frac{p'-2q'}{p-q} + \frac{p'-2r'}{p-r} - p'\left(\frac{1}{p-r} - \frac{1}{p-q}\right)\frac{\eta-\zeta}{\xi}.$$

Similarly the other two characteristic equations can be brought to the forms

$$-\frac{2q''}{q'} - \frac{Q_2 q'}{Q} = \frac{q'-2r'}{q-r} + \frac{q'-2p'}{q-p} - q'\left(\frac{1}{q-p} - \frac{1}{q-r}\right)\frac{\zeta-\xi}{\eta},$$

$$-\frac{2r''}{r'} - \frac{R_3 r'}{R} = \frac{r'-2p'}{r-p} + \frac{r'-2q'}{r-q} - r'\left(\frac{1}{r-q} - \frac{1}{r-p}\right)\frac{\xi-\eta}{\zeta}.$$

Next, logarithmic differentiation of the equation, defining the variables ξ and η, leads to the relation

$$\begin{aligned}\frac{\xi'}{\xi} - \frac{\eta'}{\eta} &= \frac{2p''}{p'} + \frac{P_1 p'}{P} - \frac{2q''}{q'} - \frac{Q_2 q'}{Q} \\ &= -\frac{p'-2r'}{p-r} + \frac{q'-2r'}{q-r} + \frac{p'+q'}{p-q} \\ &\qquad - p'\left(\frac{1}{r-p} - \frac{1}{q-p}\right)\frac{\eta-\zeta}{\xi} - q'\left(\frac{1}{q-p} - \frac{1}{q-r}\right)\frac{\zeta-\xi}{\eta}.\end{aligned}$$

Now ξ, η, ζ, are functions of p, q, r; when the values are deemed to be substituted, the equation will be satisfied by having the coefficients of p', of q', and of r', respectively equal on the two sides: and similarly for the other equations. (Otherwise, we should have a relation between p', q', r', and the variables, satisfied for all geodesics: this result cannot arise, for a direction p', q', r', through a point can be arbitrarily assumed.)

(iv) Equating the coefficients of r' in this relation, we have

$$\frac{1}{\xi}\frac{\partial \xi}{\partial r} - \frac{1}{\eta}\frac{\partial \eta}{\partial r} = \frac{2}{p-r} - \frac{2}{q-r},$$

and therefore

$$\frac{\xi(q-r)^{-2}}{\eta(p-r)^{-2}} = \text{a function of } p \text{ and } q \text{ only, not of } r.$$

Similarly from the other two equations, we find

$$\frac{\eta(r-p)^{-2}}{\zeta(q-p)^{-2}} = \text{a function of } q \text{ and } r \text{ only, not of } p,$$

$$\frac{\zeta(p-q)^{-2}}{\xi(r-q)^{-2}} = \text{a function of } r \text{ and } p \text{ only, not of } q.$$

It follows that we must have

$$\xi = (q-r)^2 f(p),$$
$$\eta = (r-p)^2 g(q),$$
$$\zeta = (p-q)^2 h(r),$$

where $f(p)$ is a function of p alone, $g(q)$ of q alone, and $h(r)$ of r alone.

Equating the coefficients of q' in the same relation, we have

$$\frac{1}{\xi}\frac{\partial \xi}{\partial q} - \frac{1}{\eta}\frac{\partial \eta}{\partial q} = \frac{1}{q-r} + \frac{1}{p-q} - \left(\frac{1}{q-p} - \frac{1}{q-r}\right)\frac{\zeta-\xi}{\eta},$$

or, substituting for ξ and for η the values which have just been obtained

$$\frac{g'(q)}{g(q)} = \frac{1}{q-r} + \frac{1}{q-p} - \left(\frac{1}{p-q} - \frac{1}{r-q}\right)\frac{\zeta-\xi}{\eta}.$$

Similarly, from the other equations,

$$\frac{h'(r)}{h(r)} = \frac{1}{r-p} + \frac{1}{r-q} - \left(\frac{1}{q-r} - \frac{1}{p-r}\right)\frac{\xi-\eta}{\zeta},$$

$$\frac{f'(p)}{f(p)} = \frac{1}{p-q} + \frac{1}{p-r} - \left(\frac{1}{r-p} - \frac{1}{q-p}\right)\frac{\eta-\zeta}{\xi}.$$

The last equation is equivalent to

$$(r-p)(q-p)f'(p) = (2p-q-r)f(p) - \frac{1}{q-r}\{(p-r)^2 g(q) - (p-q)^2 h(r)\}.$$

Let this be differentiated with respect to p; then

$$(r-p)(q-p)f''(p) = 2f(p) - \frac{2}{q-r}\{(p-r)g(q) - (p-q)h(r)\},$$

that is,

$$-(r-p)(p-q)(q-r)f''(p) = 2\{(q-r)f(p) + (r-p)g(q) + (p-q)h(r)\}.$$

Similarly

$$-(p-q)(q-r)(r-p)g''(q), \quad -(q-r)(r-p)(p-q)h''(r),$$

are, each of them, equal to the same magnitude. Consequently

$$f''(p) = g''(q) = h''(r).$$

As f is a function of p alone, g of q alone, and h of r alone, each of these three derivatives has a constant value; let it be denoted by 2γ. Thus

$$f(p) = \gamma p^2 + 2\alpha' p + \beta',$$
$$g(q) = \gamma q^2 + 2\alpha'' q + \beta'',$$
$$h(r) = \gamma r^2 + 2\alpha''' r + \beta'''.$$

When these values are substituted in the expression for $f''(p)$, we have

$$(q-r)(2\alpha' p + \beta') + (r-p)(2\alpha'' q + \beta'') + (p-q)(2\alpha''' r + \beta''') = 0,$$

while p, q, r, are independent variables; and therefore

$$\alpha' = \alpha'' = \alpha''', \quad \beta' = \beta'' = \beta'''.$$

Manifestly, we can make γ equal to unity, without any loss of generality; and therefore

$$\left.\begin{aligned} \xi &= (q-r)^2(p^2 + 2\alpha' p + \beta') = (q-r)^2(p+\alpha)(p+\beta) \\ \eta &= (r-p)^2(q^2 + 2\alpha' q + \beta') = (r-p)^2(q+\alpha)(q+\beta) \\ \zeta &= (p-q)^2(r^2 + 2\alpha' r + \beta') = (p-q)^2(r+\alpha)(r+\beta) \end{aligned}\right\},$$

where α and β can be regarded as arbitrary constants.

(v) Now let $S(\theta)$, for any argument θ, denote the sextic polynomial

$$S(\theta) = \theta(\theta+\alpha)(\theta+b)(\theta+c)(\theta+\alpha)(\theta+\beta);$$

then

$$\frac{\xi}{P} = \frac{(q-r)^2}{p^2} S(p),$$

$$\frac{\eta}{Q} = \frac{(r-p)^2}{q^2} S(q),$$

$$\frac{\zeta}{R} = \frac{(p-q)^2}{r^2} S(r).$$

Hence, as

$$\frac{Pp'^2}{\xi} = \frac{Qq'^2}{\eta} = \frac{Rr'^2}{\zeta} = T^2,$$

where T^2 is introduced to denote the common value of the fractions, we have

$$\frac{p^2 p'^2}{(q-r)^2 S(p)} = \frac{q^2 q'^2}{(r-p)^2 S(q)} = \frac{r^2 r'^2}{(p-q)^2 S(r)} = T^2;$$

and therefore

$$\frac{p\,dp}{\{S(p)\}^{\frac{1}{2}}}+\frac{q\,dq}{\{S(q)\}^{\frac{1}{2}}}+\frac{r\,dr}{\{S(r)\}^{\frac{1}{2}}}=0,$$

$$\frac{p^2\,dp}{\{S(p)\}^{\frac{1}{2}}}+\frac{q^2\,dq}{\{S(q)\}^{\frac{1}{2}}}+\frac{r^2\,dr}{\{S(r)\}^{\frac{1}{2}}}=0.$$

Moreover,

$$\begin{aligned} ds^2 &= A\,dp^2+B\,dq^2+C\,dr^2 \\ &= \Sigma(p-q)(p-r)P\,dp^2 \\ &= -T^2(p-q)(q-r)(r-p)\Sigma\{(q-r)(p+\alpha)(p+\beta)\}\,ds^2 \\ &= T^2(p-q)^2(q-r)^2(r-p)^2\,ds^2; \end{aligned}$$

and therefore

$$\Sigma\frac{dp}{\{S(p)\}^{\frac{1}{2}}}=\Sigma\frac{T\,ds}{p}(q-r)=-\frac{1}{pqr}\,ds.$$

If then we take

$$\left.\begin{aligned} &\frac{dp}{\{S(p)\}^{\frac{1}{2}}}+\frac{dq}{\{S(q)\}^{\frac{1}{2}}}+\frac{dr}{\{S(r)\}^{\frac{1}{2}}}=d\omega \\ &\frac{p\,dp}{\{S(p)\}^{\frac{1}{2}}}+\frac{q\,dq}{\{S(q)\}^{\frac{1}{2}}}+\frac{r\,dr}{\{S(r)\}^{\frac{1}{2}}}=0 \\ &\frac{p^2\,dp}{\{S(p)\}^{\frac{1}{2}}}+\frac{q^2\,dq}{\{S(q)\}^{\frac{1}{2}}}+\frac{r^2\,dr}{\{S(r)\}^{\frac{1}{2}}}=0 \end{aligned}\right\},$$

we have

$$ds=-pqr\,d\omega.$$

The full expression of p, q, r, and therefore of x, y, z, v, requires the use of hyperelliptic functions of grade two, because the quantity S under the radical sign is a sextic polynomial.

Geodesic polar coordinates.

303. In plane geometry, there can be convenience in using polar coordinates; and in the Gauss geometry of surfaces in homaloidal triple space, there can be convenience in using geodesic polar coordinates, the natural extension of the plane polar coordinates. In the solid geometry of homaloidal triple space, there can be convenience in using (spherical) polar coordinates; so, in the geometry of regions in homaloidal quadruple space, there can be convenience in using geodesic polar coordinates, an extension of the (spherical) polar coordinates.

Accordingly, we proceed to consider such a form for the expression of the arc-element of a region in the quadruple space. We shall assume that one set of lines of reference for the region, given by

$$p=\text{variable},\quad q=\text{constant},\quad r=\text{constant},$$

is a geodesic in the region. Consequently, the equations that are characteristic of geodesics must be satisfied by $q=$ constant, $r=$ constant, for such an assignment of the parametric variable p: and with this assignment, we have along the curve

$$p'^2=\frac{1}{A},\quad q'=0,\quad r'=0.$$

The initial form of the geodesic equations (§ 298) was

$$\frac{d}{ds}(Ap' + Hq' + Gr') = \tfrac{1}{2}(A_1 p'^2 + 2H_1 p'q' + 2G_1 p'r' + B_1 q'^2 + 2F_1 q'r' + C_1 r'^2),$$

$$\frac{d}{ds}(Hp' + Bq' + Fr') = \tfrac{1}{2}(A_2 p'^2 + 2H_2 p'q' + 2G_2 p'r' + B_2 q'^2 + 2F_2 q'r' + C_2 r'^2),$$

$$\frac{d}{ds}(Gp' + Fq' + Cr') = \tfrac{1}{2}(A_3 p'^2 + 2H_3 p'q' + 2G_3 p'r' + B_3 q'^2 + 2F_3 q'r' + C_3 r'^2).$$

When these equations are satisfied by $Ap'^2 = 1$, $q' = 0$, $r' = 0$, we must have

$$\frac{1}{2}\frac{A_1}{A} = \frac{d}{ds}(Ap') = \frac{d}{ds}(A^{\frac{1}{2}}) = \frac{A_1}{2A},$$

$$\frac{1}{2}\frac{A_2}{A} = \frac{d}{ds}\left(\frac{H}{A^{\frac{1}{2}}}\right) = \frac{1}{A^{\frac{1}{2}}}\frac{\partial}{\partial p}\left(\frac{H}{A^{\frac{1}{2}}}\right),$$

$$\frac{1}{2}\frac{A_3}{A} = \frac{d}{ds}\left(\frac{G}{A^{\frac{1}{2}}}\right) = \frac{1}{A^{\frac{1}{2}}}\frac{\partial}{\partial p}\left(\frac{G}{A^{\frac{1}{2}}}\right),$$

from the three equations respectively. The first relation is an identity. The second relation is equivalent to

$$\frac{\partial}{\partial p}\left(\frac{H}{A^{\frac{1}{2}}}\right) = \frac{\partial}{\partial q}(A^{\frac{1}{2}}),$$

and the third relation is equivalent to

$$\frac{\partial}{\partial p}\left(\frac{G}{A^{\frac{1}{2}}}\right) = \frac{\partial}{\partial r}(A^{\frac{1}{2}}).$$

Hence there exists a function D such that

$$A^{\frac{1}{2}} = \frac{\partial D}{\partial p}, \quad \frac{H}{A^{\frac{1}{2}}} = \frac{\partial D}{\partial q}, \quad \frac{G}{A^{\frac{1}{2}}} = \frac{\partial D}{\partial r}.$$

With these values substituted for A, H, G, the expression for the arc-element is

$$ds^2 = \left(\frac{\partial D}{\partial p}\right)^2 dp^2 + 2\frac{\partial D}{\partial p}\frac{\partial D}{\partial q}dpdq + 2\frac{\partial D}{\partial p}\frac{\partial D}{\partial r}dpdr + Bdq^2 + 2Fdqdr + Cdr^2$$

$$= dD^2 + \left\{B - \left(\frac{\partial D}{\partial q}\right)^2\right\}dq^2 + 2\left\{F - \frac{\partial D}{\partial q}\frac{\partial D}{\partial r}\right\}dqdr + \left\{C - \left(\frac{\partial D}{\partial r}\right)^2\right\}dr^2.$$

Because A does not vanish, D must involve p; and therefore D, q, r, can be taken instead of p, q, r, as parameters of the region. The arc-length along the curve $q = \text{constant}$, $r = \text{constant}$, manifestly is dD: that is, the variable D is a geodesic length, measured along the geodesic. Hence when one of the parametric curves, chosen to be the curve along which p is the variable parameter, is a geodesic in the region, the arc-element is expressible in the form

$$ds^2 = dp^2 + Bdq^2 + 2Fdqdr + Cdr^2.$$

With these variables, the direction-cosines of the geodesic are

$$\frac{\partial x}{\partial p}, \quad \frac{\partial y}{\partial p}, \quad \frac{\partial z}{\partial p}, \quad \frac{\partial v}{\partial p}.$$

When we take p equal to a definite constant, all the points in the region for which p has this definite value are such that

$$ds'^2 = Bdq^2 + 2Fdqdr + Cdr^2,$$

that is, they lie on a surface in the region. At any point on this surface, any direction tangential to the surface has direction-cosines

$$x_2\frac{dq}{ds'} + x_3\frac{dr}{ds'}, \quad y_2\frac{dq}{ds'} + y_3\frac{dr}{ds'}, \quad z_2\frac{dq}{ds'} + z_3\frac{dr}{ds'}, \quad v_2\frac{dq}{ds'} + v_3\frac{dr}{ds'}.$$

If ψ denote the inclination of this direction to the geodesic, then

$$\begin{aligned}\cos\psi &= \Sigma\frac{dx}{dp}\left(x_2\frac{dq}{ds'} + x_3\frac{dr}{ds'}\right)\\ &= \Sigma x_1\left(x_2\frac{dq}{ds'} + x_3\frac{dr}{ds'}\right)\\ &= \frac{dq}{ds'}\Sigma x_1x_2 + \frac{dr}{ds'}\Sigma x_1x_3 = 0:\end{aligned}$$

that is, every direction in the surface at the point is perpendicular to the geodesic through the point. Consequently the geodesic is normal to the surface; and therefore the surfaces

$$p = \text{constant}$$

in a region, for which the arc-element is

$$ds^2 = dp^2 + Bdq^2 + 2Fdqdr + Cdr^2,$$

are parallel surfaces to which the geodesics are orthogonal, while the geodesic distance between two surfaces is the difference between the values of p specifying them.

Two orthogonal families of geodesics not generally possible.

304. At this stage, it is natural to enquire whether there is a possibility that, in a given region, there are two orthogonal families of regional geodesics. Through any point of any one geodesic of a given family, we can draw an infinitude of regional geodesics, orthogonal to the selected geodesic. But is it possible that there shall be orthogonal trajectories, of the whole family and not merely of any selected member of the family, such that these trajectories shall themselves be geodesics? The answer is in the negative when the region is completely general: and it is attained as follows.

We proceed from the foregoing arc-element, which puts a geodesic in evidence. For that element, we have

$$A = 1, \quad H = 0, \quad G = 0, \quad \Omega^2 = BC - F^2,$$
$$a = BC - F^2, \quad b = C, \quad c = B, \quad f = -F, \quad g = 0, \quad h = 0.$$

If it is possible that there should be a second orthogonal family of geodesics in the region, given by a curve on the surface orthogonal to p, this curve can, by a change of variables, be made $r = \text{constant}$, so that the new family of geodesics would be given by

$$q = \text{variable}, \quad p = \text{constant}, \quad r = \text{constant}.$$

Reverting to the equations already quoted, and making $A = 1$, $H = 0$, $G = 0$, we have the geodesics on the surface given by the equations

$$p'' = \tfrac{1}{2}(B_1 q'^2 + 2F_1 q' r' + C_1 r'^2),$$

$$\frac{d}{ds}(Bq' + Fr') = \tfrac{1}{2}(B_2 q'^2 + 2F_2 q' r' + C_2 r'^2),$$

$$\frac{d}{ds}(Fq' + Cr') = \tfrac{1}{2}(B_3 q'^2 + 2F_3 q' r' + C_3 r'^2).$$

For the hypothetical new family of geodesics, we have

$$Bq'^2 = 1, \quad p' = 0, \quad r' = 0.$$

Then the first equation gives

$$0 = \frac{B_1}{2B},$$

that is, B is a function of q and r only. The second equation is

$$\frac{d}{ds}(B^{\frac{1}{2}}) = \frac{1}{2}\frac{B_2}{B},$$

that is,

$$\frac{1}{B^{\frac{1}{2}}}\frac{\partial}{\partial q}(B^{\frac{1}{2}}) = \frac{1}{2}\frac{B_2}{B},$$

which is an identity. The third equation is

$$\frac{d}{ds}\left(\frac{F}{B^{\frac{1}{2}}}\right) = \frac{B_3}{2B},$$

that is,

$$\frac{\partial}{\partial q}\left(\frac{F}{B^{\frac{1}{2}}}\right) = \frac{1}{2}\frac{B_3}{B^{\frac{1}{2}}} = \frac{\partial}{\partial r}(B^{\frac{1}{2}}).$$

Hence there is a function θ, which must be independent of p, and is such that

$$B^{\frac{1}{2}} = \frac{\partial\theta}{\partial q}, \quad FB^{-\frac{1}{2}} = \frac{\partial\theta}{\partial r}.$$

With these values,

$$B\,dq^2 + 2F\,dq\,dr + C\,dr^2 = d\theta^2 + \left\{C - \left(\frac{\partial\theta}{\partial r}\right)^2\right\} dr^2 = d\theta^2 + \bar{C}\,dr^2;$$

and thus the arc-element on the surface is

$$ds^2 = dp^2 + d\theta^2 + \bar{C}\,dr^2.$$

On a surface with this arc-element, the subsidiary quantities Γ, Δ, Θ, are

$$\left.\begin{matrix}\Gamma_{11}=0\\ \Delta_{11}=0\\ \Theta_{11}=0\end{matrix}\right\},\quad \left.\begin{matrix}\Gamma_{12}=0\\ \Delta_{12}=0\\ \Theta_{12}=0\end{matrix}\right\},\quad \left.\begin{matrix}\Gamma_{22}=0\\ \Delta_{22}=0\\ \Theta_{22}=0\end{matrix}\right\},\quad \left.\begin{matrix}\Gamma_{13}=0\\ \Delta_{13}=0\\ \Theta_{13}=\dfrac{\bar{C}_1}{2\bar{C}}\end{matrix}\right\},\quad \left.\begin{matrix}\Gamma_{23}=0\\ \Delta_{23}=0\\ \Theta_{23}=\dfrac{\bar{C}_2}{2\bar{C}}\end{matrix}\right\},\quad \left.\begin{matrix}\Gamma_{33}=-\frac{1}{2}\bar{C}_1\\ \Delta_{33}=-\frac{1}{2}\bar{C}_2\\ \Theta_{33}=\dfrac{\bar{C}_3}{2\bar{C}}\end{matrix}\right\},$$

while $A=1$, $H=0$, $G=0$, $B=1$, $F=0$; $a=\bar{C}$, $b=\bar{C}$, $c=1$, $f=g=h=0$. The extended Gauss relations (§ 278) become

$$\begin{aligned}
l &= NJ - I^2 = -\tfrac{1}{2}\bar{C}_{22}+\Gamma_{23}{}^2+\Delta_{23}{}^2+\Theta_{23}{}^2=-\tfrac{1}{2}\bar{C}_{22}+\frac{\bar{C}_2{}^2}{4\bar{C}},\\
n &= JL - K^2 = -\tfrac{1}{2}\bar{C}_{11}+\Gamma_{31}{}^2+\Delta_{31}{}^2+\Theta_{31}{}^2=-\tfrac{1}{2}\bar{C}_{11}+\frac{\bar{C}_1{}^2}{4\bar{C}},\\
j &= LN - M^2 = 0,\\
i &= MK - LI = 0,\\
k &= IM - NK = 0,\\
m &= KI - JM = \tfrac{1}{2}\bar{C}_{12}-\frac{\bar{C}_1\bar{C}_2}{4\bar{C}}.
\end{aligned}$$

or, if we write

$$\bar{C}=\square^2,$$

we have

$$l=-\square\square_{22},\quad m=\square\square_{12},\quad n=-\square\square_{11}.$$

It is at once clear that a region, for which such relations can hold, is not perfectly general; for

$$\Upsilon^2=\begin{vmatrix} l, & m, & k\\ m, & n, & i\\ k, & i, & j\end{vmatrix}=0,$$

and therefore the volumetric measure of the curvature of the region is zero. With the notation of § 281, we have

$$\mathfrak{S}_2=l+n,\quad \mathfrak{S}_1=(L+N)\square^2+J;$$

but

$$\frac{L}{M}=\frac{M}{N}=\frac{K}{I}=\theta,$$

so that

$$l=NJ-I^2,\quad m=-\theta(NJ-I^2),\quad n=\theta^2(NJ-I^2).$$

Thus

$$\theta=\frac{\square_{12}}{\square_{22}}=\frac{\square_{11}}{\square_{12}},$$

where $\square$ satisfies the equation

$$\square_{11}\square_{22}-\square_{12}{}^2=0;$$

and so

$$\mathfrak{G}_2 = \left(1 + \frac{\square_{11}}{\square_{22}}\right)(JN - I^2), \qquad \mathfrak{G}_1 = \left(1 + \frac{\square_{11}}{\square_{22}}\right)N + J.$$

The equation for the principal measures of linear curvature of the region is

$$0 - \left(1 + \frac{\square_{11}}{\square_{22}}\right)(JN - I^2)\frac{1}{\rho} + \left\{\left(1 + \frac{\square_{11}}{\square_{22}}\right)N + J\right\}\frac{1}{\rho^2} - \square^2\frac{1}{\rho^3} = 0.$$

As the volumetric measure is zero, we can take

$$\frac{1}{\gamma} = 0;$$

and then the other two measures of linear curvature are given by

$$\left.\begin{aligned} \frac{1}{\alpha\beta} &= \frac{1}{\square^2}\left\{J + \left(1 + \frac{\square_{11}}{\square_{22}}\right)N\right\} \\ \frac{1}{\alpha} + \frac{1}{\beta} &= \frac{1}{\square^2}(JN - I^2)\left(1 + \frac{\square_{11}}{\square_{22}}\right) \end{aligned}\right\}.$$

Moreover, as Υ vanishes, the quantities J, N, I, are not necessarily nor usually expressible (§ 286) in terms of the magnitudes of the first order—that is, here they are not expressible in terms of $\square$.

Further, when a region can be developed, geodesics remain geodesics and inclinations are unchanged. If therefore there originally are two orthogonal families of geodesics, there would remain two orthogonal families of geodesics for the surface in the developed region. This possibility arises only when a surface is a plane: that is, when the region can be developed into a flat. We therefore infer the theorem:

It is not possible to have two orthogonal families of geodesic curves in a general region, that is, a region which cannot be developed into a flat.

Ex. When the arc-element is expressed with reference to geodesic polar coordinates, so that it takes the form

$$ds^2 = A\,dp^2 + 2H\,dp\,dq + B\,dq^2 + dr^2,$$

a special (but not unimportant) instance arises when the region is of a columnar type represented by

$$x = x(p, q), \quad y = y(p, q), \quad z = z(p, q), \quad v = v,$$

the variable v not depending upon p and q. (The section of the region by a flat region, v = constant, is a surface in a homaloidal triple space.) Then we take

$$r = v,$$

and A, H, B, do not involve r (or v).

By direct calculation, we find

$$\Theta_{11} = 0, \quad \Theta_{12} = 0, \quad \Theta_{22} = 0;$$

and as

$$G = 0, \quad F = 0, \quad C = 1,$$

while A, H, B, do not involve r, we have

$$\begin{aligned} &\Gamma_{13} = 0, \quad \Delta_{13} = 0, \quad \Theta_{13} = 0, \\ &\Gamma_{23} = 0, \quad \Delta_{23} = 0, \quad \Theta_{23} = 0, \\ &\Gamma_{33} = 0, \quad \Delta_{33} = 0, \quad \Theta_{33} = 0. \end{aligned}$$

With these values, the equations of the regional geodesics are

$$p''+(\Gamma_{11},\ \Gamma_{12},\ \Gamma_{22}\}p',\ q')^2=0,$$
$$q''+(\Delta_{11},\ \Delta_{12},\ \Delta_{22}\}p',\ q')^2=0,$$
$$r''=0.$$

It must be remembered that only when r' is equal to zero (the last equation merely gives a constant value to r'), is the regional geodesic the same as the superficial geodesic in the flat $v=$constant; otherwise, the regional geodesic is of a helical type.

Partial differential equation of geodesically parallel surfaces.

305. One further inference can be made from the form to which the arc-element can be reduced when geodesic polar parameters are used; we shall obtain *the partial differential equation of the first order which is satisfied by the parallel surfaces orthogonal to a family of geodesics.*

The two expressions for the arc-element on a surface

$$ds^2=A\,dp^2+Bdq^2+Cdr^2+2F\,dq\,dr+2G\,dr\,dp+2H\,dp\,dq,$$
$$ds^2=dD^2+\bar{B}\,d\phi^2+2\bar{F}\,d\phi\,d\psi+\bar{C}\,d\psi^2,$$

are equivalent to one another, while the parallel surfaces orthogonal to the geodesics are given by

$$D=\text{constant}.$$

From the equivalence of the two forms we have

$$\begin{aligned}
A &= D_1^2 &&+\bar{B}\phi_1^2 &&+2\bar{F}\phi_1\psi_1 &&+\bar{C}\psi_1^2\ ,\\
B &= D_2^2 &&+\bar{B}\phi_2^2 &&+2\bar{F}\phi_2\psi_2 &&+\bar{C}\psi_2^2\ ,\\
C &= D_3^2 &&+\bar{B}\phi_3^2 &&+2\bar{F}\phi_3\psi_3 &&+\bar{C}\psi_3^2\ ,\\
F &= D_2D_3 &&+\bar{B}\phi_2\phi_3 &&+\bar{F}(\phi_2\psi_3+\psi_2\phi_3) &&+\bar{C}\psi_2\psi_3,\\
G &= D_3D_1 &&+\bar{B}\phi_3\phi_1 &&+\bar{F}(\phi_3\psi_1+\psi_3\phi_1) &&+\bar{C}\psi_3\psi_1,\\
H &= D_1D_2 &&+\bar{B}\phi_1\phi_2 &&+\bar{F}(\phi_1\psi_2+\psi_1\phi_2) &&+\bar{C}\psi_1\psi_2;
\end{aligned}$$

and therefore

$$\begin{vmatrix} A-D_1^2\ , & H-D_1D_2, & G-D_1D_3\\ H-D_1D_2, & B-D_2^2\ , & F-D_2D_3\\ G-D_1D_3, & F-D_2D_3, & C-D_3^2 \end{vmatrix}=0.$$

When expanded, this equation is

$$aD_1^2+2hD_1D_2+2gD_1D_3+bD_2^2+2fD_2D_3+cD_3^2=\Omega^2,$$

which accordingly is the partial differential equation of the first order satisfied by the parallel surfaces

$$D=\text{constant},$$

orthogonal to a complete family of concurrent geodesics in the region.

The quantity $\Delta(D)$ in the Beltrami notation, where

$$\Delta(D)=\frac{1}{\Omega^2}(aD_1^2+2hD_1D_2+2gD_1D_3+bD_2^2+2fD_2D_3+cD_3^2),$$

is a *differential invariant* of the region; it can be called the *first differential parameter* of the function D; and it is the natural extension of Beltrami's first differential parameter which occurs* in the general theory of surfaces in homaloidal triple space.

Any integral of the partial differential equation

$$\Delta(D) = 1$$

of the first order provides a family of geodesically parallel surfaces, which are cut orthogonally by a family of geodesic curves in the region.

Ex. Prove that the parameters of reference upon any one of these surfaces are any two functionally independent integrals of any one of three equations of the type

$$(a - CD_2^2 - BD_3^2 + 2FD_2D_3)\theta_1 + (h + HD_3^2 - FD_1D_3 - GD_2D_3 + CD_1D_2)\theta_2 + (g + GD_2^2 - HD_2D_3 - FD_1D_2 + BD_1D_3)\theta_3 = 0.$$

Torsion of a geodesic in the region.

306. The torsion of a geodesic in the region is expressible by means of quantities of the first order and of the second order.

Along a geodesic, we have $\rho x'' = X$; therefore

$$x''' = -\frac{\rho'}{\rho^2}X + \frac{1}{\rho}X',$$

and similarly for y''', z''', v'''. Hence

$$\frac{1}{\sigma^2\rho^2} + \frac{1}{\rho^4} + \frac{\rho'^2}{\rho^4} = \Sigma x'''^2 = \frac{\rho'^2}{\rho^4} + \frac{1}{\rho^2}\Sigma X'^2;$$

and therefore, for the geodesic,

$$\frac{1}{\sigma^2} + \frac{1}{\rho^2} = \Sigma X'^2 = \Sigma (X_1p' + X_2q' + X_3r')^2.$$

The values of X_1, X_2, X_3, have been obtained in § 274: using them, we have

$$-\Omega^2(X_1p' + X_2q' + X_3r') = x_1U + x_2T + x_3W,$$

where

$$\begin{aligned} U &= p'(aL + hM + gK) + q'(aM + hN + gI) + r'(aK + hI + gJ) \\ &= a(Lp' + Mq' + Kr') + h(Mp' + Nq' + Ir') + g(Kp' + Iq' + Jr'), \\ T &= h(Lp' + Mq' + Kr') + b(Mp' + Nq' + Ir') + f(Kp' + Iq' + Jr'), \\ W &= g(Lp' + Mq' + Kr') + f(Mp' + Nq' + Ir') + c(Kp' + Iq' + Jr'). \end{aligned}$$

Thus

$$\begin{aligned} \Omega^4\Sigma(X_1p' + X_2q' + X_3r')^2 &= \Sigma(x_1U + x_2T + x_3W)^2 \\ &= (A, B, C, F, G, H\,\}\,U, T, W)^2 \\ &= \Omega^2(a, b, c, f, g, h\,\}\,Lp' + Mq' + Kr',\ Mp' + Nq' + Ir',\ Kp' + Iq' + Jr')^2. \end{aligned}$$

We thus have an expression leading to a value of $1/\sigma^2$, in the form

$$\Omega^2\left(\frac{1}{\sigma^2} + \frac{1}{\rho^2}\right) = (a, b, c, f, g, h\,\}\,Lp' + Mq' + Kr',\ Mp' + Nq' + Ir',\ Kp' + Iq' + Jr')^2.$$

The symbol **D** will be used to denote the quantity on the right-hand side.

* *Mem. Acc. Bologna*, 2da ser., t. viii (1869), pp. 549–590; see my *Lectures on Differential Geometry*, §§ 115, 132.

Next, let λ, μ, ν, κ, be the direction-cosines of the trinormal to the geodesic, so that (§ 141)

$$\Sigma\lambda x' = 0, \quad \Sigma\lambda x'' = 0, \quad \Sigma\lambda x''' = 0.$$

As $\rho x'' = X$, with like values for y'', z'', v'', the last two relations can be taken as

$$\Sigma\lambda X = 0, \quad \Sigma\lambda X' = 0,$$

forms which verify the property (§ 186) that the trinormal lies in the rectifying plane of the curve, here given by the two equations

$$\Sigma(\bar{x} - x)X = 0, \quad \Sigma(\bar{x} - x)X' = 0.$$

Thus, for some quantity θ, we have

$$\theta\lambda = \begin{vmatrix} y', & z', & v' \\ Y', & Z', & V' \\ Y, & Z, & V \end{vmatrix},$$

with corresponding values for $\theta\mu$, $\theta\nu$, $\theta\kappa$. Now

$$\Sigma \begin{vmatrix} y', & z', & v' \\ Y', & Z', & V' \\ Y, & Z, & V \end{vmatrix}^2 = \begin{vmatrix} \Sigma x'^2, & \Sigma x'X', & \Sigma x'X \\ \Sigma X'x', & \Sigma X'^2, & \Sigma X'X \\ \Sigma Xx', & \Sigma XX', & \Sigma X^2, \end{vmatrix} = \begin{vmatrix} 1, & -\frac{1}{\rho}, & 0 \\ -\frac{1}{\rho}, & \frac{1}{\rho^2} + \frac{1}{\sigma^2}, & 0 \\ 0, & 0, & 1 \end{vmatrix} = \frac{1}{\sigma^2};$$

consequently $\theta = 1/\sigma$.

The expressions for λ/σ, μ/σ, ν/σ, κ/σ, are quadratic in p', q', r'. In λ/σ, the coefficient of p'^2 is

$$= -\begin{vmatrix} Y_1, & y_1, & Y \\ Z_1, & z_1, & Z \\ V_1, & v_1, & V \end{vmatrix} = \frac{1}{\Omega^2}\begin{vmatrix} y_1\lambda_1 + y_2\mu_1 + y_3\nu_1, & y_1, & Y \\ z_1\lambda_1 + z_2\mu_1 + z_3\nu_1, & z_1, & Z \\ v_1\lambda_1 + v_2\mu_1 + v_3\nu_1, & v_1, & V \end{vmatrix}$$

$$= \frac{\mu_1}{\Omega^2}\begin{vmatrix} y_2, & y_1, & Y \\ z_2, & z_1, & Z \\ v_2, & v_1, & V \end{vmatrix} + \frac{\nu_1}{\Omega^2}\begin{vmatrix} y_3, & y_1, & Y \\ z_3, & z_1, & Z \\ v_3, & v_1, & V \end{vmatrix}$$

$$= \frac{1}{\Omega^3}\{\mu_1(gx_1 + fx_2 + cx_3) - \nu_1(hx_1 + bx_2 + fx_3)\}$$

$$= \frac{1}{\Omega}\begin{vmatrix} x_1, & x_2, & x_3 \\ A, & H, & G \\ L, & M, & K \end{vmatrix}.$$

Similarly for other terms in λ/σ; the full expression is

$$\lambda\frac{\Omega}{\sigma} = x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w},$$

where

$$\mathbf{u}, \mathbf{v}, \mathbf{w} = \begin{Vmatrix} Ap' + Hq' + Gr', & Hp' + Bq' + Fr', & Gp' + Fq' + Cr' \\ Lp' + Mq' + Kr', & Mp' + Nq' + Ir', & Kp' + Iq' + Jr' \end{Vmatrix};$$

and there are corresponding expressions for $\mu\Omega/\sigma$, $\nu\Omega/\sigma$, $\kappa\Omega/\sigma$. Squaring and adding, we have

$$\frac{\Omega^2}{\sigma^2} = (A, B, C, F, G, H)(\mathbf{u}, \mathbf{v}, \mathbf{w})^2.$$

The symbol **S** will be used, when necessary, to denote the covariant on the right-hand side.

Torsion of a geodesic with parametric lines of curvature.

307. Comparatively simple expressions are derivable for the torsion, when the parametric curves of reference are the lines of curvature of the region. We then have

$$\Omega^2 = ABC, \quad F = G = H = 0, \quad I = K = M = 0;$$

and each of the formulæ in § 306 for the torsion gives

$$\frac{1}{\sigma^2} = \frac{1}{ABC}\{A(BJ - CN)^2 q'^2 r'^2 + B(CL - AJ)^2 r'^2 p'^2 + C(AN - BL)^2 p'^2 q'^2\}$$

$$= \left(\frac{1}{\beta} - \frac{1}{\gamma}\right)^2 BCq'^2 r'^2 + \left(\frac{1}{\gamma} - \frac{1}{\alpha}\right)^2 CAr'^2 p'^2 + \left(\frac{1}{\alpha} - \frac{1}{\beta}\right)^2 ABp'^2 q'^2,$$

where α, β, γ, respectively equal to A/L, B/N, C/J, denote the principal radii of curvature of the region at the point.

We also have three equivalent expressions

$$\frac{1}{\sigma^2} = A\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\left(\frac{1}{\alpha} - \frac{1}{\gamma}\right)p'^2 - \left(\frac{1}{\rho} - \frac{1}{\beta}\right)\left(\frac{1}{\rho} - \frac{1}{\gamma}\right)$$

$$= B\left(\frac{1}{\beta} - \frac{1}{\gamma}\right)\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)q'^2 - \left(\frac{1}{\rho} - \frac{1}{\gamma}\right)\left(\frac{1}{\rho} - \frac{1}{\alpha}\right)$$

$$= C\left(\frac{1}{\gamma} - \frac{1}{\alpha}\right)\left(\frac{1}{\gamma} - \frac{1}{\beta}\right)r'^2 - \left(\frac{1}{\rho} - \frac{1}{\alpha}\right)\left(\frac{1}{\rho} - \frac{1}{\beta}\right).$$

Obviously the torsion of a geodesic tangent to a line of curvature is zero at the point of contact.

Maximum and minimum geodesic torsions.

308. For all directions, which are distinct from those of the lines of curvature at any point in the region, the quantity $1/\sigma^2$ is positive; and it must be finite owing to its analytical expression. We therefore naturally enquire what are the maximum values the quantity may acquire, the minimum value (or values) being zero. To this end, we make the foregoing expression for the value of $1/\sigma^2$, being

$$\frac{1}{\sigma^2} = \left(\frac{1}{\beta} - \frac{1}{\gamma}\right)^2 BCq'^2 r'^2 + \left(\frac{1}{\gamma} - \frac{1}{\alpha}\right)^2 CAr'^2 p'^2 + \left(\frac{1}{\alpha} - \frac{1}{\beta}\right)^2 ABp'^2 q'^2$$

$$= \theta q'^2 r'^2 + \phi r'^2 p'^2 + \psi p'^2 q'^2,$$

(with manifest meanings for θ, ϕ, ψ), a maximum for possible values of p', q', r', subject to the condition

$$Ap'^2 + Bq'^2 + Cr'^2 = 1.$$

The critical conditions are

$$\left.\begin{aligned} p'(\phi r'^2 + \psi q'^2) &= \lambda A p', \\ q'(\psi p'^2 + \theta r'^2) &= \lambda B q', \\ r'(\theta q'^2 + \phi p'^2) &= \lambda C r', \end{aligned}\right\}$$

where λ is a multiplier undetermined by the critical conditions themselves. To find λ, we multiply these three equations by p', q', r', respectively, and add: then

$$\frac{2}{\sigma^2} = \lambda,$$

and thus the critical equations are

$$\left.\begin{aligned} p'(\phi r'^2 + \psi q'^2) &= \frac{2}{\sigma^2} A p', \\ q'(\psi p'^2 + \theta r'^2) &= \frac{2}{\sigma^2} B q', \\ r'(\theta q'^2 + \phi p'^2) &= \frac{2}{\sigma^2} C r' \end{aligned}\right\}.$$

(i) These equations are satisfied by the three sets of values

$$p' = 0,\ q' = 0;\quad q' = 0,\ r' = 0;\quad r' = 0,\ p' = 0.$$

They represent the respective lines of curvature; for each line of curvature, they provide a zero (minimum) value for $1/\sigma^2$.

(ii) It is conceivable that stationary values could be provided by values of p', q', r', all distinct from zero. In that event, we should have

$$\phi r'^2 + \psi q'^2 = \frac{2}{\sigma^2} A,\quad \psi p'^2 + \theta r'^2 = \frac{2}{\sigma^2} B,\quad \theta q'^2 + \phi p'^2 = \frac{2}{\sigma^2} C,$$

and therefore

$$\begin{aligned} \frac{p'^2}{\theta} &= \frac{1}{\sigma^2}(-A\theta + B\phi + C\psi)\frac{1}{\theta\phi\psi}, \\ \frac{q'^2}{\phi} &= \frac{1}{\sigma^2}(\ A\theta - B\phi + C\psi)\frac{1}{\theta\phi\psi}, \\ \frac{r'^2}{\psi} &= \frac{1}{\sigma^2}(\ A\theta + B\phi - C\psi)\frac{1}{\theta\phi\psi}. \end{aligned}$$

Now, on substitution for θ, ϕ, ψ, we have

$$-A\theta + B\phi + C\psi = 2ABC\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\left(\frac{1}{\alpha} - \frac{1}{\gamma}\right),$$

and therefore

$$\sigma^2 A p'^2 = \frac{2}{\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\left(\frac{1}{\alpha} - \frac{1}{\gamma}\right)};$$

and similarly

$$\sigma^2 Bq'^2 = \frac{2}{\left(\frac{1}{\beta}-\frac{1}{\gamma}\right)\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)},$$

$$\sigma^2 Cr'^2 = \frac{2}{\left(\frac{1}{\gamma}-\frac{1}{\alpha}\right)\left(\frac{1}{\gamma}-\frac{1}{\beta}\right)}.$$

Hence

$$\sigma^2 (Ap'^2 + Bq'^2 + Cr'^2) = \Sigma \frac{2}{\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)},$$

that is,

$$\sigma^2 = 0,$$

a value excluded by the analytical conditions. Thus no suitable maximum or minimum is provided when p', q', r', simultaneously differ from zero.

(iii) There remain three possible cases, viz. $p' = 0$, with q' and r' different from zero: $q' = 0$, with r' and p' different from zero: and $r' = 0$, with p' and q' different from zero.

For the first of these cases, we have

$$\theta r'^2 = \frac{2}{\sigma^2} B, \quad \theta q'^2 = \frac{2}{\sigma^2} C,$$

while

$$Bq'^2 + Cr'^2 = 1.$$

Hence the value of σ is given by

$$\frac{2}{\sigma} = \frac{1}{\beta} \sim \frac{1}{\gamma},$$

and the directions are

$$p' = 0, \quad q' = \pm \frac{1}{(2B)^{\frac{1}{2}}}, \quad r' = \pm \frac{1}{(2C)^{\frac{1}{2}}};$$

the directions lying in the plane $p' = 0$, that is, the plane through the directions of the lines of curvature $p' = 0$, $q' = 0$; and $p' = 0$, $r' = 0$.

For the second of the cases, we have

$$\frac{2}{\sigma} = \frac{1}{\gamma} \sim \frac{1}{\alpha},$$

with directions given by

$$p' = \pm \frac{1}{(2A)^{\frac{1}{2}}}, \quad q' = 0, \quad r' = \pm \frac{1}{(2C)^{\frac{1}{2}}},$$

lying in the plane $q' = 0$, that is, the plane through the directions of the lines of curvature $q' = 0$, $r' = 0$; and $q' = 0$, $p' = 0$.

For the third of the cases, we have

$$\frac{2}{\sigma} = \frac{1}{\alpha} \sim \frac{1}{\beta},$$

with directions given by

$$p' = \pm \frac{1}{(2A)^{\frac{1}{2}}}, \quad q' = \pm \frac{1}{(2B)^{\frac{1}{2}}}, \quad r' = 0,$$

lying in the plane $r' = 0$, that is, the plane through the directions of the lines of curvature $r' = 0, p' = 0$; and $r' = 0, q' = 0$.

Tilt of a geodesic: first expression.

309. Two different expressions, relating to the tilt of a geodesic in a region, can be derived from the formulæ relating to the various curvatures of a twisted curve in homaloidal quadruple space. Both expressions involve the regional magnitudes of the third order. We proceed to them in turn.

It has been proved (§ 145) that for a curve, which has ρ, σ, τ, for its radii of circular curvature, of torsion, and of tilt, the relation

$$-\frac{1}{\rho^3\sigma^2\tau} = \begin{vmatrix} x' & y' & z' & v' \\ x'' & y'' & z'' & v'' \\ x''' & y''' & z''' & v''' \\ x^{\text{IV}} & y^{\text{IV}} & z^{\text{IV}} & v^{\text{IV}} \end{vmatrix}$$

holds. For a geodesic in the general region, we have

$$x'' = \frac{1}{\rho} X,$$

and therefore

$$x''' = \frac{1}{\rho} X' - \frac{\rho'}{\rho^2} X,$$

$$x^{\text{IV}} = \frac{1}{\rho} X'' - 2\frac{\rho'}{\rho^2} X' + \left(2\frac{\rho'^2}{\rho^3} - \frac{\rho''}{\rho^2}\right) X;$$

and there are similar expressions for the corresponding derivatives of y, z, v, in terms of derivatives of Y, Z, V. Substituting for these derivatives of x, y, z, v, we find

$$\frac{1}{\sigma^2\tau} = \begin{vmatrix} X & Y & Z & V \\ x' & y' & z' & v' \\ X' & Y' & Z' & V' \\ X'' & Y'' & Z'' & V'' \end{vmatrix}.$$

Now any expression for a geometric magnitude is necessarily a covariant; it therefore is determinable from its leading coefficient. We have

$$x' = x_1 p' + x_2 q' + x_3 r',$$

with corresponding values for y', z', v'. Again, we have

$$X' = X_1 p' + X_2 q' + X_3 r',$$

with corresponding values for Y', Z', V'. Once again, we have

$$\begin{aligned} X'' = {} & X_1 p'' + X_2 q'' + X_3 r'' \\ & + X_{11} p'^2 + 2X_{12} p'q' + 2X_{13} p'r' + X_{22} q'^2 + 2X_{23} q'r' + X_{33} r'^2 \\ = {} & - X_1 \Sigma \Gamma_{11} p'^2 - X_2 \Sigma \Delta_{11} p'^2 - X_3 \Sigma \Theta_{11} p'^2 + \Sigma X_{11} p'^2, \end{aligned}$$

along the geodesic, that is,

$$X'' = \Sigma (X_{11} - \Gamma_{11} X_1 - \Delta_{11} X_2 - \Theta_{11} X_3) p'^2,$$

with corresponding values for Y'', Z'', V''. For brevity, we write

$$\begin{aligned} \xi_{11} &= X_{11} - \Gamma_{11} X_1 - \Delta_{11} X_2 - \Theta_{11} X_3, \\ \eta_{11} &= Y_{11} - \Gamma_{11} Y_1 - \Delta_{11} Y_2 - \Theta_{11} Y_3, \\ \zeta_{11} &= Z_{11} - \Gamma_{11} Z_1 - \Delta_{11} Z_2 - \Theta_{11} Z_3, \\ \upsilon_{11} &= V_{11} - \Gamma_{11} V_1 - \Delta_{11} V_2 - \Theta_{11} V_3, \end{aligned}$$

the values of all the coefficients of the combinations of p', q', r', in X'', Y'', Z'', V'', having already (§ 295) been obtained. Consequently the expression for $(\sigma^2 \tau)^{-1}$ becomes a covariant of the fourth order in p', q', r'; if

$$\frac{1}{\sigma^2 \tau} = \Phi p'^4 + \ldots,$$

so that Φ is the leading coefficient in the covariant, we have

$$\Phi = \begin{vmatrix} X, & Y, & Z, & V \\ x_1, & y_1, & z_1, & v_1 \\ X_1, & Y_1, & Z_1, & V_1 \\ \xi_{11}, & \eta_{11}, & \zeta_{11}, & \upsilon_{11} \end{vmatrix}.$$

Now we have proved (§ 274) that

$$- \Omega^2 X_1 = x_1 \lambda_1 + x_2 \mu_1 + x_3 \nu_1,$$

with corresponding values for Y_1, Z_1, V_1, and (§ 295) that

$$\begin{aligned} - \Omega^2 \xi_{11} &= X D_{11} + \xi_1 e_{300} + \xi_2 e_{210} + \xi_3 e_{201} \\ &= X D_{11} + x_1 \theta + x_2 \phi + x_3 \psi, \\ - \Omega^2 \eta_{11} &= Y D_{11} + y_1 \theta + y_2 \phi + y_3 \psi, \\ - \Omega^2 \zeta_{11} &= Z D_{11} + z_1 \theta + z_2 \phi + z_3 \psi, \\ - \Omega^2 \upsilon_{11} &= V D_{11} + v_1 \theta + v_2 \phi + v_3 \psi, \end{aligned}$$

where

$$\begin{aligned} \theta &= a e_{300} + h e_{210} + g e_{201}, \\ \phi &= h e_{300} + b e_{210} + f e_{201}, \\ \psi &= g e_{300} + f e_{210} + c e_{201}. \end{aligned}$$

When these values are substituted, we find

$$\Omega^4\Phi = \begin{vmatrix} X, & x_1, & x_1\lambda_1 + x_2\mu_1 + x_3\nu_1, & XD_{11} + x_1\theta + x_2\phi + x_3\psi \\ Y, & y_1, & y_1\lambda_1 + y_2\mu_1 + y_3\nu_1, & YD_{11} + y_1\theta + y_2\phi + y_3\psi \\ Z, & z_1, & z_1\lambda_1 + z_2\mu_1 + z_3\nu_1, & ZD_{11} + z_1\theta + z_2\phi + z_3\psi \\ V, & v_1, & v_1\lambda_1 + v_2\mu_1 + v_3\nu_1, & VD_{11} + v_1\theta + v_2\phi + v_3\psi \end{vmatrix}$$

$$= \begin{vmatrix} X, & Y, & Z, & V \\ x_1, & y_1, & z_1, & v_1 \\ x_2\mu_1 + x_3\nu_1, & y_2\mu_1 + y_3\nu_1, & z_2\mu_1 + z_3\nu_1, & v_2\mu_1 + v_3\nu_1 \\ x_2\phi + x_3\psi, & y_2\phi + y_3\psi, & z_2\phi + z_3\psi, & v_2\phi + v_3\psi \end{vmatrix}$$

$$= (\mu_1\psi - \nu_1\phi) \begin{vmatrix} X, & Y, & Z, & V \\ x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ x_3, & y_3, & z_3, & v_3 \end{vmatrix}$$

$$= \Omega(\mu_1\psi - \nu_1\phi).$$

In the expression $\mu_1\psi - \nu_1\phi$, the coefficient of e_{300} is

$$= g(hL + bM + fK) - h(gL + fM + cK)$$
$$= M(bg - fh) + K(fg - ch) = \Omega^2(HK - GM);$$

the coefficient of e_{210} is

$$= f(hL + bM + fK) - b(gL + fM + cK)$$
$$= L(fh - bg) + K(f^2 - bc) = \Omega^2(GL - AK);$$

and the coefficient of e_{201} is

$$= c(hL + bM + fK) - f(gL + fM + cK)$$
$$= L(ch - fg) + M(bc - f^2) = \Omega^2(AM - HL).$$

Consequently

$$\Omega\Phi = e_{300}(HK - GM) + e_{210}(GL - AK) + e_{201}(AM - HL)$$
$$= \begin{vmatrix} e_{300}, & e_{210}, & e_{201} \\ A, & H, & G \\ L, & M, & K \end{vmatrix},$$

thus giving the value of Φ, the leading coefficient in the value of $(\sigma^2\tau)^{-1}$, that is, the coefficient of p'^4.

But we have had three covariantive quantics (§§ 292, 262, 272)

$$\overline{W} = (e \,\mathfrak{X}\, p', q', r')^3,$$
$$\overline{U} = (A \,\mathfrak{X}\, p', q', r')^2,$$
$$\overline{V} = (L \,\mathfrak{X}\, p', q', r')^2;$$

and the Jacobian of three ternary quantics is a covariant of their system. So, taking the Jacobian of $\overline{W}$, $\overline{U}$, $\overline{V}$, namely

$$\begin{vmatrix} \frac{\partial \overline{W}}{\partial \overline{p'}}, & \frac{\partial \overline{W}}{\partial \overline{q'}}, & \frac{\partial \overline{W}}{\partial \overline{r'}} \\ \frac{\partial \overline{U}}{\partial p'}, & \frac{\partial \overline{U}}{\partial \overline{q'}}, & \frac{\partial \overline{U}}{\partial \overline{r'}} \\ \frac{\partial \overline{V}}{\partial \overline{p'}}, & \frac{\partial \overline{V}}{\partial \overline{q'}}, & \frac{\partial \overline{V}}{\partial \overline{r'}} \end{vmatrix},$$

and writing it in the form

$$J\left(\frac{\overline{U}, \overline{V}, \overline{W}}{p', q', r'}\right),$$

we have a covariant of the fourth order in p', q', r'; and its leading coefficient is

$$12 \begin{vmatrix} e_{300}, & e_{210}, & e_{201} \\ A, & H, & G \\ L, & M, & K \end{vmatrix},$$

which is the value of $12\Omega\Phi$. It follows, therefore, that

$$\frac{1}{\sigma^2 \tau} = \frac{1}{12\Omega} J\left(\frac{\overline{U}, \overline{V}, \overline{W}}{p', q', r'}\right),$$

an expression which, as the value of σ^2 is known (§ 306), gives an analytical expression for *the tilt of a regional geodesic.*

Tilt of a geodesic: second expression.

310. Another expression for the tilt of a regional geodesic is derived from the general formulæ connecting the directions of the lines in the orthogonal frame of a twisted curve with the circular curvature, the torsion, and the tilt, of the curve.

Along a geodesic in a region, we have

$$x'' = \frac{X}{\rho}, \quad y'' = \frac{Y}{\rho}, \quad z'' = \frac{Z}{\rho}, \quad v'' = \frac{V}{\rho};$$

and therefore

$$x''' = \frac{1}{\rho} X' - \frac{\rho'}{\rho^2} X, \quad y''' = \frac{1}{\rho} Y' - \frac{\rho'}{\rho^2} Y, \quad z''' = \frac{1}{\rho} Z' - \frac{\rho'}{\rho^2} Z, \quad v''' = \frac{1}{\rho} V' - \frac{\rho'}{\rho^2} V.$$

Hence

$$\Sigma x'''^2 = \frac{1}{\rho^2} \Sigma X'^2 + \frac{\rho'^2}{\rho^4},$$

so that, as before (§ 306), along a geodesic,

$$\Sigma X'^2 = \frac{1}{\rho^2} + \frac{1}{\sigma^2}.$$

Again, along a geodesic, we have

$$x^{\mathrm{iv}} = \frac{1}{\rho} X'' - 2\frac{\rho'}{\rho^2} X' + \left(2\frac{\rho'^2}{\rho^3} - \frac{\rho''}{\rho^2}\right) X,$$

with similar expressions for y^{iv}, z^{iv}, v^{iv}; thus

$$\Sigma x^{\mathrm{iv}2} = \frac{1}{\rho^2}\Sigma X''^2 - 4\frac{\rho'}{\rho^3}\Sigma X'X'' + \frac{2}{\rho}\left(2\frac{\rho'^2}{\rho^3} - \frac{\rho''}{\rho^2}\right)\Sigma XX''$$
$$+ 4\frac{\rho'^2}{\rho^4}\Sigma X'^2 - 4\frac{\rho'}{\rho^2}\left(2\frac{\rho'^2}{\rho^3} - \frac{\rho''}{\rho^2}\right)\Sigma XX' + \left(2\frac{\rho'^2}{\rho^3} - \frac{\rho''}{\rho^2}\right)^2.$$

But

$$\Sigma XX' = 0,$$
$$\Sigma XX'' = -\Sigma X'^2 = -\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right),$$
$$\Sigma X'X'' = -\left(\frac{\rho'}{\rho^3} + \frac{\sigma'}{\sigma^3}\right);$$

and therefore

$$\Sigma x^{\mathrm{iv}2} = \frac{1}{\rho^2}\Sigma X''^2 + 2\frac{\rho''}{\rho^3}\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) + 4\frac{\rho'}{\rho^3}\left(\frac{\rho'}{\rho^3} + \frac{\sigma'}{\sigma^3}\right) + \left(2\frac{\rho'^2}{\rho^3} - \frac{\rho''}{\rho^2}\right)^2.$$

Now, for any curve, there are the relations (§ 151), connecting the fourth derivatives of x, y, z, v, with the main direction-cosines in its orthogonal frame and its curvatures, in the form

$$x^{\mathrm{iv}} = l_1\bar{L} + l_2\bar{M}\rho + l_3\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right)\frac{1}{\rho\sigma} + l_4\frac{1}{\rho\sigma\tau},$$

(with corresponding values for y^{iv}, z^{iv}, v^{iv}), where l_1, m_1, n_1, k_1, are the direction-cosines of the tangent; l_2, m_2, n_2, k_2, those of the principal (or first) normal; l_3, m_3, n_3, k_3, those of the binormal; l_4, m_4, n_4, k_4, those of the trinormal; and

$$-\bar{M} = \frac{\rho''}{\rho^3} - 2\frac{\rho'^2}{\rho^4} + \frac{1}{\rho^2\sigma^2} + \frac{1}{\rho^4}, \quad \bar{L} = 3\frac{\rho'}{\rho^3}.$$

Hence

$$\Sigma x^{\mathrm{iv}2} = \bar{L}^2 + \bar{M}^2\rho^2 + \frac{1}{\rho^2\sigma^2}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right)^2 + \frac{1}{\rho^2\sigma^2\tau^2}.$$

When the two expressions obtained for $\Sigma x^{\mathrm{iv}2}$ are equated, and the resulting equation is reduced, we find

$$\frac{1}{\sigma^2\tau^2} = \Sigma X''^2 - \left(\frac{\rho'^2}{\rho^4} + \frac{\sigma'^2}{\sigma^4}\right) - \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)^2,$$

a first form of the indicated expression for the tilt.

311. It has been proved (§ 309) that

$$X'' = \Sigma\left(X_{11} - \Gamma_{11}X_1 - \Delta_{11}X_2 - \Theta_{11}X_3\right)p'^2,$$

where the summation is over the six terms in p'^2, $p'q'$, $p'r'$, q'^2, $q'r'$, r'^2; and so

$$-\Omega^2 X'' = \Sigma\left(XD_{11} + x_1\theta + x_2\phi + x_3\psi\right)p'^2.$$

Hence

$$\Omega^4 \Sigma X''^2 = (\Sigma D_{11} p'^2)^2 + \sum_x \{\sum_p (x_1\theta + x_2\phi + x_3\psi)\, p'^2\}^2.$$

Now

$$D_{11} = L\lambda_1 + M\mu_1 + K\nu_1$$
$$= (a, b, c, f, g, h)(L, M, K)^2,$$

so that

$$\Sigma D_{11} p'^2 = \frac{1}{4}(a, b, c, f, g, h)\left(\frac{\partial \overline{V}}{\partial p}, \frac{\partial \overline{V}}{\partial q}, \frac{\partial \overline{V}}{\partial r}\right)^2.$$

The covariant on the right-hand side has been denoted by $\mathbf{D}$; then, by the result in § 306,

$$\mathbf{D} = \Omega^2\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right);$$

consequently

$$\Omega^4\left\{\Sigma X''^2 - \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)^2\right\} = \sum_x \{\sum_p (x_1\theta + x_2\phi + x_3\psi)\, p'^2\}^2.$$

Next,

$$\sum_x \{\sum_p (x_1\theta + x_2\phi + x_3\psi)\, p'^2\}^2$$
$$= \sum_p (A, B, C, F, G, H)(\theta, \phi, \psi)^2\, p'^4.$$

On the right-hand side, the coefficient of p'^4 is

$$= A\theta^2 + 2H\theta\phi + 2G\theta\psi + B\phi^2 + 2F\phi\psi + C\psi^2.$$

Now, from the values of θ, ϕ, ψ, on p. 113, we have

$$A\theta + H\phi + G\psi = \Omega^2 e_{300}, \quad H\theta + B\phi + F\psi = \Omega^2 e_{210}, \quad G\theta + F\phi + C\psi = \Omega^2 e_{201};$$

and therefore the coefficient of p'^4

$$= \Omega^2(\theta e_{300} + \phi e_{210} + \psi e_{201})$$
$$= \Omega^2(a, b, c, f, g, h)(e_{300}, e_{210}, e_{201})^2.$$

Also

$$\frac{1}{3}\frac{\partial \overline{W}}{\partial p'} = \Sigma e_{300} p'^2, \quad \frac{1}{3}\frac{\partial \overline{W}}{\partial q'} = \Sigma e_{210} p'^2, \quad \frac{1}{3}\frac{\partial \overline{W}}{\partial r'} = \Sigma e_{201} p'^2;$$

and therefore

$$\Omega^2\left\{\Sigma X''^2 - \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)^2\right\} = \frac{1}{9}(a, b, c, f, g, h)\left(\frac{\partial \overline{W}}{\partial p'}, \frac{\partial \overline{W}}{\partial q'}, \frac{\partial \overline{W}}{\partial r'}\right)^2$$
$$= \Psi,$$

where Ψ is a new covariant; and our formula now becomes

$$\frac{1}{\sigma^2\tau^2} = \frac{\Psi}{\Omega^2} - \left(\frac{\rho'^2}{\rho^4} + \frac{\sigma'^2}{\sigma^4}\right).$$

We still require the expression of the second term on the right-hand side, in terms of the magnitudes of the region and of p', q', r'.

312. For this purpose, we proceed from the relation

$$\Omega^2\left(\frac{1}{\sigma^2}+\frac{1}{\rho^2}\right)=(a, b, c, f, g, h\,\bar{u}, \bar{v}, \bar{w})^2=\mathbf{D},$$

where

$$\bar{u}=Lp'+Mq'+Kr', \quad \bar{v}=Mp'+Nq'+Ir', \quad \bar{w}=Kp'+Iq'+Jr';$$

and thus

$$\frac{1}{\sigma^2}+\frac{1}{\rho^2}=\frac{a}{\Omega^2}\bar{u}^2+2\frac{h}{\Omega^2}\bar{u}\bar{v}+2\frac{g}{\Omega^2}\bar{u}\bar{w}+\frac{b}{\Omega^2}\bar{v}^2+2\frac{f}{\Omega^2}\bar{v}\bar{w}+\frac{c}{\Omega^2}\bar{w}^2.$$

Differentiating both sides with regard to s, and noting that

$$\frac{d}{ds}\left(\frac{a}{\Omega^2}\right)=p'\frac{\partial}{\partial p}\left(\frac{a}{\Omega^2}\right)+q'\frac{\partial}{\partial q}\left(\frac{a}{\Omega^2}\right)+r'\frac{\partial}{\partial r}\left(\frac{a}{\Omega^2}\right)$$

for each quantity such as a, we have

$$\begin{aligned}
&-2\Omega^2\left(\frac{\sigma'}{\sigma^3}+\frac{\rho'}{\rho^3}\right)\\
&=-2\bar{u}^2\ [\ (a\Gamma_{11}+h\Gamma_{12}+g\Gamma_{13})\,p'+\ldots]\\
&\quad-2\bar{u}\bar{v}\ [\{(h\Gamma_{11}+b\Gamma_{12}+f\Gamma_{13})+(a\Delta_{11}+h\Delta_{12}+g\Delta_{13})\}\,p'+\ldots]\\
&\quad-2\bar{u}\bar{w}\,[\{(g\Gamma_{11}+f\Gamma_{12}+c\Gamma_{13})+(a\Theta_{11}+h\Theta_{12}+g\Theta_{13})\}\,p'+\ldots]\\
&\qquad-2\bar{v}^2\ [\ (h\Delta_{11}+b\Delta_{12}+f\Delta_{13})\,p'+\ldots]\\
&\qquad-2\bar{v}\bar{w}\,[\{(g\Delta_{11}+f\Delta_{12}+c\Delta_{13})+(h\Theta_{11}+b\Theta_{12}+f\Theta_{13})\}\,p'+\ldots]\\
&\qquad-2\bar{w}^2[(g\Theta_{11}+f\Theta_{12}+c\Theta_{13})\ p'+\ldots]\\
&+2\,(a\bar{u}+h\bar{v}+g\bar{w})\,\{Lp''+Mq''+Kr''\\
&\quad+p'\,(L_1p'+M_1q'+K_1r')+q'\,(L_2p'+M_2q'+K_2r')+r'\,(L_3p'+M_3q'+K_3r')\}\\
&+2\,(h\bar{u}+b\bar{v}+f\bar{w})\,\{Mp''+Nq''+Ir''\\
&\quad+p'\,(M_1p'+N_1q'+I_1r')+q'\,(M_2p'+N_2q'+I_2r')+r'\,(M_3p'+N_3q'+I_3r')\}\\
&+2\,(g\bar{u}+f\bar{v}+c\bar{w})\,\{Kp''+Iq''+Jr''\\
&\quad+p'\,(K_1p'+I_1q'+J_1r')+q'\,(K_2p'+I_2q'+J_2r')+r'\,(K_3p'+I_3q'+J_3r')\}.
\end{aligned}$$

Let the coefficient of $2p'^3$ on the right-hand side be denoted by Π, so that

$$-\Omega^2\left(\frac{\sigma'}{\sigma^3}+\frac{\rho'}{\rho^3}\right)=\sum_p \Pi p'^3.$$

In the expression for Π, after substitution has been made for p'', q'', r'', for the values along a geodesic, we have

$$\begin{aligned}
\Pi=&-L^2\quad(a\Gamma_{11}+h\Gamma_{12}+g\Gamma_{13})\\
&-LM\,\{(h\Gamma_{11}+b\Gamma_{12}+f\Gamma_{13})(a\Delta_{11}+h\Delta_{12}+g\Delta_{13})\}\\
&-LN\,\{(g\Gamma_{11}+f\Gamma_{12}+c\Gamma_{13})+(a\Theta_{11}+h\Theta_{12}+g\Theta_{13})\}\\
&\qquad-M^2\quad(h\Delta_{11}+b\Delta_{12}+f\Delta_{13})\\
&\qquad-MN\,\{(g\Delta_{11}+f\Delta_{12}+c\Delta_{13})+(h\Theta_{11}+b\Theta_{12}+f\Theta_{13})\}\\
&\qquad-N^2\,(g\Theta_{11}+f\Theta_{12}+c\Theta_{13})
\end{aligned}$$

$$+(aL+hM+gK)\,(L_1-L\Gamma_{11}-M\Delta_{11}-K\Theta_{11})$$
$$+(hL+bM+fK)\,(M_1-M\Gamma_{11}-N\Delta_{11}+I\Theta_{11})$$
$$+(gL+fM+cK)\,(K_1-K\Gamma_{11}-I\Delta_{11}+J\Theta_{11}):$$

or, when we substitute for L_1, M_1, K_1, and reduce, we find

$$\Pi=(aL+hM+gK)\,e_{300}+(hL+bM+fK)\,e_{210}+(gL+fM+cK)\,e_{201}.$$

But manifestly Π is the coefficient of p'^3 in the covariant

$$\Xi=\frac{1}{6}(a,b,c,f,g,h)\left(\frac{\partial\overline{V}}{\partial\overline{p}'},\ \frac{\partial\overline{V}}{\partial\overline{q}'},\ \frac{\partial\overline{V}}{\partial\overline{r}'}\right)^3\left(\frac{\partial\overline{W}}{\partial\overline{p}'},\ \frac{\partial\overline{W}}{\partial\overline{q}'},\ \frac{\partial\overline{W}}{\partial\overline{r}'}\right);$$

consequently we have

$$-\Omega^2\left(\frac{\sigma'}{\sigma^3}+\frac{\rho'}{\rho^3}\right)=\Xi.$$

Again, we have

$$-\frac{\rho'}{\rho^2}=\overline{W},\quad \frac{1}{\rho}=\overline{V};$$

hence

$$-\Omega^2\frac{\sigma'}{\sigma^3}=\Xi-\Omega^2\overline{V}\,\overline{W}.$$

The final expression for $\frac{1}{\sigma^2\tau^2}$ requires the value of

$$\frac{\rho'^2}{\rho^4}+\frac{\sigma'^2}{\sigma^4};$$

the preceding relations give

$$\frac{\rho'^2}{\rho^4}+\frac{\sigma'^2}{\sigma^4}=\frac{1}{\Omega^2}\,\frac{\Xi^2-2\Xi\Omega^2\overline{V}\,\overline{W}+\Omega^2\overline{W}^2\mathbf{D}}{\mathbf{D}-\Omega^2\overline{V}^2}.$$

Hence, finally, we have

$$\frac{1}{\sigma^2\tau^2}=\frac{1}{\Omega^2}\left\{\Psi-\frac{\Xi^2-2\Xi\Omega^2\overline{V}\,\overline{W}+\Omega^2\overline{W}^2\mathbf{D}}{\mathbf{D}-\Omega^2\overline{V}^2}\right\},$$

or

$$\frac{1}{\tau^2}=\frac{\Psi}{\mathbf{D}-\Omega^2\overline{V}^2}-\frac{\Xi^2-2\Xi\Omega^2\overline{V}\,\overline{W}+\Omega^2\overline{W}^2\mathbf{D}}{(\mathbf{D}-\Omega^2\overline{V}^2)^2}$$
$$=\frac{\Psi-\Omega^2\overline{W}^2}{\mathbf{S}}-\left(\frac{\Xi-\Omega^2\overline{V}\,\overline{W}}{\mathbf{S}}\right)^2,$$

where $\mathbf{S}$ is the covariant on p. 109.

This value of τ, and the value of τ obtained in § 309, are formally distinct from one another; equated, they lead to a relation between the various functions. The relation is a purely algebraical identity affecting the system of concomitants that belong to the three simultaneous ternary forms $\overline{U}$, $\overline{V}$, $\overline{W}$.

CHAPTER XVIII.

GEODESIC SURFACES: SUPERFICIAL CURVATURE OF A REGION.

A regional geodesic, lying in a surface in the region, is a superficial geodesic.

313. Geodesics in a region are used by Riemann, for the construction of a measure of superficial curvature at any orientation in the region.

Two preliminary propositions may be established as an initial basis for Riemann's process. The first proposition, standing isolated, is almost a truism: the analysis, subsidiary to a mathematical proof or verification, is otherwise useful. The proposition is that *when a geodesic in a region lies wholly on a surface in the region, it is a geodesic on the surface*; and it can be verified as follows. The equations of a geodesic in a region are (§ 298)

$$\begin{aligned} Ap'' + Hq'' + Gr'' + (\alpha_1, \delta_1, \omega_1, \epsilon_1, \gamma_1, \beta_1 \mathbin{\text{\textbigtriangledown}} p', q', r')^2 = 0, \\ Hp'' + Bq'' + Fr'' + (\alpha_2, \delta_2, \omega_2, \epsilon_2, \gamma_2, \beta_2 \mathbin{\text{\textbigtriangledown}} p', q', r')^2 = 0, \\ Gp'' + Fq'' + Cr'' + (\alpha_3, \delta_3, \omega_3, \epsilon_3, \gamma_3, \beta_3 \mathbin{\text{\textbigtriangledown}} p', q', r')^2 = 0. \end{aligned}$$

Let the surface, upon which the geodesic lies wholly, be given as a surface in the region by equations which express p, q, r, the parameters of the region, as functions of u and v, two parameters for the surface; and write

$$\frac{\partial p}{\partial u} = p_1, \quad \frac{\partial p}{\partial v} = p_2, \quad \frac{\partial q}{\partial u} = q_1, \quad \frac{\partial q}{\partial v} = q_2, \quad \frac{\partial r}{\partial u} = r_1, \quad \frac{\partial r}{\partial v} = r_2.$$

The general arc-element in the region is given by

$$ds^2 = A\,dp^2 + 2H\,dp\,dq + 2G\,dp\,dr + B\,dq^2 + 2F\,dq\,dr + C\,dr^2;$$

and consequently the general arc-element on the postulated surface is

$$ds^2 = \Theta\,du^2 + 2\Phi\,du\,dv + \Psi\,dv^2,$$

where

$$\begin{aligned} \Theta &= (A, B, C, F, G, H \mathbin{\text{\textbigtriangledown}} p_1, q_1, r_1)^2, \\ \Phi &= (A, B, C, F, G, H \mathbin{\text{\textbigtriangledown}} p_1, q_1, r_1 \mathbin{\text{\textbigtriangledown}} p_2, q_2, r_2), \\ \Psi &= (A, B, C, F, G, H \mathbin{\text{\textbigtriangledown}} p_2, q_2, r_2)^2. \end{aligned}$$

Now (§ 219) the equations of a geodesic on this surface in quadruple space are

$$\left.\begin{aligned} \Theta u'' + \Phi v'' + mu'^2 + 2m'u'v' + m''v'^2 = 0 \\ \Phi u'' + \Psi v'' + nu'^2 + 2n'u'v' + n''v'^2 = 0 \end{aligned}\right\},$$

where

$$\begin{aligned} m &= \frac{1}{2}\frac{\partial \Theta}{\partial u}, \quad m' = \frac{1}{2}\frac{\partial \Theta}{\partial v}, \quad m'' = \frac{\partial \Phi}{\partial v} - \frac{1}{2}\frac{\partial \Psi}{\partial u}, \\ n &= \frac{\partial \Phi}{\partial u} - \frac{1}{2}\frac{\partial \Theta}{\partial v}, \quad n' = \frac{1}{2}\frac{\partial \Psi}{\partial u}, \quad n'' = \frac{1}{2}\frac{\partial \Psi}{\partial v}. \end{aligned}$$

In order that the proposition may be verified, the set of equations for the regional geodesic must, by the relations between p, q, r, u, v, be transformed into the equations for the superficial geodesic. We have

$$p' = p_1 u' + p_2 v', \quad p'' = p_1 u'' + p_2 v'' + p_{11} u'^2 + 2p_{12} u'v' + p_{22} v'^2,$$
$$q' = q_1 u' + q_2 v', \quad q'' = q_1 u'' + q_2 v'' + q_{11} u'^2 + 2q_{12} u'v' + q_{22} v'^2,$$
$$r' = r_1 u' + r_2 v', \quad r'' = r_1 u'' + r_2 v'' + r_{11} u'^2 + 2r_{12} u'v' + r_{22} v'^2:$$

$$Ap'' + Hq'' + Gr'' = (Ap_1 + Hq_1 + Gr_1) u'' + (Ap_2 + Hq_2 + Gr_2) v''$$
$$+ (Ap_{11} + Hq_{11} + Gr_{11}) u'^2 + 2 (Ap_{12} + Hq_{12} + Gr_{12}) u'v'$$
$$+ (Ap_{22} + Hq_{22} + Gr_{22}) v'^2,$$

$$Hp'' + Bq'' + Fr'' = (Hp_1 + Bq_1 + Fr_1) u'' + (Hp_2 + Bq_2 + Fr_2) v''$$
$$+ (Hp_{11} + Bq_{11} + Fr_{11}) u'^2 + 2 (Hp_{12} + Bq_{12} + Fr_{12}) u'v'$$
$$+ (Hp_{22} + Bq_{22} + Fr_{22}) v'^2,$$

$$Gp'' + Fq'' + Cr'' = (Gp_1 + Fq_1 + Cr_1) u'' + (Gp_2 + Fq_2 + Cr_2) v''$$
$$+ (Gp_{11} + Fq_{11} + Cr_{11}) u'^2 + 2 (Gp_{12} + Fq_{12} + Cr_{12}) u'v'$$
$$+ (Gp_{22} + Fq_{22} + Cr_{22}) v'^2.$$

Multiply the last three equations by p_1, q_1, r_1, respectively, and add: the sum of the right-hand sides is equal to

$$\Theta u'' + \Phi v''$$
$$+ \{(Ap_1 + Hq_1 + Gr_1) p_{11} + (Hp_1 + Bq_1 + Fr_1) q_{11} + (Gp_1 + Fq_1 + Cr_1) r_{11}\} u'^2$$
$$+ 2\{(Ap_1 + Hq_1 + Gr_1) p_{12} + (Hp_1 + Bq_1 + Fr_1) q_{12} + (Gp_1 + Fq_1 + Cr_1) r_{12}\} u'v'$$
$$+ \{(Ap_1 + Hq_1 + Gr_1) p_{22} + (Hp_1 + Bq_1 + Fr_1) q_{22} + (Gp_1 + Fq_1 + Cr_1) r_{22}\} v'^2,$$

while the sum of the left-hand sides, after substitution from the earlier equations of the regional geodesics, is equal to

$$-\{(p_1\alpha_1 + q_1\alpha_2 + r_1\alpha_3) p'^2 + 2 (p_1\beta_1 + q_1\beta_2 + r_1\beta_3) p'q' + \cdots$$
$$+ (p_1\omega_1 + q_1\omega_2 + r_1\omega_3) r'^2\}.$$

Thus, as one equivalent of the equations of the regional geodesic, we have

$$\Theta u'' + \Phi v''$$
$$+ u'^2 \{(Ap_1 + Hq_1 + Gr_1) p_{11} + (Hp_1 + Bq_1 + Fr_1) q_{11} + (Gp_1 + Fq_1 + Cr_1) r_{11}\} + \cdots$$
$$+ (p_1\alpha_1 + q_1\alpha_2 + r_1\alpha_3) p'^2 + \ldots + (p_1\omega_1 + q_1\omega_2 + r_1\omega_3) r'^2 = 0.$$

In this last equation, substitute the values of p', q', r'; after substitution, the aggregate coefficient of u'^2 is

$$(Ap_1 + Hq_1 + Gr_1) p_{11} + (Hp_1 + Bq_1 + Fr_1) q_{11} + (Gp_1 + Fq_1 + Cr_1) r_{11}$$
$$+ p_1^2 (p_1\alpha_1 + q_1\alpha_2 + r_1\alpha_3) + \ldots + r_1^2 (p_1\omega_1 + q_1\omega_2 + r_1\omega_3).$$

This expression is equal to

$$\begin{aligned}
&p_1\{(Ap_{11}+Hq_{11}+Gr_{11})+\alpha_1p_1^2+(\alpha_2+\beta_1)p_1q_1+(\alpha_3+\gamma_1)p_1r_1\\
&\qquad+\beta_2q_1^2+(\gamma_2+\beta_3)q_1r_1+\gamma_3r_1^2\}\\
&+q_1\{(Hp_{11}+Bq_{11}+Fr_{11})+\beta_1p_1^2+(\beta_2+\delta_1)p_1q_1+(\beta_3+\epsilon_1)p_1r_1\\
&\qquad+\delta_2q_1^2+(\epsilon_2+\delta_3)q_1r_1+\epsilon_3r_1^2\}\\
&+r_1\{(Gp_{11}+Fq_{11}+Cr_{11})+\gamma_1p_1^2+(\gamma_2+\epsilon_1)p_1q_1+(\gamma_3+\omega_1)p_1r_1\\
&\qquad+\epsilon_2q_1^2+(\omega_2+\epsilon_3)q_1r_1+\omega_3r_1^2\}\\
&=\tfrac{1}{2}p_1\{2(Ap_{11}+Hq_{11}+Gr_{11})+A_1p_1^2+2H_1p_1q_1+2G_1p_1r_1\\
&\qquad+B_1q_1^2+2F_1q_1r_1+C_1r_1^2\}\\
&+\tfrac{1}{2}q_1\{2(Hp_{11}+Bq_{11}+Fr_{11})+A_2p_1^2+2H_2p_1q_1+2G_2p_1r_1\\
&\qquad+B_2q_1^2+2F_2q_1r_1+C_2r_1^2\}\\
&+\tfrac{1}{2}r_1\{2(Gp_{11}+Fq_{11}+Cr_{11})+A_3p_1^2+2H_3p_1q_1+2G_3p_1r_1\\
&\qquad+B_3q_1^2+2F_3q_1r_1+C_3r_1^2\}\\
&=\frac{1}{2}\left(p_1\frac{\partial\Theta}{\partial p}+q_1\frac{\partial\Theta}{\partial q}+r_1\frac{\partial\Theta}{\partial r}\right)=\frac{1}{2}\frac{\partial\Theta}{\partial u}=m:
\end{aligned}$$

that is, the term in u'^2 is mu'^2. Similarly for the terms in $u'v'$ and v'^2. Thus one equation, derived from the regional equations, is

$$\Theta u''+\Phi v''+mu'^2+2m'u'v'+m''v'^2=0.$$

Similarly, another equation, derived from the regional equations, is

$$\Phi u''+\Psi v''+nu'^2+2n'u'v'+n''v'^2=0.$$

Moreover, there are (§ 300) only two independent equations defining the regional geodesics. We thus have obtained two equivalent equations: and these derived equations are those of the superficial geodesic. The proposition is therefore verified.

Two inflexional regional geodesics for any surface.

314. In the next place, *through any point on a surface in a region of quadruple space there pass, in general, two regional geodesics which are inflexional to the surface at the point.*

Let any surface in the region be given by the equation

$$f(p, q, r)=0.$$

Along any direction in the surface, the equations

$$p'f_1+q'f_2+r'f_3=0,$$

$$p''f_1+q''f_2+r''f_3+p'^2f_{11}+2p'q'f_{12}+2p'r'f_{13}+q'^2f_{22}+2q'r'f_{23}+r'^2f_{33}=0,$$

are satisfied. If then the direction, up to the second order inclusive, be a regional geodesic, the equations

$$p''+(\Gamma\mathrel{\backslash\!\!\backslash}p', q', r')^2=0,\quad q''+(\Delta\mathrel{\backslash\!\!\backslash}p', q', r')^2=0,\quad r''+(\Theta\mathrel{\backslash\!\!\backslash}p', q', r')^2=0,$$

are satisfied; in that event, we have

$$(\Phi_{11}, \Phi_{22}, \Phi_{33}, \Phi_{23}, \Phi_{31}, \Phi_{12}\mathrel{\backslash\!\!\backslash}p', q', r')^2=0,$$

where we write

$$\Phi_{lm} = f_{lm} - \Gamma_{lm} f_1 - \Delta_{lm} f_2 - \Theta_{lm} f_3,$$

for all the six combinations l, m, $= 1, 2, 3$. Accordingly, the parameters p', q', r', of such a direction satisfy a homogeneous linear equation and also a homogeneous quadratic equation, and therefore there usually are two sets of such values: that is, in general there are two such directions. Moreover, the second equation arises out of an equation implying contact of the second order: hence the two geodesics in question, belonging to the region, are inflexional to the surface.

The two directions coincide at all points on the curve which is the intersection of the surface $f = 0$ by the surface

$$\begin{vmatrix} \Phi_{11}, & \Phi_{12}, & \Phi_{13}, & f_1 \\ \Phi_{12}, & \Phi_{22}, & \Phi_{23}, & f_2 \\ \Phi_{13}, & \Phi_{23}, & \Phi_{33}, & f_3 \\ f_1, & f_2, & f_3, & 0 \end{vmatrix} = 0.$$

It may happen, however, that the second equation is satisfied for any set of values p', q', r', satisfying the first (the linear) equation. In that event, every direction through a point gives a regional geodesic which is inflexional to the surface: the necessary and sufficient conditions are

$$\left.\begin{aligned} 2\frac{\Phi_{23}}{f_2 f_3} &= \frac{\Phi_{22}}{f_2^2} + \frac{\Phi_{33}}{f_3^2} \\ 2\frac{\Phi_{31}}{f_3 f_1} &= \frac{\Phi_{33}}{f_3^2} + \frac{\Phi_{11}}{f_1^2} \\ 2\frac{\Phi_{12}}{f_1 f_2} &= \frac{\Phi_{11}}{f_1^2} + \frac{\Phi_{22}}{f_2^2} \end{aligned}\right\}.$$

These equations, manifestly partial differential equations of the second order, can be regarded as the equations characteristic of geodesic surfaces lying within the specified region.

Riemann's geodesic surfaces in a region.

315. We now proceed to consider Riemann's geodesic surfaces in a region, specially for the purpose of estimating their Gauss measure of curvature.

We take any two directions

$$p_1', \ q_1', \ r_1'; \ p_2', \ q_2', \ r_2';$$

in the region. Each of these directions, taken as an initial direction, determines a regional geodesic. Associated with those two directions and determined by them, there exists the infinitude of directions given by

$$\lambda p_1' + \mu p_2', \ \lambda q_1' + \mu q_2', \ \lambda r_1' + \mu r_2';$$

and each such direction, taken as initial, determines (§ 301) a regional geodesic. All these directions lie in one plane; for when the two initial directions are tangential to a surface $f(p, q, r) = 0$, so that

$$p_1' f_1 + q_1' f_2 + r_1' f_3 = 0, \quad p_2' f_1 + q_2' f_2 + r_2' f_3 = 0,$$

we also have

$$(\lambda p_1' + \mu p_2') f_1 + (\lambda q_1' + \mu q_2') f_2 + (\lambda r_1' + \mu r_2') f_3 = 0$$

unconditionally. All these regional geodesics, passing through one point and having their initial directions at the point lying in one plane, generate a surface; and such a surface is called a *geodesic surface.* The Gauss measure of curvature of this surface at the point is of importance in connection with the Riemann estimate of superficial curvature of the region for the surface-orientation at that point.

The parameters λ and μ in the sheaf of directions are subject to the relation

$$\Sigma A (\lambda p_1' + \mu p_2')^2 = 1,$$

that is,

$$\lambda^2 + \mu^2 + 2\lambda\mu \cos\omega = 1,$$

where ω is the inclination of the two selected geodesics.

Ex. The form of equation of a geodesic surface within a region may be illustrated by considering such a surface within a globular region $x^2+y^2+z^2+v^2=1$.

With the parametric representation in Ex. 1, § 302, let a point be taken within the region at which the value of p is κ. Through that point let two globular geodesics be taken, as the foundation of the sheaf of geodesics through the point for the construction of the geodesic surface. When these two geodesics are determined by directions p_1', q_1', r_1', and p_2', q_2', r_2', the sheaf of geodesics is determined by directions

$$\lambda_1 p_1' + \lambda_2 p_2', \quad \lambda_1 q_1' + \lambda_2 q_2', \quad \lambda_1 r_1' + \lambda_2 r_2',$$

for parametric values of λ_1 and λ_2 subject to the relation

$$(\lambda_1 p_1' + \lambda_2 p_2')^2 + (\lambda_1 q_1' + \lambda_2 q_2')^2 \sin^2\kappa + (\lambda_1 r_1' + \lambda_2 r_2')^2 \cos^2\kappa = 1.$$

But

$$p_1'^2 + q_1'^2 \sin^2\kappa + r_1'^2 \cos^2\kappa = 1, \quad p_2'^2 + q_2'^2 \sin^2\kappa + r_2'^2 \cos^2\kappa = 1,$$

$$p_1' p_2' + q_1' q_2' \sin^2\kappa + r_1' r_2' \cos^2\kappa = \cos\omega,$$

where ω is the angle between the initial directions of the two guiding geodesics; thus the foregoing relation between λ_1 and λ_2 is

$$\lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2 \cos\omega = 1.$$

The half-integrated equations of the globular geodesic through the initial point with the initial directions $\lambda_1 p_1' + \lambda_2 p_2'$, $\lambda_1 q_1' + \lambda_2 q_2'$, $\lambda_1 r_1' + \lambda_2 r_2'$, where λ_1 and λ_2 are subject to the foregoing relation, are (p. 85)

$$q' \sin^2 p = (\lambda_1 q_1' + \lambda_2 q_2') \sin^2\kappa,$$
$$r' \cos^2 p = (\lambda_1 r_1' + \lambda_2 r_2') \cos^2\kappa.$$

When λ_1 and λ_2 are eliminated from the three equations, the resultant is a relation valid for all geodesics belonging to the sheaf, that is, valid for the geodesic surface. Thus the resultant is an equation satisfied by the geodesic surface. Now this resultant is easily found to be

$$Rq'^2 \frac{\sin^4 p}{\sin^4\kappa} - 2Tq'r' \frac{\sin^2 p \cos^2 p}{\sin^2\kappa \cos^2\kappa} + Qr'^2 \frac{\cos^4 p}{\cos^4\kappa} = (q_1' r_2' - r_1' q_2')^2,$$

where

$$Q = q_1'^2 + q_2'^2 - 2q_1'q_2' \cos\omega,$$
$$T = q_1'r_1' + q_2'r_2' - (q_1'r_2' + r_1'q_2') \cos\omega,$$
$$R = r_1'^2 + r_2'^2 - 2r_1'r_2' \cos\omega.$$

Also

$$(1 - q_1'^2 \sin^2\kappa - r_1'^2 \cos^2\kappa)(1 - q_2'^2 \sin^2\kappa - r_2'^2 \cos^2\kappa)$$
$$= p_1'^2 p_2'^2 = (\cos\omega - q_1'q_2' \sin^2\kappa - r_1'r_2' \cos^2\kappa)^2,$$

so that

$$Q \sin^2\kappa + R \cos^2\kappa - \sin^2\omega = (q_1'r_2' - r_1'q_2')^2 \sin^2\kappa \cos^2\kappa;$$

and

$$p'^2 + q'^2 \sin^2 p + r'^2 \cos^2 p = 1.$$

Hence the equation satisfied by the geodesic surface is

$$(Q \sin^2\kappa + R \cos^2\kappa - \sin^2\omega)\, dp^2$$
$$= \left\{ R \sin^4 p \frac{\cos^2\kappa}{\sin^2\kappa} - (Q \sin^2\kappa + R \cos^2\kappa - \sin^2\omega) \sin^2 p \right\} dq^2$$
$$- 2T \sin^2 p \cos^2 p\, dq\, dr$$
$$+ \left\{ Q \cos^4 p \frac{\sin^2\kappa}{\cos^2\kappa} - (Q \sin^2\kappa + R \cos^2\kappa - \sin^2\omega) \cos^2 p \right\} dr^2,$$

a non-linear Pfaffian equation of geodesic surfaces in the globular region

$$x^2 + y^2 + z^2 + v^2 = 1.$$

It may be added that an integral equivalent (though in a shape that is complicated and implicit) is derivable from the results already given concerning globular geodesics (§ 302, Ex. 1). In those results, we have

$$\mu \tan(q + \alpha) = \left(\frac{\sin^2 p - c\mu}{\frac{c}{\mu} - \sin^2 p} \right)^{\frac{1}{2}} = \left(\frac{1 - c\mu}{1 - \frac{c}{\mu}} \right)^{\frac{1}{2}} \tan(r + \beta),$$

where α and β are determinable by the requirement that $q = q_0$ and $r = r_0$ when $p = \kappa$: while

$$c = (\lambda_1 q_1' + \lambda_2 q_2') \sin^2\kappa, \quad k = (\lambda_1 r_1' + \lambda_2 r_2') \cos^2\kappa,$$

and μ is given by

$$\left(1 - \frac{c}{\mu}\right)(1 - c\mu) = k^2.$$

The equation of the geodesic surface can be regarded as derivable by the elimination of λ_1 and λ_2, from these equations and $\lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2 \cos\omega = 1$.

Principal measures of curvature of a geodesic surface.

316. Let a direction P', Q', R', be taken in the region, perpendicular to the two directions p_1', q_1', r_1', and p_2', q_2', r_2'; then P', Q', R', is perpendicular to the foregoing plane which contains the sheaf of initial directions, and so is perpendicular to each direction $\lambda p_1' + \mu p_2', \lambda q_1' + \mu q_2', \lambda r_1' + \mu r_2'$; and P', Q', R', is a direction normal to the geodesic surface. We have, for the determination of P', Q', R', the two relations

$$P'(Ap_1' + Hq_1' + Gr_1') + Q'(Hp_1' + Bq_1' + Fr_1') + R'(Gp_1' + Fq_1' + Cr_1') = 0,$$
$$P'(Ap_2' + Hq_2' + Gr_2') + Q'(Hp_2' + Bq_2' + Fr_2') + R'(Gp_2' + Fq_2' + Cr_2') = 0,$$

as the conditions that P', Q', R', is perpendicular to p_1', q_1', r_1', and to p_2', q_2', r_2', respectively. We write

$$q_1'r_2' - r_1'q_2' = \xi, \quad r_1'p_2' - p_1'r_2' = \eta, \quad p_1'q_2' - q_1'p_2' = \zeta,$$

so that ξ, η, ζ, can be regarded as orientation-coordinates (§ 288) of the surface at the point in the region. Resolving the two equations in P', Q', R', we have

$$\frac{P'}{a\xi + h\eta + g\zeta} = \frac{Q'}{h\xi + b\eta + f\zeta} = \frac{R'}{g\xi + f\eta + c\zeta} = \frac{1}{T},$$

with the customary significance of a, b, c, f, g, h, and using T^{-1} to denote the common value of the three fractions. Now

$$AP'^2 + 2HP'Q' + 2GP'R' + BQ'^2 + 2FQ'R' + CR'^2 = 1,$$

because P', Q', R', give a direction in the region; hence

$$\begin{aligned} T^2 = {} & (a\xi + h\eta + g\zeta)\{A(a\xi + h\eta + g\zeta) + H(h\xi + b\eta + f\zeta) + G(g\xi + f\eta + c\zeta)\} \\ & + (h\xi + b\eta + f\zeta)\{H(a\xi + h\eta + g\zeta) + B(h\xi + b\eta + f\zeta) + F(g\xi + f\eta + c\zeta)\} \\ & + (g\xi + f\eta + c\zeta)\{G(a\xi + h\eta + g\zeta) + F(h\xi + b\eta + f\zeta) + C(g\xi + f\eta + c\zeta)\}. \end{aligned}$$

Now

$$\begin{aligned} & A(a\xi + h\eta + g\zeta) + H(h\xi + b\eta + f\zeta) + G(g\xi + f\eta + c\zeta) \\ & \quad = \xi(Aa + Hh + Gg) + \eta(Ah + Hb + Gf) + \zeta(Ag + Hf + Gc) \\ & \quad = \Omega^2\xi; \end{aligned}$$

and similarly

$$\begin{aligned} H(a\xi + h\eta + g\zeta) + B(h\xi + b\eta + f\zeta) + F(g\xi + f\eta + c\zeta) &= \Omega^2\eta, \\ G(a\xi + h\eta + g\zeta) + F(h\xi + b\eta + f\zeta) + C(g\xi + f\eta + c\zeta) &= \Omega^2\zeta. \end{aligned}$$

Thus

$$T^2 = \Omega^2(a\xi^2 + 2h\xi\eta + 2g\xi\zeta + b\eta^2 + 2f\eta\zeta + c\zeta^2).$$

But

$$\begin{aligned} a\xi^2 + 2h\xi\eta + 2g\xi\zeta + b\eta^2 + 2f\eta\zeta + c\zeta^2 &= (\Sigma Ap_1'^2)(\Sigma Ap_2'^2) - (\Sigma Ap_1'p_2')^2 \\ &= 1 - \cos^2\omega \\ &= \sin^2\omega, \end{aligned}$$

where ω is the angle between the two directions p_1', q_1', r_1', and p_2', q_2', r_2'. Hence

$$T = \Omega \sin\omega.$$

Next, any direction p', q', r', in the tangent plane to the surface is perpendicular to the direction P', Q', R', so that

$$p'(AP' + HQ' + GR') + q'(HP' + BQ' + FR') + r'(GP' + FQ' + CR') = 0.$$

From the preceding values, we have

$$\begin{aligned} T(AP' + HQ' + GR') &= A(a\xi + h\eta + g\zeta) + H(h\xi + b\eta + f\zeta) + G(g\xi + f\eta + c\zeta) \\ &= \Omega^2\xi; \end{aligned}$$

and, similarly,

$$T(HP' + BQ' + FR') = \Omega^2\eta, \quad T(GP' + FQ' + CR') = \Omega^2\zeta.$$

Consequently

$$\xi p' + \eta q' + \zeta r' = 0.$$

Now the curvature of a normal section of the surface is the same as that of a superficial geodesic through the direction; and this superficial geodesic is, by initial hypothesis, a regional geodesic. Hence the curvature of a normal section of the geodesic surface through p', q', r', is given (§ 272) by

$$\frac{1}{\rho} = Lp'^2 + 2Mp'q' + 2Kp'r' + Nq'^2 + 2Iq'r' + Jr'^2,$$

where the quantities p', q', r', satisfy the two relations

$$Ap'^2 + 2Hp'q' + 2Gp'r' + Bq'^2 + 2Fq'r' + Cr'^2 = 1, \quad \xi p' + \eta q' + \zeta r' = 0,$$

the orientation-magnitudes ξ, η, ζ, at the plane being invariable for the actually constructed geodesic surface.

We take the principal values of $1/\rho$ at the point on the geodesic surface, which is constructed within the postulated region, to be the maximum value and the minimum value of $1/\rho$ for all the directions lying in the surface; and the directions which yield this maximum value and this minimum value are the principal directions on the geodesic surface. Accordingly, to obtain these principal values and these principal directions, we must make $1/\rho$ a maximum or a minimum, subject to the two equations

$$\Sigma A p'^2 = 1, \quad \Sigma \xi p' = 0.$$

The critical equations, for this purpose, are

$$\begin{aligned} Lp' + Mq' + Kr' - \lambda(Ap' + Hq' + Gr') - \mu\xi &= 0, \\ Mp' + Nq' + Ir' - \lambda(Hp' + Bq' + Fr') - \mu\eta &= 0, \\ Kp' + Iq' + Jr' - \lambda(Gp' + Fq' + Cr') - \mu\zeta &= 0, \end{aligned}$$

where λ and μ are multipliers, undetermined by the equations themselves. The value of λ can be found at once; on multiplying the three equations by p', q', r', respectively and adding, we have

$$\frac{1}{\rho} - \lambda = 0.$$

Hence the three critical equations become

$$\begin{aligned} \left(L - \frac{A}{\rho}\right)p' + \left(M - \frac{H}{\rho}\right)q' + \left(K - \frac{G}{\rho}\right)r' - \mu\xi &= 0, \\ \left(M - \frac{H}{\rho}\right)p' + \left(N - \frac{B}{\rho}\right)q' + \left(I - \frac{F}{\rho}\right)r' - \mu\eta &= 0, \\ \left(K - \frac{G}{\rho}\right)p' + \left(I - \frac{F}{\rho}\right)q' + \left(J - \frac{C}{\rho}\right)r' - \mu\zeta &= 0. \end{aligned}$$

Moreover, we have

$$\xi p' + \eta q' + \zeta r' = 0.$$

Consequently, ρ satisfies the equation

$$\begin{vmatrix} L-\frac{A}{\rho}, & M-\frac{H}{\rho}, & K-\frac{G}{\rho}, & \xi \\ M-\frac{H}{\rho}, & N-\frac{B}{\rho}, & I-\frac{F}{\rho}, & \eta \\ K-\frac{G}{\rho}, & I-\frac{F}{\rho}, & J-\frac{C}{\rho}, & \zeta \\ \xi, & \eta, & \zeta, & 0 \end{vmatrix} = 0,$$

a quadratic equation in ρ. We use the symbols a, b, c, f, g, h, for the minors of the determinant Ω^2, as in § 266; the symbols l, n, j, i, k, m, for those of the determinant Υ, as in §§ 278, 281; and, as in § 296, we write

$$\bar{l} = BJ - 2FI + CN, \quad \bar{i} = GM + HK - FL - AI,$$
$$\bar{n} = CL - 2GK + AJ, \quad \bar{k} = HI + FM - GN - BK,$$
$$\bar{j} = AN - 2HM + BL, \quad \bar{m} = FK + GI - HJ - CM.$$

Then the equation for ρ is

$$\frac{1}{\rho^2}(a\xi^2 + 2h\xi\eta + 2g\xi\zeta + b\eta^2 + 2f\eta\zeta + c\zeta^2)$$
$$-\frac{1}{\rho}(\bar{l}\xi^2 + 2\bar{m}\xi\eta + 2\bar{k}\xi\zeta + \bar{n}\eta^2 + 2\bar{i}\eta\zeta + \bar{j}\zeta^2)$$
$$+(l\xi^2 + 2m\xi\eta + 2k\xi\zeta + n\eta^2 + 2i\eta\zeta + j\zeta^2) = 0.$$

Directions of the principal lines on a geodesic surface.

317. Let these two values of ρ, the principal values, be denoted by ρ_1 and ρ_2; to each of them, there corresponds a principal direction. Let these principal directions be denoted by $\bar{P}_1'$, $\bar{Q}_1'$, $\bar{R}_1'$, associated with ρ_1, and by $\bar{P}_2'$, $\bar{Q}_2'$, $\bar{R}_2'$, associated with ρ_2.

In the first place, *the two principal lines of curvature on the surface are perpendicular to one another.* We have

$$L\bar{P}_1' + M\bar{Q}_1' + K\bar{R}_1' - \frac{1}{\rho_1}(A\bar{P}_1' + H\bar{Q}_1' + G\bar{R}_1') - \mu_1\xi = 0,$$

$$M\bar{P}_1' + N\bar{Q}_1' + I\bar{R}_1' - \frac{1}{\rho_1}(H\bar{P}_1' + B\bar{Q}_1' + F\bar{R}_1') - \mu_1\eta = 0,$$

$$K\bar{P}_1' + I\bar{Q}_1' + J\bar{R}_1' - \frac{1}{\rho_1}(G\bar{P}_1' + F\bar{Q}_1' + C\bar{R}_1') - \mu_1\zeta = 0;$$

also, as $\bar{P}_2'$, $\bar{Q}_2'$, $\bar{R}_2'$, is a direction in the tangent plane of the geodesic surface,

$$\xi\bar{P}_2' + \eta\bar{Q}_2' + \zeta\bar{R}_2' = 0;$$

and therefore

$$L\bar{P}_1'\bar{P}_2' + M(\bar{Q}_1'\bar{P}_2' + \bar{P}_1'\bar{Q}_2') + K(\bar{R}_1'\bar{P}_2' + \bar{P}_1'\bar{R}_2') + N\bar{Q}_1'\bar{Q}_2' + I(\bar{R}_1'\bar{Q}_2' + \bar{Q}_1'\bar{R}_2') + J\bar{R}_1'\bar{R}_2'$$
$$-\frac{1}{\rho_1}\{A\bar{P}_1'\bar{P}_2' + H(\bar{Q}_1'\bar{P}_2' + \bar{P}_1'\bar{Q}_2') + G(\bar{R}_1'\bar{P}_2' + \bar{P}_1'\bar{R}_2') + B\bar{Q}_1'\bar{Q}_2' + F(\bar{R}_1'\bar{Q}_2' + \bar{Q}_1'\bar{R}_2') + C\bar{R}_1'\bar{R}_2'\} = 0.$$

Proceeding similarly from the other principal direction, we find

$$L\bar{P}_2'\bar{P}_1' + M(\bar{P}_2'\bar{Q}_1' + \bar{Q}_2'\bar{P}_1') + K(\bar{P}_2'\bar{R}_1' + \bar{R}_2'\bar{P}_1') + N\bar{Q}_2'\bar{Q}_1' + I(\bar{Q}_2'\bar{R}_1' + \bar{R}_2'\bar{Q}_1') + J\bar{R}_2'\bar{R}_1'$$
$$-\frac{1}{\rho_2}\{A\bar{P}_2'\bar{P}_1' + H(\bar{P}_2'\bar{Q}_1' + \bar{Q}_2'\bar{P}_1') + G(\bar{P}_2'\bar{R}_1' + \bar{R}_2'\bar{P}_1') + B\bar{Q}_2'\bar{Q}_1' + F(\bar{Q}_2'\bar{R}_1' + \bar{R}_2'\bar{Q}_1') + C\bar{R}_2'\bar{R}_1'\} = 0.$$

At any arbitrary point in a general region and for any arbitrarily known geodesic surface through that point, the principal radii ρ_1 and ρ_2 of the surface are unequal; hence

$$A\bar{P}_1'\bar{P}_2' + H(\bar{Q}_1'\bar{P}_2' + \bar{P}_1'\bar{Q}_2') + G(\bar{R}_1'\bar{P}_2' + \bar{P}_1'\bar{R}_2') + B\bar{Q}_1'\bar{Q}_2' + F(\bar{R}_1'\bar{Q}_2' + \bar{Q}_1'\bar{R}_2') + C\bar{R}_1'\bar{R}_2' = 0,$$

a condition which, being satisfied, shews that the two principal directions are perpendicular to one another.

In the next place, the directions of the sections containing the principal radii are deducible from the same equations. Let

$$\bar{L} = l - \frac{1}{\rho}\bar{l} + \frac{1}{\rho^2}a, \quad \bar{I} = i - \frac{1}{\rho}\bar{i} + \frac{1}{\rho^2}f,$$
$$\bar{N} = n - \frac{1}{\rho}\bar{n} + \frac{1}{\rho^2}b, \quad \bar{K} = k - \frac{1}{\rho}\bar{k} + \frac{1}{\rho^2}g,$$
$$\bar{J} = j - \frac{1}{\rho}\bar{j} + \frac{1}{\rho^2}c, \quad \bar{M} = m - \frac{1}{\rho}\bar{m} + \frac{1}{\rho^2}h,$$

so that $\bar{L}$, $\bar{N}$, $\bar{J}$, $\bar{I}$, $\bar{K}$, $\bar{M}$, are the minors of the determinant

$$\square = \begin{vmatrix} L - \frac{A}{\rho}, & M - \frac{H}{\rho}, & K - \frac{G}{\rho} \\ M - \frac{H}{\rho}, & N - \frac{B}{\rho}, & I - \frac{F}{\rho} \\ K - \frac{G}{\rho}, & I - \frac{F}{\rho}, & J - \frac{C}{\rho} \end{vmatrix}.$$

The equations give

$$\frac{1}{\xi}\left\{\left(L-\frac{A}{\rho}\right)\bar{P}'+\left(M-\frac{H}{\rho}\right)\bar{Q}'+\left(K-\frac{G}{\rho}\right)\bar{R}'\right\}$$
$$=\frac{1}{\eta}\left\{\left(M-\frac{H}{\rho}\right)\bar{P}'+\left(N-\frac{B}{\rho}\right)\bar{Q}'+\left(I-\frac{F}{\rho}\right)\bar{R}'\right\}$$
$$=\frac{1}{\zeta}\left\{\left(K-\frac{G}{\rho}\right)\bar{P}'+\left(I-\frac{F}{\rho}\right)\bar{Q}'+\left(J-\frac{C}{\rho}\right)\bar{R}'\right\},$$

for each principal radius ρ; hence

$$\frac{\bar{P}'}{\xi\bar{L}+\eta\bar{M}+\zeta\bar{K}}=\frac{\bar{Q}'}{\xi\bar{M}+\eta\bar{N}+\zeta\bar{I}}=\frac{\bar{R}'}{\xi\bar{K}+\eta\bar{I}+\zeta\bar{J}}=\frac{1}{W},$$

say. It is easy to verify that these values of $\bar{P}'$, $\bar{Q}'$, $\bar{R}'$, satisfy the relation

$$\xi\bar{P}'+\eta\bar{Q}'+\zeta\bar{R}'=0;$$

for

$$\bar{L}\xi^2+2\bar{M}\xi\eta+2\bar{K}\xi\zeta+\bar{N}\eta^2+2\bar{I}\eta\zeta+\bar{J}\zeta^2=\Sigma l\xi^2-\frac{1}{\rho}\Sigma\bar{l}\xi^2+\frac{1}{\rho^2}\Sigma a\xi^2$$
$$=\left\{\frac{1}{\rho_1\rho_2}-\frac{1}{\rho}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\frac{1}{\rho^2}\right\}\Sigma a\xi^2,$$

a quantity which vanishes when, as in present circumstances, ρ is equal either to ρ_1 or to ρ_2.

A value of W in the foregoing fractions can be obtained by using the relation

$$A\bar{P}'^2+2H\bar{P}'\bar{Q}'+2G\bar{P}'\bar{R}'+B\bar{Q}'^2+2F\bar{Q}'\bar{R}'+C\bar{R}'^2=1,$$

as follows. We write

$$l\xi+m\eta+k\zeta=u, \qquad \bar{l}\xi+\bar{m}\eta+\bar{k}\zeta=\bar{u},$$
$$m\xi+n\eta+i\zeta=v, \qquad \bar{m}\xi+\bar{n}\eta+\bar{i}\zeta=\bar{v},$$
$$k\xi+i\eta+j\zeta=w, \qquad \bar{k}\xi+\bar{i}\eta+\bar{j}\zeta=\bar{w};$$

and then, from the values of $\bar{P}'$, $\bar{Q}'$, $\bar{R}'$, we have

$$W\bar{P}'=\xi\bar{L}+\eta\bar{M}+\zeta\bar{K}=u-\frac{1}{\rho}\bar{u}+\frac{1}{\rho^2}(a\xi+h\eta+g\zeta),$$

$$W\bar{Q}'=\xi\bar{M}+\eta\bar{N}+\zeta\bar{I}=v-\frac{1}{\rho}\bar{v}+\frac{1}{\rho^2}(h\xi+b\eta+f\zeta),$$

$$W\bar{R}'=\xi\bar{K}+\eta\bar{I}+\zeta\bar{J}=w-\frac{1}{\rho}\bar{w}+\frac{1}{\rho^2}(g\xi+f\eta+c\zeta).$$

Now

$$W^2=W^2\Sigma A\bar{P}'^2=\Sigma A\,(W\bar{P}')^2$$
$$=\Sigma Au^2-\frac{2}{\rho}\Sigma Au\bar{u}+\frac{1}{\rho^2}\Sigma A\bar{u}^2+\frac{1}{\rho^4}\Sigma A\,(a\xi+h\eta+g\zeta)^2$$
$$+\frac{2}{\rho^2}\Sigma Au\,(a\xi+h\eta+g\zeta)-\frac{2}{\rho^3}\Sigma A\bar{u}\,(a\xi+h\eta+g\zeta).$$

But

$$\Sigma Au\,(a\xi+h\eta+g\zeta)=\Omega^2\Sigma l\xi^2,\quad \Sigma A\bar{u}\,(a\xi+h\eta+g\zeta)=\Omega^2\Sigma\bar{l}\xi^2,$$

$$\Sigma A\,(a\xi+h\eta+g\zeta)^2=\Omega^2\Sigma a\xi^2=T^2,$$

as in § 316; hence, using the equation satisfied by ρ, we have

$$W^2=\Sigma Au^2-\frac{2}{\rho}\Sigma Au\bar{u}+\frac{1}{\rho^2}\Sigma A\bar{u}^2-\frac{1}{\rho^4}T^2.$$

By direct substitution, we find (in the notation of § 281)

$$\Sigma Au^2=\mathfrak{G}_2\Sigma l\xi^2-\Upsilon\Sigma\bar{l}\xi^2=\left\{\frac{1}{\rho_1\rho_2}\mathfrak{F}-\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\mathfrak{R}\right\}T^2,$$

$$\Sigma Au\bar{u}=\mathfrak{G}_1\Sigma l\xi^2-\Upsilon\Sigma a\xi^2=\left\{\frac{1}{\rho_1\rho_2}\mathfrak{L}-\mathfrak{R}\right\}T^2,$$

$$\Sigma A\bar{u}^2=\mathfrak{G}_2\Sigma a\xi^2-\Omega^2\Sigma l\xi^2=\left\{\mathfrak{F}-\frac{1}{\rho_1\rho_2}\right\}T^2;$$

and therefore

$$\frac{W^2}{T^2}=\left(\frac{2}{\rho}-\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\mathfrak{R}+\left(\frac{1}{\rho^2}+\frac{1}{\rho_1\rho_2}\right)\left(\mathfrak{F}-\frac{1}{\rho^2}\right)-\frac{2}{\rho\rho_1\rho_2}\mathfrak{L}.$$

The value of ρ, which occurs here, is the value associated with the principal direction determined by $\bar{P}'$, $\bar{Q}'$, $\bar{R}'$: that is, we associate ρ_1 with $\bar{P}_1'$, $\bar{Q}_1'$, $\bar{R}_1'$, and ρ_2 with $\bar{P}_2'$, $\bar{Q}_2'$, $\bar{R}_2'$.

Alternative derivation of the two measures of curvature of a geodesic surface.

318. The preceding results can otherwise be obtained as follows, without using the special relations between the orientation-coordinates ξ, η, ζ, and the first derivatives of θ; and the process has the advantage of being directly applicable (§ 431) to a primary amplitude in a plenary homaloidal space of any number of dimensions.

Let p_1', q_1', r_1', and p_2', q_2', r_2', be any two directions in the region. Each of them determines a regional geodesic through itself, as an initial direction. Every direction in the sheaf

$$p'=\lambda p_1'+\mu p_2',\quad q'=\lambda q_1'+\mu q_2',\quad r'=\lambda r_1'+\mu r_2',$$

as an initial direction, likewise determines a regional geodesic; and the aggregate of these geodesics, for the admissible values, determines (as before) a geodesic surface which has the plane, containing all the directions, as its tangent plane. Moreover, the regional geodesic, which lies entirely in the geodesic surface (the latter being also completely enclosed in the region), is

also a geodesic* on the surface; its osculating plane gives a normal section; and the circular curvature of the superficial geodesic is equal to the circular curvature of the regional geodesic, while its regional geodesic curvature (§ 345) is zero. Moreover, as $\Sigma Ap'^2 = 1$, the parameters λ and μ satisfy the relation

$$\lambda^2 \Sigma A p_1'^2 + 2\lambda\mu \Sigma A p_1' p_2' + \mu^2 \Sigma A p_2'^2 = 1,$$

the coefficients of λ^2, $2\lambda\mu$, μ^2, being left unsimplified.

The circular curvature of the superficial geodesic is given by

$$\begin{aligned}\frac{1}{\rho} &= \Sigma L p'^2 \\ &= \lambda^2 \Sigma L p_1'^2 + 2\lambda\mu \Sigma L p_1' p_2' + \mu^2 \Sigma L p_2'^2,\end{aligned}$$

a quadratic function of the parameters λ and μ, which themselves are connected by the preceding quadratic relation. Accordingly, the maximum or minimum values of ρ, and the corresponding values of λ and μ, are given by the equations

$$\left.\begin{aligned}\lambda \Sigma L p_1'^2 + \mu \Sigma L p_1' p_2' &= \frac{1}{\rho}\{\lambda \Sigma A p_1'^2 + \mu \Sigma A p_1' p_2'\} \\ \lambda \Sigma L p_1' p_2' + \mu \Sigma L p_2'^2 &= \frac{1}{\rho}\{\lambda \Sigma A p_1' p_2' + \mu \Sigma A p_2'^2\}\end{aligned}\right\}.$$

The maximum or minimum values of ρ are therefore the roots of the equation

$$\begin{vmatrix} \Sigma L p_1'^2 - \frac{1}{\rho}\Sigma A p_1'^2, & \Sigma L p_1' p_2' - \frac{1}{\rho}\Sigma A p_1' p_2' \\ \Sigma L p_1' p_2' - \frac{1}{\rho}\Sigma A p_1' p_2', & \Sigma L p_2'^2 - \frac{1}{\rho}\Sigma A p_2'^2 \end{vmatrix} = 0.$$

Now

$$\begin{aligned}&(\Sigma L p_1'^2)(\Sigma L p_2'^2) - (\Sigma L p_1' p_2')^2 \\ &\qquad = \Sigma l\,(q_1' r_2' - r_1' q_2')^2 = (l, n, j, i, k, m \mathbin{\text{\textbigcircle}} \xi, \eta, \zeta)^2; \\ &(\Sigma A p_1'^2)(\Sigma A p_2'^2) - (\Sigma A p_1' p_2')^2 \\ &\qquad = \Sigma a\,(q_1' r_2' - r_1' q_2')^2 = (a, b, c, f, g, h \mathbin{\text{\textbigcircle}} \xi, \eta, \zeta)^2; \\ &(\Sigma L p_1'^2)(\Sigma A p_2'^2) - (\Sigma L p_1' p_2')(\Sigma A p_1' p_2') + (\Sigma L p_2'^2)(\Sigma A p_1'^2) \\ &\qquad = \Sigma\,(BJ - 2FI + CN)(q_1' r_2' - r_1' q_2')^2 = (\bar{l}, \bar{n}, \bar{j}, \bar{i}, \bar{k}, \bar{m} \mathbin{\text{\textbigcircle}} \xi, \eta, \zeta)^2;\end{aligned}$$

* If any other curve, lying on the surface and joining two points on this regional geodesic, were shorter than the included arc of the regional geodesic, it would be an arc between the two points and lying in the region; its existence would contravene the essential quality of the regional geodesic.

and therefore the equation for ρ becomes

$$\frac{1}{\rho^2}(a, b, c, f, g, h\mathcal{X}\xi, \eta, \zeta)^2 - \frac{1}{\rho}(\bar{l}, \bar{n}, \bar{j}, \bar{i}, \bar{k}, \bar{m}\mathcal{X}\xi, \eta, \zeta)^2 + (l, n, j, i, k, m\mathcal{X}\xi, \eta, \zeta)^2 = 0,$$

the same equation as before (§ 316).

For the directions of these maximum or minimum values of ρ, let the two sets of values of λ, μ, be α, β, and γ, δ. The directions, in connection with these values of λ, μ, satisfy the equation

$$\begin{vmatrix} \lambda\Sigma Lp_1'^2 + \mu\Sigma Lp_1'p_2', & \lambda\Sigma Lp_1'p_2' + \mu\Sigma Lp_2'^2 \\ \lambda\Sigma Ap_1'^2 + \mu\Sigma Ap_1'p_2', & \lambda\Sigma Ap_1'p_2' + \mu\Sigma Ap_2'^2 \end{vmatrix} = 0;$$

so that

$$\frac{\Sigma Lp_1'^2\Sigma Ap_1'p_2' - \Sigma Lp_1'p_2'\Sigma Ap_1'^2}{\beta\delta} = \frac{\Sigma Lp_1'^2\Sigma Ap_2'^2 - \Sigma Lp_2'^2\Sigma Ap_1'^2}{-(\alpha\delta + \beta\gamma)} = \frac{\Sigma Lp_1'p_2'\Sigma Ap_2'^2 - \Sigma Lp_2'^2\Sigma Ap_1'p_2'}{\alpha\gamma}.$$

The condition of perpendicularity of two directions

$$\alpha p_1' + \beta p_2', \quad \alpha q_1' + \beta q_2', \quad \alpha r_1' + \beta r_2',$$
$$\gamma p_1' + \delta p_2', \quad \gamma q_1' + \delta q_2', \quad \gamma r_1' + \delta r_2',$$

is

$$\Sigma A(\alpha p_1' + \beta p_2')(\gamma p_1' + \delta p_2') = 0,$$

that is,

$$\alpha\gamma\Sigma Ap_1'^2 + (\alpha\delta + \beta\gamma)\Sigma Ap_1'p_2' + \beta\delta\Sigma Ap_2'^2 = 0.$$

The foregoing ratios of $\alpha\gamma$, $\alpha\delta + \beta\gamma$, $\beta\delta$, satisfy this relation. Hence the two principal directions are perpendicular to one another.

We make the same inference as before (§ 316) concerning the two measures of curvature of this geodesic surface. We now proceed to make them the two measures of superficial curvature of the region with the assigned orientation ξ, η, ζ. The postulation of an estimate of superficial curvature of a region was first made by Riemann; he adopted an extension of the Gauss measure alone.

Riemann's measure of superficial curvature for a region.

319. The principal radii of curvature ρ_1 and ρ_2 of the geodesic surface, with the arbitrarily assigned orientation ξ, η, ζ, at an initial origin, are the roots of the quadratic equation in § 316. Let **K** denote the Gauss measure of curvature of the geodesic surface at the point, and let **H** denote the additive (or, as it often is called, the mean) measure of curvature of that surface, so that

$$\mathbf{K} = \frac{1}{\rho_1\rho_2}, \quad \mathbf{H} = \frac{1}{\rho_1} + \frac{1}{\rho_2}.$$

From the quadratic equation satisfied by ρ_1 and ρ_2, we have

$$\mathbf{K} = \frac{(l, n, j, i, k, m\chi\xi, \eta, \zeta)^2}{(a, b, c, f, g, h\chi\xi, \eta, \zeta)^2}$$

$$= \frac{l\xi^2 + 2m\xi\eta + 2k\xi\zeta + n\eta^2 + 2i\eta\zeta + j\zeta^2}{a\xi^2 + 2h\xi\eta + 2g\xi\zeta + b\eta^2 + 2f\eta\zeta + c\zeta^2},$$

$$\mathbf{H} = \frac{(\bar{l}, \bar{n}, \bar{j}, \bar{i}, \bar{k}, \bar{m}\chi\xi, \eta, \zeta)^2}{(a, b, c, f, g, h\chi\xi, \eta, \zeta)^2}$$

$$= \frac{\bar{l}\xi^2 + 2\bar{m}\xi\eta + 2\bar{k}\xi\zeta + \bar{n}\eta^2 + 2\bar{i}\eta\zeta + \bar{j}\zeta^2}{a\xi^2 + 2h\xi\eta + 2g\xi\zeta + b\eta^2 + 2f\eta\zeta + c\zeta^2}.$$

This quantity **K**, corresponding to the Gauss measure of curvature of a geodesic surface constructed with an arbitrary orientation at a point by means of regional geodesics through the point, is taken by Riemann* to be the measure of curvature of the region at the point with the orientation defined by the quantities ξ, η, ζ. The additive measure **H** is usually unnoticed.

Just as **K** and **H** arise through the principal radii of a geodesic surface with an arbitrary orientation, so also there are principal values of **K** and of **H** corresponding to principal orientations. To these we now proceed.

Principal values of the Riemann measure of superficial curvature.

320. The orientation of a geodesic surface, as constructed from two arbitrarily assumed initial directions p_1', q_1', r_1', and p_2', q_2', r_2', for regional geodesics, is given by means of the orientation-coordinates ξ, η, ζ, themselves equal to

$$\xi = q_1'r_2' - r_1'q_2', \quad \eta = r_1'p_2' - p_1'r_2', \quad \zeta = p_1'q_2' - q_1'p_2';$$

and it can, with equal effect, be determined by P', Q', R', the direction variables (§ 316) of the normal to the tangent plane to the surface which lies in the region. Now the relation

$$a\xi^2 + 2h\xi\eta + 2g\xi\zeta + b\eta^2 + 2f\eta\zeta + c\zeta^2 = \sin^2\omega$$

has been established (§ 266), where ω is the angle between the directions p_1', q_1', r_1', and p_2', q_2', r_2'. When the surface has been constructed, the orientation is not affected by substituting two orthogonal directions in its tangent plane for the actually assumed initial directions. The effect of this substitution is to leave the orientation unaltered; to change ξ, η, ζ, into ξ', η', ζ', where $\xi = \xi' \sin\omega$, $\eta = \eta' \sin\omega$, $\zeta = \zeta' \sin\omega$, so that we can assume, without any loss of generality,

$$a\xi^2 + 2h\xi\eta + 2g\xi\zeta + b\eta^2 + 2f\eta\zeta + c\zeta^2 = 1;$$

and to leave the quantities **K** and **H** absolutely unaltered in value, because the numerator and the denominator in each of them are, by the change, merely multiplied by the cancelling factor $\sin^2\omega$.

* *Gesammelte Werke* (1876), p. 261.

Thus, in obtaining the principal values of $\mathbf{K}$ and of $\mathbf{H}$, for varying orientations, we can assume a permanent relation

$$a\xi^2 + 2h\xi\eta + 2g\xi\zeta + b\eta^2 + 2f\eta\zeta + c\zeta^2 = 1,$$

satisfied by the orientation variables.

The principal values of $\mathbf{K}$ are obtained by determining the maximum or minimum values of

$$\mathbf{K} = l\xi^2 + 2m\xi\eta + 2k\xi\zeta + n\eta^2 + 2i\eta\zeta + j\zeta^2$$

for values of ξ, η, ζ, subject to the foregoing permanent relation. The critical equations are

$$\begin{aligned} l\xi + m\eta + k\zeta &= \lambda(a\xi + h\eta + g\zeta), \\ m\xi + n\eta + i\zeta &= \lambda(h\xi + b\eta + f\zeta), \\ k\xi + i\eta + j\zeta &= \lambda(g\xi + f\eta + c\zeta), \end{aligned}$$

where λ is a multiplier left undetermined by the critical equations. Multiply by ξ, η, ζ, respectively, and add: then

$$\mathbf{K} = \lambda.$$

Consequently, $\mathbf{K}$ is a root of the cubic equation

$$\begin{vmatrix} l - a\mathbf{K}, & m - h\mathbf{K}, & k - g\mathbf{K} \\ m - h\mathbf{K}, & n - b\mathbf{K}, & i - f\mathbf{K} \\ k - g\mathbf{K}, & i - f\mathbf{K}, & j - c\mathbf{K} \end{vmatrix} = 0.$$

In this equation, the term independent of $\mathbf{K}$ is

$$= \begin{vmatrix} l, & m, & k \\ m, & n, & i \\ k, & i, & j \end{vmatrix} = \Upsilon^2;$$

the coefficient of the first power of $\mathbf{K}$ is

$$\begin{aligned} &= -\Sigma a(nj - i^2) \\ &= -\Upsilon\Sigma aL = -\Upsilon\mathfrak{G}_1; \end{aligned}$$

the coefficient of $\mathbf{K}^2$ is

$$\begin{aligned} &= \Sigma l(bc - f^2) \\ &= \Omega^2\Sigma lA = \Omega^2\mathfrak{G}_2; \end{aligned}$$

and the coefficient of $\mathbf{K}^3$ is

$$= -\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = -\Omega^4;$$

where $\mathfrak{G}_1$ and $\mathfrak{G}_2$ have the same signification as in § 281. Thus the cubic equation satisfied by $\mathbf{K}$ is

$$\Upsilon^2 - \Upsilon\mathfrak{G}_1\mathbf{K} + \Omega^2\mathfrak{G}_2\mathbf{K}^2 - \Omega^4\mathbf{K}^3 = 0.$$

Now the principal radii of curvature α, β, γ, of normal sections of the region at the point are the roots of the cubic

$$\Upsilon - \mathfrak{C}_2 \frac{1}{\rho} + \mathfrak{C}_1 \frac{1}{\rho^2} - \Omega^2 \frac{1}{\rho^3} = 0;$$

and therefore the roots of the cubic equation in $\mathbf{K}$ are

$$\mathbf{K}_1 = \frac{1}{\beta\gamma}, \quad \mathbf{K}_2 = \frac{1}{\gamma\alpha}, \quad \mathbf{K}_3 = \frac{1}{\alpha\beta}:$$

that is, the three principal values of the Riemann measure of superficial curvature at the point are expressible in terms of the three principal measures of curvature of geodesics (or normal sections) at the point.

We also have

$$\mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3 = \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} + \frac{1}{\alpha\beta} = \mathfrak{F},$$

where $\mathfrak{F}$ is the *superficial measure of curvature* of the region (§ 281): the *spatial curvature* (or *volumetric curvature*), $\mathfrak{R}$, of the region is

$$\mathfrak{R} = (\mathbf{K}_1 \mathbf{K}_2 \mathbf{K}_3)^{\frac{1}{2}};$$

and the *linear measure of curvature*, $\mathfrak{L}$, of the region is

$$\mathfrak{L} = (\mathbf{K}_1 \mathbf{K}_2 \mathbf{K}_3)^{\frac{1}{2}} \left(\frac{1}{\mathbf{K}_1} + \frac{1}{\mathbf{K}_2} + \frac{1}{\mathbf{K}_3} \right).$$

It follows that, just as the measure $\mathbf{K}$ of superficial curvature of the region is constructed (§ 316) out of the measure of linear curvature of the geodesics in the region constituting a geodesic surface, so the principal values of that superficial measure are the appropriate combinations of the principal values of the linear measure. We proceed to shew that the orientations for the principal values of the superficial measure are determined by the corresponding combinations of the lines of curvature for the principal values of the measure of linear curvature.

Principal orientations for the Riemann measure of superficial curvature.

321. Further, the principal orientations—that is, the orientations to be associated with the respective roots of the cubic equation satisfied by $\mathbf{K}$, the Riemann measure of superficial curvature—are given by the equations

$$\left.\begin{aligned} \left(l - \frac{a}{\theta}\right)\xi + \left(m - \frac{h}{\theta}\right)\eta + \left(k - \frac{g}{\theta}\right)\zeta = 0 \\ \left(m - \frac{h}{\theta}\right)\xi + \left(n - \frac{b}{\theta}\right)\eta + \left(i - \frac{f}{\theta}\right)\zeta = 0 \\ \left(k - \frac{g}{\theta}\right)\xi + \left(i - \frac{f}{\theta}\right)\eta + \left(j - \frac{c}{\theta}\right)\zeta = 0 \end{aligned}\right\},$$

where θ denotes $1/\mathbf{K}$. We have seen that the principal values of θ are $\beta\gamma, \gamma\alpha, \alpha\beta$, where α, β, γ, are the three principal radii of linear curvature; thus the three principal measures of the Riemann superficial curvature of the region, and the three principal radii of linear curvature of the region, can be so coupled that, for each pair, we have

$$\rho\theta = \alpha\beta\gamma = \frac{\Omega^2}{\Upsilon}.$$

Now let P', Q', R', be a direction perpendicular to a principal orientation ξ, η, ζ; then, by § 316, we have

$$\frac{P'}{a\xi + h\eta + g\zeta} = \frac{Q'}{h\xi + b\eta + f\zeta} = \frac{R'}{g\xi + f\eta + c\zeta} = \frac{1}{T} = \frac{1}{\Omega}.$$

Hence

$$\begin{aligned}\Omega P' &= a\xi + h\eta + g\zeta \\ &= \theta\,(l\xi + m\eta + k\zeta),\end{aligned}$$

because the orientation ξ, η, ζ, is one of the principal orientations: that is,

$$\Upsilon P' = \frac{\Omega}{\rho}(\ l\xi + m\eta + k\zeta).$$

Similarly

$$\Upsilon Q' = \frac{\Omega}{\rho}(m\xi + n\eta + i\zeta),$$

$$\Upsilon R' = \frac{\Omega}{\rho}(\ k\xi + i\eta + j\zeta).$$

Consequently

$$LP' + MQ' + KR' = \frac{\Omega}{\rho}\xi,$$

$$MP' + NQ' + IR' = \frac{\Omega}{\rho}\eta,$$

$$KP' + IQ' + JR' = \frac{\Omega}{\rho}\zeta.$$

But, from the equations

$$\Omega P' = a\xi + h\eta + g\zeta,$$

$$\Omega Q' = h\xi + b\eta + f\zeta,$$

$$\Omega R' = g\xi + f\eta + c\zeta,$$

we have the relations

$$AP' + HQ' + GR' = \Omega\xi,$$

$$HP' + BQ' + FR' = \Omega\eta,$$

$$GP' + FQ' + CR' = \Omega\zeta;$$

and therefore the direction P', Q', R', perpendicular to a principal orientation ξ, η, ζ, satisfies the equations

$$\left(L - \frac{A}{\rho}\right) P' + \left(M - \frac{H}{\rho}\right) Q' + \left(K - \frac{G}{\rho}\right) R' = 0,$$
$$\left(M - \frac{H}{\rho}\right) P' + \left(N - \frac{B}{\rho}\right) Q' + \left(I - \frac{F}{\rho}\right) R' = 0,$$
$$\left(K - \frac{G}{\rho}\right) P' + \left(I - \frac{F}{\rho}\right) Q' + \left(J - \frac{C}{\rho}\right) R' = 0,$$

that is, it is the direction of a principal radius of linear curvature of the region. We therefore infer that *the three principal directions of linear curvature of the region, when taken in pairs, constitute the three principal orientations for the Riemann measure of superficial curvature* of the region. It has already been proved that *the three principal Riemann measures of superficial curvature are the products, in respective pairs, of the three principal measures of linear curvature.*

Principal values of the measure **H** *of superficial curvature of a region.*

322. We can deal similarly with the principal values of the measure **H** of the curvature of the geodesic surface with any arbitrary orientation through a point in the region. For this purpose, it is necessary and sufficient to obtain the maximum and the minimum value or values of **H**, where

$$\mathbf{H} = \bar{l}\xi^2 + 2\bar{m}\xi\eta + 2\bar{k}\xi\zeta + \bar{n}\eta^2 + 2\bar{i}\eta\zeta + \bar{j}\zeta^2,$$

subject to the permanent condition

$$1 = a\xi^2 + 2h\xi\eta + 2g\xi\zeta + b\eta^2 + 2f\eta\zeta + c\zeta^2.$$

The critical equations, constructed as usual, are

$$\left.\begin{aligned}(\bar{l} - a\mathbf{H})\,\xi + (\bar{m} - h\mathbf{H})\,\eta + (\bar{k} - g\mathbf{H})\,\zeta &= 0\\ (\bar{m} - h\mathbf{H})\,\xi + (\bar{n} - b\mathbf{H})\,\eta + (\bar{i} - f\mathbf{H})\,\zeta &= 0\\ (\bar{k} - g\mathbf{H})\,\xi + (\bar{i} - f\mathbf{H})\,\eta + (\bar{j} - c\mathbf{H})\,\zeta &= 0\end{aligned}\right\};$$

and they can be treated in the same manner as were the critical equations for the principal values of the Riemann measure of curvature of the geodesic surface and, therefore, of the region.

An entirely different method of proceeding can be followed: only an outline, sufficient for the deduction of the necessary results in the present instance, will be given here, as the method is only a very special example in a much wider theory which will arise later (in Chapter XXVII) for consideration. Here, and in other places, we are concerned with curvature properties, in effect dealing with invariants and covariants (contravariants have not yet occurred) in the algebraically complete system of two simultaneous ternary quadratic forms

$$(A, B, C, F, G, H \mathbin{\text{\textbigcircle}} p', q', r')^2, \quad (L, N, J, I, K, M \mathbin{\text{\textbigcircle}} p', q', r')^2.$$

To treat them, there are three sufficient processes, alternative to one another. By one process, we use the umbral notation and the umbral calculus. By the second process, we use the leading coefficients of covariants and of contravariants, fortified by the theorem that the leading coefficient is adequate for the expression of the covariant or the contravariant as the case may be, while all invariants must be taken in their full expression. By the third process, we use the complete canonical forms of the ternary quadratics and therefore also the complete canonical forms of all members in the associated concomitant system. This third process will be adopted here, for the immediate aim.

For the canonical forms, we denote the variables by

$$p_0, q_0, r_0; \quad \xi_0, \eta_0, \zeta_0;$$

for the curve-direction and the surface-orientation respectively. The two fundamental ternary quadratics are

$$\mathbf{A}p_0'^2 + \mathbf{B}q_0'^2 + \mathbf{C}r_0'^2, \quad \mathbf{L}p_0'^2 + \mathbf{N}q_0'^2 + \mathbf{J}r_0'^2,$$

respectively, with

$$\mathbf{F} = \mathbf{G} = \mathbf{H} = 0; \quad \mathbf{I} = \mathbf{K} = \mathbf{M} = 0;$$

and now

$$a = \mathbf{BC}, \quad b = \mathbf{CA}, \quad c = \mathbf{AB}, \quad f = 0, \quad g = 0, \quad h = 0;$$
$$l = \mathbf{NJ}, \quad n = \mathbf{JL}, \quad j = \mathbf{LN}, \quad i = 0, \quad k = 0, \quad m = 0.$$

The four invariants of the two forms are

$$\Omega^2 = \mathbf{ABC},$$
$$\mathfrak{G}_1 = \mathbf{LBC} + \mathbf{NCA} + \mathbf{JAB},$$
$$\mathfrak{G}_2 = \mathbf{ANJ} + \mathbf{BJL} + \mathbf{CLN},$$
$$\Upsilon = \mathbf{LNJ}.$$

The canonical equation for the principal measures of linear curvature is

$$(\mathbf{A} - \rho\mathbf{L})(\mathbf{B} - \rho\mathbf{N})(\mathbf{C} - \rho\mathbf{J}) = 0,$$

that is,

$$\mathbf{ABC} - \rho(\mathbf{LBC} + \mathbf{NCA} + \mathbf{JAB}) + \rho^2(\mathbf{ANJ} + \mathbf{BJL} + \mathbf{CLN}) - \rho^3\mathbf{LNJ} = 0;$$

expressed in terms of the invariants, and thus in general form, it is

$$\Omega^2 - \rho\mathfrak{G}_1 + \rho^2\mathfrak{G}_2 - \rho^3\Upsilon = 0,$$

the earlier form of equation (§ 281), the roots being denoted by α, β, γ.

The canonical equation for the principal values of the Riemann measure of superficial curvature is

$$(\mathbf{BC} - \theta\mathbf{NJ})(\mathbf{CA} - \theta\mathbf{JL})(\mathbf{AB} - \theta\mathbf{LN}) = 0,$$

that is,

$$\mathbf{A}^2\mathbf{B}^2\mathbf{C}^2 - \theta\mathbf{ABC}(\mathbf{ANJ} + \mathbf{BJL} + \mathbf{CLN}) + \theta^2\mathbf{LNJ}(\mathbf{LBC} + \mathbf{NCA} + \mathbf{JAB}) - \theta^3\mathbf{L}^2\mathbf{N}^2\mathbf{J}^2 = 0:$$

expressed in terms of the invariants, and thus in general form, it is

$$\Omega^4 - \theta\Omega^2\mathfrak{G}_2 + \theta^2\Upsilon\mathfrak{G}_1 - \theta^3\Upsilon^2 = 0,$$

the earlier form of equation (§ 320). As before, the principal values of θ are

$$\frac{1}{\beta\gamma}, \quad \frac{1}{\gamma\alpha}, \quad \frac{1}{\alpha\beta}.$$

The canonical equation for the principal values of ω (used temporarily in place of the former symbol $\mathbf{H}$), the second of the measures of curvature of the geodesic surface, is

$$(\mathbf{BJ} + \mathbf{CN} - \omega\,\mathbf{BC})(\mathbf{CL} + \mathbf{AJ} - \omega\,\mathbf{CA})(\mathbf{AN} + \mathbf{BL} - \omega\,\mathbf{AB}) = 0.$$

Now

$$\begin{aligned}(\mathbf{BJ} + \mathbf{CN})(\mathbf{CL} + \mathbf{AJ})(\mathbf{AN} + \mathbf{BL}) \\ &= (\mathbf{LBC} + \mathbf{NCA} + \mathbf{JAB})(\mathbf{ANJ} + \mathbf{BJL} + \mathbf{CLN}) - \mathbf{ABC}\,\mathbf{LNJ} \\ &= \mathfrak{G}_1\mathfrak{G}_2 - \Omega^2\Upsilon,\end{aligned}$$

in terms of the invariants; and, similarly,

$$\begin{aligned}\Sigma\mathbf{BC}(\mathbf{CL} + \mathbf{AJ})(\mathbf{AN} + \mathbf{BL}) \\ &= (\mathbf{LBC} + \mathbf{NCA} + \mathbf{JAB})^2 + \mathbf{ABC}(\mathbf{ANJ} + \mathbf{BJL} + \mathbf{CLN}) \\ &= \mathfrak{G}_1^2 + \Omega^2\mathfrak{G}_2,\end{aligned}$$

$$\begin{aligned}\Sigma\,\mathbf{BC}\,.\,\mathbf{CA}(\mathbf{AN} + \mathbf{BL}) &= 2\mathbf{ABC}(\mathbf{LBC} + \mathbf{NCA} + \mathbf{JAB}) \\ &= 2\Omega^2\mathfrak{G}_1.\end{aligned}$$

Thus the equation for the principal values of ω, when expressed in general form, is

$$\mathfrak{G}_1\mathfrak{G}_2 - \Omega^2\Upsilon - \omega(\mathfrak{G}_1^2 + \Omega^2\mathfrak{G}_2) + 2\omega^2\Omega^2\mathfrak{G}_1 - \omega^3\Omega^4 = 0:$$

that is,

$$\begin{vmatrix} \bar{l} - a\mathbf{H}, & \bar{m} - h\mathbf{H}, & \bar{k} - g\mathbf{H} \\ \bar{m} - h\mathbf{H}, & \bar{n} - b\mathbf{H}, & \bar{i} - f\mathbf{H} \\ \bar{k} - g\mathbf{H}, & \bar{i} - f\mathbf{H}, & \bar{j} - c\mathbf{H} \end{vmatrix} = \mathfrak{G}_1\mathfrak{G}_2 - \Omega^2\Upsilon - \mathbf{H}(\mathfrak{G}_1^2 + \Omega^2\mathfrak{G}_2) + 2\mathbf{H}^2\Omega^2\mathfrak{G}_1 - \mathbf{H}^3\Omega^4.$$

Let a quantity ρ be taken such that

$$\mathbf{H} + \frac{1}{\rho} = \frac{\mathfrak{G}_1}{\Omega^2};$$

the equation for $\mathbf{H}$ is transformed into the ρ-equation

$$\Omega^2 - \rho\mathfrak{G}_1 + \rho^2\mathfrak{G}_2 - \rho^3\Upsilon = 0,$$

which is the equation for the principal values of the radius of linear curvature for the region. As

$$\frac{\mathfrak{G}_1}{\Omega^2} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma},$$

it follows that the principal values of the measure $\mathbf{H}$ are

$$\frac{1}{\beta} + \frac{1}{\gamma}, \quad \frac{1}{\gamma} + \frac{1}{\alpha}, \quad \frac{1}{\alpha} + \frac{1}{\beta}.$$

Moreover, in the canonical forms*, the equations for the directions of the principal values of the linear curvature—that is, for the direction of the lines of curvature—are

$$(\mathbf{A}-\rho\mathbf{L})\,p_0'=0,\quad (\mathbf{B}-\rho\mathbf{N})\,q_0'=0,\quad (\mathbf{C}-\rho\mathbf{J})\,r_0'=0:$$

that is, the three principal directions, the lines of curvature, are

$$\left.\begin{matrix} q_0'=0 \\ r_0'=0 \end{matrix}\right\},\quad \left.\begin{matrix} r_0'=0 \\ p_0'=0 \end{matrix}\right\},\quad \left.\begin{matrix} p_0'=0 \\ q_0'=0 \end{matrix}\right\},$$

for the respective values $\rho=\alpha,\ \beta,\ \gamma$.

In these forms, the equations for the orientations of the principal values of the Riemann measure of the superficial curvature are

$$(\mathbf{BC}-\theta\mathbf{NJ})\,\xi_0=0,\quad (\mathbf{CA}-\theta\mathbf{JL})\,\eta_0=0,\quad (\mathbf{AB}-\theta\mathbf{LN})\,\zeta_0=0:$$

that is, the three principal orientations are

$$\left.\begin{matrix} \eta_0=0 \\ \zeta_0=0 \end{matrix}\right\},\quad \left.\begin{matrix} \zeta_0=0 \\ \xi_0=0 \end{matrix}\right\},\quad \left.\begin{matrix} \xi_0=0 \\ \eta_0=0 \end{matrix}\right\}.$$

These orientations are composed of the lines of curvature, taken in pairs.

Lastly, in these forms, the equations for the orientations of the principal values of the measure $\mathbf{H}$ of curvature of the geodesic surface are

$$(\mathbf{BJ}+\mathbf{CN}-\omega\mathbf{BC})\,\xi_0=0,\quad (\mathbf{CL}+\mathbf{AJ}-\omega\mathbf{CA})\,\eta_0=0,\quad (\mathbf{AN}+\mathbf{BL}-\omega\mathbf{AB})\,\zeta_0=0:$$

that is, these three principal orientations are

$$\left.\begin{matrix} \eta_0=0 \\ \zeta_0=0 \end{matrix}\right\},\quad \left.\begin{matrix} \zeta_0=0 \\ \xi_0=0 \end{matrix}\right\},\quad \left.\begin{matrix} \xi_0=0 \\ \eta_0=0 \end{matrix}\right\}.$$

These principal orientations are the same as the principal orientations for the Riemann measure, a result to be expected: as before, they are composed of the lines of curvature, taken in pairs.

Kinds and measures of curvature of a region: the fundamental importance of the primary magnitudes.

323. Hence for the curvature of a region at any point, we have

(i) three principal measures of linear curvature $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$, with the three corresponding directions at the point, being the lines of curvature through the point:

(ii) three principal (Riemann) measures of superficial curvature, estimated by means of geodesic surfaces in the region, and equal to $\frac{1}{\beta\gamma}, \frac{1}{\gamma\alpha}, \frac{1}{\alpha\beta}$, with the three corresponding principal orientations constituted by the lines of curvature taken in pairs:

* For a full discussion of the algebra of the concomitant system of two ternary quadratics, see Clebsch-Lindemann, *Vorlesungen über Geometrie*, bd. i, p. 290.

(iii) three principal values of the additive measure **H** of superficial curvature, also estimated by means of those geodesic surfaces in the region, and equal to $\frac{1}{\beta}+\frac{1}{\gamma}, \frac{1}{\gamma}+\frac{1}{\alpha}, \frac{1}{\alpha}+\frac{1}{\beta}$, with the three corresponding orientations, the same as for the Riemann measures and constituted by the lines of curvature taken in pairs; and

(iv) the measure of volumetric curvature of the region at the point; it is devoid of direction or orientation, and the magnitude is $\frac{1}{\alpha\beta\gamma}$, or is $\frac{\Upsilon}{\Omega^2}$, that is,

$$\begin{vmatrix} L, & M, & K \\ M, & N, & I \\ K, & I, & J \end{vmatrix} \div \begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix}.$$

Further, it has been seen that the Riemann measure, **K**, of superficial curvature of a region in any orientation is deduced from the measure of linear curvature of the region in any direction which lies in that orientation: that the principal values of the Riemann measure are the due combinations of the principal values of the linear measure along the lines of curvature: and that the orientations for the principal values of the Riemann measure are constituted from the due combinations, in pairs, of the lines of curvature.

The same inferences hold exactly, *pari passu*, for the additive measure, **H**, of superficial curvature of a region.

There is only a single measure of solid (or volumetric) curvature of any region in quadruple space. It has neither direction nor orientation, for a region in such a space; in magnitude, it is expressible in terms of the principal values of the linear measure of curvature.

It therefore follows that, in estimating the curvatures of a region in homaloidal quadruple space, a complete foundation is constituted by the measure of linear curvature of a region in any direction, as estimated by the circular curvature of the regional geodesic drawn in that direction. The two measures of superficial curvature in any orientation are derived from the general measure of linear curvature in any direction contained by the orientation: the principal orientations are compounded from the lines of curvature: and the principal values of the two measures of superficial curvature (as well as the volumetric measure) are merely the respective due combinations of the principal values of the linear measure of curvature.

Moreover, it has been proved, in § 278, that the minors in the determinant Υ are expressible in terms of the primary magnitudes A, B, C, F, G, H, and

of their first and second derivatives. In particular, using here for brevity the Riemann four-index symbol* $(\lambda\nu, \mu\tau)$, we can write

$$\left.\begin{aligned} l &= NJ - I^2 = (23, 23), & i &= MK - LI = (12, 31) \\ n &= JL - K^2 = (31, 31), & k &= IM - NK = (23, 12) \\ j &= LN - M^2 = (12, 12), & m &= KI - JM = (31, 23) \end{aligned}\right\}.$$

The determinant Υ is therefore expressible in terms of the primary magnitudes and their derivatives by the relation

$$\Upsilon^2 = \begin{vmatrix} l, & m, & k \\ m, & n, & i \\ k, & i, & j \end{vmatrix} = \begin{vmatrix} (23, 23), & (23, 31), & (23, 12) \\ (31, 23), & (31, 31), & (31, 12) \\ (12, 23), & (12, 31), & (12, 12) \end{vmatrix}.$$

Then, provided Υ does not vanish, that is, if the region is general and non-developable, the quantities L, N, J, I, K, M, are expressible also in terms of the primary magnitudes and their derivatives by the forms

$$\left.\begin{aligned} \Upsilon L &= nj - i^2, & \Upsilon I &= mk - li \\ \Upsilon N &= jl - k^2, & \Upsilon K &= im - nk \\ \Upsilon J &= ln - m^2, & \Upsilon M &= ki - jm \end{aligned}\right\}.$$

It therefore follows, for a region in homaloidal quadruple space:

(i) the circular curvature of a regional geodesic (that is, the linear curvature of the region) in any direction is expressible in terms of the variables of the direction and of the primary magnitudes A, B, C, F, G, H, with their derivatives:

(ii) the two measures of superficial curvature of the region in any orientation ξ, η, ζ, viz. the Riemann measure $\bar{K}$, and the additive measure $\bar{H}$, are expressible in terms of the orientation variables, and of the primary magnitudes with their derivatives: and

(iii) all the principal measures of curvature of the region, linear, superficial, volumetric, are expressible solely in terms of the primary magnitudes A, B, C, F, G, H, and their derivatives.

Hence for a general region (that is, a non-developable region) in homaloidal quadruple space, the primary magnitudes A, B, C, F, G, H, as they occur in the expression

$$ds^2 = \Sigma A dp^2$$

for the arc-element of the region, suffice for the expression of all the curvatures, linear, superficial, volumetric, and for the principal values of each measure of the various types of curvature.

* See § 423, *post*, for the definition of the symbol for n-fold amplitudes.

As will be seen from the brief sketch of the curvatures of a primary amplitude in an enclosing n-fold space, as given in Chapter XXIV, the same statement holds concerning the fundamental importance of the primary magnitudes which occur in the expression of the arc-element of that primary amplitude.

Regions of constant curvature.

324. As an illustration of the preceding general results, consider a triple region of constant superficial curvature, as contained in homaloidal quadruple space. (Such a region was first postulated by Riemann for a region of n dimensions.) It follows, from the summary in § 323, that the circular curvature of every geodesic is constant for all directions throughout the region: let that constant circular curvature be denoted by κ.

Let the region be referred to its lines of curvature as parametric curves (§ 284), so that

$$F=0=G=H, \quad I=0=K=M.$$

Owing to the constant value κ of geodesic curvature everywhere, we have

$$L=A\kappa, \quad N=B\kappa, \quad J=C\kappa.$$

The eight (extended) Codazzi-Mainardi relations (§ 285) are satisfied, without imposing any limitations on the values of A, B, C.

The six (extended) Gauss relations now take the form, in two sets as before (§ 285),

$$\left.\begin{aligned}
\kappa^2 BC &= -\frac{1}{2}(B_{33}+C_{22})+\frac{1}{4}\left(\frac{B_3^2}{B}+\frac{C_2^2}{C}-\frac{B_1C_1}{A}+\frac{B_2C_2}{B}+\frac{B_3C_3}{C}\right)\\
\kappa^2 CA &= -\frac{1}{2}(C_{11}+A_{33})+\frac{1}{4}\left(\frac{C_1^2}{C}+\frac{A_3^2}{A}+\frac{C_1A_1}{A}-\frac{C_2A_2}{B}+\frac{C_3A_3}{C}\right)\\
\kappa^2 AB &= -\frac{1}{2}(A_{22}+B_{11})+\frac{1}{4}\left(\frac{A_2^2}{A}+\frac{B_1^2}{B}+\frac{A_1B_1}{A}+\frac{A_2B_2}{B}-\frac{A_3B_3}{C}\right)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
2A_{23} &= \frac{A_2A_3}{A}+\frac{A_2B_3}{B}+\frac{A_3C_2}{C}\\
2B_{31} &= \frac{B_1A_3}{A}+\frac{B_3B_1}{B}+\frac{B_3C_1}{C}\\
2C_{12} &= \frac{C_1A_2}{A}+\frac{C_2B_1}{B}+\frac{C_1C_2}{C}
\end{aligned}\right\}.$$

The general solution, so far as concerns the parametric expression of the arc-element of the amplitude, is provided by the general integral of these equations. That general integral is, itself, a differential equation of all the amplitudes, which are applicable to any one amplitude characterised by the arc-element. There is no single Cartesian equation $F(x, y, z, v)=0$, free from arbitrary elements, which provides a complete equivalent.

(i) We begin with Riemann's solution*, specially symmetrical in the three parameters; its property is the equality of A, B, C. Let

$$A = B = C = \frac{1}{u^2},$$

where (with these limitations) u has to satisfy the six equations.

The second set of these equations now are

$$u_{23} = 0, \quad u_{31} = 0, \quad u_{12} = 0;$$

and therefore

$$u = P + Q + R,$$

where P is a function of p alone, Q of q alone, and R of r alone.

The first set of the equations are

$$u(u_{11} + u_{22}) - (u_1^2 + u_2^2 + u_3^2) = \kappa^2,$$
$$u(u_{22} + u_{33}) - (u_1^2 + u_2^2 + u_3^2) = \kappa^2,$$
$$u(u_{33} + u_{11}) - (u_1^2 + u_2^2 + u_3^2) = \kappa^2.$$

Hence

$$u_{11} = u_{22} = u_{33},$$

that is,

$$P'' = Q'' = R'',$$

consequently each of these equal quantities is a constant.

(α) If the constant is not zero, denote it by $2a$; then

$$P = ap^2 + 2b_1 p + c_1, \quad Q = aq^2 + 2b_2 q + c_2, \quad R = ar^2 + 2b_3 r + c_3,$$

where b_1, b_2, b_3, c_1, c_2, c_3, are constants. Each equation in the second set now is

$$4au - (u_1^2 + u_2^2 + u_3^2) = \kappa^2;$$

in order that this single equation may be satisfied, we have

$$\kappa^2 = 4a(c_1 + c_2 + c_3) - 4(b_1^2 + b_2^2 + b_3^2).$$

As a is not zero, no essential change in the region is made by changing $p + b_1/a$, $q + b_2/a$, $r + b_3/a$, into a new p, q, r, respectively: so that, in effect, we can take $b_1 = 0$, $b_2 = 0$, $b_3 = 0$. The relation becomes

$$\kappa^2 = 4a(c_1 + c_2 + c_3);$$

and now

$$u = P + Q + R = a(p^2 + q^2 + r^2) + \frac{\kappa^2}{4a}.$$

Thus

$$ds^2 = \frac{1}{u^2}(dp^2 + dq^2 + dr^2),$$

or, with a mere change of scale,

$$ds^2 = \frac{dp^2 + dq^2 + dr^2}{1 + \frac{1}{4\kappa^2}(p^2 + q^2 + r^2)},$$

which is Riemann's form.

* Given for an amplitude of n dimensions, *Ges. Werke*, p. 264.

Obviously this region of constant curvature can be conformally represented on a flat, taken as a homaloidal space of three dimensions with p, q, r, for variables. The surfaces of constant magnification in the representation are spheres in the flat.

This conformal representation of the region upon a flat is an extension of Mercator's projection in ordinary terrestrial maps.

(β) If the common value of P'', Q'', R'', be zero—a possibility equally admissible—we take

$$P = b_1 p + c_1, \quad Q = b_2 q + c_2, \quad R = b_3 r + c_3,$$

and the necessary relation is

$$-\kappa^2 = b_1^2 + b_2^2 + b_3^2.$$

If the constants b are different from zero, we can make $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, without loss of generality; and then

$$ds^2 = \frac{dp^2 + dq^2 + dr^2}{(b_1 p + b_2 q + b_3 r)^2},$$

the constant measure of curvature being $-(b_1^2 + b_2^2 + b_3^2)$. We can also satisfy the condition by taking

$$b_2 = 0 = b_3, \quad c_1 = c_2 = c_3 = 0;$$

and then

$$ds^2 = -\frac{1}{\kappa^2 p^2}(dp^2 + dq^2 + dr^2),$$

a form indicated by Bianchi*.

(γ) If all the constants b are zero, the region is a flat.

(ii) We consider, next, the form where p, q, r, are geodesic polar coordinates, the variable p being a geodesic length in the region, while the surfaces $p =$ constant are orthogonal to the geodesics. Then

$$A = 1.$$

The first of the second set of equations is now satisfied unconditionally. The other two equations are

$$\left.\begin{aligned} B_{31} &= \frac{1}{2}\frac{B_1 B_3}{B} + \frac{1}{2}\frac{C_1 B_3}{C} \\ C_{12} &= \frac{1}{2}\frac{C_1 C_2}{C} + \frac{1}{2}\frac{B_1 C_2}{B} \end{aligned}\right\}.$$

The third and the second of the first set of equations are

$$\kappa^2 B = -\frac{1}{2} B_{11} + \frac{1}{4}\frac{B_1^2}{B}, \quad \kappa^2 C = -\frac{1}{2} C_{11} + \frac{1}{4}\frac{C_1^2}{C}.$$

We write

$$B = \beta^2, \quad C = \gamma^2, \quad \kappa p = t,$$

* *Lezioni di geometria differenziale*, vol. i (1904), p. 345. Bianchi uses $-1/R^2$ as the measure of superficial curvature of the region.

and we denote derivatives $\partial\beta/\partial t$, $\partial^2\beta/\partial t^2$, ..., by β', β'', ..., and so on. The two last equations now are

$$\beta''+\beta=0, \quad \gamma''+\gamma=0,$$

and therefore

$$\beta=f\sin t+g\cos t, \quad \gamma=F\sin t+G\cos t,$$

where f, g, F, G, can be functions of both q and r, but are independent of t.

The two remaining equations of the second set now are

$$\beta_{13}=\beta_3\frac{\gamma_1}{\gamma}, \quad \gamma_{12}=\gamma_2\frac{\beta_1}{\beta};$$

and these are satisfied, by taking

$$\frac{\beta_3}{\gamma}=\theta, \quad \frac{\gamma_2}{\beta}=\phi,$$

where θ and ϕ are independent of t (and therefore of p). When the preceding values of β and γ are substituted, we have

$$\frac{1}{F}\frac{\partial f}{\partial r}=\frac{1}{G}\frac{\partial g}{\partial r}=\theta, \quad \frac{1}{f}\frac{\partial F}{\partial q}=\frac{1}{g}\frac{\partial G}{\partial q}=\phi;$$

which accordingly are limitations on the coefficients in β and γ.

The only remaining equation is the first equation in the first set, which, for $B=\beta^2$ and $C=\gamma^2$, becomes

$$\kappa^2\beta^2\gamma^2=-\beta\beta_{33}-\gamma\gamma_{22}-\beta\beta_1\gamma\gamma_1+\beta_2\gamma\frac{\gamma_2}{\beta}+\gamma_3\beta\frac{\beta_3}{\gamma}.$$

But

$$\beta_1=\kappa(f\cos t-g\sin t), \quad \gamma_1=\kappa(F\cos t-G\sin t),$$

so that

$$\kappa^2\beta^2\gamma^2+\beta\beta_1\gamma\gamma_1=\kappa^2(fF+gG)\beta\gamma.$$

Thus the equation is

$$\kappa^2(fF+gG)\beta\gamma=-\beta\beta_{33}+\beta_3\gamma_3\frac{\beta}{\gamma}-\gamma\gamma_{22}+\beta_2\gamma_2\frac{\gamma}{\beta}$$

$$=-\beta\gamma\frac{\partial}{\partial r}\left(\frac{\beta_3}{\gamma}\right)-\beta\gamma\frac{\partial}{\partial q}\left(\frac{\gamma_2}{\beta}\right),$$

or, finally

$$\kappa^2(fF+gG)=-\frac{\partial\theta}{\partial r}-\frac{\partial\phi}{\partial q},$$

with

$$\left.\begin{aligned}\frac{\partial f}{\partial r}=\theta F, \quad \frac{\partial g}{\partial r}=\theta G, \quad \frac{\partial F}{\partial q}=\phi f, \quad \frac{\partial G}{\partial q}=\phi g\end{aligned}\right\},$$

five equations involving the six quantities f, g, F, G, θ, ϕ.

(iii) The simplest example arises when $\theta=0$, $\phi=0$, so that f, g, F, G, are constants; then the only limitation is

$$fF+gG=0.$$

We take

$$g = f \tan \alpha, \quad F = -G \tan \alpha, \quad t + \alpha = \tau,$$

$$qf \sec \alpha = q_0/\kappa, \quad rG \sec \alpha = r_0/\kappa;$$

and we have

$$ds^2 = dp^2 + (\sin^2\tau)\,\kappa^{-2} dq_0^2 + (\cos^2\tau)\,\kappa^{-2} dr_0^2$$

$$= \kappa^{-2}(d\tau^2 + \sin^2\tau\, dq_0^2 + \cos^2\tau\, dr_0^2),$$

the equation characteristic (§ 302) of a globular region.

(iv) Another example occurs, when we take $\theta = 0$ and proceed to determine ϕ; by appropriate choice of parameters, the arc-element can be expressed in the form

$$\kappa^2 ds^2 = d\tau^2 + \sin^2\tau\, dq^2 + \{\sin q \sin\tau + \psi(r)\cos\tau\}\, dr^2,$$

where $\psi(r)$ is any function of r.

CHAPTER XIX.

CURVATURE COVARIANTS: TRIPLE REGION OF CENTRES OF CURVATURE.

Ternariants belonging to a system of ternary forms.

325. At this stage, it is fitting to notice the fact that certain invariants and covariants of homogeneous ternary forms are appearing with increasing frequency. Their special interpretation in a geometrical sense and their special forms in an analytical aspect will be useful later (Chapter XXVII) when the analytical concomitants of any region are considered.

The parameters p, q, r, have been taken arbitrarily in the first instance: only for particular purposes have they been specialised, such as in the reference of a region to its lines of curvature as parametric curves. Various geometrical magnitudes naturally are bound to be unchanged in value whatever coordinates are adopted, and therefore are covariantive for all systems of coordinates. As examples of the simplest character, we may cite the expression for an arc-element of the region, the spatial direction-cosines of a normal to the region, the circular curvature of a normal section of the region, the torsion of a geodesic, the tilt of a geodesic, and the various measures of the different types of curvature of a region. Some of these magnitudes are invariants, some are covariants, some are contravariants: no specimen of a mixed concomitant has been cited.

There have been two initial ternary quadratic forms, the original variables being dp, dq, dr, or (as it was more convenient to take them) p', q', r'. The two forms are

$$(A, B, C, F, G, H\,\backslash\!\backslash\, p', q', r')^2,$$

$$(L, N, J, I, K, M\,\backslash\!\backslash\, p', q', r')^2.$$

The first of the forms is equal to unity, being in fact the definition of the arc-element of the region. The second of the forms is equal to $1/\rho$, the circular curvature of a normal section of the region through the direction p', q', r', or (what has been proved to be the same thing) the circular curvature of a regional geodesic through that direction. Thus, there apparently is an absolute covariant, $U = 1$: but, effectively, it is a definition of the arc ds, taken in the direction dp, dq, dr. Accordingly, we use other variables to denote dp, dq, dr, by taking

$$dp = \bar{x}, \quad dq = \bar{y}, \quad dr = \bar{z},$$

so that we should write $\bar{x}_1, \bar{x}_2, \ldots$, for different variations $d_1p, \ldots, d_2p, \ldots$, with different arcs $ds_1, ds_2, \ldots$, and so for the other two variables $\bar{y}$ and $\bar{z}$; and now we have

$$U = (A, B, C, F, G, H)(\bar{x}, \bar{y}, \bar{z})^2,$$

$$V = (L, N, J, I, K, M)(\bar{x}, \bar{y}, \bar{z})^2.$$

The geometrical significance of both U and V is immediate, viz.

$$U = ds^2, \quad V = \frac{ds^2}{\rho}.$$

There have been the four customary irreducible invariants of these two ternary forms*, constituted by the discriminant Ω^2 of U; the discriminant Υ of V; and the two invariants, $\mathfrak{S}_1$ and $\mathfrak{S}_2$, intermediate between the two discriminants. Their full expressions, in terms of the coefficients of the two covariants, are

$$\Omega^2 = ABC + 2FGH - AF^2 - BG^2 - CH^2,$$

$$\mathfrak{S}_1 = aL + bN + cJ + 2fI + 2gK + 2hM,$$

$$\mathfrak{S}_2 = Al + Bn + Cj + 2Fi + 2Gk + 2Hm,$$

$$\Upsilon = LNJ + 2IKM - LI^2 - NK^2 - JM^2,$$

where

$$a = BC - F^2, \quad b = CA - G^2, \quad c = AB - H^2,$$

$$f = GH - AF, \quad g = HF - BG, \quad h = FG - CH,$$

$$l = NJ - I^2, \quad n = JL - K^2, \quad j = LN - M^2,$$

$$i = KM - LI, \quad k = MI - NK, \quad m = IK - JM.$$

These invariants arise in connection with the principal radii of curvature of the region, which have been denoted by α, β, γ. The linear measure, the superficial measure, and the volumetric measure, of curvature of the region are respectively

$$\frac{\mathfrak{S}_1}{\Omega^2} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}, \quad \frac{\mathfrak{S}_2}{\Omega^2} = \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} + \frac{1}{\alpha\beta}, \quad \frac{\Upsilon}{\Omega^2} = \frac{1}{\alpha\beta\gamma}.$$

Orientation-coordinates as contragredient variables.

326. The variables $\bar{x}, \bar{y}, \bar{z}$, are line-directions at a point, with a length of arc ds at the point. When two line-directions $\bar{x}_1, \bar{y}_1, \bar{z}_1$, and $\bar{x}_2, \bar{y}_2, \bar{z}_2$, with arc-lengths ds_1 and ds_2, are taken at a point, giving magnitudes p_1', q_1', r_1', and p_2', q_2', r_2', at the point, we have had orientation-coordinates

$$q_1'r_2' - r_1'q_2', \quad r_1'p_2' - p_1'r_2', \quad p_1'q_2' - q_1'p_2',$$

defining the orientation of the plane through the two directions or (what is the same thing) the orientation of the tangent plane at the point of any

* They are well known in the theory of two conics.

surface in which these two directions lie. Instead of using the former definitions of ξ, η, ζ, we now take, as defining ξ, η, ζ, the relations

$$\xi = \begin{vmatrix} \bar{y}_1, & \bar{z}_1 \\ \bar{y}_2, & \bar{z}_2 \end{vmatrix} = d_1q\, d_2r - d_1r\, d_2q,$$

$$\eta = \begin{vmatrix} \bar{z}_1, & \bar{x}_1 \\ \bar{z}_2, & \bar{x}_2 \end{vmatrix} = d_1r\, d_2p - d_1p\, d_2r,$$

$$\zeta = \begin{vmatrix} \bar{x}_1, & \bar{y}_1 \\ \bar{x}_2, & \bar{y}_2 \end{vmatrix} = d_1p\, d_2q - d_1q\, d_2p.$$

Then we have a concomitant, universal for all configurations in the region,

$$\Theta = \bar{x}\xi + \bar{y}\eta + \bar{z}\zeta = \begin{vmatrix} \bar{x}\ , & \bar{y}\ , & \bar{z} \\ \bar{x}_1, & \bar{y}_1, & \bar{z}_1 \\ \bar{x}_2, & \bar{y}_2, & \bar{z}_2 \end{vmatrix},$$

which manifestly is covariantive when the independent variables p, q, r, are transformed to any other independent set P, Q, R. Moreover, the variables ξ, η, ζ, are contragredient to the variables $\bar{x}, \bar{y}, \bar{z}$, under such transformations. A function of ξ, η, ζ, persistent in value through such transformations, is a *contravariant*, just as we call a persistent function of $\bar{x}, \bar{y}, \bar{z}$, a covariant; and a persistent function of both sets of variables $\bar{x}, \bar{y}, \bar{z}$, and ξ, η, ζ, is a *mixed concomitant*.

We have already had (§ 266) a contravariant which, with the modified definition of ξ, η, ζ, will be denoted by T, and is given by

$$T = (a, b, c, f, g, h \mathbin{\text{\textbackslash\!\!\textbackslash}} \xi, \eta, \zeta)^2.$$

Let ds_1 and ds_2 be the arc-elements determined by the constituents of ξ, η, ζ; and let ω (or ω_{12}, if other directions should enter) be the inclination of those two directions. Then

$$\begin{aligned} ds_1^2 &= (A, B, C, F, G, H \mathbin{\text{\textbackslash\!\!\textbackslash}} d_1p, d_1q, d_1r)^2 \\ &= (A, B, C, F, G, H \mathbin{\text{\textbackslash\!\!\textbackslash}} \bar{x}_1, \bar{y}_1, \bar{z}_1)^2, \\ ds_2^2 &= (A, B, C, F, G, H \mathbin{\text{\textbackslash\!\!\textbackslash}} d_2p, d_2q, d_2r)^2 \\ &= (A, B, C, F, G, H \mathbin{\text{\textbackslash\!\!\textbackslash}} \bar{x}_2, \bar{y}_2, \bar{z}_2)^2, \\ ds_1 . ds_2 . \cos\omega &= (A, B, C, F, G, H \mathbin{\text{\textbackslash\!\!\textbackslash}} d_1p, d_1q, d_1r \mathbin{\text{\textbackslash\!\!\textbackslash}} d_2p, d_2q, d_2r) \\ &= (A, B, C, F, G, H \mathbin{\text{\textbackslash\!\!\textbackslash}} \bar{x}_1, \bar{y}_1, \bar{z}_1 \mathbin{\text{\textbackslash\!\!\textbackslash}} \bar{x}_2, \bar{y}_2, \bar{z}_2); \end{aligned}$$

and therefore

$$\begin{aligned} ds_1^2 . ds_2^2 . \sin^2\omega &= ds_1^2 . ds_2^2 - (ds_1 . ds_2 . \cos\omega)^2 \\ &= (a, b, c, f, g, h \mathbin{\text{\textbackslash\!\!\textbackslash}} \xi, \eta, \zeta)^2. \end{aligned}$$

Now $ds_1 . ds_2 . \sin\omega$ is the area of the rudimentary parallelogram of which ds_1 and ds_2 are adjacent sides. Let dS denote this element of area; then we have

$$T = dS^2,$$

as the geometrical significance of the contravariant T.

There is a corresponding interpretation of the universal concomitant Θ. Let λ, μ, ν, denote a direction within the region which is perpendicular to each of the arc-directions ds_1 and ds_2 and is therefore a perpendicular (in the region) to the plane with the coordinates ξ, η, ζ; then, using the analysis of § 268, we have

$$\frac{\lambda}{a\xi+h\eta+g\zeta}=\frac{\mu}{h\xi+b\eta+f\zeta}=\frac{\nu}{g\xi+f\eta+c\zeta}=\frac{1}{\Omega ds_1 . ds_2 . \sin\omega},$$

and therefore

$$\frac{1}{\xi}(A\lambda+H\mu+G\nu)=\frac{1}{\eta}(H\lambda+B\mu+F\nu)=\frac{1}{\zeta}(G\lambda+F\mu+C\nu)=\frac{\Omega}{ds_1 . ds_2 . \sin\omega}.$$

We take any direction $\bar{x}$, $\bar{y}$, $\bar{z}$, in the region, making an angle χ with the plane specified by the coordinates ξ, η, ζ. This direction, of length ds, makes an angle $\frac{1}{2}\pi-\chi$ with the perpendicular λ, μ, ν, to that plane; hence

$$\begin{aligned} ds . \sin\chi &= A\lambda\bar{x}+H(\lambda\bar{y}+\mu\bar{x})+G(\lambda\bar{z}+\nu\bar{x})+B\mu\bar{y}+F(\mu\bar{z}+\nu\bar{y})+C\nu\bar{z} \\ &= \bar{x}(A\lambda+H\mu+G\nu)+\bar{y}(H\lambda+B\mu+F\nu)+\bar{z}(G\lambda+F\mu+C\nu) \\ &= \frac{\Omega}{ds_1 . ds_2 . \sin\omega}(\bar{x}\xi+\bar{y}\eta+\bar{z}\zeta), \end{aligned}$$

and therefore

$$\Omega\Theta = ds . ds_1 . ds_2 . \sin\chi \sin\omega.$$

Consider a rudimentary parallelepiped in the region, having ds, ds_1, ds_2, for its adjacent edges; and let $d\bar{V}$ be its volume, so that $d\bar{V}$ stands for an element of volume in the space. Then

$$d\bar{V} = (ds_1 . ds_2 . \sin\omega)\, ds . \sin\chi,$$

and therefore the geometrical significance of the concomitant Θ is

$$\Theta = \frac{1}{\Omega}\, d\bar{V},$$

where $d\bar{V}$ is an elementary volume, being the parallelepiped which has a face with variables ξ, η, ζ, and a conterminous edge with variables $\bar{x}$, $\bar{y}$, $\bar{z}$.

Two other contravariants have occurred (§§ 316, 318): they are

$$T' = (l, n, j, i, k, m\,\|\,\xi, \eta, \zeta)^2,$$
$$\Gamma = (\bar{l}, \bar{n}, \bar{j}, \bar{i}, \bar{k}, \bar{m}\,\|\,\xi, \eta, \zeta)^2,$$

in the latter of which the coefficients are (§ 316) of the type

$$\bar{l} = BJ - 2FI + CN.$$

We now denote by $\bar{K}$ the Riemann measure of superficial curvature of a geodesic surface in the region with the orientation-coordinates ξ, η, ζ, at a point, and by $\bar{H}$ the additive measure of superficial curvature of that surface; the significance of the two contravariants is given by the relations

$$\bar{K} = \frac{T'}{T}, \quad \bar{H} = \frac{\Gamma}{T},$$

that is,

$$T' = \bar{K}\, dS^2, \quad \Gamma = \bar{H}\, dS^2.$$

Again, we have

$$\bar{x}\frac{\partial U}{\partial \bar{x}}+\bar{y}\frac{\partial U}{\partial \bar{y}}+\bar{z}\frac{\partial U}{\partial \bar{z}}=2U,\quad \bar{x}\frac{\partial V}{\partial \bar{x}}+\bar{y}\frac{\partial V}{\partial \bar{y}}+\bar{z}\frac{\partial V}{\partial \bar{z}}=2V,$$

so that the two sets of variables

$$\frac{\partial U}{\partial \bar{x}},\ \frac{\partial U}{\partial \bar{y}},\ \frac{\partial U}{\partial \bar{z}};\quad \frac{\partial V}{\partial \bar{x}},\ \frac{\partial V}{\partial \bar{y}},\ \frac{\partial V}{\partial \bar{z}};$$

are cogredient with ξ, η, ζ; when either set is substituted for the orientation variables in a contravariant, the result is a new concomitant which manifestly will be a covariant. Such a covariant has arisen in the form (§ 306)

$$\mathbf{D}=\frac{1}{4}(a, b, c, f, g, h)\left(\frac{\partial V}{\partial \bar{x}}, \frac{\partial V}{\partial \bar{y}}, \frac{\partial V}{\partial \bar{z}}\right)^2;$$

and it (*l.c.*) has been proved that

$$\mathbf{D}=\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)\Omega^2 ds^2,$$

where $1/\sigma$ is the torsion of the regional geodesic in the direction $\bar{x}, \bar{y}, \bar{z}$. The covariant $\mathbf{D}$ manifestly also gives the curvature of screw of the geodesic (§ 131).

Also, we have had (§ 306) variables $\bar{u}, \bar{v}, \bar{w}$, cogredient with $\bar{x}, \bar{y}, \bar{z}$, now to be taken in the form

$$\bar{u}, \bar{v}, \bar{w}=\left\| \begin{matrix} A\bar{x}+H\bar{y}+G\bar{z}, & H\bar{x}+B\bar{y}+F\bar{z}, & G\bar{x}+F\bar{y}+C\bar{z} \\ L\bar{x}+M\bar{y}+K\bar{z}, & M\bar{x}+N\bar{y}+I\bar{z}, & K\bar{x}+I\bar{y}+J\bar{z} \end{matrix} \right\|;$$

and they lead to the covariant $\mathbf{S}$, where

$$\frac{\Omega^2 ds^4}{\sigma^2}=\mathbf{S}=(A, B, C, F, G, H)(\bar{u}, \bar{v}, \bar{w})^2.$$

Obviously

$$\mathbf{S}=\mathbf{D}ds^2-\Omega^2 V^2=\mathbf{D}U-\Omega^2 V^2.$$

Further, we have

$$\xi\frac{\partial T'}{\partial \xi}+\eta\frac{\partial T'}{\partial \eta}+\zeta\frac{\partial T'}{\partial \zeta}=2T',\quad \xi\frac{\partial \Gamma}{\partial \xi}+\eta\frac{\partial \Gamma}{\partial \eta}+\zeta\frac{\partial \Gamma}{\partial \zeta}=2\Gamma,$$

and similarly for any other contravariant: thus the two sets of variables

$$\frac{\partial T'}{\partial \xi},\ \frac{\partial T'}{\partial \eta},\ \frac{\partial T'}{\partial \zeta};\quad \frac{\partial \Gamma}{\partial \xi},\ \frac{\partial \Gamma}{\partial \eta},\ \frac{\partial \Gamma}{\partial \zeta};$$

and corresponding derivatives of any contravariant, are cogredient with $\bar{x}, \bar{y}, \bar{z}$, and can therefore be substituted for these variables in a covariant, so as to produce a new concomitant which manifestly will be a contravariant. Such a contravariant is

$$\Lambda=\frac{1}{4}(A, B, C, F, G, H)\left(\frac{\partial T'}{\partial \xi}, \frac{\partial T'}{\partial \eta}, \frac{\partial T'}{\partial \zeta}\right)^2.$$

Its leading coefficient, being the coefficient of ξ^2 in the full expression, is

$$\begin{aligned} &=(A, B, C, F, G, H)(l, m, k)^2 \\ &=l\mathfrak{C}_2+B(n^2-ln)+2F(mk-li)+C(k^2-lj)=l\mathfrak{C}_2-\Upsilon\bar{l}, \end{aligned}$$

which is the leading coefficient of ξ^2 in the contravariant

$$T'\mathfrak{G}_2 - \Upsilon\Gamma;$$

and therefore

$$\Lambda = \left\{\overline{K}\left(\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}+\frac{1}{\alpha\beta}\right) - \overline{H}\frac{1}{\alpha\beta\gamma}\right\}\Omega^2 dS^2.$$

Other concomitants will be required in the later interpretation (Chapter XXVII) of aggregates of covariants, derived analytically by the theory of groups. Many can be constructed by the ordinary processes of the theory of ternary forms*: the following illustrations are selected because of their subsequent occurrence.

327. Let a concomitant be constructed from T', so as to have a leading coefficient similar to that of the contravariant Γ constructed from U, so that the leading coefficient will be

$$bj - 2fi + cn.$$

We take

$$\left.\begin{aligned} \overline{\lambda} &= bj - 2fi + cn, & \bar{\imath} &= gn + hk - fl - ai \\ \bar{\nu} &= cl - 2gk + aj, & \bar{\kappa} &= hi + fm - gn - bk \\ \bar{\zeta} &= an - 2hm + bl, & \bar{\mu} &= fk + gi - hj - cm \end{aligned}\right\};$$

it is easy to see that the concomitant thus constructed is a covariant Γ' given by

$$\Gamma' = (\overline{\lambda}, \bar{\nu}, \bar{\zeta}, \bar{\imath}, \bar{\kappa}, \bar{\mu}\,\|\,\bar{x}, \bar{y}, \bar{z})^2.$$

Now

$$\begin{aligned} \overline{\lambda} &= bj - 2fi + cn \\ &= b(LN - M^2) - 2f(MK - LI) + c(JL - K^2) \\ &= L(aL + 2hM + 2gK + bN + 2fI + cJ) \\ &\quad - (aL^2 + 2hLM + 2gLK + bM^2 + 2fMK + cK^2) \\ &= L\mathfrak{G}_1 - (a, b, c, f, g, h\,\|\,L, M, K)^2, \end{aligned}$$

which is the leading coefficient in the covariant

$$V\mathfrak{G}_1 - \mathbf{D};$$

consequently

$$\begin{aligned} \Gamma' &= V\mathfrak{G}_1 - \mathbf{D} \\ &= \Omega^2\frac{ds^2}{\rho}\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right) - \Omega^2\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)ds^2 \\ &= \left\{\frac{1}{\rho}\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right) - \frac{1}{\rho^2} - \frac{1}{\sigma^2}\right\}\Omega^2 ds^2. \end{aligned}$$

In the preceding treatment of $\overline{\lambda}$, substitution has been made for the minors of Υ, viz. n, i, j. In that same coefficient, let substitution be made for the minors of Ω^2, viz. b, f, c; then

$$\begin{aligned} \overline{\lambda} &= j(CA - G^2) - 2i(GH - AF) + n(AB - H^2) \\ &= A\mathfrak{G}_2 - (l, n, j, i, k, m\,\|\,A, H, G)^2. \end{aligned}$$

* E. B. Elliott, *Algebra of Quantics*: and J. H. Grace and A. Young, *Algebra of Invariants*: may be consulted for the exposition of these processes.

Now take a covariant

$$\Delta = \frac{1}{4}(l, n, j, i, k, m\,\overline{\xi}\,\frac{\partial U}{\partial \bar{x}}, \frac{\partial U}{\partial \bar{y}}, \frac{\partial U}{\partial \bar{z}})^2.$$

The leading coefficient, being the coefficient of $\bar{x}^2$, is

$$= (l, n, j, i, k, m\,\overline{\xi}\,A, H, G)^2$$
$$= A\mathfrak{S}_2 - \bar{\lambda}.$$

Consequently

$$\Delta = U\mathfrak{S}_2 - \Gamma'$$
$$= \left\{\frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} + \frac{1}{\alpha\beta} - \frac{1}{\rho}\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) + \frac{1}{\rho^2} + \frac{1}{\sigma^2}\right\}\Omega^2 ds^2.$$

We thus have a number of invariants, covariants, and contravariants, constituted by the set

$$U,\ \Theta;\ V,\ \Omega^2,\ \mathfrak{S}_1,\ \mathfrak{S}_2,\ \Upsilon;\ T,\ T',\ \Gamma,\ \Gamma',$$

all belonging to the two simultaneous ternary forms U and V, which are of the second order. The respective geometrical interpretations of all these concomitants, as well as of $\mathbf{D}$, $\mathbf{S}$, Λ, Δ, which are algebraically expressible in terms of them, have been given.

Ex. Obtain the following results:

$$\frac{1}{4}(a, b, c, f, g, h\,\overline{\xi}\,\frac{\partial U}{\partial \bar{x}}, \frac{\partial U}{\partial \bar{y}}, \frac{\partial U}{\partial \bar{z}})^2 = \Omega^2 ds^2;$$

$$\frac{1}{4}(l, n, j, i, k, m\,\overline{\xi}\,\frac{\partial V}{\partial \bar{x}}, \frac{\partial V}{\partial \bar{y}}, \frac{\partial V}{\partial \bar{z}})^2 = \frac{\Omega^2 ds^2}{\alpha\beta\gamma\rho};$$

$$\frac{1}{4}(A, B, C, F, G, H\,\overline{\xi}\,\frac{\partial \Gamma}{\partial \xi}, \frac{\partial \Gamma}{\partial \eta}, \frac{\partial \Gamma}{\partial \zeta})^2$$
$$= \left\{\bar{K} + \bar{H}\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) - \left(\frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} + \frac{1}{\alpha\beta}\right)\right\}\Omega^2 dS^2.$$

Introduction of a ternary cubic: further covariants.

328. Again, in § 292, we had the equation

$$\frac{d}{ds}\left(\frac{1}{\rho}\right) = (e\,\overline{\xi}\,p', q', r')^3;$$

so we take the cubic ternary form

$$W = (e\,\overline{\xi}\,\bar{x}, \bar{y}, \bar{z})^3;$$

and then

$$\frac{d}{ds}\left(\frac{1}{\rho}\right) = \frac{W}{ds^3} = \frac{W}{U^{\frac{3}{2}}}.$$

There were (§ 296) three linear covariants: when we write

$$\Omega^2\mathfrak{L}_1 = \Sigma a e_{300}, \quad \Omega^2\mathfrak{F}_1 = \Sigma \bar{l} e_{300}, \quad \Omega^2\mathfrak{R}_1 = \Sigma l e_{300},$$

with the corresponding values (§ 296) for $\mathfrak{L}_2$, $\mathfrak{L}_3$, $\mathfrak{F}_2$, $\mathfrak{F}_3$, $\mathfrak{R}_2$, $\mathfrak{R}_3$, the covariants and their values are

$$\frac{d\mathfrak{L}}{ds} = \frac{1}{\Omega^2 U^{\frac{1}{2}}}(\Sigma a e_{300}, \Sigma a e_{210}, \Sigma a e_{201})(\bar{x}, \bar{y}, \bar{z}),$$

$$\frac{d\mathfrak{F}}{ds} = \frac{1}{\Omega^2 U^{\frac{1}{2}}}(\Sigma \bar{l} e_{300}, \Sigma \bar{l} e_{210}, \Sigma \bar{l} e_{201})(\bar{x}, \bar{y}, \bar{z}),$$

$$\frac{d\mathfrak{R}}{ds} = \frac{1}{\Omega^2 U^{\frac{1}{2}}}(\Sigma l e_{300}, \Sigma l e_{210}, \Sigma l e_{201})(\bar{x}, \bar{y}, \bar{z}),$$

with a manifest invariant

$$\begin{vmatrix} \Sigma a e_{300}, & \Sigma a e_{210}, & \Sigma a e_{201} \\ \Sigma \bar{l} e_{300}, & \Sigma \bar{l} e_{210}, & \Sigma \bar{l} e_{201} \\ \Sigma l e_{300}, & \Sigma l e_{210}, & \Sigma l e_{201} \end{vmatrix}$$

which is easily seen to vanish when there exists a permanent relation between the three principal radii of linear curvature at all points of a region *.

It was proved (§ 309) that the tilt of a regional geodesic is given by

$$\frac{1}{\sigma^2 \tau} = \frac{1}{12\Omega} J\left(\frac{\overline{U}, \overline{V}, \overline{W}}{p', q', r'}\right).$$

Accordingly we take the Jacobian of U, V, W, and we write

$$\Phi = \frac{1}{12}\begin{vmatrix} \frac{\partial U}{\partial \bar{x}}, & \frac{\partial U}{\partial \bar{y}}, & \frac{\partial U}{\partial \bar{z}} \\ \frac{\partial V}{\partial \bar{x}}, & \frac{\partial V}{\partial \bar{y}}, & \frac{\partial V}{\partial \bar{z}} \\ \frac{\partial W}{\partial \bar{x}}, & \frac{\partial W}{\partial \bar{y}}, & \frac{\partial W}{\partial \bar{z}} \end{vmatrix};$$

thus Φ is a covariant of order four, having

$$\begin{vmatrix} A, & H, & G \\ L, & M, & K \\ e_{300}, & e_{210}, & e_{201} \end{vmatrix}$$

for its leading coefficient: and now, inserting the value of σ from § 326, we have

$$\frac{1}{\tau} = \frac{\Omega\Phi}{\mathbf{S}}.$$

We obtained (§ 312) the equation

$$-\Omega^2\left(\frac{\sigma'}{\sigma^3} + \frac{\rho'}{\rho^3}\right) = \frac{1}{6}(a, b, c, f, g, h)\left(\frac{\partial \overline{V}}{\partial p'}, \frac{\partial \overline{V}}{\partial q'}, \frac{\partial \overline{V}}{\partial r'}\right)\left(\frac{\partial \overline{W}}{\partial p'}, \frac{\partial \overline{W}}{\partial q'}, \frac{\partial \overline{W}}{\partial r'}\right).$$

* Surfaces in homaloidal triple space, such that there is a permanent relation between their principal radii of curvature at every point, are usually called Weingarten surfaces; they were discussed first in his memoir, *Crelle*, t. lxii (1862), pp. 160–173. See my *Lectures on Differential Geometry*, §§ 203–208.

Accordingly, we take a covariant, of order three, given by

$$\Xi = \frac{1}{6}(a, b, c, f, g, h)\left(\frac{\partial V}{\partial \bar{x}}, \frac{\partial V}{\partial \bar{y}}, \frac{\partial V}{\partial \bar{z}}\right)\left(\frac{\partial W}{\partial \bar{x}}, \frac{\partial W}{\partial \bar{y}}, \frac{\partial W}{\partial \bar{z}}\right),$$

and having

$$(a, b, c, f, g, h)(L, M, K)(e_{300}, e_{210}, e_{201})$$

for its leading coefficient; and now

$$-\Omega^2\left(\frac{\sigma'}{\sigma^3} + \frac{\rho'}{\rho^3}\right) = \frac{\Xi}{U^{\frac{1}{2}}},$$

so that

$$-\frac{\sigma'}{\sigma^3} = \frac{1}{\Omega^2 U^{\frac{1}{2}}}(\Xi U - \Omega^2 W V).$$

We take a covariant of order four as arising through the expression for $\frac{1}{\sigma^2\tau^2}$ obtained in § 311, defined by

$$\Psi = \frac{1}{9}(a, b, c, f, g, h)\left(\frac{\partial W}{\partial \bar{x}}, \frac{\partial W}{\partial \bar{y}}, \frac{\partial W}{\partial \bar{z}}\right)^2,$$

its leading coefficient being

$$(a, b, c, f, g, h)(e_{300}, e_{210}, e_{201})^2;$$

and then

$$\frac{1}{\sigma^2\tau^2} + \frac{\rho'^2}{\rho^4} + \frac{\sigma'^2}{\sigma^4} = \frac{\Psi}{\Omega^2 U^2},$$

that is,

$$\frac{1}{\sigma^2\tau^2} = \frac{1}{\Omega^2 U^2}\left(\Psi - \frac{U\Xi^2 - 2\Omega^2 V W \Xi + \Omega^2 \mathbf{D} W^2}{U\mathbf{D} - \Omega^2 V^2}\right).$$

A comparison of the two values of τ leads to the relation

$$\Omega^2\Phi^2 = \begin{vmatrix} U, & \Omega V, & \Omega W \\ \Omega V, & \mathbf{D}, & \Xi \\ \Omega W, & \Xi, & \Psi \end{vmatrix},$$

which is merely a relation between the concomitants of three ternary forms, and expresses the square of their Jacobian (itself, a skew covariant) in terms of symmetric concomitants of the complete set.

Umbral expressions.

329. Briefer formal expressions can be obtained for these invariants, covariants, and contravariants, when the umbral method of representation is adopted. In this method, umbral coefficients are taken for the forms U, V, W, such that

$$a_1\bar{x} + a_2\bar{y} + a_3\bar{z} = a_x,$$
$$a_1'\bar{x} + a_2'\bar{y} + a_3'\bar{z} = a_x',$$
$$e_1\bar{x} + e_2\bar{y} + e_3\bar{z} = e_x,$$

while only combinations $a_1^{\lambda} a_2^{\mu} a_3^{\nu}$ (with $\lambda+\mu+\nu=2$), $a_1'^{\lambda} a_2'^{\mu} a_3'^{\nu}$ (with $\lambda+\mu+\nu=2$), $e_1^{\lambda} e_2^{\mu} e_3^{\nu}$ (with $\lambda+\mu+\nu=3$), have significance according to the interpretations

$$a_x^2 = (A, B, C, F, G, H \mid \bar{x}, \bar{y}, \bar{z})^2,$$
$$a_x'^2 = (L, N, J, I, K, M \mid \bar{x}, \bar{y}, \bar{z})^2,$$
$$e_x^3 = (e \mid \bar{x}, \bar{y}, \bar{z})^3.$$

Further, to represent powers and products of A, B, C, F, G, H, sets of umbral symbols a are introduced; and similarly for powers and products of the coefficients L, N, J, I, K, M, and of the coefficients e. Thus we have

$$l = \tfrac{1}{2}(b_1'c_2' - b_2'c_1')^2, \quad n = \tfrac{1}{2}(c_1'a_2' - c_2'a_1')^2, \quad j = \tfrac{1}{2}(a_1'b_2' - a_2'b_1')^2,$$

and so on. Again, we write

$$(l, n, j, i, k, m \mid \xi, \eta, \zeta)^2 = (\alpha_1'\xi + \alpha_2'\eta + \alpha_3'\zeta)^2 = \alpha_\xi'^2,$$

with the contragredient variables ξ, η, ζ, and the new compound umbral symbols $\alpha_1', \alpha_2', \alpha_3'$; and there are similarly composed umbral symbols $\alpha_1, \alpha_2, \alpha_3$, different sets being used in order to allow powers and products, always with the limitation that, for each set, only those combinations of umbral symbols are admissible which are interpretable in terms of literal coefficients.

Accordingly, under this representation, we have

$$\begin{aligned}
U &= a_x^2 = b_x^2 = c_x^2 = \dots, \\
V &= a_x'^2 = b_x'^2 = c_x'^2 = \dots, \\
W &= e_x^3 = f_x^3 = g_x^3 = \dots, \\
\Omega^2 &= \tfrac{1}{6}(abc)^2, \\
\mathfrak{S}_1 &= \tfrac{1}{2}(aba')^2, \\
\mathfrak{S}_2 &= \tfrac{1}{2}(aa'b')^2, \\
\Upsilon &= \tfrac{1}{6}(a'b'c')^2, \\
\Theta &= x_\xi, \\
T &= \tfrac{1}{2}(ab\xi)^2 = \alpha_\xi^2, \\
\Gamma &= (aa'\xi)^2 = \bar{\alpha}_\xi^2, \\
T' &= \tfrac{1}{2}(a'b'\xi)^2 = \alpha_\xi'^2, \\
\Gamma' &= \tfrac{1}{2}(\alpha\alpha' x)^2, \\
\mathbf{D} &= \tfrac{1}{2}(aba')(abb')\, a_x' b_x' = \alpha_{a'}\alpha_{b'} a_x' b_x', \\
\Lambda &= \alpha_a' \beta_a' \alpha_\xi' \beta_\xi', \\
\Delta &= \alpha_a' \alpha_b' a_x b_x, \\
\overline{L} &= \alpha_e^2 e_x, \\
\overline{F} &= \bar{\alpha}_e^2 e_x, \\
\overline{R} &= \alpha_e'^2 e_x, \\
\Phi &= \bar{\alpha}_e a_x a_x' e_x^2, \\
\mathbf{S} &= (aca')(bcb')\, a_x b_x a_x' b_x', \\
\Xi &= \alpha_{a'}\alpha_e a_x' e_x^2, \\
\Psi &= \alpha_e \alpha_f e_x^2 f_x^2;
\end{aligned}$$

and the values of the various concomitants are

$$U = ds^2, \quad V = \frac{ds^2}{\rho}, \quad W = -\frac{\rho'}{\rho^2} ds^3,$$

$$\mathfrak{G}_1 = \Omega^2 \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) = \Omega^2 \mathfrak{L}, \quad \mathfrak{G}_2 = \Omega^2 \left(\frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} + \frac{1}{\alpha\beta}\right) = \Omega^2 \mathfrak{F},$$

$$\mathfrak{T} = \Omega^2 \frac{1}{\alpha\beta\gamma} = \Omega^2 \mathfrak{R},$$

$$T = dS^2, \qquad \Theta = \frac{1}{\Omega} d\overline{V},$$

$$T' = \overline{K} dS^2, \quad \Gamma = \overline{H} dS^2,$$

$$\mathbf{D} = \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right) \Omega^2 ds^2,$$

$$\Lambda = \left\{\overline{K}\left(\frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} + \frac{1}{\alpha\beta}\right) - \overline{H}\frac{1}{\alpha\beta\gamma}\right\} \Omega^2 dS^2,$$

$$\Gamma' = \left\{\frac{1}{\rho}\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) - \frac{1}{\rho^2} - \frac{1}{\sigma^2}\right\} \Omega^2 ds^2,$$

$$\Delta = \left\{\frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} + \frac{1}{\alpha\beta} - \frac{1}{\rho}\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) + \frac{1}{\rho^2} + \frac{1}{\sigma^2}\right\} \Omega^2 ds^2,$$

$$\overline{L} = \frac{d\mathfrak{L}}{ds} \Omega^2 ds, \quad \overline{F} = \frac{d\mathfrak{F}}{ds} \Omega^2 ds, \quad \overline{R} = \frac{d\mathfrak{R}}{ds} \Omega^2 ds,$$

$$\mathbf{S} = \frac{1}{\sigma^2} \Omega^2 ds^4, \quad \Phi = \frac{1}{\sigma^2 \tau} \Omega ds^4,$$

$$\Xi = -\left(\frac{\sigma'}{\sigma^3} + \frac{\rho'}{\rho^3}\right) \Omega^2 ds^3,$$

$$\Psi = \left(\frac{1}{\sigma^2 \tau^2} + \frac{\rho'^2}{\rho^4} + \frac{\sigma'^2}{\sigma^4}\right) \Omega^2 ds^4.$$

Further discussion, and other illustrations, of these concomitants and of other concomitants will be deferred until the general theory is systematically investigated in Chapter XXVII.

Ex. 1. A *globular representation* of a region is provided by using the direction-cosines X, Y, Z, V, of the normal to the region at any point, as coordinates of points in a globular region

$$X^2 + Y^2 + Z^2 + V^2 = 1.$$

Prove the following results:

(i) if $d\bar{s}$ be the arc in the globular representation, which corresponds to the arc ds in the original region,

$$\frac{d\bar{s}}{ds} = \left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)^{\frac{1}{2}},$$

where ρ and σ are the radii of circular curvature and of torsion of the regional geodesic through the arc, thus shewing that the globular representation is not a conformal representation of the region;

(ii) if

$$d\bar{s}^2=(\bar{A},\ \bar{B},\ \bar{C},\ \bar{F},\ \bar{G},\ \bar{H}\|dp,\ dq,\ dr)^2,$$

the discriminant of $d\bar{s}^2$, being

$$\begin{vmatrix} \bar{A}, & \bar{H}, & \bar{G} \\ \bar{H}, & \bar{B}, & \bar{F} \\ \bar{G}, & \bar{F}, & \bar{C} \end{vmatrix},$$

is equal to $\dfrac{\Omega^2}{a^2\beta^2\gamma^2}$; and

(iii) the globular image of a line of curvature in the original region is parallel to that line of curvature.

Verify that the curvature of every normal section in the globular image is equal to unity.

Ex. 2. The *tangential coordinates* of a region are taken to be X, Y, Z, V, the direction-cosines of the normal to the region, and Π, the perpendicular from the origin of Cartesian coordinates upon the tangential flat to the region at x, y, z, v, so that

$$\Pi=xX+yY+zZ+vV.$$

Prove that the region is completely represented by these tangential coordinates, supposed given in terms of the three parameters, by means of the equations

$$\left.\begin{aligned} x&=X\Pi+\frac{1}{\bar{\Omega}^2}(\bar{a},\ \bar{b},\ \bar{c},\ \bar{f},\ \bar{g},\ \bar{h}\|\Pi_1,\ \Pi_2,\ \Pi_3\|X_1,\ X_2,\ X_3)\\ y&=Y\Pi+\frac{1}{\bar{\Omega}^2}(\bar{a},\ \bar{b},\ \bar{c},\ \bar{f},\ \bar{g},\ \bar{h}\|\Pi_1,\ \Pi_2,\ \Pi_3\|Y_1,\ Y_2,\ Y_3)\\ z&=Z\Pi+\frac{1}{\bar{\Omega}^2}(\bar{a},\ \bar{b},\ \bar{c},\ \bar{f},\ \bar{g},\ \bar{h}\|\Pi_1,\ \Pi_2,\ \Pi_3\|Z_1,\ Z_2,\ Z_3)\\ v&=V\Pi+\frac{1}{\bar{\Omega}^2}(\bar{a},\ \bar{b},\ \bar{c},\ \bar{f},\ \bar{g},\ \bar{h}\|\Pi_1,\ \Pi_2,\ \Pi_3\|V_1,\ V_2,\ V_3) \end{aligned}\right\},$$

where $\bar{\Omega}^2$ is the discriminant of $d\bar{s}^2$ in the globular representation, and $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{f}$, $\bar{g}$, $\bar{h}$, are its first minors.

Prove also that

$$\begin{aligned} \bar{a}&=(a,\ b,\ c,\ f,\ g,\ h\|\ l,\ m,\ k)^2,\\ \bar{h}&=(a,\ b,\ c,\ f,\ g,\ h\|\ l,\ m,\ k\|m,\ n,\ i),\\ \bar{g}&=(a,\ b,\ c,\ f,\ g,\ h\|\ l,\ m,\ k\|\ k,\ i,\ j),\\ \bar{b}&=(a,\ b,\ c,\ f,\ g,\ h\|m,\ n,\ i)^2,\\ \bar{f}&=(a,\ b,\ c,\ f,\ g,\ h\|m,\ n,\ i\|\ k,\ i,\ j),\\ \bar{c}&=(a,\ b,\ c,\ f,\ g,\ h\|\ k,\ i,\ j)^2, \end{aligned}$$

where the quantities on the right-hand side belong to the original region.

Ex. 3. Prove that, if a functional relation exists between the three linear curvatures alone, the necessary invariant condition can be represented in the form

$$a_e^{\ 2}\bar{a}_f^{\ 2}a_g^{\prime 2}(efg)=0.$$

Three regions of centres of curvature of a region.

330. We have seen that a measure of linear curvature of a region, taken in any direction, is given by the circular curvature of a geodesic drawn in that regional direction; and the centre of circular curvature lies on the normal to the region, drawn inwards from the external convexity of the region.

Now this normal is the same, whatever direction be assigned through a point: consequently, the locus of the centres of circular curvature, when the whole tale of directions is considered, is composed of a range or ranges along the line of the normal. At the point, there are three principal directions, defined sometimes as directions of maximum and minimum curvature (the maximum or minimum being selected by choice of direction and not by stationary value, if any, of ρ along a continued direction), defined sometimes

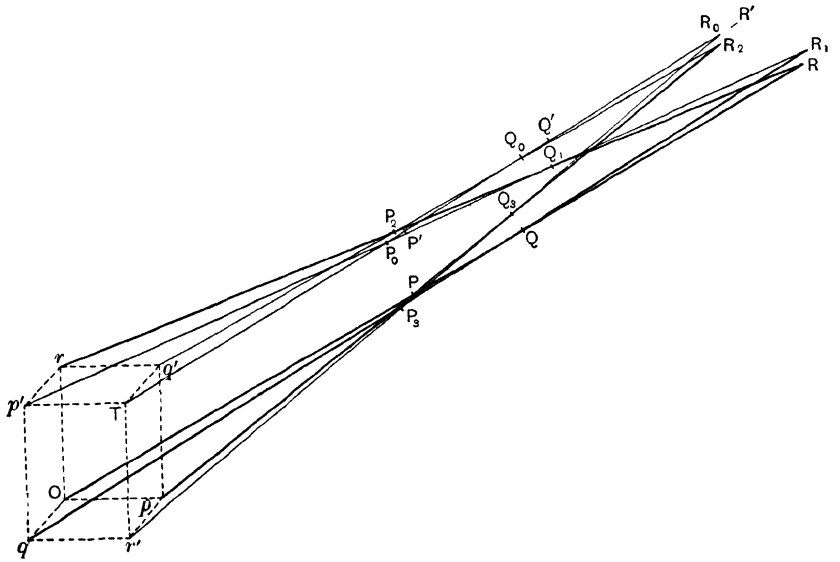


Fig. 23.

as directions along which two successive normals actually meet. From either definition, it results that there are three such principal directions in the region, and accordingly three principal radii measured along the normal. The three corresponding points, being the respective centres of circular curvature of geodesics in these principal directions, are therefore points of organic importance, when associated with a point O. They sometimes are called the centres of curvature of the region: some attention to the consideration of their respective loci, and to the relations of their loci, is naturally demanded.

In the figure, the dotted lines are elements of lines of curvature through the marked points of the region: and these are taken to be the parametric curves of reference. Thus at O, the lines are Op, Oq, Or, taken in the positive direction for positive increments of the arguments p, q, r, respectively; at p, they similarly are Op, pr', pq'; at q, they similarly are qr', Oq, qp'; at r, they similarly are rq', rp', Or; at p', they similarly are $p'T$, rp', qp'; at q', they similarly are rq', $q'T$, pq'; at r', they similarly are qr', pr', $r'T$; and at T, they similarly are $p'T$, $q'T$, $r'T$.

The normals to the region at two consecutive points on a line of curvature meet, so that there are three centres of curvature on each normal. The centre of curvature, for intersection of normals at consecutive points on a line of curvature q = constant and r = constant, will be called the *first* centre, and will be denoted by a symbol P with (or without) a suffix: the centre, for a line of curvature r = constant and p = constant, will be called the *second* centre, denoted by a symbol Q; and the centre, for a line of curvature p = constant and q = constant, will be called the *third* centre, denoted by a symbol R. Thus the region of centres of circular curvature of a region in homaloidal quadruple space is a three-volumed region.

As eight angular points are marked at the corners of the elementary rectangular parallelepiped of the region, the edges being rudimentary arcs of lines of curvature, there are eight normals $OPQR$, pPQ_3R_2, qP_3QR_1, rP_2Q_1R, $p'P_0Q_1R_1$, $q'P_2Q_0R_2$, $r'P_3Q_3R_0$, $TP_0Q_0R_0$. These normals at the eight respective points meet according to the following tableau, each point of intersection being a centre of curvature:

	O,	p,	q,	r,	p',	q',	r',	T
O	×,	P,	Q,	R,	—,	—,	—,	—
p	P,	×,	—,	—,	—,	R_2,	Q_3,	—
q	Q,	—,	×,	—,	R_1,	—,	P_3,	—
r	R,	—,	—,	×,	Q_1,	P_2,	—,	—
p'	—,	—,	R_1,	Q_1,	×,	—,	—,	P_0
q'	—,	R_2,	—,	P_2,	—,	×,	—,	Q_0
r'	—,	Q_3,	P_3,	—,	—,	—,	×,	R_0
T	—,	—,	—,	—,	P_0,	Q_0,	R_0,	×

The three points in a column lie on the normal at the point which heads the column; the three points in a row lie on the normal at the point which directs the row.

But it is necessary to specify the proper association of the three centres of curvature with the foregoing points. The limiting (or ultimate) positions of the various points P, the various points Q, and the various points R, are centres of curvature. For the purpose of description in connection with penultimate positions of the various points of all the sets, we define (as a convention) the centre of curvature at a point associated with a line of curvature through the point, to be the intersection of the regional normal at that point with the intersection of the regional normal at the consecutive point along the line of curvature given by a positive increment of the variable parameter of the line. Thus at O, which is the point p, q, r, the centre of curvature for the curve Op is P, being the intersection of the normal at O with the normal at p, which is the point $p+dp, q, r$; the centre of curvature for the curve Oq is Q, being the intersection of the normal at O with the normal at q, which is the point $p, q+dq, r$; and the centre of curvature for the curve Or is R, being the intersection of the normal at O with the normal at r, which is the point $p, q, r+dr$. Again, at p, the point $p+dp, q, r$, the second centre of curvature is Q_3, being the intersection of the normal at p with the normal at r', which is the point $p+dp, q+dq, r$; the third centre of curvature is R_2, being the intersection of the normal at p with the normal at q', which is the point $p+dp, q, r+dr$; but the first centre of curvature at p is not P, but an unindicated point which is the intersection of the normal at p with the normal at a point along the curve Op specified by an increase of the parameter beyond its value $p+dp$ at p, that is, a point along the curve Op on the side of p remote from O. In the same way, at p', the point $p, q+dq, r+dr$, the first centre of curvature is P_0, being the intersection of the normal at p' with the normal at T, which is the point $p+dp, q+dq, r+dr$: but the second centre of curvature is not Q_1, which is the intersection of the normal at p' with the normal at r, the point $p, q, r+dr$ (that is, a backward and not a forward point from p' along the positive direction of the line of curvature rp'): and similarly the third centre of curvature at p' is not R_1. The second centre of curvature at p' is a point not indicated in the figure, arising as the intersection of the normal at p' with the normal at a consecutive point in the curve rp' produced beyond p'; while the third centre of curvature at p' is a point also not indicated in the figure, arising as the intersection of the normal at p' with the normal at a consecutive point in the curve qp' produced in p'. Similarly, at T, the point $p+dp, q+dq, r+dr$, the points P_0, Q_0, R_0, are not the centres of curvature; for they are the intersections of the normal at T with the normals at the three respective backward points $p, q+dq, r+dr$; $p+dp, q, r+dr$; $p+dp, q+dq, r$; along the respective lines of curvature. Actually, at T, the first centre of curvature is P' (the intersection of the normal at T with the normal at a point consecutive to T along the curve $p'T$ produced), the second centre of curvature is Q' (the intersection of the normal at T with the normal at a point consecutive to T along the curve $q'T$ produced), and the third centre of

curvature is R' (the intersection of the normal at T with the normal at a point consecutive to T along the curve $r'T$ produced). Thus the centres of curvature at the eight points, indicated in the diagram (p. 161), are:

at O, the points P, Q, R;
at p, the points Q_3, R_2, but not P;
at q, ... P_3, R_1, ... Q;
at r, ... P_2, Q_1, ... R;
at p', the point P_0, but not the points Q_1, R_1;
at q', ... Q_0, ... P_2, R_2;
at r', ... R_0, ... P_3, Q_3;
at T, not the points P_0, Q_0, R_0.

As already stated, the centres of curvature at T are the three specified points P', Q', R'.

The principal radii of curvature at O have been denoted by α, β, γ; thus

$$OP = \alpha, \quad OQ = \beta, \quad OR = \gamma.$$

The radii of curvature at p (the point $p + dp, q, r$), are $\alpha + \alpha_1 dp$, $\beta + \beta_1 dp$, $\gamma + \gamma_1 dp$; thus

$$pP = \alpha, \quad pQ_3 = \beta + \beta_1 dp, \quad pR_2 = \gamma + \gamma_1 dp.$$

Similarly at q, the radii of curvature are $\alpha + \alpha_2 dq$, $\beta + \beta_2 dq$, $\gamma + \gamma_2 dq$, and

$$qP_3 = \alpha + \alpha_2 dq, \quad qQ = \beta, \quad qR_1 = \gamma + \gamma_2 dq.$$

At r, the radii of curvature are $\alpha + \alpha_3 dr$, $\beta + \beta_3 dr$, $\gamma + \gamma_3 dr$, and

$$rP_2 = \alpha + \alpha_3 dr, \quad rQ_1 = \beta + \beta_3 dr, \quad rR = \gamma.$$

At p', the radii of curvature are $\alpha + \alpha_2 dq + \alpha_3 dr$, $\beta + \beta_2 dq + \beta_3 dr$, $\gamma + \gamma_2 dq + \gamma_3 dr$: but

$$p'P_0 = \alpha + \alpha_2 dq + \alpha_3 dr, \quad p'Q_1 = \beta + \beta_3 dr, \quad p'R_1 = \gamma + \gamma_2 dq.$$

At q', the radii of curvature are $\alpha + \alpha_1 dp + \alpha_3 dr$, $\beta + \beta_1 dp + \beta_3 dr$, $\gamma + \gamma_1 dp + \gamma_3 dr$: but

$$q'P_2 = \alpha + \alpha_3 dr, \quad q'Q_0 = \beta + \beta_1 dp + \beta_3 dr, \quad q'R_2 = \gamma + \gamma_1 dp.$$

At r', the radii of curvature are $\alpha + \alpha_1 dp + \alpha_2 dq$, $\beta + \beta_1 dp + \beta_2 dq$, $\gamma + \gamma_1 dp + \gamma_2 dq$: but

$$r'P_3 = \alpha + \alpha_2 dq, \quad r'Q_3 = \beta + \beta_1 dp, \quad r'R_0 = \gamma + \gamma_1 dp + \gamma_2 dq.$$

At T', the radii of curvature are $\alpha + \alpha_1 dp + \alpha_2 dq + \alpha_3 dr$, $\beta + \beta_1 dp + \beta_2 dq + \beta_3 dr$, $\gamma + \gamma_1 dp + \gamma_2 dq + \gamma_3 dr$: but

$$T'P_0 = \alpha + \alpha_2 dq + \alpha_3 dr, \quad T'Q_0 = \beta + \beta_1 dp + \beta_3 dr, \quad T'R_0 = \gamma + \gamma_1 dp + \gamma_2 dq,$$

while

$$P_0P' = \alpha_1 dp, \quad Q_0Q' = \beta_2 dq, \quad R_0R' = \gamma_3 dr.$$

Coordinates of the three centres.

331. The coordinates of the respective points P, Q, R, are obtained by measuring the appropriate distances along the respective intersecting normals. Thus for P, the intersection of the normals $OPQR$ and pPQ_2R_3, the coordinates are

$$x+\alpha X, \quad y+\alpha Y, \quad z+\alpha Z, \quad v+\alpha V,$$

as being on the former normal*, and they are

$$x+x_1dp+\alpha(X+X_1dp), \quad y+y_1dp+\alpha(Y+Y_1dp),$$
$$z+z_1dp+\alpha(Z+Z_1dp), \quad v+v_1dp+\alpha(V+V_1dp),$$

as being on the latter normal: that is, we have the relations

$$x_1+\alpha X_1=0, \quad y_1+\alpha Y_1=0, \quad z_1+\alpha Z_1=0, \quad v_1+\alpha V_1=0,$$

and we take the coordinates of P to be

$$x+\alpha X, \quad y+\alpha Y, \quad z+\alpha Z, \quad v+\alpha V.$$

Similarly, the coordinates of Q are

$$x+\beta X, \quad y+\beta Y, \quad z+\beta Z, \quad v+\beta V,$$

and there are the associated relations

$$x_2+\beta X_2=0, \quad y_2+\beta Y_2=0, \quad z_2+\beta Z_2=0, \quad v_2+\beta V_2=0;$$

and the coordinates of R are

$$x+\gamma X, \quad y+\gamma Y, \quad z+\gamma Z, \quad v+\gamma V,$$

and there are the associated relations

$$x_3+\gamma X_3=0, \quad y_3+\gamma Y_3=0, \quad z_3+\gamma Z_3=0, \quad v_3+\gamma V_3=0.$$

The point P_2 is attained by measuring a distance $rP_2(=\alpha+\alpha_3dr)$ inwards along the normal at r. Thus, for P_2, the coordinate x is

$$=x+x_3dr+(\alpha+\alpha_3dr)(X+X_3dr)$$
$$=x+\alpha X+(x_3+\alpha X_3)dr+X\alpha_3dr=x+\alpha X+\left(1-\frac{\alpha}{\gamma}\right)x_3dr+X\alpha_3dr;$$

and similarly for the other coordinates y, z, v, so that P_2 is the point

$$x+\alpha X+\left(1-\frac{\alpha}{\gamma}\right)x_3dr+X\alpha_3dr, \quad y+\alpha Y+\left(1-\frac{\alpha}{\gamma}\right)y_3dr+Y\alpha_3dr, \ \ldots.$$

In the same way, P_3 is the point

$$x+\alpha X+\left(1-\frac{\alpha}{\beta}\right)x_2dq+X\alpha_2dq, \quad y+\alpha Y+\left(1-\frac{\alpha}{\beta}\right)y_2dq+Y\alpha_2dq, \ \ldots;$$

the point P_0 is

$$x+\alpha X+\left(1-\frac{\alpha}{\beta}\right)x_2dq+\left(1-\frac{\alpha}{\gamma}\right)x_3dr+X\alpha_2dq+X\alpha_3dr, \ \ldots,$$

* The direction-cosines X, Y, Z, V, are those of the normal to the region drawn inwards to the region (§ 269).

and the point P' is

$$x+\alpha X+\left(1-\frac{\alpha}{\beta}\right)x_2dq+\left(1-\frac{\alpha}{\gamma}\right)x_3dr+X\alpha_1dp+X\alpha_2dq+X\alpha_3dr,\ \ldots,$$

with corresponding values for the other three coordinates.

As the lines of reference are, by hypothesis, lines of curvature, we have

$$\Sigma x_2x_3=0,\quad \Sigma x_3x_1=0,\quad \Sigma x_1x_2=0.$$

Hence all the points P, P_2, P_3, P_0, P', lie in the flat

$$\Sigma x_1(\bar{x}-x)=0,$$

which manifestly also passes through the point O, and the three further points q, r, p': that is, it is the flat $POqp'rP'$, which is tangential to the first volume of the region of centres. Moreover, the equation of the flat shews, as is manifest from the simple geometry whereby Op is perpendicular to Oq, Or, OP, that the normal to the flat (and therefore the normal to the volume of the region of centres) is parallel to the corresponding line of curvature Op.

Ex. Shew that the x-coordinates of the points in the immediate vicinity of Q in the figure are

for Q, $x+\beta X$;

Q_1, $x+\beta X+x_3\left(1-\frac{\beta}{\gamma}\right)dr+X\beta_3dr$;

Q_3, $x+\beta X+x_1\left(1-\frac{\beta}{\alpha}\right)dp+X\beta_1dp$;

Q_0, $x+\beta X+x_1\left(1-\frac{\beta}{\alpha}\right)dp+x_3\left(1-\frac{\beta}{\gamma}\right)dr+X\beta_1dp+X\beta_3dr$;

Q', $x+\beta X+x_1\left(1-\frac{\beta}{\alpha}\right)dp+x_3\left(1-\frac{\beta}{\gamma}\right)dr+X\beta_1dp+X\beta_2dq+X\beta_3dr.$

Shew likewise that the x-coordinates of the points in the immediate vicinity of R in the figure are

for R, $x+\gamma X$;

R_1, $x+\gamma X+x_2\left(1-\frac{\gamma}{\beta}\right)dq+X\gamma_2dq$;

R_2, $x+\gamma X+x_1\left(1-\frac{\gamma}{\alpha}\right)dp+X\gamma_1dp$;

R_0, $x+\gamma X+x_1\left(1-\frac{\gamma}{\alpha}\right)dp+x_2\left(1-\frac{\gamma}{\beta}\right)dq+X\gamma_1dp+X\gamma_2dq$;

R', $x+\gamma X+x_1\left(1-\frac{\gamma}{\alpha}\right)dp+x_2\left(1-\frac{\gamma}{\beta}\right)dq+X\gamma_1dp+X\gamma_2dq+X\gamma_3dr.$

Similarly for the set of points associated with the second centres: they lie in a flat, which is tangential to the second volume and the normal to which is parallel to the line of curvature Oq. Likewise also for the set of points associated with the third centres: they lie in a flat which is tangential to the third volume and the normal to which is parallel to the line of curvature Or.

Clearly the tangential flats to the first volume and the second volume of the region of centres intersect in the line $OPQR$ and also in the line Or: that is, they intersect in the plane rOR. This plane contains the points P and P_2, and therefore touches the first volume along a tangent direction PP_2; but PP_2 and Or are not parallel, for their inclination is

$$\tan^{-1}\left(\frac{\gamma\alpha_3 C^{-\frac{1}{2}}}{\gamma-\alpha}\right).$$

This plane contains the points Q and Q_1, and therefore touches the second volume along a tangent direction QQ_1; but QQ_1 and Or are not parallel, for their inclination is

$$\tan^{-1}\left(\frac{\gamma\beta_3 C^{-\frac{1}{2}}}{\gamma-\beta}\right).$$

The tangential flats to the second volume and the third volume of the region of centres intersect in the plane pOP, with similar properties as to tangency of the second volume at Q in the direction QQ_3 and tangency of the third volume at R in the direction RR_2, neither of which directions is parallel to Op. Likewise, the tangential flats to the third volume and the first volume of the region of centres intersect in the plane qOQ, with similar properties as to tangency of the third volume at R in the direction RR_1 and tangency of the first volume at P in the direction PP_3, neither of which directions is parallel to Oq.

Moreover, each of the normals at the respective points touches each volume of the region of the centres, the points of respective contact being those points on the lines given in the tableau, when (and only when) the points are centres of curvature. Thus the points of contact of the normal at O are the three centres P, Q, R. The points of contact of the normal at p are Q_3, R_2, but not P; the point of contact with the first volume is the first centre of curvature at p which is at a distance along pP equal to $\alpha_1 dp$ beyond P. So for the points of contact of each of the other normals with the respective volumes: the points of contact of the normal at T being the points P', Q', R', respectively.

Principal magnitudes for the three volumes of the centro-region.

332. We now proceed to a more detailed consideration of the three volumes of the region of centres, each volume being a region in itself which possesses its own fundamental magnitudes of the first order and the second order in particular.

We denote the centres P, Q, R, by the coordinates $\xi, \eta, \zeta, \upsilon$; $\xi', \eta', \zeta', \upsilon'$; $\xi'', \eta'', \zeta'', \upsilon''$; respectively, so that

$$\begin{aligned}
\xi &= x+\alpha X, & \eta &= y+\alpha Y, & \zeta &= z+\alpha Z, & \upsilon &= v+\alpha V;\\
\xi' &= x+\beta X, & \eta' &= y+\beta Y, & \zeta' &= z+\beta Z, & \upsilon' &= v+\beta V;\\
\xi'' &= x+\gamma X, & \eta'' &= y+\gamma Y, & \zeta'' &= z+\gamma Z, & \upsilon'' &= v+\gamma V;
\end{aligned}$$

and there are the relations

$$x_1+\alpha X_1=0, \quad x_2+\beta X_2=0, \quad x_3+\gamma X_3=0,$$

with the corresponding relations affecting y and Y, z and Z, v and V.

For the first volume of the region of centres, we have

$$\xi_1=X\alpha_1, \quad \xi_2=\left(1-\frac{\alpha}{\beta}\right)x_2+X\alpha_2, \quad \xi_3=\left(1-\frac{\alpha}{\gamma}\right)x_3+X\alpha_3,$$

$$\eta_1=Y\alpha_1, \quad \eta_2=\left(1-\frac{\alpha}{\beta}\right)y_2+Y\alpha_2, \quad \eta_3=\left(1-\frac{\alpha}{\gamma}\right)y_3+Y\alpha_3,$$

$$\zeta_1=Z\alpha_1, \quad \zeta_2=\left(1-\frac{\alpha}{\beta}\right)z_2+Z\alpha_2, \quad \zeta_3=\left(1-\frac{\alpha}{\gamma}\right)z_3+Z\alpha_3,$$

$$v_1=V\alpha_1, \quad v_2=\left(1-\frac{\alpha}{\beta}\right)v_2+V\alpha_2, \quad v_3=\left(1-\frac{\alpha}{\gamma}\right)v_3+V\alpha_3.$$

The fundamental magnitudes of the first order for the first volume of the region of centres are

$$\bar{A}=\Sigma\xi_1^2=\alpha_1^2, \quad \bar{F}=\Sigma\xi_2\xi_3=\alpha_2\alpha_3,$$

$$\bar{B}=\Sigma\xi_2^2=\left(1-\frac{\alpha}{\beta}\right)^2B+\alpha_2^2, \quad \bar{G}=\Sigma\xi_3\xi_1=\alpha_3\alpha_1,$$

$$\bar{C}=\Sigma\xi_3^2=\left(1-\frac{\alpha}{\gamma}\right)^2C+\alpha_3^2, \quad \bar{H}=\Sigma\xi_1\xi_2=\alpha_1\alpha_2.$$

and therefore

$$\bar{\Omega}^2=\begin{vmatrix} \bar{A}, & \bar{H}, & \bar{G} \\ \bar{H}, & \bar{B}, & \bar{F} \\ \bar{G}, & \bar{F}, & \bar{C} \end{vmatrix}=BC\alpha_1^2\left(1-\frac{\alpha}{\beta}\right)^2\left(1-\frac{\alpha}{\gamma}\right)^2.$$

The direction-cosines of the normal to the first volume are

$$\frac{1}{\bar{\Omega}}\begin{Vmatrix} \xi_1, & \eta_1, & \zeta_1, & v_1 \\ \xi_2, & \eta_2, & \zeta_2, & v_2 \\ \xi_3, & \eta_3, & \zeta_3, & v_3 \end{Vmatrix}$$

$$=\frac{1}{\bar{\Omega}}\alpha_1\begin{Vmatrix} X, & Y, & Z, & V \\ x_2, & y_2, & z_2, & v_2 \\ x_3, & y_3, & z_3, & v_3 \end{Vmatrix}\left(1-\frac{\alpha}{\beta}\right)\left(1-\frac{\alpha}{\gamma}\right)$$

$$=\begin{Vmatrix} X, & Y, & Z, & V \\ x_2B^{-\frac{1}{2}}, & y_2B^{-\frac{1}{2}}, & z_2B^{-\frac{1}{2}}, & v_2B^{-\frac{1}{2}} \\ x_3C^{-\frac{1}{2}}, & y_3C^{-\frac{1}{2}}, & z_3C^{-\frac{1}{2}}, & v_3C^{-\frac{1}{2}} \end{Vmatrix}.$$

Now X, Y, Z, V; $x_1A^{-\frac{1}{2}}, y_1A^{-\frac{1}{2}}, z_1A^{-\frac{1}{2}}, v_1A^{-\frac{1}{2}}$; $x_2B^{-\frac{1}{2}}, y_2B^{-\frac{1}{2}}, z_2B^{-\frac{1}{2}}, v_2B^{-\frac{1}{2}}$; $x_3C^{-\frac{1}{2}}, y_3C^{-\frac{1}{2}}, z_3C^{-\frac{1}{2}}, v_3C^{-\frac{1}{2}}$; are an orthogonal frame of lines; and

$$\begin{vmatrix} X\ , & Y\ , & Z\ , & V \\ x_1A^{-\frac{1}{2}}, & y_1A^{-\frac{1}{2}}, & z_1A^{-\frac{1}{2}}, & v_1A^{-\frac{1}{2}} \\ x_2B^{-\frac{1}{2}}, & y_2B^{-\frac{1}{2}}, & z_2B^{-\frac{1}{2}}, & v_2B^{-\frac{1}{2}} \\ x_3C^{-\frac{1}{2}}, & y_3C^{-\frac{1}{2}}, & z_3C^{-\frac{1}{2}}, & v_3C^{-\frac{1}{2}} \end{vmatrix} = 1.$$

Hence

$$\begin{Vmatrix} X\ , & Y\ , & Z\ , & V \\ x_2B^{-\frac{1}{2}}, & y_2B^{-\frac{1}{2}}, & z_2B^{-\frac{1}{2}}, & v_2B^{-\frac{1}{2}} \\ x_3C^{-\frac{1}{2}}, & y_3C^{-\frac{1}{2}}, & z_3C^{-\frac{1}{2}}, & v_3C^{-\frac{1}{2}} \end{Vmatrix} = -x_1A^{-\frac{1}{2}}, -y_1A^{-\frac{1}{2}}, -z_1A^{-\frac{1}{2}}, -v_1A^{-\frac{1}{2}}:$$

that is, the positive direction of the normal at P, inwards to its region (§ 269), is parallel but opposite to that of the associated line of curvature.

It thus appears that, though the normal to the first volume is parallel to a line of curvature, the lines on that first volume corresponding to the lines of curvature are not even perpendicular to one another. Consequently, such lines are certainly not the lines of curvature in the first volume of the region.

The length of a rudimentary arc in the first volume is given by

$$d\bar{s}^2 = da^2 + \left(1 - \frac{\alpha}{\beta}\right)^2 B\,dq^2 + \left(1 - \frac{\alpha}{\gamma}\right)^2 C\,dr^2;$$

and therefore (§ 303) the curves

$$q = \text{constant}, \quad r = \text{constant},$$

(that is, the curves corresponding to the associated Op-set of lines of curvature in the original region), are geodesics in that first volume.

For the fundamental magnitudes of the second order, which belong to the first region, we have*

$$\xi_{11} = X\alpha_{11} - x_1\frac{\alpha_1}{\alpha},$$

$$\xi_{12} = X\alpha_{12} - x_2\frac{\alpha_1}{\beta},$$

$$\xi_{13} = X\alpha_{13} - x_3\frac{\alpha_1}{\gamma},$$

$$\xi_{22} = x_{22}\left(1 - \frac{\alpha}{\beta}\right) - x_2\left\{\frac{\partial}{\partial q}\left(\frac{\alpha}{\beta}\right) + \frac{\alpha_2}{\beta}\right\} + X\alpha_{22},$$

$$\xi_{23} = x_{23}\left(1 - \frac{\alpha}{\beta}\right) - x_2\frac{\partial}{\partial r}\left(\frac{\alpha}{\beta}\right) - x_3\frac{\alpha_2}{\gamma} + X\alpha_{23},$$

$$\xi_{33} = x_{33}\left(1 - \frac{\alpha}{\gamma}\right) - x_3\left\{\frac{\partial}{\partial r}\left(\frac{\alpha}{\gamma}\right) + \frac{\alpha_3}{\gamma}\right\} + X\alpha_{33}.$$

* Apparently, two differing values of a quantity such as ξ_{12} are given by $\frac{\partial \xi_1}{\partial q}$ and $\frac{\partial \xi_2}{\partial p}$; they are easily verified to be equal, by means of the extended Codazzi-Mainardi relations.

Now the direction-cosines of the positive direction (inward drawn) normal to the first volume are $-x_1A^{-\frac{1}{2}}, -y_1A^{-\frac{1}{2}}, -z_1A^{-\frac{1}{2}}, -v_1A^{-\frac{1}{2}}$; hence

$$\bar{L} = -\Sigma A^{-\frac{1}{2}} x_1 \xi_{11} = A^{\frac{1}{2}} \frac{\alpha_1}{\alpha},$$

$$\bar{N} = -\Sigma A^{-\frac{1}{2}} x_1 \xi_{22} = -A^{-\frac{1}{2}} \left(1 - \frac{\alpha}{\beta}\right) \Sigma x_1 x_{22}$$

$$= A^{\frac{1}{2}} \left(1 - \frac{\alpha}{\beta}\right) B_1$$

$$= -\frac{B}{A^{\frac{1}{2}}} \frac{\alpha}{\beta^2} \beta_1,$$

after reduction by means of the Codazzi-Mainardi relations; and

$$\bar{J} = -\Sigma A^{-\frac{1}{2}} x_1 \xi_{33} = -A^{-\frac{1}{2}} \left(1 - \frac{\alpha}{\gamma}\right) \Sigma x_1 x_{33}$$

$$= -\frac{C}{A^{\frac{1}{2}}} \frac{\alpha}{\gamma^2} \gamma_1.$$

Similarly

$$\bar{I} = -\Sigma A^{-\frac{1}{2}} x_1 \xi_{23} = -A^{-\frac{1}{2}} \left(1 - \frac{\alpha}{\beta}\right) \Sigma x_1 x_{23} = 0,$$

$$\bar{K} = -\Sigma A^{-\frac{1}{2}} x_1 \xi_{31} = 0,$$

$$\bar{M} = -\Sigma A^{-\frac{1}{2}} x_1 \xi_{12} = 0.$$

It follows, from the vanishing values of $\bar{I}, \bar{K}, \bar{M}$, that the lines in the first volume of the region of centres, which correspond to the lines of curvature in the original region, are conjugate to one another. We have already seen above that they are not perpendicular to one another.

Again, the circular curvature of a normal (geodesic) section of the first volume of the centro-region is equal to

$$A^{-\frac{1}{2}} \alpha \frac{A \frac{\alpha_1}{\alpha^2} dp^2 - B \frac{\beta_1}{\beta^2} dq^2 - C \frac{\gamma_1}{\gamma^2} dr^2}{d\alpha^2 + \left(1 - \frac{\alpha}{\beta}\right)^2 B dq^2 + \left(1 - \frac{\alpha}{\gamma}\right)^2 C dr^2}.$$

The three principal measures of curvature of the first volume are given by the customary formulæ of § 285; in particular, the volumetric curvature of that first volume is equal to

$$\frac{\bar{L}\bar{N}\bar{J}}{\bar{\Omega}^2} = \frac{\beta_1 \gamma_1 \alpha}{\alpha_1 A^{\frac{1}{2}}} \frac{1}{(\alpha - \beta)^2 (\alpha - \gamma)^2}.$$

The directions of the lines of curvature in the first volume of the region of centres, for which $\bar{I}=0$, $\bar{K}=0$, $\bar{M}=0$, are given by the equations (§ 283)

$$\frac{\bar{A}\,dp+\bar{H}\,dq+\bar{G}\,dr}{\bar{L}\,dp}=\frac{\bar{H}\,dp+\bar{B}\,dq+\bar{F}\,dr}{\bar{N}\,dq}=\frac{\bar{G}\,dp+\bar{F}\,dq+\bar{C}\,dr}{\bar{J}\,dr},$$

the common value of the fractions being the radius of curvature of the region along the line of curvature.

333. Similar results hold for the second volume and for the third volume of the region of centres.

In fact, for the second volume of that region, we have

$$\bar{A}'=A\left(1-\frac{\beta}{\alpha}\right)^2+\beta_1^2, \quad \bar{F}'=\beta_2\beta_3,$$

$$\bar{B}'=\beta_2^2 \quad , \quad \bar{G}'=\beta_3\beta_1,$$

$$\bar{C}'=C\left(1-\frac{\beta}{\gamma}\right)^2+\beta_3^2, \quad \bar{H}'=\beta_1\beta_2,$$

so that the element of arc in that second volume is given by

$$d\bar{s}'^2=\left(1-\frac{\beta}{\alpha}\right)^2 A\,dp^2+d\beta^2+\left(1-\frac{\beta}{\gamma}\right)^2 C\,dr^2;$$

and therefore the curves in this second volume, corresponding to the Oq-set of lines of curvature in the original region, are geodesics in this second volume. Further,

$$\bar{L}'=-\frac{A}{B^{\frac{1}{2}}}\frac{\beta}{\alpha^2}\alpha_2, \quad \bar{I}'=0,$$

$$\bar{N}'= \quad B^{\frac{1}{2}}\frac{\beta_2}{\beta} \quad , \quad \bar{K}'=0,$$

$$\bar{J}'=-\frac{C}{B^{\frac{1}{2}}}\frac{\beta}{\gamma^2}\gamma_2, \quad \bar{M}'=0,$$

so that the lines in the second volume of the region of centres, corresponding to the lines of curvature in the original region, are conjugate to one another. The circular curvature of a normal section of the second volume of the region of centres is

$$B^{-\frac{1}{2}}\beta\,\frac{-A\frac{\alpha_2}{\alpha^2}dp^2+B\frac{\beta_2}{\beta^2}dq^2-C\frac{\gamma_2}{\gamma^2}dr^2}{\left(1-\frac{\beta}{\alpha}\right)^2 A\,dp^2+d\beta^2+\left(1-\frac{\beta}{\gamma}\right)^2 C\,dr^2};$$

the volumetric measure of curvature of this second volume is

$$\frac{\gamma_2\alpha_2\beta}{\beta_2 B^{\frac{1}{2}}}\frac{1}{(\gamma-\beta)^2(\alpha-\beta)^2},$$

and so for the other principal measures of curvature of the second volume. Its lines of curvature are given by the equations

$$\frac{\bar{A}'dp+\bar{H}'dq+\bar{G}'dr}{\bar{L}'dp}=\frac{\bar{H}'dp+\bar{B}'dq+\bar{F}'dr}{\bar{N}'dq}=\frac{\bar{G}'dp+\bar{F}'dq+\bar{C}'dr}{\bar{J}'dr}.$$

Likewise for the third volume, we have

$$\bar{A}''=A\left(1-\frac{\gamma}{\alpha}\right)^2+\gamma_1^2, \quad \bar{F}''=\gamma_2\gamma_3,$$

$$\bar{B}''=B\left(1-\frac{\gamma}{\beta}\right)^2+\gamma_2^2, \quad \bar{G}''=\gamma_3\gamma_1,$$

$$\bar{C}''=\gamma_3^2 \quad , \quad \bar{H}''=\gamma_1\gamma_2,$$

$$d\bar{s}''^2=\left(1-\frac{\gamma}{\alpha}\right)^2 A\,dp^2+\left(1-\frac{\gamma}{\beta}\right)^2 B\,dq^2+d\gamma^2;$$

$$\bar{L}''=-\frac{A}{C^{\frac{1}{2}}}\frac{\gamma}{\alpha^2}\alpha_3 \quad , \quad \bar{I}''=0 \quad ,$$

$$\bar{N}''=-\frac{B}{C^{\frac{1}{2}}}\frac{\gamma}{\beta^2}\beta_3 \quad , \quad \bar{K}''=0 \quad ,$$

$$\bar{J}''=C^{\frac{1}{2}}\frac{\gamma_3}{\gamma} \quad , \quad \bar{M}''=0 \quad ,$$

with the corresponding inferences. In particular, the equations of its lines of curvature are

$$\frac{\bar{A}''dp+\bar{H}''dq+\bar{G}''dr}{\bar{L}''dp}=\frac{\bar{H}''dp+\bar{B}''dq+\bar{F}''dr}{\bar{N}''dq}=\frac{\bar{G}''dp+\bar{F}''dq+\bar{C}''dr}{\bar{J}''dr}.$$

Lines of curvature and inflexional surfaces of the centro-region.

334. It has already been noted that, in no one of the three volumes of the centro-region, do the lines of curvature correspond with the lines of curvature of the original region: a result which follows from the fact that the parametric lines in any one of the volumes are not perpendicular to one another, and which is clear also from the fact that the parametric lines do not satisfy the equations

$$\frac{\bar{A}dp+\bar{H}dq+\bar{G}dr}{\bar{L}dp}=\frac{\bar{H}dp+\bar{B}dq+\bar{F}dr}{\bar{N}dq}=\frac{\bar{G}dp+\bar{F}dq+\bar{C}dr}{\bar{J}dr},$$

characteristic of the lines of curvature in a volume.

There might, however, be the possibility that these lines of curvature in the different volumes, though they do not correspond with the lines of curvature of the original region, may correspond with one another: that is, may be

given by the same two equations between the parameters p, q, r. If this possibility were a reality, the equations

$$\left.\begin{aligned} \bar{L}\bar{H}\,dp^2+(\bar{L}\bar{B}-\bar{N}\bar{A})\,dp\,dq+\bar{L}\bar{F}\,dp\,dr-\bar{N}\bar{H}\,dq^2-\bar{N}\bar{G}\,dq\,dr &=0\\ \bar{L}\bar{G}\,dp^2+\bar{L}\bar{F}\,dp\,dq+(\bar{L}\bar{C}-\bar{J}\bar{A})\,dp\,dr-\bar{J}\bar{H}\,dq\,dr-\bar{J}\bar{G}\,dr^2 &=0 \end{aligned}\right\},$$

which hold for the first volume, would agree with the pair which hold for the second volume and would agree also with the pair which hold for the third volume: and the analytical condition is that the coefficients in the equations of the respective pairs have the same ratios. Now a comparison of the first of the equations taken from the respective pairs, in accordance with this permissive condition, requires the relations (among others)

$$\frac{\bar{H}'}{\bar{H}}=\frac{\bar{G}'}{\bar{G}}=\frac{\bar{F}'}{\bar{F}},\quad \frac{\bar{L}'}{\bar{L}}=\frac{\bar{N}'}{\bar{N}},$$

from the first and second volumes, and the relations (among others)

$$\frac{\bar{H}''}{\bar{H}}=\frac{\bar{G}''}{\bar{G}}=\frac{\bar{F}''}{\bar{F}},\quad \frac{\bar{L}''}{\bar{L}}=\frac{\bar{N}''}{\bar{N}},$$

from the first and third volumes. There is no difficulty in finding that these equations are not consistent with one another. Thus the permissive condition is not satisfied: the lines of curvature in the three volumes do not correspond with one another.

It is easy to see similarly that the inflexional surfaces of the three volumes are not alike; their equations are

$$\begin{aligned} \bar{L}\,dp^2+\bar{N}\,dq^2+\bar{J}\,dr^2&=0,\\ \bar{L}'\,dp^2+\bar{N}'\,dq^2+\bar{J}'\,dr^2&=0,\\ \bar{L}''dp^2+\bar{N}''dq^2+\bar{J}''dr^2&=0, \end{aligned}$$

no two of which are the same.

Nul surfaces in a region: a relation with geodesic surfaces.

335. We have seen (§ 314) that a surface $\theta(p,q,r)=$ constant is a geodesic surface if, for every direction in the surface as given by the equation

$$\theta_1 p'+\theta_2 q'+\theta_3 r'=0,$$

the equation

$$\gamma_{11}p'^2+2\gamma_{12}p'q'+2\gamma_{13}p'r'+\gamma_{22}q'^2+2\gamma_{23}q'r'+\gamma_{33}r'^2=0$$

is satisfied, where

$$\gamma_{lm}=\theta_{lm}-\Gamma_{lm}\theta_1-\Delta_{lm}\theta_2-\Theta_{lm}\theta_3.$$

In order that the second equation, as involving p', q', r', may be the product of two factors linear in those variables, the condition

$$\begin{vmatrix} \gamma_{11}, & \gamma_{12}, & \gamma_{13}\\ \gamma_{12}, & \gamma_{22}, & \gamma_{23}\\ \gamma_{13}, & \gamma_{23}, & \gamma_{33} \end{vmatrix}=0$$

must be satisfied. The general conditions, that one of the linear factors may be the quantity $\theta_1 p' + \theta_2 q' + \theta_3 r'$, are

$$\left.\begin{aligned} 2\frac{\gamma_{23}}{\theta_2\theta_3} &= \frac{\gamma_{22}}{\theta_2^2} + \frac{\gamma_{33}}{\theta_3^2} \\ 2\frac{\gamma_{31}}{\theta_3\theta_1} &= \frac{\gamma_{11}}{\theta_1^2} + \frac{\gamma_{33}}{\theta_3^2} \\ 2\frac{\gamma_{12}}{\theta_1\theta_2} &= \frac{\gamma_{11}}{\theta_1^2} + \frac{\gamma_{22}}{\theta_2^2} \end{aligned}\right\};$$

and it is easy to verify that these three conditions cover the former single condition. The special conditions, that both the linear factors are the same and that their common valué is $\theta_1 p' + \theta_2 q' + \theta_3 r'$, are

$$\frac{\gamma_{11}}{\theta_1^2} = \frac{\gamma_{12}}{\theta_1\theta_2} = \frac{\gamma_{13}}{\theta_1\theta_3} = \frac{\gamma_{22}}{\theta_2^2} = \frac{\gamma_{23}}{\theta_2\theta_3} = \frac{\gamma_{33}}{\theta_3^2};$$

and these five conditions cover the former three conditions.

When this last set of conditions is used, the common value of the six fractions will be denoted by Θ.

Again, it was proved (§ 268) that, if $d\Sigma$ be an element of area upon the surface $\theta(p, q, r) = \text{constant}$ with orientation-coordinates $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$, we have

$$\bar{\xi} D^{\frac{1}{2}} = \theta_1 d\Sigma, \quad \bar{\eta} D^{\frac{1}{2}} = \theta_2 d\Sigma, \quad \bar{\zeta} D^{\frac{1}{2}} = \theta_3 d\Sigma,$$

where

$$D = a\theta_1^2 + 2h\theta_1\theta_2 + 2g\theta_1\theta_3 + b\theta_2^2 + 2f\theta_2\theta_3 + c\theta_3^2.$$

Accordingly, if the surface be a *nul surface*, meaning thereby a surface everywhere of zero area—the extension to surfaces of the notion of nul lines of zero length, both being imaginary when the variables are real—then $d\Sigma$ is zero though $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$, are not zero; and the corresponding condition then is

$$D = 0:$$

that is, this equation is characteristic of nul surfaces.

Now we have

$$\frac{1}{2}\frac{\partial D}{\partial p} = \frac{1}{2}(a_1, b_1, c_1, f_1, g_1, h_1 \mathbin{\text{\textbigcircle}} \theta_1, \theta_2, \theta_3)^2$$
$$+ (a\theta_1 + h\theta_2 + g\theta_3)\,\theta_{11} + (h\theta_1 + b\theta_2 + f\theta_3)\,\theta_{12} + (g\theta_1 + f\theta_2 + c\theta_3)\,\theta_{13}.$$

By the use of the formulæ in § 273, the first line

$$= \frac{\Omega_1}{\Omega}(a, b, c, f, g, h \mathbin{\text{\textbigcircle}} \theta_1, \theta_2, \theta_3)^2$$
$$- \theta_1^2(a\Gamma_{11} + h\Gamma_{12} + g\Gamma_{13}) - \theta_2^2(h\Delta_{11} + b\Delta_{12} + f\Delta_{13}) - \theta_3^2(g\Theta_{11} + f\Theta_{12} + c\Theta_{13})$$
$$- \theta_2\theta_3\{(g\Delta_{11} + f\Delta_{12} + c\Delta_{13}) + (h\Theta_{11} + b\Theta_{12} + f\Theta_{13})\}$$
$$- \theta_3\theta_1\{(a\Theta_{11} + h\Theta_{12} + g\Theta_{13}) + (g\Gamma_{11} + f\Gamma_{12} + c\Gamma_{13})\}$$
$$- \theta_1\theta_2\{(h\Gamma_{11} + b\Gamma_{12} + f\Gamma_{13}) + (a\Delta_{11} + h\Delta_{12} + g\Delta_{13})\}$$

$$= \frac{\Omega_1}{\Omega} D - (a\theta_1 + h\theta_2 + g\theta_3)(\Gamma_{11}\theta_1 + \Delta_{11}\theta_2 + \Theta_{11}\theta_3)$$
$$- (h\theta_1 + b\theta_2 + f\theta_3)(\Gamma_{12}\theta_1 + \Delta_{12}\theta_2 + \Theta_{12}\theta_3)$$
$$- (g\theta_1 + f\theta_2 + c\theta_3)(\Gamma_{13}\theta_1 + \Delta_{13}\theta_2 + \Theta_{13}\theta_3);$$

and therefore

$$\frac{1}{2}\frac{\partial D}{\partial p} = \frac{\Omega_1}{\Omega} D + (a\theta_1 + h\theta_2 + g\theta_3)\gamma_{11}$$
$$+ (h\theta_1 + b\theta_2 + f\theta_3)\gamma_{12}$$
$$+ (g\theta_1 + f\theta_2 + c\theta_3)\gamma_{13}.$$

When the surface $\theta(p, q, r) = \text{constant}$ is a geodesic surface such that the special conditions connecting its two intrinsic equations are satisfied (p. 174), the right-hand side becomes

$$\frac{\Omega_1}{\Omega} D + \theta_1 \Theta D.$$

Similarly, for the other derivatives of D: thus we have

$$\left.\begin{aligned} \frac{1}{2}\frac{\partial D}{\partial p} &= \left(\frac{\Omega_1}{\Omega} + \Theta\theta_1\right) D \\ \frac{1}{2}\frac{\partial D}{\partial q} &= \left(\frac{\Omega_2}{\Omega} + \Theta\theta_2\right) D \\ \frac{1}{2}\frac{\partial D}{\partial r} &= \left(\frac{\Omega_3}{\Omega} + \Theta\theta_3\right) D \end{aligned}\right\},$$

for the special geodesic surfaces in question.

These three equations are satisfied by the relation

$$D = 0,$$

which is characteristic of nul surfaces; and therefore we infer that nul surfaces are, analytically, a special form of geodesic surfaces.

This property is an analogue of the property that, on a surface, nul lines satisfy the equations for geodesic lines.

Examples of nul surfaces.

336. The equation, characteristic of nul surfaces, can be obtained in another manner. Let the surface be

$$\theta(p, q, r) = 0$$

as before; and also as in § 355 (*post*), let $\bar{E}$, $\bar{F}$, $\bar{G}$, be the fundamental quantities of the first order, so that

$$\theta_3^2 \bar{V}^2 = D.$$

Now the expression for the elementary area on the surface is

$$(\bar{E}\bar{G} - \bar{F}^2)^{\frac{1}{2}} dp\, dq, \quad = \bar{V}\, dp\, dq,$$

so that a nul surface can be given by the equation

$$\overline{V} = 0.$$

Consequently, the equation $D = 0$, that is,

$$(a, b, c, f, g, h\mathcal{Q}\theta_1, \theta_2, \theta_3)^2 = 0,$$

can be regarded as the partial differential equation of nul surfaces.

Ex. 1. The globular region

$$x^2+y^2+z^2+v^2=a^2$$

is parametrically represented (§ 302, Ex. 1) by

$$x=a\sin p\sin q, \quad y=a\sin p\cos q, \quad z=a\cos p\cos r, \quad v=a\cos p\sin r;$$

and the fundamental magnitudes of the first order then are

$$A=a^2, \quad B=a^2\sin^2 p, \quad C=a^2\cos^2 p, \quad F=0=G=H.$$

The equation of nul surfaces in the globular region therefore is

$$\theta_1^2+\frac{\theta_2^2}{\sin^2 p}+\frac{\theta_3^2}{\cos^2 p}=0;$$

and the general integral of this equation is

$$\theta=i(aq+\beta r)+\int\frac{(a^2\cos^2 p+\beta^2\sin^2 p)^{\frac{1}{2}}}{\sin p\cos p}\,dp+\gamma$$

$$=i(aq+\beta r)+\frac{1}{2}\left(a\log\frac{a-P}{a+P}-\beta\log\frac{\beta-P}{\beta+P}\right)+\gamma,$$

where

$$P=(a^2\cos^2 p+\beta^2\sin^2 p)^{\frac{1}{2}}.$$

Among these nul surfaces, there are three special families, as follows:

(i) When $a=0$, the family can be taken in the form

$$e^{-ir}(\sec p-\tan p)=\text{constant};$$

(ii) When $\beta=0$, the family can be taken in the form

$$e^{-iq}(\operatorname{cosec} p+\cot p)=\text{constant};$$

(iii) When $a=\beta$, the family can be taken in the form

$$e^{i(q+r)}\tan p=\text{constant}.$$

Ex. 2. Prove that, when the sphero-conical region

$$v^2=(x^2+y^2+z^2)\tan^2 a$$

is parametrically represented by the equations

$$x=r\sin\theta\cos\phi, \quad y=r\sin\theta\sin\phi, \quad z=r\cos\theta, \quad v=r\tan a,$$

two families of nul surfaces are given by the equations

$$re^{-i\theta\cos a}=\text{constant}, \quad e^{-i\phi}\tan\tfrac{1}{2}\theta=\text{constant}.$$

Ex. 3. Obtain the equation of nul surfaces in the minimal region (§ 417) of axial revolution

$$\frac{x^2+y^2+z^2}{a^2}=\wp\left(\frac{v}{a}\right),$$

(where $\wp$ denotes the Weierstrass elliptic function, and the invariants g_2. g_3, $=4, 0$) in the form

$$(x^2+y^2+z^2)^3(u_x^2+u_y^2+u_z^2)=a^4(xu_x+yu_y+zu_z)^2;$$

and verify that the general integral of this equation is

$$u=\tfrac{1}{2}a\sin^{-1}\frac{r^2}{a^2}+\beta\phi+at-\beta\tan^{-1}\left(\frac{a}{\beta}\tan t\right),$$

where

$$(a^2-\beta^2)^{\frac{1}{2}}\cos t=a\cos\theta,$$

$$x=r\sin\theta\cos\phi,\quad y=r\sin\theta\sin\phi,\quad z=r\cos\theta,$$

and a, β, are arbitrary constants.

Shew that two particular families of nul surfaces are given by

$$e^{\psi}\tan\tfrac{1}{2}\theta=\text{constant},$$

$$2\theta+\sin^{-1}\frac{r^2}{a^2}=\text{constant}.$$

CHAPTER XX.

SURFACES IN A REGION: GEODESICS

Surfaces in a region.

337. When surfaces were discussed at an earlier stage (Chaps. XII–XIV), they were propounded by a diparametric representation

$$x = x(p, q), \quad y = y(p, q), \quad z = z(p, q), \quad v = v(p, q);$$

and all their properties were derived in relation to the existence of the surfaces in quadruple homaloidal space. In particular, the curvature properties were considered in the same way as the curvature properties of skew curves were considered: that is, ultimately by the deviation of the curve on the one hand, and of a geodesic curve in the surface on the other hand, from a line or a plane or a flat. No particular account was taken of any curved region in which the surface might lie; thus the foregoing general surface might have been taken as the intersection, complete or partial, of two columnar regions

$$f(x, y, z) = 0, \quad g(x, y, v) = 0,$$

obtained by the elimination of p and q between x, y, z, and x, y, v, respectively. But once the diparametric representation is given, without any indication of its source, no immediate advantage seems to accrue from regarding the surface as the intersection of the two columnar regions.

When, however, a curved region* is in evidence, and after its intrinsic properties have been established in connection with a triparametric representation, it is necessary to consider surfaces lying in that region. For example, it has already (§ 315) proved necessary to consider geodesic surfaces in the region, in order to obtain a measure of superficial curvature of the region in any orientation at a point; and these are a very special class of surfaces, each organically determinate by the intrinsic properties of the region. We proceed to the consideration of arbitrarily assumed surfaces, which lie within the region; they are supposed to be of a general character, without any special organic association with properties of the region.

There is an immediate necessity of admitting notions, not entirely novel when regarded in their general significance, but non-occurrent when the surface was regarded solely as a configuration in quadruple space. In con-

* It is unnecessary to consider a flat region in detail; for (as in § 228) the flat region can be transformed to $v = 0$, and the theory of surfaces in that flat becomes the usual Gauss theory.

ceptions of curvature, we usually have dealt with deviations of a curve from straightness, of a surface from evenness, or of a region from flatness. The measure of curvature of a curve, for example, is taken to be its arc-rate of deviation from a linear tangent; and the deviation is estimated as the angle between two successive linear tangents. But, in a general triple region, the straight lines and the even planes of homaloidal triple space do not exist; yet measures of deviation, occurrent within the region, must be framed in order to stage the knowledge of properties of surfaces within, and relative to, the region alone. These measures are constructed by reference to the curves which, in the region, are analogous to the lines in the homaloidal space: such curves, of course, are the geodesics of the region. Further, it will be convenient here, as before for all curvature properties, to refer the region itself to its own geodesics; so that we shall be led to consider the relation of superficial geodesics (that is, geodesics on a surface) to regional geodesics (that is, geodesics of the region in which the surface lies).

338. There is one consideration which must be noted in passing, for it cannot be ignored. It is not to be expected (and it is not the fact) that the results obtained for a surface, which exists as a configuration in free homaloidal quadruple space, are the same as the results obtained for a surface, which exists as a configuration enfolded within a region in the same quadruple space. They may not even correspond in substantial detail. Agreement can emerge, only if the spatial configuration should happen to be enfolded within the region. That this precedent condition usually is not satisfied can be seen, analytically, as follows.

Let u and w be the parameters for the spatial surface. The three equations

$$x(u, w) = x(p, q, r),$$
$$y(u, w) = y(p, q, r),$$
$$z(u, w) = z(p, q, r),$$

potentially serve to determine p, q, r, as specific functions of u and w. When these functions are substituted in the parametric expression for the fourth coordinate $v(p, q, r)$ in the region, they usually do not provide a pre-assigned function of u and w, so as to give the value of the fourth coordinate of a point on the surface as a spatial configuration.

Should the condition be satisfied, the formulæ having a common significance can be brought into agreement by sheer algebraical manipulation. Even so, there are results in the second configuration, which do not arise in the first, because there is no indication of the existence (if there be existence) of an enfolding region. The two sets of analytical investigations will therefore be distinct in general.

Analytical representation of a surface in a region.

339. Accordingly, we consider surfaces in a curved region which, initially, has been given by a non-linear equation $F(x, y, z, v) = 0$ and, for present purposes, is algebraically expressed by the preceding triparametric representation of the point-coordinates x, y, z, v, in terms of p, q, r. A general surface, lying wholly within the region, may be given as the intersection of that region by another curved region, indicated initially by another non-linear equation $G(x, y, z, v) = 0$. To secure its occurrence within the analytical scheme of the region, we substitute for x, y, z, v, in G their regional values in terms of the three parameters; if

$$G(x, y, z, v) = \theta(p, q, r),$$

so that θ is a definite function of p, q, r, then the surface is given by the relation

$$\theta(p, q, r) = 0.$$

The properties of such a surface, partly in reference to the enfolding quadruple space, partly in reference to the enclosing region, have to be established.

At any point on the surface, which is common to the two regions, let ω be the angle between the tangent flat to the new region $G = 0$ and the tangent flat to the given enclosing region $F = 0$, so that

$$XG_x + YG_y + ZG_z + VG_v = \Pi \cos\omega,$$

where

$$\Pi^2 = G_x^2 + G_y^2 + G_z^2 + G_v^2.$$

Also, we have

$$x_l G_x + y_l G_y + z_l G_z + v_l G_v = \theta_l,$$

for $l = 1, 2, 3$. Let four quantities P, Q, R, W', be taken so that

$$\begin{aligned} G_x &= x_1 P + x_2 Q + x_3 R + X W', \\ G_y &= y_1 P + y_2 Q + y_3 R + Y W', \\ G_z &= z_1 P + z_2 Q + z_3 R + Z W', \\ G_v &= v_1 P + v_2 Q + v_3 R + V W'. \end{aligned}$$

Then multiplying by x_1, y_1, z_1, v_1, and adding; by x_2, y_2, z_2, v_2, and adding; by x_3, y_3, z_3, v_3, and adding; and by X, Y, Z, V, and adding; we have

$$\begin{aligned} AP + HQ + GR &= \theta_1, \\ HP + BQ + FR &= \theta_2, \\ GP + FQ + CR &= \theta_3, \\ W' &= \Pi \cos\omega. \end{aligned}$$

Hence

$$\left.\begin{aligned} \Omega^2 P &= a\theta_1 + h\theta_2 + g\theta_3 \\ \Omega^2 Q &= h\theta_1 + b\theta_2 + f\theta_3 \\ \Omega^2 R &= g\theta_1 + f\theta_2 + c\theta_3 \end{aligned}\right\};$$

and

$$\begin{aligned}\Pi^2 &= G_x^2 + G_y^2 + G_z^2 + G_v^2 \\ &= W'^2 + (A, B, C, F, G, H \mathcal{Y} P, Q, R)^2 \\ &= W'^2 + \frac{1}{\Omega^2}(a, b, c, f, g, h \mathcal{Y} \theta_1, \theta_2, \theta_3)^2,\end{aligned}$$

so that

$$\Pi^2 \sin^2\omega = \frac{1}{\Omega^2}(a, b, c, f, g, h \mathcal{Y} \theta_1, \theta_2, \theta_3) = \left(\frac{d\theta}{d\nu}\right)^2,$$

by § 268; and therefore

$$\Pi \sin\omega = \frac{d\theta}{d\nu}.$$

Regional normal to the surface.

340. For properties, intrinsic to the surface $\theta(p, q, r) = 0$, the quantities Π and ω will usually cease to appear. Meanwhile, it is to be noted that the former concomitant (§ 268)

$$D = (a, b, c, f, g, h \mathcal{Y} \theta_1, \theta_2, \theta_3)^2$$

has recurred. Moreover, it is convenient to write

$$\begin{aligned}\Omega(x_1 P + x_2 Q + x_3 R) &= \overline{X} D^{\frac{1}{2}}, \\ \Omega(y_1 P + y_2 Q + y_3 R) &= \overline{Y} D^{\frac{1}{2}}, \\ \Omega(z_1 P + z_2 Q + z_3 R) &= \overline{Z} D^{\frac{1}{2}}, \\ \Omega(v_1 P + v_2 Q + v_3 R) &= \overline{V} D^{\frac{1}{2}},\end{aligned}$$

so that

$$\overline{X}^2 + \overline{Y}^2 + \overline{Z}^2 + \overline{V}^2 = 1,$$

and we may take $\overline{X}$, $\overline{Y}$, $\overline{Z}$, $\overline{V}$, as direction-cosines. Also

$$\Omega\Pi \sin\omega = D^{\frac{1}{2}}, \quad \Sigma X \overline{X} = 0.$$

Hence

$$\left.\begin{aligned}\frac{G_x}{\Pi} &= X\cos\omega + \overline{X}\sin\omega \\ \frac{G_y}{\Pi} &= Y\cos\omega + \overline{Y}\sin\omega \\ \frac{G_z}{\Pi} &= Z\cos\omega + \overline{Z}\sin\omega \\ \frac{G_v}{\Pi} &= V\cos\omega + \overline{V}\sin\omega\end{aligned}\right\},$$

thus giving the direction-cosines of a normal to the region $G = 0$.

The quantity denoted by $d\nu$ is a distance measured normal to the surface $\theta = 0$ within the region; and the direction magnitudes $\frac{dp}{d\nu}, \frac{dq}{d\nu}, \frac{dr}{d\nu}$, for

that normal direction are (§ 268)

$$\left.\begin{aligned}\frac{dp}{d\nu}&=\frac{1}{\Omega D^{\frac{1}{2}}}(a\theta_1+h\theta_2+g\theta_3)=\frac{\Omega}{D^{\frac{1}{2}}}P\\ \frac{dq}{d\nu}&=\frac{1}{\Omega D^{\frac{1}{2}}}(h\theta_1+b\theta_2+f\theta_3)=\frac{\Omega}{D^{\frac{1}{2}}}Q\\ \frac{dr}{d\nu}&=\frac{1}{\Omega D^{\frac{1}{2}}}(g\theta_1+f\theta_2+c\theta_3)=\frac{\Omega}{D^{\frac{1}{2}}}R\end{aligned}\right\}.$$

Hence

$$\frac{dx}{d\nu}=x_1\frac{dp}{d\nu}+x_2\frac{dq}{d\nu}+x_3\frac{dr}{d\nu}=\frac{\Omega}{D^{\frac{1}{2}}}(x_1P+x_2Q+x_3R)=\overline{X};$$

and similarly

$$\frac{dy}{d\nu}=\Sigma y_1\frac{dp}{d\nu}=\overline{Y},\quad \frac{dz}{d\nu}=\Sigma z_1\frac{dp}{d\nu}=\overline{Z},\quad \frac{dv}{d\nu}=\Sigma v_1\frac{dp}{d\nu}=\overline{V}.$$

It thus appears that the quantities $\overline{X}$, $\overline{Y}$, $\overline{Z}$, $\overline{V}$, being equal to

$$\Sigma x_1\frac{dp}{d\nu},\quad \Sigma y_1\frac{dp}{d\nu},\quad \Sigma z_1\frac{dp}{d\nu},\quad \Sigma v_1\frac{dp}{d\nu},$$

respectively, are the direction-cosines of the direction which is normal to the surface $\theta(p, q, r)=0$ within the region.

Also, from the values of $\overline{X}$, $\overline{Y}$, $\overline{Z}$, $\overline{V}$, we have

$$D^{\frac{1}{2}}\Sigma\overline{X}x_1=\Omega(AP+HQ+GR)=\Omega\theta_1,$$

and similarly for the other two results: the three results are

$$\left.\begin{aligned}\Sigma\overline{X}x_1&=\theta_1\Omega D^{-\frac{1}{2}}\\ \Sigma\overline{X}x_2&=\theta_2\Omega D^{-\frac{1}{2}}\\ \Sigma\overline{X}x_3&=\theta_3\Omega D^{-\frac{1}{2}}\end{aligned}\right\}.$$

Again, from the same values, we have

$$\begin{aligned}D^{\frac{1}{2}}\Sigma\overline{X}x_{11}&=\Omega(P\Sigma x_1x_{11}+Q\Sigma x_2x_{11}+R\Sigma x_3x_{11})\\ &=\Omega(\alpha_1P+\alpha_2Q+\alpha_3R)\\ =\Omega\{P(A\Gamma_{11}+H\Delta_{11}+G\Theta_{11})+Q(H\Gamma_{11}+B\Delta_{11}+F\Theta_{11})+R(G\Gamma_{11}+F\Delta_{11}+C\Theta_{11})\}&\\ &=\Omega(\Gamma_{11}\theta_1+\Delta_{11}\theta_2+\Theta_{11}\theta_3),\end{aligned}$$

and so for the others: the six results are

$$\Sigma\overline{X}x_{lm}=(\Gamma_{lm}\theta_1+\Delta_{lm}\theta_2+\Theta_{lm}\theta_3)\,\Omega D^{-\frac{1}{2}},$$

for the six combinations l, m, $=1, 2, 3$.

Derivatives of P, Q, R, D.

341. The first derivatives of P, Q, R, will be required later. We have

$$P=\frac{1}{\Omega^2}(a\theta_1+h\theta_2+g\theta_3),$$

and therefore

$$P_1=\frac{1}{\Omega^2}(a\theta_{11}+h\theta_{12}+g\theta_{13})+\frac{\theta_1}{\Omega^2}\left(a_1-2a\frac{\Omega_1}{\Omega}\right)+\frac{\theta_2}{\Omega^2}\left(h_1-2h\frac{\Omega_1}{\Omega}\right)+\frac{\theta_3}{\Omega^2}\left(g_1-2g\frac{\Omega_1}{\Omega}\right)$$

$$=\frac{1}{\Omega^2}(a\theta_{11}+h\theta_{12}+g\theta_{13})-2\frac{\theta_1}{\Omega^2}(a\Gamma_{11}+h\Gamma_{12}+g\Gamma_{13})$$

$$-\frac{\theta_2}{\Omega^2}\{(h\Gamma_{11}+b\Gamma_{12}+f\Gamma_{13})+(a\Delta_{11}+h\Delta_{12}+g\Delta_{13})\}$$

$$-\frac{\theta_3}{\Omega^2}\{(g\Gamma_{11}+f\Gamma_{12}+c\Gamma_{13})+(a\Theta_{11}+h\Theta_{12}+g\Theta_{13})\},$$

by the results in § 273. Now let

$$\gamma_{lm}=\theta_{lm}-\theta_1\Gamma_{lm}-\theta_2\Delta_{lm}-\theta_3\Theta_{lm},$$

for all the six combinations $l, m, =1, 2, 3$; then

$$P_1=\frac{1}{\Omega^2}(a\gamma_{11}+h\gamma_{12}+g\gamma_{13})-(P\Gamma_{11}+Q\Gamma_{12}+R\Gamma_{13}).$$

Similarly for the other derivatives of P, and for the derivatives of Q and R. The complete results are

$$\left.\begin{aligned}P_1&=\Omega^{-2}(a\gamma_{11}+h\gamma_{12}+g\gamma_{13})-(P\Gamma_{11}+Q\Gamma_{12}+R\Gamma_{13})\\P_2&=\Omega^{-2}(a\gamma_{12}+h\gamma_{22}+g\gamma_{23})-(P\Gamma_{12}+Q\Gamma_{22}+R\Gamma_{23})\\P_3&=\Omega^{-2}(a\gamma_{13}+h\gamma_{23}+g\gamma_{33})-(P\Gamma_{13}+Q\Gamma_{23}+R\Gamma_{33})\end{aligned}\right\},$$

$$\left.\begin{aligned}Q_1&=\Omega^{-2}(h\gamma_{11}+b\gamma_{12}+f\gamma_{13})-(P\Delta_{11}+Q\Delta_{12}+R\Delta_{13})\\Q_2&=\Omega^{-2}(h\gamma_{12}+b\gamma_{22}+f\gamma_{23})-(P\Delta_{12}+Q\Delta_{22}+R\Delta_{23})\\Q_3&=\Omega^{-2}(h\gamma_{13}+b\gamma_{23}+f\gamma_{33})-(P\Delta_{13}+Q\Delta_{23}+R\Delta_{33})\end{aligned}\right\},$$

$$\left.\begin{aligned}R_1&=\Omega^{-2}(g\gamma_{11}+f\gamma_{12}+c\gamma_{13})-(P\Theta_{11}+Q\Theta_{12}+R\Theta_{13})\\R_2&=\Omega^{-2}(g\gamma_{12}+f\gamma_{22}+c\gamma_{23})-(P\Theta_{12}+Q\Theta_{22}+R\Theta_{23})\\R_3&=\Omega^{-2}(g\gamma_{13}+f\gamma_{23}+c\gamma_{33})-(P\Theta_{13}+Q\Theta_{23}+R\Theta_{33})\end{aligned}\right\}.$$

Various other derived magnitudes are required. We have

$$D=a\theta_1^2+2h\theta_1\theta_2+2g\theta_1\theta_3+b\theta_2^2+2f\theta_2\theta_3+c\theta_3^2=\Sigma a\theta_1^2;$$

and therefore

$$\tfrac{1}{2}D_1=\tfrac{1}{2}\Sigma a_1\theta_1^2+(a\theta_1+h\theta_2+g\theta_3)\theta_{11}+(h\theta_1+b\theta_2+f\theta_3)\theta_{12}+(g\theta_1+f\theta_2+c\theta_3)\theta_{13}.$$

Also, substituting from § 273 for the derivatives $a_1, \ldots$, we have

$$\tfrac{1}{2}\Sigma a_1\theta_1^2=\frac{\Omega_1}{\Omega}D-\Sigma(a\Gamma_{11}+h\Gamma_{12}+g\Gamma_{13})\theta_1^2;$$

consequently

$$\tfrac{1}{2}D_1-\frac{\Omega_1}{\Omega}D=(a\theta_1+h\theta_2+g\theta_3)\gamma_{11}+(h\theta_1+b\theta_2+f\theta_3)\gamma_{12}+(g\theta_1+f\theta_2+c\theta_3)\gamma_{13}$$

$$=\Omega^2(P\gamma_{11}+Q\gamma_{12}+R\gamma_{13}).$$

Likewise

$$\tfrac{1}{2}D_2 - \frac{\Omega_2}{\Omega}D = \Omega^2(P\gamma_{12} + Q\gamma_{22} + R\gamma_{23}),$$

$$\tfrac{1}{2}D_3 - \frac{\Omega_3}{\Omega}D = \Omega^2(P\gamma_{13} + Q\gamma_{23} + R\gamma_{33}).$$

Again, we have

$$\bar{X} = \frac{\Omega}{D^{\frac{1}{2}}}(x_1P + x_2Q + x_3R),$$

and therefore

$$\frac{\bar{X}_1}{\bar{X}} + \frac{1}{D}\left(\tfrac{1}{2}D_1 - \frac{\Omega_1}{\Omega}D\right) = \frac{1}{x_1P + x_2Q + x_3R}\frac{\partial}{\partial p}(x_1P + x_2Q + x_3R).$$

Hence

$$\bar{X}_1 + \bar{X}\frac{\Omega^2}{D}(P\gamma_{11} + Q\gamma_{12} + R\gamma_{13})$$

$$= \frac{\Omega}{D^{\frac{1}{2}}}(x_1P_1 + x_2Q_1 + x_3R_1 + Px_{11} + Qx_{12} + Rx_{13})$$

$$= \frac{\Omega}{D^{\frac{1}{2}}}\left[\frac{1}{\Omega^2}\{x_1(a\gamma_{11} + h\gamma_{12} + g\gamma_{13}) + x_2(h\gamma_{11} + b\gamma_{12} + f\gamma_{13}) + x_3(g\gamma_{11} + f\gamma_{12} + c\gamma_{13})\}\right.$$
$$\left. + (PL + QM + RK)X\right],$$

and similarly for the others. The full body of results for the derivatives of $\bar{X}$ is

$$\left.\begin{aligned}
&\bar{X}_1 - X\frac{\Omega}{D^{\frac{1}{2}}}(PL + QM + RK) + \bar{X}\frac{\Omega^2}{D}(P\gamma_{11} + Q\gamma_{12} + R\gamma_{13})\\
&= \frac{1}{\Omega D^{\frac{1}{2}}}\{x_1(a\gamma_{11} + h\gamma_{12} + g\gamma_{13}) + x_2(h\gamma_{11} + b\gamma_{12} + f\gamma_{13}) + x_3(g\gamma_{11} + f\gamma_{12} + c\gamma_{13})\}\\
&\bar{X}_2 - X\frac{\Omega}{D^{\frac{1}{2}}}(PM + QN + RI) + \bar{X}\frac{\Omega^2}{D}(P\gamma_{12} + Q\gamma_{22} + R\gamma_{23})\\
&= \frac{1}{\Omega D^{\frac{1}{2}}}\{x_1(a\gamma_{12} + h\gamma_{22} + g\gamma_{23}) + x_2(h\gamma_{12} + b\gamma_{22} + f\gamma_{23}) + x_3(g\gamma_{12} + f\gamma_{22} + c\gamma_{23})\}\\
&\bar{X}_3 - X\frac{\Omega}{D^{\frac{1}{2}}}(PK + QI + RJ) + \bar{X}\frac{\Omega^2}{D}(P\gamma_{13} + Q\gamma_{23} + R\gamma_{33})\\
&= \frac{1}{\Omega D^{\frac{1}{2}}}\{x_1(a\gamma_{13} + h\gamma_{23} + g\gamma_{33}) + x_2(h\gamma_{13} + b\gamma_{23} + f\gamma_{33}) + x_3(g\gamma_{13} + f\gamma_{23} + c\gamma_{33})\}
\end{aligned}\right\}.$$

The derivatives of $\bar{Y}$, $\bar{Z}$, $\bar{V}$, follow similar obvious laws.

We shall also require the derivatives of θ_x, θ_y, θ_z, θ_v, where

$$\theta_x = x_1P + x_2Q + x_3R, \quad \theta_y = y_1P + y_2Q + y_3R,$$
$$\theta_z = z_1P + z_2Q + z_3R, \quad \theta_v = v_1P + v_2Q + v_3R.$$

Proceeding as above, we find

$$\frac{\partial\theta_x}{\partial p} - X(PL+QM+RK)$$
$$= \frac{1}{\Omega^2}\{x_1(a\gamma_{11}+h\gamma_{12}+g\gamma_{13})+x_2(h\gamma_{11}+b\gamma_{12}+f\gamma_{13})+x_3(g\gamma_{11}+f\gamma_{12}+c\gamma_{13})\},$$
$$\frac{\partial\theta_x}{\partial q} - X(PM+QN+RI)$$
$$= \frac{1}{\Omega^2}\{x_1(a\gamma_{12}+h\gamma_{22}+g\gamma_{23})+x_2(h\gamma_{12}+b\gamma_{22}+f\gamma_{23})+x_3(g\gamma_{12}+f\gamma_{22}+c\gamma_{23})\},$$
$$\frac{\partial\theta_x}{\partial r} - X(PK+QI+RJ)$$
$$= \frac{1}{\Omega^2}\{x_1(a\gamma_{13}+h\gamma_{23}+g\gamma_{33})+x_2(h\gamma_{13}+b\gamma_{23}+f\gamma_{33})+x_3(g\gamma_{13}+f\gamma_{23}+c\gamma_{33})\}:$$

there being similar expressions for the derivatives of θ_y, θ_z, θ_v.

We at once have

$$\frac{d\theta_x}{ds} - \frac{1}{2}X\left(P\frac{\partial W}{\partial p'}+Q\frac{\partial W}{\partial q'}+R\frac{\partial W}{\partial r'}\right) = \frac{1}{2}\frac{1}{\Omega^2}\left\{\begin{aligned} &x_1\left(a\frac{\partial\Psi}{\partial p'}+h\frac{\partial\Psi}{\partial q'}+g\frac{\partial\Psi}{\partial r'}\right)\\ &+x_2\left(h\frac{\partial\Psi}{\partial p'}+b\frac{\partial\Psi}{\partial q'}+f\frac{\partial\Psi}{\partial r'}\right)\\ &+x_3\left(g\frac{\partial\Psi}{\partial p'}+f\frac{\partial\Psi}{\partial q'}+c\frac{\partial\Psi}{\partial r'}\right)\end{aligned}\right\},$$

with like expressions for $\frac{d\theta_y}{ds}$, $\frac{d\theta_z}{ds}$, $\frac{d\theta_v}{ds}$.

Further,

$$\Sigma X x_1 = 0, \quad \Sigma X x_2 = 0, \quad \Sigma X x_3 = 0,$$
$$D^{\frac{1}{2}}\Sigma\bar{X}x_1 = \Omega\theta_1, \quad D^{\frac{1}{2}}\Sigma\bar{X}x_2 = \Omega\theta_2, \quad D^{\frac{1}{2}}\Sigma\bar{X}x_3 = \Omega\theta_3;$$

and therefore

$$\left.\begin{aligned}\Sigma x_1\bar{X}_1 &= \frac{\Omega}{D^{\frac{1}{2}}}\gamma_{11} - \frac{\Omega^3}{D^{\frac{3}{2}}}(P\gamma_{11}+Q\gamma_{12}+R\gamma_{13})\theta_1\\ \Sigma x_2\bar{X}_1 &= \frac{\Omega}{D^{\frac{1}{2}}}\gamma_{12} - \frac{\Omega^3}{D^{\frac{3}{2}}}(P\gamma_{11}+Q\gamma_{12}+R\gamma_{13})\theta_2\\ \Sigma x_3\bar{X}_1 &= \frac{\Omega}{D^{\frac{1}{2}}}\gamma_{13} - \frac{\Omega^3}{D^{\frac{3}{2}}}(P\gamma_{11}+Q\gamma_{12}+R\gamma_{13})\theta_3\end{aligned}\right\},$$

$$\left.\begin{aligned}\Sigma x_1\bar{X}_2 &= \frac{\Omega}{D^{\frac{1}{2}}}\gamma_{12} - \frac{\Omega^3}{D^{\frac{3}{2}}}(P\gamma_{12}+Q\gamma_{22}+R\gamma_{23})\theta_1\\ \Sigma x_2\bar{X}_2 &= \frac{\Omega}{D^{\frac{1}{2}}}\gamma_{22} - \frac{\Omega^3}{D^{\frac{3}{2}}}(P\gamma_{12}+Q\gamma_{22}+R\gamma_{23})\theta_2\\ \Sigma x_3\bar{X}_2 &= \frac{\Omega}{D^{\frac{1}{2}}}\gamma_{23} - \frac{\Omega^3}{D^{\frac{3}{2}}}(P\gamma_{12}+Q\gamma_{22}+R\gamma_{23})\theta_3\end{aligned}\right\},$$

$$\left.\begin{aligned}
\Sigma x_1 \bar{X}_3 &= \frac{\Omega}{D^{\frac{1}{2}}}\gamma_{13} - \frac{\Omega^3}{D^{\frac{3}{2}}}(P\gamma_{13} + Q\gamma_{23} + R\gamma_{33})\,\theta_1 \\
\Sigma x_2 \bar{X}_3 &= \frac{\Omega}{D^{\frac{1}{2}}}\gamma_{23} - \frac{\Omega^3}{D^{\frac{3}{2}}}(P\gamma_{13} + Q\gamma_{23} + R\gamma_{33})\,\theta_2 \\
\Sigma x_3 \bar{X}_3 &= \frac{\Omega}{D^{\frac{1}{2}}}\gamma_{33} - \frac{\Omega^3}{D^{\frac{3}{2}}}(P\gamma_{13} + Q\gamma_{23} + R\gamma_{33})\,\theta_3
\end{aligned}\right\}.$$

Tangent plane to a surface: the orthogonal plane.

342. The tangent plane to the surface is the intersection of the tangent flat to the enfolding region $F \neq 0$, which has

$$\Sigma X\,(\bar{x} - x) = 0$$

for its equation, by the tangent flat to the specific region $G = 0$, which has

$$\Sigma G_x\,(\bar{x} - x) = 0$$

for its equation. When these two equations are taken together, or when two equivalent equations of the first degree are substituted for them, they constitute the equations of the tangent plane of the surface. Hence the equations of the *tangent plane* are

$$\left.\begin{aligned}
\Sigma X\,(\bar{x} - x) &= 0 \\
\Sigma \bar{X}\,(\bar{x} - x) &= 0
\end{aligned}\right\},$$

in which form only the magnitudes of the enfolding region and magnitudes connected with the equation $\theta = 0$ occur.

The equations of the *orthogonal plane* of the surface at the point x, y, z, v, being the plane at that point orthogonal to the foregoing tangent plane, are

$$\left\|\begin{matrix}
\bar{x} - x, & \bar{y} - y, & \bar{z} - z, & \bar{v} - v \\
X\ , & Y\ , & Z\ , & V \\
\bar{X}\ , & \bar{Y}\ , & \bar{Z}\ , & \bar{V}
\end{matrix}\right\| = 0.$$

If l, m, n, k, be the direction-cosines of any line lying in the orthogonal plane, then there are parametric quantities η and $\bar{\eta}$ such that

$$l = \eta X + \bar{\eta}\bar{X}, \quad m = \eta Y + \bar{\eta}\bar{Y}, \quad n = \eta Z + \bar{\eta}\bar{Z}, \quad k = \eta V + \bar{\eta}\bar{V}.$$

These equations will be modified hereafter (§ 346).

Curvature of a normal section.

343. Let $\xi, \eta, \zeta, \upsilon$, denote a point on the surface in the immediate vicinity of x, y, z, v; and let a perpendicular, ∇ in length, be drawn from $\xi, \eta, \zeta, \upsilon$, upon the tangent plane, meeting it in a point $\bar{x}, \bar{y}, \bar{z}, \bar{v}$. The quantity

$$\Sigma\,(\xi - \bar{x})^2$$

has to be a minimum among all the possibilities $\bar{x}, \bar{y}, \bar{z}, \bar{v}$, allowed by

$$\Sigma X\,(\bar{x} - x) = 0, \quad \Sigma \bar{X}\,(\bar{x} - x) = 0;$$

consequently we have

$$\left.\begin{aligned} l\nabla &= \xi - \bar{x} = \lambda X + \mu \bar{X} \\ m\nabla &= \eta - \bar{y} = \lambda Y + \mu \bar{Y} \\ n\nabla &= \zeta - \bar{z} = \lambda Z + \mu \bar{Z} \\ k\nabla &= \upsilon - \bar{v} = \lambda V + \mu \bar{V} \end{aligned}\right\},$$

shewing (as is to be expected) that this perpendicular lies in the orthogonal plane.

The multipliers λ and μ, indeterminate in these critical equations, have to be determined. Now

$$\Sigma X(\xi - \bar{x}) = \Sigma X(\xi - x) - \Sigma X(\bar{x} - x) = \Sigma X(\xi - x),$$

and $\Sigma X \bar{X} = 0$; hence

$$\lambda = \Sigma X(\xi - x);$$

and similarly

$$\mu = \Sigma \bar{X}(\xi - x),$$

giving expressions for λ and μ, which have to be evaluated.

We have

$$\xi - x = x'\delta + \frac{1}{2}x''\delta^2 + \ldots,$$

where δ is an arc along the surface in a direction p', q', r', which is subject to the arc-element law

$$\Sigma A p'^2 = 1,$$

and to the superficial condition

$$\theta_1 p' + \theta_2 q' + \theta_3 r' = 0.$$

Now

$$\Sigma X x' = p'\Sigma X x_1 + q'\Sigma X x_2 + r'\Sigma X x_3 = 0,$$

so that the coefficient of δ in the expansion of λ is zero; and

$$\begin{aligned} \Sigma X x'' &= p''\Sigma X x_1 + q''\Sigma X x_2 + r''\Sigma X x_3 \\ &\quad + p'^2\Sigma X x_{11} + 2p'q'\Sigma X x_{12} + \ldots + r'^2\Sigma X x_{33} \\ &= Lp'^2 + 2Mp'q' + 2Kp'r' + Nq'^2 + 2Iq'r' + Jr'^2 \\ &= W, \end{aligned}$$

where $W, = 1/\rho$, denotes the linear curvature of the normal section of the given enfolding region in the direction p', q', r'. Hence

$$\begin{aligned} \lambda = \Sigma X(\xi - x) &= \frac{1}{2}W\delta^2 + \ldots \\ &= \frac{\delta^2}{2}\frac{1}{\rho} + \ldots, \end{aligned}$$

where the omitted terms involve powers of δ higher than the second.

Again, for μ, we have

$$\begin{aligned} \mu &= \Sigma \bar{X}(\xi - x) \\ &= (\Sigma \bar{X} x')\delta + \frac{1}{2}(\Sigma \bar{X} x'')\delta^2 + \ldots. \end{aligned}$$

The coefficient of δ in this expression

$$= p'\Sigma\bar{X}x_1 + q'\Sigma\bar{X}x_2 + r'\Sigma\bar{X}x_3$$

$$= \frac{\Omega}{D^{\frac{1}{2}}}\{p'\Sigma x_1(x_1P + x_2Q + x_3R) + q'\Sigma x_2(x_1P + x_2Q + x_3R) + r'\Sigma x_3(x_1P + x_2Q + x_3R)\}$$

$$= \frac{\Omega}{D^{\frac{1}{2}}}\{p'(AP + HQ + GR) + q'(HP + BQ + FR) + r'(GP + FQ + CR)\}$$

$$= \frac{\Omega}{D^{\frac{1}{2}}}(\theta_1 p' + \theta_2 q' + \theta_3 r') = 0.$$

The coefficient of $\frac{1}{2}\delta^2$ in the expression

$$= p''\Sigma\bar{X}x_1 + q''\Sigma\bar{X}x_2 + r''\Sigma\bar{X}x_3 + p'^2\Sigma\bar{X}x_{11} + 2p'q'\Sigma\bar{X}x_{12} + \ldots + r'^2\Sigma\bar{X}x_{33}$$

$$= \Omega D^{-\frac{1}{2}}(\theta_1 p'' + \theta_2 q'' + \theta_3 r'') + \Omega D^{-\frac{1}{2}}\{\theta_1(\Gamma\{p', q', r')^2 + \theta\ (\Delta\{p', q', r')^2 + \theta_3(\Theta\{p', q', r')^2\}.$$

But along the surface, we have

$$\theta_1 p'' + \theta_2 q'' + \theta_3 r'' + \theta_{11}p'^2 + 2\theta_{12}p'q' + 2\theta_{13}p'r' + \theta_{22}q'^2 + 2\theta_{23}q'r' + \theta_{33}r'^2 = 0;$$

consequently the coefficient of $\frac{1}{2}\delta^2$ becomes

$$= -\Omega D^{-\frac{1}{2}}\{\theta_{11}p'^2 + 2\theta_{12}p'q' + \ldots\} + \Omega D^{-\frac{1}{2}}\{\theta_1(\Gamma\{p', q', r')^2 + \ldots\}$$

$$= -\Omega D^{-\frac{1}{2}}(\gamma_{11}p'^2 + 2\gamma_{12}p'q' + 2\gamma_{13}p'r' + \gamma_{22}q'^2 + 2\gamma_{23}q'r' + \gamma_{33}r'^2)$$

with the definitions of γ_{lm} on p. 183. We write

$$\Psi = (\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12}\{p', q', r')^2;$$

and so the coefficient of $\frac{1}{2}\delta^2$ is $-\Psi\Omega D^{-\frac{1}{2}}$. Thus

$$\mu = \frac{1}{2}\delta^2\Sigma\bar{X}(\xi - x) = -\frac{1}{2}\delta^2\Psi\Omega D^{-\frac{1}{2}}.$$

We thus have

$$l\nabla = \xi - \bar{x} = \frac{1}{2}\delta^2(XW - \bar{X}\Psi\Omega D^{-\frac{1}{2}}),$$

$$m\nabla = \eta - \bar{y} = \frac{1}{2}\delta^2(YW - \bar{Y}\Psi\Omega D^{-\frac{1}{2}}),$$

$$n\nabla = \zeta - \bar{z} = \frac{1}{2}\delta^2(ZW - \bar{Z}\Psi\Omega D^{-\frac{1}{2}}),$$

$$k\nabla = \nu - \bar{\nu} = \frac{1}{2}\delta^2(VW - \bar{V}\Psi\Omega D^{-\frac{1}{2}}).$$

Now let ρ_0 denote the radius of circular curvature of the normal section of the surface, so that

$$\nabla \cdot 2\rho_0 = \delta^2;$$

then the foregoing formulæ become

$$\left.\begin{aligned} \frac{l}{\rho_0} &= XW - \bar{X}\Psi\Omega D^{-\frac{1}{2}} \\ \frac{m}{\rho_0} &= YW - \bar{Y}\Psi\Omega D^{-\frac{1}{2}} \\ \frac{n}{\rho_0} &= ZW - \bar{Z}\Psi\Omega D^{-\frac{1}{2}} \\ \frac{k}{\rho_0} &= VW - \bar{V}\Psi\Omega D^{-\frac{1}{2}} \end{aligned}\right\}.$$

We know that $W, = \Sigma L p'^2$, is the curvature of the normal section of the region through the same direction p', q', r'. A corresponding significance, in § 345 below, will be attached to $\Psi\Omega D^{-\frac{1}{2}}$; to derive this significance, we must first consider some properties of geodesics on the surface.

Geodesics on the surface: characteristic equations.

344. To obtain the equations, characteristic of geodesics upon the surface in the region, we have to make

$$\int S dt = \int \left\{ \Sigma A \left(\frac{dp}{dt}\right)^2 \right\}^{\frac{1}{2}} dt$$

a minimum subject to $\theta(p, q, r) = 0$, the equation of the surface. By the customary rule*, the three critical equations are

$$\frac{d}{dt}\left(\frac{\partial S}{\partial p_1}\right) - \frac{\partial S}{\partial p} = \bar{\bar{M}}\theta_1,$$

that is,

$$Ap'' + Hq'' + Gr'' + \alpha_1 p'^2 + 2\beta_1 p'q' + 2\gamma_1 p'r' + \delta_1 q'^2 + 2\epsilon_1 q'r' + \omega_1 r'^2 = \bar{\bar{M}}\theta_1,$$

with two others, where $\bar{\bar{M}}$ (and $\bar{\bar{M}}$) are multipliers, indeterminate so far as the critical equations are concerned initially, and where $\alpha_1, \ldots, \omega_1$, are the quantities of § 273. This equation may be written

$$Ap'' + Hq'' + Gr'' + (\alpha_1, \ldots \mathfrak{X} p', \ldots)^2 = \bar{\bar{M}}\theta_1;$$

and the other two critical equations are

$$Hp'' + Bq'' + Fr'' + (\alpha_2, \ldots \mathfrak{X} p', \ldots)^2 = \bar{\bar{M}}\theta_2,$$
$$Gp'' + Fq'' + Cr'' + (\alpha_3, \ldots \mathfrak{X} p', \ldots)^2 = \bar{\bar{M}}\theta_3.$$

When these relations are resolved for p'', q'', r'', they lead to the results

$$p'' + (\Gamma \mathfrak{X} p', q', r')^2 = \frac{\bar{\bar{M}}}{\Omega^2}(a\theta_1 + h\theta_2 + g\theta_3) = \bar{\bar{M}}P,$$

$$q'' + (\Delta \mathfrak{X} p', q', r')^2 = \frac{\bar{\bar{M}}}{\Omega^2}(h\theta_1 + b\theta_2 + f\theta_3) = \bar{\bar{M}}Q,$$

$$r'' + (\Theta \mathfrak{X} p', q', r')^2 = \frac{\bar{\bar{M}}}{\Omega^2}(g\theta_1 + f\theta_2 + c\theta_3) = \bar{\bar{M}}R.$$

The multiplier $\bar{\bar{M}}$ has to be made determinate.

* See my *Calculus of Variations*, § 261.

In the first place, the three equations are equivalent to only two independent equations, in virtue of the arc-relation $\Sigma A p'^2 = 1$ of the region in which the surface lies, and of the equation $\theta(p, q, r) = 0$.

Multiplying the equations by

$$Ap' + Hq' + Gr',\ Hp' + Bq' + Fr',\ Gp' + Fq' + Cr',$$

respectively and adding, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}(\Sigma A p'^2) &= \bar{M}\{P(Ap' + Hq' + Gr') + Q(Hp' + Bq' + Fr') + R(Gp' + Fq' + Cr')\} \\ &= \bar{M}(p'\theta_1 + q'\theta_2 + r'\theta_3) = 0,\end{aligned}$$

an identity.

Next, multiply the equations by θ_1, θ_2, θ_3, respectively and add: then

$$\theta_1 p'' + \theta_2 q'' + \theta_3 r'' + \theta_1(\Gamma\{p', q', r')^2 + \theta_2(\Delta\{p', q', r')^2 + \theta_3(\Theta\{p', q', r')^2$$

$$= \frac{\bar{M}}{\Omega^2}(a\theta_1^2 + 2h\theta_1\theta_2 + 2g\theta_1\theta_3 + b\theta_2^2 + 2f\theta_2\theta_3 + c\theta_3^2) = \frac{\bar{M}}{\Omega^2} D.$$

Now, for any direction on the surface, we have

$$\theta_1 p'' + \theta_2 q'' + \theta_3 r'' + \Sigma\theta_{11} p'^2 = 0;$$

hence, along a superficial geodesic, we have

$$-(\theta_{11} - \theta_1\Gamma_{11} - \theta_2\Delta_{11} - \theta_3\Theta_{11}, \ldots \{p', q', r')^2 = \frac{\bar{M}}{\Omega^2} D,$$

that is,

$$-(\gamma\{p', q', r')^2 = \frac{\bar{M}}{\Omega^2} D,$$

and therefore

$$\bar{M} = -\Omega^2 \frac{\Psi}{D}.$$

Consequently the *equations of geodesics on the surface* are

$$\left.\begin{aligned} p'' + (\Gamma\{p', q', r')^2 &= -\frac{\Psi}{D}\Omega^2 P \\ q'' + (\Delta\{p', q', r')^2 &= -\frac{\Psi}{D}\Omega^2 Q \\ r'' + (\Theta\{p', q', r')^2 &= -\frac{\Psi}{D}\Omega^2 R \end{aligned}\right\},$$

or, in an equivalent form,

$$\left.\begin{aligned} p'' + (\Gamma\{p', q', r')^2 &= -(a\theta_1 + h\theta_2 + g\theta_3)\frac{\Psi}{D} \\ q'' + (\Delta\{p', q', r')^2 &= -(h\theta_1 + b\theta_2 + f\theta_3)\frac{\Psi}{D} \\ r'' + (\Theta\{p', q', r')^2 &= -(g\theta_1 + f\theta_2 + c\theta_3)\frac{\Psi}{D} \end{aligned}\right\}.$$

Further, along the geodesic, we have

$$\begin{aligned} x'' &= x_1 p'' + x_2 q'' + x_3 r'' + \Sigma x_{11} p'^2 \\ &= \Sigma \{(x_{11} - x_1 \Gamma_{11} - x_2 \Delta_{11} - x_3 \Theta_{11}) p'^2\} - \frac{\Psi}{D} \Omega^2 (x_1 P + x_2 Q + x_3 R) \\ &= X \Sigma L p'^2 - \bar{X} \Psi \Omega D^{-\frac{1}{2}} \\ &= X W - \bar{X} \Psi \Omega D^{-\frac{1}{2}} \\ &= \frac{l}{\rho_0}, \end{aligned}$$

where l, m, n, k, are the direction-cosines of the perpendicular from a point $\xi, \eta, \zeta, \upsilon$, near x, y, z, v, upon the tangent plane to the surface, and where ρ_0 is the radius of curvature of the normal section of the surface. Similarly,

$$y'' = \frac{m}{\rho_0}, \quad z'' = \frac{n}{\rho_0}, \quad v'' = \frac{k}{\rho_0}.$$

Hence the principal normal of the geodesic and the foregoing perpendicular coincide in direction; the direction of the radius of circular curvature of the geodesic lies in the orthogonal plane of the surface at the point; and ρ_0 is the radius of circular curvature of the geodesic.

To obtain a fuller significance of these equations, it is necessary to obtain the geometrical interpretation of the concomitant denoted by Ψ: and the interpretation depends upon the introduction of the notion of curvature with respect to the region and not, primarily, with respect to the given enfolding homaloidal quadruple space.

Regional geodesic curvature of a superficial geodesic.

345. We define the angle of regional contingence of a geodesic on a surface to be the angle between consecutive regional geodesics touching the superficial geodesic; and we define the *regional geodesic curvature* of a superficial geodesic to be the arc-rate of change of the regional contingence of the superficial geodesic.

At the point x, y, z, v, the direction of the superficial geodesic and the direction of the tangential regional geodesic (being the same) are, both of them, denoted by x', y', z', v'. At a consecutive point on the superficial geodesic, distant δ from x, y, z, v, the direction of the superficial geodesic is given by

$$x' + x''\delta, \quad y' + y''\delta, \quad z' + z''\delta, \quad v' + v''\delta;$$

let the direction of the tangential regional geodesic be

$$x' + \bar{x}''\delta, \quad y' + \bar{y}''\delta, \quad z' + \bar{z}''\delta, \quad v' + \bar{v}''\delta.$$

The former of these directions is to be taken in connection with the superficial geodesics, represented by the equations of the type

$$p'' + (\Gamma \mathfrak{X} p', q', r')^2 = -\frac{\Psi}{D}\Omega^2 P;$$

the latter is to be taken in connection with the regional geodesics represented by the corresponding equations (§ 298) which will be written

$$\bar{p}'' + (\Gamma \mathfrak{X} p', q', r')^2 = 0,$$

the values of p', q', r', being the same for the two geodesics.

Let $d\epsilon$ denote the angle of regional contingence of the superficial geodesic, so that (by use of a unit globe of reference) we have

$$\frac{d\epsilon^2}{\delta^2} = (x'' - \bar{x}'')^2 + (y'' - \bar{y}'')^2 + (z'' - \bar{z}'')^2 + (v'' - \bar{v}'')^2.$$

Now

$$\bar{x}'' = x_1 \bar{p}'' + x_2 \bar{q}'' + x_3 \bar{r}'' + \Sigma x_{11} p'^2,$$
$$x'' = x_1 p'' + x_2 q'' + x_3 r'' + \Sigma x_{11} p'^2;$$

and therefore

$$\begin{aligned}\bar{x}'' - x'' &= x_1(\bar{p}'' - p'') + x_2(\bar{q}'' - q'') + x_3(\bar{r}'' - r'') \\ &= \frac{\Psi}{D}\Omega^2(x_1 P + x_2 Q + x_3 R) \\ &= \bar{X}\Psi\Omega D^{-\frac{1}{2}}.\end{aligned}$$

Similarly,

$$\frac{\bar{y}'' - y''}{\bar{Y}} = \frac{\bar{z}'' - z''}{\bar{Z}} = \frac{\bar{v}'' - v''}{\bar{V}} = \Psi\Omega D^{-\frac{1}{2}}.$$

Consequently

$$\frac{d\epsilon^2}{\delta^2} = \frac{\Psi^2\Omega^2}{D}.$$

The quantity $d\epsilon/\delta$, being the arc-rate of change of the angle of regional contingence of the superficial geodesic, is the regional geodesic curvature of the superficial geodesic. We shall denote it by $1/\gamma$, to be estimated as a positive measure when taken from the regional geodesic towards the superficial geodesic. Thus

$$\frac{1}{\gamma} = -\frac{\Omega}{D^{\frac{1}{2}}}\Psi,$$

where

$$D = (a, b, c, f, g, h \mathfrak{X} \theta_1, \theta_2, \theta_3)^2,$$
$$\Psi = (\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12} \mathfrak{X} p', q', r')^2.$$

Another mode of approaching this regional geodesic curvature shews that the quantity γ is not merely a magnitude but has direction; in fact, it is a radius of curvature in the same sense as ρ_0 and ρ are radii of curvature. Along the regional geodesic, determined by the direction p', q', r', lying in the surface, take a point T distant δ from x, y, z, v, with coordinates $\bar{\xi}, \bar{\eta}, \bar{\zeta}, \bar{v}$;

while we still denote by $\xi, \eta, \zeta, \upsilon$, the coordinates of the point P also at that distance δ from x, y, z, v, along the superficial geodesic. With the preceding notation, we have

$$\xi = x + x'\delta + \tfrac{1}{2}x''\delta^2 + \ldots, \quad \bar{\xi} = x + x'\delta + \tfrac{1}{2}\bar{x}''\delta^2 + \ldots,$$

the quantities x', y', z', v', being the same for the two curves; thus

$$\xi - \bar{\xi} = \tfrac{1}{2}\delta^2(x'' - \bar{x}'') + \ldots,$$

with similar values for $\eta - \bar{\eta}$, $\zeta - \bar{\zeta}$, $\upsilon - \bar{\upsilon}$. Hence the limiting position of the line TU gives a definite direction, measured from T on the regional geodesic to U on the superficial geodesic; and the direction-cosines of this direction are proportional to $\xi - \bar{\xi}$, $\eta - \bar{\eta}$, $\zeta - \bar{\zeta}$, $\upsilon - \bar{\upsilon}$. Let the length of TU be denoted by $\overline{\Pi}$ temporarily, and the direction-cosines by $\lambda, \mu, \nu, \kappa$, also temporarily; thus

$$\frac{\xi - \bar{\xi}}{\lambda} = \frac{\eta - \bar{\eta}}{\mu} = \frac{\zeta - \bar{\zeta}}{\nu} = \frac{\upsilon - \bar{\upsilon}}{\kappa} = \overline{\Pi}.$$

Now we proved above that

$$x'' - \bar{x}'' = -\bar{X}\Psi\Omega D^{-\frac{1}{2}} = \frac{\bar{X}}{\gamma},$$

and likewise for $y'' - \bar{y}''$, $z'' - \bar{z}''$, $v'' - \bar{v}''$; and therefore

$$\lambda\overline{\Pi} = \tfrac{1}{2}\delta^2\frac{\bar{X}}{\gamma}, \quad \mu\overline{\Pi} = \tfrac{1}{2}\delta^2\frac{\bar{Y}}{\gamma}, \quad \nu\overline{\Pi} = \tfrac{1}{2}\delta^2\frac{\bar{Z}}{\gamma}, \quad \kappa\overline{\Pi} = \tfrac{1}{2}\delta^2\frac{\bar{V}}{\gamma}.$$

The limit of the quantity

$$\frac{2\overline{\Pi}}{\delta^2},$$

as δ tends to the value zero and yields the limiting position of the line TU, is the measure of the curvature of the superficial geodesic relative to the regional geodesic: if this limit be

$$\frac{1}{\Gamma},$$

we have

$$\frac{\lambda}{\Gamma} = \frac{\bar{X}}{\gamma}, \quad \frac{\mu}{\Gamma} = \frac{\bar{Y}}{\gamma}, \quad \frac{\nu}{\Gamma} = \frac{\bar{Z}}{\gamma}, \quad \frac{\kappa}{\Gamma} = \frac{\bar{V}}{\gamma}.$$

Consequently

$$\Gamma = \gamma,$$

$$\lambda = \bar{X}, \quad \mu = \bar{Y}, \quad \nu = \bar{Z}, \quad \kappa = \bar{V}.$$

We thus have a radius, γ, of curvature of the superficial geodesic, being the measure of its deviation from the regional geodesic; and this radius of regional geodesic curvature has $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{V}$, for its direction-cosines.

Relations between the three curvatures.

346. The direction-cosines of the principal normal to the geodesic on the surface are now expressible in the form

$$\left.\begin{aligned} \frac{l}{\rho_0} &= \frac{X}{\rho} + \frac{\bar{X}}{\gamma} \\ \frac{m}{\rho_0} &= \frac{Y}{\rho} + \frac{\bar{Y}}{\gamma} \\ \frac{n}{\rho_0} &= \frac{Z}{\rho} + \frac{\bar{Z}}{\gamma} \\ \frac{k}{\rho_0} &= \frac{V}{\rho} + \frac{\bar{V}}{\gamma} \end{aligned}\right\},$$

where ρ_0 is the radius of circular curvature of the superficial geodesic through the direction p', q', r', on the surface, ρ is the radius of circular curvature of the region (and therefore of the regional geodesic) with direction-cosines X, Y, Z, V, and γ is the radius of regional geodesic curvature of the superficial geodesic with direction-cosines $\bar{X}, \bar{Y}, \bar{Z}, \bar{V}$. We at once have

$$\frac{1}{\rho_0^2} = \frac{1}{\rho^2} + \frac{1}{\gamma^2}.$$

Again, let ψ be the angle between the principal normal of the superficial geodesic and the principal normal of the regional geodesic (the actual direction of the normal to the region): then

$$\cos\psi = \Sigma l X, \quad \sin\psi = \Sigma l \bar{X},$$

the three directions l, m, n, k; X, Y, Z, V; $\bar{X}, \bar{Y}, \bar{Z}, \bar{V}$; being complanar. Hence

$$\left.\begin{aligned} \frac{\cos\psi}{\rho_0} &= \frac{1}{\rho} \\ \frac{\sin\psi}{\rho_0} &= \frac{1}{\gamma} \end{aligned}\right\},$$

being the extension of the equations (in the Gauss theory) for the two curvatures of a curve on a surface in homaloidal triple space. Moreover, we also have

$$\frac{1}{\rho_0} = \frac{\rho_0}{\rho^2} + \frac{\rho_0}{\gamma^2} = \frac{\cos\psi}{\rho} + \frac{\sin\psi}{\gamma}.$$

The analogy between the composition of the curvatures and the composition of vectors is obvious.

As a last inference for the present, we at once deduce the differential equations of a geodesic surface. If the surface $\theta(p, q, r) = 0$ be of this type, every superficial geodesic through x, y, z, v, is also a regional geodesic, and therefore its regional geodesic curvature is zero; consequently, we have

$$(\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12})(p', q', r')^2 = \Psi = 0,$$

with

$$\theta_1 p' + \theta_2 q' + \theta_3 r' = 0,$$

as determining geodesic surfaces in the region.

The various directions are as in the figure. The point P is external to the surface but near it: PF is the perpendicular to the tangent flat of the given region; PG is the perpendicular to the tangent flat of the region $G=0$; so that (§ 52) PS, the diameter of the circle FPG, is the line l, m, n, k, the perpendicular from P upon the surface in the quadruple space. The line PN (with direction-cosines $\bar{X}, \bar{Y}, \bar{Z}, \bar{V}$) is parallel to the normal through S to the surface drawn within the region: and it lies in the plane FPG.

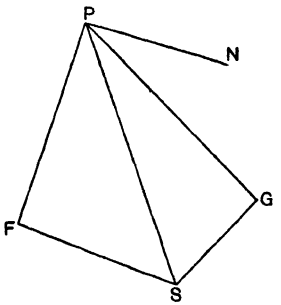


Fig. 24.

The direction-cosines of FP are X, Y, Z, V; so that, if $SPF=\psi$, and therefore $SPN=\frac{1}{2}\pi-\psi$, while $FPG=\omega$, we have

$$\cos\psi = \Sigma l X, \quad \sin\psi = \Sigma l \bar{X}.$$

The direction-cosines of GP are G_x/Π, G_y/Π, G_z/Π, G_v/Π, where

$$\frac{G_x}{\Pi} = X\cos\omega + \bar{X}\sin\omega,$$

with like expressions for the other three magnitudes.

Orthogonal frames for superficial geodesics.

347. The orthogonal frame, connected with any direction p', q', r', on the surface $\theta=0$, can be constructed as before. For this purpose, one initial requirement (§ 233) is a direction, lying in the tangent plane of the surface, and perpendicular to the direction x', y', z', v'. Let its direction-cosines be $\bar{l}, \bar{m}, \bar{n}, \bar{k}$; so that, as it lies in the plane

$$\Sigma X(\bar{x}-x)=0, \quad \Sigma \bar{X}(\bar{x}-x)=0,$$

we have

$$X\bar{l} + Y\bar{m} + Z\bar{n} + V\bar{k} = 0,$$

$$\bar{X}\bar{l} + \bar{Y}\bar{m} + \bar{Z}\bar{n} + \bar{V}\bar{k} = 0,$$

and, because it is perpendicular to x', y', z', v',

$$x'\bar{l} + y'\bar{m} + z'\bar{n} + v'\bar{k} = 0.$$

Now $\Sigma X x' = 0$, $\Sigma \bar{X} x' = 0$, and also $\Sigma X\bar{X}=0$; hence, the four directions x', y', z', v'; X, Y, Z, V; $\bar{l}, \bar{m}, \bar{n}, \bar{k}$; and $\bar{X}, \bar{Y}, \bar{Z}, \bar{V}$; are perpendicular to one another and thus constitute an orthogonal frame. We therefore have

$$\bar{l} = \begin{vmatrix} Y, & Z, & V \\ \bar{Y}, & \bar{Z}, & \bar{V} \\ y', & z', & v' \end{vmatrix},$$

with similar values for $\bar{m}$, $\bar{n}$, $\bar{k}$. But

$$X\Omega = J_x, \quad Y\Omega = -J_y, \quad Z\Omega = J_z, \quad V\Omega = -J_v,$$

$$\bar{X}D^{\frac{1}{2}} = \Sigma x_1 P, \quad \bar{Y}D^{\frac{1}{2}} = \Sigma y_1 P, \quad \bar{Z}D^{\frac{1}{2}} = \Sigma z_1 P, \quad \bar{V}D^{\frac{1}{2}} = \Sigma v_1 P;$$

and therefore

$$\bar{l}D^{\frac{1}{2}} = \begin{vmatrix} -J_y, & J_z, & -J_v \\ \Sigma y_1 P, & \Sigma z_1 P, & \Sigma v_1 P \\ \Sigma y_1 p', & \Sigma z_1 p', & \Sigma v_1 p' \end{vmatrix}$$

$$= \begin{vmatrix} q', & r' \\ Q, & R \end{vmatrix} \begin{vmatrix} J_{y_2}, & -J_z, & J_v \\ y_2, & z_2, & v_2 \\ y_3, & z_3, & v_3 \end{vmatrix} + \begin{vmatrix} r', & p' \\ R, & P \end{vmatrix} \begin{vmatrix} J_y, & -J_z, & J_v \\ y_3, & z_3, & v_3 \\ y_1, & z_1, & v_1 \end{vmatrix}$$

$$+ \begin{vmatrix} p', & q' \\ P, & Q \end{vmatrix} \begin{vmatrix} J_y, & -J_z, & J_v \\ y_1, & z_1, & v_1 \\ y_2, & z_2, & v_2 \end{vmatrix}.$$

In the third term on the right-hand side, the coefficient of $p'Q - q'P$ is equal to $-h_{12}J_y - g_{12}J_z + a_{12}J_v$, with the notation of § 263: that is, it is

$$= \begin{vmatrix} 0, & h_{12}, & -g_{12}, & -a_{12} \\ x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ x_3, & y_3, & z_3, & v_3 \end{vmatrix}$$

$$= x_1(a_{12}a_{23} + h_{12}h_{23} + g_{12}g_{23}) + x_2(a_{12}a_{31} + h_{12}h_{31} + g_{12}g_{31}) + x_3(a_{12}^2 + h_{12}^2 + g_{12}^2).$$

But the relations

$$x_1\mu_{23} + x_2\mu_{31} + x_3\mu_{12} = 0$$

are satisfied identically for $\mu = b, c, f$; hence the said coefficient is

$$= x_1(a_{12}a_{23} + b_{12}b_{23} + c_{12}c_{23} + f_{12}f_{23} + g_{12}g_{23} + h_{12}h_{23})$$
$$+ x_2(a_{12}a_{31} + b_{12}b_{31} + c_{12}c_{31} + f_{12}f_{31} + g_{12}g_{31} + h_{12}h_{31})$$
$$+ x_3(a_{12}^2 + b_{12}^2 + c_{12}^2 + f_{12}^2 + g_{12}^2 + h_{12}^2)$$
$$= x_1g + x_2f + x_3c,$$

from the last set (III) of formulæ in § 263. Similarly for the other terms; hence

$$\bar{l}D^{\frac{1}{2}} = (x_1a + x_2h + x_3g)(q'R - r'Q)$$
$$+ (x_1h + x_2b + x_3f)(r'P - p'R)$$
$$+ (x_1g + x_2f + x_3c)(p'Q - q'P)$$

$$= x_1 \begin{vmatrix} a, & h, & g \\ p', & q', & r' \\ P, & Q, & R \end{vmatrix} + x_2 \begin{vmatrix} h, & b, & f \\ p', & q', & r' \\ P, & Q, & R \end{vmatrix} + x_3 \begin{vmatrix} g, & f, & c \\ p', & q', & r' \\ P, & Q, & R \end{vmatrix}.$$

The determinants on the right-hand side can be modified. As

$$gQ-hR=\frac{1}{\Omega^2}\{(bg-fh)\,\theta_2+(fg-ch)\,\theta_3\}$$

$$=-G\theta_2+H\theta_3,$$

$$aR-gP=-F\theta_2+B\theta_3,$$

$$hP-aQ=-C\theta_2+F\theta_3,$$

we have

$$\begin{vmatrix} a, & h, & g \\ p', & q', & r' \\ P, & Q, & R \end{vmatrix} = p' \begin{vmatrix} H, & G \\ \theta_2, & \theta_3 \end{vmatrix} + q' \begin{vmatrix} B, & F \\ \theta_2, & \theta_3 \end{vmatrix} + r' \begin{vmatrix} F, & C \\ \theta_2, & \theta_3 \end{vmatrix};$$

and so for the others. Consequently,

$$\bar{l}D^{\frac{1}{2}} = \begin{vmatrix} x_1 & , & x_2 & , & x_3 \\ Ap'+Hq'+Gr', & & Hp'+Bq'+Fr', & & Gp'+Fq'+Cr' \\ \theta_1 & , & \theta_2 & , & \theta_3 \end{vmatrix},$$

with corresponding values for $\bar{m}D^{\frac{1}{2}}$, $\bar{n}D^{\frac{1}{2}}$, $\bar{k}D^{\frac{1}{2}}$, involving the first derivatives of y, of z, of v, respectively.

We thus obtain a quadruply orthogonal frame belonging to the geodesic, with special regard to directions in the quadruple space; its four axes have

$$\begin{matrix} x', & y', & z', & v'; \\ \bar{l}, & \bar{m}, & \bar{n}, & \bar{k}; \\ X, & Y, & Z, & V; \\ \bar{X}, & \bar{Y}, & \bar{Z}, & \bar{V}; \end{matrix}$$

for their direction-cosines respectively.

It is to be noted further that, in the region and specially associated with the surface, there are three directions, whose spatial direction-cosines are

$$\begin{matrix} x', & y', & z', & v'; \\ \bar{l}, & \bar{m}, & \bar{n}, & \bar{k}; \\ l, & m, & n, & k; \end{matrix}$$

respectively. These are such that

$$\Sigma x'l=0, \quad \Sigma x'\bar{l}=0,$$

$$\Sigma l\bar{l}=\Sigma\left(\frac{\rho_0}{\rho}X+\frac{\rho_0}{\gamma}\bar{X}\right)\bar{l}=\frac{\rho_0}{\rho}\Sigma X\bar{l}+\frac{\rho_0}{\gamma}\Sigma\bar{X}\bar{l}=0;$$

that is, the three directions constitute a triply orthogonal frame belonging solely to the surface in the region. Finally, another triply orthogonal frame is provided by the direction

$$p', q', r',$$

(defining x', y', z', v', along the geodesic, by the relations

$$x' = x_1 p' + x_2 q' + x_3 r'$$

and three others), the direction

$$\frac{dp}{dv}, \quad \frac{dq}{dv}, \quad \frac{dr}{dv},$$

(defining $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{V}$, normal to the surface, by the relations

$$\bar{X} = x_1 \frac{dp}{dv} + y_1 \frac{dq}{dv} + z_1 \frac{dr}{dv}$$

and three others), and the direction

$$\frac{dp}{dt}, \quad \frac{dq}{dt}, \quad \frac{dr}{dt},$$

(defining $\bar{l}$, $\bar{m}$, $\bar{n}$, $\bar{k}$, tangential to the surface and perpendicular to the geodesic direction, by the relations

$$\bar{l} = x_1 \frac{dp}{dt} + y_1 \frac{dq}{dt} + z_1 \frac{dr}{dt}$$

and three others).

Principal measures of regional geodesic curvature along a surface.

348. We have seen (§ 345) that the regional geodesic curvature of a superficial geodesic in a direction p', q', r', upon the surface is given by

$$\frac{1}{\gamma} = -\frac{\Omega}{D^{\frac{1}{2}}} \Psi = -\frac{\Omega}{D^{\frac{1}{2}}} (\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12} \mathbin{\text{\textbackslash\!\!\textbackslash}} p', q', r')^2,$$

where Ω^2 is the determinant of the primary magnitudes of the whole region; also D, denoting the magnitude

$$(a, b, c, f, g, h \mathbin{\text{\textbackslash\!\!\textbackslash}} \theta_1, \theta_2, \theta_3)^2,$$

is independent of directions p', q', r', through the point on the surface. The maximum and minimum values of this regional geodesic curvature, and the directions along which such values are attained, are obtainable by making Ψ a maximum or a minimum for values of p', q', r', that are subject to the two relations

$$\Sigma A p'^2 = 1, \quad \theta_1 p' + \theta_2 q' + \theta_3 r' = 0.$$

The critical conditions for such values are

$$\gamma_{11} p' + \gamma_{12} q' + \gamma_{13} r' = \lambda (A p' + H q' + G r') + \mu \theta_1,$$
$$\gamma_{12} p' + \gamma_{22} q' + \gamma_{23} r' = \lambda (H p' + B q' + F r') + \mu \theta_2,$$
$$\gamma_{13} p' + \gamma_{23} q' + \gamma_{33} r' = \lambda (G p' + F q' + C r') + \mu \theta_3,$$

where λ and μ are multipliers, indeterminate so far as concerns the formation of the conditions.

Multiply the three equations by p', q', r', respectively, and add: then

$$\Sigma\gamma_{11}p'^2 = \lambda\Sigma Ap'^2 + \mu(\theta_1 p' + \theta_2 q' + \theta_3 r'),$$

that is,

$$\Psi = \lambda;$$

and thus the three critical conditions become

$$\left.\begin{aligned}(\gamma_{11} - A\Psi)p' + (\gamma_{12} - H\Psi)q' + (\gamma_{13} - G\Psi)r' &= \mu\theta_1\\ (\gamma_{12} - H\Psi)p' + (\gamma_{22} - B\Psi)q' + (\gamma_{23} - F\Psi)r' &= \mu\theta_2\\ (\gamma_{13} - G\Psi)p' + (\gamma_{23} - F\Psi)q' + (\gamma_{33} - C\Psi)r' &= \mu\theta_3\end{aligned}\right\}.$$

We also have the relation

$$\theta_1 p' + \theta_2 q' + \theta_3 r' = 0.$$

Hence eliminating determinantally the four quantities p', q', r', μ, which occur linearly and homogeneously, we have

$$\begin{vmatrix} \gamma_{11} - A\Psi, & \gamma_{12} - H\Psi, & \gamma_{13} - G\Psi, & \theta_1 \\ \gamma_{12} - H\Psi, & \gamma_{22} - B\Psi, & \gamma_{23} - F\Psi, & \theta_2 \\ \gamma_{13} - G\Psi, & \gamma_{23} - F\Psi, & \gamma_{33} - C\Psi, & \theta_3 \\ \theta_1, & \theta_2, & \theta_3, & 0 \end{vmatrix} = 0$$

as an equation for the principal values of Ψ (and therefore for the principal values of γ). When the determinant is expanded, the equation becomes

$$\Psi^2\Sigma a\theta_1^2 - \Psi\Sigma\{(B\gamma_{33} - 2F\gamma_{23} + C\gamma_{22})\theta_1^2\} + \Sigma\{(\gamma_{22}\gamma_{33} - \gamma_{23}^2)\theta_1^2\} = 0.$$

We write

$$\mathbf{H} = \Sigma\{(B\gamma_{33} - 2F\gamma_{23} + C\gamma_{22})\theta_1^2\}, \quad \mathbf{K} = \Sigma\{(\gamma_{22}\gamma_{33} - \gamma_{23}^2)\theta_1^2\};$$

then, when we substitute for Ψ from the relation

$$\frac{1}{\gamma} = -\frac{\Omega}{D^{\frac{1}{2}}}\Psi,$$

the equation for γ_1 and γ_2, the principal values of γ, becomes

$$\frac{D^2}{\Omega^2}\frac{1}{\gamma^2} + \frac{D^{\frac{1}{2}}\mathbf{H}}{\Omega}\frac{1}{\gamma} + \mathbf{K} = 0.$$

We write

$$\mathfrak{G} = \frac{1}{\gamma_1\gamma_2} = \frac{\Omega^2}{D^2}\mathbf{K} = \frac{\Sigma\{(\gamma_{22}\gamma_{33} - \gamma_{23}^2)\theta_1^2\}}{(\Sigma a\theta_1^2)^2}\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix},$$

$$\mathfrak{H} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} = -\frac{\Omega\mathbf{H}}{D^{\frac{3}{2}}} = -\frac{\Sigma\{(B\gamma_{33} - 2F\gamma_{23} + C\gamma_{22})\theta_1^2\}}{(\Sigma a\theta_1^2)^{\frac{3}{2}}}\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix}^{\frac{1}{2}}.$$

Thus $\mathfrak{G}$ is a Gauss (product) measure of the geodesic curvature of the whole surface, while $\mathfrak{g}$ is a second (additive) measure of the geodesic curvature of the whole surface, at any point: these measures belonging not to any particular geodesic but to the surface generally.

Again, the equations connected with the principal values of the regional geodesic curvature can be taken in the form

$$\begin{aligned}
\gamma_{11}p' + \gamma_{12}q' + \gamma_{13}r' &= (Ap' + Hq' + Gr')\Psi + \mu\theta_1,\\
\gamma_{12}p' + \gamma_{22}q' + \gamma_{23}r' &= (Hp' + Bq' + Fr')\Psi + \mu\theta_2,\\
\gamma_{13}p' + \gamma_{23}q' + \gamma_{33}r' &= (Gp' + Fq' + Cr')\Psi + \mu\theta_3;
\end{aligned}$$

whence, by eliminating Ψ and μ determinantally, we have

$$\begin{vmatrix}
\gamma_{11}p' + \gamma_{12}q' + \gamma_{13}r', & Ap' + Hq' + Gr', & \theta_1\\
\gamma_{12}p' + \gamma_{22}q' + \gamma_{23}r', & Hp' + Bq' + Fr', & \theta_2\\
\gamma_{13}p' + \gamma_{23}q' + \gamma_{33}r', & Gp' + Fq' + Cr', & \theta_3
\end{vmatrix} = 0,$$

a homogeneous quadratic in p', q', r'. This quadratic equation is

$$\begin{vmatrix} \gamma_{11}, & A, & \theta_1\\ \gamma_{12}, & H, & \theta_2\\ \gamma_{13}, & G, & \theta_3 \end{vmatrix} p'^2 + \begin{vmatrix} \gamma_{12}, & H, & \theta_1\\ \gamma_{22}, & B, & \theta_2\\ \gamma_{23}, & F, & \theta_3 \end{vmatrix} q'^2 + \begin{vmatrix} \gamma_{13}, & G, & \theta_1\\ \gamma_{23}, & F, & \theta_2\\ \gamma_{33}, & C, & \theta_3 \end{vmatrix} r'^2$$

$$+ \left\{ \begin{vmatrix} \gamma_{11}, & H, & \theta_1\\ \gamma_{12}, & B, & \theta_2\\ \gamma_{13}, & F, & \theta_3 \end{vmatrix} + \begin{vmatrix} \gamma_{12}, & A, & \theta_1\\ \gamma_{22}, & H, & \theta_2\\ \gamma_{23}, & G, & \theta_3 \end{vmatrix} \right\} p'q'$$

$$+ \left\{ \begin{vmatrix} \gamma_{12}, & G, & \theta_1\\ \gamma_{22}, & F, & \theta_2\\ \gamma_{23}, & C, & \theta_3 \end{vmatrix} + \begin{vmatrix} \gamma_{13}, & H, & \theta_1\\ \gamma_{23}, & B, & \theta_2\\ \gamma_{33}, & F, & \theta_3 \end{vmatrix} \right\} q'r'$$

$$+ \left\{ \begin{vmatrix} \gamma_{13}, & A, & \theta_1\\ \gamma_{23}, & H, & \theta_2\\ \gamma_{33}, & G, & \theta_3 \end{vmatrix} + \begin{vmatrix} \gamma_{11}, & G, & \theta_1\\ \gamma_{12}, & F, & \theta_2\\ \gamma_{13}, & C, & \theta_3 \end{vmatrix} \right\} r'p' = 0,$$

which may be written

$$\bar{A}p'^2 + \bar{B}q'^2 + \bar{C}r'^2 + 2\bar{F}q'r' + 2\bar{G}r'p' + 2\bar{H}p'q' = 0.$$

Taken in conjunction with

$$\theta_1 p' + \theta_2 q' + \theta_3 r' = 0,$$

the quadratic equation determines the two directions of principal regional geodesic curvature.

These two directions are to be associated with the two principal values of the regional geodesic curvature. For each of the directions and the associated value of Ψ, the connecting relations are

$$\frac{1}{\theta_1}\{(\gamma_{11}-A\Psi)p'+(\gamma_{12}-H\Psi)q'+(\gamma_{13}-G\Psi)r'\}$$

$$=\frac{1}{\theta_2}\{(\gamma_{12}-H\Psi)p'+(\gamma_{22}-B\Psi)q'+(\gamma_{23}-F\Psi)r'\}$$

$$=\frac{1}{\theta_3}\{(\gamma_{13}-G\Psi)p'+(\gamma_{23}-F\Psi)q'+(\gamma_{33}-C\Psi)r'\}.$$

Moreover, it is easy to establish the property that the two directions of principal regional geodesic curvature are perpendicular to one another. Let

$$p_1', q_1', r_1', \Psi_1; \; p_2', q_2', r_2', \Psi_2;$$

denote the two directions and the associated values of Ψ. There are three equations of the type

$$\gamma_{11}p_1'+\gamma_{12}q_1'+\gamma_{13}r_1'=\Psi_1(Ap_1'+Hq_1'+Gr_1')+\mu_1\theta_1,$$

so that, multiplying these three equations by p_2', q_2', r_2', and adding, we have

$$\Sigma\gamma_{11}p_1'p_2'=\Psi_1\Sigma Ap_1'p_2'+\mu_1(\theta_1p_2'+\theta_2q_2'+\theta_3r_2')$$
$$=\Psi_1\Sigma Ap_1'p_2';$$

and there are three equations of the type

$$\gamma_{11}p_2'+\gamma_{12}q_2'+\gamma_{13}r_2'=\Psi_2(Ap_2'+Hq_2'+Gr_2')+\mu_2\theta_1,$$

so that, multiplying these three equations by p_1', q_1', r_1', and adding, we have

$$\Sigma\gamma_{11}p_1'p_2'=\Psi_2\Sigma Ap_1'p_2'+\mu_2(\theta_1p_1'+\theta_2q_1'+\theta_3r_1')$$
$$=\Psi_2\Sigma Ap_1'p_2'.$$

Now Ψ_1 and Ψ_2 are unequal in general; hence we have

$$\Sigma\gamma_{11}p_1'p_2'=0, \quad \Sigma Ap_1'p_2'=0.$$

The former relation shews that the two directions p_1', q_1', r_1', and p_2', q_2', r_2', are conjugate. The latter relation shews that these two directions are perpendicular: that is, the two directions of principal regional geodesic curvature on the surface are perpendicular.

Also, there are two directions, given by

$$\Sigma\gamma_{11}p'^2=0, \quad \theta_1p'+\theta_2q'+\theta_3r'=0.$$

They determine curves of zero regional geodesic curvature on the surface, in the sense that superficial geodesics touching these curves have inflexional contact with the regional geodesics in the same direction through the point.

We leave, as an exercise*, an analytical proof of the property that the principal directions of regional geodesic curvature, as given by the equations

$$\begin{vmatrix} \gamma_{11}p' + \gamma_{12}q' + \gamma_{13}r', & Ap' + Hq' + Gr', & \theta_1 \\ \gamma_{12}p' + \gamma_{22}q' + \gamma_{23}r', & Hp' + Bq' + Fr', & \theta_2 \\ \gamma_{13}p' + \gamma_{23}q' + \gamma_{33}r', & Gp' + Fq' + Cr', & \theta_3 \end{vmatrix} = 0$$

and

$$\theta_1 p' + \theta_2 q' + \theta_3 r' = 0,$$

bisect the angles between the two directions

$$\Sigma\gamma_{11}p'^2 = 0, \quad \theta_1 p' + \theta_2 q' + \theta_3 r' = 0.$$

Indicatrix quadric and asymptotic cone for regional geodesic curvature of a superficial geodesic.

349. In connection with the circular curvature of a regional geodesic, we had (§§ 290, 291) an indicatrix quadric and its asymptotic cone. So also, in connection with the regional geodesic curvature of a superficial geodesic, there is an indicatrix quadric and there is its asymptotic cone. As before (*l.c.*), we take

$$A^{\frac{1}{2}}p' = \lambda, \quad B^{\frac{1}{2}}q' = \mu, \quad C^{\frac{1}{2}}r' = \nu,$$

$$(BC)^{-\frac{1}{2}}F = \cos\alpha, \quad (CA)^{-\frac{1}{2}}G = \cos\beta, \quad (AB)^{-\frac{1}{2}}H = \cos\gamma,$$

so that the permanent relation becomes

$$\lambda^2 + \mu^2 + \nu^2 + 2\mu\nu\cos\alpha + 2\nu\lambda\cos\beta + 2\lambda\mu\cos\gamma = 1.$$

Further, we take quantities $\bar{x}$, $\bar{y}$, $\bar{z}$, r, such that

$$\bar{x} = \lambda r, \quad \bar{y} = \mu r, \quad \bar{z} = \nu r,$$

so that

$$r^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2 + 2\bar{y}\bar{z}\cos\alpha + 2\bar{z}\bar{x}\cos\beta + 2\bar{x}\bar{y}\cos\gamma.$$

Also, we have a quadric

$$U = \frac{\gamma_{11}}{A}\bar{x}^2 + \frac{\gamma_{22}}{B}\bar{y}^2 + \frac{\gamma_{33}}{C}\bar{z}^2 + 2\frac{\gamma_{23}}{F}\bar{y}\bar{z}\cos\alpha + 2\frac{\gamma_{31}}{G}\bar{z}\bar{x}\cos\beta + 2\frac{\gamma_{12}}{H}\bar{x}\bar{y}\cos\gamma = e,$$

where e is a constant. Then if r be the radius vector of this quadric in any direction, the radius γ of regional geodesic curvature of a superficial geodesic in that direction is given by

$$\gamma = -\frac{D^{\frac{1}{2}}}{e\Omega}r^2.$$

The quadric is an *indicatrix quadric* for the regional geodesic curvature of superficial geodesics; and its *asymptotic cone* is

$$U = 0,$$

being the locus of directions in the region where the regional geodesic has second contact with the superficial geodesic.

* The necessary algebra would apply whatever be the significance of the equation $\Sigma\gamma_{11}p'^2 = 0$.

Torsion and tilt of superficial geodesics: regional, spatial.

350. In considering the various curvatures of the surface, relative to the enfolding region and now not relative to the enclosing homaloidal quadruple space, there occurs a triply orthogonal frame specially appertaining to the region in connection with the surface. It is associated with any direction in the tangent plane of the surface and, as explained (§ 347), its three axial lines are

(i) the space-direction x', y', z', v', defining direction-coordinates p', q', r', within the region along a direction tangential to the surface:

(ii) the space-direction $\bar{l}$, $\bar{m}$, $\bar{n}$, $\bar{k}$, defining direction-coordinates $\frac{dp}{dt}$, $\frac{dq}{dt}$, $\frac{dr}{dt}$, within the region along a direction in the tangent plane to the surface perpendicular to x', y', z', v': and

(iii) the space-direction $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{V}$, normal to the surface and defining direction-coordinates $\frac{dp}{dv}$, $\frac{dq}{dv}$, $\frac{dr}{dv}$, within the region.

The geometrical relations of superficial geodesics to regional geodesics with a common tangent give rise to three kinds of regional curvature. There is the arc-rate of deviation of the tangent to the superficial geodesic from the tangent to the regional geodesic; this measure has been called (§ 345) the regional geodesic curvature. There is the arc-rate of the twist of the plane, containing the tangent to the geodesic and the normal to the surface, the twist being measured for deviation within the region; this measure will be called the *regional geodesic torsion*. Let the former measure be denoted by $1/\bar{\rho}$: the value of $\bar{\rho}$ is known (*l.c.*) to be γ, but the result will be established in another manner (§ 351). Let the measure of regional torsion be denoted by $1/\bar{\sigma}$.

A full orthogonal frame, for the quadruple space and embodying the preceding triple frame arising with the superficial geodesic within the region, is completed by associating, with this triple frame in the region, the normal to the region with direction-cosines X, Y, Z, V. The arc-rate of change of this normal direction, being the arc-rate of rotation of the tangent flat to the region, will be called the *regional tilt*; and $1/\bar{\tau}$ is used to denote this measure of regional tilt.

It is to be noted that these three regional measures of curvature for the tangential geodesic are, each of them, specific to the frame in the region. Now the region, and therefore the surface contained in the region, exist in the quadruple space: and thus there are three spatial measures of curvature for the superficial geodesic. The circular curvature of this superficial geodesic in

the homaloidal quadruple space is the arc-rate of its deviation from its linear tangent in the space; thus it is an entirely different magnitude from the regional geodesic curvature, which is the arc-rate of its deviation from the regional geodesic having the same linear tangent in the quadruple space. But torsion and tilt are arc-deviations of the inclinations of planes and of flats; and if the plane and the flat, together, are not the same for the region and for the quadruple space, the measure of spatial torsion and that of regional torsion will not be the same, and the measure of spatial tilt and that of regional tilt will not be the same.

Now for the superficial geodesic as a configuration primarily regarded with reference to the quadruple space, we already (§ 233) have had a quadruply orthogonal frame. The axes of this quadruple frame have for their respective sets of direction-cosines

$$\begin{array}{cccc} x', & y', & z', & v'; \\ l, & m, & n, & k; \\ \bar{l}, & \bar{m}, & \bar{n}, & \bar{k}; \\ A, & B, & C, & D. \end{array}$$

Here $\bar{l}$, $\bar{m}$, $\bar{n}$, $\bar{k}$, are the direction-cosines of a line in the tangent plane of the surface, perpendicular to x', y', z', v'. The magnitudes l, m, n, k, are the direction-cosines of the principal normal to the geodesic; they are connected with the directions X, Y, Z, V, and $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{V}$, by the relations

$$l = X\cos\psi + \bar{X}\sin\psi, \quad m = Y\cos\psi + \bar{Y}\sin\psi,$$
$$n = Z\cos\psi + \bar{Z}\sin\psi, \quad k = V\cos\psi + \bar{V}\sin\psi.$$

Finally, A, B, C, D, are the direction-cosines of the normal to the flat which contains the first three directions; they are connected with the directions X, Y, Z, V, and $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{V}$, by the relations

$$A = X\sin\psi - \bar{X}\cos\psi, \quad B = Y\sin\psi - \bar{Y}\cos\psi,$$
$$C = Z\sin\psi - \bar{Z}\cos\psi, \quad D = V\sin\psi - \bar{V}\cos\psi.$$

In proceeding along the superficial geodesic, the two directions

$$x',\ y',\ z',\ v';\quad \bar{l},\ \bar{m},\ \bar{n},\ \bar{k};$$

are the same for the two frames. For the frame connected specifically with the region, the two remaining directions have

$$\bar{X},\ \bar{Y},\ \bar{Z},\ \bar{V};\quad X,\ Y,\ Z,\ V;$$

for their direction-cosines; while, for the spatial frame, the two remaining directions have

$$l,\ m,\ n,\ k;\quad A,\ B,\ C,\ D;$$

for their direction-cosines. The plane, containing the two respective directions in the regional frame, can be obtained from the plane, containing the two

respective directions in the spatial frame, by means of a rotation through an angle ψ round the tangent plane, which is orthogonal to the plane in both its positions. Accordingly, the measures of torsion and of tilt will not be the same for the regional frame as for the spatial frame, though of course they are not independent of one another.

Measures of regional torsion and tilt.

351. We proceed to obtain, first, the regional measures. The generalised Frenet equations for the superficial geodesic in its regional variations are four sets of the type

$$\left.\begin{aligned} \frac{d}{ds}x' &= \frac{\bar{X}}{\bar{\rho}} \\ \frac{d}{ds}\bar{X} &= -\frac{x'}{\bar{\rho}} + \frac{\bar{l}}{\bar{\sigma}} \\ \frac{d}{ds}\bar{l} &= -\frac{\bar{X}}{\bar{\sigma}} + \frac{X}{\bar{\tau}} \\ \frac{d}{ds}X &= -\frac{\bar{l}}{\bar{\tau}} \end{aligned}\right\};$$

the other three sets similarly affect y', $\bar{Y}$, $\bar{m}$, Y; z', $\bar{Z}$, $\bar{n}$, Z; v', $\bar{V}$, $\bar{k}$, V.

From the second of these equations, used in the quartette of which it is typical, we have

$$-\frac{1}{\bar{\rho}} = \Sigma x' \frac{d\bar{X}}{ds}.$$

But we have $\Sigma x'\bar{X} = 0$, so that

$$\begin{aligned} \Sigma x' \frac{d\bar{X}}{ds} &= -\Sigma \bar{X} x'' \\ &= -\frac{1}{\rho_0}\Sigma \bar{X}\rho_0 x'' \\ &= -\frac{1}{\rho_0}\Sigma \bar{X} l = -\frac{\sin\psi}{\rho_0} = -\frac{1}{\gamma}, \end{aligned}$$

so that

$$\bar{\rho} = \gamma,$$

as is to be expected.

This result can be obtained directly, without the use of the expression for γ, by means of the formulæ for $\Sigma x_i \bar{X}_j$, for $i, j, = 1, 2, 3$. For

$$\begin{aligned} &\Sigma x' \frac{d\bar{X}}{ds} \\ &= \Sigma (x_1 p' + x_2 q' + x_3 r')(\bar{X}_1 p' + \bar{X}_2 q' + \bar{X}_3 r') \\ &= p'^2(\Sigma x_1 \bar{X}_1) + q'^2(\Sigma x_2 \bar{X}_2) + r'^2(\Sigma x_3 \bar{X}_3) \\ &\quad + q'r'(\Sigma x_2 \bar{X}_3 + x_3 \bar{X}_2) + r'p'(x_3 \bar{X}_1 + x_1 \bar{X}_3) + p'q'(\Sigma x_1 \bar{X}_2 + \Sigma x_2 \bar{X}_1): \end{aligned}$$

when the values of the various coefficients on the right-hand side are substituted, we have

$$-\frac{1}{\bar{\rho}}=\frac{\Omega}{D^{\frac{1}{2}}}\Sigma\gamma_{11}p'^2$$
$$-\Omega^3D^{-\frac{3}{2}}\{\ (P\gamma_{11}+Q\gamma_{12}+R\gamma_{13})(\theta_1p'^2+\theta_2p'q'+\theta_3p'r')$$
$$+(P\gamma_{12}+Q\gamma_{22}+R\gamma_{23})(\theta_1p'q'+\theta_2q'^2+\theta_3q'r')$$
$$+(P\gamma_{13}+Q\gamma_{23}+R\gamma_{33})(\theta_1p'r'+\theta_2q'r'+\theta_3r'^2)\}$$
$$=\frac{\Omega}{D^{\frac{1}{2}}}\Psi=-\frac{1}{\gamma},$$

from the result in § 348: that is, as before,

$$\bar{\rho}=\gamma.$$

Again using the second of the typical Frenet equations in its quartette, we have

$$\frac{1}{\bar{\sigma}}=\Sigma\bar{l}\frac{d\bar{X}}{ds}.$$

Now we have (§ 347)

$$\bar{l}D^{\frac{1}{2}}=\frac{1}{2}\begin{vmatrix} x_1, & x_2, & x_3 \\ \frac{\partial U}{\partial p'}, & \frac{\partial U}{\partial q'}, & \frac{\partial U}{\partial r'} \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix},$$

where

$$U=(A, B, C, F, G, H)(p', q', r')^2;$$

and from the values of the derivatives X_1, X_2, X_3, (§ 341), we have

$$\frac{d\bar{X}}{ds}=\frac{1}{2}X\frac{\Omega}{D^{\frac{1}{2}}}\left(P\frac{\partial W}{\partial p'}+Q\frac{\partial W}{\partial q'}+R\frac{\partial W}{\partial r'}\right)-\frac{1}{2}\bar{X}\frac{\Omega^2}{D}\left(P\frac{\partial\Psi}{\partial p'}+Q\frac{\partial\Psi}{\partial q'}+R\frac{\partial\Psi}{\partial r'}\right)$$
$$+\frac{1}{2}\frac{1}{\Omega D^{\frac{1}{2}}}\left\{(ax_1+hx_2+gx_3)\frac{\partial\Psi}{\partial p'}+(hx_1+bx_2+fx_3)\frac{\partial\Psi}{\partial q'}+(gx_1+fx_2+cx_3)\frac{\partial\Psi}{\partial r'}\right\}.$$

Also

$$\Sigma X\bar{l}=0,\quad \Sigma\bar{X}\bar{l}=0;$$

hence

$$\Omega D\Sigma\bar{l}\frac{d\bar{X}}{ds}=\frac{1}{4}\Omega^2\begin{vmatrix} \frac{\partial\Psi}{\partial p'}, & \frac{\partial\Psi}{\partial q'}, & \frac{\partial\Psi}{\partial r'} \\ \frac{\partial U}{\partial p'}, & \frac{\partial U}{\partial q'}, & \frac{\partial U}{\partial r'} \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix},$$

that is, the regional torsion of the superficial geodesic is

$$\frac{1}{\bar{\sigma}}=\frac{\Omega}{D}\begin{vmatrix} \gamma_{11}p'+\gamma_{12}q'+\gamma_{13}r', & \gamma_{12}p'+\gamma_{22}q'+\gamma_{23}r', & \gamma_{13}p'+\gamma_{23}q'+\gamma_{33}r' \\ Ap'+Hq'+Gr', & Hp'+Bq'+Fr', & Gp'+Fq'+Cr' \\ \theta_1 & \theta_2 & \theta_3 \end{vmatrix}.$$

Finally, from the fourth of the Frenet typical equations, taken in its quartette, we have

$$\begin{aligned}\frac{1}{\bar{\tau}} &= -\Sigma \bar{l} \frac{dX}{ds} \\ &= -\Sigma \bar{l} (X_1 p' + X_2 q' + X_3 r') \\ &= -\frac{1}{D^{\frac{1}{2}}} \Sigma \tfrac{1}{2} \begin{vmatrix} x_1, & x_2, & x_3 \\ \frac{\partial U}{\partial p'}, & \frac{\partial U}{\partial q'}, & \frac{\partial U}{\partial r'} \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix} (X_1 p' + X_2 q' + X_3 r').\end{aligned}$$

Now, from the relation $\Sigma X x_1 = 0$, we have

$$\begin{aligned}\Sigma x_1 X_1 &= -\Sigma X x_{11} = -L, \\ \Sigma x_1 X_2 &= -\Sigma X x_{12} = -M, \\ \Sigma x_1 X_3 &= -\Sigma X x_{13} = -K,\end{aligned}$$

and therefore

$$\Sigma x_1 (X_1 p' + X_2 q' + X_3 r') = -(Lp' + Mq' + Kr');$$

and there are similar values for

$$\Sigma x_2 (X_1 p' + X_2 q' + X_3 r') \text{ and } \Sigma x_3 (X_1 p' + X_2 q' + X_3 r').$$

Thus

$$\frac{1}{\bar{\tau}} = \frac{1}{D^{\frac{1}{2}}} \begin{vmatrix} Lp' + Mq' + Kr', & Mp' + Nq' + Ir', & Kp' + Iq' + Jr' \\ Ap' + Hq' + Gr', & Hp' + Bq' + Fr', & Gp' + Fq' + Cr' \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix},$$

giving the regional tilt of the superficial geodesic.

Before proceeding to construct the measures of the spatial torsion and the spatial tilt of a superficial geodesic, it is convenient to consider the surface, rather in relation to the plenary quadruple space than in relation to the enfolding region.

Regional surface as a configuration in quadruple space.

352. In the discussion of curves lying in surfaces themselves contained in homaloidal triple space, it proves necessary to consider the relations of the curves not merely to the surfaces but also their relations to that space. Accordingly, after the discussion of properties of surfaces lying in a region itself contained in homaloidal quadruple space, we proceed to consider the space-relations of such surfaces.

We have seen that, when a perpendicular is drawn from a point near x, y, z, v, to the tangent plane at that point to the surface, its direction lies in the orthogonal plane given by the equations

$$\begin{Vmatrix} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ X\ , & Y\ , & Z\ , & V \\ \bar{X}\ , & \bar{Y}\ , & \bar{Z}\ , & \bar{V} \end{Vmatrix} = 0,$$

or by the equivalent equations

$$\begin{Vmatrix} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ X\ , & Y\ , & Z\ , & V \\ \theta_x\ , & \theta_y\ , & \theta_z\ , & \theta_v \end{Vmatrix} = 0.$$

In particular, this direction l, m, n, k, is that of the radius of circular curvature of the geodesic as a curve in quadruple space. The centre of circular curvature lies in this plane and therefore lies in the flat

$$\Sigma(\bar{x}-x)x' = 0$$

normal to the geodesic; and it also lies in the consecutive normal flat which has, for its equation,

$$\Sigma(\bar{x}-x-x'\delta)(x'+x''\delta) = 0:$$

that is, we may take the equations of the plane of cleavage of the two flats in the form

$$\Sigma(\bar{x}-x)x' = 0, \quad \Sigma(\bar{x}-x)x'' = 1,$$

and the latter equation is

$$\Sigma(\bar{x}-x)l = \rho_0.$$

Consider now the intersection of the orthogonal plane, which lies in the normal flat, and the consecutive normal flat. The intersection is a straight line; and the straight line passes through the centre of circular curvature of the geodesic. Any point in the orthogonal plane has coordinates of the form

$$\begin{aligned} \bar{x}-x &= \alpha X + \beta\bar{X}, \\ \bar{y}-y &= \alpha Y + \beta\bar{Y}, \\ \bar{z}-z &= \alpha Z + \beta\bar{Z}, \\ \bar{v}-v &= \alpha V + \beta\bar{V}; \end{aligned}$$

and the direction-cosines l, m, n, k, are such that

$$\begin{aligned} l &= X\cos\psi + \bar{X}\sin\psi, \quad m = Y\cos\psi + \bar{Y}\sin\psi, \\ n &= Z\cos\psi + \bar{Z}\sin\psi, \quad k = V\cos\psi + \bar{V}\sin\psi. \end{aligned}$$

Thus any point in the orthogonal plane has coordinates

$$\begin{aligned} \bar{x}-x-\ l\rho_0 &= X(\alpha-\rho_0\cos\psi) + \bar{X}(\beta-\rho_0\sin\psi), \\ \bar{y}-y-m\rho_0 &= Y(\alpha-\rho_0\cos\psi) + \bar{Y}(\beta-\rho_0\sin\psi), \\ \bar{z}-z-\ n\rho_0 &= Z(\alpha-\rho_0\cos\psi) + \bar{Z}(\beta-\rho_0\sin\psi), \\ \bar{v}-v-\ k\rho_0 &= V(\alpha-\rho_0\cos\psi) + \bar{V}(\beta-\rho_0\sin\psi); \end{aligned}$$

and therefore, for appropriate values of α and β, these values give the coordinates of any point on the line of intersection of the orthogonal plane by the consecutive normal flat. Along this intersection, the equation

$$\Sigma\,(\bar{x}-x)\,l=\rho_0$$

is satisfied, that is, the equation

$$\Sigma\,(\bar{x}-x-l\rho_0)\,l=0$$

is satisfied. Substituting these values of $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{v}$, and the preceding values of l, m, n, k, we find that, for the line of intersection in question, α and β satisfy the relation

$$\Sigma\,(X\cos\psi+\bar{X}\sin\psi)\,\{X\,(\alpha-\rho_0\cos\psi)+\bar{X}\,(\beta-\rho_0\sin\psi)\}=0,$$

that is,

$$(\alpha-\rho_0\cos\psi)\cos\psi+(\beta-\rho_0\sin\psi)\sin\psi=0.$$

Accordingly, we take

$$\alpha-\rho_0\cos\psi=\Lambda\sin\psi,\quad \beta-\rho_0\sin\psi=-\Lambda\cos\psi,$$

where Λ is a new parameter; and then the foregoing coordinates, which now belong only to points on the line of intersection, are

$$\begin{aligned}
\bar{x}-x-\ l\rho_0&=\Lambda\,(X\sin\psi-\bar{X}\cos\psi),\\
\bar{y}-y-m\rho_0&=\Lambda\,(Y\sin\psi-\bar{Y}\cos\psi),\\
\bar{z}-z-\ n\rho_0&=\Lambda\,(Z\sin\psi-\bar{Z}\cos\psi),\\
\bar{v}-v-\ k\rho_0&=\Lambda\,(V\sin\psi-\bar{V}\cos\psi),
\end{aligned}$$

where Λ is a variable parameter. Thus the equations of the line of intersection of the orthogonal plane at x, y, z, v, by the consecutive normal flat are

$$\frac{\bar{x}-x-l\rho_0}{X\sin\psi-\bar{X}\cos\psi}=\frac{\bar{y}-y-m\rho_0}{Y\sin\psi-\bar{Y}\cos\psi}=\frac{\bar{z}-z-n\rho_0}{Z\sin\psi-\bar{Z}\cos\psi}=\frac{\bar{v}-v-k\rho_0}{V\sin\psi-\bar{V}\cos\psi}.$$

The direction-cosines of this line are

$$X\sin\psi-\bar{X}\cos\psi,\quad Y\sin\psi-\bar{Y}\cos\psi,\quad Z\sin\psi-\bar{Z}\cos\psi,\quad V\sin\psi-\bar{V}\cos\psi.$$

We already have three perpendicular directions

x', y', z', v', the tangent to the geodesic in the tangent plane,

l, m, n, k, the radius of circular curvature of the geodesic,

$\bar{l}, \bar{m}, \bar{n}, \bar{k}$, the tangent-plane direction, perpendicular to x', y', z', v'.

Now

$$\begin{aligned}
\Sigma\,(X\sin\psi-\bar{X}\cos\psi)\,x'&=(\Sigma Xx')\sin\psi-(\Sigma\bar{X}x')\cos\psi=0;\\
\Sigma\,(X\sin\psi-\bar{X}\cos\psi)\,l&=(\Sigma Xl\ )\sin\psi-(\Sigma\bar{X}l\ )\cos\psi\\
&=\ \cos\psi\sin\psi-\ \sin\psi\cos\psi=0;\\
\Sigma\,(X\sin\psi-\bar{X}\cos\psi)\,\bar{l}&=(\Sigma X\bar{l}\ )\sin\psi-(\Sigma\bar{X}\bar{l}\ )\cos\psi=0.
\end{aligned}$$

Thus the line of intersection is the fourth axis for a quadruply orthogonal frame in space, the direction-cosines of the axes of which are

$$\begin{array}{cccc} x' & y' & z' & v' \\ X\cos\psi + \bar{X}\sin\psi, & Y\cos\psi + \bar{Y}\sin\psi, & Z\cos\psi + \bar{Z}\sin\psi, & V\cos\psi + \bar{V}\sin\psi, \\ \bar{l} & \bar{m} & \bar{n} & \bar{k} \\ X\sin\psi - \bar{X}\cos\psi, & Y\sin\psi - \bar{Y}\cos\psi, & Z\sin\psi - \bar{Z}\cos\psi, & V\sin\psi - \bar{V}\cos\psi. \end{array}$$

Manifestly when this frame is moved, always remaining an orthogonal frame, by a rotation round the tangent plane through an angle ψ, it assumes the position of the orthogonal frame of § 347 belonging to the geodesic on the surface specially chosen in connection with the enfolding region.

In the investigation connected with an orthogonal frame for a geodesic on a surface in § 233, where the surface is defined solely in quadruple space without a postulated knowledge of any enclosing region, the direction-cosines of the fourth axis were denoted by A, B, C, D: in fact,

$$A = X\sin\psi - \bar{X}\cos\psi, \quad B = Y\sin\psi - \bar{Y}\cos\psi,$$
$$C = Z\sin\psi - \bar{Z}\cos\psi, \quad D = V\sin\psi - \bar{V}\cos\psi,$$

with, of course, the former significance for A, B, C, D. As on p. 208, we write

$$l = X\cos\psi + \bar{X}\sin\psi, \quad m = Y\cos\psi + \bar{Y}\sin\psi,$$
$$n = Z\cos\psi + \bar{Z}\sin\psi, \quad k = V\cos\psi + \bar{V}\sin\psi.$$

Measures of spatial torsion and tilt.

353. We now are in a position to use the Frenet equations in connection with this orthogonal frame, and thus to obtain the spatial curvatures (§ 350) of the superficial geodesic. Any typical four direction-cosines, one from each set, are $x', l, \bar{l}, A$; and for them, as typical of the quartette, there is the group of equations

$$\left.\begin{aligned} \frac{dx'}{ds} &= \frac{l}{\rho_0} \\ \frac{dl}{ds} &= -\frac{x'}{\rho_0} + \frac{\bar{l}}{\sigma_0} \\ \frac{d\bar{l}}{ds} &= -\frac{l}{\sigma_0} + \frac{A}{\tau_0} \\ \frac{dA}{ds} &= -\frac{\bar{l}}{\tau_0} \end{aligned}\right\},$$

where $1/\rho_0$, $1/\sigma_0$, $1/\tau_0$, are the spatial measures of circular curvature, of torsion, and of tilt, respectively.

The first quartette is, effectively, the analytical expression of the property that the principal normal of the geodesic is the radius of curvature of the normal section (§ 344) of the surface. Also from the second equation, taken with the rest of its quartette,

$$-\frac{1}{\rho_0} = \Sigma x' \frac{dl}{ds}$$
$$= -\Sigma x'' l,$$

because $\Sigma x' l = 0$: that is, as $l = \rho_0 x''$, we have the known relation

$$\frac{1}{\rho_0^2} = \Sigma x''^2.$$

Next, we have

$$\frac{dl}{ds} = -A\frac{d\psi}{ds} + X' \cos\psi + \bar{X}' \sin\psi,$$

$$\frac{dA}{ds} = \quad l\frac{d\psi}{ds} + X' \sin\psi - \bar{X}' \cos\psi.$$

Hence by the second of the Frenet equations, taken with the rest of its quartette, we have

$$\frac{1}{\sigma_0} = \Sigma \bar{l} \frac{dl}{ds}$$
$$= -(\Sigma A \bar{l})\frac{d\psi}{ds} + (\Sigma \bar{l} X')\cos\psi + (\Sigma \bar{l}\bar{X}')\sin\psi.$$

But we have

$$\Sigma A \bar{l} = 0,$$

because the frame is orthogonal; also, by § 351,

$$\frac{1}{\bar{\tau}} = -\Sigma \bar{l} X', \quad \frac{1}{\bar{\sigma}} = \Sigma \bar{l}\bar{X}',$$

where $1/\bar{\sigma}$ and $1/\bar{\tau}$ are the regional measures of torsion and tilt: consequently

$$\frac{1}{\sigma_0} = \frac{1}{\bar{\sigma}} \sin\psi - \frac{1}{\bar{\tau}} \cos\psi.$$

Similarly, from the fourth equation in its quartette,

$$-\frac{1}{\tau_0} = \Sigma \bar{l} \frac{dA}{ds}$$
$$= (\Sigma l \bar{l})\frac{d\psi}{ds} + (\Sigma \bar{l} X')\sin\psi - (\Sigma \bar{l}\bar{X}')\cos\psi$$
$$= -\frac{\sin\psi}{\bar{\tau}} - \frac{\cos\psi}{\bar{\sigma}},$$

that is,

$$\frac{1}{\tau_0} = \frac{\cos\psi}{\bar{\sigma}} + \frac{\sin\psi}{\bar{\tau}}.$$

Thus the regional measures $1/\bar{\sigma}$ and $1/\bar{\tau}$ are expressible, in terms of the spatial measures, by the equations

$$\left.\begin{aligned} \frac{1}{\bar{\sigma}} &= \frac{\sin\psi}{\sigma_0} + \frac{\cos\psi}{\tau_0} \\ \frac{1}{\bar{\tau}} &= -\frac{\cos\psi}{\sigma_0} + \frac{\sin\psi}{\tau_0} \end{aligned}\right\}.$$

Also, as

$$\Sigma l \frac{dA}{ds} = -\frac{1}{\tau_0}\Sigma l\bar{l} = 0,$$

we have

$$0 = (\Sigma l^2)\frac{d\psi}{ds} + (\Sigma l X')\sin\psi - (\Sigma l \bar{X}')\cos\psi.$$

From § 351, we have $\Sigma l X' = 0$, $\Sigma l \bar{X}' = 0$; and therefore

$$\frac{d\psi}{ds} = 0.$$

It therefore appears that a superficial geodesic and a regional geodesic, which touch one another, have their principal normals inclined to one another at an angle which is constant along the superficial geodesic.

NOTE. A particular example of the preceding relations between $\bar{\sigma}$, $\bar{\tau}$, and σ_0, τ_0, is provided by the case, when the region degenerates into a flat. The surface then belongs to a three-dimensional space: the direction of every line in the flat is perpendicular to the normal to the flat, so that ψ is $\frac{1}{2}\pi$; and, as X, Y, Z, V, then are constant, $1/\bar{\tau} = 0$. Thus $1/\tau_0$ also vanishes: manifestly there is no spatial tilt, as there also is no regional tilt. Finally, $\bar{\sigma} = \sigma_0$: the regional torsion and the spatial torsion are the same.

Principal values of regional torsion and tilt.

354. Two inferences may be noted at once.

I. The regional torsion of a superficial geodesic is given by

$$\frac{1}{\bar{\bar{\sigma}}} = \frac{1}{4}\frac{\Omega}{D}\begin{vmatrix} \frac{\partial\Psi}{\partial p'}, & \frac{\partial\Psi}{\partial q'}, & \frac{\partial\Psi}{\partial r'} \\ \frac{\partial U}{\partial p'}, & \frac{\partial U}{\partial q'}, & \frac{\partial U}{\partial r'} \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix} = \frac{1}{4}\frac{\Omega}{D}\bar{\bar{G}}.$$

The regional geodesic curvature is (§ 345) equal to $-\Omega D^{-\frac{1}{2}}\psi$; hence its directions of maximum or minimum value satisfy the equation

$$\bar{\bar{G}} = 0:$$

that is, the regional torsion of a superficial geodesic vanishes along the directions of maximum or minimum regional geodesic curvature.

II. The regional tilt of a superficial geodesic is given by

$$\frac{1}{\bar{\tau}} = \frac{1}{4} D^{-\frac{1}{2}} \begin{vmatrix} \frac{\partial W}{\partial p'}, & \frac{\partial W}{\partial q'}, & \frac{\partial W}{\partial r'} \\ \frac{\partial U}{\partial p'}, & \frac{\partial U}{\partial q'}, & \frac{\partial U}{\partial r'} \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix}.$$

The superficial directions of maximum or minimum circular curvature of a regional geodesic are determined by the equation

$$\begin{vmatrix} \frac{\partial W}{\partial p'}, & \frac{\partial W}{\partial q'}, & \frac{\partial W}{\partial r'} \\ \frac{\partial U}{\partial p'}, & \frac{\partial U}{\partial q'}, & \frac{\partial U}{\partial r'} \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix} = 0,$$

for W has to be made a maximum or a minimum subject to the relations $U = \Sigma A p'^2 = 1$, $\theta_1 p' + \theta_2 q' + \theta_3 r' = 0$: that is, the regional tilt of a superficial geodesic vanishes along those superficial directions which provide a maximum or a minimum value of the circular curvature of a regional geodesic through the direction.

The values of the torsion and the tilt of a superficial geodesic (§ 351) have been obtained, in their relation both to the region which enfolds the surface and to the quadruple space which encloses the region. Each of them is a quadratic function of p', q', r', the direction on the surface satisfying the equation $\theta_1 p' + \theta_2 q' + \theta_3 r' = 0$. The two following results are easily established and will merely be stated, as their import will be obtained by another method.

(i) The regional torsion of a superficial geodesic is a maximum or a minimum for the two directions on the surface which, in addition to $\theta_1 p' + \theta_2 q' + \theta_3 r' = 0$, satisfy the equation

$$(\vartheta_{11}, \vartheta_{22}, \vartheta_{33}, \vartheta_{23}, \vartheta_{31}, \vartheta_{12} \backslash\!\!\backslash \phi, \chi, \psi)^2 = (A, B, C, F, G, H \backslash\!\!\backslash \phi, \chi, \psi \backslash\!\!\backslash \bar{\phi}, \bar{\chi}, \bar{\psi}),$$

where

$$\phi, \chi, \psi = \left\| \begin{matrix} \frac{\partial U}{\partial p'}, & \frac{\partial U}{\partial q'}, & \frac{\partial U}{\partial r'} \\ \theta_1, & \theta_2, & \theta_3 \end{matrix} \right\|,$$

$$\bar{\phi}, \bar{\chi}, \bar{\psi} = \left\| \begin{matrix} \frac{\partial \Psi}{\partial p'}, & \frac{\partial \Psi}{\partial q'}, & \frac{\partial \Psi}{\partial r'} \\ \theta_1, & \theta_2, & \theta_3 \end{matrix} \right\|.$$

(ii) The regional tilt of a superficial geodesic is a maximum or a minimum for the two directions on the surface which, in addition to $\theta_1 p' + \theta_2 q' + \theta_3 r' = 0$, satisfy the equation, modified from the preceding equation by the substitution of the first derivatives of the quadratic ternary form W for the first derivatives of the quadratic ternary form Ψ.

CHAPTER XXI.

CURVATURE OF SURFACES IN A REGION: CURVES.

Modified expression for circular curvature of regional geodesic along a surface.

355. In estimating curvatures of a surface in a region, it is necessary to use

(i) the circular curvature $(1/\rho)$ of a regional geodesic through a direction lying on the surface: and

(ii) the regional geodesic curvature $(1/\gamma)$ of a superficial geodesic through that direction.

It will appear that, for each of these curvatures, there are two principal directions on the surface, a direction being defined as principal when it provides a maximum value or a minimum value of the measure. Moreover, these principal directions for the circular curvature of the regional geodesic are not the same as the principal directions of the regional geodesic curvature of a superficial geodesic; consequently, the principal directions for the magnitude

$$\frac{1}{\rho^2}+\frac{1}{\gamma^2},$$

(being the square of the circular curvature of the superficial geodesic) are distinct from the preceding directions and, as in Chapter XIII, are found to be four in number.

To discuss the respective curvatures, we modify the algebra by utilising the necessary condition

$$\theta_1 p' + \theta_2 q' + \theta_3 r' = 0,$$

so as to remove r' (and, later, r'') from expressions and leave r as a latent variable given by the equation $\theta(p, q, r)=0$ of the surface, thus making p and q as the parametric variables. We take quantities $\bar{E}$, $\bar{F}$, $\bar{G}$, such that

$$\left.\begin{aligned}\bar{E}\theta_3^2 &= A\theta_3^2 - 2G\theta_1\theta_3 + C\theta_1^2\\ \bar{F}\theta_3^2 &= H\theta_3^2 - G\theta_2\theta_3 - F\theta_1\theta_3 + C\theta_1\theta_2\\ \bar{G}\theta_3^2 &= B\theta_3^2 - 2F\theta_2\theta_3 + C\theta_2^2\end{aligned}\right\}.$$

Then the arc-relation becomes

$$\begin{aligned}1 = U &= (A, B, C, F, G, H \,\backslash\!\backslash\, p', q', r')^2\\ &= (\bar{E}, \bar{F}, \bar{G} \,\backslash\!\backslash\, p', q')^2;\end{aligned}$$

that is, there is now an arc-relation for the surface

$$1 = \bar{U} = (\bar{E}, \bar{F}, \bar{G} \,\backslash\!\backslash\, p', q')^2;$$

and we regard p and q as the parametric variables of the surface. Also, we need the discriminant of the quadratic form $\bar{U}$: we take

$$\bar{V}^2 = \bar{E}\bar{G} - \bar{F}^2,$$

so that

$$\begin{aligned}\theta_3^4 \bar{V}^2 &= (A\theta_3^2 - 2G\theta_1\theta_3 + C\theta_1^2)(B\theta_3^2 - 2F\theta_2\theta_3 + C\theta_2^2) \\ &\quad - (H\theta_3^2 - G\theta_2\theta_3 - F\theta_1\theta_3 + C\theta_1\theta_2)^2 \\ &= \theta_3^2 (a, b, c, f, g, h \mathbin{\text{\textbigcircle}} \theta_1, \theta_2, \theta_3)^2,\end{aligned}$$

on reduction. The concomitant on the right-hand side has been denoted (§ 341) by D; thus

$$\theta_3^2 \bar{V}^2 = D.$$

We make the same modification—the removal of r' by means of the relation $\theta_1 p' + \theta_2 q' + \theta_3 r' = 0$—in the expression for the curvature of the regional geodesic, so that we thereby consider specially those regional geodesics which are determined by directions in the surface. We take quantities $\bar{L}$, $\bar{M}$, $\bar{N}$, according to the definitions

$$\left.\begin{aligned}\bar{L}\theta_3^2 &= L\theta_3^2 - 2K\theta_1\theta_3 + J\theta_1^2 \\ \bar{M}\theta_3^2 &= M\theta_3^2 - K\theta_2\theta_3 - I\theta_1\theta_3 + J\theta_1\theta_2 \\ \bar{N}\theta_3^2 &= N\theta_3^2 - 2I\theta_2\theta_3 + J\theta_2^2\end{aligned}\right\};$$

then the circular curvature of the regional geodesic, through the direction p', q', touching the surface, is given by

$$\begin{aligned}\frac{1}{\rho} = W &= (L, N, J, I, K, M \mathbin{\text{\textbigcircle}} p', q', r')^2 \\ &= (\bar{L}, \bar{M}, \bar{N} \mathbin{\text{\textbigcircle}} p', q')^2.\end{aligned}$$

We take

$$\bar{W} = (\bar{L}, \bar{M}, \bar{N} \mathbin{\text{\textbigcircle}} p', q')^2,$$

so that $\bar{W}$ is the circular curvature of the regional geodesic through the direction p', q' in the surface. Also

$$\begin{aligned}\theta_3^4 (\bar{L}\bar{N} - \bar{M}^2) &= (L\theta_3^2 - 2K\theta_1\theta_3 + J\theta_1^2)(N\theta_3^2 - 2I\theta_2\theta_3 + J\theta_2^2) \\ &\quad - (M\theta_3^2 - K\theta_2\theta_3 - I\theta_1\theta_3 + J\theta_1\theta_2)^2 \\ &= \theta_3^2 (l, n, j, i, k, m \mathbin{\text{\textbigcircle}} \theta_1, \theta_2, \theta_3)^2,\end{aligned}$$

on reduction. The concomitant on the right-hand side is denoted by D'; thus the discriminant of $\bar{W}$ is given by

$$\begin{aligned}\theta_3^2 (\bar{L}\bar{N} - \bar{M}^2) &= (l, n, j, i, k, m \mathbin{\text{\textbigcircle}} \theta_1, \theta_2, \theta_3)^2 \\ &= D'.\end{aligned}$$

There are the two simultaneous binary quadratic forms $\bar{U}$ and $\bar{W}$, the respective discriminants of which have just been obtained. They also possess

an intermediate invariant $\bar{E}\bar{N} - 2\bar{F}\bar{M} + \bar{G}\bar{L}$; its value can be found by direct substitution, so that

$$\begin{aligned}\theta_3^4(\bar{E}\bar{N} - 2\bar{F}\bar{M} + \bar{G}\bar{L}) &= (A\theta_3^2 - 2G\theta_1\theta_3 + C\theta_1^2)(N\theta_3^2 - 2I\theta_2\theta_3 + J\theta_2^2) \\ &\quad - 2(H\theta_3^2 - G\theta_2\theta_3 - F\theta_1\theta_3 + C\theta_1\theta_2)(M\theta_3^2 - K\theta_2\theta_3 - I\theta_1\theta_3 + J\theta_1\theta_2) \\ &\quad + (B\theta_3^2 - 2F\theta_2\theta_3 + C\theta_2^2)(L\theta_3^2 - 2K\theta_1\theta_3 + J\theta_1^2) \\ &= \theta_3^2(BJ - 2FI + CN, \ldots\ldots \S\theta_1, \theta_2, \theta_3)^2 \\ &= \theta_3^2(\bar{l}, \bar{n}, \bar{j}, \bar{i}, \bar{k}, \bar{m} \S\theta_1, \theta_2, \theta_3)^2,\end{aligned}$$

with the notation for the coefficients adopted in § 296. Denoting the concomitant on the right-hand side by $\bar{\nabla}$, we have

$$\theta_3^2(\bar{E}\bar{N} - 2\bar{F}\bar{M} + \bar{G}\bar{L}) = \bar{\nabla} = (\bar{l}, \bar{n}, \bar{j}, \bar{i}, \bar{k}, \bar{m} \S\theta_1, \theta_2, \theta_3)^2.$$

Principal values and directions of circular curvature of tangential regional geodesics.

356. Thus the circular curvature of the regional geodesic is a quadratic function of p', q', where these quantities are subject to a permanent relation

$$(\bar{E}, \bar{F}, \bar{G} \S p', q')^2 = 1.$$

In order that

$$(\bar{L}, \bar{M}, \bar{N} \S p', q')^2 = \frac{1}{\rho}$$

may have a maximum or a minimum value for the admissible range of values of p' and q', we have critical equations

$$\bar{L}p' + \bar{M}q' = \frac{1}{\rho}(\bar{E}p' + \bar{F}q'),$$

$$\bar{M}p' + \bar{N}q' = \frac{1}{\rho}(\bar{F}p' + \bar{G}q').$$

Thus the magnitudes of the principal values of ρ, say ρ_1 and ρ_2, are the roots of the quadratic

$$\begin{vmatrix} \bar{L} - \frac{1}{\rho}\bar{E}, & \bar{M} - \frac{1}{\rho}\bar{F} \\ \bar{M} - \frac{1}{\rho}\bar{F}, & \bar{N} - \frac{1}{\rho}\bar{G} \end{vmatrix} = 0;$$

hence

$$\frac{1}{\rho_1\rho_2} = \frac{\bar{L}\bar{N} - \bar{M}^2}{\bar{E}\bar{G} - \bar{F}^2} = \frac{D'}{D},$$

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{\bar{G}\bar{L} - 2\bar{F}\bar{M} + \bar{E}\bar{N}}{\bar{E}\bar{G} - \bar{F}^2} = \frac{\bar{\nabla}}{D},$$

where D' and $\bar{\nabla}$ are the concomitants

$$D' = (l, n, j, i, k, m \S\theta_1, \theta_2, \theta_3)^2,$$

$$\bar{\nabla} = (\bar{l}, \bar{n}, \bar{j}, \bar{i}, \bar{k}, \bar{m} \S\theta_1, \theta_2, \theta_3)^2,$$

in the notation of § 355. But it is to be remembered that the circular curvatures in question belong to regional geodesics touching the surface and that they are not the circular curvatures of superficial geodesics.

Moreover, the two directions are given, for the respective values of ρ, by

$$\bar{L}p' + \bar{M}q' = \frac{1}{\rho}(\bar{E}p' + \bar{F}q'),$$

$$\bar{M}p' + \bar{N}q' = \frac{1}{\rho}(\bar{F}p' + \bar{G}q'),$$

or, together, by the equation

$$\begin{vmatrix} \bar{L}p' + \bar{M}q', & \bar{E}p' + \bar{F}q' \\ \bar{M}p' + \bar{N}q', & \bar{F}p' + \bar{G}q' \end{vmatrix} = 0.$$

Let p_1', q_1', correspond to ρ_1; and p_2', q_2', correspond to ρ_2. Then

$$(\bar{L}p_1' + \bar{M}q_1')p_2' + (\bar{M}p_1' + \bar{N}q_1')q_2' = \frac{1}{\rho_1}\{\bar{E}p_1'p_2' + \bar{F}(p_1'q_2' + q_1'p_2') + \bar{G}q_1'q_2'\},$$

and

$$(\bar{L}p_2' + \bar{M}q_2')p_1' + (\bar{M}p_2' + \bar{N}q_2')q_1' = \frac{1}{\rho_2}\{\bar{E}p_1'p_2' + \bar{F}(p_1'q_2' + q_1'p_2') + \bar{G}q_1'q_2'\}.$$

On a general surface, ρ_1 and ρ_2 are unequal; hence

$$\bar{E}p_1'p_2' + \bar{F}(p_1'q_2' + q_1'p_2') + \bar{G}q_1'q_2' = 0.$$

Inserting the values of $\bar{E}$, $\bar{F}$, $\bar{G}$, and using the relations

$$\theta_1 p_1' + \theta_2 q_1' + \theta_3 r_1' = 0, \quad \theta_1 p_2' + \theta_2 q_2' + \theta_3 r_2' = 0,$$

we obtain the relation

$$\begin{aligned} &Ap_1'p_2' + H(p_1'q_2' + q_1'p_2') + G(p_1'r_2' + r_1'p_2') \\ &+ Bq_1'q_2' + F(q_1'r_2' + r_1'q_2') + Cr_1'r_2' = 0, \end{aligned}$$

that is, the two directions are perpendicular. It follows that, on a surface in a region, there are two principal directions for the circular curvature of a tangential regional geodesic (that is, directions giving a maximum or a minimum value of that circular curvature), and that these two directions are perpendicular to one another.

Further, we had (§ 291) an indicatrix quadric, and its asymptotic cone, associated with the circular curvature of a regional geodesic.

As before (*l.c.*), we can take

$$\bar{E}^{\frac{1}{2}}p' = \lambda, \quad \bar{G}^{\frac{1}{2}}q' = \mu,$$

and then, if ω be the angle between the directions of p = constant, q = constant, the permanent relation has the form

$$\lambda^2 + 2\lambda\mu\cos\omega + \mu^2 = 1.$$

If there are quantities $\bar{x}$, $\bar{y}$, r, such that

$$\bar{x} = r\lambda, \quad \bar{y} = r\mu,$$

we have

$$r^2 = \bar{x}^2 + 2\bar{x}\bar{y}\cos\omega + \bar{y}^2;$$

and then

$$\frac{1}{\rho} = \frac{\bar{L}}{\bar{E}}\lambda^2 + 2\frac{\bar{M}}{\bar{F}}\lambda\mu\cos\omega + \frac{\bar{N}}{\bar{G}}\mu^2,$$

so that

$$\frac{R^2}{\rho} = \frac{\bar{L}}{\bar{E}}\bar{x}^2 + 2\frac{\bar{M}}{\bar{F}}\bar{x}\bar{y}\cos\omega + \frac{\bar{N}}{\bar{G}}\bar{y}^2 = T.$$

We now have a conic

$$\frac{\bar{L}}{\bar{E}}\bar{x}^2 + 2\frac{\bar{M}}{\bar{F}}\bar{x}\bar{y}\cos\omega + \frac{\bar{N}}{\bar{G}}\bar{y}^2 = c,$$

where c is a constant, and this conic lies in the tangent plane to the surface $\theta(p, q, r) = \text{constant}$, or in a parallel plane. If R be its radius vector in any direction λ and μ (that is, in a direction determined by p', q'), the radius of circular curvature of the regional geodesic in that direction touching the surface is given by

$$\rho = \frac{R^2}{c}.$$

It is an *indicatrix conic* for the circular curvature of regional geodesics; and it is the section of the former indicatrix quadric by the tangent plane to the surface.

Its asymptotes are given by

$$T = 0,$$

which determines the *asymptotic* (or *inflexional*) *directions* on the surface as regards regional geodesics. At any point, their directions are, of course, the section of the former asymptotic cone by the same tangent plane to the surface.

Modified expression for regional geodesic curvature of a superficial geodesic.

357. We proceed similarly for the regional geodesic curvature of a geodesic on the surface through the direction p', q'. We take quantities

$$\left.\begin{aligned}
\bar{\gamma}_{11}\theta_3^2 &= \gamma_{11}\theta_3^2 - 2\gamma_{13}\theta_1\theta_3 + \gamma_{33}\theta_1^2 \\
\bar{\gamma}_{12}\theta_3^2 &= \gamma_{12}\theta_3^2 - \gamma_{13}\theta_2\theta_3 - \gamma_{23}\theta_1\theta_3 + \gamma_{33}\theta_1\theta_2 \\
\bar{\gamma}_{22}\theta_3^2 &= \gamma_{22}\theta_3^2 - 2\gamma_{23}\theta_2\theta_3 + \gamma_{33}\theta_2^2
\end{aligned}\right\},$$

so that the concomitant denoted by Ψ, viz.,

$$\Psi = (\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12}\mathrm{X}\, p', q', r')^2,$$

becomes

$$\bar{\Psi} = (\bar{\gamma}_{11}, \bar{\gamma}_{12}, \bar{\gamma}_{22}\mathrm{X}\, p', q')^2.$$

The regional geodesic curvature of the superficial geodesic was given (§ 345) by

$$\frac{1}{\gamma} = -\frac{\Omega}{D^{\frac{1}{2}}}\Psi;$$

and therefore, by reference to the direction p', q', in the surface, this regional geodesic curvature of the superficial geodesic now is given by

$$\frac{1}{\gamma} = -\frac{\Omega}{D^{\frac{1}{2}}}\overline{\Psi}.$$

The discriminant of the binary form $\overline{\Psi}$ is $\overline{\gamma}_{11}\overline{\gamma}_{22} - \overline{\gamma}_{12}^2$. By direct substitution as before, and similar reduction, we find

$$\theta_3^2(\overline{\gamma}_{11}\overline{\gamma}_{22} - \overline{\gamma}_{12}^2) = (\gamma_{22}\gamma_{33} - \gamma_{23}^2, \ldots\ldots \mathrm{U}\theta_1, \theta_2, \theta_3)^2.$$

The concomitant on the right-hand side is the customary contravariant of the foregoing ternary quadratic form Ψ. We have denoted it (§ 348) by **K**, so that

$$\theta_3^2(\overline{\gamma}_{11}\overline{\gamma}_{22} - \overline{\gamma}_{12}^2) = \mathbf{K}.$$

Again, $\overline{U}$ and $\overline{\Psi}$ are simultaneous binary quadratic forms, the respective discriminants of which have been formed. These forms possess also an intermediate invariant $\overline{E}\overline{\gamma}_{22} - 2\overline{F}\overline{\gamma}_{12} + \overline{G}\overline{\gamma}_{11}$; its value is obtained by direct substitution, and we find

$$\theta_3^2(\overline{E}\overline{\gamma}_{22} - 2\overline{F}\overline{\gamma}_{12} + \overline{G}\overline{\gamma}_{11}) = (C\gamma_{22} - 2F\gamma_{23} + B\gamma_{33}, \ldots\ldots \mathrm{U}\theta_1, \theta_2, \theta_3)^2.$$

The concomitant on the right-hand side has already (§ 348) been denoted by **H**; so we have

$$\theta_3^2(\overline{E}\overline{\gamma}_{22} - 2\overline{F}\overline{\gamma}_{12} + \overline{G}\overline{\gamma}_{11}) = \mathbf{H}.$$

We note, in passing, that there is one invariant, which is intermediate to all the three forms $\overline{U}$, $\overline{W}$, $\overline{\Psi}$; it is

$$\begin{vmatrix} \overline{E}, & \overline{F}, & \overline{G} \\ \overline{L}, & \overline{M}, & \overline{N} \\ \overline{\gamma}_{11}, & \overline{\gamma}_{12}, & \overline{\gamma}_{22} \end{vmatrix}.$$

An expression for this invariant will be obtained later (§ 360), by taking it in the form

$$\overline{E}(\overline{M}\overline{\gamma}_{22} - \overline{N}\overline{\gamma}_{12}) - \overline{F}(\overline{L}\overline{\gamma}_{22} - \overline{N}\overline{\gamma}_{11}) + \overline{G}(\overline{L}\overline{\gamma}_{12} - \overline{M}\overline{\gamma}_{11}).$$

Further, we had (§ 349) an indicatrix quadric and its asymptotic cone, connected with the regional geodesic curvature of superficial geodesics drawn upon all surfaces existing within the region. So now, as for the circular curvature of regional geodesics touching an assumed particular surface (§ 356), we have an *indicatrix conic* for the regional geodesic curvature of the superficial geodesics, and we have the *asymptotic* (or *inflexional*) *directions* which are those directions along which the regional geodesic and the superficial geodesic have a second contact.

We take

$$\overline{E}^{-\frac{1}{2}}p' = \lambda, \quad \overline{G}^{-\frac{1}{2}}q' = \mu, \quad (\overline{E}\overline{G})^{-\frac{1}{2}}\overline{F} = \cos\omega,$$

so that

$$\lambda^2 + 2\lambda\mu\cos\omega + \mu^2 = 1;$$

also quantities $\bar{x}$, $\bar{y}$, r, such that

$$\bar{x} = r\lambda, \quad \bar{y} = r\mu,$$

and then

$$r^2 = \bar{x}^2 + 2\bar{x}\bar{y}\cos\omega + \bar{y}^2.$$

We now have a conic

$$\frac{\bar{\gamma}_{11}}{\bar{E}}\bar{x}^2 + 2\frac{\bar{\gamma}_{12}}{\bar{F}}\bar{x}\bar{y}\cos\omega + \frac{\bar{\gamma}_{22}}{\bar{G}}\bar{y}^2 = \alpha,$$

where α is a constant. If r be the radius vector of this conic in any direction, the radius γ of regional geodesic curvature of a superficial geodesic in that direction is given by

$$\gamma = -\frac{D^{\frac{1}{2}}}{\alpha\Omega}r^2.$$

The conic is an *indicatrix conic* for the regional geodesic curvature of geodesics on the surface $\theta(p, q, r) = \text{constant}$: it is the section of the indicatrix quadric (§ 348) by the tangent plane to the surface.

Its asymptotes are *asymptotic* (or *inflexional*) *directions*, such that

$$\frac{1}{\gamma} = 0;$$

they are the section of the corresponding asymptotic cone by the tangent plane to the surface.

Principal values and principal directions of regional geodesic curvature of a superficial geodesic.

358. The regional geodesic curvature of a superficial geodesic in the direction p', q', r', in the tangent plane to the surface is given by

$$-\frac{D^{\frac{1}{2}}}{\Omega}\frac{1}{\gamma} = \bar{\Psi} = (\bar{\gamma}_{11}, \bar{\gamma}_{12}, \bar{\gamma}_{22})(p', q')^2,$$

a quadratic function of p' and q'. This magnitude also has principal values and principal directions. These are obtained by deducing the maximum and minimum values, and the directions of these values, of $\bar{\Psi}$, for the range of values of p' and q' limited by the condition

$$\bar{E}p'^2 + 2\bar{F}p'q' + \bar{G}q'^2 = 1.$$

The critical equations, constructed as usual, are

$$\bar{\gamma}_{11}p' + \bar{\gamma}_{12}q' = \bar{\Psi}(\bar{E}p' + \bar{F}q'),$$
$$\bar{\gamma}_{12}p' + \bar{\gamma}_{22}q' = \bar{\Psi}(\bar{F}p' + \bar{G}q'),$$

that is,

$$\bar{\gamma}_{11}p' + \bar{\gamma}_{12}q' = -\frac{D^{\frac{1}{2}}}{\Omega\gamma}(\bar{E}p' + \bar{F}q'),$$

$$\bar{\gamma}_{12}p' + \bar{\gamma}_{22}q' = -\frac{D^{\frac{1}{2}}}{\Omega\gamma}(\bar{F}p' + \bar{G}q').$$

Thus the principal values in question are given by the equation

$$\begin{vmatrix} \bar{\gamma}_{11} + \frac{D^{\frac{1}{2}}}{\Omega\gamma}\bar{E}, & \bar{\gamma}_{12} + \frac{D^{\frac{1}{2}}}{\Omega\gamma}\bar{F} \\ \bar{\gamma}_{12} + \frac{D^{\frac{1}{2}}}{\Omega\gamma}\bar{F}, & \bar{\gamma}_{22} + \frac{D^{\frac{1}{2}}}{\Omega\gamma}\bar{G} \end{vmatrix} = 0.$$

The principal measures are the two simplest symmetric functions, being

$$\frac{1}{\gamma_1\gamma_2} = \frac{\Omega^2}{D}\frac{\bar{\gamma}_{11}\bar{\gamma}_{22} - \bar{\gamma}_{12}^2}{\bar{E}\bar{G} - \bar{F}^2} = \frac{\Omega^2}{D^2}\mathbf{K},$$

$$\frac{1}{\gamma_1} + \frac{1}{\gamma_2} = -\frac{\Omega}{D^{\frac{1}{2}}}\frac{\bar{E}\bar{\gamma}_{22} - 2\bar{F}\bar{\gamma}_{12} + \bar{G}\bar{\gamma}_{11}}{\bar{E}\bar{G} - \bar{F}^2} = -\frac{\Omega}{D^{\frac{3}{2}}}\mathbf{H},$$

where

$$\mathbf{K} = (\gamma_{22}\gamma_{33} - \gamma_{23}^2, \ldots \mathfrak{X}\theta_1, \theta_2, \theta_3)^2,$$

$$\mathbf{H} = (B\gamma_{33} - 2F\gamma_{23} + C\gamma_{22}, \ldots \mathfrak{X}\theta_1, \theta_2, \theta_3)^2.$$

The corresponding directions are given by the equations

$$\bar{\gamma}_{11}p' + \bar{\gamma}_{12}q' = \bar{\Psi}(\bar{E}p' + \bar{F}q'),$$

$$\bar{\gamma}_{12}p' + \bar{\gamma}_{22}q' = \bar{\Psi}(\bar{F}p' + \bar{G}q'),$$

or, together, by the equation

$$\begin{vmatrix} \bar{\gamma}_{11}p' + \bar{\gamma}_{12}q', & \bar{E}p' + \bar{F}q' \\ \bar{\gamma}_{12}p' + \bar{\gamma}_{22}q', & \bar{F}p' + \bar{G}q' \end{vmatrix} = 0,$$

which, it will be noted, is distinct from the equation (§ 356) for the principal directions of circular curvature of tangential regional geodesics. As before, and with the property that γ_1 and γ_2 (and therefore Ψ_1 and Ψ_2) are unequal, we find

$$\bar{E}p_1'p_2' + \bar{F}(p_1'q_2' + q_1'p_2') + \bar{G}q_1'q_2' = 0,$$

which, when transformed, becomes

$$(A, B, C, F, G, H \mathfrak{X} p_1', q_1', r_1' \mathfrak{X} p_2', q_2', r_2') = 0,$$

shewing that the two directions are perpendicular to one another. It follows that, on a surface in a region, there are two principal directions of regional geodesic curvature of a superficial geodesic (that is, giving a maximum value or a minimum value of that regional geodesic curvature), and that these two principal directions are perpendicular to one another.

Moreover, the two principal directions for the circular curvature of tangential regional geodesics are distinct from the two principal directions for the regional geodesic curvature of superficial geodesics.

Indicatrix conics.

359. Having regard to the expression for ρ in the form

$$\frac{1}{\rho} = \bar{L}p'^2 + 2\bar{M}p'q' + \bar{N}q'^2,$$

and to the equations

$$\bar{L}p' + \bar{M}q' = \frac{1}{\rho}(\bar{E}p' + \bar{F}q'),$$

$$\bar{M}p' + \bar{N}q' = \frac{1}{\rho}(\bar{F}p' + \bar{G}q'),$$

it is natural to consider the indicatrix conic (§ 357) as leading to an expression

$$\frac{1}{\rho} = \frac{\cos^2\phi}{\rho_1} + \frac{\sin^2\phi}{\rho_2},$$

where ϕ is the angle made by the direction p', q' with the direction of the principal curvature $1/\rho_1$. The result can be obtained directly, as follows.

The direction p_1', q_1' of this principal curvature is given by

$$\frac{p_1'}{\left(\bar{N} - \frac{\bar{G}}{\rho_1}\right)^{\frac{1}{2}}} = \frac{q_1'}{-\left(\bar{L} - \frac{\bar{E}}{\rho_1}\right)^{\frac{1}{2}}} = \frac{1}{\bar{V}\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)^{\frac{1}{2}}},$$

the common value of the fractions being obtained in § 364. Now

$$\frac{1}{\bar{V}^2}\sin^2\phi = (p_1'q' - q_1'p')^2$$

$$= \frac{1}{\bar{V}^2\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)}\left\{q'^2\left(\bar{N} - \frac{\bar{G}}{\rho_1}\right) + 2p'q'\left(\bar{M} - \frac{\bar{F}}{\rho_1}\right) + p'^2\left(\bar{L} - \frac{\bar{E}}{\rho_1}\right)\right\},$$

that is,

$$\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)\sin^2\phi = \bar{N}q'^2 + 2\bar{M}p'q' + \bar{L}p'^2 - \frac{1}{\rho_1}(\bar{G}q'^2 + 2\bar{F}p'q' + \bar{E}p'^2)$$

$$= \frac{1}{\rho} - \frac{1}{\rho_1},$$

and therefore

$$\frac{1}{\rho} = \frac{\cos^2\phi}{\rho_1} + \frac{\sin^2\phi}{\rho_2}.$$

Similar analysis leads to the corresponding result for the regional geodesic curvature of a superficial geodesic: it is equally natural to consider its indicatrix conic (§ 357) as leading to an expression

$$\frac{1}{\gamma} = \frac{\cos^2\theta}{\gamma_1} + \frac{\sin^2\theta}{\gamma_2}.$$

Torsion and tilt of a superficial geodesic.

360. The expressions, already obtained for the torsion and the tilt of a superficial geodesic, can be modified so as to involve only the elements of the three binary forms $\bar{U}$, $\bar{W}$, $\bar{\Psi}$.

We have

$$\begin{aligned}\overline{U}(p', q') &= U(p', q', r') \\ &= U\left(p', q', -\frac{\theta_1}{\theta_3}p' - \frac{\theta_2}{\theta_3}q'\right),\end{aligned}$$

and therefore

$$\left.\begin{aligned}\frac{\partial \overline{U}}{\partial \overline{p'}} &= \frac{\partial U}{\partial \overline{p'}} - \frac{\theta_1}{\theta_3}\frac{\partial U}{\partial \overline{r'}} \\ \frac{\partial \overline{U}}{\partial \overline{q'}} &= \frac{\partial U}{\partial \overline{q'}} - \frac{\theta_2}{\theta_3}\frac{\partial U}{\partial \overline{r'}}\end{aligned}\right\}.$$

Similarly, we have

$$\left.\begin{aligned}\frac{\partial \overline{W}}{\partial \overline{p'}} &= \frac{\partial W}{\partial \overline{p'}} - \frac{\theta_1}{\theta_3}\frac{\partial W}{\partial \overline{r'}} \\ \frac{\partial \overline{W}}{\partial \overline{q'}} &= \frac{\partial W}{\partial \overline{q'}} - \frac{\theta_2}{\theta_3}\frac{\partial W}{\partial \overline{r'}}\end{aligned}\right\}, \quad \left.\begin{aligned}\frac{\partial \overline{\Psi}}{\partial \overline{p'}} &= \frac{\partial \Psi}{\partial \overline{p'}} - \frac{\theta_1}{\theta_3}\frac{\partial \Psi}{\partial \overline{r'}} \\ \frac{\partial \overline{\Psi}}{\partial \overline{q'}} &= \frac{\partial \Psi}{\partial \overline{q'}} - \frac{\theta_2}{\theta_3}\frac{\partial \Psi}{\partial \overline{r'}}\end{aligned}\right\}.$$

Now it was proved (§ 351), for the torsion, that

$$\frac{1}{\overline{\sigma}} = \frac{1}{4}\frac{\Omega}{D}\begin{vmatrix} \frac{\partial \Psi}{\partial \overline{p'}}, & \frac{\partial \Psi}{\partial \overline{q'}}, & \frac{\partial \Psi}{\partial \overline{r'}} \\ \frac{\partial U}{\partial \overline{p'}}, & \frac{\partial U}{\partial \overline{q'}}, & \frac{\partial U}{\partial \overline{r'}} \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix}.$$

Hence

$$\begin{aligned}\frac{1}{\overline{\sigma}} &= \frac{1}{4}\frac{\Omega}{D}\begin{vmatrix} \frac{\partial \Psi}{\partial \overline{p'}} - \frac{\theta_1}{\theta_3}\frac{\partial \Psi}{\partial \overline{r'}}, & \frac{\partial \Psi}{\partial \overline{q'}} - \frac{\theta_2}{\theta_3}\frac{\partial \Psi}{\partial \overline{r'}}, & \frac{\partial \Psi}{\partial \overline{r'}} \\ \frac{\partial U}{\partial \overline{p'}} - \frac{\theta_1}{\theta_3}\frac{\partial U}{\partial \overline{r'}}, & \frac{\partial U}{\partial \overline{q'}} - \frac{\theta_2}{\theta_3}\frac{\partial U}{\partial \overline{r'}}, & \frac{\partial U}{\partial \overline{r'}} \\ 0, & 0, & \theta_3 \end{vmatrix} \\ &= \frac{1}{4}\frac{\Omega}{D}\theta_3\begin{vmatrix} \frac{\partial \overline{\Psi}}{\partial \overline{p'}}, & \frac{\partial \overline{\Psi}}{\partial \overline{q'}} \\ \frac{\partial \overline{U}}{\partial \overline{p'}}, & \frac{\partial \overline{U}}{\partial \overline{q'}} \end{vmatrix};\end{aligned}$$

that is,

$$\frac{1}{\overline{\sigma}} = \theta_3\frac{\Omega}{D}\begin{vmatrix} \overline{\gamma}_{11}p' + \overline{\gamma}_{12}q', & \overline{\gamma}_{12}p' + \overline{\gamma}_{22}q' \\ \overline{E}p' + \overline{F}q', & \overline{F}p' + \overline{G}q' \end{vmatrix},$$

involving the Jacobian of $\overline{\Psi}$ and $\overline{U}$.

Again, for the tilt, it was proved (§ 351) that

$$\frac{1}{\overline{\tau}} = \frac{1}{4}\frac{1}{D^{\frac{1}{2}}}\begin{vmatrix} \frac{\partial W}{\partial \overline{p'}}, & \frac{\partial W}{\partial \overline{q'}}, & \frac{\partial W}{\partial \overline{r'}} \\ \frac{\partial U}{\partial \overline{p'}}, & \frac{\partial U}{\partial \overline{q'}}, & \frac{\partial U}{\partial \overline{r'}} \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix}.$$

Proceeding in the same manner as with the determinant for $1/\bar{\sigma}$, we find

$$\frac{1}{\bar{\tau}} = \frac{1}{4}\frac{\theta_3}{D^{\frac{1}{2}}}\begin{vmatrix} \dfrac{\partial \overline{W}}{\partial p'}, & \dfrac{\partial \overline{W}}{\partial q'} \\ \dfrac{\partial \overline{U}}{\partial p'}, & \dfrac{\partial \overline{U}}{\partial q'} \end{vmatrix}$$

$$= \frac{\theta_3}{D^{\frac{1}{2}}}\begin{vmatrix} \overline{L}p' + \overline{M}q', & \overline{M}p' + \overline{N}q' \\ \overline{E}p' + \overline{F}q', & \overline{F}p' + \overline{G}q' \end{vmatrix},$$

involving the Jacobian of $\overline{W}$ and $\overline{U}$.

We now can interpret the Jacobian of $\overline{W}$ and $\overline{\Psi}$. Let

$$\bar{J} = \begin{vmatrix} \bar{\gamma}_{11}p' + \bar{\gamma}_{12}q', & \bar{\gamma}_{12}p' + \bar{\gamma}_{22}q' \\ \overline{L}p' + \overline{M}q', & \overline{M}p' + \overline{N}q' \end{vmatrix}.$$

The relation

$$\overline{E}(\bar{\gamma}_{11}\overline{M} - \bar{\gamma}_{12}\overline{L}) + \overline{L}(\bar{\gamma}_{12}\overline{E} - \bar{\gamma}_{11}\overline{F}) + \bar{\gamma}_{11}(\overline{L}\overline{F} - \overline{M}\overline{E}) = 0$$

is satisfied identically; hence, by the property that binariants are determined uniquely by their leading coefficients, we have

$$\overline{U}\bar{J} - \overline{W}\frac{D}{\theta_3\Omega}\frac{1}{\bar{\sigma}} + \overline{\Psi}\frac{D^{\frac{1}{2}}}{\theta_3}\frac{1}{\bar{\tau}} = 0.$$

But

$$\overline{U} = 1,$$

$$\overline{W} = \frac{1}{\bar{\rho}},$$

$$\overline{\Psi} = -\frac{D^{\frac{1}{2}}}{\Omega}\frac{1}{\bar{\gamma}};$$

and therefore

$$\frac{D}{\theta_3\Omega}\left(\frac{1}{\rho\bar{\sigma}} + \frac{1}{\gamma\bar{\tau}}\right) = \begin{vmatrix} \bar{\gamma}_{11}p' + \bar{\gamma}_{12}q', & \bar{\gamma}_{12}p' + \bar{\gamma}_{22}q' \\ \overline{L}p' + \overline{M}q', & \overline{M}p' + \overline{N}q' \end{vmatrix}.$$

Relations connecting the tilt with circular curvature of a regional geodesic, and the torsion with regional geodesic curvature of a superficial geodesic.

361. There are further relations, affecting superficial geodesics; they connect the torsion and the tilt with the circular curvature and the regional geodesic curvature of those geodesics.

It is easy to verify the three identities, similar to one another, represented by

$$(\overline{L}\overline{F} - \overline{M}\overline{E})^2 = \overline{E}\overline{L}(\overline{G}\overline{L} - 2\overline{F}\overline{M} + \overline{E}\overline{N}) - \overline{L}^2(\overline{E}\overline{G} - \overline{F}^2) - \overline{E}^2(\overline{L}\overline{N} - \overline{M}^2),$$

$$(\bar{\gamma}_{11}\overline{F} - \bar{\gamma}_{12}\overline{E})^2 = \overline{E}\bar{\gamma}_{11}(\overline{G}\bar{\gamma}_{11} - 2\overline{F}\bar{\gamma}_{12} + \overline{E}\bar{\gamma}_{22}) - \bar{\gamma}_{11}^2(\overline{E}\overline{G} - \overline{F}^2) - \overline{E}^2(\bar{\gamma}_{11}\bar{\gamma}_{22} - \bar{\gamma}_{12}^2),$$

$$(\bar{\gamma}_{11}\overline{M} - \bar{\gamma}_{12}\overline{L})^2 = \overline{L}\bar{\gamma}_{11}(\overline{N}\bar{\gamma}_{11} - 2\overline{M}\bar{\gamma}_{12} + \overline{L}\bar{\gamma}_{22}) - \bar{\gamma}_{11}^2(\overline{L}\overline{N} - \overline{M}^2) - \overline{L}^2(\bar{\gamma}_{11}\bar{\gamma}_{22} - \bar{\gamma}_{12}^2).$$

These three identities are relations between the leading coefficients of covariants and, after the known theorem that binariants are uniquely determined by leading coefficients, they lead to relations among the curvatures.

We have

$$\frac{D^{\frac{1}{2}}}{\theta_3}\frac{1}{\bar{\tau}} = \begin{vmatrix} \bar{L}p' + \bar{M}q', & \bar{M}p' + \bar{N}q' \\ \bar{E}p' + \bar{F}q', & \bar{F}p' + \bar{G}q' \end{vmatrix};$$

$$1 = \bar{E}p'^2 + 2\bar{F}p'q' + \bar{G}q'^2;$$

$$\frac{1}{\rho} = \bar{L}p'^2 + 2\bar{M}p'q' + \bar{N}q'^2;$$

$$\bar{G}\bar{L} - 2\bar{F}\bar{M} + \bar{E}\bar{N} = (\bar{E}\bar{G} - \bar{F}^2)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right);$$

$$\bar{L}\bar{N} - \bar{M}^2 = (\bar{E}\bar{G} - \bar{F}^2)\frac{1}{\rho_1\rho_2};$$

and

$$\frac{D}{\theta_3^2} = \bar{E}\bar{G} - \bar{F}^2.$$

Hence the first identity leads to the covariantive relation

$$\frac{1}{\bar{\tau}^2} = \frac{1}{\rho}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) - \frac{1}{\rho^2} - \frac{1}{\rho_1\rho_2}$$

$$= \left(\frac{1}{\rho_1} - \frac{1}{\rho}\right)\left(\frac{1}{\rho} - \frac{1}{\rho_2}\right).$$

Manifestly the tilt of a superficial geodesic vanishes for those directions which are principal as regards the circular curvature of the regional geodesics through them.

Again, we have

$$-\frac{D^{\frac{1}{2}}}{\Omega}\frac{1}{\gamma} = \bar{\gamma}_{11}p'^2 + 2\bar{\gamma}_{12}p'q' + \bar{\gamma}_{22}q'^2;$$

$$\frac{D}{\theta_3\Omega}\frac{1}{\bar{\sigma}} = \begin{vmatrix} \bar{\gamma}_{11}p' + \bar{\gamma}_{12}q', & \bar{\gamma}_{12}p' + \bar{\gamma}_{22}q' \\ \bar{E}p' + \bar{F}q', & \bar{F}p' + \bar{G}q' \end{vmatrix};$$

$$\bar{G}\bar{\gamma}_{11} - 2\bar{F}\bar{\gamma}_{12} + \bar{E}\bar{\gamma}_{22} = -\frac{D^{\frac{1}{2}}}{\Omega}(\bar{E}\bar{G} - \bar{F}^2)\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right) = -\frac{D^{\frac{3}{2}}}{\theta_3^2\Omega}\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right);$$

and

$$\bar{\gamma}_{11}\bar{\gamma}_{22} - \bar{\gamma}_{12}^2 = \frac{D}{\Omega^2}(\bar{E}\bar{G} - \bar{F}^2)\frac{1}{\gamma_1\gamma_2} = \frac{D^2}{\theta_3^2\Omega^2}\frac{1}{\gamma_1\gamma_2}.$$

Hence the second identity leads to the covariantive relation

$$\frac{1}{\bar{\sigma}^2} = \left(-\frac{1}{\gamma}\right)\left\{-\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right)\right\} - \frac{1}{\gamma^2} - \frac{1}{\gamma_1\gamma_2}$$

$$= \left(\frac{1}{\gamma_1} - \frac{1}{\gamma}\right)\left(\frac{1}{\gamma} - \frac{1}{\gamma_2}\right).$$

Manifestly the torsion of a superficial geodesic vanishes for those directions which are principal as regards the regional geodesic curvature of the superficial geodesics through them.

Before dealing with the third identity, it is desirable to note the forms acquired by these results when the containing region becomes a flat and, consequently, the surface becomes an untilted amplitude in the homaloidal triple space of the flat. In that event, the regional geodesic (like every regional geodesic) touching the superficial geodesic becomes a straight line, so that

$$\frac{1}{\rho} = 0, \quad \frac{1}{\rho_1} = 0 = \frac{1}{\rho_2},$$

and therefore

$$\frac{1}{\bar{\tau}} = 0,$$

the result to be expected because the whole surface lies in one flat. The regional geodesic curvature of the superficial geodesic measures the deviation from the regional geodesic touching the curve, that is, from its linear tangent; it therefore becomes the circular curvature of the superficial geodesic on the surface now in the flat, and so the relation

$$\frac{1}{\bar{\sigma}^2} = \left(\frac{1}{\gamma} - \frac{1}{\gamma_1}\right)\left(\frac{1}{\gamma_2} - \frac{1}{\gamma}\right)$$

becomes the customary relation between the circular curvature and the torsion of a geodesic on a surface existing in homaloidal triple space.

It may also be remarked that these results accord with the relation

$$\frac{1}{\rho_0^2} = \frac{1}{\rho^2} + \frac{1}{\gamma^2};$$

for, in the circumstances contemplated, $1/\rho$ vanishes and thus γ becomes ρ_0, the radius of circular curvature of the superficial geodesic.

Invariantive measure of the inclination of a principal direction of circular curvature of a regional geodesic to a principal direction of the regional geodesic curvature of a superficial geodesic.

362. In the third identity, there occurs the intermediate invariant $\bar{N}\bar{\gamma}_{11} - 2\bar{M}\bar{\gamma}_{12} + \bar{L}\bar{\gamma}_{22}$: we take a symbol Ξ, defined by

$$-\frac{D^{\frac{1}{2}}}{\theta_3^2\Omega}\Xi = \bar{N}\bar{\gamma}_{11} - 2\bar{M}\bar{\gamma}_{12} + \bar{L}\bar{\gamma}_{22},$$

so that Ξ is a geometrical magnitude, the significance of which has to be deduced. We have

$$\frac{D}{\theta_3\Omega}\left(\frac{1}{\rho\bar{\sigma}} + \frac{1}{\gamma\bar{\tau}}\right) = \begin{vmatrix} \bar{\gamma}_{11}p' + \bar{\gamma}_{12}q', & \bar{\gamma}_{12}p' + \bar{\gamma}_{22}q' \\ \bar{L}p' + \bar{M}q', & \bar{M}p' + \bar{N}q' \end{vmatrix};$$

$$-\frac{D^{\frac{1}{2}}}{\Omega}\frac{1}{\gamma} = \bar{\gamma}_{11}p'^2 + 2\bar{\gamma}_{12}p'q' + \bar{\gamma}_{22}q'^2;$$

$$\frac{1}{\rho} = \bar{L}p'^2 + 2\bar{M}p'q' + \bar{N}q'^2;$$

$$\bar{L}\bar{N} - \bar{M}^2 = \frac{D}{\theta_3^2}\frac{1}{\rho_1\rho_2};$$

and

$$\bar{\gamma}_{11}\bar{\gamma}_{22} - \bar{\gamma}_{12}^2 = \frac{D^2}{\theta_3^2 \bar{\Omega}^2} \frac{1}{\gamma_1 \gamma_2}.$$

Hence the third identity on p. 224 becomes the covariantive relation

$$\left(\frac{1}{\rho\bar{\sigma}} + \frac{1}{\gamma\bar{\tau}}\right)^2 = \frac{\Xi}{\rho\gamma} - \frac{1}{\gamma^2}\frac{1}{\rho_1\rho_2} - \frac{1}{\rho^2}\frac{1}{\gamma_1\gamma_2},$$

which can be modified by using the two earlier relations. We have

$$\begin{aligned}\frac{\Xi}{\rho\gamma} &= \frac{2}{\gamma\rho\bar{\sigma}\bar{\tau}} + \frac{1}{\rho^2}\left(\frac{1}{\bar{\sigma}^2} + \frac{1}{\gamma_1\gamma_2}\right) + \frac{1}{\gamma^2}\left(\frac{1}{\bar{\tau}^2} + \frac{1}{\rho_1\rho_2}\right) \\ &= \frac{2}{\gamma\rho\bar{\sigma}\bar{\tau}} + \frac{1}{\rho^2}\left\{\frac{1}{\gamma}\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right) - \frac{1}{\gamma^2}\right\} + \frac{1}{\gamma^2}\left\{\frac{1}{\rho}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) - \frac{1}{\rho^2}\right\},\end{aligned}$$

and therefore

$$\Xi = \frac{2}{\bar{\sigma}\bar{\tau}} - \frac{2}{\gamma\rho} + \frac{1}{\rho}\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right) + \frac{1}{\gamma}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right),$$

apparently a covariantive expression for Ξ.

Let the direction p', q', make an angle α with the direction of the principal regional geodesic curvature $1/\gamma_1$, so that

$$\frac{1}{\gamma} = \frac{\cos^2\alpha}{\gamma_1} + \frac{\sin^2\alpha}{\gamma_2},$$

and therefore

$$\frac{1}{\bar{\sigma}^2} = \left(\frac{1}{\gamma_2} - \frac{1}{\gamma_1}\right)^2 \sin^2\alpha\cos^2\alpha.$$

Again, let this same direction make an angle β with the direction of the principal regional geodesic with the circular curvature $1/\rho_1$, so that

$$\frac{1}{\rho} = \frac{\cos^2\beta}{\rho_1} + \frac{\sin^2\beta}{\rho_2},$$

and therefore

$$\frac{1}{\bar{\tau}^2} = \left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)^2 \sin^2\beta\cos^2\beta.$$

Now the negative sign occurs with the coefficients $\bar{\gamma}_{11}$, $\bar{\gamma}_{12}$, $\bar{\gamma}_{22}$, in the expression for $1/\gamma$, which is

$$\frac{1}{\gamma} = -\frac{D^{\frac{1}{2}}}{\Omega}(\bar{\gamma}_{11}p'^2 + 2\bar{\gamma}_{12}p'q' + \bar{\gamma}_{22}q'^2),$$

while the positive sign occurs with the coefficients $\bar{L}$, $\bar{M}$, $\bar{N}$, in the expression for $1/\rho$, which is

$$\frac{1}{\rho} = \bar{L}p'^2 + 2\bar{M}p'q' + \bar{N}q'^2.$$

Also

$$\frac{D}{\theta_3\Omega}\frac{1}{\bar{\sigma}} = \begin{vmatrix} \bar{\gamma}_{11}p' + \bar{\gamma}_{12}q', & \bar{\gamma}_{12}p' + \bar{\gamma}_{22}q' \\ \bar{E}p' + \bar{F}q', & \bar{F}p' + \bar{G}q' \end{vmatrix},$$

where the positive sign (not the negative sign, as for $1/\gamma$) is associated with $\bar{\gamma}_{11}$, $\bar{\gamma}_{12}$, $\bar{\gamma}_{22}$, while

$$\frac{D^{\frac{1}{2}}}{\theta_3}\frac{1}{\bar{\tau}} = \begin{vmatrix} \bar{L}p' + \bar{M}q', & \bar{M}p' + \bar{N}q' \\ \bar{E}p' + \bar{F}q', & \bar{F}p' + \bar{G}q' \end{vmatrix},$$

where the positive sign, as for $1/\rho$, is associated with $\bar{L}$, $\bar{M}$, $\bar{N}$. Accordingly, in extracting the square roots of the expressions for $1/\bar{\sigma}$ and $1/\bar{\tau}$, this difference of sign must be regarded: so we take

$$\left.\begin{aligned} \frac{1}{\bar{\sigma}} &= -\left(\frac{1}{\gamma_2} - \frac{1}{\gamma_1}\right)\sin\alpha\,\cos\alpha \\ \frac{1}{\bar{\tau}} &= \left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)\sin\beta\cos\beta \end{aligned}\right\}.$$

The angle $\alpha - \beta$ is the angle made by the direction of principal curvature $1/\rho_1$ with the direction of principal curvature $1/\gamma_1$, for $PO\gamma_1 = \alpha$, $PO\rho_1 = \beta$; consequently, $\alpha - \beta$ is independent of p' and q', that is, it is an invariant at O. We shall denote it by i, so that

$$\alpha - \beta = i.$$

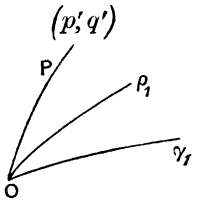


Fig. 25.

We now have

$$\left.\begin{aligned} \frac{2}{\gamma} &= \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)\cos 2\alpha \\ \frac{2}{\rho} &= \frac{1}{\rho_1} + \frac{1}{\rho_2} + \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\cos 2\beta \\ \frac{2}{\bar{\sigma}} &= -\left(\frac{1}{\gamma_2} - \frac{1}{\gamma_1}\right)\sin 2\alpha \\ \frac{2}{\bar{\tau}} &= \left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)\sin 2\beta \end{aligned}\right\};$$

and therefore

$$\begin{aligned} 2\Xi &= \frac{4}{\bar{\sigma}\bar{\tau}} - \frac{4}{\gamma\rho} + \frac{2}{\rho}\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right) + \frac{2}{\gamma}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) \\ &= \left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) - \left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\cos 2i, \end{aligned}$$

on substitution and reduction.

This expression for Ξ is manifestly invariantive and not covariantive (that is, it does not depend upon p' and q', as seemed to be the fact in the former expression, because of the occurrence of ρ, γ, $\bar{\sigma}$, $\bar{\tau}$). We also can express this invariant in the form

$$\Xi = \left(\frac{1}{\rho_1\gamma_2} + \frac{1}{\rho_2\gamma_1}\right)\cos^2 i + \left(\frac{1}{\rho_1\gamma_1} + \frac{1}{\rho_2\gamma_2}\right)\sin^2 i,$$

which will be taken as the geometrical interpretation of Ξ.

Torsion and tilt along the principal directions of curvatures.

363. (i) We note that the torsion is zero along a principal direction of regional geodesic curvature of a superficial geodesic; we shall need its values for the principal directions of circular curvature of regional geodesics.

For the direction of ρ_1, the angle α (of § 362) is i and the angle β is 0; while, for the direction of ρ_2, the angle α is $\frac{1}{2}\pi + i$ and the angle β is $\frac{1}{2}\pi$. Thus, if the two torsions are $1/\sigma_1$ and $1/\sigma_2$ respectively, we have

$$-\frac{2}{\sigma_1} = \left(\frac{1}{\gamma_2} - \frac{1}{\gamma_1}\right)\sin 2i, \quad -\frac{2}{\sigma_2} = -\left(\frac{1}{\gamma_2} - \frac{1}{\gamma_1}\right)\sin 2i.$$

Moreover, if $1/r_1$ and $1/r_2$ be the circular curvature of the regional geodesics touching the principal directions for regional geodesic curvature, we have

$$\frac{1}{r_1} = \frac{\cos^2 i}{\rho_1} + \frac{\sin^2 i}{\rho_2}, \quad \frac{1}{r_2} = \frac{\sin^2 i}{\rho_1} + \frac{\cos^2 i}{\rho_2}.$$

(ii) Again, the tilt is zero for the principal directions of circular curvature of regional geodesics; we shall need its values for the principal directions of regional geodesic curvature.

For the direction of γ_1, the angle α (of § 362) is 0 and the angle β is $-i$; while, for the direction of γ_2, the angle α is $\frac{1}{2}\pi$ and the angle β is $\frac{1}{2}\pi - i$. Thus if the two tilts are $1/\tau_1$ and $1/\tau_2$ respectively, we have

$$\frac{2}{\tau_1} = -\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)\sin 2i, \quad \frac{2}{\tau_2} = \left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)\sin 2i.$$

Moreover, if $1/g_1$ and $1/g_2$ be the regional geodesic curvature of superficial geodesics in the principal directions of circular curvature of regional geodesics touching the surface, we have

$$\frac{1}{g_1} = \frac{\cos^2 i}{\gamma_1} + \frac{\sin^2 i}{\gamma_2}, \quad \frac{1}{g_2} = \frac{\sin^2 i}{\gamma_1} + \frac{\cos^2 i}{\gamma_2}.$$

Also

$$\Xi = \frac{1}{\rho_1 g_2} + \frac{1}{\rho_2 g_1} = \frac{1}{r_1 \gamma_2} + \frac{1}{r_2 \gamma_1}.$$

The invariant $\overline{\Pi}$.

364. The significance of the invariant $\overline{\Pi}$, which is equal to

$$\begin{vmatrix} \bar{E}, & \bar{F}, & \bar{G} \\ \bar{L}, & \bar{M}, & \bar{N} \\ \bar{\gamma}_{11}, & \bar{\gamma}_{12}, & \bar{\gamma}_{22} \end{vmatrix},$$

is obtained by means of the magnitudes given in §§ 355, 356, 358.

Let p_1', q_1', and p_2', q_2', be the direction-magnitudes of the principal radii of curvature, ρ_1 and ρ_2, of regional geodesics touching the surface, so that they are given by the equation

$$\begin{vmatrix} \bar{E}p' + \bar{F}q', & \bar{F}p' + \bar{G}q' \\ \bar{L}p' + \bar{M}q', & \bar{M}p' + \bar{N}q' \end{vmatrix} = 0;$$

hence

$$\frac{q_1'q_2'}{\bar{E}\bar{M}-\bar{F}\bar{L}}=-\frac{p_1'q_2'+q_1'p_2'}{\bar{E}\bar{N}-\bar{G}\bar{L}}=\frac{p_1'p_2'}{\bar{F}\bar{N}-\bar{G}\bar{M}}.$$

The common value of the fractions is found as follows. We have

$$\bar{L}p_1'+\bar{M}q_1'=\frac{1}{\rho_1}(\bar{E}p_1'+\bar{F}q_1'),$$

$$\bar{M}p_1'+\bar{N}q_1'=\frac{1}{\rho_1}(\bar{F}p_1'+\bar{G}q_1');$$

hence

$$\frac{p_1'}{\bar{M}-\frac{\bar{F}}{\rho_1}}=\frac{-q_1'}{-\left(\bar{L}-\frac{\bar{E}}{\rho_1}\right)},$$

or, as

$$\left(\bar{M}-\frac{\bar{F}}{\rho_1}\right)^2=\left(\bar{L}-\frac{\bar{E}}{\rho_1}\right)\left(\bar{N}-\frac{\bar{G}}{\rho_1}\right),$$

we have

$$\frac{p_1'}{\left(\bar{N}-\frac{\bar{G}}{\rho_1}\right)^{\frac{1}{2}}}=\frac{q_1'}{-\left(\bar{L}-\frac{\bar{E}}{\rho_1}\right)^{\frac{1}{2}}}.$$

Equate each of these fractions to $1/\mu_1$; then, because

$$\bar{E}p_1'^2+2\bar{F}p_1'q_1'+\bar{G}q_1'^2=1,$$

we have

$$\mu_1^2=\bar{E}\left(\bar{N}-\frac{\bar{G}}{\rho_1}\right)-2\bar{F}\left(\bar{M}-\frac{\bar{F}}{\rho_1}\right)+\bar{G}\left(\bar{L}-\frac{\bar{E}}{\rho_1}\right)$$

$$=\bar{V}^2\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)-2\bar{V}^2\frac{1}{\rho_1}=\bar{V}^2\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right).$$

Similarly,

$$\frac{p_2'}{\left(\bar{N}-\frac{\bar{G}}{\rho_2}\right)^{\frac{1}{2}}}=\frac{q_2'}{-\left(\bar{L}-\frac{\bar{E}}{\rho_2}\right)^{\frac{1}{2}}}=\frac{1}{\mu_2},$$

where

$$\mu_2^2=\bar{V}^2\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right).$$

Thus

$$\mu_1^2\mu_2^2p_1'^2p_2'^2=\left(\bar{N}-\frac{\bar{G}}{\rho_1}\right)\left(\bar{N}-\frac{\bar{G}}{\rho_2}\right)$$

$$=\frac{1}{\bar{V}^2}\{\bar{N}^2(\bar{E}\bar{G}-\bar{F}^2)-\bar{N}\bar{G}(\bar{E}\bar{N}-2\bar{F}\bar{M}+\bar{G}\bar{L})+\bar{G}^2(\bar{L}\bar{N}-\bar{M}^2)\}$$

$$=-\frac{1}{\bar{V}^2}(\bar{F}\bar{N}-\bar{G}\bar{M})^2,$$

or

$$\frac{p_1'^2p_2'^2}{(\bar{F}\bar{N}-\bar{G}\bar{M})^2}=\frac{1}{\bar{V}^6\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)^2}.$$

Hence

$$\frac{q_1'q_2'}{\bar{E}\bar{M}-\bar{F}\bar{L}} = -\frac{p_1'q_2'+q_1'p_2'}{\bar{E}\bar{N}-\bar{G}\bar{L}} = \frac{p_1'p_2'}{\bar{F}\bar{N}-\bar{G}\bar{M}} = \frac{1}{\bar{V}^3\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)}.$$

We at once infer that

$$(p_1'q_2'-p_2'q_1')^2 = \frac{1}{\bar{V}^2},$$

a property otherwise derivable (§ 206) from the fact that the two directions p_1', q_1', and p_2', q_2', are perpendicular to one another.

From the expression for $\bar{\Pi}$, we have

$$\bar{\Pi} = \bar{\gamma}_{11}(\bar{F}\bar{N}-\bar{G}\bar{M}) + \bar{\gamma}_{12}(\bar{G}\bar{L}-\bar{E}\bar{N}) + \bar{\gamma}_{22}(\bar{E}\bar{M}-\bar{F}\bar{L}),$$

and therefore

$$\frac{\bar{\Pi}}{\bar{V}^3\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)} = \bar{\gamma}_{11}p_1'p_2' + \bar{\gamma}_{12}(p_1'q_2'+q_1'p_2') + \bar{\gamma}_{22}q_1'q_2'.$$

Now, identically,

$$\begin{aligned}(\bar{\gamma}_{11}p_1'^2 + 2\bar{\gamma}_{12}p_1'q_1' + \bar{\gamma}_{22}q_1'^2)(\bar{\gamma}_{11}p_2'^2 + 2\bar{\gamma}_{12}p_2'q_2' + \bar{\gamma}_{22}q_2'^2)& \\ - \{\bar{\gamma}_{11}p_1'p_2' + \bar{\gamma}_{12}(p_1'q_2'+q_1'p_2') + \bar{\gamma}_{22}q_1'q_2'\}^2& \\ = (\bar{\gamma}_{11}\bar{\gamma}_{22}-\bar{\gamma}_{12}^2)(p_1'q_2'-p_2'q_1')^2.&\end{aligned}$$

Moreover

$$-\frac{\Omega}{D^{\frac{1}{2}}}(\bar{\gamma}_{11}p_1'^2 + 2\bar{\gamma}_{12}p_1'q_1' + \bar{\gamma}_{22}q_1'^2)$$

is the geodesic curvature of a superficial geodesic in the direction p_1', q_1', that is, one of the principal directions of circular curvature of tangential regional geodesics: but this is the quantity $1/g_1$ of § 363, and is equal to

$$\frac{\cos^2 i}{\gamma_1} + \frac{\sin^2 i}{\gamma_2},$$

so that

$$-\frac{\Omega}{D^{\frac{1}{2}}}(\bar{\gamma}_{11}p_1'^2 + 2\bar{\gamma}_{12}p_1'q_1' + \bar{\gamma}_{22}q_1'^2) = \frac{\cos^2 i}{\gamma_1} + \frac{\sin^2 i}{\gamma_2}.$$

Similarly, we have

$$-\frac{\Omega}{D^{\frac{1}{2}}}(\bar{\gamma}_{11}p_2'^2 + 2\bar{\gamma}_{12}p_2'q_2' + \bar{\gamma}_{22}q_2'^2) = \frac{\sin^2 i}{\gamma_1} + \frac{\cos^2 i}{\gamma_2}.$$

Also (§ 358)

$$\gamma_{11}\bar{\gamma}_{22} - \bar{\gamma}_{12}^2 = \frac{D^2}{\theta_3^2\bar{\Omega}^2}\frac{1}{\gamma_1\gamma_2};$$

and

$$(p_1'q_2' - q_1'p_2')^2 = \frac{1}{\bar{V}^2} = \frac{\theta_3^2}{D}.$$

When the various values are substituted in the identity, we have

$$\frac{D}{\bar{\Omega}^2}\left(\frac{\cos^2 i}{\gamma_1} + \frac{\sin^2 i}{\gamma_2}\right)\left(\frac{\sin^2 i}{\gamma_1} + \frac{\cos^2 i}{\gamma_2}\right) - \frac{\bar{\Pi}^2}{\bar{V}^6\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)^2} = \frac{D}{\bar{\Omega}^2}\frac{1}{\gamma_1\gamma_2};$$

and therefore

$$\frac{\Omega^2}{D\overline{V}^6\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)^2}\overline{\Pi}^2=\frac{\cos^4 i+\sin^4 i}{\gamma_1\gamma_2}+\left(\frac{1}{\gamma_1^2}+\frac{1}{\gamma_2^2}\right)\sin^2 i\cos^2 i-\frac{1}{\gamma_1\gamma_2}$$

$$=\left(\frac{1}{\gamma_2}-\frac{1}{\gamma_1}\right)^2\sin^2 i\cos^2 i.$$

When we substitute for $\overline{V}$ from $\theta_3^2\overline{V}^2=D$, this gives

$$\overline{\Pi}^2=\frac{D^4}{\theta_3^6\Omega^2}\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)^2\left(\frac{1}{\gamma_2}-\frac{1}{\gamma_1}\right)^2\sin^2 i\cos^2 i,$$

or

$$\overline{\Pi}=\frac{D^2}{2\theta_3^3\Omega}\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)\left(\frac{1}{\gamma_2}-\frac{1}{\gamma_1}\right)\sin 2i.$$

When we introduce the torsions along the principal directions of circular curvature of regional geodesics, and the tilts along the principal directions of regional geodesic curvature of superficial geodesics, we have

$$2\overline{\Pi}^4=\frac{D^8}{\theta_3^{12}\Omega^4}\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)^2\left(\frac{1}{\gamma_2}-\frac{1}{\gamma_1}\right)^2\frac{1}{\sigma_1\sigma_2\tau_1\tau_2}.$$

Principal values and principal directions of circular curvature of a superficial geodesic.

365. There remains the determination of the directions of the maximum or minimum values of the circular curvature of a superficial geodesic, which is connected with its regional geodesic curvature and with the circular curvature of a regional geodesic through the same direction by the equation

$$\frac{1}{\rho_0^2}=\frac{1}{\rho^2}+\frac{1}{\gamma^2}.$$

Now $1/\rho$ and $1/\gamma$ are, each of them, a quadratic function of p', q', r'; and the principal values of each of them, (their maximum and minimum values for a range of p', q', r', subject to $U=1$, $\theta_1 p'+\theta_2 q'+\theta_3 r'=0$), have been obtained, separately, by a modified analysis (§§ 355–358).

But there is no rational quadratic function of p', q', r', which serves to express the value of $1/\rho_0$: instead, we have a quartic function to determine its value, as given by the equation

$$\frac{1}{\rho_0^2}=\{(L\,\mathfrak{Y}\,p', q', r')^2\}^2+\frac{\Omega^2}{D}\{(\vartheta_{11}\,\mathfrak{Y}\,p', q', r')^2\}^2=W^2+\frac{\Omega^2}{D}\Psi^2,$$

where Ω and D do not involve the variables of direction. For the present purpose, the right-hand side has to be made a maximum or a minimum subject to the two conditions

$$U=(A\,\mathfrak{Y}\,p', q', r')^2=1,\quad \theta_1 p'+\theta_2 q'+\theta_3 r'=0.$$

The critical equations are

$$W\frac{\partial W}{\partial p'}+\frac{\Omega^2}{D}\Psi\frac{\partial\Psi}{\partial p'}=2\eta\theta_1+\bar\eta\frac{\partial U}{\partial p'},$$
$$W\frac{\partial W}{\partial q'}+\frac{\Omega^2}{D}\Psi\frac{\partial\Psi}{\partial q'}=2\eta\theta_2+\bar\eta\frac{\partial U}{\partial q'},$$
$$W\frac{\partial W}{\partial r'}+\frac{\Omega^2}{D}\Psi\frac{\partial\Psi}{\partial r'}=2\eta\theta_3+\bar\eta\frac{\partial U}{\partial r'},$$

where η and $\bar\eta$ are undetermined coefficients. Multiply the equations by p', q', r', and add; then

$$W\Sigma p'\frac{\partial W}{\partial p'}+\frac{\Omega^2}{D}\Psi\Sigma p'\frac{\partial\Psi}{\partial p'}=2\eta\Sigma\theta_1p'+\bar\eta\Sigma p'\frac{\partial U}{\partial p'},$$

that is,

$$\bar\eta=W^2+\frac{\Omega^2}{D}\Psi^2=\frac{1}{\rho_0^2}=\frac{1}{\rho^2}+\frac{1}{\gamma^2}.$$

Thus the critical equations can be taken in each of two equivalent forms

$$\frac{1}{\rho}(Lp'+Mq'+Kr')-\frac{\Omega}{D^{\frac12}}\frac{1}{\gamma}(\mathfrak{H}_{11}p'+\mathfrak{H}_{12}q'+\mathfrak{H}_{13}r')-\frac{1}{\rho_0^2}(Ap'+Hq'+Gr')=\eta\theta_1,$$
$$\frac{1}{\rho}(Mp'+Nq'+Ir')-\frac{\Omega}{D^{\frac12}}\frac{1}{\gamma}(\mathfrak{H}_{12}p'+\mathfrak{H}_{22}q'+\mathfrak{H}_{23}r')-\frac{1}{\rho_0^2}(Hp'+Bq'+Fr')=\eta\theta_2,$$
$$\frac{1}{\rho}(Kp'+Iq'+Jr')-\frac{\Omega}{D^{\frac12}}\frac{1}{\gamma}(\mathfrak{H}_{13}p'+\mathfrak{H}_{23}q'+\mathfrak{H}_{33}r')-\frac{1}{\rho_0^2}(Gp'+Fq'+Cr')=\eta\theta_3,$$

together with

$$\theta_1p'+\theta_2q'+\theta_3r'=0:$$

or in the form

$$\left.\begin{aligned}\frac{1}{\rho}\frac{\partial W}{\partial p'}-\frac{\Omega}{D^{\frac12}}\frac{1}{\gamma}\frac{\partial\Psi}{\partial p'}&=\frac{1}{\rho_0^2}\frac{\partial U}{\partial p'}+2\eta\theta_1\\ \frac{1}{\rho}\frac{\partial W}{\partial q'}-\frac{\Omega}{D^{\frac12}}\frac{1}{\gamma}\frac{\partial\Psi}{\partial q'}&=\frac{1}{\rho_0^2}\frac{\partial U}{\partial q'}+2\eta\theta_2\\ \frac{1}{\rho}\frac{\partial W}{\partial r'}-\frac{\Omega}{D^{\frac12}}\frac{1}{\gamma}\frac{\partial\Psi}{\partial r'}&=\frac{1}{\rho_0^2}\frac{\partial U}{\partial r'}+2\eta\theta_3\end{aligned}\right\}.$$

Eliminating ρ_0^2 and η from the latter set of equations, we have

$$\begin{vmatrix}\frac{1}{\rho}\frac{\partial W}{\partial p'}, & \frac{1}{\rho}\frac{\partial W}{\partial q'}, & \frac{1}{\rho}\frac{\partial W}{\partial r'} & -\frac{1}{\gamma}\frac{\Omega}{D^{\frac12}} & \frac{\partial\Psi}{\partial p'}, & \frac{\partial\Psi}{\partial q'}, & \frac{\partial\Psi}{\partial r'}\\ \frac{\partial U}{\partial p'}, & \frac{\partial U}{\partial q'}, & \frac{\partial U}{\partial r'} & & \frac{\partial U}{\partial p'}, & \frac{\partial U}{\partial q'}, & \frac{\partial U}{\partial r'}\\ \theta_1, & \theta_2, & \theta_3 & & \theta_1, & \theta_2, & \theta_3\end{vmatrix}=0.$$

By the results in § 351, the coefficient of $1/\rho$ is

$$=4\frac{D^{\frac12}}{\bar\tau},$$

where $1/\bar{\tau}$ is the regional tilt of the superficial geodesic; and the coefficient of $-\frac{1}{\gamma}\frac{\Omega}{D^{\frac{1}{2}}}$ is

$$= 4\frac{D}{\Omega}\frac{1}{\bar{\sigma}},$$

where $1/\bar{\sigma}$ is the regional torsion of that superficial geodesic. Hence

$$\frac{1}{\rho\bar{\tau}} - \frac{1}{\gamma\bar{\sigma}} = 0,$$

a relation holding along a principal direction of circular curvature of a superficial geodesic. The relation will reappear later (§ 370).

Principal measures of circular curvature of a superficial geodesic.

366. Instead of proceeding from the earlier set of equations in § 365, we use the analysis of §§ 356, 357, in order to utilise the calculations already made (§ 239) in the corresponding investigation for a surface in space. We write

$$\mathbf{a} = \bar{L}^2 + \frac{\Omega^2}{D}\bar{\gamma}_{11}{}^2 \quad , \quad \mathbf{b} = \bar{M}^2 + \frac{\Omega^2}{D}\bar{\gamma}_{12}{}^2 \quad , \quad \mathbf{c} = \bar{N}^2 + \frac{\Omega^2}{D}\bar{\gamma}_{22}{}^2,$$

$$\mathbf{f} = \bar{M}\bar{N} + \frac{\Omega^2}{D}\bar{\gamma}_{12}\bar{\gamma}_{22}, \quad \mathbf{g} = \bar{L}\bar{N} + \frac{\Omega^2}{D}\bar{\gamma}_{11}\bar{\gamma}_{22}, \quad \mathbf{h} = \bar{L}\bar{M} + \frac{\Omega^2}{D}\bar{\gamma}_{11}\bar{\gamma}_{12};$$

and, by analogy, we take

$$3\mathbf{k} = \mathbf{g} + 2\mathbf{b}, \quad 3\bar{\theta} = \mathbf{g} - \mathbf{b}.$$

With these values of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{g}, \mathbf{h}$, the identical relation

$$\begin{vmatrix} \mathbf{a}, & \mathbf{h}, & \mathbf{g} \\ \mathbf{h}, & \mathbf{b}, & \mathbf{f} \\ \mathbf{g}, & \mathbf{f}, & \mathbf{c} \end{vmatrix} = 0$$

still is satisfied. Our concern, at the moment, is with these magnitudes $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{k}, \bar{\theta}$.

The radius of spatial circular curvature of the superficial geodesic is given by

$$\frac{1}{\rho_0{}^2} = \frac{1}{\rho^2} + \frac{1}{\gamma^2}$$

$$= W^2 + \frac{\Omega^2}{D}\Psi^2$$

$$= (\mathbf{a}, \mathbf{h}, \mathbf{k}, \mathbf{f}, \mathbf{c}\,\mathrm{X}\,p', q')^4.$$

We require its maximum and minimum values for directions lying in the surface, so that

$$\Sigma A p'^2 = 1, \quad \theta_1 p' + \theta_2 q' + \theta_3 r' = 0,$$

that is, for quantities p' and q' which satisfy the relation

$$(\bar{E}, \bar{F}, \bar{G}\,\mathrm{X}\,p', q')^2 = 1.$$

If then we write

$$\bar{E}^2 = a', \quad \bar{E}\bar{F} = h', \quad \bar{E}\bar{G} = g', \quad \bar{F}^2 = b', \quad \bar{F}\bar{G} = f', \quad \bar{G}^2 = c',$$
$$3k' = g' + 2b', \quad 3\theta' = g' - b',$$

the requirement, stated analytically, is that the discriminant of the binary quartic

$$\left(\mathbf{a} - \frac{1}{\rho_0^2}a', \quad \mathbf{h} - \frac{1}{\rho_0^2}h', \quad \mathbf{k} - \frac{1}{\rho_0^2}k', \quad \mathbf{f} - \frac{1}{\rho_0^2}f', \quad \mathbf{c} - \frac{1}{\rho_0^2}c'\right)(p', q')^4$$

shall vanish; and this vanishing discriminant determines the maximum values and the minimum values of $1/\rho_0^2$.

All that now is requisite is literal substitution of the foregoing magnitudes in the various invariantive combinations of § 239, which constitute *the measures of curvature of the surface.*

367. Before proceeding to this literal substitution, it is convenient to recall certain invariants of the surface and their known significance. We have

$$\left.\begin{aligned} \theta_3^2(\bar{E}\bar{G} - \bar{F}^2) &= D, \\ \theta_3^2(\bar{L}\bar{N} - \bar{M}^2) &= D' \\ \theta_3^2(\bar{G}\bar{L} - 2\bar{F}\bar{M} + \bar{E}\bar{N}) &= \bar{\nabla} \end{aligned}\right\},$$

$$\left.\begin{aligned} \theta_3^2(\bar{\gamma}_{11}\bar{\gamma}_{22} - \bar{\gamma}_{12}^2) &= \mathbf{K} \\ \theta_3^2(\bar{G}\bar{\gamma}_{11} - 2\bar{F}\bar{\gamma}_{12} + \bar{E}\bar{\gamma}_{22}) &= \mathbf{H} \end{aligned}\right\}.$$

We also have had (§ 361) the intermediate invariant $\bar{N}\bar{\gamma}_{11} - 2\bar{M}\bar{\gamma}_{12} + \bar{L}\bar{\gamma}_{22}$; and (§ 362) we have written

$$\bar{N}\bar{\gamma}_{11} - 2\bar{M}\bar{\gamma}_{12} + \bar{L}\bar{\gamma}_{22} = -\frac{D^{\frac{3}{2}}}{\theta_3^2\bar{\Omega}}\,\Xi,$$

this invariant being concerned with the inclination of a principal line of circular curvature of regional geodesics to a principal line of regional geodesic curvature of superficial geodesics.

For the geometric significance of these magnitudes, we have

$$\left.\begin{aligned} \frac{1}{\rho_1\rho_2} &= \frac{D'}{D} \\ \frac{1}{\rho_1} + \frac{1}{\rho_2} &= \frac{\bar{\nabla}}{D} \end{aligned}\right\}, \qquad \left.\begin{aligned} \frac{1}{\gamma_1\gamma_2} &= \frac{\Omega^2}{D}\frac{\mathbf{K}}{D} \\ \frac{1}{\gamma_1} + \frac{1}{\gamma_2} &= -\frac{\Omega}{D^{\frac{1}{2}}}\frac{\mathbf{H}}{D} \end{aligned}\right\}.$$

Also we take

$$\left.\begin{aligned} (\mathbf{ab} - \mathbf{h}^2)^{\frac{1}{2}} &= \frac{\Omega}{D^{\frac{1}{2}}}(\bar{L}\bar{\gamma}_{12} - \bar{M}\bar{\gamma}_{11}) \\ (\mathbf{ca} - \mathbf{g}^2)^{\frac{1}{2}} &= \frac{\Omega}{D^{\frac{1}{2}}}(\bar{L}\bar{\gamma}_{22} - \bar{N}\bar{\gamma}_{11}) \\ (\mathbf{bc} - \mathbf{f}^2)^{\frac{1}{2}} &= \frac{\Omega}{D^{\frac{1}{2}}}(\bar{M}\bar{\gamma}_{22} - \bar{N}\bar{\gamma}_{12}) \end{aligned}\right\},$$

keeping the positive signs for the radicals as in § 214 and bearing in mind the sign affixed to the expression of geodesic curvature.

We also have had (§ 364) the significance of the invariant Π, equal to

$$\begin{vmatrix} \bar{E}, & \bar{F}, & \bar{G} \\ \bar{L}, & \bar{M}, & \bar{N} \\ \bar{\gamma}_{11}, & \bar{\gamma}_{12}, & \bar{\gamma}_{22} \end{vmatrix}.$$

368. We have, in the first place,

$$\begin{aligned} 3\bar{\theta} &= \mathbf{g} - \mathbf{b} \\ &= \bar{L}\bar{N} - \bar{M}^2 + \frac{\Omega^2}{D}(\bar{\gamma}_{11}\bar{\gamma}_{22} - \bar{\gamma}_{12}^2), \end{aligned}$$

and therefore

$$3\bar{\theta}\theta_3^2 = D' + \frac{\Omega^2}{D}\mathbf{K}.$$

When the values of D' and $\mathbf{K}$ are inserted, we have

$$\frac{3\bar{\theta}}{\bar{V}^2} = \frac{1}{\rho_1\rho_2} + \frac{1}{\gamma_1\gamma_2}.$$

Now the quantity

$$\frac{3\bar{\theta}}{\bar{V}^2},$$

when the surface lies in a flat region, becomes the Gauss measure of curvature of the surface, and it is equal (§ 361) to

$$\frac{1}{\gamma_1\gamma_2}.$$

Hence the deviation, in the value of the magnitude

$$\frac{\mathbf{g} - \mathbf{b}}{\bar{V}^2},$$

from the Gauss measure of curvature of a surface in a flat, is measured by the quantity

$$\frac{1}{\rho_1\rho_2},$$

which is the Gauss measure of the geodesic surface touching the given surface, or (what is the same thing) the Riemann measure of the region in the orientation of the enclosed surface at the position considered.

Next, we have an invariant I_0, its value being

$$\begin{aligned} I_0 &= \mathbf{ac} - 4\mathbf{hf} + 3\mathbf{k}^2 \\ &= \frac{4}{3}\left\{(\bar{L}\bar{N} - \bar{M}^2)^2 - \frac{\Omega^2}{D}(\bar{L}\bar{N} - \bar{M}^2)(\bar{\gamma}_{11}\bar{\gamma}_{22} - \bar{\gamma}_{12}^2) + \frac{\Omega^4}{D^2}(\bar{\gamma}_{11}\bar{\gamma}_{22} - \bar{\gamma}_{12}^2)^2\right\} \\ &\qquad + \frac{\Omega^2}{D}(\bar{L}\bar{\gamma}_{22} - 2\bar{M}\bar{\gamma}_{12} + \bar{N}\bar{\gamma}_{11})^2, \end{aligned}$$

on substitution and reduction: that is,

$$I_0\theta_3^4 = \frac{4}{3}\left(D'^2 - \frac{\Omega^2}{D}D'\mathbf{K} + \frac{\Omega^4}{D^2}\mathbf{K}^2\right) + D^2\Xi^2.$$

Again, we have

$$I_1 = \bar{G}^2\mathbf{a} - 4\bar{F}\bar{G}\mathbf{h} + (2\bar{E}\bar{G} + 4\bar{F}^2)\,\mathbf{k} - 4\bar{E}\bar{F}\mathbf{f} + \bar{E}^2\mathbf{c}$$

$$= (\bar{G}\bar{L} - 2\bar{F}\bar{M} + \bar{E}\bar{N})^2 - \frac{4}{3}(\bar{L}\bar{N} - \bar{M}^2)(\bar{E}\bar{G} - \bar{F}^2),$$

on reduction: hence

$$I_1\theta_3^4 = \bar{\nabla}^2 - \frac{4}{3}D'D.$$

We require $I_1 - 8\bar{V}^2\bar{\theta}$; and we at once find

$$(I_1 - 8\bar{V}^2\bar{\theta})\,\theta_3^2 = \nabla^2 - 4DD' - \frac{8}{3}\Omega^2\mathbf{K}$$

$$= D^2\left\{\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^2 - \frac{8}{3}\frac{1}{\gamma_1\gamma_2}\right\}.$$

The invariant J_0 occurs, by itself, being equal to

$$J_0 = \bar{\theta}I_0 - 4\bar{\theta}^3,$$

so that

$$J_0\theta_3^4 = \bar{\theta}\left\{\frac{4}{9}\left(2D' - \frac{\Omega^2}{D}\mathbf{K}\right)\left(D' - 2\frac{\Omega^2}{D}\mathbf{K}\right) + D^2\Xi^2\right\};$$

and it occurs through the discriminant $I_0^3 - 27J_0^2$, which is equal to

$$(I_0 - 3\bar{\theta}^2)\,(I_0 - 12\bar{\theta}^2)^2,$$

where

$$\theta_3^4(I_0 - 3\bar{\theta}^2) = D^2\Xi^2 + \left(D' - \frac{\Omega^2}{D}\mathbf{K}\right)^2,$$

$$\theta_3^4(I_0 - 12\bar{\theta}^2) = D^2\Xi^2 - 4D'\frac{\Omega^2}{D}\mathbf{K}.$$

Orthogonal centre and orthogonal radius of a surface in a region.

369. We have seen that a surface in free space possesses an orthogonal centre which is the intersection of two consecutive orthogonal planes, and that the orthogonal radius is specially associated, in magnitude and in direction, with the curvatures and the principal lines of a geodesic through the direction which determines the consecutive orthogonal plane. Moreover, the locus of the orthogonal centre is a conic (Kommerell's characteristic conic), while the locus of the centre of circular curvature of the geodesic is a lemniscate curve, which is the pedal (with respect to the point on the surface) of the characteristic conic, both loci lying in the initial orthogonal plane of the surface. We proceed to the investigation of the corresponding formulæ when the surface is specified as lying within a region.

For the geodesic through the direction p', q', there is an orthogonal frame (§ 347) the four axes of which have the direction-cosines

$$\begin{array}{cccc} x', & y', & z', & v'; \\ \bar{l}, & \bar{m}, & \bar{n}, & \bar{k}; \\ X, & Y, & Z, & V; \\ \bar{X}, & \bar{Y}, & \bar{Z}, & \bar{V}; \end{array}$$

respectively. The orthogonal plane (§ 342) has, for its equations,

$$\left\| \begin{array}{cccc} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ X, & Y, & Z, & V \\ \bar{X}, & \bar{Y}, & \bar{Z}, & \bar{V} \end{array} \right\| = 0;$$

and, owing to the constitution of the orthogonal frame, equivalent equations of the orthogonal plane can be taken

$$\left.\begin{array}{l} \Sigma(\bar{x}-x)x' = 0 \\ \Sigma(\bar{x}-x)\bar{l} = 0 \end{array}\right\}.$$

In obtaining the coordinates of the orthogonal centre, as the intersection of two consecutive orthogonal planes, the second pair of equations will be used in constructing the equations of the second plane. At a consecutive point, distant δ on the surface along the direction p', q', the orthogonal plane is given by

$$\left.\begin{array}{l} \Sigma(\bar{x}-x)(x'+x''\delta) - \Sigma x' \,.\, x'\delta = 0 \\ \Sigma(\bar{x}-x)(\bar{l}+\bar{l}'\delta) - \Sigma\bar{l} \;.\, x'\delta = 0 \end{array}\right\}.$$

Now $\Sigma x'^2 = 1$, $\Sigma x'\bar{l} = 0$; hence the intersection of the two normal planes is given by associating the two equations of the first plane, which now will be taken in the earlier form, with the two equations

$$\left.\begin{array}{l} \Sigma(\bar{x}-x)x'' = 1 \\ \Sigma(\bar{x}-x)\bar{l}' = 0 \end{array}\right\}.$$

With the present notation (§§ 353, 361), in which ρ_0 denotes the radius of circular curvature of the superficial geodesic, while $1/\gamma$ is its regional geodesic curvature and $1/\rho$ is the curvature of the tangential regional geodesic, we have (§ 353)

$$x'' = \frac{l}{\rho_0} = \frac{X}{\rho} + \frac{\bar{X}}{\gamma}.$$

Again, by the third of the Frenet equations for the frame in § 351, we have

$$\frac{d\bar{l}}{ds} = \frac{X}{\bar{\tau}} - \frac{\bar{X}}{\bar{\sigma}};$$

and, therefore, the two equations to be associated with

$$\left\| \begin{array}{cccc} \bar{x}-x, & \bar{y}-y, & \bar{z}-z, & \bar{v}-v \\ X, & Y, & Z, & V \\ \bar{X}, & \bar{Y}, & \bar{Z}, & \bar{V} \end{array} \right\| = 0$$

become

$$\left.\begin{aligned}\Sigma(\bar{x}-x)\left(\frac{X}{\rho}+\frac{\bar{X}}{\gamma}\right)&=1\\ \Sigma(\bar{x}-x)\left(\frac{X}{\bar{\tau}}-\frac{\bar{X}}{\bar{\sigma}}\right)&=0\end{aligned}\right\}.$$

Any point in the plane, given by the first pair of equations, can be represented by the formulæ

$$\bar{x}-x=\alpha X+\beta\bar{X},\quad \bar{y}-y=\alpha Y+\beta\bar{Y},\quad \bar{z}-z=\alpha Z+\beta\bar{Z},\quad \bar{v}-v=\alpha V+\beta\bar{V},$$

where α and β are parametric, so far as concerns this pair of equations. To find the required point of intersection, values of α and β must be obtained satisfying the second pair of equations. Hence, as

$$\Sigma X^2=1,\quad \Sigma X\bar{X}=0,\quad \Sigma\bar{X}^2=1,$$

values of α and β are given by

$$\frac{\alpha}{\rho}+\frac{\beta}{\gamma}=1,$$

$$\frac{\alpha}{\bar{\tau}}-\frac{\beta}{\bar{\sigma}}=0,$$

and therefore

$$\frac{\alpha}{\frac{1}{\bar{\sigma}}}=\frac{\beta}{\frac{1}{\bar{\tau}}}=\frac{1}{\frac{1}{\rho\bar{\sigma}}+\frac{1}{\gamma\bar{\tau}}}.$$

Consequently, the coordinates of the point of intersection of the two normal planes—that is, of the orthogonal centre of the surface—are

$$\bar{x}-x=\frac{\frac{X}{\bar{\sigma}}+\frac{\bar{X}}{\bar{\tau}}}{\frac{1}{\rho\bar{\sigma}}+\frac{1}{\gamma\bar{\tau}}},$$

with three similar expressions for $\bar{y}-y$, $\bar{z}-z$, $\bar{v}-v$, respectively.

Let P denote the length of the orthogonal radius and λ, μ, ν, κ, its direction-cosines; thus

$$\lambda\mathrm{P}\left(\frac{1}{\rho\bar{\sigma}}+\frac{1}{\gamma\bar{\tau}}\right)=\frac{X}{\bar{\sigma}}+\frac{\bar{X}}{\bar{\tau}},$$

with three similar equations. The radius is a line in the orthogonal plane. Among other lines in this plane, there is the principal normal of the geodesic of length ρ_0, where

$$\frac{l}{\rho_0}=\frac{X}{\rho}+\frac{\bar{X}}{\gamma},$$

where ρ is the radius of curvature of the tangential geodesic, and $1/\gamma$ is the

regional geodesic curvature of the superficial geodesic. Also, we have defined (§ 346) the angle ψ by the relations

$$\frac{\cos\psi}{\rho_0} = \frac{1}{\rho}, \quad \frac{\sin\psi}{\rho_0} = \frac{1}{\gamma},$$

so that the direction-cosines of the principal normal of the geodesic are

$$l = X\cos\psi + \bar{X}\sin\psi,$$

with three similar equations; and, finally, the direction-cosines of the trinormal of the geodesic (also lying in the orthogonal plane) are

$$A = X\sin\psi - \bar{X}\cos\psi,$$

with three similar equations.

It was proved (§ 246) that the intersection of the orthogonal plane by the consecutive normal flat, which contains the consecutive orthogonal plane, is the line

$$\bar{x} - x - l\rho_0 = \Lambda(X\sin\psi - \bar{X}\cos\psi) = \Lambda A,$$

with three other equations, Λ being the current parameter of the line. To find where this meets the consecutive normal plane

$$\left.\begin{aligned} \Sigma(\bar{x}-x)\left(\frac{X}{\rho} + \frac{\bar{X}}{\gamma}\right) &= 1 \\ \Sigma(\bar{x}-x)\left(\frac{X}{\bar{\tau}} - \frac{\bar{X}}{\bar{\sigma}}\right) &= 0 \end{aligned}\right\},$$

we note that

$$\Sigma\frac{AX}{\rho} + \Sigma\frac{A\bar{X}}{\gamma} = \frac{\sin\psi}{\rho} - \frac{\cos\psi}{\gamma} = 0,$$

$$\Sigma\frac{lX}{\rho} + \Sigma\frac{l\bar{X}}{\gamma} = \frac{\cos\psi}{\rho} + \frac{\sin\psi}{\gamma} = \frac{1}{\rho_0},$$

so that the first equation is satisfied identically, as is to be expected: and the second equation gives

$$\rho_0\left(\frac{1}{\rho\bar{\tau}} - \frac{1}{\gamma\bar{\sigma}}\right) + \Lambda\left(\frac{1}{\rho\bar{\sigma}} + \frac{1}{\gamma\bar{\tau}}\right) = 0,$$

that is, the coördinates of the orthogonal centre are

$$\bar{x} - x = l\rho_0 + A\rho_0\frac{\dfrac{1}{\gamma\bar{\sigma}} - \dfrac{1}{\rho\bar{\tau}}}{\dfrac{1}{\rho\bar{\sigma}} + \dfrac{1}{\gamma\bar{\tau}}},$$

with three similar equations.

The identification of these two forms of expression for the coordinates of the orthogonal centre is immediate.

Directions of lines of curvature on a surface.

370. In the first place, we note that the orthogonal centre coincides with the centre of circular curvature of the geodesic when $\Lambda = 0$, that is, when

$$\frac{1}{\rho\bar{\tau}} - \frac{1}{\gamma\bar{\sigma}} = 0,$$

which accordingly can be regarded as an equation giving the directions of the four lines of curvature on the surface. It has already been derived (§ 365) from the conditions for maximum or minimum circular curvature in their initial form. The inference can also be derived directly from the critical equations of maximum or minimum circular curvature of a superficial geodesic, in their modified form. This curvature is given by

$$\begin{aligned}\frac{1}{\rho_0^2} &= \frac{1}{\rho^2} + \frac{1}{\gamma^2} \\ &= \overline{W}^2 + \frac{\Omega^2}{D}\overline{\Psi}^2,\end{aligned}$$

where

$$\overline{W} = \bar{L}p'^2 + 2\bar{M}p'q' + \bar{N}q'^2, \quad \overline{\Psi} = \bar{\gamma}_{11}p'^2 + 2\bar{\gamma}_{12}p'q' + \bar{\gamma}_{22}q'^2.$$

For a maximum or a minimum value of $1/\rho_0^2$ subject to the relation

$$\bar{E}p'^2 + 2\bar{F}p'q' + \bar{G}q'^2 = 1,$$

the critical equations are, as usual,

$$\left.\begin{aligned}\overline{W}(\bar{L}p' + \bar{M}q') + \frac{\Omega^2}{D}\overline{\Psi}(\bar{\gamma}_{11}p' + \bar{\gamma}_{12}q') &= \frac{1}{\rho_0^2}(\bar{E}p' + \bar{F}q') \\ \overline{W}(\bar{M}p' + \bar{N}q') + \frac{\Omega^2}{D}\overline{\Psi}(\bar{\gamma}_{12}p' + \bar{\gamma}_{22}q') &= \frac{1}{\rho_0^2}(\bar{F}p' + \bar{G}q')\end{aligned}\right\}.$$

When these are resolved linearly for $\overline{W}$ and $\frac{\Omega^2}{D}\overline{\Psi}$, they give

$$\overline{W}\begin{vmatrix} \bar{L}p' + \bar{M}q', & \bar{\gamma}_{11}p' + \bar{\gamma}_{12}q' \\ \bar{M}p' + \bar{N}q', & \bar{\gamma}_{12}p' + \bar{\gamma}_{22}q' \end{vmatrix} = \frac{1}{\rho_0^2}\begin{vmatrix} \bar{E}p' + \bar{F}q', & \bar{\gamma}_{11}p' + \bar{\gamma}_{12}q' \\ \bar{F}p' + \bar{G}q', & \bar{\gamma}_{12}p' + \bar{\gamma}_{22}q' \end{vmatrix},$$

$$\frac{\Omega^2}{D}\overline{\Psi}\begin{vmatrix} \bar{L}p' + \bar{M}q', & \bar{\gamma}_{11}p' + \bar{\gamma}_{12}q' \\ \bar{M}p' + \bar{N}q', & \bar{\gamma}_{12}p' + \bar{\gamma}_{22}q' \end{vmatrix} = \frac{1}{\rho_0^2}\begin{vmatrix} \bar{L}p' + \bar{M}q', & \bar{E}p' + \bar{F}q' \\ \bar{M}p' + \bar{N}q', & \bar{F}p' + \bar{G}q' \end{vmatrix}.$$

The three determinants are known (§ 360) in value; and

$$\overline{W} = \frac{1}{\rho}, \quad \frac{\Omega}{D^{\frac{1}{2}}}\overline{\Psi} = -\frac{1}{\gamma}.$$

When the geometrical values are substituted, the first equation becomes

$$\frac{1}{\rho}\left(\frac{1}{\rho\bar{\sigma}} + \frac{1}{\gamma\bar{\tau}}\right) = \frac{1}{\rho_0^2}\frac{1}{\bar{\sigma}},$$

and the second equation becomes

$$\frac{1}{\gamma}\left(\frac{1}{\rho\bar{\sigma}}+\frac{1}{\gamma\bar{\tau}}\right)=\frac{1}{\rho_0^2}\frac{1}{\bar{\tau}}.$$

In virtue of the relation

$$\frac{1}{\rho_0^2}=\frac{1}{\rho^2}+\frac{1}{\gamma^2},$$

each equation acquires the form

$$\frac{1}{\rho\bar{\tau}}-\frac{1}{\gamma\bar{\sigma}}=0,$$

in accordance with the result already (§ 365) indicated.

Moreover, this equation, when expressed in terms of p' and q', is

$$\overline{W}\begin{vmatrix} \bar{L}p'+\bar{M}q', & \bar{E}p'+\bar{F}q' \\ \bar{M}p'+\bar{N}q', & \bar{F}p'+\bar{G}q' \end{vmatrix}=\frac{\Omega^2}{D}\overline{\Psi}\begin{vmatrix} \bar{E}p'+\bar{F}q', & \bar{\gamma}_{11}p'+\bar{\gamma}_{12}q' \\ \bar{F}p'+\bar{G}q', & \bar{\gamma}_{12}p'+\bar{\gamma}_{22}q' \end{vmatrix},$$

the result of eliminating $1/\rho_0^2$ between the two preceding equations.

It thus appears that, for the directions of the four lines of curvature, the orthogonal centre of the surface coincides with the centre of circular curvature of the superficial geodesic along a line of curvature; and that, for each line of curvature, the relation

$$\frac{1}{\rho\bar{\tau}}-\frac{1}{\gamma\bar{\sigma}}=0$$

is satisfied, this relation being merely a disguised form of the equation giving the directions of the four lines of curvature. Also, when taken in the form

$$\frac{\cos\psi}{\bar{\tau}}=\frac{\sin\psi}{\bar{\sigma}},$$

it shews that the orthogonal radius (of course, only for these four directions) coincides with the principal normal of the superficial geodesic.

Lemniscate-locus of the centre of circular curvature of superficial geodesics.

371. In order to consider the locus of the centre of circular curvature of the superficial geodesic (which should be a lemniscate curve, after § 242) and the locus of the orthogonal centre of the surface (which should be a conic, after § 249), we change the axes of reference.

We first take

$$\Sigma x'(\bar{x}-x)=\zeta, \quad \Sigma\bar{l}(\bar{x}-x)=v.$$

For the centre of circular curvature, we have

$$\bar{x}-x=l\rho_0,$$

with three other equations: and

$$\Sigma x'l=0, \quad \Sigma\bar{l}l=0,$$

so that

$$\zeta = 0, \quad \upsilon = 0,$$

which are the equations of the orthogonal plane. For the orthogonal centre, we have

$$\bar{x} - x = l\rho_0 + A\Lambda,$$

with three other equations: and

$$\Sigma x' A = 0, \quad \Sigma \bar{l} A = 0,$$

so that

$$\zeta = 0, \quad \upsilon = 0,$$

again the orthogonal plane. Thus, as is to be expected, both loci lie in the orthogonal plane.

Next, we take

$$\Sigma X(\bar{x} - x) = \xi, \quad \Sigma \bar{X}(\bar{x} - x) = \eta;$$

thus ξ and η are current coordinates in the orthogonal plane. Also, as

$$\Sigma X^2 = 1, \quad \Sigma X\bar{X} = 0, \quad \Sigma \bar{X}^2 = 1,$$

the axes of ξ and of η are perpendicular to one another. With these coordinates, the two loci must be considered separately.

For the locus of the centre of circular curvature of the superficial geodesic, we have the four equations of the type

$$\bar{x} - x = l\rho_0 = (X \cos\psi + \bar{X} \sin\psi)\rho_0;$$

hence for this locus

$$\xi = \Sigma X(\bar{x} - x) = \rho_0 \cos\psi = \frac{\rho_0^2}{\rho},$$

$$\eta = \Sigma \bar{X}(\bar{x} - x) = \rho_0 \sin\psi = \frac{\rho_0^2}{\gamma},$$

and therefore

$$\xi^2 + \eta^2 = \rho_0^4 \left(\frac{1}{\rho^2} + \frac{1}{\gamma^2}\right) = \rho_0^2.$$

If then we take polar coordinates r and ϕ (manifestly r, the distance of the point on the locus from the origin, is equal to ρ_0, and ϕ is equal to ψ), we have

$$\frac{\cos\phi}{r} = \frac{\xi}{\xi^2 + \eta^2} = \frac{1}{\rho} = \bar{L}p'^2 + 2\bar{M}p'q' + \bar{N}q'^2,$$

$$\frac{\sin\phi}{r} = \frac{\eta}{\xi^2 + \eta^2} = \frac{1}{\gamma} = -\frac{\Omega}{D^{\frac{1}{2}}}(\bar{\gamma}_{11}p'^2 + 2\bar{\gamma}_{12}p'q' + \bar{\gamma}_{22}q'^2).$$

Also, we have

$$1 = \bar{E}p'^2 + 2\bar{F}p'q' + \bar{G}q'^2.$$

Hence, using the (invariant) determinant $\overline{\Pi}$, where

$$\overline{\Pi} = \begin{vmatrix} \bar{E}, & \bar{F}, & \bar{G} \\ \bar{L}, & \bar{M}, & \bar{N} \\ \bar{\gamma}_{11}, & \bar{\gamma}_{12}, & \bar{\gamma}_{22} \end{vmatrix},$$

we have

$$\overline{\Pi}p'^2 = (\bar{\gamma}_{12}\bar{G} - \bar{\gamma}_{22}\bar{F})\frac{\cos\phi}{r} - \frac{D^{\frac{1}{2}}}{\Omega}(\bar{F}\bar{N} - \bar{G}\bar{M})\frac{\sin\phi}{r} + (\bar{\gamma}_{22}\bar{M} - \bar{\gamma}_{12}\bar{N}),$$

$$2\overline{\Pi}p'q' = (\bar{\gamma}_{22}\bar{E} - \bar{\gamma}_{11}\bar{G})\frac{\cos\phi}{r} - \frac{D^{\frac{1}{2}}}{\Omega}(\bar{G}\bar{L} - \bar{E}\bar{N})\frac{\sin\phi}{r} + (\bar{\gamma}_{11}\bar{N} - \bar{\gamma}_{22}\bar{L}),$$

$$\overline{\Pi}q'^2 = (\bar{\gamma}_{11}\bar{F} - \bar{\gamma}_{12}\bar{E})\frac{\cos\phi}{r} - \frac{D^{\frac{1}{2}}}{\Omega}(\bar{E}\bar{M} - \bar{F}\bar{L})\frac{\sin\phi}{r} + (\bar{\gamma}_{12}\bar{L} - \bar{\gamma}_{11}\bar{M}).$$

When these values are substituted in the identity

$$(2\overline{\Pi}p'q')^2 = 4\,.\,\overline{\Pi}p'^2\,.\,\overline{\Pi}q'^2,$$

we have an equation which represents a lemniscate curve. It is the locus of the centre of circular curvature of the superficial geodesic.

Conic-locus of the orthogonal centre.

372. For the locus of the orthogonal centre of the surface, we have obtained its coordinates, referred to the initial axes of the quadruple space, in the form (§ 369)

$$\bar{x} - x = \frac{\dfrac{X}{\bar{\sigma}} + \dfrac{\bar{X}}{\bar{\tau}}}{\dfrac{1}{\rho\bar{\sigma}} + \dfrac{1}{\gamma\bar{\tau}}},$$

with three similar equations. Hence, for the new axes in the orthogonal plane,

$$\left(\frac{1}{\rho\bar{\sigma}} + \frac{1}{\gamma\bar{\tau}}\right)\xi = \left(\frac{1}{\rho\bar{\sigma}} + \frac{1}{\gamma\bar{\tau}}\right)\Sigma X\,(\bar{x} - x) = \frac{1}{\bar{\sigma}},$$

$$\left(\frac{1}{\rho\bar{\sigma}} + \frac{1}{\gamma\bar{\tau}}\right)\eta = \left(\frac{1}{\rho\bar{\sigma}} + \frac{1}{\gamma\bar{\tau}}\right)\Sigma\bar{X}\,(\bar{x} - x) = \frac{1}{\bar{\tau}}.$$

Now

$$\frac{1}{\bar{\sigma}} = \theta_3\frac{\Omega}{D}\begin{vmatrix} \bar{\gamma}_{11}p' + \bar{\gamma}_{12}q', & \bar{\gamma}_{12}p' + \bar{\gamma}_{22}q' \\ \bar{E}p' + \bar{F}q', & \bar{F}p' + \bar{G}q' \end{vmatrix},$$

$$\frac{1}{\bar{\tau}} = \theta_3\frac{1}{D^{\frac{1}{2}}}\begin{vmatrix} \bar{L}p' + \bar{M}q', & \bar{M}p' + \bar{N}q' \\ \bar{E}p' + \bar{F}q', & \bar{F}p' + \bar{G}q' \end{vmatrix},$$

$$\frac{1}{\rho\bar{\sigma}} + \frac{1}{\gamma\bar{\tau}} = \theta_3\frac{\Omega}{D}\begin{vmatrix} \bar{\gamma}_{11}p' + \bar{\gamma}_{12}q', & \bar{\gamma}_{12}p' + \bar{\gamma}_{22}q' \\ \bar{L}p' + \bar{M}q', & \bar{M}p' + \bar{N}q' \end{vmatrix};$$

and therefore a point on the locus of the orthogonal centre is given by

$$\frac{\xi}{(\bar{\gamma}_{11}\bar{F} - \bar{\gamma}_{12}\bar{E})p'^2 + (\bar{\gamma}_{11}\bar{G} - \bar{\gamma}_{22}\bar{E})p'q' + (\bar{\gamma}_{12}\bar{G} - \bar{\gamma}_{22}\bar{F})q'^2}$$

$$= \frac{\dfrac{\Omega}{D^{\frac{1}{2}}}\eta}{(\bar{L}\bar{F} - \bar{M}\bar{E})p'^2 + (\bar{L}\bar{G} - \bar{N}\bar{E})p'q' + (\bar{M}\bar{G} - \bar{N}\bar{F})q'^2}$$

$$= \frac{1}{(\bar{\gamma}_{11}\bar{M} - \bar{\gamma}_{12}\bar{L})p'^2 + (\bar{\gamma}_{11}\bar{N} - \bar{\gamma}_{22}\bar{L})p'q' + (\bar{\gamma}_{12}\bar{N} - \bar{\gamma}_{22}\bar{M})q'^2}.$$

Let the denominator of the last fraction be denoted by Θ, so that $\frac{1}{2}\Theta$ is the Jacobian of $\overline{W}$ and $\overline{\Psi}$: then we find

$$\left.\begin{aligned} \bar{L}\xi - \frac{\Omega}{D^{\frac{1}{2}}}\bar{\gamma}_{11}\eta - \bar{E} &= \frac{\overline{\Pi}}{\Theta}q'^2 \\ \bar{M}\xi - \frac{\Omega}{D^{\frac{1}{2}}}\bar{\gamma}_{12}\eta - \bar{F} &= -\frac{\overline{\Pi}}{\Theta}p'q' \\ \bar{N}\xi - \frac{\Omega}{D^{\frac{1}{2}}}\bar{\gamma}_{22}\eta - \bar{G} &= \frac{\overline{\Pi}}{\Theta}p'^2 \end{aligned}\right\}.$$

Consequently, ξ and η satisfy the equation

$$\left(\bar{L}\xi - \frac{\Omega}{D^{\frac{1}{2}}}\bar{\gamma}_{11}\eta - \bar{E}\right)\left(\bar{N}\xi - \frac{\Omega}{D^{\frac{1}{2}}}\bar{\gamma}_{22}\eta - \bar{G}\right) = \left(\bar{M}\xi - \frac{\Omega}{D^{\frac{1}{2}}}\bar{\gamma}_{12}\eta - \bar{F}\right)^2,$$

which is manifestly a conic in the orthogonal plane and is the locus of the orthogonal centre. It is Kommerell's characteristic conic, for a surface postulated in the region.

Asymptotes of the conic: they are the directions of zero tilt in space.

373. The directions of the asymptotes of this conic are indicated by the analytically infinite values of ξ and of η, that is, by the equation

$$\frac{1}{\rho\bar{\sigma}} + \frac{1}{\gamma\bar{\tau}} = 0;$$

and therefore (§ 372) the asymptotes of the conic are given by the directions

$$\begin{vmatrix} \bar{\gamma}_{11}p' + \bar{\gamma}_{12}q', & \bar{\gamma}_{12}p' + \bar{\gamma}_{22}q' \\ \bar{L}p' + \bar{M}q', & \bar{M}p' + \bar{N}q' \end{vmatrix} = 0.$$

Now we have

$$\frac{1}{\rho} = \overline{W} = \bar{L}p'^2 + 2\bar{M}p'q' + \bar{N}q'^2,$$

$$-\frac{D^{\frac{1}{2}}}{\Omega}\frac{1}{\gamma} = \overline{\Psi} = \bar{\gamma}_{11}p'^2 + 2\bar{\gamma}_{12}p'q' + \bar{\gamma}_{22}q'^2.$$

We thus have an involution of directions

$$\overline{W} - \theta\overline{\Psi} = (\bar{L} - \theta\bar{\gamma}_{11})p'^2 + 2(\bar{M} - \theta\bar{\gamma}_{12})p'q' + (\bar{N} - \theta\bar{\gamma}_{22})q'^2 = 0.$$

The double lines of the involution are given by the values of θ which are the roots of the quadratic

$$(\bar{L} - \theta\bar{\gamma}_{11})(\bar{N} - \theta\bar{\gamma}_{22}) = (\bar{M} - \theta\bar{\gamma}_{12})^2;$$

and therefore the joint equation of the double lines is

$$(\overline{W} - \theta_1\overline{\Psi})(\overline{W} - \theta_2\overline{\Psi}) = \overline{W}^2 - \overline{W}\overline{\Psi}\frac{\bar{L}\bar{\gamma}_{22} - 2\bar{M}\bar{\gamma}_{12} + \bar{N}\bar{\gamma}_{11}}{\bar{\gamma}_{11}\bar{\gamma}_{22} - \bar{\gamma}_{12}^2} + \overline{\Psi}^2\frac{\bar{L}\bar{N} - \bar{M}^2}{\bar{\gamma}_{11}\bar{\gamma}_{22} - \bar{\gamma}_{12}^2}.$$

In the foregoing equation of the asymptotes, the determinant is Θ, the Jacobian of $\overline{W}$ and $\overline{\Psi}$; and we have the identity

$$-\Theta^2 = (\bar{\gamma}_{11}\bar{\gamma}_{22} - \bar{\gamma}_{12}^2)\,\overline{W}^2 + (\overline{L}\overline{N} - \overline{M}^2)\,\overline{\Psi}^2 - (\overline{L}\bar{\gamma}_{22} - 2\overline{M}\bar{\gamma}_{12} + \overline{N}\bar{\gamma}_{11})\,\overline{W}\overline{\Psi}:$$

thus, effectively, the joint equation of the double lines is

$$\Theta = 0,$$

or the asymptotes of the characteristic conic correspond to the double lines of the foregoing involution*.

When the equation of the asymptotes is taken in the form

$$(\bar{\gamma}_{11}\bar{\gamma}_{22} - \bar{\gamma}_{12}^2)\,\overline{W}^2 - (\overline{L}\bar{\gamma}_{22} - 2\overline{M}\bar{\gamma}_{12} + \overline{N}\bar{\gamma}_{11})\,\overline{W}\overline{\Psi} + (\overline{L}\overline{N} - \overline{M}^2)\,\overline{\Psi}^2 = 0,$$

and when the values of the two discriminants and of the intermediate invariant (§ 362) are inserted, this equation takes the form

$$\frac{1}{\gamma^2}\frac{1}{\rho_1\rho_2} + \frac{1}{\rho^2}\frac{1}{\gamma_1\gamma_2} = \frac{2}{\rho\gamma}\left\{\left(\frac{1}{\rho_1\gamma_2} + \frac{1}{\rho_2\gamma_1}\right)\cos^2 i + \left(\frac{1}{\rho_1\gamma_1} + \frac{1}{\rho_2\gamma_2}\right)\sin^2 i\right\},$$

which can be brought into analytical agreement with the initial equation

$$\frac{1}{\rho\bar{\sigma}} + \frac{1}{\gamma\bar{\tau}} = 0,$$

defining the directions of the asymptotes.

One other interpretation of this equation is to be noted. We have obtained (§ 353) the expression for the spatial tilt of the superficial geodesic in the form

$$\frac{1}{\tau_0} = \frac{\cos\psi}{\bar{\sigma}} + \frac{\sin\psi}{\bar{\tau}}.$$

But (§ 346) we have

$$\frac{\cos\psi}{\rho_0} = \frac{1}{\rho}, \qquad \frac{\sin\psi}{\rho_0} = \frac{1}{\gamma};$$

and therefore

$$\frac{1}{\rho_0\tau_0} = \frac{1}{\rho\bar{\sigma}} + \frac{1}{\gamma\bar{\tau}}.$$

Consequently, along those directions on the surface, by which the asymptotes of the characteristic conic are determined, we have

$$\frac{1}{\tau_0} = 0;$$

or the asymptotes of the characteristic conic belong to those directions, the superficial geodesics through which have zero as the measure of their tilt in the quadruple space.

The lemniscate is the pedal of the conic: the four normals to the conic determine the lines of curvature.

374. We have seen (§ 369) that the radius vector from the point x, y, z, v, on the surface to a point of the conic is the orthogonal radius P corresponding to some direction p', q', on the surface. Hence a maximum or a minimum

* This property was first stated by Kommerell, *Math. Ann.*, vol. LX (1905), p. 557.

value of P is obtained by drawing a normal to the conic from x, y, z, v: from any external point, four normals can be drawn to a conic: and therefore it may be inferred that the feet of these normals, lying on the conic, correspond to the directions of the four lines of curvature on the surface.

This inference can be verified directly from the equation of the conic in § 372. Let ξ_0, η_0, be any point on the conic; the tangent at this point is

$$(\xi - \xi_0)(\bar{N}\lambda - 2\bar{M}\mu + \bar{L}\nu) - \frac{\Omega}{D^{\frac{1}{2}}}(\eta - \eta_0)(\bar{\gamma}_{22}\lambda - 2\bar{\gamma}_{12}\mu + \bar{\gamma}_{11}\nu) = 0,$$

where

$$\lambda = \bar{L}\xi_0 - \frac{\Omega}{D^{\frac{1}{2}}}\bar{\gamma}_{11}\eta_0 - \bar{E} = \frac{\overline{\Pi}}{\Theta}q'^2,$$

$$\mu = \bar{M}\xi_0 - \frac{\Omega}{D^{\frac{1}{2}}}\bar{\gamma}_{12}\eta_0 - \bar{F} = -\frac{\overline{\Pi}}{\Theta}p'q',$$

$$\nu = \bar{N}\xi_0 - \frac{\Omega}{D^{\frac{1}{2}}}\bar{\gamma}_{22}\eta_0 - \bar{G} = \frac{\overline{\Pi}}{\Theta}p'^2,$$

(by § 372), the direction p', q', being associated with the point ξ_0, η_0. The equation of the radius vector joining x, y, z, v, and ξ_0, η_0, is

$$\eta = \frac{\eta_0}{\xi_0}\xi,$$

and it will be a normal if it is at right angles to the tangent. As the axes of reference in the orthogonal plane are perpendicular, the necessary and sufficient condition, that the two lines are perpendicular, is

$$\xi_0(\bar{N}\lambda - 2\bar{M}\mu + \bar{L}\nu) + \frac{\Omega}{D^{\frac{1}{2}}}\eta_0(\bar{\gamma}_{22}\lambda - 2\bar{\gamma}_{12}\mu + \bar{\gamma}_{11}\nu) = 0.$$

The coefficient of ξ_0 on the left-hand side of this equation of condition is

$$\frac{\overline{\Pi}}{\Theta}(\bar{N}q'^2 + 2\bar{M}p'q' + \bar{L}p'^2),$$

that is, it is equal to

$$\frac{\overline{\Pi}}{\Theta}\frac{1}{\rho};$$

and the coefficient of η_0 is

$$\frac{\overline{\Pi}}{\Theta}\frac{\Omega}{D^{\frac{1}{2}}}(\bar{\gamma}_{22}q'^2 + 2\bar{\gamma}_{12}p'q' + \bar{\gamma}_{11}p'^2),$$

that is, it is equal to

$$\frac{\overline{\Pi}}{\Theta}\left(-\frac{1}{\gamma}\right).$$

Thus the requisite condition is

$$\frac{\xi_0}{\rho} - \frac{\eta_0}{\gamma} = 0.$$

But, for any point on the conic (§ 372),

$$\left(\frac{1}{\rho\bar{\sigma}} + \frac{1}{\gamma\bar{\tau}}\right)\xi = \frac{1}{\bar{\sigma}}, \quad \left(\frac{1}{\rho\bar{\sigma}} + \frac{1}{\gamma\bar{\tau}}\right)\eta = \frac{1}{\bar{\tau}};$$

and therefore the condition is

$$\frac{1}{\rho\bar{\sigma}} - \frac{1}{\gamma\bar{\tau}} = 0.$$

This equation is (§ 370) the transformed equivalent of the equation giving the directions of the four lines of curvature on the surface; and the equation also implies that the quantity Λ, of § 369, vanishes.

Hence for these points on the conic, and therefore for the four principal directions on the surface, the orthogonal centre of the surface coincides with the centre of circular curvature of the superficial geodesic through the direction; and the normals to the four principal geodesics, the same as the four principal orthogonal radii in magnitude and in direction, are the four normals from x, y, z, v, to the characteristic conic.

Further, the radius vector from x, y, z, v, to the lemniscate-curve, which is the locus of the centre of circular curvature of a superficial geodesic, is that actual radius of circular curvature which is a maximum or a minimum for the principal directions, and therefore is a normal to that locus. Thus the lemniscate locus and the characteristic conic touch at these four points, which are the feet of the normals to the conic.

This result is a corollary from the further property that, as before, the lemniscate curve is the pedal of the conic taken with respect to the point x, y, z, v, on the surface (being the origin for the representation of the conic in the orthogonal plane, taken as a plane of ξ and η)—a result, again, that can be derived from the preceding analysis. Let the tangent at the point ξ_0, η_0, of the conic make an angle ϕ with the axis of ξ; then

$$\tan\phi = \frac{\bar{N}\lambda - 2\bar{M}\mu + \bar{L}\nu}{\frac{\Omega}{D^{\frac{1}{2}}}(\bar{\gamma}_{22}\lambda - 2\bar{\gamma}_{12}\mu + \bar{\gamma}_{11}\nu)},$$

with the foregoing notation; that is,

$$\tan\phi = \frac{\frac{\bar{\Pi}}{\bar{\Theta}}\frac{1}{\rho}}{-\frac{\bar{\Pi}}{\bar{\Theta}}\frac{1}{\gamma}} = -\frac{\gamma}{\rho}.$$

The angle ψ, determining the direction of the principal normal to the superficial geodesic, is given by

$$\frac{\cos\psi}{\rho_0} = \frac{1}{\rho}, \quad \frac{\sin\psi}{\rho_0} = \frac{1}{\gamma},$$

that is,

$$\tan\psi = \frac{\rho}{\gamma}.$$

Hence

$$\tan\phi \tan\psi = -1,$$

and therefore the tangent to the characteristic conic at any point is perpendicular to the principal normal to the geodesic. The line joining the two centres (of circular curvature of the geodesic and the orthogonal centre) is, by § 369, perpendicular to this principal normal, because $\Sigma lA = 0$; hence the lemniscate-curve, which is the locus of the geodesic centre, is the pedal of the characteristic conic with respect to the point x, y, z, v, on the surface.

Three forms of the quartic equation for the directions of the lines of curvature.

375. There are various modes of approach to the equation which, quartic in its algebraical form, gives the directions of the lines of curvature on the surface, and, in a geometrical form, has been obtained as

$$\frac{1}{\rho\bar{\tau}} - \frac{1}{\gamma\bar{\sigma}} = 0.$$

For example, we already (§ 237) have established the property that, along a line of curvature on any surface in quadruple space, the torsion of a tangential geodesic in the direction of the line of curvature vanishes. Now (§ 353) the spatial torsion of a superficial geodesic is given by

$$\frac{1}{\sigma_0} = \frac{\sin\psi}{\bar{\sigma}} - \frac{\cos\psi}{\bar{\tau}},$$

so that, as

$$\frac{\cos\psi}{\rho_0} = \frac{1}{\rho}, \quad \frac{\sin\psi}{\rho_0} = \frac{1}{\gamma},$$

we have

$$\frac{1}{\rho_0\sigma_0} = \frac{1}{\gamma\bar{\sigma}} - \frac{1}{\rho\bar{\tau}}.$$

Accordingly, the vanishing of the spatial torsion of the geodesic, characteristic of a line of curvature, leads to the equation

$$\frac{1}{\gamma\bar{\sigma}} - \frac{1}{\rho\bar{\tau}} = 0.$$

Again, we have adopted (§§ 238, 365), as one of the equivalent definitions of a line of curvature, the property that its direction must be such as to make the radius of circular curvature a maximum or a minimum. Let a direction

be taken making an angle ϕ with the direction of principal circular curvature $1/\rho_1$ of the regional geodesic, and making an angle ϕ' with the direction of principal regional geodesic curvature $1/\gamma_1$ of a superficial geodesic. Then

$$\phi' = \phi + i,$$

where i is the inclination of these directions of principal curvature; also

$$\frac{1}{\rho} = \frac{\cos^2\phi}{\rho_1} + \frac{\sin^2\phi}{\rho_2}, \quad \frac{1}{\gamma} = \frac{\cos^2\phi'}{\gamma_1} + \frac{\sin^2\phi'}{\gamma_2}.$$

Thus

$$\frac{d}{d\phi}\left(\frac{1}{\rho}\right) = -2\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\sin\phi\,\cos\phi = \frac{2}{\bar{\tau}},$$

$$\frac{d}{d\phi}\left(\frac{1}{\gamma}\right) = \frac{d}{d\phi'}\left(\frac{1}{\gamma}\right) = -2\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)\sin\phi'\cos\phi' = -\frac{2}{\bar{\sigma}},$$

after § 363. Now the radius of circular curvature of the superficial geodesic is given by

$$\frac{1}{\rho_0^2} = \frac{1}{\rho^2} + \frac{1}{\gamma^2},$$

and therefore

$$\frac{d}{d\phi}\left(\frac{1}{\rho_0^2}\right) = 4\left(\frac{1}{\rho\bar{\tau}} - \frac{1}{\gamma\bar{\sigma}}\right).$$

Consequently, along the directions which provide maximum or minimum circular curvature of the superficial geodesic, we have the relation

$$\frac{1}{\rho\bar{\tau}} - \frac{1}{\gamma\bar{\sigma}} = 0,$$

the same equation as before.

376. The more directly algebraical equation for the lines of curvature admits two equivalent forms of expression, both derived from this geometrical relation between the curvatures.

In one of these expressions we take the binariants which represent the various curvatures of the superficial geodesic through the direction. We have

$$\frac{1}{\rho} = \bar{L}p'^2 + 2\bar{M}p'q' + \bar{N}q'^2 = \bar{W};$$

$$-\frac{D^{\frac{1}{2}}}{\Omega}\frac{1}{\gamma} = \bar{\gamma}_{11}p'^2 + 2\bar{\gamma}_{12}p'q' + \bar{\gamma}_{22}q'^2 = \bar{\Psi};$$

also (§ 360)

$$\frac{D}{\theta_3\Omega}\frac{1}{\bar{\sigma}} = \begin{vmatrix} \bar{\gamma}_{11}p' + \bar{\gamma}_{12}q', & \bar{\gamma}_{12}p' + \bar{\gamma}_{22}q' \\ \bar{E}p' + \bar{F}q', & \bar{F}p' + \bar{G}q' \end{vmatrix} = J_\gamma,$$

where $4J_\gamma$ is the Jacobian of the quantity $\bar{E}p'^2 + 2\bar{F}p'q' + \bar{G}q'^2$ and of the quadratic form involved in the curvature $1/\gamma$; and

$$\frac{D^{\frac{1}{2}}}{\theta_3}\frac{1}{\bar{\tau}} = \begin{vmatrix} \bar{L}p' + \bar{M}q', & \bar{M}p' + \bar{N}q' \\ \bar{E}p' + \bar{F}q', & \bar{F}p' + \bar{G}q' \end{vmatrix} = J_\rho,$$

where $4J_\rho$ is the Jacobian of the same arc-quantic and of the quadratic form for the curvature $1/\rho$. Then on substitution for ρ, γ, $\bar{\sigma}$, $\bar{\tau}$, the equation of the lines of curvature becomes

$$\Omega^2 \bar{\Psi} J_\gamma + D\overline{W} J_\rho = 0,$$

manifestly an equation of the fourth degree.

For the other expression, let the configuration be referred to the indicatrix conics. Let a line of curvature on the surface make an angle $\frac{1}{2}(\theta - i)$ with the direction $O\rho_1$ (see figure, p. 228) and therefore an angle $\frac{1}{2}(\theta + i)$ with the direction $O\gamma_1$. For such a direction, we have

$$\frac{2}{\gamma} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)\cos(\theta + i),$$

$$\frac{2}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2} + \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\cos(\theta - i),$$

$$\frac{2}{\bar{\sigma}} = \left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)\sin(\theta + i),$$

$$\frac{2}{\bar{\tau}} = \left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right)\sin(\theta - i).$$

When the direction is that of a line of curvature, along which the property

$$\frac{1}{\rho\bar{\tau}} - \frac{1}{\gamma\bar{\sigma}} = 0$$

holds, the variable θ must satisfy the equation

$$\left(\frac{1}{\rho_2^2} - \frac{1}{\rho_1^2}\right)\sin(\theta - i) - \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^2 \sin(\theta - i)\cos(\theta - i)$$
$$+ \left(\frac{1}{\gamma_1^2} - \frac{1}{\gamma_2^2}\right)\sin(\theta + i) + \left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)^2 \sin(\theta + i)\cos(\theta + i) = 0.$$

Let

$$a = \left\{\left(\frac{1}{\gamma_1^2} - \frac{1}{\gamma_2^2}\right) - \left(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2}\right)\right\}\cos i,$$

$$b = \left\{\left(\frac{1}{\gamma_1^2} - \frac{1}{\gamma_2^2}\right) + \left(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2}\right)\right\}\sin i,$$

$$2c = \left\{\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)^2 - \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^2\right\}\cos 2i,$$

$$2d = \left\{\left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)^2 + \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^2\right\}\sin 2i;$$

then the equation becomes

$$a\sin\theta + b\cos\theta + c\sin 2\theta + d\cos 2\theta = 0,$$

an equation with four distinct roots. This form is the alternative expression of the equation for the four lines of curvature.

Ex. 1. Let the equation be

$$\Gamma_1 \sin(\theta+\epsilon)+\Gamma_2 \sin(2\theta+2\eta)=0,$$

where

$$a=\Gamma_1\cos\epsilon, \quad b=\Gamma_1\sin\epsilon, \quad c=\Gamma_2\cos 2\eta, \quad d=\Gamma_2\sin 2\eta.$$

Also, let

$$\theta+\eta=\vartheta, \quad \epsilon-\eta=\omega,$$

$$k=\frac{\Gamma_1}{\Gamma_2}\cos\omega, \quad l=\frac{\Gamma_1}{\Gamma_2}\sin\omega;$$

then the equation becomes

$$\sin 2\vartheta+k\sin\vartheta+l\cos\vartheta=0.$$

Take a variable t, equal to $\tan\vartheta$; then t is a root of the quartic equation

$$(kt+l)(1+t^2)^{\frac{1}{2}}+2t=0,$$

that is,

$$k^2t^4+2klt^3+(l^2+k^2-4)t^2+2klt+l^2=0.$$

We therefore have

$$t_1+t_2+t_3+t_4=t_2t_3t_4+t_3t_4t_1+t_4t_1t_2+t_1t_2t_3,$$

that is

$$\tan(\vartheta_1+\vartheta_2+\vartheta_3+\vartheta_4)=0;$$

and therefore

$$\vartheta_1+\vartheta_2+\vartheta_3+\vartheta_4=n\pi,$$

where n is an integer.

Ex. 2. Verify that

$$4\cos\vartheta_1\cos\vartheta_2\cos\vartheta_3\cos\vartheta_4=-k^2;$$

and prove that the foregoing integer n is odd.

Comparison of results for a regional surface with those for a surface in free space.

377. Before passing from the consideration of a surface, given in a region, one remark is to be made, by way of comparison between the theory of a surface, thus given in a region, and the former theory of a surface, when given in open homaloidal quadruple space.

For a surface in free space, the direction-cosines of the principal normal to the surface l, m, n, k, and the radius of curvature ρ (now to be changed to ρ_0, solely for purposes of comparison) are determined (§§ 211, 213) by four relations of the form

$$\frac{l}{\rho_0}=(x_{11}-x_1\Gamma-x_2\Delta)p'^2+2(x_{12}-x_1\Gamma'-x_2\Delta')p'q'+(x_{22}-x_1\Gamma''-x_2\Delta'')q'^2$$
$$=\xi_{11}p'^2+2\xi_{12}p'q'+\xi_{22}q'^2.$$

There are three magnitudes L, M, N, (now to be changed to L_0, M_0, N_0, solely for purposes of comparison), such that

$$L_0=\Sigma l\xi_{11}, \quad M_0=\Sigma l\xi_{12}, \quad N_0=\Sigma l\xi_{22};$$

there are three quantities Ω, Ω', Ω'', such that

$$\Omega=\Sigma A\xi_{11}, \quad \Omega'=\Sigma A\xi_{12}, \quad \Omega''=\Sigma A\xi_{22};$$

and there are the relations

$$\frac{1}{\rho_0} = L_0 p'^2 + 2M_0 p'q' + N_0 q'^2,$$
$$0 = \Omega p'^2 + 2\Omega' p'q' + \Omega'' q'^2.$$

For a regional surface, the radius of circular curvature and its direction are determined by the same magnitudes ρ_0, l, m, n, k. But now we have the radius ρ of circular curvature of the regional geodesic, and X, Y, Z, V, the direction-cosines of ρ; also the radius γ of geodesic curvature of the superficial geodesic (that is, its curvature relative to the regional geodesic), and $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{V}$, the direction-cosines of γ. There are magnitudes $\bar{L}$, $\bar{M}$, $\bar{N}$, and magnitudes $\bar{\gamma}_{11}$, $\bar{\gamma}_{12}$, $\bar{\gamma}_{22}$, such that

$$\frac{1}{\rho} = \bar{L}p'^2 + 2\bar{M}p'q' + \bar{N}q'^2,$$

and

$$\frac{1}{\gamma} = -\frac{\Omega_0}{D^{\frac{1}{2}}}(\bar{\gamma}_{11}p'^2 + 2\bar{\gamma}_{12}p'q' + \bar{\gamma}_{22}q'^2),$$

where Ω_0^2 is used (solely for purposes of the comparison) instead of the regional magnitude Ω^2 of § 362. These magnitudes ρ and γ do not arise for surfaces in free space; the magnitudes Ω, Ω', Ω'', do not arise for regional surfaces.

Now (§ 346) we have four equations of the type

$$\frac{l}{\rho_0} = \frac{X}{\rho} + \frac{\bar{X}}{\gamma},$$

that is,

$$\xi_{11}p'^2 + 2\xi_{12}p'q' + \xi_{22}q'^2 = X(\bar{L}p'^2 + 2\bar{M}p'q' + \bar{N}q'^2) - \bar{X}\frac{\Omega_0}{D^{\frac{1}{2}}}(\bar{\gamma}_{11}p'^2 + 2\bar{\gamma}_{12}p'q' + \bar{\gamma}_{22}q'^2);$$

and therefore

$$\left.\begin{aligned} \xi_{11} &= X\bar{L} - \bar{X}\frac{\Omega_0}{D^{\frac{1}{2}}}\bar{\gamma}_{11} \\ \xi_{12} &= X\bar{M} - \bar{X}\frac{\Omega_0}{D^{\frac{1}{2}}}\bar{\gamma}_{12} \\ \xi_{22} &= X\bar{N} - \bar{X}\frac{\Omega_0}{D^{\frac{1}{2}}}\bar{\gamma}_{22} \end{aligned}\right\},$$

with three similar sets of relations for η_{11}, η_{12}, η_{22}; ζ_{11}, ζ_{12}, ζ_{22}; υ_{11}, υ_{12}, υ_{22}. Moreover, we have (§§ 346, 350)

$$\frac{\cos\psi}{\rho_0} = \frac{1}{\rho}, \quad \frac{\sin\psi}{\rho_0} = \frac{1}{\gamma};$$
$$l = X\cos\psi + \bar{X}\sin\psi, \quad A = X\sin\psi - \bar{X}\cos\psi,$$

with similar expressions for m, n, k; B, C, D.

Thus

$$L_0 = \Sigma l\xi_{11} = \Sigma (X\cos\psi + \bar{X}\sin\psi)\left(X\bar{L} - \bar{X}\frac{\Omega}{D^{\frac{1}{2}}}\bar{\gamma}_{11}\right)$$

$$= \bar{L}\cos\psi - \frac{\Omega_0}{D^{\frac{1}{2}}}\bar{\gamma}_{11}\sin\psi,$$

with

$$M_0 = \bar{M}\cos\psi - \frac{\Omega_0}{D^{\frac{1}{2}}}\bar{\gamma}_{12}\sin\psi,$$

$$N_0 = \bar{N}\cos\psi - \frac{\Omega_0}{D^{\frac{1}{2}}}\bar{\gamma}_{22}\sin\psi.$$

Similarly

$$\Omega = \Sigma A\xi_{11} = \bar{L}\sin\psi + \frac{\Omega_0}{D^{\frac{1}{2}}}\bar{\gamma}_{11}\cos\psi,$$

$$\Omega' = \Sigma A\xi_{12} = \bar{M}\sin\psi + \frac{\Omega_0}{D^{\frac{1}{2}}}\bar{\gamma}_{12}\cos\psi,$$

$$\Omega'' = \Sigma A\xi_{22} = \bar{N}\sin\psi + \frac{\Omega_0}{D^{\frac{1}{2}}}\bar{\gamma}_{22}\cos\psi.$$

From these relations, we have

$$L_0p'^2 + 2M_0p'q' + N_0q'^2$$

$$= (\bar{L}p'^2 + 2\bar{M}p'q' + \bar{N}q'^2)\cos\psi - \frac{\Omega_0}{D^{\frac{1}{2}}}(\bar{\gamma}_{11}p'^2 + 2\bar{\gamma}_{12}p'q' + \bar{\gamma}_{22}q'^2)\sin\psi,$$

that is,

$$\frac{1}{\rho_0} = \frac{\cos\psi}{\rho} + \frac{\sin\psi}{\gamma},$$

in agreement with the former result (§ 346). Also, we have

$$\Omega p'^2 + 2\Omega' p'q' + \Omega'' q'^2$$

$$= (\bar{L}p'^2 + 2\bar{M}p'q' + \bar{N}q'^2)\sin\psi + \frac{\Omega_0}{D^{\frac{1}{2}}}(\bar{\gamma}_{11}p'^2 + 2\bar{\gamma}_{12}p'q' + \bar{\gamma}_{22}q'^2)\cos\psi$$

$$= \frac{\sin\psi}{\rho} - \frac{\cos\psi}{\gamma}$$

$$= 0,$$

in agreement with the fundamental relation in the former theory.

As a final comparison, we consider the equations which, in the two expositions, determine the asymptotes of the characteristic conic. The directions of these lines are given (§§ 230, 253) by

$$\Omega p' + \Omega' q' = 0, \quad \Omega' p' + \Omega'' q' = 0,$$

really equivalent to one equation: that is, by

$$(\bar{L}p' + \bar{M}q')\sin\psi + (\bar{\gamma}_{11}p' + \bar{\gamma}_{12}q')\frac{\Omega_0}{D^{\frac{1}{2}}}\cos\psi = 0,$$

$$(\bar{M}p' + \bar{N}q')\sin\psi + (\bar{\gamma}_{12}p' + \bar{\gamma}_{22}q')\frac{\Omega_0}{D^{\frac{1}{2}}}\cos\psi = 0$$

and therefore by

$$\begin{vmatrix} \bar{L}p' + \bar{M}q', & \bar{\gamma}_{11}p' + \bar{\gamma}_{12}q' \\ \bar{M}p' + \bar{N}q', & \bar{\gamma}_{12}p' + \bar{\gamma}_{22}q' \end{vmatrix} = 0,$$

which is the equation (§ 373) for the directions of the asymptotes in the later theory.

Introduction to the analysis for a surface in free space, as the intersection of two columnar regions.

378. In Chapter XII we have discussed surfaces in free space, when they have the di-parametric representation

$$x = x(u, w), \quad y = y(u, w), \quad z = z(u, w), \quad v = v(u, w).$$

In the preceding paragraphs, we have discussed surfaces enclosed within a given tri-parametric region, when they are represented by an equation

$$\theta(p, q, r) = 0$$

between the parameters of the region; and this latter representation initially assumed that the tri-parametric region arose from an equation

$$F(x, y, z, v) = 0,$$

while the surface is given as the intersection of the region by another region represented by the equation

$$G(x, y, z, v) = 0.$$

The two can be brought into analytical relation with one another. Thus by the elimination of u and w between the values of x, y, z, we have (§ 337) a relation of the form

$$f(x, y, z) = 0,$$

where z can be regarded as a function of x and y; and by the elimination of the parameters between the values of x, y, v, we have a relation of the form

$$g(x, y, v) = 0,$$

where v can be regarded as a function of x and y. The surface then is the intersection of the two columnar regions

$$f = 0, \quad g = 0.$$

Consequently, this form can be taken as a representation of the surface; and, then, x and y can be regarded as two independent parameters for the expression of the surface. The consequent analysis will be sufficiently represented by a consideration of the conic-locus of the orthogonal centre of the surface*.

We denote by p, q, r, s, t, the customary first and second derivatives of z with respect to x and y; and by p', q', r', s', t', the corresponding derivatives of v. Then, for the square of the arc-element, we have

$$E dx^2 + 2F dx dy + G dy^2,$$

* The following analysis has a close association with Kommerell's memoir, already quoted (vol. i, p. 442).

where

$$E = 1 + p^2 + p'^2, \quad F = pq + p'q', \quad G = 1 + q^2 + q'^2,$$
$$V^2 = 1 + p^2 + q^2 + p'^2 + q'^2 + (pq' + p'q)^2.$$

We have

$$x_{11} = 0, \quad y_{11} = 0, \quad z_{11} = r, \quad v_{11} = r',$$
$$x_{12} = 0, \quad y_{12} = 0, \quad z_{12} = s, \quad v_{12} = s',$$
$$x_{22} = 0, \quad y_{22} = 0, \quad z_{22} = t, \quad v_{22} = t';$$

and so the quantities α, β, α', β', α'', β'', of § 207 are

$$\left.\begin{aligned} \alpha &= pr + p'r', \quad \alpha' = ps + p's', \quad \alpha'' = pt + p't' \\ \beta &= qr + q'r', \quad \beta' = qs + q's', \quad \beta'' = qt + q't' \end{aligned}\right\}.$$

For the quantities $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$, of § 208, we have

$$\begin{aligned} \mathbf{a}V^2 &= V^2 \Sigma x_{11}^2 - V^2 (E\Gamma^2 + 2F\Gamma\Delta + G\Delta^2) \\ &= V^2 \Sigma x_{11}^2 - (E\beta^2 - 2F\alpha\beta + G\alpha^2) \\ &= r^2 + r'^2 + (p'r - pr')^2 + (q'r - qr')^2, \end{aligned}$$

on reduction; and, similarly,

$$\begin{aligned} \mathbf{h}V^2 &= rs + r's' + (p'r - pr')(p's - ps') + (q'r - qr')(q's - qs'), \\ \mathbf{g}V^2 &= rt + r't' + (p'r - pr')(p't - pt') + (q'r - qr')(q't - qt'), \\ \mathbf{b}V^2 &= s^2 + s'^2 + (p's - ps')^2 + (q's - qs')^2, \\ \mathbf{f}V^2 &= st + s't' + (p's - ps')(p't - pt') + (q's - qs')(q't - qt'), \\ \mathbf{c}V^2 &= t^2 + t'^2 + (p't - pt')^2 + (q't - qt')^2. \end{aligned}$$

Thus the invariant $(\mathbf{g} - \mathbf{b})\, V^2$ is equal to

$$(1 + p'^2 + q'^2)(rt - s^2) - (pp' + qq')(rt' - 2ss' + tr') + (1 + p^2 + q^2)(r't' - s'^2),$$

involving invariants connected with the curvature-equations

$$\frac{\Theta^4}{\rho^4} - \Theta\{r(1 + q^2) - 2pqs + t(1 + p^2)\}\frac{1}{\rho} + (rt - s^2) = 0,$$
$$\frac{\Theta'^4}{\rho'^4} - \Theta'\{r'(1 + q'^2) - 2p'q's' + t'(1 + p'^2)\}\frac{1}{\rho'} + (r't' - s'^2) = 0.$$

for the separate columnar regions, where $\Theta^2 = 1 + p^2 + q^2$, $\Theta'^2 = 1 + p'^2 + q'^2$.

The orthogonal centre of the surface is the point of intersection of two consecutive orthogonal planes. The tangent plane at any point of the surface is given by the equations

$$\begin{Vmatrix} \bar{x} - x, & \bar{y} - y, & \bar{z} - z, & \bar{v} - v \\ 1, & 0, & p, & p' \\ 0, & 1, & q, & q' \end{Vmatrix} = 0,$$

or, in the more familiar form

$$\bar{z} - z = p(\bar{x} - x) + q(\bar{y} - y), \quad \bar{v} - v = p'(\bar{x} - x) + q'(\bar{y} - y).$$

The orthogonal plane at the point is given by the equations

$$\left.\begin{aligned} \bar{x} - x + p(\bar{z} - z) + p'(\bar{v} - v) &= 0 \\ \bar{y} - y + q(\bar{z} - z) + q'(\bar{v} - v) &= 0 \end{aligned}\right\}.$$

For the intersection of this plane with a consecutive orthogonal plane, these two equations must be associated with the two equations

$$\left.\begin{aligned}(\bar{z}-z)\,dp+(\bar{v}-v)\,dp' &= dx+p\,dz+p'dv = E\,dx+F\,dy\\ (\bar{z}-z)\,dq+(\bar{v}-v)\,dq' &= dy+q\,dz+q'dv = F\,dx+G\,dy\end{aligned}\right\}.$$

When we use the relations

$$dp = r\,dx+s\,dy, \quad dp' = r'dx+s'dy,$$
$$dq = s\,dx+t\,dy, \quad dq' = s'dx+t'dy,$$

the last two equations give

$$\frac{r(\bar{z}-z)+r'(\bar{v}-v)-E}{dy^2} = \frac{s(\bar{z}-z)+s'(\bar{v}-v)-F}{-dx\,dy} = \frac{t(\bar{z}-z)+t'(\bar{v}-v)-G}{dx^2}.$$

Let

$$W = \begin{vmatrix} dp, & dq \\ dp', & dq' \end{vmatrix} = \begin{vmatrix} r\,dx+s\,dy, & s\,dx+t\,dy \\ r'dx+s'dy, & s'dx+t'dy \end{vmatrix};$$

and let the common value of the three fractions be

$$-\frac{\Pi}{\begin{vmatrix} dp, & dq \\ dp', & dq' \end{vmatrix}};$$

then

$$\begin{aligned}\Pi = &\{E-r(\bar{z}-z)-r'(\bar{v}-v)\}(st'-ts')\\ &+\{F-s(\bar{z}-z)-s'(\bar{v}-v)\}(tr'-rt')\\ &+\{G-t(\bar{z}-z)-t'(\bar{v}-v)\}(rs'-sr')\\ =&\begin{vmatrix} E, & r, & r' \\ F, & s, & s' \\ G, & t, & t' \end{vmatrix}.\end{aligned}$$

Hence the coordinates of the point of intersection are given by

$$\bar{x}-x+p(\bar{z}-z)+p'(\bar{v}-v) = 0,$$
$$\bar{y}-y+q(\bar{z}-z)+q'(\bar{v}-v) = 0,$$
$$r(\bar{z}-z)+r'(\bar{v}-v) = E-\Pi\frac{dy^2}{W},$$
$$s(\bar{z}-z)+s'(\bar{v}-v) = F+\Pi\frac{dx\,dy}{W},$$
$$t(\bar{z}-z)+t'(\bar{v}-v) = G-\Pi\frac{dx^2}{W}.$$

Of these five equations, the first two prove that the locus lies in the orthogonal plane; and the last three, by the elimination of dx and dy, shew that the locus is a conic in that plane. The equation of the conic in that plane is

$$\begin{aligned}&(rt-s^2)(\bar{z}-z)^2+(r't-2s's+t'r)(\bar{z}-z)(\bar{v}-v)+(r't'-s'^2)(\bar{v}-v)^2\\ &\quad-(Et-2Fs+Gr)(\bar{z}-z)-(\bar{v}-v)(Et'-2Fs'+Gr')+V^2=0.\end{aligned}$$

The directions of the asymptotes of the conic are indicated by values of $\bar{z}-z$ and $\bar{v}-v$ that are indefinitely large: that is, the directions of the asymptotes of the conic-locus of the orthogonal centre are

$$\begin{vmatrix} r\,dx+s\,dy, & s\,dx+t\,dy \\ r'dx+s'dy, & s'dx+t'dy \end{vmatrix} = \begin{vmatrix} dp, & dq \\ dp', & dq' \end{vmatrix} = 0.$$

The equation of the lines of curvature of the surface, whichever of the alternative definitions be adopted, is

$$\begin{vmatrix} \mathbf{a}\,dx^3+3\mathbf{h}\,dx^2dy+(\mathbf{g}+2\mathbf{b})\,dx\,dy^2+\mathbf{f}\,dy^3, & E\,dx+F\,dy \\ \mathbf{h}\,dx^3+(\mathbf{g}+2\mathbf{b})\,dx^2dy+3\mathbf{f}\,dx\,dy^2+\mathbf{c}\,dy^3, & F\,dx+G\,dy \end{vmatrix} = 0.$$

The establishment of this result, and of the curvature and other properties of geodesics, can be effected by analysis similar to that adopted in Chapter XII.

Curve in a region, as the intersection of two surfaces.

379. Any curve lying within the region which is represented parametrically by means of the quantities p, q, r, can be regarded as the intersection of two surfaces

$$\theta(p, q, r) = 0, \quad \Im(p, q, r) = 0,$$

within that region. To identify some of the properties of the curve specially associated with the region, we shall use the results which affect the surface $\theta = 0$ alone and the analogous results for the surface $\Im = 0$ alone; but some results connecting the two surfaces must first be obtained.

We take

$$D_\theta = (a, b, c, f, g, h\,\theta_1, \theta_2, \theta_3)^2,$$
$$D_\Im = (a, b, c, f, g, h\,\Im_1, \Im_2, \Im_3)^2,$$

so that, if $\nu_\Im$ be an element of length in the region normal to the surface $\Im$, we have

$$D_\Im = \Omega^2 \left(\frac{d\Im}{d\nu_\Im}\right)^2.$$

The direction-variables for the regional normal to θ are (§ 268)

$$\Omega D_\theta^{\frac{1}{2}} \frac{dp}{d\nu_\theta} = a\theta_1 + h\theta_2 + g\theta_3,$$

with similar values for $\dfrac{dq}{d\nu_\theta}$, $\dfrac{dr}{d\nu_\theta}$; and the direction-variables for the regional normal to $\Im$ are, similarly,

$$\Omega D_\Im^{\frac{1}{2}} \frac{dp}{d\nu_\Im} = a\Im_1 + h\Im_2 + g\Im_3,$$

with similar values for $\dfrac{dq}{d\nu_\Im}$, $\dfrac{dr}{d\nu_\Im}$. Let η denote the inclination of the two surfaces so that, as their tangent planes intersect in a line, the inclination of

the tangent planes is (§ 89) the inclination of the two surface-normals in the flat enclosing the two planes; hence

$$\cos\eta = \Sigma A \frac{dp}{d\nu_\theta}\frac{dp}{d\nu_\vartheta},$$

that is,

$$\begin{aligned}\Omega^2 D_\theta^{\frac{1}{2}} D_\vartheta^{\frac{1}{2}} \cos\eta &= \Sigma A\,(a\theta_1 + h\theta_2 + g\theta_3)(a\vartheta_1 + h\vartheta_2 + g\vartheta_3)\\ &= \Omega^2 \Sigma a\theta_1\vartheta_1.\end{aligned}$$

Therefore

$$\begin{aligned}\Sigma a\theta_1\vartheta_1 &= D_\theta^{\frac{1}{2}} D_\vartheta^{\frac{1}{2}} \cos\eta\\ &= \Omega^2 \frac{d\theta}{d\nu_\theta}\frac{d\vartheta}{d\nu_\vartheta}\cos\eta,\end{aligned}$$

and (what is the equivalent)

$$\cos\eta = \frac{\Sigma a\theta_1\vartheta_1}{(\Sigma a\theta_1^2)^{\frac{1}{2}}(\Sigma a\vartheta_1^2)^{\frac{1}{2}}}.$$

Further, we have

$$\begin{aligned}D_\theta D_\vartheta \sin^2\eta &= (\Sigma a\theta_1^2)(\Sigma a\vartheta_1^2) - (\Sigma a\theta_1\vartheta_1)^2\\ &= \Omega^2 \Sigma A\,(\theta_2\vartheta_3 - \theta_3\vartheta_2)^2,\end{aligned}$$

so that

$$\sin^2\eta = \frac{\Omega^2}{D_\theta D_\vartheta}\Sigma A\,(\theta_2\vartheta_3 - \theta_3\vartheta_2)^2.$$

Next, let dS denote an element of arc along the curve which is the intersection of the two surfaces; then

$$\theta_1\frac{dp}{dS} + \theta_2\frac{dq}{dS} + \theta_3\frac{dr}{dS} = 0,$$

$$\vartheta_1\frac{dp}{dS} + \vartheta_2\frac{dq}{dS} + \vartheta_3\frac{dr}{dS} = 0.$$

Hence

$$\frac{\dfrac{dp}{dS}}{\theta_2\vartheta_3 - \theta_3\vartheta_2} = \frac{\dfrac{dq}{dS}}{\theta_3\vartheta_1 - \theta_1\vartheta_3} = \frac{\dfrac{dr}{dS}}{\theta_1\vartheta_2 - \theta_2\vartheta_1} = \frac{1}{\mu}.$$

Now, by the definition of the arc-element,

$$\Sigma A\left(\frac{dp}{dS}\right)^2 = 1;$$

hence

$$\mu^2 = \Sigma A\,(\theta_2\vartheta_3 - \theta_3\vartheta_2)^2 = \frac{D_\theta D_\vartheta}{\Omega^2}\sin^2\eta.$$

Consequently

$$\left.\begin{aligned}\frac{dp}{dS} &= \frac{\Omega}{D_\theta^{\frac{1}{2}} D_\vartheta^{\frac{1}{2}}\sin\eta}(\theta_2\vartheta_3 - \theta_3\vartheta_2)\\ \frac{dq}{dS} &= \frac{\Omega}{D_\theta^{\frac{1}{2}} D_\vartheta^{\frac{1}{2}}\sin\eta}(\theta_3\vartheta_1 - \theta_1\vartheta_3)\\ \frac{dr}{dS} &= \frac{\Omega}{D_\theta^{\frac{1}{2}} D_\vartheta^{\frac{1}{2}}\sin\eta}(\theta_1\vartheta_2 - \theta_2\vartheta_1)\end{aligned}\right\}.$$

Relation of the curve to the tangential geodesics in the region and the surfaces.

380. The curve of intersection of the θ-surface and the $\mathfrak{H}$-surface gives a direction p', q', r', at the point. Tangential to this curve at that point, there are three geodesics, being (i) the regional geodesic, (ii) the superficial geodesic in the θ-surface, (iii) the superficial geodesic in the $\mathfrak{H}$-surface.

For the curve of intersection itself, as a curve in space, let l_0, m_0, n_0, k_0, be the direction-cosines of the principal normal; and let ρ_0 be its radius of circular curvature.

For the regional geodesic, X, Y, Z, V, are the direction-cosines of the principal normal, and ρ is its radius of circular curvature.

For the θ-geodesic, let its radius of circular curvature be ρ_θ, with direction-cosines $l_\theta, m_\theta, n_\theta, k_\theta$; and let its regional geodesic curvature be $\frac{1}{\gamma_\theta}$. Also, let the direction-cosines of a line in the tangent plane to the θ-surface, perpendicular to p', q', r', be $\bar{l}_\theta$, $\bar{m}_\theta$, $\bar{n}_\theta$, $\bar{k}_\theta$.

For the $\mathfrak{H}$-geodesic, let the corresponding magnitudes be

$$\rho_\mathfrak{H};\ l_\mathfrak{H},\ m_\mathfrak{H},\ n_\mathfrak{H},\ k_\mathfrak{H};\ \frac{1}{\gamma_\mathfrak{H}};\ \bar{l}_\mathfrak{H},\ \bar{m}_\mathfrak{H},\ \bar{n}_\mathfrak{H},\ \bar{k}_\mathfrak{H}.$$

We also write

$$-\frac{1}{\gamma_\theta} = \frac{\Omega}{D_\theta^{\frac{1}{2}}}\Psi_\theta, \quad -\frac{1}{\gamma_\mathfrak{H}} = \frac{\Omega}{D_\mathfrak{H}^{\frac{1}{2}}}\Psi_\mathfrak{H}.$$

The first derivatives of p, q, r, are the same for the curve, and for the three tangential geodesics. Let the second derivatives be denoted by

p_0'', q_0'', r_0'', for the curve of intersection,
p_θ'', q_θ'', r_θ'', ... θ-geodesic,
$p_\mathfrak{H}''$, $q_\mathfrak{H}''$, $r_\mathfrak{H}''$, ... $\mathfrak{H}$-geodesic,
$\bar{P}''$, $\bar{Q}''$, $\bar{R}''$, ... regional geodesic.

Also, for brevity, we write

$$U_1 = Ap' + Hq' + Gr', \quad U_2 = Hp' + Bq' + Fr', \quad U_3 = Gp' + Fq' + Cr'.$$

We at once have one relation, derived from the equation

$$\Sigma A p'^2 = 1,$$

which is true of all curves that lie in the region. By mere differentiation and rearrangement, we have for every such curve

$$U_1(p'' + \Sigma\Gamma p'^2) + U_2(q'' + \Sigma\Delta p'^2) + U_3(r'' + \Sigma\Theta p'^2) = 0,$$

which therefore holds for each of the four sets of values of p'', q'', r'', connected with the curve and the three geodesic tangents.

For the regional geodesic, we have

$$\left.\begin{aligned} \bar{P}'' + \Sigma \Gamma p'^2 &= 0 \\ \bar{Q}'' + \Sigma \Delta p'^2 &= 0 \\ \bar{R}'' + \Sigma \Theta p'^2 &= 0 \end{aligned}\right\};$$

for the θ-geodesic, we have

$$\left.\begin{aligned} p_\theta'' + \Sigma \Gamma p'^2 &= -\frac{\Psi_\theta}{D_\theta}(a\theta_1 + h\theta_2 + g\theta_3) \\ q_\theta'' + \Sigma \Delta p'^2 &= -\frac{\Psi_\theta}{D_\theta}(h\theta_1 + b\theta_2 + f\theta_3) \\ r_\theta'' + \Sigma \Theta p'^2 &= -\frac{\Psi_\theta}{D_\theta}(g\theta_1 + f\theta_2 + c\theta_3) \end{aligned}\right\};$$

and for the ϑ-geodesic, we have

$$\left.\begin{aligned} p_\vartheta'' + \Sigma \Gamma p'^2 &= -\frac{\Psi_\vartheta}{D_\vartheta}(a\vartheta_1 + h\vartheta_2 + g\vartheta_3) \\ q_\vartheta'' + \Sigma \Delta p'^2 &= -\frac{\Psi_\vartheta}{D_\vartheta}(h\vartheta_1 + b\vartheta_2 + f\vartheta_3) \\ r_\vartheta'' + \Sigma \Theta p'^2 &= -\frac{\Psi_\vartheta}{D_\vartheta}(g\vartheta_1 + f\vartheta_2 + c\vartheta_3) \end{aligned}\right\}.$$

Circular curvature of the curve.

381. Along the curve, which is the intersection of the two surfaces, we have, on the θ-surface,

$$\theta_1 p_0'' + \theta_2 q_0'' + \theta_3 r_0'' + \Sigma \theta_{11} p'^2 = 0,$$

that is,

$$\theta_1 (p_0'' + \Sigma \Gamma p'^2) + \theta_2 (q_0'' + \Sigma \Delta p'^2) + \theta_3 (r_0'' + \Sigma \Theta p'^2) = -\Psi_\theta.$$

Similarly, on the ϑ-surface, we have

$$\vartheta_1 (p_0'' + \Sigma \Gamma p'^2) + \vartheta_2 (q_0'' + \Sigma \Delta p'^2) + \vartheta_3 (r_0'' + \Sigma \Theta p'^2) = -\Psi_\vartheta;$$

and there is the general equation

$$U_1 (p_0'' + \Sigma \Gamma p'^2) + U_2 (q_0'' + \Sigma \Delta p'^2) + U_3 (r_0'' + \Sigma \Theta p'^2) = 0.$$

Let ∇ denote the determinant

$$\nabla = \begin{vmatrix} U_1, & U_2, & U_3 \\ \theta_1, & \theta_2, & \theta_3 \\ \vartheta_1, & \vartheta_2, & \vartheta_3 \end{vmatrix},$$

the determinant of the coefficients in the last three relations: then

$$\begin{aligned} \nabla &= U_1(\theta_2\vartheta_3 - \theta_3\vartheta_2) + U_2(\theta_3\vartheta_1 - \theta_1\vartheta_3) + U_3(\theta_1\vartheta_2 - \theta_2\vartheta_1) \\ &= \frac{D_\theta^{\frac{1}{2}} D_\vartheta^{\frac{1}{2}}}{\Omega} \sin\eta\,(U_1 p' + U_2 q' + U_3 r') = \frac{D_\theta^{\frac{1}{2}} D_\vartheta^{\frac{1}{2}}}{\Omega} \sin\eta. \end{aligned}$$

With this value of ∇, we have, along the curve itself,

$$\nabla(p_0'' + \Sigma\Gamma p'^2) = -\begin{vmatrix} \Psi_\theta, & \theta_2, & \theta_3 \\ \Psi_\flat, & \Im_2, & \Im_3 \\ 0, & U_2, & U_3 \end{vmatrix},$$

$$\nabla(q_0'' + \Sigma\Delta p'^2) = -\begin{vmatrix} \theta_1, & \Psi_\theta, & \theta_3 \\ \Im_1, & \Psi_\flat, & \Im_3 \\ U_1, & 0, & U_3 \end{vmatrix},$$

$$\nabla(r_0'' + \Sigma\Theta p'^2) = -\begin{vmatrix} \theta_1, & \theta_2, & \Psi_\theta \\ \Im_1, & \Im_2, & \Psi_\flat \\ U_1, & U_2, & 0 \end{vmatrix}.$$

Now along the curve, we have

$$\begin{aligned} x_0'' &= x_1 p_0'' + x_2 q_0'' + x_3 r_0'' + \Sigma x_{11} p'^2 \\ &= x_1(p_0'' + \Sigma\Gamma p'^2) + x_2(q_0'' + \Sigma\Delta p'^2) + x_3(r_0'' + \Sigma\Theta p'^2) \\ &\qquad + \Sigma(x_{11} - x_1\Gamma_{11} - x_2\Delta_{11} - x_3\Theta_{11})\, p'^2. \end{aligned}$$

The last sum

$$= X\Sigma L p'^2 = \frac{X}{\rho};$$

and

$$\rho_0 x_0'' = l_0, \quad \rho_0 y_0'' = m_0, \quad \rho_0 z_0'' = n_0, \quad \rho_0 v_0'' = k_0;$$

hence

$$\begin{aligned} \frac{l_0}{\rho_0} = x_0'' &= \frac{X}{\rho} + x_1(p_0'' + \Sigma\Gamma p'^2) + x_2(q_0'' + \Sigma\Delta p'^2) + x_3(r_0'' + \Sigma\Theta p'^2) \\ &= \frac{X}{\rho} + \frac{1}{\nabla}\left[-\Psi_\theta \begin{vmatrix} x_1, & x_2, & x_3 \\ \Im_1, & \Im_2, & \Im_3 \\ U_1, & U_2, & U_3 \end{vmatrix} + \Psi_\flat \begin{vmatrix} x_1, & x_2, & x_3 \\ \theta_1, & \theta_2, & \theta_3 \\ U_1, & U_2, & U_3 \end{vmatrix}\right]. \end{aligned}$$

But (§ 347)

$$\bar{l}_\theta D_\theta^{\frac{1}{2}} = \begin{vmatrix} x_1, & x_2, & x_3 \\ U_1, & U_2, & U_3 \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix}, \quad \bar{l}_\flat D_\flat^{\frac{1}{2}} = \begin{vmatrix} x_1, & x_2, & x_3 \\ U_1, & U_2, & U_3 \\ \Im_1, & \Im_2, & \Im_3 \end{vmatrix};$$

hence, introducing the values of ∇, Ψ_θ, $\Psi_\flat$, we have

$$\left.\begin{aligned} \frac{l_0}{\rho_0} &= \frac{X}{\rho} + \frac{1}{\sin\eta}\left(\frac{\bar{l}_\theta}{\gamma_\flat} - \frac{\bar{l}_\flat}{\gamma_\theta}\right) \\ \frac{m_0}{\rho_0} &= \frac{Y}{\rho} + \frac{1}{\sin\eta}\left(\frac{\bar{m}_\theta}{\gamma_\flat} - \frac{\bar{m}_\flat}{\gamma_\theta}\right) \\ \frac{n_0}{\rho_0} &= \frac{Z}{\rho} + \frac{1}{\sin\eta}\left(\frac{\bar{n}_\theta}{\gamma_\flat} - \frac{\bar{n}_\flat}{\gamma_\theta}\right) \\ \frac{k_0}{\rho_0} &= \frac{V}{\rho} + \frac{1}{\sin\eta}\left(\frac{\bar{k}_\theta}{\gamma_\flat} - \frac{\bar{k}_\flat}{\gamma_\theta}\right) \end{aligned}\right\},$$

equations which give the circular curvature of the curve and the direction-cosines of the radius of circular curvature in space.

In passing, we note that

$$D_\theta^{\frac{1}{2}} D_\vartheta^{\frac{1}{2}} \Sigma \bar{l}_\theta \bar{l}_\vartheta = \sum_x \{\Sigma \theta_1 (x_2 U_3 - x_3 U_2)\} \{\Sigma \vartheta_1 (x_2 U_3 - x_3 U_2)\}.$$

The coefficient of $\theta_1 \vartheta_1$ on the right-hand side

$$= \Sigma (x_2 U_3 - x_3 U_2)^2$$
$$= B U_3^2 - 2F U_2 U_3 + C U_2^2$$
$$= -\Omega^2 p'^2 + a \Sigma A p'^2 = -\Omega^2 p'^2 + a;$$

and similarly for the other coefficients. Hence the right-hand side

$$= -\Omega^2 (\Sigma \theta_1 p')(\Sigma \vartheta_1 p') + \Sigma a \theta_1 \vartheta_1$$
$$= \Sigma a \theta_1 \vartheta_1 = D_\theta^{\frac{1}{2}} D_\vartheta^{\frac{1}{2}} \cos \eta,$$

so that

$$\Sigma \bar{l}_\theta \bar{l}_\vartheta = \cos \eta,$$

as is to be expected, because the lines $\bar{l}_\theta$, $\bar{m}_\theta$, $\bar{n}_\theta$, $\bar{k}_\theta$, and $\bar{l}_\vartheta$, $\bar{m}_\vartheta$, $\bar{n}_\vartheta$, $\bar{k}_\vartheta$, are perpendicular to the common tangent in the respective planes.

Regional geodesic curvature of the curve.

382. The regional geodesic curvature of the curve is defined as the arc-rate of deviation of its tangent from the regional geodesic. A consecutive tangent to the curve has direction-cosines

$$x' + x_0'' \delta, \quad y' + y_0'' \delta, \quad z' + z_0'' \delta, \quad v' + v_0'' \delta;$$

and the direction-cosines of the consecutive regional geodesic, tangent to the curve, are

$$x' + \xi'' \delta, \quad y' + \eta'' \delta, \quad z' + \zeta'' \delta, \quad v' + \upsilon'' \delta,$$

where

$$\xi'' = x_1 \bar{P}'' + x_2 \bar{Q}'' + x_3 \bar{R}'' + \Sigma x_{11} p'^2,$$

with like expressions for η'', ζ'', υ''. Thus

$$x_0'' - \xi'' = x_1 (p_0'' - \bar{P}'') + x_2 (q_0'' - \bar{Q}'') + x_3 (r_0'' - \bar{R}''),$$

and similarly for $y_0'' - \eta''$, $z_0'' - \zeta''$, $v_0'' - \upsilon''$. If then we denote the regional geodesic curvature of the curve by $\frac{1}{\bar{G}}$, we have

$$\frac{1}{\bar{G}^2} = \Sigma (x_0'' - \xi'')^2$$
$$= (A, B, C, F, G, H \mathbin{\text{\textbackslash\!\!\textbackslash}} p_0'' - \bar{P}'', \ q_0'' - \bar{Q}'', \ r_0'' - \bar{R}'')^2.$$

Now

$$\nabla (p_0'' - \bar{P}'') = -\Psi_\theta (\vartheta_2 U_3 - \vartheta_3 U_2) + \Psi_\vartheta (\theta_2 U_3 - \theta_3 U_2),$$

and so for the others; hence

$$\frac{\nabla^2}{\bar{G}^2} = \Psi_\theta^2 \{\Sigma A (\vartheta_2 U_3 - \vartheta_3 U_2)^2\} - 2\Psi_\theta \Psi_\vartheta \{\Sigma A (\vartheta_2 U_3 - \vartheta_3 U_2)(\theta_2 U_3 - \theta_3 U_2)\}$$
$$+ \Psi_\vartheta^2 \{\Sigma A (\theta_2 U_3 - \theta_3 U_2)^2\}$$
$$= \Psi_\theta^2 D_\vartheta - 2\Psi_\theta \Psi_\vartheta D_\theta^{\frac{1}{2}} D_\vartheta^{\frac{1}{2}} \cos \eta + \Psi_\vartheta^2 D_\theta,$$

that is,

$$\frac{D_\theta D_\lambda \sin^2\eta}{\Omega^2\bar{G}^2} = \frac{D_\theta}{\Omega^2\gamma_\theta^2}D_\lambda - 2\frac{D_\theta D_\lambda}{\Omega^2\gamma_\theta\gamma_\lambda}\cos\eta + \frac{D_\lambda}{\Omega^2\gamma_\lambda^2}D_\theta,$$

and therefore

$$\frac{\sin^2\eta}{\bar{G}^2} = \frac{1}{\gamma_\theta^2} - 2\frac{\cos\eta}{\gamma_\theta\gamma_\lambda} + \frac{1}{\gamma_\lambda^2}.$$

Again, we have

$$x_0'' - \xi'' = x_1(p_0'' - \bar{P}'') + x_2(q_0'' - \bar{Q}'') + x_3(r_0'' - \bar{R}'')$$
$$= \frac{1}{\sin\eta}\left(\frac{\bar{l}_\theta}{\gamma_\lambda} - \frac{\bar{l}_\lambda}{\gamma_\theta}\right),$$

and so for the others. Hence we can write

$$\frac{l_0}{\rho_0} = \frac{X}{\rho} + \frac{\lambda_0}{\bar{G}}, \quad \frac{m_0}{\rho_0} = \frac{Y}{\rho} + \frac{\mu_0}{\bar{G}}, \quad \frac{n_0}{\rho_0} = \frac{Z}{\rho} + \frac{\nu_0}{\bar{G}}, \quad \frac{k_0}{\rho_0} = \frac{V}{\rho} + \frac{\kappa_0}{\bar{G}},$$

where

$$\left.\begin{aligned}
\lambda_0 &= \frac{\bar{G}}{\sin\eta}\left(\frac{\bar{l}_\theta}{\gamma_\lambda} - \frac{\bar{l}_\lambda}{\gamma_\theta}\right)\\
\mu_0 &= \frac{\bar{G}}{\sin\eta}\left(\frac{\bar{m}_\theta}{\gamma_\lambda} - \frac{\bar{m}_\lambda}{\gamma_\theta}\right)\\
\nu_0 &= \frac{\bar{G}}{\sin\eta}\left(\frac{\bar{n}_\theta}{\gamma_\lambda} - \frac{\bar{n}_\lambda}{\gamma_\theta}\right)\\
\kappa_0 &= \frac{\bar{G}}{\sin\eta}\left(\frac{\bar{k}_\theta}{\gamma_\lambda} - \frac{\bar{k}_\lambda}{\gamma_\theta}\right)
\end{aligned}\right\},$$

equations connecting the circular curvature of the curve with the circular curvature of the tangential regional geodesic and the regional geodesic curvature of the curve.

We have

$$\Sigma X\lambda_0 = 0.$$

Let χ be the angle between the principal normal of the curve and the principal normal of the regional geodesic touching the curve; then

$$\Sigma l_0 X = \cos\chi, \quad \Sigma l_0\lambda_0 = \sin\chi,$$
$$\frac{\cos\chi}{\rho_0} = \frac{1}{\rho}, \quad \frac{\sin\chi}{\rho_0} = \frac{1}{\bar{G}},$$

so that χ is given by the equation

$$\tan\chi = \frac{\rho}{\bar{G}},$$

while

$$l_0 = X\cos\chi + \lambda_0\sin\chi, \quad m_0 = Y\cos\chi + \mu_0\sin\chi,$$
$$n_0 = Z\cos\chi + \nu_0\sin\chi, \quad k_0 = V\cos\chi + \kappa_0\sin\chi.$$

Orthogonal frame for the curve.

383. We require an orthogonal frame for the curve, which is not a superficial geodesic (save exceptionally) on either surface. To constitute a frame for a superficial geodesic on a surface, we take a direction which lies in the region and is perpendicular to the tangent and to the principal normal to the geodesic; so, now, we take a direction which lies in the region and is perpendicular both to the tangent and to the principal normal of the curve. The corresponding direction for the θ-surface is given (§ 347) by $\bar{l}_\theta$, $\bar{m}_\theta$, $\bar{n}_\theta$, $\bar{k}_\theta$, while for the ϑ-surface it is given by $\bar{l}_\vartheta$, $\bar{m}_\vartheta$, $\bar{n}_\vartheta$, $\bar{k}_\vartheta$: hence we take

$$\bar{l}_0 = \alpha\bar{l}_\theta - \beta\bar{l}_\vartheta, \quad \bar{m}_0 = \alpha\bar{m}_\theta - \beta\bar{m}_\vartheta, \quad \bar{n}_0 = \alpha\bar{n}_\theta - \beta\bar{n}_\vartheta, \quad \bar{k}_0 = \alpha\bar{k}_\theta - \beta\bar{k}_\vartheta,$$

so that

$$\alpha^2 - 2\alpha\beta\cos\eta + \beta^2 = 1.$$

We have

$$\Sigma X\bar{l}_\theta = 0, \quad \Sigma X\bar{l}_\vartheta = 0;$$

and so

$$\Sigma X\bar{l}_0 = 0.$$

The condition requires perpendicularity to the principal normal; hence

$$\Sigma l_0\bar{l}_0 = 0,$$

and therefore

$$\Sigma\bar{l}_0\lambda_0 = 0,$$

that is,

$$\frac{1}{\gamma_\vartheta}\Sigma\bar{l}_0\bar{l}_\theta - \frac{1}{\gamma_\theta}\Sigma\bar{l}_0\bar{l}_\vartheta = 0.$$

Thus

$$\frac{1}{\gamma_\vartheta}(\alpha - \beta\cos\eta) - \frac{1}{\gamma_\theta}(\alpha\cos\eta - \beta) = 0,$$

and therefore

$$\alpha\sin^2\eta = \bar{G}\left(\frac{1}{\gamma_\theta} - \frac{\cos\eta}{\gamma_\vartheta}\right), \quad \beta\sin^2\eta = \bar{G}\left(\frac{\cos\eta}{\gamma_\theta} - \frac{1}{\gamma_\vartheta}\right).$$

Also

$$\Sigma\bar{l}_0\bar{l}_\theta = \frac{\bar{G}}{\gamma_\theta}, \quad \Sigma\bar{l}_0\bar{l}_\vartheta = \frac{\bar{G}}{\gamma_\vartheta}.$$

Again, we denoted (§ 382) by χ the angle between the principal normal of the curve and that of the regional geodesic touching it, so that

$$\Sigma l_0 X = \cos\chi, \quad \Sigma l_0\lambda_0 = \sin\chi.$$

Then

$$\Sigma l_0(X\sin\chi - \lambda_0\cos\chi) = 0.$$

Clearly

$$\Sigma x'(X\sin\chi - \lambda_0\cos\chi) = 0,$$

for

$$\Sigma Xx' = 0, \quad \Sigma x'l_\theta = 0, \quad \Sigma x'l_\vartheta = 0;$$

and

$$\Sigma\bar{l}_0(X\sin\chi - \lambda_0\cos\chi) = -\Sigma\bar{l}_0\lambda_0 = 0;$$

while

$$\Sigma l_0 x' = 0, \quad \Sigma\bar{l}_0 x' = 0, \quad \Sigma l_0\bar{l}_0 = 0.$$

Thus we have four lines, constituting an orthogonal frame for the curve, given by

$$\begin{array}{cccc} x', & y', & z', & v'; \\ l_0, & m_0, & n_0, & k_0; \\ \bar{l}_0, & \bar{m}_0, & \bar{n}_0, & \bar{k}_0; \\ \bar{A}, & \bar{B}, & \bar{C}, & \bar{D}; \end{array}$$

where

$$\left.\begin{array}{ll} l_0 = X\cos\chi + \lambda_0 \sin\chi, & m_0 = Y\cos\chi + \mu_0 \sin\chi \\ n_0 = Z\cos\chi + \nu_0 \sin\chi, & k_0 = V\cos\chi + \kappa_0 \sin\chi \\ \bar{A} = X\sin\chi - \lambda_0 \cos\chi, & \bar{B} = Y\sin\chi - \mu_0 \cos\chi \\ \bar{C} = Z\sin\chi - \nu_0 \cos\chi, & \bar{D} = V\sin\chi - \kappa_0 \cos\chi \end{array}\right\}.$$

Torsion and tilt of the curve.

384. The Frenet equations for the curvatures $1/\rho_0$ of circular curvature, $1/\sigma_0$ of torsion, $1/\tau_0$ of tilt, are given by

$$\left.\begin{array}{rcl} \dfrac{dx'}{dS} & = & \dfrac{l_0}{\rho_0} \\ \dfrac{dl_0}{dS} & = & -\dfrac{x'}{\rho_0} \mp \dfrac{l_0}{\sigma_0} \\ \dfrac{d\bar{l}_0}{dS} & = & -\dfrac{l_0}{\sigma_0} + \dfrac{\bar{A}}{\tau_0} \\ \dfrac{d\bar{A}}{dS} & = & -\dfrac{\bar{l}_0}{\tau_0} \end{array}\right\}.$$

Now we have

$$\begin{aligned} \frac{dl_0}{dS} &= -\bar{A}\frac{d\chi}{dS} + \frac{dX}{dS}\cos\chi + \frac{d\lambda_0}{dS}\sin\chi, \\ \frac{d\bar{A}}{dS} &= l_0\frac{d\chi}{dS} + \frac{dX}{dS}\sin\chi - \frac{d\lambda_0}{dS}\cos\chi, \end{aligned}$$

and therefore

$$\left.\begin{array}{l} \dfrac{1}{\sigma_0} = \Sigma\bar{l}_0\dfrac{dl_0}{dS} = \left(\Sigma\bar{l}_0\dfrac{dX}{dS}\right)\cos\chi + \left(\Sigma\bar{l}_0\dfrac{d\lambda_0}{dS}\right)\sin\chi \\ -\dfrac{1}{\tau_0} = \Sigma\bar{l}_0\dfrac{d\bar{A}}{dS} = \left(\Sigma\bar{l}_0\dfrac{dX}{dS}\right)\sin\chi - \left(\Sigma\bar{l}_0\dfrac{d\lambda_0}{dS}\right)\cos\chi \end{array}\right\}.$$

It follows that, to find the values of σ_0 and τ_0, we must have the values of

$$\Sigma\bar{l}_0\frac{dX}{dS}, \quad \Sigma\bar{l}_0\frac{d\lambda_0}{dS}.$$

The values of $\bar{l}_0$, $\bar{m}_0$, $\bar{n}_0$, $\bar{k}_0$, are given by four relations of the form

$$\bar{l}_0 = \alpha\bar{l}_\theta - \beta\bar{l}_\gamma;$$

consequently

$$\Sigma\bar{l}_0\frac{dX}{dS} = \alpha\Sigma\bar{l}_\theta\frac{dX}{dS} - \beta\Sigma\bar{l}_\gamma\frac{dX}{dS}.$$

Now, for the surface θ, we had (§ 352)

$$-\Sigma \bar{l} X' = \frac{1}{\bar{\tau}} = \frac{1}{4} \frac{1}{D^{\frac{1}{2}}} \begin{vmatrix} \frac{\partial W}{\partial p'}, & \frac{\partial W}{\partial q'}, & \frac{\partial W}{\partial r'} \\ \frac{\partial U}{\partial p'}, & \frac{\partial U}{\partial q'}, & \frac{\partial U}{\partial r'} \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix},$$

where $1/\bar{\tau}$ is the tilt of the superficial geodesic in the direction p', q', r'. Accordingly,

$$\Sigma \bar{l}_\theta \frac{dX}{dS} = -\frac{1}{\bar{\tau}_\theta}, \quad \Sigma \bar{l}_\vartheta \frac{dX}{dS} = -\frac{1}{\bar{\tau}_\vartheta},$$

where $1/\bar{\tau}_\theta$ is the tilt of the superficial geodesic on the θ-surface touching the curve and $1/\bar{\tau}_\vartheta$ is the tilt of the superficial geodesic on the ϑ-surface touching the curve. Hence, inserting the values of α and β from the relations

$$\alpha \sin^2 \eta = \bar{G} \left(\frac{1}{\gamma_\theta} - \frac{\cos \eta}{\gamma_\vartheta} \right), \quad \beta \sin^2 \eta = \bar{G} \left(\frac{\cos \eta}{\gamma_\theta} - \frac{1}{\gamma_\vartheta} \right),$$

we have

$$\Sigma \bar{l}_0 \frac{dX}{dS} = \frac{\bar{G}}{\sin^2 \eta} \left\{ \left(\frac{1}{\gamma_\vartheta \bar{\tau}_\theta} + \frac{1}{\gamma_\theta \bar{\tau}_\vartheta} \right) \cos \eta - \left(\frac{1}{\gamma_\theta \bar{\tau}_\theta} + \frac{1}{\gamma_\vartheta \bar{\tau}_\vartheta} \right) \right\}.$$

To find the value of $\Sigma \bar{l}_0 \frac{d\lambda_0}{dS}$, we write

$$Q = \frac{\bar{G}}{\gamma_\vartheta \sin \eta}, \quad P = \frac{\bar{G}}{\gamma_\theta \sin \eta},$$

so that

$$\lambda_0 = Q \bar{l}_\theta - P \bar{l}_\vartheta;$$

and now

$$\frac{d\lambda_0}{dS} = Q' \bar{l}_\theta - P' \bar{l}_\vartheta + Q \bar{l}_\theta' - P \bar{l}_\vartheta'$$

$$= Q' \bar{l}_\theta - P' \bar{l}_\vartheta + Q \left(-\frac{l_\theta}{\sigma_\theta} + \frac{A_\theta}{\tau_\theta} \right) - P \left(-\frac{l_\vartheta}{\sigma_\vartheta} + \frac{A_\vartheta}{\tau_\vartheta} \right)$$

by using the Frenet formulæ (§ 353). But reverting to spatial direction-cosines, we have

$$\left. \begin{aligned} l_\theta &= X \cos \psi + \bar{X}_\theta \sin \psi \\ A_\theta &= X \sin \psi - \bar{X}_\theta \cos \psi \end{aligned} \right\}, \quad \left. \begin{aligned} l_\vartheta &= X \cos \phi + \bar{X}_\vartheta \sin \phi \\ A_\vartheta &= X \sin \phi - \bar{X}_\vartheta \cos \phi \end{aligned} \right\}.$$

The coefficient of Q in the value of $\frac{d\lambda_0}{dS}$ is

$$= -\frac{1}{\sigma_\theta} (X \cos \psi + \bar{X}_\theta \sin \psi) + \frac{1}{\tau_\theta} (X \sin \psi - \bar{X}_\theta \cos \psi)$$

$$= \frac{X}{\bar{\tau}_\theta} - \frac{X_\theta}{\bar{\sigma}_\theta},$$

where $1/\bar{\tau}_\theta$ and $1/\bar{\sigma}_\theta$ are (§ 353) the tilt and the torsion of the θ-surface geodesic; and similarly the coefficient of $-P$ is

$$= \frac{X}{\bar{\tau}_\vartheta} - \frac{X_\vartheta}{\bar{\sigma}_\vartheta},$$

with the corresponding significance of $\bar{\tau}_\vartheta$ and $\bar{\sigma}_\vartheta$ for the ϑ-surface geodesic.

Further,

$$\bar{l}_0 = \alpha \bar{l}_\theta - \beta \bar{l}_\vartheta$$

$$= \frac{\bar{G}}{\sin^2 \eta} \left\{ \left(\frac{1}{\gamma_\theta} - \frac{\cos \eta}{\gamma_\vartheta} \right) \bar{l}_\theta + \left(\frac{1}{\gamma_\vartheta} - \frac{\cos \eta}{\gamma_\theta} \right) \bar{l}_\vartheta \right\}.$$

Now

$$\Sigma X \bar{l}_\theta = 0, \quad \Sigma X \bar{l}_\vartheta = 0.$$

But (§§ 340, 347)

$$\Sigma X_\theta \bar{l}_\theta = \frac{\Omega}{D_\theta} \Sigma (x_1 P + x_2 Q + x_3 R) \begin{vmatrix} x_1, & x_2, & x_3 \\ U_1, & U_2, & U_3 \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix}$$

Now

$$\Sigma (x_1 P + x_2 Q + x_3 R) x_1 = AP + HQ + GR = \theta_1,$$

$$\Sigma (x_1 P + x_2 Q + x_3 R) x_2 = HP + BQ + FR = \theta_2,$$

$$\Sigma (x_1 P + x_2 Q + x_3 R) x_3 = GP + FQ + CR = \theta_3;$$

and therefore

$$\Sigma X_\theta \bar{l}_\theta = \frac{\Omega}{D} \begin{vmatrix} \theta_1, & \theta_2, & \theta_3 \\ U_1, & U_2, & U_3 \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix} = 0.$$

Similarly

$$\Sigma X_\vartheta \bar{l}_\vartheta = 0.$$

Again

$$\Sigma X_\theta \bar{l}_\vartheta = \frac{\Omega}{D_\theta^{\frac{1}{2}} D_\vartheta^{\frac{1}{2}}} \Sigma (x_1 P + x_2 Q + x_3 R) \begin{vmatrix} x_1, & x_2, & x_3 \\ U_1, & U_2, & U_3 \\ \vartheta_1, & \vartheta_2, & \vartheta_3 \end{vmatrix}$$

$$= \frac{\Omega}{D_\theta^{\frac{1}{2}} D_\vartheta^{\frac{1}{2}}} \begin{vmatrix} \theta_1, & \theta_2, & \theta_3 \\ U_1, & U_2, & U_3 \\ \vartheta_1, & \vartheta_2, & \vartheta_3 \end{vmatrix} = -\sin \eta;$$

and similarly

$$\Sigma X_\vartheta \bar{l}_\theta = \frac{\Omega}{D_\theta^{\frac{1}{2}} D_\vartheta^{\frac{1}{2}}} \begin{vmatrix} \vartheta_1, & \vartheta_2, & \vartheta_3 \\ U_1, & U_2, & U_3 \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix} = \sin \eta.$$

We now proceed to evaluate $\Sigma \bar{l}_0 \frac{d\lambda_0}{dS}$. The two terms, involving Q' and P together, are

$$= Q'(\alpha - \beta\cos\eta) - P'(\alpha\cos\eta - \beta)$$
$$= Q'\frac{\bar{G}}{\gamma_\theta} - P'\frac{\bar{G}}{\gamma_\lambda}$$
$$= \frac{\bar{G}^2}{\gamma_\theta^2\gamma_\lambda^2\sin^2\eta}(\gamma_\theta\gamma_\lambda' - \gamma_\theta'\gamma_\lambda).$$

The coefficient of Q in $\Sigma \bar{l}_0 \frac{d\lambda_0}{dS}$ is

$$= \frac{1}{\bar{\tau}_\theta}\Sigma X\bar{l}_0 - \frac{1}{\bar{\sigma}_\theta}\Sigma X_\theta(\alpha\bar{l}_\theta - \beta\bar{l}_\lambda)$$
$$= -\frac{\beta}{\bar{\sigma}_\theta}\sin\eta;$$

and the coefficient of $-P$ in the expression

$$= \frac{1}{\bar{\tau}_\lambda}\Sigma X\bar{l}_0 - \frac{1}{\bar{\sigma}_\lambda}\Sigma X_\lambda(\alpha\bar{l}_\theta - \beta\bar{l}_\lambda$$
$$= -\frac{\alpha}{\bar{\sigma}_\lambda}\sin\eta.$$

Hence the terms in Q and P together are

$$= \left(\frac{P\alpha}{\bar{\sigma}_\lambda} - \frac{Q\beta}{\bar{\sigma}_\theta}\right)\sin\eta$$
$$= \left\{\frac{\bar{G}^2}{\gamma_\theta\bar{\sigma}_\lambda}\left(\frac{1}{\gamma_\theta} - \frac{\cos\eta}{\gamma_\lambda}\right) + \frac{\bar{G}^2}{\gamma_\lambda\bar{\sigma}_\theta}\left(\frac{1}{\gamma_\lambda} - \frac{\cos\eta}{\gamma_\theta}\right)\right\}\frac{1}{\sin^2\eta}.$$

Consequently, we have

$$\Sigma \bar{l}_0 \frac{d\lambda_0}{dS}$$
$$= \frac{\bar{G}^2}{\gamma_\theta^2\gamma_\lambda^2\sin^2\eta}(\gamma_\theta\gamma_\lambda' - \gamma_\lambda\gamma_\theta') + \frac{\bar{G}^2}{\sin^2\eta}\left\{\frac{1}{\gamma_\theta^2\bar{\sigma}_\lambda} - \frac{\cos\eta}{\gamma_\theta\gamma_\lambda}\left(\frac{1}{\bar{\sigma}_\theta} + \frac{1}{\bar{\sigma}_\lambda}\right) + \frac{1}{\gamma_\lambda^2\bar{\sigma}_\theta}\right\},$$

where

$$\frac{\sin^2\eta}{\bar{G}^2} = \frac{1}{\gamma_\theta^2} - \frac{2\cos\eta}{\gamma_\theta\gamma_\lambda} + \frac{1}{\gamma_\lambda^2}.$$

As

$$\left.\begin{aligned}\frac{\cos\chi}{\sigma_0} - \frac{\sin\chi}{\tau_0} &= \Sigma\bar{l}_0\frac{dX}{dS}\\ \frac{\sin\chi}{\sigma_0} + \frac{\cos\chi}{\tau_0} &= \Sigma\bar{l}_0\frac{d\lambda_0}{dS}\end{aligned}\right\},$$

and as the values of the right-hand sides are known, we have the torsion and the tilt of the curve of intersection, the angle χ being given by

$$\bar{G}\tan\chi = \rho,$$

where $1/\rho$ is the circular curvature of the regional geodesic.

The two superficial geodesic curvatures of the curve.

385. We have obtained the regional geodesic curvature of the curve of intersection of the two surfaces.

Its superficial geodesic curvature on the θ-surface is derivable from the preceding analysis. We have

$$x_0'' = \frac{X}{\rho} + \frac{\lambda_0}{\bar{G}},$$

$$x_\theta'' = \frac{X}{\rho} + \frac{\bar{X}_\theta}{\gamma_\theta},$$

and therefore

$$x_0'' - x_\theta'' = \frac{\lambda_0}{\bar{G}} - \frac{\bar{X}_\theta}{\gamma_\theta}.$$

If then $1/\Gamma_\theta$ be the geodesic curvature on the θ-surface, and if $\lambda_\theta, \mu_\theta, \nu_\theta, \kappa_\theta$, be the corresponding direction-cosines, then

$$\frac{\lambda_\theta}{\Gamma_\theta} = \frac{\lambda_0}{\bar{G}} - \frac{\bar{X}_\theta}{\gamma_\theta},$$

with similar expressions for μ_θ/Γ_θ, ν_θ/Γ_θ, $\kappa_\theta/\Gamma_\theta$. Also

$$\frac{1}{\Gamma_\theta^2} = \Sigma \frac{\lambda_0^2}{\bar{G}^2} - \frac{2}{\bar{G}\gamma_\theta} \Sigma \lambda_0 \bar{X}_\theta + \frac{1}{\gamma_\theta^2}$$

$$= \frac{\sin^2\eta}{G^2} - \frac{2}{\bar{G}\gamma_\theta} \frac{\bar{G}}{\gamma_\theta} + \frac{1}{\gamma_\theta^2}$$

$$= \frac{1}{\gamma_\vartheta^2} - \frac{2\cos\eta}{\gamma_\vartheta \gamma_\theta}.$$

Similarly, the superficial geodesic curvature on the ϑ-surface is given by equations

$$\frac{\lambda_\vartheta}{\Gamma_\vartheta} = \frac{\lambda_0}{\bar{G}} - \frac{\bar{X}_\vartheta}{\gamma_\vartheta},$$

(with three other similar forms): and

$$\frac{1}{\Gamma_\vartheta^2} = \frac{1}{\gamma_\theta^2} - \frac{2\cos\eta}{\gamma_\theta \gamma_\vartheta}.$$

CHAPTER XXII.

OVOIDAL REGIONS.

Parametric representation of an ovoidal region.

386. Many of the preceding investigations can be illustrated collectively by a discussion of some characteristic properties of the region in homaloidal quadruple space, which is defined by the Cartesian equation

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}+\frac{v^2}{d}=1.$$

Such a region will be called *ovoidal*, merely for the sake of brevity: but no assumption is thereby implied that each of the quantities a, b, c, d, is positive. For reality, they cannot all be taken as negative. To preserve generality, it will be assumed that no two of them are equal: and, arranged in order of magnitude, they will be taken such that

$$a>b>c>d,$$

so that a certainly is positive. The simplest case, from the point of view of general description, arises when all four of the quantities are positive.

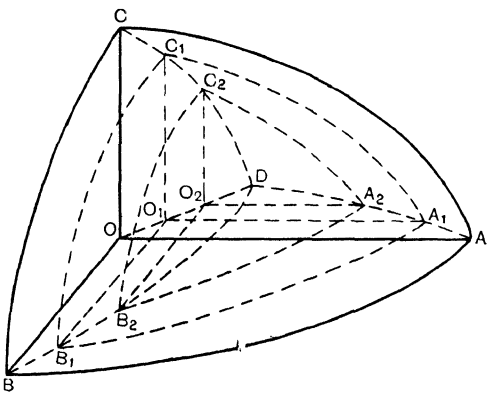


Fig. 26.

The octant $OABC$ is the section by the flat $v=0$ through O; the octant $O_1A_1B_1C_1$ is the section by the flat $v=OO_1$, through O_1 parallel to $v=0$; the octant $O_2A_2B_2C_2$ is the section by the flat $v=OO_2$, through O_2 parallel to $v=0$. The flat $v=OD$, through D parallel to $v=0$, is the tangent flat to the region at D; the section of the region by that flat is the point D. When all the coefficients are positive, the extremities of the principal axes are A, B, C, D, where $OA^2=a$, $OB^2=b$, $OC^2=c$, $OD^2=d$; and then the various octants are octants of ellipsoids in the respective flats $v=0$, $v=OO_1$, $v=OO_2$.

For the parametric representation of the region, we use an extension of the three-dimensional reference to confocal quadrics. We take a quantity

$$\frac{x^2}{a+\omega}+\frac{y^2}{b+\omega}+\frac{z^2}{c+\omega}+\frac{v^2}{d+\omega}-1,$$

which manifestly vanishes when $\omega=0$. The quantity vanishes for three other real values of ω; these are denoted by p, q, r. Thus p, q, r, are the parameters of three confocal regions; and the coordinates of any point in the given region are expressible in terms of p, q, r. It is easy to see that, for any point which is arbitrarily selected, the three parameters lie in the intervals between $-d$ and $-c$, $-c$ and $-b$, $-b$ and $-a$, respectively: we take

$$-d>p>-c>q>-b>r>-a,$$

so that

$$\begin{aligned}
&a+p>0, \quad b+p>0, \quad c+p>0, \quad d+p<0,\\
&a+q>0, \quad b+q>0, \quad c+q<0, \quad d+q<0,\\
&a+r>0, \quad b+r<0, \quad c+r<0, \quad d+r<0,
\end{aligned}$$

because of the assumption $a>b>c>d$. We now have

$$\frac{x^2}{a+\omega}+\frac{y^2}{b+\omega}+\frac{z^2}{c+\omega}+\frac{v^2}{d+\omega}-1=-\frac{\omega(\omega-p)(\omega-q)(\omega-r)}{(a+\omega)(b+\omega)(c+\omega)(d+\omega)},$$

for all values of ω; and therefore, if

$$\begin{aligned}
\mathsf{a}&=(a-b)(a-c)(a-d),\\
\mathsf{b}&=(b-c)(b-d)(b-a),\\
\mathsf{c}&=(c-d)(c-a)(c-b),\\
\mathsf{d}&=(d-a)(d-b)(d-c),
\end{aligned}$$

the coordinates of any point in the region are given by

$$\left.\begin{aligned}
x^2&=\frac{a}{\mathsf{a}}(a+p)(a+q)(a+r)\\
y^2&=\frac{b}{\mathsf{b}}(b+p)(b+q)(b+r)\\
z^2&=\frac{c}{\mathsf{c}}(c+p)(c+q)(c+r)\\
v^2&=\frac{d}{\mathsf{d}}(d+p)(d+q)(d+r)
\end{aligned}\right\}.$$

In the equation holding for all values of ω, let expansion be made in descending powers of ω; comparing the coefficients of ω^{-1}, we have

$$x^2+y^2+z^2+v^2=a+b+c+d+p+q+r.$$

Again, let that same equation be differentiated with respect to ω; and, after differentiation, let ω be made equal to zero. Then

$$\frac{1}{\Phi^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \frac{v^2}{d^2} = -\frac{pqr}{abcd}.$$

We require the direction-cosines X, Y, Z, V, of the *outward-drawn* normal* : they are

$$X = \Phi\frac{x}{a} = \left\{-\frac{bcd}{\mathbf{a}pqr}(a+p)(a+q)(a+r)\right\}^{\frac{1}{2}},$$

$$Y = \Phi\frac{y}{b} = \left\{-\frac{cda}{\mathbf{b}pqr}(b+p)(b+q)(b+r)\right\}^{\frac{1}{2}},$$

$$Z = \Phi\frac{z}{c} = \left\{-\frac{dab}{\mathbf{c}pqr}(c+p)(c+q)(c+r)\right\}^{\frac{1}{2}},$$

$$V = \Phi\frac{v}{d} = \left\{-\frac{abc}{\mathbf{d}pqr}(d+p)(d+q)(d+r)\right\}^{\frac{1}{2}}.$$

Fundamental primary and secondary magnitudes.

387. For the magnitudes of the first order, we have

$$A = \Sigma x_1^2$$

$$= \frac{1}{4}\Sigma\frac{a}{\mathbf{a}}\frac{(a+q)(a+r)}{a+p},$$

where the summation is over the four quantities a, b, c, d. If

$$f(\omega) = (a+\omega)(b+\omega)(c+\omega)(d+\omega), \quad g(\omega) = \lambda\omega^3 + \mu\omega^2 + \nu\omega + \rho,$$

then

$$\frac{g(\omega)}{f(\omega)} = \Sigma\frac{g(-a)}{-\mathbf{a}}\frac{1}{a+\omega} = \Sigma\frac{1}{\mathbf{a}}\frac{\lambda a^3 - \mu a^2 + \nu a - \rho}{a+\omega},$$

the summation being over a, b, c, d, and λ, μ, ν, ρ, denoting any quantities free from dependence on ω. Hence

$$A = \frac{1}{4}\frac{p^3 - (q+r)p^2 + qrp}{(a+p)(b+p)(c+p)(d+p)}.$$

Similarly,

$$H = \Sigma x_1 x_2$$

$$= \frac{1}{4}\Sigma\frac{a}{\mathbf{a}}\left\{\frac{(a+q)(a+r)}{a+p}\right\}^{\frac{1}{2}}\left\{\frac{(a+r)(a+p)}{a+q}\right\}^{\frac{1}{2}}$$

$$= \frac{1}{4}\Sigma\frac{a(a+r)}{\mathbf{a}} = 0.$$

* Here there is a deviation from the general investigation (§ 269), where X, Y, Z, V, are the direction-cosines of the *inward-drawn* normal: due allowance is made in the formulæ in § 285 hereafter.

So for the other *fundamental magnitudes of the first order*: the full tale of their values is

$$\left.\begin{aligned} F=0, \quad G=0, \quad H=0 \\ A=\frac{1}{4}\frac{p(p-q)(p-r)}{(a+p)(b+p)(c+p)(d+p)} \\ B=\frac{1}{4}\frac{q(q-r)(q-p)}{(a+q)(b+q)(c+q)(d+q)} \\ C=\frac{1}{4}\frac{r(r-p)(r-q)}{(a+r)(b+r)(c+r)(d+r)} \end{aligned}\right\}.$$

Again, for the fundamental magnitudes of the second order, we have

$$x_{11}=-\frac{1}{4}\left\{\frac{a}{\mathfrak{a}}\frac{(a+q)(a+r)}{(a+p)^3}\right\}^{\frac{1}{2}},$$

$$x_{12}=\frac{1}{4}\left\{\frac{a}{\mathfrak{a}}\frac{a+r}{(a+p)(a+q)}\right\}^{\frac{1}{2}},$$

and so for the others; and therefore

$$\begin{aligned} L&=-\Sigma X x_{11} \\ &=-\frac{1}{4}\left(-\frac{abcd}{pqr}\right)^{\frac{1}{2}}\Sigma\frac{1}{\mathfrak{a}}\frac{(a+q)(a+r)}{a+p} \\ &=-\frac{1}{4}\left(-\frac{abcd}{pqr}\right)^{\frac{1}{2}}\frac{(q-p)(r-p)}{(a+p)(b+p)(c+p)(d+p)}, \\ M&=-\Sigma X x_{12} \\ &=\frac{1}{4}\left(-\frac{abcd}{pqr}\right)^{\frac{1}{2}}\Sigma\frac{a+r}{\mathfrak{a}}=0, \end{aligned}$$

and so for the other magnitudes. The tale of *fundamental magnitudes of the second order* is

$$\left.\begin{aligned} I=0, \quad K=0, \quad M=0 \\ L=-\frac{1}{4}\left(-\frac{abcd}{pqr}\right)^{\frac{1}{2}}\frac{(q-p)(r-p)}{(a+p)(b+p)(c+p)(d+p)} \\ N=-\frac{1}{4}\left(-\frac{abcd}{pqr}\right)^{\frac{1}{2}}\frac{(r-q)(p-q)}{(a+q)(b+q)(c+q)(d+q)} \\ J=-\frac{1}{4}\left(-\frac{abcd}{pqr}\right)^{\frac{1}{2}}\frac{(p-r)(q-r)}{(a+r)(b+r)(c+r)(d+r)} \end{aligned}\right\}.$$

Curves of curvature: principal measures of curvature.

388. Because $F=0$, $G=0$, $H=0$, the curves of reference are perpendicular to one another; and because

$$\Sigma X x_1=0, \quad \Sigma X x_2=0, \quad \Sigma X x_3=0,$$

the frame, constituted by the tangents to the three lines of reference and the normal to the region, is an orthogonal frame. Again, because

$$I=0, \quad K=0, \quad M=0,$$

the curves of reference are conjugate directions. Finally, as the curves of reference are perpendicular to one another, and as they are conjugate to one another, these curves of reference are *the lines of curvature of the region*, which accordingly are given as the curves of intersection of the region by the respective pairs of the confocal regions

$$\Sigma \frac{x^2}{a+p}=1, \quad \Sigma \frac{x^2}{a+q}=1, \quad \Sigma \frac{x^2}{a+r}=1.$$

The principal line-curvatures of the region are the circular curvatures of the normal (geodesic) section of the region through the lines of curvature: they are

$$\left.\begin{aligned} \frac{1}{\alpha}=\frac{L}{A}&=-\left(-\frac{abcd}{p^3qr}\right)^{\frac{1}{2}} \\ \frac{1}{\beta}=\frac{N}{B}&=-\left(-\frac{abcd}{pq^3r}\right)^{\frac{1}{2}} \\ \frac{1}{\gamma}=\frac{J}{C}&=-\left(-\frac{abcd}{pqr^3}\right)^{\frac{1}{2}} \end{aligned}\right\}.$$

The negative signs are due to the fact that, for the canonical surface of reference where a, b, c, d, are positive, the convexity of the region is towards the distant parts of the quadruple space so that the curvatures, being inwards, are in a direction opposite to that of the outward-drawn normal.

The extended Gauss relations, and Codazzi-Mainardi relations.

389. The verification, that these magnitudes satisfy the extended Gauss relations (which, save for a developable region, serve to express the secondary magnitudes solely in terms of primary magnitudes and their first and second derivatives), is immediate.

A typical relation of the first set (§ 285) is

$$4LN+2(A_{22}+B_{11})=\frac{1}{A}(A_1B_1+A_2^2)+\frac{1}{B}(A_2B_2+B_1^2)-\frac{1}{C}A_3B_3.$$

To modify the right-hand side, we have

$$\frac{A_2}{A}=\frac{1}{q-p}, \quad \frac{B_1}{B}=\frac{1}{p-q}, \quad \frac{A_3}{A}=\frac{1}{r-p}, \quad \frac{B_3}{B}=\frac{1}{r-q}.$$

Again,

$$\begin{aligned} \frac{1}{4A}&=\frac{(a+p)(b+p)(c+p)(d+p)}{p(p-q)(p-r)} \\ &=p+q+r+a+b+c+d+\frac{\theta}{p}+\frac{\psi}{p-q}+\frac{\chi}{p-r}, \end{aligned}$$

where

$$\theta = \frac{abcd}{qr},$$

$$\psi = \frac{(a+q)(b+q)(c+q)(d+q)}{q(q-r)} = \frac{Q}{q(q-r)},$$

$$\chi = \frac{(a+r)(b+r)(c+r)(d+r)}{r(r-q)} = \frac{R}{r(r-q)};$$

and therefore

$$\frac{A_1}{4A^2} = -1 + \frac{abcd}{p^2qr} + \frac{Q}{q(q-r)(p-q)^2} + \frac{R}{r(r-q)(p-r)^2}.$$

Similarly

$$\frac{B_2}{4B^2} = -1 + \frac{abcd}{pq^2r} + \frac{R}{r(r-p)(q-r)^2} + \frac{P}{p(p-r)(q-p)^2},$$

while

$$\frac{1}{4A} = \frac{P}{p(p-q)(p-r)}, \quad \frac{1}{4B} = \frac{Q}{q(q-r)(q-p)}, \quad \frac{1}{4C} = \frac{R}{r(r-p)(r-q)}.$$

Thus the right-hand side

$$= AB\left[\left(\frac{1}{A} + \frac{1}{B}\right)\frac{1}{(p-q)^2} - 4\frac{R}{r}\frac{1}{(r-p)^2(r-q)^2}\right.$$
$$+ \frac{4}{p-q}\left\{-1 + \frac{abcd}{p^2qr} + \frac{Q}{q(q-r)(p-q)^2} + \frac{R}{r(r-q)(p-r)^2}\right\}$$
$$\left. + \frac{4}{q-p}\left\{-1 + \frac{abcd}{pq^2r} + \frac{P}{p(p-r)(p-q)^2} + \frac{R}{r(r-p)(q-r)^2}\right\}\right].$$

The total coefficient of R is zero. When we substitute for A, the total coefficient of P is zero; and when we substitute for B, the total coefficient of Q is zero. Thus the right-hand side is

$$= -4AB\frac{abcd}{p^2q^2r}.$$

For the left-hand side, $A_{22} = 0$, $B_{11} = 0$, and

$$\frac{LN}{AB} = -\frac{abcd}{p^2q^2r}.$$

The typical relation of the first set is therefore verified. Similarly for the other relations.

A typical relation of the second set (§ 285) is

$$2A_{23} = \frac{A_2A_3}{A} + \frac{A_2B_3}{B} + \frac{A_3C_2}{C}.$$

The right-hand side

$$= A\frac{1}{q-p}\frac{1}{r-p} + A\frac{1}{q-p}\frac{1}{r-q} + A\frac{1}{r-p}\frac{1}{q-r} = 0,$$

and the left-hand side is zero. This typical relation is also satisfied.

Similarly, the Codazzi-Mainardi relations are satisfied, without leaving any residuary conditions.

One such typical relation (§ 285) is, for the ovoidal region,

$$L_2 + \Delta_{11} N = \Gamma_{12} L,$$

that is,

$$L_2 - \frac{1}{2}\frac{A_2}{B} N = \frac{1}{2}\frac{A_2}{A} L.$$

Now, writing

$$\Theta = -\left(-\frac{abcd}{pqr}\right)^{\frac{1}{2}},$$

we have

$$\frac{L}{A} = \frac{\Theta}{p}, \quad \frac{N}{B} = \frac{\Theta}{q};$$

and so the relation becomes

$$L_2 = \frac{1}{2} A_2 \left(\frac{\Theta}{q} + \frac{\Theta}{p}\right).$$

But, by direct substitution of their values, we find

$$\frac{L_2}{L} = -\frac{1}{2q} + \frac{1}{q-p}, \quad \frac{A_2}{A} = \frac{1}{q-p},$$

so that

$$\frac{L_2}{L} = \frac{q+p}{2q}\frac{A_2}{A},$$

that is

$$L_2 = \frac{1}{2} A_2 \frac{q+p}{q}\frac{L}{A}$$

$$= \frac{1}{2} A_2 \frac{q+p}{q}\frac{\Theta}{p},$$

verifying that the typical relation is satisfied unconditionally.

Similarly for all the other Codazzi-Mainardi relations.

Magnitudes of the third order.

390. The magnitudes of the third order, in general, are required in order to express (§ 309) the tilt of a geodesic in the region, as well as other (invariantive) intrinsic magnitudes.

For the expression of these in general, when obtained (§§ 292, 293) through the derivatives of L, N, J, I, K, M, we require the quantities Γ, Δ, Θ. These are

$$\Gamma_{11} = \frac{1}{2}\frac{A_1}{A}, \quad \Gamma_{22} = -\frac{1}{2}\frac{B_1}{A}, \quad \Gamma_{33} = -\frac{1}{2}\frac{C_1}{A}, \quad \Gamma_{23} = 0, \quad \Gamma_{31} = \frac{1}{2}\frac{A_3}{A}, \quad \Gamma_{12} = \frac{1}{2}\frac{A_2}{A},$$

$$\Delta_{11} = -\frac{1}{2}\frac{A_2}{B}, \quad \Delta_{22} = \frac{1}{2}\frac{B_2}{B}, \quad \Delta_{33} = -\frac{1}{2}\frac{C_2}{B}, \quad \Delta_{23} = \frac{1}{2}\frac{B_3}{B}, \quad \Delta_{31} = 0, \quad \Delta_{12} = \frac{1}{2}\frac{B_1}{B},$$

$$\Theta_{11} = -\frac{1}{2}\frac{A_3}{C}, \quad \Theta_{22} = -\frac{1}{2}\frac{B_3}{C}, \quad \Theta_{33} = \frac{1}{2}\frac{C_3}{C}, \quad \Theta_{23} = \frac{1}{2}\frac{C_2}{C}, \quad \Theta_{31} = \frac{1}{2}\frac{C_1}{C}, \quad \Theta_{12} = 0.$$

The values of the magnitudes e_{lmn}, having regard to the facts that $I=0$, $K=0$, $M=0$, become

$$\begin{aligned} e_{300} &= L_1 - 2L\Gamma_{11}, & e_{120} &= N_1 - 2N\Delta_{12}, & e_{102} &= J_1 - 2J\Theta_{13}, \\ e_{210} &= L_2 - 2L\Gamma_{12}, & e_{030} &= N_2 - 2N\Delta_{22}, & e_{012} &= J_2 - 2J\Theta_{23}, \\ e_{201} &= L_3 - 2L\Gamma_{13}, & e_{021} &= N_3 - 2N\Delta_{23}, & e_{003} &= J_3 - 2J\Theta_{33}, \\ & & e_{111} &= 0. \end{aligned}$$

Thus

$$\begin{aligned} e_{300} &= L_1 - 2L\Gamma_{11} \\ &= A\frac{\partial}{\partial p}\left(\frac{L}{A}\right) \\ &= -A\frac{\partial}{\partial p}\left(-\frac{abcd}{p^3qr}\right)^{\frac{1}{2}} \\ &= \frac{3}{2}A\left(-\frac{abcd}{p^5qr}\right)^{\frac{1}{2}} = -\frac{3}{2}\frac{L}{p}; \\ e_{210} &= L_2 - 2L\Gamma_{12} \\ &= A\frac{\partial}{\partial q}\left(\frac{L}{A}\right) \\ &= -A\frac{\partial}{\partial q}\left(-\frac{abcd}{p^3qr}\right)^{\frac{1}{2}} \\ &= \frac{1}{2}A\left(-\frac{abcd}{p^3q^3r}\right)^{\frac{1}{2}} = -\frac{1}{2}\frac{L}{q}; \end{aligned}$$

and so for the others. The tale of the magnitudes of the third order, when thus expressed, is

$$\left.\begin{aligned} e_{300} &= -\frac{3L}{2p}, & e_{120} &= -\frac{N}{2p}, & e_{102} &= -\frac{J}{2p} \\ e_{210} &= -\frac{L}{2q}, & e_{030} &= -\frac{3N}{2q}, & e_{012} &= -\frac{J}{2q} \\ e_{201} &= -\frac{L}{2r}, & e_{021} &= -\frac{N}{2r}, & e_{003} &= -\frac{3J}{2r} \\ & & e_{111} &= 0 \end{aligned}\right\}.$$

One immediate inference may be made, in passing. With these values of the magnitudes e_{lmn}, and having regard to the source of these magnitudes, we have

$$\begin{aligned} -\frac{\rho'}{\rho^2} &= \frac{d}{ds}\left(\frac{1}{\rho}\right) = (e\chi p', q', r')^3 \\ &= -\frac{3}{2}\left(\frac{p'}{p}+\frac{q'}{q}+\frac{r'}{r}\right)(Lp'^2+Nq'^2+Jr'^2). \end{aligned}$$

Now the differentiation of $\frac{1}{\rho}$ is taken along the geodesic through the direction p', q', r'; hence, along the geodesic,

$$-\frac{\rho'}{\rho} = -\frac{3}{2}\left(\frac{p'}{p}+\frac{q'}{q}+\frac{r'}{r}\right),$$

that is,

$$\rho\,(pqr)^{-\frac{1}{2}} = \text{constant}$$

along the geodesic, a result that will otherwise be obtained later (§ 399). But the constant may (and, in fact, does) vary from one geodesic to another: all that can be asserted is the constancy in value of $\rho\,(pqr)^{-\frac{1}{2}}$ along the course of each geodesic.

Circular curvature and torsion of a line of curvature.

391. We proceed to obtain the curvatures of a line of curvature* at a point, choosing in particular a line for which $q = \text{constant}$ and $r = \text{constant}$: the curvatures of the other lines through the point are obtainable by cyclical alternation of the parameters. Let

$$\mathbf{a}\bar{a}^2 = a^3 + \lambda a^2 + \mu a, \quad \mathbf{c}\bar{c}^2 = c^3 + \lambda c^2 + \mu c,$$
$$\mathbf{b}\bar{b}^2 = b^3 + \lambda b^2 + \mu b, \quad \mathbf{d}\bar{d}^2 = d^3 + \lambda d^2 + \mu d,$$

where

$$\lambda = q + r, \quad \mu = qr, \quad \bar{a}^2 + \bar{b}^2 + \bar{c}^2 + \bar{d}^2 = 1.$$

Along the selected line of curvature, p can be taken as the current parameter; and we have

$$x^2 = \bar{a}^2(a+p), \quad y^2 = \bar{b}^2(b+p), \quad z^2 = \bar{c}^2(c+p), \quad v^2 = \bar{d}^2(d+p).$$

Thus

$$x_1 = \frac{1}{2}\bar{a}\,(a+p)^{-\frac{1}{2}}, \quad y_1 = \frac{1}{2}\bar{b}\,(b+p)^{-\frac{1}{2}}, \quad z_1 = \frac{1}{2}\bar{c}\,(c+p)^{-\frac{1}{2}}, \quad v_1 = \frac{1}{2}\bar{d}\,(d+p)^{-\frac{1}{2}};$$

the element of arc is given by

$$ds^2 = \frac{1}{4}\Sigma\frac{\bar{a}^2}{a+p}\,dp^2 = \frac{1}{4}\frac{p^3 - \lambda p^2 + \mu p}{(a+p)(b+p)(c+p)(d+p)^2}\,dp^2 = A\,dp^2;$$

and the tangent to the line of curvature has direction-cosines

$$x' = \frac{x_1}{A^{\frac{1}{2}}}, \quad y' = \frac{y_1}{A^{\frac{1}{2}}}, \quad z' = \frac{z_1}{A^{\frac{1}{2}}}, \quad v' = \frac{v_1}{A^{\frac{1}{2}}}.$$

For the (primary) circular curvature of the line, we have

$$x'' = \frac{dx'}{dp}\frac{dp}{ds} = A^{-\frac{1}{2}}\frac{d}{dp}\left(\frac{x_1}{A^{\frac{1}{2}}}\right) = \frac{x_{11}}{A} - \frac{1}{2}\frac{A_1}{A^2}x_1,$$

$$y'' = \frac{y_{11}}{A} - \frac{1}{2}\frac{A_1}{A^2}y_1,$$

$$z'' = \frac{z_{11}}{A} - \frac{1}{2}\frac{A_1}{A^2}z_1,$$

$$v'' = \frac{v_{11}}{A} - \frac{1}{2}\frac{A_1}{A^2}v_1;$$

* A curve of curvature (usually called a line of curvature) is not a geodesic; the curvatures of any geodesic are given in §§ 272, 306, 309, for any region.

and therefore

$$Xx'' + Yy'' + Zz'' + Vv'' = \frac{L}{A} = \frac{1}{\alpha}.$$

Let ρ_p be the radius of circular curvature of the line of curvature, and ϕ be the inclination of its direction to the principal normal: then the foregoing relation is

$$\frac{\cos\phi}{\rho_p} = \frac{1}{\alpha},$$

being the extension of Meusnier's theorem for this curve. Also

$$x_{11} = -\frac{1}{4}\bar{a}(a+p)^{-\frac{3}{2}}, \quad y_{11} = -\frac{1}{4}\bar{b}(b+p)^{-\frac{3}{2}},$$

$$z_{11} = -\frac{1}{4}\bar{c}(c+p)^{-\frac{3}{2}}, \quad v_{11} = -\frac{1}{4}\bar{d}(d+p)^{-\frac{3}{2}},$$

so that

$$\Sigma x_{11}^2 = \frac{1}{16}\Sigma\frac{\bar{a}^2}{(a+p)^3}.$$

But

$$A = \Sigma x_1^2 = \frac{1}{4}\Sigma\frac{\bar{a}^2}{a+p},$$

so that

$$A_{11} = \frac{1}{2}\Sigma\frac{\bar{a}^2}{(a+p)^3};$$

and therefore

$$\Sigma x_{11}^2 = \frac{1}{8}A_{11}.$$

Consequently

$$\begin{aligned}\frac{1}{\rho_p^2} &= \Sigma x''^2 \\ &= \frac{1}{A^2}\Sigma x_{11}^2 - \frac{A_1}{A^3}\Sigma x_1 x_{11} + \frac{1}{4}\frac{A_1^2}{A^3} \\ &= \frac{1}{8}\frac{A_{11}}{A^2} - \frac{1}{4}\frac{A_1^2}{A^3},\end{aligned}$$

because

$$\Sigma x_1 x_{11} = \frac{1}{2}A_1.$$

This result can be written in the form

$$\frac{1}{\rho_p^2} = -\frac{1}{8}\frac{\partial^2}{\partial p^2}\left(\frac{1}{A}\right).$$

Now we have had the relation (§ 389)

$$\frac{1}{4A} = p+q+r+a+b+c+d+\frac{\theta}{p}+\frac{\psi}{p-q}+\frac{\chi}{p-r},$$

where θ, ψ, χ, are quantities, independent of p, such that

$$\theta = \frac{abcd}{qr},$$

$$\psi = \frac{(a+q)(b+q)(c+q)(d+q)}{q(q-r)} = \frac{q-p}{4B},$$

$$\chi = \frac{(a+r)(b+r)(c+r)(d+r)}{r(r-q)} = \frac{r-p}{4C};$$

and therefore

$$\frac{1}{\rho_p^2} = -\frac{\theta}{p^3} - \frac{\psi}{(p-q)^3} - \frac{\chi}{(p-r)^3};$$

that is,

$$\frac{1}{\rho_p^2} = -\frac{abcd}{p^3 qr} + \frac{1}{4}\frac{1}{B(p-q)^2} + \frac{1}{4}\frac{1}{C(p-r)^2}.$$

For the torsion, measured by means of the radius of torsion σ_p, we use the general formula (§ 127)

$$\Sigma J_x^2 = \begin{vmatrix} \Sigma x'^2, & \Sigma x'x'', & \Sigma x'x''' \\ \Sigma x'x'', & \Sigma x''^2, & \Sigma x''x''' \\ \Sigma x'x''', & \Sigma x''x''', & \Sigma x'''^2 \end{vmatrix} = \frac{1}{\sigma^2 \rho^4}.$$

Now

$$J_x = \begin{vmatrix} y', & z', & v' \\ y'', & z'', & v'' \\ y''', & z''', & v''' \end{vmatrix},$$

generally: along the selected line of curvature,

$$y' = A^{-\frac{1}{2}} y_1,$$

$$y'' = \frac{1}{A} y_{11} - \frac{1}{2}\frac{A_1}{A^2} y_1,$$

$$y''' = \frac{1}{A^{\frac{3}{2}}} y_{111} - \frac{3}{2}\frac{A_1}{A^{\frac{5}{2}}} y_{11} - \frac{4}{\rho_p^2}\frac{y_1}{A^{\frac{1}{2}}},$$

and so for the derivatives of z and v. Thus

$$J_x = \frac{1}{A^2}\begin{vmatrix} y_1, & z_1, & v_1 \\ y_{11}, & z_{11}, & v_{11} \\ y_{111}, & z_{111}, & v_{111} \end{vmatrix}.$$

Similarly for J_y, J_z, J_v. Consequently

$$\frac{A^6}{\sigma_p^2 \rho_p^4} = A^6 \Sigma J_x^2 = \begin{vmatrix} \Sigma x_1^2, & \Sigma x_1 x_{11}, & \Sigma x_1 x_{111} \\ \Sigma x_1 x_{11}, & \Sigma x_{11}^2, & \Sigma x_{11} x_{111} \\ \Sigma x_1 x_{111}, & \Sigma x_{11} x_{111}, & \Sigma x_{111}^2 \end{vmatrix}.$$

But

$$\Sigma x_1^2 = \frac{1}{4}\Sigma\frac{\bar{a}^2}{a+p} = A,$$

$$\Sigma x_1 x_{11} = -\frac{1}{8}\Sigma\frac{\bar{a}^2}{(a+p)^2} = \frac{1}{2}A_1,$$

$$\Sigma x_{11}^2 = \frac{1}{16}\Sigma\frac{\bar{a}^2}{(a+p)^3} = \frac{1}{8}A_{11},$$

$$\Sigma x_1 x_{111} = \frac{3}{16}\Sigma\frac{\bar{a}^2}{(a+p)^3} = \frac{3}{8}A_{11},$$

$$\Sigma x_{11} x_{111} = -\frac{3}{32}\Sigma\frac{\bar{a}^2}{(a+p)^4} = \frac{1}{16}A_{111},$$

$$\Sigma x_{111}^2 = \frac{9}{64}\Sigma\frac{\bar{a}^2}{(a+p)^5} = \frac{3}{128}A_{1111},$$

all expressible in terms of A and its derivatives with regard to p, the current variable of the line of curvature. Hence

$$1024\,\frac{A^6}{\sigma_p^2\rho_p^4} = E,$$

where

$$E = 3(AA_{11} - 2A_1^2)A_{1111} - 4AA_{111}^3 + 24A_1A_{11}A_{111} - 18A_{11}^3.$$

We had

$$\frac{A^3}{\rho_p^2} = \frac{1}{8}(AA_{11} - 2A_1^2) = \frac{1}{8}D,$$

say; and therefore, with this notation,

$$\left.\begin{aligned}\frac{1}{\rho_p^2} &= \frac{1}{8}\frac{D}{A^3}\\ \frac{1}{\sigma_p^2} &= \frac{1}{16}\frac{E}{D^2}\end{aligned}\right\}.$$

The radius of spherical curvature of the line of curvature is given by

$$\begin{aligned}R_p^2 &= \rho_p^2 + \sigma_p^2\rho_p'^2\\ &= \rho_p^2 + \frac{1}{A}\sigma_p^2\left(\frac{\partial\rho_p}{\partial p}\right)^2\\ &= 8\frac{A^3}{D} + 128\frac{A^2D}{E}\left(3\frac{A_1}{A} - \frac{D_1}{D}\right)^2.\end{aligned}$$

392. We can proceed otherwise, as follows, to an expression for the torsion of the line of curvature. The result

$$A^3J_x = \begin{vmatrix} y_1 , & z_1 , & v_1\\ y_{11} , & z_{11} , & v_{11}\\ y_{111}, & z_{111}, & v_{111}\end{vmatrix}$$

has already been established. Now

$$y_1 = \frac{1}{2}\bar{b}(b+p)^{-\frac{1}{2}},\quad y_{11} = -\frac{1}{4}\bar{b}(b+p)^{-\frac{3}{2}},\quad y_{111} = \frac{3}{8}\bar{b}(b+p)^{-\frac{5}{2}},$$

and similarly for the derivatives of z and of v; thus

$$A^3 J_x = -\frac{3}{64}\frac{\overline{bcd}}{\{(b+p)(c+p)(d+p)\}^{\frac{5}{2}}}\begin{vmatrix}(b+p)^2, & (c+p)^2, & (d+p)^2\\ b+p\ , & c+p\ , & d+p\\ 1\ , & 1\ , & 1\end{vmatrix}$$

$$= \frac{3}{64}\frac{\overline{bcd}(c-d)(d-b)(b-c)}{\{(b+p)(c+p)(d+p)\}^{\frac{5}{2}}}.$$

The numerator in this fraction is equal to

$$\frac{\{(b^3+\lambda b^2+\mu b)(c^3+\lambda c^2+\mu c)(d^3+\lambda d^2+\mu d)\}^{\frac{1}{2}}}{\{(a-b)(a-c)(a-d)\}^{\frac{1}{2}}},$$

and

$$\frac{1}{(b+p)(c+p)(d+p)} = (a+p)\frac{4A}{p^3-\lambda p^2+\mu p};$$

hence

$$\frac{(p^3-\lambda p^2+\mu p)^{\frac{5}{2}}A^{\frac{1}{2}}J_x}{\prod\limits_{abcd}(a^3+\lambda a^2+\mu a)^{\frac{1}{2}}} = \frac{3}{2}\left\{\frac{(a+p)^5}{(a^3+\lambda a^2+\mu a)(a-b)(a-c)(a-d)}\right\}^{\frac{1}{2}}.$$

We have the customary formula for all curves in quadruple homaloidal space

$$\Sigma J_x^2 = \frac{1}{\sigma^2\rho^4}.$$

Applied to the line of curvature in question, this result gives

$$\frac{(p^3-\lambda p^2+\mu p)^5}{abcdQR}\frac{A}{\sigma_p^2\rho_p^4} = \frac{9}{4}\Sigma\frac{(a+p)^5}{(a^3+\lambda a^2+\mu a)(a-b)(a-c)(a-\overline{d})},$$

that is,

$$\frac{(p^3-\lambda p^2+\mu p)^6}{9abcdPQR}\frac{1}{\sigma_p^2\rho_p^4} = \Sigma\frac{(a+p)^5}{(a+q)(a+r)a(a-b)(a-c)(a-d)}.$$

Now, for any quantity ω,

$$\frac{(p-\omega)^5}{\omega(\omega-q)(\omega-r)(\omega+a)(\omega+b)(\omega+c)(\omega+d)}$$

$$= \frac{p^5}{qrabcd}\frac{1}{\omega} + \frac{(p-q)^5}{q(q-r)Q}\frac{1}{\omega-q} + \frac{(p-r)^5}{r(r-q)R}\frac{1}{\omega-r}$$

$$+\Sigma\frac{(a+p)^5}{(a+q)(a+r)a(a-b)(a-c)(a-d)}\frac{1}{\omega+a},$$

and therefore, expanding both sides in descending powers of ω, and equating the coefficients of ω^{-1}, we have

$$\Sigma\frac{(a+p)^5}{(a+q)(a+r)a(a-b)(a-c)(a-d)} = -\frac{p^5}{qrabcd} - \frac{(p-q)^5}{q(q-r)Q} - \frac{(p-r)^5}{r(r-q)R}.$$

Thus

$$\frac{1}{\sigma_p^2 \rho_p^4} = -\frac{9abcd\,PQR}{\{p(p-q)(p-r)\}^6}\left\{\frac{p^5}{qrabcd} + \frac{(p-q)^5}{q(q-r)Q} + \frac{(p-r)^5}{r(r-q)R}\right\};$$

when we substitute

$$\frac{1}{\rho_p^2} = -\left\{\frac{abcd}{p^3qr} + \frac{1}{(p-q)^3}\frac{Q}{q(q-r)} + \frac{1}{(p-r)^3}\frac{R}{r(r-q)}\right\},$$

we obtain an expression giving the torsion of the line of curvature.

Tilt of a line of curvature.

393. For the tilt of the line of curvature, we use the general expression (§ 145)

$$\frac{-1}{\rho^3\sigma^2\tau} = \Omega = \begin{vmatrix} x', & y', & z', & v' \\ x'', & y'', & z'', & v'' \\ x''', & y''', & z''', & v''' \\ x^{\text{iv}}, & y^{\text{iv}}, & z^{\text{iv}}, & v^{\text{iv}} \end{vmatrix}.$$

We have to evaluate this for the line of curvature along which the sole current parameter is p. Thus

$$x' = \frac{x_1}{A^{\frac{1}{2}}},$$

$$x'' = \frac{x_{11}}{A} + \text{a linear term in } x_1,$$

$$x''' = \frac{x_{111}}{A^{\frac{3}{2}}} + \text{linear terms in } x_{11}, x_1,$$

$$x^{\text{iv}} = \frac{x_{1111}}{A^2} + \text{linear terms in } x_{111}, x_{11}, x_1,$$

with corresponding expressions for the derivatives of y, z, v, the coefficients being the same as in x', x'', x''', x^{iv}; consequently

$$A^5\Omega = \begin{vmatrix} x_1, & y_1, & z_1, & v_1 \\ x_{11}, & y_{11}, & z_{11}, & v_{11} \\ x_{111}, & y_{111}, & z_{111}, & v_{111} \\ x_{1111}, & y_{1111}, & z_{1111}, & v_{1111} \end{vmatrix}$$

$$= \frac{15}{1024}\bar{a}\bar{b}\bar{c}\bar{d}\{(a+p)(b+p)(c+p)(d+p)\}^{-\frac{7}{2}}\Theta,$$

where

$$\Theta = \begin{vmatrix} (a+p)^3, & (b+p)^3, & (c+p)^3, & (d+p)^3 \\ (a+p)^2, & (b+p)^2, & (c+p)^2, & (d+p)^2 \\ a+p, & b+p, & c+p, & d+p \\ 1, & 1, & 1, & 1 \end{vmatrix}$$

$$= (a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$$

But

$$\Theta^2\overline{a}^2\overline{b}^2\overline{c}^2\overline{d}^2 = \Pi(a^3+\lambda a^2+\mu a)$$
$$= abcdQR,$$

and

$$4AP = p(p-q)(p-r);$$

consequently

$$\Omega = \frac{15(abcdP^3QR)^{\frac{1}{2}}}{p^5(p-q)^5(p-r)^5},$$

and therefore

$$\frac{1}{\rho_p{}^3\sigma_p{}^2\tau_p} = -15\frac{(abcdP^3QR)^{\frac{1}{2}}}{p^5(p-q)^5(p-r)^5},$$

a relation which gives the tilt of the line of the curvature because the circular curvature and the torsion are already known.

The radius of globular curvature S_p is derivable by substituting, in the general expression (§ 147)

$$\frac{S^2}{R^2}-1 = \frac{\tau^2}{\sigma^2}\left(\frac{dR}{d\rho}\right)^2$$
$$= \frac{\tau^2}{\sigma^2}\frac{R_1{}^2}{\rho_1{}^2},$$

the values of ρ_p, σ_p, R_p, τ_p, which have already been obtained.

Region of centres of curvature of an ovoid.

394. It has already been pointed out that, for any region which is not homaloidal, there are three principal linear curvatures at each point, that there are three corresponding centres of such curvature, and that the locus of the centres of that curvature is a three-volumed region. To the consideration of that region for an ovoidal region we now proceed.

The coordinates of the centre of curvature, settled by the curvature associated with the line $q=$ constant, $r=$ constant, are

$$\xi = x-\alpha X, \quad \eta = y-\alpha Y, \quad \zeta = z-\alpha Z, \quad \upsilon = v-\alpha V.$$

Now, for the ovoidal region

$$\frac{1}{\alpha} = \frac{L}{A} = -\left(-\frac{abcd}{p^3qr}\right)^{\frac{1}{2}};$$

the values of X, Y, Z, V, are given in § 386; and therefore

$$\xi = \left\{\frac{a}{\mathfrak{a}}(a+p)(a+q)(a+r)\right\}^{\frac{1}{2}} + \left(-\frac{p^3qr}{abcd}\right)^{\frac{1}{2}}\left\{-\frac{bcd}{\mathfrak{a}pqr}(a+p)(a+q)(a+r)\right\}^{\frac{1}{2}}$$
$$= \left\{\frac{1}{\mathfrak{a}}(a+p)(a+q)(a+r)\right\}^{\frac{1}{2}}\left(a^{\frac{1}{2}}+\frac{p}{a^{\frac{1}{2}}}\right),$$

that is,

$$\xi = \left\{\frac{(a+p)^3(a+q)(a+r)}{a(a-b)(a-c)(a-d)}\right\}^{\frac{1}{2}}$$

and, similarly,

$$\left.\begin{aligned}\eta &= \left\{\frac{(b+p)^3(b+q)(b+r)}{b(b-c)(b-d)(b-a)}\right\}^{\frac{1}{2}}\\ \zeta &= \left\{\frac{(c+p)^3(c+q)(c+r)}{c(c-d)(c-a)(c-b)}\right\}^{\frac{1}{2}}\\ \upsilon &= \left\{\frac{(d+p)^3(d+q)(d+r)}{d(d-a)(d-b)(d-c)}\right\}^{\frac{1}{2}}\end{aligned}\right\}.$$

For the second volume of the region of centres, belonging to the radius β associated with the line $r = \text{constant}$, $p = \text{constant}$, the coordinates are

$$\xi' = \left\{\frac{(a+p)(a+q)^3(a+r)}{a(a-b)(a-c)(a-d)}\right\}^{\frac{1}{2}},$$

with corresponding values of η', ζ', υ'. For the third volume of the region of centres, belonging to the radius γ associated with the line $p = \text{constant}$, $q = \text{constant}$, the coordinates are

$$\xi'' = \left\{\frac{(a+p)(a+q)(a+r)^3}{a(a-b)(a-c)(a-d)}\right\}^{\frac{1}{2}},$$

with corresponding values of η'', ζ'', υ''.

The Cartesian equation of the first volume of the region of centres is derivable by the elimination of p, q, r, among the four equations presented by the expressions for the coordinates $\xi, \eta, \zeta, \upsilon$; and, in the elimination, the precise significance of p, q, r, disappears. The Cartesian equation of the second volume of that region is derivable by the elimination of p, q, r, among the four equations presented by the values of ξ', η', ζ', υ'; but these differ analytically from the values of $\xi, \eta, \zeta, \upsilon$, solely by the interchange of p and q, the significance of both of which disappears in the respective eliminations: that is, the Cartesian equation of the second volume of the centro-region is the same as that of the first volume. Similarly, the Cartesian equation of the third volume is the same as that of the first volume.

Consequently there is a single Cartesian equation which represents simultaneously the three volumes of the centro-region in question; and this single equation is derivable by the elimination of p, q, r, among the four equations expressing the coordinates $\xi, \eta, \zeta, \upsilon$, in terms of the three parameters.

Order and flat sections of the centro-region.

395. The order of the region of centres (that is, the corporate degree of its Cartesian equation in the coordinates) is eighteen. To establish this result, it would be sufficient to note the intersections by a line; and we can, in fact, verify that any set of small specific values of ξ, η, ζ, gives rise to eighteen

values of v. But the shape of the centro-region is of interest; and a notion of the shape can be derived through flat sections. We therefore consider a flat section; incidentally, the order of the region will be evaluated. The section is a surface, not necessarily single-sheeted; and the aggregate order of the whole of the superficial section is the order of the region. When a particular flat is chosen, care must be exercised in framing the estimate of the aggregate order of the superficial section.

We take the section by the flat $v=0$: this section consists of three surfaces, arising respectively from the three volumes. To estimate the aggregate order, we must consider also the form of the equation of the centro-region in the immediate vicinity of that flat—a process that can be pursued by retaining only small values of v; the character, in the immediate vicinity of each of the three superficial sections, determines the respective estimate of order for each of the sections.

For the section of the region by the flat $v=0$, we have (i) $p=-d$, occurring thrice; (ii) $q=-d$, occurring once; (iii) $r=-d$, occurring once: and these must be discussed separately.

I. For (ii), when $q=-d$, we have

$$\xi^2=\frac{(a+p)^3(a+r)}{a(a-b)(a-c)}, \quad \eta^2=\frac{(b+p)^3(b+r)}{b(b-c)(b-a)}, \quad \zeta^2=\frac{(c+p)^3(c+r)}{c(c-a)(c-b)}, \quad v=0.$$

These* are the equations of the two-sheeted centro-surface of the quadric

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1, \quad v=0.$$

We have to consider the degree of occurrence of this centro-surface. For the section in the vicinity of $q=-d$, we substitute

$$q+d=\epsilon,$$

where ϵ may be taken to be a small constant. Then, approximately,

$$v_1=\frac{3}{2}\left\{\frac{(d+p)(d+r)}{d(d-a)(d-b)(d-c)}\right\}^{\frac{1}{2}}(d+q)^{\frac{1}{2}} = \text{quantity of order } \epsilon^{\frac{1}{2}},$$

$$v_2=\frac{1}{2}\left\{\frac{(d+p)^3(d+r)}{d(d-a)(d-b)(d-c)}\right\}^{\frac{1}{2}}(d+q)^{-\frac{1}{2}} = \ldots\ldots\ldots\ldots \epsilon^{-\frac{1}{2}},$$

$$v_3=\frac{1}{2}\left\{\frac{(d+p)^3}{d(d-a)(d-b)(d-c)(d+r)}\right\}^{\frac{1}{2}}(d+q)^{\frac{1}{2}} = \ldots\ldots\ldots\ldots \epsilon^{\frac{1}{2}};$$

and therefore

$$J_x=\begin{vmatrix} \eta_1, & \zeta_1, & v_1 \\ \eta_2, & \zeta_2, & v_2 \\ \eta_3, & \zeta_3, & v_3 \end{vmatrix} = \epsilon^{-\frac{1}{2}}\{(\eta_3\zeta_1-\eta_1\zeta_3)+\text{positive powers of } \epsilon\}\,\theta,$$

* The investigation follows Cayley's discussion of this centro-surface, *Coll. Math. Papers*, vol. viii, paper 520; see also my *Lectures on Differential Geometry*, § 84.

$$J_y = \begin{vmatrix} \zeta_1, & v_1, & \xi_1 \\ \zeta_2, & v_2, & \xi_2 \\ \zeta_3, & v_3, & \xi_3 \end{vmatrix} = \epsilon^{-\frac{1}{2}} \{(\xi_3\zeta_1 - \xi_1\zeta_3) + \text{positive powers of } \epsilon\}\, \theta,$$

$$J_z = \begin{vmatrix} v_1, & \xi_1, & \eta_1 \\ v_2, & \xi_2, & \eta_2 \\ v_3, & \xi_3, & \eta_3 \end{vmatrix} = \epsilon^{-\frac{1}{2}} \{(\xi_3\eta_1 - \xi_1\eta_3) + \text{positive powers of } \epsilon\}\, \theta,$$

$$J_v = \begin{vmatrix} \xi_1, & \eta_1, & \zeta_1 \\ \xi_2, & \eta_2, & \zeta_2 \\ \xi_3, & \eta_3, & \zeta_3 \end{vmatrix} = \text{finite quantity when } \epsilon \text{ is small},$$

where θ stands for

$$\frac{1}{2}\left\{\frac{(d+p)^3(d+r)}{d(d-a)(d-b)(d-c)}\right\}^{\frac{1}{2}}.$$

Hence if the direction-cosines of the normal to the volume of the region are X', Y', Z', V', then, at this section,

$$V' = 0:$$

that is, the normal to the region is perpendicular to the axis of v, and the volume of the region is cut orthogonally by the flat $v = 0$.

Thus the centro-surface of the quadric, arising through $q = -d$, is a surface occurring singly in the section of the centro-region: that is, in the numerical estimate of the order of the superficial section of the region of centres, the order of the centro-surface is to be counted only once. And that order is twelve.

For (iii), when $r = -d$, we have

$$\xi^2 = \frac{(a+p)^3(a+q)}{a(a-b)(a-c)}, \quad \eta^2 = \frac{(b+p)^3(b+q)}{b(b-c)(b-a)}, \quad \zeta^2 = \frac{(c+p)^3(c+q)}{c(c-a)(c-b)}, \quad v = 0.$$

These equations also are equations of the centro-surface of the same quadric

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1, \quad v = 0.$$

As before under (ii), when $q = -d$, this centro-surface occurs singly in the section of the centro-region: that is, in the numerical estimate of the order of the superficial section of that centro-region, the order of the centro-surface is to be counted only once. And, as before, that order is twelve.

But in regard to the two occurrences of the centro-surface of the quadric, one when $q = -d$, and the other when $r = -d$, it is to be noted that the respective values of the coordinates differ. Thus, for the former and for the latter respectively,

$$\xi^2 = \frac{(a+p)^3(a+q)}{a(a-b)(a-c)}, \quad \xi^2 = \frac{(a+p)^3(a+r)}{a(a-b)(a-c)},$$

and the range of values for q is distinct from the range of values for r. Consequently, the locus described by the point with the first set of coordinates

is one of the two sheets of the double-sheeted centro-surface of the quadric, and the locus described by the point with the second set of coordinates is the other of the two sheets of that double-sheeted centro-surface. Accordingly, the two portions of the section arising out of $q=-d$ and $r=-d$ respectively are the two sheets of the centro-surface of the quadric. The total contribution, towards the order of the superficial section of the centro-region made by the part of the flat $\upsilon=0$ arising out of $q=-d$ and $r=-d$ together, is twelve, the order of the centro-surface of the quadric.

II. For (i), when $p=-d$, we have, for the portion of the section,

$$\frac{a\xi^2}{(a-d)^2}=\frac{(a+q)(a+r)}{(a-b)(a-c)}, \quad \frac{b\eta^2}{(b-d)^2}=\frac{(b+q)(b+r)}{(b-c)(b-a)},$$

$$\frac{c\zeta^2}{(c-d)^2}=\frac{(c+q)(c+r)}{(c-a)(c-b)}, \quad \upsilon=0.$$

These are the equations of the quadric

$$\frac{a\xi^2}{(a-d)^2}+\frac{b\eta^2}{(b-d)^2}+\frac{c\zeta^2}{(c-d)^2}=1$$

in the flat (the homaloidal triple space) $\upsilon=0$.

In the immediate vicinity of this flat section of the centro-region, we take

$$p+d=\epsilon;$$

where ϵ is small; and then

$$\frac{a\xi^2}{(a-d)^2}=\frac{(a+q)(a+r)}{(a-b)(a-c)}\left(1+\frac{\epsilon}{a-d}\right)^3$$

$$=\frac{(a+q)(a+r)}{(a-b)(a-c)}\left(1+3\frac{\epsilon}{a-d}\right),$$

to the first order of small quantities. Similarly, to the same order,

$$\frac{b\eta^2}{(b-d)^2}=\frac{(b+q)(b+r)}{(b-c)(b-a)}\left(1+3\frac{\epsilon}{b-d}\right),$$

$$\frac{c\zeta^2}{(c-d)^2}=\frac{(c+q)(c+r)}{(c-a)(c-b)}\left(1+3\frac{\epsilon}{c-d}\right).$$

Let T denote $(d+q)(d+r)$, for the moment: so that we have

$$\frac{a\xi^2}{(a-d)^2}(a-b)(a-c)\left(1-3\frac{\epsilon}{a-d}\right)=a^2+(q+r)a+qr,$$

$$\frac{b\eta^2}{(b-d)^2}(b-c)(b-a)\left(1-3\frac{\epsilon}{b-d}\right)=b^2+(q+r)b+qr,$$

$$\frac{c\zeta^2}{(c-d)^2}(c-a)(c-b)\left(1-3\frac{\epsilon}{c-d}\right)=c^2+(q+r)c+qr,$$

$$T=d^2+(q+r)d+qr;$$

and therefore

$$T = \frac{a\xi^2}{(a-d)^2}(b-d)(c-d)\left(1-3\frac{\epsilon}{a-d}\right) + \frac{b\eta^2}{(b-d)^2}(c-d)(a-d)\left(1-3\frac{\epsilon}{b-d}\right) + \frac{c\zeta^2}{(c-d)^2}(a-d)(b-d)\left(1-3\frac{\epsilon}{c-d}\right),$$

to the first order of small quantities. Consequently, in the assigned immediate vicinity of the part of the superficial section due to the relation $p = -d$, we have

$$\upsilon^2 = \frac{\epsilon^3}{d(a-d)(b-d)(c-d)}(d+q)(d+r) = \frac{\epsilon^3}{d}\left\{\frac{a\xi^2}{(a-d)^3} + \frac{b\eta^2}{(b-d)^3} + \frac{c\zeta^2}{(c-d)^3}\right\},$$

approximately. Now near the section by the flat $\upsilon = 0$, we have (for this part of the relation)

$$\frac{a\xi^2}{(a-d)^2\left(1+\frac{\epsilon}{a-d}\right)^3} + \frac{b\eta^2}{(b-d)^2\left(1+\frac{\epsilon}{b-d}\right)^3} + \frac{c\zeta^2}{(c-d)^2\left(1+\frac{\epsilon}{c-d}\right)^3} = 1,$$

that is, to the first order of small quantities,

$$\frac{a\xi^2}{(a-d)^2} + \frac{b\eta^2}{(b-d)^2} + \frac{c\zeta^2}{(c-d)^2} - 1 = 3\epsilon\left\{\frac{a\xi^2}{(a-d)^3} + \frac{b\eta^2}{(b-d)^3} + \frac{c\zeta^2}{(c-d)^3}\right\}.$$

Hence, in the immediate vicinity of the flat section of the centro-region made by $\upsilon = 0$, we have

$$\left\{\frac{a\xi^2}{(a-d)^2} + \frac{b\eta^2}{(b-d)^2} + \frac{c\zeta^2}{(c-d)^2} - 1\right\}^3 = 27\upsilon^2 d\left\{\frac{a\xi^2}{(a-d)^3} + \frac{b\eta^2}{(b-d)^3} + \frac{c\zeta^2}{(c-d)^3}\right\}^2,$$

accurate up to the order υ^2 inclusive.

Thus the ellipsoid

$$\frac{a\xi^2}{(a-d)^2} + \frac{b\eta^2}{(b-d)^2} + \frac{c\zeta^2}{(c-d)^2} - 1 = 0, \quad \upsilon = 0,$$

occurs triply in the flat section of the centro-region by the flat $\upsilon = 0$. As the ellipsoid, itself, is of order two, this portion of the flat section contributes six to the whole order of the flat section.

III. The other portion of the flat section of the centro-region has contributed the number twelve to the whole order of this flat section. Consequently, the aggregate order of the flat section of the centro-region is eighteen: that is, *the order of the region of centres of the ovoidal region*

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} + \frac{v^2}{d} = 1$$

is eighteen.

396. The determination of an actual Cartesian equation of the region of centres of the ovoidal region does not seem feasible by the natural extension of Cayley's method used for the centro-surface of the ellipsoid.

From the parametric equations of the centro-region, being the equations expressing the coordinates of a point in terms of three parameters, we have

$$\Sigma \frac{a\xi^2}{(a+p)^2} = \Sigma \frac{(a+p)(a+q)(a+r)}{(a-b)(a-c)(a-d)} = 1,$$

$$\Sigma \frac{a\xi^2}{(a+p)^3} = \Sigma \frac{(a+q)(a+r)}{(a-b)(a-c)(a-d)} = 0.$$

These are the conditions that the quintic equation

$$\frac{t\xi^2}{a+t} + \frac{t\eta^2}{b+t} + \frac{t\zeta^2}{c+t} + \frac{t\upsilon^2}{d+t} = t + \theta,$$

where θ is a parameter not involving t (the variable of the equation), should have a triple root, the triple root being equal to p.

When this quintic equation is taken in the form

$$t^5 + 5a_1t^4 + 10a_2t^3 + 10a_3t^2 + 5a_4t + a_5 = 0,$$

where the coefficients are

$$\left.\begin{aligned} 5a_1 &= \theta + \Sigma a - \Sigma\xi^2 \\ 10a_2 &= \theta\Sigma a + \Sigma ab - \Sigma(b+c+d)\,\xi^2 \\ 10a_3 &= \theta\Sigma ab + \Sigma abc - \Sigma(cd+db+bc)\,\xi^2 \\ 5a_4 &= \theta\Sigma abc + abcd - \Sigma bcd\,\xi^2 \\ a_5 &= \theta abcd \end{aligned}\right\},$$

the two conditions, necessary and sufficient to secure that the quintic shall have a triple root, are two conditions among these coefficients. The two conditions involve θ and the coordinates $\xi, \eta, \zeta, \upsilon$: the θ-eliminant of the two conditions is a resulting equation in $\xi, \eta, \zeta, \upsilon$, which should be the Cartesian equation of the region of centres.

The two conditions, so far as they are known at present, appear in a form that seems to be of too high a degree in θ and ξ for the immediate purpose. Thus Cayley* gives one condition, in the vanishing quartinvariant

$$LN - M^2 = 0,$$

where

$$L = a_0a_4 - 4a_1a_3 + 3a_2^2, \quad 2M = a_0a_5 - 3a_1a_4 + 2a_2a_3,$$
$$N = a_1a_5 - 4a_2a_4 + 3a_3^2,$$

and leaves another condition to be inferred from a vanishing covariant of the fourth degree. Again, Sylvester† gives equations of condition

$$\frac{Q}{P} = \frac{R}{Q} = \frac{S}{R} = \frac{M}{L} = \frac{N}{M},$$

* *Coll. Math. Papers*, vol. ii, p. 469

† *Coll. Math. Papers*, vol. i, p. 349.

where

$$P=\begin{vmatrix} a_0, & a_1, & a_2 \\ a_1, & a_2, & a_3 \\ a_2, & a_3, & a_4 \end{vmatrix}, \quad 3Q=\begin{vmatrix} a_0, & a_1, & a_3 \\ a_1, & a_2, & a_4 \\ a_2, & a_3, & a_5 \end{vmatrix},$$

$$3R=\begin{vmatrix} a_0, & a_2, & a_3 \\ a_1, & a_3, & a_4 \\ a_2, & a_4, & a_5 \end{vmatrix}, \quad S=\begin{vmatrix} a_1, & a_2, & a_3 \\ a_2, & a_3, & a_4 \\ a_3, & a_4, & a_5 \end{vmatrix}.$$

One of these is Cayley's condition of degree four; and the others are of degree five or degree six.

It is not difficult to obtain two conditions, each of degree four, in the form

$$3(a_0a_3-a_1a_2)N-(a_0a_4+8a_1a_3-9a_2^2)M+3(a_1a_4-a_2a_3)L=0,$$

$$3(a_0a_4-a_2^2)N-(a_0a_5+9a_1a_4-10a_2a_3)M+3(a_1a_5-a_3^2)L=0.$$

But when substitution for the coefficients is effected, these equations take a shape

$$\mu_0\theta^4+\mu_2\theta^3+\mu_4\theta^2+\mu_6\theta+\mu_8=0,$$

where both sets of quantities μ_n are of order n in the variables (for $n=0$, 2, 4, 6, 8). The θ-eliminant is of too high an order (thirty-two).

There does not seem a reduction of order owing to the mode of occurrence of θ in the equations; and thus it would appear as if the two conditions for the quintic should be expressible as conditions of the third degree in the coefficients. Such conditions are not known, at present.

Geodesics in the ovoid.

397. It proves possible to obtain integrated forms of the equations of geodesic curves in the ovoidal region. Because $F=0$, $G=0$, $H=0$, the values of the magnitudes Γ, Δ, Θ, are simple: we have

$$\Gamma_{11}=\frac{1}{2}\frac{A_1}{A}, \quad \Gamma_{12}=\frac{1}{2}\frac{A_2}{A}, \quad \Gamma_{13}=\frac{1}{2}\frac{A_3}{A}, \quad \Gamma_{22}=-\frac{1}{2}\frac{B_1}{A}, \quad \Gamma_{23}=0, \quad \Gamma_{33}=-\frac{1}{2}\frac{C_1}{A},$$

$$\Delta_{11}=-\frac{1}{2}\frac{A_2}{B}, \quad \Delta_{12}=\frac{1}{2}\frac{B_1}{B}, \quad \Delta_{13}=0, \quad \Delta_{22}=\frac{1}{2}\frac{B_2}{B}, \quad \Delta_{23}=\frac{1}{2}\frac{B_3}{B}, \quad \Delta_{33}=-\frac{1}{2}\frac{C_2}{B},$$

$$\Theta_{11}=-\frac{1}{2}\frac{A_3}{C}, \quad \Theta_{12}=0, \quad \Theta_{13}=\frac{1}{2}\frac{C_1}{C}, \quad \Theta_{22}=-\frac{1}{2}\frac{B_3}{C}, \quad \Theta_{23}=\frac{1}{2}\frac{C_2}{C}, \quad \Theta_{33}=\frac{1}{2}\frac{C_3}{C}.$$

Thus the general equations of geodesics (§ 298) become, for the ovoidal region,

$$\left.\begin{aligned} -2Ap'' &= A_1p'^2+2A_2p'q'+2A_3p'r'-B_1q'^2 &&-C_1r'^2 \\ -2Bq'' &= -A_2p'^2+2B_1p'q' &&+B_2q'^2+2B_3q'r'-C_2r'^2 \\ -2Cr'' &= -A_3p'^2 &&+2C_1p'r'-B_3q'^2+2C_2q'r'+C_3r'^2 \end{aligned}\right\}.$$

These equations can also be obtained directly by forming the equations characteristic for the minimising of the integral

$$\int\left\{A\left(\frac{dp}{dt}\right)^2+B\left(\frac{dq}{dt}\right)^2+C\left(\frac{dr}{dt}\right)^2\right\}^{\frac{1}{2}}dt.$$

The values of A, B, C, have to be substituted. Let

$$P=\frac{1}{4}\frac{p}{(a+p)(b+p)(c+p)(d+p)},$$

$$Q=\frac{1}{4}\frac{q}{(a+q)(b+q)(c+q)(d+q)},$$

$$R=\frac{1}{4}\frac{r}{(a+r)(b+r)(c+r)(d+r)},$$

so that

$$A=P(p-q)(p-r),\quad B=Q(q-r)(q-p),\quad C=R(r-p)(r-q),$$

and therefore

$$\left.\begin{aligned}\frac{A_1}{A}&=\frac{P_1}{P}+\frac{1}{p-q}+\frac{1}{p-r}\\ \frac{A_2}{A}&=\frac{1}{q-p}\\ \frac{A_3}{A}&=\frac{1}{r-p}\end{aligned}\right\},\quad\left.\begin{aligned}\frac{B_1}{B}&=\frac{1}{p-q}\\ \frac{B_2}{B}&=\frac{Q_2}{Q}+\frac{1}{q-r}+\frac{1}{q-p}\\ \frac{B_3}{B}&=\frac{1}{r-q}\end{aligned}\right\},\quad\left.\begin{aligned}\frac{C_1}{C}&=\frac{1}{p-r}\\ \frac{C_2}{C}&=\frac{1}{q-r}\\ \frac{C_3}{C}&=\frac{R_3}{R}+\frac{1}{r-p}+\frac{1}{r-q}\end{aligned}\right\}.$$

To obtain integrals of the geodesic equations after these values of the coefficients have been inserted, we introduce three new variables (initially as to their ratios alone) defined by the relations

$$\frac{Pp'^2}{\xi}=\frac{Qq'^2}{\eta}=\frac{Rr'^2}{\zeta},$$

where ξ, η, ζ, are taken to be functions solely of p, q, r. Then the foregoing geodesic equations become

$$-2\frac{p''}{p'}-\frac{P_1}{P}p'=\frac{p'-2q'}{p-q}+\frac{p'-2r'}{p-r}-p'\left(\frac{1}{p-r}-\frac{1}{p-q}\right)\frac{\eta-\zeta}{\xi},$$

$$-2\frac{q''}{q'}-\frac{Q_2}{Q}q'=\frac{q'-2r'}{q-r}+\frac{q'-2p'}{q-p}-q'\left(\frac{1}{q-p}-\frac{1}{q-r}\right)\frac{\zeta-\xi}{\eta},$$

$$-2\frac{r''}{r'}-\frac{R_3}{R}r'=\frac{r'-2p'}{r-p}+\frac{r'-2q'}{r-q}-r'\left(\frac{1}{r-q}-\frac{1}{r-p}\right)\frac{\xi-\eta}{\zeta}.$$

Now from the equations defining ξ, η, ζ, we have

$$\begin{aligned}\frac{\xi'}{\xi}-\frac{\eta'}{\eta}&=2\frac{p''}{p'}+\frac{P_1}{P}p'-2\frac{q''}{q'}-\frac{Q_2}{Q}q'\\ &=\frac{p'+q'}{p-q}+\frac{q'-2r'}{q-r}+\frac{p'-2r'}{r-p}\\ &\quad-p'\left(\frac{1}{r-p}-\frac{1}{q-p}\right)\frac{\eta-\zeta}{\xi}-q'\left(\frac{1}{q-p}-\frac{1}{q-r}\right)\frac{\zeta-\xi}{\eta};\end{aligned}$$

and there are corresponding expressions for

$$\frac{\eta'}{\eta}-\frac{\zeta'}{\zeta},\quad\frac{\zeta'}{\zeta}-\frac{\xi'}{\xi}.$$

The quantities ξ, η, ζ, do not involve p', q', r', and are functions of p, q, r; hence

$$\frac{\xi'}{\xi} = \frac{1}{\xi}\frac{\partial \xi}{\partial p}p' + \frac{1}{\xi}\frac{\partial \xi}{\partial q}q' + \frac{1}{\xi}\frac{\partial \xi}{\partial r}r',$$

and so for η and for ζ. When these values are substituted, all the three equations become homogeneous and linear in p', q', r'; and all of them will be satisfied if, simultaneously in each, the coefficients of p', q', r', are respectively equal.

Comparing the coefficients of r' in the two expressions for $\frac{\xi'}{\xi} - \frac{\eta'}{\eta}$, we find

$$\frac{1}{\xi}\frac{\partial \xi}{\partial r} - \frac{1}{\eta}\frac{\partial \eta}{\partial r} = -\frac{2}{q-r} - \frac{2}{r-p};$$

and therefore

$$\frac{\xi (q-r)^{-2}}{\eta (r-p)^{-2}} = \text{a function of } p \text{ and } q \text{ alone, } = \theta(p, q).$$

Similarly, a comparison of the coefficients of p' in the expression for $\frac{\eta'}{\eta} - \frac{\zeta'}{\zeta}$ leads to the relation

$$\frac{\eta (r-p)^{-2}}{\zeta (p-q)^{-2}} = \text{a function of } q \text{ and } r \text{ alone, } = \phi(q, r);$$

and a comparison of the coefficients of q' in the expression for $\frac{\zeta'}{\zeta} - \frac{\xi'}{\xi}$ leads to the relation

$$\frac{\zeta (p-q)^{-2}}{\xi (q-r)^{-2}} = \text{a function of } r \text{ and } p \text{ alone, } = \psi(r, p).$$

Thus

$$\theta(p, q)\phi(q, r)\psi(r, p) = 1,$$

while p, q, r, are independent variables; hence there are functions, of p alone, of q alone, and of r alone, such that

$$\theta(p, q) = \frac{f(p)}{g(q)}, \quad \phi(q, r) = \frac{g(q)}{h(r)}, \quad \psi(r, p) = \frac{h(r)}{f(p)}.$$

As, hitherto, we have been concerned solely with the ratios of ξ, η, ζ, all the requirements are met by taking

$$\xi = (q-r)^2 f(p), \quad \eta = (r-p)^2 g(q), \quad \zeta = (p-q)^2 h(r).$$

In obtaining these forms, the coefficients of only one of the three quantities p', q', r', have been compared in each instance. Returning now to the equation for $\frac{\xi'}{\xi} - \frac{\eta'}{\eta}$, and comparing the coefficients of p', we have

$$\frac{1}{\xi}\frac{\partial \xi}{\partial p} - \frac{1}{\eta}\frac{\partial \eta}{\partial p} = \frac{1}{p-q} + \frac{1}{r-p} - \left(\frac{1}{r-p} - \frac{1}{q-p}\right)\frac{\eta - \zeta}{\xi},$$

which, on the substitution of the values of ξ, η, ζ, just obtained, becomes

$$\frac{f'(p)}{f(p)} = \frac{1}{p-q} + \frac{1}{p-r} - \left(\frac{1}{r-p} - \frac{1}{q-p}\right)\frac{\eta - \zeta}{\xi},$$

and therefore

$$(p-r)(p-q)f'(p) = (2p-q-r)f(p) - \frac{1}{q-r}\{(p-r)^2 g(q) - (p-q)^2 h(r)\}.$$

Let this equation be differentiated with respect to p: it yields

$$(p-r)(p-q)f''(p) = 2f(p) - \frac{2}{q-r}\{(p-r)g(q) - (p-q)h(r)\},$$

and therefore

$$-(p-q)(q-r)(r-p)f'''(p) = 2\{(q-r)f(p) + (r-p)g(q) + (p-q)h(r)\}.$$

Proceeding similarly from the corresponding coefficients in the other two equations, we find the same value for

$$-(q-r)(r-p)(p-q)g''(q), \quad -(r-p)(p-q)(q-r)h''(r),$$

respectively; and therefore

$$f'''(p) = g''(q) = h''(r),$$

so that each of these three equal quantities is constant. Let this constant be 2γ; then

$$f(p) = \gamma p^2 + 2\alpha p + \beta, \quad g(q) = \gamma q^2 + 2\alpha' q + \beta', \quad h(r) = \gamma r^2 + 2\alpha'' r + \beta''.$$

When these values are substituted in the foregoing expression for $f'''(p)$, the relation

$$(q-r)(2\alpha p + \beta) + (r-p)(2\alpha' q + \beta') + (p-q)(2\alpha'' r + \beta'') = 0$$

must be satisfied; hence

$$\alpha = \alpha' = \alpha'', \quad \beta = \beta' = \beta''.$$

Moreover, γ can be made unity without loss of generality; and then α and β are two arbitrary constants of integration. Hence

$$\left.\begin{aligned} \xi &= (q-r)^2 f(p) = (q-r)^2 (p+\kappa)(p+\lambda) \\ \eta &= (r-p)^2 g(q) = (r-p)^2 (q+\kappa)(q+\lambda) \\ \zeta &= (p-q)^2 h(r) = (p-q)^2 (r+\kappa)(r+\lambda) \end{aligned}\right\},$$

where now κ and λ, in place of α and β, are the arbitrary constants of integration.

Continuing, we write

$$S(t) = t(a+t)(b+t)(c+t)(d+t)(\kappa+t)(\lambda+t),$$

for $t = p, q, r$, successively: then

$$\frac{Pp'^2}{\xi} = \frac{p^2 p'^2}{(q-r)^2 S(p)}, \quad \frac{Qq'^2}{\eta} = \frac{q^2 q'^2}{(r-p)^2 S(q)}, \quad \frac{Rr'^2}{\zeta} = \frac{r^2 r'^2}{(p-q)^2 S(r)}.$$

These three quantities are equal to one another; hence we can take

$$\frac{p'}{\{S(p)\}^{\frac{1}{2}}} = \frac{q-r}{p}\Omega, \quad \frac{q'}{\{S(q)\}^{\frac{1}{2}}} = \frac{r-p}{q}\Omega, \quad \frac{r'}{\{S(r)\}^{\frac{1}{2}}} = \frac{p-q}{r}\Omega.$$

Thus
$$\frac{p\,dp}{2\{S(p)\}^{\frac{1}{2}}}+\frac{q\,dq}{2\{S(q)\}^{\frac{1}{2}}}+\frac{r\,dr}{2\{S(r)\}^{\frac{1}{2}}}=0,$$
$$\frac{p^2dp}{2\{S(p)\}^{\frac{1}{2}}}+\frac{q^2dq}{2\{S(q)\}^{\frac{1}{2}}}+\frac{r^2dr}{2\{S(r)\}^{\frac{1}{2}}}=0.$$

Again, we have
$$\begin{aligned}4ds^2&=4(A\,dp^2+B\,dq^2+C\,dr^2)\\&=\Sigma\frac{p(p-q)(p-r)}{(a+p)(b+p)(c+p)(d+p)}\,dp^2\\&=\Sigma\frac{p(p-q)(p-r)}{(a+p)(b+p)(c+p)(d+p)}\,S(p)(q-r)^2\frac{\overline{\Omega}^2}{p^2}\,ds^2\\&=-\overline{\Omega}^2(q-r)(r-p)(p-q)\{\Sigma(q-r)(\kappa+p)(\lambda+p)\}\,ds^2\\&=\overline{\Omega}^2(q-r)^2(r-p)^2(p-q)^2\,ds^2,\end{aligned}$$
so that we can take
$$\overline{\Omega}=\frac{2}{(q-r)(r-p)(p-q)}.$$

Finally, we have
$$\Sigma\frac{p^3p'}{2\{S(p)\}^{\frac{1}{2}}}=\frac{1}{2}\Sigma p^2(q-r)\overline{\Omega}=-1,$$
so that
$$\frac{p^3dp}{2\{S(p)\}^{\frac{1}{2}}}+\frac{q^3dq}{2\{S(q)\}^{\frac{1}{2}}}+\frac{r^3dr}{2\{S(r)\}^{\frac{1}{2}}}=-ds.$$

It thus appears that, for the integrated expression of geodesics in the ovoidal range, we require a restricted type* of Abelian functions of genus three. In order to secure reality for the geodesics, the quantities $S(p)$, $S(q)$, $S(r)$, must be positive; hence, with the assumptions
$$a>b>c>d>0,$$
we must have
$$(p+\kappa)(p+\lambda)>0,\quad(q+\kappa)(q+\lambda)<0,\quad(r+\kappa)(r+\lambda)>0,$$
so that
$$p>-\kappa>q>-\lambda>r.$$
Thus κ and c both lie between $-p$ and $-q$; we can have three cases, $\kappa>c$, $\kappa=c$, $\kappa<c$. Also λ and b both lie between $-q$ and $-r$; we can have three cases, $\lambda>b$, $\lambda=b$, $\lambda<b$. Consequently for the detailed delineation of geodesic curves in the ovoidal region, nine cases† arise for discussion.

Ex. Prove that, for the ovoidal region, the relation connecting the parameters and the element of arc along the geodesic can be taken in the form
$$\frac{dp}{2\{S(p)\}^{\frac{1}{2}}}+\frac{dq}{2\{S(q)\}^{\frac{1}{2}}}+\frac{dr}{2\{S(r)\}^{\frac{1}{2}}}=-\frac{ds}{pqr}.$$

* This type of Abelian functions was discussed by Weierstrass, *Crelle*, vol. lii (1856), pp. 285–339; *Ges. Werke*, vol. i, pp. 297–355; for functions of genus n. They do not constitute the most general type of Abelian functions of genus 3, or n, respectively.

† They are the natural extension of the three different kinds of geodesics that exist on the surface of an ellipsoid in homaloidal triple space: see my *Lectures on Differential Geometry*, § 101.

398. The integrated equations, connected with the march of a geodesic, can be exhibited in another form, which does not give explicit expressions for p, q, r, (and therefore, implicitly, for x, y, z, v) as periodic functions of the arc of the curve, but leaves only algebraic relations connecting p, q, r, p', q', r'. We have proved that

$$p(p-q)(p-r)p' = 2\{S(p)\}^{\frac{1}{2}},$$

where

$$S(p) = p(a+p)(b+p)(c+p)(d+p)(\kappa+p)(\lambda+p),$$

with corresponding values for q' and for r'; and also

$$A = \frac{1}{4}\frac{p(p-q)(p-r)}{(a+p)(b+p)(c+p)(d+p)}.$$

Hence

$$Ap'^2 = \frac{(\kappa+p)(\lambda+p)}{(p-q)(p-r)}, \quad Bq'^2 = \frac{(\kappa+q)(\lambda+q)}{(q-r)(q-p)}, \quad Cr'^2 = \frac{(\kappa+r)(\lambda+r)}{(r-p)(r-q)};$$

consequently

$$\left.\begin{aligned} Ap'^2 + Bq'^2 + Cr'^2 &= 1 \\ (q+r)Ap'^2 + (r+p)Bq'^2 + (p+q)Cr'^2 &= -(\kappa+\lambda) \\ qrAp'^2 + rpBq'^2 + pqCr'^2 &= \kappa\lambda \end{aligned}\right\},$$

symmetrical equations representing the first integrals of the general equations of ovoidal geodesics. They express, as the earlier equations express, p', q', r', as algebraic functions of p, q, r; the final equations, to express p, q, r, as functions of a parameter (taken to be the arc of the geodesic) are sextuply periodic functions of three arguments, two of which are constants along a geodesic and vary from one geodesic to another.

Conversely, the value of $\kappa\lambda$ and the value of $\kappa+\lambda$ for a geodesic can be regarded as determinate when the direction of a geodesic through a given point is assigned.

Curvature, torsion, tilt, of ovoidal geodesics.

399. The curvatures, of plane curvature, of torsion, and of tilt, of a geodesic acquire comparatively simple forms when expressed in terms of the parameters of the ovoidal region.

We have, always,

$$\frac{1}{\rho} = Lp'^2 + Nq'^2 + Jr'^2;$$

and therefore, in the ovoidal region,

$$\begin{aligned} \frac{1}{\rho} &= -\left(-\frac{abcd}{pqr}\right)^{\frac{1}{2}}\left(\frac{A}{p}p'^2 + \frac{B}{q}q'^2 + \frac{C}{r}r'^2\right) \\ &= -\left(-\frac{abcd}{p^3q^3r^3}\right)^{\frac{1}{2}}(qrAp'^2 + rpBq'^2 + pqCr'^2) \\ &= -\left(-\frac{abcd}{p^3q^3r^3}\right)^{\frac{1}{2}}\kappa\lambda, \end{aligned}$$

where $\kappa\lambda$ is a constant along the geodesic. We thus have the plane curvature of the geodesic; and its value agrees with the result already (§ 390) obtained.

Again, we have generally

$$\Omega^2\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)=\mathbf{D},$$

where $\mathbf{D}$ is the covariant as given in § 306; and therefore, when the region is referred to lines of curvature as parametric curves,

$$ABC\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)=BCL^2p'^2+CAN^2q'^2+ABJ^2r'^2.$$

Hence

$$\begin{aligned}\frac{1}{\rho^2}+\frac{1}{\sigma^2}&=\frac{L^2}{A}\quad p'^2+\frac{N^2}{B}\quad q'^2+\frac{J^2}{C}\quad r'^2\\&=\frac{L^2}{A^2}Ap'^2+\frac{N^2}{B^2}Bq'^2+\frac{J^2}{C^2}Cr'^2.\end{aligned}$$

When the region is ovoidal, we have

$$\frac{L}{A}=-\left(-\frac{abcd}{pqr}\right)^{\frac{1}{2}}\frac{1}{p},$$

with corresponding values for $\frac{N}{B}$ and $\frac{J}{C}$; thus

$$\begin{aligned}\frac{1}{\rho^2}+\frac{1}{\sigma^2}&=-\frac{abcd}{pqr}\left\{A\frac{p'^2}{p^2}+B\frac{q'^2}{q^2}+C\frac{r'^2}{r^2}\right\}\\&=-\frac{abcd}{pqr}\Sigma\frac{(\kappa+p)(\lambda+p)}{p^2(p-r)(p-q)}\\&=-\frac{abcd}{pqr}\left\{\frac{\kappa\lambda}{p^2q^2r^2}(pq+qr+rp)+\frac{\kappa+\lambda}{pqr}\right\},\end{aligned}$$

and therefore

$$\frac{1}{\sigma^2}=\frac{abcd}{p^3q^3r^3}\{\kappa^2\lambda^2-\kappa\lambda(pq+qr+rp)-(\kappa+\lambda)pqr\}.$$

We thus have the torsion of a geodesic at any point.

It is easy to verify, from this result, the known property (§ 307) that the torsion of a geodesic in a region vanishes where it touches a line of curvature. In the present instance, for a geodesic touching the line $q'=0$, $r'=0$, we have

$$q+r=-(\kappa+\lambda),\quad qr=\kappa\lambda,$$

by § 398; these values make $1/\sigma^2$ vanish for all values of p.

Again, it has been proved (§ 309) that, in any region, the tilt of a geodesic is given by

$$\frac{\Omega}{\sigma^2\tau}=\frac{1}{12}J\left(\frac{\overline{U},\ \overline{V},\ \overline{W}}{p',\ q',\ r'}\right),$$

where J is the Jacobian of the three ternary forms

$$\overline{U} = (A\,\mathfrak{Y}\,p', q', r')^2, \quad \overline{V} = (L\,\mathfrak{Y}\,p', q', r')^2, \quad \overline{W} = (e\,\mathfrak{Y}\,p', q', r')^2;$$

and therefore, in the ovoidal region, we have

$$\frac{\Omega}{\sigma^2\tau} = \frac{1}{3}\begin{vmatrix} \dfrac{\partial \overline{W}}{\partial p'}, & \dfrac{\partial \overline{W}}{\partial q'}, & \dfrac{\partial \overline{W}}{\partial r'} \\ Ap', & Bq', & Cr' \\ Lp', & Nq', & Jr' \end{vmatrix}.$$

Now

$$\begin{aligned} \frac{1}{3}\frac{\partial \overline{W}}{\partial p'} &= e_{300}p'^2 + 2e_{210}p'q' + 2e_{201}p'r' + e_{120}q'^2 + 2e_{111}q'r' + e_{102}r'^2 \\ &= -\frac{3}{2p}Lp'^2 - \frac{1}{q}Lp'q' - \frac{1}{r}Lp'r' - \frac{1}{2p}Nq'^2 - \frac{1}{2p}Jr'^2 \\ &= -\frac{1}{2p}(Lp'^2 + Nq'^2 + Jr'^2) - Lp'\left(\frac{p'}{p} + \frac{q'}{q} + \frac{r'}{r}\right) \\ &= -\frac{1}{2p\rho} - \theta Lp', \end{aligned}$$

where

$$\theta = \frac{p'}{p} + \frac{q'}{q} + \frac{r'}{r};$$

and similarly for the other two derivatives of $\overline{W}$. Consequently

$$-2\Omega\frac{\rho}{\sigma^2\tau} = \begin{vmatrix} \dfrac{1}{p}, & \dfrac{1}{q}, & \dfrac{1}{r} \\ Ap', & Bq', & Cr' \\ Lp', & Nq', & Jr' \end{vmatrix}.$$

Let

$$\frac{\overline{P}}{p} = \frac{(p+\kappa)(p+\lambda)}{(a+p)(b+p)(c+p)(d+p)}, \quad \frac{\overline{Q}}{q} = \frac{(q+\kappa)(q+\lambda)}{(a+q)(b+q)(c+q)(d+q)},$$

$$\frac{\overline{R}}{r} = \frac{(r+\kappa)(r+\lambda)}{(a+r)(b+r)(c+r)(d+r)};$$

then

$$2Ap' = -(\overline{P})^{\frac{1}{2}}, \quad 2Bq' = -(\overline{Q})^{\frac{1}{2}}, \quad 2Cr' = -(\overline{R})^{\frac{1}{2}}.$$

Substituting

$$\frac{L}{A} = -\left(-\frac{abcd}{pqr}\right)^{\frac{1}{2}}\frac{1}{p}, \quad \frac{N}{B} = -\left(-\frac{abcd}{pqr}\right)^{\frac{1}{2}}\frac{1}{q}, \quad \frac{J}{C} = -\left(-\frac{abcd}{pqr}\right)^{\frac{1}{2}}\frac{1}{r},$$

inserting the value of ρ, and reducing, we find

$$\frac{1}{\sigma^2\tau} = \frac{abcd\kappa\lambda}{8p^3q^3r^3(ABC)^{\frac{1}{2}}}\{(\overline{Q}\overline{R})^{\frac{1}{2}}(q-r) + (\overline{R}\overline{P})^{\frac{1}{2}}(r-p) + (\overline{P}\overline{Q})^{\frac{1}{2}}(p-q)\}.$$

The value of the torsion of the geodesic has been found; we thus have an expression for the tilt of the geodesic in the ovoidal region.

Parametric surfaces in the ovoidal region: curvatures of geodesics.

400. The curvatures of geodesics on any assigned surface $\theta(p,q,r)=$ constant, contained in any region which is represented parametrically by p, q, r, have already (Chapter XX) been obtained. The expression of such curvatures, when the region is ovoidal, becomes merely a matter of direct algebraic substitution.

But in an ovoidal region, there are three particular surfaces which have a close organic relation with one another and bear a special significance towards the region. Analytically, they have the importance of being the parametric surfaces for the region. These surfaces arise as the intersection of the central ovoid with the three different regions which are confocal with that ovoid. The various intrinsic magnitudes may therefore be noted; it will be sufficient to select, for this purpose, the confocal region $r=$ constant, so that the equations of the surface are

$$\Sigma \frac{x^2}{a} = 1, \quad \Sigma \frac{x^2}{a+r} = 1.$$

The values for the other two similar surfaces are, of course, derivable merely by appropriate changes of magnitudes for the interchanges of p, q, r.

Accordingly, for this surface, we take the function $\theta(p, q, r)$ to be $r-k$, where k is a parametric constant; and then

$$\theta_1 = 0, \quad \theta_2 = 0, \quad \theta_3 = 1, \quad \theta_{lm} = 0,$$

for $l, m, = 1, 2, 3$. Hence, with the values of Γ, Δ, Θ, as given in § 390, we have

$$\gamma_{11} = -\Theta_{11} = \frac{1}{2}\frac{A_3}{C}, \quad \gamma_{23} = -\Theta_{23} = -\frac{1}{2}\frac{C_2}{C},$$

$$\gamma_{22} = -\Theta_{22} = \frac{1}{2}\frac{B_3}{C}, \quad \gamma_{31} = -\Theta_{31} = -\frac{1}{2}\frac{C_1}{C},$$

$$\gamma_{33} = -\Theta_{33} = -\frac{1}{2}\frac{C_3}{C}, \quad \gamma_{12} = -\Theta_{12} = 0.$$

Accordingly, for the present instance, the quantities $\bar{\gamma}_{11}$, $\bar{\gamma}_{12}$, $\bar{\gamma}_{22}$, of § 357 become

$$\bar{\gamma}_{11} = \frac{1}{2}\frac{A_3}{C}, \quad \bar{\gamma}_{12} = 0, \quad \bar{\gamma}_{22} = \frac{1}{2}\frac{B_3}{C}.$$

Also, the concomitant D is

$$D = (a, b, c, f, g, h)(\theta_1, \theta_2, \theta_3)^2 = AB,$$

and $\Omega^2 = ABC$, so that

$$\frac{\Omega}{D^{\frac{1}{2}}} = C^{\frac{1}{2}}.$$

Thus the regional geodesic curvature of any geodesic on the surface $r=$ constant is given by

$$-\frac{1}{\gamma} = \frac{1}{2}\frac{A_3}{C^{\frac{1}{2}}}p'^2 + \frac{1}{2}\frac{B_3}{C^{\frac{1}{2}}}q'^2,$$

while

$$Ap'^2 + Bq'^2 = 1.$$

The principal lines of geodesic curvature on the surface are two directions of the lines of curvature of the ovoid which lie in the surface. The two principal radii, γ_1 and γ_2, of this regional geodesic curvature are

$$\frac{1}{\gamma_1} = -\frac{1}{2}\frac{A_3}{AC^{\frac{1}{2}}}, \quad \frac{1}{\gamma_2} = -\frac{1}{2}\frac{B_3}{BC^{\frac{1}{2}}};$$

and therefore

$$\frac{1}{\gamma} = \frac{1}{\gamma_1}Ap'^2 + \frac{1}{\gamma_2}Bq'^2.$$

For the circular curvature of the ovoidal geodesics touching the surface the quantities $\bar{L}$, $\bar{M}$, $\bar{N}$, of § 355 are

$$\bar{L} = L, \quad \bar{M} = 0, \quad \bar{N} = N;$$

hence the circular curvature of an ovoidal geodesic, whose direction touches the surface $r =$ constant, is given by

$$\frac{1}{\rho} = Lp'^2 + Nq'^2.$$

Thus the principal directions for circular curvature of ovoidal geodesics touching the surface, $r =$ constant, are the two lines of curvature of the ovoidal region lying in the surface. The principal radii, ρ_1 and ρ_2, of this circular curvature are L/A and N/B; and therefore

$$\frac{1}{\rho} = \frac{1}{\rho_1}Ap'^2 + \frac{1}{\rho_2}Bq'^2.$$

The principal directions of the preceding regional geodesic curvature of superficial geodesics, and the principal directions of the preceding circular curvature of regional geodesics touching the surface, are therefore the same. Thus the inclination i of two of these directions is zero: it is easy to verify that, in this instance, the equation in § 362 is satisfied by $i = 0$. Hence, as will be established independently in § 402, the invariant $\overline{\Pi}$ of § 364 vanishes.

Orthogonal radius of a parametric surface: lines of curvature.

401. The circular curvature $1/\rho_0$ of the superficial geodesic through the direction p', q', on the foregoing parametric surface, is given (§ 346) by

$$\frac{1}{\rho_0^2} = \frac{1}{\rho^2} + \frac{1}{\gamma^2}$$

$$= A^2\left(\frac{1}{\rho_1^2} + \frac{1}{\gamma_1^2}\right)p'^4 + 2AB\left(\frac{1}{\rho_1\rho_2} + \frac{1}{\gamma_1\gamma_2}\right)p'^2q'^2 + B^2\left(\frac{1}{\rho_2^2} + \frac{1}{\gamma_2^2}\right)q'^4.$$

The maximum and minimum values of ρ_0 and also the corresponding principal directions on the surface, are given by the equations

$$\left.\begin{aligned} A^2\left(\frac{1}{\rho_1^2}+\frac{1}{\gamma_1^2}\right)p'^3 \quad &+AB\left(\frac{1}{\rho_1\rho_2}+\frac{1}{\gamma_1\gamma_2}\right)p'q'^2=\frac{1}{\rho_0^2}Ap' \\ AB\left(\frac{1}{\rho_1\rho_2}+\frac{1}{\gamma_1\gamma_2}\right)p'^2q'+\ B^2\left(\frac{1}{\rho_2^2}+\frac{1}{\gamma_2^2}\right)\ q'^3&=\frac{1}{\rho_0^2}Bq' \end{aligned}\right\}.$$

For one of the directions of the lines of curvature on the surface, we clearly have $p'=0$, together with

$$\frac{1}{\rho_0^2}=\frac{1}{\rho_2^2}+\frac{1}{\gamma_2^2};$$

this is one of the lines of curvature of the ovoidal region, lying in the surface. Another of these directions is $q'=0$, together with

$$\frac{1}{\rho_0^2}=\frac{1}{\rho_1^2}+\frac{1}{\gamma_1^2};$$

this is the other of the lines of curvature of the ovoidal region, lying in the surface. The other two directions are given by

$$\left.\begin{aligned} A\left(\frac{1}{\rho_1^2}+\frac{1}{\gamma_1^2}\right)p'^2+B\left(\frac{1}{\rho_1\rho_2}+\frac{1}{\gamma_1\gamma_2}\right)q'^2&=\frac{1}{\rho_0^2} \\ A\left(\frac{1}{\rho_1\rho_2}+\frac{1}{\gamma_1\gamma_2}\right)p'^2+B\left(\frac{1}{\rho_2^2}+\frac{1}{\gamma_2^2}\right)q'^2&=\frac{1}{\rho_0^2} \end{aligned}\right\}.$$

Consequently these two directions are given by

$$A\left(\frac{1}{\rho_1^2}+\frac{1}{\gamma_1^2}-\frac{1}{\rho_1\rho_2}-\frac{1}{\gamma_1\gamma_2}\right)p'^2+B\left(\frac{1}{\rho_1\rho_2}+\frac{1}{\gamma_1\gamma_2}-\frac{1}{\rho_2^2}-\frac{1}{\gamma_2^2}\right)q'^2=0:$$

that is, the two directions are equally inclined to the lines of curvature of the ovoidal region which lie upon the surface. It thus is to be expected that the two values of $1/\rho_0^2$ for these directions will be one and the same.

To obtain this (repeated) principal value of $1/\rho_0^2$, we associate the relation

$$Ap'^2+Bq'^2=1$$

with the preceding equations for the other two directions. On eliminating Ap'^2 and Bq'^2, we have

$$\begin{vmatrix} \frac{1}{\rho_1^2}+\frac{1}{\gamma_1^2}, & \frac{1}{\rho_1\rho_2}+\frac{1}{\gamma_1\gamma_2}, & \frac{1}{\rho_0^2} \\ \frac{1}{\rho_1\rho_2}+\frac{1}{\gamma_1\gamma_2}, & \frac{1}{\rho_2^2}+\frac{1}{\gamma_2^2}, & \frac{1}{\rho_0^2} \\ 1, & 1, & 1 \end{vmatrix}=0,$$

leading to the single value

$$\frac{1}{\rho_0^2}\left\{\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)^2+\left(\frac{1}{\gamma_1}-\frac{1}{\gamma_2}\right)^2\right\}=\left(\frac{1}{\rho_1\gamma_2}-\frac{1}{\gamma_1\rho_2}\right)^2.$$

These magnitudes of the principal values of the radius of curvature ρ_0 can be obtained, as in § 366, by equating to zero the discriminant of the binary quartic

$$\mathbf{a}p'^4 + 6\mathbf{c}p'^2q'^2 + \mathbf{e}q'^4,$$

where

$$\mathbf{a} = A^2 \left(\frac{1}{\rho_1^2} + \frac{1}{\gamma_1^2} - \frac{1}{\rho_0^2}\right),$$

$$3\mathbf{c} = AB\left(\frac{1}{\rho_1\rho_2} + \frac{1}{\gamma_1\gamma_2} - \frac{1}{\rho_0^2}\right),$$

$$\mathbf{e} = B^2 \left(\frac{1}{\rho_2^2} + \frac{1}{\gamma_2^2} - \frac{1}{\rho_0^2}\right).$$

The quadrinvariant I and the cubinvariant J are

$$I = \mathbf{ae} + 3\mathbf{c}^2, \quad J = \mathbf{ace} - \mathbf{e}^3;$$

and so the discriminant is

$$\Delta = I^3 - 27J^2$$
$$= \mathbf{ae}(\mathbf{ae} - 9\mathbf{c}^2)^2,$$

which should provide four roots, corresponding to the four lines of curvature.

The root, arising out of $\mathbf{a} = 0$, is

$$\frac{1}{\rho_0^2} = \frac{1}{\rho_1^2} + \frac{1}{\gamma_1^2},$$

corresponding to the line of curvature $q' = 0$; and the root, arising out of $\mathbf{e} = 0$, is

$$\frac{1}{\rho_0^2} = \frac{1}{\rho_2^2} + \frac{1}{\gamma_2^2},$$

corresponding to the line of curvature $p' = 0$. The repeated root arising out of the squared factor is, as before,

$$\frac{1}{\rho_0^2}\left\{\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^2 + \left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)^2\right\} = \left(\frac{1}{\rho_1\gamma_2} - \frac{1}{\gamma_1\rho_2}\right)^2;$$

and it provides the value of $1/\rho_0^2$ which is the same for the two lines of curvature that are equally inclined to the line $q' = 0$ (and therefore to the line $p' = 0$).

Characteristic conic of a parametric surface is a point.

402. The characteristic conic, which generally is the locus of the orthogonal centres, becomes degenerate for these parametric surfaces in the ovoidal regions.

Its expression depends, in part, upon the value of the invariant which (§ 364) has been denoted by $\overline{\Pi}$. We have

$$\bar{E} = A, \quad \bar{F} = 0, \quad \bar{G} = B;$$
$$\bar{L} = L, \quad \bar{M} = 0, \quad \bar{N} = N;$$
$$\bar{\gamma}_{11} = \frac{1}{2}\frac{A_3}{C}, \quad \bar{\gamma}_{12} = 0, \quad \bar{\gamma}_{22} = \frac{1}{2}\frac{B_3}{C};$$

and therefore

$$\overline{\Pi} = \begin{vmatrix} \bar{E}, & \bar{F}, & \bar{G} \\ \bar{L}, & \bar{M}, & \bar{N} \\ \bar{\gamma}_{11}, & \bar{\gamma}_{12}, & \bar{\gamma}_{22} \end{vmatrix} = 0.$$

Consequently the coordinates of the orthogonal centre, x_0, y_0, in the orthogonal plane, referred to the former axes, now (§ 372) are given by equations

$$\bar{L}(x_0 - x) - \frac{\Omega}{D^{\frac{1}{2}}}\bar{\gamma}_{11}(y_0 - y) - \bar{E} = 0,$$

$$\bar{N}(x_0 - x) - \frac{\Omega}{D^{\frac{1}{2}}}\bar{\gamma}_{22}(y_0 - y) - \bar{G} = 0,$$

that is, by

$$\left.\begin{aligned} \frac{1}{\rho_1}(x_0 - x) + \frac{1}{\gamma_1}(y_0 - y) &= 1 \\ \frac{1}{\rho_2}(x_0 - x) + \frac{1}{\gamma_2}(y_0 - y) &= 1 \end{aligned}\right\}.$$

The orthogonal radius P of § 369, being $\{(x_0 - x)^2 + (y_0 - y)^2\}^{\frac{1}{2}}$, is given by

$$\frac{1}{\mathrm{P}^2}\left\{\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^2 + \left(\frac{1}{\gamma_1} - \frac{1}{\gamma_2}\right)^2\right\} = \left(\frac{1}{\rho_1\gamma_2} - \frac{1}{\gamma_1\rho_2}\right)^2.$$

Thus the characteristic conic is a point for the parametric surface r = constant.

The characteristic conic is a point for each member of each of the three families of parametric surfaces in the ovoidal region.

Integral equations of geodesics on a parametric surface.

403. Usually geodesics are taken as determined by the general equations of the second order, which are derived by the (Euler) characteristic test according to the calculus of variations. It has, however, proved possible to obtain an integral equivalent of the differential equations of geodesics in the ovoidal region; and it proves similarly possible to obtain an integral equivalent of the differential equations of geodesics on any parametric surface in the same region.

The general equations of geodesics on a surface enclosed in a region have been obtained in § 344. For the parametric surface, r = constant, in the ovoidal region, the first two of those three equations become

$$p'' + (\Gamma_{11}, \Gamma_{12}, \Gamma_{22}\mathfrak{X}p', q')^2 = 0,$$

$$q'' + (\Delta_{11}, \Delta_{12}, \Delta_{22}\mathfrak{X}p', q')^2 = 0,$$

while the third equation becomes an identity; and the values of the coefficients are (§ 390)

$$\left.\begin{aligned} \Gamma_{11} &= \frac{1}{2}\frac{A_1}{A}, \quad \Gamma_{12} = \frac{1}{2}\frac{A_2}{A}, \quad \Gamma_{22} = -\frac{1}{2}\frac{B_1}{A} \\ \Delta_{11} &= -\frac{1}{2}\frac{A_2}{B}, \quad \Delta_{12} = \frac{1}{2}\frac{B_1}{B}, \quad \Delta_{22} = \frac{1}{2}\frac{B_2}{B} \end{aligned}\right\}.$$

In the present instance, we have

$$\left.\begin{aligned} A &= \frac{1}{4}\frac{p(p-r)}{(a+p)(b+p)(c+p)(d+p)}(p-q) = P(p-q) \\ B &= \frac{1}{4}\frac{q(q-r)}{(a+q)(b+q)(c+q)(d+q)}(q-p) = Q(q-p) \end{aligned}\right\},$$

the parameter r being constant for the selected parametric surface. The arc-identity on the surface, which now is

$$Ap'^2 + Bq'^2 = 1,$$

becomes

$$Pp'^2 - Qq'^2 = \frac{1}{p-q}.$$

When the values are substituted, the first equation of the geodesics is

$$-2p'' = \left(\frac{P_1}{P} + \frac{1}{p-q}\right)p'^2 + \frac{2p'q'}{q-p} + \frac{Qq'^2}{P(p-q)},$$

and the second is

$$-2q'' = \left(\frac{Q_2}{Q} + \frac{1}{q-p}\right)q'^2 + \frac{2q'p'}{p-q} + \frac{Pp'^2}{Q(q-p)}.$$

To obtain an integral equivalent, take a new variable θ such that

$$\frac{Pp'^2}{\theta+p} = \frac{Qq'^2}{\theta+q},$$

so that, on using the modified form of the arc-equation, we have

$$Pp'^2 = \frac{\theta+p}{(p-q)^2}, \quad Qq'^2 = \frac{\theta+q}{(q-p)^2}.$$

When the first of these is differentiated logarithmically, we find

$$2\frac{p''}{p'} + \frac{P_1}{P}p' = \frac{\theta'+p'}{\theta+p} - 2\frac{p'-q'}{p-q}.$$

But the first of the modified equations of the geodesics, on substituting

$$\frac{Qq'^2}{P} = \frac{\theta+q}{\theta+p}p'^2$$

and reducing, gives the relation

$$-\left(2\frac{p''}{p'} + \frac{P_1}{P}p'\right) = \frac{2p'-2q'}{p-q} - \frac{p'}{\theta+p};$$

and therefore the equation in θ' becomes

$$\frac{\theta'}{\theta+p} = 0.$$

When we proceed similarly from

$$Qq'^2 = \frac{\theta + q}{(q-p)^2},$$

and use the second of the modified geodesic equations, we find

$$\frac{\theta'}{\theta + q} = 0.$$

Thus $\theta' = 0$; and therefore θ is constant along a geodesic, while varying from geodesic to geodesic. Let

$$f(p) = p\,(a+p)\,(b+p)\,(c+p)\,(d+p)\,(\theta+p)\,(p-r),$$
$$f(q) = q\,(a+q)\,(b+q)\,(c+q)\,(d+q)\,(\theta+q)\,(q-r);$$

then we have, along any geodesic on the parametric surface given by $r = \text{constant}$,

$$\{p\,(p-r)\,(p-q)\}^2 \left(\frac{dp}{ds}\right)^2 = 4f(p),$$

$$\{q\,(q-r)\,(q-p)\}^2 \left(\frac{dq}{ds}\right)^2 = 4f(q),$$

or

$$\frac{p\,(p-r)}{\{f(p)\}^{\frac{1}{2}}}\,dp - \frac{q\,(q-r)}{\{f(q)\}^{\frac{1}{2}}}\,dq = 0,$$

as the equations which determine the march of a geodesic, the quantity θ being constant along the geodesic and parametric for different geodesics.

Ex. An extension, to quadruple space, of the configuration of sphero-conics in triple space, is provided by the surfaces which are the intersection of an ovoid by a concentric globe. The Cartesian equation of such a globe is

$$x^2 + y^2 + z^2 + v^2 = \text{constant};$$

and therefore (§ 386) the intersection is given by an equation

$$\theta(p, q, r) = p + q + r = \text{constant}.$$

Verify that the two principal measures of circular curvature of regional geodesics touching the surface are

$$\frac{NJ + JL + LN}{BC + CA + AB}$$

and

$$\frac{(L+N+J)(A+B+C) - (AL + BN + CJ)}{BC + CA + AB};$$

and obtain the principal measures of regional geodesic curvature of geodesics upon the surface.

Construct the characteristic equations of geodesics on this surface: and obtain an integral of these equations.

Equation of geodesic surfaces in an ovoidal region.

404. As the general integrals of geodesic lines have been obtained in the form of transcendental relations between the regional parameters satisfied along the curves, it is possible to obtain a differential relation characteristic of a Riemann geodesic surface (§ 315) within the region. But it does not seem

feasible*, at least for quite arbitrary initial orientations, to obtain an integral equivalent of this differential relation.

For the specification of a Riemann geodesic surface as already defined, we require

(i) any arbitrary initial point of reference, which can be regarded as determinate by the assignment of the values p_0, q_0, r_0, to its parameters: and

(ii) any two independent arbitrary directions through that point, each such direction being an initial direction which determines a regional geodesic uniquely.

Let these two assumed directions be denoted by p_1', q_1', r_1', and p_2', q_2', r_2'; and let A_0, B_0, C_0, be the values of A, B, C, at the initial position p_0, q_0, r_0, so that we have

$$A_0 p_1'^2 + B_0 q_1'^2 + C_0 r_1'^2 = 1, \quad A_0 p_2'^2 + B_0 q_2'^2 + C_0 r_2'^2 = 1.$$

A sheaf of directions, all lying in the plane determined by the two assumed directions, is given by

$$\alpha p_1' + \beta p_2', \quad \alpha q_1' + \beta q_2', \quad \alpha r_1' + \beta r_2',$$

where α and β are two new parametric quantities. (These new quantities are subject to the condition

$$\alpha^2 + \beta^2 + 2\alpha\beta \cos\omega = 1,$$

where ω is the angle between the two directions: but the condition is irrelevant to the immediate purpose.) Each direction in this sheaf determines a regional geodesic: and the aggregate locus, in the region, of all these geodesics is the Riemann geodesic surface through the initial point with the assigned orientation determined by the two directions through that point. The Gauss measure of curvature of this surface at the point is taken by Riemann to be a measure of curvature of the region in the orientation assigned to the geodesic surface.

Now the general equations of geodesics in the ovoidal region are (§ 397)

$$\frac{pp'}{(q-r)\{S(p)\}^{\frac{1}{2}}} = \frac{qq'}{(r-p)\{S(q)\}^{\frac{1}{2}}} = \frac{rr'}{(p-q)\{S(r)\}^{\frac{1}{2}}} = \frac{2}{(p-q)(q-r)(r-p)},$$

where the septimic polynomials S involve two arbitrary constants of integration κ and λ. We write

$$P = (p+a)(p+b)(p+c)(p+d)$$

as before, with Q and R to denote similar functions of q and of r, so that

$$S(p) = p(p+\kappa)(p+\lambda)P,$$

with like values for $S(q)$ and $S(r)$. Let the arbitrary constants for the

* The same difficulty of manipulation has arisen (§ 315, Ex.) in the case of a Riemann geodesic surface, with arbitrary initial orientation and existing in a globular region.

direction p_1', q_1', r_1', be κ_1 and λ_1, and let those for the direction p_2', q_2', r_2', be κ_2 and λ_2: then we have

$$\left(\frac{p_0}{P_0}\right)^{\frac{1}{2}} \frac{p_1'}{\{(p_0+\kappa_1)(p_0+\lambda_1)\}^{\frac{1}{2}}} = -\frac{2}{(p_0-q_0)(p_0-r_0)},$$

$$\left(\frac{p_0}{P_0}\right)^{\frac{1}{2}} \frac{p_2'}{\{(p_0+\kappa_2)(p_0+\lambda_2)\}^{\frac{1}{2}}} = -\frac{2}{(p_0-q_0)(p_0-r_0)},$$

together with associated values of the same type for q_1' and q_2', and for r_1' and r_2'. For any geodesic of the complete sheaf, let p_3', q_3', r_3', denote the initial direction at p_0, q_0, r_0, and let κ and λ be the arbitrary constants of integration, so that

$$\left(\frac{p_0}{P_0}\right)^{\frac{1}{2}} \frac{p_3'}{\{(p_0+\kappa)(p_0+\lambda)\}^{\frac{1}{2}}} = -\frac{2}{(p_0-q_0)(p_0-r_0)},$$

with similar values for q_3' and for r_3'. But because this geodesic belongs to the sheaf, there are parameters α and β such that

$$p_3' = \alpha p_1' + \beta p_2', \quad q_3' = \alpha q_1' + \beta q_2', \quad r_3' = \alpha r_1' + \beta r_2',$$

and therefore

$$\begin{vmatrix} p_3', & p_1', & p_2' \\ q_3', & q_1', & q_2' \\ r_3', & r_1', & r_2' \end{vmatrix} = 0,$$

a condition which leads to the relation

$$\begin{vmatrix} \{(p_0+\kappa)(p_0+\lambda)\}^{\frac{1}{2}}, & \{(p_0+\kappa_1)(p_0+\lambda_1)\}^{\frac{1}{2}}, & \{(p_0+\kappa_2)(p_0+\lambda_2)\}^{\frac{1}{2}} \\ \{(q_0+\kappa)(q_0+\lambda)\}^{\frac{1}{2}}, & \{(q_0+\kappa_1)(q_0+\lambda_1)\}^{\frac{1}{2}}, & \{(q_0+\kappa_2)(q_0+\lambda_2)\}^{\frac{1}{2}} \\ \{(r_0+\kappa)(r_0+\lambda)\}^{\frac{1}{2}}, & \{(r_0+\kappa_1)(r_0+\lambda_1)\}^{\frac{1}{2}}, & \{(r_0+\kappa_2)(r_0+\lambda_2)\}^{\frac{1}{2}} \end{vmatrix} = 0.$$

From the general equations of geodesics, we have

$$\frac{pp'^2}{(p+a)(p+b)(p+c)(p+d)} \cdot \frac{1}{(p+\kappa)(p+\lambda)} = \frac{4}{(p-q)^2(p-r)^2},$$

and therefore

$$(p+\kappa)(p+\lambda) = A(p-q)(p-r)p'^2;$$

and similarly

$$(q+\kappa)(q+\lambda) = B(q-r)(q-p)q'^2,$$

$$(r+\kappa)(r+\lambda) = C(r-p)(r-q)r'^2.$$

These satisfy the relations (§ 398)

$$\kappa\lambda = Aqrp'^2 + Brpq'^2 + Cpqr'^2,$$

$$-(\kappa+\lambda) = A(q+r)p'^2 + B(r+p)q'^2 + C(p+q)r'^2,$$

together with the arc-relation

$$1 = Ap'^2 + Bq'^2 + Cr'^2.$$

Consequently

$$(p_0+\kappa)(p_0+\lambda) = A(p_0-q)(p_0-r)p'^2 + B(p_0-r)(p_0-p)q'^2 + C(p_0-p)(p_0-q)r'^2;$$

or, if we write

$$\Phi(p_0) = A(p_0-q)(p_0-r)\,dp^2 + B(p_0-r)(p_0-p)\,dq^2 + C(p_0-p)(p_0-q)\,dr^2,$$

with similar significance for $\Phi(q_0)$ and for $\Phi(r_0)$, we have

$$\begin{aligned}(p_0+\kappa)(p_0+\lambda)\,ds^2 &= \Phi(p_0),\\ (q_0+\kappa)(q_0+\lambda)\,ds^2 &= \Phi(q_0),\\ (r_0+\kappa)(r_0+\lambda)\,ds^2 &= \Phi(r_0).\end{aligned}$$

Again, we have

$$\begin{aligned}(p_0+\kappa_1)(p_0+\lambda_1) &= A_0(p_0-q_0)(p_0-r_0)p_1'^2,\\ (q_0+\kappa_1)(q_0+\lambda_1) &= B_0(q_0-r_0)(q_0-p_0)q_1'^2,\\ (r_0+\kappa_1)(r_0+\lambda_1) &= C_0(r_0-p_0)(r_0-q_0)r_1'^2,\end{aligned}$$

where, in p_1', q_1', r_1', there occurs implicitly the arc-element for the first direction: and

$$\begin{aligned}(p_0+\kappa_2)(p_0+\lambda_2) &= A_0(p_0-q_0)(p_0-r_0)p_2'^2,\\ (q_0+\kappa_2)(q_0+\lambda_2) &= B_0(q_0-r_0)(q_0-p_0)q_2'^2,\\ (r_0+\kappa_2)(r_0+\lambda_2) &= C_0(r_0-p_0)(r_0-q_0)r_2'^2,\end{aligned}$$

where, in p_2', q_2', r_2', there occurs implicitly the arc-element for the second direction.

When the values of the respective expressions are substituted in the preceding determinantal relation, it becomes

$$\begin{vmatrix} \left\{\dfrac{\Phi(p_0)}{A_0(p_0-q_0)(p_0-r_0)}\right\}^{\frac{1}{2}}, & p_1', & p_2' \\ \left\{\dfrac{\Phi(q_0)}{B_0(q_0-r_0)(q_0-p_0)}\right\}^{\frac{1}{2}}, & q_1', & q_2' \\ \left\{\dfrac{\Phi(r_0)}{C_0(r_0-p_0)(r_0-q_0)}\right\}^{\frac{1}{2}}, & r_1', & r_2' \end{vmatrix} = 0,$$

a differential relation between dp, dq, dr, p, q, r, satisfied along every geodesic of the sheaf, because it is independent of α and β. It is *the differential equation of the geodesic surface determined by the orientation arising through the two assigned directions* p_1', q_1', r_1', *and* p_2', q_2', r_2'.

Ex. Prove that nul surfaces in the ovoidal region are given by the equation

$$\int\left(\frac{ap+\beta p^2}{P}\right)^{\frac{1}{2}}dp + \int\left(\frac{aq+\beta q^2}{Q}\right)^{\frac{1}{2}}dq + \int\left(\frac{ar+\beta r^2}{R}\right)^{\frac{1}{2}}dr = \text{constant},$$

where a and β are constants,

$$P = (a+p)(b+p)(c+p)(d+p),$$

and Q, R, denote the corresponding functions of q and of r respectively.

Results for a paraboloidal region.

405. The following results, concerning the paraboloidal region

$$\frac{y^2}{a}+\frac{z^2}{b}+\frac{v^2}{c}=4x,$$

can be obtained in the same way as the results concerning the ovoidal region, and are therefore merely stated*.

The parametric expressions for the region have already been obtained (§ 302, Ex. 8) in the forms

$$\left.\begin{aligned} y^2&=-4a\frac{(a+p)(a+q)(a+r)}{(a-b)(a-c)}\\ z^2&=-4b\frac{(b+p)(b+q)(b+r)}{(b-c)(b-a)}\\ v^2&=-4c\frac{(c+p)(c+q)(c+r)}{(c-a)(c-b)}\\ x&=-a-b-c-p-q-r \end{aligned}\right\}.$$

The fundamental magnitudes of the region of the first order are

$$\left.\begin{aligned} A&=\frac{p(p-q)(p-r)}{(a+p)(b+p)(c+p)}, & F&=0\\ B&=\frac{q(q-r)(q-p)}{(a+q)(b+q)(c+q)}, & G&=0\\ C&=\frac{r(r-p)(r-q)}{(a+r)(b+r)(c+r)}, & H&=0 \end{aligned}\right\}.$$

The direction-cosines of the inward-drawn normal to the region are

$$X=\frac{2}{W},\quad Y=-\frac{y}{a}\frac{1}{W},\quad Z=-\frac{z}{b}\frac{1}{W},\quad V=-\frac{v}{c}\frac{1}{W},$$

where

$$W=2\left(-\frac{pqr}{abc}\right)^{\frac{1}{2}}.$$

The fundamental magnitudes of the region of the second order are

$$\left.\begin{aligned} WL&=-\frac{(p-q)(p-r)}{(a+p)(b+p)(c+p)}, & I&=0\\ WN&=-\frac{(q-r)(q-p)}{(a+q)(b+q)(c+q)}, & K&=0\\ WJ&=-\frac{(r-p)(r-q)}{(a+r)(b+r)(c+r)}, & M&=0 \end{aligned}\right\}.$$

* These are merely stated, because a paraboloidal region can be regarded (on the analogy of the corresponding property-relation between ellipsoids and paraboloids in triple space) as a limiting form of the ovoid when the centre is moved to infinity. But the direct process from the Cartesian equation is simple.

The three principal radii of curvature at any point are

$$\frac{W}{\rho} = -\frac{1}{p}, \quad \frac{W}{\rho'} = -\frac{1}{q}, \quad \frac{W}{\rho''} = -\frac{1}{r}.$$

The region of centres is three-volumed. The coordinates of a point in the volume corresponding to the radius of curvature ρ are

$$\left.\begin{aligned} \xi &= -a-b-c-3p-q-r \\ \eta &= 2\left\{\frac{\alpha}{a^2}(a+p)^3(a+q)(a+r)\right\}^{\frac{1}{2}} \\ \zeta &= 2\left\{\frac{\beta}{b^2}(b+p)^3(b+q)(b+r)\right\}^{\frac{1}{2}} \\ \upsilon &= 2\left\{\frac{\gamma}{c^2}(c+p)^3(c+q)(c+r)\right\}^{\frac{1}{2}} \end{aligned}\right\},$$

where

$$\alpha = -\frac{a}{(a-b)(a-c)}, \quad \beta = -\frac{b}{(b-c)(b-a)}, \quad \gamma = -\frac{c}{(c-a)(c-b)}.$$

The coordinates of a point in the volume corresponding to the radius of curvature ρ' are

$$\left.\begin{aligned} \xi' &= -a-b-c-p-3q-r \\ \eta' &= 2\left\{\frac{\alpha}{a^2}(a+p)(a+q)^3(a+r)\right\}^{\frac{1}{2}} \\ \zeta' &= 2\left\{\frac{\beta}{b^2}(b+p)(b+q)^3(b+r)\right\}^{\frac{1}{2}} \\ \upsilon' &= 2\left\{\frac{\gamma}{c^2}(c+p)(c+q)^3(c+r)\right\}^{\frac{1}{2}} \end{aligned}\right\};$$

and likewise (by interchange of q and r) for the coordinates of a point in the volume corresponding to the radius of curvature ρ''.

The complete section of the region of centres by the flat $\upsilon = 0$ is composed of (i) the full two-sheeted surface of centres of the quadric

$$\frac{y^2}{a} + \frac{z^2}{b} = 4x, \quad \upsilon = 0,$$

this centro-surface being a simple section so that it occurs singly, and (ii) the quadric

$$\frac{a}{(a-c)^2}y^2 + \frac{b}{(b-c)^2}z^2 = 4(x-2c), \quad \upsilon = 0,$$

this quadric being cuspidal so that it occurs triply.

The integrated equations of geodesics in the region, requiring the simplest class of hyperelliptic functions for their expressions, have already (§ 302, Ex. 8) been obtained.

CHAPTER XXIII.

MINIMAL SURFACES: MINIMAL REGIONS.

406. Throughout the preceding investigations, alike in those relating to regions in the quadruple space, and in those relating to surfaces (whether existing freely in that space, or confined within a region belonging to that space), the properties of geodesic curves have been fundamental.

In every instance, the determination of the geodesics, whether in a region or on a surface, has been effected by means of the known results of the calculus of variations. But geodesics, however essential their properties have proved in the analytical geometry of regions and of surfaces, do not provide the only example of minimal problems. We also require to know surfaces of minimal area and to know regions of minimal content, the respective minima arising under assigned boundary conditions, as regards the rim (or portions of rim) of a surface, or the face (or portions of face) of a region.

Only a brief and elementary discussion of each of these problems will be attempted.

Minimal surface in free space: equations and primitive.

When we investigate the equations of minimal surfaces, it is natural to expect (after the discrimination between surfaces, as shewn in Chapters XII and XX) that the analysis for a minimal surface existing freely in the homaloidal quadruple space will be distinct from the analysis for a minimal surface confined to a triple region in that space. We begin with the consideration of minimal surfaces of the former kind*. With the notation of Chapter XII for surfaces, existing freely in quadruple space without any limitation by an enclosing region, the element of area is given by

$$V\,dp\,dq, \quad = (EG - F^2)^{\frac{1}{2}}\,dp\,dq,$$

where, as usual,

$$E = \Sigma x_1^2, \quad F = \Sigma x_1 x_2, \quad G = \Sigma x_2^2.$$

For a minimal surface, the double integral

$$\iint V\,dp\,dq, \quad = \iint (EG - F^2)^{\frac{1}{2}}\,dp\,dq,$$

taken between assigned limits, must be a minimum. As the subject of

* In this connection, a memoir by Eisenhart, "Minimal surfaces in Euclidean four-space," *Amer. Journ. Math.* vol. xxxiv (1912), pp. 214–236, may be consulted.

integration does not involve x, y, z, v, explicitly, the four critical equations are of the type

$$\frac{\partial}{\partial p}\left(\frac{\partial V}{\partial x_1}\right)+\frac{\partial}{\partial q}\left(\frac{\partial V}{\partial x_2}\right)=0,$$

that is,

$$\frac{\partial}{\partial p}\left\{\frac{1}{V}(Gx_1-Fx_2)\right\}+\frac{\partial}{\partial q}\left\{\frac{1}{V}(Ex_2-Fx_1)\right\}=0.$$

As in Weierstrass's theory of minimal surfaces existing in homaloidal triple space, let the parametric curves of the required surface be nul-lines, so that, because

$$ds^2=Edp^2+2Fdp\,dq+Gdq^2,$$

and because we now take these nul-lines to be $dp=0$, $dq=0$, we must have

$$E=0, \quad G=0.$$

Thus $V=iF$; and the foregoing critical equation becomes

$$x_{12}=0.$$

Similarly the other three critical equations are

$$y_{12}=0, \quad z_{12}=0, \quad v_{12}=0.$$

Consequently we have

$$x,\ y,\ z,\ v,\ = \text{function of } p + \text{function of } q,$$

for each of the variables. So far as the critical equations of the second order are concerned, these functions are arbitrary; but the arbitrary functions must satisfy the two assumed relations

$$E=x_1^2+y_1^2+z_1^2+v_1^2=0,$$
$$G=x_2^2+y_2^2+z_2^2+v_2^2=0.$$

As x, y, z, v, are, each of them, expressible as the sum of a function of p alone and a function of q alone, we have x_1, y_1, z_1, v_1, functions of p only, and x_2, y_2, z_2, v_2, functions of q only; and we have therefore to make these functions satisfy $E=0$ and $G=0$.

Now from $E=0$, we have

$$\frac{x_1+iy_1}{z_1+iv_1}=-\frac{z_1-iv_1}{x_1-iy_1}.$$

Each of the fractions is a function of p alone, and we can substitute any function of p, for p, without affecting the analysis; accordingly, without loss of generalisation, we can take p as the common value of the fractions, so that

$$x_1+iy_1=\ \ (z_1+iv_1)\,p,$$
$$z_1-iv_1=-(x_1-iy_1)\,p.$$

Also, all the combinations $x_1\pm iy_1$, $z_1\pm iv_1$, are functions of p alone; so we take

$$x_1-iy_1=-2g''(p), \quad z_1+iv_1=2f''(p),$$

where now f and g are arbitrary functions of their common argument p. Then

$$x_1 + iy_1 = 2pf''(p), \quad z_1 - iv_1 = 2pg''(p);$$

and therefore

$$\left.\begin{aligned} x_1 &= pf''(p) - g''(p) \\ iy_1 &= pf''(p) + g''(p) \\ z_1 &= f''(p) + pg''(p) \\ iv_1 &= f''(p) - pg''(p) \end{aligned}\right\}.$$

We shall assume that we are dealing with real surfaces, so that p and q (now the parameters of nul-lines) are conjugate magnitudes. Accordingly, when $G = 0$ is resolved in the same manner as $E = 0$ has been resolved, we find

$$\left.\begin{aligned} x_2 &= q\phi''(q) - \psi''(q) \\ -iy_2 &= q\phi''(q) + \psi''(q) \\ z_2 &= \phi''(q) + q\psi''(q) \\ -iv_2 &= \phi''(q) - q\psi''(q) \end{aligned}\right\},$$

where, for reality in the surface, $f(p)$ and $\phi(q)$ are functions conjugate to one another, and $g(p)$ and $\psi(q)$ also are functions conjugate to one another. For brevity, we shall denote the four functions respectively by f, g, ϕ, ψ, their arguments being implied.

Now

$$\int pf''\,dp = pf' - f, \quad \int qg''\,dq = qg' - g;$$

hence

$$\left.\begin{aligned} x &= \text{Real part of } 2(pf' - f - g') \\ y &= \ldots\ldots\ldots\ldots \frac{2}{i}(pf' - f + g') \\ z &= \ldots\ldots\ldots\ldots 2(f' + pg' - g) \\ v &= \ldots\ldots\ldots\ldots \frac{2}{i}(f' - pg' + g) \end{aligned}\right\},$$

where f and g are arbitrary independent functions of the complex parameter p. These are the integrated equations of a minimal surface existing freely in homaloidal quadruple space*.

The surface manifestly is algebraical when the functions f and g are algebraical.

Fundamental magnitudes, with parametric nul-lines.

407. With these values of the coordinates, the fundamental magnitudes of the first order are

$$E = 0, \quad G = 0,$$
$$F = 2(1 + pq)(f''\phi'' + g''\psi''),$$
$$V^2 = -F^2;$$

* With only a slight change of notation, they are the equations apparently first due to Eisenhart (*l.c.*).

and the derived quantities Γ, Γ', Γ'', Δ, Δ', Δ'', are

$$\Gamma = \frac{F_1}{F}, \quad \Gamma' = 0, \quad \Gamma'' = 0,$$

$$\Delta = 0\ , \quad \Delta' = 0, \quad \Delta'' = \frac{F_2}{F}.$$

The differential equations of geodesics are

$$\frac{d^2p}{ds^2} + \frac{F_1}{F}\left(\frac{dp}{ds}\right)^2 = 0, \quad \frac{d^2q}{ds^2} + \frac{F_2}{F}\left(\frac{dq}{ds}\right)^2 = 0,$$

which manifestly are satisfied by $p = \text{constant}$, and $q = \text{constant}$, separately—the property that nul-lines satisfy the geodesic equations has already (§ 219, Ex. 2) been noted; and the single equivalent equation is

$$\frac{d^2q}{dp^2} + \frac{F_2}{F}\left(\frac{dq}{dp}\right)^2 - \frac{F_1}{F}\frac{dq}{dp} = 0.$$

For the magnitudes **a**, **b**, **c**, **f**, **g**, **h**, of the second order, we have

$$\left.\begin{aligned} x_{11} &= pf''' + f'' - g''' \\ iy_{11} &= pf''' + f'' + g''' \\ z_{11} &= f''' + pg''' + g'' \\ iv_{11} &= f''' - pg''' - g'' \end{aligned}\right\}, \quad \left.\begin{aligned} x_{22} &= q\phi''' + \phi'' - \psi''' \\ -iy_{22} &= q\phi''' + \phi'' + \psi''' \\ z_{22} &= \phi''' + q\psi''' + \psi'' \\ -iv_{22} &= \phi''' - q\psi''' - \psi'' \end{aligned}\right\},$$

$$x_{12} = 0 = y_{12} = z_{12} = v_{12}.$$

Hence we have

$$\mathbf{a} = 4(f'''g'' - g'''f''),$$
$$\mathbf{c} = 4(\phi'''\psi'' - \psi'''\phi''),$$
$$\mathbf{h} = 0, \quad \mathbf{b} = 0, \quad \mathbf{f} = 0;$$

and the value of **g** is

$$\mathbf{g} = 2(1+pq)\frac{(f''g''' - g''f''')(\phi''\psi''' - \psi''\phi''')}{f''\phi'' + g''\psi''} + \frac{2}{1+pq}(f''\phi'' + g''\psi''),$$

though, until special forms for particular instances are assigned to the arbitrary functions, this actual value of **g** will usually not be inserted.

Principal curvatures: lines of curvature.

408. With the notation of §§ 213, 240, we have

$$3\theta = \mathbf{g}, \quad 3\mathbf{k} = \mathbf{g};$$

so that the invariantive measure connected with θ is

$$\frac{3\theta}{V^2} = -\frac{\mathbf{g}}{F^2}.$$

The principal radii of curvature of the surface are obtained (§§ 238, 239) by equating, to zero, the discriminant of the quartic

$$\mathbf{a}p'^4 + 6\mathbf{k}p'^2q'^2 + \mathbf{c}q'^4 - \frac{1}{\rho^2}(2Fp'q')^2,$$

that is, of

$$\mathbf{a}p'^4 + 2\left(\mathbf{g} - \frac{2F^2}{\rho^2}\right)p'^2q'^2 + \mathbf{c}q'^4.$$

The invariant I of this equation is

$$I = \mathbf{ac} + \frac{1}{3}\left(\mathbf{g} - \frac{2F^2}{\rho^2}\right)^2$$

$$= \mathbf{ac} + \frac{1}{3}\mathbf{g}^2 - \frac{4}{3}\mathbf{g}F^2\frac{1}{\rho^2} + \frac{4}{3}F^4\frac{1}{\rho^4},$$

thus giving the values of I_0 and I_1 (§ 239); and its cubinvariant J is

$$J = \frac{1}{3}\mathbf{ac}\left(\mathbf{g} - \frac{2F^2}{\rho^2}\right) - \frac{1}{27}\left(\mathbf{g} - \frac{2F^2}{\rho^2}\right)^3$$

$$= \frac{1}{3}\mathbf{acg} - \frac{1}{27}\mathbf{g}^3 - \frac{2}{3}\left(\mathbf{ac} - \frac{1}{3}\mathbf{g}^2\right)\frac{F^2}{\rho^2} - \frac{4}{9}\mathbf{g}F^4\frac{1}{\rho^4} + \frac{8}{27}F^6\frac{1}{\rho^6},$$

thus giving the values of J_0 and J_1 (§ 239). The vanishing discriminant gives the relation

$$I^3 - 27J^2 = 0,$$

which, after substitution, becomes

$$\mathbf{ac}\left\{\mathbf{ac} - \left(\mathbf{g} - \frac{2F^2}{\rho^2}\right)^2\right\}^2 = 0,$$

a quartic in ρ^2, as is to be expected. Let

$$\left.\begin{aligned}\frac{2F^2}{\rho_1^2} &= \mathbf{g} + (\mathbf{ac})^{\frac{1}{2}}\\ \frac{2F^2}{\rho_2^2} &= \mathbf{g} - (\mathbf{ac})^{\frac{1}{2}}\end{aligned}\right\};$$

then the four radii are

$$\rho_1, \quad -\rho_1, \quad \rho_2, \quad -\rho_2,$$

or they consist of two pairs, equal in value and opposite in sign. Obviously

$$\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} = \frac{\mathbf{g}}{F^2},$$

$$\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2} = \frac{(\mathbf{ac})^{\frac{1}{2}}}{F^2},$$

and

$$F^4\frac{1}{\rho_1^2\rho_2^2} = \frac{1}{4}(\mathbf{g}^2 - \mathbf{ac}),$$

or

$$\frac{F^2}{\rho_1\rho_2} = (1 + pq)\frac{(f''g''' - g''f''')(\phi''\psi''' - \psi''\phi''')}{f''\phi'' + g''\psi''} - \frac{f''\phi'' + g''\psi''}{1 + pq}.$$

409. The lines of curvature on the surface are (§ 238) given by

$$\begin{vmatrix} \mathrm{a}p'^3 + 3\mathrm{h}p'^2q' + 3\mathrm{k}p'q'^2 + \mathrm{f}q'^3, & Ep' + Fq' \\ \mathrm{h}p'^3 + 3\mathrm{k}p'^2q' + 3\mathrm{f}p'q'^2 + \mathrm{c}q'^3, & Fp' + Gq' \end{vmatrix} = 0,$$

which, in the present instance, becomes

$$\mathrm{a}p'^4 - \mathrm{c}q'^4 = 0.$$

Thus they are given by the quadrature of the equation

$$(f'''g'' - g'''f'')^{\frac{1}{4}}\,dp = (\phi'''\psi'' - \psi'''\phi'')^{\frac{1}{4}}\,dq.$$

Let

$$P = (f'''g'' - g'''f'')^{\frac{1}{4}}, \quad Q = (\phi'''\psi'' - \psi'''\phi'')^{\frac{1}{4}},$$

$$\theta = (2FPQ)^{\frac{1}{2}};$$

then the four lines of curvature can be represented by

$$\left(\frac{dp}{ds}\right)_1 = \frac{Q}{\theta}, \quad \left(\frac{dq}{ds}\right)_1 = \frac{P}{\theta};$$

$$\left(\frac{dp}{ds}\right)_2 = \frac{Q}{\theta i^{\frac{1}{2}}}, \quad \left(\frac{dq}{ds}\right)_2 = \frac{P}{\theta}\,i^{\frac{1}{2}};$$

$$\left(\frac{dp}{ds}\right)_3 = \frac{Q}{\theta i}, \quad \left(\frac{dq}{ds}\right)_3 = \frac{P}{\theta}\,i;$$

$$\left(\frac{dp}{ds}\right)_4 = \frac{Q}{\theta i^{\frac{3}{2}}}, \quad \left(\frac{dq}{ds}\right)_4 = \frac{P}{\theta}\,i^{\frac{3}{2}}.$$

Consequently, denoting the angle between the lines 1 and 2 by $\widehat{12}$, and so for the others, we have

$$\cos\widehat{12} = F\left\{\left(\frac{dp}{ds}\right)_1\left(\frac{dq}{ds}\right)_2 + \left(\frac{dp}{ds}\right)_2\left(\frac{dq}{ds}\right)_1\right\}$$

$$= \frac{1}{2}\left(i^{\frac{1}{2}} + \frac{1}{i^{\frac{1}{2}}}\right) = \cos\tfrac{1}{4}\pi,$$

$$\cos\widehat{13} = \frac{1}{2}\left(i + \frac{1}{i}\right) = 0,$$

$$\cos\widehat{14} = \frac{1}{2}\left(i^{\frac{3}{2}} + \frac{1}{i^{\frac{3}{2}}}\right) = \cos\tfrac{3}{4}\pi,$$

$$\cos\widehat{23} = \frac{1}{2}\left(i^{\frac{1}{2}} + \frac{1}{i^{\frac{1}{2}}}\right) = \cos\tfrac{1}{4}\pi,$$

$$\cos\widehat{24} = \frac{1}{2}\left(i + \frac{1}{i}\right) = 0,$$

$$\cos\widehat{34} = \frac{1}{2}\left(i^{\frac{1}{2}} + \frac{1}{i^{\frac{1}{2}}}\right) = \cos\tfrac{1}{4}\pi.$$

Hence the lines 1 and 3 are perpendicular; and the angles between them are bisected by the lines 2 and 4, also perpendicular: that is, on any minimal surface, the four lines of curvature consist of two perpendicular pairs, each pair bisecting the angles between the other pair.

Orthogonal centre and radius: characteristic conic.

410. The orthogonal centre is the intersection of two consecutive orthogonal planes. The first of these is given, as usual, by the two equations

$$\Sigma(\bar{x}-x)x_1=0, \quad \Sigma(\bar{x}-x)x_2=0.$$

On the minimal surface with the parameters of nul-lines as the variables of reference, we have

$$E=0, \quad G=0, \quad x_{12}=0=y_{12}=z_{12}=v_{12};$$

hence the two equations, which are to be associated with the preceding pair in order to provide the orthogonal centre, are

$$\Sigma(\bar{x}-x)x_{11}p'=\Sigma x_1(x_1p'+x_2q')=Fq',$$

$$\Sigma(\bar{x}-x)x_{22}q'=\Sigma x_2(x_1p'+x_2q')=Fp':$$

that is, they are

$$\Sigma(\bar{x}-x)x_{11}=Ft, \quad \Sigma(\bar{x}-x)x_{22}=F/t,$$

where $tp'=q'$.

We at once have the equations of the characteristic conic: they are the pair

$$\Sigma(\bar{x}-x)x_1=0, \quad \Sigma(\bar{x}-x)x_2=0,$$

shewing that it lies in the orthogonal plane: and the further equation

$$\{\Sigma(\bar{x}-x)x_{11}\}\{\Sigma(\bar{x}-x)x_{22}\}=F^2,$$

a locus of the second degree and therefore, in the orthogonal plane section, giving a conic.

For the coordinates of any point on the conic, in terms of the parameter t, we have the four equations

$$\Sigma(\bar{x}-x)x_1=0, \quad \Sigma(\bar{x}-x)x_2=0, \quad \Sigma(\bar{x}-x)x_{11}=Ft, \quad \Sigma(\bar{x}-x)x_{22}=F/t.$$

The determinant of the coefficients on the left-hand side is

$$\begin{vmatrix} x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ x_{11}, & y_{11}, & z_{11}, & v_{11} \\ x_{22}, & y_{22}, & z_{22}, & v_{22} \end{vmatrix},$$

as in § 249; and this is equal to

$$V(\mathbf{ca}-\mathbf{g}^2)^{\frac{1}{2}}.$$

We take the quantities $e_{11}, f_{11}, g_{11}, h_{11}$; and $e_{22}, f_{22}, g_{22}, h_{22}$; as in § 231. The quantities $e_{12}, f_{12}, g_{12}, h_{12}$, now vanish. Thus we obtain, as the coordinates of the orthogonal centre,

$$V(\mathrm{ca}-\mathrm{g}^2)^{\frac{1}{2}}(\bar{x}-x)=F\left(e_{22}t-e_{11}\frac{1}{t}\right),$$

or, because $V^2=-F^2$, we have

$$\left.\begin{aligned}(\mathrm{g}^2-\mathrm{ac})^{\frac{1}{2}}(\bar{x}-x)&=e_{22}t-e_{11}\frac{1}{t}\\(\mathrm{g}^2-\mathrm{ac})^{\frac{1}{2}}(\bar{y}-y)&=f_{22}t-f_{11}\frac{1}{t}\\(\mathrm{g}^2-\mathrm{ac})^{\frac{1}{2}}(\bar{z}-z)&=g_{22}t-g_{11}\frac{1}{t}\\(\mathrm{g}^2-\mathrm{ac})^{\frac{1}{2}}(\bar{v}-v)&=h_{22}t-h_{11}\frac{1}{t}\end{aligned}\right\}.$$

The length of the orthogonal radius has been denoted (§ 248) by ρ_0, so that

$$\rho_0^2=\Sigma(\bar{x}-x)^2.$$

Also

$$\Sigma e_{11}^2=\begin{vmatrix}E,&F,&0\\F,&G,&0\\0,&0,&\mathrm{a}\end{vmatrix}=-\mathrm{a}F^2,\quad \Sigma e_{22}^2=\begin{vmatrix}E,&F,&0\\F,&G,&0\\0,&0,&\mathrm{c}\end{vmatrix}=-\mathrm{c}F^2,$$

$$\Sigma e_{11}e_{22}=\begin{vmatrix}E,&F,&0\\F,&G,&0\\0,&0,&\mathrm{g}\end{vmatrix}=-\mathrm{g}F^2.$$

Hence the length of the orthogonal radius is given by

$$\begin{aligned}(\mathrm{ac}-\mathrm{g}^2)\rho_0^2&=F^2\left(\mathrm{a}\frac{1}{t^2}-2\mathrm{g}+\mathrm{c}t^2\right)\\&=\frac{F^2}{p'^2q'^2}(\mathrm{a}p'^4-2\mathrm{g}p'^2q'^2+\mathrm{c}q'^4)\\&=4F^4\ \ (\mathrm{a}p'^4-2\mathrm{g}p'^2q'^2+\mathrm{c}q'^4).\end{aligned}$$

With these values we may compare the expression for the curvature of the geodesic as given by

$$\frac{1}{\rho^2}=\mathrm{a}p'^4+2\mathrm{g}p'^2q'^2+\mathrm{c}q'^4,$$

or

$$\frac{4F^2}{\rho^2}=\mathrm{a}\frac{1}{t^2}+2\mathrm{g}+\mathrm{c}t^2.$$

The maximum and minimum values of ρ are provided by

$$\mathrm{c}t^4-\mathrm{a}=0,$$

being the equation of the lines of curvature; and these provide the maximum and minimum values of ρ_0, as in the general theory (§ 255).

If we take

$$\Sigma(\bar{x}-x)\,\xi_{11}=\Sigma(\bar{x}-x)\,x_{11}=X\mathbf{a}^{\frac{1}{2}},$$

$$\Sigma(\bar{x}-x)\,\xi_{22}=\Sigma(\bar{x}-x)\,x_{22}=Y\mathbf{c}^{\frac{1}{2}},$$

and write

$$-\cos\epsilon=\frac{\mathbf{g}}{(\mathbf{ac})^{\frac{1}{2}}},$$

the characteristic conic is

$$XY(\mathbf{ac})^{\frac{1}{2}}=F^2,$$

referred to axes X and Y in the orthogonal plane inclined at an angle ϵ. Manifestly the centre of the conic is the X-Y origin, that is, the point x, y, z, v, on the surface itself.

The principal axes r_1 and r_2 of the conic are given by the roots of the equation

$$(\mathbf{g}^2-\mathbf{ac})\frac{F^4}{r^4}+\mathbf{gac}\frac{F^2}{r^2}+\frac{1}{4}\mathbf{a}^2\mathbf{c}^2=0,$$

so that the characteristic conic cannot be a circle.

Ex. Shew that the surface, given by

$$z+iv=f(x+iy),$$

is a minimal surface, whatever be the form of the function f. (Kommerell*.)

The Monge-Cauchy characteristics of a minimal surface.

411. In this form of the general equations of a minimal surface, two independent arbitrary functions, f and g, occur, the functions ϕ and ψ being conjugate to f and to g respectively. The precise determination of any minimal surface, satisfying postulated conditions such as a requirement of passing through a given curve and having a required orientation along that curve, is usually effected by determining the forms of the functions f and g so as to be in accord with such conditions. The expressions of the coordinates of any point on the minimal surface shew that the arguments of these arbitrary functions are, respectively, the parameters of the two sets of nul-lines on the surface. Consequently the determination of the forms both of f and of g would not be possible, if the postulated initial conditions were given along either curve

$$p=\text{constant}, \quad q=\text{constant},$$

when these curves are the nul-lines on the surface.

It follows that the Monge (or Cauchy) characteristic† of the set of simultaneous differential equations for x, y, z, v, leading to a minimal surface, is composed of the nul-lines on the surface.

* In his memoir, already cited (vol. i, p. 442, note), Kommerell calls such a surface a Riemann surface in four-dimensional space and discusses its properties.

† See the author's *Theory of Differential Equations*, vol. vi, §§ 326–329. The same result (see § 415, *post*) holds for minimal surfaces in a region.

Minimal surfaces in a region: characteristic equations.

412. To obtain the equations characteristic of a minimal surface enclosed in a region, expressed by the three parameters p, q, r, we assume these parameters to be functions of two new variables t and u, so that the arc-element of the region is

$$\begin{aligned} ds^2 &= \Sigma A\, dp^2 \\ &= \bar{E}\, dt^2 + 2\bar{F}\, dt\, du + \bar{G}\, du^2, \end{aligned}$$

where

$$\bar{E} = \Sigma A p_t^2, \quad \bar{F} = \Sigma A p_t p_u, \quad \bar{G} = \Sigma A p_u^2.$$

Variations of t and u now yield a surface, on which the element of area is

$$(\bar{E}\bar{G} - \bar{F}^2)^{\frac{1}{2}}\, dt\, du;$$

and the surface will be a minimal surface within the region when the double integral

$$\iint \bar{V}\, dt\, du, \quad = \iint (\bar{E}\bar{G} - \bar{F}^2)^{\frac{1}{2}}\, dt\, du$$

is made a minimum.

Now in this integral there are three dependent variables p, q, r; so that, if θ momentarily denotes either p, q, or r, there are three critical equations

$$-\frac{\partial \bar{V}}{\partial \theta} + \frac{d}{dt}\left(\frac{\partial \bar{V}}{\partial \theta_t}\right) + \frac{d}{du}\left(\frac{\partial \bar{V}}{\partial \theta_u}\right) = 0.$$

As in the Weierstrass theory for minimal surfaces for homaloidal triple space, it is convenient to make the independent variables (as yet left open to choice) the parameters of nul-lines on the minimal surface; so we take

$$\bar{E} = 0, \quad \bar{G} = 0,$$

and then

$$\bar{V}^2 = -\bar{F}^2,$$

or

$$\bar{V} = i\bar{F}.$$

The three-fold characteristic equation thus becomes

$$-\frac{\partial \bar{F}}{\partial \theta} + \frac{d}{dt}\left(\frac{\partial \bar{F}}{\partial \theta_t}\right) + \frac{d}{du}\left(\frac{\partial \bar{F}}{\partial \theta_u}\right) = 0,$$

for $\theta = p, q, r$, in succession.

When we take $\theta = p$, this equation becomes

$$-\Sigma A_1 p_t p_u + \frac{d}{dt}(A p_u + H q_u + G r_u) + \frac{d}{du}(A p_t + H q_t + G r_t) = 0,$$

that is,

$$\begin{aligned} &2(A p_{tu} + H q_{tu} + G r_{tu}) - \Sigma A_1 p_t p_u \\ &\quad + p_u(A_1 p_t + A_2 q_t + A_3 r_t) + q_u(H_1 p_t + H_2 q_t + H_3 r_t) \\ &\qquad\qquad + r_u(G_1 p_t + G_2 q_t + G_3 r_t) \\ &\quad + p_t(A_1 p_u + A_2 q_u + A_3 r_u) + q_t(H_1 p_u + H_2 q_u + H_3 r_u) \\ &\qquad\qquad + r_t(G_1 p_u + G_2 q_u + G_3 r_u) = 0. \end{aligned}$$

When substitution is made for the derivatives of A, H, G, in terms of the quantities of § 273, we have

$$\begin{aligned} &Ap_{tu} + Hq_{tu} + Gr_{tu} \\ &\quad + \alpha_1 p_t p_u + \beta_1 (p_t q_u + p_u q_t) + \gamma_1 (p_t r_u + p_u r_t) \\ &\qquad\qquad + \delta_1 q_t q_u + \epsilon_1 (q_t r_u + q_u r_t) + \omega_1 r_t r_u = 0. \end{aligned}$$

The other two equations similarly become

$$\begin{aligned} &Hp_{tu} + Bq_{tu} + Fr_{tu} \\ &\quad + \alpha_2 p_t p_u + \beta_2 (p_t q_u + p_u q_t) + \gamma_2 (p_t r_u + p_u r_t) \\ &\qquad\qquad + \delta_2 q_t q_u + \epsilon_2 (q_t r_u + q_u r_t) + \omega_2 r_t r_u = 0, \end{aligned}$$

and

$$\begin{aligned} &Gp_{tu} + Fq_{tu} + Cr_{tu} \\ &\quad + \alpha_3 p_t p_u + \beta_3 (p_t q_u + p_u q_t) + \gamma_3 (p_t r_u + p_u r_t) \\ &\qquad\qquad + \delta_3 q_t q_u + \epsilon_3 (q_t r_u + q_u r_t) + \omega_3 r_t r_u = 0. \end{aligned}$$

When these equations are resolved for p_{tu}, q_{tu}, r_{tu}, they yield the relations

$$\left.\begin{aligned} p_{tu} + (\Gamma_{11}, \Gamma_{22}, \Gamma_{33}, \Gamma_{23}, \Gamma_{31}, \Gamma_{12} \mathbin{\text{\textbackslash\!\!\textbackslash}} p_t, q_t, r_t \mathbin{\text{\textbackslash\!\!\textbackslash}} p_u, q_u, r_u) = 0 \\ q_{tu} + (\Delta_{11}, \Delta_{22}, \Delta_{33}, \Delta_{23}, \Delta_{31}, \Delta_{12} \mathbin{\text{\textbackslash\!\!\textbackslash}} p_t, q_t, r_t \mathbin{\text{\textbackslash\!\!\textbackslash}} p_u, q_u, r_u) = 0 \\ r_{tu} + (\Theta_{11}, \Theta_{22}, \Theta_{33}, \Theta_{23}, \Theta_{31}, \Theta_{12} \mathbin{\text{\textbackslash\!\!\textbackslash}} p_t, q_t, r_t \mathbin{\text{\textbackslash\!\!\textbackslash}} p_u, q_u, r_u) = 0 \end{aligned}\right\},$$

which accordingly are the general differential equations of a minimal surface in the region when referred to its nul-lines as parametric curves.

413. Because the nul-lines are the parametric curves, we have

$$\Sigma A p_t^2 = 0, \quad \Sigma A p_u^2 = 0.$$

From the first of these relations, we have

$$\begin{aligned} 2\{(Ap_t + Hq_t + Gr_t) p_{tu} + (Hp_t + Bq_t + Fr_t) q_{tu} + (Gp_t + Fq_t + Cr_t) r_{tu}\} \\ + \Sigma p_t^2 (A_1 p_u + A_2 q_u + A_3 r_u) = 0. \end{aligned}$$

Let the differential equations of the minimal surface, in their unresolved form, be denoted by

$$\bar{P} = 0, \quad \bar{Q} = 0, \quad \bar{R} = 0;$$

then the relation just derived from $\Sigma A p_t^2 = 0$ becomes

$$p_t \bar{P} + q_t \bar{Q} + r_t \bar{R} = 0,$$

that is, the relation

$$\Sigma A p_t^2 = 0$$

is an integral of the three characteristic equations.

Similarly the relation

$$\Sigma A p_u^2 = 0$$

leads to a condition

$$p_u \bar{P} + q_u \bar{Q} + r_u \bar{R} = 0:$$

that is, the relation is an integral of the three characteristic equations.

It follows that, in consequence of the two parametric relations

$$\Sigma A p_t^2 = 0, \quad \Sigma A p_u^2 = 0,$$

defining nul-lines on the surface, the three equations of the second order, characteristic of minimal surfaces within the region, are equivalent to a single equation of that order.

NOTE. It may be observed, in passing, that, when the containing region becomes (or degenerates into) a flat, the foregoing equations become those of a minimal surface in the usual homaloidal triple space. In such an event, we can take the flat to be given by $v =$ constant, so that we have

$$p = x, \quad q = y, \quad r = z,$$

as a possible parametric representation: then

$$A = 1, \quad B = 1, \quad C = 1, \quad F = 0, \quad G = 0, \quad H = 0,$$

and all the eighteen subsidiary magnitudes Γ, Δ, Θ, vanish. The three characteristic equations then become

$$x_{tu} = 0, \quad y_{tu} = 0, \quad z_{tu} = 0,$$

where (as v now does not vary) the parametric nul-lines are given by the equations

$$x_t^2 + y_t^2 + z_t^2 = 0, \quad x_u^2 + y_u^2 + z_u^2 = 0,$$

that is,

$$\bar{E} = 0, \quad \bar{G} = 0,$$

while

$$\bar{F} = x_t x_u + y_t y_u + z_t z_u,$$

being the initial forms of the equations of the minimal surface.

Geometrical interpretation of the equations of the minimal surface.

414. The single independent equation, which is the essential formal expression of a minimal surface in a region, can be obtained in a different form which itself admits of immediate geometrical interpretation.

Let the equation of the minimal surface in the region be

$$\theta(p, q, r) = 0.$$

Denoting the parameters of the nul-lines on the surface by t and u as before, we have

$$\theta_1 p_t + \theta_2 q_t + \theta_3 r_t = 0,$$

$$\theta_1 p_u + \theta_2 q_u + \theta_3 r_u = 0.$$

Also, because the nul-lines are of zero length and lie in the region, we have

$$\Sigma A p_t^2 = 0, \quad \Sigma A q_t^2 = 0.$$

Now eliminating r_t from the equations

$$\Sigma \theta_1 p_t = 0, \quad \Sigma A p_t^2 = 0,$$

we have

$$p_t^2(A\theta_3^2 - 2G\theta_1\theta_3 + C\theta_1^2) + 2p_tq_t(H\theta_3^2 - F\theta_1\theta_3 - G\theta_2\theta_3 + C\theta_1\theta_2) + q_t^2(B\theta_3^2 - 2F\theta_2\theta_3 + C\theta_2^2) = 0.$$

There exists precisely the same equation connecting p_u and q_u. Hence

$$\frac{p_tp_u}{B\theta_3^2 - 2F\theta_2\theta_3 + C\theta_2^2} = \frac{-\frac{1}{2}(p_tq_u + p_uq_t)}{H\theta_3^2 - F\theta_1\theta_3 - G\theta_2\theta_3 + C\theta_1\theta_2} = \frac{q_tq_u}{A\theta_3^2 - 2G\theta_1\theta_3 + C\theta_1^2}.$$

Proceeding similarly with the elimination of q_t between the same equations, and with the elimination of p_t between them, we find that each of these three equal fractions is also equal to

$$\frac{-\frac{1}{2}(p_tr_u + p_ur_t)}{G\theta_2^2 - H\theta_2\theta_3 - F\theta_1\theta_2 + B\theta_1\theta_3} = \frac{-\frac{1}{2}(q_tr_u + q_ur_t)}{F\theta_1^2 - G\theta_1\theta_2 - H\theta_1\theta_3 + A\theta_2\theta_3} = \frac{r_tr_u}{A\theta_2^2 - 2H\theta_1\theta_2 + B\theta_1^2}.$$

Along the surface, we have

$$\theta_1p_{tu} + \theta_2q_{tu} + \theta_3r_{tu} + (\theta_{11}, \theta_{22}, \theta_{33}, \theta_{23}, \theta_{31}, \theta_{12}\mathcal{X}p_t, q_t, r_t\mathcal{X}p_u, q_u, r_u) = 0.$$

On the assumption that the surface is minimal, the values of p_{tu}, q_{tu}, r_{tu}, are as given by the characteristic equations in § 412. When these values are inserted, this equation becomes

$$(\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12}\mathcal{X}p_t, q_t, r_t\mathcal{X}p_u, q_u, r_u) = 0.$$

The ratios of the coefficients of γ_{11}, γ_{22}, γ_{33}, γ_{23}, γ_{31}, γ_{12}, have just been obtained; when they are substituted, we have

$$\begin{aligned}&\gamma_{11}(B\theta_3^2 - 2F\theta_2\theta_3 + C\theta_2^2) - 2\gamma_{23}(A\theta_2\theta_3 - H\theta_1\theta_3 - G\theta_1\theta_2 + F\theta_1^2)\\ &+ \gamma_{22}(C\theta_1^2 - 2G\theta_3\theta_1 + A\theta_3^2) - 2\gamma_{31}(B\theta_3\theta_1 - F\theta_2\theta_1 - H\theta_2\theta_3 + G\theta_2^2)\\ &+ \gamma_{33}(A\theta_2^2 - 2H\theta_1\theta_2 + B\theta_1^2) - 2\gamma_{12}(C\theta_1\theta_2 - G\theta_3\theta_2 - F\theta_3\theta_1 + H\theta_3^2) = 0.\end{aligned}$$

This is the (single) partial differential equation of the second order characteristic of minimal surfaces in the region.

When it is written in the form

$$\Sigma\theta_1^2(B\gamma_{33} - 2F\gamma_{23} + C\gamma_{22}) = 0,$$

or, in the notation of § 348,

$$\mathbf{H} = 0,$$

it expresses the property that *the additive measure of regional geodesic curvature of superficial geodesics (or of a surface) is zero*, a property which is the extension, to minimal surfaces in a region of quadruple space, of the property that the additive measure of circular curvature of a surface in homaloidal triple space is zero.

It should be noted that the result is one more illustration of the fact that, for a curved region, its own geodesics have the same kind of organic influence as is exercised by the (geodesic) straight lines in a region that is homaloidal.

The Monge-Cauchy characteristics of the differential equation.

415. This differential equation for minimal surfaces within the region is a partial differential equation of the second order in three independent variables p, q, r. By Cauchy's general existence-theorem concerning integrals of such equations*, it possesses an integral containing two arbitrary functions; and these two arbitrary functions either are, or are equivalent to, an arbitrary assigned value of θ and an arbitrary assigned value of one of its first derivatives for a given assigned relation between p, q, r, that is, along a given surface. This given surface and the required minimal surface will, of course, intersect in a curve; we proceed to shew that this curve must not be a nul-line in the region, so that what is called† the characteristic surface (or the characteristic invariant) must not pass through any nul-line of the minimal surface. (It is the parameters of these lines which are made the independent variables in the establishment, in § 412, of the equations of a minimal surface in a region.)

Let the given surface, along which the supposed value of θ and the supposed value of a first derivative of θ are given, be represented by the equation

$$u(p, q, r) = 0.$$

Then in order that Cauchy's establishment of the specified integral may be effective, it is necessary that, for the initial conditions along the surface $u = 0$ as regards θ and its derivative, the equation

$$\frac{\partial E}{\partial \theta_{11}}\left(\frac{\partial u}{\partial p}\right)^2 + \frac{\partial E}{\partial \theta_{22}}\left(\frac{\partial u}{\partial q}\right)^2 + \frac{\partial E}{\partial \theta_{33}}\left(\frac{\partial u}{\partial r}\right)^2 + \frac{\partial E}{\partial \theta_{23}}\frac{\partial u}{\partial q}\frac{\partial u}{\partial r} + \frac{\partial E}{\partial \theta_{31}}\frac{\partial u}{\partial r}\frac{\partial u}{\partial p} + \frac{\partial E}{\partial \theta_{12}}\frac{\partial u}{\partial p}\frac{\partial u}{\partial q} = 0$$

must not hold, where $E = 0$ is the equation of the minimal surfaces. Now

$$\gamma_{11} = \theta_{11} - \theta_1\Gamma_{11} - \theta_2\Delta_{11} - \theta_3\Theta_{11},$$

and so for the other magnitudes γ_{lm} in E: thus, for all combinations of l and m,

$$\frac{\partial E}{\partial \theta_{lm}} = \frac{\partial E}{\partial \gamma_{lm}}.$$

Hence as E is homogeneous and linear in these coefficients γ_{lm}, the equation of the surface, along which assigned data of the kind specified would be inadequate for a solution, is

$$\left(\frac{\partial u}{\partial p}\right)^2 (B\theta_3^2 - 2F\theta_2\theta_3 + C\theta_2^2) - 2\frac{\partial u}{\partial q}\frac{\partial u}{\partial r}(A\theta_2\theta_3 - H\theta_1\theta_3 - G\theta_1\theta_2 + F\theta_1^2)$$
$$+ \left(\frac{\partial u}{\partial q}\right)^2 (C\theta_1^2 - 2G\theta_3\theta_1 + A\theta_3^2) - 2\frac{\partial u}{\partial r}\frac{\partial u}{\partial p}(B\theta_3\theta_1 - F\theta_2\theta_1 - H\theta_2\theta_3 + G\theta_2^2)$$
$$+ \left(\frac{\partial u}{\partial r}\right)^2 (A\theta_2^2 - 2H\theta_1\theta_2 + B\theta_1^2) - 2\frac{\partial u}{\partial p}\frac{\partial u}{\partial q}(C\theta_1\theta_2 - G\theta_3\theta_2 - F\theta_3\theta_1 + H\theta_3^2) = 0,$$

* See the author's *Theory of Differential Equations*, vol. vi, §§ 326–329. † *l.c.*, § 326

that is,

$$(A, B, C, F, G, H)\left(\theta_2\frac{\partial u}{\partial r}-\theta_3\frac{\partial u}{\partial q},\quad \theta_3\frac{\partial u}{\partial p}-\theta_1\frac{\partial u}{\partial r},\quad \theta_1\frac{\partial u}{\partial q}-\theta_2\frac{\partial u}{\partial p}\right)^2=0.$$

But we know (§ 379) that the direction-elements dp, dq, dr, of the curve of intersection of two surfaces $\theta =$ constant, $\Im =$ constant, are such that

$$\frac{dp}{\theta_2\Im_3-\theta_3\Im_2}=\frac{dq}{\theta_3\Im_1-\theta_1\Im_3}=\frac{dr}{\theta_1\Im_2-\theta_2\Im_1}.$$

If, then, we denote by dp, dq, dr, the elements for the arc of intersection of our minimal surface $\theta =$ constant and the surface $\Im =$ constant along which the initial data for the precise determination of θ are postulated, the equation

$$(A, B, C, F, G, H)(dp, dq, dr)^2=0$$

must not hold: that is, the arc must not be a line of zero length. We therefore infer the proposition stated: in order that postulated data given along an assigned surface for the determination of the magnitude θ may be effective, this assigned surface must not pass through a nul-line of the region lying on the minimal surface.

Minimal region: differential equation in Cartesian form: its interpretation.

416. Consider an element of volume of a region bounded by points at its angular corners, these points having coordinates

$$x, y, z;\ x+dx, y, z;\ x+dx, y+dy, z;\ x+dx, y+dy, z+dz;$$
$$x, y+dy, z;\ x, y+dy, z+dz;$$
$$x, y, z+dz;\ x+dx, y, z+dz;$$

with, in each instance, the appropriate value of v as given by the equation of the region

$$v=v(x, y, z).$$

The projection of this element of volume into the flat $v=0$ is, in volume, equal to

$$dx\,dy\,dz.$$

The cosine of the angle between the normal to the region and the normal to the flat $v=0$ (that is, the axis of v) is

$$(1+v_x^2+v_y^2+v_z^2)^{-\frac{1}{2}};$$

and therefore the element of volume of the region is

$$(1+v_x^2+v_y^2+v_z^2)^{\frac{1}{2}}\,dx\,dy\,dz,$$

or if we write

$$\Omega^2=1+v_x^2+v_y^2+v_z^2,$$

this element of volume is

$$\Omega\,dx\,dy\,dz.$$

Consequently the total volume of a part of the region, which is bounded by assigned surfaces in the region, is the value of the integral

$$\iiint \Omega \, dx \, dy \, dz,$$

when the triple integral is taken over the range included in the boundary surfaces.

As an extension of the problem in homaloidal triple space requiring the determination of the shape of a surface which shall occupy the minimum superficial area among all the surfaces that can be drawn between two given curves or portions of two given curves, it is natural to consider the problem in homaloidal quadruple space requiring the determination of the shape of a region (or the equation, or the parametric equations, of a region) which shall occupy the minimum volume among all the regions that can be selected as extending between two given surfaces or portions of two given surfaces in the quadruple space. Different analytical equations, characterised by the same intrinsic geometric property, arise according as the region (which may be called a *minimal region*) is to be represented by a single relation between the Cartesian coordinates or by equations which express those coordinates in terms of three independent parameters.

When the minimal region is to be represented by a single relation between the spatial coordinates, let this equation be

$$v = \chi(x, y, z),$$

where the form of the function $\chi(x, y, z)$ is to be determined so as to make the integral

$$\iiint \Omega \, dx \, dy \, dz, \quad = \iiint (1 + v_x^2 + v_y^2 + v_z^2)^{\frac{1}{2}} dx \, dy \, dz,$$

a minimum. In this form of the problem, there is a single dependent variable v; the variable will be determinable by a single characteristic equation.

As usual, the single characteristic equation is only one of the conditions that the integral shall acquire a minimum value. Of the other tests, it is known* that the Legendre test definitely is satisfied and that the Weierstrass test definitely is satisfied; and the Jacobi test, quantitative and not qualitative in each instance, can only be applied when a solution of the characteristic equation (or, it may happen, of the auxiliary characteristic equation) is known. We therefore proceed to the consideration of the characteristic equation, which for the selected form of the integral is

$$\frac{\partial}{\partial x}\left(\frac{v_x}{\Omega}\right) + \frac{\partial}{\partial y}\left(\frac{v_y}{\Omega}\right) + \frac{\partial}{\partial z}\left(\frac{v_z}{\Omega}\right) = 0.$$

* See my *Calculus of Variations*, pp. 643–645.

When the value of Ω is substituted, and the resulting equation is reduced to its simplest explicit form, we find that it becomes

$$(1+v_y^2+v_z^2)\,v_{xx}+(1+v_z^2+v_x^2)\,v_{yy}+(1+v_x^2+v_y^2)\,v_{zz} \\ -2v_yv_zv_{yz}-2v_zv_xv_{zx}-2v_xv_yv_{xy}=0.$$

This equation is the analytical expression of a geometrical property of the region. Any region has three principal radii of curvature at any point; and these three radii are given by a cubic equation which already (§ 282) has been constructed. If the three radii be denoted by ρ, ρ', ρ'', in general, so that these are the three roots of the cubic, the form of that equation shews that the relation

$$\frac{1}{\rho}+\frac{1}{\rho'}+\frac{1}{\rho''}=0$$

is satisfied for the minimal region: or *the arithmetic mean of the linear curvatures of a minimal region is zero**.

Minimal region of axial revolution.

417. As an example, consider the minimal region of axial revolution.

(i) Let the axis of rotation be taken as the axis of v. Any region of rotation round this axis can be represented by an equation

$$x^2+y^2+z^2=f(v),$$

and the requirement exacts the determination of the form of f when the region is minimal. We have

$$2x=v_xf', \quad 2=v_{xx}f'+v_x^2f'', \quad 0=v_{yz}f'+v_yv_zf'', \\ 2y=v_yf', \quad 2=v_{yy}f'+v_y^2f'', \quad 0=v_{zx}f'+v_zv_xf'', \\ 2z=v_zf', \quad 2=v_{zz}f'+v_z^2f'', \quad 0=v_{xy}f'+v_xv_yf''.$$

When the second derivatives of v, given by these relations, are substituted in the equation characteristic of minimal regions, it becomes

$$6+4(v_x^2+v_y^2+v_z^2)=(v_x^2+v_y^2+v_z^2)f''.$$

But

$$(v_x^2+v_y^2+v_z^2)f'^2=4(x^2+y^2+z^2)=4f,$$

and thus the characteristic equation is

$$6+16\frac{f}{f'^2}=4\frac{ff''}{f'^2}.$$

A first integral of this equation is

$$f'^2=4c^{-4}f^3-4f,$$

where c is an arbitrary constant, manifestly real, and chosen in this form for ultimate simplicity of expression; and the primitive is

$$\frac{x^2+y^2+z^2}{c^2}=\wp\left(\frac{v}{c}+B\right),$$

where $\wp$ denotes the Weierstrass elliptic function, and B is an arbitrary constant. Manifestly B, intrinsically, does not affect the form of the surface; it can be made zero, merely by change of origin along the axis of v. The constant c affects only the scale of the region. Thus a minimal region can be represented by the equation

$$x^2+y^2+z^2=\wp(v).$$

* This is the extension of the well-known property that the additive measure of linear curvature of a minimal surface in homaloidal triple space is zero.

(ii) The equation giving the principal linear curvatures of any region with an equation $v=\chi(x, y, z)$ has been shewn (§ 282, Ex. 1) to be

$$\frac{1}{\rho^3}-\frac{1}{\Omega^3\rho^2}\{\Sigma(1+v_y^2+v_z^2)v_{xx}-2\Sigma v_y v_z v_{yz}\}$$

$$+\frac{1}{\Omega^4\rho}\{\Sigma(1+v_x^2)(v_{yy}v_{zz}-v_{yz}^2)+2\Sigma v_y v_z(v_{xy}v_{xz}-v_{xx}v_{yz})\}$$

$$-\frac{1}{\Omega^5}\begin{vmatrix} v_{xx}, & v_{xy}, & v_{xz} \\ v_{xy}, & v_{yy}, & v_{yz} \\ v_{xz}, & v_{yz}, & v_{zz} \end{vmatrix}=0,$$

where

$$\Omega^2=1+v_x^2+v_y^2+v_z^2.$$

Let the values of the second derivatives of v, given by the equation

$$x^2+y^2+z^2=\wp(v),$$

be substituted in this equation, the function $\wp$ satisfying the relation

$$\wp'^2=4\wp^3-4\wp.$$

The coefficient of $\frac{1}{\rho^2}$ vanishes, as has already been noted. The coefficient of $\frac{1}{\Omega^4\rho}$ is

$$=\frac{1}{\wp'^2}\{12+4(v_x^2+v_y^2+v_z^2)-4(v_x^2+v_y^2+v_z^2)\wp''\}$$

$$=-12\frac{\Omega^2}{\wp'^2}.$$

The coefficient of $-\frac{1}{\Omega^5}$ is

$$=\frac{1}{\wp'^3}\{8-4(v_x^2+v_y^2+v_z^2)\wp''\}=-16\frac{\Omega^2}{\wp'^3}.$$

Hence the cubic equation for ρ is

$$\frac{1}{\rho^3}-12\frac{1}{\Omega^2\wp'^2}\frac{1}{\rho}+16\frac{1}{\Omega^3\wp'^3}=0;$$

and therefore the three values of ρ are

$$\frac{1}{2}\Omega\wp', \quad \frac{1}{2}\Omega\wp', \quad -\frac{1}{4}\Omega\wp',$$

which satisfy the relation

$$\frac{1}{\rho}+\frac{1}{\rho'}+\frac{1}{\rho''}=0.$$

But

$$\Omega^2=1+v_x^2+v_y^2+v_z^2$$

$$=1+4\frac{\wp}{\wp'^2}$$

$$=4\frac{\wp^3}{\wp'^2},$$

so that

$$\Omega\wp'=2\wp^{\frac{3}{2}};$$

and therefore the three principal radii are

$$\wp^{\frac{3}{2}}, \quad \wp^{\frac{3}{2}}, \quad -\frac{1}{2}\wp^{\frac{3}{2}}.$$

(iii) Let η denote the distance of any point in the region from the axis of v, so that the region can be taken in the form

$$\eta=\wp^{\frac{1}{2}}(v).$$

The intercept on the normal to the region at any point, between the point and its intersection by the axis, is easily verified to be $\wp^{\frac{3}{2}}$, as is to be expected, because that intercept on the normal is a repeated principal radius of linear curvature. The radius of curvature of the curve $\eta=\wp^{\frac{1}{2}}(v)$ is easily verified to be $\frac{1}{2}\wp^{\frac{3}{2}}$; and the curve is convex to the axis of v, so that the normal, drawn inwards to the generated region, lies in a sense opposite to that of the radius of curvature of the curve. Thus the radius of curvature is equal to half the normal intercept for the curve*. We thus again verify the descriptive curvature property, characteristic of a minimal region.

(iv) The critical tests, as to whether a minimum value is actually possessed by the region under assigned conditions, are the customary three which are additional to the characteristic equation. Of these three, we know (as a universal result for all minimal regions) that the Legendre test and the Weierstrass test are satisfied: the Jacobi test remains for consideration.

For the application of the Jacobi test, we can proceed from the primitive of the characteristic equation

$$2ff''=8f+3f'^2;$$

and this primitive can be taken in the form

$$f=\wp^{\frac{1}{2}}(\alpha x+\beta),$$

where α and β are arbitrary constants, and the invariants g_2 and g_3 of the Weierstrass elliptic function are

$$g_2=\frac{4}{\alpha^2}, \quad g_3=0.$$

For the construction of the Jacobi criterion†, we need the magnitudes $\frac{\partial f}{\partial\alpha}$ and $\frac{\partial f}{\partial\beta}$. Now

$$2f\frac{\partial f}{\partial\beta}=\wp', \quad 2f\frac{\partial f}{\partial\alpha}=x\wp'+\frac{\partial\wp}{\partial g_2}\frac{dg_2}{d\alpha}=x\wp'-\frac{2}{\alpha}g_2\frac{\partial\wp}{\partial g_2}.$$

In order to obtain the value of $\frac{\partial\wp}{\partial g_2}$, we use the well-known formulæ‡

$$4g_2\frac{\partial\wp(z)}{\partial g_2}+6g_3\frac{\partial\wp(z)}{\partial g_3}-z\frac{\partial\wp(z)}{\partial z}=2\wp(z),$$

$$12g_3\frac{\partial\wp(z)}{\partial g_2}+\frac{2}{3}g_2^2\frac{\partial\wp(z)}{\partial g_3}-2\zeta(z)\frac{\partial\wp(z)}{\partial z}=4\wp^2(z)-\frac{2}{3}g_2.$$

When g_3 is zero, the second formula shews that $\frac{\partial\wp(z)}{\partial g_3}$ is finite; and then the first formula gives

$$4g_2\frac{\partial\wp(z)}{\partial g_2}=2\wp(z)+z\frac{\partial\wp(z)}{\partial z}.$$

Thus

$$\begin{aligned}2f\frac{\partial f}{\partial\alpha}&=x\wp'-\frac{2}{\alpha}g_2\frac{\partial\wp}{\partial g_2}\\&=x\wp'-\frac{1}{\alpha}\left\{\wp+\frac{1}{2}(x\alpha+\beta)\wp'\right\}\\&=\frac{1}{2}x\wp'-\frac{1}{\alpha}\wp-\frac{\beta}{2\alpha}\wp'.\end{aligned}$$

* It is a well-known property that, if the radius of curvature of a plane curve is equal to n times the normal (n being a constant), the curve is a cycloid when $n=2$, a circle when $n=1$, a catenary when $n=-1$ (giving a minimal surface in homaloidal triple space), and a parabola when $n=-2$, the curve being convex to the appropriate axis for negative values of n. In the present instance, $n=-\frac{1}{2}$.

† See my *Calculus of Variations*, § 26.

‡ They belong to the fundamental formulæ due to Weierstrass: for their establishment, see my *Theory of Functions*, (3rd ed.), p. 312.

The quantity in the Jacobi test is

$$J=\frac{\partial f}{\partial \alpha}\div\frac{\partial f}{\partial \beta}$$

$$=\frac{1}{2}x-\frac{1}{\alpha}\frac{\varphi}{\varphi'}-\frac{\beta}{2\alpha};$$

and the requirement in the Jacobi test is that, in the range of the integral, the quantity J shall not have the same value twice. Thus, for a minimum for the region of revolution

$$x-2\frac{\varphi}{\alpha\varphi'}$$

must not acquire, in the range of the integral, the value which it has at the lower limit. Now for the curve

$$y=\varphi^{\frac{1}{2}}(\alpha x+\beta),$$

we have

$$2\frac{\varphi}{\alpha\varphi'}=\frac{y}{y'};$$

and therefore the quantity

$$x-\frac{y}{y'}$$

must not resume its initial value. But $x-\frac{y}{y'}$ is the abscissa of the point where a tangent to the curve meets the axis of x; hence if the tangent at an initial point A meet this axis in C and if CB be a tangent to the curve distinct from CA, the point B is the conjugate of A. The range of the integral, beginning at A, must not extend as far as B. When this Jacobi test is satisfied, the region of revolution is an actual minimal region.

Parametric equations of minimal region: minimal region of planar revolution.

418. When the region, which is required to be minimal, is represented by the parametric expression of the coordinates of a point, we have to make the integral

$$\iiint\Omega\,dp\,dq\,dr$$

a minimum, where

$$\Omega^2=ABC+2FGH-AF^2-BG^2-CH^2,$$

the quantities A, B, C, F, G, H, being the fundamental magnitudes of the first order (§ 262) composed of combinations of the first derivatives of x, y, z, v. Thus there are four characteristic equations, one for each of these variables. As no one of the fundamental magnitudes involves x, y, z, v, explicitly, and therefore Ω does not involve these variables explicitly, the characteristic equations are

$$\frac{\partial}{\partial p}\left(\frac{\partial\Omega}{\partial\theta_1}\right)+\frac{\partial}{\partial q}\left(\frac{\partial\Omega}{\partial\theta_2}\right)+\frac{\partial}{\partial r}\left(\frac{\partial\Omega}{\partial\theta_3}\right)=0,$$

for $\theta = x, y, z, v$, in succession. Now it has been proved (§ 274, Ex.) that

$$\left.\begin{aligned}
\frac{\partial}{\partial p}\left(\frac{\partial\Omega}{\partial x_1}\right)+\frac{\partial}{\partial q}\left(\frac{\partial\Omega}{\partial x_2}\right)+\frac{\partial}{\partial r}\left(\frac{\partial\Omega}{\partial x_3}\right)&=-\frac{X}{\Omega}\mathfrak{G}_1\\
\frac{\partial}{\partial p}\left(\frac{\partial\Omega}{\partial y_1}\right)+\frac{\partial}{\partial q}\left(\frac{\partial\Omega}{\partial y_2}\right)+\frac{\partial}{\partial r}\left(\frac{\partial\Omega}{\partial y_3}\right)&=-\frac{Y}{\Omega}\mathfrak{G}_1\\
\frac{\partial}{\partial p}\left(\frac{\partial\Omega}{\partial z_1}\right)+\frac{\partial}{\partial q}\left(\frac{\partial\Omega}{\partial z_2}\right)+\frac{\partial}{\partial r}\left(\frac{\partial\Omega}{\partial z_3}\right)&=-\frac{Z}{\Omega}\mathfrak{G}_1\\
\frac{\partial}{\partial p}\left(\frac{\partial\Omega}{\partial v_1}\right)+\frac{\partial}{\partial q}\left(\frac{\partial\Omega}{\partial v_2}\right)+\frac{\partial}{\partial r}\left(\frac{\partial\Omega}{\partial v_3}\right)&=-\frac{V}{\Omega}\mathfrak{G}_1
\end{aligned}\right\},$$

where

$$\mathfrak{G}_1 = aL + 2hM + 2gK + bN + 2fI + cJ.$$

Hence all the four characteristic equations are satisfied in virtue of the single relation

$$\mathfrak{G}_1 = 0,$$

which accordingly can be taken as the equation of minimal regions when expressed in the parametric form.

In the investigation of the various measures of linear curvature of a region (§ 281), it was seen that, if α, β, γ, be the three principal radii of curvature,

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{1}{\Omega^2}\mathfrak{G}_1;$$

consequently we infer, as in § 416, the property that the arithmetic mean of the three principal linear curvatures of a minimal region vanishes.

When no specification of particular parametric lines is made, the equation characteristic of minimal regions is

$$aL + 2hM + 2gK + bN + 2fI + cJ = 0.$$

When the minimal region is referred to its lines of curvature as parametric curves, so that $F = G = H = 0$, $I = K = M = 0$, the characteristic equation becomes

$$\frac{L}{A}+\frac{N}{B}+\frac{J}{C}=0.$$

When the parametric lines are chosen so as to be orthogonal to one another, so that $F = G = H = 0$, without an assumption that they are conjugate also, the last equation still is the form assumed by the characteristic equation. Other forms arise corresponding with each specific selection of organic curves as curves of parametric reference.

Ex. Required the minimal region of rotation round two orthogonal planes.

We take one of the planes as the plane XOY, the other of the orthogonal planes as the plane ZOV. Any region of rotation round these two planes can be represented in the form

$$t = f(r),$$

where

$$r^2 = x^2 + y^2, \quad t^2 = z^2 + v^2.$$

The equation characteristic of minimal regions, necessary to determine the form of f, is obtainable as before. It can also be constructed, as follows, by using the fundamental magnitudes of the first order and of the second order. We take

$$x=r\cos\theta, \quad y=r\sin\theta, \quad z=f\cos\phi, \quad v=f\sin\phi,$$

where f is used for $f(r)$, while f' and f'' will denote the first derivative and the second derivative of $f(r)$: and r, θ, ϕ, are to be the parameters.

The arc-element ds of the region is given by

$$\begin{aligned} ds^2 &= dx^2+dy^2+dz^2+dv^2 \\ &= dr^2+r^2d\theta^2+df^2+f^2d\phi^2 \\ &= (1+f'^2)\,dr^2+r^2d\theta^2+f^2d\phi^2, \end{aligned}$$

so that

$$A=1+f'^2, \quad B=r^2, \quad C=f^2, \quad F=0=G=H.$$

The direction-cosines X, Y, Z, V, of the inward-drawn normal are

$$X=-\frac{f'}{(1+f'^2)^{\frac{1}{2}}}\cos\theta, \quad Y=-\frac{f'}{(1+f'^2)^{\frac{1}{2}}}\sin\theta, \quad Z=\frac{1}{(1+f'^2)^{\frac{1}{2}}}\cos\phi, \quad V=\frac{1}{(1+f'^2)^{\frac{1}{2}}}\sin\phi.$$

The magnitudes of the second order are

$$L=\Sigma Xx_{11}=\frac{f''}{(1+f'^2)^{\frac{1}{2}}}, \quad N=\Sigma Xx_{22}=\frac{rf'}{(1+f'^2)^{\frac{1}{2}}}, \quad J=\Sigma Xx_{33}=-\frac{f}{(1+f'^2)^{\frac{1}{2}}},$$

$$I=\Sigma Xx_{23}=0, \quad K=\Sigma Xx_{31}=0, \quad M=\Sigma Xx_{12}=0.$$

Clearly the parametric curves are orthogonal to one another and are conjugate to one another; they therefore are the lines of curvature for all regions of rotation round two orthogonal planes.

When a region is minimal, the arithmetic mean of its principal linear curvatures is zero. For the foregoing regions of rotation, the necessary condition is

$$\frac{L}{A}+\frac{N}{B}+\frac{J}{C}=0.$$

When substitution is effected, the differential equation which must be satisfied by the function f is found to be

$$\frac{f''}{1+f'^2}+\frac{f'}{r}-\frac{1}{f}=0,$$

where

$$t=f(r).$$

Had the equation of the region been taken in the form $r=g(t)$, the equation satisfied by g would have been found to be

$$\frac{g''}{1+g'^2}+\frac{g'}{t}-\frac{1}{g}=0,$$

where t is now the independent variable in the equation.

As yet, it has not been possible to obtain the primitive of the foregoing ordinary equation of the second order which is satisfied by the function f. One exceedingly special solution is manifestly given by taking

$$f(r)=r,$$

so that the corresponding region is

$$x^2+y^2-z^2-v^2=0.$$

Nature of the primitive of the general Cartesian equation.

419. The differential equation of minimal regions is

$$(1+m^2+n^2)\,a+(1+n^2+l^2)\,b+(1+l^2+m^2)\,c-2mnf-2nlg-2lmh=0,$$

where $l,\ m,\ n,\ =v_x,\ v_y,\ v_z$, the first derivatives of v, and $a,\ b,\ c,\ f,\ g,\ h,\ =v_{xx},\ v_{yy},\ v_{zz},\ v_{yz},\ v_{zx},\ v_{xy}$, the second derivatives of v. No set of integral equations constituting the primitive of the differential equation is known.

In the first place, it is not difficult to prove that the equation possesses no intermediate integral of the first order of a form

$$\Phi(l, m, n, x, y, z, v)=0:$$

the necessary conditions are not satisfied.

In the second place, the general theory of partial differential equations* shews that the primitive of the equation contains two arbitrary functions, each of two arguments. Moreover it is known†, from Cauchy's general existence-theorem, that a primitive exists which is made uniquely determinate by the assignment of any arbitrary value $\phi(x, y, z)$ to v and of any arbitrary value $\psi(x, y, z)$ to one of the first derivatives of v, at all places on an assigned locus $\theta(x, y, z)=0$: provided this assigned locus is not a characteristic of the equation, in the Monge significance of that term. For the present equation, an outline of the Cauchy theorem is as follows.

Let the value $\psi(x, y, z)$ be assigned to the derivative v_x, that is, to l, along the locus $\theta=0$. We are to have $v=\phi(x, y, z)$ everywhere along that locus: and therefore the initial data give

$$l\,dx+m\,dy+n\,dz=d\phi$$
$$=\frac{\partial\phi}{\partial x}\,dx+\frac{\partial\phi}{\partial y}\,dy+\frac{\partial\phi}{\partial z}\,dz,$$

for all variations $dx,\ dy,\ dz$, such that

$$\frac{\partial\theta}{\partial x}\,dx+\frac{\partial\theta}{\partial y}\,dy+\frac{\partial\theta}{\partial z}\,dz=0.$$

Consequently, there exists a quantity α such that

$$l=\frac{\partial\phi}{\partial x}+\alpha\frac{\partial\theta}{\partial x},$$

$$m=\frac{\partial\phi}{\partial y}+\alpha\frac{\partial\theta}{\partial y},$$

$$n=\frac{\partial\phi}{\partial z}+\alpha\frac{\partial\theta}{\partial z},$$

* See my *Theory of Differential Equations*, vol. vi, § 184.

† *l.c.*, § 326.

for all values of x, y, z, consistent with $\theta = 0$. Now l, a first derivative of v, is given as equal to $\psi(x, y, z)$, so that α can be regarded as known over the locus; and consequently we can regard m and n, the other two first derivatives of v, as known over the locus. Accordingly, we take

$$l = \psi, \quad m = \psi_1, \quad n = \psi_2,$$

where ψ_1 and ψ_2, as well as ψ, are known functions over the locus.

In the same way, we have

$$a\,dx + h\,dy + g\,dz = dl$$
$$= \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy + \frac{\partial \psi}{\partial z}\,dz,$$

for all variations within the locus, that is, for all values of dx, dy, dz, such that

$$\frac{\partial \theta}{\partial x}\,dx + \frac{\partial \theta}{\partial y}\,dy + \frac{\partial \theta}{\partial z}\,dz = 0.$$

Consequently, there exists a variable magnitude ρ such that, along the locus,

$$a = \frac{\partial \psi}{\partial x} + \rho \frac{\partial \theta}{\partial x}, \quad h = \frac{\partial \psi}{\partial y} + \rho \frac{\partial \theta}{\partial y}, \quad g = \frac{\partial \psi}{\partial z} + \rho \frac{\partial \theta}{\partial z} = 0.$$

Similarly, from the deduced value ψ_1 possessed by m along the locus, we infer that there exists a variable magnitude σ such that, along the locus,

$$h = \frac{\partial \psi_1}{\partial x} + \sigma \frac{\partial \theta}{\partial x}, \quad b = \frac{\partial \psi_1}{\partial y} + \sigma \frac{\partial \theta}{\partial y}, \quad f = \frac{\partial \psi_1}{\partial z} + \sigma \frac{\partial \theta}{\partial z};$$

and from the deduced value ψ_2 possessed by n along the locus, we infer that there exists a variable magnitude τ such that, along the locus,

$$g = \frac{\partial \psi_2}{\partial x} + \tau \frac{\partial \theta}{\partial x}, \quad f = \frac{\partial \psi_2}{\partial y} + \tau \frac{\partial \theta}{\partial y}, \quad c = \frac{\partial \psi_2}{\partial z} + \tau \frac{\partial \theta}{\partial z}.$$

When the two values of h are equated, we have

$$\frac{\partial \psi}{\partial y} + \rho \frac{\partial \theta}{\partial y} = \frac{\partial \psi_1}{\partial x} + \sigma \frac{\partial \theta}{\partial x},$$

and therefore, on inserting the values of ψ and ψ_1,

$$\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x} + \alpha \frac{\partial \theta}{\partial x}\right) + \rho \frac{\partial \theta}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y} + \alpha \frac{\partial \theta}{\partial y}\right) + \sigma \frac{\partial \theta}{\partial x};$$

consequently

$$\left(\rho - \frac{\partial \alpha}{\partial x}\right)\frac{\partial \theta}{\partial y} = \left(\sigma - \frac{\partial \alpha}{\partial y}\right)\frac{\partial \theta}{\partial x}.$$

Similar treatment of the two values of g yields a relation

$$\left(\tau - \frac{\partial \alpha}{\partial z}\right)\frac{\partial \theta}{\partial x} = \left(\rho - \frac{\partial \alpha}{\partial x}\right)\frac{\partial \theta}{\partial z};$$

and similar treatment of the two values of f yields a relation

$$\left(\sigma - \frac{\partial \alpha}{\partial y}\right)\frac{\partial \theta}{\partial z} = \left(\tau - \frac{\partial \alpha}{\partial z}\right)\frac{\partial \theta}{\partial y}.$$

Consequently, there exists a single function β in terms of which ρ, σ, τ, can be expressed by the relations

$$\rho - \frac{\partial \alpha}{\partial x} = \beta \frac{\partial \theta}{\partial x},$$

$$\sigma - \frac{\partial \alpha}{\partial y} = \beta \frac{\partial \theta}{\partial y},$$

$$\tau - \frac{\partial \alpha}{\partial z} = \beta \frac{\partial \theta}{\partial z}.$$

When these values of ρ, σ, τ, are used, and the values of ψ, ψ_1, ψ_2, are inserted in the expressions obtained for the second derivatives of v, these second derivatives are expressible uniquely in terms of given quantities, of the magnitude α which is known, and of a magnitude β as yet unknown, in the forms

$$a = \beta \left(\frac{\partial \theta}{\partial x}\right)^2 + \frac{\partial^2 \phi}{\partial x^2} + \alpha \frac{\partial^2 \theta}{\partial x^2} + 2 \frac{\partial \alpha}{\partial x} \frac{\partial \theta}{\partial x},$$

$$b = \beta \left(\frac{\partial \theta}{\partial y}\right)^2 + \frac{\partial^2 \phi}{\partial y^2} + \alpha \frac{\partial^2 \theta}{\partial y^2} + 2 \frac{\partial \alpha}{\partial y} \frac{\partial \theta}{\partial y},$$

$$c = \beta \left(\frac{\partial \theta}{\partial z}\right)^2 + \frac{\partial^2 \phi}{\partial z^2} + \alpha \frac{\partial^2 \theta}{\partial z^2} + 2 \frac{\partial \alpha}{\partial z} \frac{\partial \theta}{\partial z},$$

$$f = \beta \frac{\partial \theta}{\partial y} \frac{\partial \theta}{\partial z} + \frac{\partial^2 \phi}{\partial y \partial z} + \alpha \frac{\partial^2 \theta}{\partial y \partial z} + \frac{\partial \alpha}{\partial y} \frac{\partial \theta}{\partial z} + \frac{\partial \alpha}{\partial z} \frac{\partial \theta}{\partial y},$$

$$g = \beta \frac{\partial \theta}{\partial z} \frac{\partial \theta}{\partial x} + \frac{\partial^2 \phi}{\partial z \partial x} + \alpha \frac{\partial^2 \theta}{\partial z \partial x} + \frac{\partial \alpha}{\partial z} \frac{\partial \theta}{\partial x} + \frac{\partial \alpha}{\partial x} \frac{\partial \theta}{\partial z},$$

$$h = \beta \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y} + \frac{\partial^2 \phi}{\partial x \partial y} + \alpha \frac{\partial^2 \theta}{\partial x \partial y} + \frac{\partial \alpha}{\partial x} \frac{\partial \theta}{\partial y} + \frac{\partial \alpha}{\partial y} \frac{\partial \theta}{\partial x}.$$

When these values of the second derivatives are substituted in the equation characteristic of a minimal region, the quantity β is thereby determinate, except under one single condition. This single condition arises when the equation

$$S = (1 + m^2 + n^2) \left(\frac{\partial \theta}{\partial x}\right)^2 + (1 + n^2 + l^2) \left(\frac{\partial \theta}{\partial y}\right)^2 + (1 + l^2 + m^2) \left(\frac{\partial \theta}{\partial z}\right)^2 - 2mn \frac{\partial \theta}{\partial y} \frac{\partial \theta}{\partial z} - 2nl \frac{\partial \theta}{\partial z} \frac{\partial \theta}{\partial x} - 2lm \frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial y} = 0$$

is satisfied.

When S does not vanish, the values of a, b, c, f, g, h, that is, of all the second derivatives of v, along the locus $\theta = 0$ are known.

By corresponding analysis—the details here are not of special additional significance—we can obtain the values of the derivatives of v of all orders

along the locus $\theta = 0$. The unknown quantity v can then be expressed uniquely in the form of a triply infinite series, the coefficients in which are known; the convergence of this series can be established in the usual manner; and consequently there results a unique value of v, which is definite through the values assigned along the locus $\theta = 0$ to v and to one of its derivatives.

Monge-Cauchy characteristics of the equation of minimal regions.

420. But this inference depends upon the assumption that the equation

$$S = 0$$

is not satisfied: it cannot be drawn when this equation is satisfied. Accordingly the initial values, which have been postulated over an assigned locus, determine a unique primitive of the characteristic equation and therefore determine a minimal region, provided the locus in question does not satisfy the equation

$$S = 0.$$

This is the equation of what usually is called a Monge characteristic.

Though this particular equation cannot be integrated at the present stage of the investigation because it contains l, m, n, which are the first derivatives of the undetermined variable v, an interpretation can be given to the equation as to a general property of the locus $\theta(x, y, z) = 0$. Let dx, dy, dz, and $\delta x, \delta y, \delta z$, denote any two infinitesimal displacements in that locus, so that

$$\frac{\partial\theta}{\partial x}dx + \frac{\partial\theta}{\partial y}dy + \frac{\partial\theta}{\partial z}dz = 0, \quad \frac{\partial\theta}{\partial x}\delta x + \frac{\partial\theta}{\partial y}\delta y + \frac{\partial\theta}{\partial z}\delta z = 0;$$

then

$$\frac{\frac{\partial\theta}{\partial x}}{dy\,\delta z - dz\,\delta y} = \frac{\frac{\partial\theta}{\partial y}}{dz\,\delta x - dx\,\delta z} = \frac{\frac{\partial\theta}{\partial z}}{dx\,\delta y - dy\,\delta x}.$$

Hence any two arbitrary directions in the Monge characteristic satisfy the relation

$$\begin{aligned}&(dy\,\delta z - dz\,\delta y)^2 + (dz\,\delta x - dx\,\delta z)^2 + (dx\,\delta y - dy\,\delta x)^2\\ &\quad + (v_y^2 + v_z^2)(dy\,\delta z - dz\,\delta y)^2 - 2v_y v_z (dz\,\delta x - dx\,\delta z)(dx\,\delta y - dy\,\delta x)\\ &\quad + (v_z^2 + v_x^2)(dz\,\delta x - dx\,\delta z)^2 - 2v_z v_x (dx\,\delta y - dy\,\delta x)(dy\,\delta z - dz\,\delta y)\\ &\quad + (v_x^2 + v_y^2)(dx\,\delta y - dy\,\delta x)^2 - 2v_x v_y (dy\,\delta z - dz\,\delta y)(dz\,\delta x - dx\,\delta z) = 0,\end{aligned}$$

where the values v_x, v_y, v_z, have been inserted in place of l, m, n, respectively. Now

$$\begin{aligned}dx\,\delta v - dv\,\delta x &= (v_x\delta x + v_y\delta y + v_z\delta z)\,dx - (v_x dx + v_y dy + v_z dz)\,\delta x\\ &= v_y(dx\,\delta y - dy\,\delta x) - v_z(dz\,\delta x - dx\,\delta z),\\ dy\,\delta v - dv\,\delta y &= v_z(dy\,\delta z - dz\,\delta y) - v_x(dx\,\delta y - dy\,\delta x),\\ dz\,\delta v - dv\,\delta z &= v_x(dz\,\delta x - dx\,\delta z) - v_y(dy\,\delta z - dz\,\delta y).\end{aligned}$$

By means of these expressions, the foregoing relation can be changed to the form

$$(dy\,\delta z - dz\,\delta y)^2 + (dz\,\delta x - dx\,\delta z)^2 + (dx\,\delta y - dy\,\delta x)^2$$
$$+ (dx\,\delta v - dv\,\delta x)^2 + (dy\,\delta v - dv\,\delta y)^2 + (dz\,\delta v - dv\,\delta z)^2 = 0.$$

Consequently along the locus $\theta(x, y, z) = 0$ in the minimal region, we have

$$(\Sigma dx^2)(\Sigma \delta x^2) - (\Sigma dx\,\delta x)^2 = 0:$$

or, if ds and δs be the arc-elements in the region along the assumed arbitrary directions and if χ be their inclination, this relation is

$$ds^2 . \delta s^2 . \sin^2\chi = 0.$$

Now χ, the inclination of the arbitrarily assumed directions, does not vanish: and therefore

$$ds^2 . \delta s^2 = 0,$$

for all arbitrary directions. Thus, everywhere along the locus $\theta = 0$ within the region, we have the arc-element equal to zero: or *the Monge characteristic is a surface of zero arc-lengths*, and its differential equation in differential elements can be taken in the form

$$A dp^2 + 2H dp dq + 2G dp dr + B dq^2 + 2F dq dr + C dr^2 = 0.$$

Arguments of the arbitrary functions in the primitive.

421. There is another source of a different nature, out of which the equation satisfied by the Monge characteristic arises: and the analytical relation between the two sources indicates not merely the fact, but the reason of the fact, that initial data for the precise determination of the arbitrary elements in the primitive must be given over a locus other than the Monge characteristic.

It is known, as stated before, that the primitive of the minimal equation contains two arbitrary functions in its full expression. Let u denote the complete argument of either of these functions; that complete argument will usually* involve two combinations of the variables, the possible forms of which are unessential for the immediate investigation. This argument u involves some, if not all, of the original independent variables x, y, z: let z be one such variable occurring in u. In that event, we change the independent variables from x, y, z, to x, y, u, and effect the corresponding transformation upon the equation. Derivatives of v with regard to the new set of variables are denoted by $\frac{\delta v}{\delta x}, \frac{\delta v}{\delta y}, \frac{\delta v}{\delta u}$; and so we have

$$l\,dx + m\,dy + n\,dz = dv = \frac{\delta v}{\delta x}\,dx + \frac{\delta v}{\delta y}\,dy + \frac{\delta v}{\delta u}\,du.$$

* *l.c.* (p. 334, footnote), § 184.

We write

$$\frac{\delta z}{\delta x} = p, \quad \frac{\delta z}{\delta y} = q, \quad \frac{\delta z}{\delta u} = \mu,$$

so that

$$dz = p\,dx + q\,dy + \mu\,du;$$

when this value of dz is substituted, and the coefficients of the independent increments dx, dy, du, are compared, we find

$$l + np = \frac{\delta v}{\delta x}, \quad m + nq = \frac{\delta v}{\delta y}, \quad n\mu = \frac{\delta v}{\delta u}.$$

Proceeding similarly from the relations

$$a\,dx + h\,dy + g\,dz = dl = \frac{\delta l}{\delta x}\,dx + \frac{\delta l}{\delta y}\,dy + \frac{\delta l}{\delta u}\,du,$$

$$h\,dx + b\,dy + f\,dz = dm = \frac{\delta m}{\delta x}\,dx + \frac{\delta m}{\delta y}\,dy + \frac{\delta m}{\delta u}\,du,$$

$$g\,dx + f\,dy + c\,dz = dn = \frac{\delta n}{\delta x}\,dx + \frac{\delta n}{\delta y}\,dy + \frac{\delta n}{\delta u}\,du,$$

we find

$$a + gp = \frac{\delta l}{\delta x}, \quad h + gq = \frac{\delta l}{\delta y}, \quad g\mu = \frac{\delta l}{\delta u},$$

$$h + fp = \frac{\delta m}{\delta x}, \quad b + fq = \frac{\delta m}{\delta y}, \quad f\mu = \frac{\delta m}{\delta u},$$

$$g + cp = \frac{\delta n}{\delta x}, \quad f + cq = \frac{\delta n}{\delta y}, \quad c\mu = \frac{\delta n}{\delta u}.$$

The three equations involving f and c require the relation

$$\frac{\delta n}{\delta y} - \frac{q}{\mu}\frac{\delta n}{\delta u} = \frac{1}{\mu}\frac{\delta m}{\delta u};$$

when the values of m and n already obtained are substituted, the relation is satisfied identically. The three equations involving g and c require the relation

$$\frac{\delta n}{\delta x} - \frac{p}{\mu}\frac{\delta n}{\delta u} = \frac{1}{\mu}\frac{\delta l}{\delta u};$$

again, this relation is satisfied identically when the values of l and n are substituted. The two values of h are

$$h = \frac{\delta m}{\delta x} - fp = \frac{\delta m}{\delta x} - p\frac{\delta n}{\delta y} + cpq,$$

$$h = \frac{\delta l}{\delta y} - gq = \frac{\delta l}{\delta y} - q\frac{\delta n}{\delta x} + cpq;$$

their equality requires the relation

$$\frac{\delta m}{\delta x} - p\frac{\delta n}{\delta y} = \frac{\delta l}{\delta y} - q\frac{\delta n}{\delta x},$$

which is satisfied identically when the values of l, m, n, are substituted. As, however, this relation does not involve the magnitude μ, it will be retained.

Now among all the derivatives with regard to u, the argument of the arbitrary function, as they occur in the expressions for the original derivatives $l, m, n, a, b, c, f, g, h$, the highest derivative with respect to u is

$$\frac{\delta^2 v}{\delta u^2};$$

and it arises directly in the expression for the original derivative c, through the term $\frac{\delta n}{\delta u}$. But that second derivative of v does not occur in the other two new derivatives of x, nor does it arise through $\frac{\delta l}{\delta x}, \frac{\delta l}{\delta y}, \frac{\delta m}{\delta x}, \frac{\delta m}{\delta y}$. It therefore is convenient to retain the quantity c, and to maintain the other surviving relations in the forms

$$f = \frac{\delta n}{\delta y} - cq,$$

$$g = \frac{\delta n}{\delta x} - cp,$$

$$h = \frac{\delta l}{\delta y} - q\frac{\delta n}{\delta x} + cpq = \frac{\delta m}{\delta x} - p\frac{\delta n}{\delta y} + cpq,$$

$$b = \frac{\delta m}{\delta y} - q\frac{\delta n}{\delta y} + cq^2,$$

$$a = \frac{\delta l}{\delta x} - p\frac{\delta n}{\delta x} + cp^2.$$

When these values are substituted in the minimal differential equation, it takes the form

$$U + cT = 0,$$

where

$$T = (1 + m^2 + n^2)\,p^2 + (1 + n^2 + l^2)\,q^2 + 1 + l^2 + m^2 + 2mnq + 2nlp - 2lmpq,$$

and where U is a linear combination of the derivatives of l, m, n, with respect to x and to y.

When the proper value of v and the values of its derivatives are substituted, the minimal equation is to be satisfied. Consider the occurrence of the arbitrary function of u in the value of v: it may occur, not merely alone, but through derivatives of successive orders. Let the derivative of highest order, which does occur in v, be that of order i, which will be assumed to be a finite integer. Then the highest derivative of the arbitrary function, occurring in the minimal equation, occurs through c: it occurs through c alone, being of order $i+2$; and all other occurrences are of order not greater than $i+1$. As the minimal equation is to be satisfied identically, the requirement can only be met (so far as the occurrence of the derivative of order $i+2$ is concerned) by having the coefficient of c actually zero: that is, we must have

$$T = 0.$$

Now u involves z certainly, and usually also x and y: moreover, p and q are the values of $\frac{\delta z}{\delta x}$ and $\frac{\delta z}{\delta y}$ respectively when u is constant, so that

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\delta z}{\delta x}=0, \quad \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\delta z}{\delta y}=0.$$

When the values of p and q, thus given, are substituted in T, the new form of the condition is

$$(1+m^2+n^2)\left(\frac{\partial u}{\partial x}\right)^2+(1+n^2+l^2)\left(\frac{\partial u}{\partial y}\right)^2+(1+l^2+m^2)\left(\frac{\partial u}{\partial z}\right)^2$$
$$-2mn\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}-2nl\frac{\partial u}{\partial z}\frac{\partial u}{\partial x}-2lm\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}=0,$$

which accordingly is *a differential equation satisfied by the argument of any arbitrary function occurring in the primitive of the minimal equation.*

This equation, for the determination of the argument u of an arbitrary function occurring in the primitive, is the same as the equation satisfied by the variable θ, when $\theta(x, y, z)=0$ (or = a constant) is a locus to be excluded as a possible locus along which initial conditions are given. One reason for the exclusion now is obvious, after the exclusion has been established. Were the initial data given as holding along such a surface, then u is zero or constant along the surface: an arbitrary function of u, and all derivatives of that arbitrary function of u, then subside into constants only, which at the best are merely arbitrary constants. Nothing then remains, to assist in the determination of the form of the arbitrary function so that the initial data shall be satisfied: those data, in the postulated circumstances, are inadequate for the precise determination of the primitive.

CHAPTER XXIV.

THE CURVATURES OF GENERAL PRIMARY AMPLITUDES.

Primary amplitudes.

422. The purpose of this chapter* is to sketch, though only in brief outline, the different kinds of measures of the curvature of an amplitude of n dimensions, existing in a homaloidal space of $n+1$ dimensions which will be called the plenary space. There will also be indicated, for each kind of measure of curvature, the invariantive principal magnitudes round which the measures in that kind are grouped.

It is assumed, specifically, that the amplitude is of one (and of only one) dimension less than the homaloidal space in which it exists. Such an amplitude is here called a *primary amplitude*. It is further assumed that the amplitude is properly primary: that is, the amplitude is not capable of being developed (in the sense of § 286, or the earlier significance of § 186) into one which exists in a homaloidal space of only n dimensions.

An amplitude of two dimensions less than the plenary homaloidal space could conveniently be called a secondary amplitude: so that our primary amplitude is assumed not to be capable of development into a secondary amplitude. We similarly could have tertiary amplitudes, and others of successively decreasing dimensions.

In dealing with primary amplitudes, there is an analytical simplicity as compared with the discussion of less extensive amplitudes in the homaloidal space: and the essential source of this simplicity is the property that, at any point of the amplitude, there exists a unique normal linear direction. Thus there is the instance of the unique direction, normal to a region in quadruple homaloidal space: whereas there is a plane locus of linear directions, normal to a surface existing freely in that region. There is also a collateral geometrical simplicity in dealing with primary amplitudes: the circular curvature of a geodesic, appertaining to the amplitude, is its curvature relative to the enclosing homaloidal space: and this circular curvature, made the measure of linear curvature of the amplitude in the direction of the geodesic curve, becomes the kernel of subsequent investigations in measures of curvature. But for secondary amplitudes, as has appeared in the discussion of surfaces in quadruple space, there will arise the circular curvature of a geodesic in the secondary manifold, estimated with reference to the external

* In connection with the earlier sections of this chapter, Bianchi's *Lezioni di geometria differenziale* (3rd edition, 1924), vol. ii, part 2, may be consulted.

homaloidal space: there can arise the curvature of that geodesic relative to a tangential geodesic belonging to an embracing primary manifold (if any), and there then can arise the curvature of that tangential geodesic relative to the external homaloidal space. The complication naturally increases with the earlier increases in the grade-number of the discussed manifold: then subsides, ultimately settling into the simplicity of the curvatures of an unfettered curve in the homaloidal space.

Accordingly, we assume that a position in the plenary homaloidal space of $(n+1)$ dimensions is determinate by its $n+1$ coordinates $y_1, y_2, \ldots, y_{n+1}$; these are not necessarily of the Cartesian type, though explanations are simplified in expression by an assumption that they are Cartesian. A primary manifold of n dimensions in that space is obtained by expressing the point-coordinates in terms of n parameters $x_1, x_2, \ldots, x_n$; and, as a condition that the manifold is essentially primary, we require that the determinants

$$\left\| \begin{array}{cccc} \frac{dy_1}{dx_1}, & \frac{dy_2}{dx_1}, & \ldots\ldots, & \frac{dy_{n+1}}{dx_1} \\ \frac{dy_1}{dx_2}, & \frac{dy_2}{dx_2}, & \ldots\ldots, & \frac{dy_{n+1}}{dx_2} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \frac{dy_1}{dx_n}, & \frac{dy_2}{dx_n}, & \ldots\ldots, & \frac{dy_{n+1}}{dx_n} \end{array} \right\|$$

do not vanish. The arc-element ds in the primary manifold is obtained by substituting the values of $y_1, \ldots, y_{n+1}$, in the definition

$$ds^2 = \sum_{r=1}^{n+1} dy_r^2;$$

and we thereby obtain the arc-element in the form

$$ds^2 = \sum_i \sum_j A_{ij} dx_i dx_j,$$

where there is only a single term in dx_i^2, being $A_{ii} dx_i^2$, (for $i = 1, \ldots, n$), and where, as $A_{ij} = A_{ji}$, the full term involving $dx_i dx_j$ is $2A_{ij} dx_i dx_j$. These quantities A, functions of the parameter x, are fundamental. Also, we write

$$\Delta = \left| \begin{array}{cccc} A_{11}, & A_{12}, & \ldots, & A_{1n} \\ A_{12}, & A_{22}, & \ldots, & A_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ A_{1n}, & A_{2n}, & \ldots, & A_{nn} \end{array} \right|;$$

and we define a_{ij}, the minor of A_{ij} in Δ, by the rule

$$a_{ii} = \frac{\partial \Delta}{\partial A_{ii}}, \quad a_{ij} = \frac{1}{2} \frac{\partial \Delta}{\partial A_{ij}}, \qquad (i \gtrless j).$$

Christoffel symbols: Riemann four-index symbols.

423. Various derivatives, and combinations of the derivatives, of the magnitudes A are required. Thus, we have

$$[ij, k] = \frac{1}{2}\left(\frac{\partial A_{ik}}{\partial x_j} + \frac{\partial A_{jk}}{\partial x_i} - \frac{\partial A_{ij}}{\partial x_k}\right),$$

where $[ij, k]$ is usually called a Christoffel symbol of the first kind; and, as

$$A_{rs} = \sum_{t=1}^{n+1} \frac{\partial y_t}{\partial x_r}\frac{\partial y_t}{\partial x_s},$$

we find, on actually substituting these values for A_{ik}, A_{jk}, A_{ij},

$$[ij, k] = \sum_{t=1}^{n+1} \frac{\partial y_t}{\partial x_k}\frac{\partial^2 y_t}{\partial x_i \partial x_j}.$$

Thus, for quadruple space, these quantities $[ij, k]$ are the former eighteen magnitudes denoted (§ 273) by $\alpha_1, \alpha_2, \alpha_3, \ldots, \omega_1, \omega_2, \omega_3$.

Further, we have

$$\{ij, k\} = \sum_{\mu=1}^{n} \frac{a_{k\mu}}{\Delta}[ij, \mu],$$

where $\{ij, k\}$ is usually called a Christoffel symbol of the second kind*. Thus, for quadruple space, these quantities $\{ij, k\}$ are the former eighteen magnitudes denoted by Γ_{ij}, Δ_{ij}, Θ_{ij}, where Γ arises when $k = 1$, Δ arises when $k = 2$, Θ arises when $k = 3$, and there are the six combinations $i, j, = 1, 2, 3$.

Finally, for the present purpose, there are the Riemann four-index symbols, given by the definition

$$(ij, kl) = \frac{1}{2}\left(\frac{\partial^2 A_{il}}{\partial x_j \partial x_k} - \frac{\partial^2 A_{ik}}{\partial x_j \partial x_l} - \frac{\partial^2 A_{jl}}{\partial x_i \partial x_k} + \frac{\partial^2 A_{jk}}{\partial x_i \partial x_l}\right)$$
$$+ \frac{1}{\Delta}\sum_{\lambda}\sum_{\mu}\{[il, \mu][jk, \lambda] - [ik, \mu][jl, \lambda]\}\, a_{\lambda\mu},$$

where, in the last summation, λ and μ separately take all values from 1 to n, so that the coefficient of a_{11} is

$$[il, 1][jk, 1] - [ik, 1][jl, 1],$$

and the full coefficient of a_{12} is

$$[il, 1][jk, 2] + [il, 2][jk, 1] - [ik, 1][jl, 2] - [ik, 2][jl, 1].$$

An alternative form, in terms of the symbols $\{ij, k\}$, can be given to the second line in the expression. We have

$$\{pq, m\} = \sum_{\mu} \frac{a_{m\mu}}{\Delta}[pq, \mu],$$

so that

$$\sum_{r} A_{rs}\{jl, r\} = \sum_{\mu}\sum_{r} A_{rs}\frac{a_{r\mu}}{\Delta}[jl, \mu]$$
$$= [jl, s],$$

* Both symbols, in slightly different notation, occur in Christoffel's memoir on the transformation of quadratic differential expressions, *Crelle*, vol. lxx (1869), pp. 46–70.

because $\Sigma A_{rs} a_{r\mu} = 0$ when s and μ are different, and $\Sigma A_{r\mu} a_{r\mu} = \Delta$. Consequently

$$\sum_{r}\sum_{s} A_{rs} \{jl, r\} \{ik, s\} = \sum_{s} \{ik, s\} [jl, s]$$
$$= \sum_{\mu}\sum_{s} \frac{a_{s\mu}}{\Delta} [ik, \mu] [jl, s];$$

and we therefore could write

$$(ij, kl) = \frac{1}{2}\left(\frac{\partial^2 A_{il}}{\partial x_j \partial x_k} - \frac{\partial^2 A_{ik}}{\partial x_j \partial x_l} - \frac{\partial^2 A_{jl}}{\partial x_i \partial x_k} + \frac{\partial^2 A_{jk}}{\partial x_i \partial x_l}\right)$$
$$+ \sum_{\lambda}\sum_{\mu} A_{\lambda\mu} [\{il, \mu\} \{jk, \lambda\} - \{ik, \mu\} \{jl, \lambda\}],$$

the summation being taken, as before, for $\lambda, \mu, = 1, \ldots, n$, independently of one another.

General properties of geodesics in a primary amplitude.

424. The following results are preliminary to the discussion of the curvatures of a primary amplitude.

(i) The direction of the radius of circular curvature at any point of a geodesic in a primary amplitude is the normal to the amplitude at that point.

Let t denote a current variable for the geodesic, distinct from the arc-variable s until the critical equations have been formed; and let

$$f(y_1, y_2, \ldots, y_{n+1}) = 0$$

be the point-coordinate equation establishing the primary amplitude. To obtain a geodesic, we must make

$$\int \left\{\sum_{r=1}^{n+1} \left(\frac{dy_r}{dt}\right)^2\right\}^{\frac{1}{2}} dt$$

a minimum, subject to the condition $f = 0$; to satisfy this requirement, the critical equations are

$$\frac{d}{dt}\left[\frac{\frac{dy_r}{dt}}{\left\{\Sigma \left(\frac{dy_r}{dt}\right)^2\right\}^{\frac{1}{2}}}\right] = \lambda \frac{\partial f}{\partial y_r},$$

for $r = 1, \ldots, n+1$, where λ is a multiplier, indeterminate so far as the formation of the equations is concerned. These equations are

$$\frac{d^2 y_r}{ds^2} = \lambda \frac{dt}{ds} \frac{\partial f}{\partial y_r},$$

for $r = 1, \ldots, n+1$. Now the quantities $\frac{d^2 y_1}{ds^2}, \ldots, \frac{d^2 y_{n+1}}{ds^2}$, are proportional to the direction-cosines of the radius of circular curvature of the geodesic; and the quantities $\frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_{n+1}}$, are proportional to the direction-cosines of the normal to the primary amplitude at the point; hence the proposition.

If the radius of circular curvature be ρ, and if the direction-cosines of the normal to the amplitude are $Y_1, \ldots, Y_{n+1}$, we have

$$\frac{d^2 y_r}{ds^2} = \frac{1}{\rho} Y_r,$$

for $r = 1, \ldots, n+1$.

(ii) We require the equations of the geodesics in parametric form. For the expression in terms of the parameters which are independent of one another, we have to make

$$\int \left\{ \sum_i \sum_j A_{ij} \frac{dx_i}{dt} \frac{dx_j}{dt} \right\}^{\frac{1}{2}} dt$$

a minimum, there being no relation among $x_1, \ldots, x_n$. The critical equations are

$$\frac{d}{dt} \left\{ \frac{\sum_l A_{kl} \frac{dx_l}{dt}}{\left(\sum_i \sum_j A_{ij} \frac{dx_i}{dt} \frac{dx_j}{dt} \right)^{\frac{1}{2}}} \right\} - \frac{1}{2} \frac{\sum_i \sum_j \frac{\partial A_{ij}}{\partial x_k} \frac{dx_i}{dt} \frac{dx_j}{dt}}{\left(\sum_i \sum_j A_{ij} \frac{dx_i}{dt} \frac{dx_j}{dt} \right)^{\frac{3}{2}}} = 0,$$

for $k = 1, \ldots, n$: that is,

$$\frac{d}{ds} \left(\sum_l A_{kl} \frac{dx_l}{ds} \right) = \frac{1}{2} \sum_i \sum_j \frac{\partial A_{ij}}{\partial x_k} \frac{dx_i}{ds} \frac{dx_j}{ds}.$$

Expressing the left-hand side in full, and then transferring to the right-hand side the terms which do not involve $\frac{d^2 x_l}{ds^2}$, for $l = 1, \ldots, n$, we have the equation in the form

$$\sum_l A_{kl} x_l'' + \sum_i \sum_j [ij, k] x_i' x_j' = 0,$$

for $k = 1, \ldots, n$. When these equations, linear in the second derivatives of $x_1, \ldots, x_n$, are resolved for those n derivatives, they yield

$$x_k'' + \sum_i \sum_j \{ij, k\} x_i' x_j' = 0,$$

which, for $k = 1, \ldots, n$, are the canonical equations of a geodesic in the primary amplitude.

(iii) It is obvious from geometrical considerations, and this obvious inference can be verified by purely analytical calculations, that if a geodesic of a primary amplitude lies in a secondary amplitude, which itself lies wholly in that primary amplitude, the said geodesic is also geodesic in that secondary amplitude. Likewise for a geodesic of a primary amplitude which lies within an amplitude of any grade-index greater than unity.

Circular curvature of an amplitudinal geodesic.

425. The circular curvature of a geodesic in the primary amplitude can be obtained as follows. Let $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{n+1}$, denote current point-coordinates

in the plenary space; the equation of the flat amplitude, tangential to the primary amplitude, is

$$\begin{vmatrix} \bar{y}_1 - y_1, & \bar{y}_2 - y_2, & \dots\dots, & \bar{y}_{n+1} - y_{n+1} \\ \frac{\partial y_1}{\partial x_1}, & \frac{\partial y_2}{\partial x_1}, & \dots\dots, & \frac{\partial y_{n+1}}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2}, & \frac{\partial y_2}{\partial x_2}, & \dots\dots, & \frac{\partial y_{n+1}}{\partial x_2} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{\partial y_1}{\partial x_n}, & \frac{\partial y_2}{\partial x_n}, & \dots\dots, & \frac{\partial y_{n+1}}{\partial x_n} \end{vmatrix} = 0.$$

Let the left-hand side be written

$$Z_1(\bar{y}_1 - y_1) + Z_2(\bar{y}_2 - y_2) + \dots + Z_{n+1}(\bar{y}_{n+1} - y_{n+1}) = 0,$$

so that $Y_1, Y_2, \dots, Y_{n+1}$, are proportional to $Z_1, Z_2, \dots, Z_{n+1}$. Also (§ 16)

$$\sum_{r=1}^{n+1} Z_r^2 = \begin{vmatrix} A_{11}, & A_{12}, & \dots, & A_{1n} \\ A_{12}, & A_{22}, & \dots, & A_{2n} \\ \dots\dots\dots\dots\dots\dots\dots \\ A_{1n}, & A_{2n}, & \dots, & A_{nn} \end{vmatrix} = \Delta,$$

so that the direction-cosines of the normal to the primary amplitude are

$$\Delta^{-\frac{1}{2}} Z_1, \ \Delta^{-\frac{1}{2}} Z_2, \ \dots, \ \Delta^{-\frac{1}{2}} Z_{n+1}.$$

Let Π denote the perpendicular from a point $\eta_1, \eta_2, \dots, \eta_{n+1}$, adjacent to $y_1, y_2, \dots, y_{n+1}$, drawn to this tangential amplitude: then, by projection,

$$\Pi = Y_1(\eta_1 - y_1) + Y_2(\eta_2 - y_2) + \dots + Y_{n+1}(\eta_{n+1} - y_{n+1}),$$

and therefore

$$\Pi \Delta^{\frac{1}{2}} = \begin{vmatrix} \eta_1 - y_1, & \dots\dots, & \eta_{n+1} - y_{n+1} \\ \frac{\partial y_1}{\partial x_1}, & \dots\dots, & \frac{\partial y_{n+1}}{\partial x_1} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{\partial y_1}{\partial x_n}, & \dots\dots, & \frac{\partial y_{n+1}}{\partial x_n} \end{vmatrix}.$$

Now, if δs be the element of arc between the two points (y) and (η),

$$\eta_r - y_r = y_r' \delta s + \frac{1}{2} y_r'' \delta s^2 + \dots,$$

for all the values of r. We have

$$y_r' = \sum_{i=1}^{n} \frac{\partial y_r}{\partial x_i} x_i',$$

for all the $n+1$ values of r, and

$$\sum_{r=1}^{n} Y_r \frac{\partial y_r}{\partial x_i} = 0,$$

for all the n values of i.

When substitution of the values of the quantities $\eta_r - y_r$ takes place in Π, the coefficient of the first power of δs vanishes. The term in δs^2 survives and becomes the governing term; and so, neglecting powers of δs higher than the second, we have

$$\frac{2\Pi}{\delta s^2}\Delta^{\frac{1}{2}} = \begin{vmatrix} y_1'', & \ldots\ldots, & y''_{n+1} \\ \dfrac{\partial y_1}{\partial x_1}, & \ldots\ldots, & \dfrac{\partial y_{n+1}}{\partial x_1} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \dfrac{\partial y_1}{\partial x_n}, & \ldots\ldots, & \dfrac{\partial y_{n+1}}{\partial x_n} \end{vmatrix}$$

$$= \Delta^{\frac{1}{2}} \sum_{r=1}^{n+1} Y_r y_r''.$$

But

$$y_r'' = \sum_i \frac{\partial y_r}{\partial x_i}\frac{d^2x_i}{ds^2} + \sum\sum_{i\ j} \frac{\partial^2 y_r}{\partial x_i \partial x_j}\frac{dx_i}{ds}\frac{dx_j}{ds}.$$

We take magnitudes L_{ij} (for $i, j, = 1, \ldots, n$) of the second kind, defined by the equations

$$L_{ij} = \sum_{r=1}^{n+1} Y_r \frac{\partial^2 y_r}{\partial x_i \partial x_j};$$

and then

$$\Sigma Y_r y_r'' = \sum_i \left\{\frac{d^2x_i}{ds^2}\left(\sum_{r=1}^{n+1} Y_r \frac{\partial y_r}{\partial x_i}\right)\right\} + \sum\sum_{i\ j} \left\{\frac{dx_i}{ds}\frac{dx_j}{ds}\left(\sum_{r=1}^{n+1} Y_r \frac{\partial^2 y_r}{\partial x_i \partial x_j}\right)\right\}$$

$$= \sum\sum_{i\ j} L_{ij}\frac{dx_i}{ds}\frac{dx_j}{ds}.$$

Moreover, in the limit as δs tends towards a zero value, the curvature $1/\rho$ of the curve along the direction $x_1', \ldots, x_n'$, is

$$\frac{1}{\rho} = \operatorname*{Limit}_{\delta s \to 0} \frac{2\Pi}{ds^2}.$$

Consequently the circular curvature of the normal section of the primary amplitude, being the circular curvature of the geodesic through the tangential direction in the normal section, is

$$\frac{1}{\rho} = \sum\sum_{i\ j} L_{ij}\frac{dx_i}{ds}\frac{dx_j}{ds}.$$

Lines of curvature: principal measures of curvature: assumption that the amplitude is not developable.

426. This circular measure of an amplitudinal geodesic is taken as the measure of linear curvature of the amplitude in any arbitrarily selected direction, represented by the parametric variables $x_1', \ldots, x_n'$, which are subject to the relation

$$\sum\sum_{i\ j} A_{ij}\frac{dx_i}{ds}\frac{dx_j}{ds} = 1.$$

There are lines of curvature for the amplitude. These may be defined as the directions in the amplitude along which (for all the possible directions) the foregoing measure of curvature has a maximum or a minimum value. They may also be defined as the directions along which the amplitudinal normals at consecutive points intersect: the two definitions being analytically equivalent.

We proceed from the former definition. The maximum or the minimum values, and the directions along which these values are attained, are determined by the critical equations

$$\sum_{i=1}^{n} L_{ri}x_i' = \frac{1}{\rho}\sum_{i=1}^{n} A_{ri}x_i',$$

for $r = 1, \ldots, n$. Thus the *principal values* of the curvature are the roots of the equation

$$\begin{vmatrix} L_{11}-\frac{1}{\rho}A_{11}, & L_{12}-\frac{1}{\rho}A_{12}, & \ldots, & L_{1n}-\frac{1}{\rho}A_{1n} \\ L_{12}-\frac{1}{\rho}A_{12}, & L_{22}-\frac{1}{\rho}A_{22}, & \ldots, & L_{2n}-\frac{1}{\rho}A_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ L_{1n}-\frac{1}{\rho}A_{1n}, & L_{2n}-\frac{1}{\rho}A_{2n}, & \ldots, & L_{nn}-\frac{1}{\rho}A_{nn} \end{vmatrix} = 0,$$

of degree n in ρ. The symmetric combinations

$$\Sigma\frac{1}{\rho_1}, \quad \Sigma\frac{1}{\rho_1\rho_2}, \quad \ldots\ldots, \quad \Sigma\frac{1}{\rho_1\rho_2\cdots\rho_{n-1}}, \quad \frac{1}{\rho_1\rho_2\cdots\rho_{n-1}\rho_n},$$

are the *principal measures of linear curvature* of the amplitude. In particular, we call Θ, where

$$\Theta = \frac{1}{\rho_1\rho_2\cdots\rho_{n-1}\rho_n} = \frac{|L_{ij}|}{|A_{ij}|},$$

the corporate measure of curvature of the amplitude.

The equivalence of the two preceding definitions of a line of curvature is easily established. The normal to the amplitude at any point is

$$\frac{\bar{y}_1-y_1}{Y_1} = \frac{\bar{y}_2-y_2}{Y_2} = \ldots = \frac{\bar{y}_{n+1}-y_{n+1}}{Y_{n+1}} = l,$$

where l is a current parameter for the normal straight line. If this normal meets the normal at a consecutive point (y_r+dy_r) of the amplitude, we must have

$$dy_r + l\,dY_r + Y_r\,dl = 0,$$

for $r = 1, \ldots, n+1$: that is, for these values of r,

$$\sum_{i=1}^{n}\frac{\partial y_r}{\partial x_i}dx_i + l\sum_{i=1}^{n}\frac{\partial Y_r}{\partial x_i}dx_i + Y_r\,dl = 0.$$

Now

$$\sum_{r=1}^{n}\frac{\partial y_r}{\partial x_i}\frac{\partial y_r}{\partial x_j} = A_{ij};$$

also

$$\sum_{r=1}^{n}\frac{\partial y_r}{\partial x_i}Y_r = 0,$$

and therefore

$$\sum_{r=1}^{n} \frac{\partial y_r}{\partial x_i} \frac{\partial Y_r}{\partial x_j} = -\sum_{r=1}^{n} Y_r \frac{\partial^2 y_r}{\partial x_i \partial x_j} = -L_{ij}.$$

Hence, multiplying the foregoing typical equation by $\frac{\partial y_r}{\partial x_i}$ and adding for all the values of r, we have

$$\sum_{i=1}^{n} A_{ij} dx_i - l \sum_{i=1}^{n} L_{ij} dx_i = 0$$

for all the values of j: that is,

$$\sum_{i=1}^{n} A_{ij} x_i' = l \sum_{i=1}^{n} L_{ij} x_i'.$$

Multiplying by x_1', ..., x_j', ..., x_n' and adding, we have

$$1 = \frac{l}{\rho},$$

and so the equations are

$$\frac{1}{\rho} \sum_{i=1}^{n} A_{ij} x_i' = \sum_{i=1}^{n} L_{ij} x_i',$$

for $j = 1, \ldots, n$. These are the same equations as before, for the lines of curvature and for the magnitude of the curvature.

The conditions, that the selected parametric curves shall be lines of curvature, are easily obtained, from the equations

$$\sum_{i=1}^{n} L_{ri} x_i' = \frac{1}{\rho} \sum_{i=1}^{n} A_{ri} x_i',$$

for $r = 1, \ldots, n$. Let the curve, for which $x_2, \ldots, x_n$, are invariable and along which x_1 is variable, be a line of curvature, and let the radius of curvature be ρ_1; then the foregoing equations must be satisfied by

$$x_2' = 0, \quad x_3' = 0, \quad \ldots, \quad x_n' = 0, \quad x_1' \neq 0, \quad \rho = \rho_1.$$

Then

$$L_{11} = \frac{1}{\rho_1} A_{11}, \quad L_{12} = \frac{1}{\rho_1} A_{12}, \quad \ldots, \quad L_{1n} = \frac{1}{\rho_1} A_{1n}.$$

Similarly, if a second line of curvature be

$$x_1' = 0, \quad x_3' = 0, \quad \ldots, \quad x_n' = 0, \quad x_2' \neq 0, \quad \rho = \rho_2,$$

we have

$$L_{12} = \frac{1}{\rho_2} A_{12}, \quad L_{22} = \frac{1}{\rho_2} A_{22}, \quad \ldots, \quad L_{2n} = \frac{1}{\rho_2} A_{2n}.$$

We have similar relations for each curve in succession, the last set being

$$L_{1n} = \frac{1}{\rho_n} A_{1n}, \quad L_{2n} = \frac{1}{\rho_n} A_{2n}, \quad \ldots, \quad L_{nn} = \frac{1}{\rho_n} A_{nn}.$$

On the assumption that the radii of curvature are unequal (otherwise, the lines of curvature would not be distinct), all these equations can coexist only if

$$\left.\begin{matrix} L_{12} = 0 \\ A_{12} = 0 \end{matrix}\right\}, \quad \left.\begin{matrix} L_{13} = 0 \\ A_{13} = 0 \end{matrix}\right\}, \quad \ldots\ldots, \quad \left.\begin{matrix} L_{n-1,n} = 0 \\ A_{n-1,n} = 0 \end{matrix}\right\};$$

that is, if

$$A_{ij}=0, \quad L_{ij}=0,$$

for all the combinations of unequal values of i and j, from 1, 2, ..., n. These equations, accordingly, are *the conditions that the parametric lines are lines of curvature.*

We then have

$$\frac{1}{\rho}=\sum_{i=1}^{n} L_{ii}x_i'^2, \quad 1=\sum_{i=1}^{n} A_{ii}x_i'^2;$$

and the principal curvatures are

$$\frac{L_{11}}{A_{11}}, \quad \frac{L_{22}}{A_{22}}, \quad \ldots, \quad \frac{L_{nn}}{A_{nn}}.$$

We assume that the corporate measure Θ does not vanish. If it should be the fact that $\Theta=0$, this property can only arise if one at least of the principal values of ρ shall vanish. In that eventuality, it follows (by an argument similar to the argument in § 286) that the amplitude has a family of lines of curvature which are straight lines meeting one another in succession. The primary amplitude (of n dimensions) can be modified, into a secondary amplitude (of $n-1$ dimensions) lying in a homaloidal amplitude of n dimensions. The necessary modification is achieved (as before, for quadruple space, though now with the necessary extensions) by turning, without stretch or compression or rupture, the homaloidal tangent n-dimensional amplitudes each successively into coincidence with the next. The ultimate position, that of what may be called the last tangent homaloidal amplitude, is the homaloidal amplitude of n dimensions containing the whole modified configuration; and the ultimate form of the original primary amplitude is now that of an amplitude of $n-1$ dimensions, contained in this form—that is, a secondary amplitude in the original homaloidal space of $n+1$ dimensions.

As already implied, we assume this possibility to be excluded by an assumption that the corporate measure Θ does not vanish. Accordingly, we assume our primary amplitude to be without restricting qualities: it is an amplitude, properly and unalterably of n dimensions; and, on that account, it will be described as *non-developable.*

Equations of the second order, satisfied by the point-coordinates.

427. Partial equations of the second order, with the parameters as deriving variables, are satisfied by the point-variables. We have had the relations

$$[ij, k]=\sum_{t=1}^{n+1} \frac{\partial y_t}{\partial x_k}\frac{\partial^2 y_t}{\partial x_i \partial x_j},$$

$$L_{ij}=\sum_{t=1}^{n+1} Y_t \frac{\partial^2 y_t}{\partial x_i \partial x_j}.$$

If therefore we take $n+1$ magnitudes $\eta, \eta_1, \eta_2, \ldots, \eta_n$, such that

$$\frac{\partial^2 y_t}{\partial x_i \partial x_j} = Y_t \eta + \frac{\partial y_t}{\partial x_1}\eta_1 + \frac{\partial y_t}{\partial x_2}\eta_2 + \ldots + \frac{\partial y_t}{\partial x_n}\eta_n,$$

for $t = 1, \ldots, n+1$, we have, first,

$$\sum_{t=1}^{n+1} Y_t \frac{\partial^2 y_t}{\partial x_i \partial x_j} = \eta \Sigma Y_t^2 + \sum_{s=1}^{n} \eta_s \left(\sum_{t=1}^{n+1} Y_t \frac{\partial y_t}{\partial x_s} \right) = \eta,$$

that is,

$$\eta = L_{ij};$$

and next, for all values of k,

$$\sum_{t=1}^{n+1} \frac{\partial y_t}{\partial x_k} \frac{\partial^2 y_t}{\partial x_i \partial x_j} = \eta \sum_{t=1}^{n+1} Y_t \frac{\partial y_t}{\partial x_k} + \sum_{s=1}^{n} \eta_s \left(\sum_{t=1}^{n+1} \frac{\partial y_t}{\partial x_s} \frac{\partial y_t}{\partial x_k} \right)$$
$$= \sum_{s=1}^{n} \eta_s A_{sk},$$

that is,

$$\sum_{s=1}^{n} \eta_s A_{sk} = [ij, k],$$

and therefore

$$\eta_s = \{ij, s\}.$$

Consequently, we have

$$\frac{\partial^2 y_t}{\partial x_i \partial x_j} = Y_t L_{ij} + \frac{\partial y_t}{\partial x_1}\{ij, 1\} + \frac{\partial y_t}{\partial x_2}\{ij, 2\} + \ldots + \frac{\partial y_t}{\partial x_n}\{ij, n\},$$

for $t = 1, \ldots, n+1$; and for $i, j, = 1, 2, \ldots, n$.

These are the partial differential equations of the second order, satisfied by the point-variables in the plenary space; the parameters of a primary amplitude in that space are the deriving variables.

The Riemann four-index symbols: curvature of geodesics.

428. From these expressions just obtained for the second derivatives of the point-coordinates, we have

$$\sum_t \frac{\partial^2 y_t}{\partial x_j \partial x_l} \frac{\partial^2 y_t}{\partial x_i \partial x_k}$$
$$= L_{jl} L_{ik} + \sum_r \sum_s A_{rs} [\{jl, r\}\{ik, s\} + \{jl, s\}\{ik, r\}],$$

where, on the right-hand side, there is the expressed double term when r and s are different, and there should be only the single term $\{jl, r\}\{ik, r\}$ when r and s are the same. We can, preferably, write the result in the form

$$\sum_t \frac{\partial^2 y_t}{\partial x_j \partial x_l} \frac{\partial^2 y_t}{\partial x_i \partial x_k} = L_{jl} L_{ik} + \sum_r \sum_s A_{rs} \{jl, r\}\{ik, s\},$$

where now on the right-hand side r and s, independently of one another, take the values $1, \ldots, n$.

Now returning to the expressions for the quantities A, we had

$$A_{il} = \sum_t \frac{\partial y_t}{\partial x_i} \frac{\partial y_t}{\partial x_l},$$

so that

$$\frac{\partial^2 A_{il}}{\partial x_j \partial x_k} = \sum_t \left(\frac{\partial^3 y_t}{\partial x_i \partial x_j \partial x_k} \frac{\partial y_t}{\partial x_l} + \frac{\partial^3 y_t}{\partial x_j \partial x_l \partial x_k} \frac{\partial y_t}{\partial x_i} + \frac{\partial^2 y_t}{\partial x_i \partial x_j} \frac{\partial^2 y_t}{\partial x_l \partial x_k} + \frac{\partial^2 y_t}{\partial x_j \partial x_l} \frac{\partial^2 y_t}{\partial x_i \partial x_k} \right).$$

Interchanging k and l, but not j and i, in this equation, we have a similar expression for

$$\frac{\partial^2 A_{ik}}{\partial x_j \partial x_l};$$

interchanging j and i, but not k and l, in the same equation, we have a similar expression for

$$\frac{\partial^2 A_{jl}}{\partial x_i \partial x_k};$$

and finally interchanging j and i, and also interchanging k and l, in the same equation, we have a similar expression for

$$\frac{\partial^2 A_{jk}}{\partial x_i \partial x_l}.$$

When we frame the combination of these second derivatives, which occurs in the Riemann four-index symbol, the third-order derivatives of the point-coordinates disappear; and the result is

$$\frac{1}{2} \left(\frac{\partial^2 A_{il}}{\partial x_j \partial x_k} - \frac{\partial^2 A_{ik}}{\partial x_j \partial x_l} - \frac{\partial^2 A_{jl}}{\partial x_i \partial x_k} + \frac{\partial^2 A_{jk}}{\partial x_i \partial x_l} \right) = \sum_t \left(\frac{\partial^2 y_t}{\partial x_j \partial x_l} \frac{\partial^2 y_t}{\partial x_i \partial x_k} - \frac{\partial^2 y_t}{\partial x_j \partial x_k} \frac{\partial^2 y_t}{\partial x_i \partial x_l} \right).$$

We have proved that

$$\sum_t \frac{\partial^2 y_t}{\partial x_j \partial x_l} \frac{\partial^2 y_t}{\partial x_i \partial x_k} = L_{jl} L_{ik} + \sum_r \sum_s A_{rs} \{jl, r\} \{ik, s\},$$

and therefore, by interchange of symbols,

$$\sum_t \frac{\partial^2 y_t}{\partial x_j \partial x_k} \frac{\partial^2 y_t}{\partial x_i \partial x_l} = L_{jk} L_{il} + \sum_r \sum_s A_{rs} \{jk, r\} \{il, s\};$$

hence

$$\sum \left(\frac{\partial^2 y_t}{\partial x_j \partial x_l} \frac{\partial^2 y_t}{\partial x_i \partial x_k} - \frac{\partial^2 y_t}{\partial x_j \partial x_k} \frac{\partial^2 y_t}{\partial x_i \partial x_l} \right) = L_{jl} L_{ik} - L_{jk} L_{il} + \sum_r \sum_s A_{rs} [\{jl, r\} \{ik, s\} - \{jk, r\} \{il, s\}],$$

where, throughout the summations on the right-hand side, r and s separately acquire all the values $1, \ldots, n$. Accordingly,

$$\begin{aligned} L_{jl} L_{ik} - L_{jk} L_{il} &= \frac{1}{2} \left(\frac{\partial^2 A_{jl}}{\partial x_j \partial x_k} - \frac{\partial^2 A_{ik}}{\partial x_j \partial x_l} - \frac{\partial^2 A_{jl}}{\partial x_i \partial x_k} + \frac{\partial^2 A_{jk}}{\partial x_i \partial x_l} \right) \\ &\quad + \sum_r \sum_s A_{rs} [\{il, s\} \{jk, r\} - \{ik, s\} \{jl, r\}] \\ &= (ij, kl). \end{aligned}$$

Consequently the Riemann four-index symbols, for all the possible combinations of i, j, k, l, represent the combinations

$$L_{jl}L_{ik} - L_{jk}L_{il}.$$

Now, by the definition in § 423, the Riemann four-index symbols are combinations of the fundamental primary magnitudes A of the arc-element and of the derivatives of these magnitudes, and they are combinations of these quantities alone. Hence we have the theorem that all the magnitudes

$$L_{jl}L_{ik} - L_{jk}L_{il},$$

for all the possible combinations of the numbers i, j, k, l, among the values $1, \ldots, n$, taken independently of one another, can be expressed solely in terms of the coefficients A in the expression for the arc-element of our primary amplitude and in terms of the derivatives of these coefficients.

Of this theorem, we had an illustration in the results of § 286 which express all the first minors of the determinant

$$\begin{vmatrix} L, & M, & K \\ M, & N, & I \\ K, & I, & J \end{vmatrix},$$

as it occurs for a non-developable region in homaloidal quadruple space, in terms of A, B, C, F, G, H, and of their derivatives. Moreover, the agreement of these particular results with the foregoing general results, where $n = 3$, is easily verified.

All secondary magnitudes of a non-developable primary amplitude are expressible by means of the primary magnitudes.

429. From the general theorem just obtained, other significant consequences follow analytically. Consider the determinant D, where

$$D = \begin{vmatrix} L_{ii'}, & L_{ij'}, & L_{ik'} \\ L_{ji'}, & L_{jj'}, & L_{jk'} \\ L_{ki'}, & L_{kj'}, & L_{kk'} \end{vmatrix}.$$

Every first minor of D, being a combination of the magnitudes $L_{\mu\nu}$ such as occurs in the preceding theorem, is expressible in terms of the fundamental magnitudes A; and therefore the determinant

$$\begin{vmatrix} L_{jj'}L_{kk'} - L_{kj'}L_{jk'}, & L_{jk'}L_{ki'} - L_{ji'}L_{kk'}, & L_{ji'}L_{kj'} - L_{jj'}L_{ki'} \\ L_{kj'}L_{ik'} - L_{ij'}L_{kk'}, & L_{kk'}L_{ii'} - L_{ki'}L_{ik'}, & L_{ki'}L_{ij'} - L_{kj'}L_{ii'} \\ L_{ij'}L_{jk'} - L_{jj'}L_{ik'}, & L_{ik'}L_{ji'} - L_{ii'}L_{jk'}, & L_{ii'}L_{jj'} - L_{ij'}L_{ji'} \end{vmatrix},$$

the value of which is D^2, is expressible in terms of those fundamental magnitudes. Consequently, D is expressible in terms of those fundamental magnitudes and their derivatives alone.

Again, taking the minors of D^2, we have

$$(L_{kk'}L_{ii'} - L_{ki'}L_{ik'})(L_{ii'}L_{jj'} - L_{ij'}L_{ji'}) - (L_{ik'}L_{jc'} - L_{ii'}L_{jk'})(L_{ki'}L_{ij'} - L_{kj'}L_{ii'}) = L_{ii'}D,$$

identically. The quantities on the left-hand side, which are the minors of D^2, and also the determinant D, are expressible in terms of the fundamental magnitudes A and their derivatives. Hence $L_{ii'}$ is similarly expressible, unless the determinant D vanishes. Similarly for the other constituents of D.

Now it may be the fact that, for some particular selection of i, j, k, and i', j', k', the particular determinant D may vanish. But if this should happen for any particular selection of i, j, k, and for all selections of i', j', k', that is, if all the determinants

$$\begin{Vmatrix} L_{i1}, & L_{i2}, & L_{i3}, & \dots, & L_{in} \\ L_{j1}, & L_{j2}, & L_{j3}, & \dots, & L_{jn} \\ L_{k1}, & L_{k2}, & L_{k3}, & \dots, & L_{kn} \end{Vmatrix}$$

should vanish, the whole determinant

$$\| L_{ij} \|$$

would vanish, that is, the corporate measure Θ of the primary amplitude would vanish, and the amplitude itself would be developable. This possibility has been excluded (§ 426). Hence, at the utmost, only some of the foregoing third-order determinants may vanish, though not sufficient in number and selection to make Θ vanish. There then remain sufficient non-vanishing combinations i', j', k', to allow the inference that the quantities $L_{ii'}$ are then determinate.

If for instance $n = 5$, and all the combinations

$$\begin{Vmatrix} L_{11}, & L_{12}, & L_{13}, & L_{14} \\ L_{12}, & L_{22}, & L_{23}, & L_{24} \\ L_{13}, & L_{23}, & L_{33}, & L_{34} \end{Vmatrix}$$

vanish, but no others, there still remain the non-vanishing combinations

$$\begin{vmatrix} L_{1i'}, & L_{1j'}, & L_{15} \\ L_{2i'}, & L_{2j'}, & L_{25} \\ L_{3i'}, & L_{3j'}, & L_{35} \end{vmatrix},$$

for $i', j', = 1, 2, 3, 4$, with unequal values for i' and j'.

Hence we have the theorem that *all the magnitudes* $L_{ii'}$, defined by the relation

$$L_{ii'} = Y_1 \frac{\partial^2 y_1}{\partial x_i \partial x_{i'}} + Y_2 \frac{\partial^2 y_2}{\partial x_i \partial x_{i'}} + \dots + Y_{n+1} \frac{\partial^2 y_{n+1}}{\partial x_i \partial x_{i'}},$$

and occurring in the expression for the circular curvature of a geodesic in a primary amplitude of the homaloidal plenary space of $n+1$ dimensions, *are expressible in terms solely of the fundamental magnitudes A which occur in the expression for the arc-element of the amplitude.*

The theorem does not hold for $n=1$, that is, for Euclidean plane geometry. It does not hold for $n=2$, that is, for the Gauss theory of surfaces as occurring in Euclidean geometry of three dimensions. In neither of these instances can the determinant D be formed; so that, although (with the usual notation of that Gauss theory) the quantity $LN-M^2$ is expressible in terms of the derivatives of the fundamental magnitudes, no inference can be drawn as to the expressibility of L, M, N, separately, in such terms. There is thus a drastic distinction between the expressibility of the curvature of primary amplitudes (that is, surfaces) in homaloidal space of three dimensions and the expressibility of the curvature of primary amplitudes in homaloidal spaces of more than three dimensions.

One further remark may be made, to emphasise the significance of this result.

We have seen (§ 426) that the principal measures of linear curvature of the primary amplitude are expressible in terms of the magnitudes L_{ij} and the magnitudes A_{ij}, for $i, j, = 1, \ldots, n$. It now appears that for a completely general primary amplitude, which is non-developable, all the magnitudes L_{ij} are expressible in terms of the fundamental magnitudes A_{ij} and of the derivatives of these magnitudes A_{ij}. Consequently, in the general case when a primary amplitude is non-developable, all the measures of linear curvature, whether specific for a particular arbitrarily selected direction, or principal as belonging to the corporate amplitude, are expressible in terms of the fundamental magnitudes A, occurring in the parametric expression for the arc-element of the amplitude, and of the derivatives of these fundamental magnitudes.

Amplitude of $(n-1)$ dimensions in a primary n-fold amplitude.

430. The following results are stated, without proof, and are left for verification.

Ex. With the foregoing notation for a primary amplitude in a homaloidal space of $n+1$ dimensions, an amplitude of $n-1$ dimensions existing within that primary amplitude can be represented by an equation

$$\theta(x_1, x_2, \ldots, x_n)=0.$$

Let

$$\vartheta_{ij}=\frac{\partial^2\theta}{\partial x_i\,\partial x_j}-\sum_{k=1}^{n}\{ij, k\}\,\theta_k,$$

$$\Theta=\sum_i\sum_j\vartheta_{ij}x_i'x_j',$$

$$D=\sum_i\sum_j a_{ij}\theta_i\theta_j,$$

where (§ 422) a_{ij} is the minor of A_{ij} in Δ, which denotes the determinant $\|A_{ij}\|$. Establish the following results:

(i) the equations of geodesics of the amplitude of $n-1$ dimensions are

$$\frac{d^2x_k}{ds^2}+\sum_i\sum_j\{ij, k\}\frac{dx_i}{ds}\frac{dx_j}{ds}=-\frac{\Theta}{D}\sum_{m=1}^{n}a_{mk}\theta_m,$$

for $k=1, \ldots, n$;

(ii) if $1/\gamma$ is the amplitudinal geodesic curvature of a θ-geodesic, that is, the rate of deviation of a θ-geodesic from a geodesic of the primary amplitude touching the θ-geodesic,

$$\frac{1}{\gamma} = -\frac{\Delta^{\frac{1}{2}}}{D^{\frac{1}{2}}}\Theta;$$

(iii) if $1/\rho_0$ be the circular curvature of the θ-geodesic, and $1/\rho$ be the circular curvature of the geodesic of the primary amplitude touching the θ-geodesic,

$$\frac{1}{\rho_0^2} = \frac{1}{\rho^2} + \frac{1}{\gamma^2}.$$

Geodesic surfaces: two specific measures of superficial curvature of a primary amplitude.

431. There are other aspects of a primary amplitude, in association with which curvatures other than linear curvatures arise.

In the first place, there is the superficial curvature of the primary amplitude in any orientation estimated, after Riemann, by the generalised Gauss measure of a surface in three-dimensional homaloidal space, when, in the amplitude, the surface for measuring the curvature is constituted by the amplitudinal geodesics touching the sheaf of directions in the orientation. To construct this geodesic surface, we take two directions in the orientation, represented by

$$p_1', p_2', \ldots, p_n'; \quad q_1', q_2', \ldots, q_n';$$

so that the orientation-variables ξ_{ij} are the magnitudes

$$p_i'q_j' - q_i'p_j',$$

for all the values of i and j. The sheaf of directions in the orientation is constituted by

$$\lambda p_1' + \mu q_1', \quad \lambda p_2' + \mu q_2', \quad \ldots\ldots, \quad \lambda p_n' + \mu q_n',$$

for all values of parameters λ and μ, admissible under the condition

$$\lambda^2 \sum_i \sum_j A_{ij} p_i' p_j' + 2\lambda\mu \sum_i \sum_j A_{ij}(p_i'q_j' + q_i'p_j') + \mu^2 \sum_i \sum_j A_{ij} q_i' q_j' = 1.$$

We take the geodesics, determined by these directions, as curves constituting a geodesic surface; the surface, thus composed of such amplitudinal geodesics, lies wholly in the amplitude. Each geodesic of the sheaf lies wholly in the geodesic surface: there cannot be, on the surface between any two points on the curve, a shorter arc than that of the amplitudinal geodesic; and it is therefore a superficial geodesic. Its circular curvature is known, because it is an amplitudinal geodesic; and therefore the circular curvature of the superficial geodesic is given by

$$\begin{aligned}\frac{1}{\rho} &= \sum_i \sum_j L_{ij}(\lambda p_i' + \mu q_i')(\lambda p_j' + \mu q_j') \\ &= \lambda^2 \sum_i \sum_j L_{ij} p_i' p_j' + 2\lambda\mu \sum_i \sum_j L_{ij}(p_i'q_j' + q_i'p_j') + \mu^2 \sum_i \sum_j L_{ij} q_i' q_j'.\end{aligned}$$

The admissible values of λ and μ are limited by the condition

$$\lambda^2 \sum_{i}\sum_{j} A_{ij} p_i' p_j' + 2\lambda\mu \sum_{i}\sum_{j} A_{ij}(p_i' q_j' + q_i' p_j') + \mu^2 \sum_{i}\sum_{j} A_{ij} q_i' q_j' = 1.$$

Consequently, the circular curvature of the geodesics on the geodesic surface, thus orientated in the amplitude, has principal values (being its maximum or its minimum values) given by the quadratic equation

$$PQ - R^2 = 0,$$

where

$$P = \sum_{i}\sum_{j} L_{ij} p_i' p_j' - \frac{1}{\rho} \sum_{i}\sum_{j} A_{ij} p_i' p_j',$$

$$R = \sum_{i}\sum_{j} L_{ij}(p_i' q_k' + q_i' p_j') - \frac{1}{\rho} \sum_{i}\sum_{j} A_{ij}(p_i' q_j' + q_i' p_j'),$$

$$Q = \sum_{i}\sum_{j} L_{ij} q_i' q_j' - \frac{1}{\rho} \sum_{i}\sum_{j} A_{ij} q_i' q_j'.$$

In this quadratic equation, when it is arranged in powers of $1/\rho$, the coefficient of the term independent of $1/\rho$ is

$$= (\sum_{i}\sum_{j} L_{ij} p_i' p_j')(\sum_{i}\sum_{j} L_{ij} q_i' q_j') - \{\sum_{i}\sum_{j} L_{ij}(p_i' q_j' + q_i' p_j')\}^2.$$

Now

$$\sum_{i}\sum_{j} L_{ij} p_i' p_j' = L_{11} p_1'^2 + 2L_{12} p_1' p_2' + L_{22} p_2'^2 + \ldots,$$

$$\sum_{i}\sum_{j} L_{ij} q_i' q_j' = L_{11} q_1'^2 + 2L_{12} q_1' q_2' + L_{22} q_2'^2 + \ldots,$$

$$\sum_{i}\sum_{j} L_{ij}(p_i' q_j' + q_i' p_j') = L_{11} p_1' q_1' + L_{12}(p_1' q_2' + q_1' p_2') + L_{22} p_2' q_2' + \ldots;$$

consequently, the foregoing coefficient is

$$\begin{vmatrix} L_{11}, & L_{12} \\ L_{12}, & L_{22} \end{vmatrix} \begin{vmatrix} p_1', & q_1' \\ p_2', & q_2' \end{vmatrix}^2 + 2 \begin{vmatrix} L_{11}, & L_{12} \\ L_{13}, & L_{23} \end{vmatrix} \begin{vmatrix} p_1', & p_2' \\ q_1', & q_2' \end{vmatrix} \begin{vmatrix} p_1', & p_3' \\ q_1', & q_3' \end{vmatrix} + \ldots\ldots,$$

the general term being

$$2 \begin{vmatrix} L_{ik}, & L_{il} \\ L_{jk}, & L_{jl} \end{vmatrix} \begin{vmatrix} p_i', & p_j' \\ q_i', & q_j' \end{vmatrix} \begin{vmatrix} p_k', & p_l' \\ q_k', & q_l' \end{vmatrix}.$$

Similarly, the coefficient of $1/\rho^2$ is

$$\begin{vmatrix} A_{11}, & A_{12} \\ A_{12}, & A_{22} \end{vmatrix} \begin{vmatrix} p_1', & q_1' \\ p_2', & q_2' \end{vmatrix}^2 + 2 \begin{vmatrix} A_{11}, & A_{12} \\ A_{13}, & A_{23} \end{vmatrix} \begin{vmatrix} p_1', & p_2' \\ q_1', & q_2' \end{vmatrix} \begin{vmatrix} p_1', & p_3' \\ q_1', & q_3' \end{vmatrix} + \ldots\ldots,$$

the general term being

$$2 \begin{vmatrix} A_{ik}, & A_{il} \\ A_{jk}, & A_{jl} \end{vmatrix} \begin{vmatrix} p_i', & p_j' \\ q_i', & q_j' \end{vmatrix} \begin{vmatrix} p_k', & p_l' \\ q_k', & q_l' \end{vmatrix}.$$

Finally, the coefficient of $-1/\rho$ is

$$(L_{11} A_{22} - 2L_{12} A_{12} + L_{22} A_{11}) \begin{vmatrix} p_1', & q_1' \\ p_2', & q_2' \end{vmatrix}^2 + \ldots.$$

Another form can be given to the quadratic equation. We write

$$\begin{aligned} T_0 &= \begin{vmatrix} A_{11}, & A_{12} \\ A_{12}, & A_{22} \end{vmatrix} \begin{vmatrix} p_1', & q_1' \\ p_2', & q_2' \end{vmatrix}^2 + \dots \\ &= \begin{vmatrix} A_{11}, & A_{12} \\ A_{12}, & A_{22} \end{vmatrix} \xi_{12}^2 + \dots \\ &= \Sigma \begin{vmatrix} A_{ik}, & A_{il} \\ A_{jk}, & A_{jl} \end{vmatrix} \xi_{ij} \xi_{kl}, \end{aligned}$$

for the summations $i, j, k, l, = 1, 2, \dots, n$, with $i < j$, and $k < l$, terms with equal values for i and j disappearing, and likewise for equal values of k and l. We write also

$$\begin{aligned} T_1 &= \Sigma\Sigma_{i\ j} \left(L_{ij} \frac{\partial}{\partial A_{ij}} \right) T_0 \\ &= \Sigma \left\{ \begin{vmatrix} A_{ik}, & A_{il} \\ L_{jk}, & L_{jl} \end{vmatrix} + \begin{vmatrix} L_{ik}, & L_{il} \\ A_{jk}, & A_{jl} \end{vmatrix} \right\} \xi_{ij} \xi_{kl}; \end{aligned}$$

and we write

$$\begin{aligned} T_2 &= \begin{vmatrix} L_{11}, & L_{12} \\ L_{12}, & L_{22} \end{vmatrix} \begin{vmatrix} p_1', & q_1' \\ p_2', & q_2' \end{vmatrix}^2 + \dots \\ &= \Sigma \begin{vmatrix} L_{ik}, & L_{il} \\ L_{jk}, & L_{jl} \end{vmatrix} \xi_{ij} \xi_{kl}, \end{aligned}$$

with the like summation for i, j, k, l, an obvious form for T_2 being

$$\frac{1}{2} \left(\Sigma\Sigma_{i\ j} L_{ij} \frac{\partial}{\partial A_{ij}} \right)^2 T_0.$$

Then the equation for the principal values for $1/\rho$, the circular curvature of a typical geodesic on the surface constructed from amplitudinal geodesics having their initial directions lying in the assigned orientation with $\xi_{11}, \xi_{12}, \xi_{13}, \dots$ as its orientation-variables, is

$$T_0 \frac{1}{\rho^2} - T_1 \frac{1}{\rho} + T_2 = 0.$$

Thus there are two principal values for ρ, as there are similarly two principal values for the curvature of a surface in homaloidal triple space. We write

$$\left. \begin{aligned} \bar{K} &= \frac{1}{\rho_1 \rho_2} = \frac{T_2}{T_0} = \frac{\Sigma (L_{ik} L_{jl} - L_{il} L_{jk}) \xi_{ij} \xi_{kl}}{\Sigma (A_{ik} A_{jl} - A_{il} A_{jk}) \xi_{ij} \xi_{kl}} \\ \bar{H} &= \frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{T_1}{T_0} = \frac{\Sigma (L_{ik} A_{jl} - L_{il} A_{jk} - L_{jk} A_{il} + L_{jl} A_{ik}) \xi_{ij} \xi_{kl}}{\Sigma (A_{ik} A_{jl} - A_{il} A_{jk}) \xi_{ij} \xi_{kl}} \end{aligned} \right\}.$$

The quantity denoted by $\overline{K}$ is a measure of superficial curvature of the primary amplitude for any orientation $((\xi_{ij}))$; and it was first defined by Riemann*. When regard is paid to the result of § 428 as regards the four-letter symbols, we have

$$\overline{K} = \frac{\Sigma (ij, kl)\, \xi_{ij} \xi_{kl}}{\Sigma (A_{ik} A_{jl} - A_{il} A_{jk})\, \xi_{ij} \xi_{kl}}$$

for $i, j, k, l, = 1, \ldots, n$, with $i < j$, $k < l$. We call this *the Riemann measure of superficial curvature* of the primary amplitude for the general orientation $((\xi_{ij}))$.

But there is also the measure $\overline{H}$ of superficial curvature of the amplitude; we shall call $\overline{H}$ *the additive measure of superficial curvature* of the primary amplitude for the same general orientation.

We have proved (§ 429) that all the coefficients L_{ij} can be expressed in terms of the fundamental magnitudes A, belonging to the arc-element of the amplitude, and their derivatives. Hence both the Riemann measure and the additive measure of superficial curvature of the amplitude in any orientation are expressible in terms of the orientation-variables and of the fundamental magnitudes A that occur in the expression of the arc-element.

Principal values and principal orientations of the two measures of superficial curvature of an amplitude.

432. Just as for the linear curvature of a primary amplitude (taken to be the curvature of a geodesic line) there are principal values and principal directions for those values, leading to central measures of the linear curvature of the amplitude, so there are principal values and principal orientations for those values of the Riemann estimate and the additive estimate of the superficial curvature of a primary amplitude, obtained as the two measures of curvature of a geodesic surface. These likewise lead to central measures of the two estimates.

In connection with any surface and with a geodesic surface in particular, we have had the quantity T_0, where

$$\begin{aligned} T_0 &= \Sigma (A_{ik} A_{jl} - A_{il} A_{jk})\, \xi_{ij} \xi_{kl} \\ &= (\Sigma A_{11} \delta p_1^2)(\Sigma A_{11} \delta q_1^2) - (\Sigma A_{11} \delta p_1 \delta q_1)^2 \\ &= ds_1^2 . ds_2^2 - (ds_1 ds_2 \cos \omega)^2, \end{aligned}$$

where ds_1 and ds_2 are the amplitudinal arcs along the geodesics $\delta p_1, \ldots, \delta p_n$, and $\delta q_1, \ldots, \delta q_n$, and ω is the angle between them. If then dS is an element of area on the geodesic surface,

$$T_0 = dS^2;$$

and if we write

$$\xi_{ij} = \lambda_{ij} dS,$$

* *Ges. Math. Werke* (1876), p. 261.

so that the quantities λ_{ij} are direction-cosines of orientation (the cosines of the inclination of dS to the planes of parametric reference at the point), we have

$$U = \Sigma (A_{ik}A_{jl} - A_{il}A_{jk})\lambda_{ij}\lambda_{kl} = 1.$$

Then

$$\left.\begin{aligned} \overline{K} &= \Sigma (L_{ik}L_{jl} - L_{il}L_{jk})\lambda_{ij}\lambda_{kl} \\ \overline{H} &= \Sigma (L_{ik}A_{jl} - L_{il}A_{jk} - L_{jk}A_{il} + L_{jl}A_{ik})\lambda_{ij}\lambda_{kl} \end{aligned}\right\}.$$

In addition to the relation

$$\Sigma (A_{ik}A_{jl} - A_{il}A_{jk})\lambda_{ij}\lambda_{kl} = 1,$$

there are certain identical relations among the orientation-variables ξ_{ij} and corresponding relations among the orientation-cosines λ_{ij}, of the forms

$$E_{1234} = \lambda_{23}\lambda_{14} + \lambda_{31}\lambda_{24} + \lambda_{12}\lambda_{34} = 0,$$

$$E_{1235} = \lambda_{23}\lambda_{15} + \lambda_{31}\lambda_{25} + \lambda_{12}\lambda_{35} = 0,$$

and so on, all comprehended in the typical relation

$$E_{ijkl} = 0.$$

Obviously the value of $\overline{K}$ and the value of $\overline{H}$ will remain unaffected if, to either of them or to each of them independently of one another, there be added or be subtracted terms of the form

$$\Sigma c_{ijkl} E_{ijkl},$$

where the coefficients c_{ijkl} are constants.

433. To obtain the principal values of the magnitudes $\overline{K}$ and $\overline{H}$, the two measures of the superficial curvature of the amplitude, we have to make $\overline{K}$ and $\overline{H}$ respectively have their maximum or minimum values, for the range of orientation-variables as limited by the one persistent condition $U = 1$ and by the identities.

The critical equations for the principal values of $\overline{K}$ are

$$\frac{\partial \overline{K}}{\partial \lambda_{12}} = M \frac{\partial U}{\partial \lambda_{12}} + c_{1234}\lambda_{34} + c_{1235}\lambda_{35} + \dots,$$

$$\frac{\partial \overline{K}}{\partial \lambda_{31}} = M \frac{\partial U}{\partial \lambda_{31}} + c_{1234}\lambda_{24} + c_{1235}\lambda_{25} + \dots,$$

$$\frac{\partial \overline{K}}{\partial \lambda_{23}} = M \frac{\partial U}{\partial \lambda_{23}} + c_{1234}\lambda_{14} + c_{1235}\lambda_{15} + \dots,$$

$$\frac{\partial \overline{K}}{\partial \lambda_{14}} = M \frac{\partial U}{\partial \lambda_{14}} + c_{1234}\lambda_{23} + \dots,$$

$$\frac{\partial \overline{K}}{\partial \lambda_{15}} = M \frac{\partial U}{\partial \lambda_{15}} + c_{1235}\lambda_{23} + \dots,$$

where, so far as these critical equations are concerned, the quantities M, c_{1234}, c_{1235}, ... are not determinate. We have already noted the property

that the value of $\overline{K}$ is unaltered when, to it, terms are added or subtracted constituting an aggregate of the form

$$\Sigma c_{ijkl} E_{ijkl}.$$

Now

$$\frac{\partial}{\partial \lambda_{12}} E_{1234} = \lambda_{34}, \quad \frac{\partial}{\partial \lambda_{12}} E_{1235} = \lambda_{35}, \ \ldots,$$
$$\frac{\partial}{\partial \lambda_{31}} E_{1231} = \lambda_{24}, \quad \frac{\partial}{\partial \lambda_{31}} E_{1235} = \lambda_{25}, \ \ldots,$$

and so on; accordingly, if we take an expression

$$\overline{K} - \Sigma c_{ijkl} E_{ijkl},$$

and still denote its (unchanged) value by $\overline{K}$, the foregoing critical equations become

$$\frac{\partial \overline{K}}{\partial \lambda_{ij}} = M \frac{\partial U}{\partial \lambda_{ij}},$$

for all the values of i, j. These equations can be taken as a canonical form of the critical equations.

To determine M, we now multiply the typical equation by λ_{ij}, and add all the results. We have

$$\Sigma \lambda_{ij} \frac{\partial \overline{K}}{\partial \lambda_{ij}} = 2\overline{K}, \quad \Sigma \lambda_{ij} \frac{\partial U}{\partial \lambda_{ij}} = 2U = 2,$$

and therefore

$$M = \overline{K}.$$

Thus the critical equations for the principal values of $\overline{K}$ are

$$\frac{\partial \overline{K}}{\partial \lambda_{ij}} = \overline{K} \frac{\partial U}{\partial \lambda_{ij}};$$

or, in full,

$$\begin{vmatrix} L_{11}, & L_{12} \\ L_{12}, & L_{22} \end{vmatrix} \lambda_{12} + \begin{vmatrix} L_{11}, & L_{13} \\ L_{12}, & L_{23} \end{vmatrix} \lambda_{13} + \ldots = \overline{K} \left\{ \begin{vmatrix} A_{11}, & A_{12} \\ A_{12}, & A_{22} \end{vmatrix} \lambda_{12} + \begin{vmatrix} A_{11}, & A_{13} \\ A_{12}, & A_{23} \end{vmatrix} \lambda_{13} + \ldots \right\},$$
$$\begin{vmatrix} L_{11}, & L_{13} \\ L_{12}, & L_{23} \end{vmatrix} \lambda_{12} + \begin{vmatrix} L_{11}, & L_{13} \\ L_{13}, & L_{33} \end{vmatrix} \lambda_{13} + \ldots = \overline{K} \left\{ \begin{vmatrix} A_{11}, & A_{13} \\ A_{12}, & A_{23} \end{vmatrix} \lambda_{12} + \begin{vmatrix} A_{11}, & A_{13} \\ A_{13}, & A_{33} \end{vmatrix} \lambda_{13} + \ldots \right\},$$

and so on. Hence the principal values of $\overline{K}$ are the roots of the algebraic equation

$$\begin{vmatrix} \begin{vmatrix} L_{11}, & L_{12} \\ L_{12}, & L_{22} \end{vmatrix} - \overline{K} \begin{vmatrix} A_{11}, & A_{12} \\ A_{12}, & A_{22} \end{vmatrix}, & \begin{vmatrix} L_{11}, & L_{13} \\ L_{12}, & L_{23} \end{vmatrix} - \overline{K} \begin{vmatrix} A_{11}, & A_{13} \\ A_{12}, & A_{23} \end{vmatrix}, \ \ldots \\ \begin{vmatrix} L_{11}, & L_{13} \\ L_{12}, & L_{23} \end{vmatrix} - \overline{K} \begin{vmatrix} A_{11}, & A_{13} \\ A_{12}, & A_{23} \end{vmatrix}, & \begin{vmatrix} L_{11}, & L_{13} \\ L_{13}, & L_{33} \end{vmatrix} - \overline{K} \begin{vmatrix} A_{11}, & A_{13} \\ A_{13}, & A_{33} \end{vmatrix}, \ \ldots \\ \ldots \ \ldots & \ldots \ \ldots \end{vmatrix} = 0.$$

If, instead of the two-row determinants, we use the four-index symbol, writing

$$L_{ik} L_{jl} - L_{il} L_{jk} = (ij, kl),$$
$$A_{ik} A_{jl} - A_{il} A_{jk} = (\overline{ij, kl}),$$

and note the properties of these symbols

$$(ij, kl) = -(ji, kl) = -(ij, lk) = (ji, lk),$$
$$= (kl, ij) = -(kl, ji) = -(lk, ij) = (lk, ji),$$

with corresponding properties for the symbols $(\overline{ij, kl})$, the equation for $\bar{K}$ can be written

$$\begin{vmatrix} (12,12)-\bar{K}(\overline{12,12}), & (12,13)-\bar{K}(\overline{12,13}), & (12,14)-\bar{K}(\overline{12,14}), \ldots \\ (13,12)-\bar{K}(\overline{13,12}), & (13,13)-\bar{K}(\overline{13,13}), & (13,14)-\bar{K}(\overline{13,14}), \ldots \\ (14,12)-\bar{K}(\overline{14,12}), & (14,13)-\bar{K}(\overline{14,13}), & (14,14)-\bar{K}(\overline{14,14}), \ldots \\ \ldots\ \ldots & \ldots\ \ldots & \ldots\ \ldots \quad , \ldots \end{vmatrix} = 0,$$

manifestly an equation of degree $\frac{1}{2}n(n-1)$.

This equation has $\frac{1}{2}n(n-1)$ roots, which accordingly are the principal values of $\bar{K}$. Those fundamental symmetric functions of the roots, constituted by the ratios of the coefficients of the different powers of $\bar{K}$ to the coefficients of the highest power of $\bar{K}$, are the principal measures of the Riemann estimate of the superficial curvature of the amplitude. It will appear, almost immediately, that these principal measures are expressible in terms of the principal measures of the linear curvature (the circular curvature of geodesics) of the amplitude.

434. Precisely similar analysis is applicable to the determination of the principal values of $\bar{H}$, the additive measure of the superficial curvature of the amplitude in any orientation; and the equations for $\bar{H}$, relating to principal values and to orientations for those values, are similar to the equations for $\bar{K}$.

We need a symbol for the typical coefficient in $\bar{H}$, taken to be that of the term involving $\lambda_{ij}\lambda_{kl}$; we write

$$[ij, kl] = L_{ik}A_{jl} - L_{il}A_{jk} - L_{jk}A_{il} + L_{jl}A_{ik}.$$

The general canonical critical equations for the principal values of $\bar{H}$ are

$$\frac{\partial \bar{H}}{\partial \lambda_{ij}} = \bar{H}\frac{\partial U}{\partial \lambda_{ij}}$$

for all the values of i, j: and the algebraic equation determining the $\frac{1}{2}n(n-1)$ principal values of $\bar{H}$, with the corresponding principal measures as the ratios of the coefficients of the various powers of $\bar{H}$ to the coefficient of the highest power of $\bar{H}$, is

$$\begin{vmatrix} [12,12]-\bar{H}(\overline{12,12}), & [12,13]-\bar{H}(\overline{12,13}), & [12,14]-\bar{H}(\overline{12,14}), \ldots \\ [13,12]-\bar{H}(\overline{13,12}), & [13,13]-\bar{H}(\overline{13,13}), & [13,14]-\bar{H}(\overline{13,14}), \ldots \\ [14,12]-\bar{H}(\overline{14,12}), & [14,13]-\bar{H}(\overline{14,13}), & [14,14]-\bar{H}(\overline{14,14}), \ldots \\ \ldots\ \ldots & \ldots\ \ldots & \ldots\ \ldots \quad , \ldots \end{vmatrix} = 0.$$

As for $\bar{K}$, so for $\bar{H}$, it is found that these principal measures are expressible in terms of the principal measures of the linear curvature (the circular curvature of geodesics) of the amplitude.

Umbral notation: expressions of the two measures of superficial curvature.

435. It is convenient to consider the substance of the foregoing investigation when umbral methods are adopted for the representation of the homogeneous quadratic forms.

We write

$$U = A_{11}p_1'^2 + 2A_{12}p_1'p_2' + \ldots + A_{nn}p_n'^2 = a_p^2 = b_p^2 = c_p^2 = \ldots,$$
$$W = L_{11}p_1'^2 + 2L_{12}p_1'p_2' + \ldots + L_{nn}p_n'^2 = \alpha_p^2 = \beta_p^2 = \gamma_p^2 = \ldots,$$

where, as usual with the umbral notation,

$$A_{ij} = a_i a_j = b_i b_j = c_i c_j = \ldots,$$
$$L_{ij} = \alpha_i \alpha_j = \beta_i \beta_j = \gamma_i \gamma_j = \ldots.$$

We require different sets of variables: so we take $q_1, \ldots, q_n$; $r_1, \ldots, r_n$; and so on, each set being cogredient with the set $p_1, \ldots, p_n$.

The amplitudinal variables for the orientation of a surface are

$$\| \xi_{ij} \| = \begin{Vmatrix} p_1', & p_2', & \ldots, & p_n' \\ q_1', & q_2', & \ldots, & q_n' \end{Vmatrix};$$

the amplitudinal variables for the orientation of a region are

$$\| \eta_{ijk} \| = \begin{Vmatrix} p_1', & p_2', & \ldots, & p_n' \\ q_1', & q_2', & \ldots, & q_n' \\ r_1', & r_2', & \ldots, & r_n' \end{Vmatrix};$$

and so for other amplitudes of successive dimensions contained in the primary amplitude. With each set of line-variables, there is an arc-element; thus

$$p_1' = \frac{dp_1}{ds_1}, \quad q_1' = \frac{dq_1}{ds_2}, \quad r_1' = \frac{dr_1}{ds_3}, \ldots.$$

The circular curvature of a geodesic line is

$$\frac{1}{\rho} = \alpha_p^2 = \beta_p^2 = \gamma_p^2 = \ldots,$$

while we have the permanent relation in the form

$$1 = a_p^2 = b_p^2 = c_p^2 \ldots,$$
$$= a_q^2 = b_q^2 = c_q^2 \ldots,$$
$$= a_r^2 = b_r^2 = c_r^2 \ldots,$$

and so on, the number of sets of umbral symbols corresponding to the degree of a concomitant in the literal coefficients.

The critical equations determining the maximum or minimum values of the curvature arise, by obtaining the maximum or the minimum of the quantity $1/\rho$, where

$$\frac{1}{\rho} = a_p^2,$$

for those values of the variables $p_1, \ldots, p_n$, which satisfy

$$1 = a_p^2.$$

The equations are

$$\alpha_i \alpha_p = M a_i a_p,$$

where M is a multiplier undetermined by the critical equations, and i has the values $1, \ldots, n$ in succession. We determine M as usual: we multiply the typical equation by p_i, and add for all the n values of p, so that

$$\alpha_p^2 = M a_p^2,$$

that is, $M = 1/\rho$, and so the equations become

$$\alpha_i \alpha_p = \frac{1}{\rho} a_i a_p.$$

When the variables $p_1, \ldots, p_n$, are eliminated, the resulting equation is

$$\begin{vmatrix} L_{11} - \frac{1}{\rho} A_{11}, & L_{12} - \frac{1}{\rho} A_{12}, & \ldots, & L_{1n} - \frac{1}{\rho} A_{1n} \\ L_{12} - \frac{1}{\rho} A_{12}, & L_{22} - \frac{1}{\rho} A_{22}, & \ldots, & L_{2n} - \frac{1}{\rho} A_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ L_{1n} - \frac{1}{\rho} A_{1n}, & L_{2n} - \frac{1}{\rho} A_{2n}, & \ldots, & L_{nn} - \frac{1}{\rho} A_{nn} \end{vmatrix} = 0.$$

When the umbral forms are maintained, the symbols α can occur only once, and they occur in quadratic combinations; so that a second occurrence of the coefficients requires a second set of symbols β, a third occurrence requires a third set γ, and so on; and likewise for the symbols $a, b, c, \ldots$. Thus the first umbral form of the equation would be

$$\begin{vmatrix} \alpha_1^2 - \frac{1}{\rho} a_1^2, & \alpha_1\alpha_2 - \frac{1}{\rho} a_1 a_2, & \ldots, & \alpha_1\alpha_n - \frac{1}{\rho} a_1 a_n \\ \beta_1\beta_2 - \frac{1}{\rho} b_1 b_2, & \beta_2^2 - \frac{1}{\rho} b_2^2, & \ldots, & \beta_2\beta_n - \frac{1}{\rho} b_2 b_n \\ \gamma_1\gamma_3 - \frac{1}{\rho} c_1 c_3, & \gamma_2\gamma_3 - \frac{1}{\rho} c_2 c_3, & \ldots, & \gamma_3\gamma_n - \frac{1}{\rho} c_3 c_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{vmatrix} = 0;$$

and the usual manipulation of umbral symbols leads to the final form of the equation

$$\frac{1}{n!}(\alpha\beta \ldots \kappa)^2 - \frac{1}{(n-1)!}(a\beta \ldots \kappa)^2 \frac{1}{\rho} + \frac{1}{2!\,(n-2)!}(ab\gamma \ldots \kappa)^2 \frac{1}{\rho^2} + \frac{(-1)^n}{n!}(ab \ldots k)^2 \frac{1}{\rho^n} = 0,$$

where $(\alpha\beta \ldots \kappa)$ denotes the determinant of n^2 constituents

$$\begin{vmatrix} \alpha_1, & \alpha_2, & \ldots, & \alpha_n \\ \beta_1, & \beta_2, & \ldots, & \beta_n \\ \ldots\ldots\ldots\ldots \\ \kappa_1, & \kappa_2, & \ldots, & \kappa_n \end{vmatrix},$$

and so for the others. Thus the principal measures of linear curvature (§426) are

$$\Sigma \frac{1}{\rho_1} = \frac{n!}{1!\,(n-1)!} \frac{(abc \dots jk)^2}{(\overline{abc \dots jk})^2},$$

$$\Sigma \frac{1}{\rho_1 \rho_2} = \frac{n!}{2!\,(n-2)!} \frac{(\alpha\beta c \dots jk)^2}{(\overline{abc \dots jk})^2},$$

up to the corporate measure

$$\frac{1}{\rho_1 \rho_2 \dots \rho_n} = \frac{(\alpha\beta \dots \kappa)^2}{(ab \dots k)^2}.$$

When we have to deal with the superficial curvature of the primary amplitudes, we have obtained the Riemann estimate and the additive estimate in the form

$$\overline{K} = \frac{T_2}{T_0} = \frac{(\Sigma L_{11} p_1'^2)(\Sigma L_{11} q_1'^2) - (\Sigma L_{11} p_1' q_1')^2}{(\Sigma A_{11} p_1'^2)(\Sigma A_{11} q_1'^2) - (\Sigma A_{11} p_1' q_1')^2},$$

$$\overline{H} = \frac{T_1}{T_0} = \frac{(\Sigma L_{11} p_1'^2)(\Sigma A_{11} q_1'^2) - 2(\Sigma L_{11} p_1' q_1')(\Sigma A_{11} p_1' q_1') + (\Sigma L_{11} q_1'^2)(\Sigma A_{11} p_1'^2)}{(\Sigma A_{11} p_1'^2)(\Sigma A_{11} q_1'^2) - (\Sigma A_{11} p_1' q_1')^2}.$$

Again by the usual processes of the umbral method, we have

$$2T_0 = a_p^2 b_q^2 + a_q^2 b_p^2 - 2a_p a_q b_p b_q,$$

that is,

$$T_0 = \frac{1}{2}(a_p b_q - a_q b_p)^2.$$

Now

$$a_p b_q - a_q b_p = \sum_i \sum_j \begin{vmatrix} a_i, & a_j \\ b_i, & b_j \end{vmatrix} \begin{vmatrix} p_i, & p_j \\ q_i, & q_j \end{vmatrix} = \sum_i \sum_j \begin{vmatrix} a_i, & a_j \\ b_i, & b_j \end{vmatrix} \xi_{ij};$$

and therefore the term in T_0 involving $\xi_{ij}\xi_{kl}$ has, for its coefficient,

$$\begin{vmatrix} a_i, & a_j \\ b_i, & b_j \end{vmatrix} \begin{vmatrix} a_k, & a_l \\ b_k, & b_l \end{vmatrix}$$

$$= (a_i b_j - b_i a_j)(a_k b_l - b_k a_l)$$

$$= a_i a_k b_j b_l - a_j a_k b_i b_l - a_i a_l b_j b_k + a_j a_l b_i b_k$$

$$= A_{ik} A_{jl} - A_{jk} A_{il} - A_{il} A_{jk} + A_{jl} A_{ik}$$

$$= 2(A_{ik} A_{jl} - A_{il} A_{jk}),$$

while the term involving ξ_{ij}^2 has, for its coefficient,

$$\frac{1}{2} \begin{vmatrix} a_i, & a_j \\ b_i, & b_j \end{vmatrix}^2$$

$$= \frac{1}{2}(a_i b_j - b_i a_j)^2$$

$$= \frac{1}{2}(a_i^2 b_j^2 - 2a_i a_j b_i b_j + b_i^2 a_j^2)$$

$$= \frac{1}{2}(A_{ii} A_{jj} - 2A_{ij}^2 + A_{ii} A_{jj})$$

$$= A_{ii} A_{jj} - A_{ij}^2.$$

We thus have the value of T_0 in the form

$$T_0 = \Sigma\,(A_{ik}A_{jl} - A_{il}A_{jk})\,\xi_{ij}\xi_{kl},$$

with the summations $i, j, k, l, = 1, \ldots, n$.

Similarly, we find

$$\begin{aligned} T_2 &= \frac{1}{2}\,(\alpha_p\beta_q - \alpha_q\beta_p)^2 \\ &= \Sigma\,(L_{ik}L_{jl} - L_{il}L_{jk})\,\xi_{ij}\xi_{kl}, \end{aligned}$$

with the same summations as in the expression for T_0.

Finally, we have

$$\begin{aligned} T_1 &= \alpha_p{}^2 a_q{}^2 - 2\alpha_p\alpha_q a_p a_q + \alpha_q{}^2 a_p{}^2 \\ &= (\alpha_p a_q - a_p\alpha_q)^2. \end{aligned}$$

Now

$$\alpha_p a_q - \alpha_q a_p = \sum_i\sum_j \begin{vmatrix} \alpha_i, & \alpha_j \\ a_i, & a_j \end{vmatrix} \begin{vmatrix} p_i, & p_j \\ q_i, & q_j \end{vmatrix} = \sum_i\sum_j \begin{vmatrix} \alpha_i, & \alpha_j \\ a_i, & a_j \end{vmatrix} \xi_{ij};$$

in T_1, the term involving $\xi_{ij}\xi_{kl}$ has, for its coefficient,

$$\begin{aligned} &2 \begin{vmatrix} \alpha_i, & \alpha_j \\ a_i, & a_j \end{vmatrix} \begin{vmatrix} \alpha_k, & \alpha_l \\ a_k, & a_l \end{vmatrix} \\ &\quad = 2\,(\alpha_i a_j - \alpha_j a_i)\,(\alpha_k a_l - \alpha_l a_k) \\ &\quad = 2\,(\alpha_i\alpha_k a_j a_l - \alpha_j\alpha_k a_i a_l - \alpha_i\alpha_l a_j a_k + \alpha_j\alpha_l a_i a_k) \\ &\quad = 2\,(L_{ik}A_{jl} - L_{jk}A_{il} - L_{il}A_{jk} + L_{jl}A_{ik}), \end{aligned}$$

while the term involving $\xi_{ij}{}^2$ has, for its coefficient,

$$\begin{aligned} &\begin{vmatrix} \alpha_i, & \alpha_j \\ a_i, & a_j \end{vmatrix}^2 \\ &\quad = (\alpha_i a_j - \alpha_j a_i)^2 \\ &\quad = L_{ii}A_{jj} - 2L_{ij}A_{ij} + L_{jj}A_{ii}. \end{aligned}$$

We thus have the value of T_1 in the form

$$T_1 = \Sigma\,(L_{ik}A_{jl} - L_{jk}A_{il} - L_{il}A_{jk} + L_{jl}A_{ik})\,\xi_{ij}\xi_{kl},$$

with the same summations as for T_0.

Hence the Riemann measure of superficial curvature in any orientation (pq) is

$$\overline{K} = \frac{(\alpha_p\beta_q - \alpha_q\beta_p)^2}{(a_p b_q - a_q b_p)^2},$$

and the additive measure of superficial curvature in that orientation is

$$\overline{H} = 2\,\frac{(\alpha_p a_q - \alpha_q a_p)^2}{(a_p b_q - a_q b_p)^2}.$$

These results can also be stated in the forms

$$\left.\begin{aligned} \overline{K} &= \Sigma\,(\alpha\beta)_{ij}(\alpha\beta)_{kl}\lambda_{ij}\lambda_{kl} \\ \overline{H} &= 2\Sigma\,(\alpha a)_{ij}(\alpha a)_{kl}\lambda_{ij}\lambda_{kl} \end{aligned}\right\},$$

while

$$\left.\begin{aligned} U = 1 = \Sigma\,(ab)_{ij}(ab)_{kl}\lambda_{ij}\lambda_{kl} \end{aligned}\right\}.$$

Principal values of $\overline{K}$ and $\overline{H}$: principal orientations: relation to principal linear curvatures and lines of curvature.

436. Returning now to the critical equations for the linear curvature of the amplitude, let a principal value of ρ be ρ_1, with direction-variables $p_1', \ldots, p_n'$, of the line of curvature, and associated umbral symbols $\alpha_1, \ldots, \alpha_n$, and $a_1, \ldots, a_n$; and let a second principal value of ρ be ρ_2, with direction-variables $q_1', \ldots, q_n'$, of the line of curvature, and associated umbral symbols $\beta_1, \ldots, \beta_n$, and $b_1, \ldots, b_n$. Thus we have, as the umbral forms of the critical equations defining ρ_1, $p_1', \ldots, p_n'$, and ρ_2, $q_1', \ldots, q_n'$, respectively,

$$\alpha_1\alpha_p = \frac{1}{\rho_1} a_1 a_p, \quad \alpha_2\alpha_p = \frac{1}{\rho_1} a_2 a_p, \quad \alpha_3\alpha_p = \frac{1}{\rho_1} a_3 a_p, \ \ldots;$$

$$\beta_1\beta_q = \frac{1}{\rho_2} b_1 b_q, \quad \beta_2\beta_q = \frac{1}{\rho_2} b_2 b_q, \quad \beta_3\beta_q = \frac{1}{\rho_2} b_3 b_q, \ \ldots;$$

where, in every instance, the constituents in a combination such as $\alpha_i\alpha_p$ or $b_j b_q$ must be maintained in coexistence in any other combination. Hence

$$\begin{vmatrix} \alpha_i\alpha_p, & \alpha_j\alpha_p \\ \beta_i\beta_q, & \beta_j\beta_q \end{vmatrix} = \frac{1}{\rho_1\rho_2} \begin{vmatrix} a_i a_p, & a_j a_p \\ b_i b_q, & b_j b_q \end{vmatrix},$$

that is,
$$(\alpha_i\beta_j - \alpha_j\beta_i)\,\alpha_p\beta_q = \frac{1}{\rho_1\rho_2}(a_i b_j - a_j b_i)\,a_p b_q.$$

Now $\alpha_i\alpha_p = \beta_i\beta_p$, $\alpha_j\alpha_p = \beta_j\beta_p$, $\beta_i\beta_q = \alpha_i\alpha_q$, $\beta_j\beta_q = \alpha_j\alpha_q$, when the literal interpretation is given to the combinations of the symbols α and β; and, similarly,

$$a_i a_p = b_i b_p, \quad a_j a_p = b_j b_p, \quad b_i b_q = a_i a_q, \quad b_j b_q = a_j a_q.$$

Hence
$$(\beta_i\alpha_j - \beta_j\alpha_i)\,\beta_p\alpha_q = \frac{1}{\rho_1\rho_2}(b_i a_j - b_j a_i)\,b_p a_q;$$

and therefore, by addition,

$$(\alpha_i\beta_j - \alpha_j\beta_i)(\alpha_p\beta_q - \alpha_q\beta_p) = \frac{1}{\rho_1\rho_2}(a_i b_j - a_j b_i)(a_p b_q - a_q b_p),$$

for all values of i and j.

The critical equations for the principal values of the Riemann measure $\overline{K}$ are

$$\frac{\partial \overline{K}}{\partial \lambda_{ij}} = \overline{K}\frac{\partial U}{\partial \lambda_{ij}},$$

for all the values $i, j, = 1, \ldots, n$. These equations, by the elimination of the variables λ, determine the quantities $\overline{K}$ which are the principal values. Let one of these be $\overline{K}_{12}$; when this value is substituted for $\overline{K}$, the equations then determine the ratios of the orientation-variables λ for that principal value $\overline{K}_{12}$. Now

$$\alpha_p\beta_q - \alpha_q\beta_p = \Sigma\,(\alpha\beta)_{kl} \begin{vmatrix} p_k, & p_l \\ q_k, & q_l \end{vmatrix},$$

$$a_p b_q - a_q b_p = \Sigma\,(ab)_{kl} \begin{vmatrix} p_k, & p_l \\ q_k, & q_l \end{vmatrix},$$

and therefore

$$\frac{\alpha_p\beta_q - \alpha_q\beta_p}{a_pb_q - a_qb_p} = \frac{\Sigma(\alpha\beta)_{kl}\lambda_{kl}}{\Sigma(ab)_{kl}\lambda_{kl}}$$

Consequently, the earlier relations, deduced from the critical equations for the principal values of the linear curvature, become

$$(\alpha_i\beta_j - \alpha_j\beta_i)\Sigma(\alpha\beta)_{kl}\lambda_{kl} = \frac{1}{\rho_1\rho_2}(a_ib_j - a_jb_i)\Sigma(ab)_{kl}\lambda_{kl},$$

that is,

$$(\alpha\beta)_{ij}\Sigma(\alpha\beta)_{kl}\lambda_{kl} = \frac{1}{\rho_1\rho_2}(ab)_{ij}\Sigma(ab)_{kl}\lambda_{kl},$$

for the values $i, j, = 1, \ldots, n$.

The foregoing critical equations, determining $\overline{K}_{12}$ and the orientations for $\overline{K}_{12}$, are

$$(\alpha\beta)_{ij}\Sigma(\alpha\beta)_{kl}\lambda_{kl} = \overline{K}_{12}(ab)_{ij}\Sigma(ab)_{kl}\lambda_{kl}.$$

Hence we have

$$\overline{K}_{12} = \frac{1}{\rho_1\rho_2};$$

and the orientation-variables for $\overline{K}_{12}$ are the orientation-variables constructed from the (principal) directions, that is, the lines of curvature, of ρ_1 and ρ_2.

Similarly, for any pair of integers, r and s, from the set $1, \ldots, n$, we have

$$\overline{K}_{rs} = \frac{1}{\rho_r\rho_s};$$

and the orientation-variables for $\overline{K}_{rs}$ are the orientation-variables constructed from the (principal) directions, that is, the lines of curvature, of ρ_r and ρ_s.

It therefore appears that the principal Riemann measures $\overline{K}$ of superficial curvature are constructed from the principal linear measures $1/\rho$ according to the law

$$\overline{K}_{ij} = \frac{1}{\rho_i\rho_j};$$

and that the (principal) orientation-plane for $\overline{K}_{ij}$ is the plane through the (principal) directions, being the lines of curvature, of ρ_i and ρ_j.

437. We obtain, by corresponding analysis, a similar result for the principal values of the additive measure $\overline{H}$ of superficial curvature and the associated orientations. From the critical equations for the principal values of linear curvature, and choosing initially the two values $1/\rho_1$ and $1/\rho_2$, we have

$$\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)(a_ib_j - a_jb_i)a_pb_q$$

$$= \frac{1}{\rho_1}a_ia_pb_jb_q - \frac{1}{\rho_1}a_ja_pb_ib_q + \frac{1}{\rho_2}b_jb_qa_ia_p - \frac{1}{\rho_2}b_ib_qa_ja_p$$

$$= \alpha_i\alpha_pb_jb_q - \alpha_j\alpha_pb_ib_q + \beta_j\beta_qa_ia_p - \beta_i\beta_qa_ja_p.$$

But, owing to their equivalent full literal expressions,

$$b_j b_q = a_j a_q, \quad b_i b_q = a_i a_q, \quad \beta_j \beta_q = \alpha_j \alpha_q, \quad \beta_i \beta_q = \alpha_i \alpha_q;$$

and therefore

$$\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)(a_i b_j - a_j b_i)\, a_p b_q$$
$$= \alpha_i \alpha_p \alpha_j \alpha_q - \alpha_j \alpha_p \alpha_i \alpha_q + \alpha_j \alpha_q \alpha_i \alpha_p - \alpha_i \alpha_q \alpha_j \alpha_p$$
$$= (\alpha_i \alpha_j - \alpha_j \alpha_i)(\alpha_p \alpha_q - \alpha_q \alpha_p).$$

On the left-hand side, we have (again owing to their equivalent full literal expression)

$$a_i a_p = b_i b_p, \quad b_j b_q = a_j a_q, \quad a_j a_p = b_j b_p, \quad b_i b_q = a_i a_q,$$

and therefore the left-hand side is

$$\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)(a_j b_i - b_j a_i)\, b_p a_q.$$

Consequently

$$2(\alpha_i \alpha_j - \alpha_j \alpha_i)(\alpha_p \alpha_q - \alpha_q \alpha_p)$$
$$= \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)(a_i b_j - a_j b_i)(a_p b_q - a_q b_p),$$

that is,

$$2(\alpha\alpha)_{ij}(\alpha_p \alpha_q - \alpha_q \alpha_p) = \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)(ab)_{ij}(a_p b_q - a_q b_p).$$

As before, we have

$$\frac{\alpha_p \alpha_q - \alpha_q \alpha_p}{a_p b_q - a_q b_p} = \frac{\Sigma (\alpha\alpha)_{kl} \lambda_{kl}}{\Sigma (ab)_{kl} \lambda_{kl}};$$

and therefore, in connection with the principal linear measures $1/\rho_1$ and $1/\rho_2$, and the orientation-coordinates of the plane through their directions, we have

$$2(\alpha\alpha)_{ij} \Sigma (\alpha\alpha)_{kl} \lambda_{kl} = \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)(ab)_{ij} \Sigma (ab)_{kl} \lambda_{kl},$$

for all the values $i, j, = 1, \ldots, n$.

Now the critical equations, for the principal values of the additive measure $\overline{H}$ of the superficial curvature of the amplitude, are

$$\frac{\partial \overline{H}}{\partial \lambda_{ij}} = \overline{H} \frac{\partial U}{\partial \lambda_{ij}},$$

for all the values of the combination $i, j, = 1, \ldots, n$. On the elimination of the variables λ_{ij}, these equations give the principal values of $\overline{H}$. Let one of these be $\overline{H}_{12}$; when this value is substituted for $\overline{H}$, the critical equations then determine the ratios of the orientation-variables for that principal value $\overline{H}_{12}$. Thus the equations, connected with $\overline{H}_{12}$, are

$$2(\alpha\alpha)_{ij} \Sigma (\alpha\alpha)_{kl} \lambda_{kl} = \overline{H}_{12} (ab)_{ij} \Sigma (ab)_{kl} \lambda_{kl},$$

for all the values $i, j, = 1, \ldots, n$. Hence

$$\overline{H}_{12} = \frac{1}{\rho_1} + \frac{1}{\rho_2};$$

and the orientation-variables for $\overline{H}_{12}$ are the orientation-variables constructed from the (principal) directions for ρ_1 and ρ_2.

Similarly, for every pair of integers, r and s, from the set $1, \ldots, n$, we have

$$\overline{H}_{rs} = \frac{1}{\rho_r} + \frac{1}{\rho_s};$$

and the orientation-variables for $\overline{H}_{rs}$ are the orientation-variables of the plane through the (principal) directions for ρ_r and ρ_s.

It follows that the principal values of the additive measure $\overline{H}$ of superficial curvature of the primary amplitude are constructed from the principal linear measures $1/\rho$, according to the law

$$\overline{H}_{ij} = \frac{1}{\rho_i} + \frac{1}{\rho_j};$$

and that the principal planes for the additive measure are the planes through the principal directions for the linear curvature of the amplitude.

Thus the principal values of the two measures of superficial curvature are constructed from the principal measures of linear curvature; the orientation-planes for the principal values of those two measures are the same, being the planes through pairs of principal directions of the measures of linear curvature.

Geodesic regions: regional curvature of an amplitude.

438. Among the measures derived through the principal values of the linear curvature of the region, there has been the measure

$$\Sigma \frac{1}{\rho_1 \rho_2 \rho_3},$$

the symmetric sum of the products of the principal curvatures, taken in threes. Manifestly, it is of the nature of a volumetric measure of the amplitude, symmetric in relation to the whole content.

But there is a volumetric measure of the amplitude, specific for a volumetric orientation; it is analogous to the superficial measure, specific for a superficial orientation. The latter is constructed by means of a surface, constituted from geodesics whose directions at the point lie in the plane of orientation. The volumetric measure is similarly constructed by means of a region (a three-dimensional amplitude), constituted from geodesics whose directions at the point lie in the flat of orientation. The construction is as follows.

In the primary amplitude, let three non-complanar directions

$$\begin{array}{llll} p_1', & p_2', & \ldots, & p_n', \\ q_1', & q_2', & \ldots, & q_n', \\ r_1', & r_2', & \ldots, & r_n', \end{array}$$

be taken, with elementary arcs ds_1, ds_2, ds_3, in those directions respectively. These three directions determine a flat; and the direction-variables of any other direction in the flat are given by

$$t_i' = \lambda p_i' + \mu q_i' + \nu r_i',$$

(for $i = 1, \ldots, n$), where λ, μ, ν, are three parameters. We have

$$\sum_i \sum_j A_{ij} t_i' t_j' = 1,$$

the permanent arc-relation in the amplitude; and therefore

$$\lambda^2 \Sigma A p'^2 + 2\lambda\mu \Sigma A p'q' + 2\lambda\nu \Sigma A p'r' + \mu^2 \Sigma A q'^2 + 2\mu\nu \Sigma A q'r' + \nu^2 \Sigma A r'^2 = 1,$$

as the sole relation limiting the variations of λ, μ, ν, which otherwise are independent.

We take the amplitudinal geodesics which, at the point, have the sheaf of these directions in the flat for their initial (and determining) directions; and we thus obtain a (three-dimensional) region, which may be called a *geodesic region*, touched by the flat. The orientation-variables of the flat, estimated in terms of the parameters of the region, are

$$\begin{Vmatrix} p_1', & p_2', & \ldots, & p_n' \\ q_1', & q_2', & \ldots, & q_n' \\ r_1', & r_2', & \ldots, & r_n' \end{Vmatrix};$$

and these are taken to be the variables of orientation of the geodesic volume We write

$$\theta_{ijk} = \begin{vmatrix} dp_i, & dp_j, & dp_k \\ dq_i, & dq_j, & dq_k \\ dr_i, & dr_j, & dr_k \end{vmatrix},$$

for all the combinations $i, j, k, = 1, \ldots, n$. Now

$$\left.\begin{aligned} \Sigma A p'^2 = 1, \quad & \Sigma A q'r' = \cos \widehat{qr} \\ \Sigma A q'^2 = 1, \quad & \Sigma A r'p' = \cos \widehat{rp} \\ \Sigma A r'^2 = 1, \quad & \Sigma A p'q' = \cos \widehat{pq} \end{aligned}\right\};$$

and therefore

$$\begin{vmatrix} \Sigma A p'^2, & \Sigma A p'q', & \Sigma A p'r' \\ \Sigma A p'q', & \Sigma A q'^2, & \Sigma A q'r' \\ \Sigma A p'r', & \Sigma A q'r', & \Sigma A r'^2 \end{vmatrix} = \begin{vmatrix} 1, & \cos \widehat{pq}, & \cos \widehat{pr} \\ \cos \widehat{pq}, & 1, & \cos \widehat{qr} \\ \cos \widehat{pr}, & \cos \widehat{qr}, & 1 \end{vmatrix}$$

$$= 1 - \cos^2 \widehat{pq} - \cos^2 \widehat{qr} - \cos^2 \widehat{rp} + 2 \cos \widehat{qr} \cos \widehat{rp} \cos \widehat{pq}.$$

Consequently

$$\begin{vmatrix} \Sigma A dp^2, & \Sigma A dp dq, & \Sigma A dp dr \\ \Sigma A dp dq, & \Sigma A dq^2, & \Sigma A dq dr \\ \Sigma A dp dr, & \Sigma A dq dr, & \Sigma A dr^2 \end{vmatrix}$$

$$= (1 - \cos^2 \widehat{pq} - \cos^2 \widehat{qr} - \cos^2 \widehat{rp} + 2 \cos \widehat{qr} \cos \widehat{rp} \cos \widehat{pq})\, ds_1^2 ds_2^2 ds_3^2 = dV^2,$$

where dV is the element of parallelepipedal volume, which has the three arcs ds_1, ds_2, ds_3, for conterminous edges. We take

$$\delta_{ijk} = \frac{\theta_{ijk}}{dV} = \frac{1}{dV} \begin{vmatrix} dp_i, & dp_j, & dp_k \\ dq_i, & dq_j, & dq_k \\ dr_i, & dr_j, & dr_k \end{vmatrix},$$

for all the values of i, j, k, from the set $1, \ldots, n$, and regard the quantities δ_{ijk} as the actual orientation-variables of the geodesic volume; and the last equation now acquires the form

$$U = \Sigma \begin{vmatrix} A_{ii'}, & A_{ij'}, & A_{ik'} \\ A_{ji'}, & A_{jj'}, & A_{jk'} \\ A_{ki'}, & A_{kj'}, & A_{kk} \end{vmatrix} \delta_{ijk} \delta_{i'j'k'} = 1,$$

a relation connecting these orientation-variables. There are also the identities arising out of the vanishing determinants

$$\begin{vmatrix} p'_i, & p'_j, & p'_k, & p'_{i'}, & p'_{j'}, & p'_{k'} \\ q'_i, & q'_j, & q'_k, & q'_{i'}, & q'_{j'}, & q'_{k'} \\ r'_i, & r'_j, & r'_k, & r'_{i'}, & r'_{j'}, & r'_{k'} \\ p'_i, & p'_j, & p'_k, & p'_{i'}, & p'_{j'}, & p'_{k'} \\ q'_i, & q'_j, & q'_k, & q'_{i'}, & q'_{j'}, & q'_{k'} \\ r'_i, & r'_j, & r'_k, & r'_{i'}, & r'_{j'}, & r'_{k'} \end{vmatrix}.$$

Now a geodesic of the primary amplitude is a regional geodesic in a geodesic region contained in that amplitude; every arc in the region lies in the amplitude, so that no arc in the region can be shorter than the geodesic arc of the amplitude. Consequently, the circular curvature of the regional geodesic is the circular curvature of the amplitudinal geodesic; and therefore, taken in the direction $t_1', \ldots, t_n'$, the circular curvature of the regional geodesic is

$$\frac{1}{\rho} = \Sigma\Sigma L_{ij} t_i' t_j'$$

$$= \lambda^2 \Sigma L p'^2 + 2\lambda\mu \Sigma L p' q' + 2\lambda\nu \Sigma L p' r' + \mu^2 \Sigma L q'^2 + 2\mu\nu \Sigma L q' r' + \nu^2 \Sigma L r'^2.$$

The parameters λ, μ, ν, are subject to the single relation

$$\lambda^2 \Sigma A p'^2 + \ldots + \nu^2 \Sigma A r'^2 = 1.$$

Hence there are maximum and minimum values of ρ given by the equations

$$\lambda\Sigma Lp'^2 + \mu\Sigma Lp'q' + \nu\Sigma Lp'r' = \frac{1}{\rho}(\lambda\Sigma Ap'^2 + \mu\Sigma Ap'q' + \nu\Sigma Ap'r'),$$

$$\lambda\Sigma Lp'q' + \mu\Sigma Lq'^2 + \nu\Sigma Lq'r' = \frac{1}{\rho}(\lambda\Sigma Ap'q' + \mu\Sigma Aq'^2 + \nu\Sigma Aq'r'),$$

$$\lambda\Sigma Lp'r' + \mu\Sigma Lq'r' + \nu\Sigma Lr'^2 = \frac{1}{\rho}(\lambda\Sigma Ap'r' + \mu\Sigma Aq'r' + \nu\Sigma Ar'^2);$$

and the actual maximum and minimum values are the roots of the cubic

$$\begin{vmatrix} (\Sigma Lp'^2) - \frac{1}{\rho}(\Sigma Ap'^2), & (\Sigma Lp'q') - \frac{1}{\rho}(\Sigma Ap'q'), & (\Sigma Lp'r') - \frac{1}{\rho}(\Sigma Ap'r') \\ (\Sigma Lp'q') - \frac{1}{\rho}(\Sigma Ap'q'), & (\Sigma Lq'^2) - \frac{1}{\rho}(\Sigma Aq'^2), & (\Sigma Lq'r') - \frac{1}{\rho}(\Sigma Aq'r') \\ (\Sigma Lp'r') - \frac{1}{\rho}(\Sigma Ap'r'), & (\Sigma Lq'r') - \frac{1}{\rho}(\Sigma Aq'r'), & (\Sigma Lr'^2) - \frac{1}{\rho}(\Sigma Ar'^2) \end{vmatrix} = 0,$$

which we denote by

$$T_3 - T_2\frac{1}{\rho} + T_1\frac{1}{\rho^2} - T_0\frac{1}{\rho^3} = 0.$$

Three specific measures of regional curvature of an amplitude.

439. Accordingly, for the geodesic volume there are three specific measures; we denote these by $\mathfrak{L}$, $\mathfrak{F}$, $\mathfrak{R}$, so that

$$\left.\begin{aligned} \mathfrak{L} &= \frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} = \frac{T_1}{T_0} \\ \mathfrak{F} &= \frac{1}{\rho_2\rho_3} + \frac{1}{\rho_3\rho_1} + \frac{1}{\rho_1\rho_2} = \frac{T_2}{T_0} \\ \mathfrak{R} &= \frac{1}{\rho_1\rho_2\rho_3} = \frac{T_3}{T_0} \end{aligned}\right\}.$$

These three main measures, $\mathfrak{L}$, $\mathfrak{F}$, $\mathfrak{R}$, are the three measures of regional curvature of the primary amplitude, estimated for the orientation in that amplitude determined by the variables

$$(\delta_{ijk}).$$

We proceed to the expression of these three measures of regional curvature, in terms of the fundamental magnitudes L and A of the amplitude (that is, ultimately, in terms of the magnitudes A and their derivatives) and of the variables of orientation at which the measures are framed.

The magnitude T_0 is given by

$$T_0 = \begin{vmatrix} \Sigma Ap'^2, & \Sigma Ap'q', & \Sigma Ap'r' \\ \Sigma Ap'q', & \Sigma Aq'^2, & \Sigma Aq'r' \\ \Sigma Ap'r', & \Sigma Aq'r', & \Sigma Ar'^2 \end{vmatrix}$$

$$= \frac{dV^2}{ds_1^2 ds_2^2 ds_3^2}\Sigma \begin{vmatrix} A_{ii'}, & A_{ij'}, & A_{ik'} \\ A_{ji'}, & A_{jj'}, & A_{jk'} \\ A_{ki'}, & A_{kj'}, & A_{kk'} \end{vmatrix} \delta_{ijk}\delta_{i'j'k'} = \frac{dV^2}{ds_1^2 ds_2^2 ds_3^2} U.$$

Similarly, the magnitude T_3 is given by

$$T_3 = \begin{vmatrix} \Sigma Lp'^2, & \Sigma Lp'q', & \Sigma Lp'r' \\ \Sigma Lp'q', & \Sigma Lq'^2, & \Sigma Lq'r' \\ \Sigma Lp'r', & \Sigma Lq'r', & \Sigma Lr'^2 \end{vmatrix}$$

$$= \frac{dV^2}{ds_1^2 ds_2^2 ds_3^2} \Sigma \begin{vmatrix} L_{ii'}, & L_{ij'}, & L_{ik'} \\ L_{ji'}, & L_{jj'}, & L_{jk'} \\ L_{ki'}, & L_{kj'}, & L_{kk'} \end{vmatrix} \delta_{ijk}\delta_{i'j'k'} = \frac{dV^2}{ds_1^2 ds_2^2 ds_3^2} \Theta_3.$$

Let Δ denote the operator

$$\Sigma A_{rs} \frac{\partial}{\partial L_{rs}},$$

for all the combinations r, s, from 1, ..., n, so that

$$\begin{vmatrix} L_{ii'}, & L_{ij'}, & L_{ik'} \\ L_{ji'}, & L_{jj'}, & L_{jk'} \\ L_{ki'}, & L_{kj'}, & L_{kk'} \end{vmatrix} = \begin{vmatrix} A_{ii'}, & A_{ij'}, & A_{ik'} \\ L_{ji'}, & L_{jj'}, & L_{jk'} \\ L_{ki'}, & L_{kj'}, & L_{kk'} \end{vmatrix} + \begin{vmatrix} L_{ii'}, & L_{ij'}, & L_{ik'} \\ A_{ji'}, & A_{jj'}, & A_{jk'} \\ L_{ki'}, & L_{kj'}, & L_{kk'} \end{vmatrix} + \begin{vmatrix} L_{ii'}, & L_{ij'}, & L_{ik'} \\ L_{ji'}, & L_{jj'}, & L_{jk'} \\ A_{ki'}, & A_{kj'}, & A_{kk'} \end{vmatrix},$$

with the similar expression (A and L being interchanged) for the operation Δ^2. Then

$$T_2 = \frac{dV^2}{ds_1^2 ds_2^2 ds_3^2} \Theta_2, \quad T_1 = \frac{dV^2}{ds_1^2 ds_2^2 ds_3^2} \Theta_1,$$

where

$$\Theta_2 = \Delta\Theta_3,$$
$$\Theta_1 = \Delta^2\Theta_3.$$

The expressions for the *three specific measures of regional curvature*, in terms of the variables of regional orientation of the amplitude, are

$$\mathfrak{L} = \Theta_1, \quad \mathfrak{F} = \Theta_2, \quad \mathfrak{R} = \Theta_3,$$

with the relation

$$U = 1.$$

Principal values and orientations of the three specific measures of regional curvature: relation to principal linear curvatures and lines of curvature.

440. There are principal values, and corresponding measures symmetrically general from those principal values, for the two specific measures, $\overline{K}$ and $\overline{H}$, of superficial curvature of the primary amplitude at any superficial orientation. So, also, there arise principal values, and corresponding measures symmetrically general from those principal values, for the three specific measures, $\mathfrak{L}$, $\mathfrak{F}$, $\mathfrak{R}$, of regional curvature of the primary amplitude at any regional orientation. The analysis, with the extensions due solely to the necessity for dealing with regional orientation instead of superficial orientation, follows the course of the analysis in §§ 436, 437; and consequently, only the results need be stated.

(i) The critical equations for $\mathfrak{L}$, $\mathfrak{F}$, $\mathfrak{R}$, that is, the respective sets of equations which lead to maximum or minimum values of the three several measures, can be brought to the forms

$$\frac{\partial \mathfrak{L}}{\partial \delta_{ijk}} = \mathfrak{L} \frac{\partial U}{\partial \delta_{ijk}},$$

$$\frac{\partial \mathfrak{F}}{\partial \delta_{ijk}} = \mathfrak{F} \frac{\partial U}{\partial \delta_{ijk}},$$

$$\frac{\partial \mathfrak{R}}{\partial \delta_{ijk}} = \mathfrak{R} \frac{\partial U}{\partial \delta_{ijk}},$$

for all the possible combinations $i, j, k, = 1, \ldots, n$; these forms corresponding to the forms (pp. 368, 370) for the superficial measures.

(ii) The principal values of these several measures are, all of them, expressible in terms of the principal linear curvatures of the primary amplitude; in each instance, they are roots of an algebraical equation of degree

$$\frac{1}{6} n (n-1)(n-2).$$

The roots of the equation for $\mathfrak{L}$ are

$$\frac{1}{\rho_i \rho_j \rho_k},$$

for the combinations of i, j, k; and the corresponding measures, symmetrically general for the amplitude, are the sums of the customary combinations one at a time, two at a time, three at a time, and so on, the last being the product of them all.

The roots of the equation for $\mathfrak{F}$ are

$$\frac{1}{\rho_i \rho_j \rho_k \rho_{i'} \rho_{j'} \rho_{k'}},$$

for all the combinations of the subscript indices; and there are the like inferences for the corresponding symmetrically general measures of $\mathfrak{F}$.

The roots of the equation for $\mathfrak{R}$ are

$$\frac{1}{\rho_i \rho_j \rho_k \rho_{i'} \rho_{j'} \rho_{k'} \rho_{i''} \rho_{j''} \rho_{k''}},$$

for all the combinations of the subscript indices; and, again, there are the like inferences for the corresponding symmetrically general measures of $\mathfrak{R}$.

(iii) For each of the three measures, the principal flats of orientation are the same: these flats arise by taking, in each instance as guiding lines for a flat, three of the principal directions of linear curvature of the region.

Geodesic amplitudes of m dimensions: measures of m-fold curvatures: fundamental significance of the arc-element of a primary amplitude.

441. There are corresponding measures of curvature of the primary amplitude, graded according to the n distinct classes of variables* which determine the distinct m-fold magnitudes and their orientations and which, formally, occur as classes of variables, in the full aggregate of all the concomitants of all types belonging to a homogeneous quantic (in particular, to a quadratic form) in n initial variables. The variables of orientation, by themselves, characterise different kinds of homaloidal spaces constituted by taking due sets of directions; each such homaloidal space is tangential to a curved space of the same number of dimensions, and its orientation-coordinates are taken to be those of the curved space. Thus we have planes, touching surfaces, by taking any two distinct tangential directions; we have flats, touching regions, by taking any three distinct tangential directions which do not lie in a plane.

For any multiple curvature, m-fold (where $m \leqslant n$), arising out of a geodesic manifold of m dimensions constituted by taking m amplitudinal geodesics not lying in a manifold of dimensions fewer than m, there are specific measures at each such multiple orientation. Of such m-fold curvature, there are m kinds of measure, of degrees 1, 2, ..., m, where linear curvature is of one degree, superficial curvature of two degrees, and so on, in the magnitudes L_{ij} that occur in the circular curvature of the amplitudinal geodesic. For each of these m kinds of measure of m-fold curvature, there are principal values and there are associated principal orientations; and these principal values give rise, for each of the m kinds of specific measure, to symmetrically general measures of the respective curvatures.

All the m symmetrically general measures of curvature belonging to each value of m, and all such measures for all the values of m, are expressible in terms of the principal linear curvatures of the primary amplitude. All corresponding principal orientations are obtained, by taking the combinations, in appropriate number, of the principal directions of linear curvature. It follows that, for a complete account of all possible curvatures of all grades of a primary amplitude, it is sufficient to know, first, the specific measures of the curvatures of different grades at the corresponding orientations of the amplitude and, secondly, the magnitudes and the directions of the principal linear curvatures of the amplitudinal geodesics. We have proved that all the coefficients L_{ij} in the expression of the circular curvature of an amplitudinal geodesic are expressible in terms of the fundamental magnitudes A_{ij}. Hence, ultimately, the magnitudes, which are essential and are adequate for the complete expression of all kinds of curvature of a primary amplitude, in homaloidal space of more than three dimensions, are the variables and the coefficients which constitute the expression of the general arc-element of distance in the amplitude.

* Clebsch, *Gött. Abhand.*, "Ueber eine Fundamentalaufgabe der Invariantentheorie," vol. xvii (1872), 66 pp.

Riemann's amplitude of constant superficial curvature.

442. We shall obtain, in illustration, Riemann's parametric expression—after the last remark, it will be the expression for the arc-element—of a primary amplitude of constant curvature. In his postulation*, Riemann took superficial curvature as the specific feature of the amplitude. After the preceding observations, it is practically obvious that, if any specific measure of curvature is constant for the amplitude, all the specific measures are constant.

We refer the amplitude to its lines of curvature as parametric curves, so that (§ 426) we have

$$L_{ij} = 0, \quad A_{ij} = 0,$$

for all combinations of unequal values of i and j from 1, ..., n: and we assume that $n > 2$. Let $1/\kappa$ be the constant value of the circular curvature of every geodesic in the amplitude, as derived from a constant superficial measure K_0 such that

$$\frac{L_{11}L_{22}}{A_{11}A_{22}} = \ldots = \frac{L_{ii}L_{jj}}{A_{ii}A_{jj}} = \ldots = K_0,$$

so that

$$\frac{1}{\kappa^2} = K_0.$$

We have

$$[ii, i] = \frac{1}{2}\frac{\partial A_{ii}}{\partial x_i};$$

$$[ii, k] = -\frac{1}{2}\frac{\partial A_{ii}}{\partial x_k}, \text{ if } k \text{ is not equal to } i;$$

$$[ij, i] = \frac{1}{2}\frac{\partial A_{ii}}{\partial x_j};$$

$$[ij, k] = 0, \text{ if } i, j, k, \text{ are distinct from one another.}$$

Also

$$a_{\lambda\mu} = 0, \text{ if } \lambda \text{ and } \mu \text{ are unequal;}$$

and

$$\frac{1}{\Delta}a_{\lambda\lambda} = \frac{1}{A_{\lambda\lambda}}.$$

On substituting these values in the expressions for Riemann's four-index symbols (ij, kl), we have the following results.

(i) All the magnitudes

$$(ij, kl) = 0,$$

if the four integers i, j, k, l, are distinct from one another.

(ii) There are non-vanishing magnitudes

$$(ij, jl),$$

where i, j, l, are distinct from one another; and their values are

$$(ij, jl) = \frac{1}{2}\frac{\partial^2 A_{jj}}{\partial x_i \partial x_l} - \frac{1}{4}\left\{\frac{1}{A_{jj}}\frac{\partial A_{jj}}{\partial x_i}\frac{\partial A_{jj}}{\partial x_l} + \frac{1}{A_{ii}}\frac{\partial A_{ii}}{\partial x_l}\frac{\partial A_{jj}}{\partial x_i} + \frac{1}{A_{ll}}\frac{\partial A_{ll}}{\partial x_i}\frac{\partial A_{jj}}{\partial x_l}\right\}.$$

* Already (p. 134) cited: *Ges. Werke*, p. 261.

(iii) The remaining non-vanishing magnitudes

$$(ij, ji),$$

where i and j are distinct from one another; and their values are

$$(ij, ji) = \frac{1}{2}\left(\frac{\partial^2 A_{ii}}{\partial x_j^2} + \frac{\partial^2 A_{jj}}{\partial x_i^2}\right) - \frac{1}{4}\left\{\frac{1}{A_{ii}}\left(\frac{\partial A_{ii}}{\partial x_j}\right)^2 + \frac{1}{A_{ii}}\frac{\partial A_{ii}}{\partial x_i}\frac{\partial A_{jj}}{\partial x_i} + \frac{1}{A_{jj}}\frac{\partial A_{ii}}{\partial x_j}\frac{\partial A_{ij}}{\partial x_i} + \frac{1}{A_{jj}}\left(\frac{\partial A_{jj}}{\partial x_i}\right)^2\right\} + \frac{1}{4}\Sigma'\frac{1}{A_{kk}}\frac{\partial A_{ii}}{\partial x_k}\frac{\partial A_{jj}}{\partial x_k},$$

where the last summation is for values of k from 1, ..., n, other than $k = i$ or $k = j$.

Also we have

$$L_{jl}L_{ik} - L_{jk}L_{il} = (ij, kl),$$

that is, in the present instance

$$\frac{1}{\kappa^2}(A_{jl}A_{ik} - A_{jk}A_{il}) = (ij, kl).$$

There thus arises a set of partial differential equations, of the second order, for the determination of the fundamental magnitudes $A_{11}, A_{22}, \ldots, A_{nn}$, of the arc-element of the amplitude.

Of these equations there are many functionally distinct solutions. The particular solution instanced by Riemann is the peculiarly symmetrical form

$$A_{11} = A_{22} = \ldots = A_{nn} = X^{-2},$$

using X^{-2} to denote the assumed common value of the magnitudes A. Then the equations for X become

$$\frac{\partial^2 X}{\partial x_i \partial x_l} = 0,$$

arising out of the equations involving the quantities (ij, jl); and

$$-\frac{1}{\kappa^2}\frac{1}{X^4} = -\frac{1}{X^3}\left(\frac{\partial^2 X}{\partial x_i^2} + \frac{\partial^2 X}{\partial x_j^2}\right) + \frac{1}{X^4}\sum_r\left(\frac{\partial X}{\partial x_r}\right)^2,$$

that is,

$$X\left(\frac{\partial^2 X}{\partial x_i^2} + \frac{\partial^2 X}{\partial x_j^2}\right) = \sum_r\left(\frac{\partial X}{\partial x_r}\right)^2 + \frac{1}{\kappa^2},$$

arising out of the quantities (ij, ji).

From the former set, we have

$$X = \xi_1 + \xi_2 + \ldots + \xi_n,$$

where ξ_i is a function of x_i alone $(i = 1, \ldots, n)$.

From the latter set, we first have

$$\frac{\partial^2 X}{\partial x_k^2} = \frac{\partial^2 X}{\partial x_l^2},$$

for all the values of k and l, that is,

$$\frac{d^2 \xi_k}{dx_k^2} = \frac{d^2 \xi_l}{dx_l^2},$$

so that each of these quantities is a constant. Consequently, if this constant (taken equal to $2a$) is not zero,

$$\xi_k = a x_k^2 + 2 b_k x_k + c_k,$$

where b_k and c_k are constants. Next, substituting in this latter set the values thus obtained, viz.

$$X = a \Sigma x_k^2 + 2 \Sigma b_k x_k + \Sigma c_k,$$

we find that the equation is satisfied if

$$\frac{1}{\kappa^2} = 4a \Sigma c_k - 4 \Sigma b_k^2.$$

Manifestly, by taking ξ_k in a form $a\left(x_k + \frac{b_k}{a}\right)^2 + c_k - \frac{b_k^2}{a}$, we can assume b_k/a absorbed into x_k, that is, we can make $b_k = 0$ without loss of generality. Further, after this modification, the quantity Σc_k manifestly affects only the scale of the amplitude; so no essential loss of generality is involved by an assumption

$$\Sigma c_k = 1.$$

We then have

$$a = \frac{1}{4\kappa^2}.$$

Thus

$$X = a \Sigma x_k^2 + 2 \Sigma b_k x_k + \Sigma c_k$$
$$= 1 + \frac{1}{4\kappa^2} \Sigma x_k^2;$$

and consequently the expression for the arc-element of the amplitude is

$$ds^2 = \frac{dx_1^2 + dx_2^2 + \ldots + dx_n^2}{\left\{1 + \frac{1}{4\kappa^2}(x_1^2 + x_2^2 + \ldots + x_n^2)\right\}^2}.$$

In this expression, $1/\kappa^2$ is equal to the assigned constant superficial curvature of the amplitude.

This final form is due to Riemann. Even as it stands in connection with the restrictive assumption

$$A_{11} = A_{22} = \ldots = A_{nn},$$

it is only one of several forms; the sole critical condition to be satisfied is

$$\frac{1}{\kappa^2} = 4a \Sigma c_k - 4 \Sigma b_k^2.$$

The form adopted has been devised on the assumption that a is not zero. Were a zero, the condition would be

$$\frac{1}{\kappa^2} = -4\Sigma b_k^2,$$

and we should have

$$X = 2\Sigma b_k x_k + \Sigma c_k:$$

and the expression for the arc-element can, without loss of generality, be written

$$ds^2 = \frac{dx_1^2 + \ldots\ldots + dx_n^2}{(\alpha_1 x_1 + \ldots\ldots + \alpha_n x_n)^2},$$

where

$$\Sigma \alpha_i^2 = -\frac{1}{\kappa^2}.$$

This last form is specially convenient when the measure of constant curvature is everywhere negative. Finally, by an orthogonal transformation, this form can have the modified shape

$$ds^2 = -\frac{\kappa^2}{x_1^2}(dx_1^2 + dx_2^2 + \ldots + dx_n^2).$$

One remark may be made concerning each of these forms. Obviously, the surfaces of constant curvature which, whatever their integral equations, have their arc-element given by any of the foregoing expressions, can be conformally represented upon an amplitude of n dimensions which is homaloidal, the magnification being the same for all directions at a point. In particular, for each of the last two forms, the magnification is constant over an $(n-1)$-fold homaloidal amplitude within the n-fold amplitude upon which the representation is depicted: and thus again (§ 324) the analytical representation of the amplitude provides a generalisation of Mercator's projection in terrestrial charts.

CHAPTER XXV.

General Theory of Invariantive Forms: Application to Curves.

Intrinsic magnitudes: parametric, orientational, local, invariance.

443. At different stages in the preceding investigations, expressions have been obtained for the fundamental magnitudes and for subsidiary magnitudes connected with curves or surfaces or regions in homaloidal quadruple space. Relations have been determined among the various magnitudes: occasionally these have been identities, the existence of which reduces the number of algebraically independent magnitudes: usually they have been equations, which express conditions or properties inherent in the respective amplitudes. All of them have been formulated in terms of parameters proper to the analytical representation of the amplitudes with which the magnitudes and the conditions have been concerned. No one of them has been dependent upon the actual position in the quadruple space, as regards either site or orientation.

Every such magnitude and every such relation is intrinsic in character. Each is specified by its fitting expression in terms of the variables used for the calculations; but each magnitude and each relation must be the same in essence, however much its expression may depend upon the representative variables adopted. Each magnitude and each relation must remain unchanged in value and in significance, however much those representative variables may be changed from stage to stage as occasion may demand or may render desirable.

Such magnitudes and such relations are of the type called *invariantive*: that is to say, they persist in value and in significance through all analytical modifications that can be contemplated. Such modifications may be of two distinct classes. The representative variables (or parameters) may be changed without affecting, and without being dependent upon, the spatial position of the configurations under consideration. The spatial position of a configuration, as regards site and orientation, may be changed—subject to the limitations that there must be no accompanying magnification or distortion or disruption—without affecting the representative variables: but the contemplated magnitudes and the contemplated relations are to persist unaltered, through all modifications of site and of orientation.

Accordingly, the invariantive magnitudes and relations (for brevity, their aggregate will be called the system of *concomitants*) must maintain unchanged their value and their significance through all modifications, that ensue from

changes of variables or changes of position, in the respective expressions. The restrictions that are thus imposed, when changes are effected upon the parameters, are called the parametric conditions; and the corresponding property is called the *parametric invariance*. When changes are effected upon the orientation, the corresponding restrictions are called the orientational conditions; and the corresponding property is called the *orientational invariance*. The conditions imposed by the requirement, that the expressions of the magnitudes and the relations are independent of the site in quadruple space, are simple and are easily secured at once. The conditions of *local invariance* are met completely at the threshold of all the investigations; and so they do not recur later for consideration.

Lie's theory of continuous groups.

444. The method that will be adopted for the construction of the concomitants, which possess all the three types of invariance admissible under the limitation that magnification and distortion and disruption are excluded, is based upon Lie's theory of continuous groups of transformations*. In that theory, one fundamental proposition is that every continuous group determines, and is completely determined by, the full set of independent infinitesimal transformations which are therefore the sole changes that need to be taken into account. In the analysis by which the theory is applied, each such infinitesimal transformation gives rise to a linear partial differential equation of the first order; consequently, the full set of the infinitesimal transformations provides a system of simultaneous partial differential equations of the first order. In the theory, another fundamental proposition establishes the property that the system of partial equations thus provided is a complete Jacobian system. Of such a system†, the essential properties are known: the two most significant being (i), the *character* of the integrals of the equations and (ii), the *minimum number* of such integrals in terms of which every integral can be expressed.

The explicit determination of the integrals in such an aggregate (which, though substantially comprehensive, is not formally unique) is a matter of practical calculation. When the integrals have once been obtained, the geometrical interpretation of each integral has to be devised. When this stage has been attained, the combined results of the whole investigation provide an aggregate of concomitants, algebraically independent of one another, in terms of which (up to the appropriate order) every intrinsic magnitude of the configuration can be expressed.

* It is systematically expounded in the treatise by Lie and Engels, *Theorie der Transformationsgruppen* (Leipzig, Teubner, 1888–1893, three volumes).

† See my *Theory of Differential Equations*, vol. v, chap. iii.

The analysis, required for the manipulative completion of the process, can be laborious. It has the compelling advantage that it provides an aggregate of independent concomitants, in terms of which all algebraical relations of the configuration can be constructed and all descriptive equations can be interpreted.

Types of concomitants.

445. The definition, of position in quadruple space and of all associated magnitudes belonging to configurations in that space and in restricted amplitudes in that space, ultimately depends upon the assignment of parametric values for the four coordinates x, y, z, v, which uniquely determine the point-element. The number of independent parameters, assigned by the definitions to amplitudes of the various dimensions, depends upon the number of degrees of freedom of the amplitude: thus there is one parameter for curves, there are two parameters for surfaces, and there are three parameters for regions.

Analytical magnitudes are constructed from the derivatives of the coordinates. Thus, to cite only a few examples, we form the direction-cosines of a line such as the tangent to a curve or the normal to a region; we form the orientation-coordinates of a surface; we construct the fundamental magnitudes of a region or of a surface. In each instance, ultimate reference is made, explicitly or implicitly, to the derivatives of x, y, z, v, with regard to the parameters which occur in the expression of the coordinates.

The orders of the derivatives that occur are of significant importance. Thus for the complete specification of a curve in homaloidal quadruple space, derivatives of the fourth order must be retained for the formation of the sufficient critical magnitudes. Similarly for surfaces and for regions in that quadruple space. Often it proves convenient to grade concomitants according to the rank of the highest derivatives which their expressions contain.

The number of independent parameters that occur is the same as the number of dimensions in the amplitude, for the expression of which they are to serve: we must have one for a curve, two for a surface, three for a region. Usually the parameters for a region will be denoted by p, q, r; those for a surface, by p, q. We have to deal with infinitesimal transformations, and these give rise to infinitesimal variations; thus we shall have variables dp, dq, dr, in a region, and variables dp, dq, on a surface.

There are concomitants of various types. Thus we have invariantive functions, which are combinations of the derivatives of x, y, z, v, and of these quantities alone; variables such as dp, dq, dr, do not occur; such functions are called *invariants*. Other invariantive functions occur which involve the variables of the kind dp, dq, dr, as well as combinations of the derivatives of x, y, z, v; such functions are called *covariants*.

Again, there are combinations of different sets of the foregoing variables, such combinations being covariantive, though their laws of change differ from the laws which govern the changes of dp, dq, dr. Such a set of variables, arising from combinations, is provided by ξ, η, ζ, where

$$\xi = \begin{vmatrix} dq, & dr \\ \delta q, & \delta r \end{vmatrix}, \quad \eta = \begin{vmatrix} dr, & dp \\ \delta r, & \delta p \end{vmatrix}, \quad \zeta = \begin{vmatrix} dp, & dq \\ \delta p, & \delta q \end{vmatrix};$$

the variables ξ, η, ζ, are contragredient to dp, dq, dr. There are invariantive functions, which involve variables ξ, η, ζ, but not dp, dq, dr, and which also involve combinations of the derivatives of x, y, z, v; such functions are called *contravariants.*

Finally, there are invariantive functions which involve a set of variables dp, dq, dr, a set of contragredient variables ξ, η, ζ, as well as combinations of the derivatives of x, y, z, v; such functions are called* *mixed concomitants.*

The foregoing invariantive functions appertain to a region. When we deal with surfaces in quadruple space, we still have invariants; we still have variables dp, dq, and so we still have covariants. But there are no contragredient variables of functional significance, such as ξ, η, ζ; and thus there are neither contravariants nor mixed concomitants to be associated with a surface.

So far as concerns quadruple space, the enumerated types of invariantive functions constitute the aggregate. In homaloidal space of n dimensions, there are $n - 2$ different kinds of variables, those which are one-fold, in a set of variables $dp, dq, dr, \ldots$; those which are two-fold, those which are three-fold, and so on, up to those which are $(n-2)$-fold, in different sets of variables of the type $dp, dq, dr, \ldots$. There are, of course, corresponding types of invariantive functions.

Conditions of local invariance.

446. The conditions, that a concomitant of any configuration in homaloidal quadruple space shall be invariantive so far as regards site alone, are simple: and they lead to a simple result, of universal range.

Any translational displacement of a configuration—that is, a displacement involving change of site alone, with a settled orientation—can be represented by a change of x, y, z, v, to x', y', z', v', such that

$$x' = x + x_0, \quad y' = y + y_0, \quad z' = z + z_0, \quad v' = v + v_0,$$

where x_0, y_0, z_0, v_0, are arbitrary independent quantities. Let C denote any concomitant which remains invariantive under this translational displacement;

* In the nomenclature, used in the absolute differential calculus (due to Ricci and Levi-Civita) and in vector geometry, the name *invariant* has the foregoing significance; a covariant is called a *covariant vector*; a contravariant is called a *contravariant vector*; a mixed concomitant is called a *tensor*; and the coefficients in each term involving variables, whether in a covariant vector or a contravariant vector or a tensor, are called *components.*

as all derivatives of x, y, z, v, are unaffected by the change, the invariance can be expressed by the relation

$$C(x', y', z', v') = C(x, y, z, v),$$

so that

$$C(x + x_0, y + y_0, z + z_0, v + v_0) = C(x, y, z, v).$$

Hence

$$x_0 \frac{\partial C}{\partial x} + y_0 \frac{\partial C}{\partial y} + z_0 \frac{\partial C}{\partial z} + v_0 \frac{\partial C}{\partial v}$$

must be equal to zero, as being the most general infinitesimal variation of the invariantive concomitant C under the most general infinitesimal transformation of pure translation. This result can be attained, when x_0, y_0, z_0, v_0, are independent of one another and are arbitrary, only if

$$\frac{\partial C}{\partial x} = 0, \quad \frac{\partial C}{\partial y} = 0, \quad \frac{\partial C}{\partial z} = 0, \quad \frac{\partial C}{\partial v} = 0.$$

Consequently, no one of the point-variables of the quadruple space can occur in the explicit expression of a concomitant: or all the invariantive concomitants involve only derivatives of the variables, but not the variables themselves, in their explicit expression.

Orientational invariance in general.

447. The orientational invariance of concomitants requires that their values be unchanged under all rotations in the quadruple space round an origin which, after the preceding result, may be taken as fixed.

In all such rotations and displacements as can take place without any kind of deformation, any orthogonal frame remains an orthogonal frame through all the displacements. It has been proved, in the discussion of rotations, that any displacement of a configuration in quadruple space can be effected by a double rotation round two orthogonal planes, of due amounts, the planes being fitly chosen. Such displacements can be achieved by a succession of similar small displacements; and it has appeared that the most general infinitesimal orientational displacement round a settled origin is represented analytically by the equations

$$\left.\begin{aligned} x' &= x + ay + hz + gv \\ y' &= -ax + y + bz + fv \\ z' &= -hx - by + z + cv \\ v' &= -gx - fy - cz + v \end{aligned}\right\},$$

where a, b, c, f, g, h, are small quantities, each arbitrary, and all of them independent of one another. (The precise assignment, of the magnitudes of rotation and of the planes round which the rotations are effected, is irrelevant for the immediate purpose.)

In every such infinitesimal rotation the parameters, used to express x, y, z, v, in any configuration, are unaffected; and therefore equations, precisely the same in form as the four preceding equations, express the changes effected upon any derivatives of x, y, z, v, the same as for the four variables. Thus for any parameter p, we should have

$$\left(\frac{\partial x}{\partial p}\right)', \quad \left(\frac{\partial y}{\partial p}\right)', \quad \left(\frac{\partial z}{\partial p}\right)', \quad \left(\frac{\partial v}{\partial p}\right)' = \begin{vmatrix} 1, & a, & h, & g \\ -a, & 1, & b, & f \\ -h, & -b, & 1, & c \\ -g, & -f, & -c, & 1 \end{vmatrix} \left(\frac{\partial x}{\partial p}, \quad \frac{\partial y}{\partial p}, \quad \frac{\partial z}{\partial p}, \quad \frac{\partial v}{\partial p}\right);$$

and so for all derivatives of x, y, z, v, the same as for the four variables. Moreover, these transformations for different derivatives are precisely the same formally as those already given; we shall therefore use the equations as typically representative of all. Denoting the modulus of the transformation temporarily by Δ, we have

$$\begin{aligned} \Delta = & \begin{vmatrix} 1, & a, & h, & g \\ -a, & 1, & b, & f \\ -h, & -b, & 1, & c \\ -g, & -f, & -c, & 1 \end{vmatrix} \\ = & 1 + a^2 + b^2 + c^2 + f^2 + g^2 + h^2 \\ & + a^2c^2 + b^2g^2 + f^2h^2 \\ & + 2abcg - 2acfh - 2bfgh, \end{aligned}$$

which differs from unity only by squares and higher powers of the small quantities of the transformation; consequently, in any equation where only first powers of the small quantities are to be retained as expressing the infinitesimal transformations, we may take Δ as equal to unity.

Among the simplest examples of magnitudes, which are invariantive under orientational transformations, are the fundamental magnitudes of the first order given by

$$A = \Sigma x_1^2, \quad B = \Sigma x_2^2, \quad C = \Sigma x_3^2,$$
$$F = \Sigma x_2 x_3, \quad G = \Sigma x_3 x_1, \quad H = \Sigma x_1 x_2;$$

their actual invariance is easily verified.

Though it is not necessary at this stage to consider the effect of the orientational transformations upon certain definite combinations of these magnitudes which are the derivatives of the variables, there may be convenience later in considering one such effect at this stage, in connection with simplification of the analysis. Thus in any triple region where the direction-

cosines (which manifestly should be covariantive quantities) of the normal are X, Y, Z, V, we know that

$$\begin{vmatrix} X, & Y, & Z, & V \\ x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ x_3, & y_3, & z_3, & v_3 \end{vmatrix}$$

is equal to

$$(ABC+2FGH-AF^2-BG^2-CH^2)^{\frac{1}{2}},$$

which is an absolute invariant. Consequently X, Y, Z, V, are cogredient with quantities such as x_1, y_1, z_1, v_1, under orientational transformations: that is,

$$\begin{aligned} X' &= X+aY+hZ+gV, \\ Y' &= -aX+Y+bZ+fV, \\ Z' &= -hX-bY+Z+cV, \\ V' &= -gX-fY-cZ+V. \end{aligned}$$

And so for other instances of cogredient variables, and also for like instances of contragredient magnitudes.

Characteristic equations of orientational invariance.

448. As the general definition of a covariantive concomitant, we postulate a relation

$$C' = \Delta^{\mu} C,$$

where C is a function of specified arguments, C' is the same function of those arguments after they have been subjected to transformation, and Δ is the modulus of transformation. As (after Lie's theory) only those transformations need be considered, which are infinitesimal and which constitute the aggregate for the group, the conditions for orientational invariance allow Δ to be taken as equal to unity (§ 447), when only small quantities of the first order are retained—and these are all that need be retained under the theory.

For brevity, we write

$$\left\{y\frac{\partial}{\partial x}\right\} = y_1\frac{\partial}{\partial x_1}+y_2\frac{\partial}{\partial x_2}+\ldots+y_{11}\frac{\partial}{\partial x_{11}}+y_{12}\frac{\partial}{\partial x_{12}}+y_{22}\frac{\partial}{\partial x_{22}}+\ldots,$$

the summation extending over all the derivatives of x and y that can occur; and similarly for other combinations of the derivatives of all the four variables x, y, z, v. Thus the foregoing relation of definition becomes

$$\begin{aligned} C &+ a\left[\left\{y\frac{\partial C}{\partial x}\right\}-\left\{x\frac{\partial C}{\partial y}\right\}\right]+b\left[\left\{z\frac{\partial C}{\partial y}\right\}-\left\{y\frac{\partial C}{\partial z}\right\}\right]+c\left[\left\{v\frac{\partial C}{\partial z}\right\}-\left\{z\frac{\partial C}{\partial v}\right\}\right] \\ &+h\left[\left\{z\frac{\partial C}{\partial x}\right\}-\left\{x\frac{\partial C}{\partial z}\right\}\right]+g\left[\left\{v\frac{\partial C}{\partial x}\right\}-\left\{x\frac{\partial C}{\partial v}\right\}\right]+f\left[\left\{v\frac{\partial C}{\partial y}\right\}-\left\{y\frac{\partial C}{\partial v}\right\}\right] \\ &+ \text{terms involving higher powers and combinations of } a, b, c, f, g, h = \Delta^{\mu} C. \end{aligned}$$

Consequently, under Lie's general theory and because a, b, c, f, g, h, are arbitrary and independent, it follows that the aggregate of partial differential equations of the first order, necessary and sufficient to secure orientational invariance, is

$$\left\{z\frac{\partial}{\partial y}\right\}-\left\{y\frac{\partial}{\partial z}\right\}=0,$$
$$\left\{x\frac{\partial}{\partial z}\right\}-\left\{z\frac{\partial}{\partial x}\right\}=0,$$
$$\left\{y\frac{\partial}{\partial x}\right\}-\left\{x\frac{\partial}{\partial y}\right\}=0,$$
$$\left\{v\frac{\partial}{\partial x}\right\}-\left\{x\frac{\partial}{\partial v}\right\}=0,$$
$$\left\{v\frac{\partial}{\partial y}\right\}-\left\{y\frac{\partial}{\partial v}\right\}=0,$$
$$\left\{v\frac{\partial}{\partial z}\right\}-\left\{z\frac{\partial}{\partial v}\right\}=0.$$

These equations, six in number, are linearly independent of one another, except in one particular instance. In themselves, and without reference to subsequent equations of a different character expressing parametric invariance, they constitute a complete Jacobian system. As however they are to be associated with the subsequent equations indicated, it is unnecessary and perhaps premature, at this stage, to proceed to the construction of an algebraically complete set of integrals. Moreover, as they do not involve the variables of covariants or the contragredient variables of contravariants, the utmost range of functions to which they could lead, considered alone, would be invariants or coefficients of covariantive functions (the 'components' of these covariantive functions); and they will arise in the ordinary construction of such functions in the subsequent analysis.

The exceptional instance of non-independence, mentioned above, emerges when the coordinates are functions of only a single parameter and when only the first derivatives of those coordinates occur: in that event, the six equations are satisfied in virtue of the three relations

$$\frac{1}{x_1}\frac{\partial}{\partial x_1}=\frac{1}{y_1}\frac{\partial}{\partial y_1}=\frac{1}{z_1}\frac{\partial}{\partial z_1}=\frac{1}{v_1}\frac{\partial}{\partial v_1}.$$

In all other instances, the number of independent equations remains equal to the formal six, as given.

Without attempting, at this stage, to form covariantive functions of even the lowest grades in the derivatives, it is useful to note at once certain functions which manifestly are covariantive. In general terms, subject to precise reservations in detail, a function of the derivatives of x, y, z, v, which is homogeneous in the variables and the parameters, and which is completely

symmetrical in the variables, satisfies all the equations of orientational invariance. Thus functions, occurring in the theory of surfaces, such as

$$\Sigma x_1^2, \quad \Sigma x_1 x_2, \quad \Sigma x_2^2,$$
$$\Sigma x_l x_{mn},$$

where $l = 1, 2$; and $0 < m + n \leqslant 2$; all are covariantive, as may be verified immediately by substitution in the six equations; and consequently also the quantities Γ, Γ', Γ'', Δ, Δ', Δ'' (of § 207), given by defining equations of the type

$$\left.\begin{aligned} \Sigma x_1 x_{11} &= \Gamma \Sigma x_1^2 + \Delta \Sigma x_1 x_2 \\ \Sigma x_2 x_{11} &= \Gamma \Sigma x_1 x_2 + \Delta \Sigma x_2^2 \end{aligned}\right\},$$

are covariantive. It follows that, when these quantities occur in other combinations submitted to the operation of the orientational equations, they behave as constants under such operation.

Parametric invariance in general: infinitesimal changes in the parameters.

449. The equations characteristic of parametric invariance are of a nature entirely different from that of the equations which characterise orientational invariance; they might be regarded as the expression of invariance, when site and orientation have been settled or are tacitly assumed. Moreover, there are varieties of form according to the number of dimensions of the amplitude the invariants of which are sought; thus there is only one parameter for a curve; there are two for a surface, while simultaneously there is only one independent parameter for a curve on the surface; there are three for a region, while simultaneously there are only two independent parameters for a surface in the region and there is only one independent parameter for a curve in the region. But it must not be assumed (and, after Chapters XII and XX, obviously it is not the fact) that every surface in quadruple space can be obtained by taking some appropriate relation between the three parameters of an assigned region within that space: nor must a corresponding assumption be regarded as justifiable for a curve in quadruple space through some relation to a given region or to a given surface in that space.

We begin with the possibility that there shall be three parameters which, as in the preceding analysis connected with a region, are to be denoted by p, q, r. These are to be subjected to the most general continuous transformations. By the results of Lie's theory of groups, the effect of such transformations will ultimately be secured by taking the most general infinitesimal transformations. These infinitesimal transformations can be represented by the relations

$$\left.\begin{aligned} p' &= p + \epsilon\phi(p, q, r) \\ q' &= q + \epsilon\chi(p, q, r) \\ r' &= r + \epsilon\psi(p, q, r) \end{aligned}\right\},$$

where ϵ is an arbitrary constant quantity, so small that all powers above the first are neglected, and where ϕ, χ, ψ, are three arbitrary independent unconditioned functions of their arguments.

Derivatives with respect to the parameters occur. For those of the first order, and for them alone, a single-suffix notation is adopted according to the conventions

$$\frac{\partial\theta}{\partial p}=\theta_1,\quad \frac{\partial\theta}{\partial q}=\theta_2,\quad \frac{\partial\theta}{\partial r}=\theta_3,$$

for any unsuffixed magnitude θ. For the derivatives of the second order when there are three parameters, and for those of order higher than the second order, a triple-suffix notation is adopted according to the convention

$$\frac{\partial^{l+m+n}\theta}{\partial p^l\partial q^m\partial r^n}=\theta_{lmn},$$

for $l+m+n\geqslant 2$, where θ denotes any unsuffixed magnitude. When $l+m+n=2$, this notation differs from the double-suffix notation which was adopted in Chapter XV; the two fall into accord by noting that

$$\theta_{11}=\theta_{200},\quad \theta_{12}=\theta_{110},\quad \theta_{13}=\theta_{101},\quad \theta_{22}=\theta_{020},\quad \theta_{23}=\theta_{011},\quad \theta_{33}=\theta_{002}.$$

Moreover, if there should be convenience at any stage in adopting the triple-suffix notation for derivatives of the first order, we note that

$$\theta_1=\theta_{100},\quad \theta_2=\theta_{010},\quad \theta_3=\theta_{001}.$$

The foregoing derivatives occur in association with regions and regional configurations. When only two parameters occur, being represented by p and q, we shall use the notation

$$\frac{\partial^{l+m}\theta}{\partial p^l\partial q^m}=\theta_{lm},$$

for all values of l and m such that $l+m\geqslant 1$, while θ denotes any unsuffixed magnitude.

In a region, there are two kinds of variables that occur in a system of concomitants. There are variables connected with elements of *arc in the region*, so that we shall need quantities dp, dq, dr; in covariants and mixed concomitants, we write

$$dp=\bar{x},\quad dq=\bar{y},\quad dr=\bar{z}.$$

(There may be sets of such quantities: the specific notations will be stated when simultaneous sets of such quantities are required.) There are variables connected with elements of *surface in the region*; these are special combinations of two sets of arc-variables, in the orientation-coordinates ξ, η, ζ, according to the definitions

$$\xi=\begin{vmatrix} d'q, & d''q \\ d'r, & d''r \end{vmatrix},\quad \eta=\begin{vmatrix} d'r, & d''r \\ d'p, & d''p \end{vmatrix},\quad \zeta=\begin{vmatrix} d'p, & d''p \\ d'q, & d''q \end{vmatrix}.$$

We do not require variables to represent elements of volume in the region. Such a variable is

$$\begin{vmatrix} dp, & d'p, & d''p \\ dq, & d'q, & d''q \\ dr, & d'r, & d''r \end{vmatrix},$$

which in itself is a concomitant, equal to

$$\xi\bar{x} + \eta\bar{y} + \zeta\bar{z};$$

manifestly it is independent of all configurations in the region. Clearly $\bar{x}$, $\bar{y}$, $\bar{z}$, correspond to point-variables in the usual algebraical theory of ternariants; and ξ, η, ζ, correspond to line-variables in that theory.

On a surface, there is only a single kind of variable corresponding to the element of arc on the surface: when there is need, we shall write

$$dp = \bar{x}, \quad dq = \bar{y}.$$

An element of area on the surface, corresponding to $dp'dq'' - dq'dp''$, is itself a concomitant.

Infinitesimal transformations of line-variables and surface-variables.

450. Let u denote an invariantive function of all the magnitudes concerned or of any selection of these magnitudes; and let u' denote the same function of the same magnitudes after they have been subjected to the most general continuous transformation. Also let J denote the Jacobian of the transformation, so that

$$J = \begin{vmatrix} \dfrac{\partial p'}{\partial p}, & \dfrac{\partial p'}{\partial q}, & \dfrac{\partial p'}{\partial r} \\ \dfrac{\partial q'}{\partial p}, & \dfrac{\partial q'}{\partial q}, & \dfrac{\partial q'}{\partial r} \\ \dfrac{\partial r'}{\partial p}, & \dfrac{\partial r'}{\partial q}, & \dfrac{\partial r'}{\partial r} \end{vmatrix};$$

then the quality of the invariance consists in the property that the relation

$$u = u'J^{\mu}$$

is satisfied identically, where μ is usually an integer and always is a constant. By Lie's theory of continuous groups, it is sufficient to take infinitesimal transformations. For such transformations, we have

$$\begin{aligned} J &= \begin{vmatrix} 1 + \epsilon\phi_1, & \epsilon\phi_2, & \epsilon\phi_3 \\ \epsilon\chi_1, & 1 + \epsilon\chi_2, & \epsilon\chi_3 \\ \epsilon\psi_1, & \epsilon\psi_2, & 1 + \epsilon\psi_3 \end{vmatrix} \\ &= 1 + \epsilon(\phi_1 + \chi_2 + \psi_3), \end{aligned}$$

when squares and higher powers of ϵ are neglected. Thus the equation, which is the analytical definition of invariance, becomes

$$u = \{1 + \mu\epsilon(\phi_1 + \chi_2 + \psi_3)\}\, u',$$

with the similar neglect of powers of ϵ above the first.

For the fuller expression of this form of the equation, we have to expand u' in powers of ϵ; and such expansion requires the infinitesimal variations of all the possible arguments of u. To these we now proceed.

We begin with the changes in the variables. Two types occur, connected with configurations in a region, viz. the arc-variables $\bar{x}$, $\bar{y}$, $\bar{z}$, and the surface-variables ξ, η, ζ.

The equations of infinitesimal transformation of p, q, r, are

$$p' = p + \epsilon\phi, \quad q' = q + \epsilon\chi, \quad r' = r + \epsilon\psi;$$

hence writing

$$\delta p', \ \delta q', \ \delta r', \ = \bar{x}', \ \bar{y}', \ \bar{z}', \text{ respectively,}$$

we have

$$\left.\begin{aligned} \bar{x}' - \bar{x} &= \epsilon(\bar{x}\phi_1 + \bar{y}\phi_2 + \bar{z}\phi_3) \\ \bar{y}' - \bar{y} &= \epsilon(\bar{x}\chi_1 + \bar{y}\chi_2 + \bar{z}\chi_3) \\ \bar{z}' - \bar{z} &= \epsilon(\bar{x}\psi_1 + \bar{y}\psi_2 + \bar{z}\psi_3) \end{aligned}\right\},$$

as the equations giving the infinitesimal changes of $\bar{x}$, $\bar{y}$, $\bar{z}$, the arc-variables.

To obtain the changes in the surface-variables, we use the definitions which express them in terms of the arc-variables. Thus

$$\begin{aligned} \xi' &= \begin{vmatrix} \delta' q', & \delta' r' \\ \delta'' q', & \delta'' r' \end{vmatrix} \\ &= \begin{vmatrix} \epsilon\chi_1\delta' p + (1+\epsilon\chi_2)\delta' q + \epsilon\chi_3\delta' r, & \epsilon\psi_1\delta' p + \epsilon\psi_2\delta' q + (1+\epsilon\psi_3)\delta' r \\ \epsilon\chi_1\delta'' p + (1+\epsilon\chi_2)\delta'' q + \epsilon\chi_3\delta'' r, & \epsilon\psi_1\delta'' p + \epsilon\psi_2\delta'' q + (1+\epsilon\psi_3)\delta'' r \end{vmatrix} \\ &= \epsilon\chi_1 \begin{vmatrix} \delta' p, & \delta' r \\ \delta'' p, & \delta'' r \end{vmatrix} + \epsilon\psi_1 \begin{vmatrix} \delta' q, & \delta' p \\ \delta'' q, & \delta'' p \end{vmatrix} + \{1 + \epsilon(\chi_2 + \psi_3)\} \begin{vmatrix} \delta' q, & \delta' r \\ \delta'' q, & \delta'' r \end{vmatrix}, \end{aligned}$$

on neglecting squares of ϵ: that is,

$$\xi' - \xi = \epsilon\xi(\chi_2 + \psi_3) - \epsilon\eta\chi_1 - \epsilon\zeta\psi_1.$$

Similarly for the changes of η and ζ: the set of values is

$$\left.\begin{aligned} \xi' - \xi &= \epsilon\{\ \ \xi(\chi_2 + \psi_3) - \eta\chi_1 \qquad - \zeta\psi_1 \qquad\} \\ \eta' - \eta &= \epsilon\{-\xi\phi_2 \qquad + \eta(\phi_1 + \psi_3) - \zeta\psi_2 \qquad\} \\ \zeta' - \zeta &= \epsilon\{-\xi\phi_3 \qquad - \eta\chi_3 \qquad + \zeta(\phi_1 + \chi_2)\} \end{aligned}\right\}.$$

Infinitesimal transformation of general typical function of the parameters, and of all its derivatives.

451. In order to obtain the infinitesimal increments of such magnitudes as may occur in a concomitant, let $\theta(p, q, r)$ denote any function of the parameters: we require the increments of θ and increments of derivatives of θ with respect to its arguments, such increments accruing from the infinitesimal transformations of the group.

Let P, Q, R, denote any arbitrary changes made in p, q, r, respectively; and let P', Q', R', denote the consequent changes in p', q', r', so that

$$p'+P'=p+P+\epsilon\phi\ (p+P,\ q+Q,\ r+R),$$
$$q'+Q'=q+Q+\epsilon\chi\ (p+P,\ q+Q,\ r+R),$$
$$r'+R'=r+R+\epsilon\psi\ (p+P,\ q+Q,\ r+R).$$

Then

$$P'-P=\epsilon\{\phi(p+P,\ q+Q,\ r+R)-\phi(p,\ q,\ r)\}$$
$$=\epsilon\Sigma\Sigma\Sigma\frac{P^lQ^mR^n}{l!\,m!\,n!}\phi_{lmn},$$
$$Q'-Q=\epsilon\Sigma\Sigma\Sigma\frac{P^lQ^mR^n}{l!\,m!\,n!}\chi_{lmn},$$
$$R'-R=\epsilon\Sigma\Sigma\Sigma\frac{P^lQ^mR^n}{l!\,m!\,n!}\psi_{lmn},$$

where the summations on the right-hand side are for positive integer values of l, m, n, such that $l+m+n>0$.

We take θ to be an absolute invariant, that is, it is to remain unchanged whatever changes be made in the parameters p, q, r. Thus θ might be any one of the four coordinates x, y, z, v, of a point in quadruple space, the values being of course unaltered by modification of the parameters expressing position; or there might be an equation such as $\theta(p,\ q,\ r)=w$, where w is a parameter, the equation being that of a family of surfaces, organic or arbitrary, in the region. The quantity θ itself must not be any derivative of any of the variables x, y, z, v, nor any non-invariantive combination of such derivatives. For all such functions θ, the relation

$$\theta(p+P,\ q+Q,\ r+R)=\theta(p'+P',\ q'+Q',\ r'+R')$$

holds, as a relation of definition.

Subsequently, after covariantive combinations of the derivatives of x, y, z, v, have been constructed, it may prove necessary to obtain the increments of the derivatives of such combinations. In that event, we should begin with a defining equation of the form

$$\theta(p,\ q,\ r)=J^\mu\theta(p',\ q',\ r'),$$

where, as before, J denotes the Jacobian of transformation

$$\frac{\partial(p',\ q',\ r')}{\partial(p,\ q,\ r)}.$$

For the immediate purpose, we choose θ to be an absolutely unchanging function, of the type indicated; and accordingly we proceed from the earlier of the foregoing equations.

We expand the two sides of the equation in series. The left-hand side is expanded in powers of P, Q, R. The right-hand side is expanded, first of all, in powers of P', Q', R': and then, in that expansion, substitution

is made for combinations of the form $P'^{\lambda}Q'^{\mu}R'^{\nu}$ in terms of the appropriate combinations of powers of P, Q, R. After the substitutions, the coefficient of any combination $P^{\alpha}Q^{\beta}R^{\gamma}$ of P, Q, R, on the left-hand side is equated to the aggregate of that same combination on the right-hand side, the equality being due to the fact that P, Q, R, are arbitrary independent variable quantities. Each such equation gives a relation of transformation for some derivative of θ.

We take the coefficient of

$$\frac{P^{\alpha}Q^{\beta}R^{\gamma}}{\alpha!\,\beta!\,\gamma!}$$

in the two expansions. On the left-hand side, the coefficient is $\theta_{\alpha\beta\gamma}$. On the right-hand side it is the coefficient of that combination in

$$\Sigma\Sigma\Sigma \frac{P'^{\lambda}Q'^{\mu}R'^{\nu}}{\lambda!\,\mu!\,\nu!}\,\theta'_{\lambda\mu\nu},$$

where $\theta'_{\lambda\mu\nu}$ denotes $\dfrac{\partial^{\lambda+\mu+\nu}\theta(p', q', r')}{\partial p'^{\lambda}\partial q'^{\mu}\partial r'^{\nu}}$. Now

$$\begin{aligned} P'^{\lambda}Q'^{\mu}R'^{\nu} - P^{\lambda}Q^{\mu}R^{\nu} &= \epsilon\lambda P^{\lambda-1}Q^{\mu}R^{\nu}\Sigma\Sigma\Sigma\frac{P^{l}Q^{m}R^{n}}{l!\,m!\,n!}\,\phi_{lmn} \\ &\quad + \epsilon\mu P^{\lambda}Q^{\mu-1}R^{\nu}\Sigma\Sigma\Sigma\frac{P^{l}Q^{m}R^{n}}{l!\,m!\,n!}\,\chi_{lmn} \\ &\quad + \epsilon\nu P^{\lambda}Q^{\mu}R^{\nu-1}\Sigma\Sigma\Sigma\frac{P^{l}Q^{m}R^{n}}{l!\,m!\,n!}\,\psi_{lmn}; \end{aligned}$$

and therefore

$$\theta_{\alpha\beta\gamma} = \text{the coefficient of } P^{\alpha}Q^{\beta}R^{\gamma} \text{ in } \Omega,$$

where

$$\begin{aligned} \Omega = \sum_{\lambda}\sum_{\mu}\sum_{\nu}\frac{\alpha!}{\lambda!}\frac{\beta!}{\mu!}\frac{\gamma!}{\nu!}\Big\{ & P^{\lambda}Q^{\mu}R^{\nu} \\ & + \epsilon\lambda P^{\lambda-1}Q^{\mu}R^{\nu}\Sigma\Sigma\Sigma\frac{P^{l}Q^{m}R^{n}}{l!\,m!\,n!}\,\phi_{lmn} \\ & + \epsilon\mu P^{\lambda}Q^{\mu-1}R^{\nu}\Sigma\Sigma\Sigma\frac{P^{l}Q^{m}R^{n}}{l!\,m!\,n!}\,\chi_{lmn} \\ & + \epsilon\nu P^{\lambda}Q^{\mu}R^{\nu-1}\Sigma\Sigma\Sigma\frac{P^{l}Q^{m}R^{n}}{l!\,m!\,n!}\,\psi_{lmn}\Big\}\,\theta'_{\lambda\mu\nu}. \end{aligned}$$

Manifestly the required coefficient, which is to be equal to $\theta_{\alpha\beta\gamma}$, is the aggregate of the contributions made by the four separate terms expressed within the multiple summation.

The contribution from the first term is obtained by taking $\lambda=\alpha$, $\mu=\beta$, $\nu=\gamma$, and is

$$\theta'_{\alpha\beta\gamma}.$$

The contribution from the second term is obtained by taking

$$\lambda - 1 + l = \alpha, \quad \mu + m = \beta, \quad \nu + n = \gamma;$$

and therefore it is equal to

$$\epsilon\Sigma\Sigma\Sigma \frac{\alpha!\,\beta!\,\gamma!}{\lambda!\,\mu!\,\nu!} \frac{\lambda}{l!\,m!\,n!} \phi_{lmn}\theta'_{\lambda\mu\nu}$$

$$= \epsilon\Sigma\Sigma\Sigma \frac{\alpha!}{(\alpha-l)!\,l!} \frac{\beta!}{(\beta-m)!\,m!} \frac{\gamma!}{(\gamma-n)!\,n!} \theta'_{\alpha+1-l,\,\beta-m,\,\gamma-n}\phi_{lmn}$$

$$= \epsilon\Sigma\Sigma\Sigma \binom{\alpha}{l}\binom{\beta}{m}\binom{\gamma}{n} \theta'_{\alpha+1-l,\,\beta-m,\,\gamma-n}\phi_{lmn};$$

where

$$\binom{\alpha}{l} = \frac{\alpha!}{(\alpha-l)!\,l!},$$

and so for the other symbols; and where the triple summation is to be taken for all values of l, m, n, such that $0 \leqslant l \leqslant \alpha$, $0 \leqslant m \leqslant \beta$, $0 \leqslant n \leqslant \gamma$, independently of one another save for the single restriction

$$l + m + n > 0.$$

Similarly the contribution from the third term is

$$\epsilon\Sigma\Sigma\Sigma \binom{\alpha}{l}\binom{\beta}{m}\binom{\gamma}{n} \theta'_{\alpha-l,\,\beta+1-m,\,\gamma-n}\chi_{lmn},$$

and the contribution from the fourth term is

$$\epsilon\Sigma\Sigma\Sigma \binom{\alpha}{l}\binom{\beta}{m}\binom{\gamma}{n} \theta'_{\alpha-l,\,\beta-m,\,\gamma+1-n}\psi_{lmn},$$

with, in each instance, the same triple summation subject to the same single restriction.

Consequently, we have

$$\theta_{\alpha\beta\gamma} = \theta'_{\alpha\beta\gamma} + \epsilon\Sigma\Sigma\Sigma \binom{\alpha}{l}\binom{\beta}{m}\binom{\gamma}{n} \Theta(\theta'),$$

where

$$\Theta(\theta') = \theta'_{\alpha+1-l,\,\beta-m,\,\gamma-n}\phi_{lmn} + \theta'_{\alpha-l,\,\beta+1-m,\,\gamma-n}\chi_{lmn} + \theta'_{\alpha-l,\,\beta-m,\,\gamma+1-n}\psi_{lmn}.$$

Thus the difference $\theta_{\alpha\beta\gamma} - \theta'_{\alpha\beta\gamma}$ varies as the first power of ϵ, when that is the only retained power; and consequently the infinitesimal transformation for $\theta_{\alpha\beta\gamma}$ can be written

$$\theta'_{\alpha\beta\gamma} - \theta_{\alpha\beta\gamma} = -\epsilon\Sigma\Sigma\Sigma \binom{\alpha}{l}\binom{\beta}{m}\binom{\gamma}{n} \Theta(\theta),$$

where

$$\Theta(\theta) = \theta_{\alpha+1-l,\beta-m,\gamma-n}\phi_{lmn} + \theta_{\alpha-l,\,\beta+1-m,\,\gamma-n}\chi_{lmn} + \theta_{\alpha-l,\,\beta-m,\,\gamma+1-n}\psi_{lmn},$$

and

$$0 \leqslant l \leqslant \alpha, \quad 0 \leqslant m \leqslant \beta, \quad 0 \leqslant n \leqslant \gamma, \quad l + m + n > 0.$$

CONCOMITANTS OF SKEW CURVES IN FREE QUADRUPLE SPACE.

452. As a first application of the theory, the algebraically complete set of concomitants of a skew curve in four dimensions will be considered. It has already (§ 169) been proved that, except for site and orientation, a skew curve is uniquely determinate by the assignment of its curvature, its torsion, and its tilt, as definite functions of a parameter which, when desirable, can be taken to be the length of the arc from some fixed point. Also we know that these sufficing magnitudes require, for their expression, derivatives of x, y, z, v, up to the fourth order inclusive. Consequently, as regards the application of the theory, it will be sufficient to consider only the first four derivatives of the point-variables.

We shall denote the current parameter by p; and we shall write

$$\frac{d^n\theta}{dp^n} = \theta_n,$$

for all positive integers n and for all quantities θ.

There is only a single variable for covariants: there is no contragredient variable. Thus

$$\bar{x} = \delta p, \quad \bar{x}' = \delta p';$$

and as

$$p' = p + \epsilon\phi,$$

so that

$$J = 1 + \epsilon\phi_1,$$

we have

$$\bar{x} = \bar{x}'\frac{dp}{dp'} = \frac{1}{J}\bar{x}',$$

so that $\bar{x}$, by itself, is covariantive (a universal covariant, that is, whatever the curve). Clearly

$$\bar{x}' - \bar{x} = \epsilon\bar{x}\phi_1.$$

It is the only variable that has to be considered.

The law of transformation for derivatives of x, y, z, v, becomes

$$\theta_\alpha' - \theta_\alpha = -\epsilon\Sigma\binom{\alpha}{l}\theta_{\alpha+1-l}\phi_l,$$

where the summation is for values of l in the range $1 \leqslant l \leqslant \alpha$. Thus, as we require only the first four derivatives of the coordinates, we have, for these in succession,

$$\theta_1' - \theta_1 = -\epsilon\theta_1\phi_1,$$

$$\theta_2' - \theta_2 = -\epsilon(2\theta_2\phi_1 + \theta_1\phi_2),$$

$$\theta_3' - \theta_3 = -\epsilon(3\theta_3\phi_1 + 3\theta_2\phi_2 + \theta_1\phi_3),$$

$$\theta_4' - \theta_4 = -\epsilon(4\theta_4\phi_1 + 6\theta_3\phi_2 + 4\theta_2\phi_3 + \theta_1\phi_4).$$

The element of arc of the curve will, of course, enter into expressions, ultimately through a definition

$$ds^2 = dx^2 + dy^2 + dz^2 + dv^2,$$

so that

$$\begin{aligned} s_1^2 &= x_1^2 + y_1^2 + z_1^2 + v_1^2 \\ &= \Sigma x_1^2. \end{aligned}$$

All derivatives of s with respect to p are expressible in terms of derivatives of the coordinates; it will be convenient to associate these derivatives of s with those of x, y, z, v, not as independent magnitudes, but as subsidiary abbreviations for the appropriate combinations of the derivatives of the coordinates. It is easy to verify that

$$\begin{aligned} s_1 s_2 &= \Sigma x_1 x_2, \\ s_1 s_3 &= \Sigma x_1 x_3 + W, \\ s_1 s_4 &= \Sigma x_1 x_4 + \tfrac{3}{2} W_1, \end{aligned}$$

where W is a magnitude (subsequently shewn to be covariantive) such that

$$W = \Sigma x_2^2 - s_2^2.$$

Characteristic equations for concomitants of the curve.

453. A function f, of the variable $\bar{x}$ and of the derivatives of the coordinates, is said to be invariantive if the relation

$$f = J^k f' = (1 + \epsilon k \phi_1) f'$$

holds, where f' is the same function of $\bar{x}'$ and of the derivatives of the coordinates with respect to p' as f is of $\bar{x}$ and of the derivatives of the coordinates with respect to p. We write

$$\theta_\alpha' - \theta_\alpha = -\epsilon \Theta_\alpha.$$

Then, retaining derivatives of order not higher than four, we have

$$\begin{aligned} f' &= f(\bar{x}', x_1', y_1', z_1', v_1', x_2', y_2', z_2', v_2', x_3', y_3', z_3', v_3', x_4', y_4', z_4', v_4') \\ &= f + \epsilon \bar{x} \frac{\partial f}{\partial x} \phi_1 - \epsilon \sum_\theta \sum_\alpha \Theta_\alpha \frac{\partial f}{\partial \theta_\alpha}, \end{aligned}$$

where on the right-hand side, the summations are for $\theta = x, y, z, v$, and for $\alpha = 1, 2, 3, 4$; powers of ϵ, above the first, being omitted. Thus the equation, defining the invariantive function, becomes

$$\left(kf + \bar{x} \frac{\partial f}{\partial \bar{x}}\right) \phi_1 - \sum_\theta \sum_\alpha \Theta_\alpha \frac{\partial f}{\partial \theta_\alpha} = 0.$$

Now the function ϕ, which enters into the equation of transformation of the variable, is completely arbitrary; and therefore the full coefficients of the

successive derivatives ϕ_1, ϕ_2, ϕ_3, ϕ_4, must vanish separately. We thus obtain equations

$$\left.\begin{aligned} \Sigma x_1 \frac{\partial f}{\partial x_1} + 2\Sigma x_2 \frac{\partial f}{\partial x_2} + 3\Sigma x_3 \frac{\partial f}{\partial x_3} + 4\Sigma x_4 \frac{\partial f}{\partial x_4} &= \bar{x} \frac{\partial f}{\partial \bar{x}} + kf \\ \Sigma x_1 \frac{\partial f}{\partial x_2} + 3\Sigma x_2 \frac{\partial f}{\partial x_3} + 4\Sigma x_3 \frac{\partial f}{\partial x_4} &= 0 \\ \Sigma x_1 \frac{\partial f}{\partial x_3} + 4\Sigma x_2 \frac{\partial f}{\partial x_4} &= 0 \\ \Sigma x_1 \frac{\partial f}{\partial x_4} &= 0 \end{aligned}\right\}.$$

As already indicated, it proves convenient, in the manipulative treatment of the equations, to introduce the associated (but not independent) quantity s through its derivatives. The laws of transformation for s_1, s_2, s_3, s_4, are the same as for the quantities x, y, z, v, so that

$$\begin{aligned} s_1' - s_1 &= -\epsilon s_1 \phi_1, \\ s_2' - s_2 &= -\epsilon (2s_2\phi_1 + s_1\phi_2), \\ s_3' - s_3 &= -\epsilon (3s_3\phi_1 + 3s_2\phi_2 + s_1\phi_3), \\ s_4' - s_4 &= -\epsilon (4s_4\phi_1 + 6s_3\phi_2 + 4s_2\phi_3 + s_1\phi_4); \end{aligned}$$

but these quantities will not be introduced explicitly until after the first grade of covariantive functions has been introduced.

We have seen that $\bar{x}$ is a covariant. It is the only covariant of grade zero, the *grade* of a covariantive function being defined as the order of the highest derivative of point-coordinates which occurs in the function. For covariants of grade zero, only a single equation arises: it is

$$\bar{x} \frac{\partial f}{\partial \bar{x}} + kf = 0,$$

so that

$$f = A\bar{x}^k,$$

where A is constant. Thus every covariant of grade zero is a power of $\bar{x}$: or $\bar{x}$ is the sole independent covariant of grade zero. There is, of course, no question of orientational invariance; for the variable $\bar{x}$ is not affected by the orientation of the curve.

Covariants of grade unity.

454. For covariants of grade unity, so that x_1, y_1, z_1, v_1, are the only derivatives of x, y, z, v, which occur, there is only a single equation

$$x_1 \frac{\partial f}{\partial x_1} + y_1 \frac{\partial f}{\partial y_1} + z_1 \frac{\partial f}{\partial z_1} + v_1 \frac{\partial f}{\partial v_1} = \bar{x} \frac{\partial f}{\partial \bar{x}} + kf.$$

Proceeding in the usual manner to obtain the primitive of this equation, we construct five independent integrals of the system

$$\frac{dx_1}{x_1} = \frac{dy_1}{y_1} = \frac{dz_1}{z_1} = \frac{dv_1}{v_1} = \frac{d\bar{x}}{-\bar{x}} = \frac{df}{kf}.$$

Five such integrals are given by

$$J\bar{x}^k,\ x_1\bar{x},\ y_1\bar{x},\ z_1\bar{x},\ v_1\bar{x};$$

and the most general integral of the equation is

$$f = \bar{x}^{-k}F(x_1\bar{x},\ y_1\bar{y},\ z_1\bar{z},\ v_1\bar{v}),$$

that is, every integral of the equation can be expressed as a combination of the five integrals

$$\bar{x},\quad x_1,\quad y_1,\quad z_1,\quad v_1,$$

which accordingly are the independent magnitudes up to the first grade (inclusive) satisfying the requirements of parametric invariance.

But only those combinations are admissible which satisfy the equations of orientational invariance. In the present instance, these equations are

$$x_1\frac{\partial f}{\partial y_1} - y_1\frac{\partial f}{\partial x_1} = 0,\quad y_1\frac{\partial f}{\partial z_1} - z_1\frac{\partial f}{\partial y_1} = 0,\quad z_1\frac{\partial f}{\partial x_1} - x_1\frac{\partial f}{\partial z_1} = 0,$$

$$z_1\frac{\partial f}{\partial v_1} - v_1\frac{\partial f}{\partial z_1} = 0,\quad x_1\frac{\partial f}{\partial v_1} - v_1\frac{\partial f}{\partial x_1} = 0,\quad y_1\frac{\partial f}{\partial v_1} - v_1\frac{\partial f}{\partial y_1} = 0,$$

which, apparently six, actually are three in number, viz.

$$\frac{1}{x_1}\frac{\partial f}{\partial x_1} = \frac{1}{y_1}\frac{\partial f}{\partial y_1} = \frac{1}{z_1}\frac{\partial f}{\partial z_1} = \frac{1}{v_1}\frac{\partial f}{\partial v_1}.$$

As there are five magnitudes, and as this complete Jacobian system contains three members, there exist two algebraically independent integrals. One such integral manifestly is $\bar{x}$, the covariant already obtained. Another such integral is

$$f = x_1^2 + y_1^2 + z_1^2 + v_1^2,$$

which will be denoted by U. Thus the two integrals are

$$\bar{x},\quad U.$$

Further, from the equation

$$\Delta f = kf + \bar{x}\frac{\partial f}{\partial \bar{x}} - \Sigma x_1\frac{\partial f}{\partial x_1} = 0,$$

we have

$$\Delta\bar{x} = (k+1)\,\bar{x},\quad \Delta U = (k-2)\,U;$$

and therefore

$$\bar{x} = \frac{1}{J}\bar{x}',\quad U = J^2U',$$

where J is the modulus of transformation

$$J = \frac{dp'}{dp};$$

that is,

$$U\bar{x}^2 = U'\bar{x}'^2,$$

or $U\bar{x}^2$ is an absolute covariant (a covariant whose rank index k is zero).

It therefore follows that we have two covariants of grade unity, viz.,

$$\bar{x},\ U,$$

and that we have one absolute covariant of rank unity. Moreover

$$U\bar{x}^2 = (x_1^2 + y_1^2 + z_1^2 + v_1^2)\, dp^2$$
$$= dx^2 + dy^2 + dz^2 + dv^2 = ds^2,$$

which accordingly is the value of the absolute covariant.

At this stage, and as a preliminary to the determination of the algebraically independent covariants of rank greater than unity, we introduce the associated subsidiary (but not independent) quantities s_1, s_2, s_3, s_4, into the equations of parametric invariance. The infinitesimal transformations of the quantities s_1, s_2, s_3, s_4, have been obtained (§ 453); and therefore the equations of parametric invariance become

$$\left.\begin{aligned}
\Sigma x_1 \frac{\partial f}{\partial x_1} + 2\Sigma x_2 \frac{\partial f}{\partial x_2} + 3\Sigma x_3 \frac{\partial f}{\partial x_3} + 4\Sigma x_4 \frac{\partial f}{\partial x_4} \\
+ s_1 \frac{\partial f}{\partial s_1} + 2s_2 \frac{\partial f}{\partial s_2} + 3s_3 \frac{\partial f}{\partial s_3} + 4s_4 \frac{\partial f}{\partial s_4} = \bar{x} \frac{\partial f}{\partial \bar{x}} + kf \\
\Sigma x_1 \frac{\partial f}{\partial x_2} + 3\Sigma x_2 \frac{\partial f}{\partial x_3} + 6\Sigma x_3 \frac{\partial f}{\partial x_4} + s_1 \frac{\partial f}{\partial s_2} + 3s_2 \frac{\partial f}{\partial s_3} + 6s_3 \frac{\partial f}{\partial s_4} = 0 \\
\Sigma x_1 \frac{\partial f}{\partial x_3} + 4\Sigma x_2 \frac{\partial f}{\partial x_4} + s_1 \frac{\partial f}{\partial s_3} + 4s_2 \frac{\partial f}{\partial s_4} = 0 \\
\Sigma x_1 \frac{\partial f}{\partial x_4} + s_1 \frac{\partial f}{\partial s_4} = 0
\end{aligned}\right\}.$$

Covariants of grade two.

455. For covariants of grade two, being covariants in which the highest entering derivatives are x_2, y_2, z_2, v_2, there are two equations of parametric invariance, being

$$\Sigma x_1 \frac{\partial f}{\partial x_2} + s_1 \frac{\partial f}{\partial s_2} = 0,$$

$$\Sigma x_1 \frac{\partial f}{\partial x_1} + 2\Sigma x_2 \frac{\partial f}{\partial x_2} + s_1 \frac{\partial f}{\partial s_1} + 2s_2 \frac{\partial f}{\partial s_2} = \bar{x} \frac{\partial f}{\partial \bar{x}} + kf.$$

This is a complete Jacobian system. There are nine arguments, $\bar{x}$, x_1, y_1, z_1, v_1, x_2, y_2, z_2, v_2; the arguments s_1 and s_2 are not independent quantities; consequently, for the complete integration of the first equation, we require eight algebraically independent integrals of that equation. As the Lagrange subsidiary equations are

$$\frac{d\bar{x}}{0} = \frac{dx_1}{0} = \frac{dy_1}{0} = \frac{dz_1}{0} = \frac{dv_1}{0} = \frac{dx_2}{x_1} = \frac{dy_2}{y_1} = \frac{dz_2}{z_1} = \frac{dv_2}{v_1}$$
$$= \frac{ds_1}{0} = \frac{ds_2}{s_1},$$

we at once have

$$\bar{x},\ x_1,\ y_1,\ z_1,\ v_1,\ (\text{and } s_1),$$

as five such integrals. Other integrals are

$$X = s_1x_2 - x_1s_2, \quad Y = s_1y_2 - y_1s_2, \quad Z = s_1z_2 - z_1s_2, \quad V = s_1v_2 - v_1s_2;$$

but

$$\begin{aligned} Xx_1 + Yy_1 + Zz_1 + Vv_1 &= s_1\Sigma x_1x_2 - s_2\Sigma x_1^2 \\ &= s_1 . s_1s_2 - s_2 . s_1^2 = 0, \end{aligned}$$

that is, X, Y, Z, V, really are only three independent integrals. Accordingly, we can take

$$\bar{x},\ x_1,\ y_1,\ z_1,\ v_1,\ (s_1),\ X,\ Y,\ Z,\ V,$$

as including eight necessary integrals, algebraically independent of one another, in terms of which every integral of the first equation can be expressed.

Before proceeding to consider the remaining equation of parametric invariance, we shall obtain those combinations of the eight independent integrals which satisfy the equations of orientational invariance. In the present instance, these equations are

$$E_{yz} = y_1\frac{\partial}{\partial z_1} - z_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial z_2} - z_2\frac{\partial}{\partial y_2} = 0,$$

$$E_{zx} = z_1\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial z_2} = 0,$$

$$E_{xy} = x_1\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial x_2} = 0,$$

$$E_{xv} = x_1\frac{\partial}{\partial v_1} - v_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial v_2} - v_2\frac{\partial}{\partial x_2} = 0,$$

$$E_{yv} = y_1\frac{\partial}{\partial v_1} - v_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial v_2} - v_2\frac{\partial}{\partial y_2} = 0,$$

$$E_{zv} = z_1\frac{\partial}{\partial v_1} - v_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial v_2} - v_2\frac{\partial}{\partial z_2} = 0,$$

apparently six in number. But the relation

$$\begin{aligned} &(x_1v_2 - v_1x_2)\,E_{yz} + (y_1v_2 - v_1y_2)\,E_{zx} + (z_1v_2 - v_1z_2)\,E_{xy} \\ &+ (y_1z_2 - z_1y_2)\,E_{xv} + (z_1x_2 - x_1z_2)\,E_{yv} + (x_1y_2 - y_1x_2)\,E_{zv} = 0 \end{aligned}$$

is satisfied identically; and therefore the system is composed of only five linearly independent members. It is a complete Jacobian system, and there are eight independent arguments; hence every integral of this system can be expressed in terms of three algebraically independent integrals.

Now two such integrals are already known, being

$$\bar{x},\ U\,(=\Sigma x_1^2);$$

and therefore one other integral must be obtained, manifestly involving second derivatives of x, y, z, v, that is, dependent upon the combinations X, Y, Z, V, of these second derivatives. We know that s_1, being

$$(x_1^2 + y_1^2 + z_1^2 + v_1^2)^{\frac{1}{2}},$$

satisfies all the equations of orientational invariance. It is easy to verify that the combination $x_1x_2 + y_1y_2 + z_1z_2 + v_1v_2$, symmetric in the quantities x, y, z, v, satisfies all the equations: and therefore s_1s_2 satisfies them all. For any operation E, we have

$$0 = E(s_1s_2) = s_2E(s_1) + s_1E(s_2) = s_1E(s_2):$$

that is, s_2 satisfies all the equations. Thus s_1 and s_2 behave like constants, in the operations E.

Now by direct substitution, we find

$$\left.\begin{aligned} E_{yz}X &= 0 \\ E_{yz}Y &= -Z \\ E_{yz}Z &= Y \\ E_{yz}V &= 0 \end{aligned}\right\}, \quad \left.\begin{aligned} E_{zx}X &= Z \\ E_{zx}Y &= 0 \\ E_{zx}Z &= -X \\ E_{zx}V &= 0 \end{aligned}\right\}, \quad \left.\begin{aligned} E_{xy}X &= -Y \\ E_{xy}Y &= X \\ E_{xy}Z &= 0 \\ E_{xy}V &= 0 \end{aligned}\right\},$$

$$\left.\begin{aligned} E_{xv}X &= V \\ E_{xv}Y &= 0 \\ E_{xv}Z &= 0 \\ E_{xv}V &= -X \end{aligned}\right\}, \quad \left.\begin{aligned} E_{yv}X &= 0 \\ E_{yv}Y &= V \\ E_{yv}Z &= 0 \\ E_{yv}V &= -Y \end{aligned}\right\}, \quad \left.\begin{aligned} E_{zv}X &= 0 \\ E_{zv}Y &= 0 \\ E_{zv}Z &= V \\ E_{zv}V &= -Z \end{aligned}\right\};$$

and therefore an obvious integral, common to all the equations, is

$$X^2 + Y^2 + Z^2 + V^2.$$

Now

$$\begin{aligned} \Sigma X^2 &= \Sigma(x_1s_2 - s_1x_2)^2 \\ &= s_2^2\Sigma x_1^2 - 2s_1s_2\Sigma x_1x_2 + s_1^2\Sigma x_2^2 \\ &= s_1^2(\Sigma x_2^2 - s_2^2), \end{aligned}$$

and s_1 itself is an integral of the equations. We therefore take

$$W = x_2^2 + y_2^2 + z_2^2 + v_2^2 - s_2^2$$

as the third required integral.

Accordingly, three algebraically independent integrals of the initial equation of parametric invariance and of the equations of orientational invariance are

$$\bar{x},\ U,\ W.$$

The remaining equation of parametric invariance is

$$kf + \bar{x}\frac{\partial f}{\partial \bar{x}} + \Delta f = 0,$$

where

$$\Delta f = -\Sigma x_1\frac{\partial f}{\partial x_1} - 2\Sigma x_2\frac{\partial f}{\partial x_2} - s_1\frac{\partial f}{\partial s_1} - 2s_2\frac{\partial f}{\partial s_2} = 0;$$

and the required integrals must be functions of $\bar{x}$, U, W, which now are the only admissible combinations of arguments. We have

$$\begin{aligned} \Delta U &= \Delta(x_1^2 + y_1^2 + z_1^2 + v_1^2) = -2U, \\ \Delta W &= \Delta(\Sigma x_2^2 - s_2^2) = -4\Sigma x_2^2 + 4s_2^2 = -4W; \end{aligned}$$

and thus the equation is

$$kf + \bar{x}\frac{\partial f}{\partial \bar{x}} - 2U\frac{\partial f}{\partial U} - 4W\frac{\partial f}{\partial W} = 0.$$

This is an equation of the linear Lagrange type; by the regular process, its most general integral is given by

$$f = \bar{x}^{-k} F(U\bar{x}^2, W\bar{x}^4),$$

that is, every integral of the equation can be expressed in terms of the three algebraically independent integrals

$$\bar{x}, \quad U\bar{x}^2, \quad W\bar{x}^4.$$

Accordingly, these constitute a complete aggregate of covariants of grade not greater than two.

The interpretation

$$ds^2 = U\bar{x}^2$$

has already been obtained. The interpretation of W is immediate; for if, as usual, x', x'', ... denote $\frac{dx}{ds}$, $\frac{d^2x}{ds^2}$, ..., and so for derivatives of the other coordinates,

$$x'' = \frac{1}{s_1^3}(s_1x_2 - x_1s_2) = \frac{1}{s_1^3}X,$$

so that

$$\Sigma x''^2 = \frac{1}{s_1^6}\Sigma X^2 = \frac{1}{s_1^4}W.$$

But

$$\Sigma x''^2 = \frac{1}{\rho^2},$$

where ρ is the radius of circular curvature of the curve; hence

$$W\bar{x}^4 = \frac{1}{\rho^2}s_1^4\bar{x}^4 = \frac{U^2\bar{x}^4}{\rho^2} = \frac{ds^4}{\rho^2},$$

thus giving the interpretation of the covariant.

It is easy to verify, by substitution in the equation $\Delta f = 0$, that $W\bar{x}^4$ has its indicial integer k equal to zero: it is an absolute covariant.

Covariants of grade three.

456. The highest derivatives of the coordinates which enter into covariantive functions of grade three are x_3, y_3, z_3, v_3. There are three equations of parametric invariance, being

$$\Delta_2 = \Sigma x_1\frac{\partial f}{\partial x_3} + s_1\frac{\partial f}{\partial s_3} = 0,$$

$$\Delta_1 = \Sigma x_1\frac{\partial f}{\partial x_2} + 3\Sigma x_2\frac{\partial f}{\partial x_3} + s_1\frac{\partial f}{\partial s_2} + 3s_2\frac{\partial f}{\partial s_3} = 0,$$

$$kf + \bar{x}\frac{\partial f}{\partial \bar{x}} = \Delta,$$

where

$$\Delta = \Sigma x_1 \frac{\partial f}{\partial x_1} + 2\Sigma x_2 \frac{\partial f}{\partial x_2} + 3\Sigma x_3 \frac{\partial f}{\partial x_3} + s_1 \frac{\partial f}{\partial s_1} + 2s_2 \frac{\partial f}{\partial s_2} + 3s_3 \frac{\partial f}{\partial s_3}.$$

The two equations $\Delta_1 = 0$ and $\Delta_2 = 0$ are a complete Jacobian system within their range. The total number of arguments is thirteen, the arguments being $\bar{x}$, x_l, y_l, z_l, v_l, (for $l = 1, 2, 3$), with s_1, s_2, s_3, as associated (but not independent) arguments. We shall thus require eleven integrals of $\Delta_1 = 0$ and $\Delta_2 = 0$, algebraically independent of one another.

The orientational equations are now six in number: and they are a complete Jacobian system within their range. As the possible arguments are the eleven integrals common to $\Delta_1 = 0$ and $\Delta_2 = 0$, there will survive five combinations of these arguments as integrals common to the orientational equations.

When these have been obtained, they lead to five distinct integrals out of which every integral of the equation

$$kf + \bar{x} \frac{\partial f}{\partial \bar{x}} = \Delta$$

can be constructed. These five distinct integrals constitute the aggregate of covariants up to grade three (inclusive).

As initially there are thirteen distinct arguments, we require twelve algebraically independent integrals of the first equation $\Delta_2 = 0$. We have the nine

$$\bar{x};\ x_1, y_1, z_1, v_1, \text{ with } s_1;\ x_2, y_2, z_2, v_2, \text{ with } s_2;$$

and the four $\xi, \eta, \zeta, \upsilon$, with s_3, where

$$\xi = s_1 x_3 - x_1 s_3, \quad \eta = s_1 y_3 - y_1 s_3, \quad \zeta = s_1 z_3 - z_1 s_3, \quad \upsilon = s_1 v_3 - v_1 s_3.$$

But

$$\Sigma x_1 \xi = s_1 \Sigma x_1 x_3 - s_1^2 s_3 = -s_1 W,$$

so that $\xi, \eta, \zeta, \upsilon$, are equivalent to only three new integrals when W (as before) is to be retained. Thus there is the full complement of the twelve necessary integrals; and it is to be noted that x_3, y_3, z_3, v_3, with s_3, enter only in the combinations $\xi, \eta, \zeta, \upsilon$.

These twelve integrals are the possible arguments of the single equation $\Delta_1 = 0$; and we therefore require eleven algebraically independent combinations of these arguments which satisfy $\Delta_1 = 0$. Now, on direct substitution, we find

$$\Delta_1 \xi = 3X, \quad \Delta_1 \eta = 3Y, \quad \Delta_1 \zeta = 3Z, \quad \Delta_1 \upsilon = 3V,$$

where, as before,

$$X = s_1 x_2 - x_1 s_2, \quad Y = s_1 y_2 - y_1 s_2, \quad Z = s_1 z_2 - z_1 s_2, \quad V = s_1 v_2 - v_1 s_2;$$

and the equation $\Delta_1 = 0$ becomes

$$\Delta_1 = \Sigma x_1 \frac{\partial f}{\partial x_2} + s_1 \frac{\partial f}{\partial s_2} + 3\Sigma X \frac{\partial f}{\partial \xi} = 0.$$

Of the necessary quota of eleven algebraically independent integrals of $\Delta_1 = 0$, five are contributed by

$$\bar{x};\ x_1,\ y_1,\ z_1,\ v_1, \text{ with } s_1.$$

Other three are contributed from the four quantities

$$X,\ Y,\ Z,\ V,$$

these being connected by the relation $Xx_1 + Yy_1 + Zz_1 + Vv_1 = 0$. A remaining three, to complete the quota, are contributed by

$$P = s_1\xi - 3s_2X, \quad Q = s_1\eta - 3s_2Y, \quad R = s_1\zeta - 3s_2Z, \quad S = s_1\upsilon - 3s_2V,$$

not more than three being independent, because

$$x_1P + y_1Q + z_1R + v_1S = s_1\Sigma x_1\xi - 3s_2\Sigma Xx_1 = -s_1^2W,$$

which is a single relation between P, Q, R, S, when W is to be retained. Thus eleven algebraically independent integrals of $\Delta_1 = 0$ and $\Delta_2 = 0$ are

$$\bar{x};\ x_1,\ y_1,\ z_1,\ v_1, \text{ with } s_1;\ X,\ Y,\ Z,\ V;\ P,\ Q,\ R,\ S;$$

each of the last two sets of four quantities contributing three independent integrals.

We now pass to the six orientational equations, in order to obtain five combinations of the eleven integrals just obtained. These orientational equations may be written

$$E_{yz} = 0, \quad E_{zx} = 0, \quad E_{xy} = 0, \quad E_{xv} = 0, \quad E_{yv} = 0, \quad E_{zv} = 0;$$

and they are linearly independent, constituting a complete Jacobian system within their range.

First of all, the quantity s_3 is invariantive for these equations, as were the quantities s_1 and s_2. For we have

$$s_1s_3 = \Sigma x_1x_3 + W:$$

it is easy to verify that Σx_1x_3 satisfies all the six equations: W is invariantive, as appeared from the preceding investigation: and therefore s_3 is invariantive, that is, s_3 behaves like a constant, so far as concerns these equations.

Clearly $\bar{x}$ is one integral of the set of equations; consequently $\bar{x}$ will be one of the complete quota of five independent integrals. For the rest, we have

$$E_{yz}x_1 = 0, \quad E_{yz}y_1 = -z_1, \quad E_{yz}z_1 = y_1, \quad E_{yz}v_1 = 0;$$
$$E_{yz}X = 0, \quad E_{yz}Y = -Z, \quad E_{yz}Z = Y, \quad E_{yz}V = 0;$$
$$E_{yz}\xi = 0, \quad E_{yz}\eta = -\zeta, \quad E_{yz}\zeta = \eta, \quad E_{yz}\upsilon = 0,$$

and therefore

$$E_{yz}P = 0, \quad E_{yz}Q = -R, \quad E_{yz}R = Q, \quad E_{yz}S = 0.$$

Similarly for the other five orientational equations. Thus there are integrals

$$\Sigma x_1^2, \quad \Sigma x_1X, \quad \Sigma x_1P, \quad \Sigma X^2, \quad \Sigma XP, \quad \Sigma P^2.$$

But we have seen that

$$\Sigma x_1 X = 0, \quad \Sigma x_1 P = -s_1^2 W = \Sigma X^2,$$

so that the second and the third do not contribute to an aggregate of algebraically independent integrals; and we therefore have the four integrals U, G, H, K, in addition to $\bar{x}$, where

$$U = \Sigma x_1^2, \quad G = \Sigma X^2, \quad H = \Sigma XP, \quad K = \Sigma P^2.$$

We now have to determine the combinations of $\bar{x}$, U, G, H, K, which suffice for the expression of the most general integral of the remaining equation

$$kf + \bar{x}\frac{\partial f}{\partial \bar{x}} = \Delta.$$

By direct substitution, we find

$$\begin{aligned} \Delta x_1 &= x_1, \text{ so that } \Delta U = 2U, \\ \Delta X &= 3X, \text{ so that } \Delta G = 6G, \\ \Delta P &= 5P, \text{ so that } \Delta H = 8H,\ \Delta K = 10K; \end{aligned}$$

and therefore the remaining equation becomes

$$kf + \bar{x}\frac{\partial f}{\partial \bar{x}} - 2U\frac{\partial f}{\partial \bar{U}} - 6G\frac{\partial f}{\partial \bar{G}} - 8H\frac{\partial f}{\partial \bar{H}} - 10K\frac{\partial f}{\partial \bar{K}} = 0.$$

The customary theory of a linear Lagrange equation of this type shews that every integral can be expressed as a function of the five quantities

$$\bar{x}, \quad U\bar{x}^2, \quad G\bar{x}^6, \quad H\bar{x}^8, \quad K\bar{x}^{10}.$$

Moreover, substitution in the final equation shews that

$$\begin{aligned} k &= -1, \text{ for the integral } \bar{x}, \\ k &= 0, \text{ for the integrals } U\bar{x}^2, G\bar{x}^6, H\bar{x}^8, K\bar{x}^{10}; \end{aligned}$$

consequently, the four quantities

$$U\bar{x}^2, \quad G\bar{x}^6, \quad H\bar{x}^8, \quad K\bar{x}^{10},$$

are absolute covariants and, with $\bar{x}$, they constitute an algebraically complete aggregate of covariants up to grade three inclusive.

For the algebraical expressions, in terms of the derivatives of x, y, z, v, s, we have

$$\begin{aligned} U &= \Sigma x_1^2 &&= s_1^2, \\ G &= \Sigma X^2 &&= s_1^2(\Sigma x_2^2 - s_2^2), \\ H &= \Sigma PX &&= s_1^3(\Sigma x_2 x_3 - s_2 s_3) - 2s_2 s_1^2(\Sigma x_2^2 - s_2^2), \\ K &= \Sigma P^2 &&= s_1^4(\Sigma x_3^2 - s_3^2) + 6s_1^3 s_2(\Sigma x_2 x_3 - s_2 s_3) \\ &&&\quad + (2s_1 s_3 + 3s_2^2)\, s_1^2(\Sigma x_2^2 - s_2^2). \end{aligned}$$

457. The geometrical interpretations are immediate. Using $x', x'', \ldots$ to denote derivatives of the coordinates with regard to the arc, we have

$$\begin{aligned} x' &= \frac{x_1}{s_1}, \\ x'' &= \frac{1}{s_1^3}(s_1 x_2 - x_1 s_2) = \frac{1}{s_1^3} X, \\ x''' &= \frac{1}{s_1^4}(s_1 x_3 - x_1 s_3) - 3\frac{s_2}{s_1^5}(s_1 x_2 - x_1 s_2) = \frac{1}{s_1^5}(s_1 \xi - 3s_2 X) = \frac{1}{s_1^5} P; \end{aligned}$$

and therefore

$$\Sigma x''^2 = \frac{1}{s_1^6}\Sigma X^2, \quad \Sigma x'' x''' = \frac{1}{s_1^8}\Sigma PX, \quad \Sigma x'''^2 = \frac{1}{s_1^{10}}\Sigma P^2,$$

so that

$$U\bar{x}^2 = (s_1\bar{x})^2\,\Sigma x'^2 = \delta s^2,$$

$$G\bar{x}^6 = (s_1\bar{x})^6\,\Sigma x''^2 = \frac{1}{\rho^2}\delta s^6,$$

$$H\bar{x}^8 = (s_1\bar{x})^8\,\Sigma x'' x''' = -\frac{\rho'}{\rho^3}\delta s^8,$$

$$K\bar{x}^{10} = (s_1\bar{x})^{10}\,\Sigma x'''^2 = \left(\frac{1}{\rho^4} + \frac{\rho'^2}{\rho^4} + \frac{1}{\rho^2\sigma^2}\right)\delta s^{10}.$$

Consequently

$$\frac{G}{U^3} = \frac{1}{\rho^2}, \quad \frac{H}{U^4} = -\frac{\rho'}{\rho^3}, \quad \frac{K}{U^5} = \frac{1}{\rho^4} + \frac{\rho'^2}{\rho^4} + \frac{1}{\rho^2\sigma^2}.$$

As the quantities are covariantive, the geometrical interpretations can also be derived by taking the arc s as the parametric variable so that $s = p$, $s_1 = 1$, $s_2 = 0$, $s_3 = 0$; and then

$$U = 1,$$

$$G = \Sigma x''^2 = \frac{1}{\rho^2},$$

$$H = \Sigma x'' x''' = -\frac{\rho'}{\rho^3},$$

$$K = \Sigma x'''^2 = \frac{1}{\rho^4} + \frac{\rho'^2}{\rho^4} + \frac{1}{\rho^2\sigma^2}.$$

Covariants of grade four.

458. The construction of covariants of grade four proceeds, almost step by step, along the same course as the construction of covariants of lower grade; the necessary analysis will, therefore, be stated more briefly.

For these covariants, there are the three equations of parametric invariance

$$\Delta_3 = \Sigma x_1 \frac{\partial f}{\partial x_4} + s_1 \frac{\partial f}{\partial s_4} = 0,$$

$$\Delta_2 = \Sigma x_1 \frac{\partial f}{\partial x_3} + 4\Sigma x_2 \frac{\partial f}{\partial x_4} + s_1 \frac{\partial f}{\partial s_3} + 4s_2 \frac{\partial f}{\partial s_4} = 0,$$

$$\Delta_1 = \Sigma x_1 \frac{\partial f}{\partial x_2} + 3\Sigma x_2 \frac{\partial f}{\partial x_3} + 6\Sigma x_3 \frac{\partial f}{\partial x_4} + s_1 \frac{\partial f}{\partial s_2} + 3s_2 \frac{\partial f}{\partial s_3} + 6s_3 \frac{\partial f}{\partial s_4} = 0;$$

and the final equation of parametric invariance is

$$kf + \bar{x}\frac{\partial f}{\partial \bar{x}} = \Delta,$$

where

$$\Delta = \Sigma x_1 \frac{\partial f}{\partial x_1} + 2\Sigma x_2 \frac{\partial f}{\partial x_2} + 3\Sigma x_3 \frac{\partial f}{\partial x_3} + 4\Sigma x_4 \frac{\partial f}{\partial x_4} + s_1 \frac{\partial f}{\partial s_1} + 2s_2 \frac{\partial f}{\partial s_2} + 3s_3 \frac{\partial f}{\partial s_3} + 4s_4 \frac{\partial f}{\partial s_4}.$$

In these equations, there are seventeen arguments, $\bar{x}$; x_l, y_l, z_l, v_l, for $l = 1, 2, 3, 4$; and, with the latter, are to be associated the variables s_1, s_2, s_3, s_4. The equations $\Delta_1 = 0$, $\Delta_2 = 0$, $\Delta_3 = 0$, are a complete Jacobian system within their own range; and consequently every integral can be expressed in terms of $(17 - 3 =) 14$ algebraically independent integrals.

We have already obtained certain covariantive or quasi-covariantive forms of the type $\Sigma x_l x_m - s_l s_m$. Let

$$\Sigma x_2^2 - s_2^2 = W, \quad \Sigma x_3^2 - s_3^2 = T,$$

the significance of W and T having been established; then we have the aggregate of results

$$\begin{aligned}
\Sigma x_1^2 - s_1^2 &= 0, \\
\Sigma x_1 x_2 - s_1 s_2 &= 0, \\
\Sigma x_1 x_3 - s_1 s_3 &= -W, \\
\Sigma x_2 x_3 - s_2 s_3 &= \frac{1}{2} W_1, \\
\Sigma x_1 x_4 - s_1 s_4 &= -\frac{3}{2} W_1, \\
\Sigma x_2 x_4 - s_2 s_1 &= \frac{1}{2} W_{11} - T, \\
\Sigma x_3 x_4 - s_3 s_4 &= \frac{1}{2} T_1;
\end{aligned}$$

and we shall obtain a form Γ, where

$$\Sigma x_4^2 - s_4^2 = \Gamma.$$

Algebraically independent integrals, adequate to form the necessary quota of sixteen integrals, of the equation $\Delta_3 = 0$, are

$$\bar{x};\ x_l, y_l, z_l, v_l, \text{ with } s_l, \text{ (for } l = 1, 2, 3);\ \alpha, \beta, \gamma, \delta,$$

where

$$\alpha = s_1 x_4 - x_1 s_4, \quad \beta = s_1 y_4 - y_1 s_4, \quad \gamma = s_1 z_4 - z_1 s_4, \quad \delta = s_1 v_4 - v_1 s_4;$$

and, because

$$\Sigma x_1 \alpha = s_1 \Sigma x_1 x_4 - s_1^2 s_4 = -\frac{3}{2} s_1 W_1,$$

which is a quantity to be retained, the four integrals $\alpha, \beta, \gamma, \delta$, are equivalent to three algebraically independent integrals. For the subsequent equations, the variables x_4, y_4, z_4, v_4, with s_4, enter only through the combinations $\alpha, \beta, \gamma, \delta$.

Next, of these sixteen quantities, we require fifteen combinations which are algebraically independent integrals of the equation $\Delta_2 = 0$. Now

$$\Delta_2\alpha = 4X, \quad \Delta_2\beta = 4Y, \quad \Delta_2\gamma = 4Z, \quad \Delta_2\delta = 4V,$$

so that the modified form of Δ_2 is

$$\Sigma x_1 \frac{\partial f}{\partial x_3} + s_1 \frac{\partial f}{\partial s_3} + 4\Sigma X \frac{\partial f}{\partial \alpha} = 0.$$

The necessary fifteen integrals are

$$\bar{x};\ x_1, y_1, z_1, v_1, \text{ with } s_1;\ x_2, y_2, z_2, v_2, \text{ with } s_2;$$
$$\xi, \eta, \zeta, \upsilon;\ I, L, M, N;$$

where

$$\left.\begin{aligned} \xi &= s_1 x_3 - x_1 s_3 \\ \eta &= s_1 y_3 - y_1 s_3 \\ \zeta &= s_1 z_3 - z_1 s_3 \\ \upsilon &= s_1 v_3 - v_1 s_3 \end{aligned}\right\}, \qquad \left.\begin{aligned} I &= s_1\alpha - 4Xs_3 \\ L &= s_1\beta - 4Ys_3 \\ M &= s_1\gamma - 4Zs_3 \\ N &= s_1\delta - 4Vs_3 \end{aligned}\right\}.$$

Because

$$\Sigma x_1 I = s_1 \Sigma x_1 \alpha - 4s_3 \Sigma x_1 X = -\frac{3}{2} s_1^2 W_1,$$

the four quantities I, L, M, N, are equivalent to three algebraically independent integrals; and because

$$\Sigma x_1 \xi = s_1 W,$$

the four quantities $\xi, \eta, \zeta, \upsilon$, are equivalent to three algebraically independent integrals.

Next, of the fifteen algebraically independent integrals of $\Delta_2 = 0$ which have been obtained (and which are integrals common to $\Delta_3 = 0$ and $\Delta_2 = 0$), we require fourteen combinations; out of these, all integrals common to the three equations $\Delta_3 = 0$, $\Delta_2 = 0$, $\Delta_1 = 0$, can be constructed. Now

$$\Delta_1\alpha = 6\xi, \quad \Delta_1\beta = 6\eta, \quad \Delta_1\gamma = 6\zeta, \quad \Delta_1\delta = 6\upsilon,$$
$$\Delta_1\xi = 3X, \quad \Delta_1\eta = 3Y, \quad \Delta_1\zeta = 3Z, \quad \Delta_1\upsilon = 3V,$$
$$\Delta_1 I = 6s_1\xi - 12s_2 X = 6(P + s_2 X),$$
$$\Delta_1 L = 6s_1\eta - 12s_2 Y = 6(Q + s_2 Y),$$
$$\Delta_1 M = 6s_1\zeta - 12s_2 Z = 6(R + s_2 Z),$$
$$\Delta_1 N = 6s_1\upsilon - 12s_2 V = 6(S + s_2 V),$$

with the former significance for P, Q, R, S. Thus the equation $\Delta_1 = 0$, for the present purpose, becomes

$$\Sigma x_1 \frac{\partial}{\partial x_2} + s_1 \frac{\partial}{\partial s_2} + 3\Sigma X \frac{\partial}{\partial \xi} + 6\Sigma (P + s_2 X) \frac{\partial}{\partial I} = 0.$$

Of this equation, there are integrals

$$\bar{x};\ x_1, y_1, z_1, v_1, \text{ with } s_1;$$
$$X, Y, Z, V;$$
$$P, Q, R, S;$$

and

$$\left.\begin{aligned} A &= s_1 I - 6s_2 (P + s_2 X) \\ B &= s_1 L - 6s_2 (Q + s_2 Y) \\ C &= s_1 M - 6s_2 (R + s_2 Z) \\ D &= s_1 N - 6s_2 (S + s_2 V) \end{aligned}\right\}.$$

Now X, Y, Z, V, are equivalent to only three independent integrals because

$$\Sigma x_1 X = 0;$$

and P, Q, R, S, similarly are equivalent to only three because

$$\Sigma x_1 P = -s_1^2 W;$$

and A, B, C, D, likewise are equivalent to only three because

$$\Sigma x_1 A = s_1 \Sigma x_1 I - 6s_2 \Sigma x_1 P - 6s_2^2 \Sigma x_1 X$$

$$= -\frac{3}{2} s_1^3 W_1 + 6s_1^2 s_2 W = -3\Sigma PX.$$

We thus have the necessary fourteen integrals, being the combinations of the former quantities constructed to satisfy $\Delta_1 = 0$.

Before proceeding to the final equation for parametric invariance, we select those combinations of our fourteen surviving quantities which satisfy the six independent orientational equations

$$E_{yz} = 0, \quad E_{zx} = 0, \quad E_{xy} = 0, \quad E_{xv} = 0, \quad E_{yv} = 0, \quad E_{zv} = 0.$$

Now

$$\left.\begin{aligned} E_{yz}\alpha &= 0 \\ E_{yz}\beta &= -\gamma \\ E_{yz}\gamma &= \beta \\ E_{yz}\delta &= 0 \end{aligned}\right\}, \quad \left.\begin{aligned} E_{yz}I &= 0 \\ E_{yz}L &= -M \\ E_{yz}M &= L \\ E_{yz}N &= 0 \end{aligned}\right\}, \quad \left.\begin{aligned} E_{yz}A &= 0 \\ E_{yz}B &= -C \\ E_{yz}C &= B \\ E_{yz}D &= 0 \end{aligned}\right\},$$

and similarly for the other five equations; and the effect of the six equations upon x_1, y_1, z_1, v_1; X, Y, Z, V; P, Q, R, S; has already (§ 456) been given. Hence, as contributions to the eight integrals $(= 14 - 6)$ which are necessary for a complete system of the six orientational equations, we have the set

$$\begin{aligned} &\bar{x};\ \Sigma x_1^2, \\ &\quad \Sigma x_1 X,\ \Sigma X^2, \\ &\quad \Sigma x_1 P,\ \Sigma XP,\ \Sigma P^2, \\ &\quad \Sigma x_1 A,\ \Sigma XA,\ \Sigma PA,\ \Sigma A^2. \end{aligned}$$

But we have obtained the relations

$$\Sigma x_1 X = 0, \quad \Sigma x_1 P = -s_1^2 W = \Sigma X^2, \quad \Sigma x_1 A = -3\Sigma PX,$$

so that the three integrals $\Sigma x_1 X$, $\Sigma x_1 P$, $\Sigma x_1 A$, are not independent; and therefore there remain eight integrals (the necessary number) independent of one another, constituted by the set

$$\bar{x},\ \Sigma x_1^2,\ \Sigma X^2,\ \Sigma XP,\ \Sigma P^2,\ \Sigma XA,\ \Sigma PA,\ \Sigma A^2.$$

The parametric equation

$$kf + \bar{x}\frac{\partial f}{\partial \bar{x}} = \Delta$$

remains to be considered. As before, we write

$$U = \Sigma x_1^2, \quad G = \Sigma X^2, \quad H = \Sigma XP, \quad K = \Sigma P^2;$$

and for the new covariantive quantities, we take

$$T = \Sigma XA, \quad \Theta = \Sigma PA, \quad \Omega = \Sigma A^2.$$

It has already (§ 456) been shewn that

$$\Delta U = 2U, \quad \Delta G = 6G, \quad \Delta H = 8H, \quad \Delta K = 10K;$$

and in the same manner we find

$$\Delta T = 10T, \quad \Delta\Theta = 12\Theta, \quad \Delta\Omega = 14\Omega.$$

Thus the remaining equation becomes

$$kf + \bar{x}\frac{\partial f}{\partial \bar{x}} = 2U\frac{\partial f}{\partial U} + 6G\frac{\partial f}{\partial G} + 8H\frac{\partial f}{\partial H} + 10K\frac{\partial f}{\partial K} + 10T\frac{\partial f}{\partial T} + 12\Theta\frac{\partial f}{\partial \Theta} + 14\Omega\frac{\partial f}{\partial \Omega}.$$

All the covariants up to (and including) the fourth grade are constituted from those of an algebraically independent set

$$U\bar{x}^2, \ G\bar{x}^6, \ H\bar{x}^8, \ K\bar{x}^{10}, \ T\bar{x}^{10}, \ \Theta\bar{x}^{12}, \ \Omega\bar{x}^{14},$$

there being the solitary covariant $\bar{x}$ independent of any curve. The interpretations of the first four of these have been given: we had

$$U\bar{x}^2 = \delta s^2,$$

$$\frac{G}{U^3} = \frac{1}{\rho^2},$$

$$\frac{H}{U^4} = -\frac{\rho'}{\rho^3},$$

$$\frac{K}{U^5} = \frac{1}{\rho^4} + \frac{\rho'^2}{\rho^4} + \frac{1}{\rho^2\sigma^2}.$$

For the remainder, in regard to explicit expression, we use the forms

$$I = s_1\alpha - 4s_3X,$$

$$P = s_1\xi - 3s_2X,$$

$$A = s_1I - 6s_2(P + s_2X)$$

$$= s_1^2\alpha - 6s_1s_2\xi + (12s_2^2 - 4s_1s_3)X.$$

Now

$$\Sigma X^2 = s_1^2W, \text{ where } W = \Sigma x_2^2 - s_2^2,$$

$$\Sigma X\xi = \frac{1}{2}s_1^2W_1 + s_1s_2W,$$

$$\Sigma\xi^2 = s_1^2V + 2s_1s_3W, \text{ where } V = \Sigma x_3^2 - s_3^2,$$

being results already established ; and

$$\Sigma x_1\alpha = -\frac{3}{2}s_1 W_1,$$

$$\Sigma X\alpha = s_1^2\left(\frac{1}{2}W_{11} - V\right) + s_1 s_2 W_1,$$

$$\Sigma\xi\alpha = \frac{1}{2}s_1^2 V_1 + \frac{3}{2}s_1 s_3 W_1 + s_1 s_4 W,$$

$$\Sigma\alpha^2 = s_1^2\Gamma + 3s_1 s_4 W_1,$$

where

$$\Gamma = \Sigma x_4^2 - s_4^2.$$

The leading quantities, for subsequent interpretation, are

$$W = \Sigma x_2^2 - s_2^2,$$

$$\frac{1}{2}W_1 = \Sigma x_2 x_3 - s_2 s_3,$$

$$V = \Sigma x_3^2 - s_3^2,$$

$$\frac{1}{2}W_{11} - V = \Sigma x_2 x_4 - s_2 s_4,$$

$$\frac{1}{2}V_1 = \Sigma x_3 x_4 - s_3 s_4,$$

$$\Gamma = \Sigma x_4^2 - s_4^2.$$

We have, by direct substitution,

$$\begin{aligned}
T &= \Sigma XA \\
&= s_1^2\Sigma X\alpha - 6s_1 s_2\Sigma X\xi + (12s_2^2 - 4s_1 s_3)\Sigma X^2, \\
\Theta &= \Sigma PA \\
&= \Sigma\,(s_1\xi - 3s_2 X)\,\{s_1^2\alpha - 6s_1 s_2\xi + (12s_2^2 - 4s_1 s_3)\,X\} \\
&= s_1^3\Sigma\xi\alpha - 3s_1^2 s_2\Sigma X\alpha + 6s_1^2 s_2\Sigma\xi^2 \\
&\qquad + (36s_1 s_2^2 - 4s_1^2 s_3)\Sigma X\xi - (54s_2^3 - 12s_1 s_2 s_3)\Sigma X^2, \\
\Omega &= \Sigma A^2 \\
&= \Sigma\,\{s_1^2\alpha - 6s_1 s_2\xi + (12s_2^2 - 4s_1 s_3)\,X\}^2 \\
&= s_1^4\Sigma\alpha^2 - 12s_1^3 s_2\Sigma\xi\alpha + s_1^2(24s_2^2 - 8s_1 s_3)\Sigma X\alpha \\
&\qquad + 36s_1^2 s_2^2\Sigma\xi^2 - 12s_1 s_2(12s_2^2 - 4s_1 s_3)\Sigma X\xi + (12s_2^2 - 4s_1 s_3)^2\Sigma X^2,
\end{aligned}$$

the full explicit values of each of which are immediately obtained by substituting the values, already given, of ΣX^2, $\Sigma X\xi$, $\Sigma\xi^2$, $\Sigma X\alpha$, $\Sigma\xi\alpha$, $\Sigma\alpha^2$.

459. As regards the geometrical interpretations, these are obtained most directly from the expressions for the absolute covariants by specialising the parameter p so that it becomes the arc s of the curve. We thus find

$$s_1 = 1,\ s_l = 0, \text{ for } l = 2, 3, 4;$$

$$U = \Sigma x_1^2 = \Sigma x'^2 = 1;$$

$$W = \Sigma x''^2 = \frac{1}{\rho^2},$$

$$\frac{1}{2} W_1 = \Sigma x'' x''' = -\frac{\rho'}{\rho^3},$$

$$V = \Sigma x'''^2 = \frac{1}{\rho^4} + \frac{\rho'^2}{\rho^4} + \frac{1}{\rho^2\sigma^2},$$

$$\frac{1}{2} W_{11} - V = \Sigma x'' x^{\text{iv}} = -\frac{\rho''}{\rho^3} + 2\frac{\rho'^2}{\rho^4} - \frac{1}{\rho^4} - \frac{1}{\rho^2\sigma^2},$$

$$\frac{1}{2} V_1 = \Sigma x''' x^{\text{iv}} = \frac{\rho'\rho''}{\rho^4} - 2\frac{\rho'^3}{\rho^5} - 2\frac{\rho'}{\rho^5} - \frac{\rho'}{\rho^3\sigma^2} - \frac{\sigma'}{\rho^2\sigma^3},$$

$$\Gamma = \Sigma x^{\text{iv}2} = \frac{1}{\rho^2\sigma^2\tau^2} + \frac{1}{\rho^2\sigma^2}\left(2\frac{\rho'}{\rho} + \frac{\sigma'}{\sigma}\right)^2 + \left(-\frac{\rho''}{\rho^2} + 2\frac{\rho'^2}{\rho^3} - \frac{1}{\rho^3} - \frac{1}{\rho\sigma^2}\right)^2 + 9\frac{\rho'^2}{\rho^2};$$

and the absolute covariants are

$$\left.\begin{aligned} \frac{G}{U^3} &= \Sigma x''^2 \\ \frac{H}{U^4} &= \Sigma x'' x''' \\ \frac{K}{U^5} &= \Sigma x'''^2 \\ \frac{T}{U^5} &= \Sigma x'' x^{\text{iv}} \\ \frac{\Theta}{U^6} &= \Sigma x''' x^{\text{iv}} \\ \frac{\Omega}{U^7} &= \Sigma x^{\text{iv}2} \end{aligned}\right\}.$$

These are the magnitudes which arose (Chapter VIII) as the essential magnitudes in the discussion of the various curvatures of a skew curve in homaloidal quadruple space.

There is one quantity which manifestly satisfies all the four equations of parametric invariance and all the six equations of orientational invariance, viz.

$$\Xi = \begin{vmatrix} x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ x_3, & y_3, & z_3, & v_3 \\ x_4, & y_4, & z_4, & v_4 \end{vmatrix}.$$

Substitution in the equation

$$kf+\bar{x}\frac{\partial f}{\partial \bar{x}}=\Delta$$

shews that the indicial integer k is 10, so that

$$\Xi\bar{x}^{10}$$

is a covariant, and therefore

$$\frac{\Xi}{\bar{U}^5}$$

is an absolute covariant.

Either by direct substitution connected with the equations

$$x'=\frac{x_1}{s_1},\quad x''=\frac{x_2}{s_1^2}-\frac{s_2}{s_1^3}x_1,\quad x'''=\frac{x_3}{s_1^3}-3\frac{s_2}{s_1^4}x_2-\left(\frac{s_3}{s_1^4}+3\frac{s_2^2}{s_1^5}\right)x_1,$$

$$x^{\mathrm{iv}}=\frac{x_4}{s_1^4}-6\frac{s_2}{s_1^5}x_3-\left(4\frac{s_3}{s_1^5}-9\frac{s_2^2}{s_1^6}\right)x_2-\left(\frac{s_4}{s_1^5}+2\frac{s_2s_3}{s_1^6}-15\frac{s_2}{s_1^7}\right)x_1,$$

so that Ξ becomes

$$s_1^{10}\begin{vmatrix} x', & y', & z', & v' \\ x'', & y'', & z'', & v'' \\ x''', & y''', & z''', & v''' \\ x^{\mathrm{iv}}, & y^{\mathrm{iv}}, & z^{\mathrm{iv}}, & v^{\mathrm{iv}} \end{vmatrix},$$

or by specialising p so that it becomes s, we have

$$\frac{\Xi}{\bar{U}^5}=\begin{vmatrix} x', & y', & z', & v' \\ x'', & y'', & z'', & v'' \\ x''', & y''', & z''', & v''' \\ x^{\mathrm{iv}}, & y^{\mathrm{iv}}, & z^{\mathrm{iv}}, & v^{\mathrm{iv}} \end{vmatrix}=-\frac{1}{\rho^3\sigma^2\tau}.$$

By the general theory, all the covariants of the curve are expressible in terms of the selected six

$$\underset{U^3}{G},\ \underset{\bar{U}^4}{H},\ \underset{U^5}{K},\ \underset{\bar{U}^5}{T},\ \underset{\bar{U}^6}{\Theta},\ \underset{\bar{U}^7}{\Omega}.$$

Hence Ξ must be expressible in terms of these covariants; the relation is

$$\Xi^2U^6=\begin{vmatrix} U, & 0, & -G, & -3H \\ 0, & G, & H, & T \\ -G, & H, & K, & \Theta \\ -3H, & T, & \Theta, & \Omega \end{vmatrix},$$

a result that can be verified by means of the actual interpreted values of the respective covariants.

Other instances of covariants can be adduced; thus there are

$$\bar{x}^6\Sigma(x_1y_2-y_1x_2)^2,\quad \bar{x}^{12}\Sigma\begin{vmatrix} y_1, & z_1, & v_1 \\ y_2, & z_2, & v_2 \\ y_3, & z_3, & v_3 \end{vmatrix}^2,$$

the former of which is actually equal to G, and the latter to $W(UV-W^2)$.

The investigation shews how the essential intrinsic magnitudes of a skew curve emerge from the consideration of parametric invariance and orientational invariance. The property (§ 167) that the point-coordinates satisfy an equation of the fourth order, and the further property (§ 169) that a curve in free quadruple space is determinate (save as to position and orientation) by the assignment of its curvature, its torsion, and its tilt, combine to establish the result that it is sufficient to take derived magnitudes of the fourth order for the formation of the tale of independent concomitants of the curve.

CHAPTER XXVI.

CONCOMITANTS OF SURFACES IN FREE SPACE.

Concomitants of a surface in free homaloidal quadruple space.

460. As a second illustration of this theory of invariantive forms, we proceed to consider the concomitants of a surface in homaloidal quadruple space, regarding it not as a sub-amplitude in some region of that space but as an unrestricted two-dimensional amplitude, specified solely by independent point-equations of the type

$$x = x(p, q), \quad y = y(p, q), \quad z = z(p, q), \quad v = v(p, q),$$

in which p and q are two parameters.

Among the concomitants to be expected, there will certainly be the class of invariantive functions which appertain to the surface as a whole or to organic curves of the surface, and there will be the class of invariantive functions which, while affected by the surface, appertain to some analytically chosen curve on the surface. For the expression of the concomitants, we shall require

(a) elements δp, δq, as the variables of covariants, which will be found to belong to a set of binary forms:

(b) the laws of parametric change of derivatives of the point-coordinates x, y, z, v:

(c) the laws of parametric change of an arbitrary function $C(p, q)$ and of its derivatives, these being needed in connection with any chosen curve on the surface represented by $C(p, q) = 0$ or constant: and

(d) the laws of orientational invariance.

There is no contravariant variable, because an element of area upon the surface is itself a covariant, being merely a portion of the whole configuration of the surface depending upon position in the surface.

We have to deal with infinitesimal transformations of the two parameters p and q; they are taken in the form

$$p' = p + \epsilon\phi(p, q), \quad q' = q + \epsilon\chi(p, q),$$

where ϕ and χ are two independent functions, each completely arbitrary.

For derivatives of all orders of any magnitude θ, which is a function of p and q, we use a double-suffix notation according to the convention

$$\frac{\partial^{m+n}\theta}{\partial p^m \partial q^n} = \theta_{mn},$$

retaining a zero suffix where it would occur under this convention.

The modulus of transformation, J, is

$$J = \begin{vmatrix} \frac{\partial p'}{\partial p}, & \frac{\partial p'}{\partial q} \\ \frac{\partial q'}{\partial p}, & \frac{\partial q'}{\partial q} \end{vmatrix} = 1 + \epsilon(\phi_{10} + \chi_{01}),$$

when powers of ϵ above the first are neglected, because we are dealing with infinitesimal transformations.

We need the variables of the covariants, defining them as t and u, where

$$\delta p = t, \quad \delta q = u.$$

Thus the law of infinitesimal transformation of the variables is

$$\left.\begin{aligned} t' = \delta p' &= \delta p + \epsilon(\phi_{10}\delta p + \phi_{01}\delta q) = (1 + \epsilon\phi_{10})\, t + \epsilon\phi_{01}\, u \\ u' = \delta q' &= \delta q + \epsilon(\chi_{10}\delta p + \chi_{01}\delta q) = \epsilon\chi_{10}\, t + (1 + \epsilon\chi_{01})\, u \end{aligned}\right\}.$$

The law of infinitesimal transformation of any function $\theta(p, q)$ and of its derivatives, under the infinitesimal transformations of the parameters, is derived from the more general law investigated in § 451, when we impose the limitations due to the non-occurrence of the third parameter r. Accordingly, the law is expressed by the relation

$$\theta'_{\alpha\beta} - \theta_{\alpha\beta} = -\epsilon\Sigma\Sigma \binom{\alpha}{l}\binom{\beta}{m} \{\theta_{\alpha+1-l,\,\beta-m}\,\phi_{lm} + \theta_{\alpha-l,\,\beta+1-m}\,\chi_{lm}\},$$

for all positive integer values of α and β, taken independently of one another, and for all positive integers l and m, such that

$$0 \leqslant l \leqslant \alpha, \quad 0 \leqslant m \leqslant \beta, \quad l + m > 0.$$

This formula is fundamental for such variables as $\theta = x, y, z, v$, and for such a function as $C(p, q)$.

It is desirable, initially, to limit the number of quantities which, being subject to variation, may enter into the expression of concomitants.

One set of variables t and u will suffice. Should there be a second set of variables, T and U, they combine with t and u to form an identical covariant: because, manifestly,

$$\begin{vmatrix} T', & U' \\ t', & u' \end{vmatrix} = J \begin{vmatrix} T, & U \\ t, & u \end{vmatrix},$$

and thus no element, characteristic of any particular configuration, is provided. Such additional variables may arise in the construction of covariants by the use of polarizing operations of the type

$$T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u}.$$

One function $C(p, q)$, in association with the selection of arbitrary curves on the surface, will suffice. Two such functions $C(p, q) = \text{parameter} = \alpha$, $D(p, q) = \text{parameter} = \beta$, merely constitute a transformation of parameters p and q into α and β, which are generically included already under the comprehensive infinitesimal transformations.

As regards derivatives of x, y, z, v, we shall retain those of the first three orders, which will prove sufficient to illustrate the application of the theory to general surfaces in quadruple space. Consequently, the number of magnitudes that may thus enter into the expression of concomitants, is

$$\begin{aligned} &8, \text{ for the first derivatives of } x, y, z, v,\\ &+12, \text{ for their second derivatives,}\\ &+16, \text{ for their third derivatives,} \end{aligned}$$

that is, 36 on this score. In addition, there may be the two first derivatives of a function C, its three second derivatives, and its four third derivatives: 9 in all. Moreover, there are the two variables t and u, as elements of covariants. Hence the whole tale of magnitudes which, being subject to infinitesimal transformation, may enter into concomitants is 47.

Equations characteristic of concomitants.

461. The fundamental definition of a concomitant f is that, if f' denote the same function of all the arguments after they have been submitted to transformation as f is of those arguments before transformation, the relation

$$f = J^{\mu} f'$$

shall be satisfied identically, where μ is a constant, usually a positive integer. By Lie's theory, only infinitesimal transformations need be considered; and therefore we take the defining equation in the form

$$f = \{1 + \mu\epsilon(\phi_{10} + \chi_{01})\} f'.$$

This equation is to be satisfied identically. In it, there occur various derivatives of ϕ, nine in number (two of the first order, three of the second, four of the third); and similarly there are nine derivatives of χ. Consequently, the coefficients of the various derivatives must balance, each separately for each of the derivatives; thus there will be eighteen equations. Two of them contain the index μ, and (by subtraction) leave a single equation free from that index; thus there are seventeen homogeneous equations. Again, it is

part of Lie's theory (and, as a check, is an easily verified fact) that these form a complete Jacobian system; consequently, the number of simultaneous algebraically independent integrals is $(47 - 17 =)$ thirty.

These thirty quantities satisfy the conditions for parametric invariance. But, further, the six equations for orientational invariance have to be satisfied; and therefore, for the full solution, it will be necessary to obtain twenty-four algebraically independent integrals. When these have been obtained, every integral of the equations (that is, every concomitant of grade not higher than three) can be expressed in terms of members of the aggregate.

We therefore proceed to form the equations of parametric invariance.

The infinitesimal variations of the derivatives of x, y, z, v, of the first three orders are given by the tableau

$$\left.\begin{aligned}
\theta'_{10} - \theta_{10} &= -\epsilon(\theta_{10}\phi_{10} + \theta_{01}\chi_{10}) \\
\theta'_{01} - \theta_{01} &= -\epsilon(\theta_{10}\phi_{01} + \theta_{01}\chi_{01})
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\theta'_{20} - \theta_{20} &= -\epsilon\{(2\theta_{20}\phi_{10} + \theta_{10}\phi_{20}) + (2\theta_{11}\chi_{10} + \theta_{01}\chi_{20})\} \\
\theta'_{11} - \theta_{11} &= -\epsilon\{(\theta_{11}\phi_{10} + \theta_{20}\phi_{01} + \theta_{10}\phi_{11}) + (\theta_{02}\chi_{10} + \theta_{11}\chi_{01} + \theta_{01}\chi_{11})\} \\
\theta'_{02} - \theta_{02} &= -\epsilon\{(2\theta_{11}\phi_{01} + \theta_{10}\phi_{02}) + (2\theta_{02}\chi_{01} + \theta_{01}\chi_{02})\}
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\theta'_{30} - \theta_{30} &= -\epsilon\{(3\theta_{30}\phi_{10} + 3\theta_{20}\phi_{20} + \theta_{10}\phi_{30}) + (3\theta_{21}\chi_{10} + 3\theta_{11}\chi_{20} + \theta_{01}\chi_{30})\} \\
\theta'_{21} - \theta_{21} &= -\epsilon\{(2\theta_{21}\phi_{10} + \theta_{11}\phi_{20} + \theta_{01}\phi_{30} + 2\theta_{20}\phi_{11} + \theta_{10}\phi_{21}) \\
&\qquad + (2\theta_{12}\chi_{10} + \theta_{02}\chi_{20} + \theta_{21}\chi_{01} + 2\theta_{11}\chi_{11} + \theta_{01}\chi_{21})\} \\
\theta'_{12} - \theta_{12} &= -\epsilon\{(\theta_{12}\phi_{10} + 2\theta_{11}\phi_{11} + \theta_{10}\phi_{12} + 2\theta_{21}\phi_{01} + \theta_{20}\phi_{02}) \\
&\qquad + (\theta_{03}\chi_{10} + 2\theta_{02}\chi_{11} + \theta_{01}\chi_{12} + 2\theta_{12}\chi_{01} + \theta_{11}\chi_{02})\} \\
\theta'_{03} - \theta_{03} &= -\epsilon\{(3\theta_{12}\phi_{01} + 3\theta_{11}\phi_{02} + \theta_{10}\phi_{03}) + (3\theta_{03}\chi_{01} + 3\theta_{02}\chi_{02} + \theta_{01}\chi_{03})\}
\end{aligned}\right\},$$

for $\theta = x, y, z, v$, separately, among the derivatives of the coordinates, and for $\theta = C(p, q)$, where $C(p, q) =$ constant represents an arbitrary selected curve or one of a family of curves on the surface.

In the ensuing parametric equations, the symbol Σ in an expression $\Sigma\theta_{rs}\dfrac{\partial f}{\partial\theta_{lm}}$ is used to denote summation over the four variables x, y, z, v.

The infinitesimal variations of the two variables t and u, which may occur in covariants, are

$$t' - t = \epsilon(t\phi_{10} + u\phi_{01}), \quad u' - u = \epsilon(t\chi_{10} + u\chi_{01}).$$

The equation characteristic of complete parametric invariance is

$$f = \{1 + \mu\epsilon(\phi_{10} + \chi_{01})\}f',$$

where f' is the same function of t', u', θ'_{lm}, for the various possibilities of θ', l, m, as f is of t, u, θ_{lm}. When substitution is made for the transformed arguments, the characteristic equation (which is to be satisfied identically) requires that the aggregate coefficients of the various derivatives of the

arbitrary functions ϕ and χ shall vanish. The term independent of ϵ merely gives $f=f$. The requirement, that the aggregate coefficients of the various derivatives of ϕ and χ shall balance, leads to the following set of equations. Subject to the assumption that no derivatives of x, y, z, v, C, of order higher than three occur, the full set of equations is

$$\mu f + t\frac{\partial f}{\partial t} = \Delta_{10}, \quad \kappa f + u\frac{\partial f}{\partial u} = \nabla_{01}, \quad u\frac{\partial f}{\partial t} = \Delta_{01}, \quad t\frac{\partial f}{\partial u} = \nabla_{10},$$

where

$$\left.\begin{aligned}
\Delta_{10} &= \Sigma\left(\theta_{10}\frac{\partial f}{\partial \theta_{10}}\right) + \Sigma\left(2\theta_{20}\frac{\partial f}{\partial \theta_{20}} + \theta_{11}\frac{\partial f}{\partial \theta_{11}}\right) + \Sigma\left(3\theta_{30}\frac{\partial f}{\partial \theta_{30}} + 2\theta_{21}\frac{\partial f}{\partial \theta_{21}} + \theta_{12}\frac{\partial f}{\partial \theta_{12}}\right) \\
&\quad + C_{10}\frac{\partial f}{\partial C_{10}} + 2C_{20}\frac{\partial f}{\partial C_{20}} + C_{11}\frac{\partial f}{\partial C_{11}} + 3C_{30}\frac{\partial f}{\partial C_{30}} + 2C_{21}\frac{\partial f}{\partial C_{21}} + C_{12}\frac{\partial f}{\partial C_{12}} \\
\nabla_{01} &= \Sigma\left(\theta_{01}\frac{\partial f}{\partial \theta_{01}}\right) + \Sigma\left(\theta_{11}\frac{\partial f}{\partial \theta_{11}} + 2\theta_{02}\frac{\partial f}{\partial \theta_{02}}\right) + \Sigma\left(\theta_{21}\frac{\partial f}{\partial \theta_{21}} + 2\theta_{12}\frac{\partial f}{\partial \theta_{12}} + 3\theta_{03}\frac{\partial f}{\partial \theta_{03}}\right) \\
&\quad + C_{01}\frac{\partial f}{\partial C_{01}} + C_{11}\frac{\partial f}{\partial C_{11}} + 2C_{02}\frac{\partial f}{\partial C_{02}} + C_{21}\frac{\partial f}{\partial C_{21}} + 2C_{12}\frac{\partial f}{\partial C_{12}} + 3C_{03}\frac{\partial f}{\partial C_{03}} \\
\Delta_{01} &= \Sigma\theta_{10}\frac{\partial f}{\partial \theta_{01}} + \Sigma\left(\theta_{20}\frac{\partial f}{\partial \theta_{11}} + 2\theta_{11}\frac{\partial f}{\partial \theta_{02}}\right) + \Sigma\left(\theta_{30}\frac{\partial f}{\partial \theta_{21}} + 2\theta_{21}\frac{\partial f}{\partial \theta_{12}} + 3\theta_{12}\frac{\partial f}{\partial \theta_{03}}\right) \\
&\quad + C_{10}\frac{\partial f}{\partial C_{01}} + C_{20}\frac{\partial f}{\partial C_{11}} + 2C_{11}\frac{\partial f}{\partial C_{02}} + {}_{30}\frac{\partial f}{\partial C_{21}} + 2C_{21}\frac{\partial f}{\partial C_{12}} + 3C_{12}\frac{\partial f}{\partial C_{03}} \\
\nabla_{10} &= \Sigma\theta_{01}\frac{\partial f}{\partial \theta_{10}} + \Sigma\left(2\theta_{11}\frac{\partial f}{\partial \theta_{20}} + \theta_{02}\frac{\partial f}{\partial \theta_{11}}\right) + \Sigma\left(3\theta_{21}\frac{\partial f}{\partial \theta_{30}} + 2\theta_{12}\frac{\partial f}{\partial \theta_{21}} + \theta_{03}\frac{\partial f}{\partial \theta_{12}}\right) \\
&\quad + C_{01}\frac{\partial f}{\partial C_{10}} + 2C_{11}\frac{\partial f}{\partial C_{20}} + C_{02}\frac{\partial f}{\partial C_{11}} + 3C_{21}\frac{\partial f}{\partial C_{30}} + 2C_{12}\frac{\partial f}{\partial C_{21}} + C_{03}\frac{\partial f}{\partial C_{12}}
\end{aligned}\right\},$$

arising from the coefficients of ϕ_{10}, χ_{01}, ϕ_{01}, χ_{10}, respectively;

$$\left.\begin{aligned}
\Delta_{20} &= \Sigma\theta_{10}\frac{\partial f}{\partial \theta_{20}} + 3\Sigma\theta_{20}\frac{\partial f}{\partial \theta_{30}} + \Sigma\theta_{11}\frac{\partial f}{\partial \theta_{21}} + C_{10}\frac{\partial f}{\partial C_{20}} + 3C_{20}\frac{\partial f}{\partial C_{30}} + C_{11}\frac{\partial f}{\partial C_{21}} = 0 \\
\nabla_{02} &= \Sigma\theta_{01}\frac{\partial f}{\partial \theta_{02}} + \Sigma\theta_{11}\frac{\partial f}{\partial \theta_{12}} + 3\Sigma\theta_{02}\frac{\partial f}{\partial \theta_{03}} + C_{01}\frac{\partial f}{\partial C_{02}} + C_{11}\frac{\partial f}{\partial C_{12}} + 3C_{02}\frac{\partial f}{\partial C_{03}} = 0
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\Delta_{11} &= \Sigma\theta_{10}\frac{\partial f}{\partial \theta_{11}} + 2\Sigma\theta_{20}\frac{\partial f}{\partial \theta_{21}} + 2\Sigma\theta_{11}\frac{\partial f}{\partial \theta_{12}} + C_{10}\frac{\partial f}{\partial C_{11}} + 2C_{20}\frac{\partial f}{\partial C_{21}} + 2C_{11}\frac{\partial f}{\partial C_{12}} = 0 \\
\nabla_{11} &= \Sigma\theta_{01}\frac{\partial f}{\partial \theta_{11}} + 2\Sigma\theta_{11}\frac{\partial f}{\partial \theta_{21}} + 2\Sigma\theta_{02}\frac{\partial f}{\partial \theta_{12}} + C_{01}\frac{\partial f}{\partial C_{11}} + 2C_{11}\frac{\partial f}{\partial C_{21}} + 2C_{02}\frac{\partial f}{\partial C_{12}} = 0
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\Delta_{02} &= \Sigma\theta_{10}\frac{\partial f}{\partial \theta_{02}} + \Sigma\theta_{20}\frac{\partial f}{\partial \theta_{12}} + 3\Sigma\theta_{11}\frac{\partial f}{\partial \theta_{03}} + C_{10}\frac{\partial f}{\partial C_{02}} + C_{20}\frac{\partial f}{\partial C_{12}} + 3C_{11}\frac{\partial f}{\partial C_{03}} = 0 \\
\nabla_{20} &= \Sigma\theta_{01}\frac{\partial f}{\partial \theta_{20}} + \Sigma\theta_{02}\frac{\partial f}{\partial \theta_{21}} + 3\Sigma\theta_{11}\frac{\partial f}{\partial \theta_{30}} + C_{01}\frac{\partial f}{\partial C_{20}} + C_{02}\frac{\partial f}{\partial C_{21}} + 3C_{11}\frac{\partial f}{\partial C_{30}} = 0
\end{aligned}\right\}$$

arising from the coefficients of ϕ_{20}, χ_{02}, ϕ_{11}, χ_{11}, ϕ_{02}, χ_{20}, respectively; and

$$\left.\begin{aligned}
\Delta_{30} &= \Sigma\theta_{10}\frac{\partial f}{\partial\theta_{30}} + C_{10}\frac{\partial f}{\partial C_{30}} = 0\\
\nabla_{03} &= \Sigma\theta_{01}\frac{\partial f}{\partial\theta_{30}} + C_{01}\frac{\partial f}{\partial C_{30}} = 0\\
\Delta_{21} &= \Sigma\theta_{10}\frac{\partial f}{\partial\theta_{21}} + C_{10}\frac{\partial f}{\partial C_{21}} = 0\\
\nabla_{12} &= \Sigma\theta_{01}\frac{\partial f}{\partial\theta_{21}} + C_{01}\frac{\partial f}{\partial C_{21}} = 0\\
\Delta_{12} &= \Sigma\theta_{10}\frac{\partial f}{\partial\theta_{12}} + C_{10}\frac{\partial f}{\partial C_{12}} = 0\\
\nabla_{21} &= \Sigma\theta_{01}\frac{\partial f}{\partial\theta_{12}} + C_{01}\frac{\partial f}{\partial C_{12}} = 0\\
\Delta_{03} &= \Sigma\theta_{10}\frac{\partial f}{\partial\theta_{03}} + C_{10}\frac{\partial f}{\partial C_{03}} = 0\\
\nabla_{30} &= \Sigma\theta_{01}\frac{\partial f}{\partial\theta_{03}} + C_{01}\frac{\partial f}{\partial C_{03}} = 0
\end{aligned}\right\},$$

from the coefficients of ϕ_{30}, χ_{03}, ϕ_{21}, χ_{12}, ϕ_{12}, χ_{21}, ϕ_{03}, χ_{30}, respectively.

In all these parametric equations, we shall find it convenient to write

$$\Delta_{lm} = D_{lm}f, \quad \nabla_{lm} = D'_{lm}f,$$

for the various combinations l, m.

Of the first set of (four) equations, the first and the second equations determine the integer μ and also yield the homogeneous equation

$$t\frac{\partial f}{\partial t} - u\frac{\partial f}{\partial u} = \Delta_{10} - \nabla_{01}.$$

This last equation, the remaining two equations of the first set, the six equations of the second set, and the eight equations of the third set, constitute the aggregate of seventeen equations. This aggregate, within its range (that is, including derivatives of the third order), is a complete Jacobian linear system.

We have seen (§ 446) that, in order to secure invariance under change of site, the variables x, y, z, v, must not explicitly occur: and a similar argument has required the non-occurrence of a function C.

The equations of orientational invariance have already (§ 448) been given in their symbolic form: they do not affect, and are not affected by, any transformations of the parameters.

Concomitants of grade unity.

462. As for skew curves, so for surfaces, it is convenient to grade covariantive functions according to the highest order of derivative which the function contains. The grade of a covariant is taken as equal to the order of the highest derivative it contains.

We begin with concomitants of grade unity: from the analysis, it will appear convenient to obtain the aggregate of algebraically independent integrals of the set of parametric equations before the orientational equations are considered.

The parametric equations, characteristic of concomitants of grade unity, are three in number, being

$$\left.\begin{aligned} \Sigma x_{10}\frac{\partial f}{\partial x_{01}} + C_{10}\frac{\partial f}{\partial C_{01}} &= u\frac{\partial f}{\partial t} \\ \Sigma x_{01}\frac{\partial f}{\partial x_{10}} + C_{01}\frac{\partial f}{\partial C_{10}} &= t\frac{\partial f}{\partial u} \\ \Sigma\left(x_{10}\frac{\partial f}{\partial x_{10}} - x_{01}\frac{\partial f}{\partial x_{01}}\right) + C_{10}\frac{\partial f}{\partial C_{10}} - C_{01}\frac{\partial f}{\partial C_{01}} &= t\frac{\partial f}{\partial t} - u\frac{\partial f}{\partial u} \end{aligned}\right\}.$$

They are a complete Jacobian system within their range. The arguments are t, u; $x_{10}, x_{01}, y_{10}, y_{01}, z_{10}, z_{01}, v_{10}, v_{01}$; C_{10}, C_{01}: twelve in number. Hence the system of three equations possesses nine algebraically independent integrals, in terms of which all integrals of the system can be expressed.

The form of the equations shews that every integral is a concomitant of the five (linear) binary forms

$$\bar{X} = (x_{10}, x_{01}\,|\,t, u), \quad \bar{Y} = (y_{10}, y_{01}\,|\,t, u), \quad \bar{Z} = (z_{10}, z_{01}\,|\,t, u), \quad \bar{V} = (v_{10}, v_{01}\,|\,t, u),$$
$$P = (C_{10}, C_{01}\,|\,t, u).$$

The simultaneous concomitants of such a system can be expressed, all of them, in terms of the five forms themselves, viz.,

$$\bar{X}, \bar{Y}, \bar{Z}, \bar{V}, P,$$

and of their Jacobians

$$\mathbf{a} = \begin{vmatrix} y_{10}, & z_{10} \\ y_{01}, & z_{01} \end{vmatrix}, \quad \mathbf{b} = \begin{vmatrix} z_{10}, & x_{10} \\ z_{01}, & x_{01} \end{vmatrix}, \quad \mathbf{c} = \begin{vmatrix} x_{10}, & y_{10} \\ x_{01}, & y_{01} \end{vmatrix},$$

$$\mathbf{f} = \begin{vmatrix} v_{10}, & x_{10} \\ v_{01}, & x_{01} \end{vmatrix}, \quad \mathbf{g} = \begin{vmatrix} v_{10}, & y_{10} \\ v_{01}, & y_{01} \end{vmatrix}, \quad \mathbf{h} = \begin{vmatrix} v_{10}, & z_{10} \\ v_{01}, & z_{01} \end{vmatrix},$$

$$\mathbf{p} = \begin{vmatrix} C_{10}, & x_{10} \\ C_{01}, & x_{01} \end{vmatrix}, \quad \mathbf{q} = \begin{vmatrix} C_{10}, & y_{10} \\ C_{01}, & y_{01} \end{vmatrix}, \quad \mathbf{r} = \begin{vmatrix} C_{10}, & z_{10} \\ C_{01}, & z_{01} \end{vmatrix}, \quad \mathbf{s} = \begin{vmatrix} C_{10}, & v_{10} \\ C_{01}, & v_{01} \end{vmatrix}.$$

These fifteen quantities are not independent of one another. There are three distinct relations represented by

$$\begin{Vmatrix} \bar{X}, & \bar{Y}, & \bar{Z}, & \bar{V}, & P \\ x_{10}, & y_{10}, & z_{10}, & v_{10}, & C_{10} \\ x_{01}, & y_{01}, & z_{01}, & v_{01}, & C_{01} \end{Vmatrix} = 0;$$

and there are three distinct relations among the five

$$\left.\begin{aligned} \mathbf{af} + \mathbf{bg} + \mathbf{ch} &= 0 \\ \mathbf{ap} + \mathbf{bq} + \mathbf{cr} &= 0 \\ -\mathbf{hq} + \mathbf{gr} + \mathbf{as} &= 0 \\ \mathbf{hp} - \mathbf{fr} + \mathbf{bs} &= 0 \\ -\mathbf{gp} + \mathbf{fq} + \mathbf{cs} &= 0 \end{aligned}\right\};$$

that is, six distinct relations in all. Consequently, there are nine quantities, independent of one another, in terms of which all are expressible.

The orientational equations are six in number, formally. They do not affect the arguments t and u, C_{10} and C_{01}. For the present purpose they contain only the eight arguments, arising out of the two first derivatives of each of the coordinates x, y, z, v; and among them, there subsists the identical relation

$$\mathbf{f}E_{yz} + \mathbf{g}E_{zx} + \mathbf{h}E_{xy} + \mathbf{a}E_{vx} + \mathbf{b}E_{vy} + \mathbf{c}E_{vz} = 0,$$

so that the number of independent equations of orientational invariants for concomitants of grade unity is five. There are nine independent magnitudes, combinations of which are to satisfy these five independent equations; consequently, we shall require four algebraically independent integrals of the equations.

Of the fifteen quantities, P (and P alone) satisfies all the equations. For the rest, we have

$$\begin{gathered} E_{yz}\bar{X} = 0, \quad E_{yz}\bar{Y} = -\bar{Z}, \quad E_{yz}\bar{Z} = \bar{Y}, \quad E_{yz}\bar{V} = 0; \\ E_{yz}\mathbf{p} = 0, \quad E_{yz}\mathbf{q} = -\mathbf{r}, \quad E_{yz}\mathbf{r} = \mathbf{q}, \quad E_{yz}\mathbf{s} = 0; \\ E_{yz}\mathbf{a} = 0, \quad E_{yz}\mathbf{b} = -\mathbf{c}, \quad E_{yz}\mathbf{c} = \mathbf{b}, \\ E_{yz}\mathbf{f} = 0, \quad E_{yz}\mathbf{g} = -\mathbf{h}, \quad E_{yz}\mathbf{h} = \mathbf{g}; \end{gathered}$$

and so for the other equations of the orientational set. We require three integrals other than P, independent of one another and of P. It is easy to see that four integrals of this set of equations, compounded from the arguments, are

$$\begin{gathered} \mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 + \mathbf{f}^2 + \mathbf{g}^2 + \mathbf{h}^2, \\ \bar{X}^2 + \bar{Y}^2 + \bar{Z}^2 + \bar{V}^2, \\ \bar{X}\mathbf{p} + \bar{Y}\mathbf{q} + \bar{Z}\mathbf{r} + \bar{V}\mathbf{s}, \\ \mathbf{p}^2 + \mathbf{q}^2 + \mathbf{r}^2 + \mathbf{s}^2. \end{gathered}$$

To express these, we define E, F, G, by the relations

$$\begin{aligned} E &= x_{10}^2 + y_{10}^2 + z_{10}^2 + v_{10}^2, \\ F &= x_{10}x_{01} + y_{10}y_{01} + z_{10}z_{01} + v_{10}v_{01}, \\ G &= x_{01}^2 + y_{01}^2 + z_{01}^2 + v_{01}^2. \end{aligned}$$

Then the first of the four integrals is

$$\Sigma\,(x_{10}y_{01} - y_{10}x_{01})^2 = EG - F^2 = V^2.$$

The second integral is

$$\Sigma (x_{10}t + x_{01}u)^2 = Et^2 + 2Ftu + Gu^2.$$

The fourth integral is

$$\Sigma (C_{10}x_{01} - C_{01}x_{10})^2 = GC_{10}^2 - 2FC_{10}C_{01} + EC_{01}^2.$$

We have also the integral

$$P = C_{10}t + C_{01}u.$$

The third integral is

$$\Sigma (x_{10}t + x_{01}u)(C_{10}x_{01} - C_{01}x_{10})$$
$$= C_{10}(Ft + Gu) - C_{01}(Et + Fu);$$

but

$$\{C_{10}(Ft + Gu) - C_{01}(Et + Fu)\}^2$$
$$= (GC_{10}^2 - 2FC_{10}C_{01} + EC_{01}^2)(Et^2 + 2Ftu + Gu^2) - V^2P^2,$$

and so the third integral is not algebraically independent of the others. Thus we can take, as an adequate system of concomitants of grade unity, the set

$$U = Et^2 + 2Ftu + Gu^2,$$
$$V^2 = EG - F^2,$$
$$P = C_{10}t + C_{01}u,$$
$$D = GC_{10}^2 - 2FC_{10}C_{01} + EC_{01}^2.$$

Geometrical interpretation of the concomitants.

463. The interpretation of the integrals is immediate. We have

$$t = dp, \quad u = dq;$$

hence

$$U = E\,dp^2 + 2F\,dp\,dq + G\,dq^2 = ds^2,$$

where ds is an elementary arc on the surface along the direction determined by dp, dq. The covariant U is a fundamental form in the geometry of the surface.

The invariant V^2 of the surface is the (non-vanishing) determinant of the form U.

The invariant P is

$$= C_{10}t + C_{01}u$$
$$= \left(C_{10}\frac{dp}{ds} + C_{01}\frac{dq}{ds}\right)ds;$$

and therefore

$$PU^{-\frac{1}{2}} = \frac{dC}{ds},$$

which is the arc-variation of the function $C(p, q)$ along the direction determined by dp, dq.

For the interpretation of D, take a direction δp, δq, perpendicular to the direction dp, dq, also in the surface; and as ds is an arc along dp, dq, let δn be an arc along δp, δq. The condition of perpendicularity of the two directions is

$$E dp \delta p + F(dp \delta q + dq \delta p) + G dq \delta q = 0,$$

so that

$$\frac{\delta p}{F dp + G dq} = \frac{\delta q}{-(E dp + F dq)} = \frac{1}{\theta}.$$

To find θ, we have

$$E \delta p^2 + 2F \delta p \delta q + G \delta q^2 = \delta n^2;$$

and therefore

$$\theta^2 \delta n^2 = V^2 ds^2.$$

Hence

$$\frac{dC}{dn} = C_{10} \frac{\delta p}{\delta n} + C_{01} \frac{\delta q}{\delta n}$$

$$= \frac{1}{V ds} \{C_{10}(F dp + G dq) - C_{01}(E dp + F dq)\},$$

so that

$$\left(\frac{dC}{dn}\right)^2 V^2 ds^2 = \{C_{10}(F dp + G dq) - C_{01}(E dp + F dq)\}^2$$

$$= D ds^2 - V^2 \left(\frac{dC}{ds}\right)^2 ds^2,$$

by the preceding formula (§ 462); consequently

$$D = V^2 \left\{\left(\frac{dC}{ds}\right)^2 + \left(\frac{dC}{dn}\right)^2\right\}.$$

In particular, if the equation $C(p, q) = \alpha$, where α is a parameter, represents a family of curves, and we take the arc ds to be tangential to the curve, so that $\frac{dC}{ds} = 0$, then

$$D = V^2 \left(\frac{dC}{dn}\right)^2 = V^2 \left(\frac{\delta \alpha}{\delta n}\right)^2,$$

and

$$\delta n = D^{-\frac{1}{2}} V \delta \alpha,$$

giving the normal superficial distance between two consecutive curves of the family.

Finally, an element of area dS, in the form of a parallelogram bounded by elementary parametric arcs on the surface is such that

$$dS = V dp dq.$$

Summarising, we have an aggregate of covariants of grade unity constituted by the set

$$U,\ V^2,\ P,\ D;$$

and a geometrical significance has been attached to each member of the set. Later, it will be found convenient, in the interpretation of covariants, to

associate with the general direction ds an element of length dl along the curve $C(p, q) = \text{constant}$ and an element of length $d\nu$ normal to that curve, instead of making them particular instances of ds, the general element of arc-length on the surface.

Surface concomitants of grade two.

464. In constructing the covariants, of grade two, which appertain to a surface and to any arbitrarily assigned curve taken on the surface, it is convenient to select an algebraically complete system in such a mode as to exhibit clearly an included aggregate of covariants of that grade which belong to the surface alone.

The nine equations characteristic of parametric invariance are

$$\left.\begin{aligned} D_{20}f &= \Sigma x_{10}\frac{\partial f}{\partial x_{20}} + C_{10}\frac{\partial f}{\partial C_{20}} = 0 \\ D_{11}f &= \Sigma x_{10}\frac{\partial f}{\partial x_{11}} + C_{10}\frac{\partial f}{\partial C_{11}} = 0 \\ D_{02}f &= \Sigma x_{10}\frac{\partial f}{\partial x_{02}} + C_{10}\frac{\partial f}{\partial C_{02}} = 0 \end{aligned}\right\}, \quad \left.\begin{aligned} D'_{20}f &= \Sigma x_{01}\frac{\partial f}{\partial x_{20}} + C_{01}\frac{\partial f}{\partial C_{20}} = 0 \\ D'_{11}f &= \Sigma x_{01}\frac{\partial f}{\partial x_{11}} + C_{01}\frac{\partial f}{\partial C_{11}} = 0 \\ D'_{02}f &= \Sigma x_{01}\frac{\partial f}{\partial x_{02}} + C_{01}\frac{\partial f}{\partial C_{02}} = 0 \end{aligned}\right\},$$

and

$$\left.\begin{aligned} u\frac{\partial f}{\partial t} &= \Sigma x_{10}\frac{\partial f}{\partial x_{01}} + \Sigma\left(x_{20}\frac{\partial f}{\partial x_{11}} + 2x_{11}\frac{\partial f}{\partial x_{02}}\right) + C_{10}\frac{\partial f}{\partial C_{01}} + \left(C_{20}\frac{\partial f}{\partial C_{11}} + 2C_{11}\frac{\partial f}{\partial C_{02}}\right) \\ t\frac{\partial f}{\partial u} &= \Sigma x_{01}\frac{\partial f}{\partial x_{10}} + \Sigma\left(2x_{11}\frac{\partial f}{\partial x_{20}} + x_{02}\frac{\partial f}{\partial x_{11}}\right) + C_{01}\frac{\partial f}{\partial C_{10}} + \left(2C_{11}\frac{\partial f}{\partial C_{20}} + C_{02}\frac{\partial f}{\partial C_{11}}\right) \\ t\frac{\partial f}{\partial t} - u\frac{\partial f}{\partial u} &= \Sigma\left(x_{01}\frac{\partial f}{\partial x_{01}} - x_{10}\frac{\partial f}{\partial x_{10}}\right) + 2\Sigma\left(x_{02}\frac{\partial f}{\partial x_{02}} - x_{20}\frac{\partial f}{\partial x_{20}}\right) \\ &\quad + \left(C_{01}\frac{\partial f}{\partial C_{01}} - C_{10}\frac{\partial f}{\partial C_{10}}\right) + 2\left(C_{02}\frac{\partial f}{\partial C_{02}} - C_{20}\frac{\partial f}{\partial C_{20}}\right) \end{aligned}\right\}.$$

In addition to these nine parametric equations, there are the six equations characteristic of orientational invariance.

We construct, first of all, an algebraically complete aggregate of integrals of the six earlier parametric equations. There are two variables t, u; there are twenty quantities of the type x_{lm}, for $0 < l + m \leqslant 2$; so that there are twenty-two arguments not appertaining to any curve, and consequently sixteen combinations of these arguments are required in the algebraically complete system. Further, there are five additional arguments C_{10}, C_{01}, C_{20}, C_{11}, C_{02}; and therefore we shall require five integrals in the algebraically complete system, two of them to contain C_{10} and C_{01} but not the derivatives of the second order, and three of them to contain the derivatives C_{20}, C_{11}, C_{02}.

Among individual integrals of the set of six equations, are the ten quantities

$$t,\ u;\ x_{10},\ y_{10},\ z_{10},\ v_{10};\ x_{01},\ y_{01},\ z_{01},\ v_{01};$$

after the preceding statement, we first seek six individual integrals of the equations, independent of the arguments C_{lm} but involving second derivatives of x, y, z, v. To this end, we use the magnitudes (of § 207)

$$\alpha = \Sigma x_{10}x_{20}, \quad \alpha' = \Sigma x_{10}x_{11}, \quad \alpha'' = \Sigma x_{10}x_{02},$$
$$\beta = \Sigma x_{01}x_{20}, \quad \beta' = \Sigma x_{01}x_{11}, \quad \beta'' = \Sigma x_{01}x_{02};$$

upon which the effect of the operators D_{20}, D_{11}, D_{02}, D'_{20}, D'_{11}, D'_{02}, is

$$\begin{aligned} D_{20}\alpha &= E, & D'_{20}\alpha &= F, & D_{20}\beta &= F, & D'_{20}\beta &= G, \\ D_{11}\alpha' &= E, & D'_{11}\alpha' &= F, & D_{11}\beta' &= F, & D'_{11}\beta' &= G, \\ D_{02}\alpha'' &= E, & D'_{02}\alpha'' &= F, & D_{02}\beta'' &= F, & D'_{02}\beta'' &= G, \end{aligned}$$

all the others such as $D_{20}\beta'$, $D_{20}\alpha'$, and the like, being zero. Hence taking the quantities Γ, Γ', Γ'', Δ, Δ', Δ'', of § 207, and noting that E, F, G are invariant for these operators, we have

$$\left.\begin{aligned} D_{20}\Gamma &= 1, & D'_{20}\Delta &= 1 \\ D_{11}\Gamma' &= 1, & D'_{11}\Delta' &= 1 \\ D_{02}\Gamma'' &= 1, & D'_{02}\Delta'' &= 1 \end{aligned}\right\};$$

and

$$D_{20}(\Gamma', \Gamma'', \Delta, \Delta', \Delta'') = 0,$$
$$D_{11}(\Gamma'', \Gamma', \Delta, \Delta', \Delta'') = 0,$$

and so on for other zero-effects. Accordingly, having regard to these results and to the forms of the equations $D_{20} = 0, \ldots, D'_{02} = 0$, we have twelve obvious solutions of the form

$$\left.\begin{aligned} \xi_{20} &= x_{20} - x_{10}\Gamma - x_{01}\Delta \\ \eta_{20} &= y_{20} - y_{10}\Gamma - y_{01}\Delta \\ \zeta_{20} &= z_{20} - z_{10}\Gamma - z_{01}\Delta \\ v_{20} &= v_{20} - v_{10}\Gamma - v_{01}\Delta \end{aligned}\right\}, \quad \left.\begin{aligned} \xi_{11} &= x_{11} - x_{10}\Gamma' - x_{01}\Delta' \\ \eta_{11} &= y_{11} - y_{10}\Gamma' - y_{01}\Delta' \\ \zeta_{11} &= z_{11} - z_{10}\Gamma' - z_{01}\Delta' \\ v_{11} &= v_{11} - v_{10}\Gamma' - v_{01}\Delta' \end{aligned}\right\}, \quad \left.\begin{aligned} \xi_{02} &= x_{02} - x_{10}\Gamma'' - x_{01}\Delta'' \\ \eta_{02} &= y_{02} - y_{10}\Gamma'' - y_{01}\Delta'' \\ \zeta_{02} &= z_{02} - z_{10}\Gamma'' - z_{01}\Delta'' \\ v_{02} &= v_{02} - v_{10}\Gamma'' - v_{01}\Delta'' \end{aligned}\right\}.$$

These twelve magnitudes are, however, subject to the six relations

$$\left.\begin{aligned} \Sigma x_{10}\xi_{20} &= 0, & \Sigma x_{10}\xi_{11} &= 0, & \Sigma x_{10}\xi_{02} &= 0 \\ \Sigma x_{01}\xi_{20} &= 0, & \Sigma x_{01}\xi_{11} &= 0, & \Sigma x_{01}\xi_{02} &= 0 \end{aligned}\right\};$$

and they therefore are equivalent to six, and not more than six, algebraically independent magnitudes. Formally, all twelve magnitudes will be retained for the sake of symmetry, their algebraical equivalence to only six independent magnitudes being borne in mind. We now have the necessary sixteen independent integrals of the kind initially specified.

Ex. Verify that integrals of this kind are given by

$$\begin{Vmatrix} x_{20}, & y_{20}, & z_{20}, & v_{20} \\ x_{10}, & y_{10}, & z_{10}, & v_{10} \\ x_{01}, & y_{01}, & z_{01}, & v_{01} \end{Vmatrix}, \quad \begin{Vmatrix} x_{11}, & y_{11}, & z_{11}, & v_{11} \\ x_{10}, & y_{10}, & z_{10}, & v_{10} \\ x_{01}, & y_{01}, & z_{01}, & v_{01} \end{Vmatrix}, \quad \begin{Vmatrix} x_{02}, & y_{02}, & z_{02}, & v_{02} \\ x_{10}, & y_{10}, & z_{10}, & v_{10} \\ x_{01}, & y_{01}, & z_{01}, & v_{01} \end{Vmatrix},$$

being the quantities $e_{11}, f_{11}, g_{11}, h_{11}$; $e_{12}, f_{12}, g_{12}, h_{12}$; $e_{22}, f_{22}, g_{22}, h_{22}$, of § 231; and prove that they are equivalent to the foregoing set, by establishing relations of the form

$$V^2\xi_{20}=\begin{vmatrix} z_{10}, & v_{10} \\ z_{01}, & v_{01} \end{vmatrix}\begin{vmatrix} x_{20}, & z_{20}, & v_{20} \\ x_{10}, & z_{10}, & v_{10} \\ x_{01}, & z_{01}, & v_{01} \end{vmatrix}+\begin{vmatrix} y_{10}, & v_{10} \\ y_{01}, & v_{01} \end{vmatrix}\begin{vmatrix} x_{20}, & y_{20}, & v_{20} \\ x_{10}, & y_{10}, & v_{10} \\ x_{01}, & y_{01}, & v_{01} \end{vmatrix}+\begin{vmatrix} y_{10}, & z_{10} \\ y_{01}, & z_{01} \end{vmatrix}\begin{vmatrix} x_{20}, & y_{20}, & z_{20} \\ x_{10}, & y_{10}, & z_{10} \\ x_{01}, & y_{01}, & z_{01} \end{vmatrix},$$

$$V^2\eta_{20}=\begin{vmatrix} z_{10}, & v_{10} \\ z_{01}, & v_{01} \end{vmatrix}\begin{vmatrix} y_{20}, & z_{20}, & v_{20} \\ y_{10}, & z_{10}, & v_{10} \\ y_{01}, & z_{01}, & v_{01} \end{vmatrix}-\begin{vmatrix} x_{10}, & v_{10} \\ x_{01}, & v_{01} \end{vmatrix}\begin{vmatrix} x_{20}, & y_{20}, & v_{20} \\ x_{10}, & y_{10}, & v_{10} \\ x_{01}, & y_{01}, & v_{01} \end{vmatrix}-\begin{vmatrix} x_{10}, & z_{10} \\ x_{01}, & z_{01} \end{vmatrix}\begin{vmatrix} x_{20}, & y_{20}, & z_{20} \\ x_{10}, & y_{10}, & z_{10} \\ x_{01}, & y_{01}, & z_{01} \end{vmatrix},$$

and the like.

Obtain the inverse relations, expressing these integrals in terms of $\xi_{20}, \eta_{20}, \zeta_{20}, v_{20}$; $\xi_{11}, \eta_{11}, \zeta_{11}, v_{11}$; $\xi_{02}, \eta_{02}, \zeta_{02}, v_{02}$.

It now is easy to obtain five additional integrals, involving the derivatives $C_{10}, C_{01}, C_{20}, C_{11}, C_{02}$, and completing the necessary tale of independent integrals of the system $D_{20}=0$, $D_{11}=0$, $D_{02}=0$, $D'_{20}=0$, $D'_{11}=0$, $D'_{02}=0$. Manifestly two such integrals are

$$C_{10},\ C_{01},$$

which would naturally be taken, after the result of § 462. Other three integrals, formed by analogy from the relations between ξ_{20} and x_{20}, ξ_{11} and x_{11}, ξ_{02} and x_{02}, are easily verified to be

$$\left.\begin{aligned} \gamma_{20} &= C_{20}-C_{10}\Gamma\ -C_{01}\Delta \\ \gamma_{11} &= C_{11}-C_{10}\Gamma'-C_{01}\Delta' \\ \gamma_{02} &= C_{02}-C_{10}\Gamma''-C_{01}\Delta'' \end{aligned}\right\}.$$

Thus the aggregate of twenty-one algebraically independent integrals of the set of six parametric equations is made up of

$$t,\ u;\ x_{10}, y_{10}, z_{10}, v_{10};\ x_{01}, y_{01}, z_{01}, v_{01};$$

which are independent of one another: of

$$\xi_{20},\ \eta_{20},\ \zeta_{20},\ v_{20};\ \xi_{11},\ \eta_{11},\ \zeta_{11},\ v_{11};\ \xi_{02},\ \eta_{02},\ \zeta_{02},\ v_{02},$$

which are connected by six linear relations: and of

$$C_{10},\ C_{01};\ \gamma_{20},\ \gamma_{11},\ \gamma_{02};$$

which are independent of one another and also are independent of the preceding constituents of the aggregate.

Introduction of the orientational equations: a system of integrals.

465. At this stage, there are alternative ways of proceeding from the foregoing aggregate of twenty-one quantities. We may construct those fifteen combinations of these quantities which constitute an algebraical system of independent integrals satisfying the six equations of orientational invariance; and then, among the fifteen members of the system, select some necessary twelve combinations which shall form an algebraically complete set

of integrals of the three remaining parametric equations. Or we may reverse the order in which the two groups of equations are successively integrated: that is, we may select eighteen magnitudes, being independent combinations of the foregoing twenty-one quantities which shall satisfy the three remaining parametric equations; and, then, we may construct those twelve combinations of the eighteen magnitudes which shall satisfy the six equations of orientational invariance.

We shall adopt the former method, now for surfaces in space, and later (Chapter XXVII) for regions. The six equations of orientational invariance are an isolated set, themselves complete in the Jacobian sense of the term; they combine also, with the two groups of parametric equations, still to constitute a complete Jacobian system. There are twenty-one possible arguments; and therefore, for an algebraically complete integral system, we shall require fifteen independent integrals of the six equations, which may be written in the form

$$\left.\begin{aligned} D_{yz} &= \left\{z\frac{\partial}{\partial y}\right\} - \left\{y\frac{\partial}{\partial z}\right\} = 0 \\ D_{zx} &= \left\{x\frac{\partial}{\partial z}\right\} - \left\{z\frac{\partial}{\partial x}\right\} = 0 \\ D_{xy} &= \left\{y\frac{\partial}{\partial x}\right\} - \left\{x\frac{\partial}{\partial y}\right\} = 0 \\ D_{xv} &= \left\{v\frac{\partial}{\partial x}\right\} - \left\{x\frac{\partial}{\partial v}\right\} = 0 \\ D_{yv} &= \left\{v\frac{\partial}{\partial y}\right\} - \left\{y\frac{\partial}{\partial v}\right\} = 0 \\ D_{zv} &= \left\{v\frac{\partial}{\partial z}\right\} - \left\{z\frac{\partial}{\partial v}\right\} = 0 \end{aligned}\right\},$$

where

$$\left\{z\frac{\partial}{\partial y}\right\} = z_{10}\frac{\partial}{\partial y_{10}} + z_{01}\frac{\partial}{\partial y_{01}} + z_{20}\frac{\partial}{\partial y_{20}} + z_{11}\frac{\partial}{\partial y_{11}} + z_{02}\frac{\partial}{\partial y_{02}},$$

and similarly for the other combinations. It is to be noted that these equations do not involve any of the arguments C_{10}, C_{01}, C_{20}, C_{11}, C_{02}.

In the first place, these six equations do not affect, and are not affected by, the last five arguments. Moreover, we have already pointed out (§ 448) that the six quantities Γ, Γ', Γ'', Δ, Δ', Δ'', severally satisfy all the equations so that, in the operation of the equations upon combinations of magnitudes, these quantities behave like constants. It follows, then, that each of the combinations

$$C_{10},\ C_{01},\ \gamma_{20},\ \gamma_{11},\ \gamma_{02},$$

satisfies the equations of orientational invariance: that is, they can be taken as five of the integrals needed to complete the tale of fifteen algebraically independent integrals of the orientational equations when satisfied at this

stage of the investigation. We therefore require ten other integrals, which manifestly (after earlier explanations) can be taken as belonging to the surface alone and are unaffected by the arguments C_{10}, C_{01}, C_{20}, C_{11}, C_{02}.

Clearly two, of these ten integrals, are

$$t, \ u,$$

which do not occur in the equations in any form. To construct other eight, we make the table of operations:

	x_{lm}	y_{lm}	z_{lm}	v_{lm}	ξ_{lm}	η_{lm}	ζ_{lm}	υ_{lm}
D_{yz}	.	z_{lm}	$-y_{lm}$	.	.	ζ_{lm}	$-\eta_{lm}$	.
D_{zx}	$-z_{lm}$	.	x_{lm}	.	$-\zeta_{lm}$	.	ξ_{lm}	.
D_{xy}	y_{lm}	$-x_{lm}$	.	.	η_{lm}	$-\xi_{lm}$	.	.
D_{xv}	v_{lm}	.	.	$-x_{lm}$	υ_{lm}	.	.	$-\xi_{lm}$
D_{yv}	.	v_{lm}	.	$-y_{lm}$	.	υ_{lm}	.	$-\eta_{lm}$
D_{zv}	.	.	v_{lm}	$-z_{lm}$	.	.	υ_{lm}	$-\zeta_{lm}$

where $l+m=1$, for the variables x_{lm}, y_{lm}, z_{lm}, v_{lm}, and $l+m=2$, for the variables ξ_{lm}, η_{lm}, ζ_{lm}, υ_{lm}. It is at once obvious that the expression

$$\Sigma\,(k_1x_{10}+k_2x_{01}+k_3\xi_{20}+k_4\xi_{11}+k_5\xi_{02})^2,$$

where k_1, k_2, k_3, k_4, k_5, are arbitrary independent constants, satisfies all the six orientational equations. Consequently, we obtain, as integrals of the six equations, the coefficients of each of the combinations k_ik_j (for $i, j = 1, 2, 3, 4, 5$). Thus we have integrals

$$\Sigma x_{10}^2 = E, \quad \Sigma x_{10}x_{01} = F, \quad \Sigma x_{01}^2 = G,$$

being three of the tale. We have

$$\Sigma x_{10}\xi_{20}, \quad \Sigma x_{10}\xi_{11}, \quad \Sigma x_{10}\xi_{02},$$
$$\Sigma x_{01}\xi_{20}, \quad \Sigma x_{01}\xi_{11}, \quad \Sigma x_{01}\xi_{02},$$

being apparently six integrals; but they are ineffective because each of them vanishes identically, owing to the quantities Γ, Γ', Γ'', Δ, Δ', Δ'', introduced into the expressions for ξ_{lm}, η_{lm}, ζ_{lm}, υ_{lm}. We have

$$\Sigma\xi_{20}^2 = \mathrm{a}, \quad \Sigma\xi_{11}^2 = \mathrm{b}, \quad \Sigma\xi_{02}^2 = \mathrm{c},$$
$$\Sigma\xi_{11}\xi_{02} = \mathrm{f}, \quad \Sigma\xi_{02}\xi_{20} = \mathrm{g}, \quad \Sigma\xi_{20}\xi_{11} = \mathrm{h}.$$

Apparently they are six integrals: but they satisfy the relation

$$\begin{vmatrix} \mathrm{a}, & \mathrm{h}, & \mathrm{g} \\ \mathrm{h}, & \mathrm{b}, & \mathrm{f} \\ \mathrm{g}, & \mathrm{f}, & \mathrm{c} \end{vmatrix} = 0,$$

which itself arises out of the relations

$$\begin{Vmatrix} \xi_{20}, & \eta_{20}, & \zeta_{20}, & v_{20} \\ \xi_{11}, & \eta_{11}, & \zeta_{11}, & v_{11} \\ \xi_{02}, & \eta_{02}, & \zeta_{02}, & v_{02} \end{Vmatrix} = 0;$$

and therefore **a**, **b**, **c**, **f**, **g**, **h**, effectively are five independent integrals of the equations.

It follows that, when the six orientational equations are satisfied at this stage in the construction of concomitants, the quantities

$$t, u; \quad E, F, G; \quad \mathbf{a}, \mathbf{h}, \mathbf{g}, \mathbf{b}, \mathbf{f}, \mathbf{c};$$
$$C_{10}, C_{01}; \quad \gamma_{20}, \gamma_{11}, \gamma_{02};$$

constitute an adequate system of fifteen independent integrals of the first set of parametric equations and of the orientational equations.

The remaining parametric equations: their integrals are concomitants of four binary forms.

466. We now require the combinations of these integrals which satisfy the three remaining parametric equations (§ 461)

$$u\frac{\partial f}{\partial t} = D_{01}f, \quad t\frac{\partial f}{\partial u} = D_{10}f, \quad t\frac{\partial f}{\partial t} - u\frac{\partial f}{\partial u} = (D_{10}D_{01} - D_{01}D_{10})f,$$

where the operators D_{01} and D_{10} are

$$D_{01} = \Sigma x_{10}\frac{\partial}{\partial x_{01}} + \Sigma\left(x_{20}\frac{\partial}{\partial x_{11}} + 2x_{11}\frac{\partial}{\partial x_{02}}\right) + C_{10}\frac{\partial}{\partial C_{01}} + \left(C_{20}\frac{\partial}{\partial C_{11}} + 2C_{11}\frac{\partial}{\partial C_{02}}\right),$$

$$D_{10} = \Sigma x_{01}\frac{\partial}{\partial x_{10}} + \Sigma\left(2x_{11}\frac{\partial}{\partial x_{20}} + x_{02}\frac{\partial}{\partial x_{11}}\right) + C_{01}\frac{\partial}{\partial C_{10}} + \left(2C_{11}\frac{\partial}{\partial C_{20}} + C_{02}\frac{\partial}{\partial C_{11}}\right).$$

For the purpose of integrating the equations, we make the foregoing integrals the arguments in these differential equations. By actual substitution, we find

$$D_{01}E = 0, \quad D_{01}F = E, \quad D_{01}G = 2F,$$
$$D_{10}E = 2F, \quad D_{10}F = G, \quad D_{10}G = 0,$$

so that

$$D_{01}V^2 = 0, \quad D_{10}V^2 = 0,$$

and therefore V^2 is an (invariant) integral of the equations. We likewise find

$$D_{01}\Gamma = -\Delta, \quad D_{01}\Gamma' = -\Delta' + \Gamma, \quad D_{01}\Gamma'' = 2\Gamma' - \Delta''.$$
$$D_{01}\Delta = 0, \quad D_{01}\Delta' = \Delta, \quad D_{01}\Delta'' = 2\Delta'$$

with similar results for the operation of D_{10} on the same six quantities; hence

$$D_{01}(\xi_{20}, \eta_{20}, \zeta_{20}, v_{20}) = (0, 0, 0, 0),$$
$$D_{01}(\xi_{11}, \eta_{11}, \zeta_{11}, v_{11}) = (\xi_{20}, \eta_{20}, \zeta_{20}, v_{20}),$$
$$D_{01}(\xi_{02}, \eta_{02}, \zeta_{02}, v_{02}) = 2(\xi_{11}, \eta_{11}, \zeta_{11}, v_{11}),$$
$$D_{10}(\xi_{20}, \eta_{20}, \zeta_{20}, v_{20}) = 2(\xi_{11}, \eta_{11}, \zeta_{11}, v_{11}),$$
$$D_{10}(\xi_{11}, \eta_{11}, \zeta_{11}, v_{11}) = (\xi_{02}, \eta_{02}, \zeta_{02}, v_{02}),$$
$$D_{10}(\xi_{02}, \eta_{02}, \zeta_{02}, v_{02}) = (0, 0, 0, 0).$$

Accordingly, we have

$$\left.\begin{aligned} D_{01}\mathbf{a} &= 0 \quad , \quad D_{10}\mathbf{a} = 4\mathbf{h} \\ D_{01}\mathbf{h} &= \mathbf{a} \quad , \quad D_{10}\mathbf{h} = 2\mathbf{b} + \mathbf{g} \\ D_{01}\mathbf{g} &= 2\mathbf{h} \quad , \quad D_{10}\mathbf{g} = 2\mathbf{f} \\ D_{01}\mathbf{b} &= 2\mathbf{h} \quad , \quad D_{10}\mathbf{b} = 2\mathbf{f} \\ D_{01}\mathbf{f} &= \mathbf{g} + 2\mathbf{b}, \quad D_{10}\mathbf{f} = \mathbf{c} \\ D_{01}\mathbf{c} &= 4\mathbf{f} \quad , \quad D_{10}\mathbf{c} = 0 \end{aligned}\right\}.$$

Now the six quantities **a**, **b**, **c**, **f**, **g**, **h**, are connected by a single relation

$$\mathbf{abc} + 2\mathbf{fgh} - \mathbf{af}^2 - \mathbf{bg}^2 - \mathbf{ch}^2 = 0;$$

so having regard to the foregoing results, we introduce a quantity **k** such that

$$2\mathbf{b} + \mathbf{g} = 3\mathbf{k},$$

we use the relation and this definition as determining **b** and **g**, and we now have **a**, **h**, **k**, **f**, **c** as five independent quantities, the effects of D_{01} and D_{10} upon which are

$$D_{01}\mathbf{a} = 0 \ , \quad D_{01}\mathbf{h} = \mathbf{a} \ , \quad D_{01}\mathbf{k} = 2\mathbf{h}, \quad D_{01}\mathbf{f} = 3\mathbf{k}, \quad D_{01}\mathbf{c} = 4\mathbf{f},$$
$$D_{10}\mathbf{a} = 4\mathbf{h}, \quad D_{10}\mathbf{h} = 3\mathbf{k}, \quad D_{10}\mathbf{k} = 2\mathbf{f} \ , \quad D_{10}\mathbf{f} = \mathbf{c} \ , \quad D_{10}\mathbf{c} = 0.$$

Also we note that

$$D_{01}(\mathbf{g} - \mathbf{b}) = 0, \quad D_{10}(\mathbf{g} - \mathbf{b}) = 0,$$

so that $\mathbf{g} - \mathbf{b}$ is an invariantive combination as it stands.

Finally, we have

$$D_{01}\gamma_{20} = 0 \ , \quad D_{01}\gamma_{11} = \gamma_{20}, \quad D_{01}\gamma_{02} = 2\gamma_{11},$$
$$D_{10}\gamma_{20} = 2\gamma_{11}, \quad D_{10}\gamma_{11} = \gamma_{02}, \quad D_{10}\gamma_{02} = 0.$$

Consequently, when we take t, u, E, F, G, **a**, **h**, **k**, **f**, **c**, C_{10}, C_{01}, γ_{20}, γ_{11}, γ_{02}, as the operating variables, we have one invariant 3θ, where (as in § 240)

$$\mathbf{g} - \mathbf{b} = 3\theta,$$

while the remaining three parametric equations become

$$\left.\begin{aligned} u\frac{\partial f}{\partial t} &= E\frac{\partial f}{\partial F} + 2F\frac{\partial f}{\partial G} + \mathbf{a}\frac{\partial f}{\partial \mathbf{h}} + 2\mathbf{h}\frac{\partial f}{\partial \mathbf{k}} + 3\mathbf{k}\frac{\partial f}{\partial \mathbf{f}} + 4\mathbf{f}\frac{\partial f}{\partial \mathbf{c}} \\ &\quad + C_{10}\frac{\partial f}{\partial C_{01}} + \gamma_{20}\frac{\partial f}{\partial \gamma_{11}} + 2\gamma_{11}\frac{\partial f}{\partial \gamma_{02}} \\ t\frac{\partial f}{\partial u} &= 2F\frac{\partial f}{\partial E} + G\frac{\partial f}{\partial F} + 4\mathbf{h}\frac{\partial f}{\partial \mathbf{a}} + 3\mathbf{k}\frac{\partial f}{\partial \mathbf{h}} + 2\mathbf{f}\frac{\partial f}{\partial \mathbf{k}} + \mathbf{c}\frac{\partial f}{\partial \mathbf{f}} \\ &\quad + C_{01}\frac{\partial f}{\partial C_{10}} + 2\gamma_{11}\frac{\partial f}{\partial \gamma_{20}} + \gamma_{02}\frac{\partial f}{\partial \gamma_{11}} \\ t\frac{\partial f}{\partial t} - u\frac{\partial f}{\partial u} &= 2\left(G\frac{\partial f}{\partial G} - E\frac{\partial f}{\partial E}\right) + 4\left(\mathbf{c}\frac{\partial f}{\partial \mathbf{c}} - \mathbf{a}\frac{\partial f}{\partial \mathbf{a}}\right) + 2\left(\mathbf{f}\frac{\partial f}{\partial \mathbf{f}} - \mathbf{h}\frac{\partial f}{\partial \mathbf{h}}\right) \\ &\quad + C_{01}\frac{\partial f}{\partial C_{01}} - C_{10}\frac{\partial f}{\partial C_{10}} + 2\left(\gamma_{02}\frac{\partial f}{\partial \gamma_{02}} - \gamma_{20}\frac{\partial f}{\partial \gamma_{20}}\right) \end{aligned}\right\}$$

We retain the invariant 3θ; when the surface is a surface lying in a flat, this invariant 3θ is connected with the fundamental Gauss measure of curvature of the surface. Thus we can regard $\mathbf{a}$, $\mathbf{h}$, $\mathbf{k}$, $\mathbf{f}$, $\mathbf{c}$, as really equivalent to four independent quantities, now that θ is a retained quantity. As, at this stage, fifteen independent invariantive functions are required, we shall need fourteen such functions in addition to θ; and they are to be arguments in the integrals of this complete Jacobian set of three equations, so that eleven integrals must be obtained.

But these are the three equations which are satisfied by (and which implicitly determine) all concomitants appertaining to the system of simultaneous concomitants of the four binary forms

$$\left.\begin{aligned} U &= (E, F, G\,|\,t, u)^2 \\ P &= (C_{10}, C_{01}\,|\,t, u) \\ \mathbf{Z} &= (\mathbf{a}, \mathbf{h}, \mathbf{k}, \mathbf{f}, \mathbf{c}\,|\,t, u)^4 \\ Q &= (\gamma_{20}, \gamma_{11}, \gamma_{02}\,|\,t, u)^2 \end{aligned}\right\}.$$

We thus require an algebraically complete set of concomitants of these four binary forms, in addition to the invariant

$$\mathbf{g} - \mathbf{b}.$$

Such a set consists of members which are unconnected by purely algebraical relations; and every concomitant of the system of binary forms is to be expressible in terms of members of the set by purely algebraical relations. In addition to the invariant $\mathbf{g} - \mathbf{b}$, eleven members are required to complete the set.

Concomitants belonging to the surface alone.

467. The complete system of U alone consists of U and its discriminant V^2. The complete system of P alone consists solely of P. The complete system of U and P, existing simultaneously, has been seen to consist of

$$U, \quad V^2, \quad P, \quad D,$$

where

$$D = GC_{10}^2 - 2FC_{10}C_{01} + EC_{01}^2.$$

It is convenient, however, to retain the Jacobian J_{12}, where

$$J_{12} = \begin{vmatrix} C_{10}, & C_{01} \\ Et + Fu, & Ft + Gu \end{vmatrix},$$

the equation of algebraical dependence being

$$J_{12}^2 = DU - V^2P^2.$$

As four members of the aggregate are thus provided, other seven members (in addition to $\mathbf{g} - \mathbf{b}$) are needed to complete the tale. Of these seven, three must be obtained so as to represent, directly or indirectly, the three arguments γ_{20}, γ_{11}, γ_{02}; consequently other four must be obtained and, preferably, they will relate to the surface in general and not be affected by a function $C(p, q)$.

To determine the necessary four concomitants (or more than four, provided there are four new independent concomitants), we take a system for the quartic form $\mathbf{Z}$, to be associated with the quadratic form U and with the invariant θ. Now for a binary quartic, existing alone, a complete system is composed of the quartic itself, its quadrinvariant, its cubinvariant, and its Hessian*. We therefore retain the quartic itself, $\mathbf{Z}$. We retain the quadrinvariant, which has been denoted (§ 239) by I_0. The cubinvariant, J_0, has been expressed (§ 240) in the form

$$J_0 = \theta (I_0 - 4\theta^2);$$

consequently, being expressible in terms of the retained concomitants θ and I_0, it cannot be used towards the tale of independent concomitants.

There remains the Hessian, temporarily denoted by $\overline{\mathbf{H}}$, where

$$\overline{\mathbf{H}} = (\mathbf{ak} - \mathbf{h}^2) t^4 + 2(\mathbf{af} - \mathbf{hk}) t^3 u + (\mathbf{ac} - 3\mathbf{k}^2 + 2\mathbf{hf}) t^2 u^2 + 2(\mathbf{ch} - \mathbf{fk}) tu^3 + (\mathbf{kc} - \mathbf{f}^2) u^4.$$

We have had a covariant

$$(\mathbf{ab} - \mathbf{h}^2)^{\frac{1}{2}} p'^2 + (\mathbf{ca} - \mathbf{g}^2)^{\frac{1}{2}} p'q' + (\mathbf{bc} - \mathbf{f}^2)^{\frac{1}{2}} q'^2,$$

the significance of which is known (§ 235): so, now, we take the covariant

$$\mathbf{W} = (\mathbf{ab} - \mathbf{h}^2)^{\frac{1}{2}} t^2 + (\mathbf{ca} - \mathbf{g}^2)^{\frac{1}{2}} tu + (\mathbf{bc} - \mathbf{f}^2)^{\frac{1}{2}} u^2,$$

with the customary positive sign for each radical. But

$$\mathbf{ab} - \mathbf{h}^2 - (\mathbf{ak} - \mathbf{h}^2) = \mathbf{a}(\mathbf{b} - \mathbf{k}) = -\theta \mathbf{a}:$$

hence, by the known theorem that concomitants in a binary system in its general form are uniquely determined by their leading coefficients (in this instance, the coefficients of t^4),

$$\mathbf{W}^2 - \overline{\mathbf{H}} = -\theta \mathbf{Z},$$

and therefore, in the algebraical system of the quartic when the additional invariant θ is retained, we may substitute $\mathbf{W}$ for the Hessian $\overline{\mathbf{H}}$. Thus there are concomitants

$$\mathbf{Z},\ I_0,\ \mathbf{W},\ \theta,$$

in our system.

There are also intermediate forms. In the equation of the fourth degree determining the principal measures of curvature of the surface, there occurred an invariant I_1. We can take

$$I_1 = \frac{1}{24}\left(E\frac{\partial^2}{\partial v^2} - 2F\frac{\partial^2}{\partial u \partial v} + G\frac{\partial^2}{\partial u^2}\right)^2 \mathbf{Z}$$
$$= G^2\mathbf{a} - 4FG\mathbf{h} + 2(EG + 2F^2)\mathbf{k} - 4EF\mathbf{f} + E^2\mathbf{c}.$$

Also we had a covariant (§ 234)

$$\begin{vmatrix} \mathbf{a}p'^3 + 3\mathbf{h}p'^2q' + 3\mathbf{k}p'q'^2 + \mathbf{f}q'^3, & Ep' + Fq' \\ \mathbf{h}p'^3 + 3\mathbf{k}p'^2q' + 3\mathbf{f}p'q'^2 + \mathbf{c}q'^3, & Fp' + Gq' \end{vmatrix};$$

* The asyzygetic system requires, further, the cubicovariant: but this is not an algebraically independent concomitant, for its square is expressible, non-fractionally, in terms of the foregoing set.

and therefore we take the Jacobian of $\mathbf{Z}$ and U in the form

$$J_{21} = \frac{1}{8}\left(\frac{\partial \mathbf{Z}}{\partial t}\frac{\partial U}{\partial u} - \frac{\partial \mathbf{Z}}{\partial u}\frac{\partial U}{\partial t}\right)$$
$$= \begin{vmatrix} \mathbf{a}t^3 + 3\mathbf{h}t^2u + 3\mathbf{k}tu^2 + \mathbf{f}u^3, & Et + Fu \\ \mathbf{h}t^3 + 3\mathbf{k}t^2u + 3\mathbf{f}tu^2 + \mathbf{c}u^3, & Ft + Gu \end{vmatrix}.$$

We thus can take, as constituting a complete algebraical system for $\mathbf{Z}$ and U, the aggregate

$$U,\ V^2,\ \theta,\ \mathbf{Z},\ I_0,\ I_1,\ \mathbf{W},\ J_{24};$$

and it is complete in the sense that every simultaneous concomitant can be expressed in terms of its members.

Thus there is a simultaneous invariant J_2 (§ 239); it is such that

$$J_2 = \frac{1}{3}V^2 I_1.$$

Also there is a covariant

$$D'' = \frac{1}{12}\left(G\frac{\partial^2}{\partial t^2} - 2F\frac{\partial^2}{\partial u \partial t} + E\frac{\partial^2}{\partial u^2}\right)\mathbf{Z}$$
$$= (G\mathbf{a} - 2F\mathbf{h} + E\mathbf{k})\,t^2$$
$$+ 2(G\mathbf{h} - 2F\mathbf{k} + E\mathbf{f})\,tu$$
$$+ (G\mathbf{k} - 2F\mathbf{f} + E\mathbf{c})\,u^2.$$

It is easy to verify that

$$(E\mathbf{h} - F\mathbf{a})^2 = -E^2(\mathbf{ab} - \mathbf{h}^2) - V^2\mathbf{a}^2 - \theta E^2\mathbf{a} + E\mathbf{a}(G\mathbf{a} - 2F\mathbf{h} + E\mathbf{k});$$

and therefore, again using the theorem that binariants are determinate through their leading coefficients,

$$J_{24}^2 = -U^2\mathbf{W}^2 - V^2\mathbf{Z}^2 - \theta U^2\mathbf{Z} + U\mathbf{Z}D'',$$

giving the expression for D'.

Again, the permanent relation

$$\begin{vmatrix} \mathbf{a}, & \mathbf{h}, & \mathbf{g} \\ \mathbf{h}, & \mathbf{b}, & \mathbf{f} \\ \mathbf{g}, & \mathbf{f}, & \mathbf{c} \end{vmatrix} = 0$$

leads to the foregoing expression for the cubinvariant J_0 in the form

$$J_0 - \theta I_0 + 4\theta^3 = 0,$$

thus leading to the omission of J_0 from the tale of independent concomitants. And so for other instances.

Concomitants belonging to a curve on the surface.

468. There remains the necessity of associating, with the members already retained, three concomitants which shall involve the arguments γ_{20}, γ_{11}, γ_{02}; they must be algebraically independent of one another and, when selected, they serve to complete the tale of concomitants of the second grade.

One such concomitant is Q, the quadratic form already noted; it is given by

$$Q = (\gamma_{20}, \gamma_{11}, \gamma_{02}\mathcal{X}t, u)^2.$$

A second concomitant can be taken, in the Jacobian of Q and U; we write

$$\begin{aligned} Q_1 &= \frac{1}{4}\left(\frac{\partial Q}{\partial t}\frac{\partial U}{\partial u} - \frac{\partial Q}{\partial u}\frac{\partial U}{\partial t}\right) \\ &= (\gamma_{20}F - \gamma_{11}E)\,t^2 + (\gamma_{20}G - \gamma_{02}E)\,tu + (\gamma_{11}G - \gamma_{02}F)\,u^2. \end{aligned}$$

A third suitable concomitant can be taken, in the covariant

$$\begin{aligned} Q_2 &= \frac{1}{4}(\gamma_{20}, \gamma_{11}, \gamma_{02}\mathcal{X}\frac{\partial U}{\partial u}, -\frac{\partial U}{\partial t}\Big)^2 \\ &= (\gamma_{20}F^2 - 2\gamma_{11}EF + \gamma_{02}E^2)\,t^2 \\ &\quad + 2\{\gamma_{20}FG - \gamma_{11}(EG + F^2) + \gamma_{02}EF\}\,tu \\ &\quad + (\gamma_{20}G^2 - 2\gamma_{11}FG + \gamma_{02}F^2)\,u^2. \end{aligned}$$

These three concomitants, which are affected by the curve-function $C(p, q)$, are to be associated with the two concomitants of the first grade also affecting that curve-function, thus constituting the five that must be introduced on account of the five arguments C_{10}, C_{01}, γ_{20}, γ_{11}, γ_{02}. Taken simultaneously with the concomitants which are affected by the surface alone without the curve-function, they constitute the aggregate

$$\begin{gathered} U,\ V^2,\ \theta,\ \mathbf{Z},\ I_0,\ I_1,\ \mathbf{W},\ J_{24}, \\ P,\ D,\ Q,\ Q_1,\ Q_2, \end{gathered}$$

the first line of which appertains to the surface and to geodesics on the surface, and the second line of which appertains to any given curve on the surface. They constitute an algebraically complete set of concomitants of the first grade and of the second grade.

It is easy to select other concomitants of the system which are algebraically expressible in terms of the members included in the foregoing set. Thus there is the discriminant Δ_Q of Q, given by

$$\Delta_Q = \gamma_{20}\gamma_{02} - \gamma_{11}^2;$$

the relation

$$U^2\Delta_Q = QQ_2 - Q_1^2$$

is easily verified. There is an invariant

$$Q_3 = \gamma_{20}G - 2\gamma_{11}F + \gamma_{02}E,$$

intermediate between the discriminants V^2 and Δ_Q; the relation

$$UQQ_3 = U^2\Delta_Q + V^2Q^2 + Q_1^2$$

is easily verified. Again, we have

$$\begin{aligned} \mathbf{Z} &= \frac{1}{4}t\frac{\partial \mathbf{Z}}{\partial t} + \frac{1}{4}u\frac{\partial \mathbf{Z}}{\partial u}, \\ U &= \frac{1}{2}t\frac{\partial U}{\partial t} + \frac{1}{2}u\frac{\partial U}{\partial u}, \\ Q &= \frac{1}{2}t\frac{\partial Q}{\partial t} + \frac{1}{2}u\frac{\partial Q}{\partial u}; \end{aligned}$$

so that, if

$$I_{24}=\frac{1}{8}\left(\frac{\partial \mathbf{Z}}{\partial t}\frac{\partial Q}{\partial u}-\frac{\partial \mathbf{Z}}{\partial u}\frac{\partial Q}{\partial t}\right),$$

we have

$$UI_{24}=QJ_{24}-\mathbf{Z}Q_1,$$

giving another concomitant I_{24}, linear in the quantities γ_{20}, γ_{11}, γ_{02}. It is

$$I_{24}=\begin{vmatrix} \mathbf{a}t^3+3\mathbf{h}t^2u+3\mathbf{k}tu^2+\mathbf{f}u^3, & \gamma_{20}t+\gamma_{11}u \\ \mathbf{h}t^3+3\mathbf{k}t^2u+3\mathbf{f}tu^2+\mathbf{c}u^3, & \gamma_{11}t+\gamma_{02}u \end{vmatrix}.$$

469. Moreover, in all the preceding examples, the variables in the covariants are t and u, that is, dp and dq, with a corresponding element of arc ds on the surface. These variables dp and dq are quite unrestricted. But there occurs a variation along the arc of the curve $C(p, q)=\text{constant}$; let the variations be d_1p and d_1q, with dl as the element of length. There occurs also a variation normal to the arc of the same curve; let the variations be d_2p and d_2q, with $d\nu$ as the element of length. Also, let ϵ be the inclination of the direction of dl to ds, and $\frac{1}{2}\pi+\epsilon$ therefore the inclination of the direction of $d\nu$ to ds. Then it is easy to shew that

$$\left.\begin{aligned} \frac{1}{dl}ds.d_1p &= dp\cos\epsilon+\frac{1}{V}(Fdp+Gdq)\sin\epsilon \\ \frac{1}{dl}ds.d_1q &= dq\cos\epsilon-\frac{1}{V}(Edp+Fdq)\sin\epsilon \end{aligned}\right\},$$

$$\left.\begin{aligned} \frac{1}{d\nu}ds.d_2p &= -dp\sin\epsilon+\frac{1}{V}(Fdp+Gdq)\cos\epsilon \\ \frac{1}{d\nu}ds.d_2q &= -dq\sin\epsilon-\frac{1}{V}(Edp+Fdq)\cos\epsilon \end{aligned}\right\};$$

and there are the relations

$$\left.\begin{aligned} dl^2 &= Ed_1p^2+2Fd_1pd_1q+Gd_1q^2 \\ d\nu^2 &= Ed_2p^2+2Fd_2pd_2q+Gd_2q^2 \end{aligned}\right\}.$$

These quantities d_1p and d_1q, d_2p and d_2q, are congruent with dp and dq (that is, with t and u) and can take their place, or can be associated with them, as arguments of covariants. For example, we have covariants

$$\gamma_{20}dpd_1p+\gamma_{11}(dpd_1q+dqd_1p)+\gamma_{02}dqd_1q,$$
$$\gamma_{20}dpd_2p+\gamma_{11}(dpd_2q+dqd_2p)+\gamma_{02}dqd_2q;$$

by direct substitution for d_1p and d_1q, the former becomes

$$\frac{dl}{ds}\left(Q\cos\epsilon+\frac{1}{V}Q_1\sin\epsilon\right),$$

and, by direct substitution for d_2p and d_2q, the latter becomes

$$-\frac{d\nu}{ds}Q\sin\epsilon.$$

Similarly, we find

$$Q(\gamma_{20}d_1p^2+2\gamma_{11}d_1pd_1q+\gamma_{02}d_1q^2)$$
$$=\left(\frac{dl}{ds}\right)^2\left(Q\cos\epsilon+\frac{Q_1}{V}\sin\epsilon\right)^2+\frac{\Delta_Q}{V^2}ds^2dl^2\sin^2\epsilon,$$
$$\frac{ds^2}{dl\,d\nu}\{\gamma_{20}d_1pd_2p+\gamma_{11}(d_1pd_2q+d_2pd_1q)+\gamma_{02}d_1qd_2q\}$$
$$=\frac{Q_1}{V}(\cos^2\epsilon-\sin^2\epsilon)+\left(\frac{Q_2}{V^2}-Q\right)\sin\epsilon\cos\epsilon,$$
$$Q(\gamma_{20}d_2p^2+2\gamma_{11}d_2pd_2q+\gamma_{02}d_2q^2)$$
$$=\left(\frac{d\nu}{ds}\right)^2Q^2\sin^2\epsilon+\frac{\Delta_Q}{V^2}ds^2d\nu^2\cos^2\epsilon.$$

And so for other covariants similarly formed.

Geometrical significance of the concomitants belonging to the surface alone.

470. We now proceed to the geometrical interpretation of the selected invariants and covariants. When this interpretation is complete, the significance of any other invariant or covariant can be obtained by substituting, in its expression in terms of the selected concomitants, the values of those concomitants.

The equation (§ 239) giving the four principal radii of curvature, or the four measures of the curvature μ_1, μ_2, μ_3, μ_4, of the surface is

$$\frac{1}{\rho^8}-\mu_1\frac{1}{\rho^6}+\mu_2\frac{1}{\rho^4}-\mu_3\frac{1}{\rho^2}+\mu_4=0,$$

where, if

$$D_0=V^4\left(I_1^2-16V^2J_1+\frac{16}{3}V^4I_0\right),$$

we have

$$\left.\begin{aligned}\mu_1D_0&=I_1^3-18V^2I_1J_1+8V^4I_0I_1-16V^6J_0\\ \mu_2D_0&=3I_0I_1^2-18V^2I_1J_0-27J_1^2+4V^4I_0^2\\ \mu_3D_0&=3I_0^2I_1-54J_0J_1\\ \mu_4D_0&=I_0^3-27J_0^2\end{aligned}\right\}.$$

Thus there are four relations, potentially sufficient to express the four invariants

$$I_0,\ I_1,\ J_0,\ J_1,$$

in terms of μ_1, μ_2, μ_3, μ_4.

The value of θ is then given by the relation

$$4\theta^3-I_0\theta+J_0=0;$$

and the value of the invariant Π is given by (§ 240)

$$\Pi^2=J_1-\theta I_1-\frac{1}{3}V^2(I_0-12\theta^2).$$

But the explicit algebraic resolution of these relations, so as to express I_0, I_1, J_0, J_1, in terms of μ_1, μ_2, μ_3, μ_4, does not seem feasible.

On the other hand, the various measures are expressed explicitly in terms of invariants of the system.

The algebraically complete system contains the covariants $\mathbf{Z}$, $\mathbf{W}$, J_{24}, which are independent of the curve $C(p, q) = \text{constant}$. Now

$$\begin{aligned}\mathbf{Z} &= (\mathbf{a}, \mathbf{h}, \mathbf{k}, \mathbf{f}, \mathbf{c}\,\mathfrak{Y}t, u)^4 \\ &= (\mathbf{a}, \mathbf{h}, \mathbf{k}, \mathbf{f}, \mathbf{c}\,\mathfrak{Y}dp, dq)^4,\end{aligned}$$

and therefore

$$\begin{aligned}\frac{\mathbf{Z}}{U^2} = \frac{\mathbf{Z}}{ds^4} &= (\mathbf{a}, \mathbf{h}, \mathbf{k}, \mathbf{f}, \mathbf{c}\,\mathfrak{Y}p', q')^4 \\ &= \frac{1}{\rho^2},\end{aligned}$$

so that

$$\mathbf{Z} = \frac{ds^4}{\rho^2},$$

where ρ is the radius of circular curvature of the geodesic through the direction p', q'.

Again,

$$\mathbf{W} = (\mathbf{ab} - \mathbf{h}^2)^{\frac{1}{2}} t^2 + (\mathbf{ca} - \mathbf{g}^2)^{\frac{1}{2}} tu + (\mathbf{bc} - \mathbf{f}^2)^{\frac{1}{2}} u^2,$$

and therefore

$$\begin{aligned}\frac{\mathbf{W}}{U} = \frac{\mathbf{W}}{ds^2} &= (\mathbf{ab} - \mathbf{h}^2)^{\frac{1}{2}} p'^2 + (\mathbf{ca} - \mathbf{g}^2)^{\frac{1}{2}} p'q' + (\mathbf{bc} - \mathbf{f}^2)^{\frac{1}{2}} q'^2 \\ &= -\frac{V}{\rho\tau},\end{aligned}$$

(by § 235): that is,

$$\mathbf{W} = -V\frac{ds^2}{\rho\tau},$$

where τ is the radius of tilt of the geodesic.

Also

$$J_{24} = \begin{vmatrix} \mathbf{a}t^3 + 3\mathbf{h}t^2u + 3\mathbf{k}tu^2 + \mathbf{f}u^3, & Et + Fu \\ \mathbf{h}t^3 + 3\mathbf{k}t^2u + 3\mathbf{f}tu^2 + \mathbf{c}u^3, & Ft + Gu \end{vmatrix},$$

so that (by § 234)

$$\frac{J_{24}}{U^2} = \frac{J_{24}}{ds^4} = \frac{V}{\rho\sigma},$$

and therefore

$$J_{24} = V\frac{ds^4}{\rho\sigma},$$

where σ is the radius of torsion of the geodesic.

Other concomitants, invariants and covariants, can be expressed in terms of these magnitudes. Thus the Hessian, say $\overline{\mathbf{H}}$, of the quartic $\mathbf{Z}$, being given by

$$\overline{\mathbf{H}} = (\mathbf{ak} - \mathbf{h}^2) t^4 + \ldots,$$

is expressible in the form (§ 467)

$$\begin{aligned}\overline{\mathbf{H}} &= \mathbf{W}^2 - \theta\mathbf{Z} \\ &= \left(\frac{V^2}{\rho^2\tau^2} - \frac{\theta}{\rho^2}\right) ds^4.\end{aligned}$$

Again, we had a covariant (§ 467)

$$D'' = (G\mathbf{a} - 2F\mathbf{h} + E\mathbf{k})\, t^2 + \ldots,$$

and it was proved that

$$D''U\mathbf{Z} = J_{24}{}^2 + U^2\mathbf{W}^2 + V^2\mathbf{Z}^2 + \theta U^2\mathbf{Z};$$

thus

$$\frac{ds^6}{\rho^2} D'' = V^2\left(\frac{1}{\rho^4} + \frac{1}{\rho^2\sigma^2} + \frac{1}{\rho^2\tau^2}\right) ds^8 + \theta\frac{ds^8}{\rho^2},$$

that is,

$$\frac{D''}{ds^2} = V^2\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2} + \frac{1}{\tau^2}\right) + \theta,$$

which really is equivalent to the relation (§ 236)

$$(EN - 2FM + GL)\frac{1}{\rho} = V^2\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2} + \frac{1}{\tau^2}\right) + 3\theta.$$

As a last example, consider the cubicovariant of $\mathbf{Z}$, which is

$$\Phi = (\mathbf{a}^2\mathbf{f} - 3\mathbf{ahk} + 2\mathbf{h}^3)\, t^6 + \ldots;$$

we have

$$\begin{aligned}\Phi^2 &= -(4\overline{\mathbf{H}}^3 - I_0\overline{\mathbf{H}}\mathbf{Z}^2 + J_0\mathbf{Z}^3)\\ &= -\frac{1}{\rho^6}\left\{4\left(\frac{\overline{\mathbf{H}}}{\mathbf{Z}}\right)^3 - I_0\frac{\overline{\mathbf{H}}}{\mathbf{Z}} + J_0\right\}\\ &= -\frac{ds^2}{\rho^6}\left\{4\left(\frac{V^2}{\tau^2} - \theta\right)^3 - I_0\left(\frac{V^2}{\tau^2} - \theta\right) + J_0\right\}.\end{aligned}$$

Geometrical significance of concomitants belonging to a curve.

471. For the interpretation of the concomitants which involve the first and the second derivatives of C, it is convenient to resume their interpretation from the beginning.

As in § 463, we denote by ds the arc-element determined by dp and dq; we denote by dl an arc-element along the curve $C(p, q) = \text{constant}$ with the increments d_1p and d_1q, and by $d\nu$ an arc-element normal to that curve in the surface with the increments d_2p and d_2q. Then

$$\begin{aligned}P &= (C_{10}, C_{01}\overline{\times}t, u)\\ &= (C_{10}, C_{01}\overline{\times}p', q')\, ds\\ &= \frac{dC}{ds} ds.\end{aligned}$$

Along the curve, we have

$$C_{10}d_1p + C_{01}d_1q = 0,$$

so that, as

$$D = GC_{10}{}^2 - 2FC_{10}C_{01} + EC_{01}{}^2,$$

we have

$$D^{\frac{1}{2}}\frac{d_1p}{dl} = C_{01}, \quad D^{\frac{1}{2}}\frac{d_1q}{dl} = -C_{10}.$$

Along a direction normal to the curve, we have

$$E\,d_1p\,d_2p + F(d_1q\,d_2p + d_1p\,d_2q) + G\,d_1q\,d_2q = 0,$$

and

$$d\nu^2 = E\,d_2p^2 + 2F\,d_2p\,d_2q + G\,d_2q^2;$$

hence

$$\left.\begin{aligned} VD^{\frac{1}{2}}\frac{d_2p}{d\nu} &= -FC_{01} + GC_{10} \\ VD^{\frac{1}{2}}\frac{d_2q}{d\nu} &= \quad EC_{01} - FC_{10} \end{aligned}\right\}.$$

Consequently

$$\frac{dC}{d\nu} = C_{10}\frac{d_2p}{d\nu} + C_{01}\frac{d_2q}{d\nu} = \frac{1}{V}D^{\frac{1}{2}},$$

which can be regarded as the dilatation of the curve; and therefore

$$D = V^2\left(\frac{dC}{d\nu}\right)^2.$$

We require the quantities

$$\frac{d}{dl}\left(\frac{dC}{d\nu}\right), \quad \frac{d^2C}{d\nu^2},$$

and we proceed from the foregoing relation

$$V^2\left(\frac{dC}{d\nu}\right)^2 = GC_{10}^2 - 2FC_{10}C_{01} + EC_{01}^2.$$

Now, for any argument u, we have

$$\begin{aligned} \frac{du}{dl} &= \frac{\partial u}{\partial p}\frac{d_1p}{dl} + \frac{\partial u}{\partial q}\frac{d_1q}{dl} \\ &= D^{-\frac{1}{2}}\left(C_{01}\frac{\partial u}{\partial p} - C_{10}\frac{\partial u}{\partial q}\right); \end{aligned}$$

so that

$$D^{\frac{1}{2}}\frac{dC_{10}}{dl} = C_{20}C_{01} - C_{11}C_{10},$$

$$D^{\frac{1}{2}}\frac{dC_{01}}{dl} = C_{11}C_{01} - C_{02}C_{10},$$

$$D^{\frac{1}{2}}\frac{dE}{dl} = 2E(\Gamma C_{01} - \Gamma' C_{10}) + 2F(\Delta C_{01} - \Delta' C_{10}),$$

$$D^{\frac{1}{2}}\frac{dF}{dl} = E(\Gamma' C_{01} - \Gamma'' C_{10}) + F\{(\Gamma C_{01} - \Gamma' C_{10}) + (\Delta' C_{01} - \Delta'' C_{10})\} + G(\Delta C_{01} - \Delta' C_{10}),$$

$$D^{\frac{1}{2}}\frac{dG}{dl} = 2F(\Gamma' C_{01} - \Gamma'' C_{10}) + 2G(\Delta' C_{01} - \Delta'' C_{10}),$$

$$D^{\frac{1}{2}}\frac{dV}{dl} = V\{(\Gamma C_{01} - \Gamma' C_{10}) + (\Delta' C_{01} - \Delta'' C_{10})\}.$$

We have

$$2V^2\frac{dC}{d\nu}\frac{d^2C}{dld\nu}+2V\frac{dV}{dl}\left(\frac{dC}{d\nu}\right)^2=\frac{d}{dl}(GC_{10}^2-2FC_{10}C_{01}+EC_{01}^2).$$

When complete substitution is made from the foregoing tableau and after the coefficients have been collected, it is found that

$$\begin{aligned}VD\frac{d^2C}{dld\nu}&=G(C_{10}C_{01}\gamma_{20}-C_{10}^2\gamma_{11})\\&\quad+F(-C_{01}^2\gamma_{20}+C_{10}^2\gamma_{02})\\&\quad+E(-C_{01}C_{10}\gamma_{02}+C_{01}^2\gamma_{11})\\&=\begin{vmatrix}\gamma_{20}C_{01}-\gamma_{11}C_{10}, & -\gamma_{11}C_{01}+\gamma_{02}C_{10}\\ EC_{01}-FC_{10}, & -FC_{01}+GC_{10}\end{vmatrix}\\&=D^{\frac{1}{2}}.VD^{\frac{1}{2}}\begin{vmatrix}\gamma_{20}\dfrac{d_1p}{dl}+\gamma_{11}\dfrac{d_1q}{dl}, & -\gamma_{11}\dfrac{d_1p}{dl}-\gamma_{02}\dfrac{d_1q}{dl}\\ \dfrac{d_2q}{d\nu}, & \dfrac{d_2p}{d\nu}\end{vmatrix}\\&=VD\left\{\gamma_{20}\frac{d_1p}{dl}\frac{d_2p}{d\nu}+\gamma_{11}\left(\frac{d_1p}{dl}\frac{d_2q}{d\nu}+\frac{d_1q}{dl}\frac{d_2p}{d\nu}\right)+\gamma_{02}\frac{d_1q}{dl}\frac{d_2q}{d\nu}\right\},\end{aligned}$$

that is,

$$\frac{d^2C}{dld\nu}=\gamma_{20}\frac{d_1p}{dl}\frac{d_2p}{d\nu}+\gamma_{11}\left(\frac{d_1p}{dl}\frac{d_2q}{d\nu}+\frac{d_1q}{dl}\frac{d_2p}{d\nu}\right)+\gamma_{02}\frac{d_1q}{dl}\frac{d_2q}{d\nu}.$$

We proceed, similarly, to obtain $\dfrac{d^2C}{d\nu^2}$. For any argument u, we have

$$\begin{aligned}\frac{du}{d\nu}&=\frac{\partial u}{\partial p}\frac{d_2p}{d\nu}+\frac{\partial u}{\partial q}\frac{d_2q}{d\nu}\\&=\frac{1}{VD^{\frac{1}{2}}}\left\{(GC_{10}-FC_{01})\frac{\partial u}{\partial p}+(-FC_{10}+EC_{01})\frac{\partial u}{\partial q}\right\}.\end{aligned}$$

We frame a tableau of the first derivatives of C_{10}, C_{01}, E, F, G, V, with regard to ν, and we substitute in the derived equation

$$2V^2\frac{dC}{d\nu}\frac{d^2C}{d\nu^2}+2V\frac{dV}{d\nu}\left(\frac{dC}{d\nu}\right)^2=\frac{d}{d\nu}(GC_{10}^2-2FC_{10}C_{01}+EC_{01}^2).$$

After reduction, we obtain the result

$$\frac{d^2C}{d\nu^2}=\gamma_{20}\left(\frac{d_2p}{d\nu}\right)^2+2\gamma_{11}\frac{d_2p}{d\nu}\frac{d_2q}{d\nu}+\gamma_{02}\left(\frac{d_2q}{d\nu}\right)^2.$$

It is convenient to associate, with these two magnitudes, the expression for the magnitude of the geodesic curvature of the curve already (§ 223) obtained in the form

$$\frac{1}{V\gamma}=\frac{(\gamma_{20},\ \gamma_{11},\ \gamma_{02}\mid C_{01},\ -C_{10})^2}{(GC_{10}^2-2FC_{10}C_{01}+EC_{01}^2)^{\frac{3}{2}}},$$

that is,

$$\frac{1}{\gamma}\frac{dC}{d\nu}=\gamma_{20}\left(\frac{d_1p}{dl}\right)^2+2\gamma_{11}\frac{d_1p}{dl}\frac{d_1q}{dl}+\gamma_{02}\left(\frac{d_1q}{dl}\right)^2.$$

The forms thus obtained lead at once to the interpretation of the surviving concomitants Q, Q_1, Q_2, of the algebraically complete system. When the values of $\frac{d_1 p}{dl}$ and $\frac{d_1 q}{dl}$, $\frac{d_2 p}{dv}$ and $\frac{d_2 q}{dv}$, in terms of p' and q', are substituted from § 469, we have

$$U\frac{1}{\gamma}\frac{dC}{dv} = Q\cos^2\epsilon + 2\frac{Q_1}{V}\sin\epsilon\cos\epsilon + \frac{Q_2}{V^2}\sin^2\epsilon,$$

$$U\frac{\omega^2 C}{dl\,dv} = \left(Q + \frac{Q_2}{V^2}\right)\sin\epsilon\cos\epsilon + \frac{Q_1}{V}(\cos^2\epsilon - \sin^2\epsilon),$$

$$U\frac{d^2 C}{dv^2} = Q\sin^2\epsilon - 2\frac{Q_1}{V}\sin\epsilon\cos\epsilon + \frac{Q_2}{V^2}\cos^2\epsilon;$$

and therefore

$$\left.\begin{aligned} Q &= \frac{U}{\cos^2 2\epsilon}\left\{\frac{1}{\gamma}\frac{dC}{dv}\cos^2\epsilon - \frac{d^2 C}{dl\,dv}\sin 2\epsilon + \frac{d^2 C}{dv^2}\sin^2\epsilon\right\} \\ Q_1 &= \frac{UV}{\cos 2\epsilon}\left\{-\frac{1}{\gamma}\frac{dC}{dv}\sin\epsilon\cos\epsilon + \frac{d^2 C}{dl\,dv} - \frac{d^2 C}{dv^2}\sin\epsilon\cos\epsilon\right\} \\ Q_2 &= \frac{UV^2}{\cos^2 2\epsilon}\left\{\frac{1}{\gamma}\frac{dC}{dv}(\sin^2\epsilon - \sin^2 2\epsilon) + \frac{d^2 C}{dl\,dv}\sin 2\epsilon + \frac{d^2 C}{dv^2}(\cos^2\epsilon - \sin^2 2\epsilon)\right\} \end{aligned}\right\},$$

being the interpretations required.

Ex Verify that the discriminant Δ_Q of Q, being $\gamma_{20}\gamma_{02} - \gamma_{11}^2$, is given by

$$\frac{\Delta_Q}{V^2} = \frac{1}{\gamma}\frac{dC}{dv}\frac{d^2 C}{dv^2} - \left(\frac{d^2 C}{dl\,dv}\right)^2;$$

and obtain the value of the invariant denoted (p. 437) by Q_3, where

$$Q_3 = \gamma_{20} G - 2\gamma_{11} F + \gamma_{02} E.$$

472. In these expressions, the angle ϵ is the angle between a tangent to the curve $C(p, q) =$ constant and the general direction on the surface determined by dp and dq. When we particularise this direction by making it coincide with the tangent to the curve, we have

$$\epsilon = 0,$$

and then $U = dl^2$; we then revert to the expressions already obtained for $\frac{1}{\gamma}\frac{dC}{dv}$, $\frac{d^2 C}{dl\,dv}$, $\frac{d^2 C}{dv^2}$. If on the other hand we particularise it by making it coincide with the normal to the curve, we have

$$\epsilon = \tfrac{1}{2}\pi,$$

and then $U = dv^2$: the values of $\frac{d_2 p}{dv}$ and $\frac{d_2 q}{dv}$ have been given, and alternative expressions for the three foregoing magnitudes arise.

It is proper to add, for the sake of reference, the former equations giving ρ_0, the radius of circular curvature of the curve, and the inclination ϖ of that radius to the radius of circular curvature of the geodesic; they are

$$\frac{\cos\varpi}{\rho_0} = \frac{1}{\rho}, \quad \frac{\sin\varpi}{\rho_0} = \frac{1}{\gamma},$$

where $1/\gamma$ is the geodesic curvature of the curve (§§ 216, 222).

CHAPTER XXVII.

CONCOMITANTS OF A REGION, OF SURFACES AND OF CURVES IN THE REGION.

Concomitants of a region in homaloidal quadruple space.

473. As a third application of this theory of invariantive forms, we shall consider the concomitants of a region in quadruple homaloidal space, restricting the consideration to those of lowest grade in the derived magnitudes. Among the concomitants to be expected, there are several obvious classes.

There will be a class, analogous to invariants in the theory of ternary forms; here, they will be found to be appropriate combinations of the derivatives of the point-coordinates, such (for example) as the fundamental magnitudes which belong to the region as a whole and are independent of all directions and all orientations within the region.

There will be another class, analogous to covariants in the theory of ternary forms; they will be appropriate for the expression of directions within the region, and may be characteristic of curves either organic (such as geodesics) or arbitrary. For such concomitants, we shall require certainly one set of variables

$$dp = t, \quad dq = u, \quad dr = w;$$

and for different directions, there will be sets of variables congruent with t, u, w, there being one set for each direction.

There will be another class analogous to contravariants in the theory of ternary forms; they will be appropriate for the expression of orientations of surfaces within the region and for the formulation of characteristic properties of such surfaces. They can occur independently, as orientated elements of area, or they can be constructed by the appropriate combinations of two sets of direction-variables: we denote them by ξ, η, ζ, and (assuming them to be constructed from variables of direction) we define them, as in § 445, by the relations

$$\xi = \begin{vmatrix} \delta q, & \delta r \\ \delta' q, & \delta' r \end{vmatrix}, \quad \eta = \begin{vmatrix} \delta r, & \delta p \\ \delta' r, & \delta' p \end{vmatrix}, \quad \zeta = \begin{vmatrix} \delta p, & \delta q \\ \delta' p, & \delta' q \end{vmatrix}.$$

Clearly such a combination as

$$\xi t + \eta u + \zeta w$$

is a covariant, for it is equivalent to

$$\begin{vmatrix} dp, & dq, & dr \\ \delta p, & \delta q, & \delta r \\ \delta' p, & \delta' q, & \delta' r \end{vmatrix}.$$

In this connection, it is to be noted that a surface within the region may be given by a relation

$$\Phi(p, q, r) = \text{constant};$$

in that event, two distinct directions in the surface are such that

$$\Phi_1 \delta p + \Phi_2 \delta q + \Phi_3 \delta r = 0,$$

$$\Phi_1 \delta' p + \Phi_2 \delta' q + \Phi_3 \delta' r = 0,$$

so that we could take

$$\Phi_1 = \xi, \quad \Phi_2 = \eta, \quad \Phi_3 = \zeta,$$

that is, derivatives of any function $\Phi(p, q, r)$ are cogredient with the surface-variables ξ, η, ζ, and are contragredient to the direction-variables t, u, w.

Finally, there will be a comprehensive class of functions, analogous to mixed concomitants in the theory of ternary forms. In their expression there will occur, simultaneously, the different kinds of variables introduced by the preceding classes: that is, there will occur magnitudes belonging to the region as a whole, there will occur the line-direction variables t, u, w, and there will occur the surface-orientation variables ξ, η, ζ; also, there may be distinct sets of such variables.

Equations characteristic of orientational invariance.

474. The invariantive equations which secure the invariance of a function for all orientational displacement within the quadruple space have already been given, their detailed form depending partially upon the sub-amplitudes within that space. In the present instance, when regions are being discussed, so that the point-coordinates of quadruple space are functions of three parameters, the equations given in § 448 require the explicit introduction of derivatives with respect to the three parameters, so that, for instance within the first grade, we should have

$$\left\{ y \frac{\partial}{\partial x} \right\} = y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + y_3 \frac{\partial}{\partial x_3}.$$

The same symbol $\left\{ y \frac{\partial}{\partial x} \right\}$ will be used to cover the occurrence of derivatives of all grades: within each grade, every derivative of x of that grade must occur, of course in combination with the appropriate derivative of y. And so

for the other variables. The orientational equations, characteristic of invariants, are formally the same as before, viz.,

$$\left.\begin{aligned} E_{xy} &= \left\{y\,\frac{\partial}{\partial x}\right\} - \left\{x\,\frac{\partial}{\partial y}\right\} = 0 \\ E_{yz} &= \left\{z\,\frac{\partial}{\partial y}\right\} - \left\{y\,\frac{\partial}{\partial z}\right\} = 0 \\ E_{zx} &= \left\{x\,\frac{\partial}{\partial z}\right\} - \left\{z\,\frac{\partial}{\partial x}\right\} = 0 \\ E_{xv} &= \left\{v\,\frac{\partial}{\partial x}\right\} - \left\{x\,\frac{\partial}{\partial v}\right\} = 0 \\ E_{yv} &= \left\{v\,\frac{\partial}{\partial y}\right\} - \left\{y\,\frac{\partial}{\partial v}\right\} = 0 \\ E_{zv} &= \left\{v\,\frac{\partial}{\partial z}\right\} - \left\{z\,\frac{\partial}{\partial v}\right\} = 0 \end{aligned}\right\},$$

being six in number.

Equations characteristic of parametric invariance.

475. The equations characteristic of parametric invariance are constructed in the same manner as before. The parameters are subjected to the general infinitesimal transformations

$$p' = p + \epsilon\phi, \quad q' = q + \epsilon\chi, \quad r' = r + \epsilon\psi,$$

where ϕ, χ, ψ, are three arbitrary independent functions of p, q, r. The law of transformation of the line-variables t, u, w, is

$$\left.\begin{aligned} t' - t &= \epsilon\,(t\phi_1 + u\phi_2 + w\phi_3) \\ u' - u &= \epsilon\,(t\chi_1 + u\chi_2 + w\chi_3) \\ w' - w &= \epsilon\,(t\psi_1 + u\psi_2 + w\psi_3) \end{aligned}\right\};$$

and the law of transformation of the surface-variables ξ, η, ζ, is

$$\left.\begin{aligned} \xi' - \xi &= \epsilon\,\{\xi\,(\chi_2 + \psi_3) \quad - \eta\chi_1 \quad - \zeta\psi_1\,\} \\ \eta' - \eta &= \epsilon\,\{\quad - \xi\phi_2 + \eta\,(\psi_3 + \phi_1) \quad - \zeta\psi_2\,\} \\ \zeta' - \zeta &= \epsilon\,\{\quad - \xi\phi_3 \quad - \eta\chi_3 + \zeta\,(\phi_1 + \chi_2)\} \end{aligned}\right\}.$$

The law of transformation for the derivatives of any one of the four space coördinates x, y, z, v, and for the derivatives of any function of the parameters which itself, like x, y, z, v, remains unaltered by the transformation of parameters, is (when θ denotes x, y, z, v, or such a function, § 451)

$$\theta'_{\alpha\beta\gamma} - \theta_{\alpha\beta\gamma} = -\epsilon\Sigma\Sigma\Sigma \binom{\alpha}{l}\binom{\beta}{m}\binom{\gamma}{n}\Theta(\theta),$$

where

$$\Theta(\theta) = \theta_{\alpha+1-l,\,\beta-m,\,\gamma-n}\phi_{lmn} + \theta_{\alpha-l,\,\beta+1-m,\,\gamma-n}\chi_{lmn} + \theta_{\alpha-l,\,\beta-m,\,\gamma+1-n}\psi_{lmn},$$

where, in the triple summation,

$$0 \leqslant l \leqslant \alpha, \quad 0 \leqslant m \leqslant \beta, \quad 0 \leqslant n \leqslant \gamma, \quad l+m+n > 0,$$

and, for the combinations α and l, β and m, γ and n,

$$\binom{i}{j} = \frac{i!}{(i-j)!\,j!}.$$

We shall deal with the concomitants of the first grade and of the second grade, quite generally.

Concomitants of grade unity.

476. For concomitants of the first grade, we shall continue to use x_1, x_2, x_3, instead of x_{100}, x_{010}, x_{001}, so as to denote the derivatives of x; likewise for y, z, v, and also for any typical function such as θ, which will be retained so as to represent a typical surface, $\theta(p, q, r) = \text{constant}$, lying in the region. Should the consideration of curves arise, they will be supposed given as the intersection of two surfaces, $\theta = \text{constant}$, $\Im = \text{constant}$. Consequently, for concomitants of the first grade, the total number of arguments entering into the characteristic equations is

3, for the linear variables t, u, w;

3, ... superficial variables ξ, η, ζ;

12, ... first derivatives of x, y, z, v, with respect to p, q, r;

3, ... the function $\theta(p, q, r)$, for surfaces;

3, ... $\Im(p, q, r)$, to admit curves;

being twenty-four in all.

The law of parametric infinitesimal transformation for the first derivatives of x, y, z, v, θ, $\Im$, is given by the equations

$$\left.\begin{aligned} \theta_1' - \theta_1 &= -\epsilon(\theta_1\phi_1 + \theta_2\chi_1 + \theta_3\psi_1) \\ \theta_2' - \theta_2 &= -\epsilon(\theta_1\phi_2 + \theta_2\chi_2 + \theta_3\psi_2) \\ \theta_3' - \theta_3 &= -\epsilon(\theta_1\phi_3 + \theta_2\chi_3 + \theta_3\psi_3) \end{aligned}\right\}.$$

We can now form the equations which are characteristic of parametric invariance. The definition of any concomitant Θ is implied in the relation

$$\Theta = \Theta' J^{\mu},$$

where Θ is a function of any set, or of all, of the possible arguments, Θ' is the same function of the included transformed arguments, μ is a constant (usually an integer), and J is the modulus of transformation of the parameters which for the infinitesimal transformation is

$$J = 1 + \epsilon(\phi_1 + \chi_2 + \psi_3).$$

Accordingly, the characteristic equations are

$$\left.\begin{aligned} \phi_2, \ \ldots\ldots\ \Sigma x_1 \frac{\partial}{\partial x_2} + \theta_1 \frac{\partial}{\partial \theta_2} + \vartheta_1 \frac{\partial}{\partial \vartheta_2} &= u \frac{\partial}{\partial t} - \xi \frac{\partial}{\partial \eta} \\ \phi_3, \ \ldots\ldots\ \Sigma x_1 \frac{\partial}{\partial x_3} + \theta_1 \frac{\partial}{\partial \theta_3} + \vartheta_1 \frac{\partial}{\partial \vartheta_3} &= w \frac{\partial}{\partial t} - \xi \frac{\partial}{\partial \zeta} \end{aligned}\right\},$$

$$\left.\begin{aligned} \chi_3, \ \ldots\ldots\ \Sigma x_2 \frac{\partial}{\partial x_3} + \theta_2 \frac{\partial}{\partial \theta_3} + \vartheta_2 \frac{\partial}{\partial \vartheta_3} &= w \frac{\partial}{\partial u} - \eta \frac{\partial}{\partial \zeta} \\ \chi_1, \ \ldots\ldots\ \Sigma x_2 \frac{\partial}{\partial x_1} + \theta_2 \frac{\partial}{\partial \theta_1} + \vartheta_2 \frac{\partial}{\partial \vartheta_1} &= t \frac{\partial}{\partial u} - \eta \frac{\partial}{\partial \xi} \end{aligned}\right\},$$

$$\left.\begin{aligned} \psi_1, \ \ldots\ldots\ \Sigma x_3 \frac{\partial}{\partial x_1} + \theta_3 \frac{\partial}{\partial \theta_1} + \vartheta_3 \frac{\partial}{\partial \vartheta_1} &= t \frac{\partial}{\partial w} - \zeta \frac{\partial}{\partial \xi} \\ \psi_2, \ \ldots\ldots\ \Sigma x_3 \frac{\partial}{\partial x_2} + \theta_3 \frac{\partial}{\partial \theta_2} + \vartheta_3 \frac{\partial}{\partial \vartheta_2} &= u \frac{\partial}{\partial w} - \zeta \frac{\partial}{\partial \eta} \end{aligned}\right\},$$

the symbols on the left-hand side indicating the derivatives of the arbitrary functions in the transformations from which the equations are derived; and

$$\phi_1, \ \ldots\ldots\ \Sigma x_1 \frac{\partial \Theta}{\partial x_1} + \theta_1 \frac{\partial \Theta}{\partial \theta_1} + \vartheta_1 \frac{\partial \Theta}{\partial \vartheta_1} - t \frac{\partial \Theta}{\partial t} + \xi \frac{\partial \Theta}{\partial \xi} = \mu \Theta,$$

$$\chi_2, \ \ldots\ldots\ \Sigma x_2 \frac{\partial \Theta}{\partial x_2} + \theta_2 \frac{\partial \Theta}{\partial \theta_2} + \vartheta_2 \frac{\partial \Theta}{\partial \vartheta_2} - u \frac{\partial \Theta}{\partial u} + \eta \frac{\partial \Theta}{\partial \eta} = \mu \Theta,$$

$$\psi_3, \ \ldots\ldots\ \Sigma x_3 \frac{\partial \Theta}{\partial x_3} + \theta_3 \frac{\partial \Theta}{\partial \theta_3} + \vartheta_3 \frac{\partial \Theta}{\partial \vartheta_3} - w \frac{\partial \Theta}{\partial w} + \zeta \frac{\partial \Theta}{\partial \zeta} = \mu \Theta,$$

leading to two homogeneous equations

$$\left.\begin{aligned} \Sigma x_1 \frac{\partial}{\partial x_1} + \theta_1 \frac{\partial}{\partial \theta_1} + \vartheta_1 \frac{\partial}{\partial \vartheta_1} - t \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial \xi} & \\ = \Sigma x_2 \frac{\partial}{\partial x_2} + \theta_2 \frac{\partial}{\partial \theta_2} + \vartheta_2 \frac{\partial}{\partial \vartheta_2} - u \frac{\partial}{\partial u} + \eta \frac{\partial}{\partial \eta} & \\ = \Sigma x_3 \frac{\partial}{\partial x_3} + \theta_3 \frac{\partial}{\partial \theta_3} + \vartheta_3 \frac{\partial}{\partial \vartheta_3} - w \frac{\partial}{\partial w} + \zeta \frac{\partial}{\partial \zeta} & \end{aligned}\right\},$$

which, together with the preceding three pairs, constitute a complete Jacobian system of eight equations. In each instance, the summation is over the four variables x, y, z, v.

Orientational equations for concomitants of grade unity.

477. The equations of orientational invariance do not affect the variables $t, u, w; \xi, \eta, \zeta$; and they do not affect the derivatives of the functions θ

and $\mathfrak{H}$. For the concomitants of the first grade, they are

$$\left.\begin{aligned}
E_{xy} &= \Sigma y_1 \frac{\partial}{\partial x_1} - \Sigma x_1 \frac{\partial}{\partial y_1} = 0 \\
E_{yz} &= \Sigma z_1 \frac{\partial}{\partial y_1} - \Sigma y_1 \frac{\partial}{\partial z_1} = 0 \\
E_{zx} &= \Sigma x_1 \frac{\partial}{\partial z_1} - \Sigma z_1 \frac{\partial}{\partial x_1} = 0 \\
E_{xv} &= \Sigma v_1 \frac{\partial}{\partial x_1} - \Sigma x_1 \frac{\partial}{\partial v_1} = 0 \\
E_{yv} &= \Sigma v_1 \frac{\partial}{\partial y_1} - \Sigma y_1 \frac{\partial}{\partial v_1} = 0 \\
E_{zv} &= \Sigma v_1 \frac{\partial}{\partial z_1} - \Sigma z_1 \frac{\partial}{\partial v_1} = 0
\end{aligned}\right\},$$

the summation in each instance being over the three suffixes 1, 2, 3. There are six equations in the set: and, within themselves, they also are a complete Jacobian system.

The two sets, being the set of eight parametric equations and the set of six orientational equations, combine to form a full Jacobian system of fourteen equations. As the total number of arguments is twenty-four, we shall ultimately require ten algebraically independent integrals in order to have the means adequate for the expression of all concomitants of the first grade.

If there were no functions θ and $\mathfrak{H}$, that is, if we consider first-grade concomitants of the region alone, there would be the eighteen arguments $t, u, w; \xi, \eta, \zeta; x_l, y_l, z_l, v_l$, for $l = 1, 2, 3$. We then should ultimately require four independent integrals, as an aggregate of concomitants of the first grade, belonging to the region alone and sufficient for the algebraical expression of any regional concomitant of the first grade.

Regional concomitants.

478. In the first place, let an adequate set of concomitants be selected, appertaining to the whole region.

There are eight equations in the complete Jacobian parametric group, and they involve eighteen regional arguments; consequently, ten algebraically independent integrals are required. Manifestly the universal concomitant t_ξ, where

$$t_\xi = t\xi + u\eta + w\zeta,$$

can be taken as one of these; and, thereafter, nine integrals are required.

The forms of the eight parametric equations shew that every integral is a concomitant of the set of linear ternary forms $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{v}$, where

$$\left.\begin{aligned} \bar{x} &= x_1 t + x_2 u + x_3 w \\ \bar{y} &= y_1 t + y_2 u + y_3 w \\ \bar{z} &= z_1 t + z_2 u + z_3 w \\ \bar{v} &= v_1 t + v_2 u + v_3 w \end{aligned}\right\}.$$

Among the simultaneous concomitants of these forms, there are all the quantities

$$\left\| \begin{matrix} \xi, & x_1, & y_1, & z_1, & v_1 \\ \eta, & x_2, & y_2, & z_2, & v_2 \\ \zeta, & x_3, & y_3, & z_3, & v_3 \end{matrix} \right\|.$$

From among these quantities, we select four invariants

$$J_x, J_y, J_z, J_v = \left\| \begin{matrix} x_1, & y_1, & z_1, & v_1 \\ x_2, & y_2, & z_2, & v_2 \\ x_3, & y_3, & z_3, & v_3 \end{matrix} \right\|,$$

with the customary notation of § 263. Then we select six contravariants

$$\alpha = \begin{vmatrix} \xi, & \eta, & \zeta \\ y_1, & y_2, & y_3 \\ z_1, & z_2, & z_3 \end{vmatrix}, \quad \beta = \begin{vmatrix} \xi, & \eta, & \zeta \\ z_1, & z_2, & z_3 \\ x_1, & x_2, & x_3 \end{vmatrix}, \quad \gamma = \begin{vmatrix} \xi, & \eta, & \zeta \\ x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \end{vmatrix},$$

$$\upsilon = \begin{vmatrix} \xi, & \eta, & \zeta \\ x_1, & x_2, & x_3 \\ v_1, & v_2, & v_3 \end{vmatrix}, \quad \iota = \begin{vmatrix} \xi, & \eta, & \zeta \\ y_1, & y_2, & y_3 \\ v_1, & v_2, & v_3 \end{vmatrix}, \quad \omega = \begin{vmatrix} \xi, & \eta, & \zeta \\ z_1, & z_2, & z_3 \\ v_1, & v_2, & v_3 \end{vmatrix}.$$

The fourteen quantities thus selected, viz.

$$\bar{x}, \bar{y}, \bar{z}, \bar{v}, J_x, J_y, J_z, J_v, \alpha, \beta, \gamma, \upsilon, \iota, \omega,$$

are not algebraically independent; for there exist identical relations

$$\begin{aligned} \bar{x}J_x + \bar{y}J_y + \bar{z}J_z + \bar{v}J_v &= 0, \\ \alpha\upsilon + \beta\iota + \gamma\omega &= 0, \\ \omega\bar{y} - \iota\bar{z} + \alpha\bar{v} &= J_x t_\xi, \\ -\omega\bar{x} + \upsilon\bar{z} + \beta\bar{v} &= J_y t_\xi, \\ \iota\bar{x} - \upsilon\bar{y} + \gamma\bar{v} &= J_z t_\xi, \\ -\alpha\bar{x} - \beta\bar{y} - \gamma\bar{z} &= J_v t_\xi, \end{aligned}$$

the last four being equivalent to three only, because of the first—that is, there are five distinct relations. Consequently nine necessary independent integrals are implicitly included in the preceding selection of fourteen quantities.

The various quantities are to be made the operating variables in the six equations of orientational invariance. To effect the transformation, we construct the tableau of operations:

	$\bar{x}$	$\bar{y}$	$\bar{z}$	$\bar{v}$	J_x	J_y	J_z	J_v	α	β	γ	υ	ι	ω
E_{xy}	$\bar{y}$	$-\bar{x}$	.	.	J_y	$-J_x$	.	.	β	$-\alpha$	.	ι	$-\upsilon$	.
E_{yz}	.	$\bar{z}$	$-\bar{y}$	.	.	J_z	$-J_y$	.	.	γ	$-\beta$	.	ω	$-\iota$
E_{zx}	$-\bar{z}$	.	$\bar{x}$	.	$-J_z$	.	J_x	.	$-\gamma$	.	α	$-\omega$	.	υ
E_{xv}	$\bar{v}$	.	.	$-\bar{x}$	J_v	.	.	$-J_x$	.	ω	$-\iota$	.	γ	$-\beta$
E_{yv}	.	$\bar{v}$	.	$-\bar{y}$	.	J_v	.	$-J_y$	$-\omega$	.	υ	$-\gamma$	.	α
E_{zv}	.	.	$\bar{v}$	$-\bar{z}$	.	.	J_v	$-J_z$	ι	$-\upsilon$	.	β	$-\alpha$	.

implying

$$E_{xy}\bar{x}=\bar{y}, \quad E_{xy}\bar{y}=-\bar{x}, \quad E_{xy}\bar{z}=0, \quad E_{xy}\bar{v}=0,$$

and so on through the tableau. Then the six equations of orientational invariance become

$$\left.\begin{aligned}
&\bar{y}\frac{\partial}{\partial\bar{x}}-\bar{x}\frac{\partial}{\partial\bar{y}}+J_y\frac{\partial}{\partial J_x}-J_x\frac{\partial}{\partial J_y}+\beta\frac{\partial}{\partial\alpha}-\alpha\frac{\partial}{\partial\beta}+\iota\frac{\partial}{\partial\upsilon}-\upsilon\frac{\partial}{\partial\iota}=0\\
&\bar{z}\frac{\partial}{\partial\bar{y}}-\bar{y}\frac{\partial}{\partial\bar{z}}+J_z\frac{\partial}{\partial J_y}-J_y\frac{\partial}{\partial J_z}+\gamma\frac{\partial}{\partial\beta}-\beta\frac{\partial}{\partial\gamma}+\omega\frac{\partial}{\partial\iota}-\iota\frac{\partial}{\partial\omega}=0\\
&-\bar{z}\frac{\partial}{\partial\bar{x}}+\bar{x}\frac{\partial}{\partial\bar{z}}-J_z\frac{\partial}{\partial J_x}+J_x\frac{\partial}{\partial J_z}-\gamma\frac{\partial}{\partial\alpha}+\alpha\frac{\partial}{\partial\gamma}-\omega\frac{\partial}{\partial\upsilon}+\upsilon\frac{\partial}{\partial\omega}=0\\
&\bar{v}\frac{\partial}{\partial\bar{x}}-\bar{x}\frac{\partial}{\partial\bar{v}}+J_v\frac{\partial}{\partial J_x}-J_x\frac{\partial}{\partial J_v}+\omega\frac{\partial}{\partial\beta}-\beta\frac{\partial}{\partial\omega}-\iota\frac{\partial}{\partial\gamma}+\gamma\frac{\partial}{\partial\iota}=0\\
&\bar{v}\frac{\partial}{\partial\bar{y}}-\bar{y}\frac{\partial}{\partial\bar{v}}+J_v\frac{\partial}{\partial J_y}-J_y\frac{\partial}{\partial J_v}-\omega\frac{\partial}{\partial\alpha}+\alpha\frac{\partial}{\partial\omega}+\upsilon\frac{\partial}{\partial\gamma}-\gamma\frac{\partial}{\partial\upsilon}=0\\
&\bar{v}\frac{\partial}{\partial\bar{z}}-\bar{z}\frac{\partial}{\partial\bar{v}}+J_v\frac{\partial}{\partial J_z}-J_z\frac{\partial}{\partial J_v}+\iota\frac{\partial}{\partial\alpha}-\alpha\frac{\partial}{\partial\iota}-\upsilon\frac{\partial}{\partial\beta}+\beta\frac{\partial}{\partial\upsilon}=0
\end{aligned}\right\}$$

Though fourteen quantities apparently are involved, they are equivalent to only nine algebraically independent quantities; and therefore, as there are six equations to be satisfied forming a complete Jacobian set in themselves, we require three independent integrals.

Obvious integrals are

$$\bar{x}^2+\bar{y}^2+\bar{z}^2+\bar{v}^2,$$

$$\bar{x}J_x+\bar{y}J_y+\bar{z}J_z+\bar{v}J_v,$$

$$J_x^2+J_y^2+J_z^2+J_v^2,$$

$$\alpha^2+\beta^2+\gamma^2+\upsilon^2+\iota^2+\omega^2.$$

Of these, the second vanishes identically. The remaining three are algebraically independent of one another: they are taken as the three necessary integrals which, with t_ξ, constitute the required aggregate of four integrals. They are

$$U = \Sigma \bar{x}^2 = (A, B, C, F, G, H)(t, u, w)^2,$$

$$\Sigma J_x^2 = \begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix} = \Omega^2,$$

$$T = \Sigma \alpha^2 = (a, b, c, f, g, h)(\xi, \eta, \zeta)^2,$$

where the symbols A, B, C, F, G, H, possess their customary significance

$$A = \Sigma x_1^2, \quad B = \Sigma x_2^2, \quad C = \Sigma x_3^2,$$
$$F = \Sigma x_2 x_3, \quad G = \Sigma x_3 x_1, \quad H = \Sigma x_1 x_2,$$

and a, b, c, f, g, h, represent their respective cofactors in the determinant denoted by Ω^2. These, with

$$t_\xi = \xi t + \eta u + \zeta w,$$

constitute an aggregate of concomitants of the first grade appertaining to the region alone.

Effect of the introduction of a surface.

479. We now consider the concomitants which must be associated with the system

$$U, \quad \Omega^2, \quad T, \quad t_\xi,$$

owing to the introduction of a single surface $\theta(p, q, r) = \text{constant}$. Additional derivatives $\theta_1, \theta_2, \theta_3$, are thus to be included: and additional surface-variables $\bar{\xi}, \bar{\eta}, \bar{\zeta}$, also arise, these being cogredient with the derivatives under relations (§ 268)

$$\frac{\bar{\xi}}{\theta_1} = \frac{\bar{\eta}}{\theta_2} = \frac{\bar{\zeta}}{\theta_3} = \frac{d\Sigma}{D^{\frac{1}{2}}},$$

where $d\Sigma$ is an element of area on the surface, and where (§ 268)

$$D = \Omega^2 \left(\frac{d\theta}{dv}\right)^2.$$

Both sets of variables are retained, for the sake of formal completeness.

The effect, upon the system, of the introduction of these variables into the parametric equations is to require the association of a new linear ternary form

$$\bar{\theta} = \theta_1 t + \theta_2 u + \theta_3 w,$$

together with its own special concomitants and with the concomitants that belong to $\bar{\theta}$ and to the earlier forms $\bar{x}, \bar{y}, \bar{z}, \bar{v}$, simultaneously. Among these are to be noted

$$t_{\bar{\xi}} = \bar{\xi} t + \bar{\eta} u + \bar{\zeta} w,$$

and the set of quantities $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$, $\bar{\upsilon}$, $\bar{\iota}$, $\bar{\omega}$, of the type

$$\bar{\alpha} = \begin{vmatrix} \bar{\xi}, & \bar{\eta}, & \bar{\zeta} \\ y_1, & y_2, & y_3 \\ z_1, & z_2, & z_3 \end{vmatrix},$$

being the result of substituting $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$, for ξ, η, ζ, in α, β, γ, υ, ι, ω. These quantities are subject to the relations

$$\begin{aligned} \bar{\alpha}\bar{\upsilon} + \bar{\beta}\bar{\iota} + \bar{\gamma}\bar{\omega} &= 0, \\ \bar{\omega}\bar{y} - \bar{\iota}\bar{z} + \bar{\alpha}\bar{\upsilon} &= J_x t_{\bar{\xi}}, \\ -\bar{\omega}\bar{x} + \bar{\upsilon}\bar{z} + \bar{\beta}\bar{\upsilon} &= J_y t_{\bar{\xi}}, \\ \bar{\iota}\bar{x} - \bar{\upsilon}\bar{y} + \bar{\gamma}\bar{\upsilon} &= J_z t_{\bar{\xi}}, \\ -\bar{\alpha}\bar{x} - \bar{\beta}\bar{y} - \bar{\gamma}\bar{z} &= J_v t_{\bar{\xi}}, \end{aligned}$$

the last four being equivalent to only three relations in virtue of the former identity

$$\bar{x}J_x + \bar{y}J_y + \bar{z}J_z + \bar{\upsilon}J_v = 0.$$

Corresponding terms in these six quantities now occur in the orientational equations. Proceeding exactly as before, we obtain, in addition to $\bar{\theta}$ and to $t_{\bar{\xi}}$, further concomitants

$$\begin{aligned} Q' &= (a, b, c, f, g, h \mathbin{\text{\textbigcirc\!\!\!\!\!\!\backslash}} \xi, \eta, \zeta \mathbin{\text{\textbigcirc\!\!\!\!\!\!\backslash}} \bar{\xi}, \bar{\eta}, \bar{\zeta}), \\ Q &= (a, b, c, f, g, h \mathbin{\text{\textbigcirc\!\!\!\!\!\!\backslash}} \xi, \eta, \zeta \mathbin{\text{\textbigcirc\!\!\!\!\!\!\backslash}} \theta_1, \theta_2, \theta_3), \\ \bar{T} &= (a, b, c, f, g, h \mathbin{\text{\textbigcirc\!\!\!\!\!\!\backslash}} \bar{\xi}, \bar{\eta}, \bar{\zeta})^2, \\ D &= (a, b, c, f, g, h \mathbin{\text{\textbigcirc\!\!\!\!\!\!\backslash}} \theta_1, \theta_2, \theta_3)^2. \end{aligned}$$

There thus arise six new concomitants sufficient to complete the system, in virtue of the introduction of the six new arguments θ_1, θ_2, θ_3, $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$; and these concomitants can be taken pair-wise, corresponding to the pair-wise equivalence θ_1 and $\bar{\xi}$, θ_2 and $\bar{\eta}$, θ_3 and $\bar{\zeta}$, of these arguments.

Geometrical interpretation.

480. It is convenient to state the geometrical interpretation of these concomitants, so far obtained: their significance has arisen through their earlier occurrence (§§ 267, 268). We have

$$\begin{aligned} U &= (A, B, C, F, G, H \mathbin{\text{\textbigcirc\!\!\!\!\!\!\backslash}} t, u, w)^2 \\ &= (A, B, C, F, G, H \mathbin{\text{\textbigcirc\!\!\!\!\!\!\backslash}} dp, dq, dr)^2 \\ &= ds^2, \end{aligned}$$

where ds is the arc-element of the region, in the direction dp, dq, dr. Also

$$\Omega^2 = ABC + 2FGH - AF^2 - BG^2 - CH^2$$

is the discriminant of U regarded as a quadratic ternary form. Again,

$$\begin{aligned} T &= (a, b, c, f, g, h\mathfrak{X}\xi, \eta, \zeta)^2 \\ &= dS^2, \end{aligned}$$

where dS is the element of area in the region, having ξ, η, ζ, for its superficial coordinates.

Again,

$$\begin{aligned} t_\xi &= \xi t + \eta u + \zeta w \\ &= \begin{vmatrix} dp\ , & dq\ , & dr \\ d_1p, & d_1q, & d_1r \\ d_2p, & d_2q, & d_2r \end{vmatrix}. \end{aligned}$$

Consider an element of volume in the region. Let the coordinates of three corners adjacent to x, y, z, v, and relative to that point, be

$$dx\ ,\ dy\ ,\ dz\ ,\ dv\ ;$$

$$d_1x,\ d_1y,\ d_1z,\ d_1v;$$

$$d_2x,\ d_2y,\ d_2z,\ d_2v.$$

The projection of this volume into the flat $v=0$ is

$$\begin{vmatrix} dx\ , & dy\ , & dz \\ d_1x, & d_1y, & d_1z \\ d_2x, & d_2y, & d_2z \end{vmatrix},$$

which is equal to

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} t_\xi :$$

that is, it is equal to $J_v t_\xi$; and similarly the projections on the other three flats of reference are $J_x t_\xi$, $J_y t_\xi$, $J_z t_\xi$. Hence if $\square$ denote this element of volume, we have

$$\begin{aligned} \square^2 &= \Sigma \begin{vmatrix} dx\ , & dy\ , & dz \\ d_1x, & d_1y, & d_1z \\ d_2x, & d_2y, & d_2z \end{vmatrix}^2 \\ &= t_\xi^2 \Sigma J_x^2 = \Omega^2 t_\xi^2 : \end{aligned}$$

that is,

$$\Omega t_\xi = \square.$$

Next, we have

$$\begin{aligned} \bar{\theta} &= \theta_1 t + \theta_2 u + \theta_3 w \\ &= \theta_1 dp + \theta_2 dq + \theta_3 dr, \end{aligned}$$

and so

$$U^{-\frac{1}{2}}\bar{\theta} = \theta_1 p' + \theta_2 q' + \theta_3 r' = \frac{d\theta}{ds},$$

being the arc-variation of the value of θ along the general direction denoted by ds. In § 268 it has been proved that

$$\begin{aligned} Q' &= (a, b, c, f, g, h\nabla\xi, \eta, \zeta\nabla\bar{\xi}, \bar{\eta}, \bar{\zeta}) \\ &= dS \,.\, d\Sigma \,.\, \cos\epsilon, \\ Q &= (a, b, c, f, g, h\nabla\xi, \eta, \zeta\nabla\theta_1, \theta_2, \theta_3) \\ &= \Omega \frac{d\theta}{d\nu} dS \,.\, \cos\epsilon, \\ T &= (a, b, c, f, g, h\nabla\bar{\xi}, \bar{\eta}, \bar{\zeta})^2 \\ &= d\Sigma^2, \\ D &= (a, b, c, f, g, h\nabla\theta_1, \theta_2, \theta_3)^2 \\ &= \Omega^2 \left(\frac{d\theta}{d\nu}\right)^2, \end{aligned}$$

where $d\nu$ is an element of length normal to the surface $\theta(p, q, r) = \text{constant}$, $d\Sigma$ is an element of area on that surface, and ϵ is the inclination of the elementary areas dS and $d\Sigma$. Manifestly, these are not independent of one another: we have

$$DQ'^2 = \bar{T}Q^2,$$

a property arising from the relations connecting $\theta_1, \theta_2, \theta_3$, with $\bar{\xi}, \bar{\eta}, \bar{\zeta}$.

Finally, if δ denote the inclination between the general arc-element ds and the arc-element $d\nu$ normal to the surface, we have

$$ds\, d\nu \cos\delta = \Sigma A\, dp\, \delta p,$$

that is,

$$\begin{aligned} ds\cos\delta &= \Sigma At \frac{dp}{d\nu} \\ &= \frac{1}{\Omega\, d\Sigma} \Sigma (At + Hu + Gw)(a\bar{\xi} + h\bar{\eta} + g\bar{\zeta}) \\ &= \frac{\Omega}{d\Sigma} t_{\bar{\xi}}, \end{aligned}$$

or

$$\Omega t_{\bar{\xi}} = ds\, d\Sigma \cos\delta,$$

thus providing one further concomitant, and completing the aggregate required for the occurrence of a single surface.

Concomitants of grade unity, for a curve in the region.

481. When a general curve is included in the configuration so that, for the immediate purpose, it is represented analytically by two equations $\theta(p, q, r) = \text{constant}$, $\Theta(p, q, r) = \text{constant}$, then further concomitants must be associated with the preceding aggregate in order to make provision for additional magnitudes.

Manifestly, concomitants appertaining to $\mathfrak{Y}$ alone, precisely analogous to those which appertain to θ alone, must be included. Accordingly, let $d\Sigma'$ be an elementary area on the surface $\mathfrak{Y}$ at which the surface-variables are denoted by $\bar{\bar{\xi}}, \bar{\bar{\eta}}, \bar{\bar{\zeta}}$; let ϵ' be the inclination of $d\Sigma'$ to dS, and let ϖ be the inclination of $d\Sigma'$ to $d\Sigma$; let $d\nu'$ be an elementary arc-length in the region normal to the surface $\mathfrak{Y}$, and let δ' denote the angle between ds and $d\nu'$. Then we have various concomitants, as follows. There is a covariant

$$\bar{\mathfrak{Y}} = \mathfrak{Y}_1 t + \mathfrak{Y}_2 u + \mathfrak{Y}_3 w,$$

so that

$$U^{-\frac{1}{2}}\bar{\mathfrak{Y}} = \frac{d\mathfrak{Y}}{ds}.$$

There are contravariants

$$\begin{aligned}
\bar{Q}' &= (a, b, c, f, g, h\mathcal{Y}\xi, \eta, \zeta\mathcal{Y}\bar{\bar{\xi}}, \bar{\bar{\eta}}, \bar{\bar{\zeta}}) \\
&= dS \,.\, d\Sigma' \,.\, \cos\epsilon', \\
\bar{Q} &= (a, b, c, f, g, h\mathcal{Y}\xi, \eta, \zeta\mathcal{Y}\mathfrak{Y}_1, \mathfrak{Y}_2, \mathfrak{Y}_3) \\
&= \Omega \frac{d\mathfrak{Y}}{d\nu'} dS \,.\, \cos\epsilon', \\
\bar{\bar{T}} &= (a, b, c, f, g, h\mathcal{Y}\bar{\bar{\xi}}, \bar{\bar{\eta}}, \bar{\bar{\zeta}})^2 \\
&= d\Sigma'^2, \\
D' &= (a, b, c, f, g, h\mathcal{Y}\mathfrak{Y}_1, \mathfrak{Y}_2, \mathfrak{Y}_3)^2 \\
&= \Omega^2 \left(\frac{d\mathfrak{Y}}{d\nu'}\right)^2, \\
t\bar{\bar{\xi}} &= \bar{\bar{\xi}} t + \bar{\bar{\eta}} u + \bar{\bar{\zeta}} w \\
&= \Omega^{-1} ds\, d\Sigma' \cos\delta'.
\end{aligned}$$

Again, we have (§ 267)

$$\begin{aligned}
d\Sigma\, d\Sigma' \cos\varpi &= (a, b, c, f, g, h\mathcal{Y}\bar{\bar{\xi}}, \bar{\eta}, \bar{\zeta}\mathcal{Y}\bar{\bar{\xi}}, \bar{\bar{\eta}}, \bar{\bar{\zeta}}) \\
&= \frac{d\Sigma\, d\Sigma'}{D^{\frac{1}{2}} D'^{\frac{1}{2}}} (a, b, c, f, g, h\mathcal{Y}\theta_1, \theta_2, \theta_3\mathcal{Y}\mathfrak{Y}_1, \mathfrak{Y}_2, \mathfrak{Y}_3),
\end{aligned}$$

and therefore

$$\Omega^2 \frac{d\theta\, d\mathfrak{Y}}{d\nu\, d\nu'} \cos\varpi = (a, b, c, f, g, h\mathcal{Y}\theta_1, \theta_2, \theta_3\mathcal{Y}\mathfrak{Y}_1, \mathfrak{Y}_2, \mathfrak{Y}_3).$$

Among concomitants, simultaneous to θ and $\mathfrak{Y}$ and belonging to the curve of intersection of the two surfaces, there is the covariant expressing its arc-element. Along this arc-element, let T', U', W', $(= d'p, d'q, d'r)$ be the direction-variables, so that

$$\theta_1 T' + \theta_2 U' + \theta_3 W' = 0, \quad \mathfrak{Y}_1 T' + \mathfrak{Y}_2 U' + \mathfrak{Y}_3 W' = 0,$$

and therefore

$$T' = \mu \begin{vmatrix} \theta_2, & \theta_3 \\ \mathfrak{Y}_2, & \mathfrak{Y}_3 \end{vmatrix} = \mu j_1, \quad U' = \mu \begin{vmatrix} \theta_3, & \theta_1 \\ \mathfrak{Y}_3, & \mathfrak{Y}_1 \end{vmatrix} = \mu j_2, \quad W' = \mu \begin{vmatrix} \theta_1, & \theta_2 \\ \mathfrak{Y}_1, & \mathfrak{Y}_2 \end{vmatrix} = \mu j_3,$$

where μ is a multiplier to be determined. Let dl denote the arc-element of the curve of intersection, so that

$$dl^2 = \Sigma A T'^2 = \mu^2 \Sigma A j_1^2.$$

Now

$$\begin{aligned} DD' \sin^2 \varpi &= DD' - (D^{\frac{1}{2}} D'^{\frac{1}{2}} \cos \varpi)^2 \\ &= (\Sigma a \theta_1^2)(\Sigma a \vartheta_1^2) - (\Sigma a \theta_1 \vartheta_1)^2 \\ &= (bc - f^2) j_1^2 + \dots \\ &= \Omega^2 \Sigma A j_1^2; \end{aligned}$$

and therefore

$$dl^2 = \frac{\mu^2}{\Omega^2} DD' \sin^2 \varpi.$$

Consequently

$$\mu = \frac{\Omega \, dl}{D^{\frac{1}{2}} D'^{\frac{1}{2}} \sin \varpi};$$

and thus the regional direction-variables $\frac{T'}{dl}$, $\frac{U'}{dl}$, $\frac{W'}{dl}$, of the arc of intersection are

$$\frac{\Omega j_1}{D^{\frac{1}{2}} D'^{\frac{1}{2}} \sin \varpi}, \quad \frac{\Omega j_2}{D^{\frac{1}{2}} D'^{\frac{1}{2}} \sin \varpi}, \quad \frac{\Omega j_3}{D^{\frac{1}{2}} D'^{\frac{1}{2}} \sin \varpi}.$$

Concomitants, generally, of grade two.

482. Derivatives of x, y, z, v, of the second order enter into the construction of concomitants of the second grade; as in § 262, we denote these by x_{11}, x_{12}, x_{13}, and so on, and likewise for y, z, v, and for any function $\theta(p, q, r)$, where

$$x_{11} = \frac{\partial^2 x}{\partial p^2}, \quad x_{12} = \frac{\partial^2 x}{\partial p \partial q}, \quad x_{13} = \frac{\partial^2 x}{\partial p \partial r}.$$

The infinitesimal changes under the infinitesimal parametric transformations in the second derivatives of x, y, z, v, θ, are formally similar to one another; those for derivatives of x are

$$\begin{aligned} x'_{11} - x_{11} &= -\epsilon \{2(x_{11}\phi_1 + x_{12}\chi_1 + x_{13}\psi_1) + (x_1 \phi_{11} + x_2 \chi_{11} + x_3 \psi_{11})\}, \\ x'_{12} - x_{12} &= -\epsilon \{ (x_{12}\phi_1 + x_{22}\chi_1 + x_{23}\psi_1) + (x_{11}\phi_2 + x_{12}\chi_2 + x_{13}\psi_2) \\ &\qquad + (x_1 \phi_{12} + x_2 \chi_{12} + x_3 \psi_{12})\}, \\ x'_{13} - x_{13} &= -\epsilon \{ (x_{13}\phi_1 + x_{23}\chi_1 + x_{33}\psi_1) + (x_{11}\phi_3 + x_{12}\chi_3 + x_{13}\psi_3) \\ &\qquad + (x_1 \phi_{13} + x_2 \chi_{13} + x_3 \psi_{13})\}, \\ x'_{22} - x_{22} &= -\epsilon \{2(x_{12}\phi_2 + x_{22}\chi_2 + x_{23}\psi_2) + (x_1 \phi_{22} + x_2 \chi_{22} + x_3 \psi_{22})\}, \\ x'_{23} - x_{23} &= -\epsilon \{ (x_{13}\phi_2 + x_{23}\chi_2 + x_{33}\psi_2) + (x_{12}\phi_3 + x_{22}\chi_3 + x_{23}\psi_3) \\ &\qquad + (x_1 \phi_{23} + x_2 \chi_{23} + x_3 \psi_{23})\}, \\ x'_{33} - x_{33} &= -\epsilon \{2(x_{13}\phi_3 + x_{23}\chi_3 + x_{33}\psi_3) + (x_1 \phi_{33} + x_2 \chi_{33} + x_3 \psi_{33})\}. \end{aligned}$$

The arguments, which occur in the equations characteristic of concomitants of the second grade, are

12, for first derivatives of x, y, z, v,

3, ... $\theta(p, q, r)$,

3, ... $\vartheta(p, q, r)$,

24, for second derivatives of x, y, z, v,

6, ... θ,

6, ... ϑ,

that is, they are 54 in number: to these must be added the three line-variables

$$t, u, w\ (= dp, dq, dr),$$

and the three general superficial variables ξ, η, ζ, making 60 in all. The superficial variables $\bar{\xi}, \bar{\eta}, \bar{\zeta}$, will also occur in connection with θ: and the superficial variables $\bar{\bar{\xi}}, \bar{\bar{\eta}}, \bar{\bar{\zeta}}$, will also occur in connection with ϑ.

Arrangement of the parametric equations in two groups.

483. As regards the equations in the complete Jacobian system arising out of the infinitesimal transformations, there will be eight from the first derivatives of ϕ, χ, ψ,—there are initially nine, but three of these (as in § 476) involve a term $\mu\Theta$ arising from

$$\Theta = \Theta' \{1 + \mu\epsilon (\phi_1 + \chi_2 + \psi_3)\},$$

and subtraction leaves two homogeneous equations for the Jacobian system: and there will be eighteen from the second derivatives of ϕ, χ, ψ. Thus, in all, there are twenty-six partial differential equations in the complete system of parametric equations.

Further, there are the usual six orientational equations, a complete system in themselves: they combine, with the parametric system, to form a complete aggregate.

Thus there are thirty-two partial differential equations in this aggregate. As there are sixty arguments in all, we shall require twenty-eight algebraically independent integrals in order to constitute an aggregate of concomitants, complete up to the second grade inclusive.

It is convenient to arrange these in two classes, independent of the universal concomitant

$$t_\xi = \xi t + \eta u + \zeta w,$$

appertaining to any region. We may regard nine of the concomitants as appertaining preferentially to θ and other nine as appertaining preferentially to ϑ: these can also be modified so as to appertain to both θ and ϑ. Thus there remain nine concomitants, belonging solely to the fundamental region.

Of these nine, we already (§ 478) have obtained the three of the first grade U, Ω^2, T: so six others, of the second grade, must be obtained.

Of the equations to be satisfied by the concomitants, there are, first, the eighteen which, out of the infinitesimal variations, arise from the second derivatives of ϕ, χ, ψ, in the equation

$$\Theta = \Theta' \{1 + \mu\epsilon(\phi_1 + \chi_2 + \psi_3)\}.$$

They are:

$$\left.\begin{aligned} &\Sigma x_1 \frac{\partial}{\partial x_{11}} + \theta_1 \frac{\partial}{\partial \theta_{11}} + \vartheta_1 \frac{\partial}{\partial \vartheta_{11}} = 0, \quad \Sigma x_1 \frac{\partial}{\partial x_{23}} + \theta_1 \frac{\partial}{\partial \theta_{23}} + \vartheta_1 \frac{\partial}{\partial \vartheta_{23}} = 0, \\ &\Sigma x_1 \frac{\partial}{\partial x_{22}} + \theta_1 \frac{\partial}{\partial \theta_{22}} + \vartheta_1 \frac{\partial}{\partial \vartheta_{22}} = 0, \quad \Sigma x_1 \frac{\partial}{\partial x_{31}} + \theta_1 \frac{\partial}{\partial \theta_{31}} + \vartheta_1 \frac{\partial}{\partial \vartheta_{31}} = 0, \\ &\Sigma x_1 \frac{\partial}{\partial x_{33}} + \theta_1 \frac{\partial}{\partial \theta_{33}} + \vartheta_1 \frac{\partial}{\partial \vartheta_{33}} = 0, \quad \Sigma x_1 \frac{\partial}{\partial x_{12}} + \theta_1 \frac{\partial}{\partial \theta_{12}} + \vartheta_1 \frac{\partial}{\partial \vartheta_{12}} = 0, \end{aligned}\right\},$$

from the six derivatives of ϕ:

$$\left.\begin{aligned} &\Sigma x_2 \frac{\partial}{\partial x_{11}} + \theta_2 \frac{\partial}{\partial \theta_{11}} + \vartheta_2 \frac{\partial}{\partial \vartheta_{11}} = 0, \quad \Sigma x_2 \frac{\partial}{\partial x_{23}} + \theta_2 \frac{\partial}{\partial \theta_{23}} + \vartheta_2 \frac{\partial}{\partial \vartheta_{23}} = 0, \\ &\Sigma x_2 \frac{\partial}{\partial x_{22}} + \theta_2 \frac{\partial}{\partial \theta_{22}} + \vartheta_2 \frac{\partial}{\partial \vartheta_{22}} = 0, \quad \Sigma x_2 \frac{\partial}{\partial x_{31}} + \theta_2 \frac{\partial}{\partial \theta_{31}} + \vartheta_2 \frac{\partial}{\partial \vartheta_{31}} = 0, \\ &\Sigma x_2 \frac{\partial}{\partial x_{33}} + \theta_2 \frac{\partial}{\partial \theta_{33}} + \vartheta_2 \frac{\partial}{\partial \vartheta_{33}} = 0, \quad \Sigma x_2 \frac{\partial}{\partial x_{12}} + \theta_2 \frac{\partial}{\partial \theta_{12}} + \vartheta_2 \frac{\partial}{\partial \vartheta_{12}} = 0, \end{aligned}\right\},$$

from the six derivatives of χ: and

$$\left.\begin{aligned} &\Sigma x_3 \frac{\partial}{\partial x_{11}} + \theta_3 \frac{\partial}{\partial \theta_{11}} + \vartheta_3 \frac{\partial}{\partial \vartheta_{11}} = 0, \quad \Sigma x_3 \frac{\partial}{\partial x_{23}} + \theta_3 \frac{\partial}{\partial \theta_{23}} + \vartheta_3 \frac{\partial}{\partial \vartheta_{23}} = 0, \\ &\Sigma x_3 \frac{\partial}{\partial x_{22}} + \theta_3 \frac{\partial}{\partial \theta_{22}} + \vartheta_3 \frac{\partial}{\partial \vartheta_{22}} = 0, \quad \Sigma x_3 \frac{\partial}{\partial x_{31}} + \theta_3 \frac{\partial}{\partial \theta_{31}} + \vartheta_3 \frac{\partial}{\partial \vartheta_{31}} = 0, \\ &\Sigma x_3 \frac{\partial}{\partial x_{33}} + \theta_3 \frac{\partial}{\partial \theta_{33}} + \vartheta_3 \frac{\partial}{\partial \vartheta_{33}} = 0, \quad \Sigma x_3 \frac{\partial}{\partial x_{12}} + \theta_3 \frac{\partial}{\partial \theta_{12}} + \vartheta_3 \frac{\partial}{\partial \vartheta_{12}} = 0, \end{aligned}\right\},$$

from the six derivatives of ψ. These eighteen equations, as their contribution towards a complete aggregate, must provide forty-two integrals that are algebraically independent of one another. Because no derivatives of x, y, z, v, θ, ϑ, of the first order occur, and because the line-variables t, u, w, and the surface-variables ξ, η, ζ, do not occur, there are twenty-four integrals at once given by

$$t, u, w; \ \xi, \eta, \zeta;$$

and

$$\pi_1, \pi_2, \pi_3,$$

for $\pi = x, y, z, v, \theta, \vartheta$. Hence other integrals, eighteen in number when algebraically independent of one another, must be obtained for these equations.

Next, there are the equations (initially nine, and ultimately eight, as already stated) which arise from the first derivatives of ϕ, χ, ψ. They are:

$$\left.\begin{aligned} O_{12} = {} & \Sigma x_{11}\frac{\partial}{\partial x_{12}} + 2\Sigma x_{12}\frac{\partial}{\partial x_{22}} + \Sigma x_{13}\frac{\partial}{\partial x_{23}} + \Sigma x_1\frac{\partial}{\partial x_2} + \theta_1\frac{\partial}{\partial \theta_2} + \vartheta_1\frac{\partial}{\partial \vartheta_2} \\ & + \theta_{11}\frac{\partial}{\partial \theta_{12}} + 2\theta_{12}\frac{\partial}{\partial \theta_{22}} + \theta_{13}\frac{\partial}{\partial \theta_{23}} + \vartheta_{11}\frac{\partial}{\partial \vartheta_{12}} + 2\vartheta_{12}\frac{\partial}{\partial \vartheta_{22}} + \vartheta_{13}\frac{\partial}{\partial \vartheta_{23}} = u\frac{\partial}{\partial t} - \xi\frac{\partial}{\partial \eta} \\ O_{13} = {} & \Sigma x_{11}\frac{\partial}{\partial x_{13}} + \Sigma x_{12}\frac{\partial}{\partial x_{23}} + 2\Sigma x_{13}\frac{\partial}{\partial x_{33}} + \Sigma x_1\frac{\partial}{\partial x_3} + \theta_1\frac{\partial}{\partial \theta_3} + \vartheta_1\frac{\partial}{\partial \vartheta_3} \\ & + \theta_{11}\frac{\partial}{\partial \theta_{13}} + \theta_{12}\frac{\partial}{\partial \theta_{23}} + 2\theta_{13}\frac{\partial}{\partial \theta_{33}} + \vartheta_{11}\frac{\partial}{\partial \vartheta_{13}} + \vartheta_{12}\frac{\partial}{\partial \vartheta_{23}} + 2\vartheta_{13}\frac{\partial}{\partial \vartheta_{33}} = w\frac{\partial}{\partial t} - \xi\frac{\partial}{\partial \zeta} \end{aligned}\right\},$$

from the coefficients of ϕ_2 and ϕ_3;

$$\left.\begin{aligned} O_{21} = {} & 2\Sigma x_{12}\frac{\partial}{\partial x_{11}} + \Sigma x_{22}\frac{\partial}{\partial x_{12}} + \Sigma x_{23}\frac{\partial}{\partial x_{13}} + \Sigma x_2\frac{\partial}{\partial x_1} + \theta_2\frac{\partial}{\partial \theta_1} + \vartheta_2\frac{\partial}{\partial \vartheta_1} \\ & + 2\theta_{12}\frac{\partial}{\partial \theta_{11}} + \theta_{22}\frac{\partial}{\partial \theta_{12}} + \theta_{23}\frac{\partial}{\partial \theta_{13}} + 2\vartheta_{12}\frac{\partial}{\partial \vartheta_{11}} + \vartheta_{22}\frac{\partial}{\partial \vartheta_{12}} + \vartheta_{23}\frac{\partial}{\partial \vartheta_{13}} = t\frac{\partial}{\partial u} - \eta\frac{\partial}{\partial \xi} \\ O_{23} = {} & \Sigma x_{12}\frac{\partial}{\partial x_{13}} + \Sigma x_{22}\frac{\partial}{\partial x_{23}} + 2\Sigma x_{23}\frac{\partial}{\partial x_{33}} + \Sigma x_2\frac{\partial}{\partial x_3} + \theta_2\frac{\partial}{\partial \theta_3} + \vartheta_2\frac{\partial}{\partial \vartheta_3} \\ & + \theta_{12}\frac{\partial}{\partial \theta_{13}} + \theta_{22}\frac{\partial}{\partial \theta_{23}} + 2\theta_{23}\frac{\partial}{\partial \theta_{33}} + \vartheta_{12}\frac{\partial}{\partial \vartheta_{13}} + \vartheta_{22}\frac{\partial}{\partial \vartheta_{23}} + 2\vartheta_{23}\frac{\partial}{\partial \vartheta_{33}} = w\frac{\partial}{\partial u} - \eta\frac{\partial}{\partial \zeta} \end{aligned}\right\},$$

from the coefficients of χ_1 and χ_3:

$$\left.\begin{aligned} O_{31} = {} & 2\Sigma x_{13}\frac{\partial}{\partial x_{11}} + \Sigma x_{23}\frac{\partial}{\partial x_{12}} + \Sigma x_{33}\frac{\partial}{\partial x_{13}} + \Sigma x_3\frac{\partial}{\partial x_1} + \theta_3\frac{\partial}{\partial \theta_1} + \vartheta_3\frac{\partial}{\partial \vartheta_1} \\ & + 2\theta_{13}\frac{\partial}{\partial \theta_{11}} + \theta_{23}\frac{\partial}{\partial \theta_{12}} + \theta_{33}\frac{\partial}{\partial \theta_{13}} + 2\vartheta_{13}\frac{\partial}{\partial \vartheta_{11}} + \vartheta_{23}\frac{\partial}{\partial \vartheta_{12}} + \vartheta_{33}\frac{\partial}{\partial \vartheta_{13}} = t\frac{\partial}{\partial w} - \zeta\frac{\partial}{\partial \xi} \\ O_{32} = {} & \Sigma x_{13}\frac{\partial}{\partial x_{12}} + 2\Sigma x_{23}\frac{\partial}{\partial x_{22}} + \Sigma x_{33}\frac{\partial}{\partial x_{23}} + \Sigma x_3\frac{\partial}{\partial x_2} + \theta_3\frac{\partial}{\partial \theta_2} + \vartheta_3\frac{\partial}{\partial \vartheta_2} \\ & + \theta_{13}\frac{\partial}{\partial \theta_{12}} + 2\theta_{23}\frac{\partial}{\partial \theta_{22}} + \theta_{33}\frac{\partial}{\partial \theta_{23}} + \vartheta_{13}\frac{\partial}{\partial \vartheta_{12}} + 2\vartheta_{23}\frac{\partial}{\partial \vartheta_{22}} + \vartheta_{33}\frac{\partial}{\partial \vartheta_{23}} = u\frac{\partial}{\partial w} - \zeta\frac{\partial}{\partial \eta} \end{aligned}\right\},$$

from the coefficients of ψ_1 and ψ_2: and

$$\left.\begin{aligned} O_{11} = {} & 2\Sigma x_{11}\frac{\partial}{\partial x_{11}} + \Sigma x_{12}\frac{\partial}{\partial x_{12}} + \Sigma x_{13}\frac{\partial}{\partial x_{13}} + \Sigma x_1\frac{\partial}{\partial x_1} + \theta_1\frac{\partial}{\partial \theta_1} + \vartheta_1\frac{\partial}{\partial \vartheta_1} \\ & + 2\theta_{11}\frac{\partial}{\partial \theta_{11}} + \theta_{12}\frac{\partial}{\partial \theta_{12}} + \theta_{13}\frac{\partial}{\partial \theta_{13}} + 2\vartheta_{11}\frac{\partial}{\partial \vartheta_{11}} + \vartheta_{12}\frac{\partial}{\partial \vartheta_{12}} + \vartheta_{13}\frac{\partial}{\partial \vartheta_{13}} = t\frac{\partial}{\partial t} - \xi\frac{\partial}{\partial \xi} + \mu \\ O_{22} = {} & \Sigma x_{12}\frac{\partial}{\partial x_{12}} + 2\Sigma x_{22}\frac{\partial}{\partial x_{22}} + \Sigma x_{23}\frac{\partial}{\partial x_{23}} + \Sigma x_2\frac{\partial}{\partial x_2} + \theta_2\frac{\partial}{\partial \theta_2} + \vartheta_2\frac{\partial}{\partial \vartheta_2} \\ & + \theta_{12}\frac{\partial}{\partial \theta_{12}} + 2\theta_{22}\frac{\partial}{\partial \theta_{22}} + \theta_{23}\frac{\partial}{\partial \theta_{23}} + \vartheta_{12}\frac{\partial}{\partial \vartheta_{12}} + 2\vartheta_{22}\frac{\partial}{\partial \vartheta_{22}} + \vartheta_{23}\frac{\partial}{\partial \vartheta_{23}} = u\frac{\partial}{\partial u} - \eta\frac{\partial}{\partial \eta} + \mu \\ O_{33} = {} & \Sigma x_{13}\frac{\partial}{\partial x_{13}} + \Sigma x_{23}\frac{\partial}{\partial x_{23}} + 2\Sigma x_{33}\frac{\partial}{\partial x_{33}} + \Sigma x_3\frac{\partial}{\partial x_3} + \theta_3\frac{\partial}{\partial \theta_3} + \vartheta_3\frac{\partial}{\partial \vartheta_3} \\ & + \theta_{13}\frac{\partial}{\partial \theta_{13}} + \theta_{23}\frac{\partial}{\partial \theta_{23}} + 2\theta_{33}\frac{\partial}{\partial \theta_{33}} + \vartheta_{13}\frac{\partial}{\partial \vartheta_{13}} + \vartheta_{23}\frac{\partial}{\partial \vartheta_{23}} + 2\vartheta_{33}\frac{\partial}{\partial \vartheta_{33}} = w\frac{\partial}{\partial w} - \zeta\frac{\partial}{\partial \zeta} + \mu \end{aligned}\right\},$$

from the coefficients of ϕ_1, χ_2, ψ_3. The last three equations provide two homogeneous equations for the complete Jacobian system in the form

$$O_{11} - t\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial \xi} = O_{22} - u\frac{\partial}{\partial u} + \eta\frac{\partial}{\partial \eta} = O_{33} - w\frac{\partial}{\partial w} + \zeta\frac{\partial}{\partial \zeta},$$

the common value of the three expressions being $\mu\Theta$, where μ is the numerical index of the concomitant.

And there are the six equations of orientational invariance symbolically represented by

$$E_{yz} = 0, \quad E_{zx} = 0, \quad E_{xy} = 0, \quad E_{xv} = 0, \quad E_{yv} = 0, \quad E_{zv} = 0,$$

which affect the derivatives, and combinations of the derivatives, of the point-coordinates x, y, z, v. They do not affect the parameters p, q, r, and functions of those parameters alone, when the functions are unconnected with point-coordinates, so that derivatives of θ and $\Im$ are unaffected. Moreover, the direction-variables t, u, w, and the orientation-variables ξ, η, ζ, are unaffected by these equations.

An aggregate of integrals of the first group of parametric equations.

484. We proceed, first, to construct an adequate aggregate of integrals, algebraically independent of one another, satisfying the first group of eighteen parametric equations.

Of these equations, we have already noted integrals

$$t, u, w; \quad \xi, \eta, \zeta; \quad \pi_1, \pi_2, \pi_3,$$

for $\pi = x, y, z, v, \theta, \Im$, all of these magnitudes being of the first grade alone. In constructing other integrals of the equations, these magnitudes can be ranked as unvarying quantities.

The portions of the eighteen equations, which affect the region alone without regard to the surfaces θ and $\Im$, are

$$\Sigma x_1 \frac{\partial}{\partial x_{lm}} = 0, \quad \Sigma x_2 \frac{\partial}{\partial x_{lm}} = 0, \quad \Sigma x_3 \frac{\partial}{\partial x_{lm}} = 0,$$

where $l, m, = 1, 2, 3$, for the different equations. As there are twenty-four variables x_{lm}, with regard to which differentiation enters into the equations, we need initially six integrals of the equations. Now the three equations

$$\Sigma x_1 \frac{\partial \Upsilon}{\partial x_{11}} = 0, \quad \Sigma x_2 \frac{\partial \Upsilon}{\partial x_{11}} = 0, \quad \Sigma x_3 \frac{\partial \Upsilon}{\partial x_{11}} = 0,$$

are equivalent to the three equations

$$\frac{1}{J_x}\frac{\partial \Upsilon}{\partial x_{11}} = -\frac{1}{J_y}\frac{\partial \Upsilon}{\partial y_{11}} = \frac{1}{J_z}\frac{\partial \Upsilon}{\partial z_{11}} = -\frac{1}{J_v}\frac{\partial \Upsilon}{\partial v_{11}}.$$

The quantities X, Y, Z, V, which are the direction-cosines of the normal to the region and are given by

$$\frac{X}{J_x} = \frac{Y}{-J_y} = \frac{Z}{J_z} = \frac{V}{-J_v} = \frac{1}{\Omega},$$

are unvarying as regards these equations; consequently, the quantity

$$Xx_{11} + Yy_{11} + Zz_{11} + Vv_{11}$$

is an integral of these three equations. It is the magnitude denoted (§ 271) by L. Similarly, for the other five groups of equations. We thus have six integrals in the form

$$\left.\begin{aligned} L &= Xx_{11} + Yy_{11} + Zz_{11} + Vv_{11} \\ M &= Xx_{12} + Yy_{12} + Zz_{12} + Vv_{12} \\ K &= Xx_{13} + Yy_{13} + Zz_{13} + Vv_{13} \\ N &= Xx_{22} + Yy_{22} + Zz_{22} + Vv_{22} \\ I &= Xx_{23} + Yy_{23} + Zz_{23} + Vv_{23} \\ J &= Xx_{33} + Yy_{33} + Zz_{33} + Vv_{33} \end{aligned}\right\}.$$

In order to simplify the construction of integrals of the whole set of equations, we shall substitute for the twenty-four second derivatives of x, y, z, v, other twenty-four quantities made up of

the six quantities L, N, J, I, K, M; and

the eighteen quantities Γ_{lm}, Δ_{lm}, Θ_{lm}, (for $l, m, = 1, 2, 3$),

the latter being defined (§ 273) in terms of the quantities

$$\alpha_l, \beta_l, \gamma_l, \delta_l, \epsilon_l, \omega_l,$$

as linear aggregates of the derivatives of x, y, z, v, of the second order.

The equations are transformed, so as to make the new set of variables the magnitudes with respect to which differentiations are performed. For example, we have

$$\left.\begin{aligned} \frac{\partial}{\partial x_{12}} &= X\frac{\partial}{\partial M} + x_1\frac{\partial}{\partial \beta_1} + x_2\frac{\partial}{\partial \beta_2} + x_3\frac{\partial}{\partial \beta_3} \\ \frac{\partial}{\partial y_{12}} &= Y\frac{\partial}{\partial M} + y_1\frac{\partial}{\partial \beta_1} + y_2\frac{\partial}{\partial \beta_2} + y_3\frac{\partial}{\partial \beta_3} \\ \frac{\partial}{\partial z_{12}} &= Z\frac{\partial}{\partial M} + z_1\frac{\partial}{\partial \beta_1} + z_2\frac{\partial}{\partial \beta_2} + z_3\frac{\partial}{\partial \beta_3} \\ \frac{\partial}{\partial v_{12}} &= V\frac{\partial}{\partial M} + v_1\frac{\partial}{\partial \beta_1} + v_2\frac{\partial}{\partial \beta_2} + v_3\frac{\partial}{\partial \beta_3} \end{aligned}\right\},$$

$$\left.\begin{aligned} \Omega^2\frac{\partial}{\partial \beta_{12}} &= a\frac{\partial}{\partial \Gamma_{12}} + h\frac{\partial}{\partial \Delta_{12}} + g\frac{\partial}{\partial \Theta_{12}} \\ \Omega^2\frac{\partial}{\partial \gamma_{12}} &= h\frac{\partial}{\partial \Gamma_{12}} + b\frac{\partial}{\partial \Delta_{12}} + f\frac{\partial}{\partial \Theta_{12}} \\ \Omega^2\frac{\partial}{\partial \delta_{12}} &= g\frac{\partial}{\partial \Gamma_{12}} + f\frac{\partial}{\partial \Delta_{12}} + c\frac{\partial}{\partial \Theta_{12}} \end{aligned}\right\}.$$

Therefore

$$\Sigma x_{11}\frac{\partial}{\partial x_{12}} = L\frac{\partial}{\partial M} + \alpha_1\frac{\partial}{\partial \beta_1} + \alpha_2\frac{\partial}{\partial \beta_2} + \alpha_3\frac{\partial}{\partial \beta_3}$$

$$= L\frac{\partial}{\partial M} + \frac{1}{\Omega^2}(A\Gamma_{11} + H\Delta_{11} + G\Theta_{11})\left(a\frac{\partial}{\partial \Gamma_{12}} + h\frac{\partial}{\partial \Delta_{12}} + g\frac{\partial}{\partial \Theta_{12}}\right)$$

$$+ \frac{1}{\Omega^2}(H\Gamma_{11} + B\Delta_{11} + F\Theta_{11})\left(h\frac{\partial}{\partial \Gamma_{12}} + b\frac{\partial}{\partial \Delta_{12}} + f\frac{\partial}{\partial \Theta_{12}}\right)$$

$$+ \frac{1}{\Omega^2}(G\Gamma_{11} + F\Delta_{11} + C\Theta_{11})\left(g\frac{\partial}{\partial \Gamma_{12}} + f\frac{\partial}{\partial \Delta_{12}} + c\frac{\partial}{\partial \Theta_{12}}\right)$$

$$= L\frac{\partial}{\partial M} + \Gamma_{11}\frac{\partial}{\partial \Gamma_{12}} + \Delta_{11}\frac{\partial}{\partial \Delta_{12}} + \Theta_{11}\frac{\partial}{\partial \Theta_{12}},$$

and

$$\Sigma x_1\frac{\partial}{\partial x_{12}} = A\frac{\partial}{\partial \beta_1} + H\frac{\partial}{\partial \beta_2} + G\frac{\partial}{\partial \beta_3} = \frac{\partial}{\partial \Gamma_{12}};$$

with similar results for the other necessary combinations.

The first set of equations, eighteen in number, thus becomes

$$\left.\begin{array}{lll} \dfrac{\partial}{\partial \Gamma_{11}} + \theta_1\dfrac{\partial}{\partial \theta_{11}} + \vartheta_1\dfrac{\partial}{\partial \vartheta_{11}} = 0, & \dfrac{\partial}{\partial \Gamma_{22}} + \theta_1\dfrac{\partial}{\partial \theta_{22}} + \vartheta_1\dfrac{\partial}{\partial \vartheta_{22}} = 0, & \dfrac{\partial}{\partial \Gamma_{33}} + \theta_1\dfrac{\partial}{\partial \theta_{33}} + \vartheta_1\dfrac{\partial}{\partial \vartheta_{33}} = 0 \\ \dfrac{\partial}{\partial \Gamma_{23}} + \theta_1\dfrac{\partial}{\partial \theta_{23}} + \vartheta_1\dfrac{\partial}{\partial \vartheta_{23}} = 0, & \dfrac{\partial}{\partial \Gamma_{31}} + \theta_1\dfrac{\partial}{\partial \theta_{31}} + \vartheta_1\dfrac{\partial}{\partial \vartheta_{31}} = 0, & \dfrac{\partial}{\partial \Gamma_{12}} + \theta_1\dfrac{\partial}{\partial \theta_{12}} + \vartheta_1\dfrac{\partial}{\partial \vartheta_{12}} = 0 \end{array}\right\},$$

$$\left.\begin{array}{lll} \dfrac{\partial}{\partial \Delta_{11}} + \theta_2\dfrac{\partial}{\partial \theta_{11}} + \vartheta_2\dfrac{\partial}{\partial \vartheta_{11}} = 0, & \dfrac{\partial}{\partial \Delta_{22}} + \theta_2\dfrac{\partial}{\partial \theta_{22}} + \vartheta_2\dfrac{\partial}{\partial \vartheta_{22}} = 0, & \dfrac{\partial}{\partial \Delta_{33}} + \theta_2\dfrac{\partial}{\partial \theta_{33}} + \vartheta_2\dfrac{\partial}{\partial \vartheta_{33}} = 0 \\ \dfrac{\partial}{\partial \Delta_{23}} + \theta_2\dfrac{\partial}{\partial \theta_{23}} + \vartheta_2\dfrac{\partial}{\partial \vartheta_{23}} = 0, & \dfrac{\partial}{\partial \Delta_{31}} + \theta_2\dfrac{\partial}{\partial \theta_{31}} + \vartheta_2\dfrac{\partial}{\partial \vartheta_{31}} = 0, & \dfrac{\partial}{\partial \Delta_{12}} + \theta_2\dfrac{\partial}{\partial \theta_{12}} + \vartheta_2\dfrac{\partial}{\partial \vartheta_{12}} = 0 \end{array}\right\},$$

$$\left.\begin{array}{lll} \dfrac{\partial}{\partial \Theta_{11}} + \theta_3\dfrac{\partial}{\partial \theta_{11}} + \vartheta_3\dfrac{\partial}{\partial \vartheta_{11}} = 0, & \dfrac{\partial}{\partial \Theta_{22}} + \theta_3\dfrac{\partial}{\partial \theta_{22}} + \vartheta_3\dfrac{\partial}{\partial \vartheta_{22}} = 0, & \dfrac{\partial}{\partial \Theta_{33}} + \theta_3\dfrac{\partial}{\partial \theta_{33}} + \vartheta_3\dfrac{\partial}{\partial \vartheta_{33}} = 0 \\ \dfrac{\partial}{\partial \Theta_{23}} + \theta_3\dfrac{\partial}{\partial \theta_{23}} + \vartheta_3\dfrac{\partial}{\partial \vartheta_{23}} = 0, & \dfrac{\partial}{\partial \Theta_{31}} + \theta_3\dfrac{\partial}{\partial \theta_{31}} + \vartheta_3\dfrac{\partial}{\partial \vartheta_{31}} = 0, & \dfrac{\partial}{\partial \Theta_{12}} + \theta_3\dfrac{\partial}{\partial \theta_{12}} + \vartheta_3\dfrac{\partial}{\partial \vartheta_{12}} = 0 \end{array}\right\}.$$

The arguments, occurring as operating variables through all the partial differential equations, still are sixty in number, being

$$L, N, J, I, K, M;$$

$$\Gamma_{lm}, \Delta_{lm}, \Theta_{lm}, \theta_{lm}, \vartheta_{lm}, \text{ (for } l, m, = 1, 2, 3);$$

$$x_n, y_n, z_n, v_n, \theta_n, \vartheta_n, \text{ (for } n = 1, 2, 3);$$

and

$$t, u, w; \xi, \eta, \zeta.$$

We therefore require 42 $(= 60 - 18)$ algebraically independent integrals of this first set of equations.

Because of their non-occurrence among the operating variables, the thirty magnitudes

$$t,\ u,\ w;\ \xi,\ \eta,\ \zeta;$$

$$x_n,\ y_n,\ z_n,\ v_n,\ \theta_n,\ \vartheta_n,\ (\text{for } n=1,\ 2,\ 3);$$

$$L,\ N,\ J,\ I,\ K,\ M;$$

manifestly are integrals of the kind required. Other twelve integrals are provided by the quantities

$$\left.\begin{aligned}\gamma_{lm} &= \theta_{lm} - \theta_1\Gamma_{lm} - \theta_2\Delta_{lm} - \theta_3\Theta_{lm}\\ \bar{\gamma}_{lm} &= \vartheta_{lm} - \vartheta_1\Gamma_{lm} - \vartheta_2\Delta_{lm} - \vartheta_3\Theta_{lm}\end{aligned}\right\},$$

for $l,\ m,\ =1,\ 2,\ 3.$

Characteristic orientational equations: their integrals.

485. We now proceed to find the combinations of these forty-two magnitudes which are algebraically independent integrals of the six equations characteristic of orientational invariance. It happens that, owing to the transformations already adopted in connection with the derivatives of the second order, the selection of the necessary combinations is comparatively easy.

First of all, we use the property already established (§§ 447, 478) that $\Omega^2,\ =\Sigma J_x^2$, satisfies all the six equations; so that, as

$$X\Omega = J_x,\quad Y\Omega = -J_y,\quad Z\Omega = J_z,\quad V\Omega = -J_v,$$

we have

$$E_{xy}X = Y,\quad E_{xy}Y = -X,\quad E_{xy}Z = 0,\quad E_{xy}V = 0,$$

and so for the others. Hence

$$E_{xy}L = E_{xy}(Xx_{200} + Yy_{200} + Zz_{200} + Vv_{200}) = 0,$$

on substitution; and likewise

$$E_{yz}L = 0,\quad E_{zx}L = 0,\quad E_{xv}L = 0,\quad E_{yv}L = 0,\quad E_{zv}L = 0.$$

Consequently L satisfies all the six equations; and the same is true of $N,\ J,\ I,\ K,\ M.$

Next, all the quantities $\alpha_l, \beta_l, \gamma_l, \delta_l, \epsilon_l, \omega_l$, for $l=1,\ 2,\ 3$, of § 273 are easily found to satisfy all the six equations; hence also all the quantities $\Gamma_{lm}, \Delta_{lm}, \Theta_{lm}$, for $l,\ m,\ =1,\ 2,\ 3$, satisfy the equations. The quantities θ, ϑ, and their derivatives of all orders are unaffected by the orientational equations; hence the quantities

$$\theta_l\ ,\ \vartheta_l\ ,\quad \text{for } l=1,\ 2,\ 3,$$

$$\gamma_{lm},\ \bar{\gamma}_{lm},\quad \text{for } l,\ m,\ =1,\ 2,\ 3,$$

also are integrals of the equations.

We thus have twenty-four of the necessary integrals. It remains to obtain six more which, when regard is paid to the analytical character of the magnitudes already obtained, must specially be associated with the twelve first derivatives of x, y, z, v. Such combinations have already been obtained (§ 478) in the forms

$$A = \Sigma x_1^2 \ , \quad B = \Sigma x_2^2 \ , \quad C = \Sigma x_3^2 \ ,$$
$$F = \Sigma x_2 x_3, \quad G = \Sigma x_3 x_1, \quad H = \Sigma x_1 x_2.$$

Accordingly, as an aggregate of algebraically independent integrals which satisfy the first set of eighteen parametric equations and satisfy also the set of six orientational equations, we have thirty necessary integrals in the aggregate of forms

$$A, B, C, F, G, H;$$
$$L, N, J, I, K, M;$$
$$t, u, w;$$
$$\xi, \eta, \zeta;$$
$$\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12};$$
$$\overline{\gamma}_{11}, \overline{\gamma}_{22}, \overline{\gamma}_{33}, \overline{\gamma}_{23}, \overline{\gamma}_{31}, \overline{\gamma}_{12}.$$

Second group of parametric equations: their transformation.

486. We now pass to the second group of parametric equations, eight in number; and we shall require 22 $(= 30 - 8)$ algebraically independent combinations of these thirty magnitudes which are integrals of the eight equations. To obtain such integrals, the simplest plan seems to be that the equations themselves shall be transformed so as to have these magnitudes as operating variables. For such a transformation, we leave $t, u, w; \xi, \eta, \zeta$; as variables for the new forms of the equations; and we form a tableau exhibiting the effect, upon the various other magnitudes, of the action of the operators O_{lm}, where $l, m, = 1, 2, 3$, in the equations as given in § 483. The tableau is:

	A	H	G	B	F	C	γ_{11}	γ_{12}	γ_{13}	γ_{22}	γ_{23}	γ_{33}
O_{12}	.	A	.	$2H$	G	.	.	γ_{11}	.	$2\gamma_{12}$	γ_{13}	.
O_{13}	.	.	A	.	H	$2G$	.	.	γ_{11}	.	γ_{12}	$2\gamma_{13}$
O_{21}	$2H$	B	F	.	.	.	$2\gamma_{12}$	γ_{22}	γ_{23}	.	.	
O_{23}	.	.	H	.	B	$2F$	.	.	γ_{12}	.	γ_{22}	$2\gamma_{23}$
O_{31}	$2G$	F	C	.	.	.	$2\gamma_{13}$	γ_{23}	γ_{33}	.	.	.
O_{32}	.	G	.	$2F$	C	.	.	γ_{13}	.	$2\gamma_{23}$	γ_{33}	.

the effect upon L, M, K, N, I, J, being the same as for A, H, G, B, F, C, mutatis mutandis, and the effect upon $\overline{\gamma}_{lm}$ being the same as upon γ_{lm}, for $l, m, = 1, 2, 3$.

By the use of these results, the operators O_{lm}, for $l, m, = 1, 2, 3$, in the eight equations of the second set of parametric equations (the eight constituting a Jacobian system, complete within itself) are changed to the respective forms

$$\left.\begin{aligned} O_{12} = {} & A\frac{\partial}{\partial \overline{H}} + 2H\frac{\partial}{\partial \overline{B}} + G\frac{\partial}{\partial \overline{F}} + L\frac{\partial}{\partial \overline{M}} + 2M\frac{\partial}{\partial \overline{N}} + K\frac{\partial}{\partial \overline{I}} \\ & + \gamma_{11}\frac{\partial}{\partial \overline{\gamma}_{12}} + 2\gamma_{12}\frac{\partial}{\partial \overline{\gamma}_{22}} + \gamma_{13}\frac{\partial}{\partial \overline{\gamma}_{23}} + \overline{\gamma}_{11}\frac{\partial}{\partial \overline{\overline{\gamma}}_{12}} + 2\overline{\gamma}_{12}\frac{\partial}{\partial \overline{\overline{\gamma}}_{22}} + \overline{\gamma}_{13}\frac{\partial}{\partial \overline{\overline{\gamma}}_{23}} \\ O_{13} = {} & A\frac{\partial}{\partial \overline{G}} + H\frac{\partial}{\partial \overline{F}} + 2G\frac{\partial}{\partial \overline{C}} + L\frac{\partial}{\partial \overline{K}} + M\frac{\partial}{\partial \overline{I}} + 2K\frac{\partial}{\partial \overline{J}} \\ & + \gamma_{11}\frac{\partial}{\partial \overline{\gamma}_{13}} + \gamma_{12}\frac{\partial}{\partial \overline{\gamma}_{23}} + 2\gamma_{13}\frac{\partial}{\partial \overline{\gamma}_{33}} + \overline{\gamma}_{11}\frac{\partial}{\partial \overline{\overline{\gamma}}_{13}} + \overline{\gamma}_{12}\frac{\partial}{\partial \overline{\overline{\gamma}}_{23}} + 2\overline{\gamma}_{13}\frac{\partial}{\partial \overline{\overline{\gamma}}_{33}} \end{aligned}\right\},$$

$$\left.\begin{aligned} O_{21} = {} & 2H\frac{\partial}{\partial \overline{A}} + B\frac{\partial}{\partial \overline{H}} + F\frac{\partial}{\partial \overline{G}} + 2M\frac{\partial}{\partial \overline{L}} + N\frac{\partial}{\partial \overline{M}} + I\frac{\partial}{\partial \overline{K}} \\ & + 2\gamma_{12}\frac{\partial}{\partial \overline{\gamma}_{11}} + \gamma_{22}\frac{\partial}{\partial \overline{\gamma}_{12}} + \gamma_{23}\frac{\partial}{\partial \overline{\gamma}_{13}} + 2\overline{\gamma}_{12}\frac{\partial}{\partial \overline{\overline{\gamma}}_{11}} + \overline{\gamma}_{22}\frac{\partial}{\partial \overline{\overline{\gamma}}_{12}} + \overline{\gamma}_{23}\frac{\partial}{\partial \overline{\overline{\gamma}}_{13}} \\ O_{23} = {} & H\frac{\partial}{\partial \overline{G}} + B\frac{\partial}{\partial \overline{F}} + 2F\frac{\partial}{\partial \overline{C}} + M\frac{\partial}{\partial \overline{K}} + N\frac{\partial}{\partial \overline{I}} + 2I\frac{\partial}{\partial \overline{J}} \\ & + \gamma_{12}\frac{\partial}{\partial \overline{\gamma}_{13}} + \gamma_{22}\frac{\partial}{\partial \overline{\gamma}_{23}} + 2\gamma_{23}\frac{\partial}{\partial \overline{\gamma}_{33}} + \overline{\gamma}_{12}\frac{\partial}{\partial \overline{\overline{\gamma}}_{13}} + \overline{\gamma}_{22}\frac{\partial}{\partial \overline{\overline{\gamma}}_{23}} + 2\overline{\gamma}_{23}\frac{\partial}{\partial \overline{\overline{\gamma}}_{33}} \end{aligned}\right\},$$

$$\left.\begin{aligned} O_{31} = {} & 2G\frac{\partial}{\partial \overline{A}} + F\frac{\partial}{\partial \overline{H}} + C\frac{\partial}{\partial \overline{G}} + 2K\frac{\partial}{\partial \overline{L}} + I\frac{\partial}{\partial \overline{M}} + J\frac{\partial}{\partial \overline{K}} \\ & + 2\gamma_{13}\frac{\partial}{\partial \overline{\gamma}_{11}} + \gamma_{23}\frac{\partial}{\partial \overline{\gamma}_{12}} + \gamma_{33}\frac{\partial}{\partial \overline{\gamma}_{13}} + 2\overline{\gamma}_{13}\frac{\partial}{\partial \overline{\overline{\gamma}}_{11}} + \overline{\gamma}_{23}\frac{\partial}{\partial \overline{\overline{\gamma}}_{12}} + \overline{\gamma}_{33}\frac{\partial}{\partial \overline{\overline{\gamma}}_{13}} \\ O_{32} = {} & G\frac{\partial}{\partial \overline{H}} + 2F\frac{\partial}{\partial \overline{B}} + C\frac{\partial}{\partial \overline{F}} + K\frac{\partial}{\partial \overline{M}} + 2I\frac{\partial}{\partial \overline{N}} + J\frac{\partial}{\partial \overline{I}} \\ & + \gamma_{13}\frac{\partial}{\partial \overline{\gamma}_{12}} + 2\gamma_{23}\frac{\partial}{\partial \overline{\gamma}_{22}} + \gamma_{33}\frac{\partial}{\partial \overline{\gamma}_{23}} + \overline{\gamma}_{13}\frac{\partial}{\partial \overline{\overline{\gamma}}_{12}} + 2\overline{\gamma}_{23}\frac{\partial}{\partial \overline{\overline{\gamma}}_{22}} + \overline{\gamma}_{33}\frac{\partial}{\partial \overline{\overline{\gamma}}_{23}} \end{aligned}\right\};$$

and the operators in the other two equations are

$$O_{12}O_{21} - O_{21}O_{12} = O_{23}O_{32} - O_{32}O_{23} = O_{31}O_{13} - O_{13}O_{31}.$$

The eight equations themselves are

$$\left.\begin{aligned} O_{12} &= u\frac{\partial}{\partial \overline{t}} - \xi\frac{\partial}{\partial \overline{\eta}} \\ O_{13} &= w\frac{\partial}{\partial \overline{t}} - \xi\frac{\partial}{\partial \overline{\zeta}} \end{aligned}\right\}, \qquad \left.\begin{aligned} O_{21} &= t\frac{\partial}{\partial \overline{u}} - \eta\frac{\partial}{\partial \overline{\xi}} \\ \omega_{23} &= w\frac{\partial}{\partial \overline{u}} - \eta\frac{\partial}{\partial \overline{\zeta}} \end{aligned}\right\}, \qquad \left.\begin{aligned} O_{31} &= t\frac{\partial}{\partial \overline{w}} - \zeta\frac{\partial}{\partial \overline{\xi}} \\ O_{32} &= u\frac{\partial}{\partial \overline{w}} - \zeta\frac{\partial}{\partial \overline{\eta}} \end{aligned}\right\},$$

$$\left.\begin{aligned} O_{12}O_{21} - O_{21}O_{12} - \left(u\frac{\partial}{\partial u} - t\frac{\partial}{\partial t}\right) - \left(\xi\frac{\partial}{\partial \xi} - \eta\frac{\partial}{\partial \eta}\right) \\ = O_{23}O_{32} - O_{32}O_{23} - \left(w\frac{\partial}{\partial w} - u\frac{\partial}{\partial u}\right) - \left(\eta\frac{\partial}{\partial \eta} - \zeta\frac{\partial}{\partial \zeta}\right) \\ = O_{31}O_{13} - O_{13}O_{31} - \left(t\frac{\partial}{\partial t} - w\frac{\partial}{\partial w}\right) - \left(\zeta\frac{\partial}{\partial \zeta} - \xi\frac{\partial}{\partial \xi}\right) \end{aligned}\right\},$$

eight in number.

As there are thirty arguments, we shall require twenty-two independent integrals of these equations: or, in addition to

$$t_\xi, = \xi t + \eta u + \zeta w,$$

we shall require twenty-one algebraically independent integrals of the system of equations.

Distribution of concomitants: association with ternary forms.

487. As the thirty arguments include the six second derivatives of θ as well as its three first derivatives, all of which occur in the six combinations γ_{lm}, there must be six among the twenty-one integrals which can be specially selected to be associated with the surface θ. Similarly, an allowance of six integrals must be made for special association with the surface $\mathfrak{S}$. Consequently there can remain an aggregate of nine concomitants, in addition to t_ξ, which can be regarded as appertaining generally to the whole region without regard to any superficial or curvilinear configuration selected in the region.

There will, of course, be composite concomitants which can be used to express combined properties, such as curvatures of a curve, the geodesic curvature of a curve whether in the region or on a contained surface: or there may be the relations of curvature between any assumed surface in the region and, at a point on that surface, the tangential geodesic surface of the region. All such magnitudes and properties can be expressed by means of concomitants and relations between concomitants. The immediate purpose is to obtain a complete system which is to be such that, by means of its constituent members, every concomitant can be expressed.

The algebraical derivation of an adequate system of concomitants is made immediate by noting, first of all, the forms of the eight differential equations in the complete system characteristic of every concomitant. The eight equations are the equations of the concomitants of the simultaneous system of quadratic ternary forms

$$\begin{aligned} A_t^2 &= (A, B, C, F, G, H \mathbin{\text{\textbackslash\!\!\textbackslash}} t, u, w)^2, \\ L_t^2 &= (L, N, J, I, K, M \mathbin{\text{\textbackslash\!\!\textbackslash}} t, u, w)^2, \\ \gamma_t^2 &= (\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12} \mathbin{\text{\textbackslash\!\!\textbackslash}} t, u, w)^2, \\ \bar{\gamma}_t^2 &= (\bar{\gamma}_{11}, \bar{\gamma}_{22}, \bar{\gamma}_{33}, \bar{\gamma}_{23}, \bar{\gamma}_{31}, \bar{\gamma}_{12} \mathbin{\text{\textbackslash\!\!\textbackslash}} t, u, w)^2, \end{aligned}$$

with t, u, w, as variables, and ξ, η, ζ, as the contragredient variables. With the system of twenty-one algebraically independent concomitants of the four forms, we have to associate the universal concomitant

$$\xi t + \eta u + \zeta w.$$

Concomitants belonging to the region alone: their identification.

488. Now the concomitants which belong to the region alone, unaffected by the postulation of surfaces such as θ and $\Im$, clearly must now be the concomitants belonging to the two simultaneous quadratic ternary forms

$$A_t^2 = (A\,\mathfrak{Y}t)^2, \quad L_t^2 = (L\,\mathfrak{Y}t)^2;$$

and, as indicated already, they consist of nine algebraically independent members in addition to the universal concomitant. The results, algebraically, are similar to the results belonging to a system of two simultaneous conics represented by $A_t^2 = 0$, $L_t^2 = 0$. The complete system, asyzygetically (and not merely algebraically) complete, of two conics has already been given by Clebsch*; with the preceding change of notation, this asyzygetic system is composed of the members

$$\begin{array}{lll}
f = A_t^2 & , & f' = L_t^2 \\
T = \xi_a^2 & , & T' = \xi_l^2 \\
\Delta = A_a^2 & , & \Delta' = L_l^2 \\
\Theta = L_a^2 & , & \Theta' = A_l^2 \\
\Gamma = \tfrac{1}{2}(AL\xi)^2 & , & \Gamma' = \tfrac{1}{2}(alt)^2 \\
\bar{B} = L_a L_t \xi_a & , & \bar{B}' = A_l A_t \xi_l \\
\bar{N} = (AL\xi) A_t L_t & , & \bar{N}' = (alt)\, \xi_a \xi_l \\
\bar{C} = (AL\xi) L_a A_t \xi_a & , & \bar{C}' = (alt) A_l A_t \xi_a \\
\bar{E} = (AL\xi) A_l L_t \xi_l & , & \bar{E}' = (alt) L_a L_t \xi_l \\
\bar{D} = (AL\xi) A_l L_a \xi_a \xi_l, & & \bar{D}' = (alt) A_l L_a A_t L_t,
\end{array}$$

where the symbols a, b, c, f, g, h, are the cofactors of the constituents of Ω^2, and l, n, j, i, k, m, are the cofactors of the constituents of Υ in § 286. For the immediate purpose, it is sufficient to select nine algebraically independent members from this aggregate.

To this end, there is an ample store of results obtained in the investigations of Chapter XIX; so identification alone is required. The quantities, there denoted by $\bar{x}, \bar{y}, \bar{z}$, are here denoted by t, u, w; and the significance of ξ, η, ζ, has remained unaltered. Accordingly, we select the concomitants, and we give their geometrical significance, as follows.

* *Vorlesungen über Geometrie*, Clebsch-Lindemann, vol. i, p. 291: the letters, used to denote the concomitants, are in accord with Clebsch's assignment.

We have

$$U = A_t^2 = (A, B, C, F, G, H \mathbin{\backslash\!\!\backslash} t, u, w)^2$$
$$= ds^2,$$

and

$$V = L_t^2 = (L, N, J, I, K, M \mathbin{\backslash\!\!\backslash} t, u, w)^2$$
$$= \frac{ds^2}{\rho},$$

where ds is the general arc-element having t, u, w, as its parametric line-variables in the region, and ρ is the radius of circular curvature of a geodesic in the direction t, u, w (that is, the radius of curvature of a normal section of the region). Also U and V are the two fundamental ternary forms for the region alone. Moreover,

$$\xi t + \eta u + \zeta w = t_\xi$$
$$= \frac{1}{\Omega} d\overline{V},$$

where $d\overline{V}$ is a general parallelepipedal element of volume of the region at the point, having t, u, w, for the variables of an edge, and ξ, η, ζ, for the variables of a face conterminous with that edge.

Again, we have

$$T = \xi_a^2 = (a, b, c, f, g, h \mathbin{\backslash\!\!\backslash} \xi, \eta, \zeta)^2$$
$$= dS^2,$$

where dS is the general element of superficial area in the region having ξ, η, ζ, for its parametric orientation-coordinates. Also

$$T' = \xi_t^2 \qquad = (l, n, j, i, k, m \mathbin{\backslash\!\!\backslash} \xi, \eta, \zeta)^2$$
$$= \overline{K} dS^2,$$

and

$$\Gamma = \tfrac{1}{2} (AL\xi)^2 = (\bar{l}, \bar{n}, \bar{j}, \bar{i}, \bar{k}, \bar{m} \mathbin{\backslash\!\!\backslash} \xi, \eta, \zeta)^2$$
$$= \overline{H} dS^2,$$

where $\overline{K}$ is the Riemann measure of superficial curvature of the region, with the orientation dS, (as estimated by the geodesic surface through dS), and $\overline{H}$ is the corresponding additive measure of superficial curvature of the region similarly estimated.

For the principal measures of linear curvature of the region, as constructed from the three principal radii of linear curvature α, β, γ, associated with the respective principal directions at a point, we need the concomitants

$$\Omega^2 = A_a^2 = \begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix} = ABC + 2FGH - AF^2 - BG^2 - CH^2,$$

(the interpretation of Ω being given in connection with the element of volume at the point and the universal concomitant ξ_t);

$$\begin{aligned}\mathfrak{G}_1 &= L_a{}^2\\ &= aL + bN + cJ + 2fI + 2gK + 2hM\\ &= \Omega^2\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right);\end{aligned}$$

$$\begin{aligned}\mathfrak{G}_2 &= A_l{}^2\\ &= Al + Bn + Cj + 2Fi + 2Gk + 2Hm\\ &= \Omega^2\left(\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}+\frac{1}{\alpha\beta}\right);\end{aligned}$$

and

$$\begin{aligned}\Upsilon = L_l{}^2 &= \begin{vmatrix} L, & M, & K\\ M, & N, & I\\ K, & I, & J\end{vmatrix}\\ &= LNJ + 2IKM - LI^2 - NK^2 - JM^2\\ &= \Omega^2\,\frac{1}{\alpha\beta\gamma}.\end{aligned}$$

Also, we have

$$\begin{aligned}\Gamma' &= \frac{1}{2}(alt)^2\\ &= (\bar\lambda, \bar\nu, \bar\zeta, \bar\imath, \bar\kappa, \bar\mu \mid t, u, w)^2,\end{aligned}$$

where (§ 327) the coefficients in Γ' are constructed from the surface-contravariant T in the same way as the coefficients in Γ are constructed from the line-covariant U, so that (e.g.)

$$\bar l = BJ - 2FI + CN, \quad \bar\lambda = bj - 2fi + cn;$$

and the geometrical value of Γ' is

$$\Gamma' = \left\{\frac{1}{\rho}\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right) - \left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)\right\} ds^2,$$

where $1/\sigma$ is the torsion of the regional geodesic in the direction ds.

We thus have an aggregate of concomitants

$$\xi_t;$$

$$U,\ \Omega^2,\ T;$$

$$V;\ \mathfrak{G}_1,\ \mathfrak{G}_2,\ \Upsilon;\ T';\ \Gamma,\ \Gamma';$$

more than adequate in number for the algebraical expression of any concomitant of the first or the second grade appertaining to the region alone. Of these concomitants, ξ_t is universal for any configuration: while U, Ω^2, and T, are of the first grade alone.

We have already had instances (§ 326) of algebraical expression in terms of these concomitants. Thus, for example,

$$U\Omega^2\left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)=\frac{1}{4}(a,\,b,\,c,\,f,\,g,\,h)\left(\frac{\partial V}{\partial t},\,\frac{\partial V}{\partial u},\,\frac{\partial V}{\partial w}\right)^2$$
$$=\mathbf{D}$$
$$=V\mathfrak{G}_1-\Gamma';$$

and therefore

$$\frac{\Omega^2}{\sigma^2}=\mathfrak{G}_1\frac{V}{U}-\Omega^2\frac{V^2}{U^2}-\frac{\Gamma'}{U}.$$

The construction of illustrations is a matter of algebraical calculation.

Ex. Establish the relations:

(i) $\overline{B}^2=TV\mathfrak{G}_1-V\Omega^2\Gamma'-T\Gamma+\Omega^2\mathfrak{G}_2\xi_t^2-\Omega^2UT'-2\Omega^2\overline{B}'\xi_t$;

(ii) $\overline{B}'^2=T'U\mathfrak{G}_2-U\Upsilon\Gamma-T'\Gamma'+\Upsilon\mathfrak{G}_1\xi_t^2-\Upsilon VT-2\Upsilon\overline{B}\xi_t$;

(iii) $\overline{N}^2=UV\Gamma+\Gamma'\xi_t^2-U(\mathfrak{G}_2\xi_t^2+T'U-2B'\xi_t)-V(\mathfrak{G}_1\xi_t^2+VT-2\overline{B}\xi_t)$;

(iv) $\overline{N}'^2=TT'\Gamma'+\Omega^2\Upsilon\Gamma\xi_t^2-T\Upsilon(\mathfrak{G}_1\xi_t^2+TV-2\overline{B}\xi_t)-T'\Omega^2(\mathfrak{G}_2\xi_t^2+UT'-2\overline{B}'\xi_t)$.

Concomitants specially affecting a surface in the region.

489. We now have to include concomitants affecting the θ-surface, up to (and including) those of grade two. The algebraical effect upon the whole system of concomitants of grade unity has already (§ 479) been considered; and, there, adequate account has been taken of the occurrence of the arguments θ_1, θ_2, θ_3. It remains to take account of the occurrence of the six arguments γ_{11}, γ_{22}, γ_{33}, γ_{23}, γ_{31}, γ_{12}. To obtain an algebraically complete system, we must therefore obtain six additional (and algebraically independent) concomitants, arising through the association of the new ternary form

$$\Psi_t=(\gamma_{11},\,\gamma_{22},\,\gamma_{33},\,\gamma_{23},\,\gamma_{31},\,\gamma_{12}\,\mathfrak{Y}\,*)^2$$

with the forms

$$U_t=(A,\,B,\,C,\,F,\,G,\,H\,\mathfrak{Y}\,*)^2,$$
$$W_t=(L,\,N,\,J,\,I,\,K,\,M\,\mathfrak{Y}\,*)^2,$$

as well as the form of the first order

$$\bar{\theta}_t=(\theta_1,\,\theta_2,\,\theta_3\,\mathfrak{Y}\,*).$$

The interpretation of the concomitants of the first grade has already (§ 480) been given.

Various arcs arise. There is the arc-element ds, general and unspecified. Any arc-element lying on the θ-surface is denoted by ds_θ, and the arc-element normal to that surface by $d\nu_\theta$; and similarly for $ds_\mathfrak{Z}$ and $d\nu_\mathfrak{Z}$ as regards the $\mathfrak{Z}$-surface. The arc-element along the curve of intersection of the θ-surface with the $\mathfrak{Z}$-surface is denoted by dS. Thus

$$ds^2=\Sigma Adt^2=\Sigma A\,dp^2,\quad ds_\theta^2=\Sigma A\,dp_\theta^2,\quad ds_\mathfrak{Z}^2=\Sigma A\,dp_\mathfrak{Z}^2,$$

where

$$\theta_1dp_\theta+\theta_2dq_\theta+\theta_3dr_\theta=0,\quad \mathfrak{Z}_1dp_\mathfrak{Z}+\mathfrak{Z}_2dq_\mathfrak{Z}+\mathfrak{Z}_3dr_\mathfrak{Z}=0.$$

Parametric magnitudes as functions of position.

490. The quantities θ and $\Im$, later, are mainly associated (in significance) with surfaces; initially, they are functions of position in the region. Thus they can be regarded in two ways; either as functions of position, which is their initial significance in the differential equations characteristic of concomitants: or as parameters of positional loci constituted by surfaces

$$\theta(p, q, r) = \text{constant}, \quad \Im(p, q, r) = \text{constant},$$

in connection with which curvature-properties are investigated. Thus invariantive functions arise involving the derivatives of θ and $\Im$; the geometrical interpretation of such functions is compelled to assign a meaning to θ, sometimes as a function of position, sometimes as the parameter (it may be the unvarying parameter) of a surface.

The linear ternary form $\bar{\theta}$, where

$$\bar{\theta} = \theta_1 \frac{dp}{ds} + \theta_2 \frac{dq}{ds} + \theta_3 \frac{dr}{ds},$$

when ds is any general element of arc in the region, is equal to $\frac{d\theta}{ds}$, the arc-rate of change of θ as a function of position. But, when the arc-element is taken in the surface (that is, in any direction in the tangent plane to the surface), the quantity $\bar{\theta}$ vanishes and provides a relation

$$\theta_1 dp + \theta_2 dq + \theta_3 dr = 0$$

of variation in the surface. Similarly, as will appear at once, there is a positional covariant

$$\frac{d^2\theta}{ds^2}$$

(with the general arc ds) which has no particular relation to the surface; there is a covariant of position

$$\frac{d^2\theta}{dv^2},$$

which is associated with the surface alone.

Further, care must be exercised in the use of repeated operations of differentiation when these are connected with directions. We can have a direction ds_θ in the surface. Tangential to such a direction, there passes a superficial geodesic and there passes a regional geodesic. But the differential operations of the second order, which imply progress along a superficial geodesic, are distinct in result from the similar operations which imply progress along a regional geodesic. Indeed, as already pointed out, the measure of the arc-rate of deviation between the two operations is the regional geodesic curvature of the curve which actually is a geodesic on the surface.

Thus, when we take a second spatial arc-derivative of θ in the form

$$\frac{d^2\theta}{ds^2} = \theta_1 \frac{d^2p}{ds^2} + \theta_2 \frac{d^2q}{ds^2} + \theta_3 \frac{d^2r}{ds^2} + \Sigma\theta_{11}\left(\frac{dp}{ds}\right)^2,$$

there is no particular significance until the quantities $\frac{d^2p}{ds^2}, \frac{d^2q}{ds^2}, \frac{d^2r}{ds^2}$, have been defined, that is, until the curve has received some definition beyond the property of touching the direction. As an illustration, let the curve be the regional geodesic which touches that direction, so that (§ 298)

$$\frac{d^2p}{ds^2} + \Sigma\Gamma_{11}\left(\frac{dp}{ds}\right)^2 = 0,$$

with two similar equations; then

$$\begin{aligned}\frac{d^2\theta}{ds^2} &= \Sigma\left(\theta_{11} - \theta_1\Gamma_{11} - \theta_2\Delta_{11} - \theta_3\Theta_{11}\right)\left(\frac{dp}{ds}\right)^2 \\ &= \Sigma\gamma_{11}\left(\frac{dp}{ds}\right)^2 \\ &= \Psi,\end{aligned}$$

in the old significance of Ψ. But now it must be borne in mind that while p', q', r', are the same for all the curves which touch the same line, the second derivative $\frac{d^2\theta}{ds^2}$ is estimated along the regional geodesic.

But when the arc-element is taken along the surface, we have

$$0 = \frac{d^2\theta}{ds_\theta^2} = \theta_1 \frac{d^2p}{ds_\theta^2} + \theta_2 \frac{d^2q}{ds_\theta^2} + \theta_3 \frac{d^2r}{ds_\theta^2} + \Sigma\theta_{11}\left(\frac{dp}{ds}\right)^2,$$

and the equation is merely a single relation limiting the values of the second derivatives p'', q'', r'', along the surface, these otherwise not being precisely definite. When, however, the direction on the surface is taken to be that of a superficial geodesic, then (§ 344)

$$\frac{d^2p}{ds_\theta^2} + \Sigma\Gamma_{11}\left(\frac{dp}{ds}\right)^2 = -\frac{\Psi_\theta}{D_\theta}(a\theta_1 + h\theta_2 + g\theta_3);$$

and we are led to the relation

$$\Psi_\theta = \Sigma\gamma_{11}\left(\frac{dp}{ds_\theta}\right)^2,$$

where the geometrical interpretation indicates that Ψ_θ, the value of the ternary form, now is a measure of the regional geodesic curvature of the superficial geodesic.

The distinction could be indicated by more elaborate notation: thus, differentiation along a regional geodesic might be indicated by ds_r, so that

$$\frac{d^2\theta}{ds_r^2} = \Sigma\gamma_{11}\left(\frac{dp}{ds_r}\right)^2 = \Psi_r,$$

while

$$\Psi_\theta = \Sigma\gamma_{11}\left(\frac{dp}{ds_\theta}\right)^2$$

would be a measure of regional geodesic curvature of the superficial geodesic in the direction $\frac{dp}{ds_\theta}, \frac{dq}{ds_\theta}, \frac{dr}{ds_\theta}$, lying in the tangent plane to the surface. Possible confusion is most simply avoided by remembering the distinction between continued differentiation along a regional geodesic and continued differentiation along a superficial geodesic, the first differentiations being the same when the two geodesics touch.

Concomitants involving θ, as parameter of a regional surface.

491. After this monitory explanation, we return to the completion of the system of concomitants of the second grade, which are to involve the magnitudes γ_{lm}, the coefficients of the new ternary form.

Some of these, suitable for our purpose, have already been obtained, in connection with the θ-surface. For each of them, arc-differentiations belong to the surface: and, where arc-continuation is necessary (as, for instance, when torsion and tilt are considered), the arc-continuation is taken along a superficial geodesic Among these magnitudes, we have

$$\frac{1}{\gamma_\theta} = -\frac{\Omega}{D_\theta^{\frac{1}{2}}}(\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12}\mathcal{X}\, p', q', r')^2$$

$$= -\frac{\Omega}{D_\theta^{\frac{1}{2}}}\Psi_\theta,$$

giving the regional geodesic curvature of a superficial geodesic. There are two principal directions of this regional geodesic curvature, perpendicular to one another; and there are (§ 348) the two principal measures,

$$\frac{1}{\gamma_1\gamma_2} = \frac{\Omega^2}{D_\theta^2}\mathbf{K}$$

$$= \frac{\Omega^2}{D_\theta^2}(\gamma_{22}\gamma_{33} - \gamma_{23}^2, \ldots\mathcal{X}\theta_1, \theta_2, \theta_3)^2,$$

$$\frac{1}{\gamma_1} + \frac{1}{\gamma_2} = -\frac{\Omega}{D_\theta^{\frac{3}{2}}}\mathbf{H}$$

$$= -\frac{\Omega}{D_\theta^{\frac{3}{2}}}(B\gamma_{33} - 2F\gamma_{23} + C\gamma_{22}, \ldots\mathcal{X}\theta_1, \theta_2, \theta_3)^2.$$

Manifestly Ψ_θ, $\mathbf{K}$, $\mathbf{H}$, are three concomitants of the kind required.

Further, the torsion and the tilt of a superficial geodesic have been obtained in the respective forms (§ 351)

$$\frac{1}{\bar{\sigma}} = \frac{\Omega}{4D_\theta} \begin{vmatrix} \frac{\partial \Psi_\theta}{\partial p'}, & \frac{\partial \Psi_\theta}{\partial q'}, & \frac{\partial \Psi_\theta}{\partial r'} \\ \frac{\partial U}{\partial p'}, & \frac{\partial U}{\partial q'}, & \frac{\partial U}{\partial r'} \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix},$$

$$\frac{1}{\bar{\tau}} = \frac{1}{4D_\theta^{\frac{1}{2}}} \begin{vmatrix} \frac{\partial W}{\partial p'}, & \frac{\partial W}{\partial q'}, & \frac{\partial W}{\partial r'} \\ \frac{\partial U}{\partial p'}, & \frac{\partial U}{\partial q'}, & \frac{\partial U}{\partial r'} \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix},$$

while (§ 360)

$$\frac{1}{\rho\bar{\sigma}} + \frac{1}{\gamma_\theta \bar{\tau}} = \frac{\Omega}{4D_\theta} \begin{vmatrix} \frac{\partial \Psi_\theta}{\partial p'}, & \frac{\partial \Psi_\theta}{\partial q'}, & \frac{\partial \Psi_\theta}{\partial r'} \\ \frac{\partial W}{\partial p'}, & \frac{\partial W}{\partial q'}, & \frac{\partial W}{\partial r'} \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix}.$$

In each expression, the direction p', q', r', lies in the θ-surface, so that

$$\theta_1 p' + \theta_2 q' + \theta_3 r' = 0,$$

and it determines the superficial geodesic in question.

Concomitants involving θ, as a function of position in the region.

492. The preceding concomitants are directly concerned with θ as the parameter, constant over the surface, and they involve a direction restricted to lie within the tangent plane to the surface.

We require, also, other concomitants not thus limited, which are concerned with θ as a function of position.

An example of such a concomitant is

$$U\frac{d^2\theta}{ds^2} = (\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12}\mathcal{X}t, u, w)^2,$$

where

$$U = (A, B, C, F, G, H\mathcal{X}t, u, w)^2,$$

the arc-element ds belonging to a regional geodesic, with $t, u, w, (= dp, dq, dr)$ as its direction-variables.

Again, we have

$$\left(\frac{d\theta}{d\nu}\right)^2 = \frac{D_\theta}{\Omega^2},$$

where $d\nu$ is an arc-element in the region, drawn normal to the θ-surface: and, here, $\dfrac{d\theta}{d\nu}$ itself is a function of position solely, though constructed along a direction. Hence

$$\frac{d\theta}{d\nu}\frac{\partial}{\partial p}\left(\frac{d\theta}{d\nu}\right) = \frac{1}{2}\frac{1}{\Omega^2}\frac{\partial D_\theta}{\partial p} - \frac{D_\theta}{\Omega^2}\frac{\Omega_1}{\Omega}$$

$$= P\gamma_{11} + Q\gamma_{12} + R\gamma_{13},$$

by the result of § 341: also

$$\frac{dp}{d\nu} = \frac{\Omega}{D_\theta^{\frac{1}{2}}}P, \quad \frac{dq}{d\nu} = \frac{\Omega}{D_\theta^{\frac{1}{2}}}Q, \quad \frac{dr}{d\nu} = \frac{\Omega}{D_\theta^{\frac{1}{2}}}R,$$

that is,

$$\frac{\partial}{\partial p}\left(\frac{d\theta}{d\nu}\right) = \gamma_{11}\frac{dp}{d\nu} + \gamma_{12}\frac{dq}{d\nu} + \gamma_{13}\frac{dr}{d\nu}.$$

Similarly

$$\frac{\partial}{\partial q}\left(\frac{d\theta}{d\nu}\right) = \gamma_{12}\frac{dp}{d\nu} + \gamma_{22}\frac{dq}{d\nu} + \gamma_{23}\frac{dr}{d\nu},$$

$$\frac{\partial}{\partial r}\left(\frac{d\theta}{d\nu}\right) = \gamma_{13}\frac{dp}{d\nu} + \gamma_{23}\frac{dq}{d\nu} + \gamma_{33}\frac{dr}{d\nu}.$$

We thence deduce two new concomitants. One is

$$\frac{d}{ds}\left(\frac{d\theta}{d\nu}\right) = p'\frac{\partial}{\partial p}\left(\frac{d\theta}{d\nu}\right) + q'\frac{\partial}{\partial q}\left(\frac{d\theta}{d\nu}\right) + r'\frac{\partial}{\partial r}\left(\frac{d\theta}{d\nu}\right)$$

$$= (\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12}\mathcal{X}p', q', r')\left(\frac{dp}{d\nu}, \frac{dq}{d\nu}, \frac{dr}{d\nu}\right):$$

or, if we write

$$\Theta = \frac{d^2\theta}{ds^2} = (\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12}\mathcal{X}p', q', r')^2,$$

then

$$\frac{d}{ds}\left(\frac{d\theta}{d\nu}\right) = \frac{1}{2}\left(\frac{\partial\Theta}{\partial p'}\frac{dp}{d\nu} + \frac{\partial\Theta}{\partial q'}\frac{dq}{d\nu} + \frac{\partial\Theta}{\partial r'}\frac{dr}{d\nu}\right)$$

$$= \frac{1}{4\Omega D_\theta^{\frac{1}{2}}}\left(\frac{\partial\Theta}{\partial p'}\frac{\partial D_\theta}{\partial\theta_1} + \frac{\partial\Theta}{\partial q'}\frac{\partial D_\theta}{\partial\theta_2} + \frac{\partial\Theta}{\partial r'}\frac{\partial D_\theta}{\partial\theta_3}\right),$$

where, now, the direction ds is the general arc-element in the region. Another form can be given to this concomitant. Let

$$T = d\Sigma^2 = (a, b, c, f, g, h)\mathcal{X}\bar{\xi}, \bar{\eta}, \bar{\zeta})^2,$$

where (§§ 267, 268) $\bar{\xi}, \bar{\eta}, \bar{\zeta}$, are the surface-variables for the θ-surface corresponding to ξ, η, ζ, as general orientation-coordinates, while

$$\frac{\bar{\xi}}{\theta_1} = \frac{\bar{\eta}}{\theta_2} = \frac{\bar{\zeta}}{\theta_3} = \frac{d\Sigma}{D_\theta^{\frac{1}{2}}} = \frac{T^{\frac{1}{2}}}{D_\theta^{\frac{1}{2}}}:$$

thus

$$\frac{d}{ds}\left(\frac{d\theta}{d\nu}\right) = \frac{1}{4\Omega T^{\frac{1}{2}}}\left(\frac{\partial\Theta}{\partial p'}\frac{\partial T}{\partial\bar{\xi}} + \frac{\partial\Theta}{\partial q'}\frac{\partial T}{\partial\bar{\eta}} + \frac{\partial\Theta}{\partial r'}\frac{\partial T}{\partial\bar{\zeta}}\right),$$

a mixed concomitant.

Next, we have

$$\begin{aligned}\frac{d^2\theta}{d\nu^2} &= \frac{d}{d\nu}\left(\frac{d\theta}{d\nu}\right)\\ &= \frac{dp}{d\nu}\frac{\partial}{\partial p}\left(\frac{d\theta}{d\nu}\right) + \frac{dq}{d\nu}\frac{\partial}{\partial q}\left(\frac{d\theta}{d\nu}\right) + \frac{dr}{d\nu}\frac{\partial}{\partial r}\left(\frac{d\theta}{d\nu}\right)\\ &= (\gamma_{11}', \gamma_{22}, \gamma_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12})\left(\frac{dp}{d\nu}, \frac{dq}{d\nu}, \frac{dr}{d\nu}\right)^2,\end{aligned}$$

a magnitude which, as thus expressed, is of the nature of an invariant appertaining simultaneously to the region and the θ-surface. It can also be expressed in a contravariant form

$$\frac{d^2\theta}{d\nu^2} = \frac{1}{4\Omega^2 T}(\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12})\left(\frac{\partial T}{\partial\bar{\xi}}, \frac{\partial T}{\partial\bar{\eta}}, \frac{\partial T}{\partial\bar{\zeta}}\right)^2,$$

where $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$, are the orientation-coördinates for the surface θ in the region.

There is a covariant of the region, connected with θ as a function of position, and given by

$$\frac{d^2\theta}{ds^2} = \Theta = (\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12})(p', q', r')^2,$$

where p', q', r', is any regional direction and, as already explained, the repeated differentiation with respect to s denotes progress along a regional geodesic. The variables of this direction are limited by the relation

$$U = \Sigma A p'^2 = 1.$$

Hence, as usual, there are three principal directions for the quantity Θ, that is, directions providing a maximum value or a minimum value for Θ: and they are given by

$$\frac{\partial\Theta}{\partial p'} - \Theta\frac{\partial U}{\partial p'} = 0, \quad \frac{\partial\Theta}{\partial q'} - \Theta\frac{\partial U}{\partial q'} = 0, \quad \frac{\partial\Theta}{\partial r'} - \Theta\frac{\partial U}{\partial r'} = 0,$$

equivalent to two equations only. Further, these principal values are the roots of the cubic equation

$$\begin{vmatrix} \gamma_{11} - A\Theta, & \gamma_{12} - H\Theta & \gamma_{13} - G\Theta \\ \gamma_{12} - H\Theta, & \gamma_{22} - B\Theta, & \gamma_{23} - F\Theta \\ \gamma_{13} - G\Theta, & \gamma_{23} - F\Theta, & \gamma_{33} - C\Theta \end{vmatrix} = 0.$$

Accordingly, there are three covariantive measures connected with these principal values, being the three rudimentary symmetric functions of the principal values. Let

$$\Delta_\gamma = \begin{vmatrix} \gamma_{11}, & \gamma_{12}, & \gamma_{13} \\ \gamma_{12}, & \gamma_{22}, & \gamma_{23} \\ \gamma_{13}, & \gamma_{23}, & \gamma_{33} \end{vmatrix},$$

and let $\overline{\Gamma}_{lm}$ denote the customary cofactor of the constituent γ_{lm} in Δ_γ, so that

$$\overline{\Gamma}_{11} = \gamma_{22}\gamma_{33} - \gamma_{23}^2, \quad \overline{\Gamma}_{23} = \gamma_{12}\gamma_{13} - \gamma_{11}\gamma_{23},$$

and so for the others. The foregoing equation becomes

$$\Delta_\gamma - (A\overline{\Gamma}_{11} + H\Gamma_{12} + G\overline{\Gamma}_{13} + B\overline{\Gamma}_{22} + F\overline{\Gamma}_{23} + C\overline{\Gamma}_{33})\,\Theta + (a\gamma_{11} + h\gamma_{12} + g\gamma_{13} + b\gamma_{22} + f\gamma_{23} + c\gamma_{33})\,\Theta^2 - \Omega^2\Theta^3 = 0;$$

consequently, there are three principal measures

$$\Theta_1\Theta_2\Theta_3 = \frac{1}{\Omega^2}\Delta_\gamma,$$

$$\Theta_2\Theta_3 + \Theta_3\Theta_1 + \Theta_1\Theta_2 = \frac{1}{\Omega^2}\Sigma a\gamma_{11},$$

$$\Theta_1 + \Theta_2 + \Theta_3 = \frac{1}{\Omega^2}\Sigma A\overline{\Gamma}_{11}.$$

Thus there are three concomitants

$$\Delta_\gamma, \quad \Sigma a\gamma_{11}, \quad \Sigma A\overline{\Gamma}_{11},$$

appertaining to θ as a function of position in the region: and their geometrical significance has been obtained.

Final selection from the concomitants obtained.

493. Among this aggregate of concomitants, it is sufficient to select six which, algebraically independent of one another, shall involve the quantities $\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12}$. Such a selection can manifestly be taken to be, the three concomitants just obtained

$$\Delta_\gamma, \quad \Sigma a\gamma_{11}, \quad \Sigma A\overline{\Gamma}_{11};$$

and three others as given by the concomitants denoted by

$$\Theta, \quad \frac{1}{\bar{\sigma}}, \quad \nabla.$$

But it is not to be assumed that all the concomitants of the system have been obtained: thus there remain the members of the complete asyzygetic system of twenty concomitants belonging to the two ternary quantics

$$W = \Sigma Lp'^2, \quad \Theta = \Sigma\gamma_{11}p'^2,$$

and likewise those of the complete system of

$$U = \Sigma A p'^2, \quad \Theta = \Sigma \gamma_{11} p'^2,$$

to mention only the simplest isolated aggregates*.

Concomitants of grade two, belonging to a curve in the region.

494. A second surface, represented by

$$\Im (p, q, r) = \text{constant},$$

was introduced primarily for the representation of a curve existing wholly within the region. Once introduced into the geometrical configuration, it causes the introduction of additional concomitants into the complete algebraical system, precisely the same in number and exactly analogous in significance, to those which were introduced through the presence of the θ-surface, but now of course connected with the derivatives $\Im_1$, $\Im_2$, $\Im_3$, of $\Im$ of the first order, and the derivatives of $\Im$ of the second order in the combinations (§ 485)

$$\bar{\gamma}_{11}, \ \bar{\gamma}_{22}, \ \bar{\gamma}_{33}, \ \bar{\gamma}_{23}, \ \bar{\gamma}_{31}, \ \bar{\gamma}_{12}.$$

Of the additional concomitants of the first grade which thus are required, adequate account has been taken (§ 481). Some of these, such as

$$\bar{D} = (a, b, c, f, g, h \mathbin{\text{\textbigcircle}} \Im_1, \Im_2, \Im_3)^2$$
$$= \Omega^2 \left(\frac{d\Im}{d\bar{\nu}}\right)^2,$$

belong solely to the $\Im$-surface, without any regard to its coexistence with the θ-surface. Others involve the parameters of both surfaces, such as

$$\Omega^2 \frac{d\theta}{d\nu} \frac{d\Im}{d\bar{\nu}} \cos\eta = (a, b, c, f, g, h \mathbin{\text{\textbigcircle}} \theta_1, \theta_2, \theta_3 \mathbin{\text{\textbigcircle}} \Im_1, \Im_2, \Im_3),$$

which gives the inclination η of the two surfaces at a point of their curve of intersection.

It is sufficient, for our algebraical aggregate, to select for the $\Im$-surface the concomitants of the second grade which are analogous to those selected for the θ-surface. Thus we can regard the system as complete by the introduction of the invariant

$$\Delta_{\bar{\gamma}} = \begin{vmatrix} \bar{\gamma}_{11}, & \bar{\gamma}_{12}, & \bar{\gamma}_{13} \\ \bar{\gamma}_{12}, & \bar{\gamma}_{22}, & \bar{\gamma}_{23} \\ \bar{\gamma}_{13}, & \bar{\gamma}_{23}, & \bar{\gamma}_{33} \end{vmatrix},$$

* There are, of course, the concomitants which involve the simultaneous expression of the three quantics

$$U, W, \Theta.$$

For a discussion of the complete asyzygetic aggregate of concomitants belonging to three ternary quadratics, reference may be made to a memoir by H. F. Baker, *Camb. Phil. Trans.*, vol. xv (1894), pp. 62–106.

and of the two invariants, which are intermediate for the region and the ϑ-surface,

$$\Sigma a\bar{\gamma}_{11},$$
$$\Sigma A\,(\bar{\gamma}_{22}\bar{\gamma}_{33} - \bar{\gamma}_{23}^2),$$

these three being the principal measures of the quantity $\frac{d^2\vartheta}{ds^2}$ estimated along a regional geodesic. There is this magnitude itself

$$\frac{d^2\vartheta}{ds^2}, = (\bar{\gamma}_{11}, \bar{\gamma}_{22}, \bar{\gamma}_{33}, \bar{\gamma}_{23}, \bar{\gamma}_{31}, \bar{\gamma}_{12}\mathcal{X}\, p', q', r')^2,$$

taken geodesically in the region along any non-superficial direction. There is the torsion of a superficial geodesic

$$\frac{1}{\bar{\sigma}_\vartheta} = \frac{\Omega}{4D_\vartheta}\begin{vmatrix} \frac{\partial\Psi_\vartheta}{\partial p'}, & \frac{\partial\Psi_\vartheta}{\partial q'}, & \frac{\partial\Psi_\vartheta}{\partial r'} \\ \frac{\partial U}{\partial p'}, & \frac{\partial U}{\partial q'}, & \frac{\partial U}{\partial r'} \\ \vartheta_1, & \vartheta_2, & \vartheta_3 \end{vmatrix}.$$

Finally, in this selection, there is the additive measure of the principal regional geodesic curvatures of ϑ-surface geodesics, in the form

$$-\frac{D_\vartheta^{\frac{3}{2}}}{\Omega}\left(\frac{1}{\bar{\gamma}_{1\vartheta}} + \frac{1}{\bar{\gamma}_{2\vartheta}}\right) = (B\bar{\gamma}_{33} - 2F\bar{\gamma}_{23} + C\bar{\gamma}_{22}, \ldots\mathcal{X}\vartheta_1, \vartheta_2, \vartheta_3)^2.$$

As for the θ-surface, so for the ϑ-surface, there are other concomitants, such as those belonging to the aggregate for

$$W = \Sigma Lp'^2, \quad \bar{\Theta} = \Sigma\bar{\gamma}_{11}p'^2,$$

and those for

$$U = \Sigma Ap'^2, \quad \Theta = \Sigma\gamma_{11}p'^2.$$

Moreover, these are only the simplest isolated aggregates, each algebraically complete in itself.

Beyond all these, there remain members of the whole system of simultaneous ternary quantics

$$U,\ W,\ \Theta,\ \bar{\Theta},$$

each of the second order. Among the aggregate, there are the four fundamental concomitants of the curve itself. As before, we denote by $\frac{dp}{dS}, \frac{dq}{dS}, \frac{dr}{dS}$, the direction-variables, where (§ 379)

$$\left.\begin{aligned} \frac{dp}{dS} &= \frac{\Omega}{D_\theta D_\vartheta^{\frac{1}{2}}\sin\eta}(\theta_2\vartheta_3 - \theta_3\vartheta_2) \\ \frac{dq}{dS} &= \frac{\Omega}{D_\theta D_\vartheta^{\frac{1}{2}}\sin\eta}(\theta_3\vartheta_1 - \theta_1\vartheta_3) \\ \frac{dr}{dS} &= \frac{\Omega}{D_\theta D_\vartheta^{\frac{1}{2}}\sin\eta}(\theta_1\vartheta_2 - \theta_2\vartheta_1) \end{aligned}\right\}.$$

With these direction-variables, we write

$$\frac{1}{\rho} = (L, N, J, I, K, M)\left(\frac{dp}{dS}, \frac{dq}{dS}, \frac{dr}{dS}\right)^2,$$

$$\frac{1}{\gamma_\theta} = -\frac{\Omega^2}{D_\theta}(\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{23}, \gamma_{31}, \gamma_{12})\left(\frac{dp}{dS}, \frac{dq}{dS}, \frac{dr}{dS}\right)^2 = -\frac{\Omega^2}{D_\theta}\Psi_\theta,$$

$$\frac{1}{\gamma_\vartheta} = -\frac{\Omega^2}{D_\vartheta}(\bar\gamma_{11}, \bar\gamma_{22}, \bar\gamma_{33}, \bar\gamma_{23}, \bar\gamma_{31}, \bar\gamma_{12})\left(\frac{dp}{dS}, \frac{dq}{dS}, \frac{dr}{dS}\right)^2 = -\frac{\Omega^2}{D_\vartheta}\Psi_\vartheta,$$

$$\frac{\sin^2\eta}{\bar G^2} = \frac{1}{\gamma_\theta^2} - 2\frac{\cos\eta}{\gamma_\theta\gamma_\vartheta} + \frac{1}{\gamma_\vartheta^2},$$

$$\bar G \tan\chi = \rho;$$

and then the concomitants of the curve of intersection of the two surfaces are as follows. The *radius* ρ_0 *of circular curvature* (estimated in the quadruple space) is given by the equation

$$\frac{1}{\rho_0^2} = \frac{1}{\rho^2} + \frac{1}{\bar G^2};$$

and the *radius of regional geodesic curvature* of the curve is $\bar G$. The radius of torsion, σ_0, and the radius of tilt, τ_0, are given by the two relations

$$\frac{\cos\chi}{\sigma_0} - \frac{\sin\chi}{\tau_0} = \frac{\bar G}{\sin^2\eta}\left\{\left(\frac{1}{\gamma_\vartheta\bar\tau_\theta} + \frac{1}{\gamma_\theta\bar\tau_\vartheta}\right)\cos\eta - \left(\frac{1}{\gamma_\theta\tau_\theta} + \frac{1}{\gamma_\vartheta\bar\tau_\vartheta}\right)\right\},$$

$$\frac{\sin\chi}{\sigma_0} + \frac{\cos\chi}{\tau_0} = \frac{\bar G^2}{\sin^2\eta}\left[\left\{\frac{1}{\gamma_\theta^2\bar\sigma_\vartheta} - \left(\frac{1}{\bar\sigma_\theta} + \frac{1}{\bar\sigma_\vartheta}\right)\frac{\cos\eta}{\gamma_\theta\gamma_\vartheta} + \frac{1}{\gamma_\vartheta^2\bar\sigma_\theta}\right\} + \frac{\gamma_\theta\gamma_\vartheta' - \gamma_\vartheta\gamma_\theta'}{\gamma_\theta^2\gamma_\vartheta^2}\right].$$

Here $\bar\sigma_\theta$, $\bar\tau_\theta$, $\bar\sigma_\vartheta$, $\bar\tau_\vartheta$, are the radii of torsion and of tilt of the θ-geodesic and the ϑ-geodesic through the direction; and their values are

$$\frac{1}{\bar\sigma_\theta} = \frac{\Omega}{4D_\theta}\begin{vmatrix} \frac{\partial\Psi_\theta}{\partial p'}, & \frac{\partial\Psi_\theta}{\partial q'}, & \frac{\partial\Psi_\theta}{\partial r'} \\ \frac{\partial U}{\partial p'}, & \frac{\partial U}{\partial q'}, & \frac{\partial U}{\partial r'} \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix}, \quad \frac{1}{\bar\sigma_\vartheta} = \frac{\Omega}{4D_\vartheta}\begin{vmatrix} \frac{\partial\Psi_\vartheta}{\partial p'}, & \frac{\partial\Psi_\vartheta}{\partial q'}, & \frac{\partial\Psi_\vartheta}{\partial r'} \\ \frac{\partial U}{\partial p'}, & \frac{\partial U}{\partial q'}, & \frac{\partial U}{\partial r'} \\ \vartheta_1, & \vartheta_2, & \vartheta_3 \end{vmatrix},$$

$$\frac{1}{\bar\tau_\theta} = \frac{1}{4D_\theta^{\frac{1}{2}}}\begin{vmatrix} \frac{\partial W}{\partial p'}, & \frac{\partial W}{\partial q'}, & \frac{\partial W}{\partial r'} \\ \frac{\partial U}{\partial p'}, & \frac{\partial U}{\partial q'}, & \frac{\partial U}{\partial r'} \\ \theta_1, & \theta_2, & \theta_3 \end{vmatrix}, \quad \frac{1}{\bar\tau_\vartheta} = \frac{1}{4D_\vartheta^{\frac{1}{2}}}\begin{vmatrix} \frac{\partial W}{\partial p'}, & \frac{\partial W}{\partial q'}, & \frac{\partial W}{\partial r'} \\ \frac{\partial U}{\partial p'}, & \frac{\partial U}{\partial q'}, & \frac{\partial U}{\partial r'} \\ \vartheta_1, & \vartheta_2, & \vartheta_3 \end{vmatrix},$$

where p', q', r', denote dp/dS, dq/dS, dr/dS, the direction-variables of the curve of intersection of the surfaces.

NOTE. Before passing, it should be noted that, owing to the introduction of the θ-surface and the ϑ-surface, we have two specific and independent

directions in which differentiation can be effected, viz. along the regional normals of the surfaces, according to the laws

$$\frac{d}{d\nu} = \frac{dp}{d\nu}\frac{\partial}{\partial p_1} + \frac{dq}{d\nu}\frac{\partial}{\partial q_1} + \frac{dr}{d\nu}\frac{\partial}{\partial r_1}, \quad \frac{d}{d\bar{\nu}} = \frac{dp}{d\bar{\nu}}\frac{\partial}{\partial p_1} + \frac{dq}{d\bar{\nu}}\frac{\partial}{\partial q_1} + \frac{dr}{d\bar{\nu}}\frac{\partial}{\partial r_1}.$$

There is also the differentiation in any general unspecified direction in the region, in either of the forms

$$\frac{d}{ds}, \quad p'\frac{\partial}{\partial p} + q'\frac{\partial}{\partial q} + r'\frac{\partial}{\partial r},$$

the former being requisite for any function of direction, the latter adequate for any function of position. These results are useful in the geometrical identification of the algebraical concomitants of grade higher than the second.

Concomitants of grade three.

495. We now shall consider concomitants of grade three; after the initial stages, the statements will be made mainly as to results, without the inclusion of the ancillary analytical details. There now will enter derivatives of x, y, z, v, of the third order, as well as those of the first order and those of the second order; and, in order to have uniformity of notation for the magnitudes that occur, we shall write

$$\frac{\partial^{l+m+n}\theta}{\partial p^l \partial q^m \partial r^n} = \theta_{lmn},$$

for all the values $l+m+n = 1$, or 2, or 3. Thus the former symbols are changed; we have x_{100} in place of x_1, x_{010} for x_2, x_{001} for x_3, x_{200} in place of x_{11}, and so on: thus reverting to the earlier usage (§ 449).

The infinitesimal changes of the third derivatives of x, y, z, v, arising under the infinitesimal parametric transformations, are particular instances of the result in § 451. The formulæ for the changes in the third derivatives of y, z, v, are analogous to those for the changes in the derivatives of x; and these changes in the third derivatives of x are:

$$\left.\begin{aligned}
\frac{1}{\epsilon}(x_{300} - x'_{300}) &= 3(x_{300}\phi_{100} + x_{210}\chi_{100} + x_{201}\psi_{100}) \\
&\quad + 3(x_{200}\phi_{200} + x_{110}\chi_{200} + x_{101}\psi_{200}) \\
&\quad + (x_{100}\phi_{300} + x_{010}\chi_{300} + x_{001}\psi_{300}) \\
\frac{1}{\epsilon}(x_{030} - x'_{030}) &= 3(x_{120}\phi_{010} + x_{030}\chi_{010} + x_{021}\psi_{010}) \\
&\quad + 3(x_{110}\phi_{020} + x_{020}\chi_{020} + x_{011}\psi_{020}) \\
&\quad + (x_{100}\phi_{030} + x_{010}\chi_{030} + x_{001}\psi_{030}) \\
\frac{1}{\epsilon}(x_{003} - x'_{003}) &= 3(x_{102}\phi_{001} + x_{012}\chi_{001} + x_{003}\psi_{001}) \\
&\quad + 3(x_{101}\phi_{002} + x_{011}\chi_{002} + x_{002}\psi_{002}) \\
&\quad + (x_{100}\phi_{003} + x_{010}\chi_{003} + x_{001}\psi_{003})
\end{aligned}\right\},$$

$$\frac{1}{\epsilon}(x_{210}-x'_{210}) = (x_{300}\phi_{010}+x_{210}\chi_{010}+x_{201}\psi_{010})+2(x_{210}\phi_{100}+x_{120}\chi_{100}+x_{111}\psi_{100})$$
$$+2(x_{200}\phi_{110}+x_{110}\chi_{110}+x_{101}\psi_{110})+(x_{110}\phi_{200}+x_{020}\chi_{200}+x_{011}\psi_{200})$$
$$+(x_{100}\phi_{210}+x_{010}\chi_{210}+x_{001}\psi_{210})$$

$$\frac{1}{\epsilon}(x_{201}-x'_{201}) = (x_{300}\phi_{001}+x_{210}\chi_{001}+x_{201}\psi_{001})+2(x_{201}\phi_{100}+x_{111}\chi_{100}+x_{102}\psi_{100})$$
$$+2(x_{200}\phi_{101}+x_{110}\chi_{101}+x_{101}\psi_{101})+(x_{101}\phi_{200}+x_{011}\chi_{200}+x_{002}\psi_{200})$$
$$+(x_{100}\phi_{201}+x_{010}\chi_{201}+x_{001}\psi_{201})$$

$$\frac{1}{\epsilon}(x_{120}-x'_{120}) = (x_{120}\phi_{100}+x_{030}\chi_{100}+x_{021}\psi_{100})+2(x_{210}\phi_{010}+x_{120}\chi_{010}+x_{111}\psi_{010})$$
$$+2(x_{110}\phi_{110}+x_{020}\chi_{110}+x_{011}\psi_{110})+(x_{200}\phi_{020}+x_{111}\chi_{020}+x_{101}\psi_{020})$$
$$+(x_{100}\phi_{120}+x_{010}\chi_{120}+x_{001}\psi_{120})$$

$$\frac{1}{\epsilon}(x_{102}-x'_{102}) = (x_{102}\phi_{100}+x_{012}\chi_{100}+x_{003}\psi_{100})+2(x_{201}\phi_{001}+x_{111}\chi_{001}+x_{102}\psi_{001})$$
$$+2(x_{101}\phi_{101}+x_{011}\chi_{101}+x_{002}\psi_{101})+(x_{200}\phi_{002}+x_{110}\chi_{002}+x_{101}\psi_{002})$$
$$+(x_{100}\phi_{102}+x_{010}\chi_{102}+x_{001}\psi_{102})$$

$$\frac{1}{\epsilon}(x_{021}-x'_{021}) = 2(x_{111}\phi_{010}+x_{021}\chi_{010}+x_{012}\psi_{010})+(x_{120}\phi_{001}+x_{030}\chi_{001}+x_{021}\psi_{001})$$
$$+2(x_{110}\phi_{011}+x_{020}\chi_{011}+x_{011}\psi_{011})+(x_{101}\phi_{020}+x_{011}\chi_{020}+x_{002}\psi_{020})$$
$$+(x_{100}\phi_{021}+x_{010}\chi_{021}+x_{001}\psi_{021})$$

$$\frac{1}{\epsilon}(x_{012}-x'_{012}) = 2(x_{111}\phi_{001}+x_{021}\chi_{001}+x_{012}\psi_{001})+(x_{102}\phi_{010}+x_{012}\chi_{010}+x_{003}\psi_{010})$$
$$+2(x_{101}\phi_{011}+x_{011}\chi_{011}+x_{002}\psi_{011})+(x_{110}\phi_{002}+x_{020}\chi_{002}+x_{011}\psi_{002})$$
$$+(x_{100}\phi_{012}+x_{010}\chi_{012}+x_{001}\psi_{012})$$

and

$$\frac{1}{\epsilon}(x_{111}-x'_{111})$$
$$=(x_{111}\phi_{100}+x_{021}\chi_{100}+x_{012}\psi_{100})+(x_{201}\phi_{010}+x_{111}\chi_{010}+x_{102}\psi_{010})$$
$$+(x_{210}\phi_{001}+x_{120}\chi_{001}+x_{111}\psi_{001})$$
$$+(x_{101}\phi_{110}+x_{011}\chi_{110}+x_{002}\psi_{110})+(x_{110}\phi_{101}+x_{020}\chi_{101}+x_{011}\psi_{101})$$
$$+(x_{200}\phi_{011}+x_{110}\chi_{011}+x_{101}\psi_{011})$$
$$+(x_{100}\phi_{111}+x_{010}\chi_{111}+x_{001}\psi_{111}).$$

Similar forms hold, as already stated, for the third derivatives of y, z, v. They also hold for the derivatives of any such functions as $\theta(p, q, r)$ and $\Theta(p, q, r)$ which, when equated severally to constants, provide level surfaces in the region.

Equations for purely regional concomitants.

496. We begin with the concomitants that affect the region alone and organic lines in the region; and, at this stage, we omit the consideration of arbitrarily assigned surfaces and arbitrarily assigned curves totally enclosed in the region.

(i) The customary process, as already used for the concomitants of grade unity and the concomitants of grade two, consists of the construction of the linear partial differential equations which must be satisfied by the concomitants. In the present instance, the full aggregate of these differential equations is composed of three sets. The first set consists of thirty equations, arising out of the thirty derivatives of the arbitrary functions ϕ, χ, ψ, of the third order. The second set consists of eighteen equations, arising out of the eighteen derivatives of ϕ, χ, ψ, of the second order. The third set arises out of the nine derivatives of ϕ, χ, ψ, of the first order: but, in three of these, the index (μ) of the concomitant occurs, as in §§ 476, 483; and thus there are eight $(=9-1)$ equations of the necessary type. Thus the total number of partial differential equations, arising for the conservation of parametric invariance, is fifty-six.

In addition, there are the six partial differential equations, arising out of the conservation of orientational invariance.

Consequently, there are sixty-two equations in all. They are a complete Jacobian system.

(ii) When restriction is made to concomitants affecting the region alone, the total number of arguments that occur in these partial differential equations, is made up from the aggregates in two categories. In the first category, we have, as before, the three variables t, u, w, of pure covariants, and the three variables ξ, η, ζ, of pure contravariants: of these, there are six, in all. In the second category, we have the derivatives of the first three orders of each of the variables x, y, z, v; of these, there are

$$4(3+6+10),$$

that is, seventy-six in all. Thus the total number of arguments, independent of one another so far as concerns the partial differential equations and occurring in those equations, is eighty-two.

(iii) As the complete Jacobian system contains sixty-two equations and involves eighty-two arguments, it possesses twenty integrals which are algebraically independent of one another; and every integral of the Jacobian system is algebraically expressible in terms of those twenty. We therefore choose an aggregate of twenty independent integrals as a complete set of integrals; every integral can be expressed algebraically in terms of the constituents of the set. Obviously, no set is unique in its content: that is, for one included integral, we may substitute another not included if the algebraical expression of the latter involves the one to be removed. But the number of integrals in a complete set is invariable.

Now this set involves concomitants of all three grades. We have seen (§ 478) that there are three concomitants specifically of the first grade, in addition to the universal concomitant t_ξ; and these three concomitants are denoted by U, Ω^2, T. We have also seen (§ 488) that, in addition to these four concomitants, there are six concomitants specifically of the second grade;

and such a set of six (not unique in the selection) is provided by the concomitants which are denoted by V; $\mathfrak{C}_1$, $\mathfrak{C}_2$, Υ; T', and Γ; being the circular curvature of the geodesic in any direction; the three principal measures of curvature; and the two measures (the Riemann measure and the additive measure) of superficial curvature in any orientation. Consequently, in a set of concomitants complete up to the third grade inclusive, we shall require ten concomitants which are specifically of the third grade.

The process of constructing the concomitants is similar to that adopted in the earlier instances of grade one and of grade two. We obtain a full aggregate of independent integrals of the set of thirty equations in (i); then we form, out of the constituents of this aggregate, another full aggregate of algebraically independent integrals of the eighteen equations in (i); and then, out of the constituents of this last aggregate, we take a full set of algebraically independent integrals of the third set in (i). The experience, acquired in the process for the earlier instances of grade one and of grade two, shews the analytical prudence of selecting, either initially or immediately after dealing with the first set of thirty equations, such combinations of constituents as automatically satisfy the six equations of orientational invariance; the selection has the resulting advantage that, for the final aggregate of integrals satisfying the third set of equations conserving parametric invariance, the equations of orientational invariance are satisfied without further modification or limitation or reduction.

Integration of the parametric equations.

497. To deal with the first set of parametric equations, thirty in number, we substitute, in place of the forty magnitudes x_{lmn}, y_{lmn}, z_{lmn}, v_{lmn}, where $l+m+n=3$, forty quantities, framed according to the combinations

$$\left.\begin{aligned} E_{lmn} &= \Sigma X x_{lmn} \\ \Omega^2 R_{lmn} &= a\Sigma x_{100}x_{lmn} + h\Sigma x_{010}x_{lmn} + g\Sigma x_{001}x_{lmn} \\ \Omega^2 S_{lmn} &= h\Sigma x_{100}x_{lmn} + b\Sigma x_{010}x_{lmn} + f\Sigma x_{001}x_{lmn} \\ \Omega^2 T_{lmn} &= g\Sigma x_{100}x_{lmn} + f\Sigma x_{010}x_{lmn} + c\Sigma x_{001}x_{lmn} \end{aligned}\right\},$$

these being the magnitudes E_{lmn}, R_{lmn}, S_{lmn}, T_{lmn}, which occur in § 294. When the thirty equations are transformed, so that these new magnitudes take the place of the old arguments, the equations become

$$\frac{\partial}{\partial R_{lmn}} = 0, \quad \frac{\partial}{\partial S_{lmn}} = 0, \quad \frac{\partial}{\partial T_{lmn}} = 0.$$

Thus ten independent integrals, involving derivatives of the third order, are provided for the thirty equations, in the form

$$E_{lmn}.$$

Moreover, for all the combinations of magnitudes which can occur, after the quantities R_{lmn}, S_{lmn}, T_{lmn}, have been substituted for the derivatives of x, y, z, v, of the third order, these equations require that the final integrals (and, consequently, all intermediate useful integrals) of the parametric equations do not involve these quantities R, S, T.

Further, as the derivatives of x, y, z, v, which are of the first order or of the second order, do not occur in the thirty equations, all such derivatives, and all combinations of such derivatives, also are integrals of the equations.

Lastly, it should be noted that, when the foregoing equations of definition for E, R, S, T are resolved for the quantities $x_{lmn}, y_{lmn}, z_{lmn}, v_{lmn}$, they yield the results of § 294; and so the values of E, R, S, T, are expressible in the forms there given. The same holds for the quantities e_{lmn} of § 292, about to be introduced analytically.

When we proceed to the second set of parametric equations, eighteen in number, we have to frame combinations of the quantities E_{lmn} (which are the only admissible arguments involving the third derivatives of x, y, z, v) with derivatives of the first order and of the second order of the point-variables. Accordingly, the first necessary step is the transformation of those equations, so that derivatives with regard to $x_{lmn}, y_{lmn}, z_{lmn}, v_{lmn}$, shall be merged into derivatives with regard to E_{lmn}, taking account of the requirements

$$\frac{\partial}{\partial R_{lmn}} = 0, \quad \frac{\partial}{\partial S_{lmn}} = 0, \quad \frac{\partial}{\partial T_{lmn}} = 0.$$

For example, we have (with $i+j+k=2$, $l+m+n=3$)

$$\begin{aligned}\Sigma x_{ijk} \frac{\partial}{\partial x_{lmn}} = \Sigma x_{ijk} \Big[X \frac{\partial}{\partial E_{lmn}} &+ \frac{1}{\Omega^2}(ax_{100} + hx_{010} + gx_{001}) \frac{\partial}{\partial R_{lmn}} \\ &+ \frac{1}{\Omega^2}(hx_{100} + bx_{010} + fx_{001}) \frac{\partial}{\partial S_{lmn}} \\ &+ \frac{1}{\Omega^2}(gx_{100} + fx_{010} + cx_{001}) \frac{\partial}{\partial T_{lmn}} \Big] \\ = (\Sigma X x_{ijk}) \frac{\partial}{\partial E_{lmn}};\end{aligned}$$

and thus an equation

$$\Sigma x_{100} \frac{\partial}{\partial x_{200}} + 3\Sigma x_{200} \frac{\partial}{\partial x_{300}} + \Sigma x_{110} \frac{\partial}{\partial x_{210}} + \Sigma x_{101} \frac{\partial}{\partial x_{201}} = 0$$

of the set of eighteen equations becomes

$$\Sigma x_{100} \frac{\partial}{\partial x_{200}} + 3L \frac{\partial}{\partial E_{300}} + M \frac{\partial}{\partial E_{210}} + K \frac{\partial}{\partial E_{201}} = 0;$$

and correspondingly for the others.

Again, when, instead of the twenty-four arguments $x_{ijk}, y_{ijk}, z_{ijk}, v_{ijk}$ (with $i+j+k=2$), we use the twenty-four magnitudes $L, N, J, I, K, M, \Gamma_{rs}, \Delta_{rs}, \Theta_{rs}$ (with $rs = 11, 12, 13, 22, 23, 33$), we have proved (§ 484) that

$$\Sigma x_{100} \frac{\partial}{\partial x_{200}} = \frac{\partial}{\partial \Gamma_{11}}, \quad \Sigma x_{100} \frac{\partial}{\partial x_{110}} = \frac{\partial}{\partial \Gamma_{12}}, \quad \Sigma x_{100} \frac{\partial}{\partial x_{101}} = \frac{\partial}{\partial \Gamma_{13}},$$

$$\Sigma x_{100} \frac{\partial}{\partial x_{020}} = \frac{\partial}{\partial \Gamma_{22}}, \quad \Sigma x_{100} \frac{\partial}{\partial x_{011}} = \frac{\partial}{\partial \Gamma_{23}}, \quad \Sigma x_{100} \frac{\partial}{\partial x_{002}} = \frac{\partial}{\partial \Gamma_{33}};$$

with similar results for combinations of x_{010}, x_{001}, respectively in the operators.

Hence, so far as the magnitudes of the third order are concerned, we take as integrals

$$E_{300} - 3(L\Gamma_{11} + M\Delta_{11} + K\Theta_{11}),$$

$$E_{210} - 2(L\Gamma_{12} + M\Delta_{12} + K\Theta_{12}) - (M\Gamma_{11} + N\Delta_{11} + K\Theta_{11}),$$

and so on: that is, we have the ten quantities

$$e_{lmn}$$

(with $l + m + n = 3$) as ten integrals of these equations which, in their expressions, involve derivatives of the third order. Moreover, the form of these equations shews that there are six integrals

$$L, N, J, I, K, M,$$

involving derivatives of the second order; they are the modified arguments of the second order which do not occur as operating variables.

Moreover, the equations do not involve derivatives of x, y, z, v, of the first order; so that, in addition, these twelve derivatives are integrals.

Now all the ten quantities e, of the third order, and all the six quantities L, N, J, I, K, M, of the second order, satisfy the six equations of orientational invariance, which accordingly impose no limitation of form or of combination in their occurrence in final integrals of the whole system. But there is restriction upon the combinations of the twelve derivatives of the first order alone; in order to satisfy the six equations of orientational invariance, we can have only six $(= 12 - 6)$ combinations of the twelve derivatives. Our earlier investigation (§ 478) shewed that six such combinations, adequate for the purpose, are provided by the magnitudes

$$A, B, C, F, G, H.$$

Accordingly, of the seventy-six magnitudes which enter into the expression of the thirty members of the first set and the eighteen members of the second set of equations of parametric invariance, and enter into the expression of the six equations of orientational invariance, we have twenty-two algebraically independent integrals

$$e_{lmn};\ L, N, J, I, K, M;\ A, B, C, F, G, H;$$

(with $l + m + n = 3$), constituting the requisite number at this stage.

For the third set of parametric equations—being all the equations that remain to be satisfied—we have these twenty-two quantities, together with t, u, w; ξ, η, ζ; as the possible arguments. They are twenty-eight in number. This third set of equations contains eight members of the homogeneous type (nine in all, but three of them have a linear term involving the index of the concomitant); and therefore, as before, we infer the necessity of twenty algebraically independent integrals.

In order to derive these integrals, the equations in the third set must be transformed so that the variables, which they contain explicitly, belong only

to these twenty-eight arguments. To express this transformation we denote certain symbolical operators, in agreement with the notation of § 483, as follows:

$$\left.\begin{aligned}
\overline{O}_{11} = &\left\{\left(3e_{300}\frac{\partial}{\partial e_{300}} + 2e_{210}\frac{\partial}{\partial e_{210}} + e_{120}\frac{\partial}{\partial e_{120}}\right) + \left(2e_{201}\frac{\partial}{\partial e_{201}} + e_{111}\frac{\partial}{\partial e_{111}}\right) + e_{102}\frac{\partial}{\partial e_{102}}\right\} \\
&+ \left(2L\frac{\partial}{\partial \overline{L}} + M\frac{\partial}{\partial \overline{M}} + K\frac{\partial}{\partial \overline{K}}\right) + \left(2A\frac{\partial}{\partial \overline{A}} + H\frac{\partial}{\partial \overline{H}} + G\frac{\partial}{\partial \overline{G}}\right) \\
\overline{O}_{22} = &\left\{\left(3e_{030}\frac{\partial}{\partial e_{030}} + 2e_{021}\frac{\partial}{\partial e_{021}} + e_{012}\frac{\partial}{\partial e_{012}}\right) + \left(2e_{120}\frac{\partial}{\partial e_{120}} + e_{111}\frac{\partial}{\partial e_{111}}\right) + e_{210}\frac{\partial}{\partial e_{210}}\right\} \\
&+ \left(M\frac{\partial}{\partial \overline{M}} + 2N\frac{\partial}{\partial \overline{N}} + I\frac{\partial}{\partial \overline{I}}\right) + \left(H\frac{\partial}{\partial \overline{H}} + 2B\frac{\partial}{\partial \overline{B}} + F\frac{\partial}{\partial \overline{F}}\right) \\
\overline{O}_{33} = &\left\{\left(3e_{003}\frac{\partial}{\partial e_{003}} + 2e_{012}\frac{\partial}{\partial e_{012}} + e_{021}\frac{\partial}{\partial e_{021}}\right) + \left(2e_{102}\frac{\partial}{\partial e_{102}} + e_{111}\frac{\partial}{\partial e_{111}}\right) + e_{201}\frac{\partial}{\partial e_{201}}\right\} \\
&+ \left(K\frac{\partial}{\partial \overline{K}} + I\frac{\partial}{\partial \overline{I}} + 2J\frac{\partial}{\partial \overline{J}}\right) + \left(G\frac{\partial}{\partial \overline{G}} + F\frac{\partial}{\partial \overline{F}} + 2C\frac{\partial}{\partial \overline{C}}\right)
\end{aligned}\right\},$$

$$\left.\begin{aligned}
\overline{O}_{12} = &\left\{\left(e_{300}\frac{\partial}{\partial e_{210}} + 2e_{210}\frac{\partial}{\partial e_{120}} + 3e_{120}\frac{\partial}{\partial e_{030}}\right) + \left(e_{201}\frac{\partial}{\partial e_{111}} + 2e_{111}\frac{\partial}{\partial e_{021}}\right) + e_{102}\frac{\partial}{\partial e_{012}}\right\} \\
&+ \left(L\frac{\partial}{\partial \overline{M}} + 2M\frac{\partial}{\partial \overline{N}} + K\frac{\partial}{\partial \overline{I}}\right) + \left(A\frac{\partial}{\partial \overline{H}} + 2H\frac{\partial}{\partial \overline{B}} + G\frac{\partial}{\partial \overline{F}}\right) \\
\overline{O}_{21} = &\left\{\left(3e_{210}\frac{\partial}{\partial e_{300}} + 2e_{120}\frac{\partial}{\partial e_{210}} + e_{030}\frac{\partial}{\partial e_{120}}\right) + \left(2e_{111}\frac{\partial}{\partial e_{201}} + e_{021}\frac{\partial}{\partial e_{111}}\right) + e_{012}\frac{\partial}{\partial e_{102}}\right\} \\
&+ \left(2M\frac{\partial}{\partial \overline{L}} + N\frac{\partial}{\partial \overline{M}} + I\frac{\partial}{\partial \overline{K}}\right) + \left(2H\frac{\partial}{\partial \overline{A}} + B\frac{\partial}{\partial \overline{H}} + F\frac{\partial}{\partial \overline{G}}\right) \\
\overline{O}_{13} = &\left\{\left(3e_{102}\frac{\partial}{\partial e_{003}} + 2e_{201}\frac{\partial}{\partial e_{102}} + e_{300}\frac{\partial}{\partial e_{201}}\right) + \left(2e_{111}\frac{\partial}{\partial e_{012}} + e_{210}\frac{\partial}{\partial e_{111}}\right) + e_{120}\frac{\partial}{\partial e_{021}}\right\} \\
&+ \left(L\frac{\partial}{\partial \overline{K}} + M\frac{\partial}{\partial \overline{I}} + 2K\frac{\partial}{\partial \overline{J}}\right) + \left(A\frac{\partial}{\partial \overline{G}} + H\frac{\partial}{\partial \overline{F}} + 2G\frac{\partial}{\partial \overline{C}}\right) \\
\overline{O}_{31} = &\left\{\left(3e_{201}\frac{\partial}{\partial e_{300}} + 2e_{102}\frac{\partial}{\partial e_{201}} + e_{003}\frac{\partial}{\partial e_{102}}\right) + \left(2e_{111}\frac{\partial}{\partial e_{210}} + e_{012}\frac{\partial}{\partial e_{111}}\right) + e_{021}\frac{\partial}{\partial e_{120}}\right\} \\
&+ \left(2K\frac{\partial}{\partial \overline{L}} + I\frac{\partial}{\partial \overline{M}} + J\frac{\partial}{\partial \overline{K}}\right) + \left(2G\frac{\partial}{\partial \overline{A}} + F\frac{\partial}{\partial \overline{H}} + C\frac{\partial}{\partial \overline{G}}\right) \\
\overline{O}_{23} = &\left\{\left(3e_{012}\frac{\partial}{\partial e_{003}} + 2e_{021}\frac{\partial}{\partial e_{012}} + e_{030}\frac{\partial}{\partial e_{021}}\right) + \left(2e_{111}\frac{\partial}{\partial e_{102}} + e_{120}\frac{\partial}{\partial e_{111}}\right) + e_{210}\frac{\partial}{\partial e_{201}}\right\} \\
&+ \left(M\frac{\partial}{\partial \overline{K}} + N\frac{\partial}{\partial \overline{I}} + 2I\frac{\partial}{\partial \overline{J}}\right) + \left(H\frac{\partial}{\partial \overline{G}} + B\frac{\partial}{\partial \overline{F}} + 2F\frac{\partial}{\partial \overline{C}}\right) \\
\overline{O}_{32} = &\left\{\left(3e_{021}\frac{\partial}{\partial e_{030}} + 2e_{012}\frac{\partial}{\partial e_{021}} + e_{003}\frac{\partial}{\partial e_{012}}\right) + \left(2e_{111}\frac{\partial}{\partial e_{120}} + e_{102}\frac{\partial}{\partial e_{111}}\right) + e_{201}\frac{\partial}{\partial e_{210}}\right\} \\
&+ \left(K\frac{\partial}{\partial \overline{M}} + 2I\frac{\partial}{\partial \overline{N}} + J\frac{\partial}{\partial \overline{I}}\right) + \left(G\frac{\partial}{\partial \overline{H}} + 2F\frac{\partial}{\partial \overline{B}} + C\frac{\partial}{\partial \overline{F}}\right)
\end{aligned}\right\}.$$

Then the members of the first set of parametric equations are

$$\overline{O}_{11} - t\frac{\partial}{\partial \overline{t}} + \xi\frac{\partial}{\partial \overline{\xi}} = \overline{O}_{22} - u\frac{\partial}{\partial \overline{u}} + \eta\frac{\partial}{\partial \overline{\eta}} = \overline{O}_{33} - w\frac{\partial}{\partial \overline{w}} + \zeta\frac{\partial}{\partial \overline{\zeta}} = \mu,$$

which (when the index, μ, of the concomitant is set aside) are two in number, and

$$\left.\begin{aligned} \overline{O}_{12} &= u\frac{\partial}{\partial \overline{t}} - \xi\frac{\partial}{\partial \overline{\eta}} \\ \overline{O}_{21} &= t\frac{\partial}{\partial \overline{u}} - \eta\frac{\partial}{\partial \overline{\xi}} \end{aligned}\right\}, \quad \left.\begin{aligned} \overline{O}_{13} &= w\frac{\partial}{\partial \overline{t}} - \xi\frac{\partial}{\partial \overline{\zeta}} \\ \overline{O}_{31} &= t\frac{\partial}{\partial \overline{w}} - \zeta\frac{\partial}{\partial \overline{\xi}} \end{aligned}\right\}, \quad \left.\begin{aligned} \overline{O}_{23} &= w\frac{\partial}{\partial \overline{u}} - \eta\frac{\partial}{\partial \overline{\zeta}} \\ \overline{O}_{32} &= u\frac{\partial}{\partial \overline{w}} - \zeta\frac{\partial}{\partial \overline{\eta}} \end{aligned}\right\},$$

being eight in all. They are a complete Jacobian system.

The integrals as concomitants of three ternary quantics: their significance.

498. These eight equations are the partial differential equations which must be satisfied by the concomitants, and which completely determine the concomitants, of the system of ternary quantics

$$\left.\begin{aligned} U &= (A, B, C, F, G, H\,\mathcal{X}t, u, w)^2 \\ V &= (L, N, J, I, K, M\,\mathcal{X}t, u, w)^2 \\ W &= (e\,\mathcal{X}t, u, w)^3 \end{aligned}\right\},$$

with ξ, η, ζ, as the contragredient variables. One of the necessary concomitants is

$$t_\xi = \xi t + \eta u + \zeta w;$$

it is universal, and is independent of all ternary quantics. In order to construct the algebraically complete set of concomitants of these ternary quantics—that is, a set in terms of the members of which any concomitant can be expressed algebraically—we need nineteen concomitants, algebraically independent of one another and not reducible by a unit through the use of t_ξ; and they must involve the coefficients of at least one of the quantics. They will be invariants, if they involve none of the variables t, u, w, ξ, η, ζ; covariants, if they involve only t, u, w; contravariants, if they involve only ξ, η, ζ; and mixed concomitants, if they involve both sets of variables.

Now of such concomitants, we already possess nine that are suitable for the purpose. We have the group of three,

$$U, \Omega^2, T;$$

which (§ 478) constitute the algebraically complete set for the quantic U alone and are of grade unity; and their geometrical significance has been stated. We have the group of six,

$$V;\ \mathfrak{C}_1, \mathfrak{C}_2, \Upsilon;\ T', \mathrm{I}';$$

which (§ 479) constitute an algebraically complete set*, in conjunction with U, Ω^2, T, of concomitants of grade two as well as of grade one; and their geometrical significance has been stated. We therefore require ten other concomitants which, while specifically involving the coefficients of the ternary cubic W, shall be algebraically independent of one another and shall not, among themselves, provide any relation among the nine concomitants already retained. Moreover, it is necessary to investigate the geometrical significance of the ten concomitants.

Some concomitants, fitted to supply these requirements, have already been obtained. We have (§ 292)

$$\frac{d}{ds}\left(\frac{1}{\rho}\right) = (e\,\mathfrak{Y}\,p', q', r')^3;$$

and therefore

$$W = (e\,\mathfrak{X}\,t, u, w)$$
$$= ds^3 \frac{d}{ds}\left(\frac{1}{\rho}\right),$$

or

$$WU^{-\frac{3}{2}} = \frac{d}{ds}\left(\frac{1}{\rho}\right).$$

Again, a concomitant connecting all three quantics is their Jacobian

$$J_0 = J\left(\frac{U, V, W}{p', q', r'}\right).$$

But, while $1/\sigma$—the torsion of a geodesic in the region—is a concomitant of grade two only, we have obtained (§ 309) an expression for the tilt of the geodesic in the form

$$\frac{1}{\sigma^2\tau} = \frac{1}{\Omega} J\left(\frac{U, V, W}{p', q', r'}\right):$$

that is, we have

$$J\left(\frac{U, V, W}{t, u, w}\right) = \frac{\Omega\, ds^3}{\sigma^2\tau}.$$

Next, we had (§ 312) a covariant Ξ, of order three in p', q', r'. We take

$$\Xi = \frac{1}{6}(a, b, c, f, g, h\,\mathfrak{X}\frac{\partial V}{\partial t}, \frac{\partial V}{\partial u}, \frac{\partial V}{\partial w}\,\mathfrak{X}\frac{\partial W}{\partial t}, \frac{\partial W}{\partial u}, \frac{\partial W}{\partial w}),$$

and therefore having

$$(a, b, c, f, g, h\,\mathfrak{X}\,L, M, K\,\mathfrak{X}\,e_{300}, e_{210}, e_{201})$$

for its leading coefficient; and the relation

$$-\frac{\sigma'}{\sigma^3} = \frac{1}{\Omega^2 U^{\frac{1}{2}}}(\Xi U - \Omega^2 W V)$$

* The concomitants V, Υ, T', are the complete set for V alone, as U, Ω^2, T are for U alone, regarded merely as ternary quantics.

was established; thus the value of $\overline{\Xi}$ is known, being given by

$$\overline{\Xi} = -\Omega^2\left(\frac{\sigma'}{\sigma^3}+\frac{\rho'}{\rho^3}\right)ds^3.$$

Again, the first derivatives, with regard to p, q, r, of the three principal measures of curvature $\mathfrak{L}$ (the linear measure), $\mathfrak{F}$ (the superficial measure) $\mathfrak{R}$ (the volumetric measure), have been obtained (§ 296). By means of these derivatives, several concomitants can be constructed.

We have obtained their first variations (§ 296) along the direction p', q', r', so that

$$\left.\begin{aligned}(\mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}_3 \mathfrak{Y} t, u, w) &= \frac{d\mathfrak{L}}{ds}.ds\\ (\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3 \mathfrak{Y} t, u, w) &= \frac{d\mathfrak{F}}{ds}.ds\\ (\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3 \mathfrak{Y} t, u, w) &= \frac{d\mathfrak{R}}{ds}.ds\end{aligned}\right\},$$

providing three concomitants of the third grade.

Next, a direction $\frac{dp}{dv_0}, \frac{dq}{dv_0}, \frac{dr}{dv_0}$, which lies in the region and is perpendicular to the surface element with ξ, η, ζ, as its orientation-coordinates, is given by

$$\frac{dp}{a\xi+h\eta+g\zeta} = \frac{dq}{h\xi+b\eta+f\zeta} = \frac{dr}{g\xi+f\eta+c\zeta} = \frac{dv_0}{\Omega T^{\frac{1}{2}}}.$$

Hence

$$\begin{aligned}\frac{d\mathfrak{L}}{dv_0} &= \mathfrak{L}_1\frac{dp}{dv_0}+\mathfrak{L}_2\frac{dq}{dv_0}+\mathfrak{L}_3\frac{dr}{dv_0}\\ &= \frac{1}{\Omega T^{\frac{1}{2}}}\{(\mathfrak{L}_1 a+\mathfrak{L}_2 h+\mathfrak{L}_3 g)\,\xi+(\mathfrak{L}_1 h+\mathfrak{L}_2 b+\mathfrak{L}_3 f)\,\eta+(\mathfrak{L}_1 g+\mathfrak{L}_2 f+\mathfrak{L}_3 c)\,\zeta\}\\ &= \frac{1}{24\Omega^3 T^{\frac{1}{2}}}(e)\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right)^3 T^2;\end{aligned}$$

and similarly

$$\begin{aligned}\frac{d\mathfrak{F}}{dv_0} &= \frac{1}{24\Omega^3 T^{\frac{1}{2}}}(e)\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right)^3 T\Gamma,\\ \frac{d\mathfrak{R}}{dv_0} &= \frac{1}{24\Omega^3 T^{\frac{1}{2}}}(e)\left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\zeta}\right)^3 TT',\end{aligned}$$

where T, Γ, T' are the contravariants (§§ 480, 488) giving the measure of a surface element, of the additive measure $\bar{H}$ and of the Riemann measure K of the superficial curvature of the region, with the orientation represented by ξ, η, ζ. Three additional concomitants are thus provided.

Finally, when we take a direction dv which is perpendicular to the direction ds and lies in the surface element with orientation-coordinates ξ, η, ζ

(so that it is perpendicular to the preceding direction dv_0), its direction-elements are given by

$$\frac{dp}{\begin{vmatrix} a, & h, & g \\ p', & q', & r' \\ \frac{\partial T}{\partial \xi}, & \frac{\partial T}{\partial \eta}, & \frac{\partial T}{\partial \zeta} \end{vmatrix}} = \frac{dq}{\begin{vmatrix} h, & b, & f \\ p', & q', & r' \\ \frac{\partial T}{\partial \xi}, & \frac{\partial T}{\partial \eta}, & \frac{\partial T}{\partial \zeta} \end{vmatrix}} = \frac{dr}{\begin{vmatrix} g, & f, & c \\ p', & q', & r' \\ \frac{\partial T}{\partial \xi}, & \frac{\partial T}{\partial \eta}, & \frac{\partial T}{\partial \zeta} \end{vmatrix}} = \frac{dv}{2\Phi}.$$

We obtain the value of Φ in the form

$$\Phi^2 = \Omega^4 \{TU - \Omega^2 (\xi p' + \eta q' + \zeta r')^2\};$$

and the significance of T and of $\xi p' + \eta q' + \zeta r'$ have been (§ 480) obtained. We write

$$\xi_1 = q' \frac{\partial T}{\partial \zeta} - r' \frac{\partial T}{\partial \eta}, \quad \eta_1 = r' \frac{\partial T}{\partial \xi} - p' \frac{\partial T}{\partial \zeta}, \quad \zeta_1 = p' \frac{\partial T}{\partial \eta} - q' \frac{\partial T}{\partial \xi},$$

so that ξ_1, η_1, ζ_1, are cogredient with ξ, η, ζ. Also we write

$$\left.\begin{aligned} T_1 &= \text{the value of } T \\ \Gamma_1 &= \ldots\ldots\ldots\ldots \Gamma \\ T_1' &= \ldots\ldots\ldots\ldots T' \end{aligned}\right\} \text{ with } \xi_1, \eta_1, \zeta_1, \text{ as arguments,}$$

so that these are mixed concomitants; and then, as in the preceding measures, we find

$$\begin{aligned} \frac{d\mathfrak{Q}}{dv} &= \mathfrak{Q}_1 \frac{dp}{dv} + \mathfrak{Q}_2 \frac{dq}{dv} + \mathfrak{Q}_3 \frac{dr}{dv} \\ &= \frac{1}{24\Omega^2\Phi} (e) \left(\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \eta_1}, \frac{\partial}{\partial \zeta_1} \right)^3 T_1^2, \\ \frac{d\mathfrak{F}}{dv} &= \frac{1}{24\Omega^2\Phi} (e) \left(\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \eta_1}, \frac{\partial}{\partial \zeta_1} \right)^3 T_1 \Gamma_1, \\ \frac{d\mathfrak{R}}{dv} &= \frac{1}{24\Omega^2\Phi} (e) \left(\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \eta_1}, \frac{\partial}{\partial \zeta_1} \right)^3 T_1 T_1', \end{aligned}$$

thus obtaining three additional concomitants.

Additional concomitants can be constructed in similar fashion. Moreover, there is the complete system which belongs solely to the ternary cubic W.

499. We now shall select, from the concomitants already constructed, an aggregate of the concomitants which are algebraically independent of one another. When they are selected, a complete aggregate of concomitants up to the third order inclusive is thereby established.

For this purpose, we use a customary theorem in ternariants, already used in § 309 and elsewhere: it is that, save as to a possible factor t_ξ^n, a ternariant is determinate through its leading coefficient, this leading coefficient belonging to the term which involves the highest power of t and the highest power of ξ that occur. When, therefore, an algebraical relation between leading

coefficients is once obtained, it determines a relation between the corresponding ternariants; occasionally, it may be necessary to incorporate a power of t_ξ into the ternariant relation, in order to secure homogeneity in t and in ξ.

Accordingly, we begin with the leading coefficients of ternariants. We take the concomitants W, J_0, Ξ, (on p. 491), and the three covariants

$$L_0 = \Omega^2 \frac{d\mathfrak{L}}{ds}\,ds, \quad F_0 = \Omega^2 \frac{d\mathfrak{F}}{ds}\,ds, \quad R_0 = \Omega^2 \frac{d\mathfrak{R}}{ds}\,ds,$$

derived from those on p. 492. We denote the leading coefficients of these six ternariants by $w_0, j_0, \xi_0, l_0, f_0, r_0$, respectively, so that

$$\begin{aligned}
w_0 &= e_{300},\\
j_0 &= (HK - GM)\, e_{300} + (GL - AK)\, e_{210} + (AM - HL)\, e_{201},\\
\xi_0 &= (aL + hM + gK)\, e_{300} + (hL + bM + fK)\, e_{210} + (gL + fM + cK)\, e_{201},\\
l_0 &= ae_{300} + 2he_{210} + 2ge_{201} + be_{120} + 2fe_{111} + ce_{102},\\
f_0 &= \bar{l}e_{300} + 2\bar{m}e_{210} + 2\bar{k}e_{201} + \bar{n}e_{120} + 2\bar{i}e_{111} + \bar{j}e_{102},\\
r_0 &= le_{300} + 2me_{210} + 2ke_{201} + ne_{120} + 2ie_{111} + je_{102}.
\end{aligned}$$

These are six equations, involving the six magnitudes $e_{300}, e_{210}, e_{201}, e_{120}, e_{111}, e_{102}$, linearly. The determinant of the coefficients of these six magnitudes on the right-hand sides is equal to the product

$$\begin{vmatrix} 1, & 0, & 0 \\ HK - GM, & GL - AK, & AM - HL \\ aL + hM + gK, & hL + bM + fK, & gL + fM + cK \end{vmatrix} \cdot \begin{vmatrix} b, & 2f, & c \\ \bar{n}, & 2\bar{i}, & \bar{j} \\ n, & 2i, & j \end{vmatrix}.$$

Neither factor in this product vanishes; thus the equations are independent of one another, that is, the six coefficients $w_0, j_0, \xi_0, l_0, f_0, r_0$, are independent of one another.

We next take the three linear contravariants

$$L_1 = \Omega^3 T^{\frac{1}{2}} \frac{d\mathfrak{L}}{d\nu_0}, \quad F_1 = \Omega^3 T^{\frac{1}{2}} \frac{d\mathfrak{F}}{d\nu_0}, \quad R_1 = \Omega^3 T^{\frac{1}{2}} \frac{d\mathfrak{R}}{d\nu_0},$$

constructed from those on p. 492, and the three mixed concomitants

$$L_2 = \frac{\Phi ds}{\Omega^2} \frac{d\mathfrak{L}}{d\nu}, \quad F_2 = \frac{\Phi ds}{\Omega^2} \frac{d\mathfrak{F}}{d\nu}, \quad R_2 = \frac{\Phi ds}{\Omega^2} \frac{d\mathfrak{R}}{d\nu};$$

and we denote the leading coefficients of these six concomitants by l_1, f_1, r_1. l_2, f_2, r_2, respectively. We thus have, at once,

$$\begin{aligned}
l_1 &= \Omega^2 (\mathfrak{L}_1 a + \mathfrak{L}_2 h + \mathfrak{L}_3 g) = l_0 a + \Omega^2 (\mathfrak{L}_2 h + \mathfrak{L}_3 g),\\
f_1 &= \Omega^2 (\mathfrak{F}_1 a + \mathfrak{F}_2 h + \mathfrak{F}_3 g) = f_0 a + \Omega^2 (\mathfrak{F}_2 h + \mathfrak{F}_3 g),\\
r_1 &= \Omega^2 (\mathfrak{R}_1 a + \mathfrak{R}_2 h + \mathfrak{R}_3 g) = r_0 a + \Omega^2 (\mathfrak{R}_2 h + \mathfrak{R}_3 g),
\end{aligned}$$

where the values of $\mathfrak{L}_2$, $\mathfrak{L}_3$, $\mathfrak{F}_2$, $\mathfrak{F}_3$, $\mathfrak{R}_2$, $\mathfrak{R}_3$, are given on pp. 73, 74. The

coefficients of t_ξ in $\frac{\Phi ds}{\Omega^2}\frac{dp}{d\nu}$, $\frac{\Phi ds}{\Omega^2}\frac{dq}{d\nu}$, $\frac{\Phi ds}{\Omega^2}\frac{dr}{d\nu}$, respectively, are found to be $0, G, -H$; hence

$$l_2 = G\mathfrak{L}_2 - H\mathfrak{L}_3,$$
$$f_2 = G\mathfrak{F}_2 - H\mathfrak{F}_3,$$
$$r_2 = G\mathfrak{R}_2 - H\mathfrak{R}_3.$$

All these six expressions for the leading coefficients $l_1, f_1, r_1, l_2, f_2, r_2$, involve the six quantities e_{300}, e_{210}, e_{201}, e_{120}, e_{111}, e_{102}, as well as the remaining four quantities e_{030}, e_{021}, e_{012}, e_{003}. Let the values already obtained for e_{300}, e_{210}, e_{201}, e_{120}, e_{111}, e_{102}, be substituted in each of the six specified equations. Such substitutions do not affect the terms involving the quantities e_{030}, e_{021}, e_{012}, e_{003}, because these do not occur in the substitutions. The six equations can therefore be taken in the form

$$h(be_{030} + 2fe_{021} + ce_{012}) + g(be_{021} + 2fe_{012} + ce_{003}) = \bar{l}_1,$$
$$G(be_{030} + 2fe_{021} + ce_{012}) + H(be_{021} + 2fe_{012} + ce_{003}) = \bar{l}_2.$$
$$h(\bar{n}e_{030} + 2\bar{i}e_{021} + \bar{j}e_{012}) + g(\bar{n}e_{021} + 2\bar{i}e_{012} + \bar{j}e_{003}) = \bar{f}_1,$$
$$G(\bar{n}e_{030} + 2\bar{i}e_{021} + \bar{j}e_{012}) + H(\bar{n}e_{021} + 2\bar{i}e_{012} + \bar{j}e_{003}) = \bar{f}_2,$$
$$h(ne_{030} + 2ie_{021} + je_{012}) + g(ne_{021} + 2ie_{012} + je_{003}) = \bar{r}_1,$$
$$G(ne_{030} + 2ie_{021} + je_{012}) + H(ne_{021} + 2ie_{012} + je_{003}) = \bar{r}_2,$$

where $\bar{l}_1$ is linear in l_1, $\bar{l}_2$ in l_2, $\bar{f}_1$ in f_1, $\bar{f}_2$ in f_2, $\bar{r}_1$ in r_1, $\bar{r}_2$ in r_2, and all six are linear in $w_0, j_0, \xi_0, l_0, f_0, r_0$.

Out of these six concomitants $L_1, F_1, R_1, L_2, F_2, R_2$, we select the four L_1, R_1, L_2, R_2, (the selection is not the only choice that is suitable for the purpose); and we associate this selected quartet with the sextet $W, J_0, \Xi, L_0, F_0, R_0$, to make an aggregate of ten. The leading coefficients of the four, thus selected, are taken in the form

$$be_{030} + 2fe_{021} + ce_{012} = \frac{1}{\mu}(g\bar{l}_2 + H\bar{l}_1),$$
$$be_{021} + 2fe_{012} + ce_{003} = \frac{1}{\mu}(-h\bar{l}_2 + G\bar{l}_1),$$
$$ne_{030} + 2ie_{021} + je_{012} = \frac{1}{\mu}(g\bar{r}_2 + H\bar{r}_1),$$
$$ne_{021} + 2ie_{012} + je_{003} = \frac{1}{\mu}(-h\bar{r}_2 + G\bar{r}_1),$$

where

$$\mu = Hh + Gg = Aa + Hh + Gg - Aa = \Omega^2 - Aa,$$

a magnitude that does not vanish. The four quantities e_{030}, e_{021}, e_{012}, e_{003}, do not occur in the six leading coefficients w_0, j_0, ξ_0, l_0, f_0, r_0, and therefore are

independent of that sextet. The determinant of the coefficients of these four quantities is equal to

$$(bj-cn)^2-4(bi-fn)(fj-ci),$$

a quantity which does not vanish. Hence the four equations are independent of one another, that is, $\bar{l}_1$, $\bar{l}_2$, $\bar{r}_1$, $\bar{r}_2$, are independent of one another; they involve the quantities e_{030}, e_{021}, e_{012}, e_{003}, which do not occur in the selected sextet of the leading coefficients, and they are therefore independent of the members of the sextet. Also $\bar{l}_1, \bar{l}_2, \bar{r}_1, \bar{r}_2$, are linear in l_1, in l_2, in r_1, and in r_2, respectively; hence the whole set of ten coefficients w_0, j_0, ξ_0, l_0, f_0, r_0, l_1, r_1, l_2, r_2, are independent of one another.

Consequently the ten concomitants, of which they are the leading coefficients, and which are

$$W, J_0, \Xi, L_0, F_0, R_0, L_1, R_1, L_2, R_2,$$

are independent of one another. They are of the third grade. We thus have obtained ten independent concomitants of the third grade: and, with them, the aggregate of third grade concomitants is complete.

500. As the selected concomitants constitute a complete system, every concomitant of the third grade is expressible in terms of the members of the system. When the actual expression is desired, it can be obtained as follows.

Because the ten equations, expressing the leading coefficients, are independent of one another, they can be resolved so as to provide values of the ten magnitudes e_{lmn}, for $l+m+n=3$. Now take any regional concomitant C of grade three. It involves the quantities e_{lmn}; and it is homogeneous, say of degree δ, in those quantities. Let $C_0 t^r \xi^s$ be its leading term, so that its leading coefficient C_0 is homogeneous of degree δ in at least some of the quantities e_{lmn}. In C_0, let substitution be made for all the occurring quantities e_{lmn} in terms of the ten selected leading coefficients; thus C_0 will acquire a form

$$C_0=\phi(w_0, j_0, \xi_0, l_0, f_0, r_0, l_1, r_1, l_2, r_2),$$

where the coefficients of the various combinations of the ten arguments are functions of magnitudes of the second order and may therefore be described as of rank two. Then the regional concomitant C is expressible in a form

$$C=\phi(W, J_0, \Xi, L_0, F_0, R_0, L_1, R_1, L_2, R_2),$$

where changes from the coefficients in C_0 have been made. It is found that these coefficients of rank two, as they occur in C_0, are leading coefficients of concomitants of grade two. In effecting the transition from the expression for C_0 to the expression for C, the coefficients in C_0 of rank two are changed into the concomitants of grade two of which they are the leading coefficients; and, where necessary, a factor t_ξ^n (for an appropriate value of the integer n) is associated with a term, in order to secure homogeneity in the variables t, u, w, and in the variables ξ, η, ζ. The last modification is admissible because unity is the leading coefficient in t_ξ.

Simple illustrations are provided by the expressions of $\mathfrak{F}_1$ and $\mathfrak{F}_2$, which are not included in the selected aggregate. From the analysis in the text, it follows at once that relations $\Theta_1=0$, and $\Theta_2=0$, express the leading coefficients $\bar{f}_1$ and $\bar{f}_2$ in terms of those of the ten selected concomitants, where

$$\Theta_1=\begin{vmatrix} \mu\bar{f}_1 & , & h\bar{n}, & 2h\bar{i}+g\bar{n}, & hj+2g\bar{i}, & g\bar{j} \\ g\bar{l}_2+H\bar{l}_1, & b, & 2f, & c, & 0 & \\ -h\bar{l}_2+G\bar{l}_1, & 0, & b, & 2f, & c & \\ g\bar{r}_2+H\bar{r}_1, & n, & 2i, & j, & 0 & \\ -h\bar{r}_2+G\bar{r}_1, & 0, & n, & 2i, & j & \end{vmatrix},$$

and Θ_2 is derived from Θ_1 by changing the first row in Θ_1 into

$$\mu f_2, \quad -H\bar{n}, \quad -2H\bar{i}+G\bar{n}, \quad -H\bar{j}+2G\bar{i}, \quad G\bar{j}.$$

When Θ_1 is expanded, the term in f_1 is $\mu S_0\bar{f}_1$. The coefficient μ is

$$\begin{aligned}\mu &= Hh+Gg \\ &= Aa+Hh+Gg=\Omega^2-Aa,\end{aligned}$$

which is the leading coefficient of the mixed concomitant $\Omega^2 t_\xi^2-UT$, of grade two. The coefficient S_0 is

$$\begin{aligned}S_0 &= (bj-cn)^2-4(bi-fn)(fj-ci) \\ &= (bj-2fi+cn)^2-4(bc-f^2)(jn-i^2) \\ &= \bar{\lambda}^2-4\Omega^2\Upsilon A L,\end{aligned}$$

which (pp. 150, 154) is the leading coefficient of a covariant

$$\Gamma'^2-4\Omega^4\mathfrak{R}UV,$$

of grade two.

In the same expansion, the coefficient of $\bar{l}_2$, after reduction, is found to be

$$2(ng^2-2igh+jh^2)\begin{vmatrix} \bar{n}, & \bar{i}, & \bar{j} \\ n, & i, & j \\ b, & f, & c \end{vmatrix}.$$

Now

$$ng^2-2igh+jh^2=a(cn-2fi+bj)-\Omega^2\{\mathfrak{S}_2-2(Al+Hm+Gk)+Al\};$$

by the results in § 488, this quantity is the leading coefficient of the mixed concomitant

$$T\Gamma'-\Omega^2(\mathfrak{S}_2 t_\xi^2-2B'+UT''),$$

which is of grade two. Also, with the notation of § 329 where the concomitants T, Γ, T'', are

$$T=\Sigma a\xi^2=a_\xi^2, \quad \Gamma=\Sigma\bar{l}\xi^2=\bar{a}_\xi^2, \quad T''=\Sigma l\xi^2=a_\xi'^2,$$

the determinant

$$\begin{vmatrix} \bar{n}, & \bar{i}, & \bar{j} \\ n, & i, & j \\ b, & f, & c \end{vmatrix}$$

is the leading coefficient of the covariant

$$(\bar{a}a't)(\bar{a}at)(a'at),$$

which, being of grade two, is expressible (§ 329) in terms of the retained concomitants of grade two.

Thus, in the transition from the expression of the leading coefficient $\bar{f}_1$, linear in f_1, to the expression of the concomitant F_1, the coefficient of $\bar{l}_2$, which as it occurs in the expres-

sion of $\bar{f}_1$ is of rank two, is the leading coefficient of a concomitant of grade two. When the transition to concomitants is made, the coefficient of rank two is changed into the concomitant of grade two of which it is the leading coefficient.

Similarly, for the coefficients of $\bar{l}_1, \bar{r}_2, \bar{r}_1$, in turn. Similarly also for the coefficients in $\bar{f}_1$ other than f_1, which occurs linearly in $\bar{f}_1$ with unity as its coefficient.

When the transition is completed, due account being taken of homogeneity in the two sets of variables, the result is to give an expression for the concomitant F_1 in terms of the constituents of the selected aggregate.

The results of the discussion may be summarised as follows. We have already obtained more than sufficient concomitants to construct an algebraically complete system of ten members, properly of the third grade. In terms of these, conjointly with the preceding set of six concomitants properly of the second grade, with the preceding set of three concomitants properly of the first grade, and with the universal concomitant $t\xi + u\eta + w\zeta$, every concomitant of grade not greater than three can be expressed algebraically; and the geometrical significance of any such concomitant is obtained by substituting, in such algebraical expression, the values of the concomitants that occur. It is not the fact that the simplest of the geometrical magnitudes of the third grade are necessarily the simplest algebraical combinations of the simultaneous concomitants of any of the three quantics; there would be an advantage in selecting some complete system which would possess at once simplicity of analytical form and simplicity of geometrical significance.

NOTE. One general result, similar to that indicated in § 494 and used (§ 498) in the preceding construction of concomitants, should be noted as providing a useful means of constructing concomitants and, simultaneously, securing their geometrical significance. The three regional operations represented by $\frac{d}{ds}, \frac{d}{d\nu_0}, \frac{d}{d\nu}$, being

$$p' \frac{\partial}{\partial p} + q' \frac{\partial}{\partial q} + r' \frac{\partial}{\partial r},$$

$$\frac{dp}{d\nu_0} \frac{\partial}{\partial p} + \frac{dq}{d\nu_0} \frac{\partial}{\partial q} + \frac{dr}{d\nu_0} \frac{\partial}{\partial r},$$

$$\frac{dp}{d\nu} \frac{\partial}{\partial p} + \frac{dq}{d\nu} \frac{\partial}{\partial q} + \frac{dr}{d\nu} \frac{\partial}{\partial r},$$

are of general utility.

Concomitants of grade three, belonging to a surface.

501. There remains the determination of the concomitants of grade three which arise through the introduction of a surface and of a curve (as the intersection of two surfaces) into the configuration of the region. We have

already determined the appropriate concomitants, of grade one and of grade two, for the surface and the curve; and the analysis indicates that, for our immediate purpose, it suffices to consider the concomitants of grade higher than two for a surface alone, those for a curve being simply derivable from those for two surfaces. We therefore proceed to the construction of those concomitants, properly of grade three, appertaining to the surface

$$\theta(p, q, r) = \text{constant},$$

which, supplementing those of the first and the second grades already (§ 489) obtained, shall complete the full algebraical system for grade three.

The changes in the derivatives of θ of the third order, due to the infinitesimal changes in the parametric variables, have already been indicated; and the effect, upon the equations which conserve parametric invariance, is the addition of terms in form analogous to the existing terms. Thus the first set of thirty, among these equations of §§ 496, 497 as now constructed, becomes

$$\left.\begin{aligned} \Sigma x_{100}\frac{\partial}{\partial x_{lmn}} + \theta_{100}\frac{\partial}{\partial \theta_{lmn}} = 0 \\ \Sigma x_{010}\frac{\partial}{\partial x_{lmn}} + \theta_{010}\frac{\partial}{\partial \theta_{lmn}} = 0 \\ \Sigma x_{001}\frac{\partial}{\partial x_{lmn}} + \theta_{001}\frac{\partial}{\partial \theta_{lmn}} = 0 \end{aligned}\right\},$$

for $l + m + n = 3$. Now, with the magnitudes E_{lmn}, R_{lmn}, S_{lmn}, T_{lmn}, defined in § 497, we have

$$\Sigma x_{100}\frac{\partial}{\partial x_{lmn}} = \frac{\partial}{\partial R_{lmn}}, \quad \Sigma x_{010}\frac{\partial}{\partial x_{lmn}} = \frac{\partial}{\partial S_{lmn}}, \quad \Sigma x_{001}\frac{\partial}{\partial x_{lmn}} = \frac{\partial}{\partial T_{lmn}};$$

and thus the equations become

$$\frac{\partial}{\partial R_{lmn}} + \theta_{100}\frac{\partial}{\partial \theta_{lmn}} = 0, \quad \frac{\partial}{\partial S_{lmn}} + \theta_{010}\frac{\partial}{\partial \theta_{lmn}} = 0, \quad \frac{\partial}{\partial T_{lmn}} + \theta_{001}\frac{\partial}{\partial \theta_{lmn}} = 0.$$

We consequently have the former (§ 497) ten integrals of these equations, given by E_{lmn}; and, as the necessary complement for a full set of integrals, we have ten additional members $\bar{\theta}_{lmn}$, where

$$\bar{\theta}_{lmn} = \theta_{lmn} - \theta_{100}R_{lmn} - \theta_{010}S_{lmn} - \theta_{001}T_{lmn},$$

for $l + m + n = 3$. Moreover, the only combinations of magnitudes, involving third-order derivatives which may survive in our integrals, are the twenty quantities E_{lmn}, $\bar{\theta}_{lmn}$.

There are corresponding additions to the equations of the other two sets in the parametric group. The effect, so far as θ is concerned in connection with concomitants properly of grade one and of grade two respectively, has

been (§§ 484, 489) to introduce magnitudes γ_{11}, γ_{22}, γ_{33}, γ_{23}, γ_{31}, γ_{12}, which, now denoted by γ_{ijk} for $i+j+k=2$, are given by the relations

$$\left.\begin{aligned}
\gamma_{11} &= \gamma_{200} = \theta_{200} - \theta_{100}\Gamma_{11} - \theta_{010}\Delta_{11} - \theta_{001}\Theta_{11} \\
\gamma_{12} &= \gamma_{110} = \theta_{110} - \theta_{100}\Gamma_{12} - \theta_{010}\Delta_{12} - \theta_{001}\Theta_{12} \\
\gamma_{13} &= \gamma_{101} = \theta_{101} - \theta_{100}\Gamma_{13} - \theta_{010}\Delta_{13} - \theta_{001}\Theta_{13} \\
\gamma_{22} &= \gamma_{020} = \theta_{020} - \theta_{100}\Gamma_{22} - \theta_{010}\Delta_{22} - \theta_{001}\Theta_{22} \\
\gamma_{23} &= \gamma_{011} = \theta_{011} - \theta_{100}\Gamma_{23} - \theta_{010}\Delta_{23} - \theta_{001}\Theta_{23} \\
\gamma_{33} &= \gamma_{002} = \theta_{002} - \theta_{100}\Gamma_{33} - \theta_{010}\Delta_{33} - \theta_{001}\Theta_{33}
\end{aligned}\right\},$$

The second set of equations contains derivatives with θ_{lmr} as arguments of derivation; and therefore, among other changes of expression, these terms must be transformed so as to become derivatives with $\bar{\theta}_{lmn}$ as arguments of derivation. When all the necessary transformations have been effected and the integrals of the equations then are constructed, it appears that, in addition to the integrals appertaining to the region alone (§ 498), and in addition to the integrals of grade two appertaining more specially to the surface as given by the quantities γ_{ijk} (with $i+j+k=2$), there are the ten further integrals δ_{lmn} (with $l+m+n=3$) given by

$$\left.\begin{aligned}
\delta_{300} &= \bar{\theta}_{300} - 3(\Gamma_{11}\gamma_{200} + \Delta_{11}\gamma_{110} + \Theta_{11}\gamma_{101}) \\
\delta_{030} &= \bar{\theta}_{030} - 3(\Gamma_{22}\gamma_{110} + \Delta_{22}\gamma_{020} + \Theta_{22}\gamma_{011}) \\
\delta_{003} &= \bar{\theta}_{003} - 3(\Gamma_{33}\gamma_{101} + \Delta_{33}\gamma_{011} + \Theta_{33}\gamma_{002})
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\delta_{210} &= \bar{\theta}_{210} - 2(\Gamma_{12}\gamma_{200} + \Delta_{12}\gamma_{110} + \Theta_{12}\gamma_{101}) - (\Gamma_{11}\gamma_{110} + \Delta_{11}\gamma_{020} + \Theta_{11}\gamma_{011}) \\
\delta_{201} &= \bar{\theta}_{201} - 2(\Gamma_{13}\gamma_{200} + \Delta_{13}\gamma_{110} + \Theta_{13}\gamma_{101}) - (\Gamma_{11}\gamma_{101} + \Delta_{11}\gamma_{011} + \Theta_{11}\gamma_{002}) \\
\delta_{120} &= \bar{\theta}_{120} - 2(\Gamma_{12}\gamma_{110} + \Delta_{12}\gamma_{020} + \Theta_{12}\gamma_{011}) - (\Gamma_{22}\gamma_{200} + \Delta_{22}\gamma_{110} + \Theta_{22}\gamma_{101}) \\
\delta_{102} &= \bar{\theta}_{102} - 2(\Gamma_{13}\gamma_{101} + \Delta_{13}\gamma_{011} + \Theta_{13}\gamma_{002}) - (\Gamma_{33}\gamma_{200} + \Delta_{33}\gamma_{110} + \Theta_{33}\gamma_{101}) \\
\delta_{021} &= \bar{\theta}_{021} - 2(\Gamma_{23}\gamma_{110} + \Delta_{23}\gamma_{020} + \Theta_{23}\gamma_{011}) - (\Gamma_{22}\gamma_{101} + \Delta_{22}\gamma_{011} + \Theta_{22}\gamma_{002}) \\
\delta_{012} &= \bar{\theta}_{012} - 2(\Gamma_{23}\gamma_{101} + \Delta_{23}\gamma_{011} + \Theta_{23}\gamma_{002}) - (\Gamma_{33}\gamma_{110} + \Delta_{33}\gamma_{020} + \Theta_{33}\gamma_{011})
\end{aligned}\right\},$$

$$\begin{aligned}
\delta_{111} = \bar{\theta}_{111} &- (\Gamma_{23}\gamma_{200} + \Delta_{23}\gamma_{110} + \Theta_{23}\gamma_{101}) \\
&- (\Gamma_{31}\gamma_{110} + \Delta_{31}\gamma_{020} + \Theta_{31}\gamma_{011}) \\
&- (\Gamma_{12}\gamma_{101} + \Delta_{12}\gamma_{011} + \Theta_{12}\gamma_{002}).
\end{aligned}$$

As regards these additional integrals, due to the introduction of the function θ and its derivatives, it is to be noted first, that no derivative of θ is affected by the equations of orientational invariance: and second, that all the coefficients in these integrals δ_{lmn} (as all the coefficients in the integrals γ_{ijk}) actually satisfy those equations. Hence the equations of orientational invariance impose neither limitation in combination nor reduction in number upon either the group of integrals δ_{lmn} or the group of integrals γ_{ijk}.

Surface concomitants of grade three: geometrical significance.

502. There remains the third set of parametric equations, eight in number. The earlier investigations (§§ 476—478) shewed that, as regards quantities of the first grade, it is necessary to take the combinations A, B, C, F, G, H, of those quantities in order that the orientational equations may be satisfied; and, consequently, the arguments, admissible for our integrals, are the former magnitudes e_{lmn}, for $l+m+n=3$; L, N, J, I, K, M; A, B, C, F, G, H; γ_{ijk}, for $i+j+k=2$; and, now, the quantities δ_{lmn}, for $l+m+n=3$. The quantities must therefore be made the arguments of derivation in this remaining set of parametric equations.

When the transformations have been effected, the form of the equations shews that every concomitant of grade one, of grade two, and of grade three, is algebraically expressible in terms of the members of an aggregate of concomitants belonging to a simultaneous system of ternary quantics, with t, u, w, as the variables of covariants, and ξ, η, ζ, as the (contragredient) variables of contravariants. The system of ternary quantics is composed of the six fundamental members, represented by

$$(A, B, C, F, G, H\,\mathfrak{X}t, u, w)^2,$$

$$(L, N, J, I, K, M\,\mathfrak{X}t, u, w)^2,$$

$$(e_{lmn}\,\mathfrak{X}t, u, w)^3;$$

$$(\theta_{100}, \theta_{010}, \theta_{001}\,\mathfrak{X}t, u, w),$$

$$(\gamma_{200}, \gamma_{020}, \gamma_{002}, \gamma_{011}, \gamma_{101}, \gamma_{110}\,\mathfrak{X}t, u, w)^2,$$

$$(\delta_{lmn}\,\mathfrak{X}t, u, w)^3.$$

The total number of constituents, in the algebraically complete system, in terms of which every concomitant can be expressed, is thirty-eight, in addition to the universal concomitant $t\xi+u\eta+w\zeta$.

Of these thirty-eight, there are six concomitants, properly of the first grade; they have been identified and interpreted (§§ 478, 480), three belonging purely to the region, and three appertaining specially to the surface contained in the region. There are twelve concomitants, properly of the second grade; they have been identified and interpreted (§§ 489, 490), six belonging purely to the region, and six appertaining specially to the surface contained in the region. Thus there remain twenty concomitants, properly of grade three. Of these twenty, ten belong purely to the region; they have been identified and interpreted (§§ 496—498).

Thus there remain, in the system which is algebraically complete up to the third grade inclusive, ten concomitants properly of the third grade and appertaining specially to the surface. For the identification and the interpretation, there is ample material in the regional first derivatives of the

magnitudes which properly are of grade two and appertain to the surface: adequate examples are provided by the quantities

$$\frac{d}{ds}\left(\frac{1}{\gamma}\right),$$

$$\frac{d}{ds}\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right), \quad \frac{d}{dv_0}\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right), \quad \frac{d}{dv}\left(\frac{1}{\gamma_1}+\frac{1}{\gamma_2}\right),$$

$$\frac{d}{ds}\left(\frac{1}{\gamma_1\gamma_2}\right), \quad \frac{d}{dv_0}\left(\frac{1}{\gamma_1\gamma_2}\right), \quad \frac{d}{dv}\left(\frac{1}{\gamma_1\gamma_2}\right),$$

$$\frac{d^3\theta}{dn^3}, \quad \frac{d^3\theta}{ds\,dn^2}, \quad \frac{d^3\theta}{ds^2\,dn},$$

where in the second and the third lines the derivatives are as in § 498, and in the fourth line dn is a regional direction normal to the surface.

Regional concomitants of grade four: Conclusion.

503. The preceding analysis provides the determination of the fundamental magnitudes, of grade not higher than three, connected with a region and a surface (or a curve, as arising by the intersection of two surfaces) in the region; and it provides also the determination of the fundamental magnitudes of like grade, connected with geodesics of the region and geodesics of a surface in the region.

Derivatives of these magnitudes might be required, taken along various directions. Such directions may be assumed arbitrarily in the region and are represented by the variables t, u, w; or they may be connected organically with arbitrarily assumed orientations in the region represented by the variables ξ, η, ζ; or they may be connected organically with a surface, such as directions tangential to a given surface or the regional direction normal to the surface. When such derivatives are to be formed, one of the methods of construction has been provided in §§ 494, 498, which furnish the analytical equivalents of the geometrical derivatives. Once formed, the analytical expressions contain (or become) concomitants of grade higher than three. And, of course, in a plenary homaloidal space of dimensions greater than four, it is necessary to obtain fundamental concomitants of a grade correspondingly higher than three, so as to provide for the expression of the additional curvatures of geodesics in configurations which (§ 194) are due to the additional dimensions.

Should an algebraically complete set of concomitants of higher grade be required for homaloidal quadruple space, a mere extension of the preceding analysis suffices. The results will be stated, without the contributory analysis, for regional concomitants of grade four, of which fifteen must be obtained (in addition to those already constructed) in order to constitute the complete

system. An initial guide to these fifteen quantities is provided by the second derivative of the circular curvature of a regional geodesic, in the form

$$\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) = \frac{d}{ds}\{(e\,\mathfrak{X}\,p', q', r')^3\}$$

$$= \Sigma \frac{4!}{l!\,m!\,n!} f_{lmn} p'^l q'^m r'^n,$$

for $l+m+n=4$; and these coefficients f_{lmn} are magnitudes of grade four.

We take, initially, sixty magnitudes as the algebraical equivalents of the fifteen derivatives of x, y, z, v, which are of the fourth order. These sixty new magnitudes are denoted by η_{lmn}, U_{lmn}, Q_{lmn}, W_{lmn}, for $l+m+n=4$; they are such that

$$\theta_{lmn} = \Theta\eta_{lmn} + \theta_1 U_{lmn} + \theta_2 Q_{lmn} + \theta_3 W_{lmn},$$

for $\theta, \Theta = x, X$; y, Y; z, Z; v, V; and $l+m+n=4$. In particular, the fifteen quantities η_{lmn} are

$$\eta_{lmn} = X x_{lmn} + Y y_{lmn} + Z z_{lmn} + V v_{lmn};$$

and the quantities U_{lmn}, Q_{lmn}, W_{lmn}, are

$$\Omega^2 U_{lmn} = a\Sigma x_{100} x_{lmn} + h\Sigma x_{010} x_{lmn} + g\Sigma x_{001} x_{lmn},$$

$$\Omega^2 Q_{lmn} = h\Sigma x_{100} x_{lmn} + b\Sigma x_{010} x_{lmn} + f\Sigma x_{001} x_{lmn},$$

$$\Omega^2 W_{lmn} = g\Sigma x_{100} x_{lmn} + f\Sigma x_{010} x_{lmn} + c\Sigma x_{001} x_{lmn}.$$

All these magnitudes η_{lmn}, U_{lmn}, Q_{lmn}, W_{lmn}, have already (§ 294, Exs. 1, 2) occurred, implicitly, in connection with derivatives of certain magnitudes of the third order. But the quantities needed for the present purpose are the magnitudes η_{lmn}, and e_{rst}, R_{rst}, S_{rst}, T_{rst}, for $r+s+t=3$.

For the construction of the fundamental magnitudes of the fourth grade, the quantities η_{lmn} are used; and these constructed magnitudes are f_{lmn}, for $l+m+n=4$, given in the following table:

$$\left.\begin{aligned} f_{400} &= \eta_{400} - 4(LR_{300} + MS_{300} + KT_{300}) \\ &\quad - 6(\Gamma_{11} e_{300} + \Delta_{11} e_{210} + \Theta_{11} e_{201}) - 3(L\,\mathfrak{X}\,\Gamma_{11}, \Delta_{11}, \Theta_{11})^2 \\ f_{040} &= \eta_{040} - 4(MR_{030} + NS_{030} + IT_{030}) \\ &\quad - 6(\Gamma_{22} e_{120} + \Delta_{22} e_{030} + \Theta_{22} e_{021}) - 3(L\,\mathfrak{X}\,\Gamma_{22}, \Delta_{22}, \Theta_{22})^2 \\ f_{004} &= \eta_{004} - 4(KR_{003} + IS_{003} + JT_{003}) \\ &\quad - 6(\Gamma_{33} e_{102} + \Delta_{33} e_{012} + \Theta_{33} e_{003}) - 3(L\,\mathfrak{X}\,\Gamma_{33}, \Delta_{33}, \Theta_{33}) \end{aligned}\right\},$$

$$
\begin{aligned}
f_{310} = \eta_{310} &- 3\,(LR_{210} + MS_{210} + KT_{210}) - (MR_{300} + NS_{300} + IT_{300}) \\
&- 3\,(\Gamma_{11}e_{210} + \Delta_{11}e_{120} + \Theta_{11}e_{111}) - 3\,(\Gamma_{12}e_{300} + \Delta_{12}e_{210} + \Theta_{12}e_{201}) \\
&- 3\,(L\,\mathcal{Y}\,\Gamma_{11},\ \Delta_{11},\ \Theta_{11}\,\mathcal{Y}\,\Gamma_{12},\ \Delta_{12},\ \Theta_{12}) \\
f_{301} = \eta_{301} &- 3\,(LR_{201} + MS_{201} + KT_{201}) - (KR_{300} + IS_{300} + JT_{300}) \\
&- 3\,(\Gamma_{11}e_{201} + \Delta_{11}e_{111} + \Theta_{11}e_{102}) - 3\,(\Gamma_{13}e_{300} + \Delta_{13}e_{210} + \Theta_{13}e_{201}) \\
&- 3\,(L\,\mathcal{Y}\,\Gamma_{11},\ \Delta_{11},\ \Theta_{11}\,\mathcal{Y}\,\Gamma_{13},\ \Delta_{13},\ \Theta_{13}) \\
f_{130} = \eta_{130} &- (LR_{030} + MS_{030} + KT_{030}) - 3\,(MR_{120} + NS_{120} + IT_{120}) \\
&- 3\,(\Gamma_{12}e_{120} + \Delta_{12}e_{030} + \Theta_{12}e_{111}) - 3\,(\Gamma_{22}e_{210} + \Delta_{22}e_{120} + \Theta_{22}e_{111}) \\
&- 3\,(L\,\mathcal{Y}\,\Gamma_{12},\ \Delta_{12},\ \Theta_{12}\,\mathcal{Y}\,\Gamma_{22},\ \Delta_{22},\ \Theta_{22}) \\
f_{031} = \eta_{031} &- 3\,(MR_{021} + NS_{021} + IT_{021}) - (KR_{030} + IS_{030} + JT_{030}) \\
&- 3\,(\Gamma_{22}e_{111} + \Delta_{22}e_{021} + \Theta_{22}e_{102}) - 3\,(\Gamma_{23}e_{120} + \Delta_{23}e_{030} + \Theta_{23}e_{021}) \\
&- 3\,(L\,\mathcal{Y}\,\Gamma_{22},\ \Delta_{22},\ \Theta_{22}\,\mathcal{Y}\,\Gamma_{23},\ \Delta_{23},\ \Theta_{23}) \\
f_{103} = \eta_{103} &- (LR_{003} + MS_{003} + KT_{003}) - 3\,(KR_{102} + IS_{102} + JT_{102}) \\
&- 3\,(\Gamma_{13}e_{102} + \Delta_{13}e_{012} + \Theta_{13}e_{003}) - 3\,(\Gamma_{33}e_{201} + \Delta_{33}e_{111} + \Theta_{33}e_{102}) \\
&- 3\,(L\,\mathcal{Y}\,\Gamma_{13},\ \Delta_{13},\ \Theta_{13}\,\mathcal{Y}\,\Gamma_{33},\ \Delta_{33},\ \Theta_{33}) \\
f_{013} = \eta_{013} &- (MR_{003} + NS_{003} + IT_{003}) - 3\,(KR_{012} + IS_{012} + JT_{012}) \\
&- 3\,(\Gamma_{23}e_{102} + \Delta_{23}e_{012} + \Theta_{23}e_{003}) - 3\,(\Gamma_{33}e_{111} + \Delta_{33}e_{021} + \Theta_{33}e_{012}) \\
&- 3\,(L\,\mathcal{Y}\,\Gamma_{23},\ \Delta_{23},\ \Theta_{23}\,\mathcal{Y}\,\Gamma_{33},\ \Delta_{33},\ \Theta_{33})
\end{aligned}
$$

$$
\begin{aligned}
f_{220} = \eta_{220} &- 2\,(LR_{120} + MS_{120} + KT_{120}) - 2\,(MR_{210} + NS_{210} + IT_{210}) \\
&- (\Gamma_{11}e_{120} + \Delta_{11}e_{030} + \Theta_{11}e_{021}) - 4\,(\Gamma_{12}e_{120} + \Delta_{12}e_{030} + \Theta_{12}e_{021}) - (\Gamma_{22}e_{300} + \Delta_{22}e_{210} + \Theta_{22}e_{201}) \\
&- 2\,(L\,\mathcal{Y}\,\Gamma_{12},\ \Delta_{12},\ \Theta_{12})^2 - (L\,\mathcal{Y}\,\Gamma_{11},\ \Delta_{11},\ \Theta_{11}\,\mathcal{Y}\,\Gamma_{22},\ \Delta_{22},\ \Theta_{22}) \\
f_{202} = \eta_{202} &- 2\,(LR_{102} + MS_{102} + KT_{102}) - 2\,(KR_{201} + IS_{201} + JT_{201}) \\
&- (\Gamma_{11}e_{102} + \Delta_{11}e_{012} + \Theta_{11}e_{003}) - 4\,(\Gamma_{13}e_{201} + \Delta_{13}e_{111} + \Theta_{13}e_{102}) - (\Gamma_{33}e_{300} + \Delta_{33}e_{210} + \Theta_{33}e_{201}) \\
&- 2\,(L\,\mathcal{Y}\,\Gamma_{13},\ \Delta_{13},\ \Theta_{13})^2 - (L\,\mathcal{Y}\,\Gamma_{11},\ \Delta_{11},\ \Theta_{11}\,\mathcal{Y}\,\Gamma_{33},\ \Delta_{33},\ \Theta_{33}) \\
f_{022} = \eta_{022} &- 2\,(MR_{012} + NS_{012} + IT_{012}) - 2\,(KR_{021} + IS_{021} + JT_{021}) \\
&- (\Gamma_{22}e_{102} + \Delta_{22}e_{012} + \Theta_{22}e_{003}) - 4\,(\Gamma_{23}e_{111} + \Delta_{23}e_{021} + \Theta_{23}e_{102}) - (\Gamma_{33}e_{120} + \Delta_{33}e_{030} + \Theta_{33}e_{021}) \\
&- 2\,(L\,\mathcal{Y}\,\Gamma_{23},\ \Delta_{23},\ \Theta_{23})^3 - (L\,\mathcal{Y}\,\Gamma_{22},\ \Delta_{22},\ \Theta_{22}\,\mathcal{Y}\,\Gamma_{33},\ \Delta_{33},\ \Theta_{33})
\end{aligned}
$$

$$
\begin{aligned}
f_{211} = \eta_{211} &- 2\,(LR_{111} + MS_{111} + KT_{111}) - (MR_{201} + NS_{201} + IT_{201}) - (KR_{210} + IS_{210} + JT_{210}) \\
&- (\Gamma_{11}e_{111} + \Delta_{11}e_{021} + \Theta_{11}e_{102}) - (\Gamma_{23}e_{300} + \Delta_{23}e_{210} + \Theta_{23}e_{201}) \\
&- 2\,(\Gamma_{12}e_{201} + \Delta_{12}e_{111} + \Theta_{12}e_{102}) - 2\,(\Gamma_{13}e_{210} + \Delta_{13}e_{120} + \Theta_{13}e_{111}) \\
&- (L\,\mathcal{Y}\,\Gamma_{11},\ \Delta_{11},\ \Theta_{11}\,\mathcal{Y}\,\Gamma_{23},\ \Delta_{23},\ \Theta_{23}) - 2\,(L\,\mathcal{Y}\,\Gamma_{12},\ \Delta_{12},\ \Theta_{12}\,\mathcal{Y}\,\Gamma_{13},\ \Delta_{13},\ \Theta_{13}) \\
f_{121} = \eta_{121} &- (LR_{021} + MS_{021} + KT_{021}) - 2\,(MR_{111} + NS_{111} + IT_{111}) - (KR_{120} + IS_{120} + JT_{120}) \\
&- (\Gamma_{22}e_{201} + \Delta_{22}e_{111} + \Theta_{22}e_{102}) - (\Gamma_{13}e_{120} + \Delta_{13}e_{030} + \Theta_{13}e_{021}) \\
&- 2\,(\Gamma_{23}e_{210} + \Delta_{23}e_{120} + \Theta_{23}e_{111}) - 2\,(\Gamma_{12}e_{111} + \Delta_{12}e_{021} + \Theta_{12}e_{102}) \\
&- (L\,\mathcal{Y}\,\Gamma_{13},\ \Delta_{13},\ \Theta_{13}\,\mathcal{Y}\,\Gamma_{22},\ \Delta_{22},\ \Theta_{22}) - 2\,(L\,\mathcal{Y}\,\Gamma_{12},\ \Delta_{12},\ \Theta_{12}\,\mathcal{Y}\,\Gamma_{23},\ \Delta_{23},\ \Theta_{23}) \\
f_{112} = \eta_{112} &- (LR_{012} + MS_{012} + KT_{012}) - (MR_{102} + NS_{102} + IT_{102}) + (KR_{111} + IS_{111} + JT_{111}) \\
&- (\Gamma_{33}e_{210} + \Delta_{33}e_{120} + \Theta_{33}e_{111}) - (\Gamma_{12}e_{102} + \Delta_{12}e_{012} + \Theta_{12}e_{003}) \\
&- 2\,(\Gamma_{13}e_{111} + \Delta_{13}e_{021} + \Theta_{13}e_{102}) - 2\,(\Gamma_{23}e_{201} + \Delta_{23}e_{111} + \Theta_{23}e_{102}) \\
&- (L\,\mathcal{Y}\,\Gamma_{12},\ \Delta_{12},\ \Theta_{12}\,\mathcal{Y}\,\Gamma_{33},\ \Delta_{33},\ \Theta_{33}) - 2\,(L\,\mathcal{Y}\,\Gamma_{13},\ \Delta_{13},\ \Theta_{13}\,\mathcal{Y}\,\Gamma_{23},\ \Delta_{23},\ \Theta_{23})
\end{aligned}
$$

When the concluding set of eight parametric equations is transformed so that the arguments of derivation are the quantities, obtained above in the form of integrals of the earlier sets of parametric equations and of the set of equations conserving orientational invariance, it appears that every concomitant of grade not higher than four, appertaining solely to the region, is a concomitant of the system of simultaneous ternary quantics

$$(A, B, C, F, G, H\,(*)^2,$$
$$(L, N, J, I, K, M\,(*)^2,$$
$$(e\,(*)^3,$$
$$(f\,(*)^4,$$

where the last two symbolical expressions represent

$$\Sigma \frac{3!}{l!\,m!\,n!}\, e_{lmn} t^l u^m w^n,$$

$$\Sigma \frac{4!}{\lambda!\,\mu!\,\nu!}\, f_{\lambda\mu\nu} t^\lambda u^\mu w^\nu,$$

respectively, for $l+m+n=3$, $\lambda+\mu+\nu=4$, the covariant variables of the quantics being t, u, w, and the (contragredient) contravariant variables for the configurations being ξ, η, ζ.

Probably a sufficient amount of the necessary calculation has been presented to establish a claim, that all concomitants of geometrical configurations of every type in homaloidal quadruple space can be constructed by direct algebraical and differential operations of similar tenor, supplemented by the integration of linear partial differential equations. Moreover, this supplementary integration is rendered superfluous, when the theory of invariantive forms, binariant and ternariant, is utilised. The method is effective, within the range of customary manipulation, for the construction of the analytical expression of all the concomitants. It does not provide (a purely analytical method cannot be expected to provide) the geometrical interpretation of each of the concomitants. Such interpretation lies, initially, outside the domain of the invariantive calculus: it belongs essentially to the domain of geometry, and it demands the investigation of geometrical measures and geometrical relations with a view to their analytical expression. Concomitants, thus obtained, are already endowed with their intrinsic significance. When once members, sufficient in number to constitute an algebraically complete system, have been constructed and their accompanying significance has been discovered, the derivation of the geometrical interpretation of each of the remaining concomitants is attained, through the formation of its expression in terms of the constituent members of the complete system of concomitants.

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