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PRACTICAL GEOMETRY

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PRACTICAL GEOMETRY

BASED ON THE VARIOUS GEOMETRY BOOKS
BY GODFREY AND SIDDONS

By

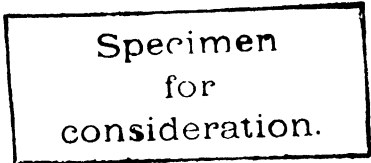
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for
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PREFACE

Practical Geometry and *Theoretical Geometry* are based on the various Geometry books by Godfrey and Siddons. It is intended that they should be used side by side; but they have been separated into two books because (i) past experience shows that, as the pupil advances, he will gain a better appreciation of the separate functions of Practical and Theoretical Geometry if they are separated in his text-book; (ii) for pupils of different ages, with different abilities and with different requirements, the blendings of the two should differ; (iii) whether looking at the whole course of Theoretical Geometry covered in this stage or examining the close logical connection of a group of theorems, the student will gain a better appreciation of Theoretical Geometry when other matter is not interposed.

The method of using the two books may be briefly described as follows:—The pupil should read *Practical Geometry* and make longer or shorter excursions into *Theoretical Geometry*. (References are given in *Practical Geometry* to the proofs of theorems that should be learnt in this stage, and the easy riders in *Theoretical Geometry* should play an important part. This blending of Theoretical and Practical Geometry has been done in detail for younger pupils in *Junior Geometry* by the same authors.) This course should give a sound knowledge of the methods of Geometry—experimental, intuitional, logical—a knowledge of most of the theorems should be acquired, and parts of the subject should be knit together with complete logical proof.

The next stage should be a continuous reading of *Theoretical Geometry* and an examination of the whole logical chain of propositions.

A complete school course for a non-specialist should cover the greater part of *Practical Geometry* and *Theoretical Geometry*. This will take him well beyond the School Certificate stage and will have broken the ground for much of the specialist's work.

For a technical school student most attention must, for many reasons, be given to the *Practical Geometry*. A course which is to be

TABLE OF CONTENTS

	PAGE
Chapter 1. ANGLES, PARALLELS	1
Chapter 2. TRIANGLES	9
Chapter 3. RULER AND COMPASS CONSTRUCTIONS .	18
Chapter 4. QUADRILATERALS. Parallelogram. Kite. Rhombus.	23
Chapter 5. SYMMETRY	30
Chapter 6. SCALE DRAWING	34
Chapter 7. AREA	42
Chapter 8. THE THEOREM OF PYTHAGORAS . . .	55
Chapter 9. LOCI	61
Miscellaneous Exercises I	67
Revision Papers A	78
Chapter 10. THE CIRCLE. CHORDS Circumscribed Circle.	86
Chapter 11. THE CIRCLE. TANGENTS. Inscribed and Escribed Circles.	92
Chapter 12. REGULAR POLYGONS	98
Chapter 13. THE CIRCLE. ANGLE PROPERTIES . . . Concyclic Points. Angle in Alternate Segment. Constant Angle Locus.	103
Chapter 14. ARCS, CIRCUMFERENCE, AREA	114
Chapter 15. CONTACT OF CIRCLES. COMMON TANGENTS	121
Chapter 16. SIMILARITY Ratio. Similar Figures. Areas of Similar Figures. Volumes of Similar Solids. Similar Triangles.	130
Chapter 17. RATIO AND PROPORTION	144
Chapter 18. APPLICATIONS OF SIMILAR TRIANGLES . Miscellaneous. Rectangle Properties of Circles. Mean Proportional. Construction of Similar Polygons.	152

	PAGE
Chapter 19. CONTINUOUS CHANGE OF A FIGURE . .	165
Maximum and Minimum Values.	
FURTHER EXAMPLES OF LOCI. ENVELOPES . .	168
Ellipse. Parabola. Hyperbola. Various Loci. Roulettes.	
Miscellaneous Exercises II	175
Chapter 20. DRAWING AND DEVELOPMENT OF SOLID FIGURES	190
Chapter 21. PRISM AND CYLINDER; SOLIDS OF UNIFORM CROSS-SECTION	193
Chapter 22. THE PYRAMID AND CONE	198
Chapter 23. INCLINATION OF LINE TO PLANE AND PLANE TO PLANE	210
Chapter 24. THE SPHERE	217
Chapter 25. THE REGULAR SOLIDS	225
Chapter 26. PLAN AND ELEVATION	227
Miscellaneous Exercises III	232
Revision Papers B	240
INDEX	259

TO THE TEACHER

¶ before an exercise implies that the exercise should be discussed in class.

Where the statement of a theorem is followed by a reference to the proof in Theoretical Geometry, the proof should be learnt and easy riders on the theorem should be done.

The exercises are all grouped under topics denoted by cross headings and carefully graded in each group. Only in the Revision Papers is no clue given to the particular topic.

Notes and Answers, published separately, contains the answers to all the numerical exercises and notes and suggestions for the teacher.

Note the summaries at the ends of the chapters.

Note the very full index at the end of the book.

CHAPTER 1

ANGLES, PARALLELS

§1. In the world a child makes its first acquaintance with Geometry by handling solid objects. In this book it is assumed that the pupil is already familiar with the appearance of the simpler geometrical solids; they are considered in detail in Chapters 21–25.

Solids are limited by surfaces; the surfaces meet in lines; the lines meet in points. The shape and arrangement of these surfaces, lines and points necessitates the examination of the properties of lines and angles, and their measurement.

It is assumed that the pupil can measure lines with a graduated scale, and angles with a protractor.

This chapter is a revision of the main facts and principles concerned with angles at a point and parallel straight lines.

ANGLE.

§2. If you hold one arm of your compasses firm and turn the other about the hinge, the two arms form an **angle**.

In the same way if two straight lines XC, XD are drawn from a point X (see fig. 1) they form an angle at X. X is called the **vertex** of the angle, and XC, XD its **arms**.

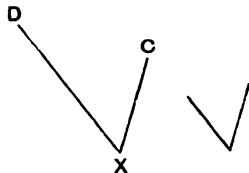


Fig. 1.

An angle may be denoted by three letters; thus we speak of the angle CXD, the middle letter denoting the vertex of the angle and the outside letters denoting points on its arms.

If there is only one angle at a point X, we call it the angle
is the abbreviation for *angle*.

It is important to notice that *the size of an angle does not depend on the lengths of its arms*. The two angles in fig. 1 are equal.

Take a piece of paper and fold it; you will get something like fig. 2; fold it again so that the edge OB fits on the edge OA; now

open the paper; you have four **right angles** made by the creases, as in fig. 3.

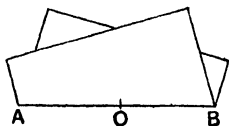


Fig. 2.

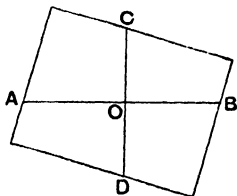


Fig. 3.

If a right angle is divided into 90 equal angles, each of these angles is called a **degree**.

25° is the abbreviation for "25 degrees."

An angle less than a right angle is called an **acute angle**. An angle greater than a right angle is called an **obtuse angle**.

ANGLES AT A POINT.

§3. You know that, in carrying out the order "about turn," you turn through two right angles. If you turn the hand of a clock from XII to VI, it turns through two right angles.

You might first of all turn it from XII to II, through an acute angle; and then again from II to VI, through

further angle, an obtuse angle. These two angles together make up two right angles. This would be equally true if you broke the journey at some other place than II. Hence



Fig. 4.

***THEOREM 1. If a straight line stands on another straight line, the sum of the two angles so formed is equal to two right angles.**

¶Ex. 1. If, in fig. 5, $\angle AOB = 57^\circ$, what is $\angle BOC$?

¶Ex. 2. (i) If, in fig. 5, $\angle BOC = 137^\circ$, what is $\angle AOB$?

(ii) " " $\angle BOC = 93^\circ$, " " $\angle AOB$?

(iii) " " $\angle AOB = 5^\circ$, " " $\angle BOC$?

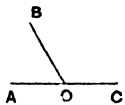


Fig. 5.

When the sum of two angles is equal to two right angles, each is called the **supplement** of the other, or is said to be **supplementary** to the other.

The numbers of Theorems in this book correspond to the numbers in *Theoretical Geometry* by the same authors.

When the sum of two angles is equal to one right angle, each is called the **complement** of the other, or is said to be **complementary** to the other.

¶Ex. 3. What are the supplements of the following angles:—(i) 60° , (ii) 54° , (iii) 0° , (iv) 90° , (v) 130° , (vi) 162° , (vii) 180° , (viii) x° ?

¶Ex. 4. What are the complements of the following angles:—(i) 30° , (ii) 72° , (iii) 45° , (iv) 0° , (v) 90° , (vi) x° ?

§4. If a number of lines meet at a point, as in fig. 6, a number of angles are formed at that point. If the clock-hand turns from OA to OB, then to OC, then to OD, and finally back to OA, how many separate angles does it turn through? What is the sum of these angles? How many degrees?

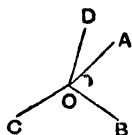


Fig. 6.

THEOREM 1. COR. If any number of straight lines meet at a point, the sum of all the angles made by the lines taken consecutively is equal to four right angles.

¶Ex. 5. In fig. 6, produce BO to B'. Now there are three angles on the one side of BOB'; two angles on the other side. What is the sum of these three angles? of these two angles?

¶Ex. 6. If three angles in fig. 6 are 30° , 140° , 110° , what is the fourth?

§5. Lay two pencils along one another; now rotate one of them so as to form a letter X: do the two parts of the pencil turn through equal angles?

The opposite angles made by two intersecting straight lines are called **vertically opposite** angles (e.g. $\angle$ s AOB, COD in fig. 7). *Vertically* is rather an unfortunate word, as it has nothing to do with vertical lines; the angles are *opposite* and have the same *vertex*: hence “**vertically opposite**.”

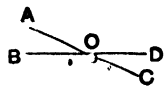


Fig. 7

THEOREM 3. If two straight lines intersect, the vertically opposite angles are equal.

¶Ex. 7. Mention two pairs of vertically opposite angles in fig. 7.

Ex. 8: Draw three straight lines at random, as in fig. 8; number all the angles in the figure, using the same numbers for angles that are equal.

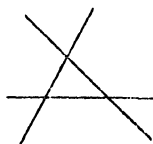


Fig. 8.

Ex. 9. If two straight lines AOC, BOD intersect at O (see fig. 7) what is the sum of $\angle$ s AOB, BOC? What is the sum of $\angle$ s BOC, COD? Hence prove the above theorem.

PARALLEL STRAIGHT LINES.

§6. It is assumed that the pupil is familiar with the following facts about parallel straight lines.

- (i) Parallel straight lines must be in the same plane.
- (ii) Parallel straight lines are everywhere the same distance apart.
- (iii) Parallel straight lines never meet however far they are produced in either direction.

It is also assumed that the pupil is able to draw parallel straight lines by sliding a set square along a straight edge (see fig. 9).

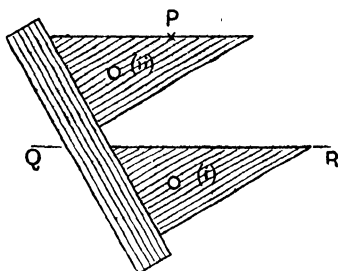


Fig. 9.

When it is required to draw a parallel to a given straight line at a given distance away, the method of fig. 10 is the best.

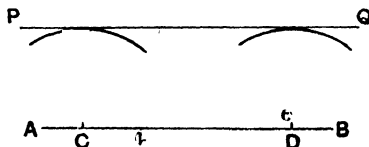


Fig. 10.

The arcs have C and D as centres and radius equal to the given distance. PQ is drawn to touch the arcs.

§7. The above properties of parallel straight lines are not generally so useful as the theorems which are developed below.

A man walks along a zig-zag road ABCD, AB being parallel to CD, so that he finishes up by walking in his original direction.

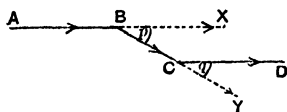


Fig. 11.

At B he turns to the *right* through $\angle p$ (N.B. *not* through $\angle ABC$); and at C he turns to the *left* through $\angle q$.*

Since his last direction is the same as his first, his right and left turns must compensate; $\therefore \angle p = \angle q$.

Hence we may see that BCY, meeting the parallel lines BX, CD, makes the **corresponding angles** p and q equal.

From the above, and by consideration of the method of drawing parallels (see fig. 9), we arrive at

THEOREM 4 (CORRESP.). When a straight line cuts two other straight lines, if a pair of corresponding angles are equal, then the two straight lines are parallel.

Conversely. **THEOREM 5 (CORRESP.).** When a straight line cuts two parallel straight lines, the corresponding angles are equal.

Also, from Theorem 3, $\angle p = \text{vert. opp. } \angle r$ and so, if $\angle p = \angle q$, then $\angle r = \angle q$.

$\angle s$ and q are called **alternate angles**.

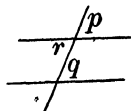


Fig. 12.

THEOREM 4 (ALT.). When a straight line cuts two other straight lines, if a pair of alternate angles are equal, then the two straight lines are parallel.

Conversely. **THEOREM 5 (ALT.).** When a straight line cuts two parallel straight lines, the alternate angles are equal.

Again, from Theorem 1, $\angle c + \angle a = 2 \text{ rt. } \angle s$ so that, if $\angle c = \angle b$, we have

$$\angle b + \angle a = 2 \text{ rt. } \angle s.$$

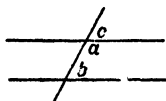


Fig. 13.

THEOREM 4 (INT.). When a straight line cuts two other straight lines, if a pair of interior angles on the same side of the cutting line are together equal to two right angles, then the two straight lines are parallel.

Conversely. **THEOREM 5 (INT.).** When a straight line cuts two parallel straight lines, the interior angles on the same side of the cutting line are together equal to two right angles.

¶Ex. 10. In fig. 14, if WX is $\parallel$ to YZ , name pairs of (i) alternate angles, (ii) corresponding angles.

¶Ex. 11. Repeat Ex. 10 for the parallels ST , UV .

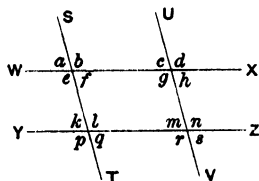


Fig. 14.

Theorem 4 leads to other methods of drawing a parallel to a given line through a given point. Draw any line through the point to cut the given line, then draw a line through the point making an equal corresponding angle or an equal alternate angle.

CONSTRUCTION OF PARALLEL LINES.

Ex. 12. Draw any line and mark off 5 equal lengths, end to end along it. Draw parallel lines through each point of division.

If these parallels are cut by any other straight line, what do you infer about the intercepted parts?

Ex. 13. Draw two lines OA , OB . Along OB mark off 5 equal parts. Join the last mark to A and through each of the other marks draw a parallel to this line. What do you infer with regard to the line OB ?

Ex. 14. Draw any triangle and through each vertex draw lines parallel to the opposite side so as to form a new triangle. Measure the angles of each triangle.

Ex. 15. Draw any triangle. Inside it, at a distance of 2 cm., draw a parallel to each side thus forming a new triangle. Draw lines joining corresponding vertices and produce them. What fact is suggested?

Ex. 16. Repeat Ex. 15, drawing the parallels outside.

Ex. 17. Draw three lines meeting in a point. Construct a triangle with its sides parallel to the lines. Measure the angles of the triangle and the angles at the point.

Ex. 18. Draw four lines meeting in a point as in fig. 6. Construct a quadrilateral with its sides parallel to the lines. Measure the angles of the quadrilateral and the angles at the point.

Ex. 19. Draw any triangle ABC and take any point O within it. From O draw OP, OQ, OR parallel to the sides, BC, CA, AB (in those directions). Measure the angles of the triangle and the angles at O.

Ex. 20. Repeat Ex. 19 for a point O outside the triangle.

Ex. 21. Draw any triangle ABC and join O, any point outside it, to A, B, C. Take P in OA; draw PQ $\parallel$ to AB meeting OB in Q; draw QR $\parallel$ to BC meeting OC in R. Join PR. Measure angles of triangles ABC, PQR. What can you state about PR?

Ex. 22. Draw two lines XOX', YOY' crossing at an angle of 60° . Draw two parallels to each line 1 in. distant, so as to form a quadrilateral. Draw the two diagonals of this quadrilateral.

EXERCISES ON THE THEOREMS.

Ex. 23. Find the remaining angles in fig. 15. (Give brief reasons.)

Ex. 24. A wheel has 12 spokes. What is the angle between consecutive spokes? If one is missing, what is the angle in the gap?

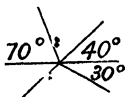


Fig. 15.

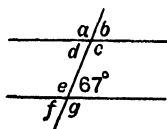


Fig. 16.

Ex. 25. Fig. 16 shows two parallel lines and a third line cutting them. Write down the sizes of the angles denoted by letters, giving reasons very shortly.

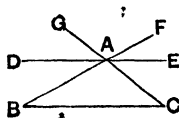


Fig. 17.

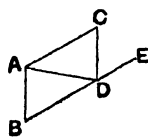


Fig. 18.

Ex. 26. In fig. 17, DAE is $\parallel$ to BC. Mention pairs of equal angles, giving reasons.

Ex. 27. In fig. 18, AB is $\parallel$ to CD and AC is $\parallel$ to BD. Mention pairs of equal angles, giving reasons.

Ex. 28. The bearing of B from A is N. 20° E. What is the bearing of A from B? See fig. 19.

Ex. 29. The bearing of B from A (Army method*) is 110° . What is the bearing of A from B?

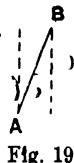


Fig. 19.

* In the Army and Navy bearings are now denoted by an angle measured clockwise from a North line, e.g. S. 20° E. is given as 150° ; S. 52° W. as 232° .

Ex. 30. In fig. 20, PQ and RS are parallel. If $\angle a = 42^\circ$ and $\angle b = 47^\circ$, find $\angle x$.

[Through O draw a parallel to PQ and RS.]

Ex. 31. In fig. 21, AB is parallel to CD.

Find the $\angle$ at E in terms of p and q .

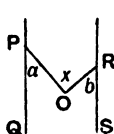


Fig. 20.

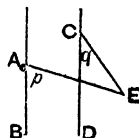


Fig. 21.

Ex. 32. In fig. 18, $\angle CDE = 70^\circ$, $\angle ADC = 60^\circ$. Find each angle in the triangle ABD. What is their sum? Repeat for the triangle ACD.

Ex. 33. In fig. 17, $\angle EAF = 30^\circ$, $\angle FAG = 98^\circ$. Find each angle of the triangle ABC and their sum.

Ex. 34. In fig. 22, PAB is $\parallel$ to OQ and OQ bisects $\angle YOZ$. Find the remaining angles.

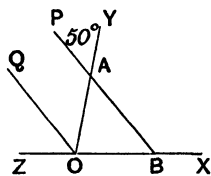


Fig. 22.

Ex. 35. Show how to draw a line through P (see fig. 22) so as to make equal angles with two intersecting lines OX and OY.

SUMMARY OF RESULTS

If a straight line stands on another straight line, the sum of the two angles so formed is equal to two right angles.

If any number of straight lines meet at a point, the sum of all the angles made by consecutive lines is equal to four right angles.

If two straight lines intersect, the vertically opposite angles are equal.

When a straight line cuts two other straight lines, if

- (i) *a pair of corresponding angles are equal,*
 - or (ii) *a pair of alternate angles are equal,*
 - or (iii) *a pair of interior angles on the same side of the cutting line are together equal to two right angles,*
- then the two straight lines are parallel.*

Conversely If a straight line cuts two parallel straight lines,

- (i) *corresponding angles are equal,*
- (ii) *alternate angles are equal,*
- (iii) *the interior angles on the same side of the cutting line are together equal to two right angles.*

CHAPTER 2

TRIANGLES

§ 1. A plane figure bounded by three straight lines is called **triangle**.

A plane figure means a figure drawn on a plane surface.

¶ Ex. 1. Draw any triangle, and measure its angles. What is the sum of its angles?

¶ Ex. 2. Repeat Ex. 1 for a triangle with one of its angles obtuse.

¶ Ex. 3. Cut out a paper triangle; mark its angles; tear off the corners and fit them together with their vertices at one point, as in fig. 23.

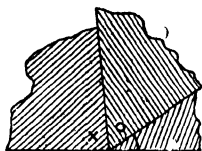


Fig 23.

What relation between the angles of a triangle is suggested by this experiment?

THEOREM 8 The sum of the angles of a triangle is equal to two right angles.

For proof of this Theorem see *Theoretical Geometry*, p. 13.

§ 2. Ex. 4. If one angle of a triangle is a right angle, the other two angles must be acute.

Ex. 5. If one angle of a triangle is obtuse, the other two angles must be acute.

A triangle which has one of its angles an obtuse angle is called an **obtuse-angled triangle**.

A triangle which has one of its angles a right angle is called a **right-angled triangle**.

The side opposite the right angle is called the **hypotenuse** (a Greek word which means "subtending" the right angle).

If a triangle has one of its angles a right angle, the other two angles are complementary.

A triangle which has *all* its angles acute angles is called an **acute-angled triangle**.

In Exx. 4, 5, we have seen that **every triangle must have at least two of its angles acute**.

A triangle which has all its sides equal is called an **equilateral** triangle.

A triangle which has two of its sides equal is called an **isosceles** triangle. (Isosceles is a Greek word meaning "equal-legged.")

A triangle which has no two of its sides equal is called a **scalene** triangle. (Scalene comes from a Greek word meaning "limping.")

A triangle which has all its angles equal is said to be **equiangular**.

§3. ¶Ex. 6. Draw a triangle ABC; produce BC to D. Measure angles A, B; what is their sum? Measure $\angle ACD$. What do you infer?

THEOREM 8. COR. 1. **If one side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior opposite angles.**

Ex. 7. Show that an exterior angle of a triangle must be greater than either interior opposite angle.

Ex. 8. If the sides of a triangle are produced in order, what is the sum of the three exterior angles thus formed?

Ex. 9. Two lines make angles of 120° and 50° with another line, what angle do they make with each other?

Ex. 10. What is the angle between two lines which make x° and y° with a given line?

¶Ex. 11. (i) Give two sets of angles which would do for the angles of a triangle. (ii) Give two sets which would not do.

¶Ex. 12. If one angle of a triangle is 36° , and the other two angles are equal, find the other two angles.

Ex. 13. Repeat Ex. 12 with the given angle (i) 90° , (ii) 132° , (iii) 108° , (iv) x° .

Ex. 14. In fig. 24, triangle ABC has $\angle A = 90^\circ$; AD is drawn perpendicular to BC. If $\angle B = 27^\circ$, find the angles marked x , y , z .

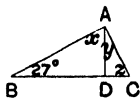


Fig. 24.

Ex. 15. Repeat Ex. 14 with (i) $\angle B = 54^\circ$, (ii) $\angle B = 33^\circ$, (iii) $\angle B = 45^\circ$, (iv) $\angle B = a^\circ$.

Ex. 16. A triangle ABC has $\angle A = 75^\circ$, $\angle B = 36^\circ$; if AD is drawn perpendicular to BC, find each angle in the figure.

Ex. 17. Draw a triangle ABC right-angled at A; through A draw XAY $\parallel$ to BC and draw AD $\perp$ to BC. Assume $\angle ACB = 30^\circ$ and find all the other angles. Give reasons for each.

Ex. 18. ABC is a triangle with BC produced to D. CE bisects $\angle ACD$ and is $\parallel$ to AB. Assume $\angle ECD = 52^\circ$ and find all the other angles in the figure.

Ex. 19. Draw an angle of 44° and from a point P outside it draw perpendiculars to each arm. Find the angle between the perpendiculars.

Ex. 20. Repeat Ex. 19, taking P inside the angle.

Ex. 21. Draw an angle of 115° and from a point within it draw perpendiculars to each arm. Find the angle between the perpendiculars.

Ex. 22. ABC is a triangle with $\angle B = 55^\circ$, $\angle C = 65^\circ$. BE , CF , perpendiculars to AC , AB respectively, meet in O . Show that the obtuse $\angle O$ and $\angle A$ are supplementary.

Ex. 23. If in fig. 25 $\angle BAC = 62^\circ$, $\angle ABC = 48^\circ$,
 $\angle BAR = \angle CBP = \angle ACQ = 20^\circ$.

Find the angles of the triangle PQR .

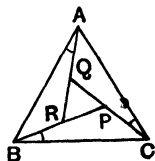


Fig. 25.

CONSTRUCTION OF TRIANGLES AND CONGRUENCE.

§ 4. [It is assumed that the pupil has some acquaintance with the subject. The following is a summary of the facts and the main line of approach. For a fuller treatment see *Junior Geometry*, Chapters 7 and 8.]

If we have a figure which is an exact copy of another, so that if one is placed on the other the two would exactly coincide, the two figures are said to be **congruent**.

The sign $\equiv$ is used to denote that figures are congruent.

A triangle has three sides and three angles; these sides and angles may be called the six **parts** of the triangle.

We have to consider how many parts are necessary to fix the triangle. This may be done by drawing a triangle and trying to copy it by copying some only of its parts.

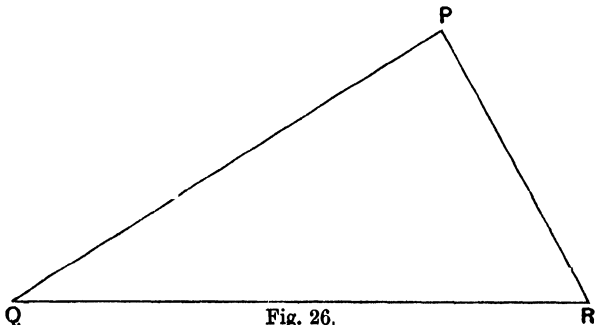


Fig. 26.

§ 5. CASE I.

¶ Ex. 24. Try to copy $\triangle PQR$, in fig. 26, freehand by copying QR , $\angle Q$ and QP .

In copying sides freehand, measure the lengths roughly along your pencil; copy angles by eye; mark on your figure the parts you have copied.

You will see that the triangle is fixed by these three parts. Hence

A triangle can be constructed if we know two sides and the contained angle. Also

THEOREM 10. If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles contained by those sides equal, the triangles are congruent.

¶Ex. 25. Try if you can copy $\triangle PQR$ freehand by copying two sides PQ , PR and the *non-included angle* R . You will find there is a difficulty here. This case is discussed in § 9.

Ex. 26. Show that the distance between G and H , the opposite corners of a house, can be found as follows. At a point P set up a post; step off HP and an equal distance PN , taking care to keep in a straight line with the post and the corner H ; step off GP and an equal distance PM , M being in the same straight line as G and P . Measure MN ; this must be equal to GH .

Draw a ground plan and prove that $MN = GH$.

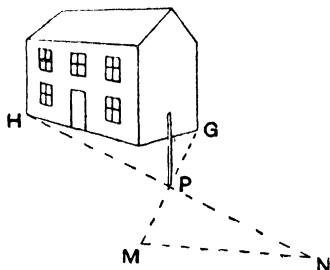


Fig. 27.

§ 6. CASE II.

¶Ex. 27. Try if you can copy $\triangle PQR$ in fig. 26, freehand, by copying QR , $\angle Q$, $\angle R$.

¶Ex. 28. Try to construct a triangle given one side and two angles *one of which is not at the end of the given side*. Say $QR = 2$ in., $\angle Q = 60^\circ$, $\angle P = 70^\circ$. If you fail at first, make a freehand sketch and think what must be the size of $\angle R$. Measure PQ when you have completed the copy.

You will see that the triangle is fixed by any two angles and a side. Hence

A triangle can be constructed if we know any two angles and a side. Also

THEOREM 11. If two triangles have two angles of the one equal to two angles of the other, each to each, and also one side of the one equal to the corresponding side of the other, the triangles are congruent.

Ex. 29. Draw two $\triangle$ s GHK , XYZ , and mark $GH=XY$, $\angle H=\angle Y$, and $\angle K=\angle X$; are the triangles congruent?

Ex. 30. The following method may be used to find the breadth of a river. Choose a place where the river is straight, note some conspicuous object T (e.g. a tree) on the edge of the other bank; from a point O opposite T measure a distance OS along the bank; put a stick in the ground at S ; walk straight on to a point P such that $SP=OS$; from P walk at right angles to the river till you are in the same straight line as S and T . PQ is equal to the breadth of the river. Prove this.

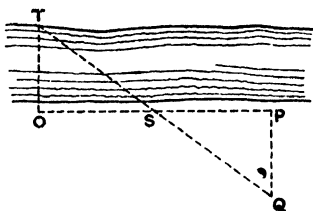


Fig. 28.

§7. CASE III.

¶Ex. 31. Copy $\triangle PQR$, in fig. 26, by making QR of the required length and then finding P by means of circles. See fig. 29 (which is not drawn full size).

Hence

A triangle can be constructed if we know the three sides. Also

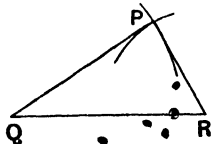


Fig. 29.

THEOREM 14. If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles are congruent.

§8. CASE IV. The special case of the **Right-angled Triangle**.

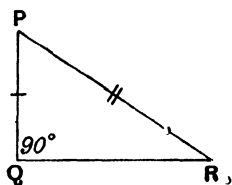


Fig. 30.

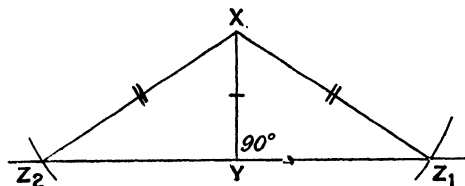


Fig. 31.

Ex. 32. Copy the right-angled triangle PQR as follows:—draw $XY=PQ$; through Y draw a line at right angles to XY ; with centre X and radius equal to PR draw a circle to cut this line at Z . You will see that there are two possible positions for Z , but $\triangle XYZ_1 \equiv \triangle XYZ_2$. Hence

A right-angled triangle can be constructed if we know the hypotenuse and one side. Also

THEOREM 15. If two right-angled triangles have their hypotenuses equal, and one side of the one equal to one side of the other, the triangles are congruent.

The Theorems on Congruent Triangles are some of the most useful in Deductive Geometry. For theoretical exercises on their use see *Theoretical Geometry*, pp. 25-29.

THE AMBIGUOUS CASE.

§9. ¶Ex. 33. Copy $\triangle PQR$ in fig. 26 as follows:—make an angle at Y equal to $\angle Q$, along one arm measure off $YX = QP$, with centre X and radius $= PR$ draw a circle to cut the other arm of the angle at Z . You will see that there are two possible positions of Z . Fig. 32 shows, on a reduced scale, fig. 26 and the figure you should have drawn.

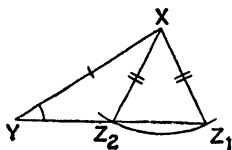
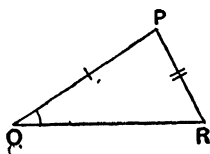


Fig. 32.

Hence

‘In general, if we know two sides of a triangle and a non-included angle, there are two possible triangles.

¶Ex. 34. If $\angle Q$ is a right angle, is there any ambiguity?

¶Ex. 35. If $\angle Q$ is obtuse, is there any ambiguity?

¶Ex. 36. If PR is greater than PQ , is there any ambiguity?

THE ISOSCELES TRIANGLE.

§10. By definition, if ABC is an isosceles triangle two sides are equal, say $AB = AC$. Suppose AD bisects the angle A . We realise that the figure is the same on each side of AD , so that $\angle ABC = \angle ACB$ and $BD = DC$ and AD is perpendicular to BC . In mathematical terms we should say that these properties follow from **symmetry**. (See Chap. 15.)

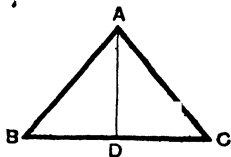


Fig. 33.

Our sense of symmetry tells us equally well that, if we started with the line BC and made equal angles at B and C , the lines would meet at A so that $AB = AC$.

The above suggests the two theorems:—

THEOREM 12. If two sides of a triangle are equal, the angles opposite to these sides are equal.

THEOREM 13. If two angles of a triangle are equal, the sides opposite to these angles are equal.

For proofs of these Theorems see *Theoretical Geometry*, p. 21.

The phrase "the sides" of an isosceles triangle is often used to mean the equal sides, "the base" to mean the other side, "the vertex" to mean the point at which the equal sides meet, and "the vertical angle" to mean the angle at the vertex.

THE EQUILATERAL TRIANGLE.

By definition, an equilateral triangle has all its sides equal. We can therefore describe an equilateral triangle ABC on a base BC by describing arcs of circles, with centres at B and C and radius BC, to meet in A (see fig. 34).

Ex. 37. Show that a triangle which is equilateral is also equiangular.

Ex. 38. Show that a triangle which is equiangular is also equilateral.

From Ex. 37 we see that all the angles are equal and therefore each is 60° .

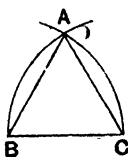


Fig. 34

CONSTRUCTION OF TRIANGLES.

When told to construct a figure to given measurements, first make a fair-sized freehand sketch and write the given measurements on the sketch. The sketch need not be accurate, but straight lines should look straight, and angles should be drawn acute, right, or obtuse according to the given measurements.

Remember that a means the side opposite A, b that opposite B, c that opposite C.

Ex. 39. Draw triangles to the following measurements and in each case measure the remaining sides and angles* :—

- | | | | |
|--------|---------------------------|------------------------------------|-------------------------|
| (i) | $\angle A = 42^\circ$, | $AB = 3.7$ in., | $AC = 3.7$ in.; |
| (ii) | $\angle C = 79^\circ$, | $b = 4.7$ cm., | $a = 6.1$ cm.; |
| (iii) | $\angle C = 116^\circ$, | $CA = 1.8$ in., | $\angle A = 78^\circ$; |
| (iv) | $AB = 7.3$ cm., | $\angle B = \angle C = 57^\circ$; | |
| (v) | $a = 6.5$ cm., | $b = 9.6$ cm., | $c = 7.2$ cm.; |
| (vi) | $BC = CA = AB = 6.9$ cm.; | | |
| (vii) | $AB = 3.5$ in., | $AC = 2.1$ in., | $\angle C = 90^\circ$; |
| (viii) | $a = 8.5$ cm., | $A = 90^\circ$, | $b = 4.5$ cm.; |
| (ix) | $A = 110^\circ$, | $b = 1.8$ in., | $a = 3.2$ in. |

Finding the remaining sides and angles of a triangle is sometimes called "solving the triangle."

Ex. 40. Draw a right-angled isosceles triangle.

Ex. 41. Draw an isosceles triangle with each base angle double the vertical angle.

Ex. 42. Draw two triangles on a base of 3 in. with the other sides 2·7 in. and 2·2 in.

Ex. 43. Draw any equilateral triangle and on each side, outwards, draw other equilateral triangles.

Ex. 44. Draw two triangles given $A=30^\circ$, $a=5\cdot5$ cm., $c=8\cdot8$ cm.

SCALE DRAWING.

For general instructions about Scale Drawing see Chap. 6, p. 35.

Ex. 45. A swing with ropes $5\frac{1}{2}$ ft. long rises till the ropes make an angle of 40° with the vertical. Through what horizontal distance will the seat move in one complete swing?

Ex. 46. To find the distance of an inaccessible object P, a base line AB of 500 yd. is marked out. The angle PAB is found to be 65° and $\angle PBA$ 72° . Find the distance of P from A.

Ex. 47. From a point A on a level road running straight towards a tower the angle of elevation of the top is found to be 21° , and that from another point 200 yd. nearer the tower is found to be 46° . Find the height of the top of the tower above the level of the road.

Ex. 48. A roof is supported by two girders 25 ft. and 20 ft. long fixed to walls at the same height above the ground and 30 ft. apart. The walls are 16 ft. high. What height is the ridge of the roof above the ground?

Ex. 49. A ladder 30 ft. long rests against a tree trunk with its foot 10 ft. from the base of the tree. What is the greatest height for a waggon 5 ft. wide to pass under the ladder without displacing it?

Ex. 50. A gun has a range of 1250 yd. and a vessel steers a straight course which brings it within 800 yd. of the gun. Over what distance is the vessel within range of the gun?

Ex. 51. A triangular field has sides 125 yd., 164 yd., 205 yd. and a pump in it is 120 yd. from each end of the longest side. How far is the pump from the opposite corner?

Ex. 52. To find the distance between two inaccessible points P and Q a base line AB of 1000 yd. is fixed. The following angles are measured:— $\angle PAB=72^\circ$, $\angle QAL=40^\circ$, $\angle PBA=38^\circ$, $\angle QBA=66^\circ$. Find the distance PQ.

Ex. 53. A framework to represent a six-pointed star is formed by fixing an equilateral framework symmetrically on an equal framework, the sides of each being 12 ft. long. Draw a plan of the star and find the distance of a point from the centre.

SUMMARY OF RESULTS

The sum of the angles of a triangle is equal to two right angles.

If one side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior opposite angles.

If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles contained by those sides equal, the triangles are congruent.

If two triangles have two angles of the one equal to two angles of the other, each to each, and also one side of the one equal to the corresponding side of the other, the triangles are congruent.

If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles are congruent.

If two right-angled triangles have their hypotenuses equal, and one side of the one equal to one side of the other, the triangles are congruent.

If two sides of a triangle are equal, the angles opposite to these sides are equal.

If two angles of a triangle are equal, the sides opposite to these angles are equal.

CHAPTER 3

RULER AND COMPASS CONSTRUCTIONS

§ 1. Hitherto we have constructed our figures mainly with the help of graduated instruments and set squares. We shall now make certain constructions with the aid of nothing but a straight edge (not graduated) and a pair of compasses. These methods are usually more accurate than those that require graduated instruments, for there is always a chance of a small error in making and transferring a measurement.

In the exercises, when you are asked to construct a figure, you should generally explain your construction in words.

In figs. 35-39 the given lines are drawn thick but you should not thicken the lines in your drawings.

In making constructions, only the necessary parts of construction circles should be drawn even though "the circle" is spoken of.

None of the construction lines should be rubbed out.

§ 2. Through a point O in a straight line OX to draw a straight line OY so that angle XOY may be equal to a given angle BAC .

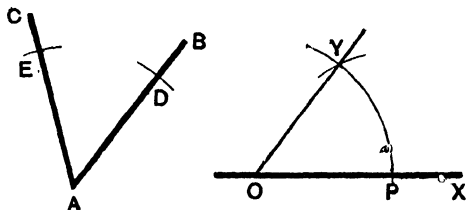


Fig. 35.

Construction With centre A and any radius describe a circle cutting AB , AC at D , E respectively.

With centre O and the same radius describe a circle cutting OX at P .

With centre P and radius $= DE$ describe a circle cutting the circle PY at Y . Join OY .

Then $\angle XOY = \angle BAC$.

The only instruments to be used in Exx. 1–22 are a straight edge and a pair of compasses.

Ex. 1. Draw an acute-angled triangle, and on a given straight line construct an equiangular triangle.

Ex. 2. Draw an obtuse-angled triangle and make a copy of it.

Ex. 3. Draw a triangle ABC; at a point Q make a copy of its angles in the manner of fig. 23.

Ex. 4. Draw a straight line EF and mark a point G (about 2 in. from the line); through G draw a line parallel to EF. [Draw any line through G cutting EF at H; make $\angle HGC = \angle GHF$.]

Ex. 5. Mark three points on one line and draw a second line crossing it. Through each point draw a parallel to the second line.

§ 3. To bisect a given angle.

Let BAC be the given angle.

Construction From AB, AC cut off equal lengths AD, AE.

With centres D and E and any convenient radius describe equal circles intersecting at F. Join AF.

Then AF bisects $\angle BAC$.

"Any convenient radius." If it is found that the equal circles do not intersect, the radius chosen is not convenient, for the construction breaks down it is necessary to take a larger radius so that the circles may intersect.

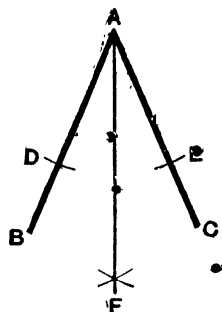


Fig. 36.

Ex. 6. If fig. 36 were folded about AF, what points would coincide? what lines?

Ex. 7. Make two equal angles, less than 60° , and bisect them; in one case join the vertex to the nearer point at which the equal circles intersect, in the other to the further point. Which gives the better result?

In the construction for bisecting an angle is there any case in which one point of intersection would coincide with the vertex of the angle?

Ex. 8. Draw a large triangle and bisect each of its angles. What fact is suggested?

Ex. 9. Draw two lines crossing and bisect a pair of adjacent angles. What is the angle between the bisectors? Why?

Ex. 10. Draw a circle and three lines touching it to form a triangle. Bisect each angle of the triangle. What fact is suggested?

Ex. 11. Draw a rectangle and bisect each angle. Show that the bisectors form another rectangle.

Ex. 12. Divide the vertical angle of any triangle into four equal parts. Is the base equally divided?

§4. To draw the perpendicular bisector of a given straight line.

To bisect a given straight line.

Let AB be the given straight line.

Construction With centres A and B and any convenient radius describe equal circles intersecting at C and D.

Join CD and let it cut AB at E.

Then CD is the perpendicular bisector of AB, and E is the mid-point of AB.

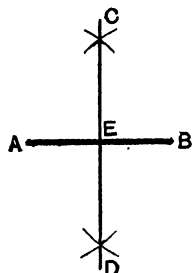


Fig. 37.

Ex. 13. If fig. 37 were folded about CD, what points would coincide? What lines?

Ex. 14. Draw a large acute-angled triangle; draw the perpendicular bisectors of its three sides. What fact is suggested?

Ex. 15. Repeat Ex. 14 for (i) a right-angled, (ii) an obtuse-angled triangle.

Ex. 16. Draw any three chords of a circle and their perpendicular bisectors. What fact is suggested?

Ex. 17. Describe a circle and on it take three points A, B, C; join BC, CA, AB. Bisect angle BAC and draw the perpendicular bisector of BC. Produce the two bisectors to meet.

§5. **Ex. 18** Draw a large obtuse angle (very nearly 180°) and bisect it.

To draw a straight line perpendicular to a given straight line AB from a given point P in AB.

Construction From P, cut off equal lengths PC, PD.

With centres C and D and any convenient radius describe equal circles intersecting at E. Join PE.

Then PE is $\perp$ to AB.

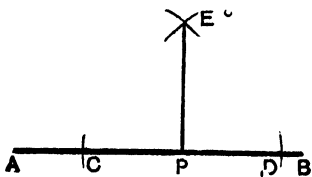


Fig. 38.

§ 6. To draw a straight line perpendicular to a given straight line AB from a given point P outside AB.

Construction. With centre P and any convenient radius describe a circle cutting AB at X and Y. With centres X and Y and any convenient radius describe equal circles intersecting at Q. Join PQ cutting AB at Z.

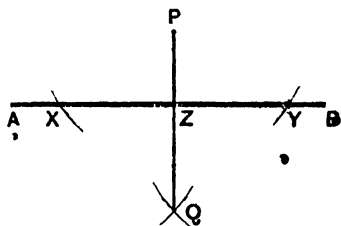


Fig. 39.

Then PZ is $\perp$ to AB.

Ex. 19. Draw a large acute-angled triangle; from each vertex draw a perpendicular to the opposite side. What fact is suggested?

Ex. 20. Repeat Ex. 19 with a right-angled triangle.

Ex. 21. Repeat Ex. 19 with an obtuse-angled triangle. [You will have to produce two of the sides.]

Ex. 22. Draw three equal chords in a circle and from the centre drop perpendiculars to the chords.

What facts are suggested about (i) the chords, (ii) their distances from the centre?

§ 7. To construct an angle of 60° .

Draw a line OX.

With centre O and any radius draw a circle to cut OX at A.

With centre A and the same radius draw an arc to cut the circle at B.

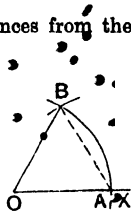


Fig. 40.

Then $\angle AOB$ is 60° .

[For by construction AOB is an equilateral triangle.]

To construct an angle of 30° .

Construct an angle of 60° and bisect it.

To construct an angle of 90° .

1st Method. See § 5.

2nd Method.

Draw a line OX.

With centre O and any radius OA draw a circle ABC.

Mark off $AB = BC = \text{radius } OA$.

Bisect $\angle BOC$ by arcs meeting at P.

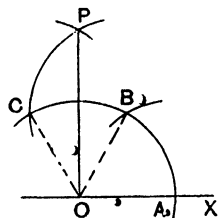


Fig. 41.

The $\angle POX$ is a right angle.

[For $\angle AOB = \angle BOC = 60^\circ$, and $\angle BOP = 30^\circ$.]

To construct an angle of 45° .

Construct an angle of 90° and bisect it.

Ex. 23. Draw an equilateral triangle and bisect one of its angles. How big is each angle in the figure?

Ex. 24. Construct angles of 240° , 15° , 150° , $22\frac{1}{2}^\circ$, 135° .

Ex. 25. Construct a rectangle, sides 10 cm. and 8 cm.

Ex. 26. Construct a right-angled triangle with hypotenuse = 3.5 in. and angles of 60° , 30° .

Ex. 27. What angles less than 180° can be constructed and trisected with ruler and compasses?

MISCELLANEOUS EXAMPLES ON CONSTRUCTIONS.

In Exx. 28-34 set squares and protractors are not to be used.

Ex. 28. Construct the perpendicular bisector of a line by drawing arcs on one side only.

Ex. 29. Draw any quadrilateral. Make a copy of it in two ways:—(i) by using sides and angles, (ii) by intersecting arcs only, using sides and diagonal.

Ex. 30. Repeat Ex. 29 for any pentagon.

Ex. 31. Draw any quadrilateral. Construct a similar quadrilateral with its sides half as long as corresponding sides in the former.

Ex. 32. At the centre of a circle divide the complete angle into 6 equal parts by lines intersecting the circle. Join the points of intersection, in order, forming a regular hexagon.

Ex. 33. As in Ex. 32 divide the angle at the centre of a circle into 8 equal parts and so form a regular eight-sided polygon.

Ex. 34. Construct the bisector of the angle between two lines which do not intersect, without producing the lines.

CHAPTER 4

QUADRILATERALS

§ I. DEF. A quadrilateral is a figure with four sides.

The sum of the angles of a quadrilateral will be seen to be four right angles if a diagonal is drawn.

DEF. A parallelogram is a four-sided figure with its opposite sides parallel.

To draw a parallelogram most easily, draw two pairs of parallel lines by sliding a set square along a ruler (see p. 4).

The following properties may be inferred from considerations of symmetry about a centre*.

THEOREM 20. (i) The opposite sides of a parallelogram are equal.

(ii) The opposite angles of a parallelogram are equal.

(iii) Each diagonal bisects the parallelogram.

(iv) The diagonals of a parallelogram bisect one another.

For proofs of this Theorem see *Theoretical Geometry*, p. 43.

§ 2. Notice that a parallelogram is *not* symmetrical about a diagonal: you see this at once if you place the parallelogram with a diagonal vertical as in fig. 44.

A parallelogram has been defined above as a quadrilateral with its opposite sides parallel, and it is convenient to choose one definition and adhere to it. But other properties of a parallelogram might have been chosen as the basis of a definition.

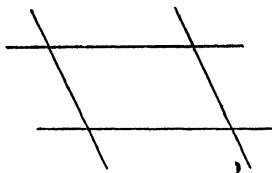


Fig. 42.

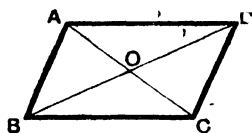


Fig. 43.



Fig. 44.

* See Chap. 5, p. 31.

For instance, a **quadrilateral with its opposite sides equal is a parallelogram**. On this property depends the action of the instrument called **parallel rulers**. PQ, RS are two flat rulers, connected by equal links AD, BC; these links are pivoted to the rulers at points such that $AB = DC$. Hence ABCD is a parallelogram and PQ is always parallel to RS.

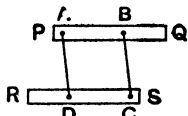


Fig. 45.

Another property that might have been taken as fundamental is illustrated if we connect the ends of two equal sticks by elastic strings (fig. 46). If the sticks are held parallel, the strings set themselves parallel. In fact, **a quadrilateral that has one pair of sides equal and parallel is a parallelogram**.

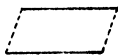


Fig. 46.

Again, if two sticks, not necessarily equal, are pivoted together at their middle points, their ends are the vertices of a parallelogram (fig. 47). Thus **a quadrilateral whose diagonals bisect one another is a parallelogram**.

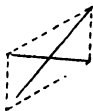


Fig. 47.

Any one of the three properties just mentioned may be used to prove that a figure is a parallelogram.

Make a deformable parallelogram, in the way illustrated in fig. 50. One more constraint is needed to make the framework rigid. If the constraint is applied by making one of the angles a right angle, it will be found that *all* the angles have become right angles; and the parallelogram has become a **rectangle**, which is a particular kind of parallelogram.

KITE.

§3. Fold a piece of paper, and cut out a triangle with one side along the crease, cutting through both thicknesses. Open out, and you have a symmetrical quadrilateral, such as fig. 48. (Two set squares may be used instead of paper.) This may be called a **kite**. The kite is symmetrical about one of its diagonals.

Ex. 1. Are any of the sides of a kite equal? What can be stated about the two diagonals? How would you describe the figures into which BD divides the kite?

Ex. 2. Construct a kite with diagonal of symmetry = 5" and sides of 3" and 4". Measure the two unequal angles, and deduce the values of the other two.

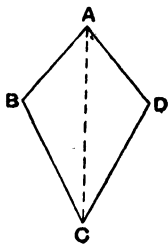


Fig. 48.

RHOMBUS.

§4. Fold a piece of paper in four, so as to form a right angle at the corner. By a straight cut through all four thicknesses, cut off a right-angled triangle from the corner; unfold it, and you have a figure called a **rhombus**. See fig. 49.

The rhombus is a symmetrical figure. What axes of symmetry has it?

It will be noticed that a rhombus is a particular kind of kite. How would it be necessary to modify the method given above of cutting out a kite in order to obtain a rhombus?

By considerations of symmetry, we see that **in a rhombus (i) all four sides are equal, (ii) the diagonals bisect one another, (iii) the diagonals are at right angles, (iv) the diagonals bisect the angles.**

Also opposite sides are parallel, so that a rhombus is a parallelogram in which all the sides are equal.

A rhombus may be made of four equal strips of cardboard, hinged together at the corners by means of paper-fasteners. This framework can be deformed; if *one* of the angles is made a right angle, it will be seen that *all* the angles become right angles and we have a **square**, which is a particular case of a rhombus.

Hence **a square has four equal sides and four equal angles each right angles.**

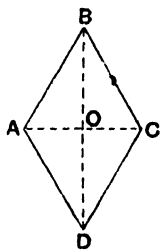


Fig. 49.

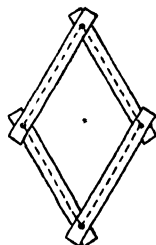


Fig. 50.

§5. Construction of Parallelograms, including the Rhombus, Rectangle and Square.

Any of the above properties may be used in the process of construction from given measurements.

Construct a Parallelogram with one side 3", distance of parallel side 2", and one angle 50° .

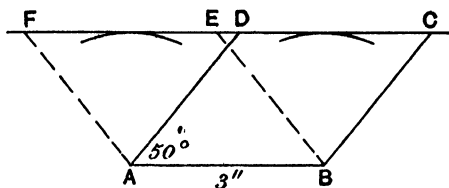


Fig. 51.

Make $AB = 3$ in., draw $DC \parallel$ to AB at a distance of 2 in.

At A make an $\angle$ of 50° . Draw $BC \parallel$ to AD , by the set square method (see p. 4).

$ABCD$ is the required parallelogram.

We might equally well have made the angle at B 50° , getting the parallelogram $ABEF$.

Construct a rhombus whose diagonals are 5 cm. and 8 cm.

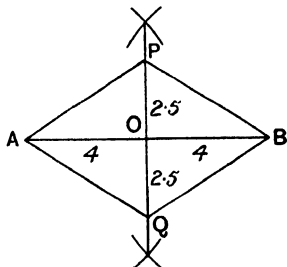


Fig. 52.

We know the diagonals of a rhombus bisect each other at rt. $\angle$ s. Hence make $AB = 8$ cm.; draw PQ the perpendicular bisector of AB .

Make $OP = OQ = 2.5$ cm.

$APBQ$ is the required rhombus.

Ex. 3. On a given line construct a rectangle with sides in the ratio of 2 to 1.

Ex. 4. Construct a parallelogram having two sides and a diagonal equal to 5 cm., 12 cm., 13 cm. Measure the other diagonal.

Ex. 5. Construct two parallelograms on a base of 3.5 in. with an angle of 50° and the other side 1.75 in.

Ex. 6. Construct a rhombus whose side is 2.8 in., the distance between opposite sides being 2.5 in. Measure its acute angle.

Ex. 7. Construct a parallelogram whose diagonals are 10 cm. and 8 cm. and one side 7 cm. Measure the other side.

Ex. 8. Construct a kite whose diagonals are 3.75 in. and 2.5 in. and one side 2 in. Measure the longer sides.

Ex. 9. Construct a rhombus with side 7 cm., acute angle 60° . Measure the diagonals.

Ex. 10. A quadrilateral is symmetrical about a diagonal 10 cm. long and has two sides of 8 cm. and 5 cm. Construct it.

Ex. 11. Trellis work is formed of laths pinned together at distances of 1 foot between each fastening on each lath. If the laths cross at an angle of 50° , what is the width of each opening?

Ex. 12. A framework of four equal rods, each 6 ft. long, is to be made rigid by a rod connecting the middle points of two adjacent rods so that the angle between them is 65° . What must be the length of this rod?

Ex. 13. Two rods of length 5 ft. are fixed together at their middle points at an angle of 70° . What length of wire can be stretched tightly once round their extremities?

CONSTRUCTIONS DEPENDING ON PARALLELOGRAMS.

Ex. 14. Draw two parallel lines 2 in. apart. Take any point between them. Through the point draw a line terminated by the parallels 3 in. long.

Ex. 15. Repeat Ex. 14 with the point outside the parallel lines.

Ex. 16. Draw two lines intersecting at 50° and any other line. To this line draw a parallel on which a length of 5 cm. is cut off by the two intersecting lines.

Ex. 17. Draw any triangle. Construct a line parallel to the base so that the part intercepted by the other two sides shall be $\frac{1}{3}$ the base.

Ex. 18. Two lines intersect at an angle of 40° . Find the distance from their point of intersection of a point which is 1.5 in. from each arm of the angle.

Ex. 19. Two vessels start from the same point at the same instant. One steams at 15 knots on a straight course N. 37° E.; the other steams on a straight course S. 50° E. and observes that the former vessel is always due N. of her. Find after what time the vessels will be 30 sea miles apart; also find the speed of the second vessel.

Ex. 20. From a point O, A bears N. 65° E., B is due E. and C bears S. 40° E. B is known to be in a straight line with A and C and the same distance from each. Find the bearing of A from C.

Ex. 21. Construct a $\triangle ABC$ with the median $AM = 85$ mm., $\angle BAM = 68^\circ$, $\angle CAM = 42^\circ$.

ANY QUADRILATERAL.

§6. ¶Ex. 22. Show that **the sum of the angles of a quadrilateral is equal to four right angles.** (Join two opposite corners.)

¶Ex. 23. How many parts has a quadrilateral?

¶Ex. 24. Try how many parts of a quadrilateral you must copy in order to copy the figure. (Either use tracing-paper, or else copy the sides and angles freehand*.)

FRAMEWORKS.

Fig. 53 represents a gate, but a bad gate. If anyone sat on the top bar, there would be a danger of the gate collapsing, and assuming

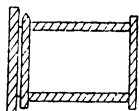


Fig. 53.

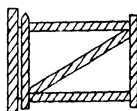


Fig. 54.

this shape . To resist a strain of this nature, there is nothing but the stiffness of the corners; and to rely upon this is bad carpentry. To make this a good gate, an additional bar must be introduced as in fig. 54. The fifth bar makes it a **rigid framework**, not relying upon the stiffness of the corners.

Take 4 strips of cardboard, of any lengths. Punch a hole at each end of each strip (with a paper-punch if possible) and joint the strips together with paper-fasteners, or preferably with bootmaker's eyelets†. The result is a quadrilateral with 4 sides given. The quadrilateral can be *deformed* into various shapes, if the hinges at the corners work freely.

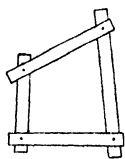


Fig. 55.

To convert this into a *rigid* framework, you may add a fifth strip along a diagonal.

This shows that a quadrilateral is fixed given 4 sides and 1 diagonal, 5 data in all.

But the original deformable framework might have been made rigid in other ways. One angle might be

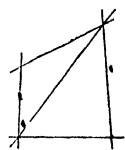


Fig. 56.

* In copying sides freehand, measure the lengths roughly along your pencil; copy angles by eye; mark on your figure the parts you have copied.

† Or Meccano-strips might be used.

fixed by pinching or gluing one corner: fixing one angle will fix all the angles. Here again 5 data fix the quadrilateral.

Notice that three strips jointed together at their ends form a rigid framework.

¶Ex. 25. Try if the deformable quadrilateral is made rigid by adding a 5th strip (i) joining points on adjacent sides, (ii) joining points on opposite sides.

¶Ex. 26. By freehand drawing, see if a quadrilateral is fixed if 3 sides are given, together with the 2 angles between these sides—5 data in all.

¶Ex. 27. Is a quadrilateral fixed by the following 5 data:—4 angles and 1 side?

§7. Construction of quadrilaterals, etc., from sufficient data.

Here it is especially important that, before beginning the construction, a freehand sketch should be made showing the given parts.

Note that the letters must be taken in order round the quadrilateral; e.g. the quadrilateral in fig. 57 is called ABCD and not ABDC.



Fig. 57.

Ex. 28. Construct quadrilaterals ABCD to the following measurements:—

In each case measure the remaining sides.

- (i) $AB = 3.4$ in., $BC = 2.2$ in., $AD = 2.9$ in., $\angle A = 68^\circ$, $\angle B = 86^\circ$.
- (ii) $\angle A = 67^\circ$, $\angle B = 113^\circ$, $\angle D = 46^\circ$, $AB = 5.3$ cm., $AD = 8.6$ cm.
- (iii) $AD = 3.0$ in., $\angle D = 118^\circ$, $\angle DAC = 27^\circ$, $\angle BAC = 35^\circ$, $AB = 2.4$ in.
- (iv) $AB = 1.9$ in., $BD = 1.7$ in., $CD = 2.0$ in., $\angle ABD = 118^\circ$, $\angle BDC = 23^\circ$

Ex. 29. Construct quadrilaterals ABCD to the following measurements:—

In each case measure the angles not given.

- (i) $AB = AD = 1.9$ in., $CB = CD = 2.9$ in., $BD = 2.5$ in.
- (ii) $AB = 5.3$ cm., $BC = 6.3$ cm., $CD = 6.7$ cm., $\angle B = 70^\circ$, $\angle C = 48^\circ$.
- (iii) $AB = 2.7$ cm., $BC = 7.5$ cm., $AD = 8.4$ cm., $\angle C = 98^\circ$, $\angle DBC = 28^\circ$.

SUMMARY OF RESULTS

The opposite sides and angles of a parallelogram are equal.

The diagonals of a parallelogram bisect one another.

A quadrilateral with its opposite sides equal is a parallelogram,

A quadrilateral that has one pair of sides equal and parallel is a parallelogram.

A quadrilateral whose diagonals bisect one another is a parallelogram.

CHAPTER 5

SYMMETRY

SYMMETRY ABOUT AN AXIS.

§ I. In the earlier chapters we have on several occasions appealed to an intuitional sense of symmetry; in this chapter we propose to examine and develop the properties of symmetry.

Let us first recall the cases in which we have appealed to symmetry. In the case of the isosceles triangle and in most of the ruler and compass constructions, the feature of the figure is that the two sides of the figure are alike—a somewhat indefinite statement which we must make more precise.

In a symmetrical figure there is a central line, called the **axis of symmetry**, and, if the figure is folded about this line, the two halves of the figure can be made to coincide.

¶ Ex. 1. Which are the axes of symmetry in figs. 33, 36, 37, 39, 52? Solids may have symmetry. The human body is more or less symmetrical about a plane. Consider the reflection in a mirror of the interior of a room. The objects in the room together with their reflections form a symmetrical whole; the surface of the mirror is the **plane of symmetry**.

¶ Ex. 2. Give four instances of solids possessing planes of symmetry.

Fold a sheet of paper once. Prick a number of holes through the double paper, forming any pattern. On opening the paper you will find that the pin-holes have marked out a symmetrical figure.

Join **corresponding points** as in fig. 58. Notice that when the figure was folded NA' fitted on to NA . This shows that $NA' = NA$. Also $\angle ANA'$ is $\perp$ to PQ . This fact may be stated as follows:—

The line joining any pair of corresponding points, in a figure which is symmetrical about an axis, is bisected by and is perpendicular to the axis of symmetry.

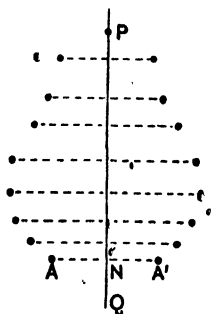


Fig. 58.

Corresponding points (e.g. A and A' in fig. 58) are said to be **images*** of one another in the axis of symmetry.

In the same way corresponding lines (see AB and A'B' in fig. 59) are said to be images of one another, and **a line and its image are equal**, for they coincide when the figure is folded about the axis.

Similarly **an angle and its image are equal**, and **an area and its image are equal**.

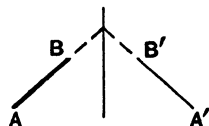


Fig. 59.

By moving a point along BA closer and closer to the axis and considering its image, we see that **a line and its image intersect on the axis of symmetry; they also make equal angles with the axis.**

SYMMETRY ABOUT A CENTRE.

§ 2. The capital letter S has no axis of symmetry; if viewed in a mirror it becomes 2. But though there is no axis of symmetry, this letter appears to possess symmetry of a kind. Thus, if through the middle point O any lines be drawn such as AOA', BOB', COC' then each of these lines is bisected at O.

Cover the S in fig. 60 with a piece of tracing-paper, and copy the outline of the letter. Then stick a pin through the point O of both. Now turn the traced S through 180° , and you will find that it again fits the original.

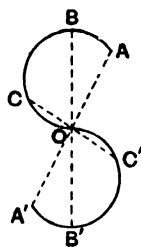


Fig. 60.

A plane figure which covers its original trace if turned in its own plane through 180° round the centre is said to be **symmetrical about a centre**.

The line joining any pair of corresponding points in a figure which is symmetrical about a centre is bisected by the centre of symmetry.

¶ Ex. 3. Draw any triangle. Construct another triangle which has central symmetry with it about the mid-point of its longest side.

¶ Ex. 4. Has a parallelogram symmetry about the intersection of its diagonals?

* The word "image" is borrowed from Light. The line joining a point and its image in a plane mirror is bisected by and is perpendicular to the plane of the mirror.

SYMMETRY ABOUT AN AXIS.

Ex. 5. Sketch freehand the following figures, showing by dotted lines all the axes of symmetry:—rectangle, rhombus, square, isosceles trapezium, regular pentagon, regular hexagon, semicircle, quadrant of circle.

Ex. 6. Sketch the following solids and indicate their planes of symmetry:—cube, square pyramid, cylinder, cone.

Ex. 7. Sketch freehand a quadrilateral which has (i) one axis, (ii) two axes of symmetry.

Ex. 8. Sketch freehand the symmetrical figure of which fig. 61 is one-half, the axis being the dotted line.



Fig. 61.

Ex. 9. Draw any quadrilateral, and to one diagonal draw a parallel outside the figure. Construct the image of the quadrilateral in this line as axis of symmetry.

Ex. 10. Sketch fig. 62 freehand and its image in the dotted line as axis of symmetry.



Fig. 62.

Ex. 11. Show in a freehand sketch the line about which a triangle must be folded to bring the vertex to the middle point of the base.

Ex. 12. Show in a freehand sketch the axis about which the figure must be folded to bring one vertex of a square (i) to an adjacent vertex, (ii) to an opposite vertex, (iii) to the point of intersection of the diagonals.

Ex. 13. Repeat Ex. 12 for (i) a rhombus, (ii) a parallelogram.

Ex. 14. About what axes must the corners of a square be folded so as to bring each corner into the centre of the square?

Ex. 15. A rectangle with sides 8 cm. and 5 cm. is folded about one diagonal. Find the new distance between those corners which were at the ends of the other diagonal.

Ex. 16. A square is folded about a diagonal AB which is 3 in. long. AB is trisected and on the middle part as base a right-angled isosceles triangle is drawn and the folded paper cut along its sides. The paper is now unfolded and laid out flat. Draw the figure obtained.

Ex. 17. A rectangular piece of paper 10 cm. by 4 cm. is folded so that the shorter edges coincide. With base along the crease and centre at its middle point a semicircle is cut out from the folded paper. The paper is now unfolded and laid out flat. Draw the figure obtained.

CENTRAL SYMMETRY.

Ex. 18. Draw any triangle and take a point outside it. Construct another triangle which has central symmetry with it about the point.

Ex. 19. A square is rotated through 180° about a point on a diagonal produced. Draw the square in its new position.

Ex. 20. Repeat for a rectangle and parallelogram.

Ex. 21. An equilateral triangle is rotated through half a complete turn about the intersection of the perpendiculars. Draw the triangle in its new position.

Ex. 22. Repeat Ex. 21 for any triangle.

IMAGES*.

Ex. 23. P and Q are two points on the same side of a line. P' is the image of P in the line, and QP' meets the line in R. Show that PR and RQ are equally inclined to the line.

Ex. 24. In fig. 63 show that the sum of PR and RQ is less in the position shown than for any other position of R on the line.

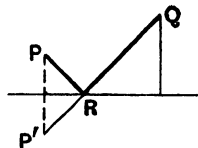


Fig. 63.

Ex. 25. In fig. 63, if a billiard ball is at Q, at what point must it be aimed to hit the cushion and then go through P? You may assume that the angle of rebound is equal to the angle at which it hits the cushion; this is approximately the case if there is no "side" on the ball.

Ex. 26. A billiard table is 12 ft. by 6 ft. Taking as axes of x and y the long and short edges respectively and the ball in a position (1, 3), construct the path of the ball so as to go into the pocket opposite the origin off the x -axis.

Ex. 27. Rays of light proceed from a point 1' in front of a vertical mirror. The eye is in a position 2' in front of the mirror and 3' vertically below the point. Construct the ray of light which enters the eye†.

SUMMARY OF RESULTS

The line joining any pair of corresponding points, in a figure which is symmetrical about an axis, is bisected by and is perpendicular to the axis of symmetry.

A line and its image are equal.

An angle and its image are equal.

An area and its image are equal.

A line and its image intersect on the axis of symmetry; they also make equal angles with it.

The line joining any pair of corresponding points in a figure which is symmetrical about a centre, is bisected by the centre of symmetry.

* Should be omitted at a first reading.

† The incident and reflected rays make equal angles with the mirror.

CHAPTER 6

SCALE DRAWING

§ 1. In all Scale Drawing Exercises the following points are essential:—

(i) **Make a freehand sketch*** of the required figure. This should be made on the same sheet as the finished drawing; it should be neat and fairly large.

(ii) **Mark all the data** in the sketch. If this is done in ink, it is then distinguished from any deductions made from the data.

(iii) **Mark any deductions** that you can make that will help you in the drawing (do this in pencil); e.g. equal alternate angles, or, the third angle of a triangle.

(iv) **State the scale** to be used. Choose this wisely (see § 2 below).

(v) **Make the accurate drawing**, working from your free-hand sketch.

(vi) **Do not rub out any construction lines.** They show how you made the figure.

(vii) **State your answer**, which should be the actual length required and *not* the scaled length.

NOTE. In the freehand sketch, the actual lengths of the data should be shown, the scaled lengths may be shown in brackets.

In the finished drawing the required length should be shown, the scaled length may be shown in brackets.

Notice that a scale drawing of a figure is similar to the figure it represents, the corresponding angles in the two are equal, and corresponding lengths are proportional or in the same ratio.

* It should be noted that in later work the freehand sketch is often all that is required. From the freehand sketch the required lengths and angles can be calculated without making an accurate drawing.

§2. In **choosing suitable scales** for drawing a figure, remember the following points:—

(i) **Choose as large a scale as possible.**

(ii) **Scales which involve only division or multiplication by 10 or 100 are the simplest;** the next best are those that involve division or multiplication by 2, 20, 5, or 50. Avoid division or multiplication by 3, 6, 7, 8, etc.

¶Ex. a. What scales would you choose if the longest line in the figure is to be 10 ft., 20 ft., 45 ft.? And what lengths would represent these in the scale drawing?

¶Ex. b. Repeat Ex. a for the following:—(i) 78 miles, (ii) 15 miles, (iii) 1410 ft., (iv) 5278 metres.

¶Ex. c. What lengths are represented by (i) 3·4 inches, when the scale is 1 inch to 1 mile, (ii) 7·4 cm., scale 1 cm. to 10 miles, (iii) 4·5 in., scale 2 in. to 1 mile, (iv) 8·3 cm., scale 1 cm. to 20 miles?

§3. The **bearing** of a place A from a second place B is the point of the compass towards which a man at B would be facing if he were looking in the direction of A.

By “N. 10° W.” or “10° W. of N.” is meant the direction in which you would be looking if you first faced due north and then turned through an angle of 10° towards the west.

In bearing questions a N. and S. line must first be drawn through the points from which bearings are to be measured.

It is now becoming customary, e.g. in the Army and the Navy, to give bearings by an angle only, viz. the angle turned through from the north in a clockwise direction (i.e. turning from N. to E.). Note that any west bearing must be greater than 180°.

If a man who is looking at a tower through a telescope holds the telescope horizontally, and then raises (or “elevates”) the end of it till he is looking at the top of the tower, the angle he has turned the telescope through is called the **angle of elevation** of the top of the tower.

The angle of elevation of the sun is called the **altitude of the sun**.

If a man standing on the edge of a cliff looks through a horizontal telescope and then lowers (or “depresses”) the end of it till he is

looking straight at a boat, the angle he has turned the telescope through is called the **angle of depression** of the boat.

Remember that the angle of elevation and the angle of depression are always angles at the observer's eye, and are always measured from the horizontal.

If A is an observer and X and Y two points, the angle XAY is said to be the **angle subtended** at A by XY.

DISTANCES AND BEARINGS.

Ex. 1. A is $15\frac{1}{2}$ miles N. 23° W. of B and C is 25 miles N. 52° E. of B. Find the distance and bearing of A from C.

Ex. 2. B bears N. 20° E. from A and A bears N. 70° W. from C, AB = 200 yards and AC = 300 yards. Find the distance and bearing of C from B.

Ex. 3. C is due E. of A, and N. $20\frac{1}{2}^\circ$ E. of B. Find how far A is from B if each is 10 miles from C. What is the bearing of A from B?

Ex. 4. A ship is steaming due East at 20 knots. The bearing of a lighthouse is observed to be N. 56° E., and $\frac{1}{4}$ hr. later it is observed to be N. 22° E. At what distance will the ships pass the lighthouse?

Ex. 5. An Admiral signals to his cruiser squadron (bearing N. 40° W. 50 miles from him) to meet him at a place N. 50° E., 70 miles from his present position. Find bearing and distance of the meeting place from the cruisers.

Ex. 6. A and B are two buoys 800 yards apart, B due N. of A. A vessel passes close to B and, steering due E., observes that after 5 minutes the bearing of A is 57° W. of S. Find the distance the vessel has moved.

Ex. 7. A straight sand bank, 1500 yards long, runs due E. and W., and has a buoy at each end. From a ship the eastward buoy bears N. 41° E., and the bank subtends an angle of 72° . Find her distance from the bank. Explain your construction.

Ex. 8. A line of telegraph poles runs 8° East of North, and the poles are 150 feet apart. Standing at one of these poles A, an observer determines by a compass that a certain building lies 35° East of North; on proceeding northwards to the next pole B, the observer finds that the building lies 41° East of North. Determine the distance of the building from the first pole A.

Ex. 9. The Mumbles Lighthouse is 25 miles N. $15\frac{1}{2}^\circ$ E. of Ilfracombe and the Nash Point Light is N. $61\frac{1}{2}^\circ$ E. of Ilfracombe. If Nash Point is S. 58° E. of the Mumbles, what is its distance from Ilfracombe?

Ex. 10. From a ship at sea the bearings of two points on the coast are found to be S. 14° E. and S. 49° W. If one of these points is found from the chart to be 10.4 miles N. 61° W. of the other, find, by drawing to scale, the distance of the ship from each point.

Ex. 11. From a ship at sea the following observations are made:—Dover bears N. 16° E., and Boulogne S. 81° E. From the chart it is found that Dover is 26 miles N. 24° W. of Boulogne. Find the distance of the ship from Dover.

Ex. 12. O and P are points on a straight stretch of shore. P is 4.5 miles N. 74° E. of O. From O a ship at sea bears S. 58° E., and from P the ship bears S. 32° W. Find the distance of the ship from P, and also its distance from the nearest point of the shore.

Ex. 13. The position of an inaccessible point C is required. From A and B, the ends of a base line 200 yards long, the following bearings are taken:—

From A	{	Bearing of B is N. $70^{\circ} 30'$ E.
	,	,, C is N. $30^{\circ} 20'$ E.
From B		,, C is N. $59^{\circ} 40'$ W.

Find the distances of C from A and B.

Ex. 14. From A to B, two railway stations, the line runs S. 67° W. A person at A observes two church spires X and Y to be in a line and to bear N. 23° W. From B the spires bear N. 37° E. and N. 15° E. respectively. If a train, travelling at 40 miles per hour, can run from A to B in two minutes, find the distance between X and Y.

Ex. 15. From the entrance to a harbour a beacon bears East, 4 miles. Steaming at 10 knots a ship leaves harbour at noon with orders to steer S. 48° E. until the beacon bears N. 50° E., and then to alter course N. 68° E. How near will she pass to the beacon on the new course and when will it be due West?

Ex. 16. A ship is sailing due N. at 8 miles an hour. At 3 o'clock a lighthouse is observed to be N.E. and after 90 minutes it is observed to bear $82\frac{1}{2}^{\circ}$ E. of S. How far is the ship from the lighthouse at the second observation, and at what time (to the nearest minute) was the ship nearest to the lighthouse?

Ex. 17. A and B are two lightships at opposite sides of a harbour, A being due West of B and the distance AB being 500 yd. An observer on a ship entering the harbour on a straight course S. 30° W. sights B in the direction S. 10° W. and at the same time observes that A is S. 50° W. Find the distances of the ship from A and B when she crosses the line AB.

Ex. 18. A lighthouse and a conspicuous tower are five miles apart and the tower bears S. 40° E. from the lighthouse. At 1.30 p.m. their bearings from a ship steering N. 25° W. are N. 20° E. and S. 81° E. respectively. Draw a figure to show the ship's position at 1.30 p.m. and measure her distance from the tower.

At 1.50 p.m. the lighthouse bears N. 70° E.; show the ship's probable position then and measure her distance from the lighthouse.

HEIGHTS AND DISTANCES.

Ex. 19. A man whose eye is 5 ft. 6 in. from the ground stands $\frac{1}{2}$ ft. in front of a wall which is 20 ft. high. Find the angle of elevation of the top of the wall.

Ex. 20. The angle of elevation of the top of a flagstaff is read off on a theodolite. Find the height of the flagstaff from the following data:—reading of theodolite = 11° ; distance of flagstaff from theodolite = 130 yd.

Ex. 21. A vertical tower stands on uniformly sloping ground. From a point 50 yd. from the foot of the tower the angle of depression of the foot is found to be 15° and the angle of elevation of the top 35° . Find the height of the tower.

Ex. 22. From a point A the top of a church tower is just visible over the top of a tree 50 feet high. If the distance from A to the foot of the tower is known to be 160 yards, and from A to the foot of the tree 60 yards, find in feet the height of the tower, and the angle of elevation of its top as seen from A.

Ex. 23. Two observations are made to find the height of a certain monument. From the first station the angle of elevation of the top is found to be 32° and from the second station, which is 27 yards from the first and exactly between it and the foot of the monument, the angle of elevation is 43° . If the telescope of the theodolite with which these observations are made is 5 feet above the ground, what is the height of the monument in feet?

Ex. 24. From the window of a house on the edge of a lake a man observes the elevation of the top of a mast of a vessel on the lake to be 15° . He also observes the angle of depression of the water-line of the vessel to be 10° . The window is 20 feet above the water. Find the distance of the vessel and the height of the top of the mast above the water (neglect the width of the ship).

Ex. 25. From the top of a mast 80 feet high the angle of depression of a buoy is 24° . From the deck it is $5\frac{1}{2}^\circ$. Find the distance of the buoy from the ship.

Ex. 26. Two stays on opposite sides of a mast make angles of 48° and 66° with the ground and the points to which they are fixed to the ground are 120 feet apart. What is the height of the point on the mast to which they are fixed?

Ex. 27. From a point on the line joining the feet of two equal masts 210 yards apart, on level ground, the angles of elevation of the tops of the masts are 30° and 42° . Find the height of the masts.

Ex. 28. From the top of a tower 120 ft. high the angles of depression of the nearer and farther edges of a flat roof of a building are 38° and 20° respectively and of the foot of the building nearest to the tower 57° , all angles being measured in the same vertical plane. Find the height of the building and its width measured away from the tower.

Ex. 29. An aeroplane flying at an altitude of 6000 feet is simultaneously revealed by two searchlights which are at the moment due south of it. The rearer searchlight is inclined at an angle of 42° with the horizontal. At what angle is the farther inclined if the distance between the searchlights is 1000 yards?

Ex. 30. An aeroplane is flying horizontally so as to pass over the head of a man with a theodolite. The latter observes the angle of elevation of the aeroplane to be 20° and then 1 minute later he observes it to be 50° .

Assuming that it is travelling at 60 miles per hour, find at what height above the ground it is flying. Give your answer in yards.

Ex. 31. An observer in a balloon, one mile high, observes the angle of depression of a church to be 35° . After ascending vertically for 20 minutes, he observes the angle of depression to be now $55\frac{1}{2}^\circ$. Find the rate of ascent in miles per hour.

Ex. 32. The angles of depression of two boats in a line with the foot of a cliff are $25^\circ 15'$ and $38^\circ 30'$ as observed by a man at the top of the cliff. If the man is 250 feet above sea-level, how far apart are the boats?

Ex. 33. A man on the top of a hill sees a level road in the valley running straight away from him. He notices two consecutive mile-stones on the road, and finds their angles of depression to be 30° and 13° respectively. Find the height of the hill as a decimal of a mile.

MISCELLANEOUS.

Ex. 34. A river is 60 ft. wide and a boat is being towed up the middle by horses on opposite banks with ropes of length 60 ft. and 90 ft. How far apart are the horses?

Ex. 35. A crane has a vertical post AB 8 ft. long, a tie rope BC 8 ft. long, and a jib AC 12 ft. long. What is the height of C above the horizontal through A? How much must the tie BC be lengthened to lower C 2 ft.?

Ex. 36. A conical plug 10 in. long and base diameter 4.5 in. is used to stop up a circular hole of diameter 2 in. What length of the plug remains exposed?

Ex. 37. The legs of a pair of compasses are 9 in. long and the pencil leg has a joint 5 in. from the hinge. The compasses are used to describe a circle of radius 3 in., the pencil leg being bent at the joint so that the pencil is perpendicular to the paper. Find the angle between the legs of the compasses.

Ex. 38. A rectangular table 6 ft. by 4 ft. is tilted through an angle of 30° about the line joining the feet of two legs at the corners of a shorter edge. If the legs are 2 ft. 9 in. long, what is now the height of the mid-point of the table above the floor?

Ex. 39. A rectangular framework 15 ft. by 6 ft. rests in a vertical plane with its 6 ft. edge along a slope whose gradient is 1 in 3 (1 vertical to 3 horizontal). Determine the height of its highest point above the horizontal through the lower corner. Find also the angle which the diagonal through this corner makes with the vertical.

Ex. 40. A small plot of grass is in the shape of a quadrilateral ABCD with AD parallel to BC and the angle at A a right angle; AB=16 ft., AD=16 ft., BC=25 ft. The grass is surrounded by a gravel path which is everywhere 10 ft. wide. Draw a plan and determine the lengths of CD and of the edges of the gravel path which are parallel to CD and BC.

Ex. 41. The essential parts of a steam governor consist of two rods AB , AD each of length 5 in., freely jointed at B , D to two rods CB , CD each of length 7 in. A is fixed and C is free to move along the line AC and the whole rotates about AC as a vertical axis. The distance BD is 7 in. when at rest and at a high pressure of steam B and D have moved outwards till the distance BD is 9 in. Find through what distance C has moved along AC .

Ex. 42. The jointed framework $ABCD$ in fig. 64 is hinged to a vertical wall at A and B , and CD is horizontal. AB is 6 ft. Find the length of AD and the distance of D from the wall.

Ex. 43. A wireless aerial has to stretch straight between two points 60 ft. and 20 ft. high, at a horizontal distance apart of 35 yards. If the wire must not exceed 100 ft., what length of rope must be used between the points of suspension?

Ex. 44. A ravine whose two sides slope at angles of 40° and 30° to the horizontal has a flat bed 30 yd. wide. The ravine is spanned by a horizontal bridge 110 yd. long. Find the height of the bridge above the bed of the ravine.

Ex. 45. In fig. 65, the rod BD turns about D and A slides along the horizontal line AD so that its greatest and least distances from D are 3 ft. 6 in. and 2 ft.

$$AB = BD = 2 \text{ ft.}, \quad BC = 3 \text{ ft.}$$

Find the greatest and least heights of C above the level of AD .

Ex. 46. Is it possible to carry a stretcher measuring 10 ft. by 2 ft. round a right angle bend in a passage 5 ft. wide, if the stretcher must be kept horizontal?

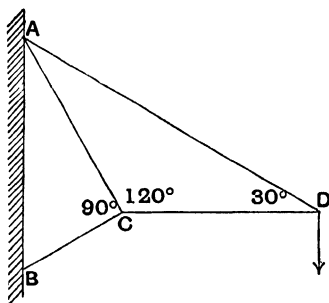


Fig. 64.

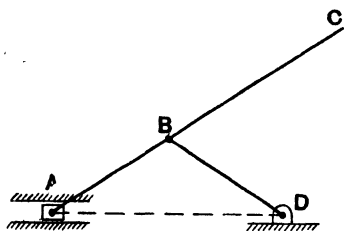


Fig. 65.

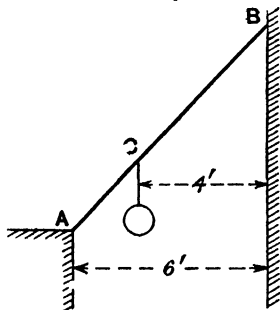


Fig. 66.

Ex. 47. Fig. 66 is a rough sketch of a beam 15 ft. long resting on a pier at A and hinged to B , a point in a vertical wall. A weight is hung from C . Find the height of B and C above the ground if the height of A is 12 ft.

If the weight at C has to be raised 1 foot higher by turning the beam about B , what is now the distance of C from the wall and what is the height through which A must be raised?

Ex. 48. AB, BC and CD (see fig. 67) are three bars linked at B and C, and capable of turning about fixed points A and D. In a certain mechanism the bar BC oscillates up and down so that the greatest value of the angle BAD on one side and of the angle ADC on the other is 45° . Find, by drawing, the distance between the extreme positions of the middle point O of BC; having given

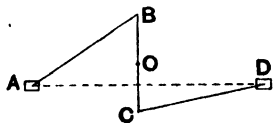


Fig. 67.

$AD = 24$ in., $AB = CD = 13$ in., $BC = 10$ in.

Ex. 49. Three bars AB, BC, CD are hinged to one another at B and C and to fixed centres at A and D. If $AB = 10$ in., $BC = 8$ in., $CD = 4$ in., $AD = 14$ in., find the angle through which AB oscillates as CD makes complete revolutions.

Ex. 5. Find the other dimension of a rectangle, given that the area = 140 sq. ft., one dimension = 35 ft. Now state a rule for finding the other dimension, given the area and one dimension.

AREA OF RIGHT-ANGLED TRIANGLE.

§3. By drawing a diagonal of a rectangle we divide the rectangle into two equal right-angled triangles. Hence the area of a right-angled triangle may be found by regarding it as half a certain rectangle.

Ex. 6. Find the areas of right-angled triangles in which the sides containing the right angle are (i) 2", 3", (ii) 6.5 cm., 4.4 cm., (iii) a' , b' .

AREA OF ANY RECTILINEAR FIGURE (ON SQUARED PAPER).

§4. With the aid of rectangles and right-angled triangles we can find the area of any figure contained by straight lines (i.e. any rectilinear figure). This way is especially convenient when one side of the figure runs along a line of the squared paper.

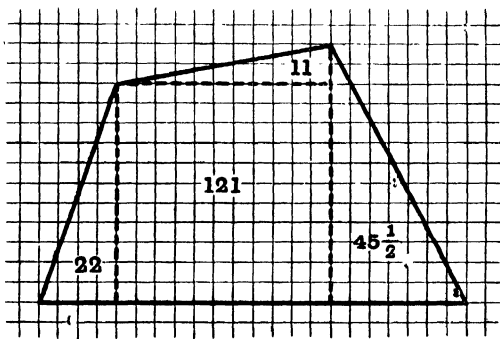


Fig. 72.

Fig. 72 shows how a 4-sided figure may be divided up into rectangles and right-angled triangles; the number inside each rectangle and triangle indicates the number of squares it contains; and the complete area is $199\frac{1}{2}$ or 199.5 squares.

If there is no side of the figure which coincides with a line of the paper, it is generally convenient to draw lines *outside* the figure, parallel to the axes.

Ex. 7. Find the area (in squares of your paper) of each of the following figures by dividing up the figures into rectangles and right-angled triangles:—

- (i) $(2\frac{1}{2}, 1), (11, 1), (8, 6), (2, 6)$.
- (ii) $(5, 0), (3, 4), (-5, 4), (-6, 0)$.
- (iii) $(0, 6), (-3, 2), (-3, -2), (0, -3)$.
- (iv) $(0, 0), (1, 4), (6, 0)$.
- (v) $(1, 4), (6, 3), (1, -3)$.
- (vi) $(-4, -3), (-3, 3), (5, 6), (10, -3)$.
- (vii) $(3, 0), (0, 6), (-3, 0), (0, -6)$.
- (viii) $(2, 5), (5, 2), (5, -2), (2, -5), (-2, -5), (-5, -2), (-5, 2), (-2, 5)$.

AREA OF PARALLELOGRAM.

§5. Take a sheet of paper (a rectangle) and call the corners P, B, C, Q; BC being one of the longer sides (fig. 73). Mark a point A on the side PQ. Join BA, and cut (or tear) off the right-angled triangle PBA. You now have two pieces of paper; you will find that you can fit them together to make a parallelogram (ABCD in fig. 73).

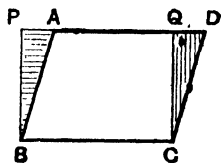


Fig. 73.

Notice (i) that the rectangle you had at first and the parallelogram you have now made are composed of the same paper, and *therefore have the same area*. Figures which are equal in area are said to be **equivalent**.

(ii) that the rectangle and the parallelogram are on the same base BC, and both lie between the same pair of parallel lines BC and PAQD. Or, we may say that they have *the same height*.

Hence, a parallelogram is equivalent to a rectangle on the same base and between the same parallels (or, of the same height).

The area of the rectangle is measured by the product of the base and height; and as the parallelogram has the same height as the rectangle, we have

The area of a parallelogram is measured by the product of the base and the height.

If two parallelograms be constructed on the same base and between the same parallels, they are each equal to the same rectangle. Hence

THEOREM 25. Parallelograms on the same base and between the same parallels (or, of the same height) are equivalent.

For example $ABCD$ and $PBCQ$ in fig. 74 are equivalent.

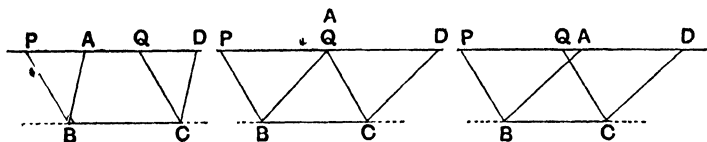


Fig. 74.

For proof of this Theorem see *Theoretical Geometry*, p. 52.

Any side of a parallelogram may be taken as the base. The perpendicular distance between the base and the opposite (parallel) side is called the height, or **altitude**.

Thus in fig. 75 if BC be taken as base, MN (which may be drawn from any point of the base) is the height (or altitude). If AB be taken as base, GH is the height.

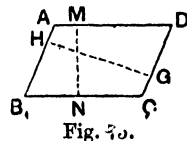


Fig. 75.

Ex. 8. In fig. 75 what is the height if CD be taken as base? if AD be taken?

Ex. 9. Find the area of a parallelogram (i) of sides 3.5 and 4.5 in. and of angle 70° , (ii) of sides 2 in. and 3 in. and of angle 30° , (iii) of sides 7.83 cm. and 8.75 cm. and of angle 45° .

Ex. 10. Draw a rectangle on base 12 cm. and of altitude 10 cm.; on the same base construct an equivalent parallelogram of angle 60° ; and measure its longer diagonal.

Ex. 11. The angle between two adjacent sides of a parallelogram of lengths 10 cm. and 8 cm. is 150° . If the side 10 cm. long remains unaltered but the 8 cm. is reduced to 6 cm., what is now the angle between them if the area is the same? What is the least length of the shorter side for constant area?

AREA OF TRIANGLE.

§6. To find the area of a triangle, $\triangle ABC$ in fig. 76, draw through A and C lines parallel to BC , LA respectively, forming a parallelogram. The triangle is half the parallelogram. But the area of the parallelogram is $BC \times AN$. Therefore the area of the triangle is $\frac{1}{2} BC \times AN$. Hence

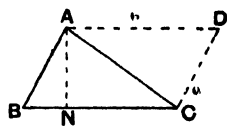


Fig. 76.

The area of a triangle is measured by half the product of the base and the height.

Or, the area of a triangle is half the area of a rectangle on the same base and of the same altitude.

THEOREM 26. Triangles on the same base and between the same parallels (or, of the same height) are equivalent.

For proof of this Theorem see *Theoretical Geometry*, p. 53.

Any side of a triangle may be taken as base. The line drawn perpendicular to the base from the opposite vertex is called the height, or altitude.

There will be three different altitudes according to the side which is taken as base.

Ex. 12. Draw an acute-angled triangle and draw the three altitudes. (*Freehand.*)

Ex. 13. Repeat Ex. 12 for a right-angled triangle. (*Freehand.*)

Ex. 14. Repeat Ex. 12 for an obtuse-angled triangle. (*Freehand.*)

Ex. 15. In what case are two of the altitudes of a triangle equal?

Ex. 16. A triangle ABC has AB = 3 in. and BC = 4 in.; trace the changes in its area as $\angle ABC$ increases from 0° to 180° .

Ex. 17. A parallelogram PQRS has PQ = 10 cm. and QR = 7 cm.; trace the changes in its area, as $\angle PQR$ increases from 0° to 180° .

Since any one of the three sides may be taken for base, there are three different ways of forming the product of a base and the corresponding altitude. Thus the area may be calculated in three different ways; and of course, theoretically, the result is the same in each case. Practically, none of the measurements will be quite exact, and the results will generally differ slightly. To get the best possible value for the area **take the average of the three results.**

Ex. 18. Find, to three significant figures, the areas of the following triangles, taking the average of three results in each case:—

(i) sides 3, 4, 4.5 in.

(iv) sides 2, 3, 4.5 in.

(ii) sides 6, 8, 9 cm.

(v) sides 3, 4 in., included $\angle 120^\circ$.

(iii) sides 3, 4, 5 in.

(vi) $BC = 7.2$ cm., $\angle B = 20^\circ$, $\angle C = 40^\circ$.

Ex. 19. Find the surface (i.e. the sum of the areas of all the faces):—

(i) of a tetrahedron (three-sided pyramid) each of whose edges is 4 inches. (See fig. 272.)

(ii) of a square pyramid whose base edges are each 2 inches and whose slant edges are each 2.5 inches. (See fig. 272.)

(iii) of a regular 3-sided prism whose base is an equilateral triangle of side 2 inches and whose height is 3.5 inches. (See fig. 265.)

Ex. 20. A prism stands on a triangular base of sides 5 in., 12 in., 13 in. and is 1 ft. high. Find the whole surface area.

Ex. 21. In Ex. 20 a plane through the longest side of the base cuts the opposite parallel edge at its mid-point. Find the lengths of sides of the triangular section and thence the area of the section.

AREA OF TRAPEZIUM.

§7. A **trapezium** is a quadrilateral with two of its sides parallel.

The parallel sides of the trapezium ABCD in fig. 77 are a and b ; the height is h .

The area of the trapezium

$$= \text{area of } \triangle ABC + \text{area of } \triangle ADC$$

$$= \frac{1}{2} a \cdot h + \frac{1}{2} b \cdot h$$

$$= \frac{1}{2} (a + b) \cdot h.$$

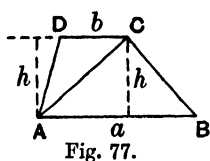


Fig. 77.

Thus **the area of a trapezium is measured by the product of the height and the average of the two parallel sides.**

Ex. 22. In fig. 77 draw perpendiculars to AB through the mid-points of AD, EC. Hence justify the expression for the area of the trapezium.

Ex. 23. Draw a trapezium on a doubled sheet of paper, cut it out and place the two parts together so as to form a parallelogram. Hence derive the above expression for the area of a trapezium.

Ex. 24. Draw a trapezium ABCD with $AB \parallel$ to CD , given $AB = 3.5$ in., $CD = 2.5$ in., $AD = 2$ in., $\angle A = 90^\circ$. Find its area.

Ex. 25. Draw a trapezium with one parallel side 10 cm., angles at its extremities 70° and 55° , and height 5 cm. Measure the parallel side and find the area.

Ex. 26. Draw a trapezium ABCD with $AB \parallel$ to CD , given $AB = 4$ in., $AD = 2.8$ in., $DC = 2$ in., $\angle A = 70^\circ$. Find its area.

Ex. 27. A symmetrical railway cutting has a cross-section 20 ft. wide at bottom, 80 ft. at top, and sloping sides 45 ft. long. Find, by drawing and measurement, the area of the section.

Ex. 28. Fig. 78 represents an oval boiler plate. To find its area the central line is divided into 8 equal parts each of length 3". The cross-lines numbered i...vii are of lengths 8", 11.5", 13", 14", 13", 11.5", 8" respectively. By regarding each portion formed by these cross-lines as a trapezium, find the area of the plate.

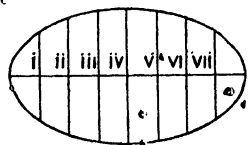


Fig. 78.

Ex. 29. Find the area of the figure (1, 2), (1, 10), (6, 13), (6, 2).

Ex. 30. Find the area of the triangle (1, 2), (2, 5), (3, 1) by adding two trapezia and subtracting one. [Draw perpendiculars to the x-axis.]

AREA BY FIELD-BOOK METHOD.

§8. The following method is used by land-surveyors. It is required to find the area of the field $ABCDEFG$ (fig. 79). The field is treated as a polygon, the sides of the polygon being chosen so that the small irregularities may roughly compensate one another. The longest diagonal AE is chosen as **base-line**. In AE points L, M, N, P are determined, namely the points where the perpendiculars from the corners meet AE . The field is thus divided up into right-angled triangles, trapezia and rectangles, whose areas can be calculated as soon as the necessary measurements have been made. The surveyor now measures with a chain the different distances along the base-line, AL, AM, AN, AP, AE ; also the distances to the different corners, right and left of the base-line*, namely, LB, MC, MG, NF, PD . These measurements are recorded in the Field-Book in the following form:—

Yards		
	To E	
	600	
240	460	
	360	50
240	300	120
200	100	
From	A	go North

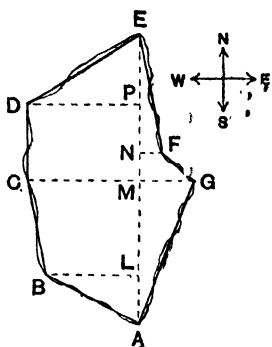


Fig. 79.

This record is to be read *upwards*. In the middle column are set down the distances from A of the different points on the base-line; on the right and left are set down the offsets as they occur; e.g. L is 100 yards North of A , and B is 200 yards to the left of L ; and so on.

Ex. 31. On inch paper draw a plan of the field represented in fig. 79 from the measurements given (scale, 1 inch to represent 100 yards); calculate its area in square yards. (Ignore the small irregularities.)

Ex. 32. Give the coordinates of the corners of the field in fig. 79, taking AE as axis of y and A as origin.

* The distances at right angles to the base-line are called **offsets**; in practice they are never allowed to exceed a few yards, on account of the difficulty of determining accurately the feet of the perpendiculars.

Ex. 33. Draw a plan and find the area of the fields in the following surveys:—(*Freehand*)

Yards		
	To D	
	400	
	340	50
70	300	
90	200	
30	100	50
From	A	go North

Yards		
	To B	
	500	
	470	100
70	400	140
	250	150
100	100	
From	A	go North

In practice, distances are measured with a **chain** of 100 **links**. The length of the surveyors' chain is the same as the length of a cricket pitch, namely 22 yards. A square whose side is 1 chain has area 22^2 or 484 sq. yards. Now an acre contains 4840 sq. yards; hence **10 sq. chains = 1 acre**.

Ex. 34. Draw plans and find the area (in acres) of the fields whose dimensions are recorded below:—(*Freehand*)

(i)

Links		
	To B	
	800	
	600	400
500	300	
100	200	
From	A	go East

(ii)

Links		
	To B	
	1100	
400	1000	
	800	800
400	600	
From	A	go S.E.

(iii)

Links		
	To B	
	800	
150	700	
100	500	200
	400	350
300	300	
100	200	
From	A	go N.W.

(iv)

Links		
	To B	
	1020	
20	900	
30	810	
330	480	300
270	270	
90	150	180
From	A	go West

AREA OF ANY RECTILINEAR FIGURE.

§ 9. This may be determined in various ways.

METHOD I. By dividing up the figure into triangles.

Ex. 35. Find the areas of the quadrilaterals in Exx. 28, 29, Ch. 4.

METHOD II. By a base-line and offsets as in field-book method.

METHOD III. It is often convenient to construct a single triangle equivalent to the given figure.

§ 10. Draw a triangle ABC , and XY parallel to BC cutting AB , AC at X , Y ; let CX , BY intersect at O . Find reasons for the equivalence of the following pairs of triangles:—(i) XBC , YBC , (ii) BCX , BCY , (iii) OBX , OCY , (iv) ABY , ACX .

¶ Ex. 36. Show how to transform a given quadrilateral $ABCD$ into an equivalent quadrilateral $ABCD'$, so that the three vertices A , B , C may be unchanged, and $\angle BAD' = 170^\circ$.

¶ Ex. 37. Repeat Ex. 36, making $\angle BAD' = 180^\circ$. What kind of figure is produced?

To construct a triangle equivalent to a given quadrilateral $ABCD$.

Construction Join CA . Through D draw

$DD' \parallel$ to CA , meeting BA produced in D' .

Join CD' .

Then $\triangle BCD' = \text{quadrilateral } ABCD$.

Proof Since DD' is $\parallel$ to AC .

$\triangle ACD' = \triangle ACD$.

Th. 26.

Add to each $\triangle ACB$.

$\therefore \triangle BCD' = \text{quadrilateral } ABCD$.

In a similar way a pentagon may be reduced, first to an equivalent quadrilateral and then to an equivalent triangle: and so for figures of more sides. The area of the triangle can then be found as already explained. A convenient method of dealing with the pentagon is shown in fig. 81.

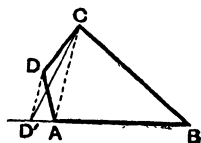


Fig. 80.

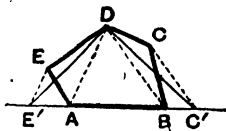


Fig. 81.

Ex. 38. Explain the construction of fig. 81, and show that

$\triangle C'DE' = \text{figure } ABCDE$.

Ex. 39. Reduce the quadrilateral ABCD to a triangle and find its area, when

(i) $DA = 1$ in., $\angle A = 100^\circ$, $AB = 2.3$ in., $\angle B = 64^\circ$, $BC = 1.5$ in.*

(ii) $AB = 5.7$ cm., $BC = 5.2$ cm., $CD = 1.7$ cm., $DA = 3.9$ cm., $\angle A = 76^\circ$.

Ex. 40. Find the area of a pentagon ABCDE, given $AB = 6.5$ cm., $BC = 2.4$ cm., $CD = DE = 4$ cm., $EA = 2.5$ cm., $\angle A = 80^\circ$, $\angle B = 133^\circ$. (Reduce the pentagon to a triangle.)

Ex. 41*. A rectangular sheet of cardboard 30 cm. by 24 cm. weighs 52 gm. An irregular figure is traced on to it, cut out, and found to weigh 21.4 gm. What is the area of the figure?

MISCELLANEOUS EXERCISES ON AREA.

Ex. 42. ABCD is a rectangle, P is the mid-point of AB, and Q and R are points on BC, AD such that $AR = \frac{1}{2}AD$, $BQ = \frac{1}{3}BC$. Compare the area of PRDQC with the area of the rectangle.

Ex. 43. ABCD is a square, side 10 cm. A, D are joined to P, the mid-point of BC, and B, C are joined to Q, the mid-point of AD. Find the area of the portion common to the two triangles APD, BQC.

Ex. 44. A field in the shape of a quadrilateral ABCD has the following measurements:— $AB = 60$ yards, $BC = 25$ yards, the angle $ABC = 90^\circ$, $CD = 39$ yards, $DA = 52$ yards. Draw a plan of the field, and find its area.

Ex. 45. ABCD is a trapezium whose parallel sides, BC and DA, are 37 and 100 feet long, respectively; $AB = 51$, $CD = 30$. Find by drawing and measurement the distance between the parallel sides, and also the area.

[Draw a parallel to AB.]

Ex. 46. Find the area in acres of a 4-sided field ABCD in which $AD = 170$ yd., $DC = 150$ yd., $BC = 190$ yd., $AC = 260$ yd., and the angle ABC is a right angle.

Ex. 47. A square of 4 jointed rods 20 cm. long is deformed into a rhombus with an angle of 50° . Find the decrease in area of the figure.*

Ex. 48. The front and back walls of a building are 20 ft. and 17 ft. high. The ridge of the roof is 24 ft. above ground level. The front slope of the roof measures 15 ft. from ridge to front wall and the back slope 12 ft. from ridge to back wall. What is the area of an end elevation?

Ex. 49. The plate of a copper voltameter has the shape shown in fig. 82. Find the area of the plate. Copper is deposited over the whole of A and on B to a height one-half the height of B. Find the area covered with copper.

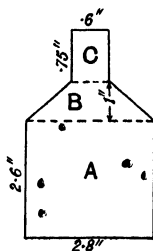


Fig. 82.

* This gives a method of finding areas by weighing.

Ex. 50. The figure represents a field to scale, 1 centimetre denoting a chain. Estimate the area of the field in acres.

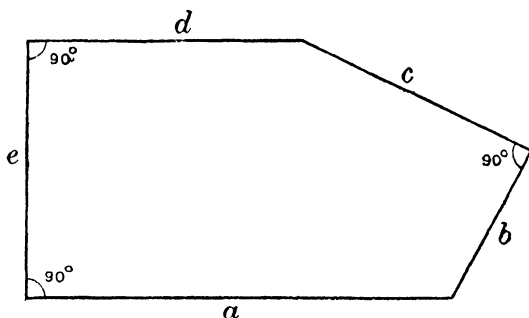


Fig. 83.

Ex. 51. ABCD is a quadrilateral whose diagonals intersect in O. Make DE on BD produced = BO. Show that $\triangle ABC = \triangle ADE + \triangle CDE$. Hence show that the quadrilateral ABCD is equivalent to the triangle ACE.

CONSTRUCTIONS.

Ex. 52. Fig. 84 represents a square ABCD with a side 5 in. long. BE = 2 in., CF = 3 in. Find the area of the figure ABEF.

Draw the figure *full size*, and by a geometrical construction find a point G in AD such that the area of ABEG = the area of ABEF. Measure DG.

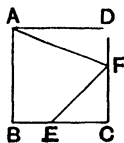


Fig. 84.

Ex. 53. Draw a square ABCD with side 8 cm. long. Join E, the middle point of AD, to a point F in AB such that AF = 6 cm. Find the area of the figure BCDEF in square cm., and by a geometrical construction find a point H in CB produced such that the area of the triangle HCD equals the area of BCDEF. Measure CH, and check the accuracy of your result by calculation.

Ex. 54. Without making any calculations, construct a parallelogram with one angle 45° and one side of length 1.4 in., equal in area to an equilateral triangle whose side is 2 in. in length.

Ex. 55. ABCD is a field with two parallel sides, the other sides not being parallel nor equal. Show how to divide it into two parts equal in area by one straight line. Justify your method.

SUMMARY OF RESULTS

The area of a parallelogram is measured by the product of the base and the height.

Parallelograms on the same base and between the same parallels (or, of the same height) are equivalent.

The area of a triangle is measured by half the product of the base and the height.

Triangles on the same base and between the same parallels (or, of the same height) are equivalent.

The area of a trapezium is measured by the product of the height and the average of the two parallel sides.

CHAPTER 8

THE THEOREM OF PYTHAGORAS

§ I. Fig. 85 represents an isosceles right-angled triangle with squares described upon each of the sides. The dotted lines divide up the squares into right-angled triangles, each of which is obviously equal to the original triangle. This sub-division shows that the square on the hypotenuse of the above right-angled triangle is equal to the sum of the squares on the sides containing the right angle. (Some tiled pavements show this fact very clearly.)

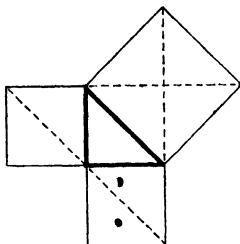


Fig. 85.

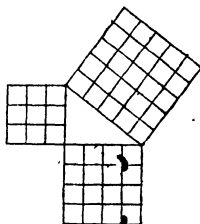


Fig. 86.

¶Ex. 1. Construct a right-angled triangle with sides of 3 cm. and 4 cm. containing the right angle. Measure the length of the hypotenuse. What relation is there between the square on the hypotenuse and the squares on the other two sides? See fig. 86.

Ex. 2. Draw a good-sized scalene right-angled triangle ABC, right-angled at A. Measure the three sides and calculate the areas of the squares upon them. Add together the areas of the two smaller squares, and arrange your results like this—

$$\begin{aligned} AB &= \dots \text{cm.}, & \text{sq. on } AB &= \dots \text{sq. cm.}, \\ AC &= \dots \text{cm.}, & \text{sq. on } AC &= \dots \text{sq. cm.}, \\ \text{sum of sqq. on } AB, AC &= \dots \text{sq. cm.}, \\ BC &= \dots \text{cm.}, & \text{sq. on } BC &= \dots \text{sq. cm.} \end{aligned}$$

The above exercises lead up to the Theorem of Pythagoras.

THEOREM 29. In a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares on the other two sides.

This famous theorem was discovered by Pythagoras, a Greek, who lived in Sicily, B.C. 570–500.

If ABC is a triangle right-angled at A , Pythagoras' theorem states that

$$BC^2 = CA^2 + AB^2,$$

$$\text{or } a^2 = b^2 + c^2.$$

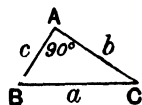


Fig. 87.

For proof of this Theorem see *Theoretical Geometry*, p. 59.

Ex. 3. Draw (on paper or, better, on thin cardboard) a right-angled triangle and the squares on the three sides (see fig. 88). Take the larger of the two smaller squares and cut it up in the following manner. First find the centre of the square by drawing the diagonals. Then, through the centre, make a cut across the square parallel to BC , the hypotenuse, and a second cut perpendicular to BC . It will be found that the four pieces of this square together with the other small square exactly make up the square on the hypotenuse. (Perigal's dissection.)

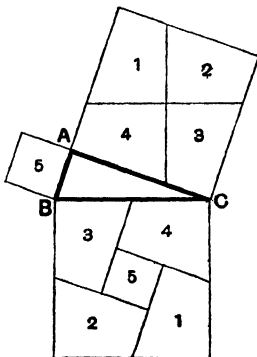


Fig. 88.

Calculate the side of a rhombus whose diagonals are of lengths 20 cm. and 12 cm.

$ABCD$ is the rhombus. The diagonals AC and BD bisect one another at right angles at O .

Hence $\triangle AOB$ is right-angled at O and $OA = 6$ cm., $OB = 10$ cm.

By Pythagoras' theorem,

$$\begin{aligned} AB^2 &= AO^2 + OB^2 \\ &= 6^2 + 10^2 \\ &= 36 + 100 \\ &= 136. \end{aligned}$$

$$\therefore AB = \sqrt{136} = 11.66 \text{ (to 2 places).}$$

each side = 11.66 cm.

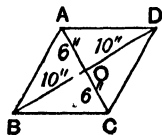


Fig. 89.

Calculate the length of a diagonal of a rectangular solid whose edges are 10", 8" and 7".

Fig. 90 represents the solid; HB is the diagonal whose length is required. All the diagonals are equal.

In the $\triangle HDB$, $\angle D$ is a rt. $\angle$.

$\therefore$ by Pythagoras' theorem,

$$HB^2 = HD^2 + DB^2 \dots\dots\dots(1).$$

In the $\triangle DAB$, $\angle A$ is a rt. $\angle$.

$\therefore$ by Pythagoras' theorem,

$$DB^2 = DA^2 + AB^2 \dots\dots\dots(2).$$

Substituting in (1) from (2)

$$\begin{aligned} HB^2 &= HD^2 + DA^2 + AB^2 \\ &= 7^2 + 8^2 + 10^2 \\ &= 49 + 64 + 100 \\ &= 213. \end{aligned}$$

$$\therefore HB = \sqrt{213} = 14.6'' \text{ (to 1 place).}$$

$$\therefore \text{length of diagonal is } 14.6 \text{ in.}$$

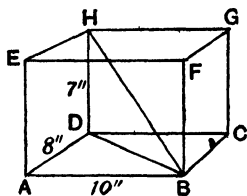


Fig. 90.

Ex. 4. What is the square on the hypotenuse of a right-angled triangle if the sides containing the right angle are 5 cm. and 12 cm.? Hence calculate the length of the hypotenuse.

Ex. 5. Find the hypotenuse of a right-angled triangle when the sides containing the right angle are (i) 7 in., 24 in., (ii) 2.34 in., 4.65 in., (iii) 65 mm., 83.5 mm., (iv) a mm., b mm.

Ex. 6. Find the remaining side and the area of a right-angled triangle, given the hypotenuse and one side, as follows:—(i) hyp. = 10 cm., side = 6 cm., (ii) hyp. = 6 in., side = 4 in., (iii) hyp. = 143 mm., side = 71.5 mm., (iv) hyp. = x in., side = y in.

Ex. 7. Find the diagonal of a rectangle whose sides are (i) 20 ft. and 30 ft., (ii) a cm. and b cm.

Ex. 8. Find the remaining side and the area of a rectangle, given (i) diagonal = 2.5 in., one side = 2 in., (ii) diagonal = x in., one side = y in.

Ex. 9. Find the diagonal of a square whose side is (i) 15 mm., (ii) x'' . What is the ratio of the diagonal to the side?

Ex. 10. Find the side and area of a square whose diagonal is (i) 10 in., (ii) x feet.

Ex. 11. The side of a rhombus is 11 in. and one diagonal is 6 in. Calculate the length of the other diagonal.

Ex. 12. Find the altitude of an isosceles triangle, given (i) base = 9 cm., side = 6 cm., (ii) base = $2x''$, side = y'' .

Ex. 13. Find the altitude of an equilateral triangle of side 10 cm. Thence find the area of the triangle.

Ex. 14. In fig. 91, ABCD represents a square of side 3 in.; AE=AH=CF=CG=1 in. Show that EFGH is a rectangle; find its perimeter and diagonal.

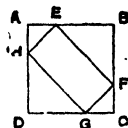


Fig. 91.

Ex. 15. If the coordinates of a point P are (1, 2) and of Q (3, 5), find the distance PQ. (PQ is the diagonal of a certain rectangle.)

Ex. 16. In each of the following cases find the distance between the pair of points whose coordinates are given:—(i) (2, 0) and (0, 3); (ii) (−1, −1) and (2, 1); (iii) (−2, 2) and (1, −2); (iv) (x, y) and (a, b).

Ex. 17. Find by drawing and measurement the value of (i) $\sqrt{2}$, (ii) $\sqrt{5}$, (iii) $\sqrt{15}$.

Ex. 18. Draw any two squares. Find, by construction, a square whose area is (i) the sum, (ii) the difference, of the two squares.

Ex. 19. Find how far a traveller is from his starting point after the following journeys:—(i) first 10 miles N., then 8 miles E., (ii) first 8 miles E., then 10 miles N., (iii) 43 km. S.W. and 32 km. S.E., (iv) 24 miles S., 10 miles E., 14 miles N. (try to complete a right-angled triangle having the required line for hypotenuse), (v) 4 miles W., 6 miles S., 3 miles W., 1 mile N.

Ex. 20. A ladder 30 ft. long rests on the top of a wall 12 ft. high with its foot on the ground 10 ft. from the wall. What length of ladder projects over the wall?

Ex. 21. A gun on shore has a range of 6 miles and a vessel steers a course parallel to the shore so as to pass within $2\frac{1}{2}$ miles of the gun. Over what distance is the vessel within range of the gun?

Ex. 22. A, B, C are three towns in a straight line running east to west. P is another town due N. of B. If PA=8 miles, PB=6 miles, BC=4.5 miles, find the distances PC and AB.

Ex. 23. Three vertical masts, 40 ft., 100 ft., 25 ft. high, are erected in a straight line, the middle one being 80 ft. from each of the others. Calculate the lengths of cable required to connect the tops of the three masts.

Ex. 24. A mountain railway up a straight incline crosses the 200 and 250 ft. contours at points 500 yd. distance apart. Find by calculation the gradient in the form 1 in n, where n is measured along the horizontal.

Ex. 25. A roof truss is constructed as follows. AB is horizontal and 24 ft. long, CD is vertical from C, the mid-point of AB, and 12 ft. long. CE and CF are perpendicular to AD and DB. Calculate the lengths of the members AD and CE.

Ex. 26. A lighthouse shows a light through a sector from E. to S.E. Steering N.E. the captain of a 10-knot vessel first sees the light at 1 a.m. At 1.30 a.m. the light disappears. How far off is it then?

Ex. 27. From A I walk 2 miles due north to B; from B I walk 1 mile to C in direction south-east; from C I walk 2 miles due east to D; from D I walk in a south-westerly direction to a point E situated due east of A. Calculate in miles (to the nearest hundredth) the distances DE and AE. Check your results by drawing and measurement.

Ex. 28. A haystack of which the edge of the thatch is 15 ft. vertically above the ground overhangs its base by 3 ft. What length of ladder will be required so that it will just reach to the edge of the thatch if placed at a point on the ground 11 ft. from the base of the haystack?

THREE DIMENSIONS.

Ex. 29. A room is 18 ft. long, 14 ft. wide, 10 ft. high. Find the diagonals of the walls. Find the diagonal of the floor and the length of a string stretched across the room from one corner of the floor to the opposite corner of the ceiling.

Ex. 30. Find the length of a string stretched from the middle of the ceiling to a corner of the floor of the room in Ex. 29.

Ex. 31. Find the diagonal of the face of a cubic decimetre. Also find the diagonal of the cube.

Ex. 32. Find the slant side of a cone of (i) height 5", base-radius 3"; (ii) height 25 mm., base-diameter 56 mm., (iii) height h feet, base-diameter $2r$ feet.

Ex. 33. Find the height of a cone of (i) slant side 20", base-radius 6"; (ii) slant side 6.2 m., base-diameter 10 m., (iii) slant side l , base-radius r .

Ex. 34. Find the base-radius of a cone of (i) slant side 11.3 cm., height 57 mm., (ii) slant side l ", height h ".

Ex. 35. A pyramid of height 5" stands on a square base of side 4". Find the length of a sloping edge of the pyramid. (See fig. 272.)

Ex. 36. At two points, one due E. and the other due S. of a tower 100 ft. high, the tower subtends an angle of 45° . What is the distance between the two points?

Ex. 37. ABC is the base of a right prism, $AB=9"$, $BC=8"$, $CA=12"$. P is on the edge through A perpendicular to the base and $AP=5"$. Calculate the sides of the triangle PBC.

Ex. 38. In Ex. 37, Q is on the perpendicular edge at B and $BQ=4"$, R on that at C and $CR=6"$. Calculate the sides of the triangular section PQR.

Ex. 39. Show that the length of the diagonal of a cube is $\sqrt{3}$ times the length of a side.

Ex. 40*. AD is the altitude of a triangle ABC. Show that

$$AB^2 - AC^2 = BD^2 - CD^2.$$

* This Ex. leads to one method of "solving a triangle" by calculation when the lengths of the sides are given. (See Ex. 41.)

Ex. 41. In Ex. 40 let $AB=3$ in., $AC=2$ in., $BC=3$ in. Calculate $BD^2 - CD^2$. Hence find $BD - CD$.

$$[BD^2 - CD^2 = (BD - CD)(BD + CD) = (BD - CD)BC]$$

Knowing $BD - CD$ and $BD + CD$, you may now find BD and CD . Hence find AD . Hence find area of $\triangle ABC$.

Ex. 42. ABC is a triangle right-angled at B and BN is perpendicular to AC . If $BA=8''$, $BC=6''$, find AN , CN .

Ex. 43. The sides of a triangle are 6 cm., 12 cm., 15 cm. Calculate the length of the perpendicular to the greatest side from the opposite vertex.

§ 2. Pythagoras' theorem asserts that if a triangle is right-angled a certain relation holds between the lengths of its sides. The **converse** theorem is also true; if the sides of a triangle are connected by the relation $a^2 = b^2 + c^2$, then the angle opposite to a is a right angle. Thus a triangle whose sides are 3, 4, 5 units is right-angled, for $5^2 = 3^2 + 4^2$.

This leads to a method of constructing a right angle which was perhaps used by the ancient Egyptians as early as 2000 B.C., in constructing the ground plan of temples and pyramids. It is still the best method for marking out the right angles at the corners of a lawn tennis court. Suppose that the base-line has been marked. On a piece of rope mark off three lengths of 3, 4, 5 yards. Put in a peg at one end of the base-line, another peg on the base-line 4 yards from the corner. Stretch the rope round these two pegs, so that the ends of the 4-yard length coincide with the two pegs. Now bring the two ends of the rope together so that a triangle is formed with sides of 3, 4, 5 yards. This will give a right angle at the corner, and the side line can be marked out.

Ex. 44. Are the triangles right-angled whose sides are

(i) 8, 17, 15; (ii) 12, 36, 34; (iii) 25.5, 25.7, 3.2;

(iv) $4n$, $4n^2 - 1$, $4n^2 + 1$; (v) $m^2 + n^2$, $m^2 - n^2$, $2mn$; (vi) a , b , $a + b$?

Ex. 45. Bristol is 71 miles due W. of Reading; Reading is 55 miles from Northampton; Northampton is 92 miles from Bristol. Ascertain whether Northampton is due N. of Reading.

SUMMARY OF RESULTS

In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the sides containing the right angle.

CHAPTER 9

LOCI

§ I. Mark two points **A** and **B**, say about 2 inches apart. Mark a point equidistant from **A** and **B** (i.e. at equal distances from **A** and **B**), judging the distance by eye.

In a similar way mark about 10 points equidistant from **A** and **B**; some above and some below **AB**.

Notice what pattern this set of points seems to form. Draw a line passing through all of them.

Find a point on **AB** equidistant from **A** and **B**; this belongs to the set of points.

The pattern formed by all possible points equidistant from two fixed points **A** and **B** is called the **locus** of points equidistant from **A** and **B**. (Keep this figure to use again when you have done the following set of exercises.)

¶Ex. 1. What is the locus of points at a distance of 1 inch from a fixed point **O** (i) on the paper, (ii) in space?

¶Ex. 2. Draw a straight line right across your paper. What is the locus of points distant 1 inch from this line? Remember that the distance is reckoned perpendicular to the line.

¶Ex. 3. A bicyclist is riding straight along a level road. What is the locus of the hub of the back wheel?

¶Ex. 4. What is the locus of the tip of the hand of a clock?

¶Ex. 5. What is the locus of a man's hand as he works the handle of a common pump?

¶Ex. 6. A stone is thrown into still water and causes a ripple to spread outwards. What is the locus of the points which the ripple reaches after one second?

¶Ex. 7. What is the locus of all points at which a thunderclap could be heard at the same instant? Sound travels about 1100 ft. per second.

¶Ex. 8. One cog-wheel rolls round the circumference of another wheel whose centre is fixed. What is the locus of the centre of the rolling wheel?

§ 2. The above exercises suggest the following definition of a locus.

DEF. If a point moves so as to satisfy certain conditions, the **path** traced out by the point is called its **locus**.

Ex. 9. A man stands on the middle rung of a ladder against a wall. The ladder slips down; find the locus of the man's feet.

[Do this by drawing two straight lines at right angles to represent the wall and the ground; take a length of, say, 4 inches to represent the ladder; draw the ladder (the line of 4 inches) on **tracing-paper**; then bring the ends of the ladder on to the two lines of the paper below, and prick through the middle point. Or you may use the edge of your ruler.]

Ex. 10. Draw two unlimited lines, intersecting near the middle of your paper at an angle of 60° . By eye, mark a point equidistant from the two lines. Mark a number of such points, say 20, in various positions. The pattern formed should be *two* straight lines. How are these lines related to the original lines? How are they related to one another? (Keep this figure for reference later.)

Ex. 11. (On squared paper.) Draw a pair of lines at right angles (OX , OY); plot a series of points each of which is twice as far from OX as from OY . What is the locus? (Keep your figure for the next Ex.)

Ex. 12. Using the figure of Ex. 11, plot the locus of points 3 times as far from OX as from OY ; also the locus of points $\frac{1}{2}$ as far from OX as from OY .

Ex. 13. (On squared paper.) Plot the locus of a point which moves so that the sum of its distances from two lines at right angles is always 4 inches.

Ex. 14. (On squared paper.) Plot the locus of a point which moves so that the difference of its distances from two lines at right angles is always 1 inch.

Ex. 15. Draw a line, and mark a point O 2 inches distant from the line. Let P be a point moving along the line. Experimentally, plot the locus of the mid-point of OP .

Ex. 16. A point O is 3 cm. from the centre of a circle of radius 5 cm. Plot the locus of the mid-point of OP , when P moves round the circumference of the circle.

Ex. 17. A rod OP is freely hinged at O ; what is the locus of P ? At P another rod PQ is freely hinged and always hangs vertically; what is the locus of Q ?

Ex. 18. Draw $ABCD$ a parallelogram, AB being its base; on CD construct an isosceles right-angled triangle CDP , $\angle P$ being the right angle. Now regard the figure as a framework freely jointed at A, B, C, D . If AB is fixed, what are the loci (i) of C , (ii) of D , (iii) of P ?

For further Exs. on loci, especially those involving the use of tracing-paper, see Ch. 19, p. 163.

§3. You may now refer to the figure you constructed at the beginning of this chapter, showing the locus of points equidistant from two given points A and B. What does this locus appear to be?

In fig. 92, PNQ is the perpendicular bisector of AB .

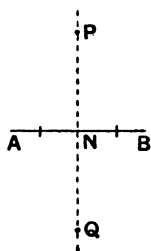


Fig. 92.

The figure is symmetrical about PQ .

A and B are corresponding points: by symmetry, any point on the axis of symmetry is equidistant from A and B.

And again, no point off this line can be equidistant from A and B. A point to the left of the axis is nearer to A; a point to the right of the axis is nearer to B. Hence

THEOREM 34. The locus of points equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points.

§4. Refer to the figure you constructed for Ex. 10, showing the locus of points equidistant from two unlimited lines. What did the locus appear to be?

Two intersecting straight lines form four angles, generally two acute and two obtuse.

Let us for the moment fix our attention on the two *acute* angles formed by the unlimited straight lines AOA' , BOB' in fig. 93.

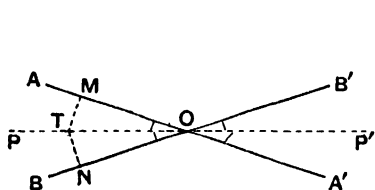


Fig. 93.

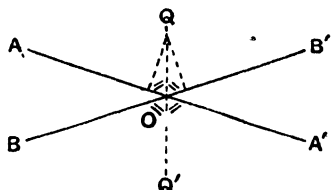


Fig. 94.

The straight line POP' bisects these angles.

The figure is now symmetrical about POP' as axis.

By symmetry, any point T on the axis is equidistant from AOA' and BOB' .

By precisely the same argument as before, we see that any point on QOQ' , the bisector of the *obtuse* angle, is equidistant from AOA' and BOB' .

A point that lies on neither bisector must be nearer to the one line than to the other. For instance, a point inside the angle AOP (fig. 93) is nearer to AOA'.

Hence

THEOREM 35. The locus of points equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines.

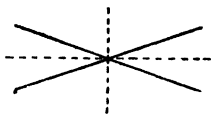


Fig. 95.

INTERSECTION OF LOCI.

§ 5. *Given a line l and a point P , find a point (or points) distant in. from the line and 1.8 in. from the point.*

The locus of all points 2 in. from the line consists of the parallel lines 1 in. from the line.

$\therefore$ draw m and $n \parallel$ to l 1 in. away from it.

The locus of all points 1.8 in. from P is a circle, centre P .

$\therefore$ draw a circle, centre P , radius 1.8 in.

The points of intersection of the circle and the parallels m and n are the points required.

A, B, C and D are the points required.

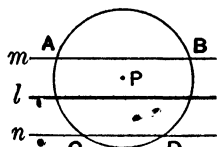


Fig. 96.

In Exs. 19–46 brief explanations must be given, as above.

Ex. 19. Draw two unlimited straight lines intersecting at an angle of 45° . Find a point, or points, distant 1 in. from each line.

Ex. 20. Draw an unlimited straight line and mark a point O 2 in. from the line. Find two points distant 4 inches from O and from the line. Measure the distance between them.

Ex. 21. In Ex. 20 find as many points as you can distant 1 inch from both point and line.

Ex. 22. If A, B are two points 3 inches apart, show how to find a point (or points) distant 4 inches from A and 5 inches from B .

Ex. 23. Make an angle of 45° on one of the arms mark a point A 3 inches from the vertex of the angle. Find a point (or points) equidistant from the arms of the angle, and 2 inches from A . Measure the distance between the two points found.

Ex. 24. On the diagonal BD (produced if necessary) of a quadrilateral $ABCD$, show how to find a point (i) equidistant from A and C , (ii) equidistant from AB and AD , (iii) equidistant from AB and DC .

Ex. 25. Find a point on the base of an equilateral triangle (side 10 cm.) which is 4 cm. from one side. Measure the two parts into which it divides the base.

Ex. 26. Find a point on the base of a scalene triangle equidistant from the two sides. Is this the middle point of the base?

Ex. 27. Draw a circle, a diameter AB, and a chord AC through A. Find a point (or points) on the circle equidistant from AB and AC.

Ex. 28. Find a point (or points) on a given circle equidistant from a pair of parallel straight lines.

Ex. 29. Find a point (or points) equidistant from two given straight lines and at a given distance from a third line.

Ex. 30. Find a point (or points) equidistant from two sides of a triangle and also equidistant from the ends of the third side.

Ex. 31. Draw a triangle ABC; find a point O which is equidistant from B, C; and also equidistant from C, A. Test by drawing a circle with centre O to pass through A, B, C.

Ex. 32. Draw a triangle ABC. Inside the triangle find a point P which is equidistant from AB and BC; and also equidistant from BC and CA. From P draw perpendiculars to the three sides; with P as centre and one of the perpendiculars as radius draw a circle.

Ex. 33. P is a moving point on a fixed line AB; O is a fixed point outside the line. P is joined to O, and PO is produced to Q so that $OQ = PO$. Show by symmetry that the locus of Q is a line parallel to AB.

Ex. 34. Use the locus of Ex. 33 to solve the following problem. O is a point in the angle formed by two lines AB, AC. Through O draw a line, terminated by AB, AC, and bisected at O.

Ex. 35. Draw a figure like fig. 97, making radius of circle 2 in., $CO = 3$ in., $ON = 5$ in. Through O draw a line (or lines), terminated by AB and the circle, and bisected at O. (See Ex. 33.)

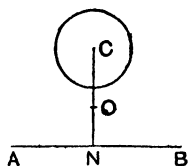


Fig. 97.

Ex. 36. A river with straight banks is crossed, slantwise, by a straight weir. Sketch a figure representing the position of a boat which finds itself at the same distance from the weir and the two banks.

Ex. 37. A town X is 2 miles from a straight railway; but the two stations nearest to X are each 3 miles from X. Find the distance between the two stations.

Ex. 38. Two breakwaters, A and B, each 950 yards long, are built out from the same point of a shore which runs due E. and W. A runs S.W. and B due S. A ship is at anchor, 200 yards from A, and 110 yards from B. How far is she from the shore?

Ex. 39. At noon, from a ship steering East and making 10 knots a lighthouse bears N. 32° E. The navigator draws through the lighthouse, as marked on his chart, a line on which the ship is known to lie at noon. At 12.30 the ship must lie on a parallel line shifted 5 sea-miles to the East.

At 12.30 p.m. the lighthouse is found to bear N. 25° W.; accordingly another line is drawn on the chart. Thus the position of the ship at 12.30 p.m. is determined; find its distance from the lighthouse at 12.30 p.m.

CONSTRUCTION OF TRIANGLES, ETC. BY MEANS OF LOCI.

Ex. 40. Construct $\triangle ABC$, given

- (i) base $BC=14$ cm., height $=9$ cm., $\angle B=65^{\circ}$. Measure AB .
- (ii) $AB=59$ mm., $AC=88$ mm., height $AD=49$ mm. (Draw height first.) Measure base BC .
- (iii) $BC=4$ in., $\angle B=80^{\circ}$, median $CN=4$ in. Measure BA .
- (iv) base $BC=12$ cm., height $AD=4$ cm., median $AL=5$ cm. Measure AB, AC .

Ex. 41. Construct a right-angled triangle, given

- (i) longest side $=10$ cm., another side $=5$ cm. Measure the smallest angle.
- (ii) side opposite right angle $=4$ in., another side $=3$ inches. Measure the third side.

Ex. 42. Construct a right-angled triangle ABC , given $\angle A=90^{\circ}$, $AB=7$ cm., distance of A from $BC=2.5$ cm. Measure the smallest angle.

Ex. 43. Construct a triangle, given height $=2$ in., angles at the extremities of the base $=40^{\circ}$ and 60° . Find length of base.

Ex. 44. Construct a quadrilateral $ABCD$, given diagonal $AC=9$ cm., diagonal $BD=10$ cm., distances of B, D from AC 5 cm. and 4 cm. respectively, side $AB=7$ cm. Measure CD .

Ex. 45. Construct a trapezium $ABCD$, given base $AB=10$ cm., height $=4$ cm., $AD=4.5$ cm., $BC=4.2$ cm. Measure angles A and B . (There are 4 cases.)

Ex. 46. Construct a trapezium $ABCD$, given base $AB=3.5$ in., height $=1.7$ in., diagonals $AC, BD=2.5, 3.5$ in. respectively. Measure CD .

SUMMARY OF RESULTS

The locus of a point which is equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points.

The locus of a point which is equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines.

MISCELLANEOUS EXERCISES I

Ex. 1. ABC is an isosceles triangle with $AB = AC$. X is a point on AC produced such that $BX = BA$. If $\angle BXC = 34^\circ$, calculate all the angles, giving reasons.

Ex. 2. PQR is an isosceles triangle with $PQ = PR$, $\angle P = 118^\circ$; QXP is another isosceles triangle with $QP = QX$, X being on RP produced. Calculate all the angles in the figure and show that $\angle QXR = 2\angle QRP$.

Ex. 3. To trisect a given angle. Much time was devoted to this famous problem by the Greeks and the geometers of the Middle Ages; it has now been shown that it is impossible with only the aid of a pair of compasses and a straight edge (ungraduated).

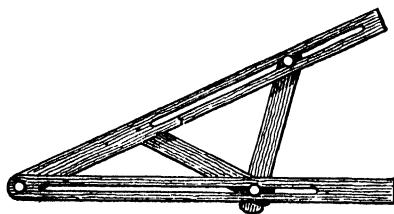


Fig. 98.

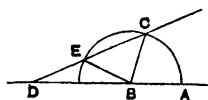


Fig. 99.

Fig. 98 shows a simple form of trisector; the instrument is opened until the angle between the rods corresponding to BA and BC can be made to coincide with the given angle; then the angle between the long rods (corresponding to $\angle ADC$) is one-third of the given angle. To prove this, take $DE =$ the radius of the circle (in fig. 99), and show that $\angle ADC = \frac{1}{3}\angle ABC$.

With a ruler, marked on its edge in two places, and a pair of compasses, it is possible to trisect an angle as follows:—

Let ABC be the angle. With B as centre and radius equal to the distance between the two marks describe a circle cutting BC at C ; place the ruler so that its edge passes through C and has one mark on AB produced, the other on the circle (this must be done by trial; a pin stuck through the paper at C will help); rule the line DEC , then $\angle D = \frac{1}{3}\angle ABC$.

Ex. 4. Call one of the short edges of your paper AB ; construct its perpendicular bisector by folding. Fold the paper again so that the new crease may pass through A , and B may fall on the old crease; mark the point C on which B falls and join CA , CB . What kind of triangle is ABC ?

Ex. 5. Is it possible to make a pavement consisting of equal equilateral triangles? Is it possible to do so with equal regular figures of (i) 4, (ii) 5, (iii) 6, (iv) 7 sides?

Ex. 6. How many diagonals can be drawn through one vertex of (i) a quadrilateral, (ii) a hexagon, (iii) a n -gon?

¶Ex. 7. If one side of a triangle is *double* another, is the angle opposite the former double the angle opposite the latter?

In order to answer this question, take the following instances:—

(i) Consider a triangle whose angles are 45° , 45° , 90° .

(ii) Draw $\triangle ABC$ in which $AB = 8.2$ cm., $BC = 4.1$ cm., $CA = 6$ cm. Measure the angles. Is C double A ?

(iii) Show that in $\triangle ABC$ whose angles are 30° , 60° , 90° , the *longest side* AB is double the *shortest* BC .

[Make $\angle CAD = 30^\circ$ and produce BC to meet AD in D .

How many degrees are there in $\angle D$?

What kind of a triangle is ABD ?]

Ex. 8. A triangle ABC has $\angle B = 30^\circ$, $BC = 8$ cm.; what is the least possible size for the side CA ? What is the greatest possible size for $\angle C$?

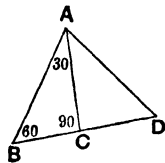


Fig. 100.

Ex. 9. The gable end of a house is in the form of a pentagon, of which the three angles at the ridge and eaves are equal to each other: show that each of these angles is equal to 120° .

Ex. 10. Through a given point P , not situated symmetrically with respect to OX , OY , draw a straight line to cut off equal parts from the arms of a given angle XOY .

Ex. 11. Draw a triangle ABC in which $\angle B$ is less than $\angle C$. Show how to find a point P in AB , such that $PB = PC$.

Ex. 12. A rough sketch of part of a range-finder is shown in fig. 101. A ray of light proceeding from a point A as shown by the arrow strikes the mirror B , is reflected to the mirror C , and thence to the eye E of an observer.

Given that the angles marked m are equal and also that the angles marked n are equal, find the relation between the angles x and y ; x being the angle between the original ray and its direction after the second reflection, and y the angle between the mirrors B and C ; first consider the special case in which $m = 55^\circ$, $n = 75^\circ$, and write the values of the various angles on a large figure (which need not be drawn to scale); then give a general proof of your statement.

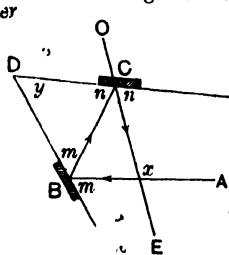


Fig. 101.

When the range-finder is in use the line of sight from E to the mirror C has to be perpendicular to AB . Show what must be the angle of inclination of the mirrors to each other in this case.

Ex. 13. In fig. 102, Q , R are the edges of two mirrors whose planes are perpendicular to the plane of the paper, a is the angle between the two mirrors. You are to suppose a ray of light from P to strike the mirror Q at an angle b , thence to be reflected to R and thence again reflected to S , its path being shown by the arrows; note that the angles marked alike are equal.

Denoting the angles between the first and last paths of the ray by x , first take the special case $a = 33^\circ$, $b = 38^\circ$; calculate the angles marked in the figure and show them in a sketch. Then, for the general case, express the various angles in terms of a and b , show them in a sketch, and find the relation between a and x . Show your procedure.

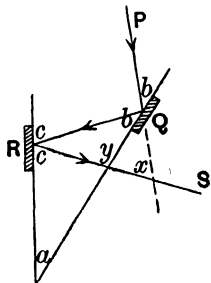


Fig. 102.

Ex. 14. The sides of a triangle are 3 in., 4 in., 5 in. respectively. Show that the angle opposite the longest side is a right angle. Two pieces of paper ABC , $A'B'C'$ are cut out in the form of this triangle, the right angles being at C , C' , and they are fitted together so that the point B' is at the point A and the side $B'C'$ lies along the side AC ; the pieces are side by side and do not overlap. Show that they make a square corner at A . Draw an accurate full-size figure.

Ex. 15. A star-shaped figure is made by drawing a regular seven-sided polygon and then describing an equilateral triangle on each side. The figure is then cut out along the outer sides of the triangles. Find the sum of the interior angles of this star.

Ex. 16. A rectangle of sides 10" and 6" is folded in such a way that one vertex is brought to the other extremity of the diagonal through that vertex. It is then flattened out. Calculate the length of the crease.

Ex. 17. A triangle ABC is folded so that the vertex C falls on B . Show in freehand figures the relative positions of the original triangle, the crease, and the folded figure for the cases (i) $AB < AC$, (ii) $AB = AC$, (iii) $AB > AC$.

Ex. 18. A triangle ABC with sides $AB = 2.5$ ", $BC = 4$ ", $CA = 3$ " is folded so that C falls on B . Find, by drawing, the length of the crease. If folded so that B falls on C , find the distance of the new position of A from its old position.

Ex. 19. $ABCD$ is a square, side 2 inches. The diagonal AC is produced to E , CE being also 2 inches. BE is joined, and EF drawn perpendicular to AB produced. Obtain by calculation the lengths AE , BE , EF (to the nearest hundredth of an inch), and the angles AEB , $A\hat{B}E$.

Verify your results by constructing the figure.

Ex. 20. What length of ribbon is needed to trim a belt 30 inches long according to the pattern shown in fig. 103? The angles are right angles and half right angles. (Don't worry about the ends.)



Fig. 103.

Ex. 21. In fig. 104 $ABCD$, $EFGH$ are two squares with their diagonals coincident; $AB = x + y$, $EF = x - y$.

(i) Find the lengths of AL , LB , BM , MC , CN , ND , DS , SA ;

(ii) Deduce from the figure the algebraical identity

$$(x + y)^2 - (x - y)^2 = 4xy.$$

Given a wire of length 12 inches, show how to bend it into a rectangle of the greatest possible area.

Ex. 22. ABC is a triangle having $AC = 7.2''$, $BC = 9.6''$. AX , BY are drawn perpendicular to BC , AC respectively. If $AX = 2.4''$, find the length of BY .

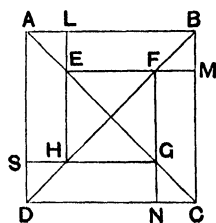


Fig. 104.

Ex. 23. Fig. 105 shows the plan of a passage in a house drawn to a scale of an inch to 3 yards. Find the area of the floor of the passage in sq. yd., taking the necessary measurements to the nearest tenth of an inch and explaining with the aid of a figure what measurements you take. Also, find the volume of wood, in cubic feet, required to pave the floor with blocks 2 in. deep, and the cost of paving at 8s. 9d. per sq. yd.

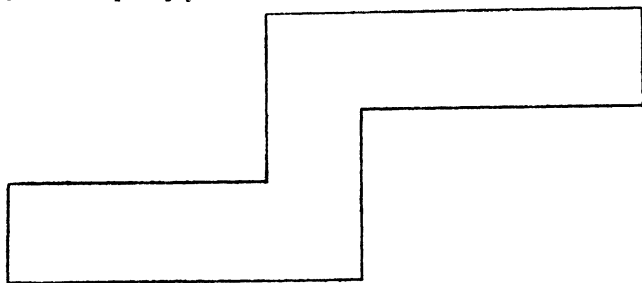


Fig. 105.

Ex. 24. ABC is a triangle, D is a point in AB and E a point in AC such that $AD = \frac{1}{3}AB$ and $AE = \frac{1}{3}AC$. Join DE and BE . What fraction is $\triangle ADE$ of $\triangle ABE$? What fraction is $\triangle ADE$ of $\triangle ABC$? Give reasons.

Ex. 25. The side of a square $ABCD$ is one yd. AB is produced to E making BE one foot. The lines EC , ED are drawn. By calculation or drawing find, in square inches, the areas of the four pieces that compose the figure.

Ex. 26. Find the area of the quadrilateral $ABCD$ in fig. 106.

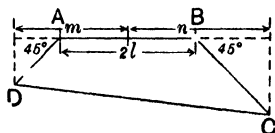


Fig. 106.

Ex. 27. Half the area of a square is cut away in the form of a square about a diagonal, one corner being common to the two squares. Draw an accurate figure and show that the remaining figure is divided by the diagonals of the original square into four parts which may be re-united into a square.

Ex. 28. The end view of a table looks like a letter X with its top corners joined. The cross-pieces are each 34 in. long and are hinged at their mid-points; the top is 23 in. long. Find the height of the table above the floor.

If the crossing-point of the legs of the table were altered so that the upper parts were 15 inches each and the lower parts 19 inches, how much would the top of the table be lowered?

Ex. 29. A square made of jointed rods, each 4 inches long is deformed into a rhombus having half the area of the square. Calculate the lengths of the diagonals of the resulting figure and check by drawing.

If the square is part of a lattice-work the original height of which is 6 times the diagonal of one of these squares (see rough sketch in fig. 107), find by calculation how much the height of the lattice-work could be increased if each square were reduced as above to half its area.



Fig. 107.

Ex. 30. Fig. 108 shows an arrangement for raising or lowering a pole AB while keeping it in a horizontal position.

Cords ACDE, BDE, attached to the ends A, B of the pole, pass over pulleys at C and D and are finally knotted together. The knot can be slipped over any one of a number of hooks (of which E, F are two) in a vertical wall KH. Other indications are marked on the figure and you are given further that, when the knot is slipped over the hook F, the value of x is 10.

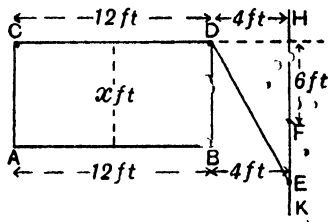


Fig. 108.

Find, by calculation or by drawing, the length of the shorter of the two cords.

Find also the depth below F of a hook E such that when the knot is slipped over it, the value of x is $\frac{1}{2}$.

Ex. 31. A rhombus is made of four rods each of length a cm., jointed together at their extremities. If the length of one diagonal be x cm., show that the length of the other is $\sqrt{4a^2 - x^2}$ cm., and that the area of the rhombus is $\frac{1}{2}x\sqrt{4a^2 - x^2}$ sq. cm.

Ex. 32. A small drawbridge AB over a stream turns on a hinge at A, and is raised by a rope RPB fastened to the end B and passing over a pulley at P, which is attached to an upright post AP. AB is 4 ft. long and AP 8 ft.

- When AB is horizontal, what is the length of the portion BP of the rope?
- How much must the end R of the rope be lowered in order to bring AB into a vertical position?
- How much would R have been lowered when AB had been raised to make the angle ABP a right angle?

Ex. 33. ABC (fig. 109) represents a flower-bed in the form of an isosceles right-angled triangle whose sides AB, AC are each 12 feet long. Parallels DE, FG are drawn at equal distances x feet from AC, AB respectively, to divide ABC into four areas, those shaded being each y square feet.

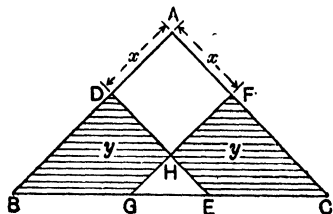


Fig. 109.

Express the lengths BD, GF, GH in term. of x , and then express the four separate areas in terms of x alone. Draw a graph showing y as a function of x , and carefully mark the scales. Hence or otherwise find how x must be chosen to make y as large as possible.

Ex. 34. CDEF is a strip of paper 5 cm. wide (fig. 110). AB is a crease parallel to CF. A point P is taken in CA, x cm. from A, and the paper is folded again as in fig. 110, so that a crease is formed along PB. In this position PD

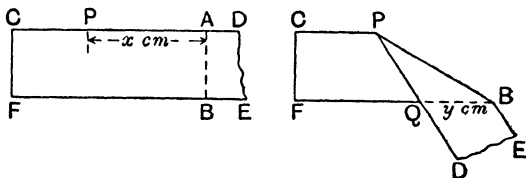


Fig. 110.

crosses FB at Q, y cm. from B. Prove that $PQ = QB$, and show that y and x are connected by the formula $y = \frac{x^2 + 25}{2x}$.

Calculate the value of y when x is 15, and the value of x when y is 15. Draw a graph showing how y changes when x ranges between the values 1 and 15. Read off those values of x which make $y = 7$. Then obtain those values more accurately by calculation.

CONSTRUCTIONS.

Ex. 35. Draw two triangles with two sides 1.2" and 2.6", having in one case 90° as the contained angle, and in the other 150° . Making any measurements you like, find the area of each triangle. [You may use the same 2.6" line for both triangles.]

Ex. 36. Construct a triangle ABC having AB = 3 in., BC = 5 in., and the median to BC = 2.5 in. Measure AC and check by calculation.

Ex. 37. Construct a triangle ABC having given AB = 10 cm., AC = 8 cm., and the perpendicular from A to BC = 7.5 cm. Is there any ambiguity? [First draw the line of the base, and the perpendicular.]

Ex. 38. Construct a triangle ABC having given AB = 11.5 cm., BC = 4.5 cm., and the perpendicular from A to BC = 8.7 cm. Is there any ambiguity?

Ex. 39. Show that, if two pairs of tram-lines of the same gauge cross, the figure formed by their intersection is a rhombus.

If the angle of intersection is 30° and the gauge of the tram-lines is 4 feet, find the side of the rhombus.

Ex. 40. In the quadrilateral ABCD, $AB=8$ cm., $AD=5$ cm., $CD=6$ cm., and the angle B is 80° . Draw the quadrilateral when the diagonal AC has the lengths 9 cm., 10 cm., 10.5 cm. (all the cases may be drawn in one figure), and find the greatest length of this diagonal.

Ex. 41. If the sides of a convex quadrilateral are $AB=2.2$ in., $BC=1.6$ in., $CD=1.9$ in., $DA=2.7$ in., find, by accurate construction, the least possible lengths of the diagonals AC, BD. What are their greatest possible lengths?

Ex. 42. ABCD is a plane quadrilateral; $AB=5$ cm., $BC=7$ cm., $CD=8$ cm., $DA=9$ cm., $AC=10$ cm. Replace the quadrilateral by a rectangle of equal area with a side of 10 cm., and deduce the area of the quadrilateral.

Ex. 43. Construct a square containing an area of 2 square inches.

Ex. 44. Draw two parallel straight lines AB, CD 9 cm. apart (the line CD should be near the right-hand edge of your paper and roughly parallel to it). To the left of AB draw a straight line PQR parallel to AB with P at the top of the line and $PQ=4$ in., $QR=2$ in. Take a point O to the left of PQR and join OP, OQ, OR. Through any point X in AB draw XY parallel to OQ to cut CD at Y. Through X draw XZ parallel to OR and through Y draw YZ parallel to OP; let these lines cut at Z. Measure the distances of Z from AB and CD.

Ex. 45. A rectangle ABCD, $AB=4$ in., $BC=2.5$ in., moves with its vertices A, B on two lines OX, OY at right angles. M is the mid-point of AB. Show that OD is greatest when the points O, M, D are in the same straight line. Construct the rectangle in this position.

Ex. 46. The following is an approximate construction for drawing a regular polygon of n sides with its vertices on the circumference of a given circle. Draw a circle of radius 5 cm. Let AB be a diameter. On AB construct an equilateral triangle ABC. Divide AB into n equal parts. Let D be the second division mark from A. Join CD and produce to meet the circle in E. Then AE is the side of the required regular polygon. By stepping off lengths AE round the circle, complete the polygon. The construction is exact for $n=3, 4, 6$, and approximate for other values of n .

SCALE DRAWING. (HEIGHTS AND DISTANCES.)

Ex. 47. A lies 7 miles N. 32° W. of B; C is 5 miles S. 67° E. of A. Find the distance and bearing of C from B.

Ex. 48. A landmark bears N. 34° W. from a ship. After the ship has sailed 7.2 miles N. 22° E. the landmark is observed to bear N. 71° W. How far is it then from the ship?

Ex. 49. In a boat 200 yd. from the foot of a vertical cliff, on the top of which stands a tower, the elevations of the base and top of the tower are 20° and 30° . Find the height of the tower.

Ex. 50. A house 50 ft. high stands at the foot of a hill of uniform slope of 20° . Two objects are noted in a straight line with the house along a line of greatest slope of the hill. From the top of the house the one is seen horizontally and the other at an angle of depression of 40° . Find the distance between the objects.

Ex. 51. Two aeroplanes P, Q are simultaneously observed from two points A, B, 1 mile apart on a horizontal plane. The points on the ground vertically below P and Q are in AB produced; the angles of elevation of P from A and B being 21° , 36° and of Q, 46° , 70° respectively, determine the difference in the vertical heights of the aeroplanes.

Ex. 52. An observer finds that the line joining two forts A and B subtends a right angle at a point C; from C he walks 100 yards towards B and finds that AB now subtends an angle of 107° ; find the distance of A from the two points of observation.

Ex. 53. You are required to find the distance between two inaccessible objects C and D. A base-line AB is chosen, and the following readings taken:—

Length of base-line AB = 740 yards. Bearing of B from A = due E.

Bearing of C from A = N. 36° E. „ „ C „ B = N. 11° W.

„ „ D „ A = N. 63° E. „ „ D „ B = N. 8° E.

Ex. 54. A is 1 mile due W. of B. From A, C bears N. 28° W. and D bears N. 34° E. From B, C bears N. 44° W. and D bears N. 9° W., Find the distance of D from C.

Ex. 55. Some positions were to be staked out on a common from the dimensions on a rough sketch which was made before any measurements were taken. The sketch shows a quadrilateral ABCD; $\angle ABC$ is 80° , $\angle BCD$ 100° , and B is due north of A. The lengths in yards of AB, BC and CD are 360, 275, 240 respectively.

- (i) Give a correct map of the positions on a scale of 1 inch to 80 yards.
- (ii) Express the scale in the form 1 : x.
- (iii) Give the distance on your map and on the common between A and D.
- (iv) Give the bearing of C and of D from A, B.
- (v) Mention any obvious rough checks on the accuracy of your drawing.
- (vi) Show that if the length of AD had been given instead of the length DC there might be uncertainty as to what was intended.

Ex. 56. A ship A leaves port at 12 noon and steams at 15 knots on the course N. 35° W. until 1.50 p.m.; she then alters her course to N. 50° E. and proceeds at the same speed. A second ship B leaves the same port at 12.45 p.m. and steams at 20 knots N. 78° E. for 25 sea-miles: she then alters course to N. 15° W. and proceeds at the same speed until she reaches the second course of A. The ship that is the first to reach the point where the courses cross is to wait there for the other. Find which ship has to wait for the other, and for how long it must wait. [A knot is a speed of one sea-mile per hour.]

Ex. 57. A and B are two observing stations on the east coast of England, A being 50 miles due north of B. A ship proceeds to a station off the Belgian

coast on her way to Zeebrugge, which is marked on the ship's chart, and may be marked on your paper by laying off bearings of $S. 55^{\circ} E.$ and $S. 80^{\circ} E.$ from A and B respectively. To fix her own position, the ship drops a depth-charge; the detonation, conveyed through the water, is heard at listening stations on the English coast, and enables A and B to wireless the ship that their distances from the ship are 98 miles and 79 miles respectively. Find the range and bearing of Zeebrugge from the ship.

Ex. 58. The entrance to a harbour is protected by a boom running $N. 57^{\circ} W.$ from A to B, with a gate in the middle. A buoy is placed bearing $S. 75^{\circ} W.$ from A and $S. 11^{\circ} E.$ from B. The navigator of a ship wishing to enter harbour rounds the buoy but is unable to see the gate. What course should he steer?

Ex. 59. Being anxious to fix his position a navigator keeps a sharp look-out for a light, and at 10.30 p.m. sights it bearing 20° on the port bow (i.e. on the left side, on a bearing 20° from that on which the ship is proceeding). The fog then comes down and when next the light is seen at 10.50 p.m. it is 60° on the bow. The ship is making 10 knots. A shoal is known to extend to a distance of $1\frac{1}{2}$ miles from the light. Will the ship clear it if he holds on his course?

Ex. 60. AB and CD (see the rough sketch in fig. 111) are two roads running due east and west, and separated by a wood which lies between B and C. A house P lies on the north side of both roads. PA, PB, PC, PD are 850 yd., 500 yd., 580 yd., 1500 yd. respectively. AB is 450 yd., CD is 1040 yd. From these data draw the two triangles with a common vertex P, but without trying to give AB and CD the same direction.

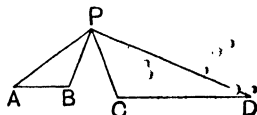


Fig. 111.

Then determine, in any way you please, whether the road AB, if continued in its own direction through the wood, will coincide with CD.

SCALE DRAWING. (MISCELLANEOUS.)

Ex. 61. ABCD is a rectangular field. $AB = CD = 120$ yd., $BC = DA = 80$ yd. A straight path PQ leads across the field from a point P in AB to a point Q in CD. $AP = 35$ yd., $QC = 50$ yd. Find the length of the path to the nearest yard and the areas in square yards of the two parts into which the path divides the field.

Ex. 62. The seat of a chair is a square of side 16 inches and it is supported by four legs, at the corners of the seat at right angles to it, each 18 inches long. If three inches are cut off each of the two back legs what angle would the seat make with the horizontal?

Ex. 63. In fig. 112, which is not to scale, FEDGMN represents a strip of felt 10 in. wide, laid in a passage, one wall of which is represented by ABC; the felt is 5 in. away from AB and 2 in. away from BC; the angles at B, M, E are 135° and AB is 36 in. AGD is perpendicular to AB, GM and DE. From a drawing made to scale find the lengths of GM and DE. Verify your results by calculation.

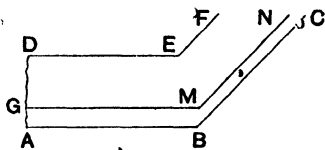


Fig. 112.

Ex. 64. A cubical tank of edge 3 ft. rests with its base inclined at 20° to the horizontal, and water is poured in till it just overflows. The tank is then placed with its base horizontal; find the depth of the water.

Ex. 65. ABCD (fig. 113, which is not drawn to scale) is the end view of a desk. The lid DC is hinged at C, and can be kept open by a strip of wood SR which is pivoted at S, S being 1 inch below D and 1 inch to the right from DA. $AD=6$ in., $AB=17$ in., $BC=10$ in., $SR=16$ in. Make an accurate scale drawing of ABCD.

Measure the angle BCD, and deduce the angle of slope of DC to the horizontal.

Imagine the desk opened and the lid propped as wide open as possible by SR. Draw the figure to scale, showing this position, and state your construction. What is the length of CR in this position?

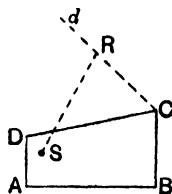


Fig. 113.

Ex. 66. Fig. 114 shows the plan of an open window hinged at O; when the window is shut OA coincides with OB. PQ is a bar hinged to the window at P; there are holes in PQ which fit over a peg at T to fix the window. $OP=8$ in., $OT=10$ in. The centres of the holes in PQ are 2, 4, 6, 8, 10, 12 inches from P.

On a single diagram show OB and the positions of OA when the 1st, 2nd, 3rd, 4th, 5th, 6th holes are over the peg. Measure and tabulate the $\angle BOA$ in each case. Draw a graph showing the relation between $\angle BOA$ and PT.

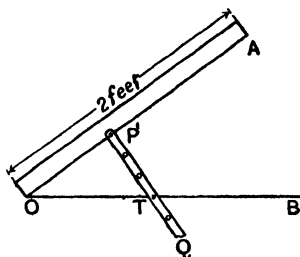


Fig. 114.

Ex. 67. Refer to fig. 230, representing the working parts of a locomotive. Find the distance BC when $\angle BDC$ has the following values, 30° , 90° , 150° .

Ex. 68. Fig. 115, which is not drawn to scale, shows a fly-wheel, centre O, which is driven by the rod AH, the block A being moved to and fro along the groove PQ by the engine. The rod KB moves as the wheel turns, and draws the block B to and fro along the line XY. It is given that $AH=BK=5$ ft., $\angle HQK=90^\circ$, and the radius of the wheel is 2 ft.

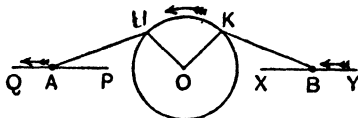


Fig. 115.

State the lengths of PQ and XY, and the distances of P, Q, X, Y from O.

Imagine A to move from P to Q. Draw to scale in one figure, and indicate clearly by letters or numbers, the positions of B when A is at P, and 1 ft., 2 ft., ... from P. Copy and complete the following table :—

PA, in feet	0	1	2		
Angle turned by wheel, in degrees					
XB, in feet... ..					

REVISION PAPERS A

Paper 1.

1. A piece of wood with parallel edges is sawn through as shown in the first figure; the cut makes an angle of 50° with the long edge. One piece is then turned over and the two sawn edges are fitted together as in the second figure. Find the angle between the two parts in the new position.

What would this angle be if the cut made an angle of x° with the parallel edges?

At what angle would the cut have to be made so that in the new position the two parts should be (i) at right angles, (ii) at an angle of 120° ?



2. Show that, if one side of a triangle is produced, the exterior angle so formed is greater than either of the interior opposite angles. In what case is the exterior angle double each of the interior opposite angles?

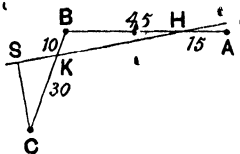
3. In a triangle $AB = 12$ cm., $BC = 9$ cm. and the perpendicular from B to AC is 5.7 cm. Show that there are two non-congruent triangles that fulfil these conditions and draw them both. State how the two triangles are related.

4. Draw a parallelogram $ABCD$ having $AB = 9.4$ cm., $AD = 7.2$ cm., and $\angle A = 110^\circ$. Determine its area.

5. A wire stay is attached to a telegraph post at a point 16 feet above the ground, and made fast to a peg on the ground at a distance of 10 feet from the foot of the post. Calculate the length of the stay.

6. A quadrilateral $ABCD$ can be folded so that A coincides with D and B with C . What does this tell you about the shape of the quadrilateral?

7. A, B, C are three trees: HK a straight path. The distances in yards are shown in the figure and $\angle ABQ$ is 107° . Draw the plan to scale and find by measurement the shortest distance CS from C to the path. Find the area of the triangle HCK .



8. Explain what locus is represented by (i) a line parallel to a given line, (ii) the surface of a sphere, (iii) the middle page of a book, (iv) the line bisecting an angle.

Paper 2.

1. Draw a parallelogram ABCD; from AB, AD cut off equal lengths AX, AY; through X, Y draw parallels to the sides to meet the opposite sides. Indicate what lines and angles are equal.

2. In a triangle whose angles are 90° , 60° , 30° the longest side is double the shortest.

[Complete an equilateral triangle.]

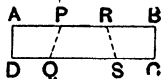
3. A and B are two points on a straight road 400 yards apart; C is a cottage which is 350 yards from A and 200 yards from B. Find how far the cottage is from the nearest point of the road.

4. Find the size, in degrees, of each angle of a regular octagon (8-sided figure), and state a general method of finding the angles of any regular polygon.

Explain, or show by figures, why a level floor can be completely paved with tiles all of the same form, that form being either equilateral triangle, square, or regular hexagon.

If you were given regular octagonal tiles, what other shape of tile would you require to use with them in order to cover a floor completely? Give reasons for your answer.

5. The figure shows a straight rectangular rod, ABCD, 3 cm. wide and 20 cm. long which is to be cut into 3 pieces by two straight cuts PQ and RS equally inclined to the sides of the rod but in opposite directions, so that, when the middle piece is turned over and the other pieces fitted to it, the two outside pieces shall be at right angles to each other. Find the angle which the cuts must make with the sides.



Draw a full-sized diagram showing the pieces so fitted, and making the 3 pieces all of the same area. Explain your method.

6. Find, in centimetres, the base-radius of a cone of slant side 3 decimetres and height 12 cm.

7. A man walking along a straight road PQX observes two hills A, B on the left of the road. When he is at P he sees the hills at angles $\angle XPA = 25^\circ$, $\angle XPB = 40^\circ$. When he reaches Q, a mile along the road from P, he sees the hills in line at an angle $\angle XQA = 63^\circ$. Find the distance AB between the hills.

8. Construct a triangle whose base is 7.3 cm. long, with vertex distant 3 cm. from the base-line and 4 cm. from the middle point of the base; measure the sides of the triangle.

Paper 3.

1. In marking out a football ground, after having made the opposite sides equal, I was uncertain if it was rectangular. What test should I apply to find out if it is so?

2. Draw an isosceles triangle ABC with $AB = AC$ and $BC < AB$. In AB take a point D such that $AD = DC$; join CD. Denote $\angle A$ by a° ; find all the angles in the figure in terms of a .

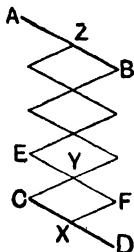
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3. The area of a triangle is 24 square inches, the altitude is 8"; find the length of the base, and on it describe a parallelogram, the area of which shall be 48 square inches.

4. A volcanic mountain is in the shape of a cone 4000 ft. high: the base is a circle of 8000 ft. radius. Calculate (to the nearest tenth of a mile) the length of a rack-and-pinion railway which takes the shortest way to the top.

5. Draw a quadrilateral ABCD having $AB = 2.8$ in., $\angle ABC = 100^\circ$, $BC = 1.4$ in., $\angle BAD = 50^\circ$, $AD = 2.1$ in. Measure CD. Find the area of ABCD by two methods.

6. The figure gives a side view of a collapsible periscope. It is jointed where the bars meet or cross, and all the parts such as XC, XF, CY, FY, YE, BZ are equal in length. AB and CD are the mirrors. Prove that any quadrilateral with its opposite sides equal must also have its opposite sides parallel, and deduce that for all positions of the framework of the periscope the mirrors AB, CD are parallel.



Each of the lengths XC, XF, etc. is 3 inches. Calculate the distance of Z from X for the position in which the angle X is 90° . If a similar periscope of larger size were required, what increase in the total distance XZ would there be (in the position in which the angle X is 90°) for every inch added to the lengths XC, XF, etc.?

7. A regular pentagon, side 10 cm., rests with its base along a line. The pentagon is now rotated in its own plane about one end of the base through an angle of 20° . What is now the distance from the line of the vertex opposite to the base?

8. A vessel steaming at uniform speed finds that the bearings of a lighthouse at 3, 4 and 5 p.m. are N. 20° E., N. 25° W., and N. 70° W. respectively. Its distance from the lighthouse at 4 p.m. is 10 miles. Find, by drawing, the ship's course.

Paper 4.

1. A traveller observes two distant landmarks both of which are marked on his map. Explain what method he should adopt to fix his position on his map, assuming that he has a prismatic compass.

2. ABCD is a quadrilateral. If angle $A = x^\circ$, angle $B = (x + 10)^\circ$, and angle $C = 3x^\circ$, express in terms of x the number of degrees in the angle D.

If the angle $D = 50^\circ$, find the angles A, B, C.

3. Construct a triangle, given $BC = 9.2$ cm., $CA = 8.2$ cm., $AB = 10$ cm.

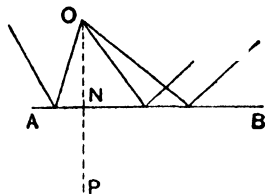
On AB construct an equivalent isosceles triangle. Measure the equal sides and find the area.

4. ABC, DBC are two congruent triangles on opposite sides of the same base BC; draw figures to show that ABDC is either a parallelogram or a kite.

5. With ruler and compass only, construct a rhombus with each side 3" and an angle of 75° . Find its altitude.

6. Two guns A and B fire at the same target, the shell from A carrying 4100 yards and that from B 4200. While the direction of each shell is accurate, A's shell falls 100 yards, and B's 200 yards, beyond the target, and the distance between the points where the shells strike the ground is found to be 150 yards. By drawing, find the angle between the lines of fire and the distance between the guns.

7. Rays of light proceeding from a point, O fall on a mirror AB and are reflected, making an equal angle with the mirror. Show that the reflected rays, if produced backwards, would all pass through a point P, the image of O in AB.



8. The two ends of a piece of string 17 in. long are attached to two points A, B on a board that are 13 in. apart, so that the string forms a loop. The point of a pencil is placed inside the loop and the string tightened so as to form two sides AP, PB of a triangle APB, the pencil point being at P. The pencil point can now be moved so as to draw an ellipse on the board, the string being kept taut. At a certain instant the triangle will be found to have a right angle at P. What are the lengths of AP, PB at this instant?

Paper 5.

1. In a quadrilateral ABCD, $\angle A = \angle B$ and $\angle C = \angle D$; what is the quadrilateral? Justify your conclusion.

2. You are asked to make a triangle, having two sides 8 cm. and 12 cm., long; one angle, which may be opposite to either of the given sides, or included by them, to be 30° . Show that four different forms are possible for the triangle, and construct them.

With the sides the same length as before, ascertain how many triangles can be constructed if the given angle instead of being 30° is 60° , also if it is 150° . Show how you arrive at your conclusions.

3. Given a point and a straight line, show how to find the image of the point in the line, using compasses only. (No proof required.)

4. The section of a trough is a trapezium, 6" wide at the bottom, 15" at the top, and sloping sides 9" long. Find the sectional area by drawing and measurement (or by calculation).

5. In a triangle ABC, AB measures 15 cm., AC 13 cm., and BC 14 cm. From A a perpendicular is drawn to BC meeting it in P. Let $BP = x$ cm., and $CP = y$ cm.

(i) Prove that $x^2 - y^2 = 56$.

(ii) Solve the simultaneous equations $x^2 - y^2 = 56$, $x + y = 14$.

(iii) Calculate the length of AP.

[See next page

6. A man on a wreck observes the bearing* of two places A and B on a straight shore to be 40° and 55° . He knows the places to be 3 miles apart and that the line AB is due east. Find his distance from the shore.

7. Draw any triangle ABC and take a point O outside it. Construct another triangle A'B'C' which has central symmetry with ABC about the point O.

8. Two houses are 50 yd. and 85 yd. from a straight road and their carriage drives, which are the shortest possible, meet the road at points 100 yd. apart. A lamp is to be placed on the road equidistant from the two houses. How far will it be from the entrance to each drive?

Where must the lamp be placed on the road so that the two drives subtend equal angles at the lamp?

Paper 6.

1. How would you instruct a person to place a chessman on a particular square of a chessboard? How would you fix the position of a point on a triangular metal plate?

2. Define a "trapezium." How is its area found?

Find the area of the cross-section of an embankment 10 ft. high, which is 20 ft. wide at the top and 30 ft. at the bottom.

3. In a four-sided field ABCD, $AB=470$ yd., $BC=238$ yd., $CD=327$ yd., $AC=525$ yd., and $DA=356$ yd. Draw a plan of the field to the scale of 100 yd. to an inch and construct a triangle of equal area, giving a few words to explain our construction.

Find the area of the field by means of the triangle.

4. A ship is situated 4.5 miles from a straight shore. Two beacons on the shore are respectively 6 miles and 8.9 miles from the ship. Calculate the distance between the beacons.

5. Construct (without any calculation) a square whose area shall be equal to the difference between the areas of two squares of sides 7 and 4 cm.

6. A right cylinder of diameter 6.8 cm. and height 7.6 cm. is divided into two parts by a plane through its centre inclined at 28° to its base; measure the length of the section.

7. The angle of elevation of a peak is 48° , and after walking 400 yd. up a slope inclined at 15° to the horizontal, the angle of elevation is 62° . Find the vertical height of the peak above the first point of observation.

8. Find a point P in a straight line XY such that the lines joining P to two given points A, B are equally inclined to XY. Consider the two cases (i) A, B on the same side, (ii) A, B on opposite sides of XY.

Paper 7.

1. Draw a small acute-angled triangle ABC . Through each vertex draw parallels to the opposite sides forming a triangle $A_1B_1C_1$. Repeat the process for the triangle $A_1B_1C_1$ forming the triangle $A_2B_2C_2$. Compare the sides and angles of $A_2B_2C_2$ with those of ABC .

2. Verify by drawing and measurement that the bisector of the vertical angle of a triangle and the perpendicular bisector of the base make with each other an angle which is half the difference of the base angles of the triangle.

3. Explain why, if we are given two sides of a triangle and the angle opposite to one of them, there may be two distinct triangles satisfying these conditions or possibly no such triangle.

If $a=37$, $b=52$, $A=35^\circ$, determine how many, if any, triangles satisfy these conditions and complete their solution.

4. Place a rectangular sheet of paper (about 25 cm. by 20 cm.) on your desk with a short side nearest to you. Fold the paper so that one of the bottom corners just reaches the opposite long edge of the paper at a point P 10 centimetres from the top of the paper; measure the length of the crease.

Take P at different distances 11 cm., 12 cm., etc., from the top of the paper, measure the length of the crease (y) in each case, and show in a graph this length as a function of the distance of P (x) from the top of the paper.

By means of the graph find when the length of the crease is least

Measure the width of your paper and *calculate* the length of the crease when $x=10$ cm.; compare the result with what you obtained by measurement.

5. A destroyer, steaming N. 10° E. at 30 knots, sights a cruiser, 11 miles on, bearing N. 62° E. Half an hour later the cruiser is 5 miles away, bearing N. 70° E. Find the course and speed of the cruiser. If the destroyer then alter course to N. 70° E., how far astern of the cruiser will she pass?

6. A rhombus is formed by jointing together four rods each 1 foot long. Show that the area of the rhombus may be anything up to 1 square foot.

7. State and criticise the converses of the following facts:—

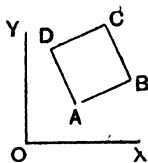
(i) Congruent triangles have the same area.

(ii) The perpendiculars from the vertices of a triangle to the opposite sides are three concurrent lines.

8. In the given figure OX , OY are two roads at right angles and $ABCD$ is a square field. The following table gives the distances of A and B from each road:—

	A	B
Distance from OX	260 metres	490 metres
“ “ OY	620 “	1180 “

By calculation find the distances of C and D from the two roads, and the area of the field.



Paper 8.

1. Are the following statements true? Give the missing number in the last two lines.

If a figure has 3 sides the sum of the angles is 2 rt. $\angle$ s.

"	"	4	"	"	"	"	"	4	"
"	"	5	"	"	"	"	"	6	"
"	"	6	"	"	"	"	"	?	"
"	"	n	"	"	"	"	"	?	"

2. Calculate the area of a right-angled triangle which has a hypotenuse of 7 inches and a side of 5 inches.

3. ABCD is a square; the bisector of the angle ACB cuts AB at P; PQ is drawn perpendicular to AC and meets AC at Q.

- What pairs of congruent triangles are there in the figure?
- Which triangles in the figure are isosceles?
- State in degrees the size of the angle PCD.
- Which angles in the figure are equal to PCD?
- Prove $AQ = PB$.

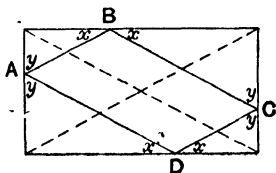
4. "Draw a freehand plan and find the area of the field in the following survey. Give your answer in square links and also in acres.

Links

	To B	
	800	
100	700	300
	400	200
	300	300
100	250	
From	A	go North

5. In the figure ABCD is a parallelogram whose sides are parallel to the diagonals of the rectangle in which it is inscribed. Prove that the angles marked x are all equal to one another, as are also the angles marked y . What is the relation between x and y ?

Show also that the perimeter of the parallelogram is equal to the sum of the diagonals of the rectangle.



Hence give a solution of the following problem taken from an examination paper set 60 years ago:—"Trace the course of a billiard ball which after striking the four sides of the table returns to its first position." [The ball is supposed to rebound from each side of the table at the same angle as that at which it strikes it.]

6. An estate is in the form of a pentagon $ABCDE$. D lies 2 miles 3 furlongs N. of A , and B lies 1 mile 2 furlongs 5 chains E. of A ; C is 1 mile 2 furlongs 5 chains N. of the line AB and 2 miles 3 chains E. of the line AD , while E is 7 furlongs 3 chains N. of AB and 1 mile 7 chains W. of AD . Draw the plan of the estate to the scale of 1 in. to 5 furlongs, and find the area of the estate in acres correct to the nearest acre.

7. A square, side 10 cm., is folded flat so that one vertex falls on the middle point of either of the opposite sides. Find the length of the crease from an accurate drawing.

8. Draw two lines AB, BC ; mark D, F their mid-points; complete the parallelogram $BDEF$. The figure represents a pantograph formed of four rods AB, BC, DE, EF , jointed at B, D, E, F . Suppose it to be pivoted down at A over a sheet of paper but otherwise quite free to move in the planes of the paper.

Assuming that $AD = DB = EF$ and $DE = BF = FC$, prove

- (i) that in every position the opposite sides of $BDEF$ are parallel,
- (ii) that A, E, C are always in line with one another,
- (iii) that AC is always equal to twice AE .

CHAPTER 10

THE CIRCLE. CHORDS

§ 1. A **circle** is a line, lying in a plane, such that all points in the line are equidistant from a certain fixed point, called the **centre** of the circle.

In view of what has been said already about loci we may give the following alternative definition of a circle:—

A **circle** is the locus of points in a plane that lie at a fixed distance from a fixed point (the centre). The fixed distance is called the **radius** of the circle.

The word “circle” has been defined above to mean a certain kind of curved line. The term is, however, often used to indicate the part of the plane inside this line. If any doubt exists as to the meaning, the line is called the **circumference** of the circle.

To cut a circular hoop into two pieces needs two cuts. Similarly, if two points are taken on the circumference of a circle, the circumference is divided into two **arcs** (Latin *arcus* = bow). Thus, in fig. 116, C and D divide the circumference into two arcs; CABD is called a **major arc** and CTD is called a **minor arc**.

A and B divide the circumference into two equal arcs. Each arc is then a **semicircle**. The line joining the two extremities of a semicircle clearly passes through the centre: it is called a **diameter**.

The term “semicircle” like the term “circle” is used in two different senses; sometimes in the sense of an arc (as in the definition); sometimes as the part of the plane bounded by a semi-circumference and the corresponding diameter.

§ 2. A straight line cannot cut a circle in more than two points.

(i) An unlimited straight line may cut a circle in two points, e.g. AB or CD in fig. 116. In this case the part of the line which lies inside the circle is called a **chord** of the circle.

(ii) The line may meet the circle in one point only; thus EF meets the circle

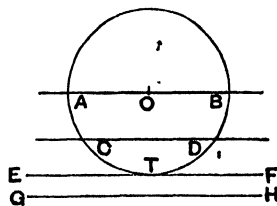


Fig. 116.

in T . In this case the line is said to **touch** the circle; it is called a **tangent**. T is called the **point of contact**.

(iii) The line may lie outside the circle, and have no common with the circle, i.e.

A **segment** of a circle is the part of the circle and its chord (fig. 1) bounded by two radii.

SYMMETRY

§3. It is

Ex. 1. Draw a circle and a diameter.

Ex. 2. Draw a circle and a chord. What is the locus of the mid-point of the chord?

Ex. 3. Draw a circle and a diameter. Draw freehand four pairs of corresponding points on the circle. Which pair of corresponding points coincide?

Ex. 4. AB is a chord of a circle. What is the locus of points equidistant from A and B ? What conclusion does this lead to about the position of the centre?

From considerations of symmetry we arrive at the following theorem, which may be stated in three forms.

THEOREM 36. (i) **A straight line, drawn from the centre of a circle to bisect a chord which is not a diameter, is at right angles to the chord.**

(ii) **The perpendicular to a chord from the centre bisects the chord.**

(iii) **The perpendicular bisector of a chord of a circle will, if produced, pass through the centre of the circle.**

For proof of this Theorem see *Theoretical Geometry*, p. 76.

Ex. 5. The locus of the mid-points of a system of parallel chords of a circle is the diameter perpendicular to the chords. (Give reasons.)

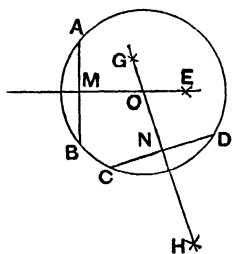
Ex. 6. Draw two concentric circles (i.e. with the same centre) and any chord cutting them in A, B and C, D . Why are the parts intercepted, AC and BD , equal to one another?

PLANE GEOMETRY

three chords each 1.2" from the centre.

Through any point on the diameter
diameter. Verify that they are equal

a circle.



9.

and GH,

d CD

ing through three given points
the straight line; in other words, To find a point
ant from three given points.

is done as in the preceding construction.

Ex. 9. How would the construction fail if the three points were in a straight line?

Ex. 10. With a fine-pointed pencil trace round part of the edge of a penny, or better still the curved edge of your protractor, so as to obtain an arc of a circle. (Take care to keep the pencil perpendicular to the paper.) Complete the circle by finding the centre.

CIRCUMSCRIBED CIRCLE.

§5. If a circle passes through all the vertices of a polygon, the circle is said to be **circumscribed** about the polygon (Latin *circum* = around); and the polygon is said to be **inscribed** in the circle (fig. 119).

To circumscribe a circle about a given triangle.

This is the same problem as that of describing a circle to pass through three given points, namely the three vertices of the triangle.

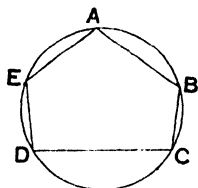


Fig. 119.

The centre of the circle circumscribed about a triangle is called the **circumcentre** of the triangle.

Notice that, though the perpendicular bisectors of all *three* sides pass through the circumcentre, it is not necessary to draw more than *two* of these bisectors in order to find the centre.

Ex. 11. Find the radius of a circle circumscribing a triangle whose sides are (i) $2\frac{1}{2}$ " , 3" , 3" , (ii) 6 cm., 8 cm., 10 cm., (iii) $2\frac{1}{2}$ " , $3\frac{1}{2}$ " , 5" .

Ex. 12. (Inch paper.) Draw a circle to pass through the points (0, 2), (3, 0), (1, 0), and measure its radius.

Ex. 13. (Inch paper.) Draw the circumcircle of the triangle whose vertices are (4, 0), (0, 2), (0, -1), and find its radius.

Ex. 14. (Inch paper.) Find the circumradius, and the coordinates of the circumcentre of the triangle whose vertices are (1, 0), (0, 3), (0, -3).

Ex. 15. A ship's position is equidistant from a lighthouse and 2 buoys. The buoys are 2.4 miles apart and each is 1.3 miles from the lighthouse. Draw an accurate figure to scale and measure the distance of the ship from the lighthouse.

Ex. 16. A Dutch barn has a roof whose section is an arc of a circle. The framework supporting the roof is triangular with two beams of 15 ft. length meeting at an angle of 130° . What is the radius of the circular arc?

Ex. 17. A circular hoop hangs over three nails in a vertical wall. The horizontal distances between the nails are each 1 foot and their heights above the ground are 4 ft., 4 ft. 5 in., 4 ft. 3 in. What is the diameter of the hoop?

Ex. 18. A window is in the shape of a rectangle surmounted by an arc of a circle. The width of the window is 6 feet. Its least height is 10 feet, and its greatest height 12 feet. Draw the outline of the upper part of the window to a scale of 1" to 2'.

Ex. 19. From Paignton Pier (P) the Green Light (T) of Torquay Harbour bears N. 40° E., 1.6 miles, and the Green Light (B) of Brixham Harbour, S. 36° E., 2.9 miles. A ship is found by observation to be at X, equidistant from P, T and B. Her required anchorage is at a point (A) which bears S. 20° E. from T and N. 70° E. from P. Find by plotting on a scale 2" to 1 mile:—(i) the position of A and X, (ii) the bearing and distance of A from X.

Ex. 20. Can a circle be circumscribed about a rectangle?

Ex. 21. Draw a parallelogram (not rectangular) and try if a circle can be circumscribed about it.

Ex. 22. The perpendicular bisectors of the three sides of a triangle meet in a point. Give reasons.

LENGTH OF CHORD.

§6. The perpendicular from the centre upon a chord bisects the chord; therefore the radius, distance of chord, and half-length of chord are connected by Pythagoras' theorem. (See fig. 120.)

Ex. 23. Calculate the distances from the centre of a circle (radius 5 cm.) of chords whose lengths are (i) 7 cm., (ii) $2a$ cm.

Ex. 24. Calculate the lengths of chords of a circle (radius 2.5 in.) whose distances from the centre are (i) 1.5 in., (ii) d in.

Ex. 25. A chord CD of a circle, whose centre is O , is bisected at N by a diameter AB . $OA = OB = 5$ cm., $ON = 4$ cm. Calculate CD , CA , CB .

Ex. 26. The lengths of two parallel chords of a circle of radius 6 cm. are 10 cm. and 6 cm. respectively. Calculate the distance between the chords. (There are two cases.)

Ex. 27. Calculate the radius of a circle, given that a chord 3 in. long is 2 in. from the centre.

Ex. 28. What is the radius of a circle when a chord of length $2l$ is at distance d from the centre?

Ex. 29. Given that a chord 12 cm. long is distant 2.5 cm. from the centre, calculate (i) the length of a chord distant 5 cm. from the centre, (ii) the distance from the centre of a chord 6 cm. long.

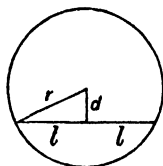


Fig. 120.

Exx. 30-34 require a figure showing a section of a sphere by a plane, see fig. 121. The section is a circle with diameter AB and centre N . If C is the centre of the sphere, CN bisects AB at right angles, for AB is a chord of the circle APB .

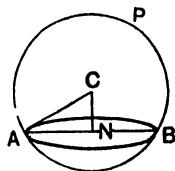


Fig. 121.

Ex. 30. A wooden ball of 4" radius is planed down till there is a flat circular face of radius 2". If the ball is now made to stand on the flat face, how high will it stand?

Ex. 31. The distance of the plane of the Arctic circle from the centre of the earth is 5700 miles (to the nearest 100 miles); the radius of the earth is 4000 miles. Find the radius of the Arctic circle.

Ex. 32. A ball of radius 4 cm. floats in water immersed to the depth of $\frac{3}{4}$ of its diameter. Calculate the radius of the water-line circle.

Ex. 33. The diameter of an orange is 4", and the thickness of the rind is $\frac{1}{4}$ ". A piece is sliced off just grazing the flesh; find the radius of the piece.

Ex. 34. A sphere of 6" diameter rests on the top of an open hollow cylinder whose inner diameter is 4". To what distance will the sphere project above the top of the cylinder? (See fig. 122.)

Ex. 35. Equal chords of a circle are equidistant from the centre. Give reasons.

Ex. 36. Chords of a circle that are equidistant from the centre are equal. Give reasons.

Ex. 37. Find the locus of the mid-points of chords 6 cm. in length in a circle of radius 5 cm. Generalise your result.

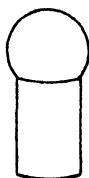


Fig. 122.

SUMMARY OF RESULTS

A straight line, drawn from the centre of a circle to bisect a chord which is not a diameter, is at right angles to the chord.

The perpendicular to a chord from the centre bisects the chord.

The perpendicular bisector of a chord of a circle will, if produced, pass through the centre of the circle.

CHAPTER 11

THE CIRCLE. TANGENTS

§ I. The meaning of the term tangent has been explained on p. 87. Note that the line in fig. 123 is *not* a tangent and is *not* said to touch the circle; if produced it would cut it in two points.



Think of a wheel with spokes. The roadway is a tangent to the circle; a spoke that happens to be vertical and downwards is the radius to the point of contact. The radius, then, is perpendicular to the tangent at its extremity.

Fig. 123.

A tangent may be regarded as the **limiting position of a produced chord** whose two points of intersection with the circle have come to coincide.

Fig. 124 suggests the way in which the chord may approach its limiting position—the tangent.

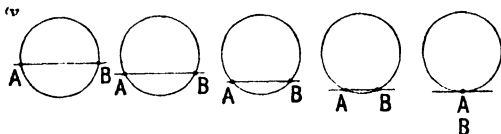


Fig. 124.

¶ Ex. 1. In fig. 124, what becomes of the theorem that “the perpendicular from the centre on a chord bisects the chord” when B comes to coincide with A?

In fig. 125, i, ii, the triangle OAB is isosceles;

$$\therefore \angle OAB = \angle OBA, \therefore \angle OAP = \angle OBQ.$$

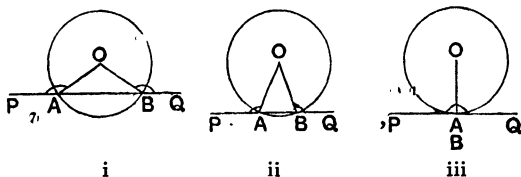


Fig. 125.

When the chord attains its limiting position, the tangent, in fig. iii, we are left with $\angle OAP = \angle OAQ$. Hence again

THEOREM 41. The tangent at any point of a circle and the radius through the point are perpendicular to one another.

A straight line drawn through the point of contact of a tangent at right angles to the tangent will, if produced, pass through the centre of the circle.

§ 2. To draw the tangent to a circle at a given point on the circle, join the point to the centre, and draw a straight line through the point perpendicular to the radius.

The tangent from an external point may be drawn with the ruler (by eye). It is not possible to distinguish the point of contact accurately without further construction; to find this point, drop a perpendicular upon the tangent from the centre; the foot of this perpendicular is the point of contact.

This method is accurate enough for many purposes; another construction is given in Ex. 13, Ch. 13 and in *Theoretical Geometry*, Ch. XI, p. 99.

By symmetry it is easy to see that

The two tangents drawn to a circle from a point A are (i) equal, (ii) equally inclined to AO.

Ex. 2. In fig. 126 show that AO bisects PQ at right angles.

Ex. 3. If each tangent AP, AQ is equal to the radius, what is the angle between the tangents and what is the angle at the centre subtended by the chord of contact (PQ)?

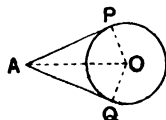


Fig. 126.

TANGENTS (CALCULATIONS).

§ 3. In fig. 126 the triangle APO is right-angled at P. Hence, when the radius and the distance AO are known, the length of the tangent AP can always be found by Pythagoras' theorem.

Ex. 4. Calculate the lengths of the tangents to a circle of radius r from a point distant d from the centre when (i) $r=6$ cm., $d=8$ cm.; (ii) $r=1$ in., $d=5$ in.

Ex. 5. Give a formula for the length of the tangent in terms of r , d .

Ex. 6. The tangents from a point P to a circle of radius r are of length t . What is the distance of P from the centre?

Ex. 7. A line 6 cm. long always touches a circle of radius 4 cm. at its middle point. What is the locus of each end of the line?

Ex. 8. Two circles, of radii 3 in. and 2 in., are concentric. Calculate the length of a chord of the outer circle which touches the inner.

Ex. 9. Show that all chords 8 cm. long of a circle of radius 5 cm. touch a certain concentric circle; find its radius.

Ex. 10. A continuous string passes round a pulley of 4 cm. radius and supports it by passing over a nail 12 cm. above the centre of the pulley. What is the length of each straight portion of the string?

Ex. 11. A straight shaft to a horse roller is hinged to the axle of the roller and is 10 ft. long; it rests on the ground 9 ft. 3 in. from the line along which the roller touches the ground. What is the diameter of the roller?

Ex. 12. A circular lake of 70 yards diameter has a round island in the centre of diameter 20 yards. What is the longest possible straight course to sail a model yacht on this lake?

Ex. 13. Two 30 ft. wide roads meet at right angles. A man cycling on the extreme outer edge of the roads takes the bend in a circle which just grazes the inner corner. Find the radius of the circle in which he turns.

[Use Pythagoras' theorem to form an equation for r .]

¶Ex. 14. If fig. 126 were spun about OA , what figure would be generated (i) by the circle, (ii) by AP , (iii) by PQ ? Hence find the locus of the points of contact of tangents from a fixed point to a fixed sphere.

Ex. 15. If P is a point 25 cm. from the centre of a sphere of radius 20 cm., what is the length of a tangent line from P to the sphere? Are all the tangent lines from P equal?

Ex. 16. What is the distance of the horizon (i.e. the length of the tangent line) from a point 5 miles above the surface of the earth, the radius of the earth being taken as 4000 miles?

TANGENTS (DRAWING).

Ex. 17. Draw two tangents from a point to a circle of radius 1.5 in., the length of each tangent being 2 in.

Measure the distance of their point of intersection from the centre.

Ex. 18. Draw tangents from a point to a circle of radius 5 cm., so that the chord of contact may be 7 cm. long. What is the angle between the tangents?

Ex. 19. Draw a circle and a line outside it. Draw tangents to the circle (i) parallel to the given line, (ii) perpendicular to the given line. Explain your construction.

Ex. 20. Draw a circle and any line. Draw a tangent to the circle, making an angle of 55° with the given line. State and justify your method.

Ex. 21. Draw two tangents to a circle of radius 5 cm., so that the angle between the tangents is 80° . State and justify your method.

Ex. 22. Draw the locus of the mid-points of chords, 8 cm. long, of a circle of radius 5 cm.

Through a point distant 4 cm. from the centre draw a chord of length 8 cm.

Ex. 23. From a point 3 in. from the centre of a circle of radius 2 in., draw a line so that the part intercepted by the circle is 2.8 in. long. Give reasons for your construction.

INSCRIBED AND EScribed CIRCLES.

§ 4. If a circle touches all the sides of a polygon, the circle is said to be **inscribed** in the polygon; and the polygon is said to be **circumscribed** about the circle (fig. 127).

¶ **Ex. 24.** What is the locus of the centres of circles touching two lines which cross at an angle? (Remember that two lines form four angles at a point.) Draw a number of such circles.

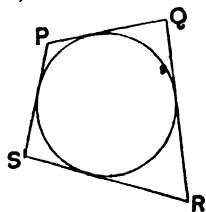


Fig. 127.

To inscribe a circle in a given triangle.

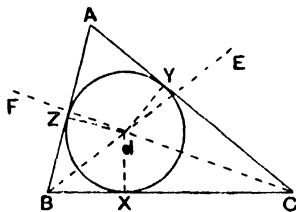


Fig. 128.

Analysis and construction. The centre is equidistant from the three sides.

Since it is equidistant from BA and BC, and is obviously inside the triangle, it lies somewhere on the internal bisector of $\angle ABC$. Draw this line, BE.

Similarly, it lies somewhere on CF the internal bisector of $\angle ACB$.

The centre therefore is the point of intersection of BE and CF, namely I.

To find the radius, draw $IX \perp$ to BC. X is the point of contact and IX is the radius. The circle can now be drawn.

This circle is the **inscribed circle** of the triangle ABC; its centre is called the **incentre** of the triangle.

Ex. 25. The point of intersection of the internal bisectors of two angles of a triangle is equidistant from the three sides. Give reasons.

Ex. 26. The bisectors of the three angles of a triangle meet in a point. Give reasons.

The escribed circles of a triangle.

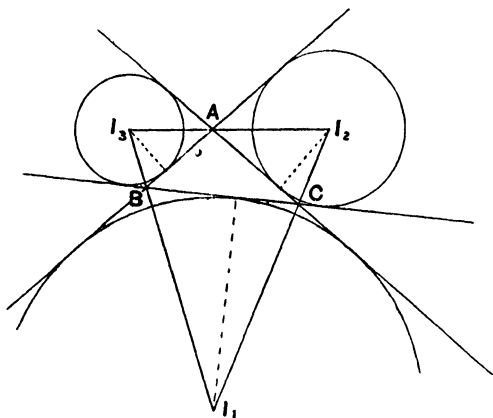


Fig. 129.

Draw BI_1 , CI_1 to bisect the angles exterior to ABC and BCA .

Then I_1 is equidistant from BC and AB , AC (produced).

Drop a perpendicular from I_1 to BC . A circle drawn with I_1 as centre and this perpendicular as radius will touch the side BC and the sides AB , AC produced. This circle is called an **escribed** circle of the triangle. There are three such circles (see fig. 129).

Ex. 27. Draw the inscribed circle of a triangle whose sides are (i) 5, 6, 7 in., (ii) 8, 6, 8 cm. Measure the radii of the circles.

Ex. 28. Draw the escribed circles of a triangle whose sides are 3, 4, 5 cm. Measure the radii.

Ex. 29. Draw an equilateral triangle and compare the radius of an escribed circle with the radius of the inscribed circle.

Ex. 30. What is the radius of the largest disc that can be stamped out of a triangular sheet of metal with sides 7.5 in., 8.6 in., 9.3 in.?

Ex. 31. A triangle has two equal sides of 8 cm, enclosing an angle of 110° . Find the distance between the centres of its inscribed and circumscribed circles.

Ex. 32. Draw a triangle ABC with $BC=6$ cm., $CA=5$ cm., $AB=7$ cm. Find I the centre of its inscribed circle and I_1 the centre of the escribed circle touching BC . Show by drawing that the circle through I_1 , BC also passes through I .

Ex. 33. AB , BC , CD are three lengths of stone forming the top of a large window $ABCD$. $AB=CD=3$ ft. 6 in., $BC=5$ ft., $\angle B = \angle C = 120^\circ$. What is the radius of the circular wooden framework used to place the slabs in position?

Ex. 34. A, B, C are three holes of a golf course, AB is 440 yd., AC 330 yd. and BC 520 yd. A shelter is to be erected equidistant from the lines joining each hole. What is its distance from A?

Ex. 35. Inscribe a circle in a sector of a circle.

[Draw a tangent to the circle where the bisector of the angle meets it.]

Ex. 36. Draw a sector of a circle of radius 2" and angle 30° . Find the radii of the two circles which each touch the curve and the radii, or radii produced.

Ex. 37. Two lines meet at an angle of 45° . Construct a circle of radius 5 cm. to touch the arms.

Inscribe a circle in the space between the lines and this circle, touching both the lines and the circle. What is its radius?

Ex. 38. Draw four equal circles, radius 1.75", each touching two long intersecting lines.

Ex. 39. When the sun's altitude at mid-day is 50° the edge of the shadow of a large cylindrical pipe lying E. and W. on flat ground is 10 ft. from the line of contact of the cylinder with the ground. What is the diameter of the pipe?

Also find the diameter of the largest pipe which can lie completely in the shadow parallel to and touching the former.

Ex. 40. A sphere rests on a triangular framework the sides of which are 8 cm., 9.5 cm. and 7.3 cm. Find the radius of the circular section of the sphere by the plane of the triangle.

Ex. 41. A wineglass is in the shape of a cone of depth 3.5" and base radius 1.75". An orange, which is a sphere of diameter 3", is placed in the wineglass. How far is the top of the orange from the bottom of the wineglass?

SUMMARY OF RESULTS

The tangent at any point of a circle and the radius through the point are perpendicular to one another.

The two tangents drawn to a circle from a point are equal and equally inclined to the line joining the point to the centre.

CHAPTER 12

REGULAR POLYGONS

§ 1. A plane figure bounded by straight lines is called a **polygon**.
(From two Greek words meaning “many” “angles or corners.”)

A polygon which has all its sides equal and all its angles equal is called a **regular polygon**.

A polygon of 5 sides is called a **pentagon** (5-gon).

„ 6 „ „ **hexagon** (6-gon).

„ 7 „ „ **heptagon** (7-gon).

„ 8 „ „ **octagon** (8-gon).

§ 2. To inscribe a regular polygon in a circle or to draw a regular polygon when the length of a side is not given.

¶ Ex. 1. Describe a circle of radius 5 cm. (see fig. 130); at its centre O draw two lines at right angles to cut the circle at A, B, C, D. Join AB, BC, CD, DA. Is the figure ABCD regular? (See definition; there are two questions to be settled.)

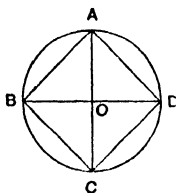


Fig. 130.

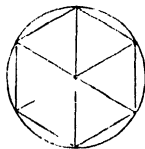


Fig. 131.

¶ Ex. 2. Describe a circle of radius 5 cm. (see fig. 131); at its centre make a set of angles each equal to 60° (i.e. $\frac{360^\circ}{6}$); join the points where the arms cut the circle. Is the resulting figure regular?

Exx. 1 and 2 suggest the following method of inscribing a regular polygon of n sides in a circle:—Make an angle at the centre of $\frac{360^\circ}{n}$; the chord intercepted by the arms of the angle is one side of the polygon.

If it is required simply to draw a regular polygon, no length of side being given, begin by drawing any circle and in this inscribe the polygon.

Ex. 3. Calculate the angle at the centre for each of the following regular polygons:—(i) 5-gon, (ii) 8-gon, (iii) 9-gon, (iv) triangle, (v) 10-gon, (vi) 16-gon, (vii) n -gon. Arrange your results in the form of a table.

Number of sides	3	4	5	
Angle at centre	120°	90°		

Ex. 4. Draw regular polygons with (i) 5, (ii) 8, (iii) 9, (iv) 10 sides.

To inscribe a regular hexagon in a circle.

Analysis and construction. Suppose the figure drawn; join two vertices A, B to the centre O. The 6 equal sides subtend 6 equal angles at the centre: the sum of these angles is 360° ; $\therefore \angle AOB = 60^\circ$.

But $OA = OB$, $\therefore \angle x = \angle y$.

And $\angle x + \angle y = 120^\circ$, $\therefore \angle x = \angle y = 60^\circ = \angle O$.

$\therefore \triangle AOB$ is equilateral and $AB = AO$.

Hence the following construction:—using compasses, place end to end in the circle six chords each equal to the radius.

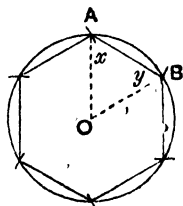


Fig. 132.

THE ANGLE OF A REGULAR POLYGON.

§3. The method in §2 of constructing a regular polygon by drawing a circle and making an angle of $\frac{360^\circ}{n}$ at the centre will not apply when a side is given. In this case it is necessary to know an angle of the polygon.

Consider a yacht sailing from A round the pentagon BCDEF back to A (see fig. 133). Sketch the figure freehand, and mark the angle the yacht turns through at B, C, D, E, F. When it gets back to A, it has headed towards every point of the compass; what then is the sum of the angles through which it has turned?

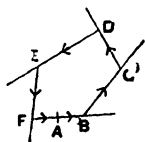


Fig. 133.

This suggests

THEOREM 9. If the sides of a convex* polygon are produced in order, the sum of the angles so formed is equal to four right angles.

Ex. 5. Show that the sum of the interior angles of a polygon is $(2n - 4)$ right angles where "n" is the number of sides.

¶**Ex. 6.** What is the size of each exterior angle of a regular octagon (8-gon)? Hence find the size of each interior angle.

Ex. 7. What are the exterior angles of regular polygons of 12, 10, 5, 3, n sides? Hence find the interior angles of these polygons.

Hence to calculate the angle of a Regular Polygon: first find the exterior angle by dividing 360° by the number of sides, then subtract the result from 180° .

The angle may also be calculated by dividing the sum of the interior angles by the number of sides. (See Ex. 5.)

¶**Ex. 8.** The exterior angle of a regular polygon is 60° ; how many sides has the polygon?

¶**Ex. 9.** How many sides have the regular polygons whose exterior angles are (i) 10° , (ii) 1° , (iii) $2\frac{1}{2}^\circ$?

¶**Ex. 10.** Is it possible to have regular polygons whose exterior angles are (i) 15° , (ii) 7° , (iii) 11° , (iv) 6° , (v) 5° , (vi) 4° ?

¶**Ex. 11.** Can there be regular polygons with exterior angles obtuse?

¶**Ex. 12.** Is it possible to have regular polygons whose interior angles are (i) 108° , (ii) 120° , (iii) 130° , (iv) 144° , (v) 60° ? (Think of the exterior angles.) In the cases which are possible, find the number of sides.

CONSTRUCTION OF REGULAR POLYGONS.

§4. **Ex. 13.** Construct a regular pentagon having each side 2 in. long.

[Calculate its angle, draw $AB = 2$ in., at B make $\angle ABC =$ the angle of the regular pentagon, cut off $BC = 2$ in., etc., etc.]

The method suggested in Ex. 13 is long and liable to be very inaccurate. The work can be shortened thus:—

Calculate the angle of the polygon and draw two sides at the required angle. Construct a circle to pass through the three vertices already found. The length of the side can now be marked off, with compasses, the required number of times round the circumference of the circle.

The polygon in fig. 133 is **convex**; whereas such a polygon as that of fig. 134 is called **re-entrant**.



Fig. 134.

Ex. 14. Construct a regular octagon having each side 2 in. long.

Ex. 15. Inscribe in a circle an equilateral triangle of side 6 cm. Give the radius.

Ex. 16. A wallpaper design is made up of regular hexagons joined side to side. One side of a hexagon is 3.5 in. What is the width of a room which has 40 complete hexagons in a horizontal line on a short wall? (Each hexagon has two sides vertical.)

Ex. 17. The base of a monument is a flat block in the shape of a regular octagon of side 5 ft. On this is placed another octagonal block so as to form a step 1 ft. wide all round. What is the length of a side of the smaller octagon?

Ex. 18. A framework is made of 5 equal rods each of length 8 ft. with each adjacent pair at the same angle. What will be the length of the rods joining one corner to the two opposite corners in order to make the frame rigid?

Ex. 19. The vertical section of the roof of a building consists of four sides of a regular 9-gon each 6 ft. long. Find the greatest width spanned by the roof.

Ex. 20. The roof of a building is a portion of a circular cylinder and is supported on 4 beams each 5 ft. long which form 4 sides of a regular 10-sided polygon. Find the radius of the cylinder and the width of the roof.

Ex. 21. A round table of 4' 6" diameter is supported on four legs symmetrically placed 6" from the edge of the table. What is the distance between adjacent legs?

Ex. 22. A round table of 5' diameter folds about two parallel lines; the flaps fold over so that they just meet in the centre of the table. What is now the width of the table?

CIRCUMSCRIBED POLYGONS.

§ 5. To circumscribe a regular polygon about a given circle.

The points of contact of the sides with the circle obviously form a regular polygon of the same number of sides inscribed in the circle.

Hence the construction consists of finding the vertices of an inscribed polygon of the given number of sides and drawing the tangents at these points.

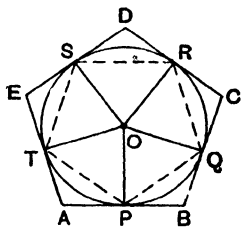


Fig. 135.

Ex. 23. Circumscribe a regular 5-gon about a circle of radius 1.5 in. Measure the side.

Ex. 24. Find the length of a side of a regular 12-gon, circumscribed about a circle of radius 3 in.

Ex. 25. Find the length of a line joining opposite vertices of a regular 20-gon whose sides touch a circle of radius 10 cm.

[Only two sides need be drawn.]

Ex. 26. By how much does the perimeter of a regular hexagon circumscribed to a circle of radius 5 cm. exceed the perimeter of the inscribed hexagon?

Ex. 27. Repeat Ex. 26 for a regular octagon (8-gon).

Ex. 28. A coil of cable of diameter 5 ft. is enclosed in a frame whose shape is that of a regular octagon. What is the length of the side of this octagon?

SUMMARY OF RESULTS

The angle subtended at the centre of a circle by a side of an inscribed regular polygon of n sides is $\frac{360^\circ}{n}$.

If the sides of a convex polygon are produced in order, the sum of the angles so formed is equal to four right angles.

CHAPTER 13

THE CIRCLE. ANGLE PROPERTIES

§ I. ¶Ex. 1. Draw a figure like fig. 136, making the radius of the circle about 2 in. Measure angles x and y . Notice that they are subtended* by the same arc.

¶Ex. 2. Do the same for figs. 137, 138. What relation do you notice between the angle x and the angle y in the three experiments?

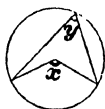


Fig. 136.

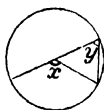


Fig. 137.

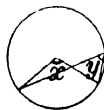


Fig. 138.

THEOREM 44. The angle which an arc of a circle subtends* at the centre is double that which it subtends at any point on the remaining part of the circumference.

That is, in the above figures, the angle x is double the angle y .

For proof of this Theorem see *Theoretical Geometry*, p. 31. -

A **reflex angle** is an angle greater than two right angles and less than four right angles. Fig. 139 shows two straight lines OA, OB forming a reflex angle (marked), and also an obtuse angle (unmarked).

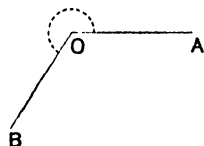


Fig. 139.

¶Ex. 3. Account for the necessity of the phrase "less than four right angles" in the above definition.

¶Ex. 4. Express the following bearings† by reflex angles:—(i) S. 20° W., (ii) N. 50° W., (iii) W.N.W.

* For "subtend" see Index.

† See p. 35.

A **segment** of a circle is the part of the plane bounded by an arc and its chord.

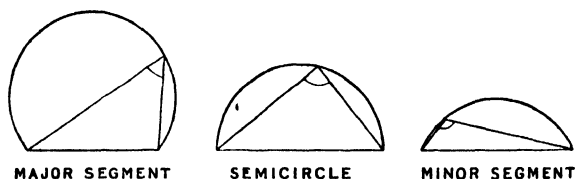


Fig. 140.

An **angle in a segment** of a circle is an angle subtended by the chord of the segment at a point on the arc (fig. 140).

§ 2. ¶Ex. 5. Draw a circle; place in it a chord AB ; mark four points P, Q, R, S in the major arc. What relation do you notice between the angles subtended at these points by AB (angles in the same segment)?

¶Ex. 6. In the figure of Ex. 5 mark three points X, Y, Z in the minor arc. What do you notice about the angles subtended at X, Y, Z by AB ?

THEOREM 45. Angles in the same segment of a circle are equal.

For they are all halves of the same angle at the centre.

NOTE. Since all the angles in a segment are equal, we may in future speak of *the* angle in a segment when we mean the magnitude of any angle in the segment.

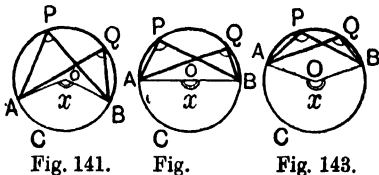


Fig. 141.

Fig.

Fig. 143.

¶Ex. 7. Are $\angle PAQ, \angle PBQ$ in figs. 141–143 equal? Give a reason.

Ex. 8. Copy fig. 144 (freehand); join BC, BD . Find all the angles in your figure.

Ex. 9. AC, BD are perpendicular diameters of a circle, P is a point on the minor arc AB ; show that the arcs AD, DC, CB each subtend 45° at P .

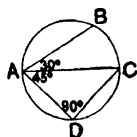


Fig. 144.

§ 3. ¶Ex. 10. Draw a circle and a diameter. Mark four points on the circle, at random. What are the angles subtended by the diameter at each of these points?

¶Ex. 11. What kind of angle is subtended at the centre by a major arc? a semicircle? a minor arc?

What kind of angles then are subtended by these arcs at a point on the remaining part of the circumference?

Notice that in fig. 145 $\angle APB$ is *subtended by* a minor arc, and is *the angle in* a major segment.

What similar statement can be made about figs. 146, 147?

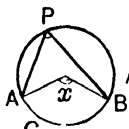


Fig. 145.

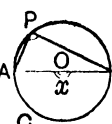


Fig. 146.

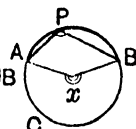


Fig. 147.

THEOREM 49. The angle in a semicircle is a right angle.

For the angle at the centre is two right angles.

Ex. 12. What can be said about the angle in a major segment? in a minor segment?

Ex. 13. Justify the following **construction for drawing a tangent to a circle from an external point P**. Let C be the point midway between P and the centre O. With centre C and radius CO draw a circle cutting the original circle in R and S. Then PR, PS are tangents.

How does the construction fail if P is inside the circle?

§ 4. ¶Ex. 14. Find the magnitude of all the marked angles in fig. 148. What is the sum of the angles at the centre? of $\angle s$ ACB and ADB? of $\angle s$ CAD and CBD? What property of a quadrilateral inscribed in a circle does this suggest?

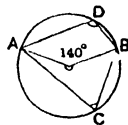


Fig. 148.

THEOREM 47. The opposite angles of any quadrilateral inscribed in a circle are supplementary.

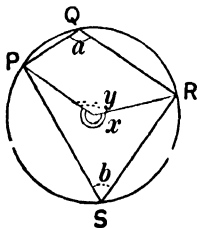


Fig. 149.

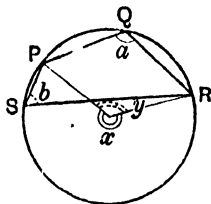


Fig. 150.

For $a = \frac{1}{2}x$, $b = \frac{1}{2}y$ and $x + y = 4 \text{ rt. } \angle s$.

COR. If a side of a quadrilateral inscribed in a circle is produced, the exterior angle so formed is equal to the interior opposite angle of the quadrilateral.

Ex. 15. Give reasons for the corollary.

¶ **Ex. 16.** What is the relation between the angles subtended by a chord at a point in its minor arc, and at a point in its major arc?

Ex. 17*. From the given angles, find all the angles in fig. 151.

Ex. 18*. Repeat Ex. 17, taking

$$\angle B = 71^\circ, \angle ACD = 35^\circ, \angle AOB = 72^\circ.$$

Prove that in this case AD is $\parallel$ to BC.

Ex. 19*. Repeat Ex. 17, taking

$$\angle B = x^\circ, \angle BCO = y^\circ, \angle AOD = z^\circ.$$

Ex. 20*. A, B, C, D are points on a circle, AC a diameter and O its middle point. Given $\angle BOD = 140^\circ$ and $\angle BOC = 75^\circ$, find all angles when all the points are joined.

Ex. 21*. ABCD is a quadrilateral inscribed in a circle, DA and CB meet in Q, AB and DC meet in P. Join AC, BD. If $\angle QAB = 75^\circ$, $\angle ACD = 36^\circ$, $\angle QBF = 58^\circ$, find the remaining angles.

Ex. 22. Two circles intersect in P and Q. APC, BQD are two straight lines meeting the one circle in A and B, and the other in C and D. Join PQ, PB, AQ, CD. If $\angle APQ = 70^\circ$, $\angle AQB = 32^\circ$, $\angle PBQ = 40^\circ$, find the remaining angles in the figure.

Ex. 23. P, Q, R, S are points in order on a circle whose centre is C.

$$\angle PCQ = 30^\circ \quad \angle SCR = 150^\circ$$

Show that PR and QS cross at right angles.

Ex. 24. AB and DC are two parallel chords, in the same half, of a circle. AC and BD meet within the circle at an angle of 48° . Find the angles which BC and AD subtend at the centre.

Ex. 25. Two circles intersect at P and Q, and APB, CQD are lines cutting one circle in A, C and the other in B, D. AC and BD are joined. If

$$\angle APQ = 110^\circ, \angle CQP = 98^\circ,$$

find the other angles and show that AC and BD are parallel. Does the last fact depend on the size of the given angles?

Ex. 26. Two circles intersect at P and Q, and APB, CQD are two parallel lines meeting one circle in A, C and the other in B, D. AC and BD are joined. If $\angle APQ = 70^\circ$, find all the other angles and show that AC = BD.

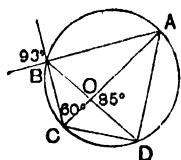


Fig. 151.

Brief reasons should be given, e.g.

$$\angle ADB = 60^\circ \text{ (same segment as } \angle ACB),$$

$$\angle OAD = 35^\circ \text{ (angles of a } \Delta = 180^\circ).$$

Ex. 27. Circumscribe about a given circle a triangle of given angles.

[Take 50° , 60° , 70° . Suppose the figure drawn and the points of contact joined to the centre; what are the angles at the centre?]

Ex. 28. Draw a circle and any triangle. With ruler and compass only, circumscribe about the circle a triangle equiangular to the given triangle.

Ex. 29. Show that the only parallelograms that can be inscribed in a circle are rectangles or squares.

CONCYCLIC POINTS.

§5. If *three* points are taken at random, a circle can be drawn to pass through them (except in one case; what is this case?).

Mark *four* points at random and try (freehand) if a circle can be drawn to pass through all four. A circle can as a rule be drawn through any three of them; is this bound to pass through the fourth? For instance, can a circle be circumscribed about a rhombus?

In general, a circle cannot be drawn to pass through four points in a plane; it can be so drawn only if, on joining the points, there are certain relations between the angles; either that (i) equal angles are obtained as in fig. 152 (a) or (ii) a quadrilateral with opposite angles supplementary is formed as in fig. 152 (b). If either of these conditions holds, a circle can be described to pass through the four points*.



Fig. 152.

These conditions follow from the converses of Theorems 45 and 47.

A set of points such that a circle can be described to pass through them are called **concyelic**. A quadrilateral through whose vertices a circle can be described is called **cyclic**.

Ex. 30. BE, CF are altitudes of the triangle ABC. Why are B, F, E, C concyclic? Sketch in the circle; where is its centre?

Ex. 31. BE, CF, two altitudes of $\triangle ABC$, intersect at H. Why is AEHF a cyclic quadrilateral? Sketch in the circle; where is its centre?

Ex. 32. Using only a protractor and a ruler, find 4 points which must lie on a circle. Test by drawing a circle through 3 of them.

Ex. 33. ABC is a triangle, I its in-centre, I_1 , I_2 , I_3 the centres of its escribed circles. Why are B, C, I, I_1 concyclic? Name two other sets of 4 concyclic points.

Ex. 34. ABCD is a parallelogram. Without using compasses, find where the circle round ABC will cut AD. Explain how you find the point.

* A third condition for four points to be concyclic follows from Th. 59, Ch. 8.

Ex. 35. ABC is a triangle right-angled at B. Find, without drawing a circle, the points where (i) the perpendicular from B to AC, (ii) the median from B, would meet the circle ABC.

Ex. 36. Draw any triangle ABC. At A make any angle BAO within the triangle. At C make an angle BCO equal to $\angle BAO$ without the triangle. Verify that O lies on the circle ABC.

Ex. 37. Construct a cyclic quadrilateral ABCD given $AB = 4$ cm., $BC = 7$ cm., $\angle A = 80^\circ$, $\angle B = 120^\circ$, without drawing the circle.

ANGLE IN ALTERNATE SEGMENT.

§6. Looking at the tangent as the limit of a chord produced, you will see from fig. 153 that as B moves round towards A the constant

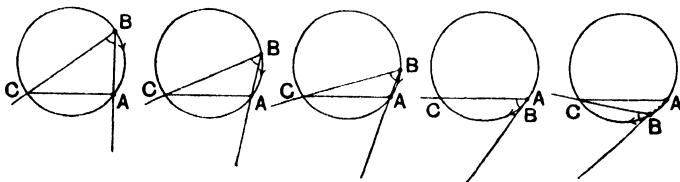


Fig. 153.

angle CBA becomes, in the limit, the angle between the chord and the tangent at A. This suggests the following theorem:—

THEOREM 51. If a straight line touch a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the angles in the alternate segments.

In fig. 154, ACB is the tangent and CD the chord.

Then $\angle BCD = \angle CED$ in the alt. seg.,

$\angle ACD = \angle CFD$ in the alt. seg.

For proof of this Theorem see *Theoretical Geometry*, p. 97.

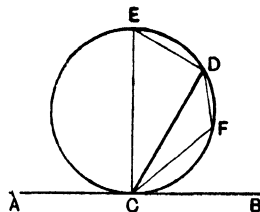


Fig. 154.

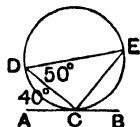


Fig. 155.



Fig. 156.

Ex. 38. Find all the angles of fig. 155.

Ex. 39. Find all the angles of fig. 156.

Ex. 40. If in fig. 154 arc $CD=2$ arc DE , what is $\angle BCD$?

Ex. 41. If in fig. 154 arc ED were $\frac{1}{2}$ arc DC , what would be the magnitude of $\angle BCD$?

Ex. 42. In fig. 154, $\angle DCB=62^\circ$, F is the mid-point of arc CD , find all the remaining angles in the figure.

Ex. 43. ACB is a tangent to a circle and CD a chord of the circle such that $\angle ACD=74^\circ$. CF bisects the $\angle BCD$ meeting circle in F . Find the angles of the triangle CFD .

Ex. 44. ABC is a triangle inscribed in a circle, $\angle A=42^\circ$, $\angle B=84^\circ$. Show that the tangent to the circle at C is parallel to the bisector of the angle B .

Ex. 45. A, B, C are three points on a circle, $\angle ACB=110^\circ$. At what angle will the tangents at A and B intersect?

Ex. 46. Inscribe in a given circle a triangle of given angles. (Take 40° , 55° , 85° . Supposing the figure drawn, and the tangent at one vertex also drawn the three angles at this point are known.)

Ex. 47. (*Without protractor.*) Draw any obtuse-angled triangle. In a circle of radius 3 in. inscribe a triangle equiangular to this triangle. Test the accuracy of the angles.

Ex. 48. In a circle of radius 2 in. inscribe a triangle having its sides parallel to three given straight lines.

CONSTANT ANGLE LOCUS.

§7. ¶**Ex. 49.** Stick two pins into the paper 2 in. apart at A and B ; place the set square on the paper so that the side containing the 60° are in contact with the pins; mark the point where the vertex of the angle rests. Now slide the set square about, keeping the same two sides against the pins, and plot the locus of the 60° vertex. What is the locus? are A, B points in the locus?

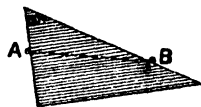


Fig. 157.

¶**Ex. 50.** Repeat the experiment of Ex. 49 with (i) the 90° vertex, (ii) the 30° vertex, (iii) an angle of 120° cut out of paper.

¶**Ex. 51.** Draw, on tracing-paper, two straight lines intersecting at P . On your drawing paper mark two points A, B . Move your tracing-paper about so that the one line may always pass through A , and the other through B . Plot the locus of P by pricking through.

The foregoing exercises will have prepared the reader for the following theorem:—

THEOREM 46 (COR.). **The locus of points (on one side of a given straight line) at which the line subtends a constant angle is an arc of a circle, the given line being the chord of the arc.**

¶Ex. 52. What is the theorem on the converse of which the above depends?

¶Ex. 53. What kind of arc is obtained if the constant angle is (i) acute, (ii) a right angle, (iii) obtuse? On which side of the base is the centre in each case?

¶Ex. 54. If the constant angle is 45° , what angle is subtended by the given line at the centre of the circle? Using this suggestion, show how to draw the locus of points at which a line of 5 cm. subtends 45° , without actually determining any of the points.

An important special case of this theorem is:—

The locus of the vertices of all right-angled triangles with the same hypotenuse is the circle on the hypotenuse as diameter.

The constant angle locus has many practical applications. For example, an observer may use it to fix his position on a map provided three prominent objects can be identified on the map. Suppose A, B, C are these objects and that AB is found to subtend an angle of 74° and BC an angle of 48° at the point of observation. If segments of circles be described on AB and BC to contain these angles, their second point of intersection P must be the position of the observer.

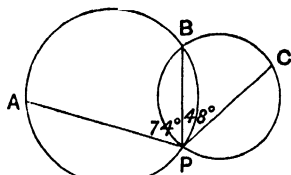


Fig. 158.

Ex. 55. In what special case would this method fail to give a position for P?

§ 8. To construct the constant angle locus.

I.e. On a given base to construct a segment of a circle to contain a given angle.

1st Method. First make a freehand sketch of the required figure. The centre is the vertex of an isosceles triangle whose angles you can calculate. Consider the three cases in which the centre of the circle is (i) on the same side of the line as the locus, (ii) on the line, (iii) on the side of the line opposite to the locus.

On a line AB , 2 in. long, draw a segment of a circle to contain an angle of 110°

Draw a freehand figure.

$$\angle \text{ at circumference} = 110^\circ,$$

$$\therefore \angle \text{ at centre } O = 220^\circ \text{ (reflex),}$$

$$\therefore \angle AOB = 140^\circ,$$

$$\therefore \angle OAB = \angle OBA = 20^\circ$$

Hence to find centre of circle make angles of 20° at A and B below the line.

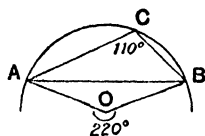


Fig. 159.

¶Ex. 56. If the constant angle is 90° , where is the centre of the circle?

¶Ex. 57. If the constant angle is x° ($x < 90$), what is the angle at the vertex of the isosceles triangle? What are the base angles of the triangle?

¶Ex. 58. Repeat Ex. 57 for the case in which $x > 90$.

2nd Method.

At A make $\angle BAP$ equal to the given angle.

Draw $AO \perp$ to AP .

Draw OD the $\perp$ bisector of AB .

The circle centre O , radius OA , gives the required segment.

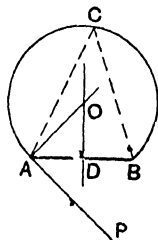


Fig. 160.

Ex. 59. Justify this construction (see Th. 51, p. 108).

To construct a triangle with given base, given altitude, and given vertical angle.

Let the base be 7 cm.; the altitude 6.5 cm.; the vertical angle 46°

Draw the given base. Draw the locus of points at which the given base subtends 46° . Draw the locus of points distant 6.5 cm. from the given base (produced if necessary).

The intersections of these loci will be the required positions of the vertex.

How many solutions are there to this problem?

Ex. 60. On a chord of 8 cm. construct a segment of a circle to contain an angle of 72° . Measure the radius. (Use both methods.)

Ex. 61. Repeat Ex. 60 with a chord of 3 in. and angle of 124° .

Ex. 62. Draw a rectangle sides 10 cm. and 7 cm. and find all points at which both longer sides subtend right angles.

Ex. 63. Construct a triangle having

- (i) base = 4 in., altitude = 1 in., vertical angle = 90° .
- (ii) base = 10 cm., altitude = 2 cm., vertical angle = 120° .
- (iii) base = 8 cm., altitude = 5 cm., vertical angle = 90° .
- (iv) base = 3.5 in., altitude = 1 in., vertical angle = 54° .

In each case measure the base angles.

Ex. 64. Construct a triangle having base = 8 cm., vertical angle = 64° median = 5 cm.

Ex. 65. Construct a triangle on a base of 3 in. whose vertical angle is 40° and whose area is 4 sq. in.

Ex. 66. Construct a quadrilateral ABCD given AB = 5 cm., BC = 7 cm., CA = 10 cm., AD = 6 cm., $\angle BDC = 48^\circ$.

Ex. 67. Draw a triangle ABC with BC = 4 in., CA = 2.8 in., AB = 3.5 in. Find the distance from A of a point O within it such that each side subtends an angle of 120° at O.

Ex. 68. The distances of C from two points B and D are required, but direct measurement is impossible. C is known to be 350 yd. from a point A, on the opposite side of BD to C, and the following measurements are made:—
B = 270 yd., AD = 220 yd., $\angle BAD = 124^\circ$, $\angle ACB = 50^\circ$. Find CB and CD.

Ex. 69. A rotating conical beam of light illuminates two walls which meet at right angles at O; in one position it illuminates a width AB on one wall, in another position it illuminates a width CD on the other. B, A, O, C, D are in the same horizontal plane and OA = 6 ft., OB = 10 ft., OC = 2 ft., OD = 7 ft. The angle of the cone of light is 20° . Find the possible distances of the source of light from each wall.

Ex. 70. A ship steers a straight course N. 10° E. from a place A. B and C are two landmarks N.W. of A and distant 5 and 15 miles from A. For what distance of the ship's run will the angle subtended at the ship by BC be not less than 30° ?

Ex. 71. A and B are two conspicuous points on the shore, B being 6 miles due E. of A, and C is a point inland, 2 miles N. of the mid-point of AB. A navigator observes with his sextant that AC subtends 40° and AB 90° . Fix the ship's position. Find the course to be steered in order to clear B by 2 miles.

Ex. 72. A ship is torpedoed and sunk while steaming near a straight coast line on which there are two lighthouses, A and B, 10 miles apart. There are two survivors: the first states that the ship was between 4 and 6 miles from the coast: the second, that the angle subtended at the ship by the line AB was between 60° and 90° . Draw a plan on a scale 1 cm. to 2 miles, and shade the area within which the wreck must lie.

Ex. 73. In fig. 158, A is 7.5 m. S.W. of B, and C is 5 m. S. 65° E. from B. Find the distance and bearing of P from B for the stated angles.

SUMMARY OF RESULTS

The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

Angles in the same segment of a circle are equal.

The angle in a semicircle is a right angle.

The opposite angles of any quadrilateral inscribed in a circle are supplementary.

If a side of a quadrilateral inscribed in a circle is produced, the exterior angle so formed is equal to the interior opposite angle of the quadrilateral.

The locus of points (on one side of a given straight line) at which the line subtends a constant angle is an arc of a circle, the given line being the chord of the arc.

If a straight line touch a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the angles in the alternate segments.

CHAPTER 14

ARCS, CIRCUMFERENCE, AREA

§ I. The circumference of a semicircular protractor is graduated in equal arcs, 180 in number. Each of these equal arcs subtends at

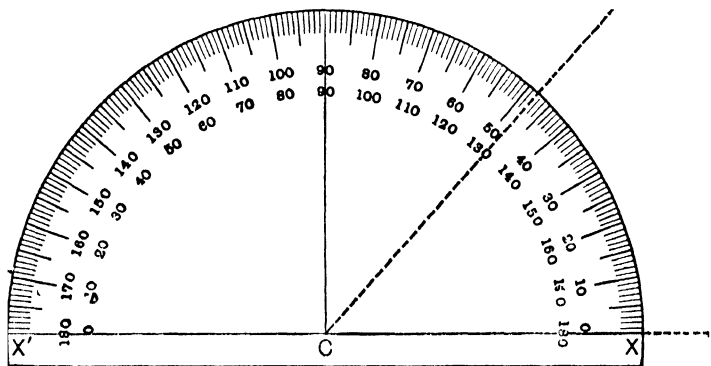


Fig. 161.

the centre an angle of 1° . For this reason, the degree is commonly used as a measure of arc as well as of angle: thus an arc that subtends 30° at the centre is called an **arc of 30°** . There is no danger of confusing these two meanings of the word "degree."

It is clear that **equal arcs of a circle subtend equal angles at the centre.**

In fact, **arcs of a circle are proportional to the angles which they subtend at the centre.**

Hence an arc subtending D° is $\frac{D}{360}$ of the whole circumference of the circle.

Note that the length of a *chord* of a circle is not proportional to the angle which it subtends at the centre.

Ex. 1. Draw a chord of a circle and the angle it subtends at the centre. Prove that the chord which subtends half the angle at the centre is greater than half the original chord.

CIRCUMFERENCE OF CIRCLE. LENGTH OF ARC.

You will have found in the laboratory that the circumference of a circle contains the diameter about 3·14 times*. The ratio of circumference to diameter has been worked out to 700 places of decimals and begins thus

$$3\cdot1415926535\dots$$

For the sake of brevity this number is denoted by the Greek letter π ; a useful approximation is $\frac{22}{7}$ or $3\frac{1}{7}$.

$$\begin{aligned}\text{Thus circumference of circle} &= \text{diameter} \times \pi \\ &= \text{radius} \times 2\pi \\ &= 2\pi r, \text{ where } r \text{ is the radius.}\end{aligned}$$

Ex. 2. Calculate the circumference of a circle whose radius is (i) 7 in., (ii) 35 miles. (Take $\pi = \frac{22}{7}$.)

Ex. 3. How far does a wheel roll in one revolution if its diameter is 28 in.?

Ex. 4. In fig. 162, AD is divided into three equal parts and all the arcs are semicircles; show that the four curved lines which connect A with D are of equal length.

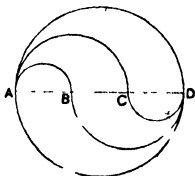


Fig. 162.

Ex. 5. Calculate the radius of a circle of circumference (i) 22 ft., (ii) 40 ft. (to three significant figures).

Ex. 6. A bicycle wheel makes 7200 turns in an hour while the cyclist is riding 10 miles an hour; what is the diameter of the wheel (to the nearest inch)?

Ex. 7. What fractions of a circumference are arcs of 90° , 60° , 120° , 1° , 35° , 300° ?

Ex. 8. Calculate the lengths of arcs of 40° and 170° in a circle of radius 63 cm.

Ex. 9. The circumference of a circle is 7·82 in. and the length of a certain arc is 1·25 in. What decimal of the circumference is the arc? What angle does the arc subtend at the centre?

Ex. 10. If the length of an arc is equal to the radius of the circle, find what angle it subtends at the centre. Is this angle the same for any circle?

Ex. 11. Each of the rockers of a cradle is an arc of a circle of length 22 in., the radius of the arc being 5 ft. What is the greatest angle through which the cradle should be allowed to rock?

* You may verify this fact very roughly by rolling a halfpenny along a straight line and thus measuring the circumference in inches. The diameter of a halfpenny is 1 inch

Ex. 12. A driving-wheel of diameter 14", containing 64 teeth, is connected by a continuous chain to a small gear-wheel containing 8 teeth. What length of chain passes round while the small wheel makes 1 revolution?

Ex. 13. A beam resting across a cylindrical pipe of diameter $3\frac{1}{2}$ ft. can rock through an angle of 20° without slipping. What length of the beam comes into contact with the pipe?

Ex. 14. A marble, diameter 1.5 cm., rolls down an arc of 60° of a hemispherical bowl of diameter 30 cm. How many revolutions does it make?

Ex. 15. A large wheel of diameter 3 ft. 6 in. is fixed to an axle of diameter $3\frac{1}{2}$ in. round which a rope is coiled with a weight attached; the wheel and axle can turn about a line through their common centre. By turning the wheel, the weight is raised 10 in. Through what angle has the wheel turned?

Ex. 16. A wheel radius 2" rolls round the outside of another wheel radius 5" When the small wheel has made $2\frac{1}{2}$ revolutions, what is the angle subtended at the centre of the large circle by the arcual distance traversed on the large circle?

Ex. 17. What continuous length of rope is required to pass round a wheel of 2 ft. diameter and a thin rod distant $3\frac{1}{2}$ ft. from the centre of the wheel? (See fig 126.)

Ex. 18. In fig. 163 what fractions of the circumference are the arcs AB, BC, CD, DA, BCD?

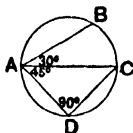


Fig. 163.

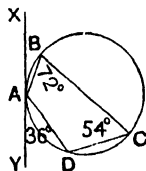


Fig. 164.

Ex. 19. In fig. 164 what fractions of the circumference are the arcs AB, BC, CD, DA?

Ex. 20. The two tangents OA, OB from a point O are inclined at an angle of 48° . How many degrees are there in the minor and major arcs AB respectively? What is the ratio of the major to the minor arc?

Ex. 21. Without assuming the value of π , prove that the circumference of a circle is $>$ three times the diameter, by inscribing a hexagon in the circle.

Ex. 22. Without assuming the value of π , prove that the circumference is $<$ four times the diameter by circumscribing a square about the circle.

The ratios of the perimeters (or circumferences) of regular polygons to the diameters of their circumscribing circles are shown in the following table:—

Table showing the perimeters of regular polygons inscribed in a circle of radius 5 cm.

No. of sides	Perimeter in centimetres	Ratio of perimeter to diameter
3	25.98	2.598
4	28.29	2.829
5	29.39	2.939
6	30.00	3.000
7	30.38	3.033
8	30.61	3.061
9	30.78	3.078
10	30.90	3.090

It will be noticed that the ratio increases with the number of sides, being always less than π . If the number of sides is very great, the ratio is very nearly equal to π . E.g. for a polygon of 384 sides the ratio is 3.14156.....

Ex. 23. By how much per cent. does the perimeter of a regular decagon inscribed in a circle differ from the perimeter of the circle? Give the result to one significant figure.

Ex. 24. Calculate the length of an arc of 60° in a circle of radius 7 cm. What is the length of the chord of this arc? Find, to three significant figures, the ratio of arc to chord; also the difference of arc and chord.

Ex. 25. The radius of a circle is 10 cm.; a piece of string as long as the radius is laid along an arc of the circle; what angle does it subtend at the centre? Also find the angle subtended at the centre by a chord of 10 cm.

Ex. 26. Find the lengths of the minor and major arcs cut off from a circle of radius 7 cm. by a chord of 7 cm.

AREA OF CIRCLE.

§ 2. Let PQRS... be a regular polygon of n sides circumscribing a circle.

Join the vertices of the polygon to the centre of the circle. The circle is thus divided into n triangles, having for bases the sides of the polygon, and for common vertex the centre of the circle.

Draw perpendiculars from the centre to the sides of the polygon. These meet the sides at their points of contact and are radii of the circle. Thus the triangles OPQ, OQR, etc. are all of height equal to the radius of the circle.

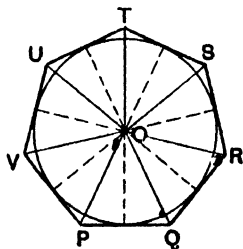


Fig. 165.

Let r be the radius of the circle, a the side of the polygon.

The area of $\triangle OPQ = \frac{1}{2}ar$,

$\therefore$ area of polygon $= n \times$ area of $\triangle OPQ$

$$= n \times \frac{1}{2}ar$$

$$= \frac{1}{2}r (na)$$

$$= \frac{1}{2} \text{ radius} \times \text{perimeter of polygon.}$$

This is true for any polygon circumscribing the circle.

If we draw a regular polygon of a very great number of sides, it is difficult to distinguish it from the circle itself. The area of the polygon approaches closer and closer to the area of the circle; and the perimeter of the polygon to the circumference of the circle. Hence

area of a circle $= \frac{1}{2}$ radius $\times$ circumference of circle

$$= \frac{1}{2}r \times 2\pi r$$

$$= \pi r^2.$$

In the following exercises it will generally be sufficient if answers are given correct to three significant figures.

Ex. 27. Calculate the area of a circle whose radius is (i) 1 inch, (ii) 2 inches.

Ex. 28. The radius of one circle is twice the radius of another; how many times does the area of the greater contain the area of the smaller? Fig. 166 shows that the area of the greater is more than double the area of the smaller. Find the area of the shaded part of fig. 166, taking the diameter of the small circles to be 1 cm.

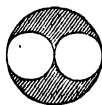


Fig. 166.

Ex. 29. Find the ratio of the area of a circle to the area of the circumscribing square.

Ex. 30. In the centre of a circular pond of radius 100 yards is a circular island of radius 20 yards. Find the area of the surface of the water.

Ex. 31. Find whether the area in Ex. 30 is greater or less than the area of a circular sheet of water of 80 yards radius.

Ex. 32. The radius of the inside edge of a circular running track is a feet; and the width of the track is b feet; find the area of the track.

Ex. 33. From a point P , on the larger of two concentric circles, a tangent PT is drawn to the smaller. Show that the area of the circular ring between the circles is equal to that of a circle with radius PT .

Ex. 34. Show how to draw a circle whose area is equal to (i) the sum, (ii) the difference of the areas of two given circles.

Ex. 35. Four equal circles are described with centres at the corners of a square of side 8 cm. so that each touches two others at the mid-points of the sides. Find the area of that portion of the square not in any circle.

Ex. 36. Calculate the radius of a circle whose area is 1 sq. in.

Ex. 37. Calculate the diameter of a circular field whose area is 1 acre (= 4840 sq. yards).

Ex. 38. Prove that in fig. 162 the three portions into which the circle is divided by the curved lines are of equal area.

Ex. 39. In fig. 167 $\angle BAC$ is a right angle, and the curves are semicircles. Prove that the two shaded areas are together equivalent to the triangle.



Fig. 167.

AREA OF SECTOR OF CIRCLE.

§ 3. If through the centre of a circle 360 radii were drawn making equal angles with one another, 360 angles of 1 degree would be formed at the centre of the circle. The area of the circle would be divided into 360 equal sectors. A sector of angle 1° has therefore $\frac{1}{360}$ of the area of the circle; and a sector of angle, say, 53° contains $\frac{53}{360}$ of the area of the circle.

Ex. 40. Find the area of a sector of 40° in a circle of radius 5 in.

Ex. 41. Prove that the area of a sector of a circle is half the product of the radius and the arc of the sector.

Ex. 42. A wheel 8" in diameter runs in an oil-bath to a depth of 1". Find the area of the wheel immersed at any moment.

Ex. 43. Fig. 168 shows a window 5 feet wide with a semicircular arch. It consists of 16 panes; namely, 12 rectangular panes equal in all respects, 1 semicircular pane A, and 3 panes B, C, D, equal in all respects. Compare the area of B with that of A.

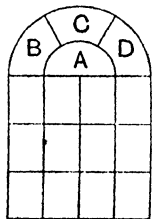


Fig. 168.

Show that it is possible for all 16 panes to be of equal area. When this is the case, calculate the central height of the window, and compare the perimeters of the various panes.

Ex. 44. A circle radius 12 cm. is cut into two unequal parts by a chord at a distance of 4 cm. from the centre. Find the area of each portion.

Ex. 45. Tangents are drawn to a circle of radius 2" from a point 4" from the centre. What is the area of the sector contained by the radii to the points of contact?

Ex. 46. A section of a buoy consists of the figure formed by two tangents to a circle and the major arc intercepted as in fig. 311. If the circle be $2\frac{1}{2}$ ft. radius and the point where the tangents intersect 4 ft. from the centre, find the area of the section.

SUMMARY OF RESULTS

Arcs of a circle are proportional to the angles which they subtend at the centre.

The circumference of a circle = $2\pi r$, where r is the radius and $\pi = 3\frac{1}{7}$ (approx.) or 3.14 (approx.).

The area of a circle = πr^2 .

CHAPTER 15

CONTACT OF CIRCLES. COMMON TANGENTS

§ I. The different relative positions which are possible for two circles are shown in fig. 169.

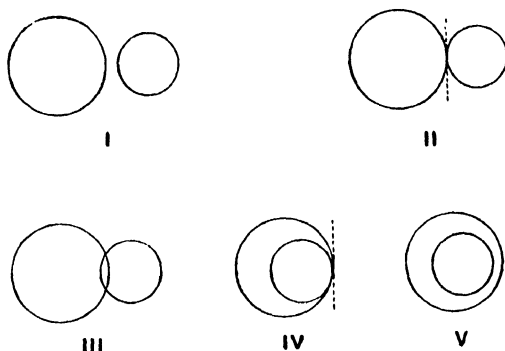


Fig. 169.

In II the circles are said to **touch externally**, in IV to **touch internally**. The formal definition of contact of circles is as follows:—

If two circles touch the same line at the same point, they are said to touch one another.

¶Ex. 1. Draw a figure showing the different relative positions possible for two equal circles

¶Ex. 2. Describe in words each of the relative positions shown in fig. 170.

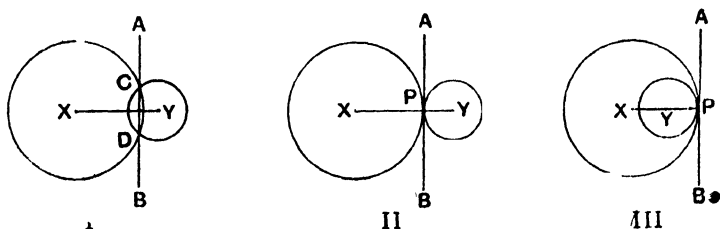


Fig. 170.

Fig. 170 (I) shows two intersecting circles, their common chord produced and their centres X, Y. The whole figure is, symmetrical about XY; hence XY bisects CD at right angles.

Let the small circle move to the *right* until the position of external contact is reached (II). The produced chord becomes the common tangent AB, the points C and D coincide at P; and XY which formerly bisected CD now passes through P.

Now let fig. I change by moving the small circle to the *left*, until the position of internal contact is reached (III). By similar reasoning we see that XY produced passes through P. Hence

THEOREM 43. If two circles touch, the point of contact lies in the straight line joining the centres.

For proof of this Theorem see *Theoretical Geometry*, p. 82.

If two circles touch externally the distance between their centres is equal to the sum of their radii; if they touch internally the distance between their centres is equal to the difference of their radii.

Ex. 3. Describe the relative position of the two circles in each of the following cases (d is the distance between the centres, R and r are the radii). Do this, if you can, without drawing the circles.

- (i) $d = 4.1$ cm., $R = 2.1$ cm., $r = 1.4$ cm.
- (ii) $d = 0.7$ cm., $R = 2.2$ cm., $r = 1.2$ cm.
- (iii) $d = 3.4$ cm., $R = 2.0$ cm., $r = 1.4$ cm.
- (iv) $d = 0.8$ cm., $R = 2.1$ cm., $r = 1.3$ cm.
- (v) $d = 0$ cm., $R = 1.9$ cm., $r = 1.2$ cm.
- (vi) $d = 1.5$ cm., $R = 2.0$ cm., $r = 1.5$ cm.
- (vii) $d = 2.5$ cm., $R = 1.7$ cm., $r = 1.7$ cm.

Ex. 4. What is the distance between the centres of two circles of radii 15 and 14 in. (i) if they have external contact, (ii) if they have internal contact?

Ex. 5. Draw three circles having for centres the vertices of an equilateral triangle of side 2 in., so that each circle may touch the two others externally.

Ex. 6. Three circles, of radii 1, 2, 3 in., touch externally, each circle touching the other two. What are the distances between the centres? Draw the circles.

Ex. 7. On the same side of the straight line ADC two semicircles are drawn, touching at A; AKC has radius R and AED has radius r . Find expressions for the radii of the following circles:—(i) a circle which touches the two semicircles and has its centre equidistant from A and C, (ii) a circle which touches the two semicircles and has its centre equidistant from A and D.

Ex. 8. Two cylindrical pipes of radii 2.5" and 1.5" lie inside another pipe of radius 4.5" and touching each other. What is the distance between the centres of the pipes and the distance of the centre of the large pipe from the line joining the centres of the other two?

Ex. 9. A sphere of diameter 30 cm. rests on two cylinders of diameters 10 cm. and 5 cm. lying on the ground with their axes parallel and 20 cm. apart. Find by drawing and measurement the height of the centre of the sphere above the ground.

Ex. 10. Two circles of radii 3 cm. and 2 cm. have their centres 6 cm. apart. Find all the positions of the centre of a circle of radius 7 cm. which touches both circles in all possible ways.

CONSTRUCTION OF CIRCLES TO SATISFY GIVEN CONDITIONS.

§ 2. ¶Ex. 11. What is the locus of the centres of all circles of radius 1 in., which touch externally a fixed circle of radius 2 in.? Draw the locus, and draw a number of the touching circles.

¶Ex. 12. If required to draw a circle to touch a given circle *at a given point*, where would you look for the centre of the touching circle? What is the locus of the centres of circles touching a given circle at a given point? Draw a number of such circles, some enclosing the given circle, some inside it, some external to it.

¶Ex. 13. What is the locus of the centres of circles which touch a given line at a given point?

¶Ex. 14. What is the locus of the centres of circles of radius 1 in., touching a given circle of radius 2 in., and lying inside it? Draw a number of such circles.

¶Ex. 15. Repeat Ex. 14 with 1 in. radius for the touching circles, and 3 in. radius for the fixed circle.

¶Ex. 16. Draw a number of circles of radius 3 in. to touch a circle of radius 2 in. and enclose it.

¶Ex. 17. Draw a number of circles of radius 4 in. to touch a given circle of radius 2 in. and enclose it.

¶Ex. 18. What is the locus of centres of circles of given radius passing through a given point?

¶Ex. 19. What is the locus of centres of circles (i) passing through two given points, (ii) touching two given lines?

Each of the following problems is to be solved by finding the centre of the required circle (generally by the intersection of loci). Some of the group have been solved already; they are recapitulated below for the sake of completeness. In several cases a numerical instance is given which should be attempted first, the radius of the resulting circle being measured.

Ex. 20. Draw a circle (or circles) to satisfy the following conditions :—

- (i) To pass through three given points (solved already).
- (ii) Of given radius, to pass through two given points.
- (iii) Of given radius, to pass through a given point and touch a given line, e.g. take radius 2.5 in. and a point distant 1 in. from the line. (What is the locus of centres of 2.5 in. circles passing through given point? touching given line?) When is the general problem impossible?
- (iv) To touch a given line AB at a given point P, and to pass through a given point Q outside the line. (What is the locus of centres of $\odot$ s touching the line at P? passing through P and Q? Let $PQ = 5$ cm., $\angle QPA = 35^\circ$.)
- (v) To touch a given circle at a given point P, and to pass through a given point Q not on the circle. In what case is this impossible?
- (vi) To touch a given line AB at P, and also to touch a given line CD, not parallel to AB. (What is the locus of centres of circles touching AB and CD?)
- (vii) Of given radius, to pass through a given point P and touch a given circle, e.g. let given radius = 4 cm., radius of given circle = 3 cm., distance of P from centre of given circle = 5 cm. (Compare (iii).)
- (viii) Of given radius, to touch a given circle at a given point (how many solutions are there?).
- (ix) To touch three given lines (solved already).
- (x) Of given radius to touch two given lines, e.g. let the lines intersect at an angle of 60° , and radius = 1 in. (How many solutions are there?)
- (xi) Of given radius, to touch a given line and a given circle (e.g. given radius = 3 cm., radius of given circle = 5 cm., distance of line from centre of circle = 6 cm.). What is the condition that the general problem may be possible?
- (xii) To touch three equal circles (a) so as to enclose them all, (b) so as to enclose none of them. (Begin by drawing a circle through the three centres.)
- (xiii) Of given radius, to touch two given circles (e.g. let given radius = 2 in., radii of given circles = 1 in., 1.5 in., distance between centres = 3.5 in.).

MISCELLANEOUS CONTACT PROBLEMS.

§ 3. Ex. 21. Fig. 171 shows parts of two straight roads on a scale of 4 inches to the mile. They are to be united by a third road in the form of a circular arc less than a semicircle, which is to be tangential to both the roads and to have a radius of a quarter-mile.

Prick the figure on to your paper, produce the lines to the left, and draw the third road by a geometrical construction.

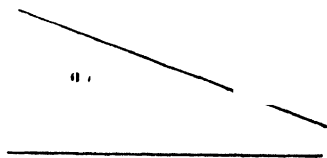


Fig. 171.

Ex. 22. In a semicircle of radius 5 cm. inscribe a circle of radius 2 cm. Measure the parts into which the diameter of the semicircle is divided by the point of contact. (See fig. 172.)

Ex. 23. Draw four circles of radius 2 in., touching a fixed circle of radius 1 in., and also touching a straight line 2 in. distant from the centre of the fixed circle.



Fig. 172.

Ex. 24. Show how to inscribe a circle in a sector of 80° of a circle whose radius is 4 in.

Ex. 25. Show how to draw three equal circles, each touching the other two; and how to circumscribe a fourth circle round the other three.

Ex. 26. In fig. 173, BAC is an equilateral arch, B being the centre of the arc AC and C the centre of arc BA; BED, CFD are similar arches, B being the centre of DE and C the centre of DF and D of BE and CF. What is the locus of the centres of circles touching (i) arcs AB and AC, (ii) arcs DE and DF, (iii) arcs BA and DF?

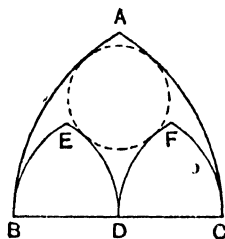


Fig. 173.

Hence explain how to construct with your instruments a circle (shown dotted in the figure) which will touch the arcs BA, AC, DE, DF.

Draw the figure carefully, taking BC 5 inches long.

Ex. 27. Fig. 174 represents a light railway running along a straight line from A to B, and thence through C to D. BC, CD are arcs of circles touching at C, the arc BC touching AB at B. The lengths marked are those of the chords. Find, by drawing, the radius of each arc. Explain your construction.

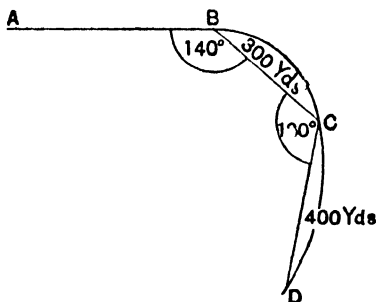


Fig. 174.

Ex. 28. The railway track shown in fig. 175 consists of two straight parallel pieces AB, DE, and two circular arcs BC, CD, whose radii are 700 yards and 1400 yards respectively. Find the length of track from B to D. Draw the figure to a scale of $1''$ to 500 yards.

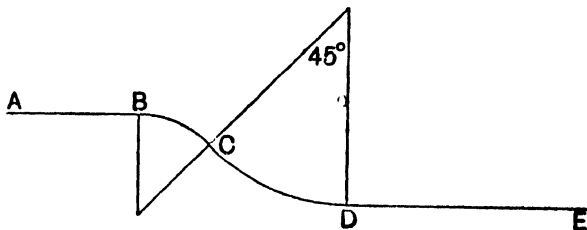


Fig. 175.

Ex. 29. Fig. 176 represents an iron scroll ABCDEF with two supports OC, OD. The arcs AB, BC are semicircles with a common tangent at B; the arcs DE, EF are semicircles with a common tangent at E. The circular arc CD touches OC and OD at C and D; the semicircles BC and DE also touch OC and OD at C and D respectively. AB and EF each have diameter $1\frac{1}{2}$ ft., BC and DE each have diameter 2 ft.; OC is of length 3 ft. and the angle COD is a right angle.

Calculate the length of the scroll ABCDEF, and also the distance AF. Check the latter answer by drawing to scale.

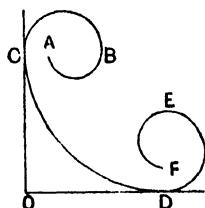


Fig. 176.

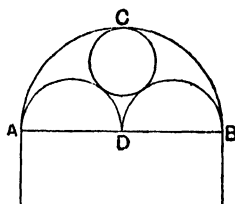


Fig. 177.

Ex. 30. Fig. 177 represents a doorway over which are a circular and two semicircular windows, the curve ACB being also a semicircle. If the radius of each of the two smaller semicircles is R inches and the radius of the circular window r inches, find by calculation the ratio of r to R .

Taking AB as 50 inches, make a drawing to scale of the part above AB, representing 10 inches by 1 inch. Give the radius of the circular window, and state the construction and the geometrical principles assumed.

Ex. 31. In fig. 178, which is not drawn to scale, the diameter OA of the semicircle has a length of 10 cm., and the length of ON is x cm., which is greater than 5 cm. PN is perpendicular to OA. The smaller circle, of radius r cm., touches NA at Q, NP at R, and the arc of the semicircle at S. Obtain an algebraic relation connecting r with x .

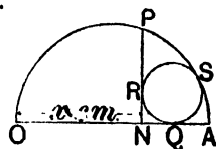


Fig. 178.

Calculate (i) r when $x = 6$, and (ii) x when $r = 1.5$.

Ex. 32. An oval is formed of four circular arcs, which have common tangents at their points of contact (see fig. 179). The length EF is 4.6 in. The radius of the arc AB is 1.1 in., that of CD 1.6 in., that of AC and of BD 4.2 in. Construct the oval full size, stating your method.

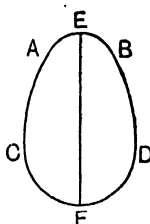


Fig. 179.

Ex. 33. A road 40 ft. wide has a right-angled turn in it, the sides being straight. Construct the course of a bicycle taking the turn, such that it is part of a circle and of as large a radius as the road permits. Calculate the radius of this circle.

Ex. 34. The front wheel of a motor near the kerb touches the ground at A and is moving in direction AB (see fig. 180); the line CD represents the kerb-stone. Find the greatest radius allowed to the circular track of the wheel as the motor turns, if it is to avoid the kerb.

Ex. 35. In turning, a motor-car rotates about some point O on the back axle produced. All four wheels describe different circles, to which their ground plans are tangents.

The front wheels are coupled together by a mechanical device which makes them deviate from parallelism by the correct angle (COD in fig. 181).

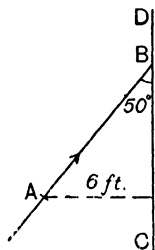


Fig. 180.

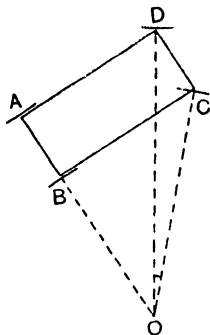


Fig. 181.

A, B, C, D are the four points of contact of the wheels of a motor-car with the ground, A, B for the back wheels, C, D for the front wheels. The car is turning in a circle whose centre is on the line AB produced; the largest circle is described by the outer front wheel; if the radius of this is 20 ft. and $AB = 5$ ft., $BC = 12$ ft., draw a figure to show the four circles; calculate the four radii, and measure the angle by which the front wheels are out of parallel.

Ex. 36. Given the three circular tracks of the two back wheels and one front wheel, show that it is possible to find the "wheel base" AD and "track" AB of a car and also to supply the track of the other wheel.

Ex. 37. A car is turning round, and the lock (i.e. the angle between a front wheel and the side of the chassis) is 30° . If the distance between the points of contact with the ground of the front and back wheels on one side of the car is 12 ft., and it is known that the centre of rotation for a car is some point on the back axle produced, find the radius of the circle described by the front wheel.

COMMON TANGENTS TO TWO CIRCLES.

§ 4. A straight line which touches two circles is called a **common tangent** to the two circles.

Fig. 182 shows that when the circles do not intersect there are four common tangents.

If the two circles lie on the same side of a common tangent, it is called an **exterior common tangent**; thus AB, CD (fig. 182) are exterior common tangents. If the two circles lie on opposite sides of a common tangent, it is called an **interior common tangent**; thus EF, GH are interior common tangents.

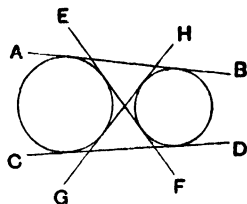


Fig. 182.

Ex. 38. Draw sketches to show how many common tangents can be drawn in Cases II, III, IV, V, of fig. 169; in each case state the number of exterior and of interior common tangents.

For practical purposes, the common tangents may be drawn “by eye” with a ruler; this method will not however give the points of contact with any accuracy, and to obtain these, *perpendiculars must be drawn from the centre*.

To calculate the length of the common tangents. By symmetry it is obvious that the two exterior common tangents are equal, as are the two interior common tangents. If the radii (r , R) and the distance between the centres (l) are given, the lengths (i.e. the distance between the points of contact) are calculated as follows:—

Exterior common tangents. ST is a common tangent, AS, BT the perpendiculars from A and B to ST. Draw BP $\perp$ to AS. Then BTSP is a rectangle. (Why?) $\therefore$ ST = PB.

But in the right-angled $\triangle APB$, we have $AB = l$ and $AP = R - r$. Hence BP may be calculated by Pythagoras’ theorem.

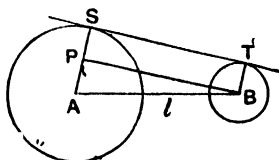


Fig. 183.

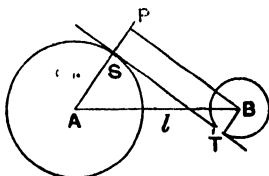


Fig. 184.

Interior common tangents are calculated by a similar method, the figure being as in fig. 184. Notice that $AP = R + r$

¶Ex. 39. Can you suggest an easier method of constructing the interior common tangents to two equal circles?

¶Ex. 40. If the radius of the smaller circle diminishes till the circle becomes a point, what becomes of the four common tangents?

Ex. 41. Draw two circles of radii 1.5 in. and 0.5 in., the centres 2.5 in. apart. Measure and calculate the length of the exterior common tangents.

Ex. 42. Draw the two circles of Ex. 41, and draw the interior common tangents. Measure and calculate the length of these tangents.

Ex. 43. The diameters of the wheels of an old-fashioned bicycle are 4 ft. and 1 ft., and the distance between the points where the wheels touch the ground is $2\frac{1}{2}$ ft. Calculate the distance between the centres of the wheels.

Ex. 44. A thin endless cord is wrapped round two equal pulley wheels so as to cross midway between their centres. Each wheel is 12 inches in diameter, and the distance between the centres is 20 inches. Find the length of the cord.

[For curved portion see Ch. 14, § 1. You will have to measure an angle.]

Ex. 45. A driving wheel of 20" diameter is connected by an endless chain to a wheel of 8" diameter with its centre 3' 6" distant from the centre of the driving wheel. Find the length of the chain.

Ex. 46. A shaft of 6" diameter is driven by crossed belting from a wheel of 3' diameter, the distance between the centres being 5'. Find the length of the belt, assumed taut.

Ex. 47. Three circular logs of diameters 10", 10", 6" are bound together by wire. What is the length of one coil?

Ex. 48*. Construct, with ruler and compass only, the exterior common tangents to two circles, radii 5 cm. and 4 cm., with their centres 11 cm. apart.

[See fig. 183; draw a circle with centre A and radius equal to the difference of the radii. Find P by drawing the tangent from B.]

Ex. 49. Construct the interior common tangents to the circles in Ex. 48.

SUMMARY OF RESULTS

If two circles touch, the point of contact lies in the straight line joining the centres.

If two circles touch externally the distance between their centres is the sum of their radii; if internally, the difference of their radii.

* See *Theoretical Geometry*, p. 100,

CHAPTER 16

SIMILARITY

RATIO.

§ 1. Consider a model of an engine and the engine itself. If the model is 3 ft. long and the engine 15 ft. long, we say that the model is to the scale of 3 to 15 or $\frac{3}{15}$. Also it is clear that if the width of the model is 1 ft., the width of the engine is 5 ft. In fact the ratio* of any length in the model to the corresponding length in the engine is 1 : 5 or $\frac{1}{5}$.

§ 2. In the next chapter we shall develop the meaning of the word "ratio" more fully and in a slightly different way, but we may define the ratio of a length PQ to length RS as the quotient

$$\frac{\text{measure of PQ}}{\text{measure of RS}},$$

the two measurements being made with respect to the same unit of length. Thus the ratio of 2 ft. to 9 in. is the ratio of 24 in. to 9 in. $= \frac{24}{9} = \frac{8}{3}$.

¶ Ex. 1. What are the ratios of the following lengths?

(i) 3 yd. to 2 yd., (ii) 4 yd. to 12 yd., (iii) 3 yd. to 10 ft., (iv) 2 ft. 6 in. to 5 ft., (v) 3 ft. to 1 ft. 6 in., (vi) 15 cm. to 5 cm., (vii) 1 metre to 1 cm., (viii) 6 miles to $1\frac{1}{2}$ miles.

In order to compare two ratios it is often necessary to express the vulgar fractions obtained in the form of decimals. Thus

the ratio of 2.78 in. to 3.59 in. $= \frac{2.78}{3.59} = 0.774\dots$ (by division), and

the ratio of 64 ft. to 83 ft. $= \frac{64}{83} = 0.771\dots$ (by division).

We see that these two ratios are nearly equal, the first ratio being slightly larger than the second.

* The ratio of a to b is written $\frac{a}{b}$, or a/b , or $a : b$, or $a \div b$.

Ex. 2. Compare the following pairs of ratios by expressing each as a decimal:—

- (i) 1·4 in. to 4·5 in., 7 in. to 22 in.; (ii) 25 cm. : 2 cm., 64 cm. : 5 cm.;
(iii) 7·65 in. : 4·25 in., 5·7 in. : 3·15 in.

¶Ex. 3. Find the ratio $\frac{PQ}{RS}$ (fig. 185); measure (i) in inches, (ii) in centi-

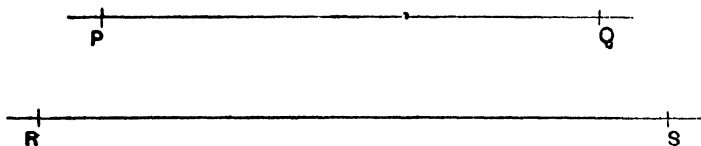


Fig. 185.

metres. Work out the ratios to three significant figures. Why might you expect your results to differ?

The ratio of two magnitudes is independent of the unit chosen.

For example, the ratio of a length of 5 yd. to a length of 2 yd. is 5 : 2; if these lengths are measured in feet the measures are 15 and 6, and the ratio is 15 : 6. Now we know that 5 : 2 = 15 : 6.

¶Ex. 4. Fill in the missing terms in the following:—

$$(i) \frac{2}{3} = \frac{\quad}{6}, \quad (iii) 7 : \quad = 3 : 11, \quad (v) \frac{\quad}{2p} = \frac{3}{2},$$

$$(ii) \frac{\quad}{5} = \frac{1}{2}, \quad (iv) 5 : 2 = 7 : \quad, \quad (vi) \frac{a}{b} = \frac{\quad}{d}.$$

SIMILAR FIGURES.

§3. The principle of similarity is one of the most fundamental in Geometry. From our earliest childhood we try to draw pictures of things we see. Of a figure in two dimensions we can draw an exact copy, or 'we can make a similar figure, i.e. a figure of the same shape but on a different scale.' Of a solid figure we can make an exact model, or a model to scale which will be similar to the given figure.

You have frequently used maps and plans and made drawings to scale; in all these cases you have made use, consciously or unconsciously, of the principle of similarity. We must now consider more definitely what we mean by the principle of similarity.

If we place side by side two scale drawings of the same thing, we say at once that they are of the same shape, or that they are similar. But let us get down to detail. What can you say about corresponding

angles in the two figures? They are obviously equal. It remains to consider corresponding lengths.

If the scale chosen for the one drawing is double that chosen for the other, you will agree that every length in the one figure is double the corresponding length in the other; and whatever scales are chosen, it is clear that the ratios of any one pair of corresponding lengths in the two figures will equal the ratios of any other pair.

Again, look at figs. 186, 187, are they the same shape? or, in other

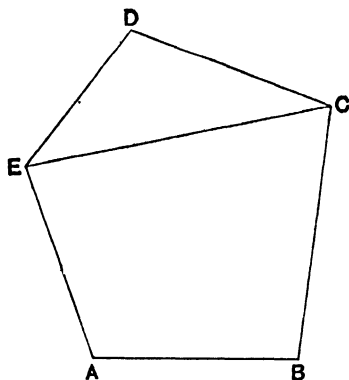


Fig. 186.

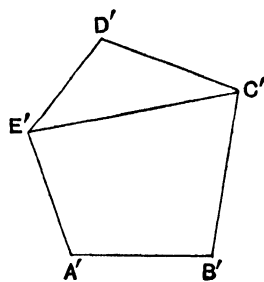


Fig. 187.

words, are they similar? What test must we apply? If AB were double A'B', what could you say about BC and B'C', about CD and C'D', etc.?

Measure the lengths of the lines in the figures and calculate the ratios $\frac{A'B'}{AB}$, $\frac{B'C'}{BC}$, etc. Are they all equal (to the degree of accuracy justified by your measurements)?

Is the equality of these ratios enough?

If the figures were made of jointed rods, could you alter the shape of the quadrilateral A'B'C'E'? Would any of the ratios $\frac{A'B'}{AB}$, $\frac{B'C'}{BC}$, etc. be altered? Would A'B'C'E' still be similar to (of the same shape as) ABCE*?

* See Notes and Answers.

We see that **for figures to be similar** two conditions must be satisfied:—

- (i) **the ratios of corresponding sides must be equal,**
- (ii) **corresponding angles must be equal.**

Take two different-sized maps of England. It is clear that the ratios of the lengths in the two maps that represent (i) London to Bristol, (ii) London to Liverpool, will be equal, also the angles Birmingham—London—Bristol in the two maps will be equal to one another.

Again, if a small photograph be taken of a large picture, the photograph is similar to the picture: the lengths are altered in a certain ratio but the angles are unaltered.

Also, a cinematograph projects on the screen a large picture similar in every respect to the small picture on the film.

¶Ex. 5. Give further instances* of practical processes in which similar figures occur.

Similar figures are not necessarily flat, i.e. in two dimensions. They may be solid, i.e. in three dimensions. A model of a ship is a solid figure similar to the ship.

¶Ex. 6. Give further instances of similar solid figures.

¶Ex. 7. Give instances of two similar curved figures, plane or solid.

¶Ex. 8. In fig. 188, (i) Are the angles equal? (ii) Measure the lengths of the sides in centimetres and calculate the ratios $\frac{A'B'}{AB}$, $\frac{B'C'}{BC}$, $\frac{C'A'}{CA}$

Are the triangles similar?

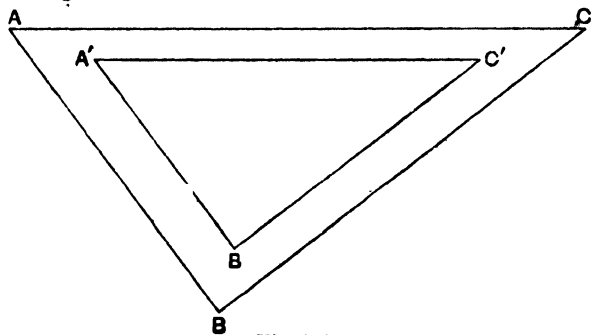


Fig. 188.

* See Notes and Answers.

Ex. 9. In fig. 189 the sides of $P'Q'R'S'T'$ are $\parallel$ to the sides of $PQRST$.
 (i) Show that corresponding angles of the two figures are equal. (ii) Measure the sides of the two figures and find the ratios of each pair of corresponding sides. Are $PQRST$ and $P'Q'R'S'T'$ similar?

Ex. 10. Are the following pairs of figures similar:—(i) a square and a rectangle, (ii) any two squares, (iii) any two rectangles, (iv) two equiangular parallelograms, (v) two equiangular rhombuses, (vi) two circles, (vii) any two regular pentagons?

Ex. 11. Draw any rectangle and four lines parallel to the sides one inch away to form another rectangle outside (or inside). Find whether the two rectangles are similar.

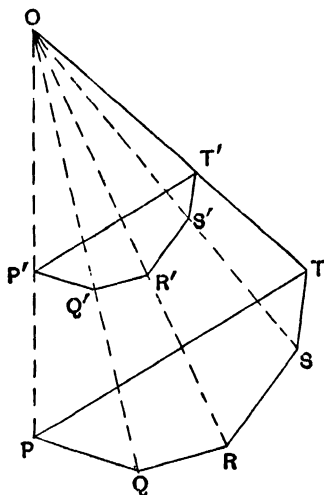


Fig. 189.

Ex. 12. The length, breadth and height of a shed are 20 ft., 12 ft., 8 ft. In a model the length is 4"; what is the breadth and height?

Ex. 13. Another model of the shed in Ex. 12 is 7" long, what will the other measurements be?

Ex. 14. In the ground floor plan of a building the actual length of which is 110 ft., the length is a line of 55 in. In the plan one room measures 22 in. by 18 in. What are the actual measurements of the room?

Ex. 15. A quadrilateral has sides 3.2", 2.5", 3.8", 4.1". A similar quadrilateral has the side of 3.2" represented by a length of 2". Find the lengths of the other sides.

Ex. 16. The height of a lantern slide is $3\frac{1}{4}$ in. and on the screen it gives a line 5 ft. 6 in. high. The heights of three trees on the slide are 1.6", 0.7", 0.5". What heights will they be on the screen?

AREAS OF SIMILAR FIGURES.

§ 4. **Ex. 17.** If the sides of a square are doubled, how is the area altered? Illustrate with a freehand sketch.

Ex. 18. If the sides of a square are trebled, how is the area altered? Illustrate with a freehand sketch.

Ex. 19. If the sides of a square are halved, how does the small square compare with the large? Illustrate with a freehand sketch.

Ex. 20. If each side of a rectangle is doubled, how does the area of the new figure compare with that of the old? Illustrate with a freehand sketch.

Consider any figure, rectilinear or curvilinear; let it be divided up into small equal squares for purposes of counting the number of

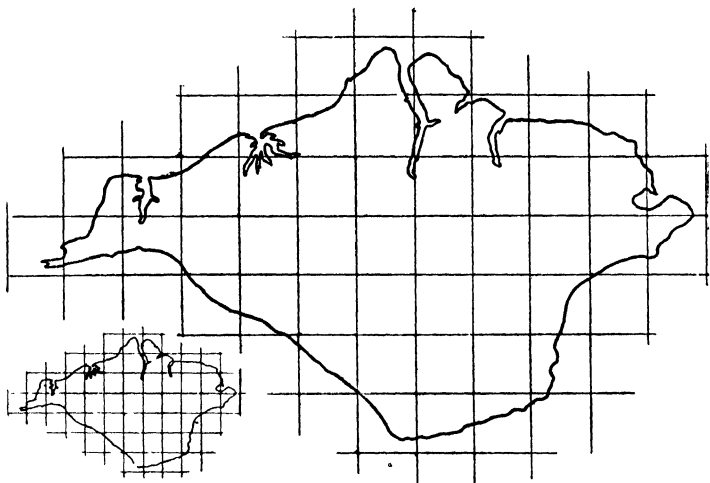


Fig. 190.

units of area. Now suppose that the figure, squares and all, is magnified, say 3 times; we shall have a figure similar to the former, the linear dimensions of the two being in the ratio 3 : 1. The area of each of the small squares will be magnified 9 times; and as the number of squares is unchanged, the area of the whole figure will be magnified 9 times. This suggests

The ratio of the areas of similar figures is equal to the square of the ratio of their linear dimensions.

This result will be clearer if you will remember that an area is the product of some *two* linear dimensions and that if two figures are similar *both* of these dimensions are increased or decreased in the same ratio. Consequently this ratio occurs twice in the product.

¶Ex. 21. Two similar triangles stand on bases of (i) 4 in. and 5 in., (ii) a in. and b in. What is the ratio of their areas?

¶Ex. 22. The area of a triangle on a base of 10 cm. is 30 sq. cm. find the area of a similar triangle on a base of 8 cm. What is the area of a similar triangle on a base of 8 in.?

Ex. 23. The areas of two similar figures are 169 sq. in., 81 sq. in. One side of the greater is 3 in.; find the corresponding side in the smaller.

Ex. 24. What is the ratio of the areas of two circles whose radii are (i) R, r ,
(ii) 3 in., 2 in.?

Ex. 25. A plan of a hall is drawn on a scale of 1 in. to 10 ft. What is the ratio of corresponding areas on the plan and in the hall? What area is represented by 2.5 sq. in. on the plan?

Ex. 26. A model of a monument is made on a scale of 2 in. to 5 ft. The whole surface area of the model is found to be 14 sq. in. What is the surface area of the monument?

Ex. 27. An exact model of a ship 450 ft. long is of length 10 in. The surface area below the water line of the model is 50 sq. in. What is the surface area exposed to water in the ship?

Ex. 28. Two similar windows are glazed with small lozenge-shaped panes of glass, these panes being all identical in size and shape. The heights of the windows are 10 ft. and 15 ft. The number of panes in the smaller window is 1200; what is the number in the larger?

Ex. 29. If it cost £5 to paint the outside of a hut whose length was 10 ft., what should it cost to paint in the same way the outside of an exactly similar hut of length 18 ft.?

Ex. 30. If it required 350 sq. ft. of planking to make a packing case of length 9 ft., what would be the length of a similar packing case that could be made from 250 sq. ft. of planking?

Ex. 31. The load that can be carried by an aeroplane travelling at a given velocity depends on the area of its wings. Experiments with a model show that it can carry a weight w lb.; the full-sized machine is 10 times the length of the model; what weight can the machine carry at the same velocity?

Ex. 32. Similar figures are described on the side and diagonal of a square; prove that the ratio of their areas is 1 : 2.

Ex. 33. Similar figures are described on the side and altitude of an equilateral triangle; prove that the ratio of their areas is 4 : 3.

Ex. 34. Show how to draw a straight line parallel to the base of a triangle so as to bisect the triangle.

VOLUMES OF SIMILAR SOLIDS.

§ 5. Ex. 35. What is the volume of a cube with edge 2 in.? How many times does it contain a cube with edge 1 in.?

Ex. 36. What is the volume of a cube with edge 3 in.? How many times does it contain a cube with edge 1 in.?

Ex. 37. If the edges of a cube are doubled, how is the volume altered?

Ex. 38. If the edges of a cube are trebled, how is the volume altered?

Ex. 39. If the edges of a cube are made n times as big, how is the volume altered?

¶Ex. 40. If the ratio of the edges of two cubes is 2:3, what is the ratio of their volumes?

¶Ex. 41. How is the volume of a rectangular solid changed

- (i) when its length only is doubled?
- (ii) when its length and breadth are doubled?
- (iii) when its length, breadth and height are doubled?

Are any of these solids similar to the original solid?

Consider any solid; its volume is a certain number of cubic units, which number depends on the length, breadth, and height of the solid. If we have a similar solid its length, breadth and height will each be altered in the same ratio. Hence if this ratio is $n:1$ the volume will be increased in the ratio of $n^3:1$, i.e. the new volume will be n^3 times the original volume. If the ratios of the lengths, breadths and heights are each $\frac{a}{b}$ then the volume will be $\frac{a^3}{b^3}$ times the original volume. This suggests

The ratio of the volumes of similar solids is equal to the cube of the ratio of their linear dimensions.

¶Ex. 42. What is the ratio of the volumes of two similar solids whose heights are 2 ft. and 3 ft.?

Ex. 43. What is the ratio of the volumes of two similar solids whose heights are 4 ft. and 7 ft.?

Ex. 44. Two similar solids have volumes of 125 cu. in. and 512 cu. in. A side of one is 3.5 in., what is the corresponding side of the other?

Ex. 45. The shape of a box is to be kept the same but its capacity is to be doubled. By what must each side be multiplied to get the new dimensions?

Ex. 46. Sand falling for 1 min. has formed a certain conical heap. After what additional time will the conical heap be twice as high?

Ex. 47. A measure of diameter $3\frac{1}{2}$ in. holds 1 pint. What is the diameter of an exactly similar measure which holds 1 gallon?

Ex. 48. A model of a dock is constructed on a scale of 1 in. to 50 ft. The model holds 1 pint of water; how many gallons of water could the dock hold?

Ex. 49. Two exactly similar cylinders have volumes of 350 c.c. and 500 c.c. Find the ratio of their diameters in the form 1 to n .

Ex. 50. An hotel possesses coffee pots of different sizes for serving coffee at breakfast to one, two or three visitors respectively. If the pot for one visitor is 5 inches high, how high should the other two be if all are exactly similar in shape?

SIMILAR TRIANGLES.

§6. In the case of two congruent triangles we know that corresponding angles are equal and also that corresponding sides are equal. But we also know that we need not be told all these six facts to be sure that the triangles are congruent.

If two triangles are similar, we know that they have their angles equal and their corresponding sides in the same ratio. But the question arises—can we argue that the triangles are similar if we are given less information than the above?

I. It will be remembered that to fix a triangle three data are needed and that three data are generally sufficient. But if the three data are *three angles* (adding up to 180° of course), then the triangle is not fixed. Why is this? It may be answered that two angles of a triangle give us the third, and to know three angles is really not more than to know two. But what do we really know if the three angles are given? We know the *shape*, but not the *size* of the triangle.

Draw any triangle freehand, and then draw, freehand, lines parallel to the sides; produce these parallels to form a second triangle. The two triangles are equiangular*; obviously they are of the same shape. If the base of one triangle is double that of the other, will the sides of the one triangle be doubles of the sides of the other?

In Ex. 8 we had two equiangular triangles and we found their corresponding sides in the same ratio. This suggests

THEOREM 54. If two triangles are equiangular, their corresponding sides are in the same ratio†.

In other words, if you can prove that two triangles have their angles equal, you may infer that the *corresponding* sides are in the same ratio. Thus if (fig. 191) $\angle A = \angle P$, $\angle B = \angle Q$, $\angle C = \angle R$, then

$$\frac{a}{p} = \frac{b}{q} = \frac{c}{r}.$$

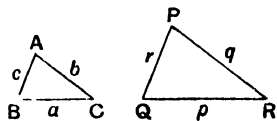


Fig. 191.

* (Note the two uses of the word “equiangular.” A triangle is equiangular when each of its angles is 60° . Two triangles are equiangular when the angles of the one are respectively equal to the angles of the other.

† The pupil has not yet been introduced to the word “proportional”; hence the use of the phrase “in the same ratio.” See note on p. 148.

II. Construct a triangle of sides $1''$, $1\frac{1}{2}''$, $2''$. Now construct a second triangle with sides double those of the former. The two triangles have their sides in the same ratio. Are they similar? Observe that no attention has been paid to the angles in the drawing.

A triangle must exist similar to the first and on twice the scale. It must have sides of $2''$, $3''$, $4''$.

But three sides fix a triangle, and this is the triangle you have drawn. This suggests

THEOREM 55. If two triangles have their corresponding sides in the same ratio, the triangles are equiangular.

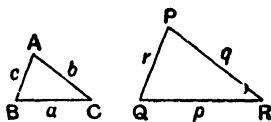


Fig. 192.

Thus, if in fig. 192

$$\frac{a}{p} = \frac{b}{q} = \frac{c}{r}, \text{ then } \angle A = \angle P, \angle B = \angle Q, \angle C = \angle R.$$

Here again is an evident property of triangles that is not true for polygons generally. The sides of the rhombus in fig. 193 are $\frac{2}{3}$ of the sides of the square; the sides of the two figures are proportional, but the figures are not similar.

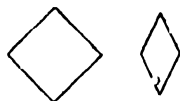


Fig. 193.

III*. Draw a triangle ABC with an angle A of say 40° , the sides containing the angle being called x in. and y in. Now make a triangle $A'B'C'$, having $\angle A' = \angle A$, the sides containing $\angle A'$ being $\frac{2}{3}x$ in. and $\frac{2}{3}y$ in.

Observe that nothing has been said about the base of $\triangle A'B'C'$ or the other two angles.

Now there must exist a triangle similar to $\triangle ABC$, and on a $\frac{2}{3}$ scale. And this similar triangle must have an angle of 40° and containing sides of $\frac{2}{3}x$ in., $\frac{2}{3}y$ in. But two sides and the included angle fix a triangle. The $\triangle A'B'C'$ must therefore be similar to $\triangle ABC$. This suggests

* This section may be postponed at a first reading.

THEOREM 56. If two triangles have one angle of the one equal to one angle of the other and the sides about these equal angles in the same ratio, the triangles are similar.

The action of **proportional compasses** depends on this theorem.

Fig. 194 represents a pair of proportional compasses. $AB = AC$ and $AH = AK$,

$$\therefore \frac{AH}{AB} = \frac{AK}{AC}, \text{ and } \angle HAK = \angle BAC,$$

$\therefore \triangle s AHK, ABC$ are similar.

Hence $\frac{HK}{BC} = \frac{AH}{AB}$, which is constant for any fixed position of the hinge. In fig. 194 the hinge is adjusted so that $\frac{AH}{AB} = \frac{1}{2}$; thus, whatever the angle to which the compasses are opened, $HK = \frac{1}{2} BC$.

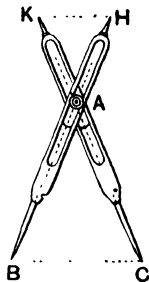


Fig. 194.

NUMERICAL EXAMPLES ON SIMILAR TRIANGLES.

Ex. 51. Test the accuracy of Th. 54 by drawing any two equiangular triangles ABC, PQR , measuring their sides and calculating the ratios $\frac{a}{p}, \frac{b}{q}, \frac{c}{r}$.

$\triangle ABC, PQR$ are two equiangular triangles; if $a = 4, b = 3, c = 6$ and $r = 8$, find p .

Draw a figure and mark in the given lengths.

Since $\triangle ABC$ is equiangular with $\triangle PQR$,

$$\therefore \frac{a}{p} = \frac{c}{r}.$$

$$\therefore \frac{4}{p} = \frac{6}{8}.$$

$\therefore$ Multiplying both sides by $8p$,

$$32 = 6p. \quad \therefore p = \frac{32}{6} = 5\frac{1}{3}.$$

Ex. 52. $\triangle ABC, PQR$ are equiangular triangles. $c = r$

(i) If $a = 4, b = 6, c = 8, p = 12$, find q, r .

(ii) If $a = 3, p = 7, q = 5, r = 9$, find b, c .

(iii) If $a = b = 7.5, c = 8.5, r = 10$, find p, q .

Ex. 53. Draw a triangle sides $2'', 3'', 4''$. Construct a triangle equiangular to it, having its shortest side $1.5''$, by making angles equal. Measure the remaining sides and verify by calculation.

Ex. 54. A triangle has sides 4 in., 5 in., 6 in. The smallest side of a similar triangle is 5 in.; what is the length of its longest side?

Ex. 55. The sides of a triangle are in the ratio of 4:5:7. The side of intermediate length is 6 in. long. What are the lengths of the other two sides?

Ex. 56. Are the triangles similar whose sides are

- | | |
|--------------------------------|--|
| (i) 3, 4, 5 and 4.5, 6, 7.5; | (ii) 4, 6, 8 and 4, 3, 2; |
| (iii) 2, 3, 4 and 3, 4, 5; | (iv) p, q, r and $(p+1), (q+1), (r+1)$; |
| (v) 9.1, 13, 6.5 and 10, 5, 7; | (vi) 6, 11, 7 and 2.8, 4.4, 2.4? |

Ex. 57. Give three different sets of numbers which could be used to construct triangles similar to a triangle whose sides are 3, 4, 5.

Ex. 58. The shadow of a flagstaff is 110 ft. long and at the same moment the shadow of a man 5 ft. 6 in. high is 12 ft. What is the height of the flagstaff?

Ex. 59. By a geometrical method draw (i) the triangle PQR given in fig. 26, and (ii) a copy of the triangle on such a scale that 1 cm. in the new figure represents 0.53 cm. in the given figure.

Tabulate the lengths of the sides and the size of the angles of both triangles.

¶Ex. 60. I, II, III above give three different methods of constructing a triangle similar to a given triangle and on a given scale. Discuss which method is preferable (i) if corresponding sides are to be parallel, (ii) if they are not to be parallel. Different conclusions may be reached according to whether (a) protractor, (b) proportional compasses may be used. Remember that measurement is always a source of error.

ABC is a triangle, DE parallel to AC cuts AB, BC in D, E; if AB = 20 cm., AC = 12 cm., BD = 8 cm., calculate DE.

In the two Δ s ABC, DBE, since DE is $\parallel$ to AC,

$$\angle BAC = \text{cor. } \angle BDE,$$

$$\angle ACB = \text{cor. } \angle DEB,$$

$$\angle B \text{ is common.}$$

$\therefore$ the Δ s are equiangular.

$$\therefore \frac{BA}{BD} = \frac{AC}{DE}.$$

$$\text{Let } DE = x \text{ cm. } \therefore \frac{20}{8} = \frac{12}{x}. \therefore 20x = 96. \therefore x = \frac{96}{20}.$$

$$\therefore DE = 4.8 \text{ cm.}$$

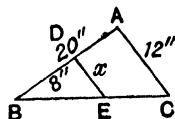


Fig. 195.

Ex. 61. In a triangle ABC, DE is drawn parallel to AC as in fig. 195.

- If BE = 12 cm., EC = 5 cm., ED = 4 cm., calculate AC.
- If BC represents 100 ft., CA 45 ft., ED 30 ft., calculate BE.

Ex. 62. A scaffold pole 40 ft. long rests on the ground leaning over a wall 6 ft. high. If the length of the pole from ground to wall is 10 ft., what height above the ground is the further end of the pole?

Ex. 63. A man whose eye is 6 ft. from the ground, when standing 14 ft. in front of a wall 12 ft. high, can just see the top of a tower which is 300 ft. away from the wall. Find the height of the tower.

Ex. 64. ABC is a triangular frame having $AB=14$ ft., $BC=10$ ft., $CA=7$ ft. A point P on AB 1.5 ft. from A is linked by a rod to a point Q on AC 3 ft. from A. Calculate the length of PQ.

Ex. 65. Two equal thin rods 15 cm. long are hinged together at a point 4 cm. from the end of each. The nearer ends just fit into a tube, touching its sides. The distance between the other ends is measured and found to be 6.5 cm. What is the diameter of the tube?

Ex. 66. Two similar rods 4' long have to be hinged together to support a table 4' 6" wide, the ends of the rods being at opposite edges of the table. The rods have to be fixed to two fastenings on the floor 2' 6" apart. Find the position of the hinge. [Let x ft. be distance from one end; form an equation for x .]

Ex. 67. In fig. 196, ABCD is a parallelogram and $\angle BOP$ is equal to $\angle BAD$.

(i) Name all the triangles similar to $\triangle ARP$.

(ii) Name two triangles similar to $\triangle BPO$ and write down the equal ratios of their sides.

(iii) Are the quadrilaterals ADOP, CBOQ similar?

Ex. 68. ABCD is a parallelogram. A line APQ meets DC in P and BC produced in Q. Name three similar triangles in this figure.

If $CP = \frac{1}{3} CD$ and $AB=6$ cm., $BC=10$ cm., $AQ=18$ cm., find CQ , PQ , AP .

Ex. 69. ABC is a triangle inscribed in a circle; KAM is the tangent at A; BK, CM are parallel to CA and BA respectively.

(i) Show that the three triangles KAB, BCA, $\triangle MC$ are equiangular.

(ii) If $BC=4''$, $CA=2''$, $AB=3''$, find AK, AM, BK, CM.

(iii) If $BC=a''$, $CA=b''$, $AB=c''$, find AK, AM, BK, CM, and show that

$$AK \cdot AM = BC^2 \quad \text{and} \quad BK \cdot CM = AB \cdot AC.$$

Ex. 70. A square board ABCD, side 4 ft., rests in a vertical plane with A on a table and B 2 ft. above the table. Calculate the height of C and D above the table.

Ex. 71. Through the vertex A of a rectangle ABCD of sides $AB=4$ in., $AD=3$ in., a line is drawn whose distance from B is 1 in. Calculate the distances of C and D from the line.

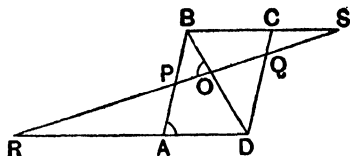


Fig. 196.

Ex. 72. A boy asserted that with only a piece of thread and squared paper he could easily cut off a half, a third, a fourth, or any other such part of a given line. His method is suggested in fig. 197. The horizontal line OQ is the line of which he has to find the required part, and OY, QZ are two lines running at right angles to OQ . Any convenient length is taken for the equal divisions OA, QB, BC, CD , etc., and he stretches the thread across from A to B, C , or D , etc., according as he is asked to cut off a half, a third, a fourth, etc., of the line OQ .

Test this method on inch paper, taking 3" for OQ and 1" each for OA, QB , etc. Give the result and justify the method, by geometrical reasoning, for the finding of a half and of a third of the given line. Also briefly give reasons for considering the method a general one.

Ex. 73. In fig. 198,

$$\angle C = 90^\circ = \angle BED = \angle END;$$

$$\angle EBD = \angle CBD,$$

$$AC = 9'', CD = 4'',$$

calculate ED, EA, AN, ND, BC .

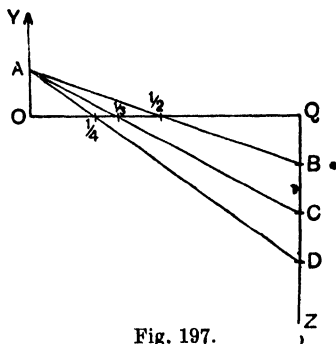


Fig. 197.

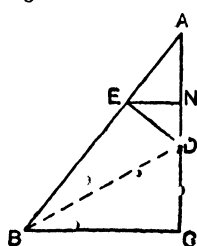


Fig. 198.

Further Exx. on the same principles will be found in Chap. 17, p. 150 and Chap. 18, p. 152.

SUMMARY OF RESULTS

If two triangles are equiangular, they are similar.

If two triangles have their sides in the same ratio, they are similar.

If two triangles have one angle of the one equal to one angle of the other and the sides about these equal angles in the same ratio, the triangles are similar.

The ratio of the areas of similar figures is equal to the square of the ratio of their linear dimensions.

The ratio of the volumes of similar solids is equal to the cube of the ratio of their linear dimensions.

CHAPTER 17

RATIO AND PROPORTION

DIVISION OF STRAIGHT LINES INTO ANY NUMBER OF EQUAL PARTS.

§ I. Ordinary ruled paper is ruled in parallel equidistant lines. If a slanting line be drawn, at any angle, cutting across these lines, it is evidently divided into equal parts.

Ex. 1. On the edge of a piece of paper, mark any length (say about 3 or 4 in., but do not measure it exactly); by laying this across the parallel lines of ruled paper, divide it into 7 equal parts. Mark off $\frac{3}{7}$ of the length.

Ex. 2. As in Ex. 1, cut off $\frac{4}{9}$ of a given line.

If you have no ruled paper or if you doubt the accuracy of the ruling, a set of equidistant parallels may be easily drawn as follows. Draw a line of convenient length; with compasses step off a number of equal lengths on it; through the points of division draw a set of parallel lines. This set of parallels may now be used for sub-dividing any other length (provided that it is not too short).



Fig. 199.

¶Ex. 3. What is the length of the shortest line that can be divided into 5 equal parts by means of the parallels in fig. 199?

If the line to be divided is already in position on paper, the construction is performed as follows:—

To divide a given straight line into five equal parts.

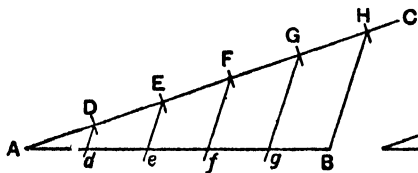


Fig. 200.

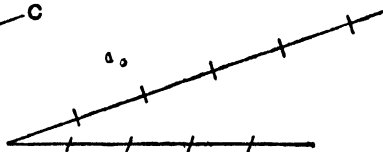


Fig. 201.

Let AB be the given straight line.

Construction Through A draw AC making any angle with AB.

With a pair of compasses, on AC mark off 5 equal parts AD, DE, EF, FG, GH.

Join BH. Through D, E, F, G draw st. lines $\parallel$ to BH.

Then **AB** is divided into 5 equal parts.

Note that the "auxiliary" line AC has *not* been sub-divided: all that has been done is to step off equal lengths along it.

In practice, it is not necessary to show the whole length of the parallels, but only the bits where they cross AB (fig. 201). Also the short arcs cutting AC should be shown.

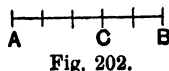
The principle underlying the above construction may be stated, as a theorem, thus:—

THEOREM 24. If there are three or more parallel straight lines, and the intercepts made by them on any one straight line that cuts them are equal, then the corresponding intercepts on any other straight line that cuts them are also equal.

RATIO OF SEGMENTS OF A LINE.

§ 2. If the line AB is divided into 5 equal parts and if AC contains 3 of those parts, we say that

$$AC = \frac{3}{5} AB, \quad \text{or} \quad \frac{AC}{AB} = \frac{3}{5}.$$



Again, if $AC = \frac{3}{8} AB$, it is clear that $CB = \frac{5}{8} AB$; then

$$AC : CB = 3 : 2.$$

To divide a given straight line in a given ratio.

Suppose that the given ratio is 3 : 7.

Call the given line AB . Divide AB into 10 equal parts. If AX contains 3 of those parts then XB contains 7 of them; and $AX : XB = 3 : 7$.

If the given ratio is $x:y$ where x and y are integers, we must divide the line into $x + y$ equal parts.

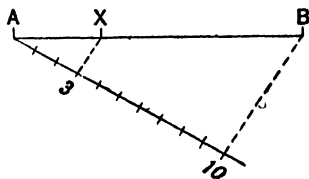


Fig. 203.

CONSTRUCTIONS.

Ex. 4. Trisect a given straight line by eye; check by making the construction.

Ex. 5. Divide a straight line of 4.5 in. into six equal parts; check.

Ex. 6. From a straight line 10 cm. long cut off a part equal to $\frac{2}{9}$ of the whole line.

Ex. 7. Divide a straight line of 13.8 cm. in the ratio of 3 : 5.

Ex. 8. From the vertex A of a triangle ABC cut off $AP = \frac{3}{8}AB$, $AQ = \frac{3}{8}AC$. Find, by measurement, the ratio of PQ : BC.

Ex. 9. Divide two adjacent sides of any triangle into four equal parts and join corresponding points. Find, by measurement, the ratio of each joining line to the base of the triangle.

Ex. 10. Divide any line AB into five equal parts and join each point of division, including A, B, to any point O outside the line. Across the joining lines draw two other lines, one parallel to AB and the other not parallel. What can you say about the parts of these lines?

CALCULATIONS.

Ex. 11. Draw a line 5 in. long. Divide it by eye in the ratio of 3 : 1. Now calculate the lengths of the segment and check by measurement.

Ex. 12. Repeat Ex. 11 for a line 8 cm., in the ratio 2 : 5.

Ex. 13. If a line l in. is divided in the ratio of $a : b$, what is the length of each segment?

Ex. 14. A line 12 in. long is divided into three segments in the ratio of 1 : 2 : 3. What is the length of each?

Ex. 15. A line AB is divided at P in the ratio of 5 : 3 and at Q in the ratio of 5 : 4. What fraction of AB is PQ?

Ex. 16. P was found on AB such that $\frac{AP}{PB} = \frac{3}{5}$ instead of $\frac{AP}{PB} = \frac{5}{3}$. What percentage error would result in the length of AP?

Ex. 17. The base of a triangle is 24 cm. and the sides are 12 cm. and 20 cm. A point divides the base in the ratio of the sides; how far is it from each end?

Ex. 18. The perimeter of a triangle is 20" and the sides are in the ratio of 3 : 2 : 4. What are the lengths of the sides?

Ex. 19. Two circles have radii 3.5 in. and 1.25 in. and their centres are 5.5 in. apart. How far from each centre is a point which divides the line joining their centres in the ratio of the radii, the larger segment being measured from the centre of the larger circle?

§3. In fig. 204 (which is part of fig. 200), XY is parallel to BH and it is clear that $\frac{AY}{AH} = \frac{2}{5} = \frac{AX}{AB}$; also $\frac{AY}{YH} = \frac{2}{3} = \frac{AX}{XB}$;

also
$$\frac{AH}{HY} = \frac{5}{3} = \frac{AB}{BX}.$$

In fact, the ratio of any two parts along AB is equal to the ratio of the corresponding parts along AH .

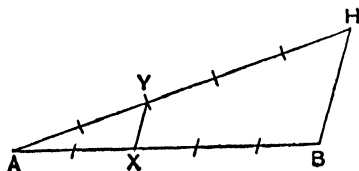


Fig. 204.

In fig. 204, X, Y coincide with e, E of fig. 200. It is clear that the ratios would still be equal if X, Y had coincided with any pair of corresponding points of division; it is also clear that this would be true if the lines, instead of being divided into 5 equal parts, had been divided into 7 (or any other number of) equal parts.

Hence we see that, when two lines (such as AB, AH) are divided (at X, Y) in the same ratio, the ratio of *any* pair of parts on the one line is equal to the ratio of the corresponding parts on the other. The lines are then said to be **divided proportionally**.

From the above we also infer

THEOREM 52. *If a line is drawn parallel to one side of a triangle, it divides the other two sides proportionally.*

The converse theorem is also true, namely

THEOREM 53. *If a line divides two sides of a triangle proportionally, it is parallel to the third side.*

Ex. 20. A variable line, drawn through a fixed point O , cuts two fixed parallel straight lines at P, Q ; prove that $OP : OQ$ is constant.

Ex. 21. O is a fixed point and P moves along a fixed line. OP is divided at Q in a fixed ratio. Find the locus of Q .

Ex. 22. Prove that a line drawn parallel to the parallel sides of a trapezium cuts the other sides proportionally.

NOTES ON EQUAL RATIOS.

1. It should be noted that, though we cannot have the ratio of two different kinds of quantity, we can deal with the ratio of two lengths measured in different units by changing to the same unit. For example the ratio

5 kilometres : 10 miles = $5 \times \frac{5}{8}$ miles : 10 miles (for 1 km. = $\frac{5}{8}$ m.); and we can test the equality of two ratios with mixed units as in the following example.

Is it true that 15 in. : 25 cm. = 9 in. : 15 cm.?

It is clear that 15 : 25 = 9 : 15, but 15 in. : 25 cm. is not equal to 15 : 25. Suppose 1 in. = x cm. Then 15 in. = $15x$ cm.; and the question is equivalent to "Is it true that $15x$ cm. : 25 cm. = $9x$ cm. : 15 cm.?" or "Is it true that $\frac{15x}{25} = \frac{9x}{15}$?" which is obviously true.

2. If two ratios $\frac{a}{b}$, $\frac{c}{d}$ are equal, then we have the equation $\frac{a}{b} = \frac{c}{d}$.

$\therefore$ multiplying both sides by bd , $ad = bc$.

The equality of the products is often a useful test of the equality of the ratios.

¶ Ex. 23. If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{a}{c} = \frac{b}{d}$. Verify by taking numerical values for a , b , c , d .

Ex. 24. Find x if

(i) $\frac{3}{5} = \frac{x}{20}$, (ii) $6'' : 13'' = 5 \text{ yd.} : x \text{ yd.}$, (iii) $30 \text{ cm.} : x = 25 \text{ cm.} : 45 \text{ in.}$

Ex. 25. Are the following true?

(i) $13 \text{ cm.} : 0.7 \text{ in.} = 32.5 \text{ cm.} : 1.75 \text{ in.}$, (ii) $\frac{315}{276} = \frac{4.2}{3.68}$

(iii) $\frac{41}{90} = \frac{42}{91}$, (iv) $\frac{a}{b} = \frac{a+1}{b+1}$, (v) $\frac{a}{b} = \frac{ma}{mb}$

§ 4. If $a : b = c : d$, the four magnitudes a , b , c , d are said to be **in proportion***

If x is such a magnitude that $\frac{a}{b} = \frac{c}{x}$ (or $a : b = c : x$), x is called the **fourth proportional** to the three magnitudes a , b , c .

If x is such a magnitude that $\frac{a}{b} = \frac{b}{x}$, then x is called a **third proportional** to a , b .

* If a , b , c , p , q , r are such that $\frac{a}{p} = \frac{b}{q} = \frac{c}{r}$, then a , b , c are said to be proportional to p , q , r . Thus, instead of saying "corresponding sides of two triangles are in the same ratio," we might say "the sides of two triangles are proportional." See Theorems 54, 55, 56.

To find the fourth proportional to three given straight lines.

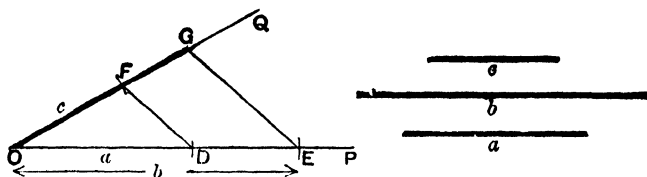


Fig. 205.

Let a, b, c be the three given straight lines.

Construction. Make an angle POQ . From OP cut off $OD = a$, and $OE = b$. From OQ cut off $OF = c$. Join DF . Through E draw $EG \parallel$ to DF , cutting OQ in G .

Then OG , by Theorem 52, is the fourth proportional to a, b, c .

Ex. 26. Find graphically the fourth proportional to 3, 4, 5. Check by calculation.

Ex. 27. Find graphically a fourth proportional to 4.5, 3, 2. Check by calculation.

Ex. 28. Find graphically a third proportional to 5, 8. Check by calculation.

Ex. 29. Find the value of x when $\frac{2.5}{4.2} = \frac{x}{3.7}$ (i) graphically, (ii) by calculation.

[Invert each and find a 4th proportional to 4.2, 2.5, 3.7.]

Ex. 30. Find, both graphically and by calculation, the value of x in the following cases:—

$$\begin{array}{ll} \text{(i)} \quad \frac{2.25}{3} = \frac{3}{x}, & \text{(ii)} \quad \frac{0.75}{x} = \frac{1.25}{5.75}, \\ \text{(iii)} \quad x : 2.6 = 5.5 : 2.9, & \text{(iv)} \quad 8.4 : 0.5 = x : 0.3. \end{array}$$

Ex. 31. If $\frac{y}{x} = \frac{5}{3}$ find x , when $y = 7$.

Ex. 32. A rectangle with one side 3 in. is equal in area to a square on a side of 4 in. Find graphically the other side of the rectangle.

Ex. 33. Construct a rectangle on a side of 5 cm. equal in area to a rectangle with sides 4.3 cm. and 9.4 cm.

Ex. 34. Solve graphically the problem:—if 6 acres produce 20 cwt., what should 14 acres produce?

Ex. 35. Solve graphically:—if 8 kilometres equal 5 miles, how many miles are there in 19 kilometres?

PROPORTIONAL DIVISION AND SIMILAR TRIANGLES.

The following examples can all be solved by similar triangles; but the work is often shortened by using Theorem 52 and that theorem should be used when possible.

It may be noted that in *Theoretical Geometry* the proofs of the theorems about similar triangles are made to depend on Theorem 52, which is proved first.

ABC is a triangle with AB = 3", AC = 2.5", BC = 4". If AF = 2.1" and FE is drawn parallel to BC meeting AC in E, calculate the length of AE.

$$\begin{aligned} \text{Since FE is } \parallel \text{ to BC, } \quad \frac{AF}{AB} &= \frac{AE}{AC}, \\ \therefore \frac{2.1}{3} &= \frac{AE}{2.5}. \end{aligned}$$

Multiplying both sides by 3×2.5 ,

$$2.5 \times 2.1 = 3 \times AE,$$

$$\therefore AE = \frac{2.5 \times 2.1}{3} = 1.75".$$

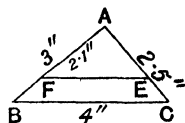


Fig. 206.

Ex. 36. *ABC is a triangle with BC = 9 cm., CA = 8 cm., AB = 12 cm., and FE parallel to BC meeting AB, AC in F, E;*

- (i) *if BF = 5 cm., find EC;*
- (ii) *if AF = $\frac{3}{5}$ AB, find EC;*
- (iii) *if AF : FB = 3 : 5, find AF, AE;*
- (iv) *if CE = $\frac{1}{4}$ CA, find AF;*
- (v) *if CE : EA = 2 : 3, find AE, AF.*

Ex. 37. *ABC is a triangle with FE parallel to BC as in fig. 206. AB = 3.75", AC = 2.8", AF = 1.5". Calculate AE, EC.*

Ex. 38. *ABC is a triangle with AB = 8.2", AC = 6.4". AB is produced to E so that BE = 2". EF parallel to BC meets AC produced in F. Calculate CF.*

Ex. 39. *ABC is a triangle, FE $\parallel$ to BC meets AB, AC (or AB, AC produced) in F, E respectively. AB = 15", BC = 20", CA = 12".*

- (i) *Find AE and FE when AF = 10".*
- (ii) *If CE : EA = 3 : 5, find EF.*
- (iii) *If EF = 14, find AF.*
- (iv) *If BC : EF = 7 : 3, find AE.*
- (v) *If AF = 20", find FE and EC.*
- (vi) *If FE = 30", find AF and AE.*

Ex. 40. ABC is a triangle having $BC=10$ cm., $CA=5$ cm., $AB=8$ cm. BA, CA are produced and a parallel to BC meets them in X, Y respectively.

- (i) If $AX=3$ cm., find AY , XY .
- (ii) If $XY=8$ cm., find AX , AY .

Ex. 41. ABCD is a quadrilateral with $AB \parallel$ to CD . The diagonals AC, BD meet in O.

- (i) If $CD=2AB$, prove O is a point of trisection of each diagonal.
- (ii) If $AB=5$ ft., $CD=15$ ft., prove $AO=\frac{1}{4}AC$.
- (iii) If $AB=8$ in. and $AO:OC=2:3$, find CD .
- (iv) If $AO:OC=5:2$ and $BD=3\cdot5''$, find OB , OD .

Ex. 42. P, Q, R are three points on a line. PA, QB, RC are perpendiculars to another line ABC, and are of lengths p , q , r respectively.

- (i) If Q is the mid-point of PR, prove $q = \frac{1}{2}(p+r)$.
- (ii) If $PQ:QR=1:2$, prove $q = \frac{1}{3}(r+2p)$.
- (iii) If $PQ:QR=5:3$, prove $q = \frac{1}{8}(5r+3p)$.
- (iv) If $PQ:QR=l:m$, prove $q = \frac{lr+mp}{l+m}$.
- (v) If $PR=4''$, $PQ=2\cdot5''$, $p=1''$, $r=5''$, calculate q .

[Join PC.]

Further Exx. will be found in Chapter 18.

SUMMARY OF RESULTS

If there are three or more parallel straight lines, and the intercepts made by them on any one straight line that cuts them are equal, then the corresponding intercepts on any other straight line that cuts them are also equal.

If a line is drawn parallel to one side of a triangle, it divides the other two sides proportionally.

If a line divides two sides of a triangle proportionally, it is parallel to the third side.

CHAPTER 18

APPLICATIONS OF SIMILAR TRIANGLES

§1. Ex. 1. OQPR is a rectangle with OQ = 4 in., OR = 3 in.; APB is a straight line cutting OQ, OR, both produced, at A, B; QA = x in., RB = y in. Show that the area of the rectangle OQPR = xy sq. in. Hence show that the area of the triangle OAB = $\frac{3}{2}x + \frac{24}{x} + 12$ sq. in.

Ex. 2. ABC is a triangle right-angled at C, and CD is drawn perpendicular to AB. Given BC = 12 cm. and AC = 5 cm., calculate AB. Specify the triangles in the figure that are similar, and hence calculate the lengths of BD, DA, CD, clearly indicating your method and reasons.

Ex. 3. B is a point visible but inaccessible from another point A, from which it is required to find its distance. A point C is taken 135 yards from A and such that CA is at right angles to AB. Along BA produced E is taken, 80 yards from A. From E a line is drawn at right angles to EB, and a point F is taken on it such that F, C, B are in a straight line. It is found that EF is 150 yards long. Calculate the distance AB and state clearly the geometrical theorem upon which your working depends.

Ex. 4. A straight rod AB 3 ft. 9 in. long is held under water, A being 2 ft. 6 in. and B 9 in. below the surface. Calculate the distance below the surface of a point C on the rod which is 12 in. from A.

Ex. 5. Two circles have equiangular triangles inscribed in them. Show that the ratio of a pair of corresponding sides is equal to the ratio of the radii. [Join the ends of corresponding sides to the centre.]

Ex. 6. (Freehand.) From A go 6 miles E. to B, from B 4 miles N. to C, from C 3 miles E. to D and from D 2 miles N. to E. Are A, C, E in one straight line? Give reasons.

Ex. 7. A plank is placed straight up stairs which are 10 in. wide and 6 in. deep. What is the greatest width of a rectangular box 4 in. deep which can be placed flat on a stair under the plank?

Ex. 8. Determine the length of the longest rectangular strip of width 6" which can be cut from a triangular piece of paper measuring 20", 20", 30" along its edges.

Ex. 9. Find the length of the shadow of a man 6 ft. high standing 20 ft. from the foot of a lamp-post, the light on which is 15 ft. above the ground. If he walks 10 ft. nearer the lamp-post, what is now the length of his shadow?

Ex. 10. A point source of light is placed 8 in. in front of a vertical cylinder of radius 5 in., the distance being measured from the nearest point of the cylinder. What is the width of the shadow 23 in. behind the centre of the cylinder?

Ex. 11. The section of a railway cutting is an isosceles trapezium 100 ft. wide at the top and 30 ft. at the bottom, and its vertical depth is 20 ft. If the sides of the section were produced to form a triangle, what would be the depth of its vertex below the bed of the embankment?

Ex. 12. The sides of a triangular frame are 3 ft., 4 ft., 5 ft. What are the lengths of the sides in a photograph in which the angular points just fall on the edges of a circular mount of 2" radius?

Ex. 13. A, B, C are three landmarks on a map in a straight line. A is on the 100 ft. contour, B on the 200 ft. and C on the 500 ft. AB is 1 mile and BC 3 miles measured from the map. Determine whether A is visible from C.

Ex. 14. A, B, C are three landmarks in a straight line down a slope, A on the 300 ft. contour, B on the 200 ft., and C on the 150 ft. Measured on the map AB is 500 yd. and BC 250 yd. Determine whether the slope ABC is uniform.

Ex. 15. A straight funicular railway of uniform gradient has to pass by three points A, B, C respectively 50 ft., 150 ft., 350 ft. above sea-level. The distances between the points on a map are 140 yd. and 300 yd. respectively. Is it necessary to excavate or make an embankment at C and to what depth?

Ex. 16. A triangle has base 3 in. and height 2.8 in. Find the areas of the two portions into which the triangle is divided by a line parallel to the base and 1.2 in. away from it.

Ex. 17. Regular hexagons are inscribed in and circumscribed about the same circle. Find the ratio of their areas.

Ex. 18. Prove the validity of the following method of solving a quadratic equation graphically:—

Suppose that $ax^2 + bx + c = 0$ is the equation; on squared paper, mark the origin, from OX cut off $OP = a$, from P draw a perpendicular PQ upwards of length b , from Q draw to the left $QR = c$ (regard must be paid to the signs of a, b, c ; e.g. if b is negative PQ will be drawn downwards); on OR describe a semicircle cutting PQ at S, T; the roots of the equation are $-\frac{PS}{OP}$ and $-\frac{PT}{OP}$ [Consider $\triangle$ s OPS, SQR.]

Ex. 19. Solve the following equations graphically as in Ex. 18, and check by calculation:—(i) $2x^2 + 5x + 1 = 0$, (ii) $x^2 + 3x - 2 = 0$, (iii) $2x^2 - x + 1 = 0$.

THE TRIGONOMETRICAL RATIOS.

§2. ¶Ex. 20. If P is any point on either arm of an angle XOY, and PN is drawn perpendicular to the other arm, $\frac{PN}{OP}$ has the same value for all positions of P.

[Take several different positions of P and prove that $\frac{PN}{OP} = \frac{P_1N_1}{OP_1} = \dots$]

$\frac{PN}{OP}$ is the **sine** of $\angle XOY$; this exercise might have been stated as follows:—the sine of an angle depends only on the magnitude of the angle.

¶Ex. 21. Prove that the **cosine** $\left(\frac{ON}{OP}\right)$ and **tangent** $\left(\frac{PN}{ON}\right)$ of an angle depend only on the magnitude of the angle.

THE DIAGONAL SCALE.

§3. The diagonal scale (fig. 207) depends in principle on the properties of similar triangles.

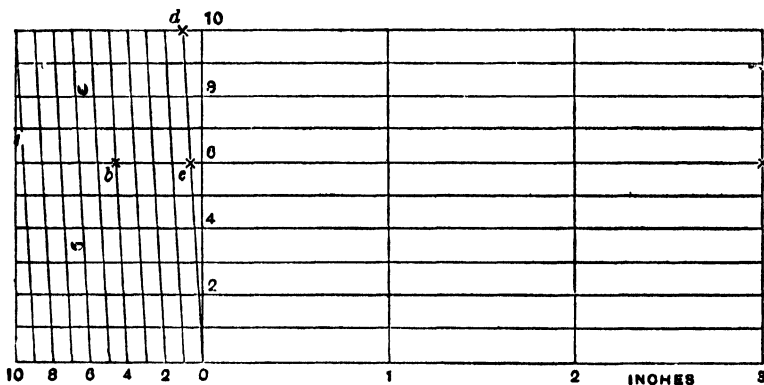


Fig. 207.

¶Ex. 22. Are the triangles whose corners are marked 0, d , 10 and 0, c , 6 equiangular?

¶Ex. 23. What fraction is the distance between the points 6, c of the distance between 10, d ?

The distance between 10, d is 0.1 in.; what is the distance between 6, c ?

¶Ex. 24. What are the distances between the points (i) a , 6, (ii) 6, c , (iii) c , b ? What is the whole distance between a , b ?

LENSES.

§4. Ex. 25. FCO is the axis of an optical system, L a convex lens, and AO an object. Rays from AO parallel to the axis pass through F after passing through the lens; and rays from AO through C, the centre of the lens, pass on unaltered in direction.

Make a freehand sketch of fig. 208 and show the path of two rays from A, one parallel to the axis and the other through C; these rays after passing through the lens intersect at A' , the image of A; let O' be the image of O and assume that AO and $A'O'$ are both perpendicular to the axis

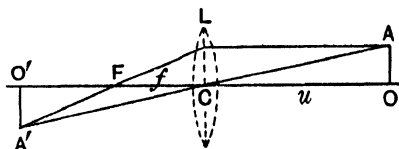


Fig. 208.

If $CF=f$ and $CO=u$, find (a) v , the distance CO' , (b) the size of the image in the following cases :—

- (i) $u=20$ cm., $f=10$ cm., $AO=1$ cm.;
- (ii) $u=30$ cm., $f=10$ cm., $AO=1$ cm.;
- (iii) $u=5$ cm., $f=10$ cm., $AO=1$ cm. (the image is virtual).

What happens when $u=10$ cm. and $f=10$ cm.?

Ex. 26. If the lens L , in Ex. 25, is changed to a concave lens, the point F is now on the same side of the lens as the object and the rays of light follow the same laws as in the case of the convex lens (except that the path of the rays parallel to the axis do not pass through F after passing through the lens, but their paths produced backwards pass through F).

Find, as in Ex. 25, (a) the distances of the virtual images from C , (b) the size of the image in the following cases :—

- (i) $u=20$ cm., $f=10$ cm., $AO=1$ cm.;
- (ii) $u=5$ cm., $f=10$ cm., $AO=1$ cm.

THREE DIMENSIONS.

§ 5. ¶Ex. 27. Any section of a right pyramid by a plane parallel to the base is a figure similar to the base.

A right pyramid stands on a base 10 ft. square and is 15 ft. in height. It is cut by a plane 10 ft. above the base and parallel to the base. Calculate the sides of the figure formed.

$PQRS$ is the plane cutting the pyramid.

$OV=15'$, $ON=10'$. $\therefore NV=5'$.

$PQRS$ is a figure similar to $ABCD$, that

is, a square. Let x' be its side.

The $\triangle s$ VNP and VOA are similar,

$$\therefore \frac{AO}{PN} = \frac{OV}{NV} = \frac{15}{5}.$$

The $\triangle s$ PNQ and AOB are similar,

$$\therefore \frac{AO}{PN} = \frac{AB}{PQ} = \frac{10}{x}.$$

Hence
$$\frac{15}{5} = \frac{10}{x}.$$

$$\therefore x = \frac{10}{3}.$$

$$\therefore \text{side of section} = 3\frac{1}{3} \text{ ft.}$$

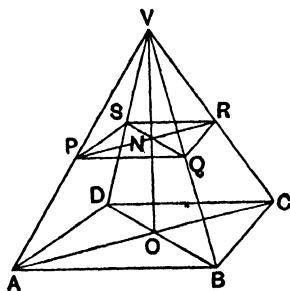


Fig. 209.

¶Ex. 28. Show that the areas of parallel sections of a right pyramid are proportional to the squares of their distances from the vertex.

Ex. 29. The base of a right pyramid is a square of side 12 cm. and a section parallel to the base and 8 cm. from it has a side of 5 cm. Find the height of the whole pyramid.

[Use x and form an equation.]

Ex. 30. The height of a cone is 20" and the base radius 8". Calculate the radius of a right section 7" from the vertex.

Ex. 31. A point is taken dividing a slant height of a cone in the ratio of 2 : 3, measured from the vertex. Through this point a section is cut parallel to the base. If the radius of the base is 18", what is the area of the section?

Ex. 32. In a model of a pyramid one face is a triangle on a base of 2 in with equal sides of 14.5 in. The actual length of the base-line is 8' 6". What is the actual length of the slant sides?

Ex. 33. A triangular wooden pyramid each of whose edges is 5 ft., floats in water base downwards, with one-quarter of its height immersed. What is the length of the water line on each face? Compare the volume of the portion not immersed with the whole volume of the pyramid.

Ex. 34. Light radiates from a point within a box through three small holes at the corners of a triangle of sides 3", 3", 4" and forms three spots of light on a vertical wall, the greatest distance between two of them being 5 ft. What are the distances of the other spot from these two?

Ex. 35. A pinhole camera consists of a light proof rectangular box with a minute hole pierced in one face, through which rays of light enter and form an image on the opposite inside face of the box. The distance between these two faces is 2' 6". What is the length of the image of a line of length 10" parallel to the faces and 6" from the pierced face?

Find also the dimensions of the image of an isosceles triangle with sides 3' 6", 2', 2', placed with its plane parallel to the faces and 10' away.

§ 6. RECTANGLE PROPERTIES OF CIRCLES.

§ 6. ¶Ex. 36. Draw a circle and a diameter AOB; in OB mark a point P. Now rotate AB about P; note that at first PB increases and PA decreases. What will happen to the product PA . PB? Take two positions of the chord and compare the products.

¶Ex. 37. Repeat Ex. 36 for the case in which P is in AB produced.

Ex. 38. AB, CD, two chords of a circle, intersect at P inside the circle. (See fig. 210.)

(i) Show that the triangles ADP, CBP are equiangular.

(ii) Write down two equal ratios of lengths measured from P.

Ex. 38 will prove to you

THEOREM 59 (i). If AB, CD, two chords of a circle, intersect at a point P inside the circle, then $PA \cdot PB = PC \cdot PD$.

Suppose that EPF is the chord which is bisected at P. O is the centre.

Then $PA \cdot PB = PE \cdot PF = PE^2 = OE^2 - OP^2$. (Why?)

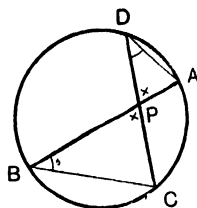


Fig. 210.

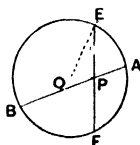


Fig. 211.

Ex. 39. BA, DC, two chords of a circle, intersect at P outside the circle.

- (i) Show that the triangles ADP, CBP are equiangular.
- (ii) Write down two equal ratios of lengths measured from P.

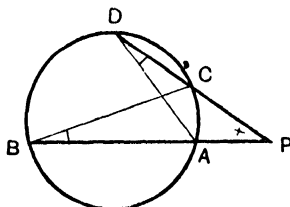


Fig. 212.

Ex. 39 will prove to you

THEOREM 59 (ii). If AB, CD, two chords of a circle, intersect at P outside the circle, then $PA \cdot PB = PC \cdot PD$.

PARTICULAR CASE. If PT is a tangent to a circle and AB a chord of the circle passing through P, then $PT^2 = PA \cdot PB$. (See fig. 213.)

NOTE. The above theorems are really two different cases of the same theorem.

To calculate the value of $PA \cdot PB$ in fig. 213.

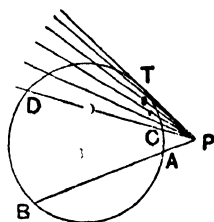


Fig. 213.

Use the fact that $PA \cdot PB = PT^2 = OP^2 - OT^2$, where O is the centre of the circle.

NOTE. The product $PA \cdot PB$ (or $PC \cdot PD$) mentioned in the above theorems is the measure of the area of a rectangle whose sides are equal to PA and PB (or PC and PD). Euclid enunciated these theorems in the following form:—"the rectangle contained by the segments of the one chord is equal to the rectangle contained by the segments of the other."

The pupil must beware of muddling $PA \cdot PB$ (which means $PA \times PB$) and $PA : PB$ (which means $PA \div PB$).

¶Ex. 40. Prove the particular case (i) by the method of limits, (ii) by similar triangles.

¶Ex. 41. What becomes of the above theorems when P is a point on the circle? When P is the centre?

Ex. 42. Calculate the areas of the rectangles contained by the segments of chords passing through P when (i) $r = 5$ in., $OP = 3$ in., (ii) $r = 5$ cm., $OP = 13$ cm.

Ex. 43. Find an expression for the areas in Ex. 42, r being the radius, and d the distance OP (i) when $d < r$, (ii) when $d > r$.

Ex. 44. Draw two straight lines APB, CPD intersecting at P; make PA = 4 cm., PB = 6 cm., PC = 3 cm. Describe a circle through ABC, cutting CP produced in D. Calculate PD. (*Freehand.*)

What would be the result if the exercise were repeated with the same lengths, but a different angle between APB, CPD?

Ex. 45. From a point P draw two straight lines PAB, PC; make PA = 4 cm., PB = 9 cm., PC = 6 cm. Describe a circle through ABC; let it cut PC again at D. Calculate PD. (*Freehand.*)

Ex. 46. Use Th. 59 to find a fourth proportional to 2.5, 1.3, 2.1.

Ex. 47. From a point O draw two straight lines OAB, OPQ in such a way that the angles OAP, OQB may be equal, and that the length of OA is 3.3 cm., of OB 5.6 cm., and of OQ 7.7 cm. State exactly the steps you take to secure a suitable figure. Find the length of OP by calculation, and verify by measurement.

Ex. 48. Circles touch the axis of x in a fixed point. Show that they cut the axis of y in points whose distance from the origin are inversely proportional.

Ex. 49. The span of a circular arch is $2a$ and the height h . Find the radius of the circle in terms of a and h .

Ex. 50. AB is a diameter of a circle radius r and PN is a perpendicular to it from any point P on the circle; if PN = y , AN = x , show that $x^2 + y^2 = 2rx$. If $y = 2.2''$ and $x = .05''$, find the error in taking $r = \frac{y^2}{2x}$ instead of its real value.

Ex. 51. A weight at the end of a string 1 metre long swings through a horizontal distance of 20 cm. each side the vertical. Calculate the vertical height through which it rises.

Ex. 52. A brick 4" thick is placed against a wheel to act as a stop. The face of the brick is 12" from the point where the wheel touches the ground. Find the radius of the wheel.

Ex. 53. OT is the distance of the horizon visible from a point O, h feet above the surface of the earth (radius 4000 miles).

Prove that OT is approximately $\sqrt{\frac{3h}{2}}$ miles. From what height can one see a mile out to sea? 4 miles? 9 miles? Show that the height of the observer's eye above sea-level varies as the square of the distance of the horizon.

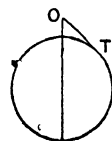


Fig. 214.

Ex. 54. An observer is on top of a mountain, 2640 ft. high, on the edge of the sea. Is it possible for him to see anything of a mountain 1320 ft. high on an island 100 miles away? If so, what approximate height is visible?

Ex. 55. Fig. 215 represents a hand-cart; the member OBC supports the body and is connected with the axle at D. The diameter of the wheels is 3 ft.; BC is 2 ft.; and the shafts rest on the ground at O, 6 ft. from A. Form an equation for OB (x ft.) and solve this equation.

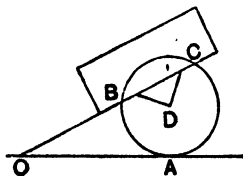


Fig. 215.

MEAN PROPORTIONAL.

§7. If x is such a quantity that $a : x = x : b$, then x is called the **mean proportional**, or the Geometric Mean, between a and b .

If $\frac{a}{x} = \frac{x}{b}$, $x^2 = ab$, and therefore $x = \sqrt{ab}$; thus the mean proportional between two numbers is the square root of the product.

If ABC is a triangle right-angled at A and AD is an altitude of the triangle, $\triangle s$ DBA , DAC are both equiangular with $\triangle ABC$. Hence

- (i) $AD^2 = BD \cdot DC$, (ii) $BA^2 = BD \cdot BC$,
(iii) $CA^2 = CD \cdot CB$.

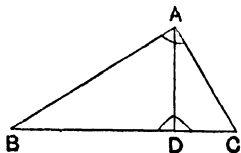


Fig. 216.

Ex. 56. Prove the triangles equiangular and write down the equal ratios. Hence prove (i), (ii), (iii) above.

To find the mean proportional between two given straight lines.

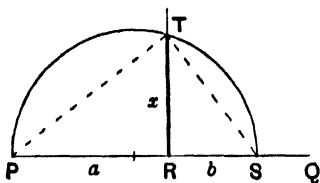
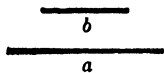


Fig. 217.



Let a, b be the two given straight lines.

Construction Draw a straight line PQ .

From PQ cut off $PR = a$, and $RS = b$.

On PS as diameter describe a semicircle.

Through R draw $RT \perp$ to PS to cut the semicircle at T .

Then $RT (x)$ is the mean proportional between a, b .

Proof The $\angle PTS$ in a semicircle is a right angle

Hence $TR^2 = PR \cdot RS$,

i.e. $x^2 = ab$.

$\therefore x$ is the mean proportional between a and b .

Notice that this construction solves the problem of constructing a square equal to a given rectangle. It also gives a means of finding geometrically $\sqrt{N}$ where N is any number, for if $b = 1$ and $a = N$ then x is $\sqrt{N}$.

¶Ex. 57. Find by calculation the mean proportional between

- (i) 4 and 9, (ii) 1 and 100, (iii) $\frac{1}{2}$ and 2,
(iv) $\frac{3}{4}$ and $\frac{4}{3}$, (v) 1 and 2, (vi) 2 and 4.

Ex. 58. Find by construction and measurement a mean proportional to 3.5 and 2.8.

Ex. 59. Find by construction the mean proportional to two lengths a , b both measured from the same point in the same sense.

[Use Ex. 56 (ii).]

Ex. 60. Construct a square equal in area to a rectangle whose sides are 2.7 in., 1.9 in. What is the length of a side of the square?

Ex. 61. Find by geometrical construction the values of

- (i) $\sqrt{2}$, (ii) $\sqrt{5}$, (iii) $\sqrt{35}$.

Ex. 62. Construct a square of area 15 sq. cm. Give reasons for your construction.

Ex. 63. Draw a straight line AB of length 7.5 cm.; in it, find by construction a point P such that the area of the rectangle AP.PB may be equal to the area of a square whose side is 3.2 cm. Measure AP.

Ex. 64. Construct a rectangle of perimeter 10" equal to a square of side 2.2".

Ex. 65. Divide a line 2 inches long into two parts which will contain a rectangle of area equal to a quarter of a square inch. Compare the lengths of the parts, as found by construction and measurement, with those found by calculation.

Ex. 66. Solve graphically the equations

- (i) $x + y = 6$, $xy = 4$, (ii) $x^2 - 3x + 1 = 0$.

To construct a square equivalent to any rectilinear figure.

- (i) Reduce the figure to an equivalent triangle.
- (ii) Transform the triangle into any equivalent rectangle.
- (iii) Construct a square equal to the rectangle.

Ex. 67. (On inch paper.) Find the side of the square equivalent to the triangles whose angular points are (1, 0), (5, 0), (4, 3).

Ex. 68. Construct a square equivalent to a regular hexagon of side 2 in.; measure the side of the square.

[An equivalent rectangle may be obtained without reducing the figure to a triangle.]

Ex. 69. Draw any equilateral triangle and make by a geometrical construction, *without any calculation*, a square whose area is that of the triangle. Now construct an equilateral triangle whose area is 4 square inches. Measure the length of a side of the triangle.

To construct a circle to pass through two given points and to touch a given line.

1st Method. Let A, B be the given points and let BA meet the given line XY in P .

Suppose the circle touches the line at T .

Then by Th. 59 (ii) $PT^2 = PA \cdot PB$.

Hence find PT , a mean proportional to PA, PB .

Draw the circle through A, B, T .

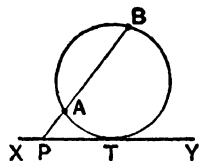


Fig. 218.

2nd Method. Draw any circle through A, B to cut XY in C, D . From P , the point where AB produced cuts XY , draw a tangent PK to this circle; with centre P and radius PK , draw a circle cutting XY in T, T' . Then the required circle touches XY at T or T' .

Ex. 70. Justify the second method of construction above.

Ex. 71. A circle through two points A, B touches a line at T . Show that the angle ATB is greater than any other angle subtended by AB at any point on the line.

[Join Q , any other point on this line, to A, B . Let QB or QA cut the circle in C . Join CA or CB .]

Ex. 72. The centre arch of a bridge over a river is 100 ft. wide and each end of it is 50 ft. from the banks of the river. Find the distance from the bridge of that point on either bank at which the centre arch appears the widest.

Ex. 73. Two blocks of height 3" and 1" at a distance apart of $5\frac{1}{2}$ " lie on a table. Find the radius of the largest circular hoop which can rest between the blocks and just touch the table.

CONSTRUCTION OF SIMILAR POLYGONS.

§8. A polygon may be divided by diagonals into triangles; and if two polygons are similar, the corresponding triangles must be similar.

On a given straight line to construct a figure similar to a given rectilinear figure.

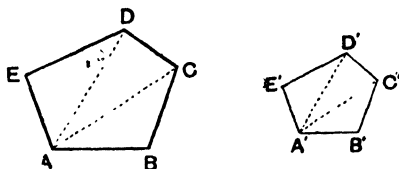


Fig. 219.

Let $ABCDE$ be the given figure and $A'B'$ the given straight line.

Construction

Join AC, AD.

On A'B' make $\triangle A'B'C'$ equiangular to $\triangle ABC$.On A'C' make $\triangle A'C'D'$ equiangular to $\triangle ACD$.On A'D' make $\triangle A'D'E'$ equiangular to $\triangle ADE$.

Then A'B'C'D'E' is similar to ABCDE.

Note. If A'B' is parallel to AB as in fig. 219, the whole figure may be constructed by drawing parallels: otherwise, some other method must be used to make the triangles equiangular.

§9. Mechanical description of similar figures.

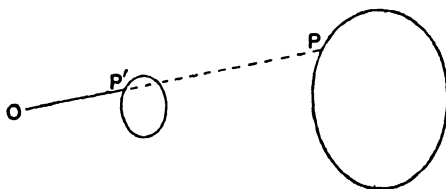


Fig. 220.

Suppose that it is desired to draw an ellipse similar to a given ellipse and reduced in the ratio 3 : 1. Attach one end of an elastic string to a fixed point O, and let the other end P move along the ellipse that is to be reduced; the string must be short enough to remain stretched throughout the motion. Mark a point P' on the string so that $OP' : OP = 1 : 3$; this ratio will persist however the string is stretched. P' will always describe a curve similar to that traced by P, and on $\frac{1}{3}$ the scale.

A more practical instrument for the same purpose is the **pantograph**. The principle of this instrument is as follows. In fig. 221

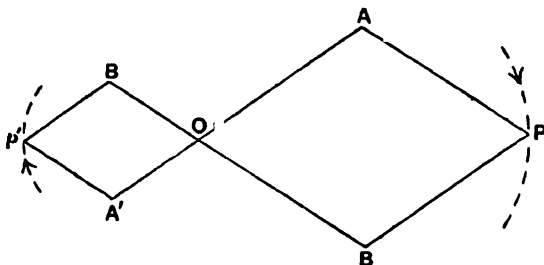


Fig. 221.

there are six rods, freely jointed at O, A, P, B, A', P', B' . The lengths are so chosen that $OAPB, OA'P'B'$ are rhombi; and since the angles at O are equal, the rhombi are similar. If $OA = 2OA'$, then $OP = 2OP'$; and obviously POP' is a straight line. If the point O is fixed and P moved along any path (e.g. a map, a building-plan, etc.), then P' (to which a pencil is attached) will describe a similar path on half the scale. By altering the relative sizes of the rhombi, any desired ratio of reduction or enlargement may be obtained.

Ex. 74. An elastic string with one end fastened to a peg at O carries at its other extremity a ring which is free to slide along a fixed straight rod AB . If the string is always kept taut, what is the locus of its middle point as the ring slides along AB ?

§ 10. The principle of the method of § 9 may be used geometrically thus:—

Suppose it is required to draw a figure similar to $PQRST$ (fig. 222). Take any point O . (O may be outside or inside the figure.) Join O to P, Q, R, S, T . Take any point P' in OP (or OP produced). Draw $P'Q' \parallel$ to $PQ, Q'R' \parallel$ to $QR, R'S' \parallel$ to $RS, S'T' \parallel$ to ST .

Then $P'Q'R'S'T'$ is always similar to $PQRST$ and is also said to be similarly situated.

Ex. 75. Use this method to draw a figure similar to a given figure, inside it, and with its sides half those of the given figure.

Ex. 76. Draw a polygon similar and similarly situated outside a given polygon, with its sides half as large again as the sides of the given polygon.

§ 11. To construct a polygon similar to a given polygon such that the areas of the polygons are in a given ratio.

Suppose a is a side of the given polygon and x a side of a polygon which is to be constructed similar to the given polygon and $\frac{1}{4}$ of it in area.

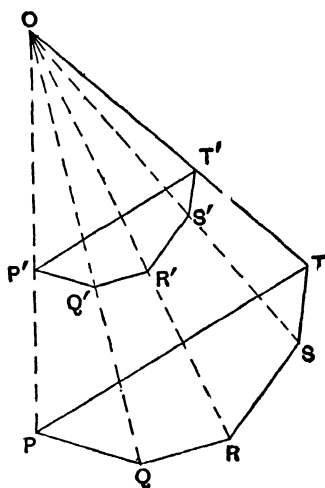


Fig. 222.

Then by § 4, p. 134, $\frac{x^2}{a^2} = \frac{\frac{3}{4}A}{A}$ where A is the area of the given polygon.

$$\therefore x^2 = \frac{3}{4}a^2,$$

$$\text{i.e. } x^2 = \frac{3}{4}a \times a,$$

or x is a mean proportional between $\frac{3}{4}a$ and a .

Hence x can be found geometrically.

Ex. 77. Draw any triangle. Bisect its area by a line parallel to the base. Justify your method.

Ex. 78. Construct a circle whose area is half that of a circle of radius 2". Give reasons for your work.

Ex. 79. Draw a semicircle on a diameter of 10 cm. Construct another semicircle to touch it at one end of the diameter and to be $\frac{2}{3}$ of it in area.

Ex. 80. Draw a triangle sides 3", 3.5", 2.5". Construct another triangle whose area bears to the former the ratio 5 : 3.

Ex. 81. Copy the figure PQRST (fig. 222). Find the length of the line corresponding to PT when the area has been increased by three-quarters the original area.

Ex. 82. A triangle ABC has sides 2, 3, and 4 inches. Construct accurately (without using calculation) a triangle which shall be similar to ABC, but whose area shall be equal to that of a square of side 1 inch. No proof of the construction is required, but every step should be carefully stated.

SUMMARY OF RESULTS

If two chords AB, CD of a circle intersect at a point P inside or outside the circle, then $PA \cdot PB = PC \cdot PD$.

If PT is a tangent to a circle and AB a chord passing through P, then $PT^2 = PA \cdot PB$.

If ABC is a triangle right-angled at A, and AD is perpendicular to BC, then $AD^2 = BD \cdot DC$.

CHAPTER 19*

CONTINUOUS CHANGE OF A FIGURE FURTHER EXAMPLES OF LOCI. ENVELOPES

§ I. The student should accustom himself to thinking of figures changing their dimensions continuously; he should vary some dimensions and trace how other dimensions change in consequence; in some cases a graph might be drawn.

For instance, let O be a fixed point outside a fixed straight line; let P be a point moving continuously along the straight line; trace the change in the length of OP .

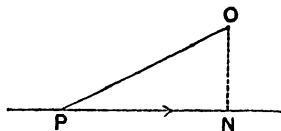


Fig. 223.

If P lies a long way to the left, OP is great; as P moves to the right, OP becomes smaller and smaller, until it reaches its least, or **minimum**, length when OP coincides with the perpendicular ON . When P passes N , OP begins to increase again; and as P moves further and further to the right, OP increases without limit.

¶Ex. 1. $X'OX$ is a fixed line of unlimited length; a line OP of fixed length rotates about O ; PN is drawn $\perp$ to $X'OX$. Trace the change in PN as $\angle XOP$ increases from 0° to 180° . Show the changes in PN by means of a graph. (Represent degrees horizontally; mark values of PN for 0° , 10° , 20° , etc.) Use the principle of symmetry to shorten your work.

¶Ex. 2. The sides AB , AC of $\triangle ABC$ are of constant lengths; trace the change in BC as $\angle BAC$ increases from 0° to 180° .

¶Ex. 3. The base of a triangle is fixed and the height is constant; how may the vertex move? Trace the consequent change in (i) the vertical angle, (ii) the remaining sides. When is the vertical angle a maximum?

¶Ex. 4. AB and $\angle BAC$ (acute) are fixed dimensions of a $\triangle ABC$, and the side AC increases continuously from zero; trace the change in the length of BC . When is BC a minimum?

* It is not intended that pupils should work straight through this chapter; it is rather a chapter to dip into from time to time.

¶Ex. 5. Make a parallelogram with strips of wood or cardboard hinged at the corners; use elastic for the diagonals.

As the figure changes shape, trace the change in the lengths of the diagonals. Are they ever equal? What are their maximum and minimum lengths?

Trace the change in the angle between the diagonals. Can it ever be a right angle?

Trace the change in height. What are its maximum and minimum values?

How would the above be modified if the parallelogram were a rhombus?

¶Ex. 6. The base and height of a parallelogram remain constant; trace the change in (i) the other side, (ii) the diagonals.

¶Ex. 7. The base of a rectangle remains constant; trace the change in the angle between the diagonals as the height increases from zero.

Ex. 8. A swinging pendulum QP (fig. 224) intersects a horizontal line RX at M so that the lengths of RM and QM are continually altering. Draw a graph that will enable you to read off the lengths of QM for values of RM ranging from 0 to 12 cm. If QM is given, can RM be determined uniquely? Make a rough sketch to illustrate your answer.

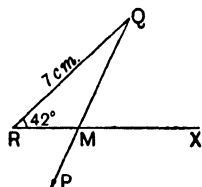


Fig. 224.

Ex. 9. A rod OQ , 12 in. long, is pivoted about O ; and a rod QR , 4 in. long, is hinged to OQ at Q , while the end P is constrained to move along a straight line OA passing through O . Discuss the range of movement of Q and P .

Ex. 10. Fig. 225 shows three jointed rods, BC having a length of 6 cm., CD of 10 cm., and DA of 3.6 cm. B moves in a slot XY and BC is always perpendicular to XY . The rod AD revolves in the plane of the paper about the point A , which is fixed.

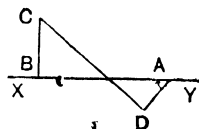


Fig. 225.

Draw the figure full size, with C at its greatest possible distance from A . Now imagine AD to revolve clockwise at the rate of 60° per second, and show the position of B at the end of each second. Give the greatest and the least distances of C from A and the range of movement possible for B . Show in a table the distance of B from its original position at the end of each second, and illustrate by a graph.

Ex. 11. Fig. 226 represents a fixed set square ABC , and an equal set square PQR which moves with its base on XAY drawn parallel to BC . One position is shown with the overlap $AKPH$ of the two triangles shaded; note that $AKPH$ is a rectangle and that H and K always denote the points of intersection of the sides of the two set squares. Draw $\triangle ABC$ assuming $AB = 6$ cm., $\angle A = 90^\circ$, $\angle B = 60^\circ$. Make a table showing the values of AH and AK when $BK = 0, 1, 2, \dots, 6$ cm.; also show the area of $AKPH$ for these values of BK . Draw a graph showing how the area $AKPH$ varies as BK changes and find the value of BK for which the area is greatest.

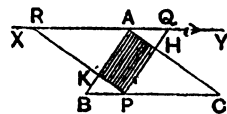


Fig. 226.

Ex. 12. A chord CD of a circle cuts the diameter AB at right angles at X . Now suppose a line $YCXDZ$ of unlimited length to revolve about X and to start from the position perpendicular to AB . By a rapidly sketched graph, give a general idea of the changes in the length of the intercepted chord CD as YZ rotates through an angle of 180° , plotting the length of the chord against the angle turned through. The simplest way is to set off the measurements direct from your figure and to take advantage of the symmetry of the circle.

MAXIMUM AND MINIMUM VALUES.

§2. By the aid of a graph it is easy to see that maximum and minimum values of any magnitude generally have equal values on either side of them. By considering equal values that are near to one another, it is often possible to determine when a maximum or minimum value will occur.

Given the base and area of a triangle, when is its perimeter a minimum?

As the base and area are given, the vertex of the triangle must lie on a line parallel to the base.

Imagine the vertex to be moving along the parallel; when it reaches the perpendicular bisector of the base, by symmetry it is obvious that points on either side of that position give equal values for the perimeter.

$\therefore$ the symmetrical position must give either a maximum or a minimum; general considerations will show that it must give a minimum value.

To find at what point on a given straight line XY a line AB subtends the maximum angle.

Draw an arc of a circle to pass through A and B and to cut XY at P and Q . Then $\angle APB = \angle AQB$.

$\therefore$ a maximum or minimum value for the angle must occur for a point between P and Q .

If we imagine the circle through A and B to grow smaller and smaller, P and Q will move closer and closer together. When they coincide the angle will be a maximum or a minimum, and it is obvious that the circle now touches XY .

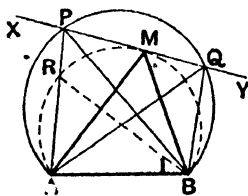


Fig. 227.

From the figure it is obvious that

$$\angle AMB = \angle ARB > \angle APB.$$

$\therefore \angle AMB$ is the maximum angle.

Hence to find the point M , we have to construct the circle through A and B that touches XY . (See p. 161.)

Ex. 13. Draw $OABX$ a horizontal line and OY at right angles to it. OY represents a tower and $OABX$ a road through its foot; A and B represent objects 80 ft. and 300 ft. from the foot of the tower. Find at what points on the tower, AB subtends angles of 32° ; also find the point at which AB subtends the maximum angle.

Ex. 14. The base and vertical angle of a triangle are given. Find when the triangle has a maximum area. Give an explanation.

Ex. 15. In Ex. 14, when is the perimeter a maximum?

Ex. 16. The sides of a parallelogram are 12 in. and 8 in. Find its maximum area. Give an explanation.

Ex. 17. A ruler, 12 in. long, moves with its ends on each of two lines at right angles. What is the maximum area of the triangle formed?

Ex. 18. Show that the greatest rectangle which can be inscribed in a circle is a square. [Draw a diagonal.]

Ex. 19. P moves round a semicircle on AB as diameter and PN is perpendicular to AB . When is the product $AN \cdot NB$ a maximum? Justify your answer.

FURTHER EXAMPLES OF LOCI. ENVELOPES

THE ELLIPSE.

§ 3. Ex. 20. Plot the locus of points the sum of whose distances from two fixed points remains constant.

Mark two points S , H , say, 4 in. apart. Suppose that the point P moves so that $SP + HP = 5$ in. Then the following are among the possible pairs of values:—

SP	4.5	4.0	3.5	3.0	2.5	2.0	1.5	1.0	0.5
HP	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5

Plot all the points corresponding to all these distances, by means of intersecting arcs. Why were not values such as $SP = 4.7$, $HP = 0.3$ included in the above table? Draw a neat curve, freehand, through all these points. The locus is an oval curve called an **ellipse**. The two fixed points are called the **foci** (Latin *focus*, a fireplace).

Ex. 21. Describe an ellipse mechanically as follows. Stick two pins into the paper about 4 in. apart; make a loop of fine string, gut or cotton and place it round the pins (see fig. 228). Keep the loop extended by means of the point of a pencil, and move the point round the pins. It will, of course, describe an ellipse.



Fig. 228.

If the same foci (the two pins) be used and a shorter string, another ellipse will be obtained inside the former; a more elongated ellipse, *not* similar to the former. By taking the string shorter and shorter we shall draw thinner and thinner ellipses; until with a certain length of the string the ellipse degenerates into the straight line joining the foci. What is this length?

* Now let us try the effect of making the string longer. We shall obtain an ellipse outside the former, of a shape more closely approaching the circle. If we take a very long string (i.e. long compared with the line joining the foci) we shall have an ellipse that is almost a circle; in fact the distance between the pins will hardly count and we might almost as well have used one pin.

The ellipse then can vary in shape from a finite straight line to a circle.

The earth describes an ellipse round the sun, one focus being almost exactly at the centre of the sun. But the earth's orbit approaches very closely to the circular shape. If the earth's orbit were drawn to scale on this sheet of paper, it could not be distinguished by eye from a circle.

All the planets describe ellipses, almost circular, round the centre sun as focus. Many comets on the other hand describe very elongated ellipses about the sun as focus. The moon describes an almost elliptical orbit round the centre of the earth as focus.

Ex. 22. What kind of symmetry has an ellipse?

Ex. 23. Draw a circle of radius 3" on inch paper. Draw a diameter AA. Let P be a variable point on the circle and PN the perpendicular (or **ordinate**) to AA'. Bisect PN at Q, using the rulings on the paper. Now take a number of positions of P and mark the corresponding positions of Q (PN need not be drawn as the lines on the paper will serve the purpose). Sketch in the locus of Q.

Ex. 24. Repeat Ex. 23 but so that Q divides PN in some other fixed ratio, e.g. $NQ = \frac{1}{4}NP$.

The two preceding exercises are connected with the fact that the sun-shadow of a circle is an ellipse.

Ex. 25. With the same axes, plot

$$(i) \frac{x^2}{9} + \frac{y^2}{9} = 1 \quad (\therefore y = \pm \sqrt{9 - x^2});$$

$$(ii) \frac{x^2}{9} + \frac{y^2}{4} = 1 \quad (\therefore y = \pm \frac{2}{3}\sqrt{9 - x^2}).$$

[Use square root tables, and take values of x at intervals of $\frac{1}{2}$; 1 inch scale for x and y .]

A plane section of a circular cylinder is an ellipse, as may be seen by cutting a lead pencil obliquely. The beam of a searchlight is almost cylindrical, and the patch of light on the plane underside of the clouds is an ellipse. As a matter of fact the searchlight beam is a circular cone of very small angle: and a **plane section of a cone** under certain circumstances is an ellipse.

Under other circumstances (which may be examined with the help of a dissected cone) the section may cease to be a closed oval; it may become a **parabola** or a **hyperbola***. Float a wooden cone on water and notice the curve of the water line.

THE PARABOLA.

§ 4. This curve is familiar as the path of a cricket ball through the air, or a jet of water from a hose. Sometimes however a cricket ball swerves to one side and the path is not a parabola: this is due to spin and the action of the air. A golf ball is generally spinning fast and the swerve is very noticeable: even if it is not sliced or pulled, its path is far from parabolic; it comes down much more steeply than it goes up. The same is true of a bullet or shell; the resistance of the air deforms the orbit from the true parabolic shape: but the shells fired into Paris from 70 miles away rose into a region where the atmospheric resistance was small, and there the orbit was almost parabolic.

You have probably often plotted the graph $y = x^2$ which is a parabola; if you are familiar with it, describe the symmetry of the parabola.

Ex. 26. A point moves so that its distance from a fixed point S is always equal to its distance from a fixed line MN : find its locus.

(This is best done on inch paper. Take the point S 2 in. distant from the line MN . Then plot points as follows. What is the locus of points distant 3 in. from MN ? distant 3 in. from S ? The intersection of these two loci gives two positions of the required point. Similarly find other points.)

The curve obtained is a parabola. It is the same curve as would be obtained by plotting the graph $y = \frac{x^2}{4} + 1$, taking for axis of x the line MN , and for axis of y the perpendicular from S to MN . Certain comets move in parabolic orbits, the sun being situated at the point S , the single focus.

* See Notes and Answers.

THE HYPERBOLA.

§5. This curve you have already met as the graph of $y = \frac{\text{const.}}{x}$. In physics it occurs as $pv = \text{const.}$, the Boyle's law curve.

Ex. 27. Plot the locus of points the difference of whose distances from two fixed points remains constant. This curve is a hyperbola. (For example, let the two fixed points S, H be 4 in. apart, and let the constant difference be 2 in. Make a table as in Ex. 20. Remember to make $SP > HP$ for some points, $HP > SP$ for other points.)

The beam of light from an electric torch is a cone; if it be thrown on to the wall, the patch of light is a plane section of a conic, or a **conic section**. By moving the torch about, you may watch the shape changing from circle to ellipse, to parabola and to hyperbola. This can be shown, less conveniently, with the cone of light emerging from a billiard-shade.

VARIOUS LOCI.

§6. Ex. 28. Plot the locus of points the product of whose distances from two fixed points remains constant.

(For example, mark two points S, H exactly 4 in. apart. First, plot the locus $SP \cdot HP = 5$.

Fill up the blanks in the following table:—

SP	5	4.8	4	3	$\sqrt{5}$	9				
HP							3	4	4.8	5

Secondly, plot the locus $SP \cdot HP = 4$; thirdly, plot the locus $SP \cdot HP = 3$. All three loci should be drawn in the same figure.

The first locus will be found to resemble a dumb-bell, the second a figure of 8; the third consists of two separate ovals.)

Ex. 29. Plot the locus of a point which moves so that the ratio of its distances from two fixed points remains constant.

(For example, let the two fixed points S, H be taken 3 in. apart; and let $\frac{SP}{HP} = 2$.)

Ex. 30. OP is a variable chord passing through a fixed point O on a circle; OP is produced to Q so that $PQ = OP$; find the locus of Q.

Ex. 31. A point moves in a plane subject to the condition that its distance from a fixed point S is always in a fixed ratio to its distance from a fixed straight line MN. Plot the curve described.

(i) Let the distance from S be always half the distance from MN. Take S 3 in. from MN.

(ii) Let the distance from S be always twice the distance from MN. Take S 3 in. from MN.

These curves will be recognised as having been obtained already.

Ex. 32. The ends of a rod slide on two wires which cross at right angles. Find the locus of a point on the rod. See fig. 229.

(Represent the rod by a line of 10 cm.; take the point 3 cm. from one end of the rod; also plot the locus of the mid-point. Draw the wires on your drawing paper; the rod on tracing-paper. Keep the ends of the rod on the wires, and prick through the different positions of the point.)

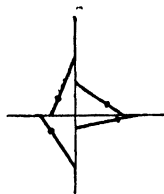


Fig. 229.

Ex. 33. Two points A, B of a straight line move along two intersecting lines at right angles. Plot the locus of a point P, in AB produced. [Tracing-paper.]

Ex. 34. Plot the locus of a point on the connecting rod of a steam engine.

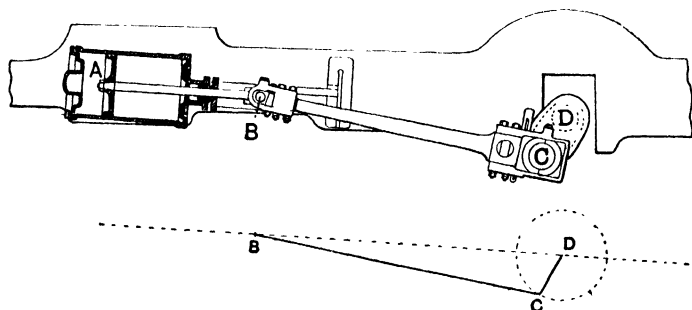


Fig. 230.

(The upper diagram in fig. 230 represents the cylinder, piston rod (AB), connecting rod (BC), and crank (CD) of a locomotive. In the lower diagram the different parts are reduced to lines. B moves to and fro along a straight line, C moves round a circle. Take $BC = 10$ cm., $CD = 3$ cm. Plot the locus of a point P on BC, where BP is (i) 1 cm., (ii) 5 cm., (iii) 9 cm. This may be done, either by drawing a large number of different positions of BC; or, much more easily, by means of tracing-paper.)

Ex. 35. A rod moves so that it always passes through a fixed point while one end always lies on a fixed circle. Plot the locus of the other end.

(Tracing-paper should be used. A great variety of curves may be obtained by varying the position of point and circle, and the length of the rod. It will be seen that this exercise applies to the locus of a point on the piston rod of an oscillating cylinder; also to the locus of a point on the stay-bar of a casement window.)

Ex. 36. Draw two intersecting lines. On tracing-paper mark three points A, B, C. Make A slide along one line and B along the other; plot the locus of C.

Ex. 37. Draw two equal circles of radius 4 cm., their centres being 10 cm. apart. The two ends of a line PQ , 10 cm. in length, slide one along each circle. Plot the locus of the mid-point of PQ ; also of a point 1 cm. from P .

(Most quickly done with tracing-paper. It is easy to construct a model machine to describe the curve.)

Ex. 38. Draw two circles. On tracing-paper mark three points A , B , C . Make A slide along one circle, B along another, and plot the locus of C . (Experiment with different circles and arrangements of points. You will find that in at least one case the locus-curve shrinks to a single point.)

Ex. 39. OA , AP are two rods jointed at A . OA revolves about a hinge at O , and AP revolves twice as fast as OA , in the same direction. Find the path of a point on AP . (Make $OA=2$ in., $AP=2$ in. Plot the locus of P ; also of Q

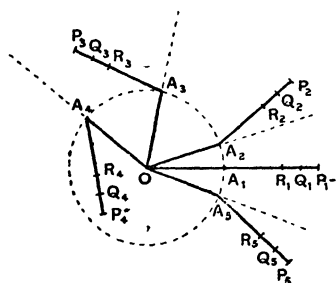


Fig. 231.

and R , taking $AQ=1$ in., $AR=\frac{1}{2}$ in. To draw the different positions of the rod, notice that when OA has turned through, say, 30° , AP has turned through 60° and therefore makes an angle of 30° with OA produced.)

The loci are different forms of the **limaçon**; the locus of Q is heart-shaped, and is called a **cardioid**. The locus of P has a small loop in it.

Ex. 40. Repeat Ex. 39, with the difference that, as OA revolves, AP remains parallel to its original position.

Ex. 41. Two equal rods OA , AP , jointed as in Ex. 39, revolve in *opposite* directions at the same rate. Find the locus of P and of the mid-point of AP .

Ex. 42. O is a fixed point on a circle of radius 1 in. OP , a variable chord, is produced to Q , PQ being a fixed length; also PQ' ($=PQ$) is marked off along PO . Plot the locus of Q and Q' when PQ is (i) $2\frac{1}{2}$ in., (ii) 2 in., (iii) $1\frac{1}{2}$ in.

(Draw a long line on tracing-paper, and on it mark P , Q and Q' .)

Ex. 43. Through a fixed point S is drawn a variable line SP to meet a fixed line MN in P . From P a fixed length PQ is measured off along SP (or SP produced). Find the locus of Q .

(Use tracing-paper. Take S 1 in. from MN . Plot the locus of Q

- when $PQ=1$ in., measured from P away from S ,
- when $PQ=1$ in., measured from P towards S ,
- when $PQ=2$ in., measured from P towards S .)

The curves obtained are different forms of the **conchoid**.

Ex. 44. A company of soldiers are extended in a straight line. At a given signal, they all begin to move towards a certain definite point, at the regulation pace. Are they in a straight line after 3 minutes? If not, what curve do they form?

Ex. 45. XOX' , YOY' are two fixed straight lines, C is a fixed point (see fig. 232). A variable line PQ is drawn through C to meet XOX' , YOY' in P , Q respectively. Plot the locus of the mid-point of PQ .

(Let XOX' , YOY' intersect at 60° , and take C on the bisector of $\angle XOY$, 5 cm. from O .)

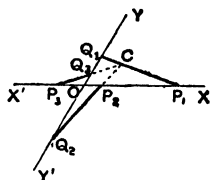


Fig. 232.

Ex. 46. (Inch paper.) Draw a circle of radius 2 in. and a straight line distant 6 in. from the centre of the circle. P is a variable point on the circle; Q is the mid-point of PN , the perpendicular from P upon the line. Plot the locus of Q .

ROULETTES.

§7. The locus of a point on the edge of a wheel rolling along a road is a particular case of a **roulette**, a term which means the locus of a point rigidly connected to a curve that is rolling on another curve.

Roulettes are conveniently plotted by means of tracing-paper. Draw a circle on tracing-paper, of radius say 2". Draw a straight line on ordinary paper, as long as possible so that there may be room for more than a complete turn of the wheel (what length of line is needed for one turn?). Place the circle so that it touches the line at A_1 ; mark a point P on the circle whose locus you propose to plot. Prick through P ; now pin A_1 on to the paper below, and give the tracing-paper a small turn about A_1 . Prick through the new position of P . The circle now does not exactly touch the line, but cuts it in A_1 and a second adjacent point A_2 . Now remove the pin from A_1 and pin through A_2 ; give the tracing a small turn about A_2 , prick through the new position of P , and proceed in like manner till the locus of P is plotted. Sketch in the locus in pencil. The locus is called a **cycloid**.

This method, though not exact, gives a good approximation to rolling.

Ex. 47. Plot the locus of (i) a point on the spoke of a rolling wheel, (ii) a point on the flange of a railway wheel, that projects beyond the rail. [It will be sufficient to mark two points on the tracing-paper, Q , $\frac{1}{2}$ " from the centre of the circle of radius 1"; and R , $1\frac{1}{2}$ " from the centre. The two loci can be plotted at the same time.]

Ex. 48. Plot the three types of locus connected with the rolling of a circle on an equal circle (point inside, on, or outside circle). Take $r=1$ ".

Ex. 49. Plot the three types of roulette for a circle of radius $1\frac{1}{2}$ " rolling *inside* a circle of radius 3".

Ex. 50. Plot the three types of roulette for a circle of radius $\frac{1}{2}$ " rolling *outside* a circle of radius 2".

ENVELOPES.

§8. We have seen that a set of points, plotted in any regular way, marks out a curve which is called the *locus* of the points.

In a rather similar manner, a set of lines (straight or curved), drawn in any regular way, marks out a curve which is called the **envelope** of the lines. Each of the lines touches the envelope.

Let a piece of paper be cut out in the shape of a circle, and a point **S** marked on it. Then fold the paper so that the circumference of the circle may pass through **S**. If this is done many times, the creases left on the paper will envelope an ellipse (fig. 233).

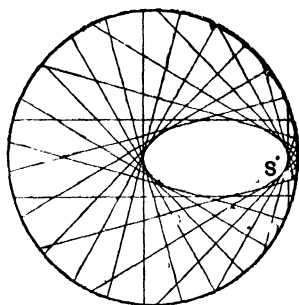


Fig. 233.

Ex. 51. Take a piece of cardboard with one edge straight; drive a pin through the cardboard into the paper underneath; then turn the cardboard round the pin, and in each position use the straight edge of the cardboard to rule a line. What is the envelope of these lines?

Ex. 52. One edge of a flat ruler is made to pass through a fixed point, and lines are drawn with the other edge. Find their envelope.

Ex. 53. Find the envelope of equal circles whose centres lie on a fixed straight line.

Ex. 54. Find the envelope of a set of equal circles whose centres are on a fixed circle when the radius of the equal circles is (i) less than, (ii) equal to, (iii) greater than, the radius of the fixed circle.

Ex. 55. Draw a straight line **MN** and drive a pin into your paper at a point **S** $\frac{1}{2}$ in. from **MN** (see fig. 234). Keep the short edge (**AB**) of your set square pressed against the pin, and keep the right angle (**B**) on the line **MN**. Rule along **BC**; and thus plot the envelope of **BC**, as the set square slides on the paper. (Lines must of course be drawn with the set square placed on the left of **S**, as well as on the right.)

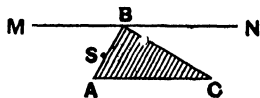


Fig. 234.

Ex. 56. Repeat Ex. 55 using the 30° angle instead of the right angle, and putting the pin 1 in. from **MN**.

Ex. 57. Draw a circle of radius 5 cm. and mark a point **S** 4 cm. from the centre. Let a variable line **SP** meet the circle in **P** and let **PQ** be drawn perpendicular to **SP**. Find the envelope of **PQ**. (The part of **PQ** *inside* the circle is the important part.)

Ex. 58. Repeat Ex. 57 with the point **S** on the circle.

Ex. 59. Find the envelope of circles passing through a fixed point **O**, and having their centres on a fixed circle.

(i) Take radius of fixed circle = 4 cm., distance of **O** from centre of fixed circle = 3.2 cm.

(ii) Take radius of fixed circle = 4 cm., distance of **O** from centre of fixed circle = 4 cm.

(iii) Take radius of fixed circle = 3 cm., distance of **O** from centre of fixed circle = 5 cm.

Ex. 60. Find the envelope of circles passing through a fixed point, and having their centres on a fixed straight line.

Ex. 61. Plot the envelope of a straight line of constant length whose ends slide upon two fixed lines at right angles.

MISCELLANEOUS EXERCISES II

CIRCLE (MAINLY CALCULATIONS).

Ex. 1. A piece of wood, whose cross-section is a rectangle, just fits into a pipe of 6 in. internal diameter. If the width of the piece is 5 in., what is its thickness?

Ex. 2. Draw a triangle ABC having $AB=4$ in., angle $BAC=55^\circ$ and angle $ABC=75^\circ$. Draw AD perpendicular to BC and BE perpendicular to AC , using a geometrical construction. Let these perpendiculars meet in O . Calculate the size of the angle AOB and check by measurement from your figure.

Give reasons why it is possible to draw a circle round the quadrilateral $ODCE$. Find the centre of this circle and measure its diameter.

Ex. 3. (i) With centre A , a point on the circumference of a circle, a second circle is drawn to intersect the first circle in P and Q . Show that P and Q are equidistant from B , the point on the first circle diametrically opposite to A .

(ii) Show by consideration of symmetry that this property can be extended to points A, B which lie on a diameter but are not its extremities.

Ex. 4. In fig. 235 the angles marked are supposed to have the values assigned to them. Find the angle BAC , and prove that $AB=BC$.

Ex. 5. From the given angles, find all the angles of fig. 236.

Ex. 6. Show how to find a point O inside $\triangle ABC$ so that

$$\angle AOB=150^\circ, \angle AOC=130^\circ.$$

Ex. 7. A triangle ABC is inscribed in a circle. The angles of the triangle are $A=62^\circ, B=55^\circ, C=63^\circ$. At A, B, C tangents to the circle are drawn and these are produced to meet at U, E, F . Calculate the angles D, E, F .

Ex. 8. O is the centre of the circle in fig. 237. If $\angle POS=130^\circ$ and $\angle OPR=20^\circ$, find the remaining angles, stating your reasons clearly.

Ex. 9. A trapezium $ABCD$ is inscribed in a circle ($AB \parallel CD$). The diagonals intersect at X and $\angle BXC=64^\circ$. Find as many angles in the figure as you can from the data.

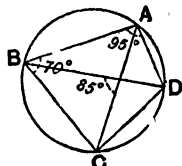


Fig. 235.

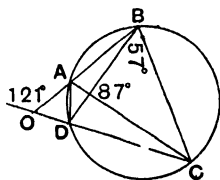


Fig. 236.

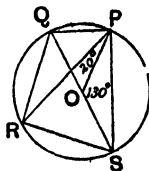


Fig. 237.

Ex. 10. AB , BC , CD are three equal chords of a circle, each subtending an angle of 25° at the centre. Show, by finding other angles, that $ABCD$ is a trapezium.

Ex. 11. P is any point on the circumference of a circle centre M ; Q is any point on the circumference of a circle centre N . The circles cut at A and B ; the angle $AMB = 110^\circ$, and $ANB = 150^\circ$. Find all possible values for the sum of the angles APB , AQB ; give an explanation of your work.

Ex. 12. ABC is an isosceles triangle having $AB = AC = 2\frac{1}{2}$ in., and $BC = 4$ in. The bisector of the angle A meets BC in D and the circumcircle of ABC in E . Calculate the length of DE .

Ex. 13. A pendulum swings through an angle of 10° on either side of the vertical: calculate the length of a scratch made on the clock-case by the back of the pendulum-weight, given that the pendulum is $4' 4''$ long. If the pendulum were to swing through twice as large an angle, would the scratch be twice as long? Would the distance between the ends of the scratch be doubled?

Ex. 14. A wire is in the form of a circle of area A square inches. If the wire were bent into an equilateral triangle, find the area of this triangle in terms of A and π .

Ex. 15. From a point P on a circle, a line PQ of constant length is drawn parallel to a fixed line. Plot the locus of Q , as P moves round the circle. Having discovered experimentally the shape of the locus, prove it theoretically.

Ex. 16. Draw a circle of radius 4 cm., and in it place a chord AB of length 6.8 cm. Then find all the points whose perpendicular distance from this chord (produced if necessary) is 2.6 cm., while the length of the tangent from each to the circle is 3.6 cm., explaining how you do so.

Ex. 17. An endless belt runs over two wheels, one 6 ft. and the other 1 ft. in radius, whose centres are 10 ft. apart. Draw an accurate figure showing carefully all construction lines. Measure, or calculate, the length of the belt, explaining the method adopted.

Ex. 18. Power is being transmitted from one shaft to another parallel shaft by means of a belt passing over two wheels. The radii of the wheels are 2 ft. and 1 ft. and the distance between the shafts is 6 ft. Assuming the belt to be taut, find its length (i) when it does not cross between the shafts, (ii) when it does cross.

Ex. 19. Two pulley wheels of radii 24" and 5" respectively are connected by leather belting. The belting is not crossed. If the portion of leather in contact with the small wheel subtends 120° at its centre, calculate (i) the distance apart of the centres of the wheels, (ii) the total length of belting used.

Ex. 20. A circle is described to touch two lines at right angles and to pass through the point whose distances from the lines are 3" and 4". Calculate the radius of this circle. Consider both cases.

Ex. 21. A rectangle measures a by b , and it is required to cut out of it two equal circular discs as large as possible. Find a quadratic equation for its radius r and verify that $2r = a + b - \sqrt{2ab}$.

Ex. 22. In fig. 238, O is the centre of the arc AB ; and Q is the centre of the arc BC ; $\angle ADC$ is a right angle. $DA = 3$, $DC = 5$, $DQ = 3$. Find OA and QC . (Let $OD = x$.)

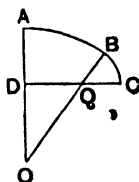


Fig. 238.

Ex. 23. Draw a minor arc of a circle and its chord AB . The figure represents a central section of a plano-convex lens (i.e. a lens one side of which is a plane and the other a part of a sphere). The distance AB is 6.4 cm. and the greatest thickness of the lens is 1 cm. Determine the radius of the spherical surface.

Ex. 24. In the fig. on p. 246 a circle is inscribed in the triangle ABF , cut off by AB . If the side FC of the square is 10 in. calculate the radius of the circle.

Ex. 25. A surveyor has to lay out a circular railway track (of large radius) through three points A, B, C on level ground. He proceeds as follows. On the side of AB remote from AC he lays out a straight line AB_1 making the angle $BAB_1 = 5^\circ$ and from C he lays out CB_1 on the same side of CB as CA making $BCB_1 = 5^\circ$. Then B_1 where AB_1 and CB_1 intersect is a point on the curve. He proceeds similarly to obtain other points. Prove the correctness of the construction.

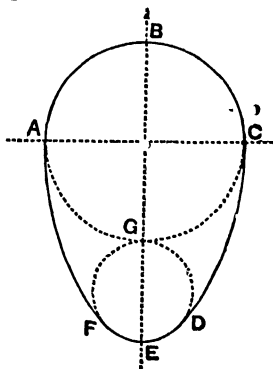


Fig. 239.

Ex. 26. An egg-shaped channel, as shown in section in fig. 239, is formed by four circular arcs. The circles ABC and DEF touch in G , and the circular arcs AF and CD touch both circles, AC being a diameter of the larger circle. If the radii of ABC and DEF are 2 ft. and 1 ft. respectively, find the radius of the arcs AF and CD .

Find also the area enclosed by the figure.

Ex. 27. Two circles each of radius r have their centres at a distance c apart. Assume, without proof, any properties obvious from symmetry, and show that when the circles intersect the length of their common chord is $\sqrt{4r^2 - c^2}$.

The guns of two forts A, B at a distance c miles apart have an equal maximum range of $\left(\frac{c}{2} + x\right)$ miles. Assume that the area common to two equal intersecting circles is approximately $\frac{3}{4}$ of the rectangle enclosing it as in fig. 240, and prove that the area which can be under fire from both A and B is $3x\sqrt{x(x+c)}$ square miles.



Fig. 240.

What is this area, to the nearest square mile, when the maximum range is 6 miles and the forts are 10 miles apart?

Ex. 28. The circle of fig. 241 represents a section through the centre of the earth. In a method of map projection the eye of the observer

is supposed to be at O , which is so situated that $OE = r \times \frac{\sqrt{2}}{2}$,

being the radius EC of the earth. AB is a diameter perpendicular to OD . The arc DP would be represented on the map by a line equal to CX and the arc PB by XB .

Choose P so that the angle BCP is equal to the angle PCD , and find the relation between the lengths of CX and BX . For the general proof it may help if you draw PQ perpendicular to OD , and consider the similar triangles OCX , OQP .

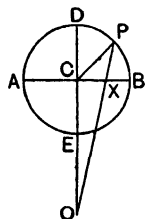


Fig. 241.

Ex. 29. Find, as a decimal of a square foot, the area of the shaded quadrant ACD (fig. 242), and also the area of the unshaded part K , which together make up the square $ABCD$, whose side AB is 1 ft.

The scale drawing of fig. 243 represents a window, each side of which is 2 ft., the shaded parts representing coloured glass, and the unshaded parts clear glass. Draw the figure to scale and find, by calculation, the area of the coloured glass and also the area of the clear glass. The construction and the method of calculation should be evident from fig. 242.

If the amount of light passing through the coloured glass is half that passing through an equal area of clear glass, what fraction of the light that would pass through the window of fig. 243 if it were all of clear glass is intercepted by using the coloured glass as there shown? Give your answer as a decimal fraction.

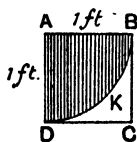


Fig. 242.

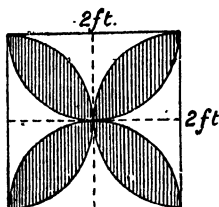


Fig. 243.

Ex. 30. A system of wheels is shown in fig. 244. If a non-slipping band passes round A and B , compare the number of revolutions made by B with the number made by A if B 's diameter is 56 in. and A 's 21 in.

Now think of C as being keyed to B and thus revolving at the same rate. Find the circumference of C if its diameter is 21 in., taking π as $3\frac{1}{7}$.

The weight W is raised by a rope which is attached to and wound round C . Find how many revolutions of A will raise W through 3 ft.

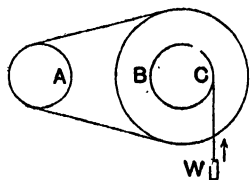


Fig. 244.

Ex. 31. The two circles in fig. 245 touch externally at O . The tangents at A and B are parallel. Prove AOB a straight line.

If the curve AOB represents one rail of a crossing from one railway track to another and the distance between the rails AD and BE is 11 ft., calculate the distance of O from each of these rails, it being given that the radii of the portions AO and OB are respectively $15\frac{1}{2}$ chains and $17\frac{1}{2}$ chains.

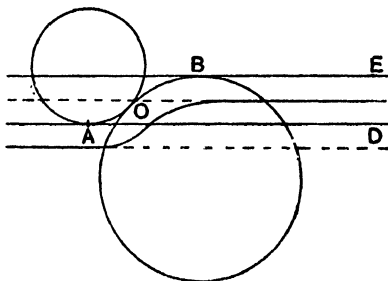


Fig. 245.

SIMILARITY (CALCULATIONS).

Ex. 32. ABC is an equilateral triangle of sides 4 ft. A line is drawn through the mid-point of one side inclined at 30° to this side. Calculate the distance of each vertex from this line.

Ex. 33. A rectangle $ABCD$ is such that a right-angled triangle AEB with sides of 3, 4, 5 in. can be cut from it, with E on CD . Calculate the sides of the remaining triangles ADE and BCE .

Ex. 34. A right-angled triangle has sides of 18 cm. and 13.5 cm. At a point on the hypotenuse 9 cm. from the end of the longer side a perpendicular is drawn to the hypotenuse. Calculate the length intercepted on this perpendicular by the two sides.

Ex. 35. A plank 25 ft. long is hinged to the floor and rests on a box 4 ft. high, distant 3 ft. from the hinge. How high above the floor is the free end of the plank?

Ex. 36. Two beams ACB , COD each 10 ft. long are hinged at O , 6 ft. from the ends A , C , and rest with A , C fixed to the ground 5 ft. apart. One end PQ of a heavy horizontal platform 3 ft. wide rests in the upper angle between the beams. What is the distance of P and Q from the hinge?

Ex. 37. The section of a gutter is triangular, the two sides, each 18 in. long, being equally inclined to the vertical. A foot rule, placed horizontally, touches the sides 8 in. from the vertex. What is the greatest width of the gutter?

Ex. 38. The two side walls of an alpine hut are each 4 metres high and 5 m. apart. The eaves of the roof each project a horizontal distance of 75 cm. from the walls and their ends are 3.5 m. above the ground. Calculate the height of the ridge of the roof above the ground.

Ex. 39. A parallelogram ABCD has AB = 12 cm., AD = 8 cm. From P in AB such that AP = 4 cm., a line is drawn parallel to AD meeting the opposite side in Q and the diagonal AC in R. Calculate the lengths PR, RQ.

Ex. 40. A rectangular board measures 30 in. by 50 in. The corners are cut off along lines drawn from the middle points of the shorter edges perpendicular to the diagonals. Calculate the lengths of the sides of the hexagonal board thus formed.

Ex. 41. The coordinates of three points A, B, C in a straight line are (x_1, y_1) , (x_2, y_2) , (x_3, y_3) . Show that

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_3}{x_2 - x_3} = \frac{y_3 - y_1}{x_3 - x_1}.$$

Ex. 42. A straight line cuts the rectangular axes OX, OY in points A and B respectively. PN, PM are perpendiculars from a point P in AB to OA and OB respectively. Show that $\frac{BM}{ON} = \frac{PN}{NA}$. Hence show that, if the coordinates of P are (x, y) , and OA = a, OB = b, then $\frac{x}{a} + \frac{y}{b} = 1$.

Ex. 43. Use similar figures to find the relation between x and y in figs. 246–249.

In figs. 246, 247. ABCD is a square.

In fig. 248 the short lines are parallel.

In fig. 249 the long lines are parallel.

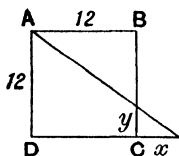


Fig. 246.

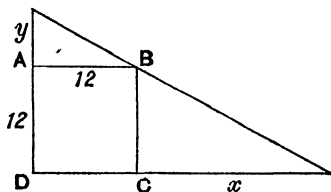


Fig. 247.

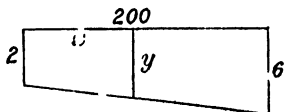


Fig. 248.

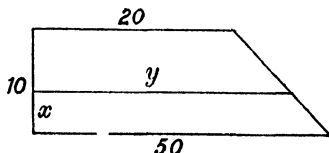


Fig. 249.

Ex. 44. A room has a striped wall paper, the stripes of which are without breadth, vertical and at regular intervals a inches apart. A man standing with his back against a wall of the room, and looking with his left eye only at a vertical electric light wire hanging from the ceiling, notices that it is just in line with one of the strips on the opposite wall. This wall is parallel to the one against

which he is standing and is b feet distant from it. If he looks with the right eye only, the wire just obscures the second stripe to the left of the previous one. If c inches denote the distance apart of the man's eyes and x feet denote the distance of the wire from the wall against which he is standing, show that $c = 2ax/(b - x)$. Hence find x if $a = 2.25$, $b = 14$, $c = 2.5$.

Ex. 45. O is the middle point of the hypotenuse AB of a right-angled triangle ABC . BD is drawn through B parallel to CA , and OD through O perpendicular to AB , meeting at D ; also the straight line AD is drawn. Show that the triangle AOD has the same shape as the original triangle ACB .

Let a denote the length of BC and b of CA . Find in terms of a and b an expression for the length of AD .

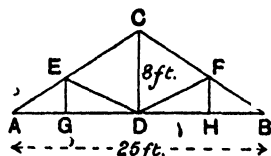
Estimate the length of AD when $a = 8$ cm. and $b = 5$ cm.

Ex. 46. Two ships are on courses that appear to converge. Show that if the bearing of one ship from the other remains constant, the ships will collide unless one of them alters course.

Ex. 47. In a pin-hole camera the rays of light from an object pass through the hole and striking the plate form two cones with vertices at the hole, whose bases are the object, and the picture on the plate respectively. Draw a diagram (scale a foot to the half-inch) showing the vertical section of these rays, from a window 4 feet high to a camera 6 feet away, passing through the hole of the camera and falling on the plate. The camera is 1 foot from hole to plate. Calculate the height of the window in the picture.

Ex. 48. Draw a rectangle $OCDE$ with $OC = 10$ units and $OE = 8$ units. If any straight line BDA is drawn through D to meet OC and OE , both produced, in A and B there is a simple relation between the area of the rectangle whose sides are equal to CA , EB , and the area of the original rectangle OD . Find this relation experimentally by taking a few positions for BDA , and recording the area of the rectangle $CA \cdot EB$ in each case. State the relation, and give a general proof of your statement.

Ex. 49. Fig. 250, which is not drawn to scale, shows the section of a roof and the beams supporting it. The span AB is 25 ft. and is horizontal; the vertical post DC is 8 ft. and rests on the middle point D of AB . A strut ED connects D to a point E of AC , which is 6 ft. 6 in. from A . A vertical post GE completes the left half of the section.



Calculate the length of GE and the distance GD . Hence obtain the length of the strut ED . Verify your calculations by an accurate drawing made to scale.

Ex. 50. Draw a straight line ANB with $AN = 5$ cm., $NB = 3$ cm.; draw NP perpendicular to ANB ; produce AP , BP to cut perpendiculars to ANB through B , A in Y , X respectively. If $AX = x$ cm., $BY = y$ cm., find the algebraic relation between x and y .

Ex. 51. Draw an isosceles trapezium $ABQP$ with PQ parallel to AB to represent the end view of a trough 25 in. long. If $PQ=16$ in., $AB=12$ in. and the depth of the trough is 10 in., find the volume of water in the trough when it is x inches deep.

Calculate how deep the water is when there is a volume of 1400 cubic inches in the trough. [Answer in inches correct to one place of decimals.]

Ex. 52. Water flows out of a cylindrical cistern of diameter 4 ft. through a pipe of diameter 6 in. Compare the rates of flow of the water in the pipe and the cistern.

Ex. 53. The flywheel of an engine is 3 ft. 6 in. diameter and weighs 18 cwt. What is the weight of the flywheel in a smaller pattern of the same engine in which the flywheel has diameter of 2 ft.?

Ex. 54. A rectangular plank measures 10 ft. by 9 in. What are the dimensions of a similar plank of half the area?

Ex. 55. A rectangular box 6 in., by 3 in., by 2 in. holds 100 marbles. What are the dimensions of a similar box holding 200 of the same marbles?

Ex. 56. ABC is an equilateral triangle with side 6 in. E divides BC in the ratio of $1:3$. If AE produced meets the circumcircle of the triangle ABC in D , calculate the lengths AE , ED .

Ex. 57. P is a point distant 12 in., 9 in. from A , B respectively. A circle is described through A , B and the mid-point of PA . Calculate the length of the segments into which it divides PB and the length of the tangent from P to the circle.

Ex. 58. Given a ruler for measuring lengths only, how can you determine whether four points are concyclic?

Ex. 59. Calculate the distance of the horizon seen from an aeroplane 15,000 ft. above the earth's surface. [Answer to nearest mile, take the earth's radius as 4000 miles.]

Ex. 60. The tangent at A to a circle ABC meets the chord BC produced in T . Find equations giving the lengths of $TA(x)$, $TC(y)$ in terms of a, b, c , the sides of the triangle ABC .

Find x, y when $a=2, b=3, c=4$.

Ex. 61. PQ, PR are two lines at right angles. A point X moves along PQ , A, B, C being three of its positions at distances 2 cm., 5 cm., 10 cm. from P . O is a point on PR distant 5 cm. from P . Y is a point on OX such that the rectangle $OX \cdot OY$ is equal to the square on OP . Show that Y is the foot of the perpendicular drawn from P to OX . Draw the figure and construct the positions A', B', C' of Y which correspond to the positions A, B, C of X .

What is the locus of Y , and how do you know?

Ex. 62. Fig. 251 shows an end view of a cylinder of circular section resting on a table between two rectangular blocks. Write down an equation connecting the radius r of the cylinder with the distance d between the blocks and the height h of the smaller block. By aid of the equation, calculate the numerical value of r when $d=8.2$ cm. and $h=2.9$ cm.

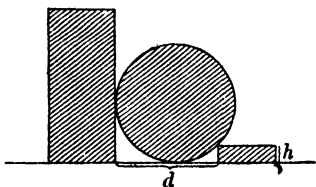


Fig. 251.

Check your result by drawing a figure full size, using the given values of d and h and your calculated value of r .

Ex. 63. A plank of length L rests with one end on the floor, over a cylinder of radius r , with its other end, beyond the cylinder, at a height h above the floor. The foot of the plank is distant l from the line of contact of the cylinder with the floor. Prove that $h = \frac{2lrL}{(r^2 + l^2)}$.

Ex. 64. Fig. 252 shows a square described in a semicircle, with two vertices on the curve and two vertices on the diameter. By noting the slope of the line OR , or otherwise, deduce a geometrical construction for drawing the square and perform the construction, taking the radius of the semicircle = 5 cm. Measure all the sides of your square and find their mean value.

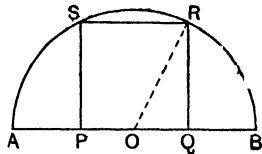


Fig. 252.

What relation exists between AQ , QR , QB ? Obtain from this relation an equation connecting x and a , where x is the side of the square and a is the radius of the semicircle, and hence calculate x to three decimal places when $a=5$ cm.

Ex. 65. OA and OB are two perpendicular radii of a circle, centre O , radius r cm.

Two points P and Q , each at a distance of y cm. from O , are taken on OA and OB respectively. What is the greatest value of y possible for which the line PQ will meet the circle?

PQ cuts the circle in the points C and D , and the length of CD is x cm. Express the length of PC or QD in terms of x and y .

Write down a relation between the lengths PC , PD , PA , OA .

From these two relations, or otherwise, show that x and y are connected by the relation $x^2 = 4r^2 - 2y^2$.

Hence (or otherwise) find the length of OP , when the chord CD is equal to the radius. Construct the figure in this case, taking $r=5$ cm.

CONSTRUCTIONS (CIRCLE).

Ex. 66. For finding the centre line of a circle without using compasses the following construction has been given. "AB and CD are parallel chords; AD and BC meet at K, AC and BD produced meet at L. Then LK produced is a line passing through the centre of the circle."

Draw a circle of radius about 8 cm. (if possible, without marking its centre), place in it two parallel chords roughly at distances 2 and 6 cm. from the centre, carry out this construction, and justify the construction. Then state in your own words a method of finding the centre of a circle by using only a ruler and a set square.

Ex. 67. Show how to trisect a given semicircular arc.

Ex. 68. A man has to fit three thin rectangular pieces of glass each of width 6 cm. within a cylinder, touching along their lengths so that an end view is a triangle inscribed in a circle. What diameter of cylinder would he require?

Ex. 69. Three ships A, B and C are moored so that B is 7 cables due E. of A, and C is 4 cables S. 44° E. of A. Another ship D is to take up a position so as to be at the same distance from each of the ships A, B and C. Find, by drawing, what angle AC will subtend at D.

What relation is there between this angle and the angle ABC?

State, without proof, how you might have known that they would be so related.

Ex. 70. Construct an isosceles triangle, given the radius of the inscribed circle, and the base.

Ex. 71. Construct a right-angled triangle, given that the radius of the inscribed circle is 2 cm. and that one of the sides about the right angle is 5 cm.

Ex. 72. The railway from P to Q consists of a circular arc AB and two tangents PA, BQ. AB is an arc of 28° of a circle whose radius is $\frac{1}{2}$ mile; PA = 1 mile, BQ = $\frac{1}{2}$ mile. Draw the railway, on a scale of 2 inches to the mile, and measure the distance from P to Q as the crow flies. Also calculate the distance as the train goes.

Ex. 73. Inscribe three equal circles to touch one another in an equilateral triangle, of side 6 in. (fig. 253).

Ex. 74. Construct all the points at which the sides AB, CD of the square ABCD (side 2 in.) both subtend the angle 35° , explaining all your work.

Ex. 75. Draw a circle with centre O and a chord of the circle; produce the chord both ways and mark points A and B on the produced chord on opposite sides of the circle. The figure represents a circular wire. A cord is to be stretched tightly between A and B to meet the wire at P and leave it at Q, lying along the wire between P and Q.

There are two possible positions of P; give a geometrical construction for them, and prove the validity of your construction.

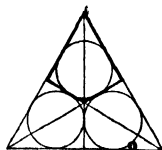


Fig. 253.

Ex. 76. (i) Two straight lines meet and terminate at O , forming an angle of 40° . A circle S of radius 1 in. touches the two lines. Draw the figure and state your construction briefly.

(ii) Draw two other circles, each of them to touch S and the two lines (not produced past O). Measure the distance between the centres of the two circles so drawn.

Ex. 77. Draw a quadrilateral $ABCD$, given $AB=2.1$ in., $BC=1.8$ in., $AC=3.4$ in., $AD=3.5$ in., $DC=3.2$ in.

Draw a circle which shall touch the three straight lines AB , BC , CD . Give reasons for the construction. Does the circle touch DA ? If so, why? Could you have foretold this?

Ex. 78. The radius of a circle is $\sqrt{3}$ in. and a rhombus with an angle of 60° is circumscribed about it. Find, by drawing, the length of a side of the rhombus. Verify by calculation.

CONSTRUCTIONS (SIMILARITY).

Ex. 79. ABC is a triangle in which $\angle ABC$ is 50° and $\angle ACB$ is 70° . CB is produced beyond B to D , so that $BD=BA$, and BC is produced beyond C to E , so that $CE=CA$. Determine from theoretical considerations the angles of the triangle ADE .

Hence construct the triangle ABC when its perimeter is 3" and two angles are 50° and 70° .

Ex. 80. Find three points the ratio of whose distances from two intersecting lines OX , OY is 2 to 1 respectively. Verify that the points lie on a straight line through O .

Ex. 81. Find a point in the base of any triangle the ratio of whose distances from the sides is 2 : 1. [See Ex. 80.]

Ex. 82. Find a point the ratio of whose distances from the sides of a triangle is 1 : 2 : 3.

Ex. 83. Draw any quadrilateral, and using only two set squares construct a similar quadrilateral.

Ex. 84. Draw any triangle ABC , its altitude AD , and a line PQ outside it. Construct a triangle similar to ABC having PQ as altitude corresponding to AD .

Ex. 85. $ABCD$ is a rectangle; $AB=4$ cm., $AD=7$ cm. P is the mid-point of AB , Q a point in AD , $PQ=5$ cm. Make an enlarged copy of this figure in which the length corresponding to PQ is 8 cm.

Ex. 86. ABC is an equilateral triangle and BDC a semicircle which does not cut the sides of the triangle. Construct this figure with ruler and compasses only and then make an enlarged copy of twice the area.

Ex. 87. Divide a circle into three parts of equal area by two concentric circles.

SCALE DRAWING.

Ex. 88. Each of three mountain peaks A, B, C is $5\frac{1}{2}$ miles distant from a fourth mountain peak D. The angle ABC is observed to be 23° and the angle BAC 48° . Find, by measurement of a carefully drawn diagram, the distance between A, B, and C.

Ex. 89. Solve the following problem by accurate drawing and measurement. A, B, C are three points such that AB is 5 miles, BC is 10 miles and CA is 7 miles. P is a point in the plane of ABC such that BPA is 80° and CPA is 80° . Find the distances of P from A, B, and C.

Ex. 90. A road bounded by walls curves through a right angle, the walls being concentric circular arcs AB, CD. The radius of the inner curve AB is 60 yards. It is just possible to have an uninterrupted view from C to D. How wide is the road, to the nearest yard?

Ex. 91. A vessel X steaming towards a port due S. at a speed of 16 miles an hour is at a point A when a hostile vessel Y appears at P, 12 miles to the E. The vessel Y steams 24 miles an hour in such direction as to remain 12 miles distant from X and to get between X and the port.

In a diagram show the four positions A, B, C, D of X at intervals of half-an-hour. Show the initial position P of the vessel Y. Show its position Q after half-an-hour, placing it 12 miles from P and from B. Show its positions R and S after an hour and after an hour and a half, placing R 12 miles from Q and C, and placing S 12 miles from R and D.

Draw a smooth curve to show the course of the hostile vessel.

Can you offer any criticism of the method of fixing the points Q, R, S?

Ex. 92. Two equal bars CH and HK, each of length 1 foot, are hinged together at H and placed flat upon a table (fig. 254). The bar CH can be turned round the end C, which is kept fixed upon the table. At first the bars are placed so that the point K is at a distance of 1 foot from the fixed point C. Then K is moved in the direction KL perpendicular to CK. Determine the length of the path of the hinge H when K is moved 1 foot from its initial position. How much further can K be moved in the same direction, and how does H move during this further motion of K?

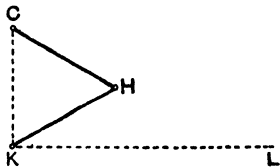


Fig. 254.

Ex. 93. The arch of a stone bridge is a circular arc (see fig. 255). In order to repair such an arch it was necessary to measure the radius of curvature of the arch. A string was stretched tightly across from C to D, and the vertical height from the centre of the string to the crown of the arch was found to be 3 feet. The length CD was 16 feet.

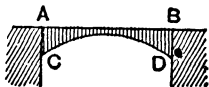


Fig. 255.

Make a drawing showing the positions of C, D and the crown of the arch to a scale of $\frac{1}{2}$ cm. to 1 foot, and determine by geometrical construction the radius of curvature of the arch. Describe the construction which you make, but do not give a proof.

Ex. 94. Make a freehand copy of fig. 65 but shorten BC so that $BC = BA$. Your figure now represents a rod AC 5 in. long and a rod BD 2.5 in. long hinged to B , the mid-point of AC , and to D . A is guided so that it moves in the horizontal line AD . Show that C will always move in a vertical line. If the travel of C is limited to 1 in. above and 1 in. below D , calculate the length which A can travel.

Ex. 95. In fig. 256, OA is the plan of an open window, which is at OA' when shut. The window is held open by a bar BC , free to turn about B and carrying at the other end a hook which fits into an eye screwed on to the window at C . It is desired to move the eye to a position which will allow the window to stand more widely open. How far from O must the eye be to allow the widest possible opening? And at what angle will the window finally stand open?

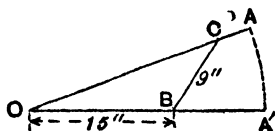


Fig. 256.

Ex. 96. Fig. 257 represents a wheel on an incline with a brick pushed up against it. The dotted line AD is vertical. The diameter (CD) of the wheel is 2 ft. 6 in. The thickness of the brick (AB) is 3 in. Determine by drawing to scale, or otherwise, the distance BC and the angle of the slope (x°).

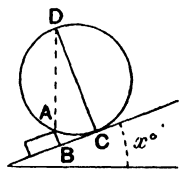


Fig. 257.

Ex. 97. Fig. 258 shows a simplified diagram of a vehicle consisting of four wheels (represented as two circles) connected by a framework ABC . The vehicle is propelled by means of the lever FBE , which pivots round B while F is moved backwards and forwards by hand-power. The end E is connected by a bar ED to a pin D which is rigidly fastened to the wheel C . By drawing to scale or by calculation, find the angle through which the lever moves.

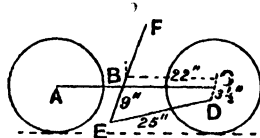


Fig. 258.

Ex. 98. A weight W is supported by a rope AOB 44 ft. long (see fig. 259). AO and OB are equally inclined to the vertical. The horizontal and vertical distances of the end B of the rope from A are 30 ft. and 10 ft. Find the lengths of the two portions AO and OB .

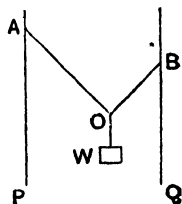


Fig. 259.

CHAPTER 20

DRAWING AND DEVELOPMENT OF SOLID FIGURES

§ 1. In many examples we have already dealt with solid figures. It is important to realise that much of the plane geometry that we have done applies to three dimensions, for it applies to any plane section of a solid figure.

§ 2. A little practice at interpreting a few simple drawings is very helpful*.

What does fig. 260 represent?

(i) It represents a cube with the middle corner in front.

(ii) It represents three planes at right angles to one another with the middle corner the furthest point.

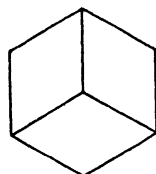


Fig. 260.

Practise seeing the above and practise changing quickly from (i) to (ii) and back again.

Consider fig. 261. It represents (i) a cone with the vertex towards you, or (ii) a hollow cone with the vertex the furthest point. Practise changing from (i) to (ii) and back again.

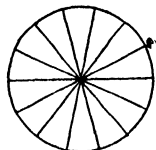


Fig. 261.

Again, consider fig. 262. It represents either

(i) A framework of rods forming a tetrahedron (or a triangular pyramid) with the edge AC in front; or

(ii) A tetrahedron with the edge BD in front.

To make (i) clearer rub out the parts of BD that are close to AC. Similarly to make (ii) clearer rub out the parts of AC that are close to BD.

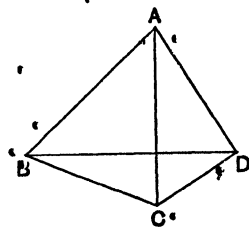


Fig. 262.

As soon as (i) and (ii) can easily be

* The first difficulty in teaching solid geometry is to get the boy to visualise a solid figure from a flat drawing. Of course a shaded figure, or a photograph, or a stereoscope make it easier for the boy: but he has got to learn to see three dimensions in a drawing that he has made himself, so that clever drawings made by the master, far from being a help, may be a hindrance.

visualised, practise changing from (i) to (ii) and back again as quickly as possible.

These simple exercises will help you to realise a solid figure, even from a comparatively poor drawing.

§ 3. After practice in seeing three dimensions in a two-dimensional drawing, the next thing is to learn to make simple drawings of solid figures.

¶ Ex. 1. Draw a figure of a box or brick as follows:—

On squared paper draw a rectangle along the lines of the paper; draw an equal rectangle slightly to the right and slightly higher up on the paper, but well overlapping the first; then join corresponding corners.

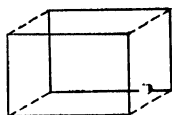


Fig. 263.

Repeat this on plain paper.

Here are a few points that will help you in making drawings of solid figures.

(i) All lines that are vertical in a solid figure should be represented by lines that are parallel to the side edges of 'your' paper. Verify this by looking at any pictures.

(ii) Note that horizontal lines are not necessarily parallel to the top and bottom edges of your paper. See fig. 264, which represents an E. and W. road crossing a N. and S. road with a vertical flagstaff OT at the crossing.

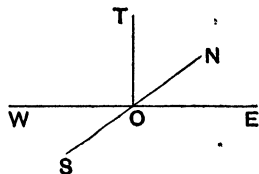


Fig. 264.

(iii) Note also that in a drawing A rectangle generally looks like a parallelogram.

A circle generally looks like an oval figure called an ellipse.

Right angles generally look like acute or obtuse angles.

(iv) In drawing a figure of a regular pyramid on a horizontal base such as figs. 272, 273, it is generally best first to draw the base and the various lines on it and then to draw the line from the centre of the base to the vertex; this line should be parallel to the edge of the paper.

Ex. 2. Make a freehand sketch of a horizontal drawing board, with a vertical line standing at the centre of the board. Show the angles of elevation of the top of the line from (i) each corner of the board, (ii) the middle of the nearest edge.

Ex. 3. Make a freehand sketch of the following:—

Two roads run, one E. and W. and the other N. and S.; A is the point of intersection, B is a point two miles due E. of A, C is a point one mile due N. of B. At C there is a vertical tower CD. Mark the angles of elevation of the top of the tower from A and B. Also name all the right angles in the figure.

Ex. 4. Make a freehand sketch of a regular tetrahedron, showing the altitude of the tetrahedron and the altitudes of the base.

[Glance at fig. 273, then try to draw it.]

Ex. 5. Make a freehand sketch of a pyramid on a square base, showing the altitude and the diagonals of the base.

[Glance at fig. 272.]

Ex. 6. Make sketches of a cube in various positions.

Ex. 7. Repeat Ex. 6 for (i) a triangular prism, (ii) a hexagonal prism, (iii) cylinder, (iv) a cone.

DEVELOPMENT OF SOLIDS.

Ex. 8. Imagine a chalk box placed on the table and cut down each upright edge so that the side faces can be rotated about their bottom edges until they are flat on the table. Make a freehand figure of the resulting shape, noting which lines in the figure are equal.

Ex. 9. Imagine a triangular prism laid on the table with the triangular faces upright; now suppose it cut along all the edges not on the table and flattened out. Sketch the resulting shape.

Ex. 10. Sketch patterns for cutting out to make (i) a pentagonal prism, (ii) an oblique triangular prism.

Ex. 11. If the curved surface of a right cylinder is slit down a generator and opened out flat, what figure is produced? How are the dimensions of the developed figure related to those of the cylinder?

Ex. 12. If a right circular cylinder is cut in two by a plane inclined to its axis, make a sketch of the figure that would be obtained by developing (i.e. laying out flat) the curved surface of one part of the cylinder.

Ex. 13. Sketch the figure obtained by cutting a cone along a generator and developing the curved surface.

Ex. 14. Sketch the figure obtained by cutting a square pyramid along a slant edge and developing the surface (ignore the base).

Ex. 15. Make a sketch of a cube and on it indicate a plane section by a vertical plane; letter the section and make a separate sketch of the section lettered as in the first sketch. Repeat for various other sections.

CHAPTER 21

PRISM AND CYLINDER; SOLIDS OF UNIFORM CROSS-SECTION

§ I. If a solid is sawn straight through, the new surface exposed is called a **section** or **cross-section**.

If a solid, say the leg of a chair or table, is cut by a series of parallel planes and if the cross-sections are all congruent the solid is said to be of uniform cross-section.

Figs. 265, 266 show solids of uniform cross-section.

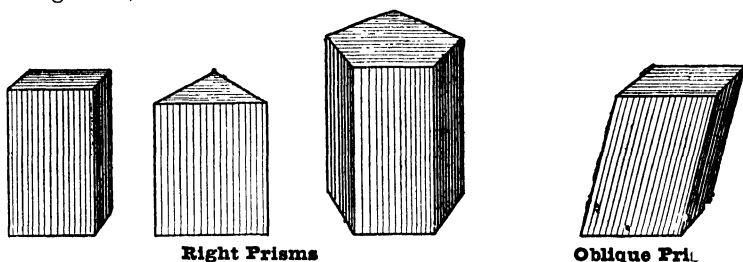


Fig. 265.

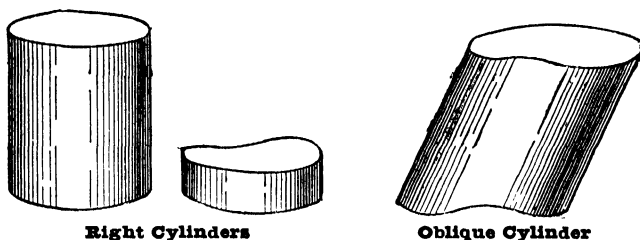


Fig. 266.

Notice that the side edges of a prism are all parallel, and that each of the side faces is parallel to these edges. Also notice that on the side faces of a prism or on the curved surface of the cylinder we can draw a series of straight lines which are all in the same direction (or all parallel to one another); these lines are called **generators**. We can think of the surface of a prism or cylinder as generated* by the motion of a straight line fixed in direction and moving round the boundary of a plane figure.

*See Notes and Answers.

We can also think of a prism or cylinder as swept out by one of its cross-sections moving along a straight line, e.g. the end of the piston sweeps out the cylinder.

Just as a curve is the limiting form of a polygon with an infinite number of infinitely short sides, so **a cylinder is the limiting form of a prism** with an infinite number of infinitely narrow side faces.

A section of a prism or cylinder **normal** (or perpendicular) to the generators is called a **normal section**.

If a prism (or cylinder) has its ends normal to its generators, it is called a **right** prism (or cylinder). If the ends are not at right angles to the generators it is called an **oblique** prism (or cylinder). See figs. 265, 266.

¶Ex. 1. What surfaces are generated when a vertical line moves round (i) a horizontal circle? (ii) a triangle whose plane is horizontal? (iii) a square drawn on the top of a sloping desk?

¶Ex. 2. What are the normal sections of (i) a lead pencil, (ii) a railway sleeper, (iii) a railway line?

¶Ex. 3. Give other instances of prisms and cylinders and state what is the normal cross-section in each case.

¶Ex. 4. Which would have the larger area, the normal section or an oblique section of a prism (or cylinder)?

Ex. 5. Sketch the shape of the shadow cast on the ground by (i) a penny held horizontally, (ii) a sphere, the sun not being overhead.

Ex. 6. A piece of wood in the form of a circular cylinder has a diameter of 4 in. It is sawn across and the section makes an angle of 60° with the axis of the cylinder. Find the greatest and least lengths of the section. Sketch the shape of the section.

Ex. 7. A cuboid of edges 5, 8, 10 cm. is cut in half, diagonally, by a plane through two opposite edges of length 8 cm. Draw the section accurately.

Ex. 8. A cube is cut by a plane passing through two diagonally opposite corners and the mid-points of two opposite parallel edges. If the side of the cube is 2 in., make an accurate drawing of the section.

Ex. 9. A beam of rectangular section 12 in. by 9 in. is used to shore up a wall. If it is inclined at 60° to the wall, find the shape of the section which has to be cut if the beam rests flat against the wall and is in a vertical plane at right angles to the wall.

Ex. 10. A hollow right prism of height 3 ft. stands on a triangular base with equal sides of 1 ft. A straight rod just rests in an angle at the top with its other edge at the middle point of the opposite edge of the base. Find its length. Find also the length of the longest rod which can rest in the prism without projecting over the edge.

Ex. 11. A line is drawn on the surface of a cylinder making a constant angle with all generators. The cylinder is then cut along a generator and opened out flat. Sketch the figure obtained.

Ex. 12. AB, CD are two opposite generators of a circular cylinder of height 4 ft. and diameter 3 ft. P is on AB, 1 ft. from the top and Q on CD, 1 ft. from the bottom of the cylinder. Find the length of the shortest line joining PQ along the surface of the cylinder.

[Imagine the curved surface cut along AB and laid out flat.]

VOLUME OF PRISM AND CYLINDER.

§2. The volume of a cuboid is measured by the product of the length, breadth and height of the cuboid or by the product of the base-area and the height. These rules are particular cases of a rule that applies to any right solid of uniform cross-section (e.g. prisms, rails for railways, T-irons, H-irons, cylinders, any metal goods made by rolling or pulling through a die).

The volume of a right solid of uniform cross-section is measured by the product of the area of the normal section and the height (or length).

A right prism (or cylinder) can be built up with a number of thin cards, congruent with the base. If these cards are made to slide over one another as in fig. 267, an oblique prism (or cylinder) will be formed of the same base, height* and volume as the right prism. Hence **the volume of an oblique prism or cylinder is the product of the base-area and the height.**

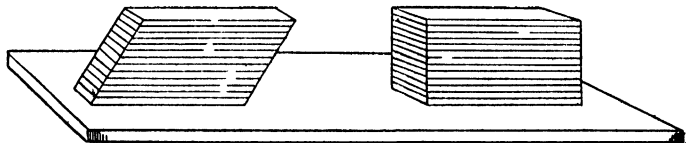


Fig. 267.

The height of an oblique prism or cylinder is the perpendicular distance between the ends.

Ex. 13. Find the surface and volume of the prism whose plan and elevation are given in fig. 268.

¶**Ex. 14.** What is the ratio (i) of the surfaces, (ii) of the volumes of similar cuboids, the linear dimensions of the one being λ times those of the other?

Ex. 15. Show that the **surface of a right circular cylinder** $= 2\pi r(r + h)$, where r is the radius of the base, h the height. (See Chap. 20, Ex. 11.)

Ex. 16. Show that the **volume of a right circular cylinder** $= \pi r^2 h$.

Ex. 17. In fig. 269 is shown the section of a wall 40 feet long, built of masonry weighing 150 lb. per cubic foot. AB and CD are horizontal; BC and DE are vertical. Find the weight of the wall.

Ex. 18. A canvas tent has a floor a ft. square. Its top is a horizontal ridge at a height of b ft., the back and front are vertical, and the side walls are vertical for a height of c ft. from the ground. Find an expression for the area of the surface, and give a numerical result when $a=8$, $b=7$, and $c=2$. Also find the cubic content of a tent with these dimensions.

Ex. 19. The end view of a trough is an isosceles trapezium whose horizontal sides are 12" and 16", the short side being the lower; the depth of the trough is 10", its length is 3' and the ends are vertical. Find in cubic feet the volume of water in the trough when it is x " deep.

Ex. 20. The cross-section of a solid tyre is a rectangle whose sides are a inches and b inches, the former being the thickness of the tyre from road to wheel and the latter its breadth measured across the top of the wheel. If the outer diameter of the tyre is D inches, what is its inner diameter?

Assuming the formula for the volume of a cylinder, and considering the tyre as the difference of two solid cylinders, prove the formula $V = \pi ab(D - a)$ for the volume of the tyre in cubic inches.

Calculate D to the nearest tenth of an inch, having given $V=600$; $a=2.3$; $b=4.2$.

Ex. 21. From a rectangular piece of cardboard measuring a in. by b in. ($a > b$), a cylinder with closed ends is to be made as follows:—A strip is to be cut off from one end, from which two complete discs of circumference b to form the ends can be taken; the remainder is rolled so as to form the body of the cylinder, the two sides of length b becoming circles of circumference b . There will of course be some waste of cardboard.

Find the length and radius of this cylinder, and prove that its volume is $b^2(\pi a - b)/4\pi^2$ cubic feet. It may help you if you make a drawing in the case when $a=3$, $b=1$.

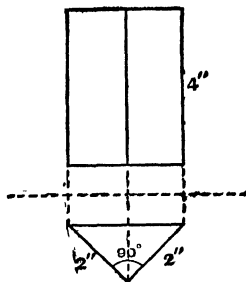


Fig. 268.

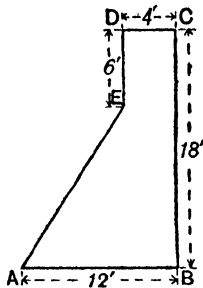


Fig. 269.

Ex. 22. The shaded rectangle in fig. 270 is rotated about AB so as to generate a ring. Prove that the whole surface and the volume of this ring are the same as the side-surface and volume of a right prism whose cross-section is the rectangle, and whose height is the length of path described by the centre of the rectangle during a complete turn. (This is a particular case of an important theorem due to Pappus, who taught at Alexandria towards the end of the 3rd century A.D.; the theorems were republished by Guldinus (1577-1643).)

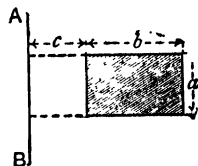


Fig. 270.

Ex. 23. The section of a uniform wooden beam by a plane perpendicular to its length is a triangle ABC, of which the sides BC, CA, AB are respectively a , b , c inches long.

If an oblique section be taken cutting the parallel edges through A, B, C in points A', B', C' respectively, and such that AA' is x inches, BB' is y inches and CC' is z inches, obtain expressions for the lengths of the sides of the triangle A'B'C' by drawing each of the side faces between the sections. Then in the numerical case when

$$a=2, \quad b=2.3, \quad c=4.2, \quad x=12, \quad y=13.5, \quad z=14.3,$$

draw the triangle A'B'C', and determine its area.

Ex. 24. Make a scale drawing of a vertical section of the wall in Ex. 17 inclined at an angle of 45° to the face of the wall.

For further exercises on Rectangular Solids see Ch. 23, p. 212.

SUMMARY OF RESULTS

The volume of a solid of uniform cross-section is measured by the product of the area of the normal section and the height (or length).

The volume of an oblique prism or cylinder is the product of the base-area and the height.

The surface of a right circular cylinder = $2\pi r(r + h)$.

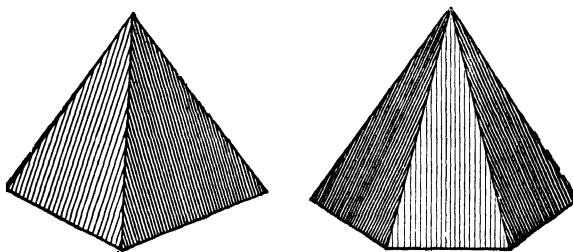
The volume of a right circular cylinder = $\pi r^2 h$.

CHAPTER 22

THE PYRAMID AND CONE

THE PYRAMID.

§ I. Fig. 271 shows examples of **pyramids**.



Pyramids

Fig. 271.

A pyramid is called **regular** if its **base** is a regular polygon, and the line joining its **vertex** to the centre of the base is perpendicular to the base.

¶Ex. 1. What are the side faces of a regular pyramid?

¶Ex. 2. If the base of a pyramid is an n -gon, what is the number of (i) side faces, (ii) faces, (iii) edges, (iv) corners?

Ex. 3. In a pyramid $OABCD$, the base $ABCD$ is a square 6 cm. in the side (see fig. 272). N is the middle point of AB . The centre of the base is M , and the line OM is perpendicular to all lines lying in the base, such as MA and MN ; OM is 4 cm. long. Draw the square base; then draw the right-angled triangles ONN and OMA , and measure the lengths ON and OA . Then draw the face OAB and give its area.

Without further reference to fig. 272, draw the pyramid. First draw the base and the lines on it and then draw MO parallel to the edge of your paper.

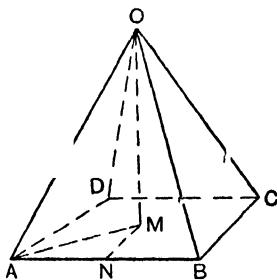


Fig 272.

§2. A pyramid whose base is a triangle (a triangular pyramid) is also called a **tetrahedron**, as it has four faces. A tetrahedron whose four faces are equilateral triangles is called a **regular tetrahedron**; the four faces are obviously congruent.

Ex. 4. Consider fig. 273, which represents a regular tetrahedron. Each face is an equilateral triangle, VO is its altitude, the three lines through O are the altitudes of the triangle ABC. Make accurate drawings of

- (i) Triangle ABC and the lines AD, BE, CF;
- (ii) Triangle VAC, showing VE;
- (iii) Triangle VBE, getting the lengths of BE and VE from (i) and (ii). Draw VO perpendicular to BE and check that $OE = \frac{1}{3} BE$.

Without further reference to fig. 273, draw a general view of the tetrahedron like the figure.

It is best first to draw the base and the various lines on it and then draw OV parallel to the edge of your paper.

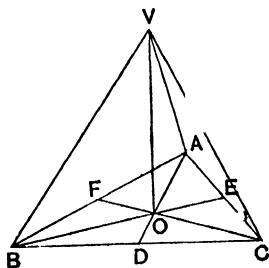


Fig. 273.

§3. A section of a pyramid by a plane parallel to the base is similar to the base.

The pyramid $OA'B'C'D'$ is clearly similar to the pyramid $OABCD$.

In accordance with the general principles of similarity:—

(i) Any line associated with $OA'B'C'D'$ is to the corresponding line associated with $OABCD$ in the ratio of the linear dimensions of the pyramids (e.g. the heights of the pyramids are proportional to corresponding side edges, or to corresponding base edges).

(ii) Corresponding areas in the two pyramids are in the ratio of the squares of the linear dimensions (e.g. the areas of $A'B'C'D'$ and $ABCD$ are proportional to the squares of the heights of the pyramids).

(iii) Corresponding volumes in the two pyramids are in the ratio of the cubes of the linear dimensions (e.g. the volumes of the pyramids are proportional to the cubes of their heights).

If a pyramid (or a cone) is divided by a plane parallel to the base the two pieces are respectively a pyramid (or cone) and a **frustum**

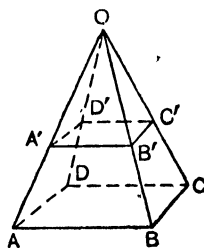


Fig. 274.

of a pyramid (or cone). The small pyramid (or cone) is obviously similar to the whole figure.

In dealing with a frustum, regard it as the difference between the two similar pyramids (or cones) referred to above.

Ex. 5. In what ratio must the height of a pyramid be divided to obtain a section whose area is half that of the base?

Ex. 6. A model of a pyramid is made on a scale of 1 to 10. Compare (i) the surfaces, (ii) the volumes of the pyramid and model.

Ex. 7. A pyramid with vertex O stands on a triangular base ABC . A plane parallel to the base cuts the edges OA, OB, OC , in A', B', C' , respectively. Show that the triangles $A'B'C'$ and ABC are similar.

If $OA' = \frac{1}{3} OA$, what is the ratio of (i) the surfaces, (ii) the volumes of the pyramids $OA'B'C', OABC$?

Ex. 8. Draw a figure suitable for cutting out to make a regular tetrahedron of edge 6 cm.

Ex. 9. A right pyramid with a base 1 in. square has to be made from a circular piece of cardboard of 5 in. diameter. Draw a figure with the square in the centre of the circle and show where the cardboard must be cut. What is the length of a slant edge of the pyramid?

Ex. 10. A pyramid is to have a rectangular base, the sides being 8 cm. and 4 cm., and the length of each slant edge is to be 6 cm. Draw a figure suitable for cutting out to make a model of the pyramid, and find what the height of the pyramid would be.

Ex. 11. A pyramid on a base 3 in. square has slant edges of 4 in. A plane section cuts the slant edges taken in order 2 in., 2 in., 3 in., 3 in. from the vertex. Find, by drawing, the true shape of the section and deduce its area. (See fig. 298.)

Ex. 12. $ABCD$ is a regular tetrahedron each edge of which measures 12 cm. $AE = AF = 4$ cm. (see fig. 275). The shaded area represents a section through C, E, F . Draw the side faces of the tetrahedron; then draw the triangle CEF and find its area.

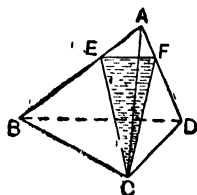


Fig. 275.

Ex. 13. In fig. 275, G is on AC such that $AG = 6$ cm. Find, by drawing, the dimensions of the triangle EFG .

Ex. 14. The ends $ABCD$ and $PQRS$ of a rectangular block are equal squares of side 3 in. The edges PA, QB, RC, SD are each 3.6 in. A pyramid $VABCD$ is inscribed in the block, having its vertex V at the intersection of the diagonals of the square $PQRS$.

Draw full size the section of the figure by the plane $PRCA$ and draw also the true shape of VAB , one of the sloping faces of the pyramid. Calculate the length of VA , one of the sloping edges of the pyramid, and check by measurement from your figure.

Ex. 15. Find the height of a regular square pyramid whose base edge is 2" and whose slant edge is 5".

Ex. 16. A tripod is composed of three rods, each of length 6 ft. It is placed so that the three points of support form an equilateral triangle of side 2 ft. Find the angle between two rods, and the height of the apex of the tripod above the ground.

Ex. 17. A pyramid of height 5" stands on a square base of side 2.75". It is cut by a plane, at a distance of 1.5" from the vertex, parallel to the base. Find the side of the section.

Ex. 18. A frustum of a pyramid stands on an octagonal base of side 4" and the top has a side of 3.5", the height of the frustum being .75". What was the height of the complete pyramid from which the frustum was formed?

Ex. 19. Find the whole surface area of a frustum of a pyramid cut off by a section which bisects the height of the pyramid whose base is a square of side 6 ft. and height 4 ft.

Ex. 20. A pyramid standing on a base 20 cm. square and of height 12 cm. is placed vertex downwards in a cubical box of side 15 cm. What height is the vertex above the bottom of the box?

Ex. 21. Find the ratio of the height of a regular tetrahedron to its edge.

Ex. 22. A weight is suspended from three points in a roof at the vertices of an equilateral triangle of side 12 ft. The suspending chains are 15 ft. long. What depth is the weight below the level of the points of suspension and what angle does each chain make with the vertical?

Ex. 23. Three equal spheres of radii 5 in. are held in contact with each other and a fourth equal sphere is placed upon them. What is the height of this fourth sphere above the plane on which they rest, and the angle made with the vertical by the line joining the centres of the upper and a lower sphere?

Ex. 24. Four thin rigid rods 6 ft. long are fixed to the floor at the corners of a square of side 2 ft. 9 in. and are hinged together at a height of 3 ft. above the floor. Their other ends support a square board, each end being at a corner of the board. Calculate the length of an edge of this board and its height above the floor.

Ex. 25. Three equal bars of length 6 ft. are hinged together at one end with their other ends fixed to three points in a horizontal plane at the vertices of a triangle of sides 4 ft., 4 ft., 3 ft. Find the height of the hinge above the horizontal plane.

Ex. 26. The base of a pyramid has edges of 6, 8, 10 cm. and the other edges are of length 7 cm. Find by calculation the height of the pyramid.

Ex. 27. A tripod is formed with three equal rods of length 5 ft. fastened together at one end with their other ends stuck into the ground at the vertices of a triangle of sides 3 ft. 6 in., 3 ft., 2 ft. 6 in. Find the height above the ground of the ends fastened together.

Ex. 28. Two beams PA , PB , each 20 ft., are fastened at one end P and their other ends A , B rest on the ground 12 ft. apart. They are supported by another beam PC 25 ft. long fixed to the ground at C , 30 ft. from both A and B . Find, by drawing or calculation, the height of P above the plane ABC .

Ex. 29. Two beams AB , AC , each 10 ft. long resting on horizontal ground at points B and C , 8 ft. apart, are prevented from falling by a rope 25 ft. long fixed to the ground 20 ft. from the mid-point of BC and in a line perpendicular to BC . Find, by drawing or calculation, the height of A above the ground.

THE VOLUME OF A PYRAMID.

§4. If two pyramids of equal height and on bases of equal area are placed on a horizontal plane and cut by another horizontal plane the sections will be of equal area. Hence

Pyramids of equal height and on bases of equal area have the same volume.

(Think of the pyramids as built up of cards as in fig. 267.)

A triangular prism may be divided into three pyramids of equal volume.

Figs. 276, 277 show a triangular prism divided into three triangular pyramids. Any two of these pyramids may be shown to have equal heights and equal bases, and therefore equal volumes.

For pyramids marked I and III have congruent bases PQR , ABC and the same height—namely the height of the prism.

Pyramids marked II and III have congruent bases PQA , BAQ and the same height—namely the perpendicular from C upon $PQBA$.

Hence the three pyramids are of equal volume, and each volume is $\frac{1}{3}$ that of the prism.

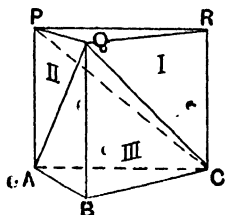


Fig. 276.

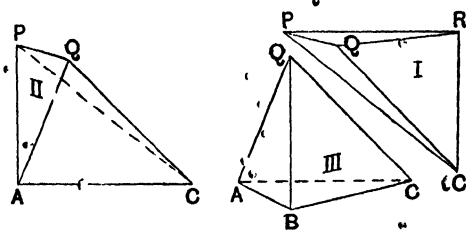


Fig. 277.

We have thus shown, in the case of pyramid $QABC$, that the volume of a triangular pyramid is $\frac{1}{3}$ of that of a prism on the same base, and of the same height.

This may be extended, at once, to any pyramid. Divide the base of the pyramid into triangles; then the pyramid can be divided into a number of triangular pyramids with a common vertex. If S_1, S_2, S_3 , etc., be the areas of the triangles, the volumes of the triangular pyramids are $\frac{1}{3} S_1 h$, $\frac{1}{3} S_2 h$, etc., and the volume of the whole pyramid is $\frac{1}{3} h (S_1 + S_2 + S_3 + \dots)$, i.e. $\frac{1}{3}$ of the volume of the prism having the same base and height.

Hence the volume of a pyramid is one-third of the volume of a prism of the same base and height, or one-third of the product of base and height.

Ex. 30. A pyramid, 4 feet vertical height, stands on a horizontal square base of 2 feet side. The top of the pyramid is cut off by a horizontal plane which bisects the axis. Find the volume of the whole pyramid and of the part cut off.

Ex. 31. A vessel containing water is in the form of an inverted hollow pyramid; its base is a square of side 6 ft., and the altitude of the pyramid is 10 ft. How much water will the vessel hold? If the depth of the water is x ft., what is its volume?

Ex. 32. A pyramid stands on a square base of sides b in. and the sides of the pyramid are isosceles triangles. The top of the pyramid is cut away by a plane parallel to the base, exposing a square section of side a in. Make a sketch of the solid that remains. If the perpendicular distance between the square faces is h inches, prove that V , the volume in cubic inches, may be calculated from the formula $V = \frac{1}{3} h (a^2 + ab + b^2)$.

What does the solid become when $a=b$ and when $a=0$? Deduce the formula for the volume in each of these special cases from the more general formula just given.

Ex. 33. Cleopatra's Needle consists approximately of a pyramid cut off horizontally and surmounted by a smaller pyramid. The lower base is $7\frac{1}{2}$ ft. square, and the upper base $4\frac{1}{2}$ ft. square; the distance between these bases is 61 ft. and the height of the upper pyramid $7\frac{1}{2}$ ft. Calculate the weight to the nearest ton, it being estimated that 1 cubic foot weighs about 170 lb.

Ex. 34. Fig. 278 represents a heap of stones. The base is a rectangle 10 ft. by 4 ft.; the ridge is 6 ft. long and 2 ft. above the base. Find the volume of the heap.

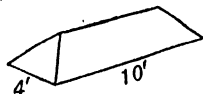


Fig. 278.

Ex. 35. OX, OY, OZ are three straight lines at right angles to one another; a plane cuts them at A, B, C , so that $OA=3$ in., $OB=OC=4$ in. Find the volume of the tetrahedron $OABC$, the area of the triangle ABC , and hence the length of the perpendicular from O on ABC .

Also show the true shape of ABC , and find the length of the perpendicular from O on ABC , both by geometrical construction; state your method.

Ex. 36. Fig. 279 represents a cube. The corners A, F, C, H are cut off by the planes BDE, BEG, BDG, DGE respectively. Show that the remaining solid (fig. 280) is a regular tetrahedron.

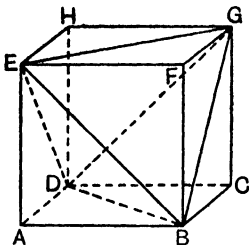


Fig. 279.

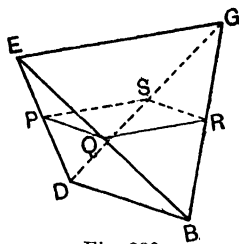


Fig. 280.

If the length of an edge of the original cube is a , find the volume of the tetrahedron in terms of a . Find also the volume of the tetrahedron in terms of the length l of one of its own edges.

If P, Q, R, S are the middle points of ED, EB, GB, GD , what is the shape of the quadrilateral $PQRS$?

Ex. 37. Fig. 281 shows a square $ABCD$ and an equilateral triangle ABE on opposite sides of a common base AB . The diagonals BD and AC of the square cross at O , and the triangle DOC is cut away. Suppose that the triangles AOD and BOC are turned about AO and BO respectively, each through a right angle, thus bringing OD and OC together, at right angles to the plane AOB , which is supposed horizontal. In this way the points C and D will coincide at a position F situated vertically over the point O . Then suppose that the triangle ABE is turned about AB until AE and BE lie along AF and BF respectively. Determine the number of degrees through which the triangle ABE must be turned round AB .

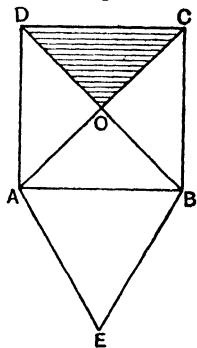


Fig. 281.

Ex. 38. $ABCD$ (fig. 282) represents a square of paper, K and L are the mid-points of BC and CD , and the dotted lines show creases made by folding the paper along AK, AL, KL . A pyramid could be formed by folding the paper along the creases and bringing the edges AB, AD together and also CK, CL against the edges BK, DL .

If the length of the side of the square is 15 cm., calculate (i) the lengths of the edges AK, KL ; (ii) the areas of the separate faces I, II, III, IV of the pyramid; (iii) the volume of the pyramid (you will observe that when the pyramid is formed the edge AB is perpendicular to the face LDL).

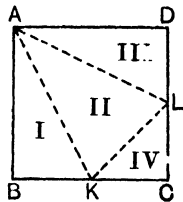


Fig. 282.

Ex. 39. A rectangular sheet of paper, $ABCD$, measures 6 inches by 4 inches. E is the middle point of AB , which is one of the longer sides. The paper is folded along CE and DE , and the triangles AED and BEC are folded up until EA coincides with EB , A and B meeting at a point which we shall call P . Find (i) the perpendicular distance of E from the plane DPC , (ii) the volume of the tetrahedron $DECP$, (iii) the perpendicular distance of P from the plane DEC .

For further exercises on Pyramids see Ch. 23, p. 214.

THE CONE.

§5. In fig. 283 we have some examples of **cones**, standing on plane bases.

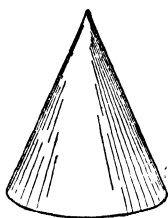


Fig. 283.

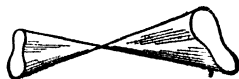
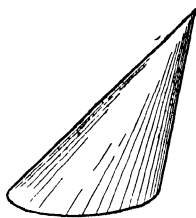


Fig. 284.

NOTE. A cone is sometimes conceived as **infinite**, i.e. having no plane base, but prolonged infinitely from the vertex. A still wider conception is that of the **infinite double cone**, see fig. 284.

Ex. 40. Describe how an infinite cone can be generated by the motion of a straight line. Such a straight line is called a **generator**.

Ex. 41. What can be asserted with regard to sections of a cone parallel to the base?

Cone as limiting form of pyramid. Just as a cylinder is the limiting form of a prism, so a cone is the limiting form of a pyramid.

Right circular cone. If the base of a cone is circular, and the **axis**, i.e. the line joining the vertex to the centre of the base, is perpendicular to the base, the cone is said to be a **right circular cone**. The generators in this case are all equal, and their length is called the **slant height** of the cone.

The section of a right circular cone by a plane through the axis is an isosceles triangle; the angle at the vertex of this triangle is called the **vertical angle** of the cone.

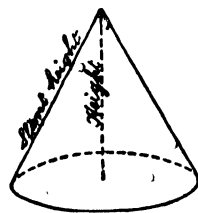


Fig. 285.

¶Ex. 42. Describe how a right circular cone can be swept out by the rotation of a plane figure.

¶Ex. 43. Express the vertical angle of a cone in terms of two measurable dimensions of the cone.

¶Ex. 44. Describe in words the figures generated by the rotation of

- (i) an obtuse-angled triangle about its longest side,
- (ii) an obtuse-angled triangle about a shorter side,
- (iii) a trapezium about the longer of the parallel sides,
- (iv) a trapezium about the shorter of the parallel sides.

¶Ex. 45. What is the section of a frustum of a cone by a plane containing the axis? (See fig. 286.)

¶Ex. 46. If the curved surface of a right circular cone is slit down a generator, and opened out flat, what figure is produced? How are the various measurements of this figure related to measurements of the cone?

¶Ex. 47. Repeat Ex. 46 for a frustum of a right circular cone.

¶Ex. 48. If the curved surface of a cone of radius r and slant height l is developed into a sector, what fraction of 360° is the angle of the sector?

¶Ex. 49. How do (i) the circumferences, (ii) the areas of sections of a cone vary with the distances of the section from the vertex?

Ex. 50. A ring, diameter 24 in., is suspended by six equal strings from a point 5 in. above its centre, the strings being attached at equal intervals round its circumference. Find the length of each string and the angle between consecutive strings.

Ex. 51. A halfpenny (diameter 1 in.) is placed in a cone (vertex downward) of height 3 in. and base-radius 2 in. so that its plane is perpendicular to the axis. What is the depth of the vertex below its plane?

Ex. 52. AB is one-quarter of the circumference of the base of a cone of base-radius 5 cm. and height 8 cm. P is the middle point of the generator through B. Find the length of the shortest line on the surface of the cone joining A to P. Find also the length of the direct line AP.

Ex. 53. The two ends of the frustum of a cone are of diameter 5 cm. and 12 cm. and its height is 9 cm. Find the length of the longest straight stick that could be placed inside it.

Ex. 54. Find the slant height of the frustum in Ex. 53 and the slant height of the cone of which it is the remaining part.

Hence, by drawing the figure obtained on developing the surface, find the length of an elastic band stretched tightly from a point on one end, to the diametrically opposite point on the other end.

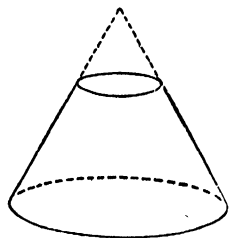


Fig. 286.

Ex. 55. A conical beam of light illuminates a circle of 1 ft. radius at a distance of 5 ft. What is the area illuminated at a distance of 12 ft.?

At what distance from the light must a screen 30 ft. by 18 ft. be placed so that the whole of it is illuminated?

SURFACE AND VOLUME OF CONE.

§6. The surface of a right circular cone is usually taken to include the base. If the base is excluded, the term **curved surface** is generally used.

By slitting the curved surface of a right circular cone along a generator and developing it, a sector of a circle is formed, the radius of which is the slant height of the cone and the arc of which is the circumference of the base. Hence, the area of the curved surface is half the product of the slant height and the circumference of the base*. (See §3, p. 119.)

Ex. 56. Find the whole surface of a cone of height h and base-radius r .

Ex. 57. Find, in terms of the curved surface of the original cone, the curved surface of a frustum of a cone whose height is $\frac{1}{n}$ that of the original cone. Use the principle of similarity.

Volume of cone. The volume of a pyramid is $\frac{1}{3}$ that of a prism of the same base and height. Proceeding to the limit, we see that the volume of a cone (of any shape) is $\frac{1}{3}$ that of a cylinder of the same base and height, and is therefore **one-third of the product of the base area and the height**.

¶Ex. 58. Express the volume of a right circular cone in terms of the base-radius and the height.

¶Ex. 59. Express the volume of a right circular cone in terms of (i) height and semi-vertical angle, (ii) height and slant height.

Ex. 60. Find the surfaces and volumes of the three solids generated by rotating a triangle of sides 3", 4", 5" about its sides

Ex. 61. A solid is generated by rotating a parallelogram of given base and height about its base. Show that the volume is independent of the angle of the parallelogram.

* The area of the sector may also be obtained thus:—If l is the slant height and r the radius of the base,

$$\frac{\text{Area of sector}}{\text{Area of whole circle } (\pi r^2)} = \frac{\text{arc of sector}}{\text{circumference of whole circle}} = \frac{2\pi r}{2\pi l}.$$

Area of sector, that is, the area of the curved surface of the cone $= \pi r l$.

Ex. 62. What area of canvas will be required for a conical tent, the diameter of the base being 20 feet and the vertical height 15 feet? How many cubic feet of air will the tent hold?

Ex. 63. A conical funnel rests inside a cylindrical jar as in fig. 287; the circumferences of the jar and the funnel edge are 10" and 14" respectively, and the vertex of the funnel is 5" below the rim of the jar. How much water would the funnel hold?

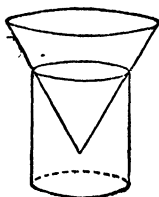


Fig. 287.

Ex. 64. The mouth of a stable bucket (fig. 288) is 13 inches in diameter, the base $8\frac{1}{2}$ inches in diameter, and the slant side measures 9 inches. Draw a vertical section through the axis of the bucket, and find by calculation the height of the bucket and the height of the cone formed by producing the slant sides beyond the base. How many gallons will the bucket hold? (1 cubic foot = $6\frac{1}{4}$ gallons.)



Fig. 288.

Ex. 65. Find the surface and volume of the frustum of a cone in terms of the radii of the ends, a and b , and the height h .

Ex. 66. A canvas conical tent, of capacity 600 cubic feet, is to be constructed on a circular base of area 160 sq. feet. Find, to the nearest square foot, the number of square feet of canvas required.

Also find an expression for the number of square feet required in terms of C , the capacity in cubic feet, and of A , the area of the base in square feet.

Ex. 67. A sector of a circle of radius 10 in. has an angle of 240° . This is cut out of cardboard and bent to form a conical lampshade. Ignoring the overlap, determine the radius of the circular base of the shade and its height.

Ex. 68. A hole is to be dug in loose sand to a depth of 10 feet and with a flat circular bottom 12 feet in diameter. As the hole is being dug the sand continually falls in, and settles down at a slope of 34° to the horizontal. The digging is continued until the bottom is cleared with the face sloping up from it at this angle, the hole forming the frustum of a cone. Calculate the volume of sand in cubic feet which must be dug out.

Ex. 69. Fig. 289 shows a section of a cylinder of radius r and height h inscribed in a cone of radius R and height H .

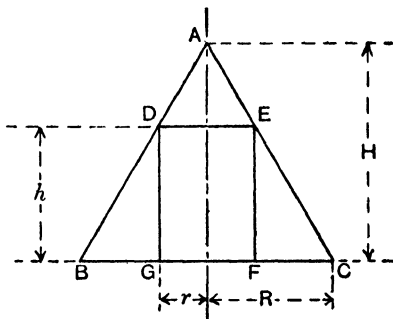


Fig. 289.

Show that the volume of the cylinder can be expressed in the form :—

$$V = \frac{\pi R^2}{H^2} (H^2 h - 2Hh^2 + h^3).$$

If R and H are both equal to unity, find the values of V when $h=0, 0.1, 0.3, 0.5, 0.7, 0.9, 1$; plot a graph showing the relation of V and h . State from the graph the approximate value of h which makes V a maximum.

SUMMARY OF RESULTS

The volume of a pyramid or cone is one-third of the product of base-area and height.

CHAPTER 23

INCLINATION OF LINE TO PLANE AND PLANE TO PLANE

PLANE AND LINE AT RIGHT ANGLES.

§ I. It is obvious that **if a line is perpendicular to a plane, it is perpendicular to any line that lies in that plane and meets it.**

It is sufficient, however, if the line is perpendicular to *two lines* in the plane. The reader should convince himself that it is not sufficient if the line is perpendicular to one line in the plane. Hold a set square with one of the short edges in a plane; need the other short edge be perpendicular to the plane?

Now take a piece of paper with a straight edge; fold this edge upon itself so as to form two right angles AOB , COB ; open out the paper

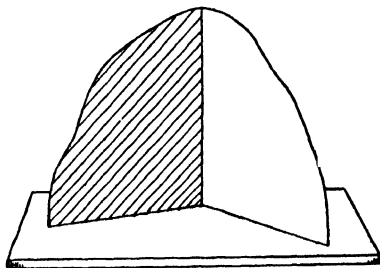


Fig. 290.

into two planes, and stand it upon the table with OA , OC on the table (see fig. 290). Then OB is perpendicular to OA , OC in the plane of the table; and is seen to be perpendicular to the table.

¶Ex. 1. Use two set squares to erect a line perpendicular to a plane.

¶Ex. 2. If a right angle be rotated about one leg, what does the other leg generate?

¶Ex. 3. Explain how to use the 3, 4, 5 right-angled triangle to set up a pole vertically.

¶Ex. 4. What can be asserted (i) about a set of straight lines perpendicular to a plane? (ii) about a set of planes perpendicular to a straight line?

The distance of a point from a plane is the length of the straight line drawn from the point perpendicular to the plane.

It is obvious that parallel planes are equidistant, and that the locus of points at a constant distance from a plane is a pair of planes parallel to the original plane.

¶Ex. 5. What is the locus of points in space at a constant distance from a straight line?

¶Ex. 6. What is the locus of points in space equidistant from two points? from two parallel lines? from two parallel planes? from two intersecting lines? from two intersecting planes?

LINE INCLINED TO PLANE.

§ 2. The inclination of a straight line to a plane may be defined as follows:—Suppose that OP is the given line (see fig. 291); from P draw PN perpendicular to the plane; then the angle NOP is the angle which the line makes with the plane.

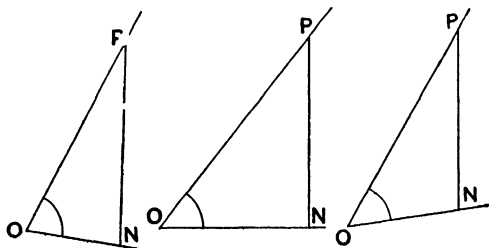


Fig. 291.

¶Ex. 7. A straight line AB meets a plane in A ; and a straight line AC , lying always in the plane, revolves round A . Between what limits does the angle BAC vary?

Ex. 8. A straight line CA always passes through a fixed point C , and moves so as to meet a fixed plane at a constant angle in A . Find the locus of A . What does CA generate?

Ex. 9. Find the inclinations of a diagonal of a cube to the various faces.

Ex. 10. What is the inclination of the generators of a right circular cone to the base, if the slant height is 5 in., the diameter of the base 8 in.?

INCLINATION OF TWO PLANES.

§ 3. Take any point O on the line of intersection of the planes; draw ON , OP in the two planes at right angles to the line of intersection (see fig. 292). The **inclination** of the planes (or the **angle between** these planes) is measured by the angle between these two lines. This angle is clearly independent of the position of O ; if another pair of perpendiculars $O'N'$, $O'P'$ were drawn, these lines would be parallel respectively to ON , OP ; and all the $\angle$ s NOP , $N'O'P'$, ACB , $A'C'B'$ would be equal.

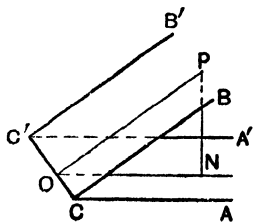


Fig. 292.

¶Ex. 11. Open a book to a right angle: try if you can fit the acute angle of a set square (or an acute angle formed by a pair of dividers) into the opening, so that the legs, though they include an acute angle, lie one on each of two planes at right angles. Given the position of the vertex of the acute angle, is the position of the set square determinate?

¶Ex. 12. Name the angles between the following pairs of planes in fig. 293:—(i) HEB , DAB , (ii) HAB , DAB , (iii) HGB , EFG .

¶Ex. 13. Name the angle between the planes OAB , $ABCD$ in fig. 296.

¶Ex. 14. Name the angle between the line HB in fig. 293 and the planes (i) $ABCD$, (ii) $BCGF$, (iii) $CDHG$.

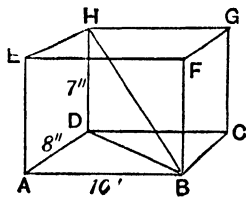


Fig. 293.

¶Ex. 15. Name the angle OA makes with the base $ABCD$ in fig. 296.

Ex. 16. AB , CD are opposite edges of a cube. Find the inclination of the plane $ABDC$ to the faces of the cube.

Ex. 17. A room is 20 ft. long, 16 ft. wide, and 12 ft. high. A string is stretched diagonally from one corner of the floor to the opposite corner of the ceiling. By drawing and measurement determine approximately, in degrees, the inclinations of the string (i) to the floor of the room, (ii) to one of the longer sides of the floor.

Ex. 18. A cubical block of edge 4 ft. rests on a table; the base is $ABCD$, and P , Q , R , S are the corners above A , B , C , D respectively. If the edge CD is raised 2 ft., AB remaining on the table, find the height of R above the table, and the inclination of AR to the table.

Ex. 26. In fig. 296, $OABCD$ is a pyramid on a square base. N is the middle point of AB and M is the centre of the base. $AB=6$ cm., $OM=4$ cm. Find the inclination to the base (i) of the plane OAB , (ii) of the line OA .

Ex. 27. The height of the Pyramid of Cheops is 149 metres, and the base is a square of side 233 metres. Find the slope of a face, and the slope of an edge.

Ex. 28. The top of a desk slopes at an angle of 10° ; and a chalk line is drawn upon it, inclined at 60° to the lines of steepest slope. What angle does the chalk line make with the horizontal plane?

Ex. 29. A set square has angles 30° , 60° , 90° . It is placed with its shortest edge, 4 inches in length, in contact with a table; the third corner is raised 4 inches above the table. Find the inclinations to the table of the two remaining edges.

Ex. 30. A 60° set square is inclined at an angle of 40° to a horizontal plane, its longest side being in the plane. Find the inclination of each of the other sides to the horizontal plane.

Ex. 31. Calculate the angle between two faces (i) of a regular tetrahedron, (ii) of a regular octahedron.

Ex. 32. A regular pyramid, vertex V , stands on a triangular base ABC , side $3''$. Its height is $10''$. Find the angle between

- (i) The face VAB and the base ABC , (iii) The edge VA and the base ABC ,
(ii) The faces VAB and VBC , (iv) The edge VA and the face VBC .

Ex. 33. A regular pyramid, whose vertex is V , stands on a square base $ABCD$. Each side of base $=12$ ft., and edge $VA=15$ ft. P is a point in VA such that $PA=10$ ft. O is the middle point of the square base $ABCD$. Find (i) distance OP , (ii) angle which OP makes with the face PAB .

Ex. 34. $ABCD$ is a cardboard square with O as centre, and the triangle AOD is cut away. The cardboard that remains is bent along OB and OC , so that the edges OA and OD come together. The cardboard now forms three faces of a triangular pyramid, or tetrahedron: let it be placed with the fourth face (an equilateral triangle) on a horizontal surface. Calculate (i) the height of this pyramid if $AB=1$ ft., (ii) the inclination of a sloping edge to the base-plane, (iii) the inclination of a sloping face to the base, (iv) the angle between two side faces.

Ex. 35. A hanging lamp is supported by three chains of equal length. The chains are fixed to points A , B , C in the ceiling which form an equilateral triangle of side 16 in., and the lower ends are connected at a point 3 ft. below the ceiling. Calculate the distance from A , B , or C of the point on the ceiling which is vertically above the point at which the chains are connected; hence, find the length of each of the chains, and its inclination to the vertical.

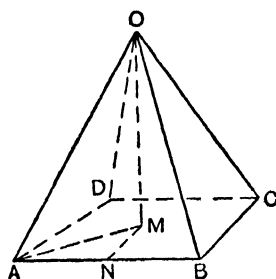


Fig. 296.

Ex. 36. The jib of a crane makes an angle of 30° with the vertical; if the crane swing through an angle of 90° about its vertical axis, find the angle between the first and last positions of the jib.

Ex. 37. Fig. 297 represents a symmetrical roof; find the angle between the two planes that meet at the ridge.

Ex. 38. Two roof ridges meet at right angles and each roof slopes at an angle of 30° to the horizon; find the inclination to the horizon of their line of intersection.

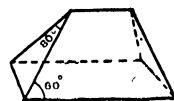


Fig. 297.

Ex. 39. $ABCDE$ (fig. 298) represents a wooden pyramid on a square base. Each edge of the pyramid is 12 cm., so that each of the triangular faces meeting at the apex A is an equilateral triangle of side 12 cm.

Imagine points K, L, M, N taken along the edges running from A , such that AK and AL are each 8 cm., while AM and AN are each 4 cm. By cutting away the top part of the pyramid, so that the edge of the saw passed through K, L, M, N , the section $KLMN$ shown shaded would be obtained. Find the lengths of the sides of this section either by drawing the equilateral triangles full size, or by calculation, making your method clear; then give a full-size drawing of $KLMN$. How does the symmetry of the figure help you?

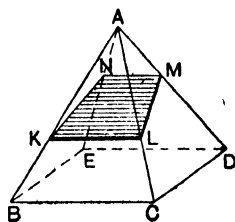


Fig. 298.

Let P, Q be the mid-points of the edges BC, ED . APQ is the plane of symmetry: draw to full scale the section made by this plane: and hence determine the inclination of the shaded section to the base of the pyramid.

Ex. 40. Cut a sheet of paper into some such shape as $ACDE$ (fig. 299). Then fold AB at right angles to BC , and place the paper before you on the desk or table (supposed horizontal) with AB, BC on the desk.

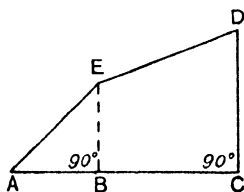


Fig. 299.

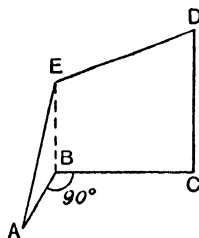


Fig. 300.

Imagine now that AED (fig. 300) represents a path on a hill side, the slope AE having a vertical rise of 1 ft. for every 5 ft. horizontal and ED a rise of 1 ft. in every 12 ft. If AB is a mile and BC three-quarters of a mile, find how many feet E is higher than A , and D than E ; also calculate the length of AC . What would be the average rise in passing direct from A to D ? give the result in the form 1 ft. vertical to x ft. horizontal.

Ex. 41. A hill surface may be assumed to be a plane with a uniform slope of 25° to the horizontal. You climb the hill by following a path, the centre line of which makes an angle of 20° with the line of greatest slope of the hill surface. Find how many feet you rise vertically when you have walked 1000 feet along the path, that is, determine the gradient of the path.

Ex. 42. Standing due S. of a tower 180 ft. high, the angle of elevation of the top is 30° . What is the angle of elevation after walking a distance of 100 yards due E.?

Ex. 43. See fig. 301, and suppose that A, C, and H are three points on the level and B an observation balloon directly above H. You are to determine the distance CB and the height HB of the balloon by drawing. For the triangle BAC use the data $AC = 850$ ft., $\angle BAC = 48^\circ$, $\angle BCA = 90^\circ$, scale 1 cm. to 100 ft. and for the triangle BCH use the additional data $\angle BCH = 32^\circ$, $\angle H = 90^\circ$. State your construction.

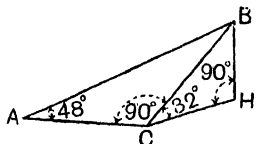


Fig. 301.

Ex. 44. Two searchlights A and B are placed on the same level 1 mile apart, B being N. 40° E. of A. They are simultaneously trained on to a Zeppelin, which is discovered to be due north of A at an elevation of 20° , and to be due west of B. Find in feet the height of the Zeppelin.

Ex. 45. In order to determine the height of the top P of a hill above a station A, a horizontal base-line AB is set off 2000 feet long. By means of a theodolite at A the angle of elevation of P is found to be $20^\circ 30'$, and the angle NAB is found to be $75^\circ 20'$, N being the point vertically below P on the same level as AB. The theodolite is then set up at B, and the angle NBA is found to be $50^\circ 30'$. What is the height of the hill-top P above the station A?

Ex. 46. A wall is 10 ft. high and its direction makes an angle of 40° with the N.-S. line; find the width of the shadow cast by the wall on the ground at noon, when the sun's altitude is 55° .

SUMMARY OF RESULTS

If a straight line is perpendicular to a plane, it is perpendicular to any straight line that lies in that plane and meets it.

If a straight line is perpendicular to two straight lines in a plane, at their point of intersection, it is perpendicular to the plane.

The inclination of a line to a plane is the angle which it makes with its projection on the plane.

The inclination of two planes is the angle between two lines drawn, one in each plane, through a point on their line of intersection at right angles to that line.

CHAPTER 24

THE SPHERE

§ 1. The reader should already know the meaning of the word **sphere**. Before attempting to define it, he should be warned that the word is commonly used in two different senses: sometimes as meaning a surface, sometimes the space enclosed by a surface. This does not, as a rule, cause inconvenience. Ambiguity may be avoided by speaking of **spherical surface** if necessary.

¶Ex. 1. Define a spherical surface as a locus.

¶Ex. 2. How may a sphere be generated by the rotation of a plane figure?

A sphere is symmetrical about any plane through the centre. Such a plane cuts the surface in a circle, called a **great circle**, whose radius is equal to that of the sphere.

¶Ex. 3. Give instances of great circles on the surface of the earth.

Plane and sphere. (i) A plane **cuts** a spherical surface if the distance of the plane from the centre is less than the radius. The section is a circle: a **great circle** if the plane passes through the centre of the sphere; otherwise a **small circle**.

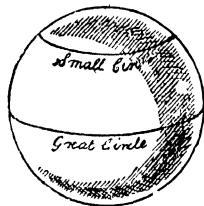


Fig. 302.

(ii) A plane **touches** a spherical surface if the distance of the plane from the centre is equal to the radius; it is said to be a **tangent plane**. Touching may be regarded as a **limiting case** of cutting, the small circle having shrunk to a point-circle. Obviously the perpendicular from the centre to a tangent plane meets the plane at the point of contact, and at radius distance.

(iii) A plane does not meet a spherical surface if the distance of the plane from the centre is greater than the radius.

If a straight line touches a sphere, it is called a **tangent line**.

¶Ex. 4. Give instances of small circles on the surface of the earth.

Ex. 5. Find the area of a small circle whose plane is distant h from the centre of a sphere of radius r .

Ex. 6. Find the length of a chord of a sphere distant d from the centre.

¶Ex. 7. What is the locus of the mid-point of a chord of length $2l$, placed in a sphere of radius r ?

¶Ex. 8. How many lines can be drawn to touch a sphere at a given point? On what surface do all these lines lie?

¶Ex. 9. TA, TB are tangents to a circle; the figure is rotated about the diameter through T ; what is generated by (i) TA, TB , (ii) A, B , (iii) AB ?

Ex. 10. The tangents to a sphere (radius r) from a point T , distant l from the centre, form a right circular cone: find the curved surface of this cone, bounded by its circle of contact.

Ex. 11. Find the radius of the horizon visible from the top of Snowdon (3560 ft.). [Take radius of earth 4000 miles. Begin by finding the distance of horizon from point at height h , where h is small compared with R : prove that it is approximately $\sqrt{2hR}$.]

Ex. 12. Through a fixed point T is drawn a variable line, meeting the surface of a fixed sphere in A, B . Prove that $TA \cdot TB$ is constant; and discuss any particular cases that may arise.

Ex. 13. A heavy sphere sinks into wet sand; when removed the diameter of the hole is found to be 10" and its greatest depth 4". What was the diameter of the sphere?

Ex. 14. A heavy sphere of diameter 7 ft. rests on three supports which form an equilateral triangle of side 2 ft. How high is the top of the sphere above the level of the supports?

Ex. 15. A triangular framework made of two equal rods of length 4 ft. and a third of length 3 ft. is held in a horizontal position 6 in. above a table. A sphere rests on the framework and just touches the table. What is the radius of the sphere?

Ex. 16. Three balls, each 5 cm. in diameter, lie on a floor in contact, and a fourth equal ball is placed on them. Calculate the height of the centre of the fourth ball above the floor.

Ex. 17. Find the radius of a sphere which just encloses a cube of edge 6 in.

Ex. 18. Find the edge of the greatest cube which could be cut from a solid sphere of diameter 20 cm.

Ex. 19. Find the radius of a sphere circumscribing a cuboid of edges 3 in., 4 in., 5 in.

Ex. 20. Find the radius of the sphere circumscribing a regular tetrahedron with edges 4 in. long.

Ex. 21. A cube just fits into a sphere of radius 4.2 centimetres. Draw a section of the sphere and cube parallel to a face of the cube and passing through the centre of the sphere.

Ex. 22. A sphere of radius 4 in. rests partly inside a hollow cubical box (whose edge is 4 in.) placed on a table; how high is the top of the sphere above the table?

Ex. 23. Three mutually perpendicular planes meet at a point O . A sphere, whose centre is C and whose radius is 2 feet, is placed so that it is in contact with the planes at the points P, Q, R . Find the lengths OP, OC . Find also the radius of the small circle PQR and the distance of its centre from O .

Ex. 24. An ordinary drinking glass, 5 inches in internal vertical height, has internal diameters of 3 inches at the top and 2.5 inches at the bottom. The inside of the bottom is flat. A sphere has to be made which, when inserted in the glass, will rest on the bottom and just touch the sides of the glass. By drawing or calculation, find the diameter of the sphere; and the height above the bottom of the glass of the circle of contact between the sphere and the glass.

Ex. 25. Find the radius of a sphere which touches the base and four slant sides of a right pyramid of height 4 in. on a base 3 in. square. Find also the distance from the base edge of a point of contact with one side.

Ex. 26. A sphere touches the base and each side face of a regular tetrahedron of side 10 cm. Find the lengths of a tangent line to the sphere (i) from the mid-point of a base edge, (ii) from the vertex. Hence find the radius of the sphere.

Ex. 27. A sphere can be placed in a framework of rods in the shape of a tetrahedron so as to touch each rod; show that the sum of the lengths of each pair of opposite rods is the same.

Ex. 28. A **Spherometer** consists of three equal legs fixed at the vertices of an equilateral triangle and a fourth leg which screws up and down in the middle of the triangle and which is adjusted so as just to touch the sphere when the other three legs are resting on any spherical surface. In fig. 303 A, B, C are the

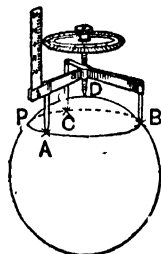


Fig. 303.

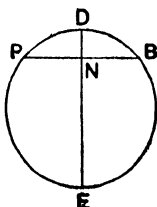


Fig. 304.

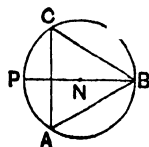


Fig. 305.

feet of the fixed legs and D the foot of the movable leg. The instrument registers the height of D above the plane ABC . Fig. 304 represents a central section through D and one point B .

Fig. 305 shows the circle through ABC .

Assuming that the sides of ABC are each 4 cm. long and that DN is found to be 1 cm., find (i) the radius of the section ABC , (ii) the radius of the sphere.

If the distance between the fixed feet is a cm., find a formula for the radius of the sphere when the height DN is h cm.

§ 2. Great circles. As the earth is approximately a sphere, the geometry of lines on a sphere is of practical importance, especially in navigation.

In general, **two points A, B on a sphere determine a great circle uniquely.** For a plane is determined by A, B and the centre of the sphere; and this plane cuts the surface in a great circle. The points A and B can be joined by two great-circle arcs, the minor arc and the major arc of the great circle.

The **shortest line** on the surface of a sphere connecting two given points is the minor arc of the great circle through the points, just as the shortest line in a plane connecting two points is the straight line. Accordingly, in the geometry of the spherical surface—**spherical geometry** as it is called—the great circle takes the place of the straight line in plane geometry.

The shortest-distance property of the great circle is used by navigators in **great-circle sailing.** Melbourne is approximately in the same latitude as Cape Town; but to sail along the parallel would be a much longer voyage than the great-circle route; this would run far to the south, as may be verified by stretching a piece of cotton on a globe between the two points. It is, in fact, impossible to follow the great-circle route throughout this voyage as it runs into the high latitudes, where navigation is unsafe.

Again, the great-circle route from the west coast of Mexico to Japan runs up the coast of America past San Francisco.

If two points A, B on a sphere are **at opposite ends of a diameter**, it is no longer the case that only one great circle can be drawn through them. It is possible to draw an infinite number of planes containing the diameter, and each one of these cuts the surface in a great circle. We see therefore that through diametrically opposite points on a sphere an infinite number of great circles can be drawn (e.g. the meridians of the earth, drawn through N. and S. poles).

Poles of a great or small circle.

The diameter of the sphere perpendicular to the plane of a great or small circle cuts the spherical surface in two points, called the **poles** of the circle; e.g. the North and South poles of the earth are the poles of the equator and of the parallels of latitude.

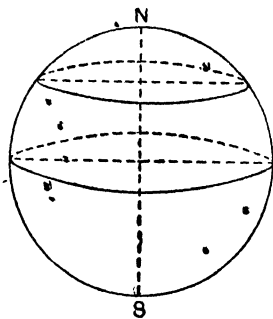


Fig. 306.

The intersection of two spheres may be shown to be a circle.

¶ **Ex. 29.** The altitude of the sun, seen from a ship at sea, is found to be α ; and, knowing the Greenwich time, the navigator can determine the point on the earth which has the sun vertically overhead. What does he now know as to the position of the ship?

Ex. 30. A small nail is driven into the highest point of a smooth spherical globe of radius 3 in. A hoop, radius 2 in., of fine wire, hangs on the nail and rests on the surface of the sphere. What angle does the plane of the hoop make with the vertical? What is the radius of the horizontal circle on the sphere which just touches the lowest point of the hoop?

Ex. 31. In order to find the radius of a portion of a hollow sphere two circular discs of radii 2 in. and 3 in. are placed inside, parallel to each other, and the distance between their planes is found to be 1 in. Calculate the radius of the sphere.

Ex. 32. A small circle is described on the surface of a sphere of radius 12 in. with a pair of compasses whose feet are 3 in. apart. Find the radius of the small circle.

Ex. 33. A billiard ball, diameter 2 in., moves on a table along a line making an angle of 20° with the cushion. The cushion overhangs so that the point at which the ball strikes it is 1.4 in. above the table. Find the distance along the cushion, between the point of contact and the point apparently arrived at.

§3. Latitude and longitude are simply coordinates on the surface of a sphere.

The equator and the meridian of Greenwich, NAS (fig. 307), are taken as rectangular great-circle axes; through the point P is drawn a great circle NPM, at right angles to the equator and therefore a meridian; AM and PM, measured in degrees, are the longitude and latitude. Longitude is measured E. and W. of A through 180° ; latitude is measured N. and S. through 90° .

Ex. 34. What is the latitude of the Equator? of the Poles?

Ex. 35. Calculate the value of a degree of Longitude (i) at the equator, (ii) on the parallel of Latitude 60° N.

[Take radius of earth as 4000 m.]

Calculate the value of a nautical mile (1 minute of arc at the equator).

Ex. 36. What is the distance between two places on meridian of Longitude 5° W. whose Latitudes are 30° N. and 55° N.?

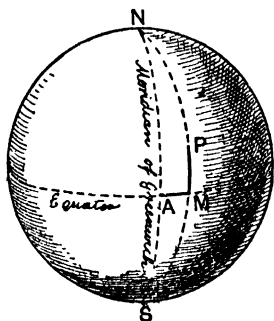


Fig. 307.

Ex. 37. Without assuming the value for the radius of the earth, compare the lengths of parallels of Latitude 30° N. and 60° N.

Ex. 38. Supposing it possible to travel along a great circle from a point 70° N. 90° E. to a point 70° N. 90° W., find the ratio of the length of the great-circle route to that of the route travelling due east.

SURFACE AND VOLUME OF SPHERE.

§4. Let the diameter NS (fig. 308) be at right angles to the parallel chords $AB, A'B'$; and let the figure rotate about NS .

The circle generates a sphere; A, B a small circle; the chords $AB, A'B'$ parallel circular areas; the arcs ANB, ASB **caps** of the spherical surface, or **spherical caps**; the arcs AA', BB' a **spherical belt**; the segments ANB, ASB **spherical segments**; the sectors $AOBN, AOBS$ **spherical sectors**.

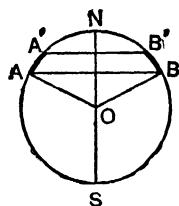


Fig. 308.

It can be proved that

A spherical belt cut off by two parallel planes has the same surface as a belt cut by the same planes from a cylinder circumscribing the sphere and having its generators perpendicular to the planes. (See fig. 309.)

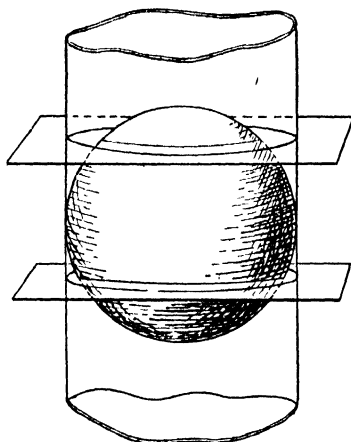


Fig. 309.

A spherical cap is a particular case of a spherical belt.

The surface of a sphere $= 4\pi r^2$.

Let σ be the surface of any small region on the surface of the sphere; S the whole surface. A radius whose extremity travels round the boundary of this region will sweep out a cone whose vertex is at the centre. To the first order of small quantities, we may regard this as a cone of height r standing on a plane base of area σ ; its volume is $\frac{1}{3}r\sigma$.

Hence the volume of the whole sphere $= \sum \frac{1}{3}r\sigma = \frac{1}{3}r \sum \sigma = \frac{1}{3}rS = \frac{4}{3}\pi r^3$.

The volume of a sphere $= \frac{4}{3}\pi r^3$.

Ex. 39. Prove that half the earth's surface is included between parallels 30° N. and 30° S.

Ex. 40. Find the ratio of the volumes of a sphere and the circumscribed cube.

Ex. 41. To double the gas-capacity of a spherical balloon, in what ratio must the area of material be increased?

Ex. 42. A boiler has the form of a right circular cylinder with two convex hemispherical ends. Show that the area of its external surface is equal to the product of its greatest length and the circumference of the circular section of the cylinder.

Ex. 43. Find the latitude of the four parallels, two N. of the equator, two S. of it, which divide the earth's surface into five zones of equal area.

Ex. 44. Find an expression for the volume of metal in a hollow metal shell, spherical inside and out. If the thickness t is small compared with the external diameter $2r$, deduce that the volume of metal is approximately $4\pi r^2 t$.

Ex. 45. A piece of lead piping 3 ft. long, having an external diameter of 1 inch and an internal diameter of $\frac{7}{8}$ of an inch, is melted down and formed into 100 spherical shot of equal size. Find the diameter of one of the shot.

Ex. 46. AB is a diameter of a given circle, and AC is a chord inclined 30° to AB . The area bounded by AC , AB and the arc BC revolves about AB . Find the volume swept out by this area, and the area of the surface generated, in terms of the radius of the circle.

Ex. 47. A spherical segment is 1 metre high, and the radius of its base is 2 metres. This segment is cut into two pieces by a plane parallel to its base and 50 cm. from it. Find the radius of the sphere and the area of the section.

* $\sum \frac{1}{3}r\sigma$ means the sum of all terms like $\frac{1}{3}r\sigma$, the surface of the sphere being divided up into a great number of small areas like σ .

Ex. 48. A sphere rests on a table, and is covered by a hollow cone which also rests on the table and just touches the sphere. If the radius of the sphere be 3 inches, and the height of the cone 8 inches, find the volume of the cone.

Find also the radius of the circle of contact between sphere and cone, and the ratio in which the surface of the sphere is divided by this circle.

Ex. 49. A plano-concave lens, shown in section in fig. 310, has the following dimensions:— $AC = 3.6$ cm., $AB = CD = 0.8$ cm., $EF = 0.4$ cm. Calculate, or find by construction and measurement, the radius of the spherical surface BFD .

Assuming that the volume of a segment of a sphere $= \frac{1}{3} \pi h^2 (3R - h)$, where h is the height of the segment and R is the radius of the sphere, calculate the volume of the lens.

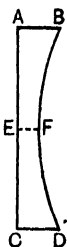


Fig. 310.

Ex. 50. A hollow buoy is made of sheet steel, and its dimensions are those shown in fig. 311.

The base of the buoy is part of a sphere and the upper portion a cone which touches the sphere.

The steel is $\frac{3}{8}$ inch thick and has a density of 480 lb. per cubic foot.

Assume that the dimensions are "mean" values between the internal and the external, and find the weight of the buoy.

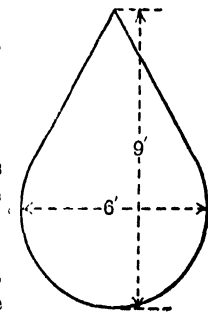


Fig. 311.

Ex. 51. Find formulae for the curved surface of a segment of a sphere in terms of (i) R , the radius of the sphere, and h , the height of the segment; (ii) h and r , the radius of the base of the segment.

Ex. 52. Find formulae for the volume of the segment of the sphere in Ex. 51 in terms (i) of R and h , (ii) of r and h .

SUMMARY OF RESULTS

The shortest line on the surface of a sphere connecting two given points is the minor arc of the great circle through the points.

A spherical belt cut off by two parallel planes has the same surface as a belt cut by the same planes from a cylinder circumscribing the sphere and having its generators perpendicular to the planes.

The surface of a sphere $= 4\pi r^2$.

The volume of a sphere $= \frac{4}{3} \pi r^3$.

CHAPTER 25

THE REGULAR SOLIDS

It is impossible to have more than five solids all the faces of which are congruent regular polygons and all the solid angles of which are bounded by the same number of plane angles. Such solids are called **regular solids**.

The figures below show the five regular solids.

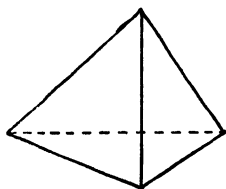


Fig. 312.

Tetrahedron (i.e. 4-faced).

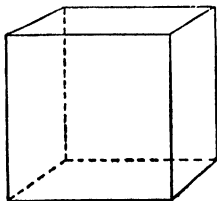


Fig. 313.

Cube.

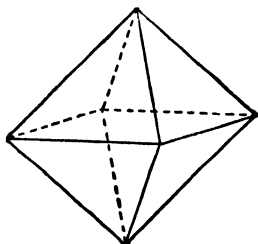


Fig. 314.

Octahedron (i.e. 8-faced).

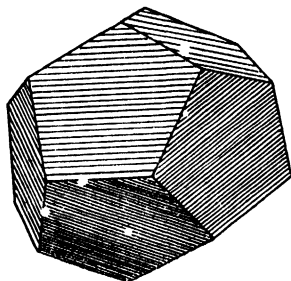


Fig. 315.

Dodecahedron (i.e. 12-faced).

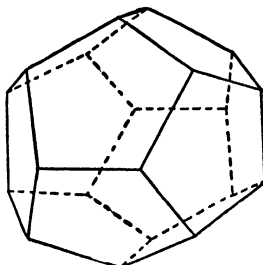


Fig. 316.

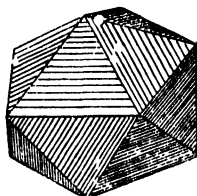


Fig. 317.

Icosahedron (i.e. 20-faced).

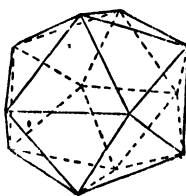


Fig. 318.

It is interesting to note that many substances crystallise in the form of some of the regular solids, e.g.

Form	Substance
Tetrahedron	Blende, Tetrahedrite (grey copper ore).
Cube	Fluor-spar, Galena (lead sulphide).
Octahedron	Alum, Diamond*, Gold, Magnetite.

Ex. 1. Make sketches of a cube showing how to cut it so as to get sections of 3, 4, 5, or 6 sides. Is it possible to have a section of more than 6 sides?

Ex. 2. If AB is a diagonal of a cube, show that the mid-points of the edges which do not pass through A or B are the vertices of a regular hexagon.

Ex. 3. What numbers of sides are possible for sections of a tetrahedron? Make sketches.

Ex. 4. Calculate the length of the diagonal of a regular octahedron in terms of the length of the edges.

¶Ex. 5. Prove that a regular octahedron can be divided into two square pyramids in three different ways.

¶Ex. 6. If each corner of a cube were planed off symmetrically (see fig. 319), and the process continued till the faces of the cube disappeared, what solid figure would be constructed?

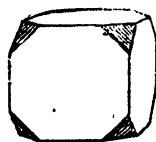


Fig. 319.

¶Ex. 7. Prove that the mid-points of the faces of a cube are the vertices of a regular octahedron.

¶Ex. 8. Prove that the mid-points of the edges of a regular tetrahedron are the vertices of a regular octahedron.

SUMMARY OF RESULTS

There are only five regular solids.

The natural faces are not to be confused with the artificial faces seen in cut stones. As a rule the crystals of diamond and gold are waterworn and not very well defined in shape.

CHAPTER 26

PLAN AND ELEVATION

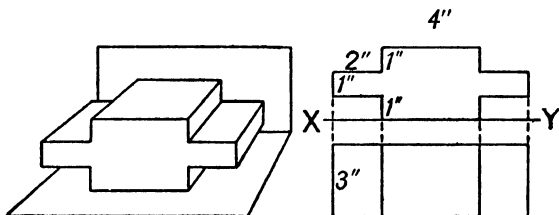


Fig. 320.

Fig. 321.

§ I. Fig. 320 shows a perspective view of a horizontal plane and a vertical plane with a piece of wood resting on the horizontal plane.

If you look down on the piece of wood, you will get a bird's-eye view of it something like the lower part of fig. 321; if you take a distant horizontal view from straight in front of the vertical plane, you will see something like the upper part of fig. 321.

The lower part of fig. 321 is called the **plan** of the wood and may be considered as made from the actual solid by drawing perpendiculars from the various points of the solid to the horizontal plane (abbreviated to "the H.P."). In the same way the upper part of the figure, called the **elevation**, may be considered as made by drawing perpendiculars to the vertical plane (the V.P.). The line XY where the H.P. and V.P. meet is called the **ground line**.

With practice a good idea of the shape of a solid may be formed from its plan and elevation. Plans and elevations are used by architects and engineers; they have the advantage over perspective drawings that measurements can be made from them.

Fig. 322 shows a perspective sketch of a house, its plan and its elevation on two V.P.s.

Note that the line joining any point on the plan to the corresponding point on the elevation cuts the ground line at right angles. Also note that the distance of any point from the ground line in the front elevation is equal to the distance of the corresponding point from the ground line in the side elevation.

The plan of a point A is denoted by a and its elevation by a' .

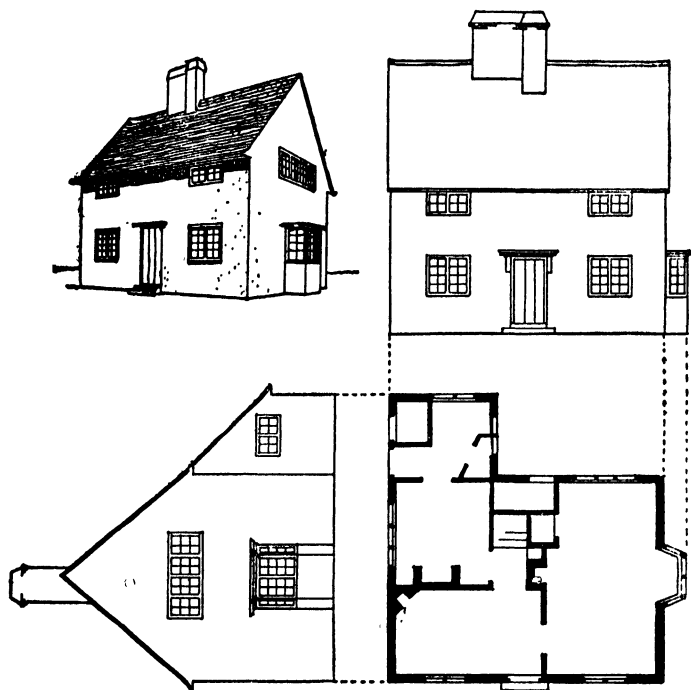


Fig. 322.

Ex. 1. With the aid of models, make freehand sketches of the plan and elevation of various solids, e.g. a brick, a triangular prism, a tetrahedron, cylinder.

[It will be convenient to open a book with one half horizontal and the other half vertical, these to be regarded as the H.P. and V.P., and to place the solid on the horizontal part.]

Ex. 2. Find the volume of the solid in fig. 320.

Ex. 3. Find the volume of the solid whose plan and elevation are given in fig. 268.

Ex. 4. Draw a square ABCD with sides 2 in. long; outside the square draw a straight line parallel to one of its edges and 1 in. from it. Take this line as the ground line and the square as the base of a regular square pyramid resting on the H.P. Complete the plan and draw the elevation on the V.P. assuming the height of the pyramid to be 1.5 in. Call the vertex of the pyramid T and letter the plan and elevation.

Ex. 5. A plane slopes at 20° to the horizontal. A cubical tank, without lid, edge 3 ft., rests on the plane, with an edge of the base horizontal. Draw, on a scale of an inch to the foot, a side elevation of the tank, that is, an elevation

on a vertical plane at right angles to the inclined plane. Draw also a plan of the tank. The thickness of the walls of the tank may be disregarded.

Ex. 6. A tub used for greenhouse plants is shown in a rough sketch in fig. 323. It measures 2 ft. square at the base and 3 ft. square at the top, and its height is 2 ft. 6 in. Draw a plan and elevation of the tub, on the scale of 1 in. to the foot. On the same scale draw a figure of one of the slanting faces as it would look if laid flat on the plane of the paper. Find the slant height (the distance between the parallel edges) of a slanting face, the length of an edge such as AB, the total surface area and the volume of the tub (outside measurement).

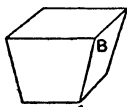


Fig. 323.

§ 2. If a line is parallel to either plane of projection its length is equal to its projection on that plane and its inclination to the other plane can be found from the drawing. The lengths of lines not parallel to either plane of projection can often be determined by making a separate drawing of some part of the solid figure.

Ex. 7. Fig. 324 shows the plan and elevation of a piece of wire; find its true length. Also draw its elevation on a v.p. through $X'Y'$.

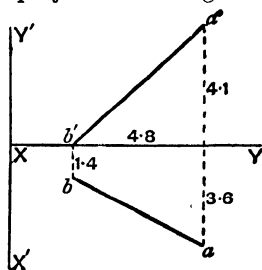


Fig. 324.

Ex. 8. A heap of stones by the roadside stands on a rectangular base measuring 10 ft. by 4 ft., and the four faces slant upwards at an angle of 43° with the horizontal, fig. 278 being a rough sketch of the arrangement. (i) Draw a plan of the base and an elevation on a v.p. parallel to the shorter edges of the base; what is the height of the heap? (ii) Draw an elevation on a v.p. parallel to the longer edges (you have already found the height of the heap). (iii) Complete the plan.

Ex. 9. Fig. 325 shows the plan and elevation of a box. Find the length of a diagonal of the box and its inclination to (i) the bottom, (ii) the side, (iii) the end, of the box. (Draw a vertical section through a diagonal of the plan.)

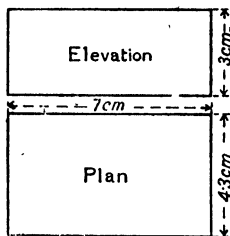


Fig. 325.

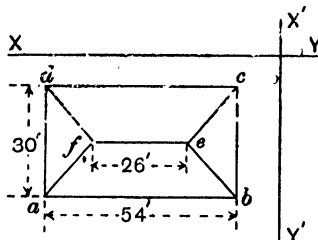


Fig. 326.

Ex. 10. Fig. 326 shows the plan of the roof of a house, ef being the plan of the ridge and a, b, c, d the plans of the corners of the eaves. The ridge is 15 ft.

vertically above the eaves. Draw the plan to scale, and then draw side and end elevations on vertical planes parallel to cd and bc respectively.

Determine (i) the angles of inclination of the sides and ends of the roof to the ground, (ii) the length of one of the four hip rafters (the plan of one is af) and its inclination to the ground. (This can be done by drawing in a separate figure a vertical section through af .)

Ex. 11. Fig. 327 represents the plan of a roof which consists of five surfaces, all inclined to the horizontal at an angle of 35° .

The surfaces all pass through B, the point where the ridge poles BC and DE meet. Draw the triangle CFG and determine the height of C above F or G. Then give a drawing of one of the surfaces which pass through the ridge pole BC, and determine (i) the true length of the valley rafter AB, (ii) the angle which this rafter makes with the horizontal.

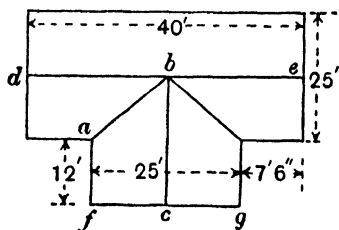


Fig. 327.

Ex. 12. Figs. 328, 329 show the front and side elevations of a blower and pipe for a military shoeing forge. The horizontal sections are rectangular; the back of the blower rests against a vertical wall; the slopes of the other three faces are as shown. Draw the plan and find (i) the length of the intersections of two sloping faces, (ii) the areas of the four faces of the blower.

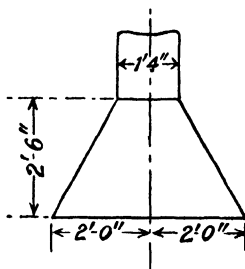


Fig. 328.

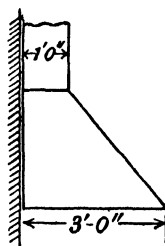


Fig. 329.

Ex. 13. Draw the plan and two elevations of the tent in Ex. 18 of chap. 21.

Ex. 14. Draw the plan and two elevations of the trough in Ex. 19 of chap. 21.

Ex. 15. The plan of the roof of a house is rectangular in outline, and two adjacent surfaces are each inclined at 50° to the horizontal. Find the inclination of the hip rafter (see Ex. 10) to the horizontal. What is the length of the hip rafter, if 20 ft. is the length of the bottom edge of one of the slopes which is triangular in shape?

Ex. 16. The plan of a roof of a house is rectangular in outline (see Ex. 10); the side slopes are inclined to the horizontal at 30° , the end slope at 40° and the span is 24 ft. Find the length of the hip rafter and its inclination to the horizontal.

Ex. 17. A right cone whose height is $2\frac{1}{2}$ in. and whose base is 2 in. in diameter stands on its base on a table in the sunshine, the sun's altitude being 50° . Draw a plan of the cone and shadow and an elevation on a v.p. parallel to the plane through the axis of the cone and the apex of the shadow. Indicate on the plan and on the elevation the portions of the slant surface of the cone which are in the shadow.

Ex. 18. A lamp shade is in the form of the slant surface of a frustum. 1 ft. high, of a right cone, the diameters at the bottom and top being 18 in. and 6 in. respectively. Draw a plan and elevation of the lamp shade. Find the length of a slant side of the frustum. Supposing the lamp shade to be made of cardboard, make a rough sketch of its appearance when cut through and spread flat.

Ex. 19. A square a, b, c, d of 2-inch side is the plan of a square prism of height 4 inches standing upright on the horizontal plane. The prism is cut obliquely by a plane which intersects the vertical edges through a, b, c, d in the points E, F, G, H respectively. E, F, G are $3\frac{1}{2}"$, $2\frac{1}{2}"$, $\frac{1}{2}"$ respectively above the horizontal plane.

Draw the plan of the prism and an elevation on a vertical plane inclined at 30° to ab (and at 75° to ac). Show the elevations of E, F, G ; construct the elevation of the point H , and complete the elevation of the section. By obtaining the true lengths of the necessary lines, construct the true shape of the oblique section. Construction lines must be clearly shown.

Ex. 20. A regular tetrahedron, edge 3", stands on its base ABC . Draw the plan of the tetrahedron and its elevation on two vertical planes, one parallel to AB , the other perpendicular to AB . If the tetrahedron is cut obliquely by a plane which intersects DA, DB, DC in points a, b, c such that $Da=1"$, $Db=1.5"$, $Dc=2"$, show the plan and the elevations, on the two vertical planes described above, of the section abc and construct its true shape. All construction lines must be clearly shown.

MISCELLANEOUS EXERCISES III

Ex. 1. Sketch, freehand, a vertical section through the diagonal of the top face of a cube, if the edges of the cube are bevelled off.

Ex. 2. A cube of edge 10 cm. has each corner cut off (see fig. 319) so as to expose a triangular section of side 3 cm. Draw a vertical section of the cube through the diagonal of the top face of the cube.

Ex. 3. OA, OB, OC are adjacent edges of a cubical block of side 4 inches; P, Q, R are the middle points of these edges. The corner O is removed by cutting through the plane PQR; and the other corners are treated similarly. Find (i) the volume of the remaining solid, (ii) the inclination of any triangular face to an adjacent square face.

Ex. 4. From a rectangular sheet of paper CDEF, four equal rectangles CP, DQ, ER, FS are cut out (see fig. 330); the remainder is folded so as to form a closed box on PQRS as base, each of the flaps LPSM, NQRO forming a side and half the lid. $CD = a''$, $CF = b''$, $CH = x''$, $CL = y''$, where $a > b$ and $x > y$.

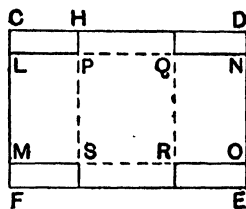


Fig. 330.

(i) Express x in terms of a and y .

(ii) Find the volume of the box in terms of a , b , y .

(iii) Find the ratio of a to b if the box is a cube.

Ex. 5. OA, OB, OC are three edges (each 2 inches long) of a cube whose diagonals are OO' , AA' , BB' , CC' . Draw (i) the section of the cube passing through the points A, B, C; (ii) the section through C, O, C', marking where it cuts the former section, and showing the perpendicular from O on the line common to these two sections. Measure the length of this perpendicular.

Ex. 6. A railway cutting 8 metres deep has to be made, with one side vertical and the other inclined at 30° to the vertical; the bottom is to be 9.4 metres broad. How many cubic metres (to the nearest integer) of material must be removed per kilometre?

Ex. 7. Sand lies against a wall, covering a strip of ground 4 ft. wide. If the sand will just rest with its surface inclined at 30° to the horizon, how much sand may lie on this strip per foot length of wall? Give the quantity to the nearest tenth of a cubic foot.

Ex. 8. OABCD is a regular pyramid on a square base ABCD. Imagine its surface cut along OA, AD, DC, CB and opened out flat. Draw the flat figure you would get if $AB = 2$ in. and $OA = 2.7$ in. If this is cut out of a rectangular piece of cardboard with D, one of the corners of the base, at one of the corners of the cardboard, find the least possible dimensions for the cardboard and the height of the pyramid.

Ex. 9. Find the area of the section of the regular tetrahedron OABC made by a plane which cuts the edges OA, OB and OC in points distant 1, 2 and 3 inches respectively from O.

Ex. 10. A pyramid stands symmetrically on its base, which is a regular pentagon of side 3 inches. The height of the pyramid is 7 inches. Find, by drawing, the length of one of the sloping edges and the angle between a sloping face and the base.

Ex. 11. A wall is illuminated by light from a very narrow vertical slit 2 in. high from which light issues in all directions. If a vertical screen 1 ft. high and $1\frac{1}{2}$ ft. wide is placed parallel to the wall with its centre exactly opposite the centre of the slit, find the dimensions of the perfect shadow and of the partial shadow cast on the wall, the distances of the slit from the screen and from the wall being 4 ft. and 12 ft. respectively.

Ex. 12. A circular table, one foot in diameter, has three legs attached to its edge at equal distances apart. Each leg is 5 feet long. Find the distance between their points of contact with the ground, supposed horizontal, when the table is at a height of $4\frac{1}{2}$ feet from the ground.

If the diameter of the table is a feet and the length of each leg is b feet, and the height of the table is h feet, show that, if d feet is the distance between the ends of the legs, d is given by the formula

$$d = \left(\frac{1}{2}a + \sqrt{b^2 - h^2}\right)\sqrt{3}.$$

Ex. 13. Fig. 331 represents a pair of shear legs OA, OB and a back leg OC, such as are used for lifting heavy weights out of a ship. A, B, C are in a horizontal plane. OA = OB = 41 feet; AB = 18 feet; OC = 60 feet; CN is perpendicular to AB where N is the mid-point of AB. The plane OAB is inclined at 60° to the horizontal.

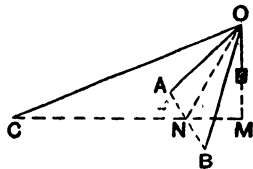


Fig. 331.

(i) Calculate ON, NM, OM, CN.

(ii) Calculate the inclination of OA to the horizontal.

(iii) In the general case, let $OC = c$, $ON = a$, $OM = x$, $NC = z$, the angle ONM being undefined. To raise O, C must be moved backwards so that x and z are connected by an equation. Prove that this is $(c^2 - a^2 - z^2)^2 = 4z^2(a^2 - x^2)$.

Ex. 14. A block of lead is bounded by six plane faces, as shown in fig. 332. The base TUVW and the top PQRS are rectangular. The slant faces PSWT and QRVU are equal in all respects, and so are the faces RSWV and PQUT. Certain dimensions are shown in inches upon the figure. The height is 4 inches, and the base is horizontal.

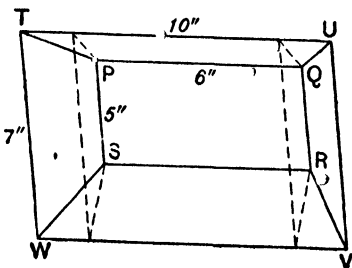


Fig. 332.

Imagine the block to be cut in three by vertical planes through PS and QR, the central piece to be removed and the two end pieces united with their out faces coincident. Sketch the two resulting blocks, showing their dimensions. (See next page.)

In the block made of the two end pieces, P and Q coincide, and so do R and S. Imagine this block cut in three by vertical planes through P and S parallel to VW, the central piece removed, and the two end pieces united with the cut faces coincident. Sketch the two resulting blocks, and show their dimensions.

Use the blocks into which the lead has been rearranged to find the volume of lead in the original block.

Ex. 15. The accompanying fig. 333, which is not drawn to scale, is symmetrical with respect to the line EF and with respect to the perpendicular bisector of EF. Imagine that the rectangle is the base of a tent

bounded by four sloping walls, and that the parts of the figure outside the rectangle are turned about its sides to form the walls of the tent; the four walls terminate in a horizontal ridge which stands at a height h over the line EF. The volume V of air-space enclosed by the tent is given by the formula

$V = \frac{1}{3}(2a + c)D$, where D denotes

the area of the vertical section

which bisects EF at right angles. Express V in terms of a , b , c , h .

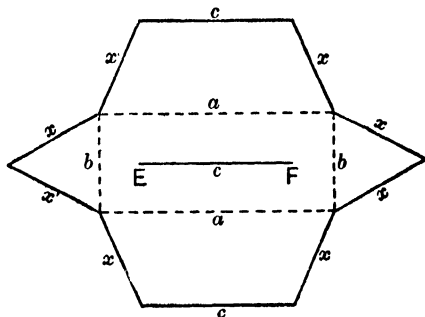


Fig. 333.

Prove the formula $V = \frac{1}{3}(2a + c)D$.

Ex. 16. Each edge of a triangular pyramid, vertex V, base ABC, is 6 cm. long; R, P, Q are points on AB, BC, VC respectively, such that BR = 2 cm., BP = 3.5 cm., VQ = 1.5 cm. A straight tunnel is bored through the pyramid from R to Q, and it is desired to bore another, intersecting this one, from P to the edge VA.

First draw, freehand, a large sketch in perspective; then construct in turn the following triangles, and from them find (i) the length of each tunnel, (ii) the distances of the crossing point from R and P, and (iii) the distance from V of the exit on VA:—

- (1) Triangle ABC; denote the intersection of AP and CR by O.
- (2) Triangle VAB, with VR drawn.
- (3) Triangle VRC; denote the intersection of RQ and VO by X, obtaining O from triangle ABC.
- (4) Triangle VBC, with VP drawn.
- (5) Triangle VAP; obtain the point O from triangle ABC, and then the point X from triangle VRC.

Ex. 17. A right circular cone has a base of 1 in. radius and a slant side of 3 in. Draw figures suitable for cutting out to make a paper model of the surface of the cone.

Ex. 18. The slant height of a cone is 12 in. and radius of base 8 in. Find the angle of the sector of a circle formed by developing the curved surface.

Ex. 19. A cone of slant height 12 in. and base-radius 3 in. rolls on a table with its vertex always on a fixed point on the table. When the base of the cone has traced out a horizontal circle, how many revolutions has a point on the base made about the axis of the cone?

Ex. 20. A cup in the form of a hollow right circular cone with its axis vertical is just full of water; its height is 8 in. and its slant side is 10 in. A ball is lowered slowly into the cup, of such a size that when it rests on the sides of the cup it is *just* covered with water. Find by a geometrical construction the radius of this ball, and measure it.

If VBC is a section of this cup through the vertex V and the axis of the cone and O is the centre of the ball, show that the areas of the triangles OVB , $C'VC$, OBC are in the ratios $5:5:6$; and hence, or otherwise, calculate the radius of the ball.

Ex. 21. A tin cup is in shape the frustum of a right cone; the internal diameters of the cup at the top and bottom are respectively 3 in. and 4 in., and the internal depth is 6 in. Find by calculation how many pints the cup will hold. (1 cu. ft. = 6.25 gallons.)

Ex. 22. A hollow cone, the height of which is 12 in. and the radius of the base 9 in., is fixed with its vertex downwards and axis vertical and is filled with water. A sphere is then lowered into the water until it rests on the inner surface of the cone, its centre being 5 in. above the vertex. I. the sphere is withdrawn, find the distance through which the surface of the water will fall.

Ex. 23. A spherical ball of radius 5 inches rests on a table; find the height of the smallest hollow cone, the diameter of whose base is 15 inches, which when placed on the table will cover the ball.

Ex. 24. A sphere of radius 4 ft. is supported by a horizontal triangular framework the lengths of whose sides are 6 ft., 8 ft., 10 ft. Find the height of the top of the sphere above the plane of the wire.

Ex. 25. A circle of radius 1 yard is drawn on a horizontal plane and a tangent PT , 2 yards long, is drawn from any point P on the circle. At the centre O a vertical post OQ , 2 yards high, is erected. Calculate the length of QT and find its inclination to the ground.

Ex. 26. I have a little writing desk, $3\frac{1}{2}$ in. high at the back, $2\frac{1}{4}$ in. high at the front; the case is $13\frac{1}{2}$ in. from back to front and 8 in. broad.

Find the slope of the writing board in degrees and the length of the line joining its top right-hand corner to its bottom left-hand corner. What is the slope of this line?

Ex. 27. $ABCD$ is the floor of a room, $A'B'C'D'$ is the ceiling, A being below A' and so on; $AB=18$ ft., $AD=12$ ft., $AA'=14$ ft. Find the angles between (i) the diagonals AC' , BD' , (ii) the diagonal AC' and the floor of the room

Ex. 28. ABCD is a wall, 20 ft. long. One end AD is 6 ft. high; the other end BC is 10 ft. high. The sun casts a shadow of the wall on level ground, the shadow of the end AD being 8 ft. long and making a right angle BAE with BA, the intersection of the wall and the ground.

Determine the length of the shadow of BC. Draw the actual shape of the shadow of the wall.

Ex. 29. Two intersecting walls 10 feet high run one due east and west and the other north-east and south-west. The shadows of the walls are respectively 7 and 11 feet broad, each on the side of the wall which is northwards. Draw a plan of the shadows, and from it find the altitude of the sun and how many degrees the sun is east or west of south.

Ex. 30. A vertical pole is set up on level ground. Two guy ropes are attached to the pole at a height of 20 ft. above the ground. One of these ropes is attached to a peg at ground level and 24 ft. east of the pole. The other rope is attached to a peg, also at ground level, 30 ft. south and 4 ft. east of the pole. Find the angle between the two ropes, both of which may be assumed to be straight, and the length of each rope.

Ex. 31. A church spire is in the form of a cone 30 feet high and 10 feet in diameter at the base. Supposing the sun's altitude to be 55° , draw a plan showing the base of the spire and the shadow which it would throw on a horizontal plane through the base. Find what proportion of the slant surface of the spire is in shadow.

Ex. 32. An aeroplane is observed to be due north of a point A, at an altitude of 40° . At the same instant, it is observed to be north-west of a point B, situated 10,000 feet east of A. Find (i) the distance from A of the point vertically below the aeroplane, (ii) the height of the aeroplane in feet.

Ex. 33. An airship A is sighted at the same time from two stations P and Q which are one mile apart and at the same level. Its elevation at P is 15° and the angles APQ and AQP are 106° and 42° respectively. Find the height of the airship in feet and the elevation at Q.

Ex. 34. A railway cutting is to be made in a hill which slopes from north to south, the rate of slope being 1 vertical to 6 horizontal. The line is to run due E., the width of the cutting at the bottom is to be 20 ft., the sides are to slope at $1\frac{1}{2}$ vertical to 1 horizontal, and the depth of the cutting is to be 25 ft. at the centre line of the cutting. Draw a figure showing the cross-section of the cutting, and calculate the volume in cubic yards of the excavation between two points on the line 60 yards apart.

Ex. 35. The side of a certain hill may be treated as a plane making an angle of 20° with the horizontal. A road is to be constructed on the hill-side with a slope of 1 in 10 (i.e. 1 foot fall for every 10 feet along the road). What angle will the road make with the lines of steepest slope on the hill-side?

Ex. 36. A horizontal line AB 500 feet long is set out. The theodolite is set up at A and at B , observations of points C and D being made and the following angles recorded :—

$$\begin{array}{l} \text{From } A \left\{ \begin{array}{l} \text{Bearing of } B, N. 30^\circ E.; \\ \text{,, ,, } C, N. 70^\circ W.; \\ \text{,, ,, } D, N. 25^\circ W. \end{array} \right. \quad \text{From } B \left\{ \begin{array}{l} \text{Bearing of } C, S. 60^\circ W.; \\ \text{,, ,, } D, N. 75^\circ W. \end{array} \right. \end{array}$$

Altitudes of C and D from A , 20° and 16° respectively.

Determine the heights of C and D above the level of AB , and the horizontal distance of C from D .

Ex. 37. Find by drawing and calculation the distance in nautical miles measured along a small circle of constant latitude (i.e. a parallel) between the points $40^\circ N.$ lat., $125^\circ W.$ long. (on the coast of California) and $40^\circ N.$ lat., $145^\circ E.$ long. (off the coast of Japan).

[1 nautical mile = 1 minute of arc on the equator.]

How would the shortest distance, e.g. for a submarine cable or a voyage, between these points lie with regard to the small circle?

Ex. 38. A roof consists of four sloping planes, two of which meet in a ridge AB ; the lowest points of the roof form a rectangle $CDEF$, 30 ft. by 14 ft. (See fig. 297, but ignore the angles in that figure.) If each plane slopes at an angle of 65° to the horizontal, find the length of AB and its height above $CDEF$.

Ex. 39. Two roof ridges meet at right angles, and the angle of slope of each roof is 45° . Find the angle of inclination to the horizontal of the line in which the roofs intersect.

Ex. 40. Fig. 334 shows the roof plan of part of a house over a bay window which projects 4 feet in front of the house. $AD = 16$ feet, $BC = 3$ feet. FE is horizontal and 4 ft. long. The horizontal distance of E from BC is 8 ft. The faces of the roof with the exception of the triangles AEB , CED are inclined at angles 45° to the horizontal. Find the inclinations to the horizontal of these two latter faces and also of the ridges AE , BE . Draw a front and a side elevation of the roof.

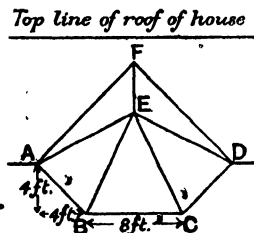


Fig. 334.

Ex. 41. The roof of a rectangular building, 63 ft. by 31 ft., consists of four portions, two equal larger portions which meet in the roof ridge and two equal smaller triangular portions which close the ends; the ridge is 48 ft. long and is 22 ft. above the walls. Draw the side and end elevations and the plan of the roof. Find the horizontal distances of the end of the ridge from (i) the nearest end wall, (ii) the nearest corner of the building. Also find the inclination of each portion of the roof to the horizontal.

Ex. 42. The hopper shown in fig. 335 has a closed bottom 1 foot square, and is open at the top. It is made of plate weighing 4 lb. per square foot. Calculate the weight of the hopper.

Ex. 43. ABC and DEF are thin 60° and 45° set squares; angles A and D are the right angles and $\angle C = 60^\circ$, $AB = EF = 3$ in. Imagine ABC, DEF placed flat on the H.P. so that B is at the mid-point of EF and EF parallel to CA. Now suppose ABC rotated about AB till its plane is vertical, and DEF rotated about EF so as to rest against the edge of the other set square. Draw the plan of the set squares and elevations of them on the vertical planes parallel to AB and EF.

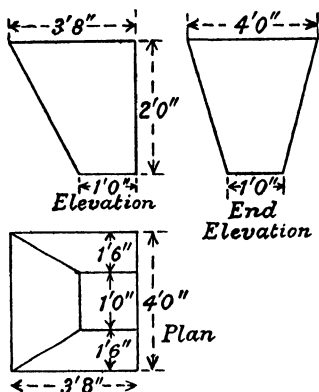


Fig. 335.

Ex. 44. A thin 30° - 60° set square ABC, whose long edge AB measures 4 inches, is laid on the horizontal plane and is then turned about AB through an angle of 58° :—(i) draw the plan of the set square; (ii) determine the inclinations to the horizontal of the edges AC and BC; (iii) draw an elevation of the set square on a vertical plane which makes an angle of 45° with AB.

Ex. 45. Draw a triangle abc , sides $ab = 8$ cm., $bc = 5$ cm., $ca = 4$ cm. abc represents the projection on a horizontal plane of a right-angled set square whose hypotenuse ab is lying in the plane. Find the height of the right angle above the plane.

Ex. 46. The sides of a triangle ABC are $AB = 3''$, $BC = 2.5''$, $CA = 2.75''$.

The triangle rests with its base AB in the horizontal plane, and the plane of the triangle is parallel to and $1.5''$ in front of the vertical plane of projection.

Draw the projections of the triangle in this position. Also construct its projections, (i) after a rotation of 30° about a vertical axis through A, (ii) after a further rotation of 30° about the base; the direction of rotation in each case being such as to move C further from the vertical plane of projection. From the necessary measurements obtained from your drawing, calculate the area of the triangle and the area of the plan in the final position. Find the ratio of these areas. How could this ratio have been obtained from the inclination of the plane of the triangle?

Ex. 47. A straight rod 15 feet long rests with its upper end in a corner of a rectangular room, the lower end resting on the floor at a point 7 feet from each of the adjacent walls. To a scale of 1 inch to 3 feet, draw the projections of the rod on the floor as horizontal plane and one of the walls as vertical plane.

Determine the height of the upper end and the inclination of the rod to the floor. What is the surface generated by the rod as the lower end moves over the

floor between the walls, while the upper end remains fixed? Draw the plan of the path of the lower end.

A plane passes through the ground line of one of the walls forming the corner, inclined at 45° to the floor. Construct the true shape of the section made by this plane with the surface generated by the rod.

Ex. 48. Construct a full size drawing of the shadow cast upon a horizontal plane by a spherical ball of radius 2 inches, when the sun is due south at an altitude of 50° . Show on your drawing the point of contact of the ball.

Loc.

¶Ex. 49. What is the locus, in three dimensions, of a point at a constant distance from (i) a fixed point, (ii) a fixed plane, (iii) a fixed line?

¶Ex. 50. What is the locus of a point, in three dimensions, equidistant from (i) two parallel planes, (ii) two parallel lines, (iii) two fixed points, (iv) two intersecting planes, (v) two intersecting lines?

¶Ex. 51. What is the locus of a point equidistant from three fixed points in a given plane?

¶Ex. 52. Show how to determine the point in a plane which is equidistant from three given points not in the plane.

¶Ex. 53. What is the locus of a point equidistant from three non-parallel lines in the same plane?

¶Ex. 54. What is the locus, in three dimensions, of points at which a given line subtends a right angle?

Can the result be extended to any angle?

¶Ex. 55. What is the locus of the mid-points of plane sections of a sphere at a constant distance from the centre?

¶Ex. 56. How many spheres can have the same circle as a plane section? What is the locus of their centres?

¶Ex. 57. A, B, C, D are four points in space. What is the locus of points equidistant from A, B, C? What is the locus of points equidistant from A, D? Hence, how many spheres can in general be drawn through four points? In what case is this untrue?

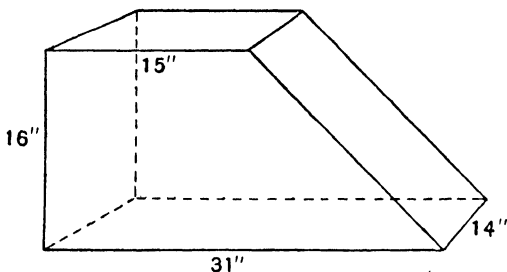
¶Ex. 58. The cover of a book is partly opened and marbles of various sizes placed so as each to touch the cover and the first leaf. What is the locus of the centres of the marbles?

REVISION PAPERS B

Paper 9.

1. On what property of a circle does the use of the semicircular protractor depend?

2. The figure represents a coal-box. Find the volume of the solid figure shown.



3. What would be the radius of a circle in which an arc 11 in. in length subtended an angle of $31\frac{1}{2}^\circ$ at the circumference?

4. If AB is a tangent to a circle of radius 5 in., where A is any point on the circumference and B is 12 in. from A, find the locus of B as A moves round the circle.

5. ABCD is a rectangle; through A a straight line XAY is drawn at right angles to AC; DE (x), BF (y) are drawn perpendicular to XY. Prove

$$EA = AF; x + y = AC; x^2 + y^2 = AC^2 - 2AF^2.$$

Given AC = 5 cm. and AF = 2.4 cm., find the value of x and y, x to be less than y.

6. If a stationary anti-aircraft gun has a range of 12,000 ft., and a Zeppelin passes directly over it at a height of 8000 ft. and at a speed of 60 miles an hour, find for what length of its course and for what time it would be within range of the gun. As a check on the results obtained by calculation, make a scale drawing.

For how much longer could a warship making 30 miles an hour and with the same range of gun keep the Zeppelin under fire? Give clear explanations.

7. (i) Draw two lines OX, OY cutting at O; on OX mark a point C, and on OY mark two points A, B. On OX construct a point D such that

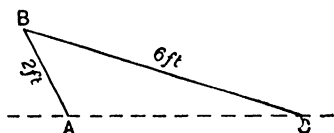
$$OA : OC = OD : OB.$$

(ii) What statement can be made about the points A, B, D, C?

Paper 10.

1. What is meant by "an angle in a segment of a circle"? Why is it legitimate to speak of the angle in a segment?

2. The crank AB turns about the pivot A; the end C of the connecting rod BC slides backwards and forwards along the line AC. The extreme positions of C to the left and right respectively are C_1 and C_2 . Find the distance C_1C_2 . At half-stroke (i.e. when AB is perpendicular to AC) the position of C is O; calculate the distance of O (i) from A, (ii) from the mid-point of C_1C_2 to the nearest tenth of a foot.



3. Construct a triangle OPR, given $OP=6$ cm., $OR=5$ cm., angle $O=74^\circ$. Draw a circle to pass through O, P, and R, and measure its radius.

4. A model boat sails in a straight line across a circular pond, towards a point 50 yd. away. The greatest distance across the pond is 70 yd. How near to the centre of the pond will the boat go? What will be the boat's least distance from the point on the pond's edge exactly opposite the starting point? [Both answers by calculation.]

5. ABCD is a rectangular sheet of paper with $AB=18$ cm., $AD=10$ cm. Along AB measure off $AH=6$ cm., $HK=8$ cm., $KB=4$ cm. and at H, K fold creases HX, KY at right angles to AB. Arrange the paper before you so that the middle part HKYX is horizontal and the two end parts AHXD, BKYC are vertical. Find by calculation the lengths of AK, AY, AB, AC and give freehand dimensioned sketches of the various triangles dealt with.

6. A, B are two places on opposite sides of a straight railway; M, N are the points on the railway nearest to A, B. $AM=2\frac{1}{2}$ miles, $MN=2$ miles, $NB=1$ mile. Draw a diagram to scale, and on it determine the position P of a railway station which shall be equidistant from A and B. Show that the station could be placed nearer to both places than P.

If $MP=x$ miles, find an algebraic equation for x and solve it. Compare with the distance measured on your diagram. Calculate the distance of P from A and B, and find how much distance would be saved from A and from B if the station were placed (i) at the point nearest to A, (ii) at the point where the sum of its distances from A and B is least.

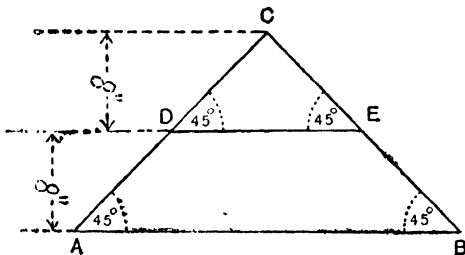
7. State the rules for comparing (i) the areas of similar figures, (ii) the volumes of similar solids, in terms of their linear dimensions.

Paper 11.

1. What do you know about the exterior angle formed by producing a side of a cyclic quadrilateral?

2. What is the locus of centres of (i) circles which touch a fixed line PQ at a fixed point P ; (ii) circles of radius $3''$ which touch a fixed line PQ ? Also of the following points:—(iii) the points of contact of tangents drawn from a fixed point to a fixed sphere; (iv) points on the earth which are 3000 nautical miles N. of the equator?

3. A corner shelf ABC is to be made from a board and to consist of two pieces $ABED$ and CDE glued together along DE . The depth $8''$ of each piece is to be the same as the breadth of the board. Determine the greatest breadth AB of the shelf. What is the shortest length of board which will suffice for the job?



4. The face of a building is 136 feet long. A photographer wants to take the building from a point at which the face subtends an angle of 37° , and for this purpose he starts off from one corner of the building in a direction making an angle of 127° with the face in question. Find, by drawing, the distance from the corner at which he must take the photograph.

Calculate also the area of ground in the triangular space between his position and the face of the building.

5. A stick, $4'$ long, is leant up against a cylindrical wooden roller of $18''$ diameter. The axis of the cylinder is perpendicular to the vertical plane in which the stick lies. The point where the stick touches the ground is $3'$ away from the point of contact of the cylinder with the ground. Without drawing to scale, find (i) the distance between the two points of contact which the stick makes respectively with the ground and the roller, (ii) the distance of the axis of the roller from the point of contact of the stick with the ground.

6. A broken lamp-shade has to be replaced. The circular edge has to rest on three supports which form the corners of an equilateral triangle; the distance between any two of the supports is 7 inches. Find, by a drawing on a scale of 1 cm. to 1 inch, what is the diameter of the circular edge and state your construction, then obtain the result more accurately by calculation.

7. You are to suppose a submarine to sight, directly north of it, an enemy battleship steaming east, and the submarine to fire a torpedo which travels at twice the rate of the battleship. Prove that the torpedo should be aimed N. 30° E.

Paper 12.

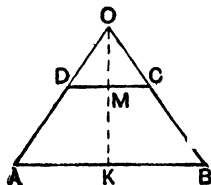
1. Explain the term "mean proportional."
2. A triangular field ABC has to be divided into four parts which are to be equal to one another in area. Draw any triangle to represent the field and show how to divide it so that the given conditions may be satisfied. Justify your work.
3. A paper cone (like an electric light shade) is slit down straight from the vertex to the base, and opened out flat; sketch the figure produced, and name it.
4. A position X lies 4000 yd. N. 68° E. of Y, whilst Z lies 3000 yd. due S. of Y. Find the distance and bearing from X of a position which is equidistant from the three positions X, Y and Z.
5. AB is a fixed line. Through A a line AC is drawn, of length 2.4 in., making an angle of 40° with AB. Draw the figure to full scale, and construct a circle to touch AB at A and to pass through C. Measure the radius of this circle.
6. Circles of radii 3 cm. and 7 cm. respectively have their centres 14 cm. apart. Calculate the length of the interior common tangents.
7. (i) Draw a freehand figure to show that two quadrilaterals may be equiangular and yet not similar.
(ii) Draw any quadrilateral ABCD and produce DA to any point A'. Now, using no instrument but ruler and set square, construct a polygon similar to ABCD on a base equal to A'D. Describe your construction briefly.

Paper 13.

1. Explain the meaning of the phrase "the angle in the alternate segment" with the aid of freehand figures.
2. Show how to draw through a given point within a given angle a straight line to be terminated by the arms of the angle, and divided in a given ratio (say $\frac{2}{3}$) at the given point.
3. In a circle a chord 2.1 in. long is 5 in. distant from the centre. Calculate (i) the radius of the circle; (ii) the length of a chord which is 10 in. distant from the centre.
4. Describe a triangle with sides 4.5, 6, 7.5 cm.; find the centres of the inscribed and circumscribed circles, and measure the distance between them.
5. Two circles of radii 4 cm. and 7 cm. have their centres 9 cm. apart. Calculate the length of the common tangents to the two circles.
6. Draw any circle and by geometrical construction draw another with the same centre and of twice the area. What is the ratio of the circumferences of the two circles?

[See next page

7. The figure represents a central section $ABCD$ of a metal lamp-shade, which, for purposes of calculation, you may imagine as obtained by cutting away the upper part of ODC from the conical surface represented by OAB . The radius R of the lower part of the shade is 5 inches, the radius r of the upper part is 2 inches, and the height MK of the shade is 4 inches.



Calculate the surface of the lamp-shade. State the geometrical principles which you assume.

Paper 14.

1. A gun is fired at noon and the sound (which travels with known velocity) is heard by two listening stations at observed times. How can the position of the gun be laid down on a chart?

2. In a quadrilateral $ABCD$ the angles at A and C are right angles. $AB=6$ cm., $AD=8$ cm., $BC=2.8$ cm. Construct the quadrilateral. Measure CD . Find the length of the side CD by calculation.

3. AB is a diameter of a circle of radius 5 cm. Draw a chord CD of the circle perpendicular to AB and 6 cm. in length. Also, through O , the point of intersection of AB and CD , draw a chord of the circle 8 cm. long. State the steps of your construction.

4. Draw a circle of radius 7 cm. and a chord PQ distant 4 cm. from the centre. Now draw a circle of radius 5 cm. to touch your first circle internally, and also to touch PQ . State your construction.

5. A triangle ABC is inscribed in a circle (centre O , and radius 4").

If the angles of the triangle are $A=72^\circ$, $B=55^\circ$, $C=53^\circ$, what are the angles BOC , COA , AOB ? Hence find the sides of the triangle by drawing.

6. Show how to draw a plane parallel to the base of a pyramid so that the area of its intersection with the pyramid may be half that of the base.

7. Draw a triangle ABC right-angled at A . Take P in AB and Q in AC ; join PQ . The figure represents a triangular field, divided into two equal portions by a straight fence PQ . The lengths in metres are as follows:— $AB=a$, $AC=b$, $AP=x$, $AQ=z$, $PQ=y$. Express y in terms of x and z . Write down a relation between x , z , a , b , and use it to show that $y^2 = x^2 + \frac{a^2 b^2}{4x^2}$.

What are the smallest and largest possible values of x ?

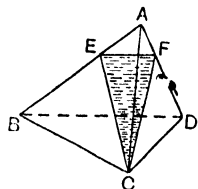
Taking $a=30$, $b=40$, draw a graph of y^2 as a function of x for possible positions of the point P . Hence or otherwise determine where the points P and Q must be if the fence is to be as short as possible. Find also the length of the fence in this case.

Paper 15.

1. How can the position of an electric light bulb be described with reference to two walls and the floor?

2. Draw a circle of radius 5 cm. and take a point O at a distance of 10 cm. from its centre. From O draw a line cutting the circle in P and Q such that P is the middle point of OQ.

3. ABCD is a regular tetrahedron, each edge of which measures 12 cm. The shaded triangle represents a section through C, E, F. $AE=AF=5$ cm. Make an accurate drawing of the triangle CEF and find its area.



4. Show that it is usually possible to draw two circles each of which touches two given sides of a given triangle (produced if necessary) and has its centre on the third side or that side produced, but that under certain circumstances only one such circle can be drawn.

5. A circle is circumscribed round a triangle ABC right-angled at A. Its radius is found to be 6". AN is drawn perpendicular to the base BC. O is the mid-point of BC and $ON=2.4$ ". Calculate the lengths of AN, AB, and AC.

6. An acute-angled triangle ABC resting with BC along a straight line XY rolls without sliding along the straight line, turning about C, A, B in succession till BC again lies along XY. Draw the paths of A, B, and C.

If O be the centre of the circumcircle of the triangle ABC, prove that the length of the path of O is equal to the circumference of the circumcircle.

7. (i) Draw a large triangle with angles 60° , 70° , 50° . Determine the ratio of the area of the triangle to the area of the square on its longest side. Why do you suppose that this ratio is independent of the size of the triangle?

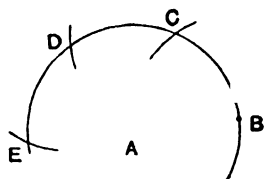
(ii) Prove that, if three similar triangles are constructed with the sides of a right-angled triangle as bases (i.e. as corresponding sides for the three triangles), then the areas of the triangles are connected by a relation similar to that of Pythagoras' theorem.

Paper 16.

1. How is the inclination of a line to a plane determined?

2. ABC is a triangle having $AC=7.2$ ", $BC=9.6$ ". AX, BY are drawn perpendicular to BC, AC respectively. If $AX=2.4$ ", find the length of BY.

3. With centre A and radius AB a circle is drawn. With centre B and equal radius an arc is drawn intersecting the first circle at C. Similarly from centre C the point D is determined, and from centre D the point E. Prove that BAE is a straight line. (This shows how a point may be obtained on a straight line "produced" without the use of a straight edge.)



[See next page]

4. A penny falls into a cup whose shape is an exact hemisphere of radius 5 cm. If the penny lies symmetrically at the bottom and its diameter is 3 cm., calculate how far the lowest point of the cup is below the penny.

5. Draw a circle of 4 cm. radius to touch two circles of radii 3 cm. and 2 cm. respectively, whose centres are 6 cm. apart. The 3-cm. circle is to lie entirely inside the 4-cm. circle, and the 2-cm. circle is to lie entirely outside it.

6. O and P are points 1000 yd. apart, P being due east of O. At Q the line OP subtends an angle of 63° . Q is 450 yd. from the line OP. Draw a figure to scale, and find the distance and bearing of Q from O.

In the triangle ABC, PQ is parallel to the base and ARD is perpendicular to the base meeting it in D and meeting PQ in R. If $AR = x$ and $RD = y$, prove that similar triangles that $PQ/BC = x/(x+y)$. Then prove that

$$\text{area APQ}/\text{area PQCB} = x^2/(2xy + y^2).$$

If the area PQCB is twice the area APQ, find the value of the ratio x/y correct to two decimal places.

Paper 17.

1. How is the angle between two intersecting planes determined?

2. Show how to construct a triangle similar to and of double the area of a given triangle.

3. ABCD is a rectangle in which $AB = 4$ in., $BC = 3$ in. A circle with A as centre passes through the middle point of BC and cuts AD at F. Calculate the length of CF.

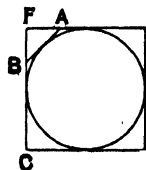
4. P and Q are two points on the circumference of a circle, and the tangents to the circle at P and Q intersect at an angle of 56° . What fraction of the whole circumference is the minor arc PQ? and what is the ratio of the major arc PQ to the minor arc PQ?

5. Construct a triangle ADE such that AD, DE, EA measure 5, 6.1, 9.7 cm. respectively. Construct the circumscribing circle and the circle escribed to DE. Measure the radii of the circles and the distance between their centres.

6. The back and front wheels of a carriage have radii 2 ft. 6 in. and 1 ft. 6 in., and the distance between their centres is 5 ft. 6 in. Find by drawing the distance between the points at which they touch the ground.

Verify your result by calculation.

7. To make a cylindrical wooden rod, a square-section rod is first of all planed down till its section is a regular octagon. The figure shows the square section, the circular section of the final cylinder, and the side of the octagon that results from planing the first edge of the square rod. Calculate the ratio of AB to BC.



Paper 18.

1. When is a straight line said to be perpendicular to a plane?

2. What is the relation between the volumes of a prism and a pyramid of the same base and height? To what other pair of solids does this relation apply?

3. A point R is taken on the side AB of a triangle ABC of area z , so that $AR = x \cdot AB$, where $x > 1/2$. RQ is drawn parallel to BC to meet AC at Q, RH parallel to AC to meet BC at H, and QK parallel to AB to meet BC at K. Prove that the areas of ARQ and BRH are x^2z and $(1-x)^2z$ respectively, and find in similar form the area of QK; use these results to find the area of QRHK. Verify your result for QRHK by giving x the values 1 and $1/2$.

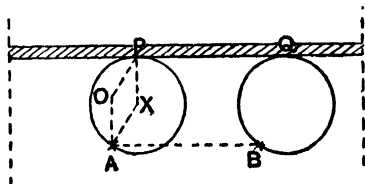
4. A dirty football was found to leave a circle of mud, 11" in circumference, when bouncing; if the radius of the ball was $6\frac{1}{2}$ ", find the depth to which it was squashed in by the impact.

5. Two halfpennies (diameter 1 in.) lie on a sheet of paper with their circumferences touching. A semicircle is drawn to circumscribe the two halfpennies, so that its diameter and its circumference touch each coin. Make an accurate drawing of this arrangement, and determine the diameter of the semicircle.

6. Show how to draw a tangent to a circle from an external point.

If the length of this tangent is 10 cm. and the shortest distance from the external point to the circumference of the circle is 4 cm., find the radius of the circle. State without proof the enunciation of any theorem which you may use in answering this question.

7. The diagram shows a horizontal rod whose ends slide in vertical guides. The rod is raised and lowered by two equal discs radius 2" turning about horizontal axes through the points A, B on their rims with equal speeds so as to keep the rod horizontal, A and B being at the same level. P is the point of contact of the rod with the left-hand disc.



(i) Find the distance between the extreme positions of P on the rod.

(ii) If each disc starts with its centre on AB or AB produced and revolves steadily at the rate of one revolution per minute, construct the position of P at intervals of 10 seconds.

(iii) If X is the centre of the disc AP and O is 2" vertically above A, prove that AXPO is a rhombus.

(iv) Prove that the locus of P is a circle.

Paper 19.

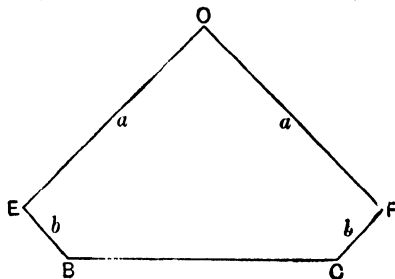
1. What is the rule for calculating the volume of a prism? To what other classes of solid does the same rule apply?

2. Calculate the lengths of the interior and exterior common tangents of two circles of radii 2 in. and 6 in., the centres being 12 in. apart. Also find to the nearest degree the angle between the exterior tangents.

3. The figure shows the plan of a corner cupboard. The angles at O, E, F are right angles; OE and OF are each of length a ; BE and CF are each of length b . The cupboard has vertical walls of height h . Its top and bottom are each as shown in the figure.

Find in terms of a and b an expression for the area A of the floor of the cupboard. Notice that the floor is in the form of a square, from which a right-angled isosceles triangle has been removed.

Calculate the volume, in cubic feet, when $a = 1$ ft. 8 in., $b = 5$ in., $h = 2$ ft. 3 in.



4. P is a point on the circumference of a circle of centre O and radius $1\frac{1}{2}$ in. Q is taken so that $\angle POQ = 40^\circ$ and $OQ = 3$ in. Construct a circle to touch the given circle at P and to pass through Q. Measure the radius of this circle.

5. A, B, C, D are four points on a circle of which O is the centre. AC is a diameter and $\angle BAC = 35^\circ$, $\angle DBC = 40^\circ$. Find $\angle ODC$, $\angle ODB$, giving your reasons briefly.

6. On a fixed line AB, 8 cm. long as base, construct a triangle ABC, whose area shall be 24 sq. cm., and vertical angle 63° . Measure (i) the smallest angle, (ii) the radius of the circumcircle of the triangle.

7. The following construction claims to divide a given straight line into three equal parts without making use of parallel lines:—

“If AB be the given line, draw AC of indefinite length at a convenient angle to AB. Along AC set off any two convenient equal parts AD, DE. Along BE, produced indefinitely, set off EH equal to BE. Then HD produced cuts AB in K, so that AK is a third of AB.”

Carry out the construction, taking AB as 8 cm. Test the result by measurement, then justify the validity of the construction, giving adequate reasons for the statements you make. In working out your proof you may draw the line EL parallel to BA; the restriction as to parallel lines applies only to the practical construction.

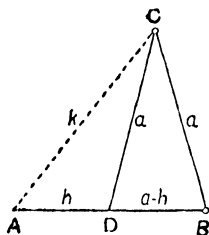
Paper 20.

1. An aviation course is in the form of an equilateral triangle, measuring $1\frac{1}{2}$ kilometres along each side, and marked out by posts at the vertices. Find the distance that must be traversed in one round by an aeroplane which turns in arcs of 50 metres radius just grazing the posts. Give a sketch to illustrate your answer.

2. Draw any triangle ABC. It is required to inscribe in this triangle an equilateral triangle of which one side shall be parallel to AB, and the opposite vertex shall lie on AB.

Show how this can be done by employing the properties of similar figures.

3. The figure shows three equal bars AB, BC, CD, jointed at B and C. The three are placed on a table, and the bar AB is kept fixed while the point D is gradually moved along AB from A to B, the joint C moving in consequence across the table. Prove that, if the straight line AC is drawn, in all positions of D, the triangle ADC has one of its angles double of another.



If each bar is of length a , obtain an expression for the length k of AC when D has been moved a distance h from A towards B. Also calculate k when $h = \frac{1}{2}a$, taking $a = 10$ cm. (Draw a $\perp$ from C to AB.)

4. CD and EF are two given parallel straight lines, and A is a given point in CD. B is a given point on the side of EF remote from A. It is required to determine the position of a point P in CD, such that, when the straight line PB is drawn crossing EF at Q, then PQ may be equal to AP. Prove that the perpendicular distance of the line PB from the point A is equal to the distance between the parallel lines. Hence solve the problem, and show that there are in general two possible positions for the point P.

Draw the figure, making the perpendicular distance between the parallel lines equal to 3 cm., AB = 8 cm. and angle DAB = 60° . Determine from your drawing the two possible lengths of PQ.

5. A lawn-mower has two cylindrical rollers in contact with the ground. The smaller has a diameter of 3 in. and the larger of 13 in. The lines in contact with the ground are 1 ft. apart. Calculate the distance between the axes of the rollers.

6. ABC is a tangent to a circle at B, BD is a diameter and BE, BF are chords such that $\angle ABE = 20^\circ$, $\angle CBF = 60^\circ$; DE, DF, EF are joined. Find all the angles of the figure.

7. Take four points A, B, C, D in a straight line, making AB = 2.8 cm., BC = 4.5 cm., CD = 6.6 cm. Draw two circles, one through A and D, the other through B and C, to cut one another in two points P and Q. Join PQ meeting DA produced in R. Measure AR.

[See next page

Verify by drawing that if either of the circles is replaced by another circle through the same two points on $ABCD$, the common chord passes through the same point R .

From a geometrical property obtain an equation for the length of AR , and solve the equation.

Paper 21.

1. What is a great circle on a sphere? Mention some properties of great circles.

2. Prove that the area of a trapezium is obtained by multiplying half the sum of the parallel sides by the altitude.

Draw a trapezium having its parallel sides 8 cm. and 6 cm., and altitude 5 cm., and one of its acute angles 68° . Transform the trapezium into an equivalent triangle. Describe your construction briefly and show how the above rule for finding the area of a trapezium follows directly from your new figure.

3. Complete the following statements:—

(i) The locus of the centre of a circle of given radius which passes through a fixed point is...

(ii) The locus of the centre of a circle of given radius which touches a fixed straight line is...

(iii) The locus of the centre of a circle of given radius which touches a fixed circle is...

Draw a circle of radius 2 inches and a straight line distant 1 inch from the centre of the circle. Then construct all the circles of radius half an inch which touch both the straight line and the circle, explaining your method carefully.

4. State, without actually performing any construction, how you would solve the problem of drawing two tangents to a circle, which should include a given angle and intersect upon a given straight line. How many solutions of the problem would you expect to get?

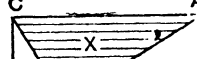
5. AB is an arc of a circle, of radius 4 in., subtending an angle of 45° at its centre C . Let the tangents at A and B meet at T , and produce CA and BT to meet at S . Prove that $AS = AT = TB$, and, denoting each of these equal lengths by x in., calculate the value of x .

Now suppose that AT and STB represent two railway lines crossing each other at T . The points A and B are connected by a loop line represented by the arc AB of radius 400 yd. Determine in yards the distances of the points A and B from the crossing T .

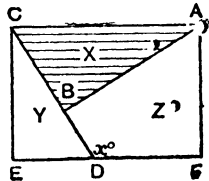
6. Draw a straight line AB of length 5 cm. Find a point P at which AB subtends an angle of 54° and such that AP is 4 cm. Measure the distance PB .

7. The height of the Great Pyramid is 149 metres; an exact model of the pyramid is made of height 1.49 metres, its side faces being triangles similar to the side faces of the pyramid. What is the ratio of the total slant surface of the pyramid to that of the model?

Paper 22.

1. What do you know about regular solids?
2. A circle has radius of length 3 in., and chords of length 1.4 in. are placed in it; on these chords as bases are drawn isosceles triangles of side 2 in., on the sides of the chords remote from the centre of the circle. Show that the locus of their vertices is a circle, and calculate the length of its radius to the nearest hundredth of an inch.
3. Take a line AB, 9 cm. long, and through B draw CD at right angles, making BC=6 cm., and BD=3 cm. Join AC and complete the rectangle AFDE. Denote by x the number of degrees in the angle BDF, and write in each angle of the figure its value in degrees.
- 

You are told that if the parts of the figure marked Y and Z were cut out they could be placed against the part marked X so that the three parts would form a square. Give the area of the square.



4. Draw two straight lines AB, AC enclosing an angle of 48° . Take a point D in AB such that $AD = 2.6$ in. Construct a circle DEF to touch AB in D and also to touch AC.

Construct another circle to touch **AB**, **AC** and also to touch the circle **DEF**.

5. Draw an isosceles triangle whose base, BC , is 3 in., and vertical angle A is 36° . Construct a circle to pass through A and touch BC at C . Let the circle cut AB at P . Measure AP . Prove that $CP = CB$.

6. The diameter AB of a circle bisects the chord EF in G . If $AE=a$, $AG=b$, $EG=c$, and the radius $=r$, obtain an algebraic equation connecting a , b and r , and also one connecting b , c and r .

Prove that the area of a circle whose radius is AE is equal to the area of a rectangle whose height is AG , and whose base is equal to the circumference of the given circle.

7. Show that the radius of a sphere which lies inside a right circular cone (whose height is h and the radius of whose base is a) so as to touch the base and the curved surface of the cone is

$$\frac{ah}{a + \sqrt{a^2 + h^2}}$$

Paper 23.

1. Assuming that the line joining the centres of two intersecting circles bisects the common chord at right angles, examine what this theorem becomes in the limiting case when the circles touch.

- 2.** C is the centre of a circle of radius 5 cm. and O is a point distant 7 cm. from C. Find as accurately as you can the greatest angle that a straight line drawn through O to meet the circle can make with OC, and find also the length of the chord intercepted by the circle on the straight line drawn through O to make with OC an angle equal to half of this greatest angle.

[See next page

3. In the accompanying diagram of a balance the beam is supported on a knife-edge at O, the pointer OP is fixed to the beam at right angles, the tip of the pointer moves along the scale CD, the scale pans hang from A and B.

$OA = OB = 4''$, $OP = 7''$, arc $CD = 3''$.

Calculate the angle through which the pointer swings as P moves from C to D; also find the vertical height through which each scale pan moves.

4. ABCD is a circle: AC, BD meet in X. Given that

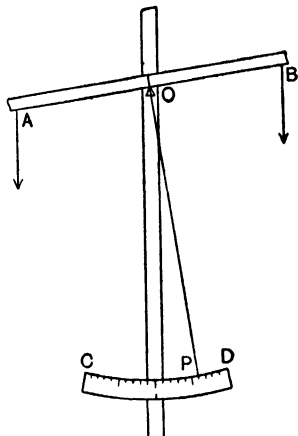
$\angle ABD = 33^\circ$, $\angle ADB = 27^\circ$, $\angle BAC = 45^\circ$,

calculate the angles BXC, ACD, ABC, showing your reasoning clearly but shortly.

5. A, B, C are 3 landmarks. B is 200 yd. due E. of A, and C is 200 yd. N. 26° E. of B. An observer in a ship which is due N. of B, observes that AC subtends an angle of 90° at his eye. Find, by drawing, the distance of the ship from A, B, and C.

6. Draw two straight lines making an angle of 60° with one another and intersecting at O. On one of the lines take the two points X, Y on opposite sides of O such that $OX = 2$ in., $OY = 4.5$ in. Draw the circle through the points X and Y which will cut the other line in two points equidistant from O. Calculate the distance of each of these points from O.

7. A right prism has for its base a parallelogram of sides 4 in., 2 in., including an angle of 50° at the corner O. Draw the section of the prism made by a plane which passes through O and whose intersections with the faces of the prism through O make angles of 35° with the sides of the base.



Paper 24.

1. Draw, on plain or squared paper, a sketch of the plot of ground whose measurements are recorded in the table; then calculate its area.

2. Draw a circle 3 in. in diameter and place in it a chord AB 2.5 in. in length; draw the diameter BC and produce BC to D so that D is 1 in. distant from the circle; and through D draw DE perpendicular to BD. Then draw a circle touching DE and also touching the former circle at A. State the steps of your construction.

3. OA is the vertical line which is the junction between two walls of a room, OB and OC the horizontal lines running along the junction between the walls and the floor. What is the locus of the point P which moves about in the room

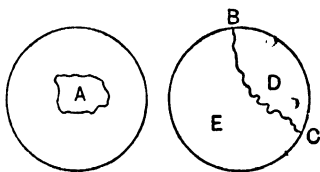
Yards

	To B	
30	80	2°
	60	
	30	
28	20	
	A	

under the following conditions:—(i) so as to be always 5 ft. above the floor? (ii) so that the angle AOP is always 50° ? (iii) so as to be always equidistant from OB and OC? (iv) so as to be 4 ft. from O and 2 ft. from the plane AQC?

4. A man in a rowing boat considers that a straight promenade 2 miles long subtends an angle of 90° at the boat. He knows he is at a distance of $1\frac{1}{4}$ miles from one end of the promenade. How far must he row parallel to the shore till he is at the same distance from the other end?

5. If a closed loop of thread is placed on a soap film that covers a ring of wire (see the figure), and the film within the loop is pierced, the film outside takes up as small an area as possible and thus pulls the thread at A into a circle. Calculate the diameter and the area of the circle formed by the thread if the total length of the thread forming the loop is 6 cm.



If the ends B, C of the thread are attached to the ring, and the film on the side marked D is pierced, the thread again becomes a circular arc. If the thread BC is 6 cm. long and the angle it subtends at the centre of the circle of which it forms an arc is 120° , calculate the length of the chord BC.

6. Make a table showing the number F of faces, the number V of corners, and, the number E of edges for a cube, a prism and a pyramid.

Name of Solid	F	V	E	$F + V - E$
---------------	---	---	---	-------------

What inference seems probable from the table?

Imagine the vertex cut off a square pyramid, one face being thus added. By how many are the corners increased? and the edges? Give the value of $F + V - E$ for this new solid. Also suppose a different corner of the square pyramid cut away, and see what effect this has on the number $F + V - E$.

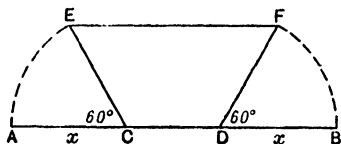
7. From a point P outside a circle of radius a , are drawn the tangent PQ (of length x), and the straight line PAB through the centre cutting the circle in points A and B, A being nearer to P and PA being of length y . Write down the relation connecting the lengths of the lines PQ, PA, PB, and express it in terms of x, y, a .

If the circle is taken as representing a section of the earth through its centre, PQ will be the range of vision of a person situated at a height y above the surface. Take $a = 4000$ miles, use the relation in the approximate form $2ay = x^2$, and find, in miles and in feet, to what height it is necessary to ascend in order to have a range of vision of 50 miles.

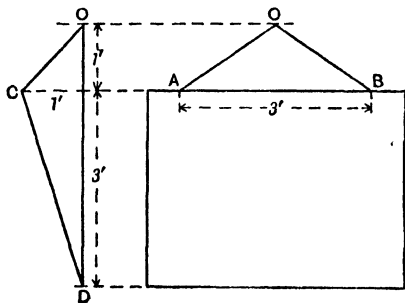
The fact that, in general, $F + V - E = 2$ for any solid figure bounded by plane surfaces is known as Euler's Theorem. Euler stated it in 1738.

Paper 25.

1. In the figure AB is a line l in. long, and $AC = DB = x$ in. CA is turned through 60° into position CE and similarly DB into position DF ; and EF is joined. Prove that the area of $ECDF$ is $\frac{1}{4}\sqrt{3}x(2l - 3x)$ sq. in.



2. The figure shows the front and side elevation of a picture, supported at the bottom by a nail at D and hung by a wire passing over a hook O in the wall. By calculation or otherwise find the length of the wire.



3. A and B are the doors of two houses and they are 52 ft. apart. A straight road is to be constructed so that A is 30 ft. from it and B is 50 ft. Using a scale of $\frac{1}{20}$ th inch to the foot, draw a figure showing how the road must be made.

4. A rectangular piece of paper $ABCD$ is folded so that one corner D is brought on to the opposite corner B . Given that the length of AD is 11 cm. and of AB 7 cm., find, either by algebra or from a geometrical construction, the distance from A of the point in which the crease meets AD . State and justify your method.

5. PX , PY are two straight lines intersecting at an angle 45° : A , B are points on PX such that $PA = AB = 5$ cm. Construct in one figure the points on PY at which the segment AB subtends angles 20° , 30° , 40° , explaining your method. How would you find the point K on PY at which AB subtends the greatest angle? Construct this point in any way you please, and measure this greatest angle.

6. In order to measure the distance away of a man who is 6 feet high, I hold a foot rule pointing vertically upwards at a distance of 3 feet from my eye, and observe the length on the scale which subtends the same angle at my eye as the man. If this length be x inches give a formula for the distance d of the man in yards.

Calculate the value of x when the man is 20, 30, 40 yards away

7. Given the radii of two circles, show how to mark off on one of them an arc equal to the circumference of the other.

A piece of wire is clamped at one end and is bent into a circular arc, the tangent at the clamped end being fixed. Construct the locus of the extremity of the wire when the radius of the arc is varied till the wire is bent into a complete circle, the wire being of such a length that the complete circle has a radius of 2 cm. Accompany your construction by a brief explanation of your method.

Paper 26.

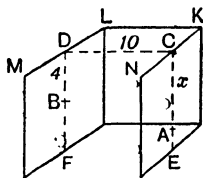
1. At noon on a certain day the altitude of the sun in London is 60° . A window in a London house is 5 ft. high, and 3 ft. wide and faces south, and the sun shining through the window at that instant makes a patch of sunlight on the floor of the room. What is the area of this patch?

2. A circle touches a straight line AA' at its middle point O . Chords OB , OC are drawn making angles of 36° and 72° respectively with the tangent OA ; and chords OB' , OC' making 36° and 72° with OA' .

State which of the arcs OB , BC , OB' , $B'C'$, CC' are equal by reason of the symmetry of the curve about its diameter through O ; also which of them are equal for other reasons, giving proofs of their equality.

3. The figure represents 3 walls of a cubical room measuring 10 ft. each way. CE and DF are vertical lines bisecting two of the walls. $DB = 4$ ft. and $CA = x$ ft.

The question has arisen which is the shorter route from A to B , (i) that across the ceiling shown in the figure, or (ii) that round the walls which becomes a straight line when the three walls shown are opened out flat. Take half a dozen different values of x , find by drawing the length of the second route for each value, and so decide the question. Sketch a graph in illustration.



Then give an expression for the second route in terms of x , and by algebraic calculation determine for what values of x this route is the shorter.

4. OY , OX are two straight lines at right angles. On OX two points A , B are marked so that $OA = 1''$, $OB = 3''$. By construction find a point (or points) on OY at which AB subtends an angle of 25° . Explain your construction and measure the distance of the point (or points) from O . Find, by drawing or otherwise, the position of the point on OY at which AB subtends the greatest possible angle.

5. The two equal circles, centres P and Q , are so drawn that each passes through the centre of the other: they intersect at A and B . The radius of each circle is r . Prove that (i) arc AQB subtends an angle of 120° at P ; (ii) the sector of arc AQB and centre P is of area $\frac{1}{3}\pi r^2$; (iii) $AB = r\sqrt{3}$; (iv) area of $\triangle PAB = \frac{r^2\sqrt{3}}{4}$; (v) the area common to the two circles $= r^2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$; (vi) the ratio of this common area to the area of each circle is 0.39.

6. A ship is end on to a straight quay with its bows 200 ft. from the quay. The ship is 280 ft. long. At what distance along the quay, measured from the point nearest the ship, will the ship subtend the greatest angle?

[See next page

7. ABCD is a square sheet of cardboard out of which it is proposed to make a box without a lid by cutting out small squares at the corners and turning up the edge all round. If the edge of the sheet is 40 inches long, find an expression for the capacity of the box in cubic inches when the depth is x inches.

Write down the capacity in cubic inches when $x=3$ inches, 4 inches, etc. up to 10 inches, and show by a graph how the capacity of the box depends on x .

From the graph find the capacity of the largest box which can be made in this way out of the given sheet of cardboard.

Paper 27.

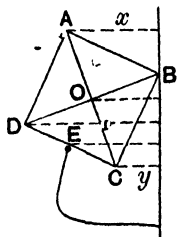
1. XY is an unlimited straight line parallel to AB and 2 cm. from it; AB is 8 cm. long and C is its middle point. P is a point on XY, PN is perpendicular to AB, and CN is x cm. long. Find expressions for AN, BN, AP, BP, and simplify the last two as far as possible. Hence find an equation for x when AP is three times BP; solve it, and test the accuracy of your result by drawing a figure to scale.

2. AB is a breakwater 2000 yd. long, B being due E. of A. The breakwater subtends an angle of 50° at each of two ships, X and Y. If X bears N. 10° E. from A, and Y is 800 yd. to the eastward (not due E.) of X, find the distance of each ship from the breakwater. [Scale 400 yd. to 1 in.]

3. (a) (i) A, B are two points on the same side of a straight line XY; C is the mid-point of AB. The distances of A and B from XY are 3 in. and 7 in. respectively. Find the distance of C from XY.

(ii) If the distances of A and B from XY are a in. and b in. respectively, what is the distance of C from XY?

(b) The figure represents a cubical box, section ABCD, resting in the rack of a railway carriage with E the mid-point of CD in contact with the rack rail and the corner B against the wall. The distances of A and C from the wall are x in. and y in. respectively. O is the centre of the box.



(i) Express in terms of x, y the distances of O, D, E from the wall.

(ii) Find how far O is from the vertical line through E.

(iii) Find the height of O above the level of E.

(iv) If the length of an edge of the box is a in., write down an equation connecting x, y, a .

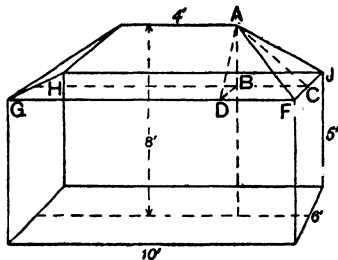
4. A small rectangular wooden platform measures 4 ft. by 6 ft.; the corners are rounded off into quarter-circles of 6 in. radius; the edge is bound all round with a brass strip. Draw a plan of the platform on squared paper.

Calculate the area of the platform in square feet to three significant figures, and the length of the brass strip, in feet and inches, to the nearest inch.

5. The figure shows a symmetrical glass skylight, with four vertical rectangular panes, two sloping triangular panes and two sloping panes of trapezium shape. The dimensions are shown in the figure.

From A is drawn AC perpendicular to FJ, AD perpendicular to GF and AB perpendicular to the centre line HC. What is the length of BC? of BD?

Denote the lengths, in feet, of AC and AD by x, y respectively. Form an expression containing x and y , for the number of square feet of glass in the skylight. Find x and y to three significant figures, and calculate the area of glass to the nearest square foot.



6. The figure represents a bridge, whose span AB is 80 ft., supported on an arch in the form of an arc of a circle. AC = 16 ft. = BD. Let r ft. denote the radius of the circular arc and b ft. the height of the roadway AB above the highest point of the arch. Find an algebraic equation connecting r with b . Then use it to calculate (i) the value of b when $r = 65$, (ii) the value of r when $b = 1$.



7. A and B are two forts on a straight coast 5 miles apart. The effective range of A's guns is $3\frac{1}{2}$ miles, and of B's, 3 miles. Draw the circles bounding the area covered by both forts, and let C be one of the points of intersection of these circles. An enemy's ship comes to C so as to be able to bombard a town lying between A and B without being within the range of the guns from either fort. By measurement, find how far C is from the coast-line. Measure the angles CAB and CBA and hence calculate approximately the number of square miles covered by the zone of effective fire from both forts.

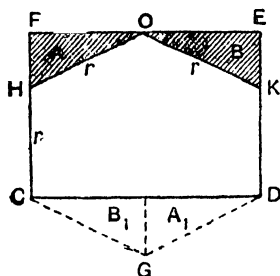
Paper 28.

1. In playing with pennies a boy observed that he could arrange six coins round a centre one, each touching the centre one and two others. Show the possibility of this by drawing a careful figure in which each circle has a radius of 2 cm. State clearly how you determine the centres of the circles and what help you get in this construction from considerations of symmetry; then justify by general reasoning the method you have adopted.

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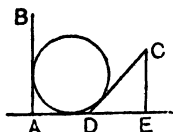
2. CDEF is a rectangle in which $CD = a$ and $CF = b$. A circle, whose centre is at O, the middle point of EF, is described to cut CF at H and DE at K. If the radius r of the circle is such that $CH = HO$, express r in terms of a and b .

Suppose that the two right-angled triangles A and B are cut away from the rectangle and placed in the positions A_1 and B_1 , thus converting the rectangle into an equilateral hexagon. Show that if $a/2 = b/\sqrt{3}$, the resulting hexagon is regular.



3. Draw a straight line OBC, making $OB = 2.5$ cm., $OC = 6.4$ cm. Through O draw a line OA making the angle $AOB = 42^\circ$. Then draw a circle passing through B and C and touching OA. (Describe the steps of your construction.)

4. The figure, which is not drawn to scale, shows the cross-section of a cylinder which is to be raised against the wall AB by pushing the wedge CDE to the left. The radius of the cylinder is 3 ft. and the angle CDE is 25° . Draw the figure to scale when the wedge has been pushed in as far as possible. Find the height through which the cylinder has been raised, and the necessary length of the face CD of the wedge.



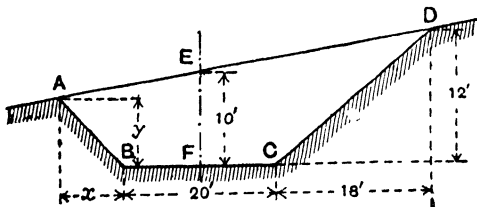
5. A line of trenches runs due north and south, and a pilot to the east of the line at a height of 1000 ft. observes the angle of depression of the nearest point in the trenches to be 26° , when his engine fails to work. If he glides down at an angle of 10° to the horizon and wishes to land on the line of trenches south of his present position, determine the direction of the point of the compass for which he must steer (in degrees west of south) and the distance that his landing place is south of his present position.

6. OT is a tangent to a circle at T, and PQ is a diameter, OPQ being a straight line; $OP = 2b$ and the radius of the circle is $a - b$. Prove that

$$OT^2 = (a + b)^2 - (a - b)^2.$$

Simplify this expression, and deduce that $OT^2 = OP \times OQ$.

7. In the figure ABCD is a cross-section showing a railway cutting made in ground, the surface of which slopes in a direction at right angles to the cutting, as shown by the line AED. BC is the trace of the horizontal plane on which the track will be laid, and EF is a vertical centre line bisecting BC. The side AB of the cutting is to have the same slope to the horizontal as the side CD.



Calculate the dimensions x and y . Find the volume, in cubic yards, of the earth which must be excavated per chain length of track.

INDEX

- Acute angle.** p. 2.
Acute-angled triangle. p. 9.
Adjacent angles. When three straight lines are drawn from a point, if one of them is regarded as lying between the other two, the angles which the line makes with the other two are called adjacent angles.
Alternate angles. p. 5.
Alternate segment. p. 108.
Altitude. See **triangle, parallelogram.**
Altitude of sun. p. 35.
Ambiguous case. p. 14.
Angle. When two straight lines are drawn from a point, they are said to form, or contain, an angle. The point is called the **vertex** of the angle, and the straight lines are called the **arms** of the angle. p. 1.
Angle in a segment. pp. 104, 108.
Angle in a semicircle. p. 105.
Angle of depression. p. 36.
Angle of elevation. p. 35.
Angle properties of circle. p. 103.
Arc of a circle. pp. 86, 114.
Area of circle. p. 117.
 cone. p. 207.
 cylinder. p. 196.
 parallelogram. p. 45.
 rectangle. p. 43.
 rectilinear figure. pp. 44, 49, 51.
 sector of circle. p. 119.
 similar figures. p. 134.
 sphere. p. 222.
 trapezium. p. 48.
 triangle. p. 46.
Base. See **triangle, parallelogram.**
Base-line. p. 49.
Bearing. p. 35.
Bisect. Divide into two equal parts.
Cardioid. p. 173.
Centre of circle, To find. p. 88.
Chain. p. 50.
Change of a figure. p. 165.
Chord of a circle. p. 86.
Circle. p. 86.
Circumcentre. p. 89.
Circumference of a circle. pp. 86, 115.

- Circumscribed circle.** p. 88.
Circumscribed polygon. pp. 95, 101.
Common tangents, exterior and interior. p. 128.
Complementary angles. p. 3.
Concentric circles are circles having the same centre.
Conchoid. p. 173.
Concyclic. p. 107.
Core, p. 205.
Congruent. Figures which are equal in all respects are said to be congruent.
 p. 11.
Congruent triangles. p. 11.
Conic section. pp. 189-191.
Constant angle locus. p. 109.
Constructions:—angle equal to given angle. p. 18,
 bisection of lines and angles. pp. 19, 20,
 centre of circle. p. 88.
 circles to satisfy given conditions. pp. 123, 161.
 circumcircle. p. 88.
 circumscribed polygon. p. 101.
 common tangents. p. 129.
 constant angle locus. p. 110.
 contact problems. pp. 122, 124.
 division of straight line. pp. 144, 145.
 equilateral triangle. p. 15.
 equivalent triangle. p. 51.
 escribed circle. p. 96.
 fourth proportional. p. 149.
 hexagon. p. 99.
 inscribed circle. p. 95.
 intersection of loci. p. 64.
 isosceles triangle. p. 14.
 mean proportional. p. 159.
 parallelograms. p. 26.
 parallels. pp. 4, 6.
 perpendiculars. p. 20.
 quadrilaterals. p. 29.
 regular polygons. p. 98.
 rhombus. p. 26.
 right-angled triangle. p. 13.
 ruler and compass. p. 18,
 segment of circle. p. 110.
 similar polygons. pp. 161, 163.
 special angles. p. 21.
 square equivalent to given figure. p. 160.
 tangents. pp. 93, 129.
 triangles. pp. 11, 66, 111.

- Contact of circles.** p. 121.
Continuous change of a figure. p. 165.
Convex. p. 100.
Corresponding angles. p. 5.
Cross-section. p. 193.
Cube, cuboid. p. 225.
Cyclic quadrilateral. pp. 105, 107.
Cycloid. p. 174.
Cylinder. p. 191.
Degree. p. 2.
Development of solid figure. p. 190.
Diagonal. See **quadrilateral.**
Diagonal scale. p. 154.
Diameter. p. 86.
Distance of point from plane. p. 211.
Division of straight line. pp. 144, 147
Dodecahedron. p. 225.
Drawing of solid figures. p. 190.
Elevation. p. 227.
 Angle of. p. 35.
Ellipse. p. 168.
Envelope. p. 175.
Equiangular. This word is used in two senses:—(i) a figure is said to be equiangular when all its angles are equal; (ii) two figures are said to be equiangular when the angles of the one figure are equal respectively to the angles of the other figure taken in the same order.
Equilateral triangle. pp. 10, 15.
Equivalent. Figures which are equal in area are said to be equivalent. p. 45.
Escribed circles of a triangle. p. 96.
Euler's theorem. p. 253 (Ex. 6).
Field-book. p. 49.
Focus. p. 168.
Fourth proportional. p. 148.
Frameworks. p. 28.
Frustum. p. 199.
Generator. p. 193.
Great circle. pp. 217, 220.
Ground line. p. 227.
Height. See **triangle. parallelogram.**
Heptagon. p. 98.
Hexagon. p. 98.
Hyperbola. p. 171.
Hypotenuse. p. 9.
Icosahedron. p. 225.
Image. pp. 31, 33, 154.
Incentre. p. 95.
Inclination of line to plane and plane to plane. p. 210.

Inscribed circle of a polygon. p. 95.

Inscribed circle of a triangle. p. 95.

Inscribed polygon. p. 88.

Intersecting chords. p. 156.

Isosceles trapezium. A trapezium with its two non-parallel sides equal.

Isosceles triangle. pp. 10, 14.

Kite. p. 24.

Latitude, longitude. p. 221.

Length of chord. p. 90.

Lenses. p. 154.

Limaçon. p. 173.

Limit. pp. 92, 108, 118.

Link. p. 50.

Locus. pp. 61, 86, 109, 110, 123, 168, 239.

Major arc, minor arc. p. 86.

Major segment, minor segment. p. 104.

Maximum and minimum values. p. 167.

Mean proportional. p. 159.

Median. See **triangle**.

Miscellaneous exercises. pp. 67, 177, 232.

Normal. p. 194.

Oblique. p. 194.

Obtuse angle. p. 2.

Obtuse-angled triangle. p. 9.

Octagon. p. 98.

Octahedron. p. 225.

Offsets. p. 49.

Pantograph. pp. 85 (Ex. 8), 162.

Parabola. p. 170.

Parallel straight lines are straight lines in the same plane, which do not meet however far they are produced in either direction. p. 4.

Parallelogram. pp. 23, 45.

Any side of a parallelogram may be taken as the **base**. The perpendicular distance between the base and the opposite (parallel) side is called the **height**, or **altitude**. p. 46.

Pentagon. p. 98.

Perigal's dissection. p. 56.

Perimeter. The perimeter of a figure is the sum of its sides.

Perpendicular. See **right angle**.

to plane. p. 211.

Plan and elevation. p. 227.

Poles. p. 220.

Polygon. p. 98.

Prism. p. 193.

Projection. The projection of a point on a line (or plane) is the foot of the perpendicular from the point to the line (or plane).

The projection of a line is the locus of the projection of the points of that line.

Proportion. p. 148.

Proportional compasses. p. 104.

Proportional division. p. 147.

Pyramid. p. 198.

Pythagoras' theorem. p. 55.

Quadrilateral. pp. 23, 51, 105, 107.

The straight lines which join opposite corners of a quadrilateral are called its **diagonals**.

Radius. p. 86.

Ratio. pp. 130, 145, 148.

Rectangle. p. 24.

Rectangle properties of circle. p. 156.

Rectilinear figure. A figure bounded by straight lines.

Re-entrant. p. 100.

Reflex angle. p. 103.

Regular polygon. p. 98.

Regular pyramid, tetrahedron. pp. 198, 199.

Regular solids. p. 225.

Revision papers. pp. 78, 240.

Rhombus. A parallelogram which has two adjacent sides equal is called a rhombus. p. 25.

Right angle, perpendicular. When one straight line stands on another straight line and makes the adjacent angles equal, each of the angles is called a right angle; and the two straight lines are said to be at right angles, or perpendicular to one another.

Right-angled triangle. pp. 9, 13, 55.

Roulette. p. 174.

Ruler and compass constructions. p. 18.

Scale drawing. pp. 16, 34, 73, 188; also pp. 65, 89, 94, 96, 101, 112. Notes on p. 34.

Scalene triangle. p. 10.

Scalars, Choice of. p. 35.

Section. p. 193.

Sector of a circle. p. 87.

Segment of a circle. pp. 87, 104. **Angle in a segment.** See p. 104.

Semicircle. p. 86.

Similar figures. p. 131.

Similar triangles. pp. 138, 152.

Slant height of cone. p. 205.

Small circle. p. 217.

Sphere. p. 217.

Spherical belt, cap, sector, segment. p. 222.

Spherometer. p. 219.

Square. p. 25.

Subtend. If P is a point outside a line AB (curved or straight), $\angle APB$ is the angle which AB subtends at P. p. 36.

Supplementary angles. p. 2.

Surface of cone. p. 207.

cylinder. p. 196 (Ex. 15).

sphere. p. 222.

Survey. p. 49.

Symmetry. p. 30.

Tangent. A tangent to a circle is a straight line which, however far it may be produced, has one point, and one only, in common with the circle. pp. 87, 92.

The tangent is said to **touch** the circle; the common point is called the **point of contact**.

Tangents, common. p. 128.

Tetrahedron. A solid with four triangular faces. pp. 199, 225.

Third proportional. p. 148.

Trapezium. p. 48.

Triangle. p. 9.

Any side of a triangle may be taken as **base**. The straight line drawn perpendicular to the base from the opposite vertex is called the **height**, or **altitude**.

The straight line joining a vertex of a triangle to the mid-point of the opposite side is called a **median**.

Trigonometrical ratios. p. 153.

Trisection of an angle. p. 67.

Vertically opposite angles. p. 3.

Vertices. The corners of a triangle or polygon are called its vertices.

Volume of cone. p. 207.

cylinder. p. 195.

prism. p. 195.

pyramid. p. 202.

similar solids. p. 136.

solid of uniform cross-section. p. 195.

sphere. p. 223.

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