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ENGINEERING GRAPHICS

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ENGINEERING GRAPHICS

A Definition of the Field to Which This Book Is Intended as an Introduction

The field of engineering graphics is defined by the totality of postulates, definitions, and theorems of synthetic, coordinate, and descriptive geometry that pertains to constructions.

Synthetic geometry is a study of the metric and projective properties of figures by purely geometric methods, that is, by straightedge and compass methods.

Coordinate geometry is concerned with the position of a geometric magnitude in a reference system based on numbers. It provides a method of interpreting algebra by geometric procedures.

Descriptive geometry is concerned with the methods of representing space configurations on a single plane and solving the problems of space geometry by means of these representations.

In this book, the chapters on geometric constructions, conic sections, projective geometry, and vectors deal primarily with the problems and ideas of synthetic geometry. The chapters on graphical scales, empirical curves, and graphical calculus deal with material in the field of coordinate geometry. The final chapters on orthographic projection and axonometry are concerned with the general field of descriptive geometry.

PREFACE

For many an engineering problem a graphical solution is quicker than its algebraic counterpart, less susceptible of accidental errors, more illuminating with respect to the natural processes involved, and, most important, more suggestive and fruitful for predictive purposes. Certainly, the engineer unacquainted with graphical methods is operating with one of his two analytical eyes always shut.

The great majority of engineering and scientific data are initially graphical. Time, mass, and distance, being continuous in nature, must be measured by graphical devices: clocks, scales, and rulers. The achievable accuracy is determined by that of the devices used. No absolute accuracy is ever possible. A completely graphical solution will be substantially as accurate as a numerical or algebraic solution, provided that its accuracy is held well within that of the original data. When this can be achieved, the choice of method should depend upon the considerations mentioned above. Under any condition, the engineer or scientist should be equally competent in both methods, so that he may make the more intelligent choice.

Formal courses in graphical methods of attack on physical problems are almost nonexistent in American universities. Fragments are included in a few courses on engineering drawing and mechanics and in engineering problems courses, but the full power of graphics has scarcely been suggested to our engineering students.

The authors feel that this is a serious gap in our educational system. The opportunity and responsibility for filling it belongs with a graphics department, which is nothing more than the present engineering drawing department expanded in philosophy and content.

The representational side of graphics, its function as the "language of the engineer," has to date dominated drawing courses given by drawing departments. The personal ability to make working drawings of machine parts has, however, become increasingly less important to the engineer as specialization and the demand for intellectual power have increased.

The modern graphics course should develop the basic theories of both the analytical and representational sides of graphics. It should concern itself with the power to understand and use without allowing itself to be dominated by professional details and standard practices which can be

1

THE STRAIGHT LINE AND CIRCLE

BRIEF HISTORY OF CONSTRUCTIVE GEOMETRY

1-1. Historical

Our knowledge of the beginnings of geometry is somewhat fragmentary. A register of Greek mathematicians was written about 330 B.C. by Endemus, a pupil of Aristotle. This record is lost, but a summary of it is preserved in the works of Procolus, a Roman historian of the fifth century.

Certain empirical geometric results were known to the Egyptians long before the time of the Greeks. The Egyptians, however, were interested only in practical calculation and showed no talent for developing an abstract mathematical science. The register attributes the beginning of theoretical geometry to Thales, who lived in the Greek colony of Miletus about 600 B.C. Thales is said to have collected the early geometric material and to have developed the idea of a logical demonstration.

The next eminent geometer mentioned in the ancient register was Pythagoras, the founder of a school in the Greek colony of southern Italy. To Pythagoras is ascribed the famous triangle theorem, which bears his name, and a theory of irrational numbers. One of the central ideas of Pythagorean philosophy was the relation of numbers to geometric solids and natural phenomena. In this connection Pythagoras discovered the relation between the length of a vibrating cord and the musical tone emitted. The Pythagorean school founded mathematics as an abstract science and promoted the idea that the natural world could be explained by physical laws.

The Pythagorean school continued until about 400 B.C., when the Academy of Athens came into prominence under Plato. This was the Golden Age of Greek literature and philosophy. While Plato was primarily a philosopher, he contributed to geometry by exacting accurate definitions, clearly stated assumptions, and logical proofs. A knowledge of geometry was required for admission to the Academy. In fact, above its door was inscribed "Let no one ignorant of geometry enter under my roof." Geometry was regarded as a discipline in logical reasoning and therefore a prerequisite for philosophical studies as well as an introduction to mathematics.

Practical application of mathematics was largely ignored, such a study being regarded as having little intellectual content. The purpose of the curriculum of the Academy was to sharpen the mind, deepen the understanding, and provide a moral code. One of its most renowned students was Aristotle, who was at one time the tutor of Alexander the Great.

Alexander, however, was no credit to his teacher. His fame rests on his subjugation of Greece in 330 B.C., which brought to an end any further development of civilization on Greek soil. In 323 B.C., Alexander died, and his empire was divided up by his generals. One of them, Ptolemy, acquired Egypt and its new capital, Alexandria. This young capital soon became the center of the intellectual world, and by 300 B.C. the famous Museum had been founded as a university of Greek learning. Here in Egypt, Greek science had its home for the next seven hundred years.

It was at Alexandria that Euclid compiled his systematic treatment of geometry, "The Elements." Almost nothing is known about Euclid. His "Elements," however, is the most famous textbook of mathematics ever written and has been the basis of elementary geometry for 2,000 years. "The Elements" was written as an introduction to mathematics and as a study in logical reasoning. It was required as a preparation for philosophical studies. Applications are not mentioned. Euclid does not even describe the straightedge and compasses used for constructions in the problems. A student once asked Euclid what was the profit in studying geometry. He turned to a servant and said, "Give this man three obols (coins) that he may have profit from geometry." Other works by Euclid include a study of conic sections, surfaces, and optics.

The next great geometer of the Alexandrian school was Archimedes, born in Syracuse, Italy, about 287 B.C. He was a scientist and engineer as well as a mathematician. His outlook was far different from that of his predecessors. He believed in combining theory with practice, as we do today. The contributions of Archimedes to mathematics and science were substantial. One of his greatest mathematical accomplishments was in determining a close approximation of π, the ratio of the circumference of a circle to its diameter. Archimedes perished in the capture of Syracuse by the Romans in 212 B.C.

The last great geometer of the Alexandrian school was Apollonius (250 B.C.). He is known chiefly for his comprehensive work on the conic sections. With Apollonius we come to the end of the great mathematicians of antiquity. Egypt was subjugated by Rome in 1 B.C., and the Alexandrian school declined, finally disappearing. The practical Romans were not interested in scientific studies and, later, the Christians were hostile to pagan learning. The spirit of inquiry which characterized Greek learning died, and the knowledge of mathematics and science gained by the Greeks lay buried for a thousand years.

Modern geometry began with the Renaissance, and the men connected with its development are mentioned with their work in later chapters.

It is suggested that a technical student should supplement this meager outline of the work of centuries by keeping at hand for leisure reading a book on the history of science and mathematics. A few are listed at the end of this chapter. However abstract mathematics and science may appear, they are essentially of human origin, and it is hard to see how an appreciation of these areas of knowledge can be had without some understanding of what has gone before. This view is well expressed by George Sarton in his "History of Science," sponsored by the Carnegie Foundation, from which the following quotation is taken:

> The only way of humanizing scientific labor is to inject into it a little of the historical spirit. . . . Show the deep human interest of science and the study of it becomes the best vehicle of humanism one could devise; exclude that interest, teach scientific knowledge only for the sake of information and professional instruction, and the study of it, however valuable from a purely technical point of view, loses all educational value. Without history, scientific knowledge may become culturally dangerous; combined with history . . . it will nourish the highest culture.

1-2. The theoretical basis of geometric drawing

THE POSTULATES OF CONSTRUCTION. All constructive geometry is based on the following postulates stated by Euclid:

1. A straight line can be drawn from any one point to any other point.
2. A straight line can be produced to any length in a straight line.
3. A circle can be described with any given point as a center and any given line as a radius.

THE INSTRUMENTS OF THE GEOMETER. The basic instruments by means of which the process of drawing is actually performed are the pencil, straightedge, and compasses. Using these tools, the following fundamental problems of plane geometry can be solved:

1. To draw a straight line through two points.
2. To find the point in which two straight lines intersect.
3. To draw a circle of given center and radius.

Any compass and straightedge construction consists of a finite succession of these three elementary operations.

The modern geometer also has at his disposal other instruments, including the protractor, dividers, draftsman's triangles, board, and T square, which increase the rapidity and accuracy of many constructions. However, in the classical geometry, as established in the Academy of Plato, the only instruments allowed were the unmarked straightedge and the compasses. Also, the field of mechanics and mechanical devices was separated from geometry, and the possibility or impossibility of solving a geometric prob-

lem was based on the tradition that only the two fundamental instruments could be used.

In the course of time a great amount of information has been accumulated on geometric constructions. It is surprising how much can be done with these instruments, both in combination and separately. It has been shown by Mascheroni (1797) that if a desired point can be established by straightedge and compass methods, it can be established by the compasses alone. Many problems can be solved with the straightedge alone. For example, by use of a theorem stated by Desargues (1636) one can draw a line through a given point to pass through the inaccessible intersection of two given lines, using only a straightedge (see Art. 3-13 and Prob. 3-12). Poncelet (1822) and Steiner (1833) studied the use of the straightedge and one given circle and showed that any point that can be established with the straightedge and compasses can be established with the straightedge and fixed circle.

One of the most famous of the classical constructions is the problem of Apollonius, in which it is required to draw a circle tangent to any three given circles. This is a problem of some difficulty and was solved with the straightedge and compasses.

The problem of constructing a polygon of n sides, using straightedge and compasses only, has always been of great interest. For values of n equal to 3, 4, 5, and 6, the solutions have been known since antiquity. For $n = 7$, the solution is impossible.

There are three other famous problems of ancient geometry that cannot be solved by the straightedge and compasses. They are (1) to trisect any angle; (2) to double the cube (*i.e.*, to find the cube having double the volume of a given cube); (3) to square the circle (to construct a square having the area of a given circle).

After centuries of fruitless effort to discover solutions for these problems, some mathematicians began to suspect that these constructions were impossible by unmarked straightedge and compass methods. An analysis of such methods shows that, when line segments are interpreted analytically as numbers, the only possible operations are addition, subtraction, multiplication, division, and extraction of the square root. The constructibility of a proposed problem is tested by stating the problem in analytical form. If the solution leads, say, to a cubic equation having rational coefficients but no rational roots, the construction has been proved to be impossible by straightedge and compasses. Thus, in the problem of doubling the cube, if a is the edge of the given cube and x the edge of the required cube, then a^3 and x^3 are their volumes and $x^3 = 2a^3$. Solving for x, we have $x = a\sqrt[3]{2}$. Since a line equal to $\sqrt[3]{2}$ cannot be constructed by straightedge and compasses, the construction is impossible by straightedge and compasses. The problem of trisecting an angle and the construction of the heptagon fail for the same reason.

The problem of squaring the circle involves a difficulty of a different nature. Since a circle of radius 1 has an area of π, the side of a square having the same area is $\sqrt{\pi}$. Hence the square can be constructed if the number π is constructible (*i.e.*, if π is a rational number or obtainable from square roots of rational numbers). Lindemann (1882) succeeded in proving π to be transcendental, *i.e.*, not a root of an algebraic equation with rational coefficients, and thus the impossibility of squaring the circle was proved.

In spite of the work done by Gauss, Lindemann, and others in proving the impossibility of solving the three classic problems of antiquity by straightedge and compasses, angle trisectors and circle squarers still persist.

DRAFTING TOOLS

The first step in developing graphical ability is an understanding of the instruments used. The following discussion is very brief. It assumes that the student has the materials before him and that proper manual minutiae will be supplied by the instructor.

1-3. Pencils

Drawing pencils are graded according to hardness: HB and F are medium soft; H to 3H, medium hard; 4H to 6H, hard; 7H to 9H, very hard. Medium-soft pencils are suitable for freehand sketching, medium-hard for instrument drawings on soft paper, hard for geometric constructions where a high degree of precision is required. Hard pencils require a hard-surfaced drawing paper.

When the highest possible precision is required, as in the layout of airfoils, etc., a needle point is used on a metal sheet which has a specially prepared painted surface.

1-4. Drawing boards and T squares

The T square is used for drawing horizontal lines and as a base for the draftsman's triangles in drawing other lines. The head of the T square is kept in contact with the left edge of the drawing board for a right-handed person. Only the top edge of the square is ever used.

1-5. The draftsman's triangles

Triangles are convenient straightedges which allow, in combination with the T square, the drawing of lines at the most common angles with the horizontal. The usual pair of triangles has angles of 45° and 90° for one and of 30°, 60°, and 90° for the other. With these in combination any angle which is a multiple of 15° may be formed.

1-6. Drawing parallels and perpendiculars with the draftsman's triangles

Let it be required to draw lines parallel or perpendicular to the given line MN, Fig. 1-1. Fit accurately to MN one leg, 1, of triangle A, and place against its hypotenuse another triangle, B. Hold B firmly in place with one hand, and slide triangle A along it to position A'. Lines drawn along edge 1 are parallel to MN. Lines drawn along edge 2 are perpendicular to MN. In this construction the blade of the T square can be substituted for triangle B.

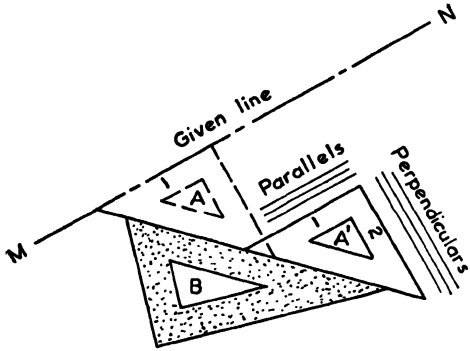

Fig. 1-1.

1-7. Scales

Scales are of many types (see Chap. 6). Specialized ones are made for different professions. For general graphical work a scale dividing the inch down to 32nds and another dividing to 50ths for decimal work are indispensable.

1-8. The dividers

The dividers are used for transferring distances from one line to another or for dividing a line into a number of equal parts. In spacing along a line, either straight or curved, the dividers should be rotated first one way and then the other, as shown in Fig. 1-2. The handle should be held between the thumb and forefinger for ease in rotation.

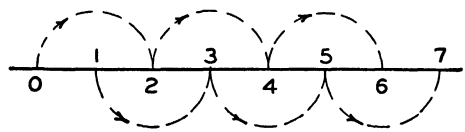

Fig. 1-2.

1-9. To divide a line by trial

To divide line AB, Fig. 1-3, into a number of equal parts, say five, open the dividers at an estimated one-fifth the distance AB, and space along the line. If the fifth spacing falls short of B, leaving an error CB, increase the opening by one-fifth of CB and repeat the spacing. This operation is repeated until the spaces are exact. A small freehand circle should be used to identify the final points of division. Do not make large holes in the paper; apply only enough pressure to make the points of the dividers take hold.

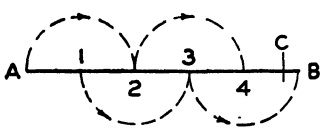

Fig. 1-3.

1-10. The compasses

The compasses are held in the same manner as are the dividers and are rotated in a clockwise direction. The needle point should be adjusted with the shoulder end for insertion in the paper, thus ensuring a small hole in the paper when a number of concentric circles are drawn. Remove the pencil leg, insert the pen, and adjust the needle point slightly longer than the pen. Replace the pencil leg, and adjust the lead to the needle point. The lead should be sharpened to a chisel point. When a circle of more than 2 in. radius is drawn, the joints in the legs should be bent so as to keep the lead and the needle point perpendicular to the paper. When the lengthening bar is used, the compasses should be held with both hands, one on each vertical leg. In drawing concentric circles, always draw the smaller ones first. When a radius is less than $\frac{3}{4}$ in., use the bow compasses.

PRECISION IN DRAFTING

1-11. Accuracy

In spite of the purity of geometric theory a graphical solution is not capable of absolute accuracy. At this point graphics departs from geometry by considering the problem of accuracy. The tolerances within which results can be obtained are dependent on a variety of factors, including the scale at which the work is done, the accuracy of the instruments used, the effects of weather on the drawing paper, and the eyesight and care of the draftsman.

When a quantity is to be found by graphical methods, the simplest construction possible should be used. The drawing should be compact, and overlapping constructions may not be objectionable. Drafting mistakes may be avoided by carefully lettering all points of construction.

Since most engineering data are obtained from instruments which contain scales of one sort or another and which are themselves subject to physical errors, the possible accuracy with which problems using these data can be solved is predetermined before the numerical or the graphical solution is begun. The problem of the draftsman, therefore, becomes one of choosing his scale and his working materials and adjusting his own care to ensure that the graphical solution itself is more accurate than the empirical data. If this is done, the graphical solution will be effectively as accurate as a numerical or algebraic solution. Therefore, the draftsman using graphical methods should be acquainted with the tolerances which he may expect from all the factors involved in his drafting. Some of these factors are discussed in the following paragraphs.

1-12. Lines

A line, as defined by Euclid, is "that which has length without breadth." Hence a mathematical line cannot have material existence, and any "line"

on a drawing is a rough approximation to a Euclidean line. With reasonable care, however, a pencil line may be made to serve most practical purposes. In geometric constructions, using a hard pencil, a line need not exceed 0.005 in. in width. It is possible to reduce the width to about half of this amount by use of special materials.

An interesting test is to determine how many separate parallel lines one can draw between two parallel lines a given distance apart. If the lines are drawn as close as possible but clearly separated, it will be found that the distance between them will be approximately equal to their width. If the widths of the lines and spaces are about 0.005 in., 10 lines can be drawn in a space 0.10 in. wide.

The distance between two points can be measured with a probable error as small as the width of the scale mark, which is about 0.005 in.

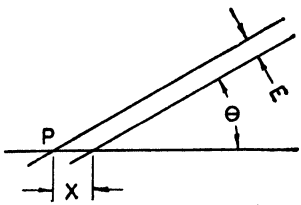

Fig. 1-4.

The possible error in the position of a point P determined by two intersecting lines depends on the error in the lateral position of one of the lines and the size of the angle of intersection. If E is the amount of lateral error in locating one of the lines and X is the resulting error in the position of the point, then $X = E/\sin\theta$, where θ is the angle between the two lines, Fig. 1-4. When $E = 0.005$ in. and $\theta = 5°$, $X =$ 0.05 in. (nearly $\frac{1}{16}$ in.). This is an example of a small initial error leading to a much larger final error.

1-13. Parallel lines

Suppose it is desired to draw a line parallel to a given line and at given distance from it. This can be done by the construction of Art. 1-22. However, in practice the draftsman usually places a scale perpendicular to the given line by eye, marks the other end of the distance, and draws the parallel line with the T square or with triangles by the method of Art. 1-6.

Let θ be the error of angle in estimating the perpendicular AB, Fig. 1-5, and L the desired distance between lines. If E is the resulting error in L, then

$$E = L - L\cos\theta$$

The eye is very sensitive to the perpendicular relationship. An angular error of 2° would be a gross one; yet in this case if L were 10 in., E would only be 0.006 in. This is an example of a relatively large initial error leading to a small final error.

Thus on drawings of the usual drawing-board sizes and where very high accuracy is not required, the very common operation of determining the distance between desired parallel lines may be safely done by laying off the required distance with a scale perpendicular to the given line by eye from

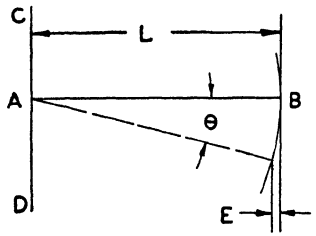

Fig. 1-5.

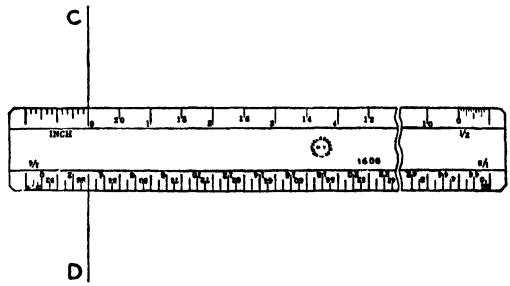

Fig. 1-6.

any random point on the line. The edge of a flat scale is practically perpendicular to a line, CD, when two opposite scale marks are placed on the line, Fig. 1-6.

It is obvious from the two examples given above that the draftsman should be conscious of the end results of his possible errors and adjust his methods accordingly.

1-14. Angles

An angle can be measured or constructed by use of a table of tangents or a table of chords.

TANGENT METHOD. To measure angle θ, Fig. 1-7, lay off on one side of the angle and from the vertex a convenient length AB. Draw BC at right angles to AB, cutting the second side of the angle at C. Measure BC. Then $\theta = \tan^{-1}(BC/AB)$. The most convenient length for AB is evidently 10 units.

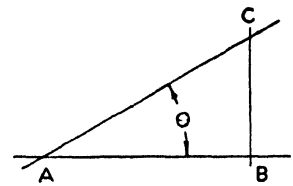

Fig. 1-7.

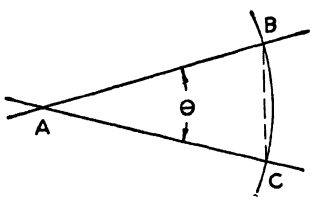

Fig. 1-8.

CHORD METHOD. Let it be required to measure angle θ, Fig. 1-8. Describe the arc BC, with A as center and a convenient radius R. Measure chord BC. Then, referring to a table of chords, θ is the angle whose chord is BC/R. Again, 10 units is a convenient radius.

The use of a 10-in. base gives a probable error of $\frac{1}{10}$ to $\frac{1}{15}$ of one degree. The usual draftsman's protractor is accurate to about $\frac{1}{4}$ of one degree.

1-15. Scale

If distances are set off with a needle point, measurements may be read to the nearest 0.01 in. The needle-point impression must be very small. A measuring scale on which the inch is divided into 50 parts is suitable for this work. Fiftieths can be read without the aid of a magnifying glass and the hundredths estimated very closely.

that the sides of similar triangles are proportional, and that in a right triangle the square of the hypotenuse is equal to the sum of the squares of the sides."

1-50. To represent a number

If line AB be taken to represent the number 1, then line CD, which is n times the length of AB, represents the number n. AB is called the *unit* and is made equal to some unit of the drawing scale.

1-51. Addition and subtraction

To construct $a + b$, line segments representing the numbers a and b are laid off end to end on a straight line, Fig. 1-40. Similarly, $a - b$ is constructed by laying off segments to represent a and b, but with b measured in a direction opposite to a, Fig. 1-41.

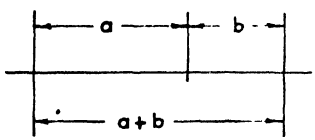

Fig. 1-40.

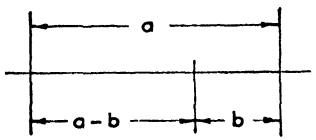

Fig. 1-41.

1-52. Multiplication and division

To construct the product ab, lay off from the vertex of any angle, Fig. 1-42, the segments $OA = a$ and $OB = b$. Also, on OA lay off $OC = 1$ unit. Draw BC. Draw AD parallel to BC. Then $a/1 = OD/b$. Thus, $OD = ab$. The ray-polygon method of graphical integration is an important application of this principle. It is described in Art. 9–3.

To construct a/b, lay off from the vertex of any angle, Fig. 1-43, $OA = a$ and $OB = b$. Also, on OB lay off $OC = 1$ unit. Draw AB. Draw CD parallel to AB. Then, $1/b = OD/a$. Thus, $OD = a/b$.

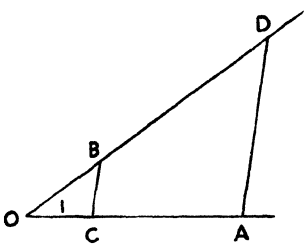

Fig. 1-42.

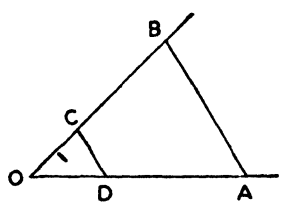

Fig. 1-43.

1-53. To find the mean proportional between two given distances

Let a and c be the given quantities. Then if $a:b = b:c$, b is the mean proportional. On a straight line lay off $OA = a$ and $OC = c$, Fig. 1-44.

ELEMENTARY CONSTRUCTIONS

1-18. Scope

The following constructions should be familiar to the student from elementary geometry. They are limited to those most frequently encountered in practical graphical work. By means of the three fundamental problems of geometry (Art. 1-2), we can immediately bisect a line segment or angle and construct a perpendicular or a parallel to a given line. In practice, however, perpendiculars and parallels are usually constructed by means of the draftsman's triangles (Art. 1-6).

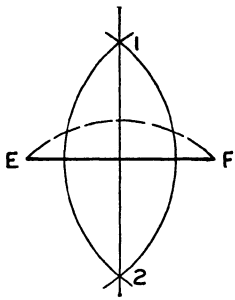

Fig. 1-10.

1-19. To bisect a line segment

First method (Fig. 1-10). Let EF be the given line. With any appropriate radius, centers E and F, draw equal arcs intersecting at points 1 and 2. Draw line 1-2, which bisects EF.

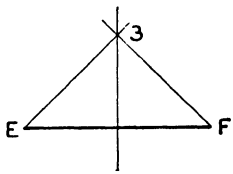

Fig. 1-11.

Second method (Fig. 1-11). If EF is parallel or perpendicular to the T square, the following construction is shorter: Using a 45° triangle, draw lines through points E and F, meeting at point 3. Through point 3 draw a line perpendicular to EF, using the T square and triangle. Practical method: see Art. 1-9.

1-20. To bisect an angle (Fig. 1-12)

Let ABC be the given angle. With any appropriate radius, center B, describe an arc, cutting the lines forming the angle at points A and C. With any radius, centers A and C, describe arcs intersecting at point D. Draw the bisector BD.

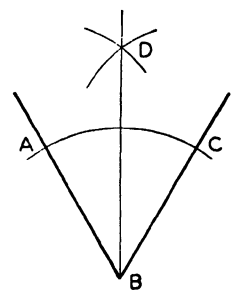

Fig. 1-12.

1-21. Through a point to draw a line parallel to a given line (Fig. 1-13)*

Through point P to draw a line parallel to line AB.

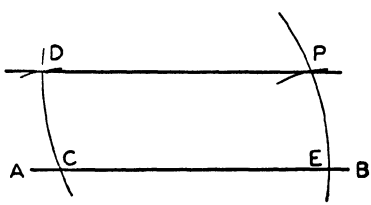

Fig. 1-13.

With P as center and any convenient radius, draw arc CD. With C as center and the same radius, draw arc PE. With PE as radius and C as center, strike an arc intersecting the first one at D. Draw line DP. Practical method: use draftsman's triangles (Arts. 1-6 and 1-13).

1-22. To draw a line parallel to a given line at a given distance from it (Fig. 1-14)

Let AB be the given line. Set the compasses to a radius equal to the given distance. With any two points of AB, as C and D, for centers, describe two arcs. Draw the required line EF tangent to the arcs (by eye). Only one arc need be drawn if the draftsman's triangles are used to draw the parallel. Practical method: see Art. 1-13.

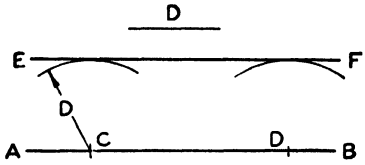

Fig. 1-14.

1-23. To divide a line into any number of equal parts (Fig. 1-15)

Let AB be the given line and the number of parts *five*. Draw AC, making any angle with AB. Lay off on AC any unit distance as many times as the required number of parts. Join the last point of division, 5, with B. Parallel to line 5-B, draw lines through points 4, 3, 2, 1, cutting AB into the required number of equal parts.

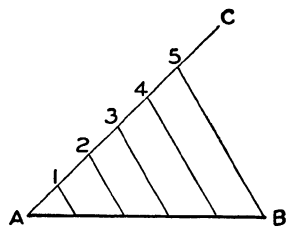

Fig. 1-15.

1-24. To divide the space between two parallel lines into any number of equal spaces (Fig. 1-16)

Let AB and CD be the given lines and the number of spaces seven. Draw any transversal between AB and CD. Divide the intercepted segment into seven equal parts by Art. 1-23. Draw lines parallel to AB through the points of division. Practical method: place a scale so that seven equal divisions are included between lines AB and CD. Mark the points of division, and through them draw lines parallel to AB.

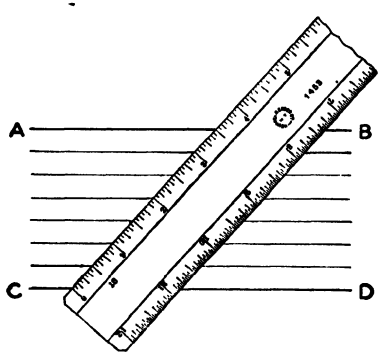

Fig. 1-16.

1-25. At any point on a line to erect a perpendicular to the line (Fig. 1-17)

At point P to erect a perpendicular to line AB. With P as center and any convenient radius, strike arcs intersecting AB at points C and D. With C and D as centers, strike arcs intersecting at E or F. Draw EP or FP. Practical method: use draftsman's triangles.

1-26. At a point near the end of a line to erect a perpendicular to the line (Fig. 1-18)

At point C on line AB to erect a perpendicular to AB. With any point D as center and DC as radius, draw the arc ECF intersecting AB at F. Draw

line FD, and produce it to intersect arc ECF at E. Draw EC. Practical method: use draftsman's triangles (see Art. 1-6).

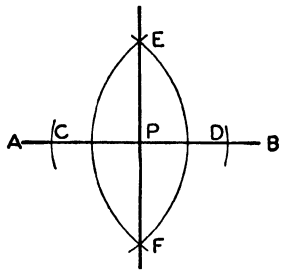

Fig. 1-17.

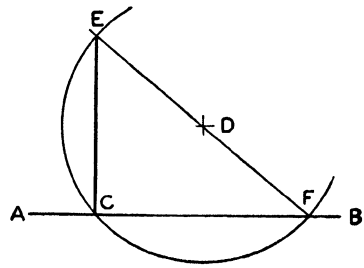

Fig. 1-18.

1-27. To construct a triangle, the lengths of the sides being given (Fig. 1-19)

Let AB, CD, and EF be the given sides and AB the base of the triangle. With B as a center and CD as radius, strike an arc at G. With A as a

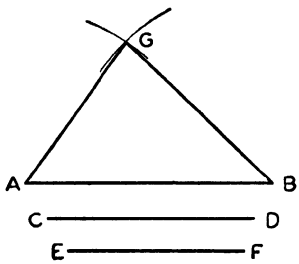

Fig. 1-19.

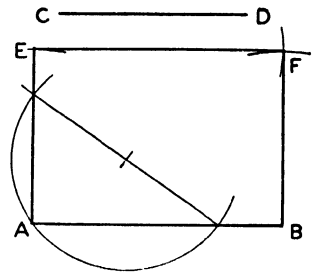

Fig. 1-20.

center and EF as radius, strike an arc intersecting the first arc at G. Draw AG and GB.

1-28. To construct a rectangle, the lengths of its sides being given (Fig. 1-20)

Let the given sides be AB and CD. At A draw a line perpendicular to line AB (Art. 1-26). Lay off AE equal to CD. With AB as radius, center E, describe an arc at F. With CD as radius, center B, cut preceding arc in point F. Draw EF and BF. Practical method: draw AE, EF, and BF with draftsman's triangles.

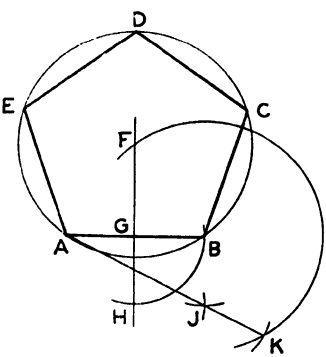

Fig. 1-21.

1-29. To construct a regular pentagon, the length of one side being given (Fig. 1-21)

Let AB be the given side. Draw FGH, the perpendicular bisector of AB. Make GH equal to GB. With radius GB, centers B and H, draw arcs

intersecting at J. Draw AJ produced, and make JK equal to GB. With radius KB, center B, intersect FH at F. Using the same radius, center F, describe circle ABD. With radius AB, and starting at B, cut the circle in points C, D, and E. Draw BC, CD, DE, EA, completing the required pentagon.

1-30. To construct a regular hexagon, the length of one side being given (Fig. 1-22)

Let AB be the given side. With radius AB, centers A and B, describe arcs intersecting at point C. With radius AB, center C, describe circle

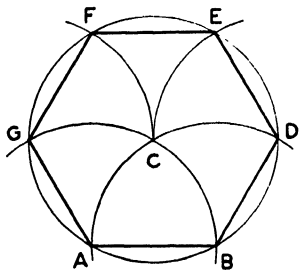

Fig. 1-22.

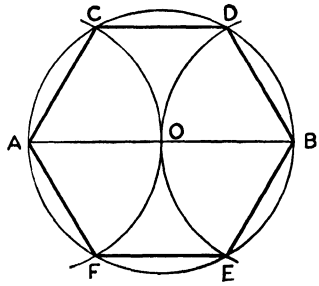

Fig. 1-23.

ABE. With same radius, starting at B, cut the circle in points D, E, F, G. These are the remaining vertices of the hexagon.

1-31. To draw a regular hexagon, given its long diameter (Fig. 1-23)

Let AB be the given diameter. Bisect AB, point O, and draw a circle on AB as a diameter. With A and B as centers and radius OA, describe arcs intersecting the circle at points C, D, E, F. Join points A, C, D, B, E, F, A.

1-32. To draw a regular hexagon, given its short diameter (Fig. 1-24)

Let AB be the given diameter. Bisect AB, and on AB as a diameter draw a circle. Using the T square and 30°-60° triangle, draw the sides of the hexagon tangent (by eye) to this circle.

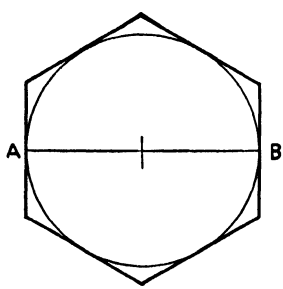

Fig. 1-24.

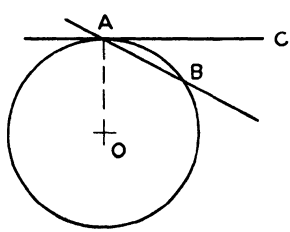

Fig. 1-25.

1-33. Tangent and normal to a curve

If a line cuts a curve in two points, A and B, it is called a secant (Fig. 1-25). Consider that the line is revolved about A as a fixed point until the second point of intersection coincides with A. The line now assumes the position AC and is *tangent* to the curve at point A. A perpendicular AO to the tangent at the point of tangency is a *normal* to the curve.

1-34. To draw a circle through three given points (Fig. 1-26)

Let A, B, and C be the given points. Draw AB and AC. Draw lines EF and GH, the perpendicular bisectors of AB and AC, respectively. The intersection, D, is the center of the required circle.

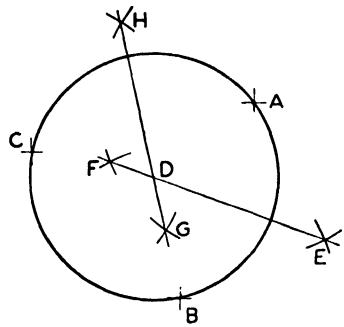

Fig. 1-26.

1-35. To draw a tangent to a circular arc at a given point on the arc (Fig. 1-27)

Let C be the given point on the arc, center O. Draw radius OC. At C draw the required tangent AB perpendicular to OC. If a tangent AB is

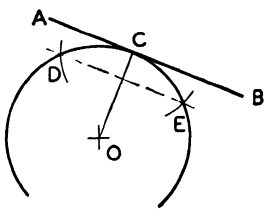

Fig. 1-27.

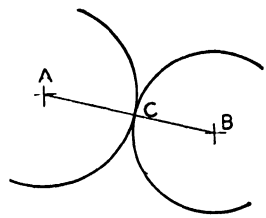

Fig. 1-28.

given, the point of tangency C is found by drawing a radius perpendicular to AB. If the center is not given, strike equal arcs from C cutting the given arc at D and E. Draw DE. Draw the required tangent parallel to DE.

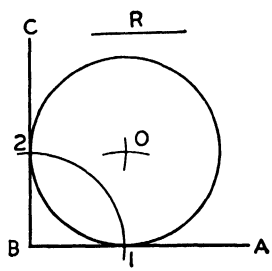

Fig. 1-29.

1-36. To find the point of tangency of two circular arcs (Fig. 1-28)

Let A and B be the centers of the given arcs. The point of tangency, C, lies on the line joining the centers of the given arcs.

1-37. To draw an arc of given radius tangent to two given lines at right angles (Fig. 1-29)

Let AB and BC be the given lines. Set the compasses to the given radius. With B as cen-

ter, intersect AB and BC at points 1 and 2. With the same radius, centers 1 and 2, draw arcs intersecting at point O, the center of the required arc. Points 1 and 2 are the points of tangency. *Note:* To set a given radius on the compasses, lay off the radius on a straight line, say B-1, place center of compasses on B, and adjust so that pencil passes through point 1.

1-38. To draw an arc of a given radius tangent to two intersecting straight lines (Fig. 1-30)

Let AB and CD be the given lines. Set the compasses to the given radius R. At the distance R from AB draw a line parallel to AB. At the distance

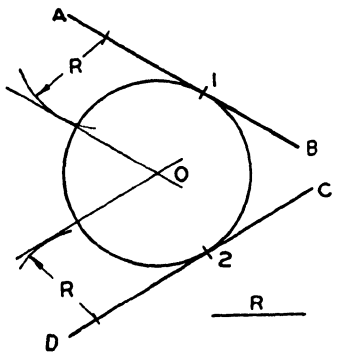

Fig. 1-30.

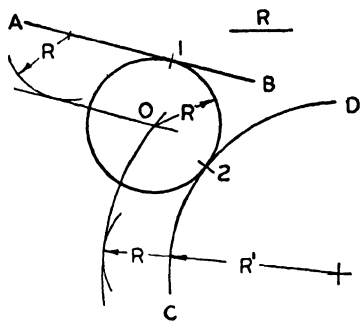

Fig. 1-31.

R from CD draw a parallel to CD. The intersection, O, is the center of the required arc. Points 1 and 2 are the points of tangency (see Art. 1-35).

1-39. To draw an arc of a given radius tangent to a given straight line and a circular arc (Fig. 1-31)

Let AB be the given line and CD the given arc. Set the compasses to the given radius R. At the distance R from AB draw a parallel to AB. At the distance R from arc CD draw a circular arc, radius $R + R'$, concentric with it. The intersection, O, is the center of the required arc. Points 1 and 2 are the points of tangency.

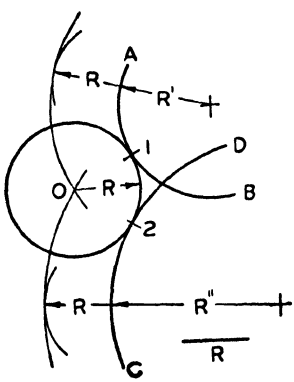

Fig. 1-32.

1-40. To draw an arc of given radius tangent to two given circular arcs (Fig. 1-32)

Let AB and CD be the given arcs and R the given radius. Draw a circular arc, radius $R + R'$, concentric with arc AB. Draw a circular arc, radius $R + R''$, concentric with arc CD. The intersection, O, is the center of the required arc. Points 1 and 2 are the points of tangency.

1-41. To draw a reversed curve tangent to two parallel lines (Fig. 1-33)

Given the parallel lines AB and CD. At B and C erect the perpendiculars BE and CF. Draw line BC, and assume any point G on it for the reversal point of the curve. Draw the perpendicular bisectors of BG and CG intersecting BE and CF at O and O'. O and O' are the centers of the reversed curve.

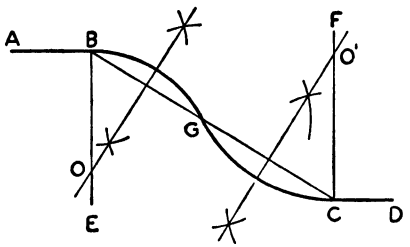

Fig. 1-33.

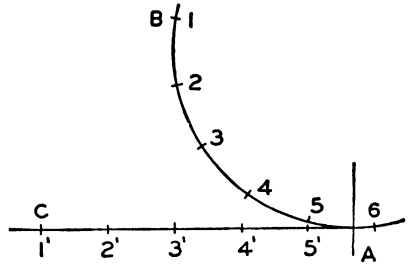

Fig. 1-34.

APPROXIMATE METHODS

1-42. To lay off on a straight line the length of a given curve (Fig. 1-34)

Let AB be the given curve. At one end of the curve, say A, draw the tangent AC. From B step off equal spaces with the bow dividers until a point at or beyond A is reached. Without removing the dividers from the paper, step off the same number of spaces along the tangent.

1-43. To lay off on a straight line the length of a given circular arc (Fig. 1-35)

Let AB be the circular arc. Draw AC tangent to AB. Draw chord BA produced. Bisect chord AB at F, and lay off AD equal to AF. With D as center, radius BD, intersect AC in point E. Then AE is very nearly equal to arc AB. This is known as Rankine's method.

If the arc subtends 30°, the error is about 1 part in 14,000; if 60°, about 1 part in 900. The error varies as the fourth power of the angle subtended. For arcs greater than 60° the arc should be bisected and the results doubled.

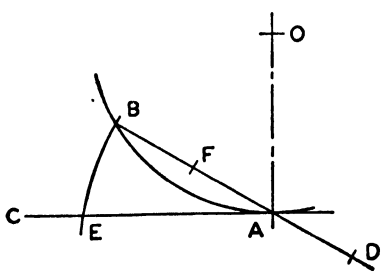

Fig. 1-35.

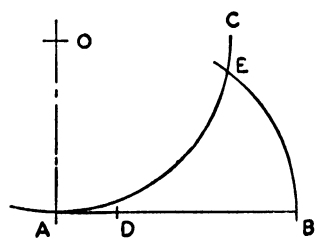

Fig. 1-36.

1-44. To lay off on a circular arc the length of a given straight line (Fig. 1-36)

Let AB be the given line. Draw the circular arc AC tangent to AB at A. Make AD equal to one-fourth of AB. With center D and radius BD, intersect arc AC at point E. Then arc AE is very nearly equal to line AB. The error is of the same order as in the preceding problem.

1-45. To rectify a circle (Fig. 1-37)

Let APB be the given circle. Draw tangent AD, making AD equal to three times AB. Draw the diameter at right angles to AB, cutting the circle in P. With radius OP, center P, cut circle at point C. Draw CE perpendicular to AB. Join points D and E. Line DE is slightly longer than the circumference of the circle, the error being about 1 in 22,000.

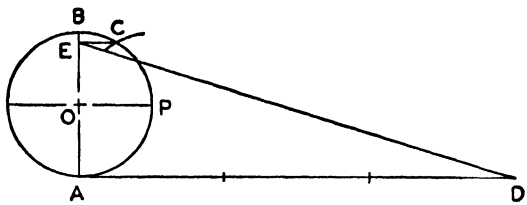

Fig. 1-37.

1-46. Use of templets

The constructions for many practical problems can be simplified by the use of tracing-paper templets, or "overlays." Constructions which represent certain mechanical motions are easily handled in this way.

1-47. To rectify a curve

Let it be required to lay off on a straight line the length of the curve between points A and B, Fig. 1-38.

Draw a straight line on a piece of tracing paper, and mark a point, A, on this line. Place point A of the tracing paper over point A of the curve, holding them together with a needle point. Now rotate the tracing paper until the straight line cuts the curve at another point, say 1. Move the needle point from A to point 1. Rotate the tracing paper again about the needle point until the straight line cuts the curve at a second point, 2. Place the needle point at 2, etc. These operations are continued until point B is reached. The resulting rectification is quite close when the chords are short. This procedure is evidently equivalent to setting off the length of a curve on a straight line by means of a pair of

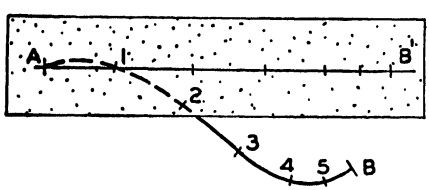

Fig. 1-38.

dividers. It has the advantage, however, of allowing chords of varying lengths to be used—short chords where the curve changes rapidly, longer chords where the curvature is small.

It is obvious that this method can be used to lay off a given distance along a curved line.

1-48. To find the path of one vertex of a given triangle when the other two vertices move along given curves

The given triangle ABC, Fig. 1-39, moves so that the vertex A always lies on line MN and B on the arc QR. To find the curve traced by point C, draw the triangle ABC on a transparent templet. Place the templet with A on MN, holding in place with a needle point. Rotate the templet to bring point B on arc RQ. Mark point C with the needle point. Repeat this operation for other positions of ABC. A smooth curve, S, drawn through the various positions of C represents the required locus.

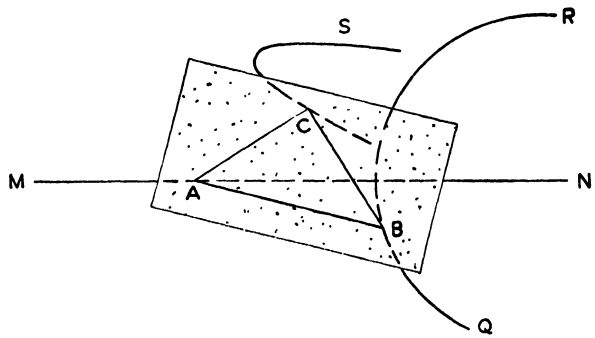

Fig. 1-39.

GRAPHICAL ARITHMETIC

1-49. Scope

The rational algebraic processes of addition, subtraction, multiplication, and division can be carried out by geometric constructions. It is also possible to extract the square root of an algebraic number graphically. Graphic addition, subtraction, multiplication, and division are given here not as substitutes for numerical computation but for their theoretical interest when the possibilities of straightedge and compass constructions are being analyzed.

Excepting the first two operations, these constructions are based on what is probably the most fundamental theorem of Euclidean geometry, that of similarity: the sides of similar triangles are proportional. It is pointed out that when Descartes (1637) wrote his "Geometry" unifying geometric methods, he stated in a letter "I use no theorems except those which assert

that the sides of similar triangles are proportional, and that in a right triangle the square of the hypotenuse is equal to the sum of the squares of the sides."

1-50. To represent a number

If line AB be taken to represent the number 1, then line CD, which is n times the length of AB, represents the number n. AB is called the *unit* and is made equal to some unit of the drawing scale.

1-51. Addition and subtraction

To construct $a + b$, line segments representing the numbers a and b are laid off end to end on a straight line, Fig. 1-40. Similarly, $a - b$ is constructed by laying off segments to represent a and b, but with b measured in a direction opposite to a, Fig. 1-41.

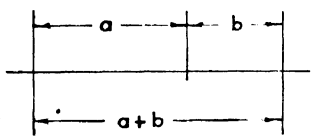

Fig. 1-40.

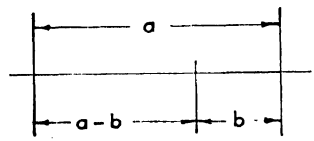

Fig. 1-41.

1-52. Multiplication and division

To construct the product ab, lay off from the vertex of any angle, Fig. 1-42, the segments $OA = a$ and $OB = b$. Also, on OA lay off $OC = 1$ unit. Draw BC. Draw AD parallel to BC. Then $a/1 = OD/b$. Thus, $OD = ab$. The ray-polygon method of graphical integration is an important application of this principle. It is described in Art. 9–3.

To construct a/b, lay off from the vertex of any angle, Fig. 1-43, $OA = a$ and $OB = b$. Also, on OB lay off $OC = 1$ unit. Draw AB. Draw CD parallel to AB. Then, $1/b = OD/a$. Thus, $OD = a/b$.

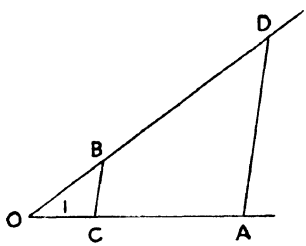

Fig. 1-42.

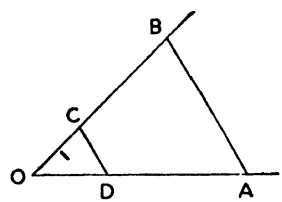

Fig. 1-43.

1-53. To find the mean proportional between two given distances

Let a and c be the given quantities. Then if $a:b = b:c$, b is the mean proportional. On a straight line lay off $OA = a$ and $OC = c$, Fig. 1-44.

Bisect AC, point D, and with radius AD draw semicircle ABC. At O draw the perpendicular, cutting semicircle at B. Then $OB = b$, the required mean proportional.

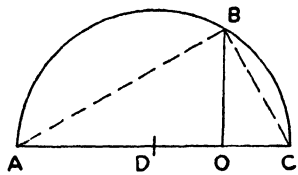

Fig. 1-44.

1-54. Square root

To find the square root of a number a. In Fig. 1-44, let $OA = a$, and let $OC = 1$ unit. Then $a:OB = OB:1$. Therefore, $OB = \sqrt{a}$.

1-55. Graphical algebra

Many graphical methods exist for the solution of algebraic equations. The field of nomography mentioned in Art. 6-16 deals exclusively with such solutions. Graphical methods of a constructional, straightedge, and compass nature for the solution of equations may also be found in the literature. Many of these are of only academic interest. The following is presented as an interesting example of the possibilities.

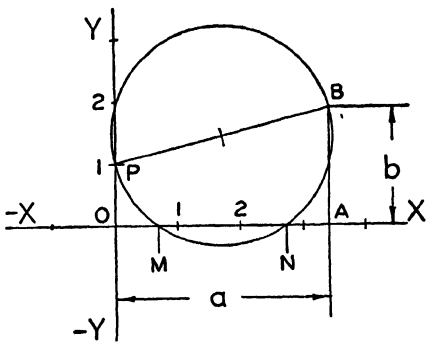

Fig. 1-45.

1-56. To determine the roots of a quadratic equation (Fig. 1-45)

Write the equation in the form $x^2 - (+a)x + b = 0$. On the X axis (see Art. 6-7), lay off $OA = a$, paying attention to sign. Erect a perpendicular at A, and on it lay off $AB = b$, paying attention to sign. Mark point $P(0, 1)$. Describe a circle on PB as a diameter intersecting the X axis in M and N. Then OM and ON are the roots of the equation. If the circle is tangent at point M, then the equation is a perfect square, OM being the root. If no intersection occurs, then the roots are imaginary.

The proof of this construction is a simple exercise in similar triangles and is left to the student.

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BALL, W. W.: "Mathematical Recreations and Essays," Chap. 12, Classical Problem, The Macmillan Company, New York, 1931.

COURANT and ROBBINS: "What Is Mathematics?" Chap. 3, Oxford University Press, New York, 1941.

DAMPIER, W.: "A History of Science," 3d ed., The Macmillan Company, New York, 1949.

SEDGWICK, TYLER, and BIGELOW: "A Short History of Science," The Macmillan Company, New York, 1939.

PROBLEMS

Location dimensions for plotted points are given to the right of and above the lower left-hand corner of the working space of a blank 8½- by 11-in. sheet. Thus point $A(2, 5)$ is 2 in. to the left and 5 in. above the lower left corner. Problem sheet numbers refer to the workbook accompanying the text.

1-1. *Precision in drawing.* Use Problem Sheet 1-1. The following problem will serve to test the student's skill in the use of drafting instruments.

EXAMPLE 1. Lay off the following consecutive lengths along line OX. $OA = 1\frac{9}{16}$ in.; $AB = 1.22$ in.; $BC = \frac{27}{32}$ in.; $CD = 2.09$ in.; $DE = 1\frac{13}{16}$ in. Measure the length OE.

EXAMPLE 2. Measure angle θ, using (*a*) the protractor, (*b*) the tangent method, (*c*) the chord method.

EXAMPLE 3. Bisect angles A, B, and C. Draw the inscribed circle, and mark the points of tangency. The diameter of the circle is equal to $AB + AC - BC$. Check with measured diameter. Divide the circle into 12 equal parts, using the 30°-60° triangle. Check the equality of the divisions with bow dividers.

EXAMPLE 4. Divide line DE into seven equal parts by proportional division. Check the equality of the divisions with bow dividers.

EXAMPLE 5. Rectify the arc FG on the tangent GH. Use two different methods. Check each with calculated lengths.

1-2. *Precision in drawing.* Use a blank sheet, long edge horizontal.

EXAMPLE 1. Set the dividers to 0.25 in. Lay off 32 lengths along a straight line starting at $A(1, 7\frac{1}{2})$. Scale the length of the result to 0.01 in. What is the accumulated error? What was the error in the original setting?

EXAMPLE 2. Draw a circle of 4 in. diameter, center (5, 5). Construct one 30° central angle with triangles, and bisect it to obtain a 15° arc. Set the dividers to the chord of 15°. Step off 24 of these chords around the circle. What is your error of closure expressed as a chord length? What was the error in your original setting?

EXAMPLE 3. Draw a horizontal line AB 8 in. long starting at $A(1, 1\frac{3}{4})$. At A erect a perpendicular AC upward ¼ in. long. At B erect a perpendicular BD downward ⅛ in. long. Draw CD, and mark by eye the point E where it intersects AB. Then EB should equal $\frac{1}{3}AB$. Set your dividers to EB, and step off three lengths from B toward A. What is your total accumulated error? What was your error in locating E? Now erect a perpendicular 3 in. long upward from A and 1½ in. long downward from B, and repeat the problem.

1-3. *Tangent arcs.* Use Problem Sheet 1-3.

1-4. *Reverse curves.* Use a blank sheet.

a. Construct a reverse curve between the two horizontal lines, points $A(2, 6)$ and $B(6, 8)$, tangent at A and B and passing through $C(3\frac{1}{2}, 6\frac{3}{4})$.

b. Construct a reverse curve between the two horizontal lines through $A(1, 1)$ and $B(1, 3)$ such that the radii shall be equal and each arc shall be 60°.

1-5. Divide 8- by 10-in. working space into two rectangles 5 by 8 in. Lay out graphical solutions for the following problems, letting the unit = 1 in.

EXAMPLE 1. *Division.* Find OB/OD. $OB = 4.5$ in.; $OD = 5.75$ in.

EXAMPLE 2. *Multiplication.* Find $OA \times OD$. $OA = 1.5$ in.; $OD = 4.0$ in.

1-6. EXAMPLE 1. *Arithmetic.* Find $\frac{AC/CB}{AD/DB} \cdot AB = 2.1$ in.; $AC = 3.9$ in.; $AD = 6.0$ in. A,B,C,D are successive points on a straight line.

EXAMPLE 2. *Square root.* Find $\sqrt{4.52}$.

1-7. *Approximate constructions.* Use a blank sheet with the long side vertical.

EXAMPLE 1. Let $A(1, 9\frac{3}{4})$ be the center of an arc of 4 in. radius. Through $B(1, 5\frac{3}{4})$ draw a horizontal tangent 5 in. long. Starting from B, lay the tangent length on the circular arc by the method of Art. 1–44.

EXAMPLE 2. Draw a circle of 1 in. radius, center $(1\frac{1}{4}, 3)$. Rectify the circle by the method of Art. 1-45. Scale the result, and compare with the calculated value.

1-8. On the circular arc AB having a 4-in. radius lay off a distance of 3 in., using the templet method. Check by Rankine's method.

1-9. Find the circumference of a 3-in.-diameter circle, using the templet method. Check by calculation.

1-10. A straight bar, OA, is $2\frac{1}{2}$ in. long and rotates about an axis at O. A bar, BAC, 5 in. long, is fastened to OA at point A. A is the mid-point of BAC. Point B is required to move along a vertical line which passes through O. Find the locus of the mid-point of AC. $O(4, 5)$. Use a templet.

1-11. *Quadratic equations.* Use a sheet of rectangular graph paper 20 squares to the inch. Divide the working space into four equal parts. Find the roots of the following quadratic equations by the method of Art. 1-56. Unit = $\frac{1}{2}$ in.

$$x^2 - 3x + 2 = 0$$
$$x^2 - 0.55x - 4.44 = 0$$
$$x^2 + 3.2x + 1.92 = 0$$
$$x^2 - 3.2x + 2.56 = 0$$

2

THE CONSTRUCTION OF THE CONIC SECTIONS

2-1. Historical

The three conic sections other than the circle—the ellipse, parabola, and hyperbola—have had a long history. Their discovery is attributed to Menaechmus, who flourished about 350 B.C. and was a pupil of Plato. Menaechmus found a solution to the problem of doubling the cube by using the parabola and hyperbola. Many other ancient Greeks contributed to the writings on conics, among whom were Euclid, Archimedes, and Apollonius. The only extensive work on conics that has survived the centuries is that of Apollonius. This work, together with some additions by Pappus (A.D. 300), one of the last mathematicians of the Alexandrian school, constitutes an almost complete treatment of the conic sections. Very few fundamental metric properties of these curves have been added since.

2-2. Sections of a cone

The conic sections can be derived from the intersection of a plane with a cone of revolution. The intersection formed by a plane Q, Fig. 2-1, which cuts every element of such a cone is an ellipse. In the one situation, where the plane is perpendicular to the axis of the cone, the section is a circle. In

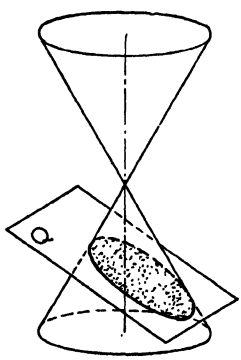

Fig. 2-1.

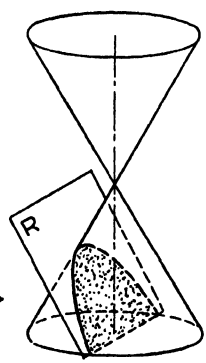

Fig. 2-2.

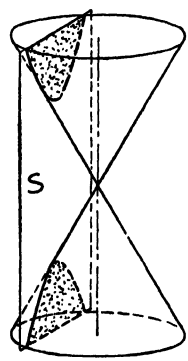

Fig. 2-3.

the particular case where the plane R, Fig. 2-2, is parallel to an element of the cone, the section is a parabola. The intersection formed by a plane S, Fig. 2-3, which cuts both nappes of the cone is a hyperbola. Thus, a plane which rotates about a suitably chosen line, Fig. 2-4, can cut from a cone one circle P, two parabolas, as R, and an infinite number of ellipses and hyperbolas, as Q and S.

2-3. The ellipse

Locus definition: *The ellipse is the locus of a point, the sum of whose distances from two fixed points, called foci, is a constant.* This constant is equal to the length of the major axis, the diameter through the foci. The minor axis

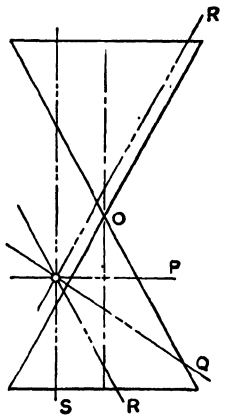

Fig. 2-4.

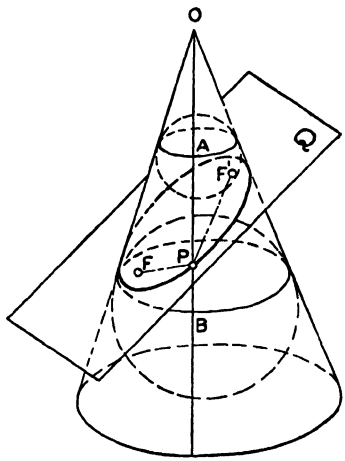

Fig. 2-5.

is defined as the diameter perpendicular to the major axis at its mid-point.

The ellipse as a section cut from a cone by a plane is related to the above locus definition by a demonstration given by the Belgian mathematician Dandelin in 1822.

The cutting plane Q, Fig. 2-5, is tangent at F and F' to two spheres inscribed in a cone of revolution. Each sphere is tangent to the cone along a circle, and the two circles are parallel. Point P is any point on the curve cut from the cone by plane Q. An element of the cone through P cuts the parallel circles in points A and B. Now, PF and PB are two tangents from an external point to a sphere.

Hence,

$$PF = PB$$

Similarly,

$$PF' = PA$$

Adding,

$$PF + PF' = PB + PA$$

But $PB + PA$ is the segment of the element lying between the parallel circles and is a constant for any position of P. Therefore, $PF + PF' = a$ constant. This is the locus definition of an ellipse. Definitions for the parabola and hyperbola can be derived in a similar manner.

The ellipse is the most familiar form of the conic section. Any circle viewed obliquely appears as an ellipse. The circle, of course, is a special case of an ellipse. The planets and periodic comets travel in paths about the sun that are ellipses, the sun being at one focus of each orbit. In Bohr's theory of atomic structure it is deduced that electrons revolve about their nuclei in elliptical orbits.

If an ellipse is rotated about its major axis, a surface called the ellipsoid of revolution is generated. This surface has the peculiar property of reflecting light or sound waves that originate at one focus so that they pass through the other focus. This produces the "whispering gallery" in a room which has an ellipsoidal form. An example of this is given by Statuary Hall in Washington, D.C.

2-4. To draw an ellipse, given the major and minor axes (Fig. 2-6)

Let AB be the major axis and CD the minor axis. With radius equal to one-half the major axis, center C or D, describe an arc cutting AB in points F and F', the *foci*. Divide FO, by eye, into any number of parts, 1, 2, 3, etc. To locate points of the curve: With A-1 as radius, center F, describe an arc at P. With radius B-1 and center F', intersect the preceding arc at point P, a point of the ellipse. Four symmetrical points can be found with

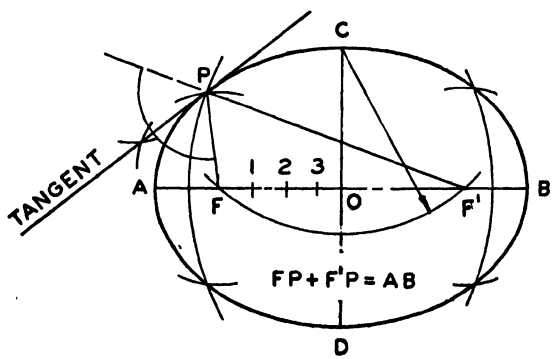

Fig. 2-6.

the same pair of radii. Point Q is found by using radii equal to A-2 and B-2, centers at F and F', respectively. As many points as desired can be found in a similar manner.

A tangent to the ellipse at a given point on the curve bisects the angle between the focal radii drawn to the point. Thus, the tangent at point P

bisects the angle between lines FP and $F'P$. For proof see Appendix II, Proof II.

2-5. To draw an ellipse, using a trammel (Fig. 2-7)

Let AB and CD be the given axes. On the edge of a strip of paper, mark the distance EF equal to the semimajor axis AO. Mark the distance EG equal to the semiminor axis CO. Place the trammel so that F lies on CD, produced if necessary, and point G lies on AB. Then point E will be a point of the required ellipse. Find as many points as desired by shifting the position of the trammel. When the axes are nearly alike, the trammel can be constructed as in Fig. 2-8. For proof see Appendix II, Proof III.

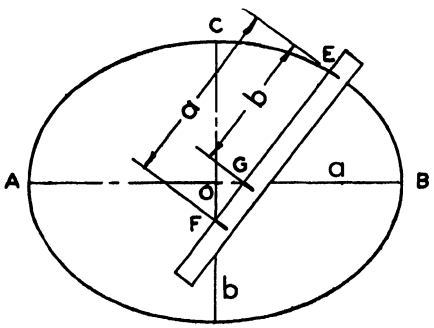

Fig. 2-7.

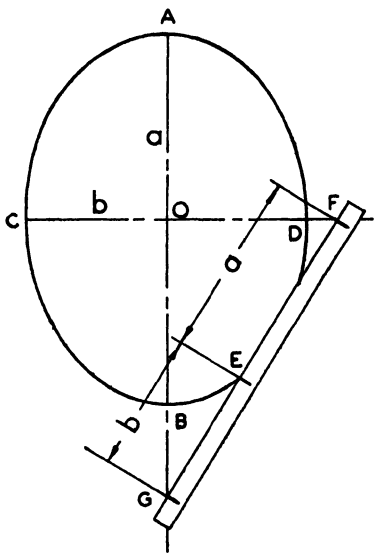

Fig. 2-8.

2-6. To draw an ellipse by the concentric-circle method (Fig. 2-9)

Let AB and CD be the given axes. With OA and OC as radii, center O, draw the major and minor auxiliary circles AEB and FDG. Assume points on the outer circle, as J, K, L. Draw radii OJ, OK, OL. Draw lines parallel to AB through points J', K', L', intersected by lines parallel to CD from points J, K, L. The points of intersection are points on the required ellipse. Additional points are found in the same manner. For proof see Appendix II, Proof IV.

To draw a tangent to the ellipse at any point, as Q, draw a tangent to the outer circle at point L, cutting AB, produced, at point M. Draw a tangent to the inner circle at L', cutting CD at point N. The line MN is the required tangent.

2-7. To draw an ellipse, given a pair of conjugate diameters (see Art. 2-19)

First method (Fig. 2-10). Let AB and CD be the given diameters. Through the ends of each diameter, draw lines parallel to the other diam-

eter, forming a parallelogram. Divide CE and CO into the same number of equal parts and number the points from C, as 1, 2; 1′, 2′. Draw A-1′ and B-1 to intersect at P. Draw A-2′ and B-2 to intersect at Q. Points

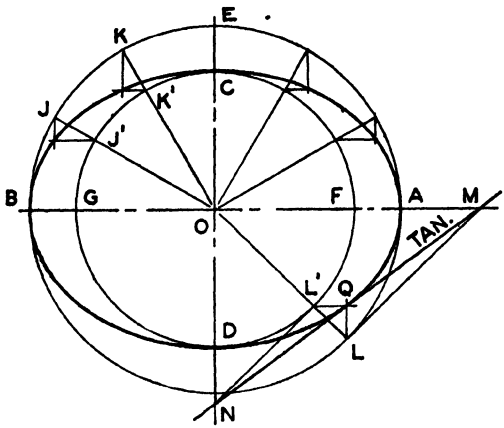

Fig. 2-9.

P and Q lie on the required ellipse. Additional points are found in a similar manner. For proof see Appendix II, Proof V.

SECOND METHOD, USING A TRIANGULAR TEMPLET (FIG. 2-11). AB and CD are the conjugate axes. Draw DE perpendicular to AB and AG per-

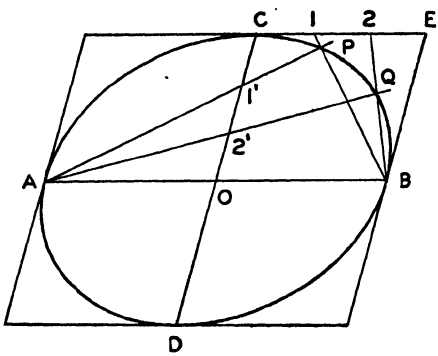

Fig. 2-10.

pendicular to CD. Lay off DH equal to AG. Draw the line EH. Trace the triangle DEH on transparent paper to form a trammel. Move the triangle so that E always lies on AB and H on CD. Point D will trace the required ellipse.

2-8. The parabola

Locus definition: *The parabola is the locus of a point whose distance from a fixed point, the focus, is always equal to its distance from a fixed line, the*

The graphs of many physical laws are parabolas. The paths of many comets are parabolas with the sun at the focus. The most economical form for a bridge arch is often a parabola.

If the parabola is rotated about its axis, a surface called the paraboloid of revolution is generated. This surface has the property of reflecting in a

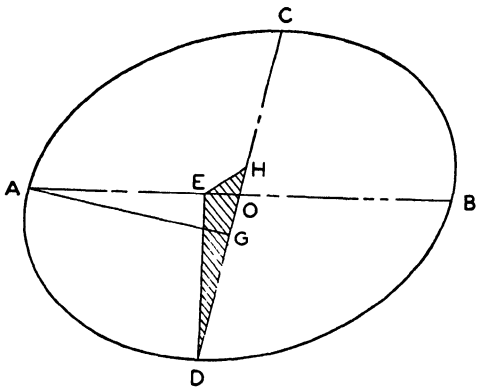

Fig. 2-11.

parallel beam rays which originate at the focus. A paraboloid cut along its axis gives a surface which, when used as the ceiling of an auditorium, for example, reduces reverberation to a minimum when the speaker is near the focus.

Galileo found that a projectile moves in a parabolic path if the effect of air resistance is neglected. This is discussed in his "Two New Sciences" (1638). As in the case of Descartes, it is noteworthy that Galileo in his work on projectiles seized on what was basic in Apollonius' "Conics," by taking from it only two theorems:

1. The length of the axis of a parabola varies as the square of the width.
2. If tangents are drawn to a parabola from an external point on the axis, then the vertex of the parabola is equidistant from this point and the chord joining the points of tangency.

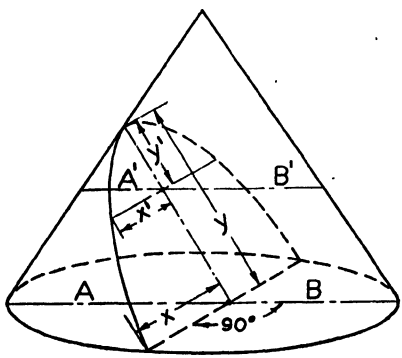

Fig. 2-12.

2-9. To construct a parabola by the offset method

Theorem 1 above follows from elementary geometry, applying the theorem on similar triangles and the theorem on the mean proportional. In Fig. 2-12, $x^2 = A \times B$ and $(x')^2 = A' \times B'$ (mean proportional, see Art. 1-53). Since the cutting plane is

parallel to the generating element of the cone, $B = B'$. Hence, $y/y' = A/A' = x^2/(x')^2$ (similar triangles).

Civil engineers usually plot the parabola by use of this fundamental property, stating it in this form: The offset varies as the square of the distance. Let the height and width of the parabola be AB and AD, respectively, Fig. 2-13, with the vertex O at the center of AD. For this parabola $y/AB = x^2/(OA)^2$. Hence $y = [x^2/(OA)^2]\ (AB)$. Divide OA into a number of equal parts, say four. Then the offset from OA for point E on the curve is equal to $(\frac{1}{4})^2(AB)$ or $\frac{1}{16}(AB)$. The offset for point F is $(\frac{2}{4})^2(AB)$; point G is $(\frac{3}{4})^2(AB)$. If OA were divided into six equal parts, the factors would be $\frac{1}{36}$, $\frac{4}{36}$, $\frac{9}{36}$, etc.

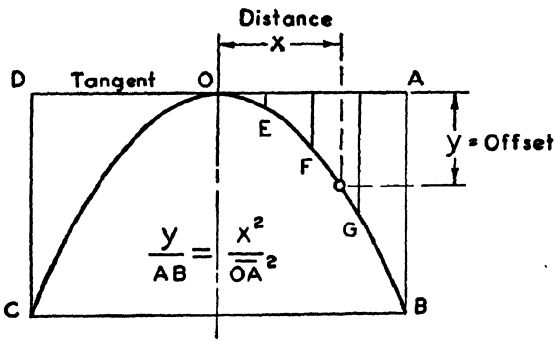

Fig. 2-13.

The area OBC is called a parabolic segment. This area is equal to $\frac{2}{3}$ base $\times$ height $= \frac{2}{3}(BC)(AB)$. The area OBA is called a parabolic spandrel and is equal to $\frac{1}{3}$ base $\times$ height $= \frac{1}{3}(OA)(AB)$.

2-10. To draw a parabola, given the focus and the directrix (Fig. 2-14)

Let AB be the directrix of the parabola and F its focus. The axis of the curve passes through F and is perpendicular to AB at C. Bisect CF at point V, the vertex of the parabola. Assume any point on the axis, as M, and draw a line parallel to AB. With CM as a radius, center F, intersect this line at points P and Q. These are points on the required parabola. This process is repeated for parallels to AB taken through different points of the axis. Thus, $CN = FR$; $CO = FS$, etc.

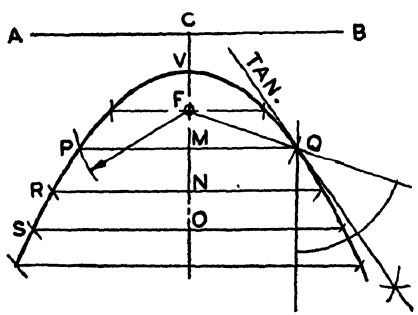

Fig. 2-14.

A line tangent to the curve at any point, as Q, bisects the angle be-

tween the focal radius drawn through the point and a line through the point parallel to the axis.

2-11. To draw a parabola, given the axis, the vertex, and a point on the curve (Fig. 2-15)

Let CD be the given axis, C the vertex, and F a point on the curve. Complete the rectangle $ABFE$. Divide BC into any number of equal parts, as four, by points 1, 2, 3. Divide BF into the same number of equal parts

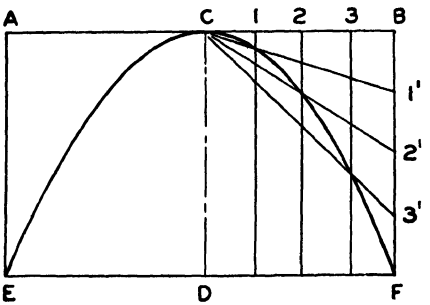

Fig. 2-15.

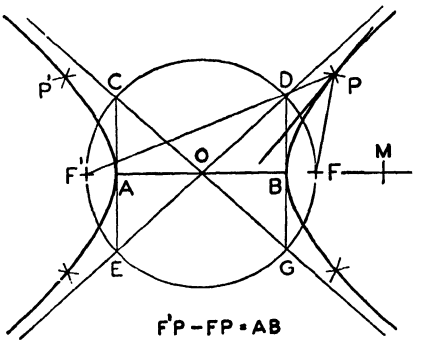

Fig. 2-16.

by points 1′, 2′, 3′. Through points 1, 2, 3 draw lines parallel to CD. Draw C-1′, C-2′, C-3′, intersecting the parallels through 1, 2, 3, at points of the parabola. The curve CE is symmetrical to CF. For proof see Appendix II, Proof VI.

2-12. The hyperbola

Locus definition: *The hyperbola is the locus of a point which moves so that the difference of its distances from two fixed points, called foci, is a constant.* This constant is equal to the length of the transverse axis. In Fig. 2-16, F and F' are the foci, and line AB, which joins the vertices A and B, is the transverse axis. Point O is the center of the curve. Point P is any point of the curve and by definition $F'P - FP = AB$. The curve is composed of two infinite branches. The hyperbola contains two infinite points, and therefore the curve has two asymptotes (see Art. 3-14). The asymptotes pass through the center of the curve, O. When the asymptotes are at right angles, Fig. 2-17, the curve is called a rectangular hyperbola. In this example only one branch of the curve is drawn.

2-13. To draw a hyperbola, given the transverse axis and the foci (Fig. 2-16)

Let AB be the transverse axis and F, F' the foci. Mark any point M on the axis outside FF'. With radius MA, centers F' and F, strike arcs at P

and P'. With radius MB and the same centers, intersect these arcs at points P' and P. These are points on the required hyperbola.

A tangent to the hyperbola at P will bisect the angle FPF'.

To DRAW THE ASYMPTOTES. Draw a circle through F and F', centered on AB. Draw perpendiculars to AB at A and B intersecting the circle in C, D, E, and G. Lines CG and DE are the asymptotes of the hyperbola.

2-14. To draw an equilateral hyperbola, given one point of the curve (Fig. 2-17)

Let the given asymptotes be OX and OY; the given point, A.

FIRST METHOD. Draw lines AB and AC parallel to OX and OY, respectively. Through any point on AC, as 1, draw O-1 cutting AB at point

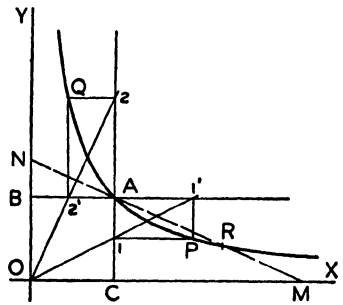

Fig. 2-17.

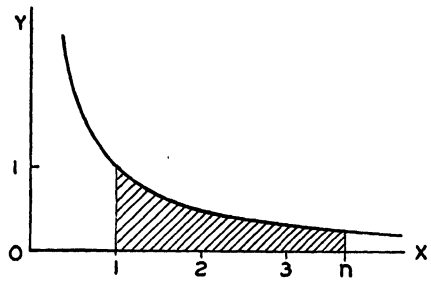

Fig. 2-18.

1′. Parallels to the asymptotes drawn through points 1 and 1′ intersect at P, a point on the required curve. A second point Q is found in a similar manner.

SECOND METHOD. Through the given point A draw any line which cuts the asymptotes in points M and N. Lay off $MR = AN$. Point R lies on the required hyperbola.

2-15. Applications of the hyperbola

While the hyperbola has fewer practical applications than the parabola, these applications are important ones. The graph of an equation in which one quantity varies inversely as a second has the form of a rectangular hyperbola. For example, the pressure-volume relationship for an ideal gas has the equation $P = \text{constant}/V$, and its graph is a rectangular hyperbola.

The rectangular hyperbola is also connected with the derivation of hyperbolic or natural logarithms. The natural logarithm of any positive number n is equal to the area under the hyperbola between the ordinates at 1 and n, Fig. 2-18.

Hyperbolic trigonometry, developed about 1875, is intimately connected with the rectangular hyperbola. The difference between circular and hyperbolic functions is illustrated in Figs. 2-19 and 2-20. Figure 2-19

represents the circular functions of θ. If OP = unity, then $PS = \sin\theta$, $OS = \cos\theta$, and $AT = \tan\theta$. Figure 2-20 represents the hyperbolic functions of ϕ, where ϕ is not an angle but is a number equal to twice the area of the hyperbolic sector POA. Line OA is the semitransverse axis of the rectangular hyperbola and is equal to unity. The hyperbolic sine of ϕ is equal to SP and is written $SP = \sinh\phi$; $OS = \cosh\phi$ and $AT = \tanh\phi$.

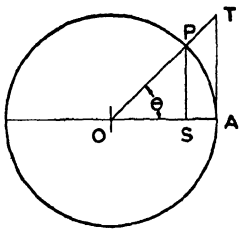

Fig. 2-19.

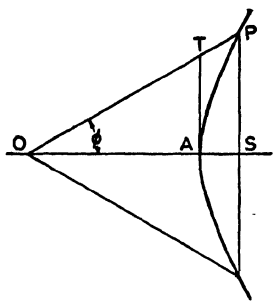

Fig. 2-20.

The lengths of certain curves are more conveniently expressed by hyperbolic functions than by other means.

A method for locating the point at which a sound wave originates is based on the properties of the hyperbola. The exact time T at which a distant report is heard is recorded by each of two listening stations, A and B. The difference of the two time readings, $T_A - T_B$, when converted to distance (from a knowledge of the speed of sound) represents the length of the transverse axis of a hyperbola having points A and B as foci. If A and B are plotted on a map and the hyperbola is constructed, the origin of the sound will lie somewhere on this curve, Fig. 2-21.

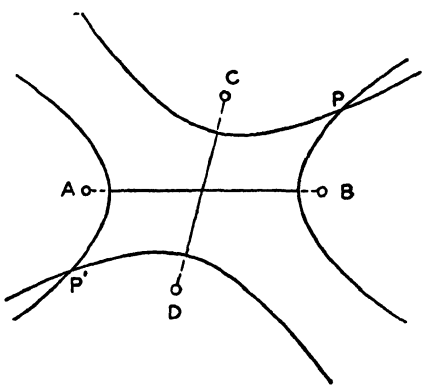

Fig. 2-21.

If this operation is now repeated for a second pair of listening stations, C and D, the origin of the sound will be on the hyperbola having C and D as foci and a transverse axis calculated from the time difference for C and D. Hence, the required origin will be at one of the points of intersection, P or P', of the two hyperbolas, the correct point being the one nearest the station having the earlier time reading.

An ingenious system of navigation called Loran, developed during the Second World War, used this basic method in reverse. Simplified, the

system operates in the following manner: Each of two radio stations, given by points on a chart (A and B, Fig. 2-21), transmit a signal simultaneously. A ship at sea notes the time difference in the reception of the two signals. This time difference, converted to distance from knowledge of the speed of the signal, is equal to the length of the transverse axis of a hyperbola having points A and B as foci. Therefore the ship lies somewhere on one branch of this hyperbola. This observation is now repeated for a second pair of stations, C and D, determining a second hyperbola for which C and D are foci. The two sets of hyperbola may intersect at four points. The point nearest the ship's position as indicated by dead reckoning is taken as the true position of the ship. The above observations are actually made by electronic instruments which yield readings that identify the correct hyperbola on previously prepared charts. Loran charts are distributed by the Hydrographic Office, Washington, D.C.

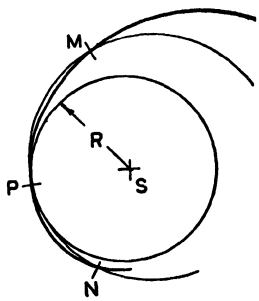

Fig. 2-22.

2-16. Circle of curvature

If M, P, and N are three points of a curve and a circle is passed through these points, then as points M and N move up to point P the circle approaches a definite limit called the *circle of curvature* at P, Fig. 2-22. The center of this circle, S, is the *center of curvature* at point P. The reciprocal of the radius of curvature at P is the *curvature* at P.

2-17. Center of curvature of a conic

The following construction will determine the center of curvature at any point of a conic except the vertex: To find the center of curvature at point P, Fig. 2-23, draw the normal, PN, to the curve at P, cutting the major axis at Q. Draw line PR through the focus F. Through Q draw QR per-

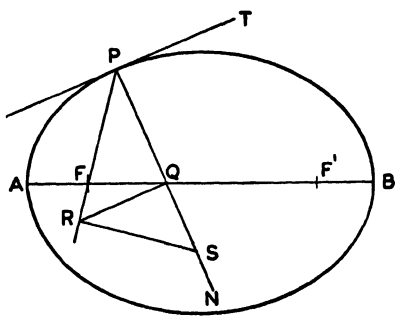

Fig. 2-23.

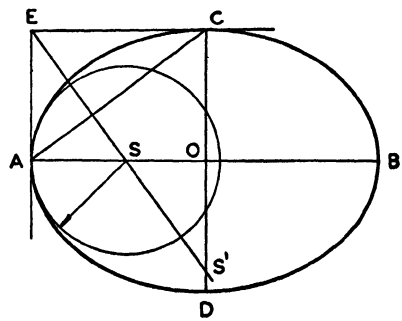

Fig. 2-24.

pendicular to PN and meeting PR at R. Through point R draw RS perpendicular to PR and meeting PN at point S, the center of curvature at P. PS is the radius of curvature at P.

CENTER OF CURVATURE AT VERTEX OF ELLIPSE (FIG. 2-24). AB and CD are the major and minor axes of an ellipse. Draw CE and AE parallel to AB and CD, respectively. Draw AC. Construct ES' perpendicular to AC. ES' meets AB in point S and CD in point S'. S and S' are the centers of curvature at A and C, respectively.

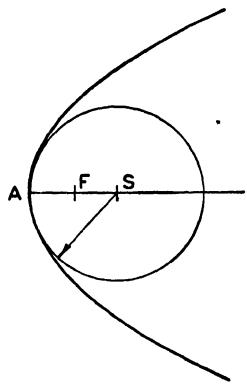

Fig. 2-25.

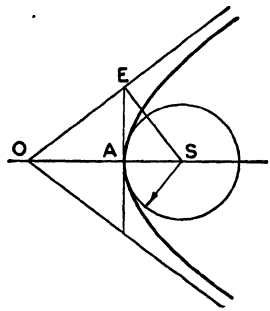

Fig. 2-26.

CENTER OF CURVATURE AT VERTEX OF PARABOLA (FIG. 2-25). Point F is the focus of the curve. Make $AF = FS$. S is the center of curvature at the vertex A.

CENTER OF CURVATURE AT VERTEX OF HYPERBOLA (FIG. 2-26). Point O is the center of the curve, and OE is an asymptote. Draw a line through the vertex A and perpendicular to the axis OA, meeting the asymptote at E. Through E draw ES perpendicular to OE, cutting the axis at S. S is the center of curvature at A.

2-18. The evolute of a curve

The locus of the center of curvature of a curve is the evolute of the curve. A given curve has but one evolute.

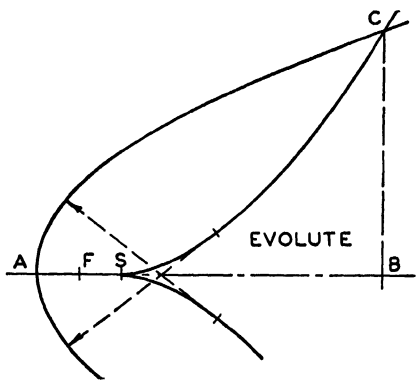

Fig. 2-27.

To find the evolute for a conic, apply the construction for center of curvature to a suitable number of points of the curve. A smooth curve through the centers obtained is the evolute. The evolute of a parabola is shown in Fig. 2-27. The evolute cuts the parabola in two points, one of which is C, where $AB = 8AF$.

2-19. Diameters and chords of conics

Chords through the center of an ellipse or hyperbola are diameters. The diameters of a parabola are at right angles to the directrix. The mid-points

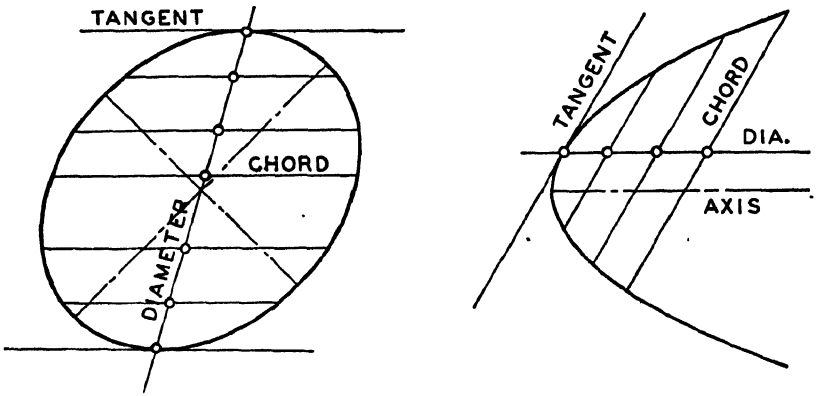

Fig. 2-28.

of any system of parallel chords of a conic lie in a straight line which is a diameter, Fig. 2-28.

This property can be used to find the point of tangency for any assumed tangent. Draw two chords parallel to the given tangent. A line through their mid-points cuts the arc at the point of tangency.

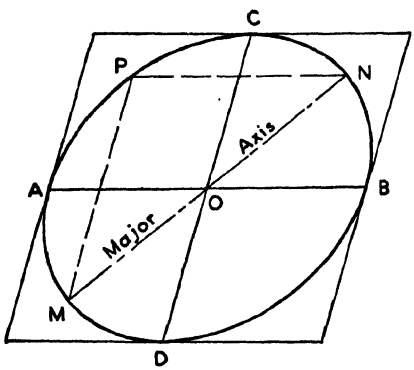

Fig. 2-29.

CONJUGATE DIAMETERS. When each of two diameters of an ellipse is parallel to the tangent at the extremity of the other, the diameters are said to be conjugate, Fig. 2-29. Each of a pair of conjugate diameters bisects every chord parallel to the other. AB and CD are conjugate diameters.

SUPPLEMENTARY CHORDS. Chords drawn from any point on the curve to the ends of the major axis are supplementary chords. Any pair of such chords, as MP and NP, establish directions for a pair of conjugate diameters.

2-20. Axes and foci

To find axes and foci when the curve only is given.

ELLIPSE (FIG. 2-30). Draw a pair of parallel chords. Their bisector is a diameter of the ellipse, and its mid-point is the center of the ellipse.

From the center of the ellipse, strike a circle which cuts the ellipse at four points. The axes of the ellipse are drawn through its center and parallel to the sides of the rectangle determined by the four points.

An arc struck off from the end of the minor axis, with radius equal to the semimajor axis, intersects the major axis at the foci.

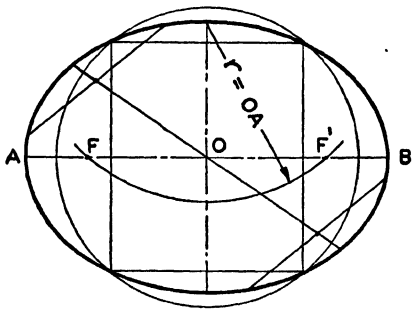

Fig. 2-30.

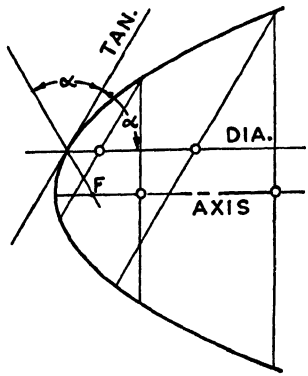

Fig. 2-31.

PARABOLA (FIG. 2-31). A diameter is obtained by bisecting a pair of parallel chords. The parallel to these chords at the end of the diameter determined by them will be the tangent at this point. The axis is found by bisecting a pair of chords which are perpendicular to the diameter. The focus is determined by the property that the tangent bisects the angle between a diameter and the focal radius through the point of tangency.

HYPERBOLA. The axes and foci are found as in the case of the ellipse.

2-21. The eccentricity of a conic

A conic may be defined as the locus of a point which moves so that the ratio of its distances from a fixed point and a fixed straight line remains a constant. The fixed point is called the *focus* and the fixed line the *directrix* of the conic. The constant ratio is called the eccentricity, e.

When $e = 0$, the conic is a circle (directrix at ∞).
$e < 1$, the conic is an ellipse.
$e = 1$, the conic is a parabola.
$e > 1$, the conic is a hyperbola.

Given the focus, F, and the directrix, d, Fig. 2-32, to construct an ellipse, a parabola, and a hyperbola. Draw the axis through the focus F perpendicular to the directrix d, intersecting d at O.

Draw at random a line, such as OA through O, making an angle θ with the axis. Assume point K on the axis, and draw line y parallel to the directrix. Set the bow compasses to the length KL, the portion of line y intercepted by the axis and line OA. With F as a center, swing an arc inter-

cepting line y at P and P'. These are points on the conic. Additional points are found by repeating this operation for other assumed lines.

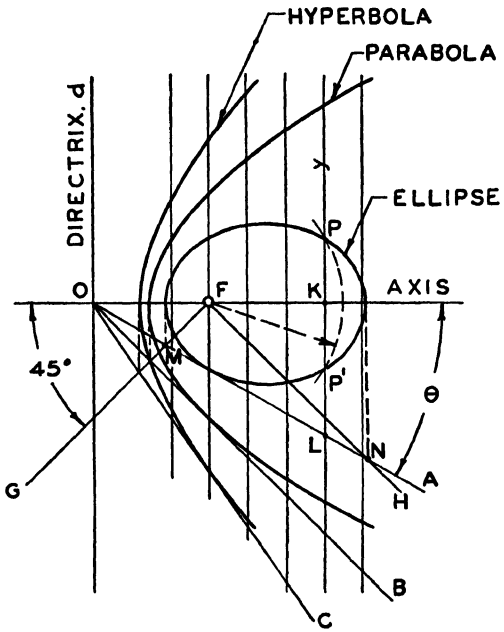

Fig. 2-32.

It is evident that the ratio of the distance of point P from the focus and directrix is $FP/KO = KL/KO = \tan\theta$. This ratio is maintained as a constant throughout the construction. Consequently, $e = \tan\theta$.

If $\theta < 45°$, then $e < 1$. Thus the conic is an ellipse.
$\theta = 45°$, then $e = 1$. Thus the conic is a parabola.
$\theta > 45°$, then $e > 1$. Thus the conic is a hyperbola.

The lines OA, OB, and OC yield the three conics shown.

In the case of the hyperbola only, points may be found to the left of the directrix which satisfy the required ratio. Thus the hyperbola has two branches.

The points in which the curves cut the axis are found as follows: Through point F, draw lines FG and FH at angles of 45° with the axis. These lines intersect line OA in points M and N. Points M and N, projected onto the axis, determine the vertices of the ellipse. The vertices of the parabola and hyperbola are found in a similar manner.

PROBLEMS

2-1. *The ellipse.* Use Problem Sheet 2-1.

2-2. *The hyperbola and parabola.* Use Problem Sheet 2-2.

2-3. *Properties of conics.* Use Problem Sheet 2-3.

2-4. *The parabola.* Use Problem Sheet 2-4.

The following problems are to be plotted in a space 8 in. wide by 10 in. high. The location dimensions are given to the right of and above the lower left-hand corner.

2-5. *The ellipse.* Draw an ellipse having a major axis of 6 in. and the distance between foci equal to 5 in. Draw a tangent to the curve at a point 2 in. to the right of the minor axis. Draw the circles of curvature at the ends of the major axis. Draw the major axis horizontal, placing the center of axis at 4, 5½.

2-6. *The ellipse.* Using the *trammel* method, draw an ellipse having a major axis of 7 in. and a minor axis of 3 in. Verify the fact that the bisector of a system of parallel chords passes through the center of the ellipse. Draw tangents at the points in which this bisector cuts the ellipse. Draw the major axis vertical, and place the center of the curve at 4, 5½.

2-7. A pair of conjugate axes of an ellipse are 3 in. and 4 in. long, respectively, and intersect at an angle of 60°. Construct the ellipse by means of a triangular trammel.

2-8. *The parabola.* Construct the parabola having a vertical axis 6 in. long and the focus ½ in. from the directrix. Locate the focus at 4, 3. Verify the fact that the bisector of a system of parallel chords is parallel to the axis of the curve. Draw the tangent at the point in which this bisector cuts the curve. Draw the circle of curvature at the vertex.

2-9. *The parabola.* A parabolic reflector 6 in. in diameter and 3 in. deep is formed by rotating a parabola about its axis. Draw a cross section taken through the axis, and find the focus of the curve. Assuming a light source at the focus, draw the path of a ray of light and its reflection. Place the axis horizontal with the vertex of the curve at 3, 5½. Construct the parabola by the parallelogram method.

2-10. *The hyperbola.* The transverse axis of a hyperbola is 1½ in. long, and the foci are 2 in. apart. Draw the curve, extending it 3 in. on each side of the axis. Construct a tangent at a point ½ in. from the axis. Locate the center of the hyperbola at 4, 5½, placing the axis in a horizontal position. Draw the circles of curvature at the vertices.

2-11. *The hyperbola.* A rather ingenious application of the hyperbola can be made in the problem of locating the point of origin of a distant sound or report.

Three listening stations, *A*, *B*, and *C*, record the exact time a distant report is heard. The records are as follows: *A*, 10 hr, 00 min, 8.4 sec; *B*, 10 hr, 00 min, 16.3 sec; *C*, 10 hr, 00 min, 19.9 sec. The difference in time between *A* and *B* is 7.9 sec. The velocity of sound is 1,086 fps. If *A* and *B* are regarded as foci of a hyperbola, the origin of sound lies on the curve for which the difference in length of the focal radii is equal to $7.9 \times 1{,}086 = 8{,}579$ ft, or 2,860 yd, the length of the transverse axis of this hyperbola. A second hyperbola having *B* and *C* as foci and the transverse axis equal to the time difference between *B* and *C* will cut the first hyperbola at a point which represents the origin of the sound.

This point is checked by plotting a third hyperbola, using *A* and *C* as foci. Owing to observational errors, the three curves will in general intersect to form a "triangle," and the center of this figure is taken as the best location.

Station B is 5,300 yd due east of A. Station C is S 13° W from B at a distance of 2,750 yd. Locate point A 1 in. from the left and 5½ in. from the lower border. Scale for drawing: 1 in. = 1,000 yd.

2-12. *Conics by eccentricity.* Using a vertical line 2 in. from the left border as a directrix and a point 1¾ in. to the right of the directrix and 5 in. from the lower border as a focus, construct conics having eccentricities of tan 30°, tan 45°, tan 60°.

3

PROJECTIVE CONSTRUCTIONS

3-1. Projective geometry

Little advance was made over classical geometry until about the beginning of the seventeenth century, when modern geometry began. The new geometry developed along two lines—analytical and nonmetrical, or projective. The development of analytical geometry was led by René Descartes (1596–1650), from whom we get our cartesian coordinates. Gérard Desargues (1593–1662), and Blaise Pascal (1623–1662) were leaders in developing projective geometry, a geometry which deals with the theory of transversals and of transformations of geometrical figures by projection.

A complete geometry based on the projective properties of figures was developed by the French soldier and mathematician Jean Poncelet in 1813, while he was in a Russian prison, having been captured during Napoleon's retreat. The field of projective geometry was extended by many others in the nineteenth century.

In this chapter we are concerned not with the systematic development of projective geometry but rather with the presentation of certain concepts in this field which are of interest in constructive geometry.

Modern geometry introduces the concept of a space of infinite extent, and therefore a point, line, or plane may lie at infinity. Euclid always thought of a line as the segment drawn between two points, and he was careful to state when it was to be produced beyond the given segment. A line was never given as a geometric element of indefinite length. Euclid's definitions were limited in a manner which excludes the idea of an infinity not apparent to the senses.

In Greek mathematics as well as in Greek art and literature there is a sort of horror of infinity and infinite processes resulting from interpretive difficulties of the type cleverly pointed out by Zeno, the Stoic philosopher.

The difference between the classical and the modern viewpoints is illustrated by the following propositions stated by Desargues, an engineer and architect of Lyons who did important work in geometry.

1. A straight line can be considered as produced to infinity, and then the two opposite ends are united.

2. Parallel lines are lines meeting at infinity, and conversely.

3. A straight line and a circle are two varieties of the same species.

Pascal, a French mathematician, theologian, and philosopher, studied geometry by himself at the age of twelve. At eighteen he had produced an essay on conics which contained the rather beautiful theorem since named for him. He had also invented the first calculating machine.

3-2. Elements at infinity

Many statements of Euclidean geometry are not completely general, containing as they do certain special cases or exceptions. These special cases are a consequence of defining two parallel lines as two lines which never meet. For example, two straight lines of a plane intersect in one point, *except* when the lines are parallel; a circle can be passed through any three points, *except* when the points are collinear; a line segment can be divided internally and externally into any two ratios, *except* the one in which the segments of each division are equal.

The above statements and many others made in a similar limited form can be stated *without exceptions* by introducing the mathematical concept of an element at infinity.

3-3. The point at infinity

Let the infinite straight line g and point O be given, Fig. 3-1. Through O pass any line k, cutting g in point P.

Now, if line k rotates clockwise about O as a center, point P will recede to the left, becoming more distant as k approaches a position parallel to g.

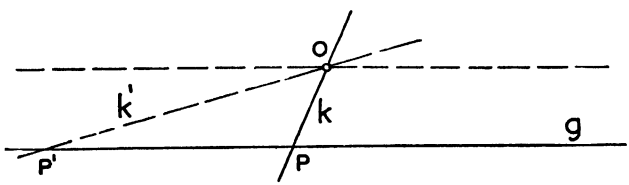

Fig. 3-1

When k is parallel to g, it no longer cuts g, according to the definition of parallelism of elementary geometry.

It is convenient to introduce the mathematical fiction of infinity here and to say that k cuts g in the *point at infinity* on g. This point is sometimes called the *ideal* point of the line. It will be seen that ideal points possess many, but not all, of the properties of real points. If line k is allowed to rotate counterclockwise from its original position OP, P recedes to the right and the point of infinity is at the right. Thus, it would appear that there are two ideal points on a line. This assumption would lead to saying that one straight line can intersect another in two points and thus enclose

space. This difficulty is avoided by agreeing that the ideal points found in Fig. 3-1 are identical and that there is only *one* ideal point on a given line. It may be objected that this assumption is not reasonable since it allows a point to be infinitely distant from itself. It is not claimed, however, that an ideal point enjoys all of the properties of a real point. It should be noted that a continuous rotation through the parallel position causes the point to recede in one direction and approach from the other.

3-4. The line at infinity

Each set of parallel lines in a plane will intersect in the point at infinity for that set. Since there are an infinite number of sets of parallels in a plane, there are a singly infinite number of points at infinity, one for each set. The question now arises as to the nature of this locus. One property is that it is cut in one point by each line of the plane, the point being the point at infinity for the line. Hence, the locus must be regarded as a straight line. To regard the locus as a circle would lead to inconsistent statements since any straight line cuts a circle in two points.

The locus therefore is called the *line at infinity* of the plane.

3-5. The circle of infinite radius

The circle $ABCD$ is tangent to line k at point A, Fig. 3-2. Let the center O retreat to infinity while the circle remains tangent to k at A. As the radius AO increases, arc BAD tends to coincide with the straight line k. Points B, C, and D approach infinity along lines through AB, AC, and AD.

In the limit, points B, C, and D become the points at infinity for the lines through AB, AC, and AD, respectively. Therefore, the circle of infinite radius is represented by an infinite line, k, and the line at infinity.

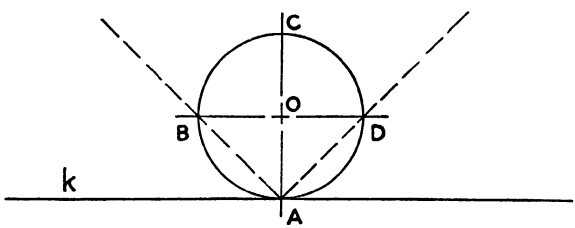

Fig. 3-2.

3-6. Dividing a line into segments

If a line segment AB is divided at point C, which lies between A and B, the line is said to be divided *internally* into the segments AC and CB, Fig. 3-3. If the given segment is divided at D, which lies on the prolongation of AB, the line AB is said to be divided *externally* into the segments AD and DB.

In each case, the length of the segment is the distance from the point of division to an extremity of the given line. When the line is divided internally, the *sum* of the segments is equal to the given line; when the line is divided externally, the *difference* of the segments is equal to the given line.

When a line segment is divided internally and externally into segments having the same ratio, the division is said to be *harmonic*. In Fig. 3-3, line AB is divided harmonically if AC/CB is equal to AD/DB.

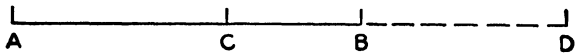

Fig. 3-3.

In harmonic division, as point C approaches the center of AB, point D approaches infinity. In the limit, C becomes the center of AB, and D the point on AB at infinity. Thus, the concept of infinity allows the line to be divided internally and externally in *any* ratio. It should be noted that all the exceptions to Euclidean geometry mentioned in Art. 3-2 have been eliminated by the above concepts.

3-7. Cross ratio

One of the most useful concepts of projective geometry is that of cross ratio. If A, B, C, D be four points on a straight line and if C and D be regarded as dividing AB (internally or externally), then the quantity $AC/CB \div AD/DB$ is called the *cross ratio* of A, B, C, D and is written as (AB, CD), Fig. 3-4. The sense of the line segments is taken into account. The distance is positive when the points are read from left to right. Thus, AB is positive; BA is negative.

The points can be paired in several ways. Thus, for (CD, AB), the segments are CA, AD and CB, BD. In this case the given line segment is assumed to be CD; A and B are the points of division.

3-8. Pencil of lines

Definition: *A system of coplanar straight lines through a point is called a pencil; the point is the vertex of the pencil.*

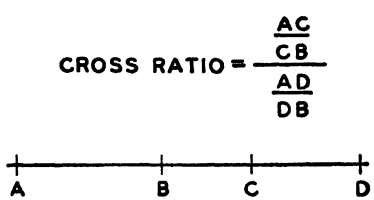

Fig. 3-4.

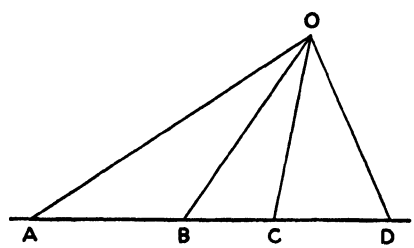

Fig. 3-5.

If OA, OB, OC, OD be a pencil of four lines, Fig. 3-5, the ratio

$$\frac{\sin AOC}{\sin BOC} : \frac{\sin AOD}{\sin BOD}$$

is the cross ratio of the pencil and is written as $O(AB, CD)$.

3-9. Cross-ratio theorem

The cross ratio of four collinear points is equal to the cross ratio of the pencil formed by the four lines which join them to any point. For proof see Appendix II, Proof I.

Thus in Fig. 3–5

$$\frac{\sin AOC}{\sin BOC} : \frac{\sin AOD}{\sin BOD} = \frac{AC}{CB} : \frac{AD}{DB}$$

Two results immediately follow from the theorem:

1. For a given pencil of four lines on center O, Fig. 3-6, any straight line cutting the pencil will intersect it in four points having the same cross ratio as that of the pencil.

2. For a given set of four collinear points A, B, C, D, Fig. 3-6, any pencil formed by joining these points to any fifth point, O', will have the same cross ratio as that of the four collinear points.

It follows from these that any transversal cutting either the pencil from O or O' will establish four points having the same cross ratio.

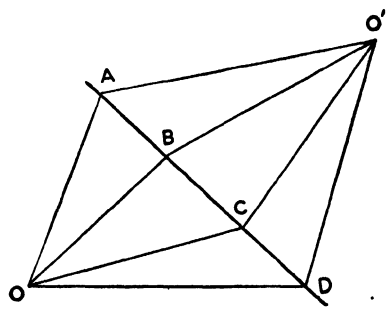

Fig. 3-6.

The concept of cross ratio can be extended to space. It can be shown that the cross ratio of four points in which a line cuts four coaxial planes (containing a common line) will remain the same, however the line may be taken. Equivalently, the cross ratio of a pencil of four lines cut from the four coaxial planes by a fifth plane is independent of the position of the plane. The cross ratio of four planes is thus defined.

3-10. Projection

An intuitive notion of projection is given by the photograph of an object.

A projection system is defined by two elements—a plane of projection, P, and a center of projection, O, Fig. 3-7. The center may be at a finite or an infinite distance from the plane. If the center is finite, the projection is a perspective one. The triangle ABC is projected onto plane P by drawing rays through points A, B, C from the center O. The points A', B', C' in which the rays intersect the plane are the projections of

points A, B, C, respectively. The projection of a straight line is the locus of the projections of its points and is, in general, a straight line.

In Figure 3-8, a straight line carrying a point range $ABCD$ is projected from the center O onto the plane P. The rays form a pencil of lines. The projection $A'B'C'D'$ of the point set is a transversal of this pencil. Hence the cross ratio $(A'B', C'D')$ is equal to the cross ratio (AB, CD).

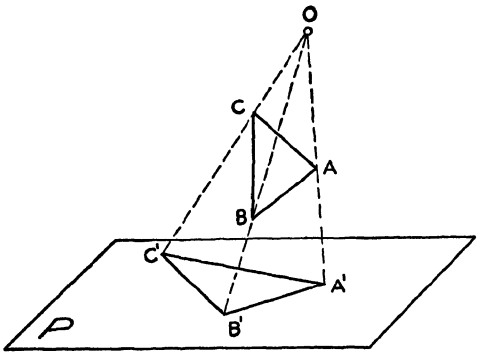

Fig. 3-7.

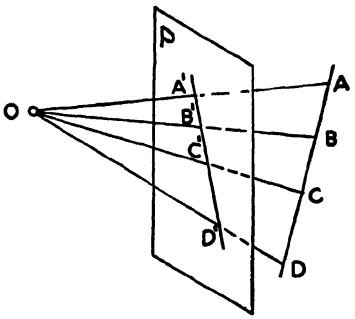

Fig. 3-8.

Geometric figures can be classified according to their properties as either *metric* or *projective*. Metric properties are those which involve magnitude—length, angle and areas; the congruence and similarity of figures. We are familiar with concepts of this nature from high-school geometry, a metric geometry. Metric properties of a figure are generally destroyed under perspective projection. The cross ratio of a set of points or a pencil, however, is the one metric property of a figure that remains unchanged in its projection. This is the reason for the importance of cross ratio. The properties of a figure that are unaltered in projection are called *projective properties*, and the branch of geometry which deals with these properties is called *projective geometry*. In practical application, projective geometry is the geometry of photography and perspective drawing.

3-11. Two projectively related pencils

If two pencils of four rays each have the same cross ratio, they are said to be projectively related, or in projective correspondence. One can be regarded as the perspective projection of the other. If two plane pencils of three lines each are given and the lines are in the same sequence, a projective correspondence between the two figures can be established by the use of cross ratio. Let the pencils abc and $a'b'c'$ be given, Fig. 3-9. Draw any fourth ray, as d, in one figure, and mark the points A, B, C, D, in which the rays a, b, c, d cut a paper strip laid across the pencil. Place the strip on the second pencil so that points A, B, C are incident with rays

a', b', c'. The ray d' which passes through O' and point D of the strip is the correspondent of ray d of the first pencil.

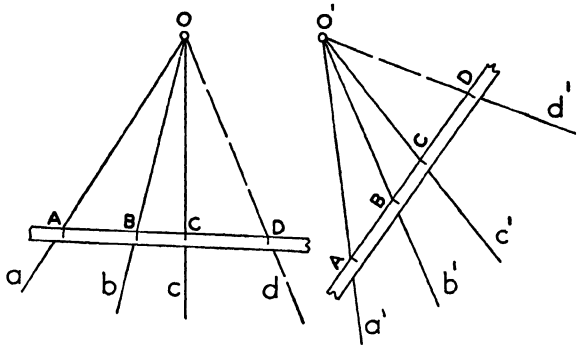

Fig. 3-9.

3-12. Map nets

Aerial surveying presents a problem which can be readily solved by projective methods. If a photograph of a flat area is taken and the exact altitude and orientation of the camera are unknown, a map of the area can be constructed from the single photograph provided that it includes four points for which the map coordinates are known. The only instrument needed for the construction of the map is a straightedge.

Let the points A, B, C, D of a photograph, Fig. 3-10, correspond to the known points A', B', C', D' of a map, Fig. 3-11. It is required to find point E' of the map which corresponds to point E of the photograph. Through A and B, C, D, E draw rays AB, AC, AD, AE. Also, through A' and B', C', D' draw rays $A'B'$, $A'C'$, $A'D'$. Mark the points C'', E'', D'', B'' in which a paper strip placed at random cuts the pencil from A on the

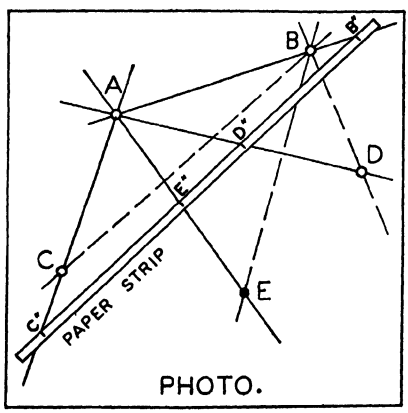

Fig. 3-10.

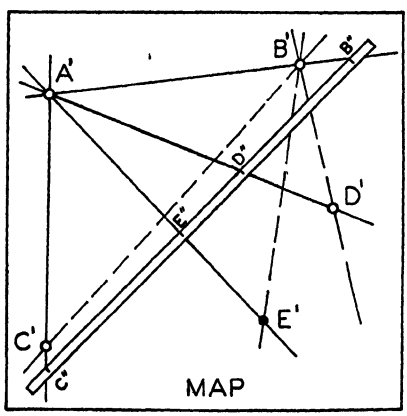

Fig. 3-11.

photograph. Adjust C'', D'', and B'' on the corresponding rays of the pencil from A' on the map, and draw ray $A'E''$ through A' and point E'' of the paper strip. Point E' of the map lies on this ray.

This operation is repeated, using a second pair of pencils drawn on points B and B', respectively. The intersection of rays $A'E''$ and the corresponding ray from B' locates point E'.

If the number of rays in the two pairs of pencils is increased, the photograph and the map become covered with two nets of lines, each line of the photograph corresponding to a definite line of the map. Since the mesh of the nets can be made as fine as desired, every point of the photograph can be transferred to the map.

3-13. Desargues' theorem

One of the oldest and most famous theorems of projective geometry is the triangle theorem set forth in 1636 by Gérard Desargues, who was interested in the application of geometry to architecture and painting.

THEOREM. *If two triangles lying in a plane are so placed that the straight lines joining the corresponding vertices intersect at a point, then the corresponding sides intersect at three points which lie on a straight line, called the axis of homology.* For proof see Appendix II, Proofs VII and VIII.

Desargues' theorem is illustrated in Fig. 3-12. Point O is the center of projection and line PR the axis of homology. The triangles ABC and $A'B'C'$ are said to be in perspective position when placed as shown.

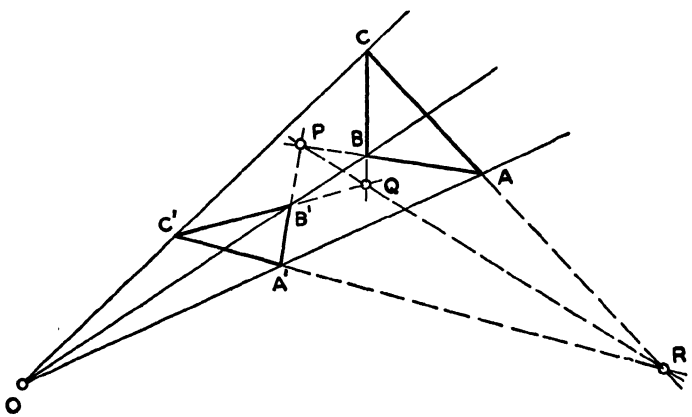

Fig. 3-12.

The problem of drawing a line through a given point and the inaccessible intersection of two given lines can be solved by Desargues' theorem, using only a straightedge. The execution of this construction is left to the student (see Prob. 3-12).

Desargues' theorem is probably verified most easily by referring to the space configuration of Fig. 3-13. Triangle ABC is projected onto a plane into triangle $A'B'C'$. Triangles ABC and $A'B'C'$ may be regarded as sections of a pyramid with vertex O. It is evident that the corresponding sides of the triangle meet in points PQR which lie on the line of intersection of the planes determined by the triangles.

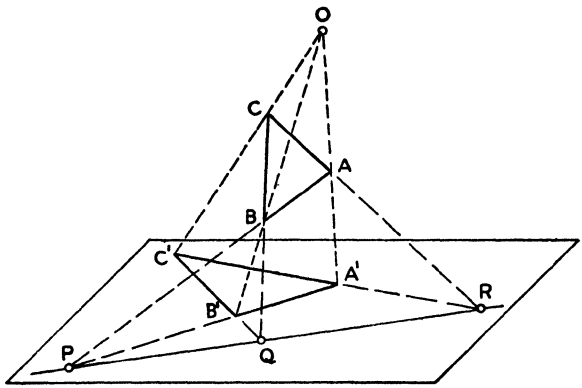

Fig. 3-13.

3-14. Projections of a circle

In projective geometry a conic is defined as a projection of a circle on a plane. The circle $ABCD$, Fig. 3-14, lying in plane P, is projected from point O as a center of projection onto the planes Q, R, and S. The projection rays form a cone having its vertex at O. Each projection of the circle is a section cut from the cone of rays by a plane. Every ray is cut by plane Q, and the projection is an ellipse. Plane R is parallel to one ray,

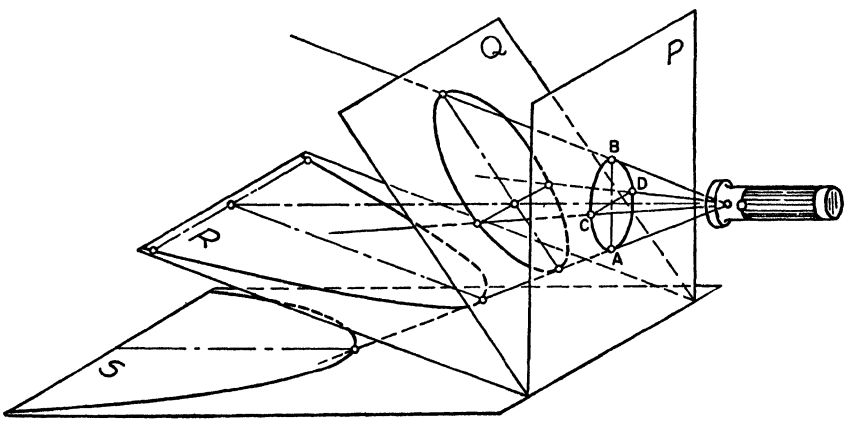

Fig. 3-14.

OB, and the projection on R is a parabola. Plane S fails to cut all of the cone of rays, and the projection on S is a hyperbola. Strictly speaking, there is only one conic in projective geometry since parallelism is a metric notion. The three cases will be distinguished here, however, since this treatment is concerned with practical constructions.

It is evident that, by suitable choice of a projection plane and center of projection, any given conic can be projected into any other conic. For example, any hyperbola can be projected into a circle, a parabola, a pair of straight lines, or a point. These two latter cases, straight lines and a point, are called degenerate conics.

3-15. Projective properties of conics

Point O is the center of projection, and $ABCD$ is a circle lying in plane P, Fig. 3-15. The circle is projected on a plane Q, so selected that every point of the projection $A'C'B'D'$ is a real point. The projection is an *ellipse*. Lines tangent to the circle, as CE and DF, project into corresponding tangents to the ellipse $C'E$ and $D'F$.

In Fig. 3-16, plane R is selected so that point B of the circle projects into an infinite point B' of plane R (plane R is parallel to line OB). Hence the projection $C'A'D'$ has one infinite point and is a *parabola*. A line tangent to the circle, as CE, projects into a tangent $C'E$ to the parabola. A tangent at point B on the circle becomes the line at infinity in plane R.

In Fig. 3-17 plane S is selected so that two points of the circle, as C and D, project into infinite points, C' and D', in plane S (plane S is parallel to OC and OD). Hence, the projection of the circle on S has two infinite points and is a *hyperbola*. Any line tangent to the circle projects into a tangent to the hyperbola. Tangents to the circle at C and D project into two particular tangents to the hyperbola. These lines are tangent at the infinite points of the curve and are the *asymptotes*.

3-16. Conics formed by intersecting pencils

With points A and B on the circumference of the circle as vertices, Fig. 3-18, draw the pencils of lines determined by four points, as 1, 2, 3, 4, on the circumference. Since the arc subtended is the measure of an inscribed angle, angles $1A2$ and $1B2$, etc., are equal and the pencils are congruent and therefore have the same cross ratios.

If the circle and accompanying pencils are now projected onto a plane Q, a conic results, as Fig. 3-19, together with the pencils $A'(1'2'3'4')$ and $B'(1'2'3'4')$. These pencils are no longer congruent, but the equality of their cross ratios is preserved. Thus, if two pencils having the same cross ratio are made to intersect, the intersections of corresponding rays will lie on a conic. The vertices of the pencils are also points on the conic. Hence

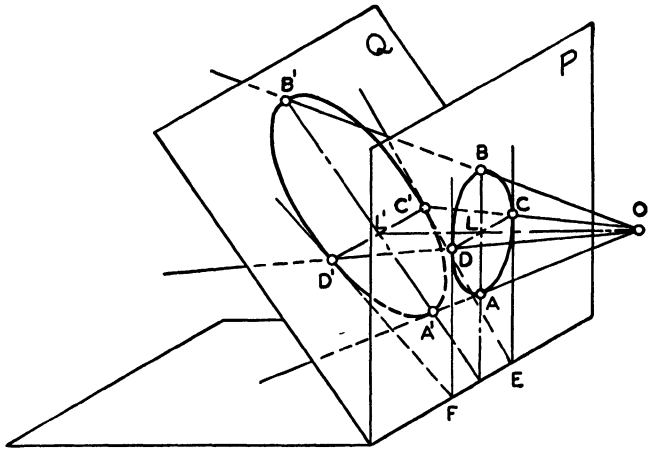

Fig. 3-15.

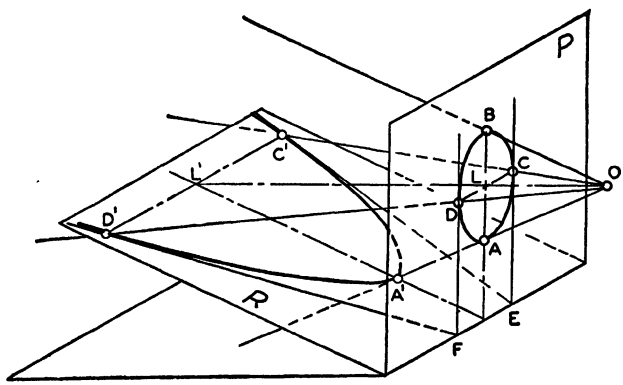

Fig. 3-16.

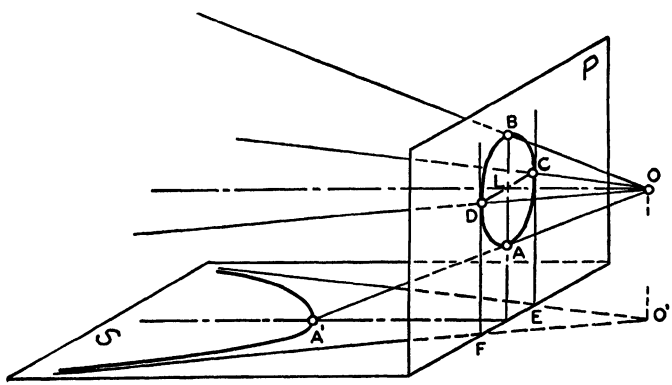

Fig. 3-17.

we may define a conic: *A conic is a set of points consisting of the intersections of the corresponding rays of two projectively related pencils of rays.*

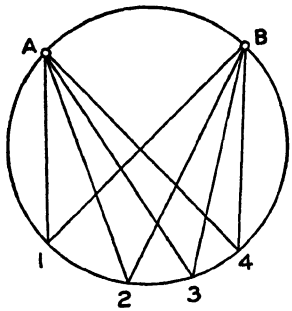

Fig. 3-18.

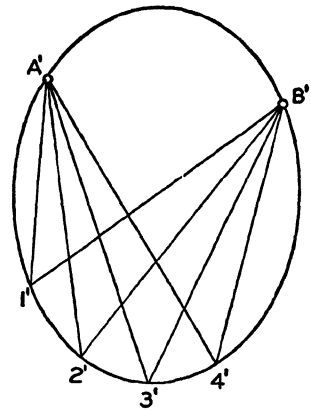

Fig. 3-19.

3-17. Projective pencils

Two pencils of the same cross ratio are called projective pencils since one can be obtained from the other by projection. Let points 1, 2, 3, 4 be any four points of line MN, Fig. 3-20. Assume point A as the vertex of a pencil and draw rays A-1, A-2, etc. Assume any second point, B, as a vertex of the second pencil, and draw rays B-1, B-2, etc. The cross ratio of each pencil is equal to that of the four points 1, 2, 3, 4. Each pencil can have an infinite number of rays so long as corresponding rays meet on MN.

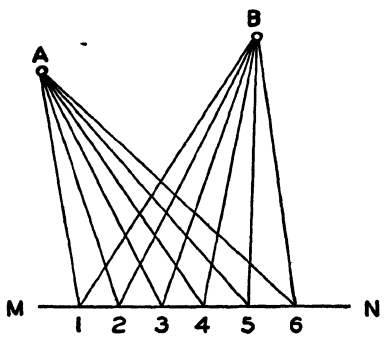

Fig. 3-20.

To find a set of points which lie on a conic, trace the pencil B on transparent paper, and lay it across pencil A. Mark the intersections of the corresponding rays, as A-1 and B-1, A-2 and B-2, etc., Fig. 3-21. The smooth curve through them is a conic. Points A and B are also on the conic.

3-18. Determining the type of conic from a portion of the curve

Draw any chord AB of the curve and tangents to the curve at points A and B, Fig. 3-22. The tangents intersect in point O. Determine the mid-point M of AB, and draw OM. If the curve cuts OM in its mid-point D, the curve is a parabola. Any curve cutting OM between M and D is an ellipse. Any curve cutting OM between O and D is a hyperbola.

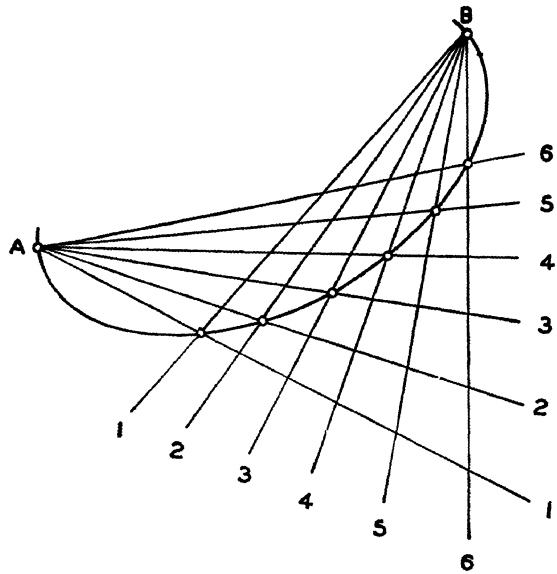

Fig. 3-21.

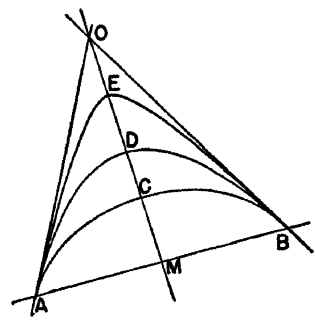

Fig. 3-22.

3-19. The hexagon

A hexagon is a plane figure composed of six points, no four of which are collinear, and the six lines which connect them, Fig. 3-23. Since the six lines which form the sides of the hexagon may be taken in any order, 60 different hexagons can be constructed on six given points.

3-20. Pascal's theorem

The three pairs of opposite sides of a hexagon inscribed in a conic intersect in three points which lie on a straight line, Fig. 3-24. The straight line is often called *Pascal's line*. This construction can readily be checked by the student, using a circle as the conic. For proof see Appendix II, Proofs VII and IX.

3-21. To construct a conic passing through five given points

Given: Points 1, 2, 3, 4, 5, Fig. 3-25. Consider them to be five vertices of an inscribed hexagon. The opposite sides of the hexagon may be paired as follows:

1-2	and	4-5
2-3	and	5-6
3-4	and	6-1

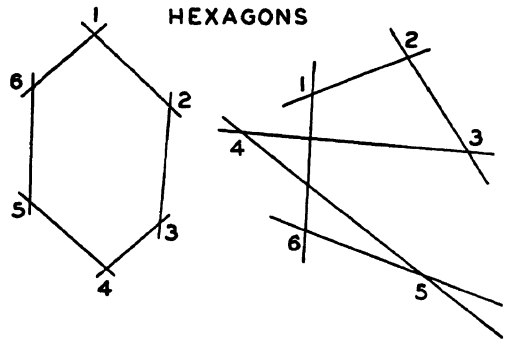

Fig. 3-23.

Draw lines 1-2 and 4-5, the only given pair of opposite sides, producing them to intersect at O.

Draw lines 2-3 and 3-4 as control axes, labeling them X and Y, respectively. Through point O, draw the random Pascal line Z, cutting the control axes in points A and B. From the theorem, the opposite sides 2-3 and 5-6 must intersect on the Pascal line at A and 3-4 and 6-1 at B.

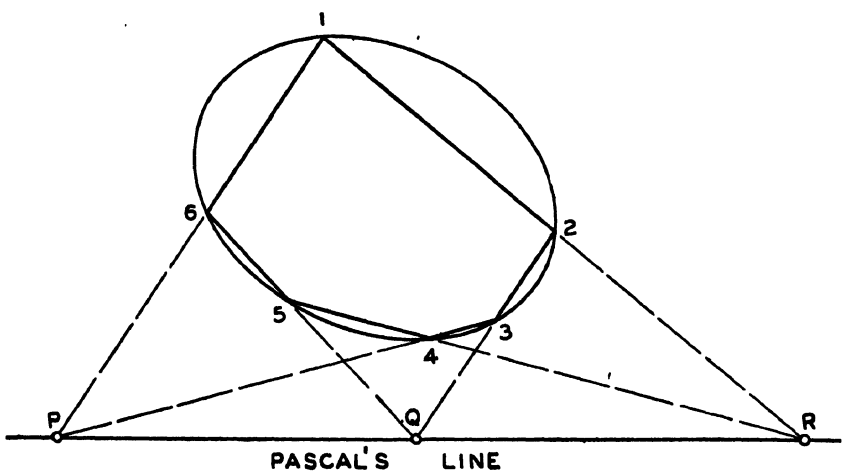

Fig. 3-24.

Draw lines 5-A and 1-B. These lines intersect at point 6. Points 1, 2, 3, 4, 5, 6 form an inscribed hexagon, and point 6 is on the conic. This construction is repeated after moving the Pascal line to a new position.

If point 1 moves along the conic until it coincides with point 2, the chord 1-2 becomes a tangent to the conic at point 2. The conic is now determined

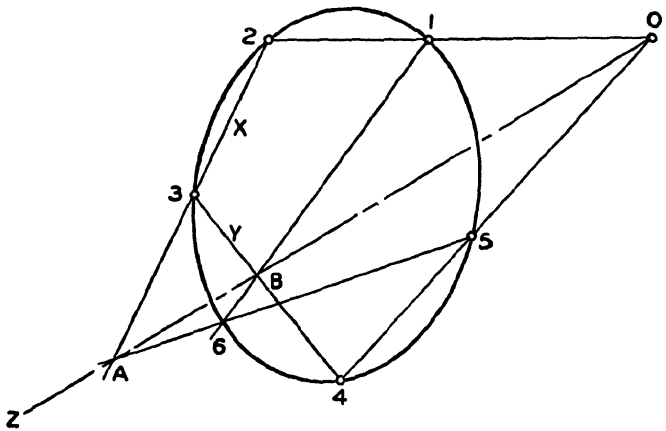

Fig. 3-25.

by three points and one point tangent. Similarly, if point 5 moves until it coincides with point 4, chord 4-5 becomes a tangent to the conic at point 4 and the conic is now determined by one point and two point tangents.

In each case, it is seen that the conic is determined by five elements which may be points or tangents.

3-22. To construct a conic having given one point and two point tangents.

Given: Point 3, tangents T_1 and T_2; points of tangency 2 and 4, Fig. 3-26. Produce tangents T_1 and T_2 to intersect at point O. Draw control axes X and Y, determined by lines 2-3 and 3-4, respectively. Draw one position of the Pascal line, cutting the control axes at points A and B. Draw lines 2-B and 4-A intersecting at point 5, a point on the required conic. Repeat with new positions of the Pascal line.

3-23. Remarks on duality

In plane geometry the point and line are called *dual* elements. Marking a point on a line or passing a line through a point are dual or reciprocal operations. Two statements are dual if one can be obtained from the other by interchanging each element by its dual. For example, if the words "point" and "line" are interchanged in the definition, "A straight line is determined by two points," the result is a second true definition, "A point

is determined by two straight lines"—also, "A triangle is determined by three points not on a straight line," "A triangle is determined by three lines not passing through the same point."

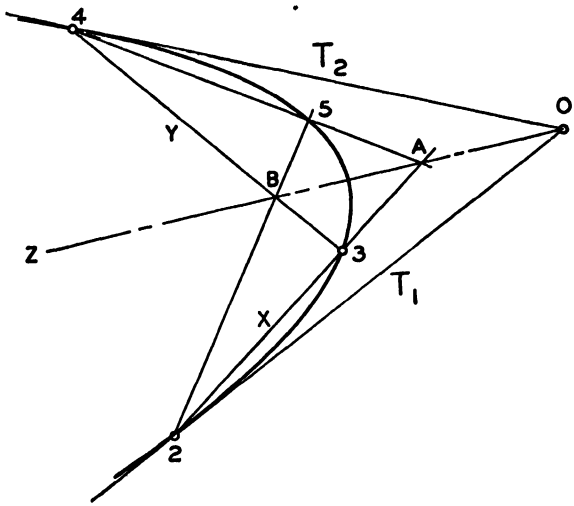

Fig. 3-26.

3-24. Line conics

The concept of duality is important in projective geometry, and the following relationships are of immediate interest:

A tangent to a curve at a point is the dual of the point of tangency. A "point" curve (the set of all of its points) is the dual of a "line" curve (the set of all of its tangents). The dual of a pencil of lines is a range of points.

Thus, the definition discussed in Art. 3-16, *A conic is a set of points consisting of the intersections of the corresponding lines of two projectively related pencils of lines*, has as its dual, *A conic is a set of lines consisting of the lines joining the corresponding points of two projectively related ranges of points.*

If the given ranges, as those of X and Y, Fig. 3-27, are congruent (or similar), the conic is a parabola. A curve determined in this manner is an envelope.

3-25. Parabolic envelope

Let it be required to construct a smooth curve tangent to lines AC and BC at points A and B, Fig. 3-28. Divide AC and BC into any number, say six, of equal parts, numbering the points from A to C and from C to B. Draw lines 0-0, 1-1, 2-2, etc. Draw a fair curve tangent to these lines. Since the point ranges are similar, the curve is a parabola.

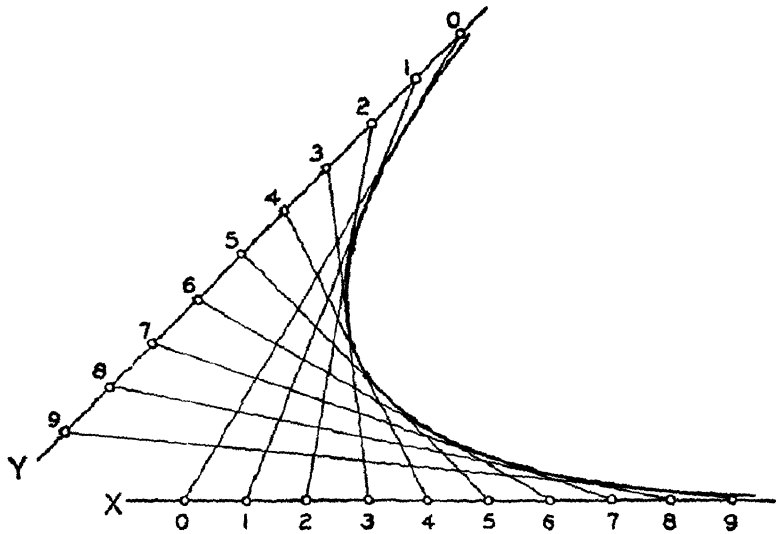

Fig. 3-27.

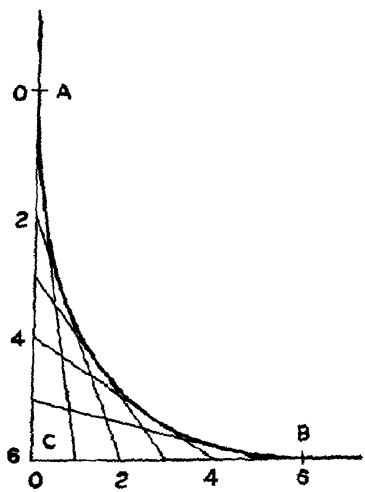

Fig. 3-28.

3-26. Brianchon's theorem

A line conic is determined by five tangents, and when these are given, a sixth tangent can be constructed by use of Brianchon's theorem: *If a hexagon is circumscribed to a conic, the lines joining the opposite pairs of vertices pass through a point* (Fig. 3-29). This theorem on the line conic is analogous to Pascal's theorem on the point conic.

To construct a conic having given tangents 1, 2, 3, 4, 5 and vertices A, B, C, D, Fig. 3-30. Join the opposite vertices A, D. Through point C draw any line cutting AD at point X and line 1 in point F. A line through B and X

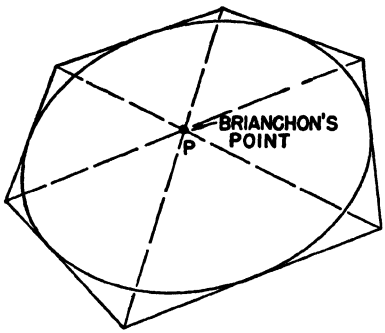

Fig. 3-29.

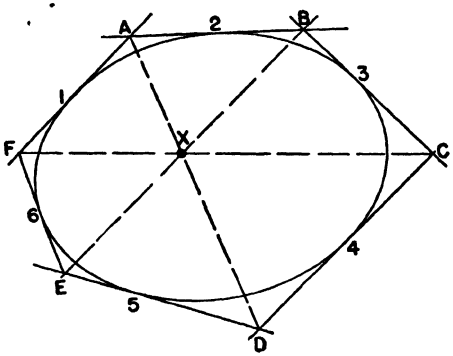

Fig. 3-30.

cuts tangent 5 at point E. The line determined by points E and F is a sixth tangent to the conic.

Other tangents may be found by altering the position of point X on line AD.

3-27. Fair curves

The conic constructions which have been described are useful in laying out molds or templets for structures requiring fair or smooth curves. For example, the fuselage cross section of Fig. 3-31 is made up of conics. Four

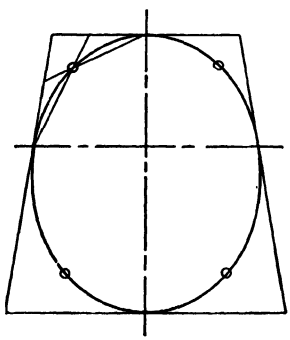

Fig. 3-31.

tangents forming a trapezoid enclose the cross section. The conic which forms any quarter is determined by two point tangents and a control point.

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PROBLEMS

3-1. *Cross ratio.* Use a blank sheet with the long edge horizontal. Locate point A ¼ in. from the top border in the center of the sheet. Draw a vertical line through A. Locate point B 2.50 in. below A. Locate point C 3.75 in. below A. Locate point D 7.50 in. below A. Calculate the cross ratio $AC/CB:AD/DB$.

Take any two random nonsymmetrical points O and O' near the left and right borders of the sheet, respectively. Draw the pencils OA, OB, etc., and $O'A$, $O'B$, etc. Draw two random transversals, one cutting each pencil of lines. Let the intersection with OA be A', with OB be B', etc., and with $O'A$ be A'', $O'B$ be B'', etc. Measure distances to the nearest $\frac{1}{100}$ in., and calculate the cross ratios.

$$\frac{A'C'}{C'B'}:\frac{A'D'}{D'B'}$$

$$\frac{A''C''}{C''B''}:\frac{A''D''}{D''B''}$$

Record these cross ratios with the original cross ratio to two decimal places in some convenient space on the sheet. Mark the points A', B', C', D' on the edge of a strip of paper, and locate the congruent transversal on the pencil of lines from O'.

3-2. *Cross ratio.* Use a blank sheet. Points A, B, C, D, E are five points easily distinguishable on an oblique aerial photograph of flat terrain. $A(\frac{1}{2}, 10)$, $B(5\frac{1}{2}, 10)$, $C(7\frac{1}{2}, 6\frac{1}{2})$, $D(2, 7)$, $E(4, 6)$.

Points A', B', C', D' on an available map are found to represent A, B, C, D. Point E is not represented on the map. It is desired to locate this point as E' on the map. $A'(1, 5)$, $B'(6\frac{1}{2}, 5)$, $C'(5\frac{1}{2}, 1\frac{1}{2})$, $D'(\frac{1}{2}, 2\frac{1}{2})$.

Using A and A', B and B', D and D' as centers for pairs of corresponding pencils of lines, locate E' on three lines. Due to error these lines may not be concurrent but may form a small triangle called the triangle of error.

Assume E' at the center of the triangle of error. Scale the distances $A'E'$ and $D'E'$ to the nearest $\frac{1}{100}$ in., and record them in some convenient space on the sheet.

3-3. *Cross ratio.* Use a blank sheet with the long edge vertical. In the top half of the sheet draw a $4\frac{1}{2}$-in. square $ABCD$. Assume any random point E inside the square. In the bottom half of the sheet, draw any random quadrilateral

$A'B'C'D'$ (letter in the same order as the square). Assuming the quadrilateral to be a projection of the square, find point E' corresponding to point E. Check by using the four pencils of rays from A, B, C, and D.

3-4. *Cross ratio.* Use a blank sheet with the long edge vertical. In the top half draw a circle of 4½ in. diameter. Assume four points A, B, C, D on the circle. In the bottom half of the sheet assume a random quadrilateral A', B', C', D' (lettered in the same order as A, B, C, D) as the projection of A, B, C, D. By assuming points on the circle and finding their projections, develop the conic which is the projection of the circle. Note that pencils containing as many lines as desired may be transferred at once as in Art. 3-17.

3-5. *Pascal's theorem.* Draw the conic tangent to AC at A, tangent to BC at B, and passing through D. Determine the form of the conic. $A(1, 1)$, $B(3, 9)$, $C(7, 5)$, $D(4, 4)$.

3-6. *Pascal's theorem.* Draw the conic from A to E passing through A, B, and C and tangent to DE at E. Obtain at least two points between A and B. $A(1, 2)$, $B(1, 4)$, $C(2, 6)$, $D(2, 8)$, $E(6, 6)$. Draw Pascal lines through point O, the intersection of AB and DE.

3-7. *Pascal's theorem.* Draw the conic tangent at $A(2, 3)$ and $B(5, 3)$ to the vertical lines through A and B and passing through $C(4, 5)$. *Note:* Since the two vertical lines meet at infinity, Pascal lines through their intersection are taken vertical. Take such lines from 1½ to 6½. Complete the resultant ellipse.

3-8. *Brianchon's theorem.* Draw the conic tangent to lines AB, BC, CD, DE, and EF. $A(0, 8)$, $B(4, 8)$, $C(7, 6)$, $D(6, 2)$, $E(3, 2)$, $F(1, 6)$.

3-9. *Pascal's theorem.* Use Problem Sheet 3-9.

3-10. *Conic lofting.* Use Problem Sheet 3-10.

3-11. *Conic lofting.* Use Problem Sheet 3-11.

3-12. *Desargues' theorem.* Given line K established by $D(0, 3)$, $E(5¾, 5)$ and line L established by $F(1, 9½)$, $G(7, 9½)$. Draw a line through $P(2½, 7)$ which shall meet K and L at their inaccessible point of intersection. Use a single straightedge only. Referring to Fig. 3–12, apply Desargues' theorem by choosing a random center of projection, O, near the bottom of the sheet. Construct the pair of projectively related triangles, ABC and $A'B'C'$, such that segments of K and L correspond to CA and $C'A'$.

3-13. *Projective pencils.* Construct two pencils of 10 or 12 lines each which are projectively related. Construct a few conics by the method of Art. 3-16, and find the type of each curve. Note that if congruent pencils are used the curve is a circle. If the angles of the pencils are equal, but in reversed order, the curve is an equilateral hyperbola.

4

ROULETTES, GLISSETTES, AND SPIRALS

4-1. Roulettes

When two curves lie in the same plane and one rolls without sliding on the other, which is assumed fixed, any point connected with the rolling curve describes a curve called a *roulette.*

4-2. To draw a roulette

The best practical construction for any roulette is probably by use of a transparent templet. Let it be required to draw the path of a point P on a circle, center C, as this circle rolls on a fixed circle, center D. Draw the rolling circle on a piece of tracing paper, and mark the point P, Fig. 4-1*a*. Draw the fixed circle on the drawing paper and mark the initial point O. Place the rolling circle on the fixed circle with P at O. Put a needle point through P. Rotate the templet until the rolling circle cuts the fixed circle at some point near O, say point 1, Fig. 4-1*b*. Move the needle point from P, and insert it at point 1. Now rotate the templet until the rolling circle is tangent to the fixed circle, Fig. 4-1*c*. The tracing point P will now have moved from O to the position P'. Mark point P'. This is a point on the roulette. Repeat this operation for additional points. It is evident that

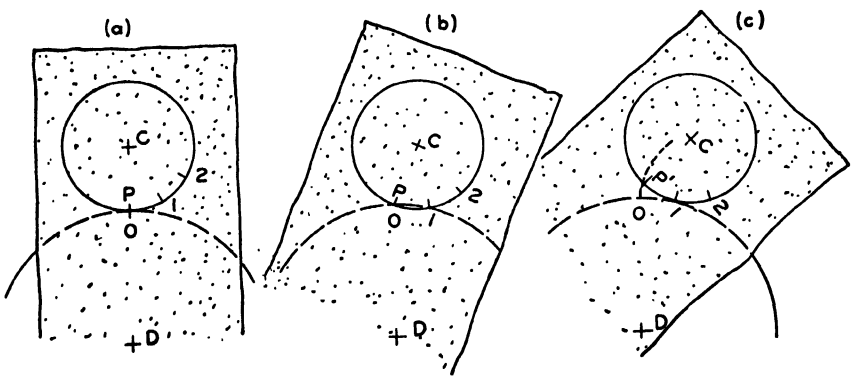


Fig. 4-1.

this method involves an approximation. The length of the arc P-1 of the rolling circle is considered equal to the length of the arc O-1 of the fixed circle.

4-3. The cycloid

When a circle rolls on a straight line, the path described by any point on the circumference of the circle is a *cycloid*. Let the generating circle, center O, roll on the straight line AB, Fig. 4-2. Point P is the generating point. In its initial position it coincides with A.

First method. Find the path of point P by means of a transparent templet as described in the preceding article.

Second method. Lay off AB equal to the circumference of the rolling circle. Through point O draw line OC equal and parallel to AB. Starting at P, divide the circle into any number of equal parts, say 12. Divide

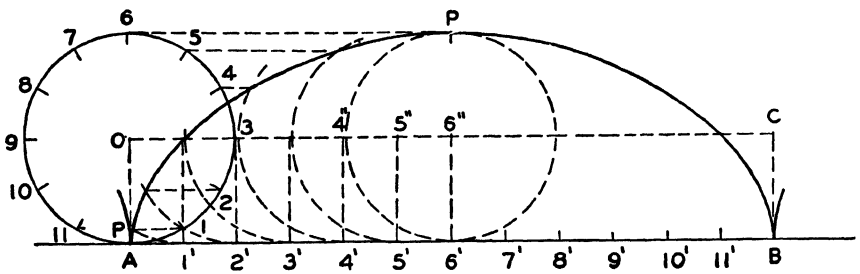


Fig. 4-2.

AB and OC into the same number of parts, determining points 1′, 2′, etc., on AB and 1″, 2″, etc., on OC. Points on the cycloid are found by constructing successive positions of point P as the circle rolls. With OP as radius, centers 1″, 2″, 3″, etc., draw arcs tangent to AB at points 1′, 2′, 3′, etc., intersected by lines drawn parallel to AB through points 1, 2, 3, etc., of the circle. The intersections are points on the cycloid.

Third method. Refer to Fig. 4-3. With centers 1′, 2′, 3′, 4′, etc., determined as in the second method, and radii P-1, P-2, P-3, P-4, etc.,

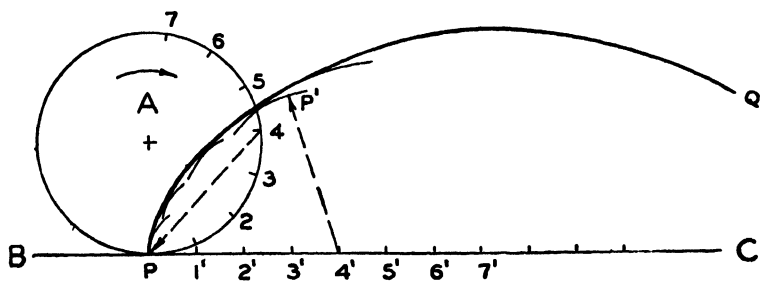


Fig. 4-3.

respectively, draw a series of arcs. Draw the curve tangent to these arcs. This is the cycloid which is the envelope of the arcs.

Each complete revolution of the rolling circle produces one branch of the cycloid. These branches meet in points, where they are tangent to a line at right angles to BC. Such points on a curve are called *cusps*.

4-4. The epicycloid

When a circle rolls on the *outside* of a fixed circle, the path of any point of the circumference of the rolling circle is an *epicycloid*. Let the circle A, Fig. 4-4, roll on the fixed circle CD. The generating point P describes

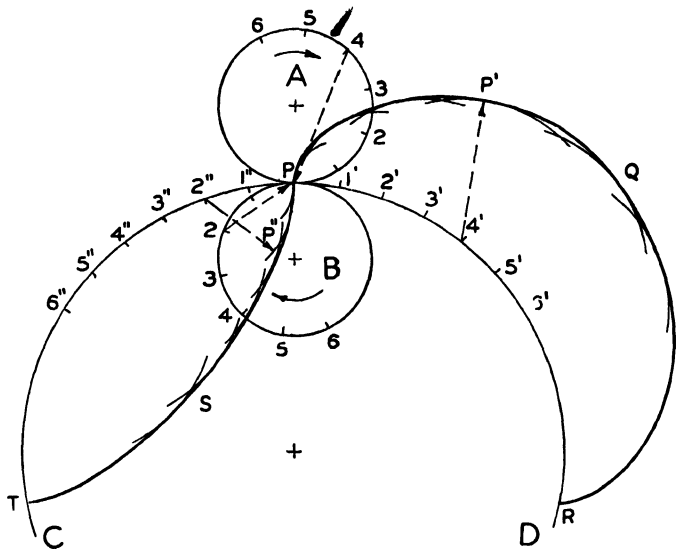


Fig. 4-4.

the epicycloid curve PQR. Points on the curve are found in a manner similar to the cycloid.

First method. Use a templet as described in Art. 4-2.

Second method. Set off equal distances, as P-1, 1-2, 2-3, etc., on circle A. Lay off these same distances as P-1′, 1′-2′, 2′-3′, etc., on CD. With centers 1′, 2′, 3′, etc., and radii P-1, P-2, P-3, etc., describe arcs. Draw the curve tangent to these arcs.

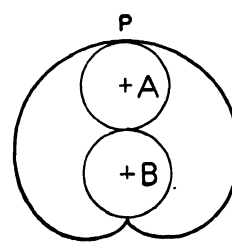


Fig. 4-5.

A special case of the epicycloid occurs when the diameter of the rolling circle A is equal to that of the fixed circle B, Fig. 4-5. The epicycloid determined by point P has but one cusp and is sometimes called the *cardioid*.

4-5. The hypocycloid

When a circle rolls on the *inside* of a fixed circle, the path of any point of the circumference of the rolling circle is a *hypocycloid.* In Fig. 4-4, the circle B rolls on the fixed circular arc CD. The path of point P on circle B is the hypocycloid PST. The construction is similar to that for the epicycloid.

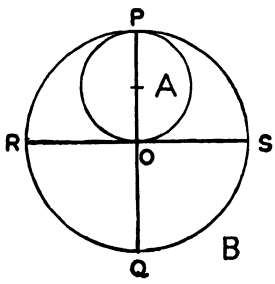


Fig. 4-6.

A special case of the hypocycloid occurs when the diameter of the fixed circle B is twice that of the rolling circle A, Fig. 4-6. In this case the hypocycloid for any point on the circumference of A is a straight line. The path of point P, on A, is the line POQ; the path of O is ROS.

4-6. The trochoid

When a circle rolls on a fixed line, the path described by any point in the plane of the circle but not on its circumference is a *trochoid.* If the generating point is outside the circumference, the curve is called superior or curtate; if the point is within the circumference, the curve is inferior or prolate.

In Fig. 4-7, the circle, radius OA, rolls on line AB. The generating point P describes a superior trochoid. At point S on this curve, two tangents can be drawn. Such a point is called a *double point* or *node.* Point Q describes an inferior trochoid. The constructions described for the cycloid are easily adapted to the trochoid.

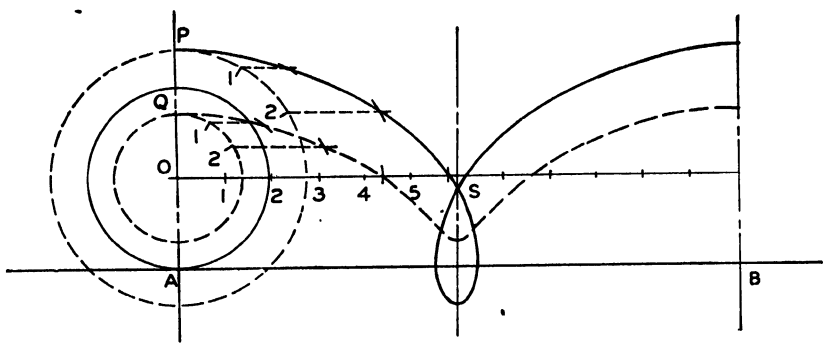


Fig. 4-7.

When a fixed circle is substituted for the fixed straight line, the epitrochoid and hypotrochoid can be generated. An important special case of the hypotrochoid occurs when the fixed circle is twice the diameter of the rolling circle, Fig. 4-8. Points, as P and Q, not on the circumference of the rolling circle, describe ellipses. Then center O, of course, describes a circle.

4-7. The envelope roulette

The curves described thus far are *point* roulettes. They represent the path of a moving point. If the rolling circle carries a fixed line, the curve to which this line is always tangent is called an *envelope* roulette.

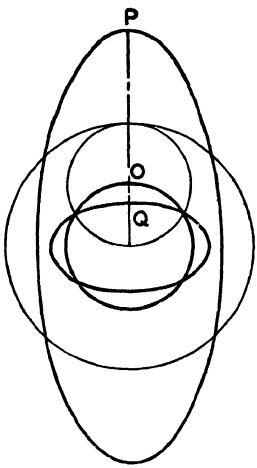


Fig. 4-8.

In Fig. 4-9, the circle, center O, carries a line tangent at point P. Let it be required to find the envelope of this tangent as the circle rolls on line AB.

Through point O draw a line O-6′ parallel to AB and equal to the semicircumference of the circle. Starting at point P, divide the semicircle into any number of equal parts, say six, and divide the line O-6′ into the same number. With radius OP and center 1′ draw an arc. Through point 1 draw a line parallel to AB, intersecting the arc at Q. Draw 1′-Q. The new position of the tangent is perpendicular to 1′-Q. Other positions are found by locating the tangent points R, S, T, U, V.

Draw a smooth curve tangent to the lines just found. There is one cusp on the half branch shown.

4-8. The catenary as a roulette

The shape taken by a completely flexible and inextensible cord or chain suspended at two points is a *catenary*. The catenary can be considered also as the roulette described by the focus F of a parabola which rolls on

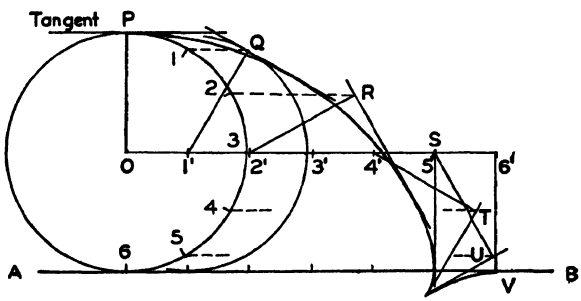


Fig. 4-9.

a fixed straight line AB, Fig. 4-10. This roulette is most easily constructed by means of a transparent templet. It can also be determined by the following geometric method: Starting at point O, the vertex of the parabola, lay off equal spaces on the parabola, as O-1, 1-2, etc. Set

these same spaces off on AB, as O-1′, 1′-2′, etc. With radius 1-F and center 1′, describe an arc; with radius 2-F and center 2′, describe an arc. Continue this operation for the remaining points. Draw the catenary tangent to these arcs.

The catenary, rotated about AB as an axis, generates a surface known as the *catenoid*. This is the minimal surface of revolution. A soap bubble

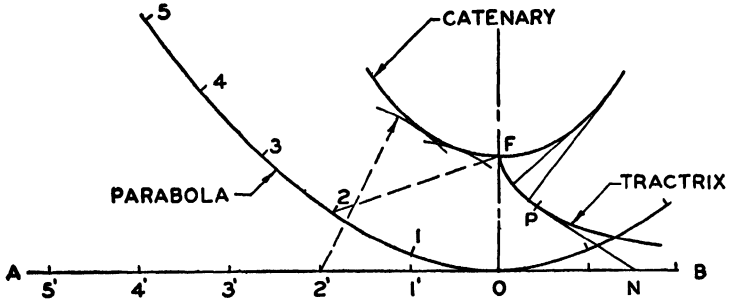
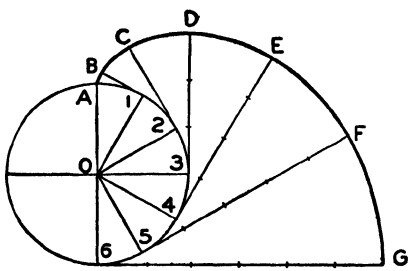


Fig. 4-10.

formed between two parallel wire circles has the form of the catenoid. The surface area is the smallest possible for any single surface connecting the wires.

4-9. The involute

When the rolling element is a straight line and the fixed curve a circle, the path of a point on the straight line describes a curve known as the *involute* of the circle. The form of the curve can be easily visualized by considering it to be the path traced by a point of a taut string as the string unwinds from a circular form.



INVOLUTE OF CIRCLE

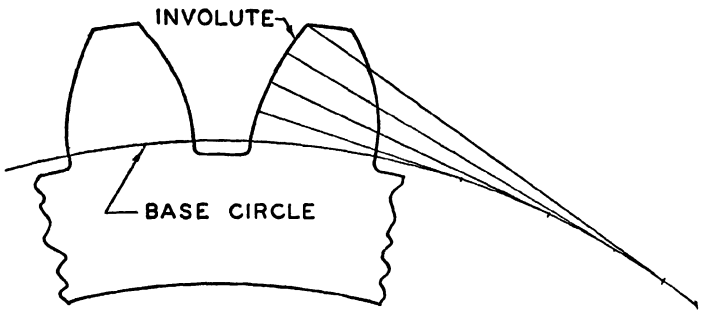
Fig. 4-11.

Let A-3-6, Fig. 4-11, be the given circle. Lay off on the circle equal parts as indicated by the points 1, 2, . . . , 6, and at each point construct a tangent to the circle. On the tangent at point 6 lay off 6-G equal to the length of the arc A-3-6. Divide 6-G into six equal parts. The length of the tangent at point 1 is made equal to one of these parts; the tangent at 2 is equal to two parts, etc. The end points B, C, . . . , G of these tangents lie on the required curve.

When only a small portion of the involute is required, a point of the curve can be found as follows: Starting at point A, set off an equal number of spaces with dividers along arc A-1, passing point 1 by a fraction of

a space. Without lifting the dividers from the drawing, set off the same number of spaces along the tangent at point 1, thus locating point B on the involute. This operation is repeated for each tangent.

To draw a tangent to the involute. A line tangent to the involute at any point is perpendicular to the generating line which passes through that point. The tangent at point F is perpendicular to line F-5.



INVOLUTE GEAR TEETH

Fig. 4-12.

The most important application of the involute of the circle is in the design of gear teeth. The tooth curve is the involute of the base circle of the gear, Fig. 4-12.

4-10. Involute of the catenary

The involute of the catenary is the *tractrix*. This curve is asymptotic to line AB. Also, $PN = FO$, Fig. 4-10, where PN is the portion of any tangent between the point of tangency and AB.

The curve may also be described as follows: If a weight placed at F on a horizontal plane is attached to a horizontal taut string OF, then as the end of the string O is moved along line AB the weight will be pulled along the tractrix. PN is one position of the string. This curve, rotated about AB as an axis, generates the *tractroid*, or *pseudosphere*, a surface connected with non-Euclidean geometry.

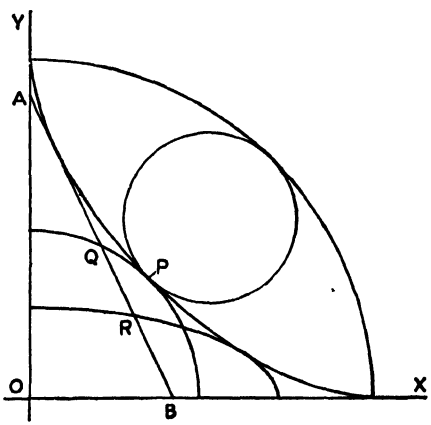


Fig. 4-13.

4-11. Glissettes

When a line, straight or curved, slides between two fixed lines (or fixed points or fixed line and point), a point carried by the moving line

describes a *point glissette*. The envelope of the sliding line is an *envelope glissette*.

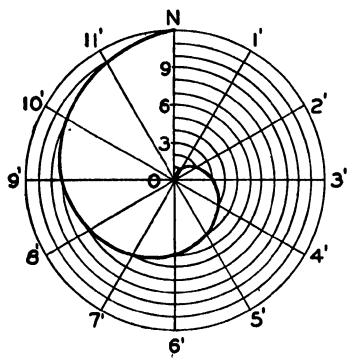
In Fig. 4-13, the line AB slides between the fixed perpendicular lines OX and OY. The mid-point Q of AB describes a circle. Any other intermediate point, R, describes an ellipse. The envelope of AB is a hypocycloid. The diameters of rolling and fixed circles of this curve, regarded as a roulette, are AQ and $2AB$, respectively. Glissettes are easily constructed by the templet method.

4-12. Spiral curves

When a line OA rotates in a plane and about O as a center, the path of a point moving along the line is a spiral curve. The form of the spiral depends on the relation between the movement of the point along OA and the angular motion of OA. The position of a point P of the curve is given by its distance r from O and the angle θ between line OA and a fixed reference line OX. Point O is the pole. The length r is the radius vector, and θ is the vectorial angle.

4-13. The Archimedean spiral

A point moving uniformly along the line OA as the line rotates with uniform angular motion about the center O describes an *Archimedean spiral*. Thus, there are equal increments of r for equal increments of θ. The polar equation of the curve is $r = a\theta$, where a is a constant. When $\theta = 0$, $r = 0$.



SPIRAL OF ARCHIMEDES

Fig. 4-14.

Let point O be the pole and line ON the radius vector of the spiral when θ has reached 360° clockwise, Fig. 4-14. With center O and radius ON draw a circle. Divide the line ON and the circle into the same number of equal parts, as 12. Draw lines O-1′, O-2′, . . . , O-11′. With O as center, draw the arcs through each division point 1, 2, . . . , 11, of line ON to intersect, respectively, the radial lines O-1′, O-2′, . . . , O-11′. A smooth curve ON drawn through the points of intersection represents the spiral.

A plate cam cut to the form of an Archimedean spiral can be used to convert uniform rotary motion into uniform reciprocating motion.

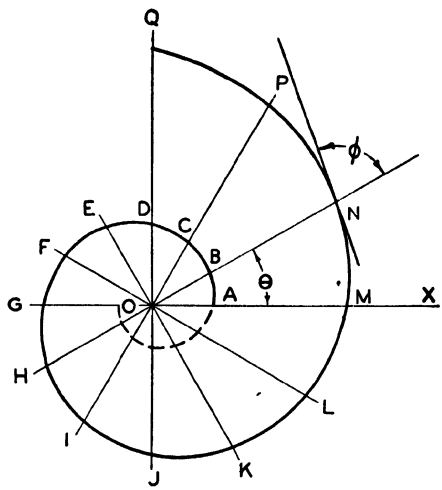
4-14. The logarithmic spiral

Point O, Fig. 4-15, is the pole and OX the initial line of a logarithmic spiral. If the values of the vectorial angle θ increase in arithmetic pro-

gression and the length of the radius increases in geometric progression, the curve traced by the end of the radius vector is a logarithmic spiral.

If the radius vectors OB and OD are drawn to two points of the curve, the radius vector which bisects the angle BOD is a mean proportional between OB and OD. This property can be used to construct the curve geometrically when two of its points and the pole are known.

The general equation of the logarithmic spiral is $r = ab^{\theta}$. Angle θ is measured in radians and a and b are constants. When θ is zero, r is equal to a. As θ increases, r increases and the curve winds around the pole. When θ is negative and increases indefinitely, r approaches zero and the curve approaches the pole as an asymptote. The dash line of Fig. 4-15 represents negative values of θ. A property of the logarithmic spiral is that it intersects the radius vectors at a constant angle ϕ. The angle $\phi = 0.434/\log b$.



LOGARITHMIC SPIRAL

Fig. 4-15.

Points on the curve can be plotted from the equation by writing it in the form $\log r = \log a = \theta \log b$, calculating the values of r which correspond to the assumed values of θ.

The loxodrome, or rhumb line, a curve which cuts the meridians of a sphere at a constant angle, when projected on the plane of the equator of the sphere by rays from one of its poles, is a logarithmic spiral.

REFERENCE FOR READING

LOWE, D. A.: "Practical Geometry and Graphics," Longmans, Green & Co., Inc., New York, 1912.

PROBLEMS

4-1. *Cycloid and involute.* Use Problem Sheet 4-1

4-2. *Trochoid and epicycloid.*

EXAMPLE 1. A wheel 3 ft in diameter is concentrically attached to one 2 ft in diameter. The latter rolls on a rail. Starting with the center at $O(\frac{1}{2}, 7\frac{3}{4})$, draw one cycle of the point on the circumference of the larger wheel directly above O at the start. Scale: 1 in. = 1 ft.

EXAMPLE 2. A $1\frac{1}{2}$-in. wheel rolls around the circumference of a 3-in. wheel centered at $C(4, 3)$. Draw the path of a point on the circumference of the rolling

wheel. Take 15° units on the circumference of the large wheel, which is 30° on the small wheel. Start at $P(4, 4\frac{1}{2})$.

4-3. *Cycloid.* A 2-in.-diameter circle rolls on a straight line and makes one revolution. Draw the curve described by a point on its circumference.

4-4. A 2-in.-diameter circle rolls on the outside of an equilateral triangle $2\frac{3}{4}$ in. on a side. Draw the path of a point on the circumference of the circle. Let the curve start at one vertex of the triangle. Go around the circle once clockwise. Use a templet.

4-5. A $1\frac{1}{2}$-in.-diameter circle, center $O(4\frac{1}{4}, 4)$ is tangent to the vertical line AB at $A(5, 4)$. The length of AB is equal to the circumference of the circle. The circle rolls on AB, making one complete revolution as the line AB rotates about B through an angle of 360°. In each rotation the angular velocity is uniform and clockwise. Draw the path of the point of the circle which has its initial position at A.

4-6. *Trochoid.* A 2-in.-diameter circle rolls on a straight line. Points P and Q lie on a line passing through the center of the circle. P and Q are $1\frac{3}{4}$ in. and $\frac{1}{2}$ in., respectively, from the center of the circle. Draw the paths of P and Q.

4-7. *Epicycloid.* A 2-in.-diameter circle rolls on the outside of a 4-in.-diameter circle. Draw the curve traced by a point on its circumference.

4-8. A 6-in.-diameter circle rolls on the outside of a 4-in.-diameter circle. The 4-in. circle is inside of the 6-in. circle. Draw the curve described by a point on the circumference of the rolling circle.

4-9. *Hypocycloid.* A 2-in.-diameter circle rolls on the inside of a 6-in.-diameter circle. Draw the curve traced by a point on the circumference as the circle makes one revolution.

4-10. *Hypocycloid.* A 4-in.-diameter circle rolls on the inside of a 6-in.-diameter circle. Draw the curve traced by a point on the circumference.

4-11. *Envelope roulette.* A 2-in.-diameter circle rolls on the outside of a 4-in.-diameter circle. The rolling circle carries a tangent. Draw the envelope roulette of this tangent. *Note:* This curve will have two cusps.

4-12. *Involute.* Draw an involute of a 2-in. circle. Center at $O(5\frac{1}{2}, 3\frac{3}{4})$. Start at $P(6\frac{1}{2}, 3\frac{3}{4})$. Draw one revolution rotating clockwise.

4-13. *Catenary.* The vertex of a parabola is 1 in. from the focus. This parabola rolls on a straight line. Draw the curve traced by the focus.

4-14. *Glissette.* Two lines XOX' and YOY' intersect in a 60° angle at O. Point B is the mid-point of the line AC, which is $2\frac{1}{2}$ in. long. Line ABC slides with A on XOX' and B on YOY'. Draw the curve traced by point C.

4-15. *Spirals.* *a.* On the right side of AB draw an Archimedean spiral to pass through points A and B. The pole is at O. Repeat the curve on the left side to form a heart-shaped cam outline. $A(4, 8\frac{3}{4})$, $O(4, 8)$, $B(4, 5\frac{3}{4})$.

b. O is the pole and OX the initial line of a logarithmic spiral. Draw a 360° turn of the curve which passes through points A and B. $A(2, 3\frac{1}{2})$, $O(3\frac{1}{2}, 3\frac{1}{2})$, $B(6\frac{1}{2}, 3\frac{1}{2})$.

5

VECTOR GEOMETRY

5-1. Scalars and vectors

A quantity having magnitude only and capable of having its measure described by a single number is called a scalar quantity. Temperature, volume, mass, etc., are scalar quantities and can be represented graphically on a linear scale.

A quantity which has both magnitude and direction is a *vector* quantity. Force, displacement, velocity, and acceleration are vector quantities. A vector quantity can be represented graphically by a straight line having a length proportional to the magnitude of the quantity and a direction identical with that of the line of action of the quantity. The sense of direction of this line segment is indicated by an arrowhead.

If a given system of vectors lies in a plane, the system is *coplanar*. If the vectors lie in different planes, the system is *noncoplanar*, or three-dimensional.

Graphical solutions by means of vector representation are useful in problems concerned with the resultant effect of a group of forces, velocities, or other vector quantities which may act on a given rigid body.

5-2. Composition of vectors

The geometric addition of two or more vectors yields a single vector which is their equivalent. This vector sum is called the *resultant*. For a coplanar system of vectors the resultant is found graphically either by the parallelogram law or the triangle law.

THE PARALLELOGRAM LAW. If two nonparallel vectors V_1 and V_2 are drawn so that both act either toward or away from their common point O, as shown in Fig. 5-1, the diagonal V_R of the parallelogram constructed on these two vectors as sides is the magnitude and direction of their resultant.

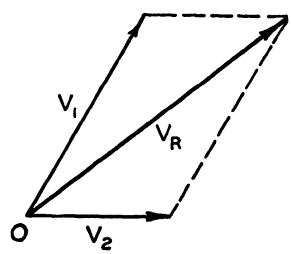


Fig. 5-1.

The resultant of three vectors drawn through a common point can be found by constructing the resultant for any two of them and then com-

bining this resultant with the third vector by means of a second parallelogram. The diagonal of the second parallelogram will represent the resultant of all three given vectors. This process can be extended to find the resultant of any number of nonparallel vectors.

Although the parallelogram law is simple in application, it produces a rather complex diagram in cases where a large number of vectors is involved. The triangle law which follows will be found to be more convenient generally.

THE TRIANGLE LAW. Since a vector is characterized only by direction and magnitude and not by position, it may be moved as desired by translation. Accordingly, if the given vectors V_1 and V_2 of Fig. 5-1 are assembled "head to tail" as in Fig. 5-2, the closing line V_R which completes

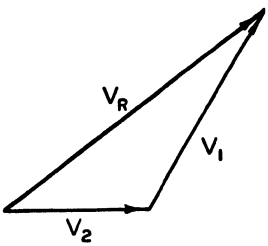
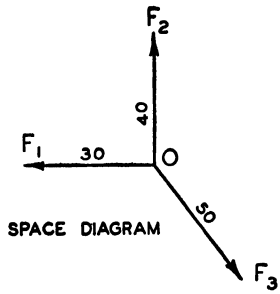


Fig. 5-2.



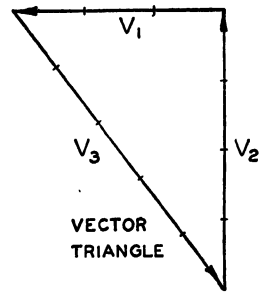


Fig. 5-3.

the triangle is the resultant vector of V_1 and V_2. In this vector triangle it should be noted that the given vectors follow each other in direction around the triangle but that the resultant has a contrary direction. Also, if V_R were replaced by a vector of the same magnitude but opposite in direction, the geometric sum of all three vectors would be zero.

Thus, in Fig. 5-3, if vectors V_1, V_2, and V_3 (drawn in length and direction to represent, respectively, the concurrent forces F_1, F_2, and F_3) form a closed triangle when assembled head to tail, the resultant is of zero magnitude and the three forces are in equilibrium. From the above considerations it may be noted that, if the direction of any one vector of this triangle is reversed, it becomes the resultant of the other two vectors.

5-3. The vector polygon

The triangle law can be extended to find the resultant of three or more vectors. Let the forces F_1, F_2, F_3, and F_4 act at point O, Fig. 5-4(A). If the vectors V_1, V_2, V_3, and V_4, respectively representing these forces, are assembled consecutively, head to tail, the resulting figure is a *vector polygon*, Fig. 5-4(B). It can be shown that the order in which the vectors are placed is immaterial. It is evident that the vector polygon is composed of a series

of vector triangles. Therefore, the vector V_R which closes the polygon represents the magnitude and direction of the resultant of the given forces. Since these forces act at point O, the line of action of their resultant will pass through this same point. The resultant force F_R is therefore drawn through O and parallel to V_R.

This method of finding the resultant of a system of forces (or other vector quantities) is general and will be found to be applicable to nonconcurrent

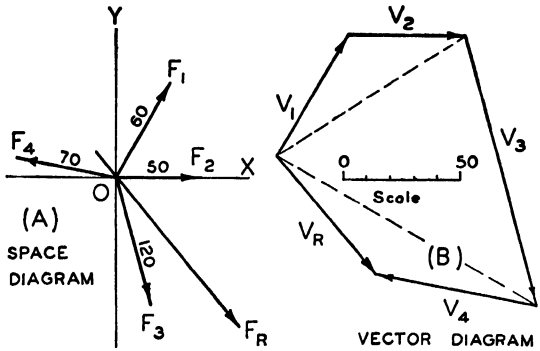


Fig. 5-4.

systems as well. If the forces are parallel, the vector polygon will degenerate into a straight line.

5-4. Resolution of a vector

If two or more vectors are combined to form the resultant R, the vectors are components of R. The operation of finding the resultant of a system of vectors, discussed above, is the *summation*, or *composition*, of vectors. The converse operation of breaking up a given vector into components is the *resolution* of the vector.

A vector can be resolved into any required number of components by constructing a polygon on the vector. The most common case is that of resolving a vector into two components having specified directions. The components can be found by an inverse application of either the parallelogram law or the triangle law.

Vector R, Fig. 5-5, is resolved into components having directions OX and OY by drawing AB and BC parallel to OX and OY, respectively and completing the parallelogram by extending OX and OY. The required vector components are given by OA and OC. If the triangle law is used, a triangle is constructed on R by drawing sides OA and AB in the directions OY and OX, respectively, Fig. 5-6. The magnitudes and directions of the components are represented by vectors OA and AB. The direction of the components, going around the triangle, is opposite to the direction of R.

5-5. The funicular polygon

Assume that the coplanar forces F_1, F_2, and F_3 are applied to a rigid body and that their lines of action do not pass through a common point, Fig. 5-7. It is desired to find (1) the magnitude and direction of the resultant force and (2) the position of the line of action of this resultant on the rigid body.

1. The vector polygon 1-2-3-4, drawn as described in Art. 5-3, gives the magnitude and direction of the resultant V_R.

2. The position of the resultant force F_R in the space diagram is found by the following method:

Assume any point O in the vector diagram, and draw the rays O-1, O-2, O-3, and O-4, joining point O and the vertices of the vector polygon. Now,

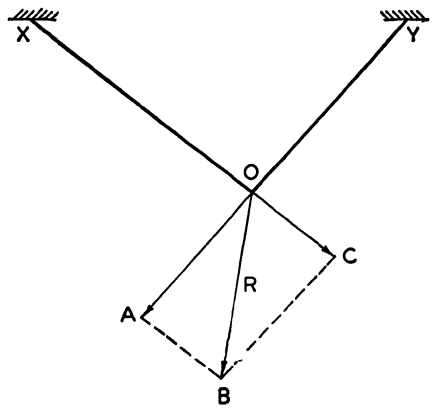


Fig. 5-5.

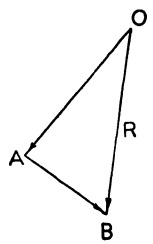


Fig. 5-6.

rays 1-O and O-2 are components of V_1; 2-O and O-3 are components of V_2; 3-O and O-4 are components of V_3. It will be noted that component O-2 of V_1 is equal but opposite in direction to component 2-O of V_2. Likewise, component O-3 of V_2 is equal and opposite to 3-O of V_3.

At any convenient point on F_1 draw lines (called strings) S_1 and S_2 parallel to the rays 1-O and O-2 of the vector diagram. At the point at which string S_2 cuts F_2, draw string S_3 parallel to ray O-3; and where S_3 cuts F_3, draw string S_4 parallel to ray O-4. These strings, considered as force components, replace the forces F_1, F_2, and F_3. The components of F_1 and F_2 which lie along string S_2 are equal in magnitude but opposite in direction. Since they are collinear, their sum is zero. Similarly, the components lying on string S_3 cancel each other. This leaves only the components which lie on string S_1 and S_4. These are the components of the resultant force and intersect at point P, a point in its line of action. *The resultant force F_R is accordingly drawn through point P and parallel to V_R.*

The figure formed by strings S_1, S_2, S_3, and S_4 is the string, or funicular, polygon of the forces F_1, F_2, etc., with respect to point O. The shape of the string polygon obviously will depend on the choice of point O. It can be shown, however, that the strings of the resultant force will intersect at a point on the same line of action irrespective of the position of O.

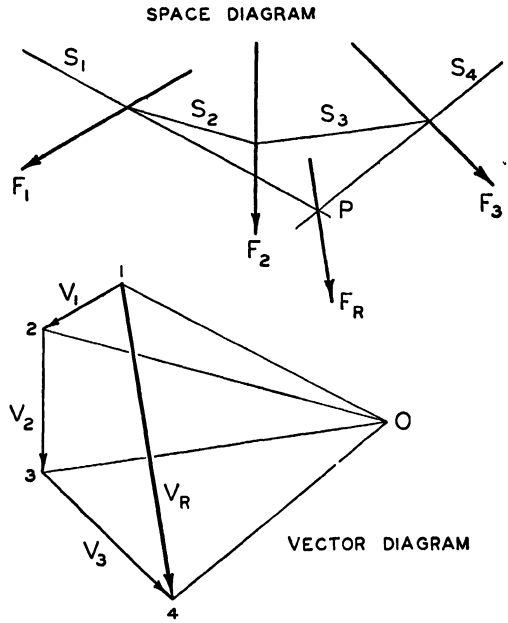


Fig. 5-7.

5-6. Resolution of a vector, using the string polygon

Let the beam AB, Fig. 5-8, be acted on by the force F_O. It is desired to find the magnitude of a vertical force at A and the magnitude and direction of an inclined force at B which will balance F_O and thus establish equilibrium among the three forces. The required forces will be components of F_O.

Produce the action lines of F_O and F_A to intersect at point C. The line of action of the force at B must pass through C. Draw CB to obtain the line of action of the force at B. Start the vector triangle A by drawing V_O to represent F_O. Resolve V_O into the components V_A and V_B parallel to AC and BC, respectively. Vectors V_A and V_B represent the magnitudes and directions, respectively, of the required forces F_A and F_B.

In cases where the force F_O is vertical or point C is inaccessible, a solution can be obtained by use of the string polygon.

Start the construction of vector triangle B by drawing vector V_O to represent force F_O. Also, draw line 1-K as shown, parallel to the line of

action of F_A. Assume point O, and draw the rays 1-O and O-2. Use point B of the beam as the starting point for a string polygon, since B is the only known point on the line of action of F_B. Through B, draw string S_2 parallel to ray O-2. At the point where S_2 cuts F_O, draw S_1 parallel to 1-O. Join

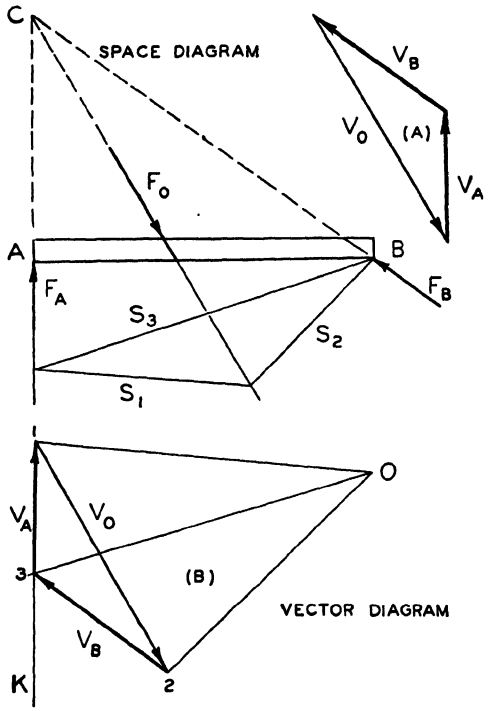


Fig. 5-8.

point where S_1 cuts F_A with the starting point B. This is string S_3, which completes the polygon. Now through O in the vector triangle, draw a ray parallel to S_3, cutting line 1-K at point 3. Then lines 3-1 and 2-3 complete the triangle and represent the required components V_A and V_B, respectively.

5-7. Relative motion

When two bodies, A and B, are moving in a given reference system, the velocity of B with respect to an observer on A is called the relative velocity of B. This is a vector quantity. Let the velocities and directions of movement of bodies starting at A and B be represented by the vectors V_A and V_B, respectively, Fig. 5-9. To find the relative velocity of B with respect to A, apply to each body a velocity vector $-V_A$ equal to that of A, but opposite in direction. This has the effect of bringing A to rest without changing the relative velocity between the two bodies. The resultant V_R, of vectors $-V_A$ and V_B, represents the relative velocity of B with respect to A.

EXAMPLE. Ship A, Fig. 5-10, is steering a course 042° true at 20 knots. This is represented by the vector AC. Ship B, 8 miles from A, bearing 057° to the starboard from A, is steering a course 330° true at 17 knots, represented by BD. An observer on A will observe the relative motion of B with respect to A, which will not be in the direction BD. This relative motion is best determined by assuming A at rest.

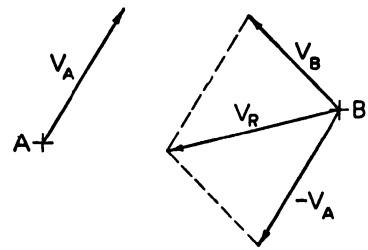


Fig. 5-9.

A is brought to rest by assuming a head current equal and opposite to the motion of A, in this case 20 knots. Since this current must be applied to B also, the vector BE is drawn equal and opposite to AC. The resultant of BD and BE, found by the parallelogram law, is represented by BF. Thus, the relative velocity of B is measured as 22 knots, and the apparent direction as 273° true.

If the apparent track BF is produced to cross the track of A, it is seen that ship B will pass astern of A. Ship B will be nearest A when it reaches

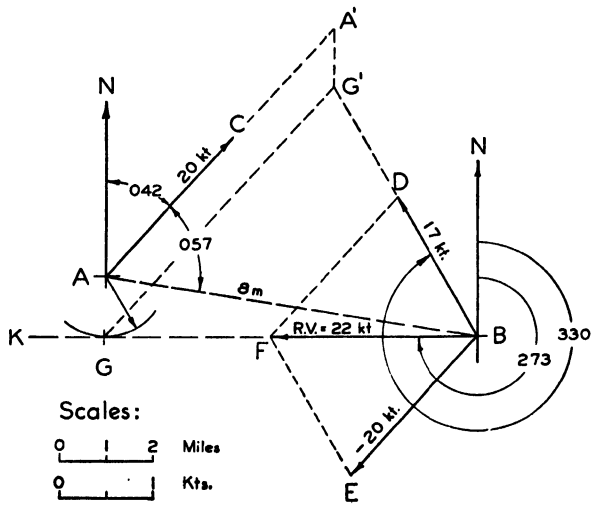


Fig. 5-10.

point G. The distance away is measured by the perpendicular AG. The time required for B to reach G is equal to the distance BG divided by the apparent velocity of B. If the track BG passes through A, the ships will collide. The true positions of the ships at the point of nearest approach will be at A' and G'.

The distance and velocity scales are independent.

REFERENCE FOR READING

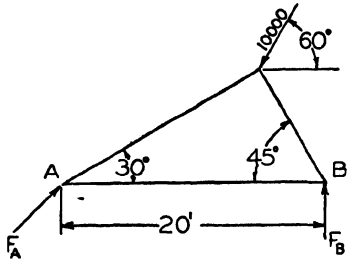
LOWE, D. A.: "Practical Geometry and Graphics," Longmans, Green & Co., Inc., New York, 1912.

PROBLEMS

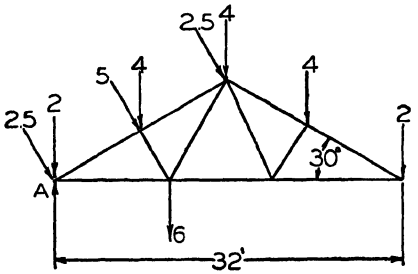
5-1. *Vector polygon.* Use Problem Sheet 5-1.

5-2. *Funicular polygon.* A horizontal beam AB is 14 ft long. It is acted on by the following vertical forces: 2,000 lb, 1 ft from A; 1,500 lb, 4 ft from A; 1,750 lb, 8 ft from A; 2,500 lb, 13 ft from A. Find the resultant force and its position. Space diagram: ½ in. = 1 ft. Vector diagram: 1 in. = 1,000 lb. Locate A at 1, 6½.

5-3. *Funicular polygon.* The structure in the figure is in equilibrium. Find the components F_A and F_B. F_B is vertical. Use the funicular-polygon method, and check with the parallelogram method. Space diagram: ¼ in. = 1 ft. Vector diagram: 1 in. = 2,000 lb. Locate A at 1, 1½.



Prob. 5-3.



Prob. 5-4.

5-4. *Funicular polygon.* Use an 11- by 17-in. sheet. The forces in the figure are in thousands of pounds. Find the resultant force and its position. Find the reactions if the left reaction is vertical. Space diagram: ¼ in. = 1 ft. Vector diagram: ¼ in. = 1,000 lb. Locate A at ½, 6¼.

5-5. *Relative motion.* Ship A is steaming on a course 023° true at 20 knots. Ship B, 6½ miles on the starboard bow, bearing 045° from the bow, is following a course 335° true at 15 knots. Find (*a*) the relative velocity of B; (*b*) the point of nearest approach to A; (*c*) the time to make this distance good. Scales: 1 in. = 5 knots; 1 in. = nautical mile. Courses given are true. Locate point A at (1, 1¼).

5-6. *Relative motion.* A carrier is steaming on a course 028° true at 18 knots. A destroyer keeping station 5 miles on the starboard beam of the carrier is ordered to take station 3 miles astern of the carrier, using a speed of 20 knots. Find (*a*) the destroyer's true course; (*b*) the distance to be made good; (*c*) the time to get to the new station. Scales: 1 in. = 5 knots; 1 in. = 1 nautical mile. Courses are true. Locate point A at (2, 6).

5-7. *Relative motion.* A liner at A(1, 3) is steaming at 20 knots on a course of 020° true. A power launch leaving a dock 6 miles due east of the liner at 25 knots

wishes to deliver a passenger to the liner. (*a*) What direction shall it take? (*b*) What distance will it go, and how long will it take to reach the liner? Scales: 1 in. = 10 knots; 1 in. = 1 mile.

5-8. *Relative motion.* A liner at $A(7, 2)$ is steaming a course 330° true at 20 knots. A launch at $B(1, 3)$ wishes to catch the liner. Distance scale: 1 in. = 2 miles. (*a*) What is the minimum speed the launch must make to catch the liner at all? (*b*) If the launch is capable of 16 knots, what course should it take to go the minimum distance and how much gasoline will it need for the round trip if it gets 1 mile per gal? 1 in. = 10 knots.

6

GRAPHICAL SCALES

6-1. Scales

The graduated scale is one of the most familiar devices of everyday life. A scale on a ruler measures length in terms of some standard unit; on a gauge dial, it may measure pressure; on a clock, it measures the passage of time. The fundamental purpose of a scale is to convert the continuous graphical character of nature into the discontinuous step symbolism of numbers. The graduations in terms of some unit determine the size of the steps. We are concerned here chiefly with using scales to show the relationship between two (or more) quantities that vary according to some physical law.

A scale can be defined as a series of marks or graduations laid off in some systematic manner along a line or scale axis which may be either straight or curved. When the spaces which represent units are equal, the scale is *uniform*. When the spaces which represent units vary in length, the scale is *nonuniform*.

6-2. Scale modulus

Consider the straight scale shown in Fig. 6-1. The scale is uniform, and the length of the space between units is equal to 0.50 in. This unit length is the *scale factor*, or *modulus*, and is usually denoted by M. If the zero

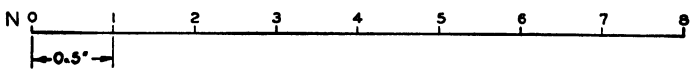


Fig. 6-1.

point of this scale is taken as an origin and the *distance* from 0 to the calibration mark for any number N is S, then $S = 0.50N$. This is the *equation* of the scale.

In general, if $f(u)$ is a function of a variable u and if $f(u)$ has a single value for each value of u, the function can be represented by a graphical scale having for its equation

$$S = Mf(u)$$

The modulus M can be any convenient length where S is the distance along the scale to the calibration marked with the corresponding value of u.

If the equation is linear (first degree), the scale will be uniform. If the equation involves powers of u other than 1, or trigonometric or logarithmic functions of u, the scale will be nonuniform.

If the range of the scale is from $u = a$ to $u = b$, the total length of the scale is $L = M[f(b) - f(a)]$.

EXAMPLE 1. *Uniform scale.* Construct a scale for $f(u) = u$ for values of u between 0 and 16. The length of the scale is to be 4 in. The length of the scale modulus $M = 4$ in./16 $= 0.25$ in.

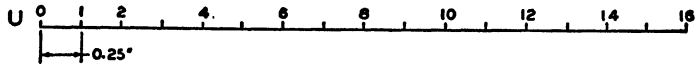


Fig. 6-2.

The *scale equation* is $S = 0.25u$. The required scale is uniform and is represented by Fig. 6-2. Each unit of u can be subdivided as finely as desired.

EXAMPLE 2. *Nonuniform scale.* Construct a scale for the function $f(u) = 0.50u^2$ for values of u between 0 and 10. Let $M = 0.20$. The scale equation is $S = 0.20(0.50u^2) = 0.10u^2$. The length of the scale will be $0.10 \times 10^2 = 10$ in. Prepare a table of corresponding values of S and u.

u	1	2	4	6	8	10
u^2	1	4	16	36	64	100
$S = 0.10u^2$	0.1	0.4	1.6	3.6	6.4	10.0

The values of u are plotted according to the distances given by S (Fig. 6-3).

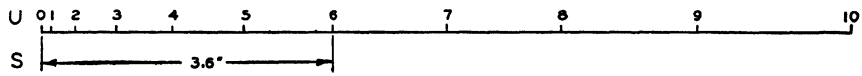


Fig. 6-3.

If one thinks of a particle moving along the scale axis according to the law $S = \frac{1}{2}gT^2$, its displacement S from O at any time T is given by the scale mark for T. Thus, the scale may be viewed as a graph of the motion.

EXAMPLE 3. *Logarithmic scale.* Construct a scale for $f(u) = \log u$ for values of u between 1 and 10 (one cycle). Make the length of one cycle equal to 5 in. Hence, $M = 5$, and the *scale equation* is $S = 5.0 \log u$. The following table is prepared, and these values are plotted as in Fig. 6-4.

u	1	2	4	6	8	10
$\log u$	0	0.301	0.602	0.778	0.903	1
$5 \log u$	0	1.50	3.01	3.89	4.51	5

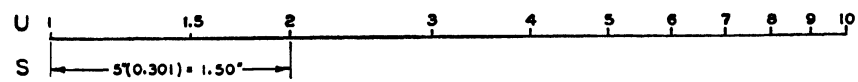


Fig. 6-4.

The numbers from 10^n to 10^{n+1} constitute one logarithmic cycle. Thus, every cycle of a logarithmic scale has the same length.

6-3. Expanding or contracting the scale modulus

A scale may be designed by assuming a convenient length as the modulus and allowing the total length of the scale to be determined by the range of the variable. In most cases the length of the scale for a given range is fixed in advance. In such a case the modulus is selected to fit the condition. The modulus should be a simple number if possible.

The most widely used scales are the uniform and the logarithmic. Since these are generally available on standard scales and slide rules, it is often

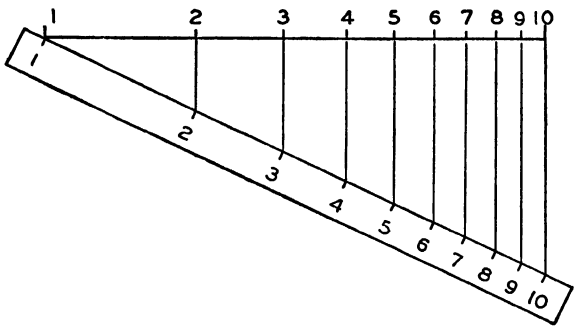


Fig. 6-5.

convenient to lay out a new scale of specified modulus or length by projecting the graduations from the standard scale by the method of proportional division (Art. 1-23) as shown in Fig. 6-5.

If a given nonuniform scale is frequently used, it is worth while to make a special scale chart as shown in Fig. 6-6. A base line is drawn perpendicular to a given scale, AB. The nonuniform scale AB is logarithmic in this illustration and was taken from a slide rule. Rays are drawn from O on the base line to the logarithmic divisions. Any line, as CD, drawn parallel to AB will be divided by the rays in the same proportion as AB. Thus, a scale of any length less than AB can be taken from this diagram by folding it along a line parallel to the logarithmic scale or by drawing a line on the diagram to represent the required scale and pricking points through to the new scale.

A remark on the language of geometry may be useful here. If point O, Fig. 6-6, is considered a center of projection (see Art. 3-9), the scale CD can be described as a projection of the scale AB on the line CD. Since AB and CD are parallel, the scales are similar and the projection is a *similarity transformation*. The modulus, M_1, of AB is thus transformed into a modulus of a different size, M_2, on CD.

6-4. Adjacent scales

A relation between two varying quantities, as u and v, can be represented graphically by means of two adjacent scales. A scale for each variable is

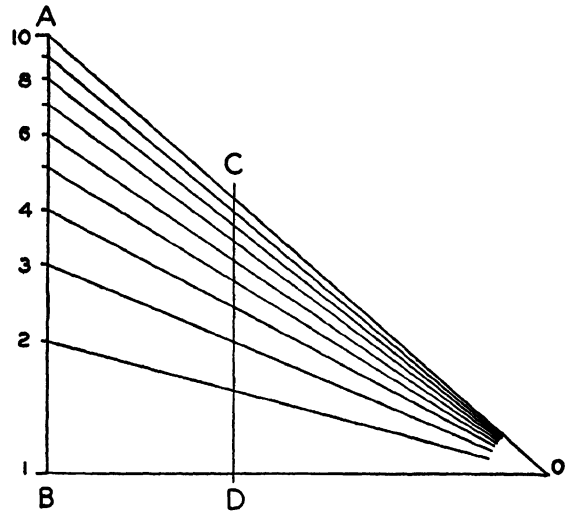


Fig. 6-6.

constructed, using the same modulus and starting each scale at corresponding values.

EXAMPLE. Construct two adjacent scales to represent the relationship between the readings of the centigrade and Fahrenheit thermometers stated by the equation: $1.8C = F-32$. The range is to be 0°-100°C. Assume a scale modulus of 0.05 in. The equation of the centigrade scale is

$$S_C = 0.05(1.8C) = 0.09C$$

The equation of the Fahrenheit scale is

$$S_F = 0.05(F-32)$$

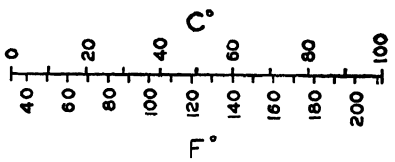


Fig. 6-7.

These are uniform scales and are plotted in Fig. 6-7. Obviously the conversion from one system to the other can be immediately performed by the use of these scales.

If the length of the scale is fixed in advance instead of the modulus, the graduations can be obtained by the method shown in Fig. 6-5.

6-5. Sliding scales

UNIFORM SCALES. Graphical addition can be performed on two adjacent uniform scales having the same modulus by sliding one along the other as in Fig. 6-8, where three units of scale B are added linearly to four units

of scale A. The scale index (zero) of A is placed over number 3 on B. Below number 4 on A, the sum, 7, is read on B. Thus, the operation $u + v = w$ is performed. Subtraction can also be carried out by means of two uniform scales.

LOGARITHMIC SCALES. The above seemingly trivial operation becomes important when logarithmic scales are used in place of uniform scales.

The product of two numbers, $u \times v = w$, is written as $\log u + \log v = \log w$. If logarithmic scales having the equations $S_u = M \log u$, $S_v = M \log v$ are chosen, we may add the distances to corresponding calibration

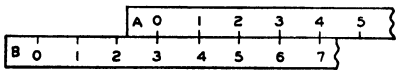


Fig. 6-8.

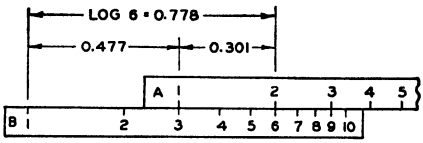


Fig. 6-9.

marks for values of u and v on the two scales to obtain a distance $S_u + S_v$ or $\log u + \log v$ equal to $\log w$ at a calibration mark reading the value of w. Hence, multiplication can be performed as addition by use of two logarithmic scales. The logarithmic scales A and B, Fig. 6-9, must have the same modulus. To multiply 3 by 2, place the index (1) of A over number 3 of B. Below number 2 of A the product, 6, is read on B. The length of the scale segment on B between its index and the number 3 equals 0.477 unit, the logarithm of 3. The length of the segment on A between its index and the number 2 is equal to 0.301, the logarithm of 2. The sum of these two segments is equal to 0.778 unit, the logarithm of 6, which corresponds to the distance on scale B between the index and number 6.

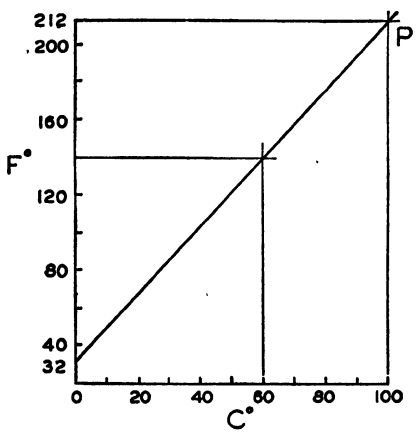


Fig. 6-10.

It is evident that division also can be performed by two sliding logarithmic scales. The slide rule employs this principle.

6-6. Perpendicular scales

The relation between the two variable quantities, centigrade and Fahrenheit degrees, described in Art. 6-4, can also be represented by the use of two perpendicular scales, Fig. 6-10. The uniform centigrade and Fahrenheit scales placed at right angles intersect in the figure where $C = 0°$ and $F = 20°$. If perpendiculars are drawn to each scale at each

pair of points which represent equal temperatures, as 100°C and 212°F, the intersection of each pair of perpendiculars will determine a point, as P, and these points will lie on a line that is the graph of the relationship. In this example, since the scales are uniform and the function is linear, the graph is a straight line. In a plot of this type it is not necessary that the scale moduli be equal. This is a method of representation that leads to important results.

6-7. Coordinates of a point

Point O, Fig. 6-11, is assumed as an origin, and the coordinate axes OX and OY are drawn at right angles, as shown. Each axis bears a scale. The position of any point P of the plane is given by the measurements x and y. The value x is plotted on OX, and a perpendicular to OX is drawn at this point. The value of y is plotted on OY, and a perpendicular to OY is drawn at this point. These two perpendiculars meet in point P. Thus, a pair of values locate one point of the plane, and, conversely, each point of the plane corresponds to one pair of values. The distance x is called the *abscissa* and the distance y the *ordinate* of the point. The distances, written as (x, y) and always in that order, are the *rectangular*, or *cartesian, coordinates* of P. The abscissa x is positive in sign when measured to the right of the origin, negative when to the left. The ordinate y is positive when measured above the origin, negative when below. The scale factors, or moduli, are independent and may be equal or unequal.

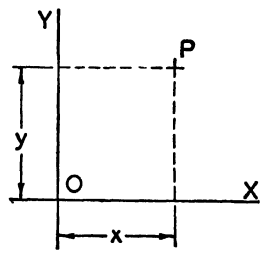


Fig. 6-11.

UNIFORM GRAPH PAPER. Rectangular coordinates are very commonly used. When the coordinates of a point are referred to without further description, it is understood that the coordinate system is rectangular. Furthermore, standard graph paper carrying uniform divisions on both scale axes is manufactured for this purpose. It is suitable for the majority of engineering graphs.

6-8. History of rectangular coordinates

Coordinate, or analytic, geometry was invented by René Descartes (1596–1650), a French mathematician and philosopher. Descartes started his career as a soldier with the French Army in Holland. Leaving the Army in 1621 to pursue his mathematical interests, he settled in Holland, where, in 1637, he wrote his famous "Discourse on the Method of Good Reasoning and of Seeking Truth in Science." This essentially philosophical treatise contains, in the appendix, a method of relating algebra and

geometry by means of a coordinate system consisting of two intersecting lines, since called the cartesian axes.

The use of coordinates was not entirely new. The location of a point on the earth by its longitude and latitude was in use. Also, relationships between geometry and algebra had been known since the time of Euclid, and the Greeks had a graphic solution for a quadratic equation. Descartes, however, established a systematic procedure by which an equation was shown to be the equivalent of the locus of the points of a geometric curve having coordinates which satisfied the equation. His interest was in interpreting algebra by means of geometry, rather than the converse.

6-9. Representing a relation between two variables by perpendicular uniform scales

To represent the equation $v = f(u)$, a table of corresponding values of u and v is made. The perpendicular uniform scales OX and OY are drawn to accommodate the desired range of the variables. Point O is the origin of each scale. These scales should have approximately the same total length. Pairs of corresponding values of u and v are plotted as in Fig. 6-11. A smooth curve through point P and other points similarly determined represents the graph of $v = f(u)$. Note that the scales are arbitrarily uniform, and thus the function is not represented by a scale as in Art. 6-2 but by the curve through the points taken in relationship to the perpendicular scales.

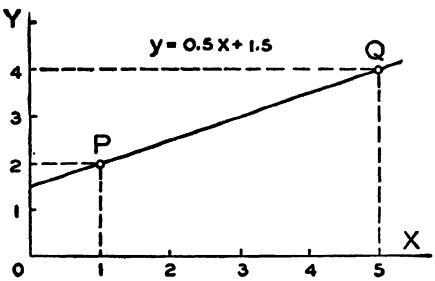


Fig. 6-12.

EXAMPLE 1. Construct a graph for the linear equation $y = 0.5x + 1.5$. The graph of a linear equation is a straight line. Its graph is drawn by plotting two points, as $P(1, 2)$ and $Q(5, 4)$.

The scale equation for OX is $S_x = M_x x$. Let $M_x = \frac{1}{2}$ in.

The scale equation for OY is $S_y = M_y y$. Let $M_y = \frac{3}{4}$ in.

The plot of P and Q yields the graph of Fig. 6-12.

EXAMPLE 2. Construct a graph using uniform scales for the equation $y = 0.125x^2$. Corresponding values of x and y are as follows:

x	0	1	2	4	6	8	10
y	0	0.125	0.5	2	4.5	8	12.5

The scale equation for OX is $S_x = M_x x$. $M_x = 0.25$ in.
The scale equation for OY is $S_y = M_y y$. $M_y = 0.25$ in.
The x, y plot yields the graph of Fig. 6-13.

EXAMPLE 3. Construct a graph using uniform scales for the equation $y = 12x^{-0.7}$. When the exponent is a fraction, corresponding values of x and y are more readily determined if the equation is written in loga-

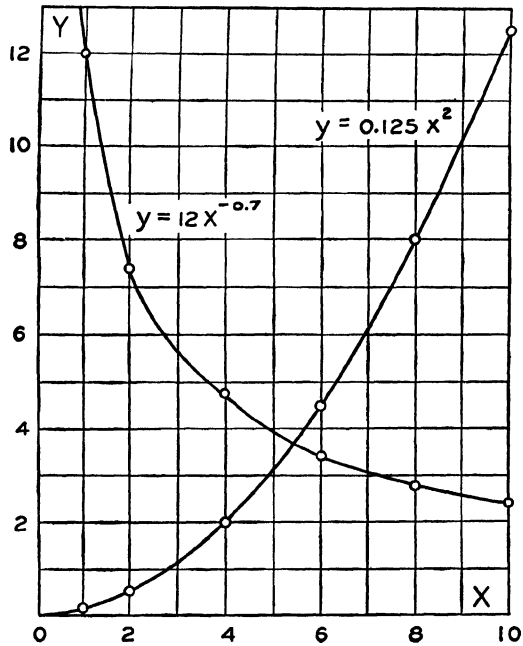


Fig. 6-13.

rithmic form as $\log y = \log 12 - 0.7 \log x$. Corresponding values of x and y are as follows:

x	0	1	2	4	6	8	10
$\log x$	0	0	0.301	0.602	0.778	0.903	1.00
$\log y$	..	1.079	0.868	0.677	0.534	0.447	0.379
y	..	12	7.38	4.75	3.42	2.80	2.39

The scale equations are as in Example 2, and the plot is also shown in Fig. 6-13.

6-10. Effect of scale transformations on a graph

A study of the graph of $y = 0.125x^2$, Fig. 6-13, will suggest that the nature of the scales used on the rectangular axes has some influence on the shape of the graph. In this example if the modulus of the X scale were one-sixteenth of that used for the Y scale, the graph would lie close to the Y axis. If nonuniform scales were used on one or both axes the graph would assume a still different shape. It is conceivable, in certain cases

at least, that by properly designing the scales one might represent any equation in two variables by a straight-line graph. If this could be done, it would simplify the construction of the graph since only two points need be plotted to determine a straight line, which is much easier to draw than a curve. The operation of transforming a curve into a straight line is called *rectifying* the curve.

Consider the equation $v = f(u)$, where $f(u)$ is an algebraic function, as u^2, u^3, $1/u$, etc. If we let $x = f(u)$ and $y = v$ and substitute these values

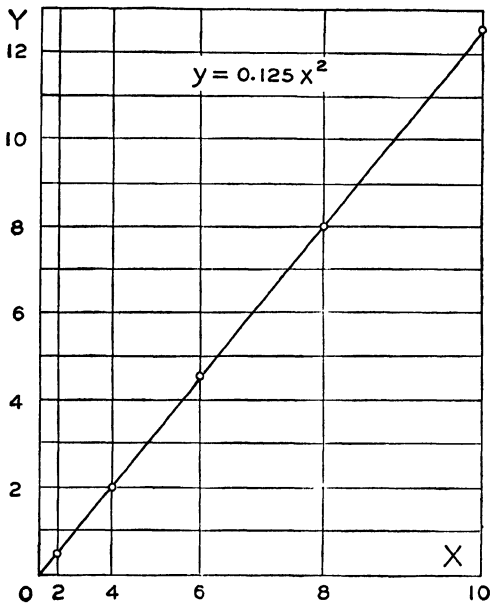


Fig. 6-14.

in $v = f(u)$, we have $y = x$. This is a first-degree equation, and its plot is a straight line. Therefore, to plot $v = f(u)$ as a straight line in terms of u and v, the scale equation for the U axis will be $S_u = M_u f(u)$. The scale for the V axis will be $S_v = M_v v$.

EXAMPLE. Construct functional scales that will yield a straight-line graph for the equation of Example 2, Art. 6-9, $y = 0.125x^2$. Corresponding values of x and y are given in that example.

The scale equation for OX is $S_x = 0.125\, M_x x^2$. Let $M_x = 0.216$.

The scale equation for OY is $S_y = M_y y$. Let $M_y = 0.25$.

The plot of x and y yields the graph of Fig. 6-14.

6-11. Perpendicular logarithmic scales

Certain forms of equations are capable of rectification by the use of logarithmic scales.

Consider the equations $v = au^m$, where a and m are any constants. Taking the logarithms of each member of this equation, we have

$$\log v = \log a + m \log u \tag{1}$$

If we let $x = \log u$ and $y = \log v$ and substitute in (1),

$$y = \log a + mx \tag{2}$$

This is a first-degree equation, and its graph is a straight line. Consequently if we use the scale equations yielding logarithmic scales, $S_v = M \log v$ and $S_u = M \log u$, we shall obtain a straight-line plot.

EXAMPLE. Construct a logarithmic graph for $y = 0.125x^2$ for the same range shown in the example of Art. 6-10. Taking the logarithms of each member of this equation, we have

$$\log y = \log 0.125 + 2 \log x$$

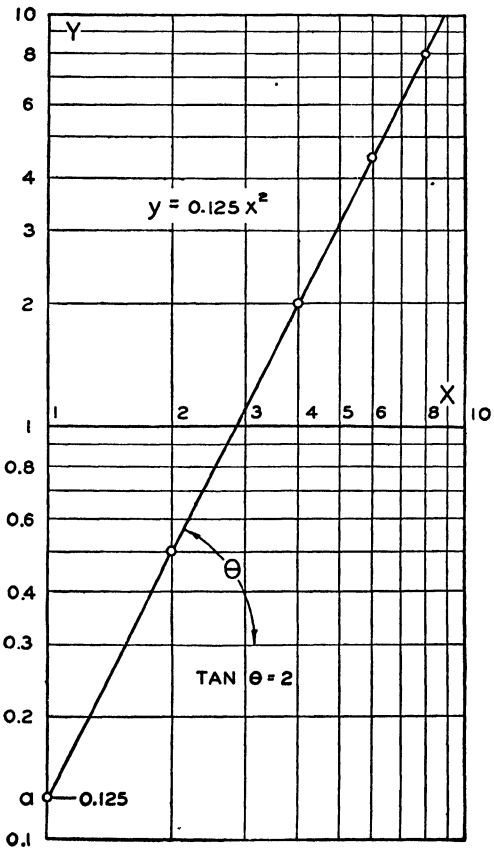


Fig 6-15.

The scale equations are

$$S_x = M \log x$$
$$S_y = M \log y$$

The same value of M is used on both. The plot of x and y yields a straight line, Fig. 6-15.

6-12. Logarithmic graph paper

A special type of graph paper which carries logarithmic scales is convenient when logarithmic plots such as the preceding example are desired. The X and Y axes of a logarithmic plotting sheet intersect in point (1, 1),

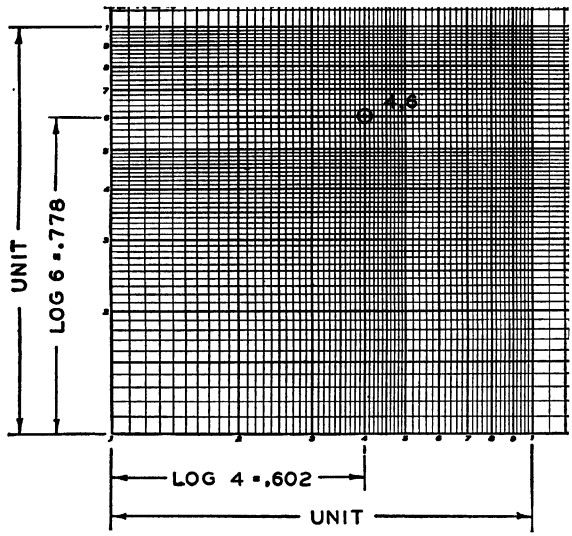


Fig. 6-16.

Fig. 6-16. A unit of length M is laid off on each axis. This represents $\log 10 - \log 1 = 1$. Hence each unit may be divided logarithmically by 10 marks to represent the numbers from 1 to 10 (or powers of 10) as on a slide rule. The distance from point (1, 1) to any number is equal to the logarithm of that number multiplied by M. Thus, if M is 1 unit, the number 4 is $1 \log 4 = 0.602$ unit from point 1. It is evident that a logarithmic plotting sheet could be constructed by taking the proper divisions from the face of a slide rule. Each 10- by 10-in. block is a cycle and is usually repeated two or more times on a sheet of standard logarithmic plotting paper.

6-13. Scales for an exponential equation

An equation of the form $y = ab^x$ in which a constant b is raised to a variable power x is called an exponential equation. The constant b is always positive.

To construct a graph for $y = 2^x$, the corresponding values of x and y are as follows:

x	0	1	2	3	4
y	1	2	4	8	16

Two types of graphs are shown for this equation, one bearing uniform scales and the other bearing one uniform and one nonuniform scale.

EXAMPLE 1. The scale equations are $S_x = M_x x$ and $S_y = M_y y$. Let $M_x = 1$ in. and $M_y = \frac{1}{4}$ in. The x, y plot yields a curve, Fig. 6-17.

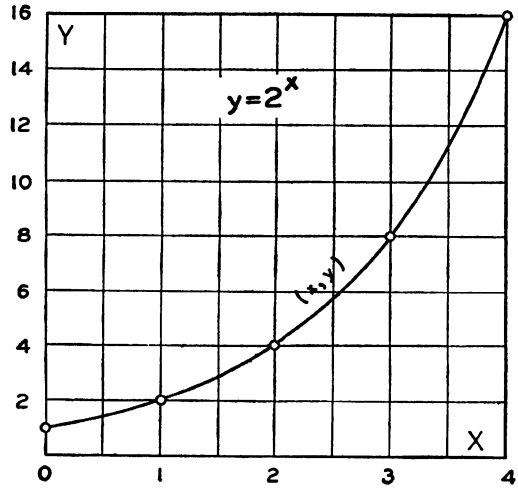


Fig. 6-17.

EXAMPLE 2. If the logarithms of each member of the equation are taken, the equation becomes $\log y = x (\log 2)$. This is a linear equation.

The equation of the OY scale is $S_y = M_y \log y$, a logarithmic scale, and the OX scale is $S_x = M_x x$, a uniform scale. Let $M_x = 5$ and $M_y = 2$. The axes are graduated according to these scales, Fig. 6-18. The values of x and y from the table are plotted on the axes to yield a straight line.

Thus, an exponential equation when plotted on a uniform scale for the exponential variable and a logarithmic scale for the other variable yields a straight line. Such equations are frequent enough to warrant the use of a special form of graph paper having one uniform and one logarithmic scale. It is called *semilogarithmic* paper.

6-14. Representation of a relation between three variables by perpendicular scales

An equation in three variables can be represented graphically by an adaptation of the method used for an equation in two variables. In the equation $f(x, y, z) = 0$, a value, say k, is assigned to one variable, z. The

equation is now one in two variables, x and y, and is plotted in the usual manner, yielding a Z curve. By assigning a series of different values to the

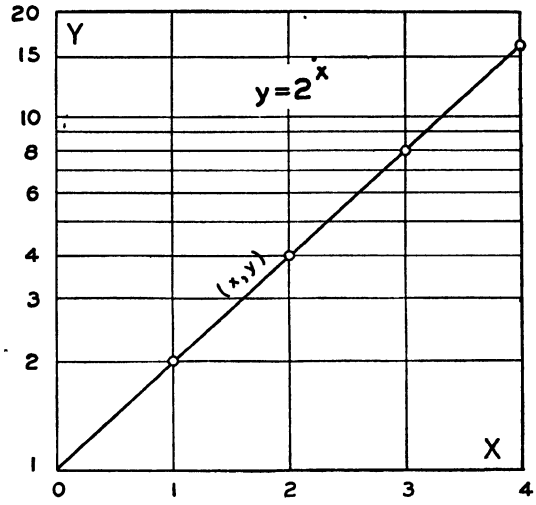


Fig. 6-18.

variable Z, a series of Z curves is determined. The resulting graph is called a *network chart.* An illustration of a network chart for Ohm's law $E = I \times R$ is given in Fig. 6-19. The scales for I and R are uniform, and

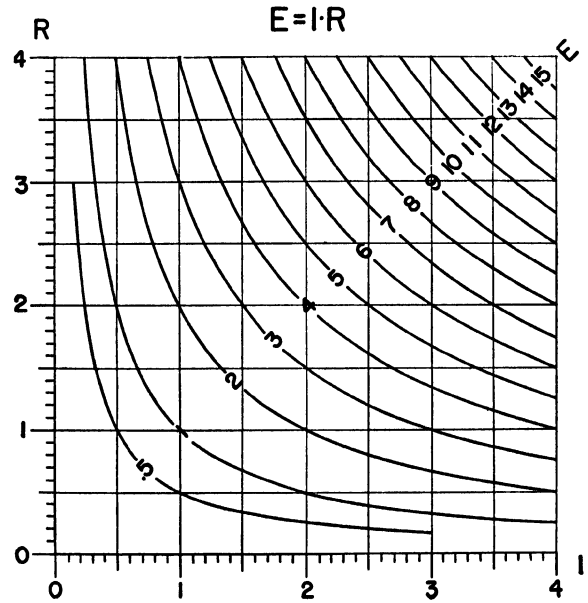


Fig. 6-19.

the variable E is represented by the set of E curves. Even though many values of E are given, this type of representation is bound to be discontinuous and interpolation is rather difficult. The network chart is common in engineering practice, however.

The E curves for this equation are equilateral hyperbolas and are somewhat difficult to draw. The form of this graph can be simplified by assigning the constants to the variable R in place of E. When R is made a constant, the equation $E = I(R_1)$ plots as a straight line. The variables E

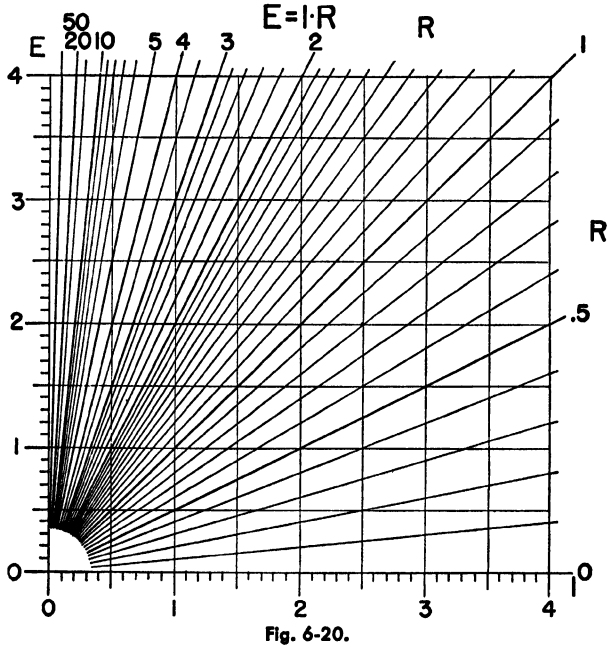


Fig. 6-20.

and I are now plotted on the perpendicular axes, and the network becomes a system of straight lines passing through the origin, Fig. 6-20. While this chart is easily drawn, values near the origin are hard to read.

The best form for the network is obtained by using nonuniform scales for R and I and assigning step values to E, Fig. 6-21. In order to determine the scales the equation is written in logarithmic form as $\log E = \log I + \log R$. The scale $S_1 = M \log I$ is laid off on the X axis; the scale $S_R = M \log R$ is laid off on the Y axis.

6-15. Parallel-scale alignment chart

We have seen that the operations of addition and multiplication can be performed with properly selected sliding scales. These same operations can be performed also on a diagram consisting of three properly chosen

parallel scales. A chart for addition, as $U + V = W$, is made as follows: Two parallel uniform scales, U and V, are constructed, Fig. 6-22. The

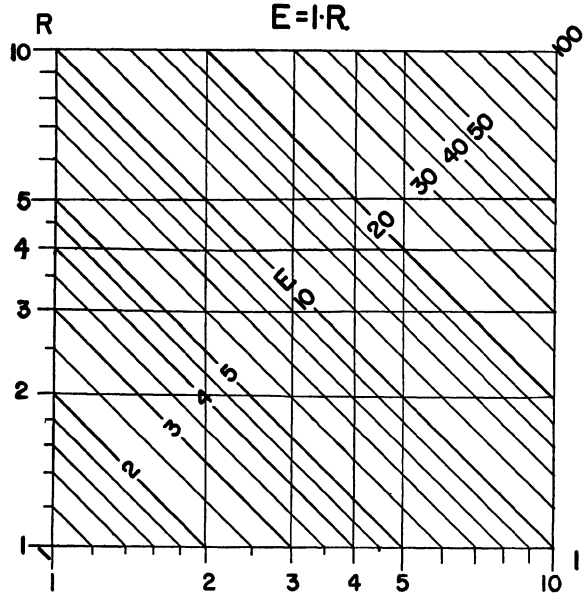


Fig. 6-21.

moduli M_u and M_v for the two scales are equal. The base line EF is drawn through the origins of the scales.

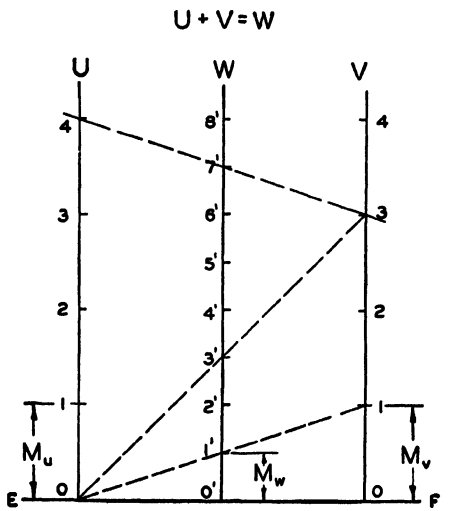


Fig. 6-22.

At the center of EF assume an origin 0, and draw the scale axis W parallel to U and V. The modulus of this center scale is found in the following manner: From 0 on the U scale draw a transversal to point 1 on the V scale, cutting the W scale in point 1′. Now, the segment 0′-1′ = ½(0-1) or $M_w = \frac{1}{2}M_v$. It is also true that $M_w = \frac{1}{2}M_u$. The W scale is now graduated, using the modulus M_w. The diagram is then ready for use.

To add two numbers, as 4 and 3, draw a straight line through points 4 and 3 of the U and V scales, respectively. This line cuts the W scale in point 7′, the sum.

This result is analyzed as follows: Draw line 0-3 through 0 on U and 3 on V, cutting W in point 3′. Thus, the length 0-3 (three units) on V is transformed into three units of W. Considering the scale segments intercepted by the lines 0-3 and 3-4, four units of U are transformed into four units of W. The units intercepted on W are contiguous, and their sum is equal to seven units.

Hence, addition is performed by transforming a given number of units of U and V into a corresponding number of units on W and reading the total on W.

Subtraction can be performed by reading the scales so that $W - V = U$. A subtraction diagram for $U - V = W$ is constructed by reversing the direction of the V scale with respect to the base line EF, Fig. 6-23.

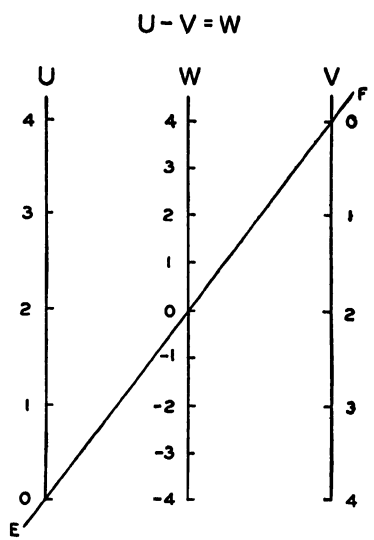


Fig. 6-23.

The scales for U, V, and W can represent any functions in u, v, and w, but each scale can have only one variable. Thus, the scale for $f(u)$ can represent log u, u^2, sin u, etc. The equation of the diagram, $U + V = W$, is usually written in more general form as

$$f_1(u) + f_2(v) = f_3(w)$$

6-16. Nomography

The diagram described in the preceding article is a simple case of a type of chart known as a nomogram, or alignment chart. A nomogram represents an equation in three variables (u, v, w) by three scales which may be on straight or curved lines. These scales are so designed that any straight line cuts them in three points, which are values of u, v, w, that satisfy the equation. The theory underlying the design of alignment charts is treated in an interesting branch of applied mathematics called *nomography*. The fundamental work in the field was done by M. d'Ocagne,

professor at the Polytechnic School in Paris, who published the first treatise on nomography in 1895.

6-17. A nomogram for $f_1(u) + f_2(v) = f_3(w)$

The nomogram of Fig. 6-22 is of this type and will now be described in a more general manner. In the general case the moduli of the U and V scales are assumed to be unequal, and the W scale is not in the center of the diagram.

Let the parallel U- and V-scale axes be drawn at any convenient distance k apart, Fig. 6-24. Also, let the modulus M_u for the U scale be equal to the segment AB and the modulus M_v for the V scale be equal to CD.

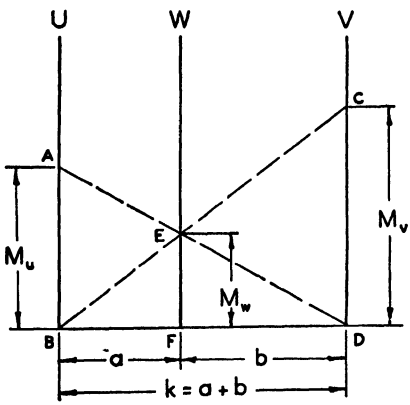


Fig. 6-24.

The problem is to find the location of the W scale and its modulus so that moduli of U and V can be added together on the W scale.

The W scale must be located so that M_u and M_v are transformed into equal units of length on the W axis. Draw the lines AD and BC meeting in point E. Draw the W axis through E parallel to the other axes, cutting BD in point F. The segment EF represents the modulus of the W scale. If the scales are now calibrated with their corresponding moduli, the sum of the units on U and V between line BD and any transversal, as AC, will be equal to the number of units of W included between BD and AC.

The form of a nomogram varies with the character of the equation to be represented. A chart for $f_1(u) + f_2(v) = f_3(w)$ is composed of three parallel straight lines, and its construction will be described. Equations of other types have other forms. The student is referred to standard works on nomography listed at the end of this chapter for a complete development of this subject.

This chart can be adapted for an equation of the form

$$f_1(u) \times f_2(v) = f_3(w)$$

by writing the equation in the logarithmic form

$$\log f_1(u) + \log f_2(v) = \log f_3(w)$$

and using logarithmic scales for the scales of the chart.

If the equation is of the form

$$\frac{f_1(u)}{f_2(v)} = f_3(w)$$

the logarithmic form is

$$\log f_1(u) - \log f_2(v) = \log f_3(w)$$

This subtraction of scales is provided for on the chart by inverting the V scales, Fig. 6-23.

A graphical method for designing an addition chart has just been described. It is common in practice to lay out the scales by means of their equations. From Fig. 6-24 the following formulas between the moduli can be deduced by considering similar triangle relationships:

Location of W scale:

$$\frac{M_u}{M_v} = \frac{a}{b} \tag{1}$$

Modulus for W scale:

$$M_w = \frac{M_v M_u}{M_v + M_u} \tag{2}$$

Relations between the moduli:

$$\frac{1}{M_u} + \frac{1}{M_v} = \frac{1}{M_w} \tag{3}$$

Equations for the scales:

$$U \text{ scale: } S_u = M_u f_1(u)$$
$$V \text{ scale: } S_v = M_v f_2(v)$$
$$W \text{ scale: } S_w = M_w f_3(w)$$

EXAMPLE. Construct a nomogram for Ohm's law, $E = IR$. The range of I is from 1 to 15. The range of R is from 1 to 20. The chart is to be about 20 in. high and 15 in. wide. This chart, shown in Fig. 6-25, is reduced 3 to 1. Write the given equation in the form

$$\log E = \log I + \log R$$

The scales are designed as follows:

I scale:

Length = 20 in. = $M_I(\log 15 - \log 1)$. Hence, $M_I = 17$ in.
Scale equation is $S_I = 17 \log I$.

R scale:

Length = 20 in. = $M_R(\log 20 - \log 1)$. Hence, $M_R = 15$ in.
Scale equation is $S_R = 15 \log R$.

E scale:

$$M_E = \frac{17 \times 15}{17 + 15} = 7.97 \qquad \text{from (2)}$$

Scale equation is $S_E = 7.97 \log E$.

Location of E scale:

Width of chart = 15 in.

$$\frac{M_I}{M_R} = \frac{17}{15} = \frac{a}{b} \qquad \text{from (1)}$$

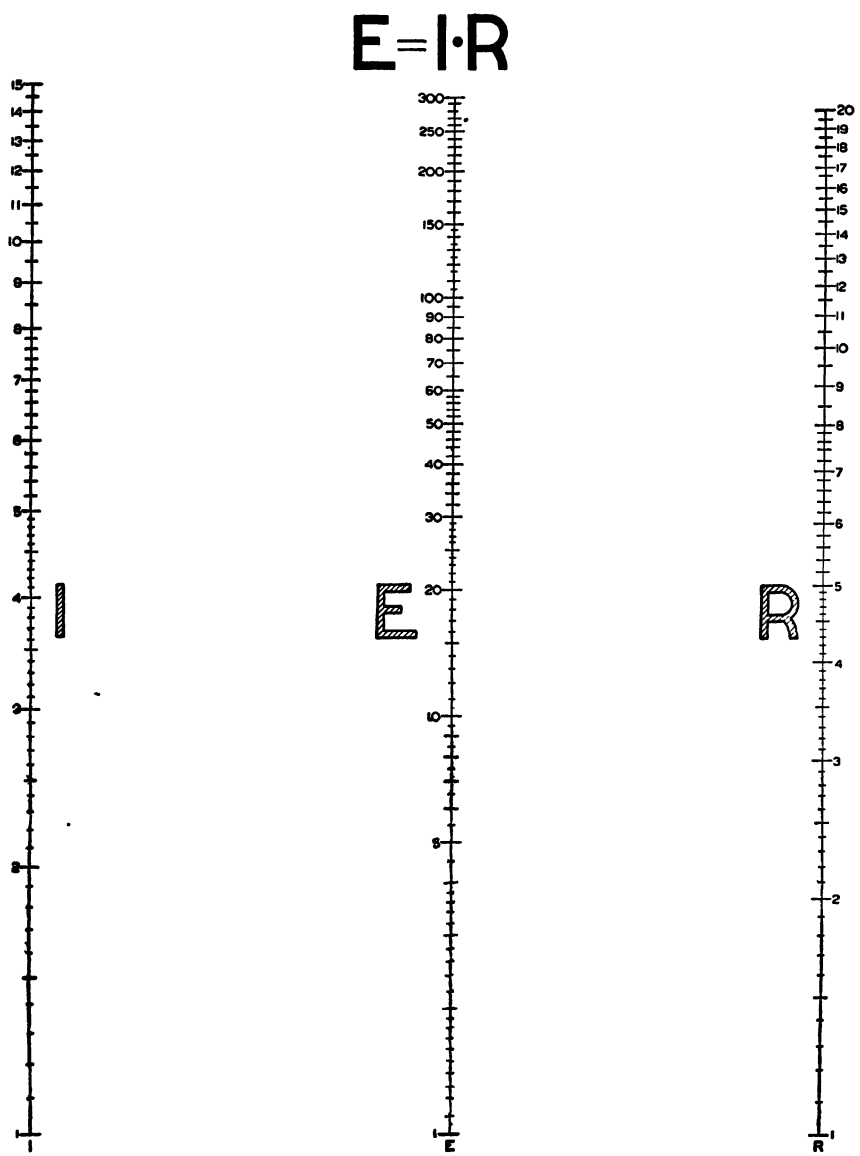


Fig. 6-25.

Therefore, the distance from the I scale to the E scale is

$$15 \times \frac{17}{32} = 7.97 \text{ in.}$$

Since $E = 1$ when I and R are each equal to 1, point 1 of the E scale will be collinear with point 1 on both I and R.

REFERENCES FOR READING

DOUGLASS and ADAMS: "Elements of Nomography," McGraw-Hill Book Company, Inc., New York, 1947.

LEVENS, A. L.: "Nomography," John Wiley & Sons, Inc., New York, 1948.

PROBLEMS

6-1. Construct scales for the following functions of the given variable. Scale lengths can be 5 to 6 in.

(*a*) $f(u) = u^3$

(*b*) $f(t) = 16t^2$

(*c*) $f(x) = \log x$

(*d*) $f(v) = \dfrac{1}{v}$

6-2. Rectify the curves represented by the following equations by designing proper functional scales and/or by using logarithmic plotting paper. Plot to fit in an 8- by 10-in. space.

(*a*) $v = u^{1.5}$

(*b*) $S = 16T^2$

(*c*) $P = 2(10)^{0.5x}$

(*d*) $S = e^{0.7t}$

(*e*) $y = \dfrac{1}{x}$

6-3. Construct a chart with adjacent scales to convert centimeters to inches. 1 in. = 2.54 cm. Range is to be from 0 to 5 in.

6-4. Construct a chart with adjacent scales to give the common logarithms of numbers from 1 to 10. Scale to be 5 in. long.

6-5. Construct a chart with adjacent scales to give the natural logarithms of numbers from 1 to 10. Scale to be 5 in. long. $\log_e N$ or $\ln N = 2.303 \log_{10} N$.

6-6. Solve the equations $y = 0.16x^2$ and $y = 6.5x^{-0.8}$ simultaneously, using logarithmic plotting paper.

6-7. Represent the equation $R = PQ$ by a network chart consisting of straight lines. Let Q be the independent variable, ranging from 1 to 10. Range of P is from 1 to 10. R is to have values 5, 10, 15, 20, 30, 40, 50. Make P and Q logarithmic scales.

6-8. Represent the equation $R = E/I$ by a network chart consisting of straight lines. Let I be the independent variable, ranging from 0 to 10. Let E range from 0 to 100. Use cartesian coordinate paper.

6-9. Construct a nomogram for $E = IR$ for values of I from 1 to 25 and values of R from 1 to 50.

6-10. Construct a nomogram for $R = E/I$ for values of E from 1 to 30 and values of I from 1 to 20.

6-11. Construct a nomogram for $PV^{1.41} = T$ for values of P and V from 1 to 10.

6-12. Construct a nomogram for $V = 0.785D^2H$, the volume of a cylinder. Range of D is from 1 to 15. Range of H is from 1 to 30. Write this equation in the form

$$\log V - \log 0.785 = 2 \log D + \log H$$

Neglect the constant, log 0.785, when designing the scale equation for V. Find the anchorage point for the V scale by solving equation for $D = 1$ and $H = 1$. Note that the effect of the constant is simply to move the V scale up or down.

6-13. Construct a nomogram for $Q = 3.33bH^{1.5}$. Range of b is from 1 to 10. Range of H is from 0.2 to 3.0. This is Francis' formula for discharge of water, Q (cubic feet per second), over a submerged weir of width b (feet) for the head H (inches).

7

EMPIRICAL CURVES

7-1. Graphs of engineering data

Many of the data that the engineer or scientist uses are empirical data obtained by a series of observations in the laboratory. Such data in general involve the relationship between two physical quantities, one of which is allowed to vary in some desired manner, while the corresponding values of the other are measured. For example, the effect of temperature change on the electrical resistance of a given copper rod can be found by measuring its resistance at various intervals within a given temperature range. A table is usually prepared, showing the results of the observations. The relationship disclosed by the experiment is now put into graphical form by plotting on rectangular-coordinate paper, using suitable scales for each of the variables. This graph is called the *direct plot.*

If the points lie approximately on a smooth curve, the possibility of representing the relationship between the variables by an equation or formula is suggested. It is evident that all of the operations involved in this process, including the original measurements, contain approximations. Therefore, the resulting curve and its equation can represent the data only approximately, and also only within the range of the experimental values. Such a curve is called *empirical,* and its equation is an empirical equation or formula. Many of the formulas used in engineering design are of this nature.

7-2. Engineering curves

The following treatment of curves and their equations will be limited to those which arise most frequently in engineering problems. These are the curves or equations that represent the laws of change which occur in nature. Since the engineer is concerned fundamentally with natural phenomena, they enter into a great many of his calculations.

The laws of change are of three general types: the power law, the exponential law, and the periodic law. The equations which describe these laws are represented graphically by the power curves (which include the straight line, parabola, and hyperbola), the exponential curves, and the

periodic curves. The purpose of this and the following chapter is to develop methods of finding equations of these types to fit empirical laboratory data satisfactorily.

7-3. The straight line in rectangular coordinates

Let k, Fig. 7-1, be any straight line which cuts the Y axis at the distance a from the origin and makes the angle θ, measured counterclockwise, with the X axis. The distance a is called the *intercept* of k on OY; the slope of the line is defined as the tangent of angle θ. If the coordinates of any point, P, of k are given by x and y, the slope m of the line is equal to $(y - a)/x$. Hence,

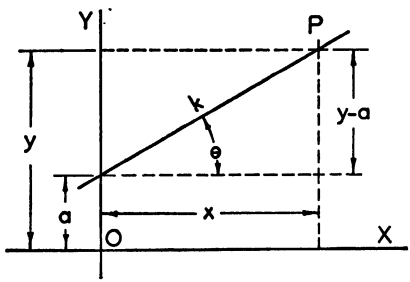


Fig. 7-1.

$$y = a + mx \qquad (1)$$

The student will recognize this equation as a straight-line or linear equation. When the unit of measurement is the same for both abscissas and ordinates, the angle of slope appears in true size on the graph. If the units are unequal, the slope m is not at its true angle but is given by $(y - a)/x$, which must be calculated by using the scales assigned to the axes.

It can be shown that any equation of the first degree, as $Ax + By + C = 0$, is represented graphically by a straight line.

7-4. The direct plot

The graphical method of fitting a line to experimental data will be illustrated by following through a specific problem step by step.

EXAMPLE (FIG. 7-2). The following data were collected in order to find the temperature coefficient of a copper rod. The temperature C is centigrade, and the resistance R is in microhms. Find the equation of the "best" line representing these data, the equation to have the form $R = f(C)$.

C, deg	19.1	25.0	30.1	36.0	40.0	45.1	50.0
R, microhms ...	76.30	77.80	79.75	80.80	82.35	83.90	85.10

1. *Choice of abscissas and ordinates.* It is standard practice to plot values of the independent variable of an equation as abscissas and values of the dependent variable as ordinates. Plot C as the independent variable, R as the dependent variable.

2. *Choice of scales.* For convenience in plotting, the scale should be 1, 2, 4, 5, or 10 squares per unit. The scale may be different on each axis. In general, it is not feasible to use the same size units on both axes. The

sizes of the units used for plotting the abscissas and ordinates should be selected so that the line will make an angle with the X axis somewhere between 30° and 60°. It is not necessary that the origin appear on the graph.

For the abscissas, let one square equal 2°C. For the ordinates, let one square equal 0.50 microhm. Let the axes intersect at 0°C and 70 microhms.

3. *Plotting the data.* A point of the data is plotted as follows: First locate the abscissa of the point on the X axis. Place a vertical straight-

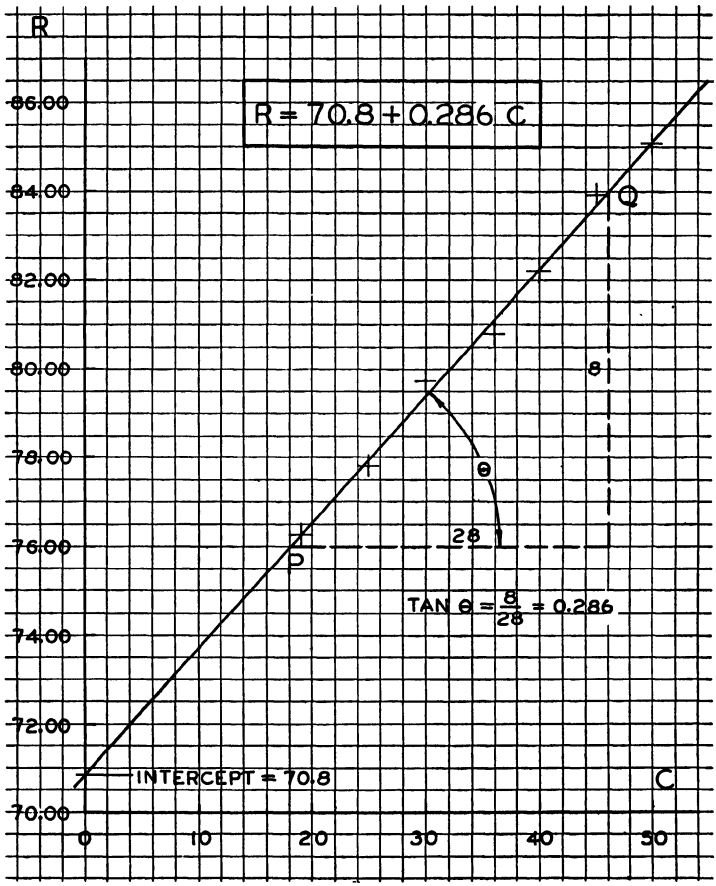


Fig. 7-2.

edge at this point, and draw a fine line about ⅛ in. long where the ordinate is to be located. Then place a horizontal straightedge at the ordinate of the point on the Y axis, and draw a short line to intersect the first line.

Dots should not be used to represent points, as the operation of locating both coordinates of a point at the same time is open to error. When a graph is finished in ink for reproduction, the points are in general indicated by small circles if they represent experimental observations. If the points are those of a mathematical curve, they are not shown on the inked curve.

4. *Drawing the best representative line.* The best representative line is found by adjusting a line among the points so that the points either lie on the line or are about equally distributed on each side of it. The deviation of a single point from this line, usually measured along the ordinate, is called a *residual.* The residuals are positive when the plotted point is above the line, negative when below. The best line, determined graphically, is substantially one for which the sum of the positive or negative residuals is a minimum and the total is zero. A practical way of achieving this is to divide the points into two roughly equal groups and place the line so that the sum of the residuals for each group is substantially zero. The exact criterion for the location of the best line is based on the *method of least squares,* a numerical method which postulates that the best value of a series of observations is that which makes the sum of the squares of the deviations a minimum.

7-5. Determination of the constants of the straight line

The disposition of the plotted points of Fig. 7-2 indicates that the relationship between C and R can be represented by a straight line of the form $R = a + mC$, where a and m are the constants to be determined.

From the graph, the slope m of the line is 8/28, or 0.286. The intercept a is 70.8. Hence, the equation connecting R and C is

$$R = 70.8 + 0.286C$$

If the ordinate at $X = 0$ is not available, the constants can be found as follows: The coordinates of any two points such as P and Q of the line are read from the graph. These points, in general, will not be observed points. The coordinates of each point are substituted in the general equation $R = a + mC$. The two equations thus formed are then solved for a and m.

Points P and Q of the line have the coordinates ($C = 18$, $R = 76.00$) and ($C = 46$, $R = 84.00$), respectively. Substituting these values in the general equation, we have

$$76.00 = a + 18m$$

and

$$84.00 = a + 46m$$

Solving the two equations simultaneously,

$$a = 70.84 \qquad m = 0.286$$

Hence, the equation connecting C and R is

$$R = 70.84 + 0.286C$$

7-6. The power law

The power law represents a situation where one quantity varies according to some power of a second quantity. This kind of relationship is most common in the study of motion and energy. The equation of the power law when the curve passes through the origin is

$$y = ax^m \tag{2}$$

where a and m are constants.

The student will recall immediately several examples of the power equation—for example, the law describing the relation between distance and time for a freely falling body, $S = \frac{1}{2}gt^2$; the pressure-volume relationship, $PV = k$; the period of a simple pendulum, $T = 2\pi(L/g)^{\frac{1}{2}}$.

7-7. Power curves

In practical problems it is usually necessary to consider only positive values of x and y, and therefore only the portions of the curves which lie in the first quadrant. In Fig. 7-3, several of these curves are drawn for $a = 1$ and values of m ranging from 3 to -3. The graph of $y = ax^m$ is a parabola when $m = 2$. For all other positive values of m, the curves resemble parabolas. The limiting case of a straight line occurs when $m = 1$. For all positive values of m, the curves pass through the origin.

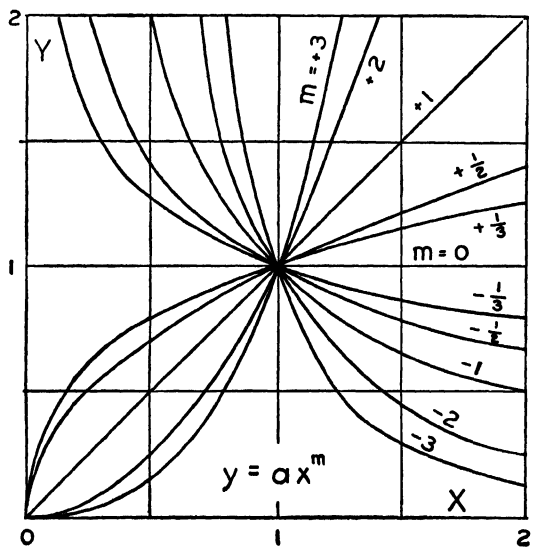


Fig. 7-3.

When $m = -1$, the curve is an equilateral hyperbola. For other negative values of m, the curves resemble hyperbolas. For all negative values of m, the curves have the X and Y axes as asymptotes. Examples 2 and 3 of Art. 6-9 and Fig. 6-13 are power curves.

7-8. Logarithmic graphs of power curves

If the logarithms of both members of the equation $y = ax^m$ are taken, we have

$$\log y = \log a + m \log x$$

as shown in Art. 6-11. This is a straight-line equation. Hence, the curves of the power group become straight lines when the logarithms of x and y are plotted. The intercept of the line is $\log a$; the slope is m.

EXAMPLE. Rectify the curve which has the equation $y = 0.125x^2$. The following values are calculated:

x	0	1	2	4	6	8	10
y	0	0.125	0.5	2	4.5	8	12.5
$\log x$	..	0	0.301	0.602	0.778	0.903	1
$\log y$	..	$\bar{1}.097$	$\bar{1}.699$	0.301	0.653	0.903	1.097

The plot of $(\log x, \log y)$ yields a straight line, Fig. 7-4. From the graph the intercept $\log a = \bar{1}.097$; thus, $a = 0.125$. The slope $m = \dfrac{\log y_2 - \log y_1}{\log x_2 - \log x_1}$. Using the coordinates of the end points of the line, $m = \dfrac{1.10 - (-0.9)}{1 - 0} = 2$.

7-9. Logarithms of numbers less than 1

The divisions of the negative Y axis, Fig. 7-4, may be denoted by any one of the three systems available for writing the logarithms of numbers less than 1. The mantissa of a number read from a table of logarithms is positive and must be used with a negative characteristic when the number is a fraction. Thus, the mantissa of 5 is 0.699 and $\log 0.5$ is $\bar{1}.699$ (which means $0.699 - 1$, or -0.301 for plotting purposes). The value of the characteristic of a number between 1 and 0 is always *one* greater than the number of zeros to the right of the decimal point. Many workers avoid the use of a negative characteristic by adding and subtracting 10 so that $\bar{1}.699$ becomes $9.699 - 10$.

Recalling that a logarithm is an exponent, the equation $\log 0.5 = \bar{1}.699$ can be put in the exponential form, $0.5 = (10)^{\bar{1}.699} = (10)^{-0.301}$.

The exponent has an actual value of $0.699 - 1 = -0.301$. This is the true value of $\log 0.5$.

Hence, the division of the Y axis, Fig. 7-4, which is two lines below the origin, may be marked $\bar{1}.8$, $9.8 - 10$, or -0.2. This graph should be compared with Fig. 6-15, where the same equation is plotted on logarithmic

paper. It will be seen that the same line is obtained in each case. The logarithmic paper has eliminated the necessity of looking up the logarithms

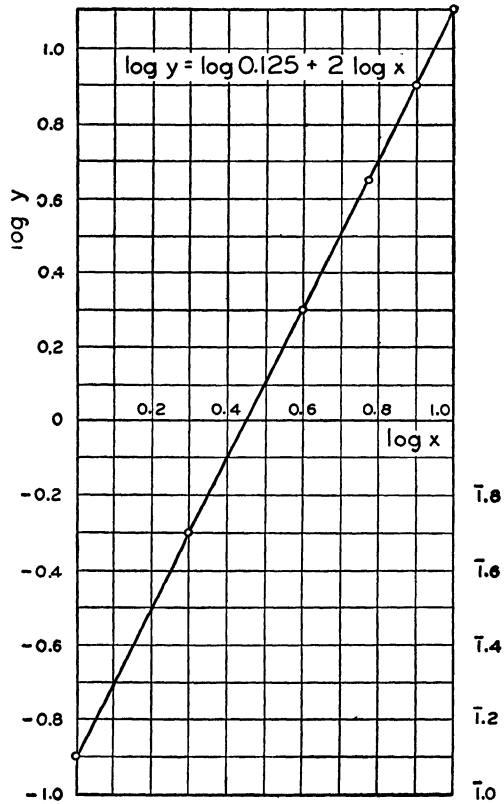


Fig. 7-4.

of the values of x and y in a table of logarithms. It thus saves a great deal of work.

7-10. The straight line on logarithmic paper

The plotting of a power equation on logarithmic paper has been shown to yield a straight line. Conversely any straight line on such paper represents a power equation the constants of which can be determined. The straight line k, drawn on logarithmic paper, Fig. 7-5, cuts the Y axis at a. The distance from point (1, 1) to a is equal to $\log a$. If $P(x, y)$ is any point of line k, the length of the abscissa of P is $\log x$ and the length of the ordinate of P is $\log y$. Hence, the slope m of k can be written as

$$m = \frac{\log y - \log a}{\log x}$$

Therefore,

$$\log y = \log a + m \log x$$

Taking the antilogarithms of each side of this equation, we have the power equation $y = ax^m$

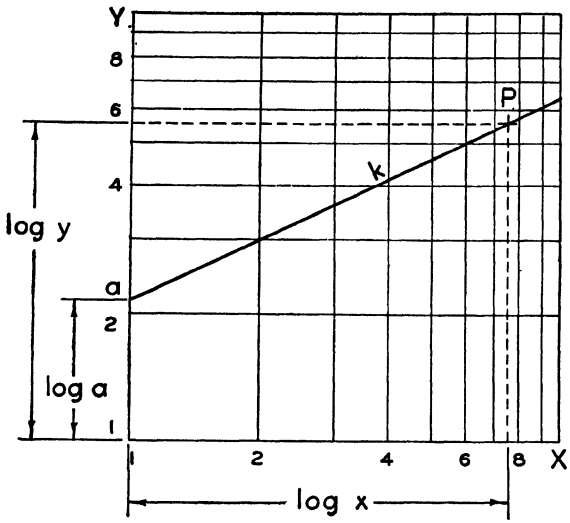


Fig. 7-5.

7-11. To find the constants of a power curve

Since a straight line on logarithmic paper can be transformed into a curve which is represented by $y = ax^m$ in cartesian coordinates, it can be concluded that, if a set of empirical data fall approximately on a straight line when plotted on logarithmic paper, the cartesian form of a satisfactory equation connecting the quantities will have the form of $y = ax^m$. The constants a and m can be deduced from the graph.

EXAMPLE. Plot the points whose coordinates are given in the following table, and find the equation of a curve which best represents these data:

x	1	3	5	7	9	10
y	10	5.77	4.77	3.78	3.33	3.16

The rectangular plot of the data, Fig. 7-6, suggests a hyperbolic curve of the form $y = ax^m$. This is verified by replotting on logarithmic paper where a straight line closely fits the points, Fig. 7-7.

The constants a and m are determined from the logarithmic plot, as described in Art. 7-10. When $x = 1$, $y = 10$. This value of y is equal to the constant a. The slope m is equal to $\tan \theta = -0.50$. Since the two scale units are equal, the slope angle can be measured directly on logarithmic paper. Hence, the required equation is $y = 10x^{-0.50}$.

The constants may also be determined if the logarithms of the two variables are plotted on uniform paper. Thus for the preceding example ($\log x$, $\log y$) is plotted on the same sheet used for the (x, y) plot, Fig. 7-8.

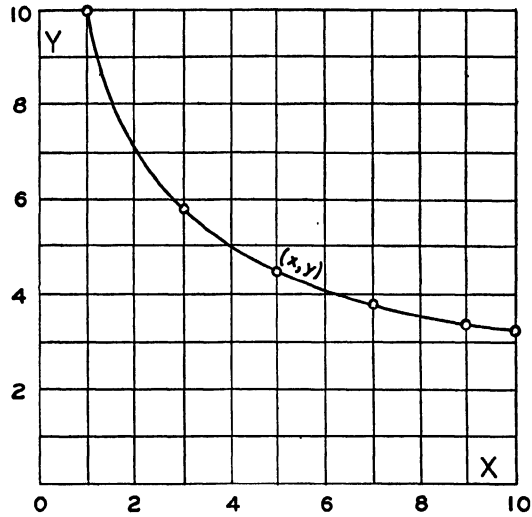


Fig. 7-6.

The data lie on a straight line, and its equation is $\log y = \log a + m \log x$. When reading the abscissa and ordinate of a point from the logarithmic plot, it should be noted that the values represent $\log x$ and $\log y$.

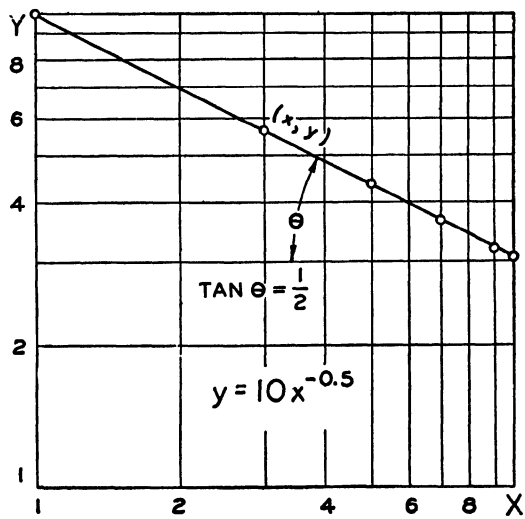


Fig. 7-7.

Since equal units are used for plotting values of $\log x$ and $\log y$, the true slope of the line appears on the graph and the value of m can be measured directly. If unequal units were used, the slope would be calculated using

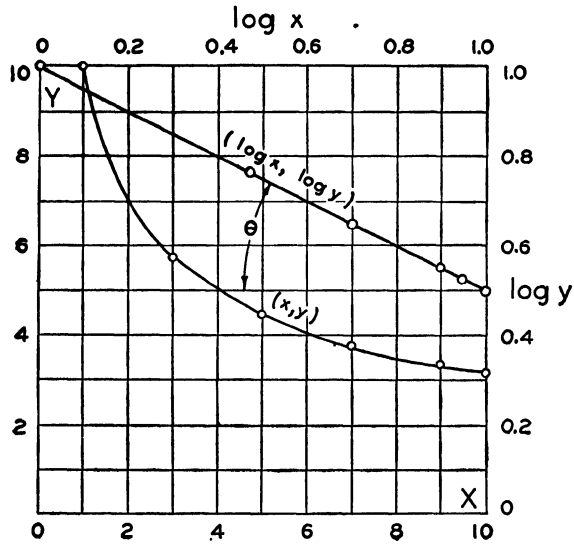


Fig. 7-8.

the scales assigned to the axes. In Fig. 7-8, $m = -0.50$. The intercept on the Y axis, using the $\log y$ scale, is 1.0. Hence, the equation of this line is $\log y = 1 - 0.50 \log x$.

Taking antilogarithms of each side of the equation

$$y = 10x^{-0.50} \quad \text{or} \quad yx^{0.50} = 10$$

7-12. To extend the range of a logarithmic graph

In Fig. 7-9 the graph for $y = x^{3/2}$ is plotted on a sheet two cycles wide and three cycles high. The line runs through four blocks, A to B, B to C, C to D, D to E. If each block is considered to be superposed on the lower left block, the entire line can be shown on a sheet carrying but one block, Fig. 7-10. This arrangement permits the use of a larger cycle and thus allows a closer reading of any value. Care must be taken, however, to assign the proper scale values for each segment of the line. Thus, for any point on AB the values are (x, y); for any point on BC $(x, 10y)$; on CD $(10x, 10y)$; on DE $(10x, 100y)$.

In power equations, known values frequently fall into a cycle other than that of 1 to 10. In such a case the range of the graph can be extended in the above manner to the ordinate at $x = 1$ in order to read from the graph the value of the constant a in the equation $y = ax^m$.

7-13. The exponential equation

An equation in which a constant is raised to a variable power is called an exponential equation. This equation has the general form of

$$y = a(b)^x \tag{3}$$

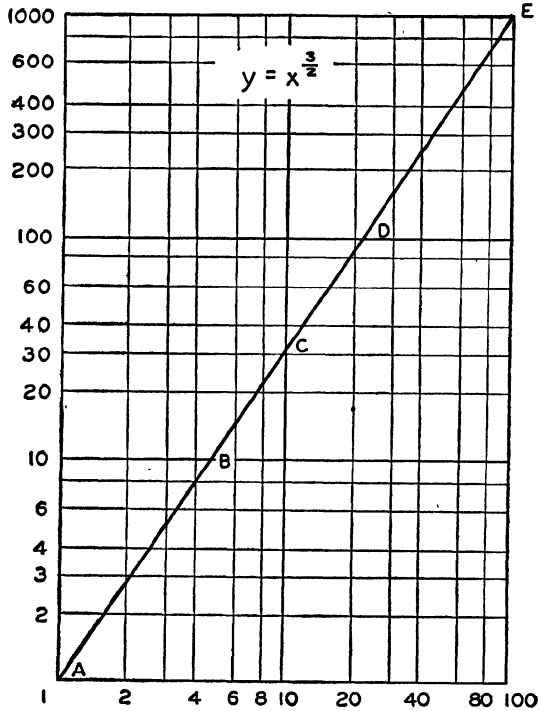


Fig. 7-9.

where a and b are constants, with b always positive. The manner in which the function $a(b)^x$ increases is shown in the following table for $a = 1$ and $b = 2$:

x	0	1	2	3	4	5	6
y	1	2	4	8	16	32	64

As successive values of x increase by a constant difference, 1, corresponding values of y increase by a constant factor, 2. The values of x form an arithmetic series, the values of y a geometric series. The constant a is the value of y when $x = 0$. The constant b is the fixed ratio between any term and the preceding term of the geometric series.

In the equation $y = a(b)^x$, if the value e^m be substituted for the constant b, the equation becomes

$$y = a(e)^{mx} \tag{4}$$

This is the most common form of the exponential equation. The constant e is the number 2.718 . . ., the base of natural logarithms; m is any power of e. Since $b = e^m$, it follows that $\ln b = m$. Thus, m is the natural logarithm of b, the fixed ratio of the geometric progression. It is shown

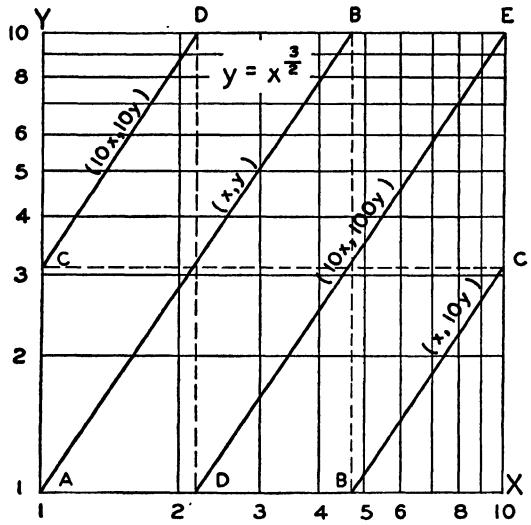


Fig. 7-10.

in analytic geometry that m is the *rate of increase* of the function. Natural logarithms are discussed in Art. 7-19.

7-14. The law of organic growth

In nature many quantities grow (or decay) continuously at a rate which is a fixed proportion of the amount present at any instant. In other words, as time progresses in an arithmetic ratio, the quantity increases (or decreases) in a geometric ratio. Consider a cell which grows by fission. First we have one cell, then two, then four, then eight, etc. This type of growth is expressed by the exponential equation. The exponential law describes such diverse phenomena as the growth of bacteria in a culture, the increase of a sum of money at continuous compound interest, the decay of radium emanation, the die-away effect of an interrupted electric current, the depreciation of a building or a machine.

In the application of the equation $y = ae^{mx}$ to each of these growth problems, y is the size of the quantity at the end of the time interval x. The constant a is the initial size of the quantity, and m is the rate of increase.

A typical exponential equation is that of Halley's law, which states that, if P is the atmospheric pressure in inches of mercury and h the elevation above sea level in feet, then

$$P = 29.92e^{-h/26,200}$$

7-15. Exponential curves

In Fig. 7-11, several exponential curves are drawn for values of m ranging from 4 to -4. For each curve, $a = 1$. Each curve is asymptotic to the X axis and cuts the Y axis at $y = a$. When m is positive, y increases as x increases, representing growth; when m is negative, y decreases as x increases, representing decay.

To construct a graph for a given equation, a range of values for x is assumed, and the corresponding values of y are calculated. For example,

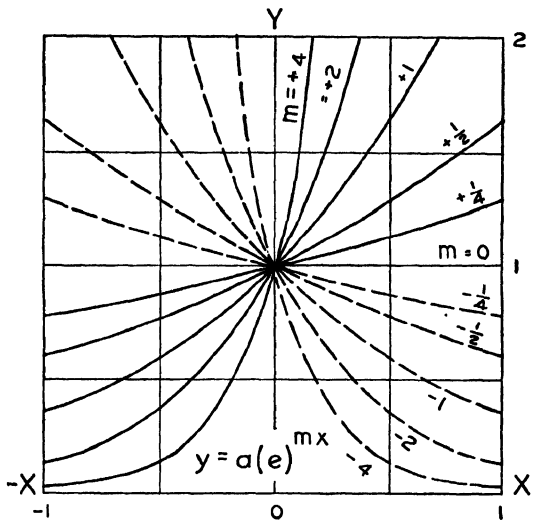


Fig. 7-11.

if \$1 is allowed to increase continuously at the nominal rate of 6.93 per cent per year, the total amount S at the end of any time t, in years, is given by

$$S = 1e^{0.0693t}$$

Let the range in t be 40 years, and solve for 10-year intervals. In order to do this conveniently, the equation is put into logarithmic form. Taking the natural logarithm of each member of the equation, we have

$$\ln S = \ln 1 + 0.0693t(\ln e)$$

Since $\ln e = 1$ and $\ln 1 = 0$,

$$\ln S = 0.0693t$$

The corresponding values of S and t are as follows:

$t = 0$	10	20	30	40
$\ln S = 0$	0.693	1.386	2.079	2.772
$S = 1$	2	4	8	16

Figure 7-12 is the resulting graph. It will be seen that the values of S form a geometric progression having a constant ratio of 2. The natural logarithm of 2 is 0.693. Thus, the *nominal rate* of increase is 0.693 per decade, or 0.0693 per year.

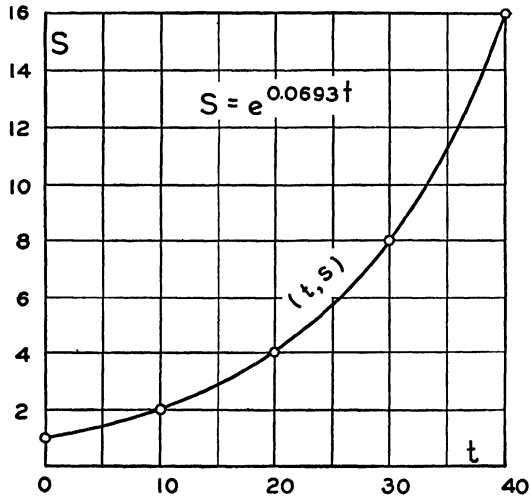


Fig. 7-12.

7-16. Semilogarithmic graphs

When the logarithms of both members of the equation $y = ae^{mx}$ are taken, we have

$$\log y = \log a + mx \log e \tag{5}$$

If natural logarithms are used, $\ln e = 1$ and Eq. (5) reduces to

$$\ln y = \ln a + mx \tag{6}$$

Letting $\ln y = y'$ and $\ln a = a'$, Eq. (6) becomes $y' = a' + mx$, the equation of a straight line. The intercept and slope of the line are equal to a' and m, respectively. Thus, the exponential curve can be rectified by plotting $\log y$ against corresponding values of x or by using a semilogarithmic paper as explained in Art. 6-13. Figure 7-13 shows the equation of Fig. 7-12 plotted on semilogarithmic paper.

In determining the constants of Fig. 7-13 the intercept, $a = 1$, may be read directly from the graph. The slope m may be calculated from any two points of the line, such as (0, 1) and (40, 16). Thus,

$$m = \frac{\ln 16 - \ln 1}{40 - 0} = \frac{2.772}{40} = 0.0693$$

However, the width of one cycle is a constant equal to $\ln 10 - \ln 1 = 2.3026$. Thus, it is usually most convenient to take this distance for the side of

the triangle used to measure slope. Calling the base of this triangle x', then $m = 2.3026/x'$, or in this instance $m = 2.3026/33.33 = 0.0693$

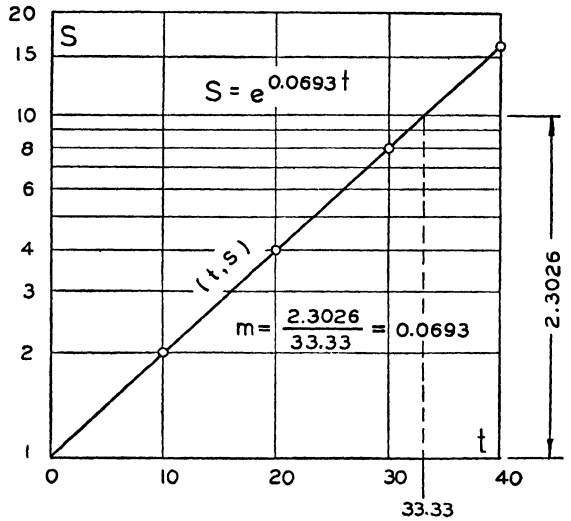


Fig. 7-13.

7-17. To find the constants for an empirical curve of exponential form

If a set of empirical data, in x and y, lie approximately on a straight line when plotted on semilogarithmic graph paper, a satisfactory equation connecting the quantities and having the form $y = ae^{mx}$ may be determined.

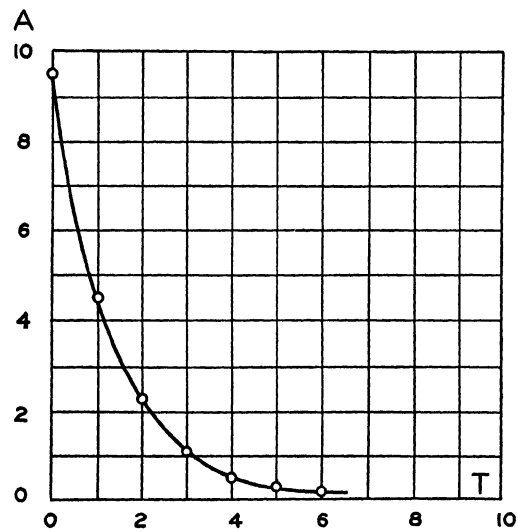


Fig. 7-14.

The constants a and m can be deduced from the graph as described in Art. 7-16.

EXAMPLE. The relationship between the amplitude of vibration, A, of a long pendulum and the time, T, since the pendulum was set in motion was observed as follows:

T, min ...	0	1	2	3	4	5	6
A, in.	9.00	4.47	2.22	1.10	0.55	0.27	0.13

Find the equation that represents this relationship between T and A.

The direct plot, Fig. 7-14, suggests an exponential curve. These data, plotted on semilogarithmic paper, with T as the independent variable,

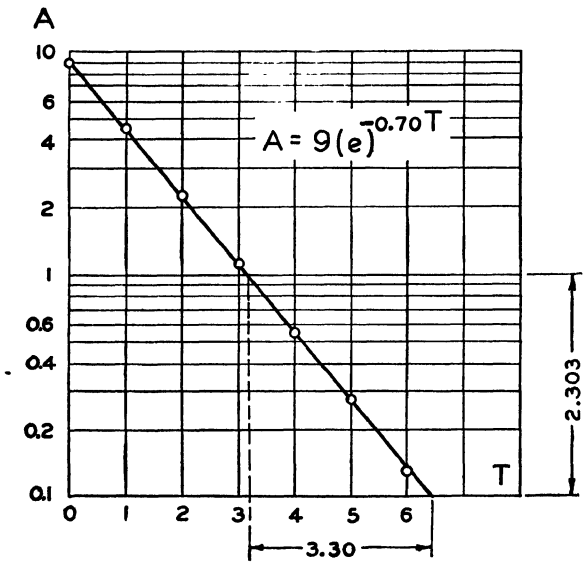


Fig. 7-15.

give a set of points which lie on a straight line, Fig. 7-15. A trial using A as the independent variable does not rectify the curve. This result indicates that the relationship desired is an exponential equation of the form $A = ae^{mT}$. The intercept, $a = 9$, is read at the point where the line cuts the ordinate at $T = 0$.

The slope m is equal to $-2.303/3.30 = -0.70$. Hence the equation is $A = 9e^{-0.70T}$.

7-18. The logarithmic curve

The exponential equation $x = b^y$ differs from $y = b^x$ in that its graph is asymptotic to the Y axis instead of the X axis. This equation is usually written in the form $y = \log_b x$, and its graph is called the *logarithmic* curve.

In Fig. 7-16, curves are drawn for $b = e$ and $b = 10$, representing the natural and common systems of logarithms, respectively. The ordinate at any point of a logarithmic curve is equal to the logarithm of the abscissa of the point. The common logarithm of any number is equal to its natural logarithm multiplied by the constant 0.434. The important property of the logarithmic curve is that $\log m + \log n = \log mn$, where m and n are any positive values of x. This fact can be verified on the graph by

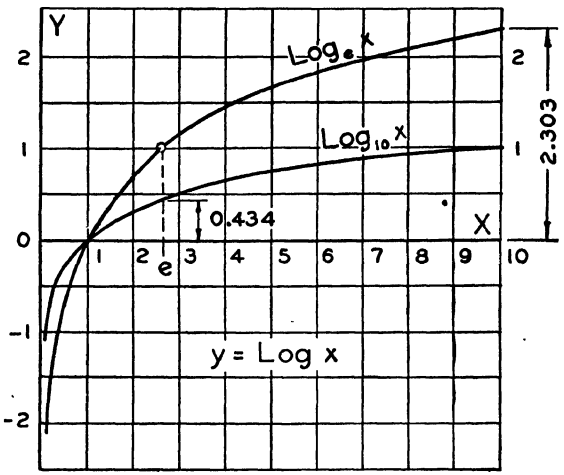


Fig. 7-16.

adding the ordinates for $m = 2$ and $n = 3$. The sum is equal to the ordinate of $x = 6$.

The following facts are also illustrated in the graph:

1. $\log 1 = 0$ in either system.
2. The logarithms of numbers between 1 and 0 are negative.
3. Since no part of the curve lies on the left side of the Y axis, the logarithms of negative numbers cannot be defined in terms of real numbers.
4. As x approaches zero, $\log x$ decreases without limit.

7-19. Natural logarithms

It is often convenient to use natural logarithms in calculations involving equations in which the constant e is an inherent part. A table of natural logarithms is included in Appendix VI.

The natural logarithm of a number can be read directly on the Log-Log slide rule. Using the scale LL1, LL2, or LL3, the hairline is set on a number on the Log-Log scale; its logarithm is read on the D scale. When the LL0 or LL00 scale is used, the logarithm is read on the A scale. Readings on the D and A scales include the characteristic of the logarithm.

The natural logarithm of a number N can be found also from a table of common logarithms by using the conversion equation

$$\log_e N = \log_{10} N \times \log_e 10$$

Since $\log_e 10 = 2.303 \ldots$, this reduces to.

$$\log_e N \text{ or } \ln N = 2.303 \log_{10} N$$

The relation between natural and common logarithms is shown graphically in Fig. 7-16.

7-20. Other uses for semilogarithmic graphs

If the rate of growth of the given quantity is not uniform, the graph is no longer a straight line, but a curve. The direction of curvature indicates

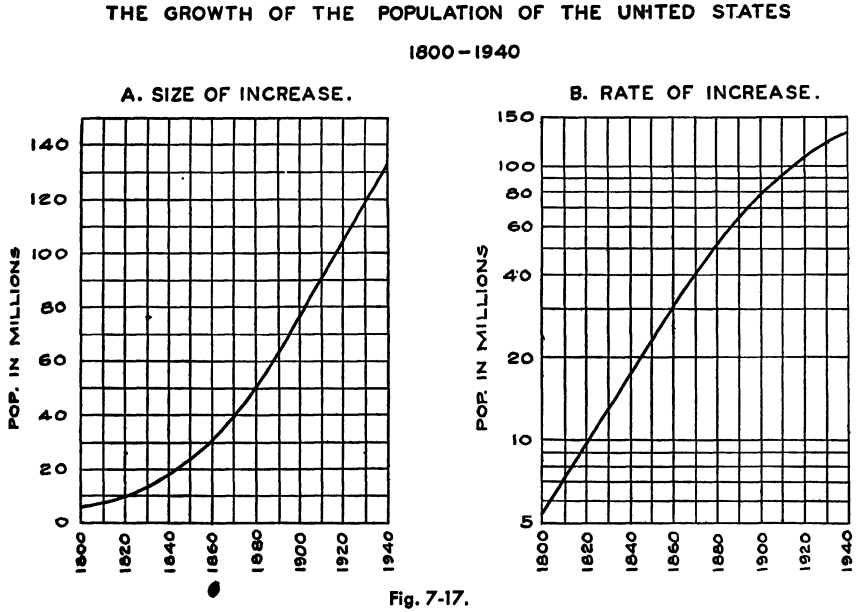


Fig. 7-17.

whether the rate of growth is increasing or decreasing. The semilogarithmic graph is therefore valuable in the study of business statistics, population trends, etc. The principal difference between a uniform plot and semilogarithmic plot is illustrated by the population study of Fig. 7-17. The uniform plot (A) shows the amount of population increase for any period; the logarithmic plot (B) shows the *rate* of change.

7-21. General procedure for data of unknown form

In general, it may not be known whether laboratory data can best be represented by a straight line, a power equation, an exponential equation,

or some other form. The procedures outlined in this chapter only indicate methods of determining an equation which closely fits the data within limited ranges taken in the laboratory. A closely fitting equation does not necessarily indicate that the natural phenomena involved are governed by a law of the character of the equation, though it establishes a strong presumption in that direction. Segments of power curves may be almost identical with segments of exponential curves, and thus either form may satisfactorily fit a limited range of data. Thus, an equation determined in the manner of this chapter may be safely interpolated but should not be extrapolated except with extreme caution.

If laboratory data of unknown form are obtained, certain inferences may be drawn which may shorten the procedure of obtaining a satisfactory equation.

1. Consider the nature of the phenomenon. An element of growth or decay strongly suggests an exponential equation.
2. Make a direct plot. Then
   a. The straight line is obvious.
   b. The power curve of the form $y = ax^m$ either goes through the origin or, for negative values of m, is asymptotic to both coordinate axes.
   c. The exponential curve of the form $y = ae^{mx}$ is asymptotic to the X axis and crosses the Y axis at the value $y = a$. Also, the exponential curve of the form $x = ae^{my}$ is asymptotic to the Y axis and crosses the X axis at the value $X = a$.
   Thus, if the curve passes through the origin, a power curve is probable. If it cuts one axis only, an exponential curve is probable.
3. Three nonuniform plots may be necessary.
   a. For a power curve on logarithmic paper. (Note that the equation may be obtained in two forms: $x = ay^m$ or $y = a'x^{m'}$. However, either of these will appear as a straight line on logarithmic paper.)
   b. For an exponential curve of the form $y = ae^{mx}$ on semilogarithmic paper with the uniform scale along the X axis.
   c. For an exponential curve of the form $x = ae^{my}$ with the uniform scale along the Y axis.

Plot first the form inferred from (2) above. If this yields a straight line, no further plot need be made unless it is desired to determine whether or not either form may be usable. Under any conditions one of the three plots should yield a straight line. In case no straight line is obtained, the equation is of some other form or contains an initial constant. It is then beyond the scope of this text, and more advanced texts should be referred to.

REFERENCES FOR READING

LIPKA, J.: "Graphical and Mechanical Computation," John Wiley & Sons, Inc., New York, 1918.

RUNGE, C.: "Graphical Methods," Columbia University Press, New York, 1912.

PROBLEMS

For these problems, except where logarithmic paper is necessary, use cartesian-coordinate paper, 20 squares to the inch. Calibrate and label the axes of each graph, stating the units used. Place an appropriate title at the top of each sheet. Determine the "best" equation in each problem, lettering the equation in an appropriate place on the sheet.

7-1. *Temperature coefficient.* The following observations were made to determine the temperature coefficient of a copper rod. Temperature C is centigrade. Resistance R is in microhms.

C, deg	19.1	25.0	30.1	36.0	40.0	45.1	50.0
R, microhms ...	76.30	77.80	79.75	80.80	82.35	83.90	85.10

7-2. *Lifting device.* f is the force in pounds which will just lift a number of pounds w.

w, lb	100	200	300	400	500	600	700
f, lb	9.0	12.6	17.4	22.0	25.4	29.6	34.2

7-3. *Angular deceleration.* In decelerating a certain machine it is essential that a uniform decrease in speed be maintained. The following observations were made. The speed v is in rps, the time t in seconds.

V, rps	13.50	12.40	10.10	8.50	6.60	4.50	2.10	.90
t, sec	0.1	1.1	2.5	3.8	4.6	6.3	7.7	8.8

7-4. *The power curve, $y = ax^m$*

1. DIRECT PLOT. Make a plot of three curves, using the following constants: (1) $a = 2$, $m = 2$; (2) $a = 2$, $m = \frac{1}{2}$; (3) $a = 2$, $m = -1$. Represent the curves in a single graph, letting the range of values for the X and Y axes vary from 0 to 5 units each. Scale: Unit = 1 in.

(2) LOGARITHMIC PLOT. Rectify the curves of graph 1 by plotting the x and y values on logarithmic paper. Check the intercept and slope of each curve with the constants a and m.

7-5. *The exponential curve, $y = ae^{mx}$*

1. DIRECT PLOT. Make a cartesian plot for three exponential curves, using the following constants: (1) $a = 1$, $m = 2$; (2) $a = 1$, $m = -1$; (3) $a = 1$, $m = \frac{1}{2}$. Represent the three curves in a single graph, letting the range of values for the X axis vary from -2 to $+2$; the Y axis, from 0 to e. Scale: Unit = 1 in.

2. SEMILOGARITHMIC PLOT. Rectify the curves of graph 1 by plotting the x and y values on semilogarithmic paper. Check the intercept and slope of each curve with the constants a and m.

The data of the following problems can be represented by power or exponential curves. In each problem first make a direct plot and then rectify the curve on suitable functional paper.

1. Plot on cartesian coordinates the equations $y = 0.1e^{2x}$ and $x = 0.1e^{2y}$ between values of 0 and 1 for both x and y.

2. Plot these equations on semilogarithmic paper, using the logarithmic scale for y.

3. Plot the equations on semilogarithmic paper, using the logarithmic scale for x.

7-7. Plot the following data on a single sheet of cartesian-coordinate paper in four ways, using the following four scales:

Left scale: y, from 0 to 14, one unit = ½ in.

Right scale: $\log y$, from −0.3 to +1.1, $\frac{1}{10}$ unit = ½ in.

Bottom scale: x, from 0 to 10, one unit = ½ in.

Top scale: $\log x$, from 0 to 1, $\frac{1}{10}$ unit = ½ in.

Plot x against y; x against $\log y$; $\log x$ against y; $\log x$ against $\log y$. One of these will be a straight line. From this determine the original equation.

x	0	1	2	3	4	5	6	7
y	0.5	0.79	1.26	1.99	3.16	5.00	7.93	12.56

7-8. *Tungsten Filament.* The following table gives the percent of normal luminous flux, F (lumens), of a tungsten filament at various percents of normal voltage.

V, % Normal.....	50	70	90	100	110	120	130	140
F, % Normal.....	11	31	71	100	135	185	232	300

7-9. *Amplitude of long pendulum.* The table gives the amplitude of vibration of a long pendulum. The amplitude y is in inches. The time in minutes since the beginning of the motion is t.

t, min	0.5	1	2	3	4	5	6
y, in.	6.99	4.90	2.53	1.20	0.60	0.31	0.16

7-10. *Electrical conductivity.* The results of measuring the electrical conductivity γ in mhos per cubic centimeters of glass at θ°F are as follows:

θ, °F	58	86	121	148	166	188	202	210
γ, mhos/cm³	No reading	0.004	0.009	0.018	0.029	0.051	0.073	0.090

7-11. *Volume-pressure relationship.* For saturated steam, the pressure P in pounds per square inch corresponds to the following volumes V in cubic feet per pound:

V, ft³/lb	53.92	26.36	14.00	6.992	4.280	2.748	1.853
P, psi	6.86	14.70	28.83	60.40	101.3	163.3	250.3

8

PERIODIC CURVES

THE SINUSOID

8-1. The periodic, or harmonic, law

This law represents cyclic change and is expressed by the circular functions. A general form of the periodic law is given by

$$y = a \sin (x + b)$$

This law describes the motion of reciprocating and rotating machine parts, alternating currents, and wave motion in general.

8-2. The sinusoid, or sine curve

Imagine a point P, Fig. 8-1, which moves with uniform velocity in a circular path of radius r. The axis OX passes through the center of the circle, and O is the initial position of point P. At any instant when the radius vector of P makes an angle of θ with OX, the vertical distance between P and OX is y, or $r \sin \theta$. The displacement of P from OX with respect to the angle θ as P travels around the circle can be represented in rectangular coordinates by plotting values of angle θ as abscissas and the corresponding values of y as ordinates. Angle θ is generally measured in radians.

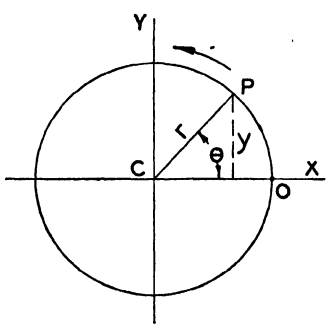


Fig. 8-1.

To construct a graph representing the relation between θ and y, draw a circle of radius r and center C, Fig. 8-2. The X axis is drawn through the center of the circle, and point O is the origin and initial position of P. To represent one complete revolution of P, set off on OX the distance OL equal to 2π, using any convenient scale. Divide OL and the circle into the same number of equal parts, as 12. Thus, each division of OL represents $2\pi/12$, or 30° of arc on the circle, starting at O. Through each division point of OL draw an ordinate having a length equal to the distance that P lies above or below OX for the cor-

responding value of the angle. This is accomplished by drawing the horizontals 1-1″, 2-2″, 3-3″, etc.

The smooth curve through the ends of the ordinates represents the sine curve or sine wave. Its equation is $y = r \sin \theta$. It is evident that for each additional complete revolution of P the curve will repeat itself at intervals or periods of 2π, extending to the right. This periodicity is indi-

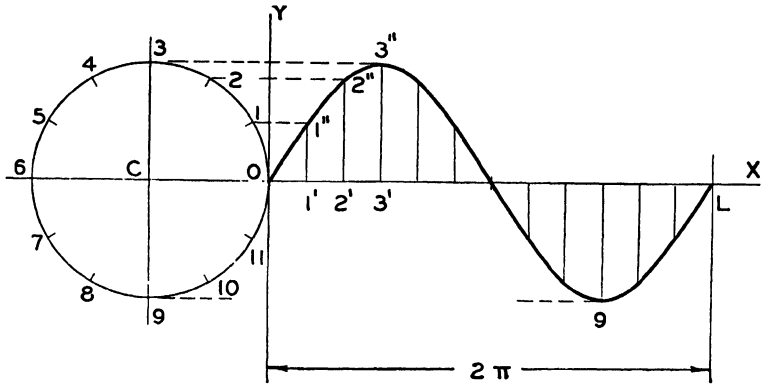


Fig. 8-2.

cated by the equation $y = r \sin (2\pi + \theta) = r \sin (4\pi + \theta) = r \sin (6\pi + \theta)$ etc., the period thus being equal to 2π.

The constant r enters the equation as a multiplier of y. The maximum value of y is equal to r and is called the amplitude of the curve.

In Fig. 8-2, point O of the circle is taken as the beginning point of a period. Let another point, as Q, Fig. 8-3, be taken as the starting point, and the angle between the X axis and the radius vector drawn to Q be denoted by α. When α is measured counterclockwise from CO, it is considered positive and is called the angle of advance; when α is measured

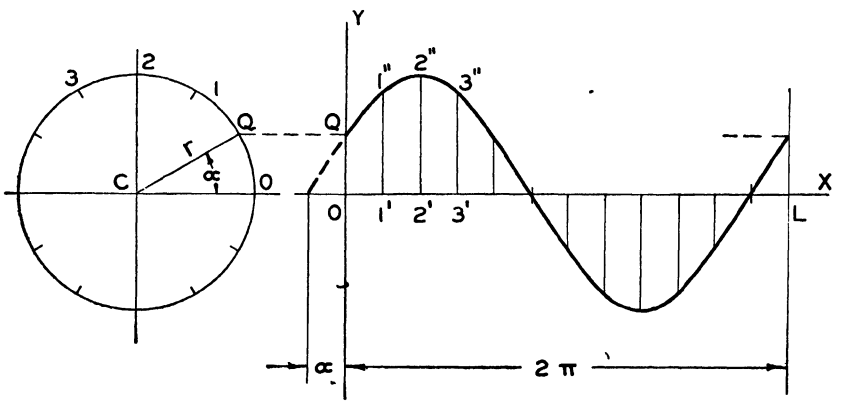


Fig. 8-3.

clockwise, it is negative and is called the angle of lag. The curve cuts OX at a distance α to the left of the origin. The effect of angle α is to move the curve in translation along the X axis. The angle α enters the equation of the curve as a constant. Thus, $y = r \sin(\theta + \alpha)$. When $\alpha = \pi/2$, $y = r \sin[\theta + (\pi/2)] = r \cos\theta$. Therefore, a cosine curve is a sine curve having a translation along OX equal to 90°.

8-3. Periodic motion

If a body moves in such a manner that every characteristic feature of its motion is repeated at equal intervals, the motion is said to be periodic. The motion may be rotary as in the case of the earth revolving about the sun, or it may be vibratory as in the vibrating string of a musical instrument. Other examples of periodic phenomena are found in wave propagation (heat, sound, radio) and in many machine motions. The rotating point of Fig. 8-1 represents the simplest form of periodic motion.

8-4. Simple harmonic motion

Let point P, Fig. 8-4, rotate uniformly in the circular path of radius r. At the same time let point M move up and down along the diameter YY' in such a manner that line PM is always perpendicular to YY'. Under these conditions point M is said to have *simple harmonic motion.* The motion is periodic and the displacement y of M from its mean position C is equal to $r \sin\theta$. Thus, simple harmonic motion can be represented graphically by the sine curve.

In practical problems the periodic time is usually given in seconds. In Fig. 8-4, if the angular velocity of P is ω radians per second, the angle θ swept out in a time interval of t seconds is ωt. Thus $\theta = \omega t$, and the equation of the sine curve becomes $y = r \sin \omega t$. The period of the motion is then equal to $2\pi/\omega$ seconds.

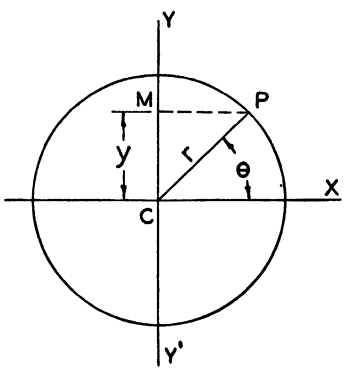


Fig. 8-4.

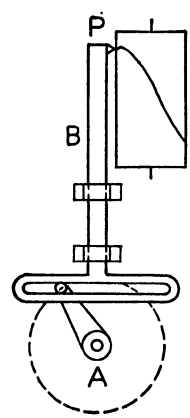


Fig. 8-5.

The number of periods per second is called the *frequency* of the motion. The frequency may be given in revolutions per second or cycles (or reversals) per second, one cycle in vibratory motion being equivalent to one revolution in rotary motion. In the equation $y = r \sin \omega t$, ω represents the angular velocity in radians. Therefore, if n is the frequency of a motion in revolutions or cycles per second, $\omega = 2\pi n$ and the equation of the sine or harmonic curve becomes $y = r \sin 2\pi nt$.

Harmonic motion can be produced mechanically by the well-known mechanism shown in Fig. 8-5. The crank A rotates with uniform velocity. The crankpin which operates in the slot of rod B gives the rod a reciprocating motion. If a pencil P is attached to the rod, the curve traced on the drum, which rotates uniformly, will be a harmonic curve.

8-5. Summary of equations for harmonic motion

$$y = r \sin (\theta + \alpha) \qquad \text{(sine law)} \tag{1}$$

In angular motion, $\theta = \omega t$. (This is analogous to the expression $s = vt$ in straight-line motion.)

Substituting in (1), we have

$$y = r \sin (\omega t + \alpha) \tag{2}$$

If n = frequency in radians per second (or revolutions or cycles per second), then

$$\omega = 2\pi n \tag{3}$$

Substituting (3) in Eq. (2), we have

$$y = r \sin (2\pi nt + \alpha) \tag{4}$$

Where T is the periodic time,

$$T = \frac{1}{n} \tag{5}$$

and

$$\omega = \frac{2\pi}{T} \tag{6}$$

Substituting (6) in Eq. (2), we have

$$y = r \sin \left(\frac{2\pi}{T} t + \alpha\right) \tag{7}$$

8-6. Composition of collinear harmonic curves

The harmonic curve p, Fig. 8-6, represents the motion of the radius vector or crank CP, which rotates uniformly and starts 30° in advance of the initial line CO. The harmonic curve q represents crank CQ, which rotates with the same velocity as CP but has a lead of 60° from the initial line.

The resulting effect of these two motions applied to a single body is found by combining the two curves. This may be done by adding the ordinates with proper regard for their signs. For example, ordinate 2-*a* plus ordinate 2-*b* is equal to ordinate 2-*c*. The resulting curve is given by *s*.

It can be shown that, when two or more collinear simple harmonic curves of the same period are combined, the resulting curve is a simple harmonic curve.

The curve *s* can be drawn directly by completing the parallelogram *CPSQ* in the polar diagram. The diagonal *CS* is the radius vector or crank

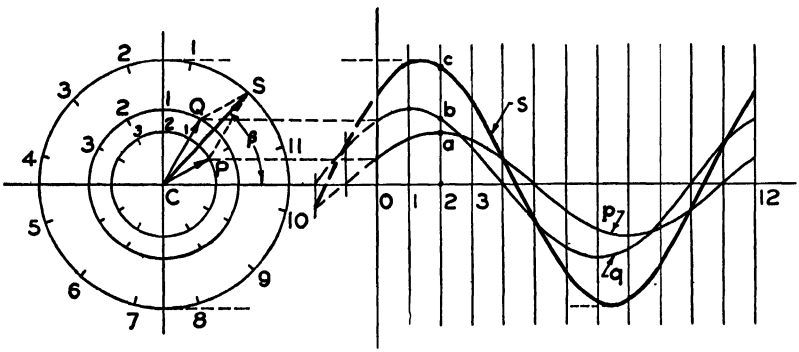


Fig. 8-6.

arm for the curve *s*, and the angle of advance is equal to β. The equation of this curve is therefore $y = CS \sin(\theta + \beta)$.

When the motions to be combined have different periods, the resulting motion is not simple harmonic motion. Hence, the curve representing such a motion is a periodic curve but not a simple harmonic curve.

Point *Q*, Fig. 8-7, rotates about the center *C* with uniform velocity. It has a lead of 60° and is represented by the curve *q*. Point *P* rotates

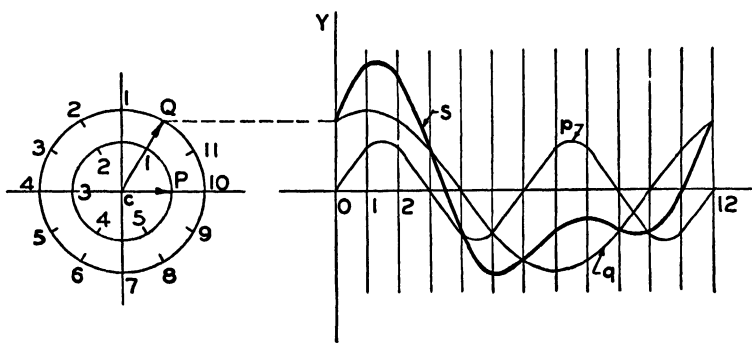


Fig. 8-7.

about C with a velocity of twice that of Q and starts on the initial line. The graph of this motion is shown by curve p, in which point P completes two cycles while point Q makes one. The curve s represents the combination of the two motions and is found by adding the ordinates of the two component curves. In this example the component curves, q and p, are the first and second harmonics, respectively, of the periodic curve s.

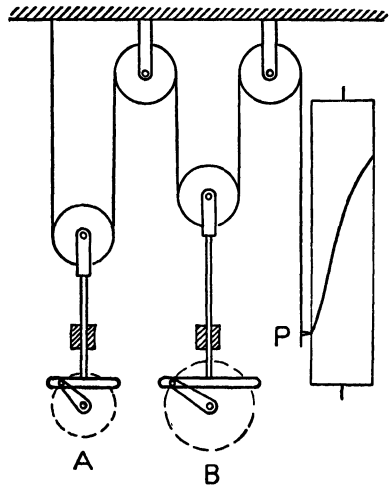


Fig. 8-8.

A device for combining two or more harmonic motions is shown in diagrammatic form in Fig. 8-8. The individual motions produced by the uniformly rotating cranks A and B are combined by means of the pulleys and string and recorded by the pencil P on the rotating drum. Owing to the action of the pulleys, the amplitude of each component curve is doubled. This mechanism was the basis of one of the early tide-predicting machines by Lord Kelvin.

Combinations of electrical wave forms can be represented visually by the *oscilloscope*, a well-known testing instrument embodying the cathode-ray tube.

HARMONIC ANALYSIS

8-7. The Fourier series

It has been shown (Art. 8-1) that, if a point has simple harmonic motion along an axis OY, its displacement from mean position is given by $y = r \sin(\theta + \alpha)$. Again, if a point has a periodic motion which is compounded of several simple harmonic motions, as $y_1 = r_1 \sin(\theta + \alpha)$, $y_2 = r_2 \sin(2\theta + \alpha_2)$, $y_3 = r_3 \sin(3\theta + \alpha_3)$, etc., the resultant displacement is given by the equation

$$y = r_1 \sin(\theta + \alpha_1) + r_2 \sin(2\theta + \alpha_2) + r_3 \sin(3\theta + \alpha_3) + \cdots + r_n \sin(n\theta + \alpha_n) \quad (1)$$

Also, it has been shown that the resultant displacement for such a periodic motion can be found graphically by plotting the component sine curves on a time, T, or an angle, θ, base and adding their corresponding ordinates to obtain the resultant curve.

Each term of Eq. (1) is called a harmonic of the resultant curve. The first term is the fundamental, or first harmonic. Succeeding terms repre-

sent curves having periods of one-half, one-third, etc., of the fundamental. These are second, third, etc., harmonics.

When the resultant displacement is measured from a base line parallel to OX and at a distance c below it, then Eq. (1) becomes

$$y = c + r_1 \sin(\theta + \alpha_1) + r_2 \sin(2\theta + \alpha_2) + r_3 \sin(3\theta + \alpha_3) + \cdots + r_n \sin(n\theta + \alpha_n) \quad (2)$$

The right-hand member of this equation is known as a Fourier series. The continuous periodic curves which depict natural phenomena have been found to be composed of various combinations of sine curves and hence can be represented by a Fourier series. This is an interesting and important finding.

8-8. Harmonic analysis

If a curve representing a periodic motion is supplied by a recording device, it can be resolved into a number of sine curves called simple harmonic components. This operation of breaking up a given periodic curve into simple harmonic components and determining the equation which represents it is called *harmonic analysis*. It is the opposite of the composition of collinear simple harmonic curves. The process of harmonic analysis is of great importance in the study of wave motions.

A method of graphical analysis is given here without proof. A discussion of the mathematical foundation for this method is given in Appendix II, Proof X. The graphical analysis for the Fourier series was presented by Harrison in "Engineering" (London, 1906).

8-9. A graphical solution for a Fourier series

Each harmonic has for its equation

$$Y_k = r_k \sin(k\theta + \alpha_k)$$

where $k = 1, 2, 3$, etc., the number of the kth term. The constants of this equation are r and α and are determined independently for each harmonic. The method of finding these constants will be illustrated by carrying out a solution, step by step, for a given periodic curve.

EXAMPLE. Let it be required to find the first three harmonic components for the curve of Fig. 8-9. OX is a convenient base line assumed so that all ordinates are positive.

Draw any number, say 12, of equally spaced ordinates of the curve, dividing the period of 360° into 30° intervals in this case.

First harmonic, $Y_1 = r_1 \sin(\theta + \alpha_1)$

1. Assume an origin O, Fig. 8-10, and draw axes OA and OB at right angles.

2. Starting at the origin, plot each ordinate of the curve as a vector, except ordinate 12 which is a duplicate of ordinate O, taking the ordinates

placed end to end to form a vector polygon. The angle between the axis OB and any given vector is equal to the angle θ for the corresponding

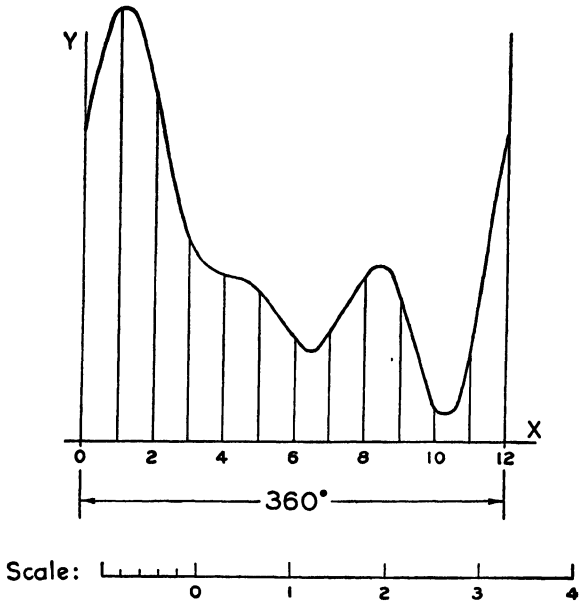


Fig. 8-9.

ordinate. Thus, for y_0, angle $\theta = 0°$; for y_1, angle $\theta = 30°$; for y_2, angle $\theta = 60°$; etc.

3. Draw the vector OP, which closes the polygon and balances the plotted vectors.

4. Measure the constants r and α as follows: The number of ordinates used $= n = 12$.

$$r_1 = \frac{2}{n} OP = \frac{1}{6} OP$$

α_1 = angle between vector OP and axis OA, measured positive in a clockwise direction from OA.

5. The constant c [Eq. (2), Art. 8-7] is equal to the average of all of the ordinates used. It is the distance from the base line OX to the axis of the curve. Thus,

$$c = \tfrac{1}{12} (y_0 + y_1 + \cdots + y_{11})$$

Second harmonic, $Y_2 = r_2 \sin (2\theta + \alpha_2)$

1. Assume the origin O, Fig. 8-11, and the rectangular axes OA and OB.

2. Lay off each ordinate of the given curves as a vector, the angle between each vector and axis OB being equal to twice the angle θ which

corresponds to each ordinate. Thus, for y_0, angle $2\theta = 0°$; for y_1, angle $2\theta = 60°$; for y_2, angle $2\theta = 120°$; etc.

3. Draw the closing vector OQ, and measure the values of r and α as for the first harmonic term.

Third harmonic, $Y_3 = r_3 \sin(3\theta + \alpha_3)$

1. Assume the origin O, Fig. 8-12, and draw the rectangular axes OA and OB.

2. Lay off each ordinate of the given curve as a vector, the angle between each vector and OB being three times the angle θ which corresponds

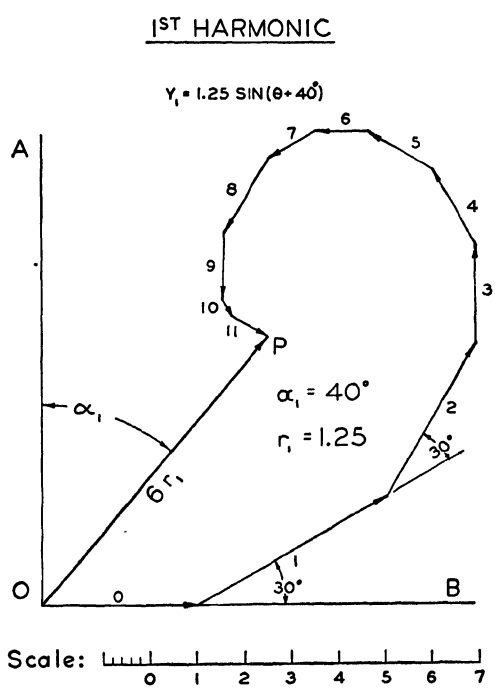


Fig. 8-10.

to each ordinate. Thus, for y_0, angle $3\theta = 0°$; for y_1, angle $3\theta = 90°$; for y_3, angle $3\theta = 180°$; etc.

3. Draw the closing vector OR, and measure the values of r and α as before.

The results of this example are checked by plotting and adding the three harmonics found, Fig. 8-13. The resulting curve Y_R closely resembles the given curve.

Missing harmonics. If the vector polygon for $k\theta$ closes by having the head of vector y_{11} fall on the origin O, the resultant is zero and the kth harmonic is missing.

The number of harmonics required in a given solution varies with the nature of the problem. In some engineering problems 3 or 4 might suffice.

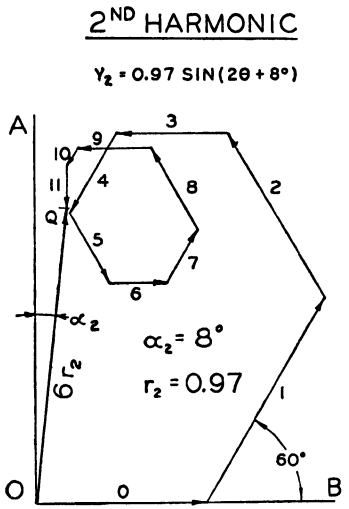


Fig. 8-11.

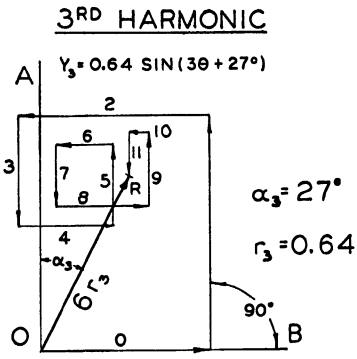


Fig. 8-12.

In electrical work it is seldom necessary to obtain as many as 12. Harmonic analyzers occasionally obtain more than this, but it is difficult to imagine a situation in which more than 50 would be useful.

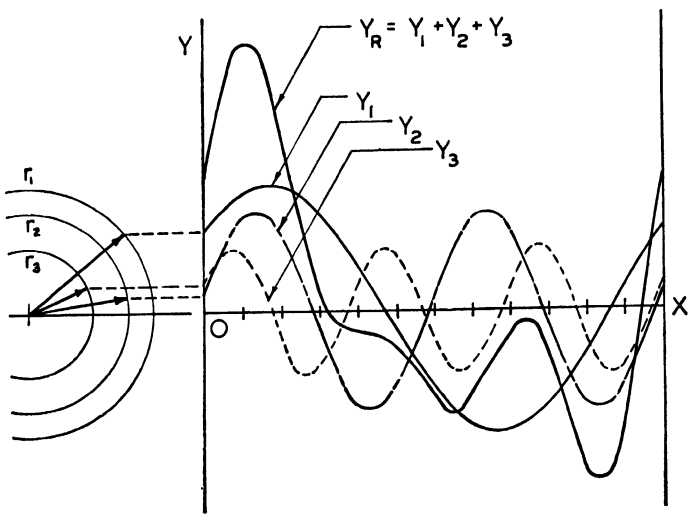


Fig. 8-13.

REFERENCE FOR READING

LIPKA, J.: "Graphical and Mechanical Computation," John Wiley & Sons, Inc., New York, 1918.

PROBLEMS

Plot the following problems on cartesian-coordinate paper, 20 squares to the inch.

8-1. Draw a graph for the equation $y = 1.5 \sin \theta$. Show one period. Scales: y, unit = 1 in.; θ, 2π radians = 6 in.

8-2. A weight hangs from a spring and moves up and down in simple harmonic motion. The amplitude is 1.75 in., and the period P is 2 sec. The angle of advance α is $\pi/2$ radians (time is reckoned from this point). The displacement y of the weight from mean position at any time t is therefore given by the equation

$$y = a \sin\left(\frac{2\pi}{P} t + \alpha\right) = 1.75 \sin\left(\pi t + \frac{\pi}{2}\right)$$

Draw a graph which shows the relation between y and t for any value of t between 0 and 2 sec. Scales: Time, 3 in. = 1 sec; y, full size.

8-3. Plot the collinear sinusoidal waves

$$Y_1 = 1.00 \sin \frac{\pi}{3} t$$

and

$$Y_2 = 1.40 \sin\left(\frac{3}{\pi} t + 60°\right)$$

These waves are combined. Draw the resultant wave. Write the equation. Scales: y, 1 in. = 1 unit; time, 1 in. = 1 sec.

8-4. Plot the collinear simple harmonic motions having the following characteristics:

I	II
Amplitude = 1.50 in.	Amplitude = 1.10 in.
Frequency = $\frac{1{,}000}{6}$ cps	Frequency = $\frac{1{,}000}{6}$ cps
Angle of lag = 30°	Angle of advance = 45°

Scales: Amplitude, full size; time, 1 in. = 0.001 sec.

8-5. Use Problem Sheet 8-5.

8-6. Use Problem Sheet 8-6.

8-7. Determine the first three harmonics of the periodic curve determined by the following measurements: curve scale: one inch = one unit; plot vectors at half scale.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
y	6.05	4.60	4.59	5.25	4.87	4.25	4.30	3.37	1.11	0.40	3.12	6.09

9

GRAPHICAL CALCULUS

9-1. Derived curves

In previous chapters many of the curves which arise in engineering practice have been classified into power, exponential, and periodic curves. Within each of these classifications there are groups or families of curves that are related geometrically. For example, graphs of the power equation $y = ax^m$ become, for positive values of m, a family of curves which include the straight line, parabola, cubic parabola, and curves of higher degree, portions of which resemble parabolas. The curves comprise the parabolic family and are so related that, if one curve of the family is given, every other member of the family can be derived from it. It is the purpose of this chapter to discuss the laws which apply to derived curves and to develop methods for applying these laws to empirical data.

9-2. The area law and graphical integration

The process of finding the area under a curve is called integration. It consists in dividing the area under the curve into a series of narrow strips and finding, by approximate methods, the area of each strip. The sum of the areas of the strips approximates the area under the curve. The following example will serve to illustrate the graphical method.

EXAMPLE. Let it be required to find the area between $x = 0$ and $x = 10$ for the following tabulated function:

x	0	1	2	3	4	5	6	7	8	9	10
y	30.0	23.0	18.2	14.7	11.8	9.5	7.6	6.0	4.5	3.5	2.5

An (x, y) plot is made, using convenient scales for the axes OX and OY, Fig. 9-1. The area under the curve is divided into a series of narrow strips by drawing ordinates at $x = 1, 2, 3$, etc. The number and spacing of the ordinates in any given case is determined by the shape of the curve. The spacing need not be uniform. The mean ordinate for each strip is approximated by drawing a horizontal line which cuts the arc between two consecutive ordinates so as to form a pair of equal "triangles." This line is drawn by trial, using the edge of a strip of thin transparent material.

When the areas under consideration are small, the eye is quite sensitive to differences and a good approximation of the mean ordinate is obtained. The integral curve is then constructed. It represents the running sum of the strip areas between $x = 0$ and $x = 10$.

The integral curve, represented by x and y' coordinates, is plotted directly above the given curve, using the same scale for x. In order to

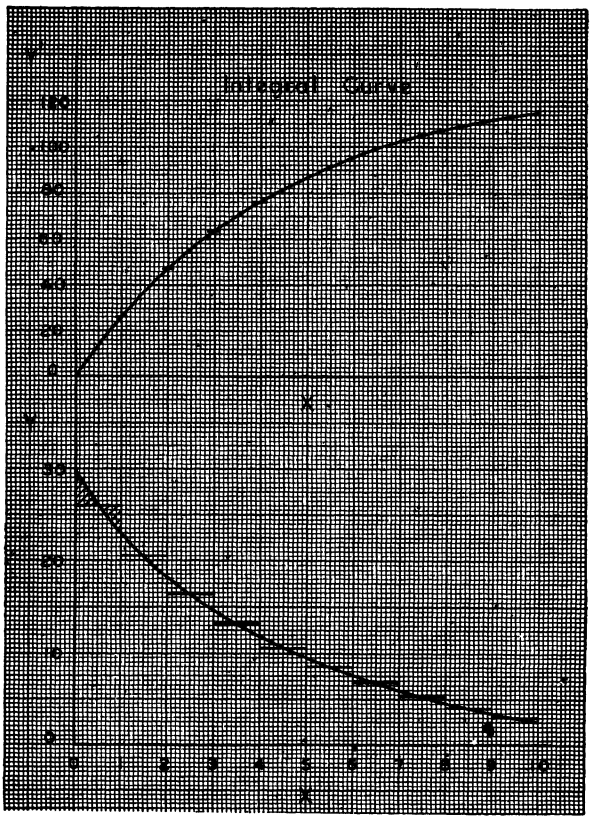


Fig. 9-1.

choose a suitable scale for y', it is necessary to know, approximately the maximum value to be plotted. This value is equal to the total area under the given curve. The mean ordinate of the entire curve, using the procedure described above, is equal to about 12 units. Hence, the total area is roughly 120 square units. A scale unit is selected to make the range $y' = 0$ to $y' = 120$ a convenient length.

The integral curve is started at $x = 0$. The ordinate at $x = 1$ is equal to the area of the strip between 0 and 1. This value is 26. The ordinate

at $x = 2$ is equal to the sum of the areas for strips 0-1 and 0-2. Thus, $y_2' = 26 + 20.5 = 46.5$. The remaining ordinates are plotted in the same manner, and a smooth curve is drawn through the points. If the scales are equal, these additions can be made graphically with the dividers. The total area under the given curve is equal to 115 square units. It is evident that the difference in length between any two ordinates of the integral curve is equal to the area under the given curve between the corresponding ordinates.

In drawing a family of curves, it is customary to arrange the curves on the sheet one above the other according to degree. Since an ordinate of an integral curve represents the product of two quantities taken from the given curve, it is one degree higher than that curve.

The principle that relates a curve and its integral curve can be stated as follows:

THE AREA LAW. *The area between any two ordinates of a given curve is equal to the difference in length between the corresponding ordinates of the next higher curve.*

9-3. Graphical integration by use of a ray polygon

The area under a curve can be found by a completely graphical process. Let it be required to find the area bound by the figure *YZTOY*, Fig. 9-2.

Divide the area into vertical strips by drawing ordinates at points *Q*, *R*, *S*, *T*. Find the mean ordinate of each strip by drawing the horizontal lines *AB*, *CD*, *EF*, *GH*, cutting the arcs between consecutive ordinates so as to form pairs of equal "triangles."

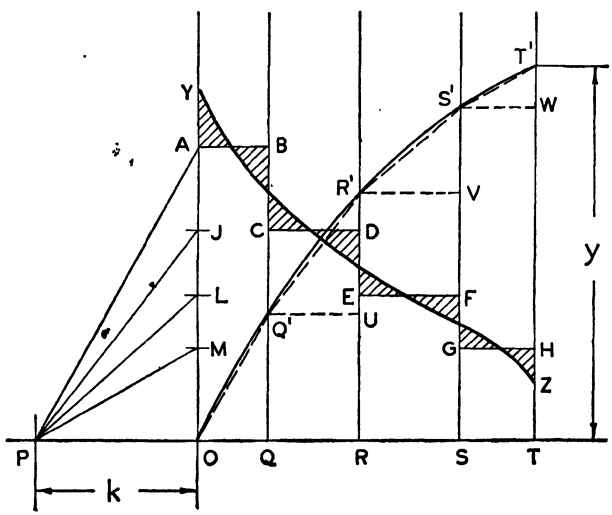


Fig. 9-2.

On OY mark the height of the mean ordinate of each strip, as OA, OJ, OL, OM. On OX produced assume any point P as a pole, and draw the rays PA, PJ, PL, PM.

To construct the integral curve, start at point O, and draw OQ' parallel to PA; $Q'R'$ parallel to PJ; $R'S'$ parallel to PL; $S'T'$ parallel to PM. A smooth curve through points O, Q', R', S', T' represents the integral curve. In this method the curve is superimposed on the original axes.

The area under the given curve, YZ, is equal to $PO \times TT'$ or $k \times y$. The factor k is measured in the units assigned to the X axis, the ordinate y in units of the Y axis.

PROOF. Triangles POA and OQQ' are similar. Hence,

$$\frac{k}{OQ} = \frac{OA}{QQ'}$$

Therefore,

$$k \times QQ' = OQ \times OA = \text{area of } OQBA$$

Similarly,

$$k \times R'U = \text{area of } QRDC$$
$$k \times S'V = \text{area of } RSFE$$
$$k \times T'W = \text{area of } STHG$$

Now

$$y = TT' = QQ' + R'U + S'V + T'W$$

Hence

$$k \times y = \text{area } OQBA + QRDC + RSFE + STHG$$
$$= \text{area under the given curve (approx)}$$

9-4. Uses of an integral curve

Any ordinate of the integral curve of Fig. 9-1 represents the area under the given curve between the corresponding ordinate and the origin. This area is the product of the average value of y for the interval and the length of the interval, x. If the variables x and y represent physical units, the ordinates of the integral curve represent a quantity which is the product of these units.

For example, if acceleration is plotted as y and time as x, the ordinates of the integral curve represent velocity ($v = at$). If velocity is plotted as y and time as x, the ordinates of the integral curve represent displacement ($s = vt$). If force is plotted against distance, the integral curve represents moment ($M = fd$).

Thus, by means of integration one is able to deduce certain new relations from given relations.

9-5. To derive a displacement curve from a velocity curve

A body moves with the velocity shown for a time interval of 9 sec, Fig. 9-3. It is desired to construct a graph which will show the displacement of the body with respect to time.

The distance the body will move during any time interval is equal to the product of its average velocity and the length of the time interval. Therefore, if the area under the velocity curve is divided into narrow strips to represent short time intervals, the area of each strip will represent the displacement of the body during that time interval. The summation of the strips will represent the total displacement in 9 sec.

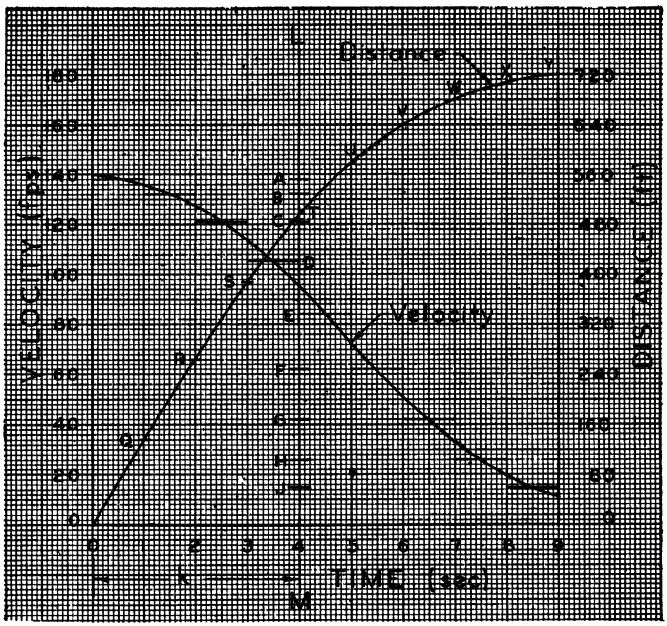


Fig. 9-3.

The area is divided into nine strips, and the mean ordinate of each strip is found. Since k (ray-polygon method) and y are measured according to the scales assigned to the X and Y axes, respectively, a value of k is selected that will yield an integral curve that will fit the space available. Estimating the average ordinate of the curve to be 75, the total area under the curve is roughly 675. If k be made equal to 4 sec, the final ordinate of the integral curve will be about 169 (as read on the V scale), a satisfactory size in this case.

In practical problems it is generally convenient to superpose the ray polygon, as POA of Fig. 9-2, on the graph, allowing P to coincide with the origin O. In Fig. 9-3, the base of the ray polygon, k, is made equal to 4 sec. Draw line LM through $k = 4$, and mark off the height of each mean ordinate on LM, as A, B, C, D, etc.

To construct the integral curve, draw OA, cutting the ordinate for $x = 1$ at Q. Draw chord QR parallel to OB; draw chord RS parallel to

OC, etc. A smooth curve through points O, Q, R, S, etc., represents the required integral curve.

The ordinate for $x = 9$, read in the original velocity units, is equal to 180. Hence the distance at $x = 9$ is 4 sec $\times$ 180 fps = 720 ft.

In order to read values on the distance curve directly, without reference to the factor k, a distance scale is laid off on the right. Each scale reading is k times the corresponding number on the velocity scale. In this case, the scale factor is 4.

9-6. The slope law and graphical differentiation

A second class of derived curves represents the slope of a given curve at every point. The slope of a curve at a given point is defined as the tangent of the angle formed by the X axis of the graph and the tangent to the curve at the given point. This value is calculated in terms of the scale units of the X and Y axes.

In Fig. 9-4, A is the given curve and B the derived curve. At a point P on curve A, the slope is equal to $\Delta y/\Delta x$. The corresponding point P'

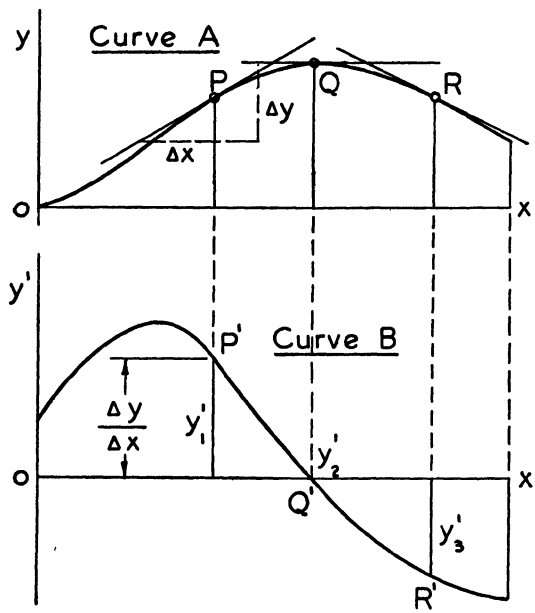


Fig. 9-4.

on curve B has an ordinate y_1', equal to the numerical value of $\Delta y/\Delta x$. Any convenient scale units can be assigned to the axes. When the slope of curve A is negative as at R, the ordinates of curve B are negative. When the slope of A is zero, as at Q, the corresponding point Q' of B is on the X axis.

A simple application of the slope law is given in Fig. 9-5. Since the slope of OA is constant and positive, the corresponding segment of the derived curve is parallel to and above OX. The slope of AB is also constant and positive. The slope of BC is negative, and the corresponding segment of the derived curve lies below OX.

The process of obtaining the slope of a curve at any point is called *differentiation*. The graph which represents the slope of a given curve at

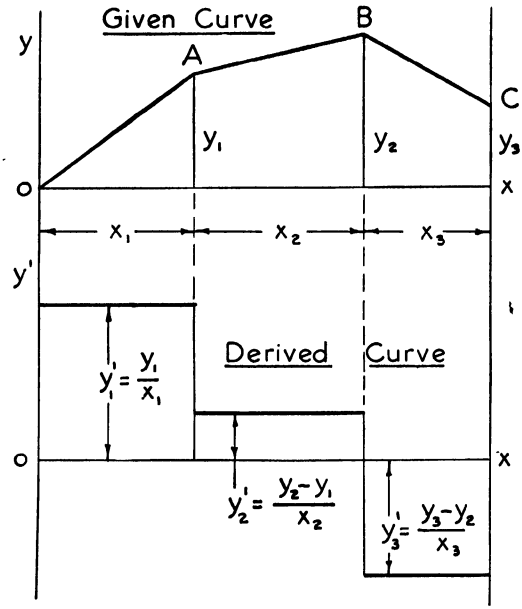


Fig. 9-5.

any point is called a *derivative* curve. The ordinates of the derivative curve B, Fig. 9-4, result from the division of two quantities read from graph A. Therefore, curve B is one degree lower than curve A, and so when a family of curves is being represented, a derivative curve is usually placed below the curve which is being differentiated.

It will be shown that there is a complementary relationship between integration and differentiation. In Fig. 9-4, curve A is the integral curve of B, and, conversely, curve B is the derivative curve of A.

The relation of a given curve to its derivative curve is as follows:

THE SLOPE LAW. *The slope at any point of a given curve is equal to the ordinate of the corresponding point of the next lower derived curve.*

9-7. Uses of a derivative curve

In any given engineering problem certain information is available. Other information pertinent to the solution of the problem is deduced

from it by various means. Many problems involve finding the change in a certain physical quantity as a second and related quantity varies. For example, the displacement of a body with respect to time may be known and its velocity at any instant desired.

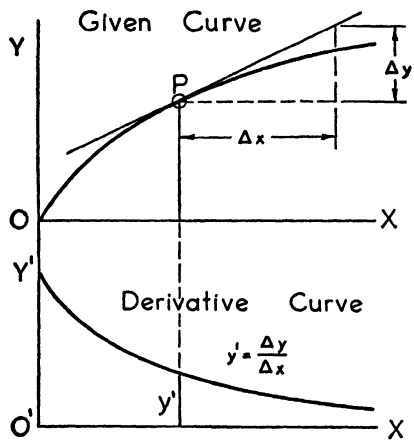


Fig. 9-6.

Let the ordinates of the given curve, Fig. 9-6, represent distance y plotted against time x for a moving body. The derivative curve is constructed on axes $XO'Y'$. The slope of the tangent at any point, as P, on the given curve now represents distance/time, which is velocity. Hence, ordinates of the derivative curve represent the velocity of the body at any time, x.

Similarly, if the ordinates of the given curve represent the quantity of water discharged from a pipe plotted against time, the ordinates of the derivative curve represent quantity/time, which represents the velocity of the water in the pipe at any given instant.

If the given curve represents velocity plotted against time, the derivative curve represents acceleration against time.

9-8. Graphical differentiation by use of the slope law

DRAWING A TANGENT. The principal difficulty in graphical differentiation is that of accurately drawing a tangent line to a curve of unknown analytical form. One practical method is to place a transparent strip, marked at the edge with two dots about $\frac{1}{10}$ in. apart, so that the dots lie on the curve. The edge of the strip will practically coincide with the tangent, the point of tangency lying midway between the dots.

The second method locates a normal to the curve by means of a plane mirror. A mirror, placed across the curve and perpendicular to the paper, is rotated until the curve and its image appear unbroken at the edge of the mirror. The normal then lies along the reflecting surface and can be drawn. The tangent is drawn at right angles to the normal. If a common glass mirror is used, the thickness of the glass should be allowed for in drawing the normal.

9-9. Graphical differentiation by use of the area law

In Fig. 9-7, curve B is the derivative of curve A. The slope of the tangent to A at point P is equal to the mean ordinate y_m' at point P' of B.

Now, draw the chord QR parallel to the tangent. The ordinates of point Q and R are y_1 and y_2, respectively. The slope of the chord is the same as the slope of the tangent. Therefore,

$$\frac{y_2 - y_1}{x_2 - x_1} = y_m'$$

and

$$y_2 - y_1 = y_m'(x_2 - x_1) = \text{area } x_1x_2DC$$

Hence, the difference between the ordinates y_2 and y_1 of A is equal to the area between the corresponding ordinates of B and the line CD, drawn through P' and parallel to OX. CD will cut the derivative curve so that the two shaded "triangular" areas are equal.

This indicates that a derivative curve can be constructed by reversing the method described in Art. 9-2, for integration. This method avoids the necessity of drawing tangents to a general curve.

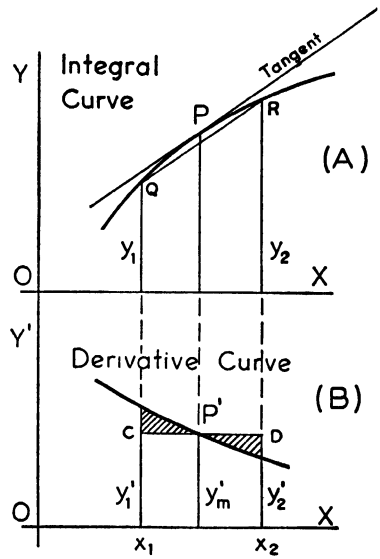


Fig. 9-7.

EXAMPLE (FIG. 9-8). Let it be required to draw the derivative curve for the following tabulated function:

x	0	1	2	3	4	5	6	7	8	9	10	11
y	0	16	26	32	36.5	39	40	39	36.5	32.5	26	16

Plot the curve, and divide it into intervals by drawing ordinates $x = 1, 2, 3$, etc. For any interval as Δx of the given curve find the difference in length of the boundary ordinates, Δy. This length divided by Δx is the length of the mean ordinate of the corresponding strip of the derived curve. Thus for the interval between $x = 1$ and $x = 2$ of the given curve we have a mean ordinate for the derived curve equal to $(26 - 16)/1 = 10$. Plot this value as the height of the corresponding strip of the derivative graph.

In this example to avoid a flat derived curve its ordinates are taken at twice the scale used for the given curve. Since the strips are of uniform width, this operation can be carried out graphically by setting Δy on the dividers and stepping it off twice to obtain the mean ordinate of the derived curve. In this manner draw all the horizontal lines which represent the mean ordinates of the derived curve. Draw a smooth curve through these horizontals so that the triangular areas above and below the curve are approximately equal. This curve is the derivative curve.

It is not necessary that the given curve be divided into equal intervals. A mean ordinate of the derived curve can be found for any pair of ordinates of the given curve and its value plotted to scale on the derivative graph.

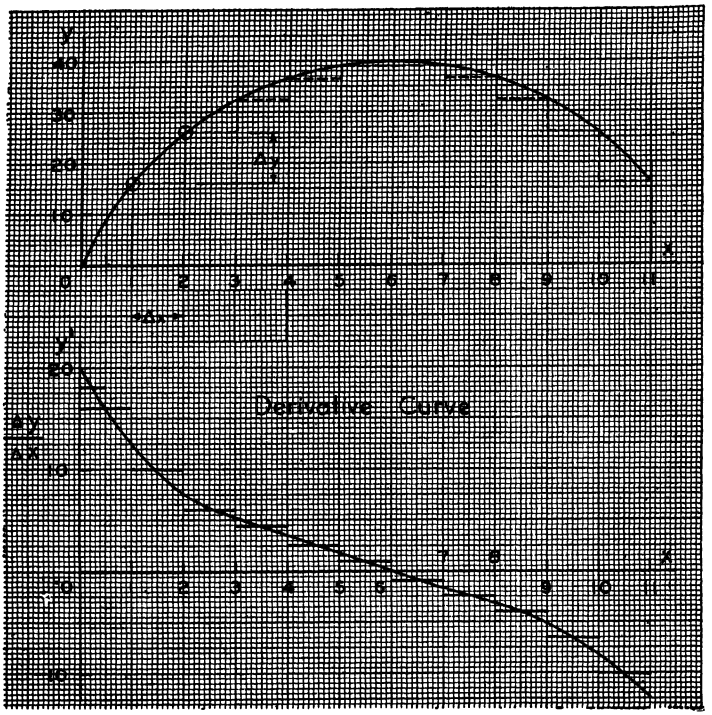


Fig. 9-8.

9-10. A family of curves

The following simple example of a family of three derived curves will further illustrate the principles involved.

EXAMPLE 1 (FIG. 9-9). Construct the derived curves which show the acceleration-time, velocity-time, and distance-time relationships for a body moving as follows:

First time interval: Time = 14 sec, acceleration = 4 fpsps, initial velocity = 0.

Second time interval: Time = 8 sec, velocity varies uniformly from its maximum at the beginning of this time interval to zero at the end of the interval.

Write the equation for each curve.

SOLUTION. *First time interval:*

1. Lay off the time intervals of 14 sec and 8 sec to scale, and draw the axes for the acceleration, velocity, and distance curves.

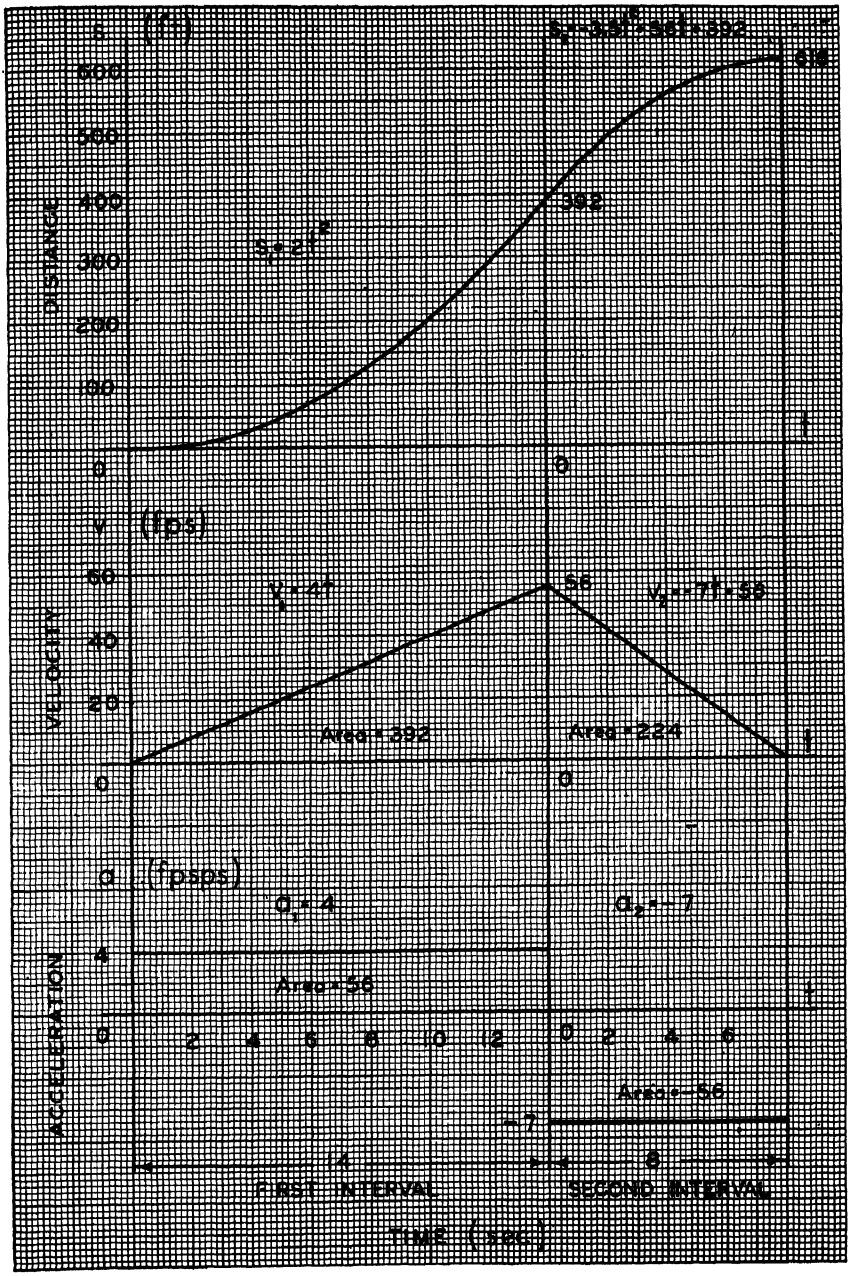

s (ft)
DISTANCE
600
500
400
300
200
100
0
$s_1 = 2t^2$
$s_2 = 3t^2 - 36t + 392$
392
v (fps)
VELOCITY
60
40
20
0
$v_1 = 4t$
56
$v_2 = -7t + 56$
Area = 392
Area = 224
(fpsps)
ACCELERATION
4
0
$a_1 = 4$
$a_2 = -7$
Area = 56
Area = -56
-7
0 2 4 6 8 10 12
0 2 4 6
8
FIRST INTERVAL
SECOND INTERVAL
TIME (sec.)

Fig. 9-9.

2. *The acceleration curve.* Draw the line $a_1 = 4$ to represent the given acceleration. The area under this line between $t = 0$ and $t = 14$ is equal to 56 fps.

3. *The velocity curve.* The initial velocity = 0. Therefore the ordinate at $t = 14$ is equal to 56 fps (the area under the acceleration curve). Since the ordinates of the acceleration curve are constant, the slope of the velocity curve is constant and equals 4. It is thus a straight line of equation $V_1 = 4t$.

4. *The distance curve.* The initial distance = 0. The ordinate of the distance curve at any point t is equal to the area under the velocity curve between the origin and point t. Since the slope of the velocity curve everywhere = 4, this area, s_1, is triangular and is equal to $\frac{1}{2}t \times 4t = 2t^2$. Thus, the distance curve is the parabola $S_1 = 2t^2$. At $t = 14$ the area is $\frac{1}{2} \times 14$ sec $\times$ 56 fps = 392 ft.

Second time interval: Take the origin for the second group of curves at the beginning of this time interval.

5. *The velocity curve.* The problem states that the velocity for the second interval changes uniformly from its maximum at the beginning of the interval to zero at the end. Therefore, draw a straight line from $V = 56$ when $t = 0$ to $V = 0$ when $t = 8$.

The area under this line is equal to 224 ft. The slope of the line is $-56/8 = -7$; its intercept is $+56$. Hence, its equation is $V_2 = -7t + 56$.

6. *The distance curve.* The ordinate of the distance curve at any point t is equal to area under the velocity curve between zero and t plus the initial distance, 392 ft. This area S_2 is represented by

$$\tfrac{1}{2}t[(-7t + 56) + 56] = -3.5t^2 + 56t.$$

Therefore, the distance curve is the parabola $S_2 = -3.5t^2 + 56t + 392$. The two parabolas are tangent at $S = 392$. The ordinate at $t = 8$ is equal to 392 + 224 = 616 ft.

7. *The acceleration curve.* The slope of the velocity curve = -7. Draw the line $a_2 = -7$. The area under this line between $t = 0$ and $t = 8$ is -56 fps, which is equal to the difference in the corresponding ordinates of the velocity curve.

9-11. Numerical integration

The following methods for numerical integration are useful when it is required to find only an area and not an integral curve. The numerical method becomes partly graphical when the data used are measured on a drawing.

1. TRAPEZOIDAL RULE. To find the approximate area bounded by the curve $y = f(x)$, the X axis, and the ordinates $x = x_0$ and $x = x_n$, Fig. 9-10, divide the interval between $x = x_0$ and $x = x_n$ into n strips of uniform

width w and measure the ordinates $y_0, y_1, y_2 + \cdots + y_{n-1}, y_n$. If the arcs between the ends of the ordinates are replaced by chords P_0P_1, P_1P_2, etc., the area of the trapezoids thus formed is

$$A = w[\tfrac{1}{2}(y_0 + y_n) + y_1 + y_2 + \cdots + y_{n-1}]$$

One or both end ordinates may be equal to zero.

2. SIMPSON'S RULE. Divide the interval between $x = x_0$ and $x = x_n$ into an even number of strips of uniform width w, Fig. 9-10. If the given

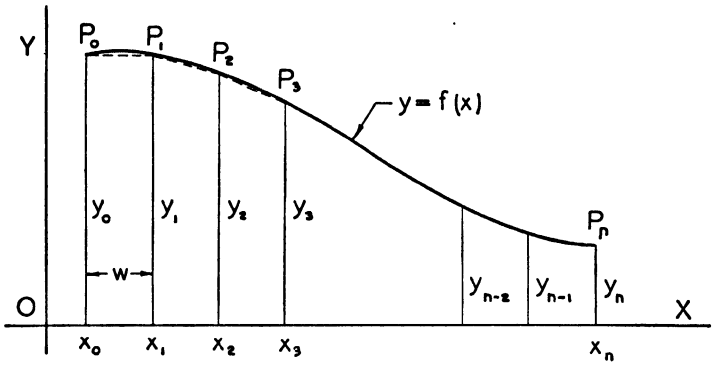


Fig. 9-10.

curve is replaced by a series of parabolic arcs $P_0P_1P_2$, $P_2P_3P_4$, etc., the area under the line thus formed, the X axis and ordinates $x = x_0$ and $x = x_n$, is

$$A = \tfrac{1}{3}w[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \cdots + y_{n-2})]$$

or the area is equal to the sum of the end ordinates plus four times the sum of the odd-numbered ordinates plus twice the sum of the even-numbered ordinates, all multiplied by one-third the width of one strip.

3. VOLUME OF A SURFACE OF REVOLUTION. The trapezoidal rule or Simpson's rule can be used to find the volume of a solid of revolution by allowing the ordinates y_0, y_1, etc., to represent the radii of plane cross sections taken at a uniform distance w apart.

9-12. The formal calculus

DIFFERENTIATION. The differential calculus is concerned with methods of calculating for a given function, the limit of the ratio of the change in the dependent variable to the change in the independent variable as this change approaches zero. For the equation $y = f(x)$ this limit is written as dy/dx. This ratio is represented graphically by the slope of a tangent to the curve having the equation $y = f(x)$, as in Fig. 9-4. Any ordinate, y', of the derived curve represents the ratio dy/dx, and the equation of the

derived curve becomes $dy/dx = f'(x)$ or $y' = f'(x)$. The slope law of derived curves is the graphical equivalent of the process of differentiation of the calculus.

For a given curve, the descending curves of the family can be obtained by the process of differentiation. When the equation of the curve is known, the best method is that of the differential calculus. If the data upon which the curve is based are empirical, the graphical method provides a practical solution. In many cases where the equation is known, the solution of a problem may be clarified by sketching the derivative curves approximately.

INTEGRATION. The area law of derived curves represents the process of integration of the calculus. The integral of the curve $y' = f'(x)$ between two specified limits is defined as the total change in the variable y when the rate of change (slope at any point) is given. This total change in y is represented graphically by the area under the derived curve between the corresponding limits. In Fig. 9-4, the slope dy/dx of each point of curve A is given by an ordinate of curve B. The equation of curve B is therefore $dy/dx = f'(x)$. The total change in dy/dx between y_2 and y_1, curve A, is represented by the area under curve B between y_2' and y_1'. Curve A is the graph of the integral of curve B, and its equation is $y = \int f'(x)\,dx$ or $y = f(x) + C$. Since the area under B between any pair of ordinates gives only the difference between the lengths of the corresponding ordinates of A, the process of integration applied to curve B gives only the shape of A. Its position with respect to the X axis is determined by initial conditions which must be known or assumed. Therefore, when a curve is integrated by the calculus and limits are not stated, a quantity, C, called the constant of integration, must be added to the resulting equation to provide for initial conditions, *i.e.*, to fix the curve with respect to the coordinate axes.

9-13. To evaluate $\int \frac{1}{x}\,dx$.

An important problem in the calculus is to integrate the equation $y = 1/x$, the rectangular hyperbola. The following table lists a series of integrals and their derivatives which are obtained by the general formula for integration:

Integral	x^3	x^2	x^1	x^0	?	x^{-1}	x^{-2}
Derivative ...	$3x^2$	$2x$	1	0	x^{-1}	$-x^{-2}$	$-2x^{-3}$

All the whole powers of x are given in both tables. However, no integral appears for x^{-1}. The formula fails in this case and leaves a gap in the upper row.

The problem here is to determine the form of this integral experimentally by means of the graphical methods developed in this and the preceding chapters.

1. Construct a portion of the rectangular hyperbola $y = x^{-1}$, Fig. 9-11.

2. Construct the integral curve, making the ordinate zero at $x = 1$. In general, integration may be started at any point, since the starting point affects only the constant of integration. Complete the curve between the limits of $x = \frac{1}{2}$ and $x = 4$. The ordinates are measured on the Y' scale.

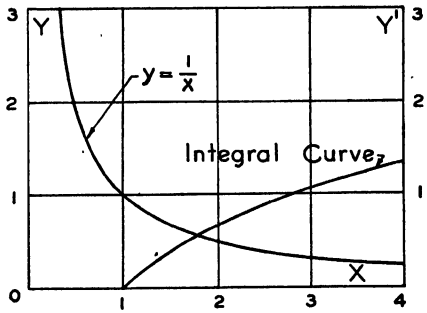


Fig. 9-11.

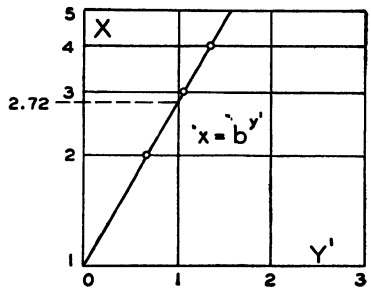


Fig. 9-12.

3. Rectify the integral curve, and determine its equation, Fig. 9-12. As a result of several trials it is found that the curve is rectified on semi-logarithmic paper when x is plotted on the logarithmic scale and y' on the uniform scale. This indicates that the curve is an exponential curve having an equation of the general form $x = a(b)^{y'}$. The intercept $a = 1$. Therefore the equation is $x = b^{y'}$. It remains to find the value of the constant, b. When $y' = 1$, $x = b$ and this value, read from the graph, is the number 2.72. The value of this number computed to three decimal places is 2.718. It is an irrational number known as Euler's number and is denoted by e. Therefore, the equation of the integral curve is $x = e^{y'}$ or $y' = \log_e x$. Hence, the integral of $y = 1/x$ is equal to $\log_e x$. This is commonly written $\ln x$.

This is an excellent example of the predictive and illuminative power of graphics. The meaning of Euler's numbers is little understood by beginning students of mathematics. Though the above derivation has, of course, no mathematical validity as a proof, its illustrative value is very high.

PROBLEMS

For these problems, except where noted, use cartesian-coordinate paper, 20 squares to the inch.

9-1. *Derived curves.* Construct curves showing the distance-time, velocity-time, and acceleration-time relationships for a body moving as follows:
First time interval:
Time = 12 sec
Acceleration = 6 fpsps throughout the interval
Initial velocity = 0
Second time interval:
Time = 9 sec
Acceleration = −3 fpsps throughout the interval
Write the equation for each curve for each interval. The curves need not be drawn to the same ordinate scale.

9-2. *Derived curves.* A locomotive approaching a station coasts at a velocity of 50 fps for 20 sec. The brakes are then applied, and the train is brought to a stop with a constant negative acceleration of 3 fpsps. Draw the curves showing the acceleration-time, velocity-time, and distance-time relationships. Write the equation for each curve for each interval.

9-3. *Derived curves.* An automobile travels 150 ft in 5 sec with a constant velocity. It is then brought to a stop in 5 sec with a constant negative acceleration. Draw the distance-time curves, and find the equation of each curve for each interval.

9-4. *Derived curves.* A body has a constant velocity of 12 fps for 6 sec. It is then brought to rest in a distance of 10 ft with uniform deceleration. Draw the distance-time, velocity-time, and acceleration-time curves, and find the equation of each curve.

9-5. *Graphical differentiation.* The distance-time relationship for a moving body is given by the following table. Find the velocity-time curve.

t, sec	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
s, in.	0	1.0	2.5	4.0	6.0	8.0	10.0	11.5	12.5	13.0

9-6. *Graphical differentiation.* Points on the curve $y = f(x)$ are as follows:

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0
y	0	14.6	24.0	29.0	30.0	28.4	24.0	18.2	12.0	5.8	0	−4.5	−6.0	−4.9	0	9.5	24.0

Draw a smooth curve through the points and construct the derivative curve.
Scales:
Given curve: x scale: 1 in. = 2 units. Origin at 2, 6½.
y scale: 1 in. = 10 units.
Derivative curve: y' scale: 1 in. = 20 units. Origin at 2, 4.
Second derivative curve: y'' scale: 1 in. = 20 units. Origin at 2, 2.
Note: The following equations are given for checking purposes.
Given curve: $y = x^3 - 12x^2 + 35x$
Derivative curve: $y' = 3x^2 - 24x + 35$
For point of inflection: $y'' = 6x - 24$
It is suggested that the student check several points on both the given curve and the derivative curve, using the above given equations.

9-7. *Graphical differentiation.* The velocity of a body at the end of certain time intervals is as follows:

T, sec ...	0	1	2	3	4	5	6	7	8	9
V, fps ...	0	1.400	2.325	2.800	3.000	3.070	3.000	2.720	2.130	0.900

Draw the velocity-time curve and the acceleration-time curve. Origin of both curves at 1, 5. Velocity = acceleration × time. $dv = a\,dt$, $a = dv/dt$.

9-8. *Graphical integration.* The average acceleration for a moving body is recorded for certain time intervals in the following table. Draw the velocity-time curve.

Time interval, sec .	0–0.5	0.5–1.0	1.0–1.5	1.5–2.0	2.0–2.5	2.5–3.0	3.0–3.5	3.5–4.0	4.0–4.5
Av. acceleration ..	2.80	2.05	1.10	0.50	0.14	−0.14	−0.44	−1.30	−2.80

9-9. *Hydraulic press.* The piston of a hydraulic press has a 10-in. stroke. The force exerted in various points of the stroke is as follows:

Distance, in.	0	1	2	3	4	5	6	7	8	9	10
Force, tons	6.0	7.0	8.0	8.8	9.0	8.4	7.0	5.4	4.0	3.0	2.4

a. Draw the force-distance curve. Origin at 1, 1. Distance scale: 1 in. = 2 in. of stroke. Force scale: 1 in. = 4 tons.

b. Construct the work curve, and find the work done in one stroke. Work = force × distance.

9-10. *Riveting machine.* In a certain riveting machine the force with which a plunger strikes a pin is measured for one stroke as follows:

Force, lb	0	6	19	37	52	55	52	40	22	10	2	0
Time, sec	0	0.01	0.02	0.03	0.04	0.045	0.05	0.06	0.07	0.08	0.09	0.10

a. Draw the force-time curve. Origin at 1, 1½. Time scale: 1 in. = 0.02 sec. Force scale: 1 in. = 20 lb.

b. Construct the impulse curve. J = impulse = force × time. Impulse scale: 1 in. = 0.4 lb-sec.

9-11. *Graphical integration.* Problem sheet 9-11. Use ray polygon.

9-12. *Graphical integration.* Problem sheet 9-12. Use ray polygon.

9-13. *Graphical differentiation.* Problem Sheet 9-13.

9-14. *Trapezoidal and Simpson's rules.* Problem Sheet 9–14.

9-15. *The integral of* $1/x$

a. Construct a portion of the rectangular hyperbola $y = x^{-1}$. Unit = 2 in.

b. Construct the integral curve, making the ordinate of the integral curve zero at $x = 1$. Complete the curve between the limits of $x = \frac{1}{2}$ and $x = 4$. In general, integration may be started at any point, since the starting point affects only the constant of integration.

c. Rectify the resultant integral curve, and determine its equation.

d. Determine the value of x when $y = 1$. This is the constant e. This number, e, is known as Euler's number and is the base of natural logarithms. It is an irrational number whose value is calculated more precisely by means of a series.

10

THE GEOMETRY OF PROJECTION DRAWING

10-1. Orthographic projection

A complete discussion of the geometry of orthographic projection will be more illuminating if its general nature and standardized procedures are discussed briefly before the fundamental geometric theory is developed. The primary purpose of a drawing of an object may be to create a visual image of the object, or it may be to show accurately the shape of the object for purposes of manufacture. Under any conditions a drawing may be explained as a process of projection. If one holds a transparent plane in front of an object and traces on it the lines forming the object as seen by one eye, the result is a projection drawing, Fig. 10-1. Any point P of the object is represented on the plane by the point P', where the sight line from the object point to the eye pierces the plane. The position of the eye is called the *center of projection.* The sight lines are called *projectors,* and the plane is called a *projection plane.* In Fig. 10-1, the projectors converge to a point representing the eye or point of sight. This type of drawing, depicting the object as it actually appears to the observer, is called *perspective* drawing and is useful for pictorial purposes. However,

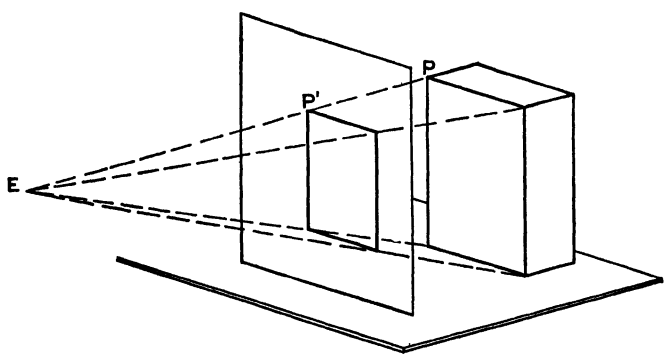


Fig. 10-1.

as a result of the convergence of the sight lines, the true proportions of the object do not appear in the projection, and the drawing is thus not a suitable shape description for manufacturing purposes.

A second type of drawing, called orthographic, is produced if the projector from each point of the object is made perpendicular to the plane, Fig. 10-2. Theoretically, the observer is assumed to be an infinite distance

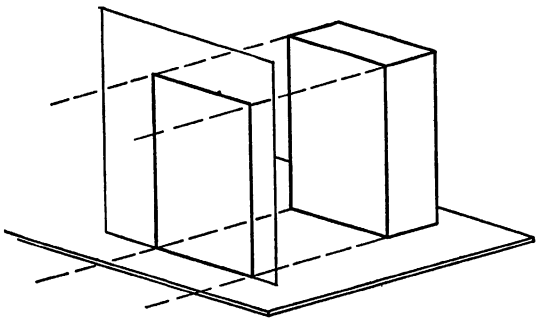


Fig. 10-2.

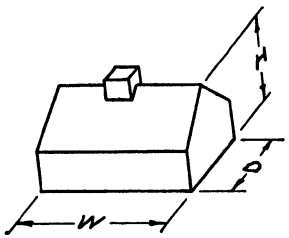


Fig. 10-3.

from the plane. In orthographic projection every edge or plane face of the object that is parallel to the plane projects in its true size. Since the true proportions of an object can be maintained in orthographic projection, it is the more important for technical purposes.

10-2. Objects

The principal axes of the majority of objects are three mutually perpendicular lines. The axes of an object are usually related to the earth by making one vertical and the other two horizontal. The principal dimensions of an object are measured parallel to the principal axes and are commonly called the *width*, *height*, and *depth*, Fig. 10-3. This orientation is accepted as standard in American practice.

10-3. Principal views

The principal projections, or views, of an object are those which show its principal dimensions. It is evident that but two of these dimensions can appear in true size in a single orthographic projection. Thus, the width and depth appear in true size on the *horizontal* projection plane, Fig. 10-4. This projection is called the *top view* and represents the object as it would appear to an observer looking down through the horizontal plane. The width and height appear in true size on the *vertical* projection plane taken parallel to these dimensions. This projection is the *front view*. The profile projection plane is perpendicular to both the vertical and the horizontal planes. The projection on the profile plane shows the height

and depth of the object in true size and is called the *profile* or *side view*. The profile plane may be taken on either the right or the left side of the

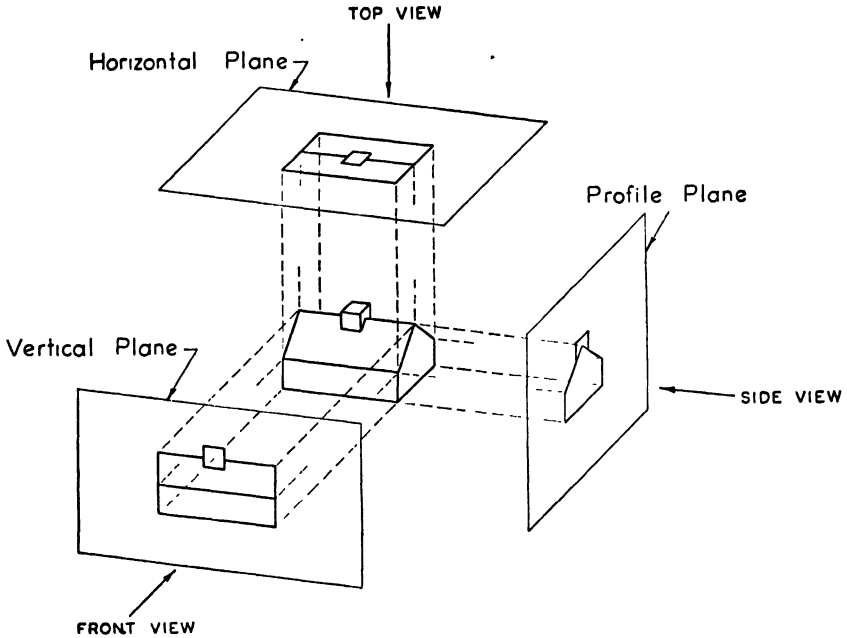


Fig. 10-4.

object. The resulting projections are called right side or left side views. In every case the projection plane is assumed to be between the object and the observer.

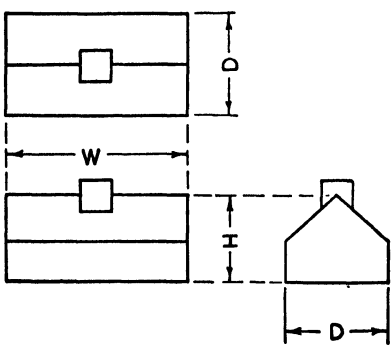


Fig. 10-5.

10-4. Arrangement of the views

The top, front, and side views of an object are arranged on the drawing paper in a standard manner. The top view is placed above and in line with the front view, and the width shows as the dimension common to both views, Fig. 10-5. The right side view is placed to the right of and in line with the front view, and the height appears as the dimension common to the front and side views. A left side view would be placed to the left of the front view. Any given view is a view of the *near* side of the adjacent view.

10-5. Visualization

Reading a drawing is the mental process of visualizing the object from its views. A view should be regarded as the object itself. The observer should assume that his direction of sight is vertically downward when looking at a top view and horizontal when looking at a front view. A single view shows two principal dimensions of an object; the third dimension appears in the *adjacent* view.

Thus, visualizing a top view requires that one build up a mental picture of the object by estimating the relative height of each point and line of the drawing. A front view is visualized by estimating the relative depth of each part of the drawing.

While the foregoing description of drawing as a process of projection is convenient for technical purposes, it should be noted that the method of representing an object by its top and front views is a natural one. In setting down on paper a concept of an object, one intuitively draws views which show its various faces in true proportions. The architects of the ancient world made plans and elevations of their buildings without recourse to any formal theory of projection, so far as we know.

10-6. Views of simple objects

A few examples of objects represented in orthographic projection will serve to illustrate the principles developed in the preceding paragraphs.

In general, an object is placed so that the front view shows its characteristic shape, Fig. 10-6. In general the longest dimension is taken as horizontal. Only the views which are necessary clearly to describe the object should be drawn. A view in which most of the edges of the object are visible is preferred to a view in which a large number of lines are hidden. When the right side view and left side view are of equal value, the right side view is preferred.

Bottom views and rear views of an object are rarely used. The arrangement for these views is shown in the ASA Standards, Sec. 1 (Appendix I).

A fixed convention is followed in the drawing of a cylinder, Fig. 10-6*D*. Two center lines at right angles are shown in the circular view; the axis of the cylinder is drawn in the rectangular view.

Since only a portion of the surface of a solid can be seen from a given position, the view of a solid usually contains both visible and hidden lines. The matter of correctly representing these visible and hidden edges is an essential part of the drawing of any solid. The correct forms for hidden and center lines are shown in the ASA Standards, Sec. 17, (Appendix I).

10-7. Sketching in orthographic projection

Facility in the use of orthographic projection as a method of representing objects is best developed by practice in drawing top, front, side views, etc.,

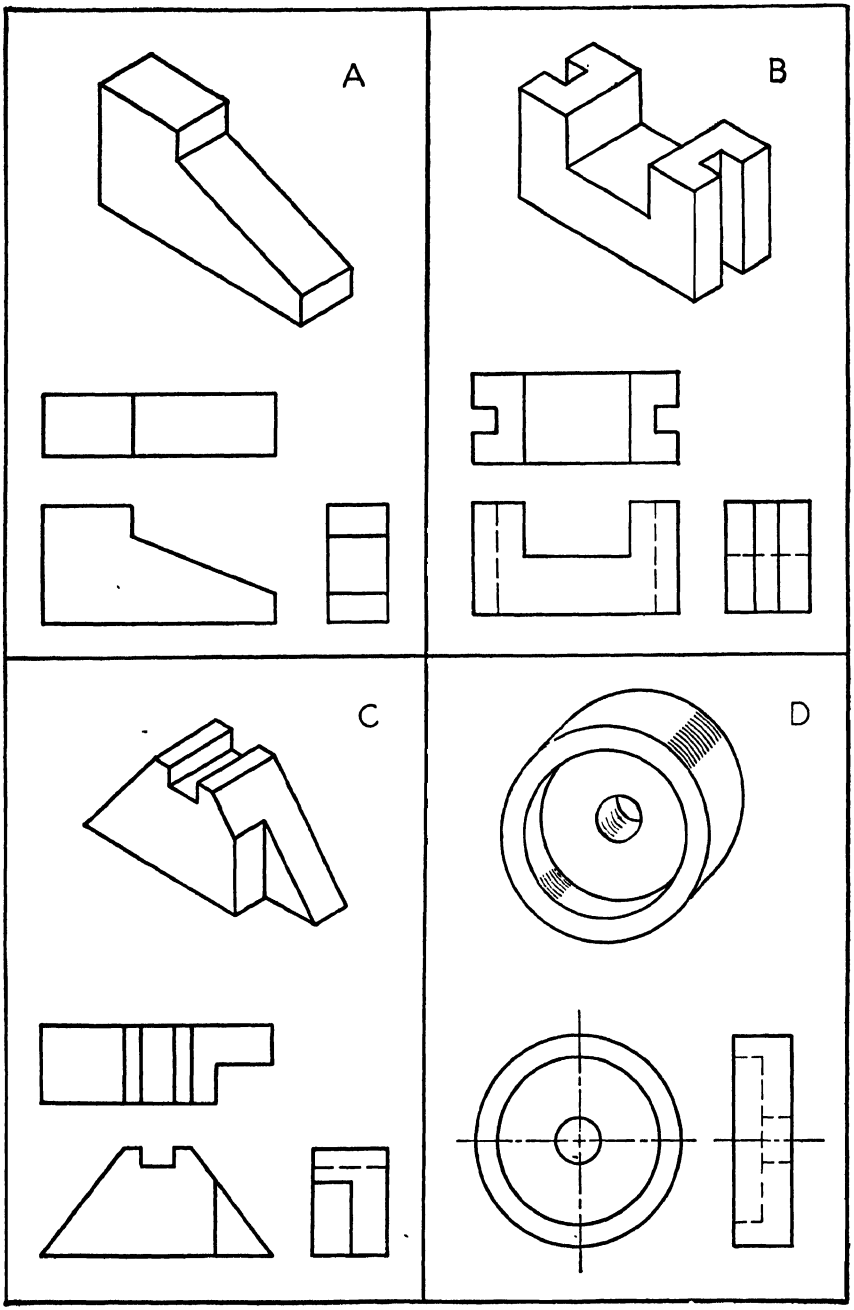


Fig. 10-6.

of a given object. Freehand sketching is a suitable medium for this study. Where it is not convenient to use actual objects, a pictorial representation of the object is a satisfactory substitute. The sketch may be made on

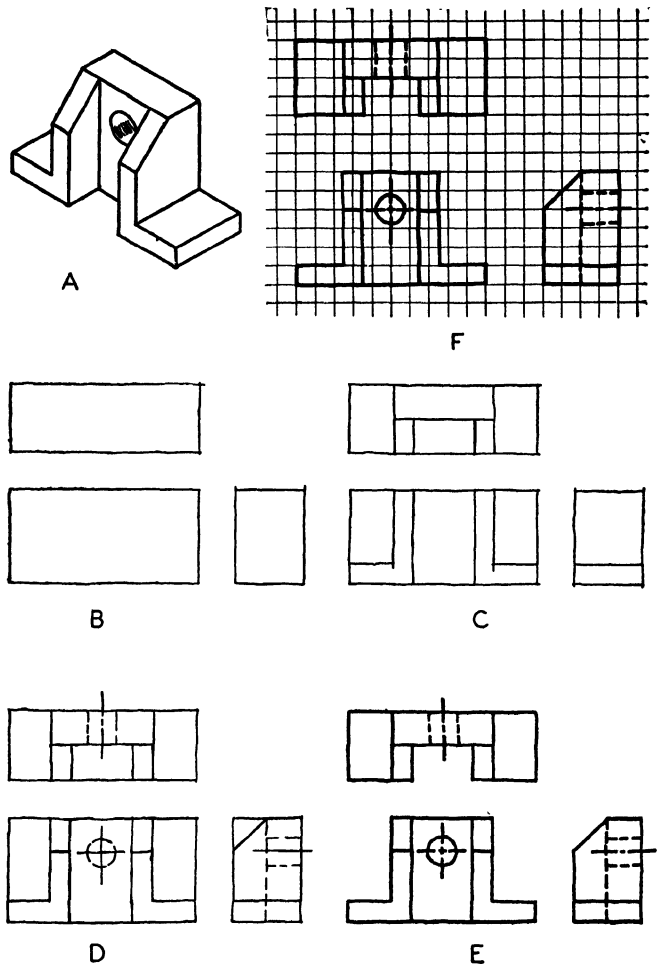


Fig. 10-7.

either plain or "squared" paper. The views are drawn in proportion to dimensions of the object, but not to exact scale.

The object of Fig. 10-7*A* is represented best by its top, front, and right side views. Block in the three views, as in (*B*). The lines are made light at this stage, using an F or 2H pencil. The detail is filled in in the order shown in (*C*) and (*D*), still using lightweight lines and developing the three views together. Finally, the sketch is cleaned with an eraser, removing

excess lines, overruns at the corners, etc., and the visible, hidden, and center lines are brightened with a bold line (*E*). All finish lines are of the same weight. The use of squared paper (*F*) helps in proportioning the views and simplifies the technique of freehand drafting.

10-8. Pictorial sketching

The interpretation of an orthographic drawing by means of a pictorial sketch provides valuable experience in reading a drawing. A simple type

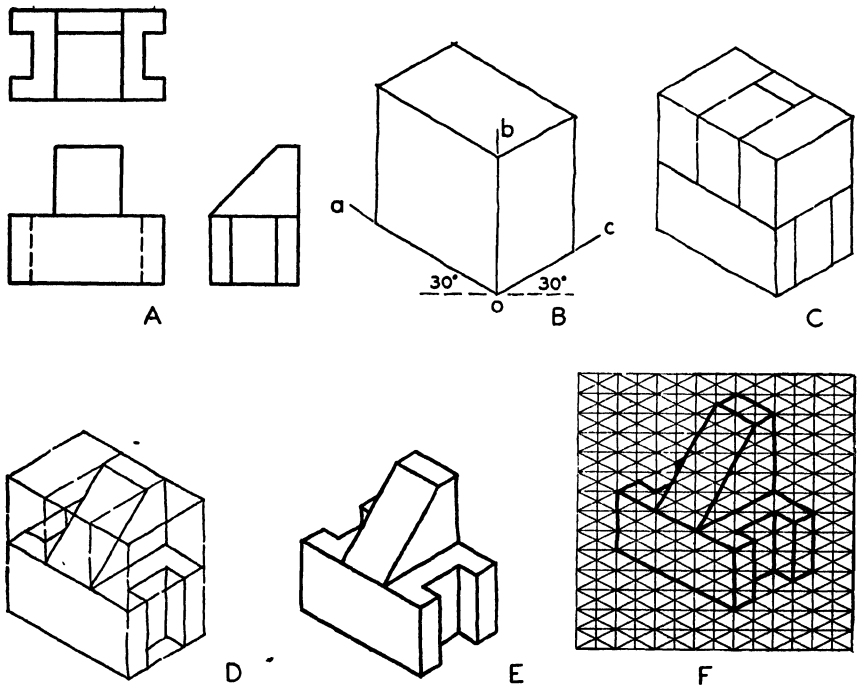


Fig. 10-8.

of sketch is that known as isometric (Ref. Chapter 12). Let it be required to make an isometric sketch from the orthographic projections of Fig. 10-8*A*.

Through an assumed point *O*, Fig. 10-8*B*, three lines *a*, *b*, *c* are drawn as axes, one vertical, the other two making 30° angles with the horizontal. For a sketch these angles can be estimated. These lines are taken to represent three mutually perpendicular lines of space. If the proportionate width, height, and depth of the object are marked off on or parallel to these axes, a view of the "box" which contains the object results. Each face of the object is now outlined on the corresponding face of the pictorial box and the necessary connecting lines inserted to yield the pictorial view

of the object, Fig. 10-8*C*, *D*, *E*. Hidden lines are omitted. A specially ruled paper called isometric paper is useful for this type of sketching, Fig. 10-8*F*.

10-9. Projection

With the brief discussion given above as a background, a more formal discussion of the geometric theory of projection drawing may now be undertaken. We shall distinguish two types of projection.

1. CENTRAL PROJECTION (Fig. 10-9). When the center of projection is a finite distance from the projection plane, the system is called *central* or

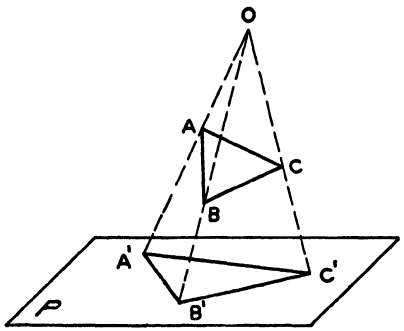


Fig. 10-9.

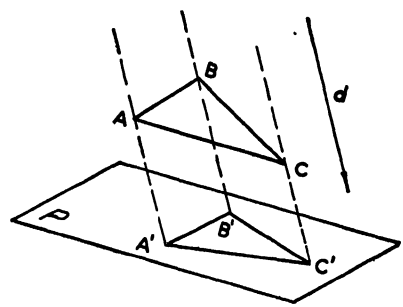


Fig. 10-10.

conical projection (see Art. 3-9). A point A is projected on a plane P by passing a projector through the point and the center of projection O. The point in which this projector cuts the plane is the projection A' of the given point. The projection of a straight line is the locus of the projections of its points and is, in general, a straight line.

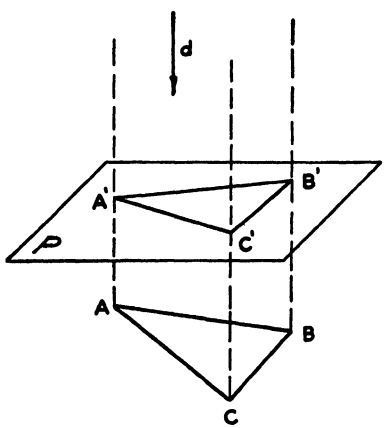


Fig. 10-11.

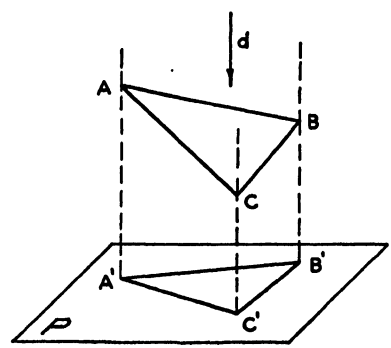


Fig. 10-12.

2. PARALLEL PROJECTION. If the center of projection retreats in a given direction to infinity, the elements of the projection system are the projection plane and a given direction of projection, not parallel to the plane. When this direction d makes an angle other than 90° with the plane, the projection system is called oblique, Fig. 10-10. When the direction d is perpendicular to the plane, the system, as explained in Art. 10-1, is called orthographic, Figs. 10-11 and 10-12.

The orthographic projection of an object on a given plane is equivalent to the *view* or image of the object as seen by an observer infinitely distant from the plane and with his direction of sight perpendicular to the plane. Theoretically, the projection plane may be taken either on the near side or on the far side of the object from the observer.

10-10. Correspondence

It is evident that a single projection of a point on a plane does not completely represent the position of the point with respect to the plane. Every point lying in the projector has the same projection. In order that a projection system be useful for technical purposes, it is necessary that a given point of space have a unique graphical representation and, conversely, that the graphical representation of a point correspond to a single point of space. When this occurs, it is said that a *one-to-one correspondence* exists between the point in space and its graphical representation.

One method for accomplishing this correspondence is to give the distance between the space point and its orthographic projection by means of an index number attached to the projection, Fig. 10-13. This method is called *horizontal projection* and has its application in mapping. For example, the summit of a hill is located on a map by its geographical coordi-

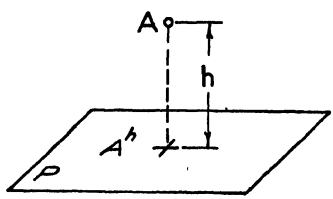


Fig. 10-13.

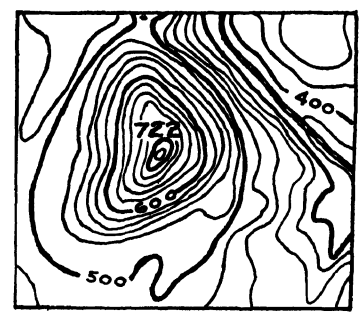


Fig. 10-14.

nates, and its elevation above mean sea level is given as a number, Fig. 10-14.

A second method for establishing a one-to-one correspondence between a space point and its graphical representation employs two mutually perpendicular projection planes. The point of space A is projected orthog-

onally on each plane, Fig. 10-15. The projections A_1 and A_2, considered as marks on the plane, are the graphical representations of the point. Since the space point A lies at the intersection of the two projectors, the pair of projections A_1 and A_2 can correspond only to A. Thus, to every point of space (except points on the intersection of the planes) there corresponds a unique pair of projections. This method of orthographic projection is the standard one for engineering drawings. The following discussion develops its principles.

10-11. Quadrants

When the projection planes are indefinite in extent, they divide all space into four quadrants numbered as shown in Fig. 10-16. This is a right

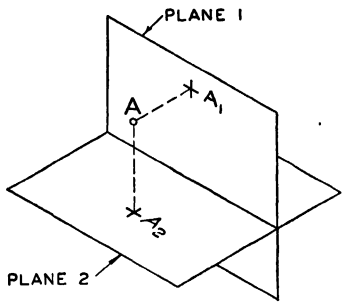


Fig. 10-15.

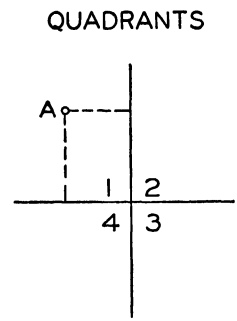


Fig. 10-16.

side view of the planes. They appear edgewise with their intersection as a point. It is standard practice in American drawing to place the object in the third quadrant. In third-quadrant projection the projection plane is assumed to be between the observer and the object. The first quadrant is used in European drawing. In the first quadrant the object is always between the observer and the projection plane.

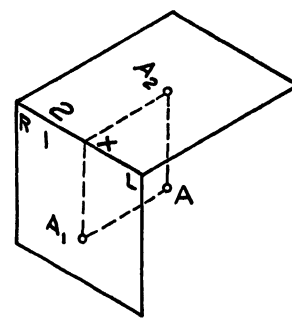


Fig. 10-17.

10-12. To represent on a single plane (the drawing paper) the projections of a point on two mutually perpendicular planes

Third-angle projection will be followed. Planes 1 and 2, taken at right angles to each other, are assumed as projection planes, Fig. 10-17. These planes intersect in a line called the *reference line*, R.L. The given space point A is projected orthogonally on each plane, yielding the projections A_1 and A_2. The projections A_1 and A_2 are the graphical repre-

sentation of point A and establish the necessary one-to-one correspondence. Point A and its projections, A_1 and A_2, lie in a plane which is perpendicular to R.L. and which cuts planes 1 and 2 in the lines A_1X and A_2X respectively. A_1X is the image on plane 1 of the projector AA_2; A_2X is the image on plane 2 of the projector AA_1.

10-13. The drawing plane

The important feature of projection drawing is the method for representing on a single plane (the drawing paper) the projections A_1 and A_2 which lie in two mutually perpendicular planes. This is achieved by assuming planes 1 and 2 to be revolved about the reference line R.L. as an axis so that they lie in a single plane (the drawing paper), Fig. 10-18.

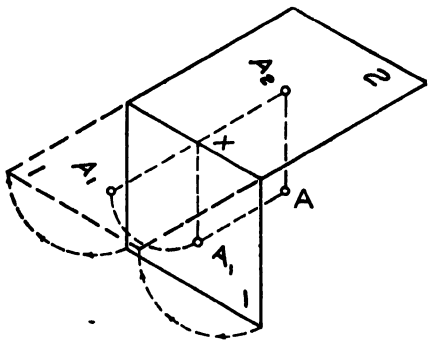


Fig. 10-18.

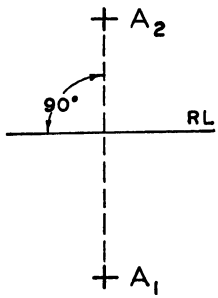


Fig. 10-19.

The projections A_1 and A_2 now lie on a straight line A_1A_2 which is perpendicular to R.L. On the drawing board the planes appear as in Fig. 10-19.

Now, A_1 represents point A of space as seen by an observer with a line of sight perpendicular to plane 1. From this position, R.L. represents the *edge view* of plane 2. The projection A_2 represents point A of space as seen by an observer looking perpendicular to plane 2. From this position, R.L. represents the *edge view* of plane 1. In each view the plane is between the observer and the point.

This system of representation was developed by Gaspard Monge (1746–1818), a professor in the military academy at Mézières, France. He was an eminent teacher and mathematician and made important contributions in the field of differential geometry. V. Poncelet, who established projective geometry, was one of his pupils. Monge was active in the Revolution and was temporary head of the republican government on the day the king was executed. One of his most important achievements was the founding of the Polytechnic School of Paris, the original model of all of our technological institutes. Monge was director of the school and pro-

fessor of descriptive geometry. A great many famous mathematicians and scientists, such as Laplace, Lagrange, and Ampère, were connected with this school. Monge became a member of the Academy and also the French Senate, but with the fall of Napoleon he was stripped of his honors and dismissed as head of the Polytechnic School. He died shortly thereafter.

10-14. Fundamental principle

The two projections of a point on two mutually perpendicular planes lie on a line which is perpendicular to the intersection (reference line) of the planes.

This principle leads to the following constructions, which are a restatement of the preceding article. They are involved in every orthographic drawing.

1. *To represent two mutually perpendicular projection planes:* Draw R.L., Fig. 10-19. The presence of a reference line on a drawing indicates that there is a projection plane on each side of this line and that they were at right angles to each other before being revolved into a single plane.

2. *To represent any space point:* Draw a line perpendicular to R.L. The projections of an assumed space point A must lie on this line. Projections A_1 and A_2 represent a point of space, A.

10-15. Horizontal and vertical projection planes

Definitions: *A vertical line is defined as the direction of a plumb line. This is a fundamental direction determined by gravitation. A horizontal line is any line perpendicular to a vertical line. A horizontal plane is any plane which is perpendicular to a vertical line. A vertical plane is any plane which contains a vertical line.*

Since the principal axes of structures are generally horizontal and vertical lines, the two principal projection planes are generally taken one horizontal and one vertical. The projection of an object on the horizontal projection plane is the *top* view. The projection on the vertical projection plane is the *front* view. The vertical projection plane which yields the front view is sometimes called the *frontal* projection plane to distinguish it from other possible vertical planes.

10-16. Projections of solids

A few examples will serve to illustrate the principles involved in drawing the projections of solids.

EXAMPLE 1. *A triangular right prism* (Fig. 10-20). The bases are parallel to the horizontal plane and appear in true shape in the top view. The back lateral face of the prism is parallel to the vertical plane. The lateral edges are vertical, and the prism projects in its true height in the front view.

EXAMPLE 2. *A hexagonal right pyramid* (Fig. 10-21). The base is parallel to the vertical projection plane and appears in true shape in the front view. The axis of the pyramid is perpendicular to the vertical plane

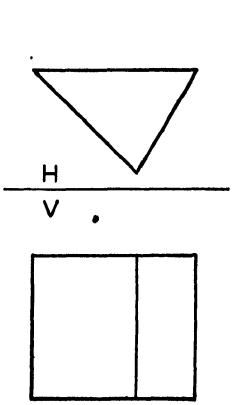


Fig. 10-20.

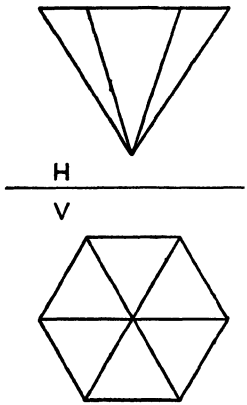


Fig. 10-21.

and appears in true length in the top view. One diagonal of the base is horizontal and thus parallel to the reference line.

10-17. Visibility

Since only a portion of the surface of a solid can be seen from a given position, the projection of a solid usually contains both visible and hidden lines. The matter of correctly representing these visible and hidden edges is an essential part of the drawing of any solid. In any projection of a solid, the boundary lines of the projection are always visible lines. Also,

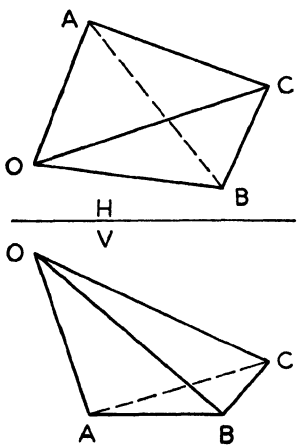


Fig. 10-22.

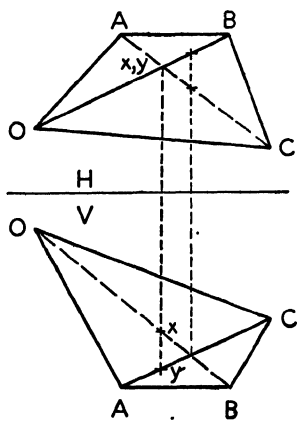


Fig. 10-23.

the highest edge of a solid is visible in the top view, and the foremost edge is visible in the front, Figs. 10-22 and 10-23.

The general method for determining visibility is illustrated by the tetrahedron, Fig. 10-22. In the top view the edges OA, OB, BC, AC are boundary lines of the view and are therefore visible. Edge OC is the highest edge (as can be seen in the front view) and is visible in the top view. Edge OB is the foremost edge (as can be seen in the top view) and is visible in the front view. The boundary lines of the front view are visible.

In the event that the visibility of certain interior lines of a view cannot be readily determined by inspection, the following procedure is useful. In the tetrahedron of Fig. 10-23, lines OB and AC are interior lines in each view. In the top view, assume points x and y on lines AC and OB at their crossing. Find the front views of these points. The higher point, x, is on the line OB. Therefore OB is visible in the top view. It also follows that planes OBA and OBC are above line AC and render it invisible. The visibility of lines OB and AC in the front view is determined in a similar fashion. The visible line is the one which contains the point that is nearer the vertical projection plane.

10-18. Basic theorems of orthographic projection

The drawing-board constructions employed in orthographic projection are based on the following geometric theorems true of this type of projection:

1. A line projects in its *true length* on a plane parallel to the line.
2. A line projects as a *point* on a plane perpendicular to the line.
3. If two lines of space are parallel, their projections on any plane are parallel.
4. If a space line is divided in a given ratio, its projection is divided into the same ratio.
5. If two lines of space are perpendicular, their projections on a plane parallel to either one of the lines are perpendicular.
6. A plane projects as a *straight line* on a projection plane perpendicular to it, *i.e.*, on any projection plane taken perpendicular to any line in the given plane.
7. A plane figure projects in true size on a plane taken parallel to the figure.

10-19. Basic constructions of orthographic projection

The fundamental principle of Art. 10-14 is involved in every orthographic drawing: The two projections of a point on two mutually perpendicular planes lie on a line which is perpendicular to the intersection (reference line) of the planes.

In addition to this principle there are three basic drawing-board constructions which provide the graphic machinery for finding any required projection of an object when two projections are given.

1. Given the projections of a point on two mutually perpendicular planes; to find a third projection of the point on any third projection plane which is perpendicular to one of the given planes.

2. Given two projections of a line which is oblique to the projection planes; to find (*a*) the *true-length* projection of the line; (*b*) the *point* projection of the line.

3. Given two projections of a plane figure; to find (*a*) the projection of the plane as a *line*; (*b*) the *true-size* projection of the figure.

10-20. Projections of a point on three planes

Let planes 1, 2, 3 and space point A be given, Fig. 10-24. The adjacent planes are at right angles—*i.e.*, planes 1 and 3 are each perpendicular to

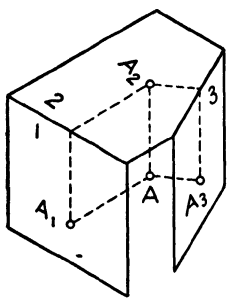


Fig. 10-24.

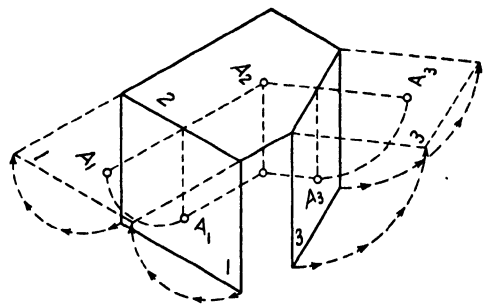


Fig. 10-25.

plane 2. Point A is projected on each plane, yielding the projections A_1, A_2, and A_3. The distance between space point A and plane 2 appears on planes 1 and 3 as the distance between the reference line and the projection of A.

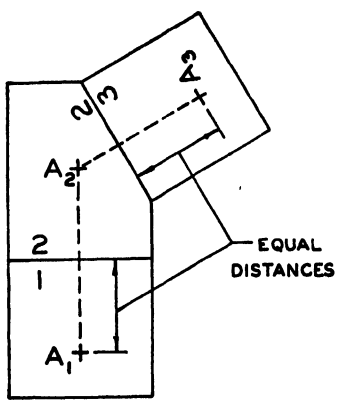


Fig. 10-26.

The system of three planes is represented on the drawing paper by assuming that planes 1 and 3 are rotated about their intersections with plane 2 until they lie in a plane with 2, Fig. 10-25. Hence, if the drawing paper itself is regarded as plane 2, planes 1 and 3 will occupy the positions shown in Fig. 10-26.

The student can easily verify these relationships by folding a sheet of paper to represent a series of three planes and then marking on the planes the projections of an assumed space point.

10-21. Construction 1

Given the two projections of a point on two mutually perpendicular projection planes; to find the third projection of the point on a third projection plane taken perpendicular to either of the given planes.

CONSTRUCTION 1

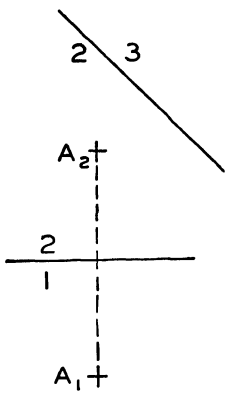


Fig. 10-27.

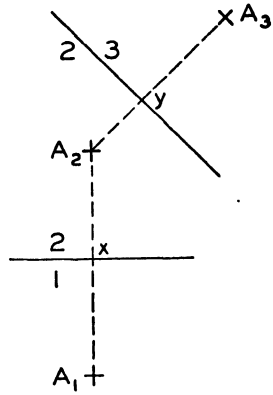


Fig. 10-28.

EXAMPLE 1. Given the projections A_1 and A_2 on planes 1 and 2, Fig. 10-27. Take plane 3 perpendicular to plane 2 by drawing the reference line 2-3 in the desired position across plane 2.

To find the projection A_3.

1. Through A_2 draw the projector at right angles to R.L.$_{2\text{-}3}$, Fig. 10-28.
2. Lay off yA_3 equal to xA_1.

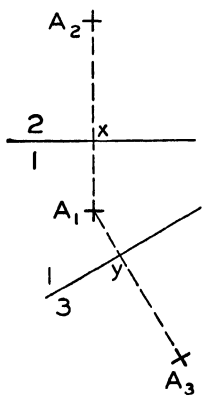


Fig. 10-29.

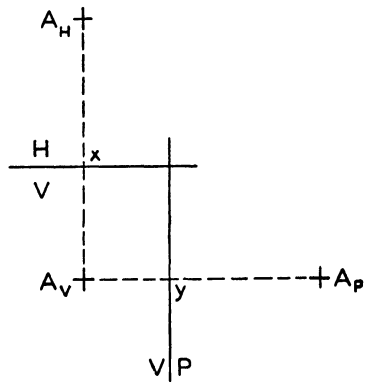


Fig. 10-30.

EXAMPLE 2. The third plane is taken perpendicular to plane 1 by drawing the reference line 1-3 across plane 1 in the desired position, Fig. 10-29.

To find the projection A_3.

1. Through A_1 draw the projector at right angles to R.L.$_{1\text{-}3}$.
2. Lay off yA_3 equal to xA_2.

EXAMPLE 3. The given planes are horizontal, H, and vertical, V, and the given projections A_H and A_V. The third plane is perpendicular to both H and V and is called the profile plane P. The reference line VP is perpendicular to reference line HV, Fig. 10-30.

To find the projection A_P.

1. Through A_V draw the projector at right angles to R.L.$_{VP}$.
2. Lay off yA_P equal to xA_H. Note that this yields the three principal views of Art. 10-3.

SUMMARY. The principle of construction 1 can be stated as follows: Consider any group of three consecutive projection planes. Since adjacent planes are always taken at right angles to each other in space, each of the two extreme planes is perpendicular to the middle plane. Therefore, the

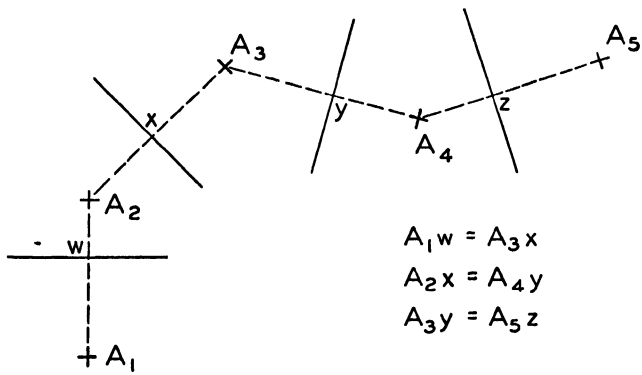


Fig. 10-31.

projections of any given point on the extreme planes will be equally distant from the reference lines. This principle can be applied to obtain any number of projections, Fig. 10-31.

10-22. Series of views

In Fig. 10-32 five successive views of a square right pyramid are shown. The principle described above applies to any three successive views of the series.

It should be noted that any given projection is a view of the *near* side of an adjacent view. The correct visible and hidden lines are thus determined. Views other than the principal views are generally called auxiliary views.

10-23. Position of observer with respect to an auxiliary projection

An auxiliary projection is visualized as a view of the object by turning the drawing paper so as to bring any vertical lines of the object into a

vertical position with respect to the reader of the drawing. The correct orientation for a given view can be indicated on the drawing by lettering a subtitle at right angles to a line of the view that is known to be vertical

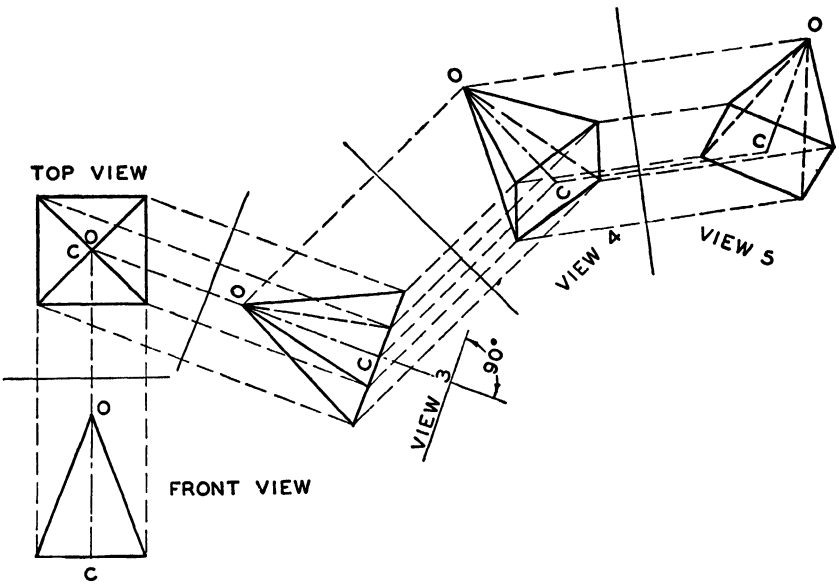


Fig. 10-32.

in space. In Fig. 10-32, the axis OC of the pyramid is vertical. The subtitle for each auxiliary view is lettered at right angles to the projection of line OC in that view.

10-24. Fundamental views of the straight line

A line segment parallel to a plane projects in its true length on the plane (Theorem 1). When projection A_1B_1, Fig. 10-33*a*, is parallel to the refer-

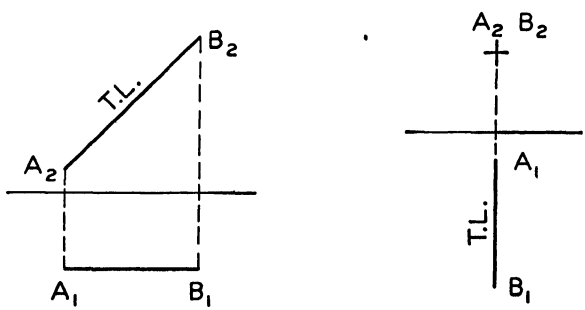


Fig. 10-33.

ence line, the line AB (in space) is parallel to plane 2 and the projection A_2B_2 is the *true-length* view of AB, labeled T.L.

If a line is perpendicular to a plane, its projection on the plane is a point (Theorem 2). Line AB, Fig. 10-33*b*, is perpendicular to plane 2 and therefore parallel to plane 1. The view A_1B_1 is its true-length view; the view A_2B_2 is its *point* view. The true-length and point views of a line are regarded as its fundamental views. Problems which are primarily concerned with straight lines can usually be solved by simple procedures if the fundamental views of the lines can be obtained.

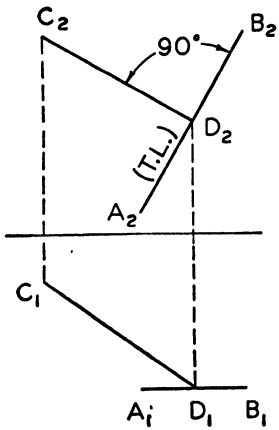


Fig. 10-34.

If two lines of space are perpendicular, their projections on a plane parallel to either one of the lines are perpendicular (Theorem 5). In Fig. 10-34, lines AB and CD are perpendicular in space. They project at right angles in view 2, where line AB appears in true length.

10-25. Construction 2

Given two views of a line inclined to both projection planes; to find (*a*) *its true-length view, and* (*b*) *its point view.*

a. Given the views A_1B_1 and A_2B_2, Fig. 10-35. To find the true-length view of AB.

PRINCIPLE. A line projects in its true length on a plane parallel to the line (Theorem 1).

CONSTRUCTION. Take plane 3 parallel to the line AB by drawing R.L.$_{2\text{-}3}$ parallel to A_2B_2. Project AB onto plane 3 by construction 1, obtaining the view A_3B_3. A_3B_3 is the true-length view of AB.

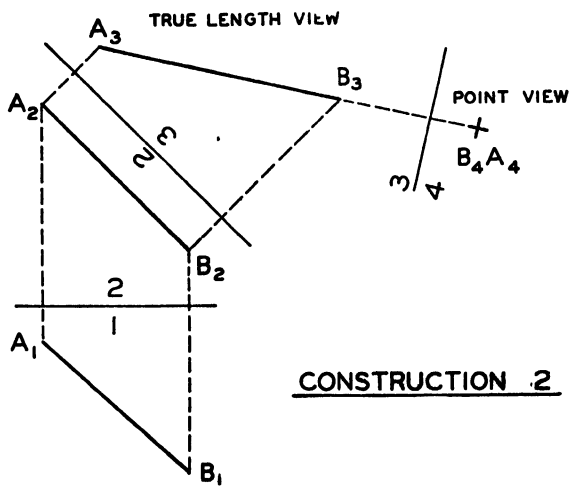


Fig. 10-35.

b. To find the point view of line AB.

PRINCIPLE. A line projects as a point on a projection plane taken perpendicular to the line (Theorem 2).

CONSTRUCTION. Take plane 4 perpendicular to plane 3 by drawing R.L.$_{3\text{-}4}$ at right angles to A_3B_3 (the true-length view of AB). Project AB on plane 4 by construction 1. A_4B_4 is the point view of AB.

10-26. Views in specified directions

One of the most important applications of construction 2 is in drawing the view of an object as seen in a specified direction. The specified direction of sight may be along the axis of the object or in the direction of one of its edges or some other given line. In each case the required new view is one in which the line representing the direction of sight projects as a point.

EXAMPLE. Given the top and front views of a prism, Fig. 10-36. To draw a view of the prism as seen in the direction of its axis. This is equiva-

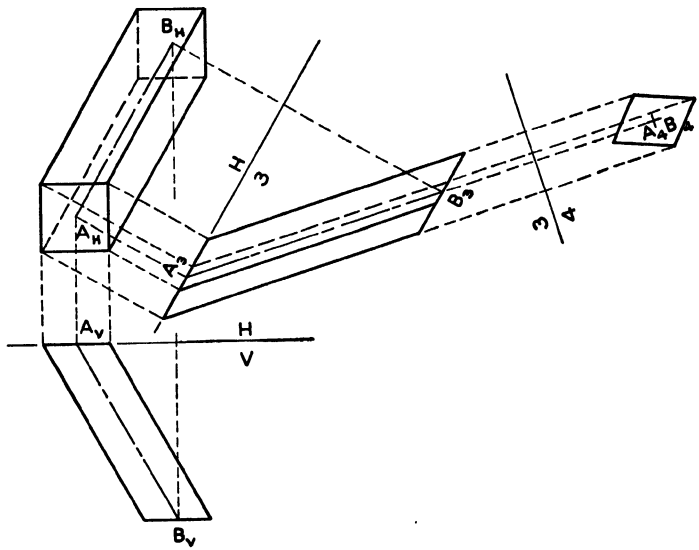


Fig. 10-36.

lent to saying that we wish to project the prism on a plane taken perpendicular to its axis.

a. Take R.L.$_{H\text{-}3}$ parallel to A_HB_H, and project the prism on plane 3 (construction 2).

b. Take R.L.$_{3\text{-}4}$ perpendicular to A_3B_3, and project the prism on plane 4. This is the required view.

10-27. Drawing the principal views of an object having an inclined axis

Another important application of construction 2 is to a large class of problems that require drawing the principal views of an object which has its axis inclined to the horizontal and vertical projection planes. While the details of the process vary in particular cases, the general principle always remains the same. The following example is typical, and the solution is given step by step.

EXAMPLE. Let points A and B be the centers of the bases of a square right prism, Fig. 10-37. Point C locates one corner of the lower base.

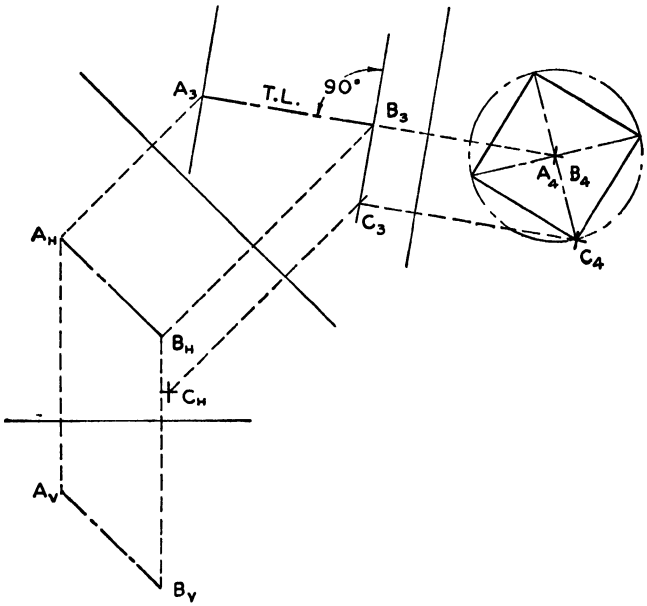


Fig. 10-37.

The projections A_HB_H, A_VB_V, and C_H are given. Draw the top and front views of the solid and the necessary auxiliary views.

a. Draw A_3B_3 and A_4B_4, the true-length and point views, respectively, for line AB, as in Fig. 10-35 (construction 2).

b. The bases being perpendicular to the axis will be perpendicular to plane 3 and thus by Theorem 6 will project as lines perpendicular to the true-length view of the axis. Draw a line of indefinite length through B_3 to represent the plane of the lower base.

c. In view 3 one corner of the lower base is found by drawing a projector through C_H at right angles to R.L.$_{H\text{-}3}$, cutting the plane of the base in C_3.

d. Project C to plane 4, thus locating C_4.

e. Line B_4C_4 is a semidiagonal of the square base. Complete the diagonals of the square, and draw the sides of the square.

f. By projecting from view 4, complete view 3 of the prism, Fig. 10-38.

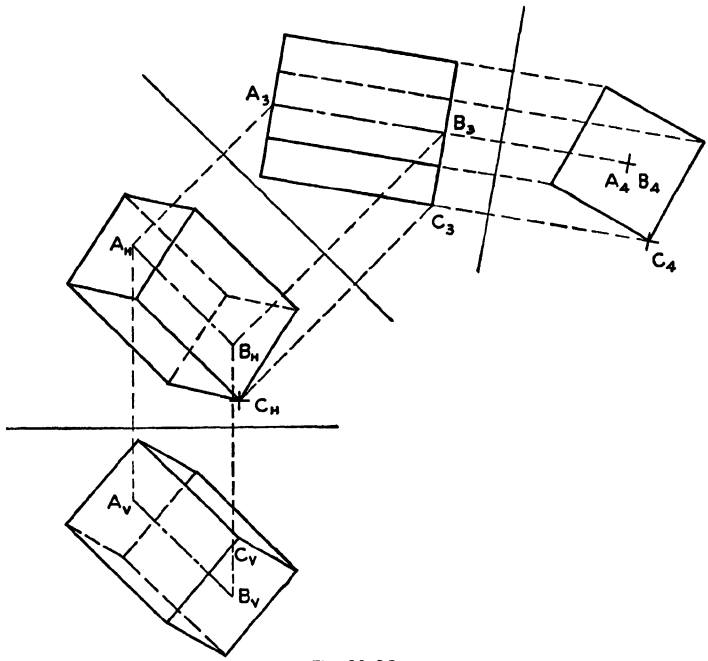


Fig. 10-38.

g. Complete the top and front views of the pyramid by projecting from views 3 and 4 (construction 1).

10-28. Fundamental views of a plane

A plane is in its simplest position with respect to two projection planes when it is parallel to one and therefore perpendicular to the second. In this position one view is a *true-shape* view and the other an *edge* view, Fig. 10-39. These are considered the fundamental views of a plane, and constructions involving the plane can usually be executed readily when these views are given. Any view of a plane that is adjacent to its true-shape view will be a straight-line or edge view parallel to the reference line between the two views, Fig. 10-40. When the given views of a plane are oblique views, the fundamental views can be found by construction 3.

10-29. Construction 3

Given two projections of a plane figure; to find (*a*) *the projection of the plane as a line;* (*b*) *the true-size projection of the figures.* Given: Projections $A_1B_1C_1$ and $A_2B_2C_2$, Fig. 10-41.

a. To find the edge view of ABC

PRINCIPLE. A plane projects as a straight line on a projection plane taken perpendicular to any line in the plane (Theorem 6).

CONSTRUCTION. Draw the line CD of the triangle parallel to plane 2 by making C_1D_1 parallel to R.L.$_{1\text{-}2}$. Thus, C_2D_2 is the true-length view

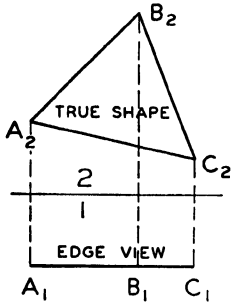


Fig. 10-39.

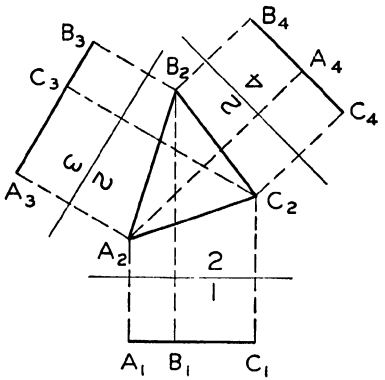


Fig. 10-40.

of CD. Take plane 3 perpendicular to CD by drawing R.L.$_{2\text{-}3}$ at right angles to C_2D_2. Project the triangle on plane 3, obtaining CD as a point and consequently the edge view $A_3B_3C_3$.

b. To find the true-size projection of ABC

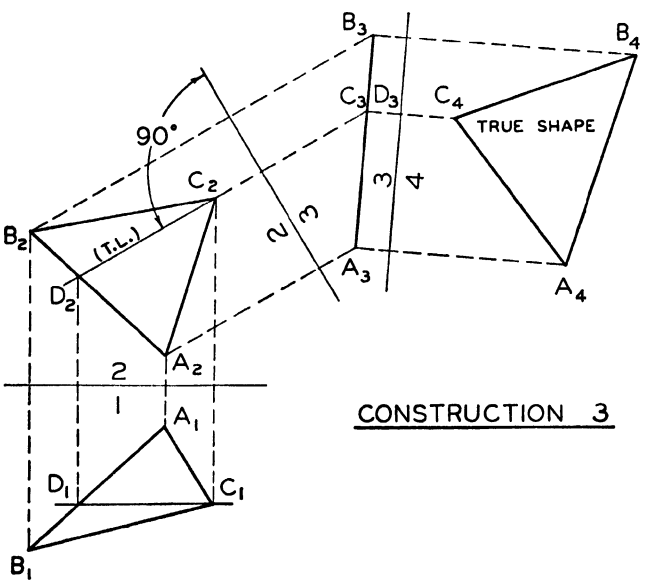


Fig. 10-41.

PRINCIPLE. A plane figure projects in true size on a plane taken parallel to the figure (Theorem 7).

CONSTRUCTION. Take plane 4 parallel to ABC by drawing R.L.$_{3\text{-}4}$ parallel to $A_3B_3C_3$. Project ABC on plane 4, obtaining the true-size view $A_4B_4C_4$.

10-30. True-shape view of face of an object

The top, front, and side views of the object are given, Fig. 10-42. It is required to draw a view in which the true shape of face 3-5-6-4 appears.

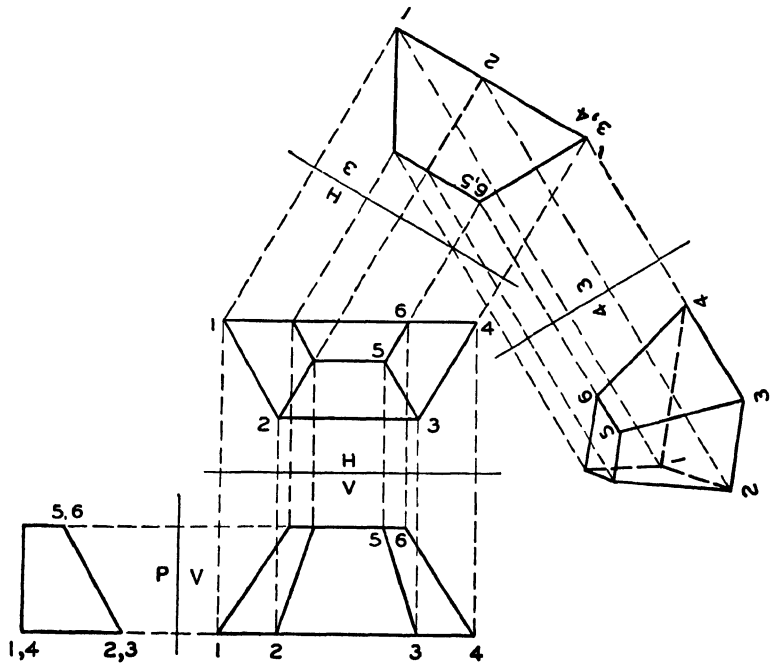


Fig. 10-42.

a. Line 3-4 projects in true length in the top view. Take R.L.$_{H\text{-}3}$ perpendicular to line 3-4, and draw the object showing an edge view of face 3-5-6-4.

b. Take R.L.$_{3\text{-}4}$ parallel to edge view 3-5-6-4, and draw the true-shape view (construction 3).

10-31. To execute a plane construction on an oblique plane of space

The upper leg CDE of a bent plate, Fig. 10-43, is to be rounded to a given radius. Find the views of the arc.

a. Draw the true-shape view of triangle ABE (construction 3).

b. In view 4, the true-shape view, lay out the given radius tangent to lines AE and BE.

c. Divide the arc into any number of parts, say six, and project the points of division to views 3, 2, and 1.

d. In each view draw a smooth curve through the points.

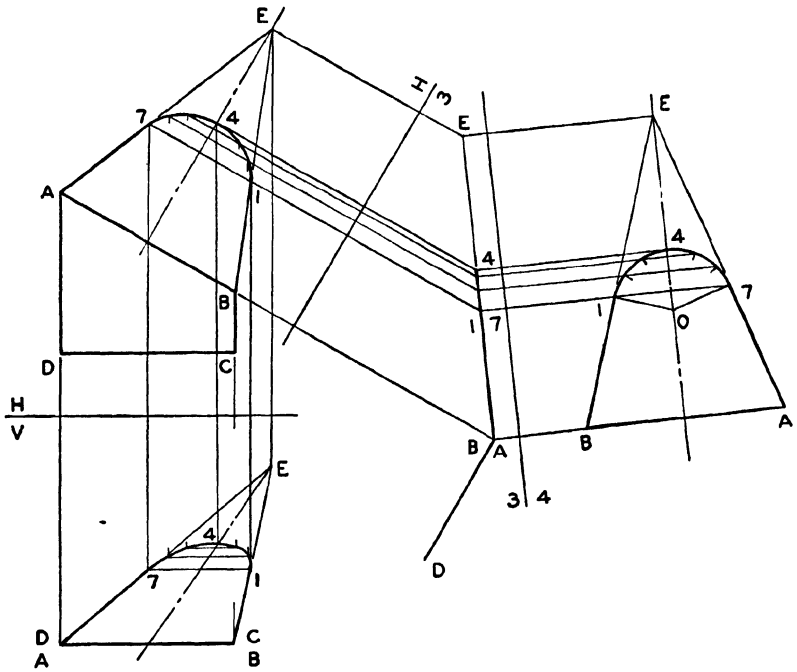


Fig. 10-43.

It will be seen that the accuracy of the drawing is improved by taking the points on the arc in view 4 symmetrical with respect to the center line EO.

10-32. Construction 4. Rotation of a point about an axis

An alternate method for finding the true shape of a plane figure is to rotate the figure into parallelism with a projection plane. The fundamental construction of the method of rotation consists in rotating a space point about an axis and into some specified position.

10-33. Rotation of a point about a vertical axis

Point P rotated about the vertical axis k, Fig. 10-44, travels in a circle which is perpendicular to the axis. The circular path shows in true shape in the top view and as a straight line at right angles to k in the front view.

The parallelogram $ABCD$, Fig. 10-45, is perpendicular to the H plane. It is rotated about a vertical axis k, taken through A, until it lies parallel to the V plane. The paths of points B, C, and D project as circular arcs

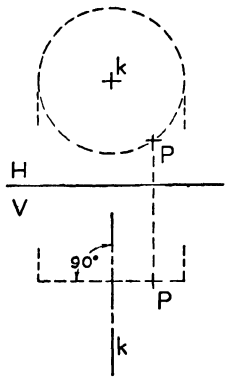


Fig. 10-44.

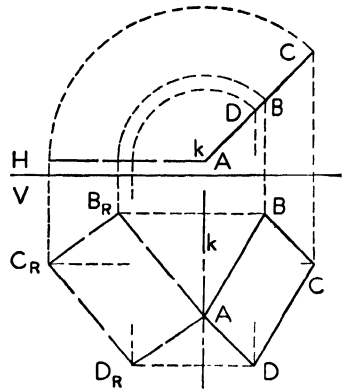


Fig. 10-45.

in the top view. In the front view the path of each point projects as a straight line at right angles to k. The rotated view $AB_RC_RD_R$ is the true-shape view of the figure.

10-34. Rotation of a point about a horizontal axis

Point P, rotated about the horizontal line k, Fig. 10-46, describes a circle perpendicular to k. The auxiliary view in which k projects as a point shows the circular path of P as a circle. This circle projects as a straight line in the view where k projects in true length. The front view of the path is an ellipse.

The triangle ABC, Fig. 10-47, is to be rotated about a horizontal axis BD, lying in the plane ABC, until it becomes parallel to the H plane. An auxiliary view of ABC is drawn in which the axis BD projects as a point. The edge view of the triangle is rotated about BD until it becomes parallel to the reference line H-3. The result of the rotation is to bring points A and C in the top view into the positions A_R and C_R. Point B lies on the axis and does not move. The rotated view A_RBC_R is the true-shape view of the triangle.

10-35. Simple sections of solids

The section cut from a solid by a plane is determined by finding points in which lines on the surface of the solid pierce the plane, Figs. 10-48 through 10-51. In each instance the section plane Q is perpendicular to the V plane. The section may be represented in either of two ways:

a. The line of intersection may be drawn on the surface of the solid, Figs. 10-48 and 10-49.

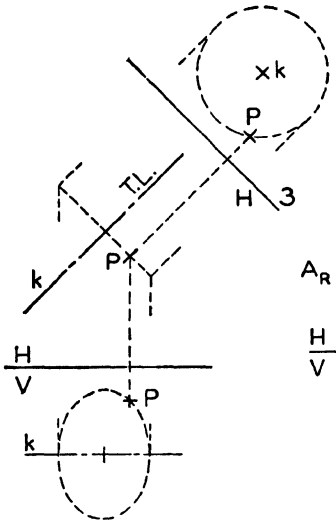


Fig. 10-46.

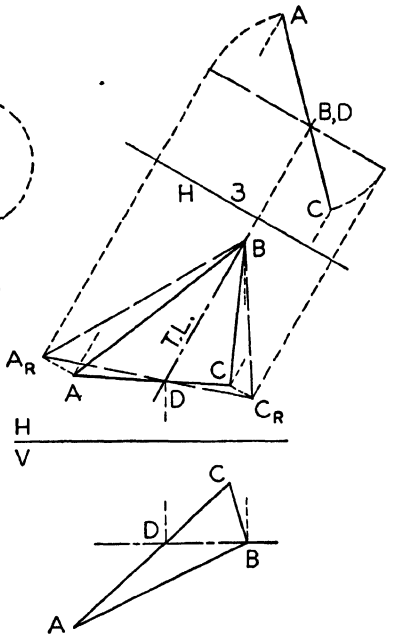


Fig. 10-47.

b. The portion of the solid between the cutting plane and the observer can be considered removed and the cut surface shaded by section lines, Figs. 10-50 and 10-51.

The intersection of a plane with a cone or cylinder is found by drawing a series of elements of the surface. A smooth curve drawn through the

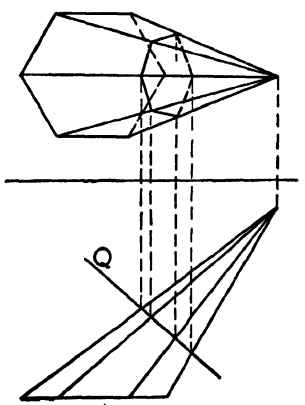


Fig. 10-48.

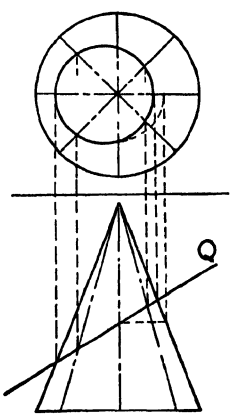


Fig. 10-49.

points in which the elements pierce the cutting plane represents the section, Figs. 10-49 and 10-51.

When a section plane is in an oblique position, the intersection can be found by constructing an auxiliary view in which the plane is edgewise by means of construction 3, Art. 10-29.

It is frequently convenient to regard a section of a set of converging lines as a central projection. Thus, in Fig. 10-48, the hexagonal base of

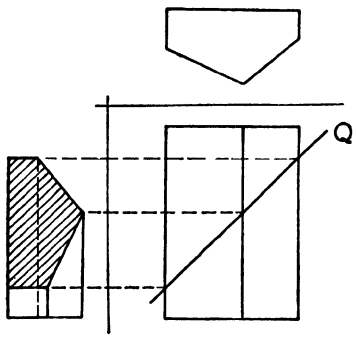


Fig. 10-50.

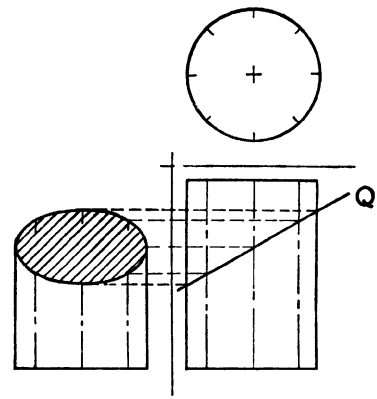


Fig. 10-51.

the pyramid can be considered as projected onto plane Q, using the vertex of the pyramid as the projection center and the edges as rays. Similarly, the circular base of the cone, Fig. 10-49, can be considered as projected onto plane Q.

If we regard the cylinder as a set of parallel rays passing through the circumference of a circle, a section of these rays can be considered as the *parallel* projection of the circle on the plane.

10-36. The problems of space geometry

In general, the shape of an object or engineering structure may be regarded as a combination of elementary geometric forms. These forms are commonly prisms and cylinders but may also include pyramids, cones, surfaces of revolution, and warped or twisted surfaces. These various space figures can be analyzed into points, lines, and surfaces as the basic geometric elements.

Space problems of interest to the engineer are concerned with the graphical representation of these figures and the geometric problems connected with their representation. These problems include methods for describing the positions of points, lines, and planes; determination of distances and angles; finding the intersections and developments of surfaces.

Graphical solutions to these space problems are obtained by application of the preceding theorems and drawing-board constructions. The fundamental concepts of the geometry of projection drawing are treated in this chapter. The detailed study of the solution of space problems by orthographic projection comprises the subject matter of *descriptive geometry*, and the reader should consult textbooks in this field for further development of this topic.

PROBLEMS

10-1. *Pictorial to orthographic.* Use Problem Sheet 10-1. Sketch the required views freehand.

Note: In each of the following problems, 10-2 to 10-6, the form of an object is given by a description. The required views are to be drawn freehand. In order that the student may develop a feeling for proportion, it is essential that all sizes be estimated. Use sketching paper.

10-2. From a block 4 in. wide, 2½ in. deep, and 2 in. high, discard the right and left thirds of the upper half, forming a T-shaped object. Draw the top, front, and right side views. Show a ⅞-in.-diameter vertical hole through the center of the object.

10-3. The shape of a cast-iron base is described as follows: From a block 5 in. wide, 2½ in. deep, and 1 in. high, remove the left and right quarters of the upper half. In the center of the top face of the block, cut a 1½-in.-square hole through the block. The sides of the square are parallel to the outside edges of the block. The four outside corners of the block, as seen in the top view, are rounded, using a ½-in. radius. On these same centers, four ½-in.-diameter holes are drilled.

10-4. A cylinder 3½ in. in diameter and ½ in. long has its axis horizontal. A second cylinder 1½ in. in diameter and 1¼ in. long is centered on the front face. A circular recess 2 in. in diameter and ⅛ in. deep is centered in the back face of the 3½-in. cylinder. A ¾-in.-diameter hole is drilled axially through both cylinders. Four ⅜-in.-diameter holes are drilled through the 3-in. cylinder. These holes are equally spaced on a 2¾-in. circle which is concentric with the 3½-in. circle. Draw the top and front views.

10-5. A cylinder 2½ in. in diameter and 3 in. high has the upper two-thirds of the right and left sides cut away to produce two flat surfaces which are parallel and 1½ in. apart. A 1-in.-diameter hole passes through the centers of the flattened surfaces.

10-6. From a block 4 in. wide, 2 in. high, and 1½ in. deep the center three-fourths of the upper three-fourths is removed, leaving a piece that appears U-shaped in the front view. The tops of the left and right arms of this piece, as seen in the side view, are made semicircular. A ½-in. hole is centered ¾ in. from the top of each arm. A cylindrical lug 1 in. in diameter and 1¼ in. high is centered on the bottom of the block.

10-7. *Orthographic to pictorial.* Use Problem Sheet 10-7. Sketch the required views freehand.

10-8. *Projections.* Illustrate by sketches the central, oblique, and orthogonal projections of a triangle.

10-9. *Central projection.* Illustrate by sketches. Under what conditions will a plane project as a line? Parallel lines as parallel? A plane figure as a similar but smaller plane figure?

10-10. *Central projection.* Find a center of projection that will project the unequal segments formed by three points of line k into equal segments. Illustrate with a sketch.

10-11. *Orthogonal projection.* Illustrate with sketches and a simple geometric proof each of the theorems of Art. 10-18.

10-12. *Orthogonal projection.* Show that if a triangle is projected orthogonally the centroid of the triangle projects into the centroid of the projection.

10-13. Divide the working space into four equal rectangles, and place one example in each. Draw the top and front views of the following solids:

a. Triangular right prism. The bases are parallel to the V plane and 1¾ in. on a side. The rear lateral face is parallel to the H plane. The axis is 1 in. long.

b. Rectangular right pyramid. The base is parallel to the H plane with one diagonal parallel to the V plane. Each diagonal is 2 in. long; the shorter edges of the base are 1 in. long. The axis of the solid is 1¼ in. long, and the vertex is below the base.

c. Frustum of rectangular right pyramid. The lower base is 1 by 2 in. and is horizontal, with its long edges parallel to the V plane. The upper base is ½ by 1 in. The axis is 1 in. long.

d. Hexagonal right pyramid. The base is 1 in. on each side and is parallel to the V plane, with one diagonal of the base vertical. The vertex is in back of the base. The axis is 1¼ in. long.

In the following problems a point is located on the drawing paper by three dimensions, as follows: To locate the point P; 2, 3, 5. The first dimension, 2 in., is the distance from the left border line to the projector which joins P_H and P_V. The second dimension, 3 in., is the distance from the lower border line to P_V. The third dimension, 5 in., is the distance from the lower border line to P_H.

10-14 to 10-18. *Orthographic-projection studies.* Use Problem Sheets 10-14 to 10-18.

10-19. *Series of views.* Use Problem Sheet 10-19.

10-20. *Tetrahedron.* Draw the top and front views of the tetrahedron: A(2½, 3¾, 8½), B(4½, 3¾, 6½), C(5½, 2¾, 9½), D(4½, 1¼, 6½).

a. Draw a view of the tetrahedron on a projection plane taken perpendicular to the H plane and parallel to $A_HB_HD_H$.

b. Draw a view on a projection plane taken parallel to face BCD.

10-21. *Tetrahedron.* Draw the top, front, and right side views of the tetrahedron determined by the points A(¼, 2, 7), B(1¼, 5, 8), C(2½, 1, 8), D(4¾, 3, 6).

10-22. *Tetrahedron.* Draw the top, front, and right side views of the tetrahedron determined by the points A(0, 1, 7), B(1, 4, 9), C(3, 5, 8), D(4, 2, 6).

10-23. *Auxiliary view.* The line A(5, 2, 7½), B(8, 2, 7½) is a diagonal of a regular hexagon which is parallel to the vertical projection plane. This hexagon is the common base of two right pyramids, the axis of each being 2 in. long. Draw the top, front, and left side views of the object. Draw a view of the object on a projection plane taken perpendicular to the profile plane and parallel to the upper front face of the object.

10-24. *Auxiliary view.* Line $A(2, 5\frac{1}{4}, 8)$, $B(2, 1, 8)$ is the axis of a thin cylindrical tube 2 in. in diameter. The cylinder is truncated by a plane which is perpendicular to the vertical projection plane and makes an angle of 30° with AB, sloping down to the right.

Draw the top and front views of the portion of the tube which is between the cutting plane and a right section through B.

Draw a view of the tube on a projection plane taken perpendicular to the V plane and parallel to the cutting plane.

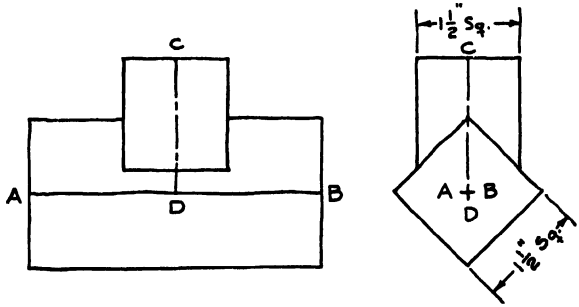
10-25. *True length of a line.* Use Problem Sheet 10-25.

10-26. *True length by rotation.* Use Problem Sheet 10-26.

10-27. *Oblique prism.* Lines $A(2\frac{1}{4}, 3, 7\frac{1}{4})$, $B(3\frac{1}{2}, 3, 7\frac{1}{4})$ and $C(1, \frac{1}{2}, 6)$, $D(2\frac{1}{4}, \frac{1}{2}, 6)$ are, respectively, the front edges of two horizontal squares which form the ends of an oblique prism. Complete the top and front views of the prism. Draw (*a*) a view in which the lateral edges project in true length and (*b*) a view in which the lateral edges project as points.

10-28. *Oblique pyramid.* Line $A(\frac{1}{2}, 4, 9)$, $B(2\frac{1}{2}, 2, 8)$ is the axis of a square right pyramid. The base is centered on point B, one corner being located by projection $C_H(3\frac{1}{2}, X, 8\frac{1}{2})$. Draw the top, front, and necessary auxiliary views.

10-29. *Tee connector.* The lines AB and CD are center lines of a tee connector made from 1½-in.-square tubing. CD is perpendicular to AB. Draw the true-size projections of the tee and also its projections on the H and V planes. $A(2\frac{1}{2},$



Prob. 10-29.

$2\frac{3}{4}, 6\frac{1}{4})$, $B(5\frac{1}{4}, 1\frac{1}{8}, 9)$, $C(4\frac{7}{8}, 3\frac{5}{8}, 7\frac{3}{8})$. Use 11- by 17-in. sheet. Scale: Full size. Method of solution: Construct a view in which line AB projects as a point. In this view line CD will project in its true length.

10-30. *View in specified direction.* Use Problem Sheet 10-30.

10-31. *Normal view of a plane.* Use Problem Sheet 10-31.

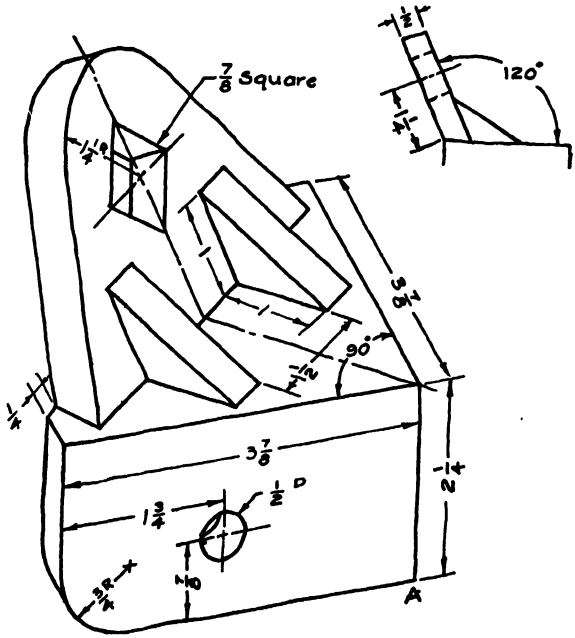
10-32. *Normal view of a plane.* Line AD is a diagonal of a regular hexagon which lies in the plane of points $A(6, 3, 4)$, $K(8, 2, 4)$, $D(7, 1, 6)$. Draw the top, front, and true-shape views of the hexagon.

10-33. *Normal view by rotation.* Use Problem Sheet 10-33.

10-34. *Corner bracket.* Draw the top and front views of the casting, together with such additional views as are necessary to complete its shape description.

The metal is ½ in. thick. Lay out the drawing with one vertical face of the casting parallel to the vertical projection plane. The two vertical faces are symmetrical. Use 11- by 17-in. sheet.

Locate point *A* at 14, ½, 5¾. Scale: Full size.



Prob. 10-34.

11

AXONOMETRY

11-1. Axonometric projection

When a projection plane is inclined to the three principal axes of an object, the projection of the object on the plane bears a resemblance to its visual image. The closeness of this resemblance depends on the position of the projection plane with respect to the three principal axes and the type of projection used.

In Fig. 11-1, let the cube be given. Assume three mutually perpendicular axes, OX, OY, OZ through one corner of the cube, as O, and coinciding with the edges. Assume any point S of space as a center of projection,

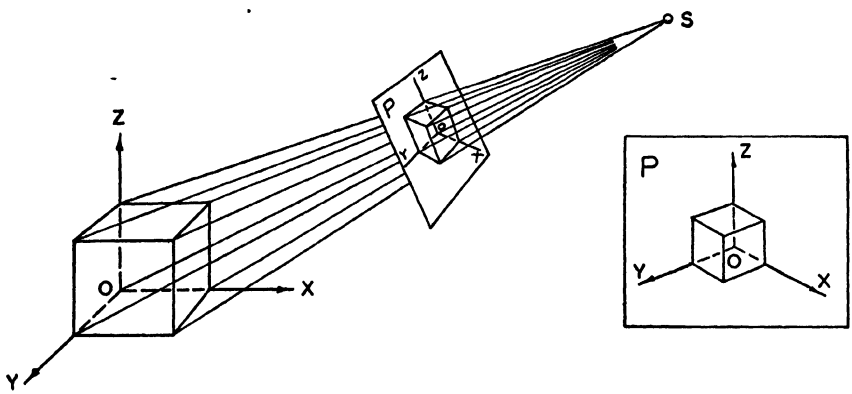


Fig. 11-1.

and draw, from S, rays to each corner of the cube. The aggregate of these rays is called a "cone" of rays, or a *perspection*. A *section* of these rays by a plane, as P, is an *axonometric projection*.

Three types of axonometric projection can be distinguished.

1. When the center S is at a finite distance from the object, the section is a *central axonometric projection*, or *perspective*.

2. When the center S recedes to infinity in a given direction not parallel to an axis, the rays of the perspection become parallel, forming a "cylinder"

of rays, and any nonperpendicular section of the perspection is an *oblique axonometric projection.*

3. A special case of (2) occurs when the section plane is perpendicular to the cylinder of rays. This right section is an *orthographic axonometric projection.*

Axonometry is a branch of descriptive geometry, and a complete discussion of the theory will be found in textbooks on that subject. Highly developed practical procedures will be found in books on pictorial drawing and production illustration.

11-2. Orthographic axonometry

The orthographic system is the simplest of the three types of axonometry and will be discussed first.

Orthographic axonometry is based on the following theorems on the straight line:

1. *The projections of parallel lines are parallel.*
2. *The projections of lines parallel to the projection plane are in true length.*
3. *The projections of parallel lines which are inclined to the projection plane are foreshortened by the same ratio.*
4. *Any two conjugate axes of an ellipse and its minor axis can be regarded, so far as directions are concerned, as the orthographic projections on the plane of the ellipse of three mutually perpendicular lines of space.* This is known as Schwartz's theorem (see Art. 11-7).

The ratio of foreshortening for each axis depends on the angle the axis makes with the projection plane. If all three axes make the same angle with the plane, the foreshortening is the same for each direction and the projection is called *isometric.* If two axes make the same angle with the plane, the projection is *dimetric.* If the angles are unequal for all three axes, the projection is *trimetric.*

11-3. Isometric projection

Let a cube be placed so that three intersecting edges make equal angles with a projection plane, Fig. 11-2. As a result of symmetry, edges AB, AC,

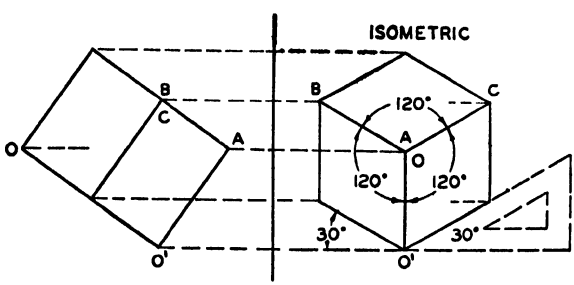


Fig. 11-2.

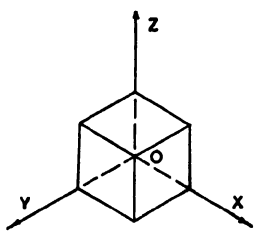


Fig. 11-3.

and AO' of the projection make 120° angles with each other. If edge AO' projects as a vertical line, edges AB and AC make 30° angles with a horizontal line. Since every edge of the cube makes the same angle (in space) with the projection plane, every edge is equally foreshortened. The ratio is equal to $\sqrt{2}/\sqrt{3}$.

As a result of these relationships, the isometric view of a cube can be constructed directly with the T square and 30°–60° triangle.

11-4. Isometric axes

The basic problem of axonometry is to construct an axonometric projection of an object from the projections of the object as given on two mutually perpendicular planes, for example, the top and front views.

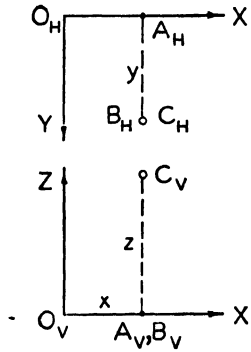


Fig. 11-4.

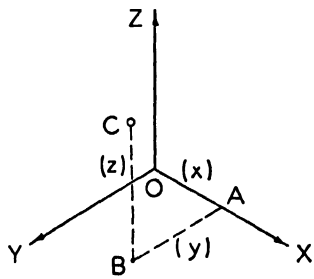


Fig. 11-5.

Three axes, OX, OY, OZ, taken along three concurrent edges of the cube are called isometric axes. They represent three mutually perpendicular space directions, Fig. 11-3. The isometric projection of an object

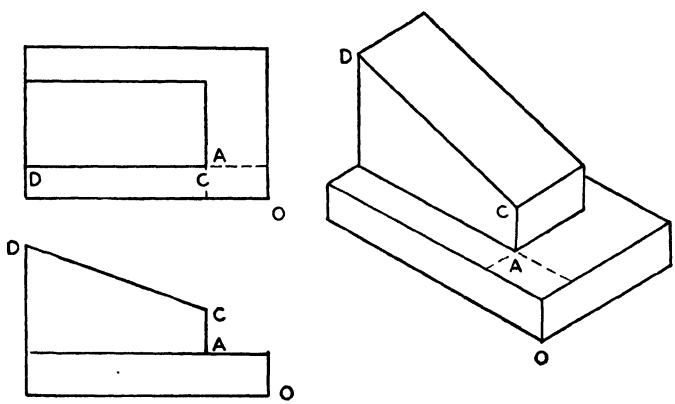


Fig. 11-6.

is constructed by plotting its principal dimensions on or parallel to these axes. The foreshortening ratio is generally disregarded in practice, and the measurements are plotted in true length along or parallel to each of the isometric axes. The method of plotting a point describes the entire process. Point C of space, given by its orthographic projection, Fig. 11-4, is located by the three measurements OA, AB, BC. The axes OX, OY, OZ are drawn in isometric, and the segments OA, AB, and BC are plotted, Fig. 11-5. The distances OA, AB, and BC are the cartesian coordinates of point C and are written as x, y, z and always in that order.

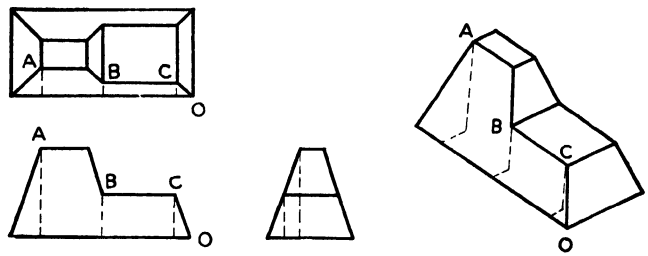


Fig. 11-7.

Axes can be assumed through any convenient corner of the object. In Figs. 11-6 and 11-7, the front corner of the object is taken as the origin of the axes. Nonisometric lines, as CD, Fig. 11-6, and OC, Fig. 11-7, are located by plotting their end points. Curves are plotted by taking offsets to points of the curve from the isometric axes, Fig. 11-8.

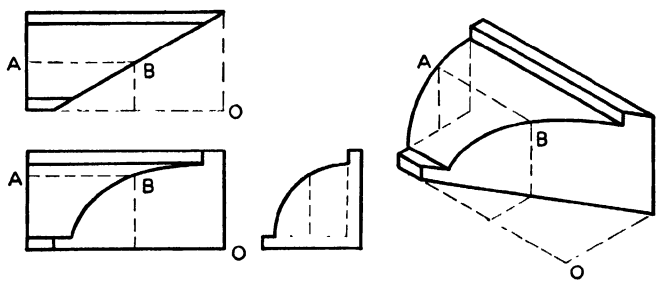
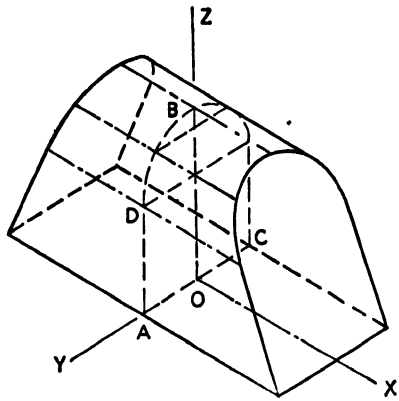


Fig. 11-8.

An object having a constant cross section, Fig. 11-9, is drawn in isometric drawing by use of a right section, as ABC, from which other points of the object are located by offsets.

Isometric drawings may be very complex, but no principles are involved other than those stated. Such drawings are very useful, for the pictorial effect is of importance. They are extensively used in "exploded drawings," which show the parts of an assembly separated from each other but in their proper order.



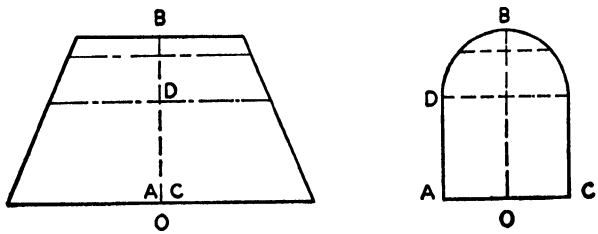


Fig. 11-9.

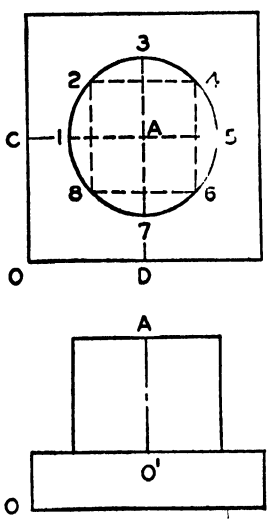


Fig. 11-10.

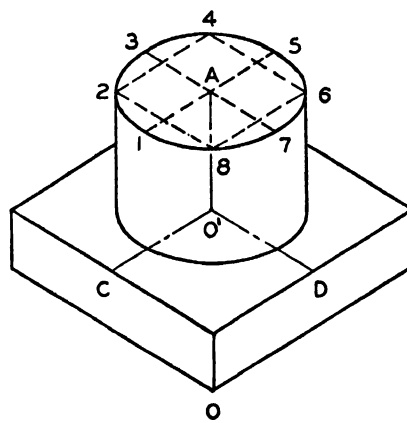


Fig. 11-11.

11-5. Circles in isometric

The isometric projection of a circle is in general an ellipse. If the circle lies parallel to a horizontal, vertical, or profile projection plane, the following method will plot it accurately. In Fig. 11-10, chords, 2-4, 2-8, etc., become isometric lines in the pictorial view and are plotted by their offsets from point A, Fig. 11-11. Sufficient points should be taken to draw a smooth ellipse. This method is seldom used in practice, the following approximate construction being simpler and sufficiently accurate for a pictorial drawing.

Construct the rhombus $ABCD$, center O, which is the isometric drawing of the circumscribed square, Fig. 11-12. Find the mid-points E, F, G, H

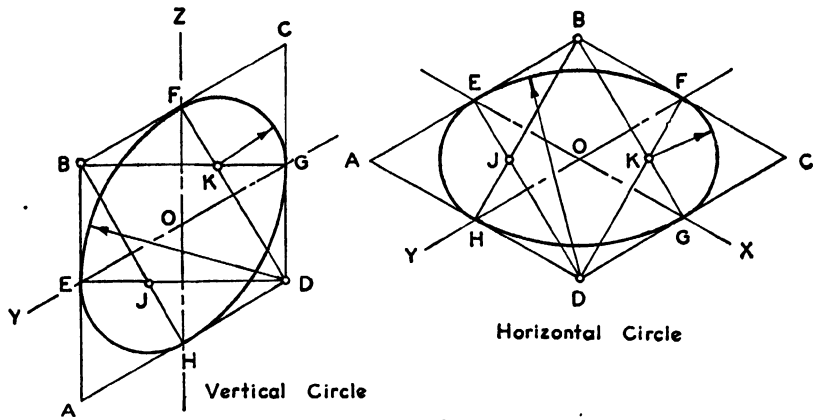


Fig. 11-12.

of each side. Draw DE and BH, intersecting at J. Draw DF and BG, intersecting at K. With radius JE, center J, draw arc EH. With radius DE, center D, draw arc EF. With radius KF, center K, draw arc FG. With radius BG, center B, draw arc GH.

11-6. Dimetric and trimetric projections

The distortion that is apparent in isometric drawings of solids is due to failure of receding parallel lines to converge. This distortion can be seemingly reduced by choosing axes that have foreshortening ratios other than 1:1:1 and by placing the object in an advantageous position with respect to these axes. Dimetric axes, in which two of the axes have the same ratio, are suitable in many cases.

The most common presentation of the object in dimetric is one which yields foreshortenings in the ratio of 1:1:½, Fig. 11-13. The axes then appear on the drawing paper as vertical, at 7°+, and at 41°+ with the horizontal. These are generally made 7½° and 45° for ease of construction with triangles.

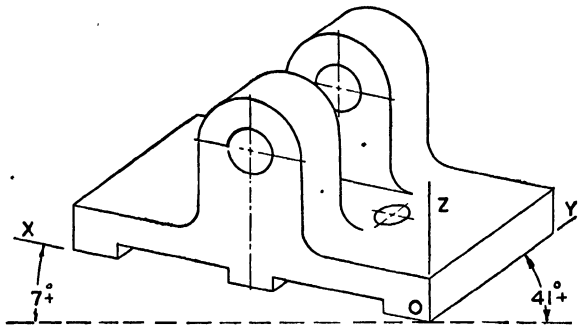


Fig. 11-13.

Other useful ratios are given by Fig. 11-14 and the following table.

Ratios $l:m:n$	a	b
$1:\frac{1}{2}:1$	1	7
$1:1:\frac{1}{2}$	7	7
$1:1:\frac{3}{4}$	6	6
$\frac{3}{4}:\frac{3}{4}:1$	2	2

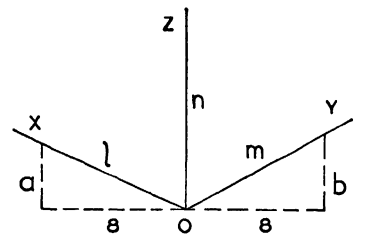


Fig. 11-14.

11-7. The theorem of Schwartz

Any two conjugate axes of an ellipse and its minor axis can be regarded, so far as directions are concerned, as the orthographic projections on the plane of the ellipse of three mutually perpendicular lines of space.

If any two noncollinear vectors, OZ and OX, are assumed in the drawing plane, it is possible by means of Schwartz's theorem to find a third vector, OY, such that the triple, OX, OY, OZ, will represent the orthographic projection on the drawing plane of three concurrent unit vectors of space. This is the fundamental theorem of orthographic axonometry of which isometric projection is a special case.

Schwartz's theorem is deduced from a consideration of three mutually perpendicular radii of a sphere. In Fig. 11-15, let the front and side views of the sphere be given. The polar axis ZN is inclined toward the vertical projection plane. In the side view the equatorial circle $ADBC$ is at right angles to OZ. The projection of the equatorial circle on the vertical plane is the ellipse $ADBC$. The major axis AB is equal to the diameter of the sphere. The polar radius of the sphere, OZ, projected onto the V plane, lies along the minor axis of the ellipse.

Lines OX and OY represent any pair of perpendicular radii of the equatorial circle. When this circle is projected on the V plane into the above-described ellipse, this pair of radii become conjugate semiaxes of this ellipse. This relationship can be verified by imagining that the ellipse is rotated about its major axis, AB, until it lies parallel to the V plane and becomes a circle on diameter AB. In this operation, points X and Y rotate into the points X' and Y', respectively, and the rotated radii OX' and OY' are at right angles. When the circle DEC is drawn on the minor axis as a

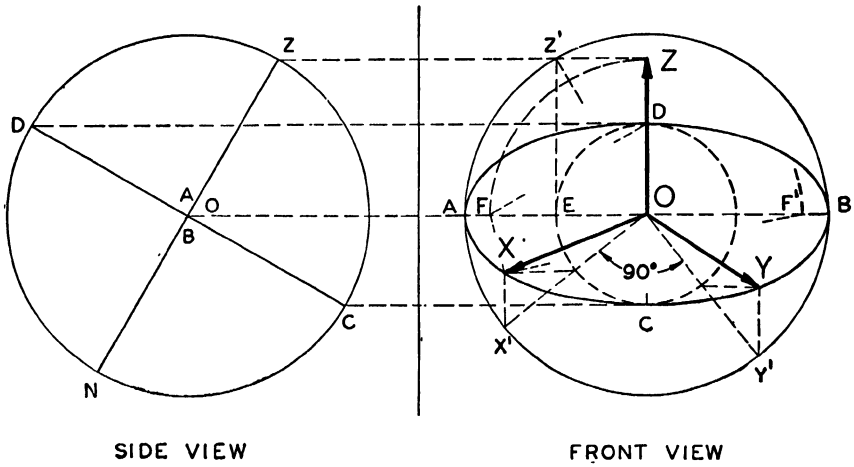


Fig. 11-15.

diameter, it is evident that the concentric-circle construction described in Art. 2-6 applies to points X and Y of this ellipse.

The relationship between the vectors OX, OY, OZ and the foci F and F' of the ellipse is also important in this construction. The distance $AO = FD = OZ'$. Hence, the right triangles DOF and OEZ' are congruent. Also, $OZ = Z'E$. Hence, $OZ = OF$. Again, since X is a point of the ellipse, the sum of the distances FX and $F'X$ is equal to AB. The true lengths of vectors OX, OY, OZ are equal to OX', OY', OZ', respectively.

11-8. To construct a set of axonometric axes

Let vectors OZ and OX and the horizontal line AB be given, Fig. 11-16. To find vector OY by means of Schwartz's theorem.

1. Lay off OF and $OF' = OZ$.
2. Find radius $OA = (XF + XF')/2$, and draw circle, center O.
3. Draw ZZ' parallel to FF'. On center O, draw circle of radius $OE = ZZ'$.
4. Draw XX' parallel to OZ, and draw OX'.
5. Lay off OY' at right angles to OX', cutting circle ECD at D.

6. Draw YY' perpendicular to AB and DY parallel to AB.
7. Draw OY, the required third vector.

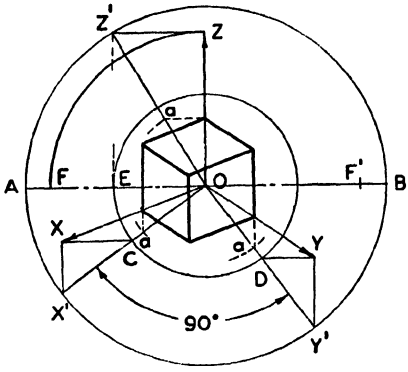


Fig. 11-16.

To represent the cube, side Oa, on these axes, lay off the length Oa on OX', OY', OZ'. Project these segments onto the axonometric axes OX, OY, and OZ as shown.

11-9. The theorem of Gauss

There is an interesting application of the foregoing construction of axonometry in a field quite outside that of pictorial drawing.

If the vectors OX, OY, OZ are regarded as three vectors which lie in a plane, they can be considered as representing three complex numbers in such a manner that the sum of their squares is equal to zero. This relation, discovered by Gauss, is stated as follows.

THEOREM. *A necessary and sufficient condition that a set of three coplanar vectors of common origin and not all null which represent the three complex numbers*

$$z_1 = a_1 + b_1 i \qquad z_2 = a_2 + b_2 i \qquad z_3 = a_3 + b_3 i$$

may be regarded as the orthographic projection on this plane of a set of three mutually perpendicular space vectors of common origin and common length is that the relation

$$z_1^2 + z_2^2 + z_3^2 = 0$$

be satisfied.

11-10. Oblique axonometry

POHLKE'S THEOREM. *Any three intersecting lines can be assumed as the direction of a set of oblique axonometric axes.*

These axes can be described as the projection of three mutually perpendicular lines of space onto a plane by means of parallel oblique projectors. The axes commonly used for a type of axonometric drawing called *oblique projection* are horizontal, vertical, and 45° to the horizontal, Fig. 11-17. Any given object is plotted with reference to these axes just as in isometric projection. Oblique projection is particu-

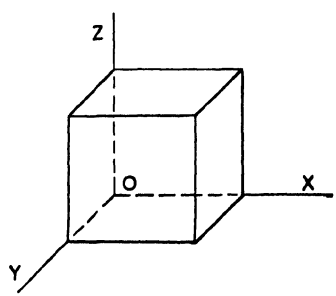


Fig. 11-17.

larly useful for the representation of an object having a large number of circles in the same or parallel planes, because the faces of the object parallel

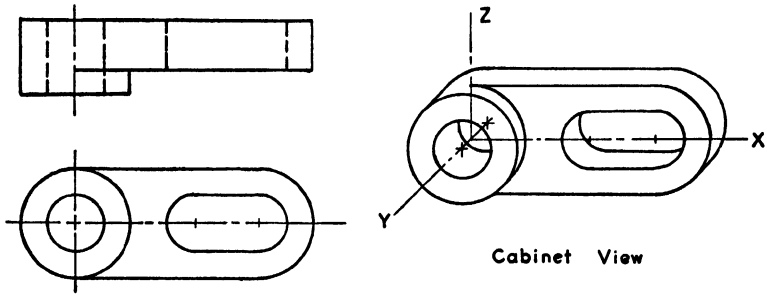


Fig. 11-18.

to the vertical projection plane are in true shape in the oblique view, Fig. 11-18.

Two general procedures should be observed. Place the object with the irregular or curved face parallel to the projection plane, and, if consistent with this, the distortion is reduced if the longest dimension is placed parallel to the projection plane. The foreshortened dimension is frequently drawn one-half size, in which case the drawing is called a "cabinet drawing."

11-11. Central axonometric projection

We now return to the general case first mentioned in Art. 11-1. The projection system for a central or perspective projection is composed of a projection plane and a center of projection. In Fig. 11-19, let the top and front views of the square prism be given, and also the center of projection, S. In this example, the vertical projection plane V is used as the perspective projection plane.

One edge, AE, of the prism is in the V plane and therefore coincides with its central projection. To find other points on the central projection, draw the "cone" of rays from S to the corners of the prism. The points in which these rays pierce the V plane

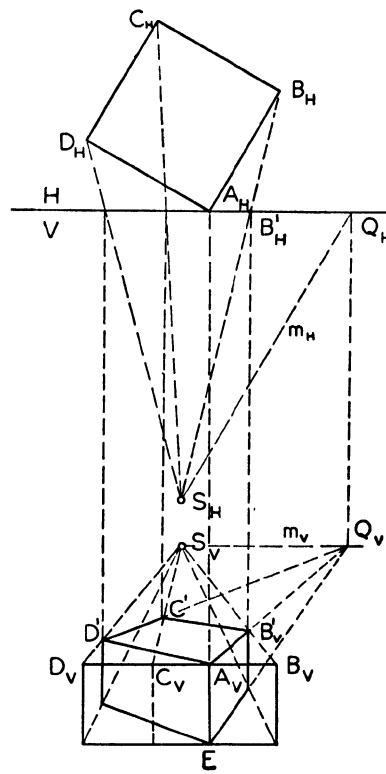


Fig. 11-19.

define the central projection on V. Thus, S_HB_H and S_VB_V represent the top and front views of the ray to point B. In the top view this ray pierces the V plane at B_H'. The projector through B_H' cuts S_VB_V at B_V', which is the central projection of point B on the V plane. Other

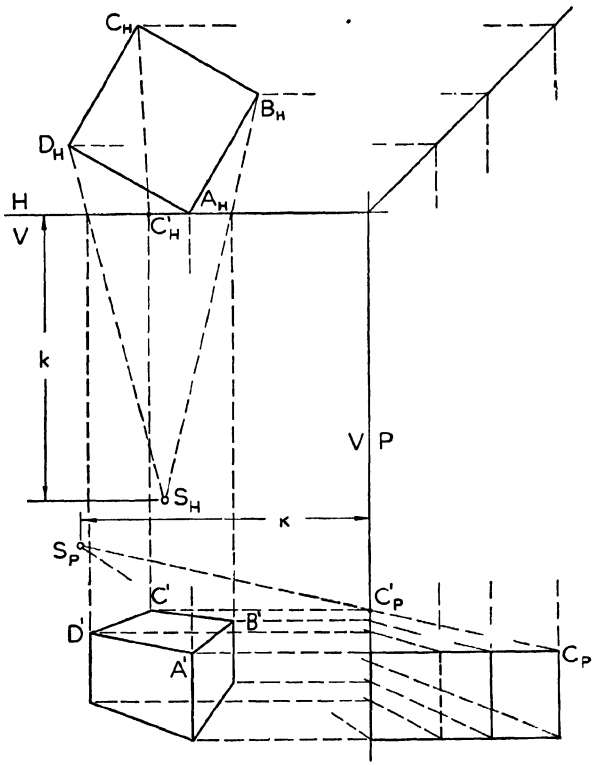


Fig. 11-20.

points of the prism are projected onto V in a similar manner, defining the central or perspective projection $A_VB_V'C'D'E$.

While the construction arrangement shown in Fig. 11-19 is simple, it is awkward to have the perspective view overlap the given front view. This situation can be avoided by omitting the front view of the prism and working with the top and side views, as shown in Fig. 11-20. In this arrangement the side view S_p of the center of projection must be used in place of S_V.

11-12. Vanishing point

Let the line k be projected from S as a center onto plane P, Fig. 11-21. A', B', C' are points on the projection. Line k pierces plane P at point T. As a point Q moves along k, its projector m approaches a position of

parallelism to line k. In the limit, where Q becomes the point at infinity on line k, the projector m becomes parallel to k and cuts plane P at Q'. Hence, Q' is the projection on P of the infinite point of k. Point Q' is called the *vanishing point* of k. Note that Q' will also be the vanishing point for

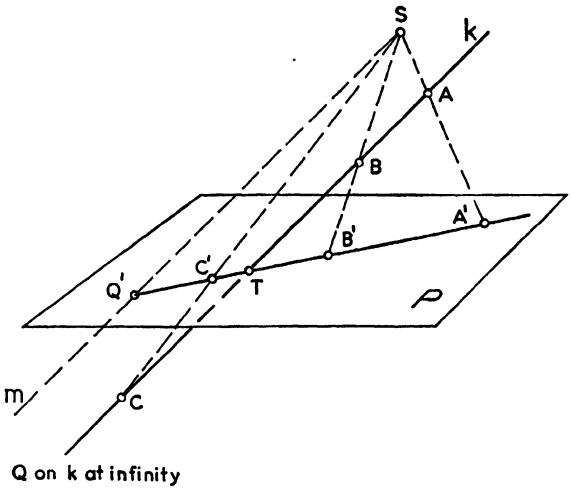


Fig. 11-21.

any line parallel to k. Thus, it can be said that in central projection every *set* of parallel lines will have one vanishing point.

The cross ratio of any set of four points of k is preserved in the projection of k. Thus,

$$\frac{AC/CB}{AQ/QB} = \frac{A'C'/C'B'}{A'Q'/Q'B'}$$

EXAMPLE. Find the vanishing point for line AB in Fig. 11-19.

The line m through S and parallel to AB cuts the V plane in Q, the vanishing point for AB. This is also the vanishing point for CD, EF, etc.

CONSTRUCTION. Through S_H draw m_H parallel to A_HB_H, cutting V in Q_H. Through S_V draw m_V parallel to A_VB_V. Through Q_H draw the projector cutting m_V in Q_V. Q_V is the required vanishing point for the perspective view.

By constructing the vanishing point for each set of parallel lines, the accuracy of a perspective projection can be improved and certain short-cut methods of perspective drawing can be employed.

The student is referred to standard works on perspective for the practical development of this subject.

REFERENCES FOR READING

GUPTILL, A. L.: "Sketching and Rendering in Pencil," Reinhold Publishing Corporation, New York, 1922.

HOELSCHER, SPRINGER, AND POHLE: "Industrial Production Illustration," McGraw-Hill Book Company, Inc., New York, 1946.

LUBSCHEZ, B. J.: "Perspective," D. Van Nostrand Company, Inc., New York, 1926.

PROBLEMS

11-1 to 11-4. *Isometric drawings.* Use Problem Sheets 11-1 to 11-4.

11-5. *Isometric.* Make an isometric drawing of the casting described in Prob. 10-4.

11-6. *Isometric.* Make an isometric drawing of the casting described in Prob. 10-5.

11-7. *Oblique.* Make an oblique drawing of the casting described in Prob. 10-4.

11-8. *Oblique.* Make an oblique drawing of the casting described in Prob. 10-5.

11-9. *Trimetric.* Design a set of trimetric axes by means of Schwartz's theorem.

11-10. *Perspective drawing.* Use Problem Sheet 11-10.

11-11. Using point $S(4, 3\frac{1}{2}, 4)$ as a center of projection, draw the central projection of the line $A(227)$, $B(519)$. The vertical projection plane is 1 in. in front of point A. Find (a) the vanishing point Q of AB, produced; (b) the point T in which AB, produced, pierces the V plane. (c) Calculate the cross ratio of points A_H, B_H, T_H, Q_H and points A, B, T, Q of the central projection.

11-12. Using point $S(3, 4, \frac{1}{2})$ and line $A(236)$, $B(539)$, solve for same quantities described in the preceding problem.

12

CONVENTIONS OF PRACTICAL DRAFTING

12-1. Working drawings

The information necessary for the manufacture or construction of a machine or structure is presented in the form of a technical drawing called a *working drawing*. A set of working drawings for a given machine includes an *assembly* drawing showing the relation of the various parts to each other and a *detail* drawing of each individual part. In a detail drawing the machine part is invariably represented by its orthographic projections. Pictorial views are sometimes used in the assembly drawing. Information concerning sizes, material, and shop operations is indicated by dimensions and notes. Since many conventions, symbols, and abbreviations are used to simplify a working drawing for practical use, it is evident that this system of representation must be uniform throughout the United States. Accordingly, the American Standards Association (ASA), a group formed by the professional engineering societies to establish uniform standards, has published a code for engineering drawing. This code describes approved practice in drafting procedure, use of thread and other shop symbols, dimensioning, lettering, etc., and it is included as Appendix I of this book. Those conventions and symbols most frequently employed in making detail drawings of machine parts are outlined in the following articles. The reader is referred to the ASA Standards for additional information on these and related topics.

12-2. Sectional views

When the interior form of an object does not show clearly in a standard view, a *sectional* view should be used. A sectional view supposes that the object is cut by a plane and that the portion of the object on the near side of the plane is removed. The surfaces produced on the object by the section plane are shaded by *section lines*.

The cylindrical object of Fig. 12-1 is cut in two by a vertical plane AA which passes through its center in the top view. The portion of the object in front of plane AA is now imagined to be removed and the surfaces actually cut by the plane section lined, Fig. 12-2.

Thus, the section plane AA converts the front view of the object to a sectional front view, Fig. 12-3, in which all interior lines are visible. Hidden lines are omitted. Arrows are placed on the section plane in the top view in order to show the direction in which the section is viewed. The special representation of the section plane is generally omitted when the plane coincides with the center line of the view.

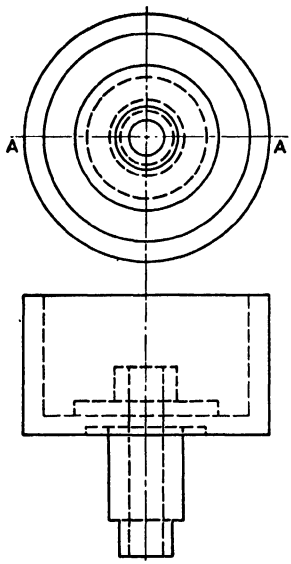


Fig. 12-1.

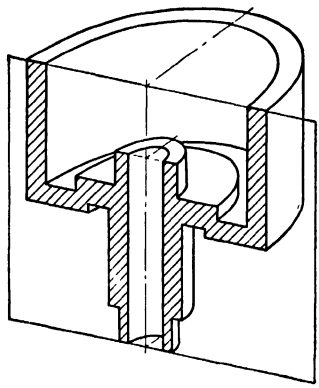


Fig. 12-2.

When the top view of an object is a symmetrical view, a *half section* is sometimes used, Fig. 12-4. In this case, the section plane cuts only one half of the view and only the left front quarter of the object is assumed to be removed. The result is a combination interior and exterior front view. Since the cutting of the object is only imaginary, the center line is not altered. Sectional views are very common in engineering drawing, and many examples of their uses are described in the American Drafting Standards, (Appendix I).

12-3. Terms relating to threads

SCREW THREAD. A ridge of uniform section in the form of a helix on the external or internal surface of a cylinder, Fig. 12-5.

EXTERNAL THREAD. A thread on the outside of a member.

INTERNAL THREAD. A thread on the inside of a member.

MAJOR DIAMETER. The largest diameter of a screw thread.

MINOR DIAMETER. The smallest diameter of a screw thread.

PITCH. The distance from a point on a screw thread to a corresponding point on the next thread measured parallel to the axis.

LEAD. The distance a screw thread advances axially in one turn. On a single-thread screw, the lead and pitch are identical; on a double-thread

screw, the lead is twice the pitch; on a triple-thread screw, the lead is three times the pitch, etc.

ANGLE OF THREAD. The angle included between the sides of the thread measured in an axial plane.

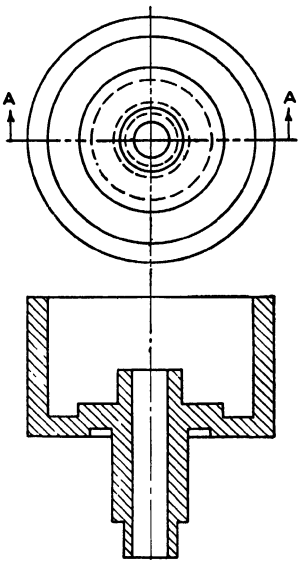


Fig. 12-3.

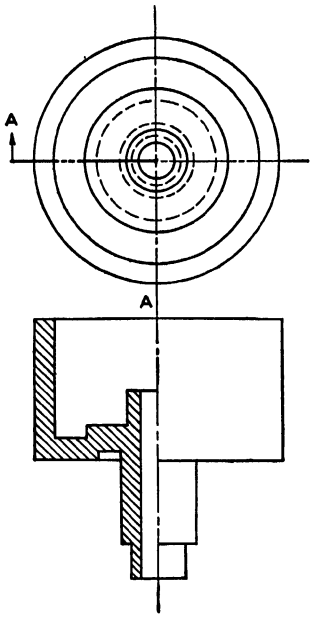
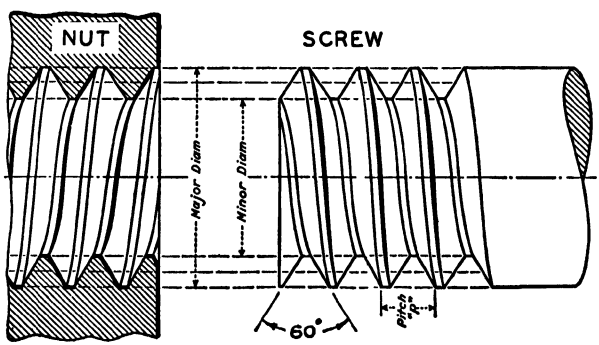


Fig. 12-4.

12-4. Classification of threads

Four standard classes of screw-thread fit are provided, class 1, class 2, class 3, and class 4. Of these class 1 fit is the loosest and class 4 the closest, with classes 2 and 3 graduated between. Class 2 represents a high quality



NATIONAL THREAD FORM

Fig. 12-5.

of commercial screw-thread product and is recommended for the great bulk of interchangeable screw-thread work.

12-5. Regular thread symbols

The true orthographic projection of a screw thread is time-consuming to draw. Since such threads are standardized, standard symbols can conveniently be used to represent them. Figure 12-6 shows the standard thread symbols in section and elevation. Figure 12-6*a* shows the regular

Coarse-Thread and Fine-Thread Series

Size	Basic Major Diameter (Inches)	Coarse-Thread Series (Threads per Inch)	Fine-Thread Series (Threads per Inch)
0	0 0600	...	80
1	0 0730	64	72
2	0 0860	56	64
3	0 0990	48	56
4	0.1120	40	48
5	0.1250	40	44
6	0.1380	32	40
8	0.1640	32	36
10	0.1900	24	32
12	0 2160	24	28
1/4	0 2500	20	28
5/16	0 3125	18	24
3/8	0 3750	16	24
7/16	0.4375	14	20
1/2	0 5000	13	20
9/16	0 5625	12	18
5/8	0 6250	11	18
3/4	0 7500	10	16
7/8	0 8750	9	14
1	1.0000	8	14
1 1/8	1.1250	7	12
1 1/4	1.2500	7	12
1 3/8	1.3750	6	12
1 1/2	1.5000	6	12
1 3/4	1.7500	5	...
2	2 0000	4 1/2	...
2 1/4	2.2500	4 1/2	...
2 1/2	2.5000	4	...
2 3/4	2.7500	4	...
3	3 0000	4	...
3 1/4	3.2500	4	...
3 1/2	3.5000	4	...
3 3/4	3 7500	4	...
4	4.0000	4	...

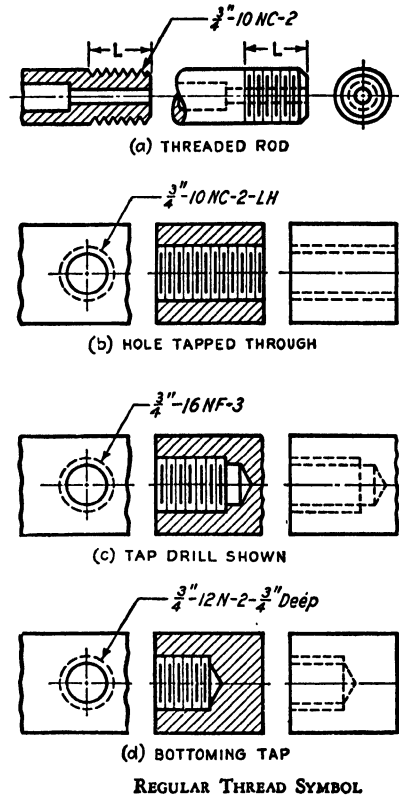


Fig 12-6.

method of representing external screw threads recommended for general use on assembly and detail drawings. Except in sections of external threads and hidden internal threads, the threads are indicated by alternate long and short lines at right angles to the axis representing the crests and roots of the thread, the short lines thicker than the long lines, or of equal width if preferred. They need only approximate the actual pitch, being spaced by eye to look well. Figure 12-6*b* shows internal threads tapped

through. Figure 12-6*c* shows internal threads where the tap drill point does not go through. The drill point should be drawn at 60° with the center line. The method shown in Fig. 12-6*d* should be used to indicate a bottoming tap, when the depth of thread is the same as the depth of drill, or when it is not necessary to specify both the depth of drill and the depth of thread. The size and length of the thread and the depth of tap should be given on the drawing. Threads are always considered to be "right-hand" unless specified as "left-hand," or L.H. The letters L.H. should be added after the class of fit number as shown in Fig. 12-6*b*. Bolt ends should be shown as flat and chamfered to the thread depth at 45° with the flat surface.

12-6. Rules for dimensioning

1. Dimensions state the true size of the object regardless of the scale of the drawing.

2. Numerals and arrowheads should be of standard size and form, Fig. 12-7.

Numerals are $\frac{1}{8}$ in. high.

Fractions are $\frac{1}{4}$ in. high.

Arrowheads are $\frac{1}{8}$ in. long. The length of the arrow is at least three times the width.

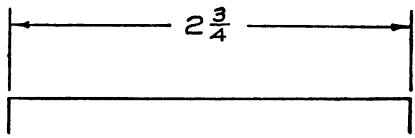


Fig. 12-7.

Inch marks may be omitted when all figures represent inches.

Extension lines run $\frac{1}{8}$ in. beyond the arrow and within $\frac{1}{16}$ in. of the object line.

The arrow is placed $\frac{1}{2}$ in. from the view.

3. Numerals are aligned with the arrow (except for angle dimensions of only two digits, which read from the bottom of the sheet), Fig. 12-8.

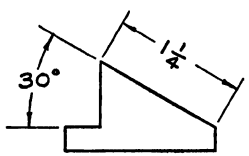


Fig 12-8.

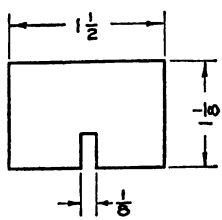


Fig. 12-9.

4. Numerals should read from the bottom or the right side of the sheet. Dimensions are arranged so as to allow full-sized figures and arrowheads to be used, Fig. 12-9.

5. A series of dimensions are placed in a single line, Fig. 12-10.

6. Parallel dimensions are arranged with the smaller dimensions nearer the view, Fig. 12-11. The first arrow is $\frac{1}{2}$ in. from view. Other arrows are $\frac{3}{8}$ in. apart.

7. *Avoid* (*a*) crossing arrows; (*b*) referring dimensions to hidden lines; (*c*) placing an arrow in line with an object line, Fig. 12-12.

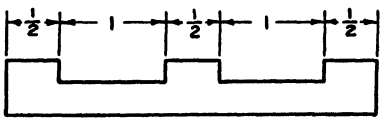


Fig. 12-10.

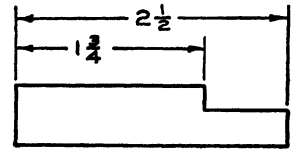


Fig. 12-11.

8. Dimensions are not repeated either in the same view or in different views.

No dimensioning rule is inviolable. Situations constantly arise where preferred practices are in conflict or impossible to follow. Consequently good dimensioning requires a continual use of judgment and compromise. Furthermore, practical minutiae frequently vary to suit the peculiarities of various industries.

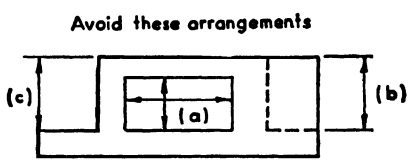


Fig. 12-12.

12-7. The elements of dimensioning

The shapes of machine and structural parts can be considered as being made up of a number of simple geometric forms. These forms are usually prisms and cylinders but may occasionally include pyramids, cones, and double-curved and warped surfaces. If the size of each of these elemental forms is given and the parts located with respect to each other, the object will be completely dimensioned from a geometric point of view. Thus, there are two types of dimensions to be considered, (1) size dimensions and (2) location dimensions.

In the practical dimensioning of a machine part, many considerations beyond its geometric form must be taken into account. The material to be used in the manufacture of the part, the nature of the machining or other shop operations, the precision required in certain of the measurements, all enter into the problem of correctly dimensioning a part for production. These are all engineering design considerations rather than questions of elementary representation and are more properly dealt with in courses of study in machine drawing and shopwork.

12-8. Size dimensions

The most common mechanical shapes are prisms, cylinders, and cylindrical end pieces. These shapes are largely a consequence of the machinery used in shop operations. The dimensions given are those used in the production of the object.

DIMENSIONING THE PRISM. Dimension the base in the view in which its true shape appears. This is the characteristic, or contour, view.

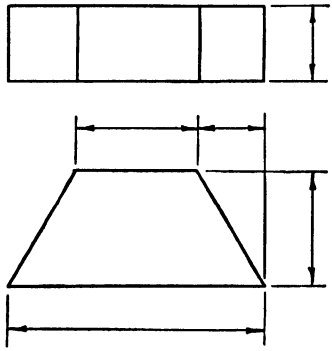


Fig. 12-13.

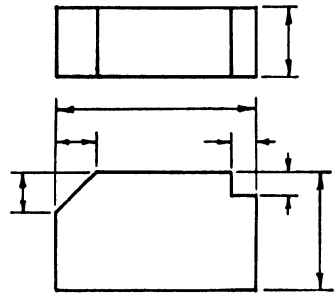


Fig. 12-14.

Give the thickness of the prism in the adjacent view, Fig. 12-13. Describe a bevel or a notch by dimensioning the piece cut off, Fig. 12-14.

DIMENSIONING THE CYLINDER. Give the diameter and length of the cylinder on its rectangular view, Fig. 12-15.

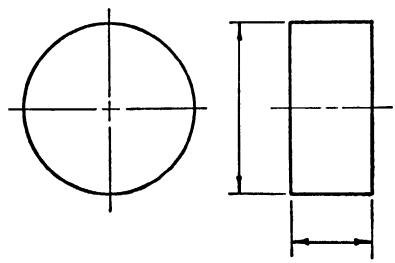


Fig. 12-15.

CYLINDRICAL-END SHAPES. In general, give the end radii and the distance between the centers. For a cylindrical-end slot give the width of the slot and the distance between centers, Figs. 12-16 to 12-19.

CIRCULAR HOLES. Use a leader and note on the circular view of the hole. The leader if extended should pass through the center of the circle,

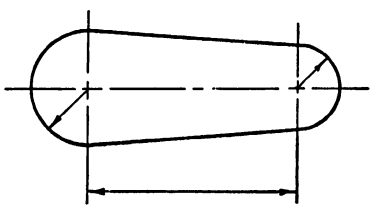


Fig. 12-16.

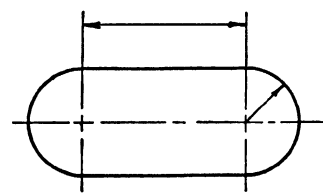


Fig. 12-17.

Fig. 12-20. The note specifies the *diameter* and the shop method of forming the hole (drill, bore, ream, etc.). When several holes of the same size

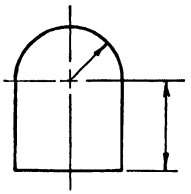


Fig. 12-18.

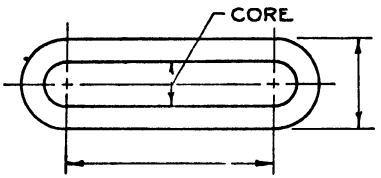


Fig. 12-19.

comprise a group, a single leader is used. The note gives the diameter and the number of holes to which this applies.

12-9. Location dimensions

Location dimensions fix the various geometric shapes which compose an object with respect to each other, or to some center line or surface,

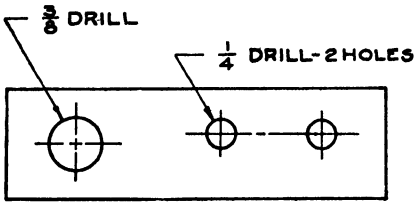


Fig. 12-20.

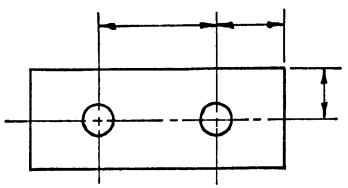


Fig. 12-21.

Fig. 12-21. These dimensions for a machine part require careful selection, since the matter of fitting or registration with other parts is involved. The selection of reference or starting points must be considered first. In cases where registration is controlled by shafts, bolts, pins, or screws

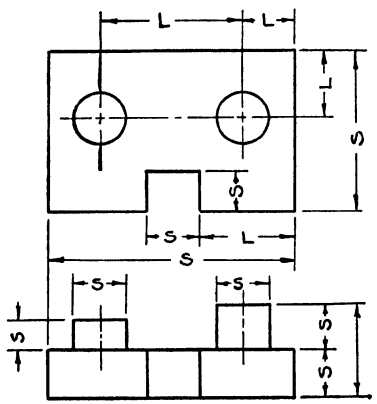


Fig. 12-22.

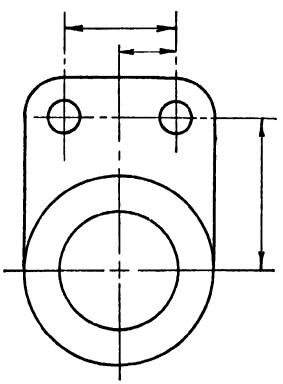


Fig. 12-23.

the starting points are usually center lines. Where pieces mate by surface contact, the reference points are usually on surfaces that are finished.

LOCATION OF HOLES. Holes are located by their centers and preferably on the view in which the hole shows as a circle. A center may be located by its distances from two intersecting faces of the object, Fig. 12-22. (The letters S and L indicate size and location dimensions, respectively.) If any of the faces are finished, they should be used for reference edges. If registration is controlled by a shaft, holes are located from the center lines of the piece, Fig. 12-23. When a center line is an axis of symmetry, a pair of holes may be dimensioned "across" the center line.

12-10. Finished surfaces

In cases where machine parts mate by surface contact, such surfaces should be located with respect to each other and also with respect to any shaft or bolt that may guide the piece, Fig. 12-24.

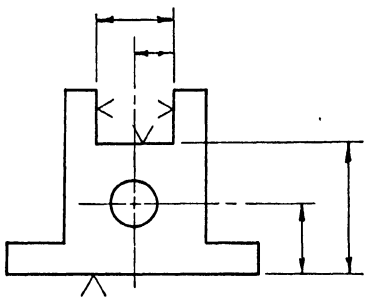


Fig. 12-24.

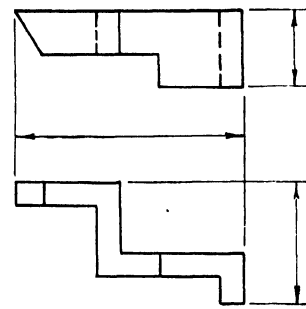


Fig. 12-25.

12-11. Over-all dimensions

The over-all dimensions give the total width, height, and depth of the object. These three over-all dimensions are given for every object except a cylindrical-end shape, Fig. 12-25.

12-12. The finish mark

The dimensions on a drawing specify the *finished* size of the object. If the object is cast, certain surfaces generally require finishing by some shop operation such as boring, milling, etc., which removes material from the casting. In order to provide sufficient material for this finishing operation, a *finish mark*, $\vee$ or f, is placed on every view where the surface to be finished appears as a line. This mark tells the patternmaker to increase the size of the object at this place to allow for finishing.

12-13. Detail drawings

A detail drawing is a drawing of a single part. The examples which follow illustrate the application of the dimensioning rules to three typical machine parts.

EXAMPLE 1. *Guide block* (*Fig.* 12-26). This object is essentially a prism and is dimensioned according to the prism rule, Art. 12-8. The front view is the characteristic view here, and the dimensions which describe

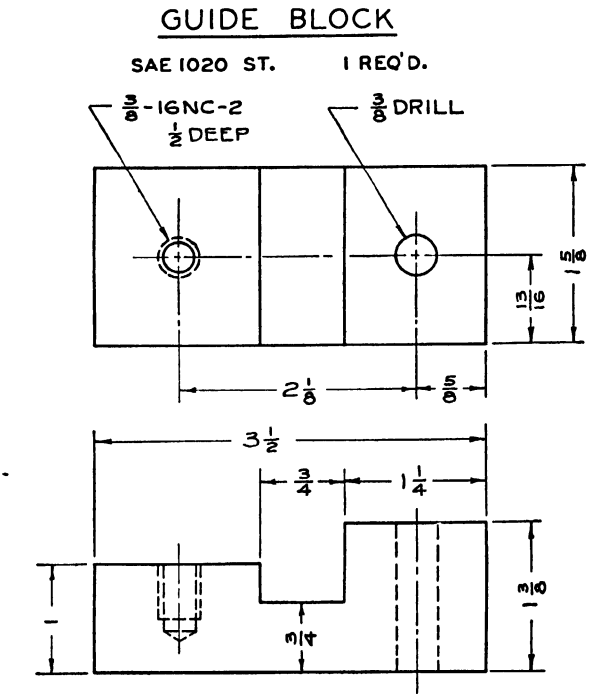


Fig. 12-26.

this shape appear on this view. While one generally avoids placing dimensions on the view itself, in this case it seems desirable to put the ¾-in. dimension on the front view in order to avoid a long extension line through the view. The thickness of the prism appears on the top view.

The holes are located from two faces of the block, and these dimensions are on the top view. The dimensions are placed between the views and on the right so far as possible.

Shop operations:

1. *Drilled hole.* ⅜ drill is specified on the circular view of the hole.

2. *Tapped hole.* ⅜-16NC-2, ½ deep is specified on the circular view of the hole. The threaded part of the hole is ½ in. deep. The tap drill extends beyond this.

The leaders for both holes are made parallel. No indication for surface finish is necessary since this block is cut from bar stock.

EXAMPLE 2. *Step pulley* (*Fig.* 12-27). This object is composed of a series of cylinders. The arrangement of the dimensions is according to the cylinder rule, Art. 12-8.

Shop operations:

1. *Shaft hole.* Holes for shafts, pins, etc., are generally finished by reaming or boring. Reamed holes are limited to those standard sizes

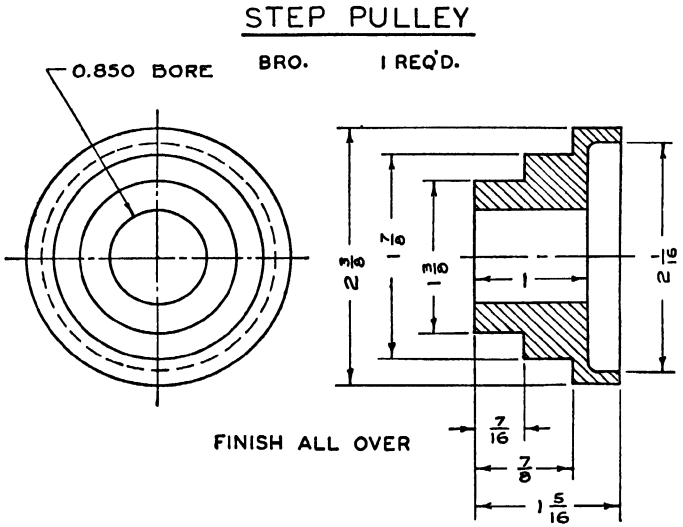


Fig. 12-27.

for which reamers are available. Odd-sized holes are bored. Thus, in this case the shaft hole is specified as "0.850 bore."

2. *Surface finish.* Every surface of the pulley is to be finished, and this is indicated by the note "finish all over." This operation would be performed by turning the pulley on a lathe.

EXAMPLE 3. *Rocker arm* (*Fig.* 12-28). For the purpose of dimensioning, this object can be analyzed into three cylinders and a cylindrical-end arm. The diameters of the cylinders are placed according to the cylinder rule. The dimensions for the height of each cylinder includes the thickness of the arm. This dimension is given because the face of each cylinder (hub) and the lower face of the arm are finished. The distance that each cylinder projects above the arm is not a necessary manufacturing dimension and therefore is not given. The end radii of the arm are not given separately because these dimensions are the same as for the cylinders, to which they are tangent. The center-to-center dimensions of the holes are given. No over-all length is required.

Shop operations:

1. *Shaft and pinholes.* The fillets are specified by a note. The holes are of standard size and are finished by reaming. Since the holes at the ends of the arm are alike a single leader is sufficient. The note is "¾ ream-

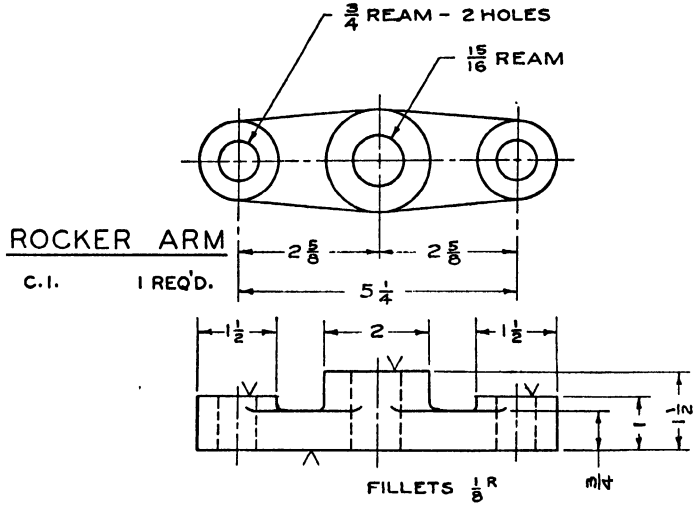


Fig. 12-28.

2 holes." The shaft hole is specified as "$1\frac{5}{16}$ ream." The leaders for both holes are made parallel.

2. *Surface finish.* The bottom face of the arm and the top faces of the hubs are finished. This is indicated by the finish mark, V. These finishing operations would generally be performed on a milling machine.

12-14. Detail drawing of toolmaker's vise

An assembly or design drawing of a toolmaker's vise is given in Fig. 12-29. The name of each part, material to be used, and number of pieces required for one vise is specified in the following table, called the Bill of Material:

Bill of Material

Piece No.	Name	No. req'd	Material
1	Body	1	Cast iron
2	Movable jaw	1	Cast iron
3	Screw	1	SAE 3140 steel
4	Knurled knob	1	SAE 1020 steel
5	Sliding plate	1	SAE 1020 steel
6	Pin ⅛ by 1 in.	1	SAE 1020 steel
7	Pin ⅛ by 1⅜ in.	1	SAE 1020 steel
8	¼ in. machine screw	1	Steel

The vise is assumed to be disassembled, and a detail drawing is made of each piece, Figs. 12-30 and 12-31. Standard items such as screws and pins are not drawn. Each detail drawing carries a subtitle which gives the part number and name, the material to be used, and the number of pieces required. Details may be drawn one to a sheet, or several may be included on one large sheet.

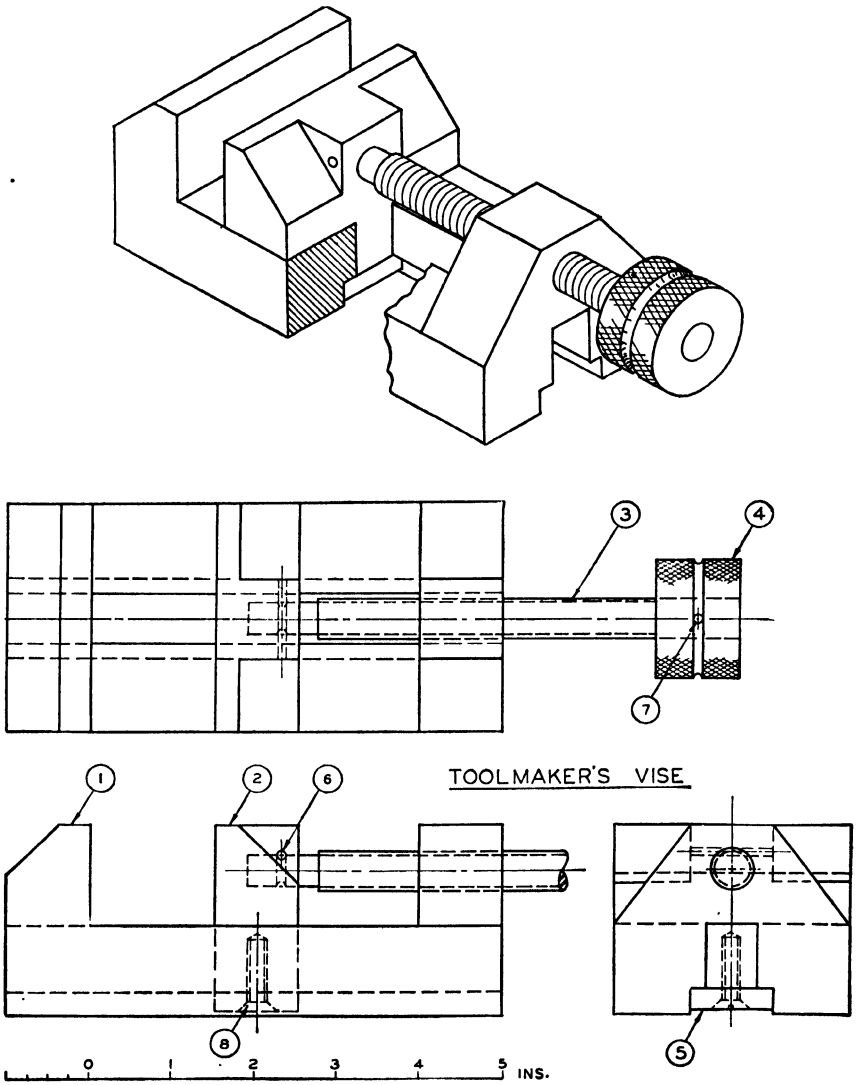


Fig. 12-29.

In the dimensioning of each detail, attention must be paid to the fitting of mating parts and to surface finish. For the purpose of this problem, a clearance of $\frac{1}{64}$ in. between sliding parts is specified. This allowance is rather coarse but avoids entering the subject of metal fits at this stage.

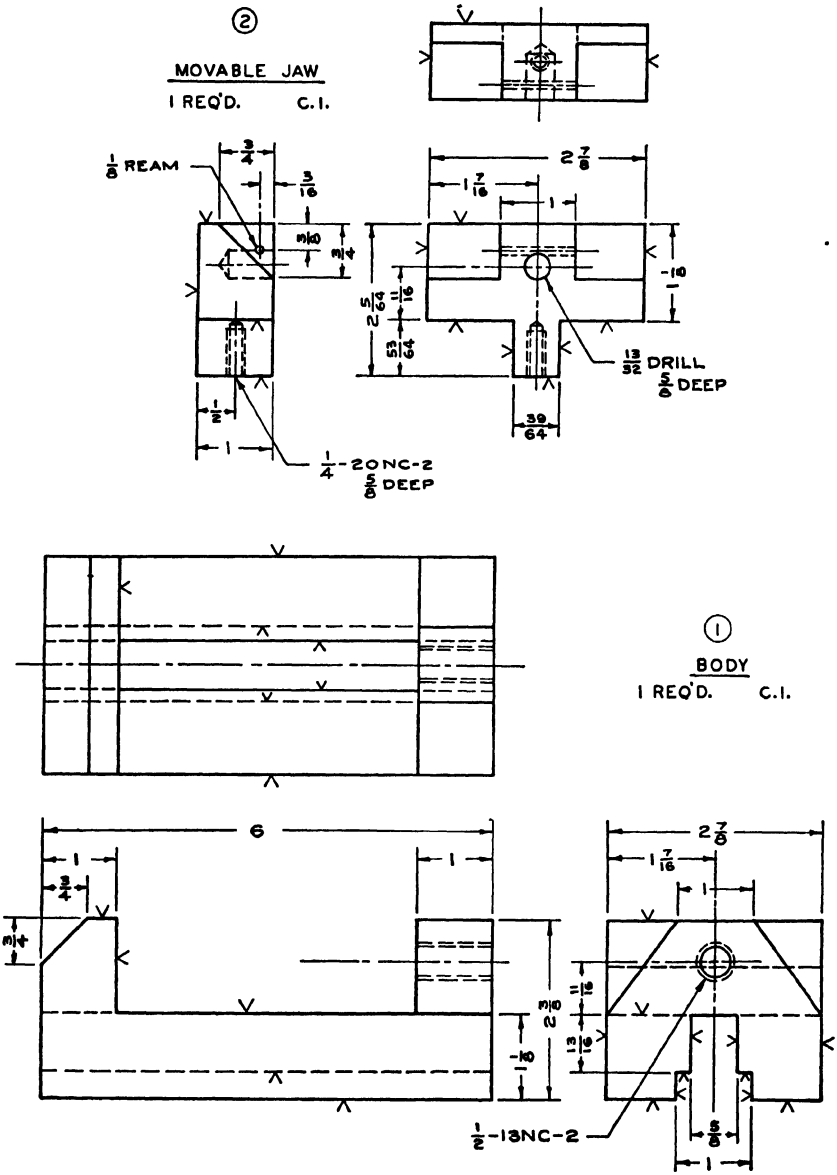
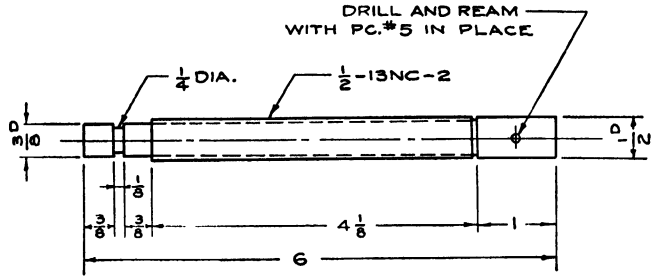


Fig. 12-30.

12-15. Dimensioning the vise details (Figs. 12-30 and 12-31)

In starting the dimensioning of any detail, the first dimensions selected are those which control the registration of the mating part.

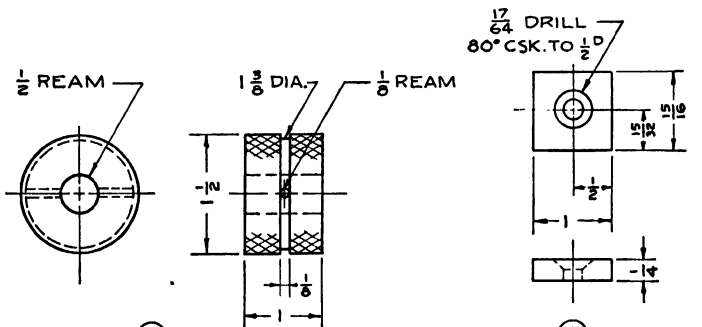
PART 1, BODY. The slot for the movable jaw is made $\frac{5}{8}$ in. wide by $1\frac{3}{16}$ in. deep. This is considered to be the basic size. Allowance for



③

SCREW

1 REQ'D. SAE 3140



④ KNOB

1 REQ'D. SAE 1020

⑤ SLIDING PLATE

1 REQ'D. SAE 1020

Fig. 12-31.

clearance on the lug of the movable jaw is made from this basic size. The surfaces on which the jaw slides are finished. The center line of the tapped hole is located from the upper finished surface. The sides and bottom of the casting are finished, and also the top of the jaw.

PART 2, MOVABLE JAW. For clearance, the width of the sliding lug is made $\frac{1}{64}$ in. less than the width of the slot in the body. Since the depth of the slot in the body is $1\frac{3}{16}$ in., the lug in the jaw is made $\frac{53}{64}$ in. high

TABLE 1

Tap-drill Sizes

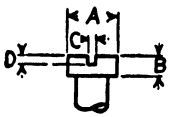
Fine- and Coarse-thread Series Standard

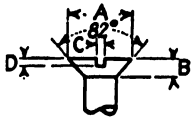
Size of Screw	Threads per Inch	Major Diameter of Screw	Clearance Drill	Tap-Drill Sizes		
				75 to 80 Per Cent of Full Thread	70 to 75 Per Cent of Full Thread	65 to 70 Per Cent of Full Thread
				Sheet Brass Sheet Nickel Babbitt Dural Fiber White Metal Hard Rubber	Mild Steel Aluminum Cast Iron Cast Brass	Bronze Tool Steel Drop Forging Stainless Steel Cast Steel Nickel Copper
00	96	0.047	55 (0.052)	64 (0.036)	63 (0.037)	63 (0.037)
0	80	0.060	51 (0.067)	56 (0.046)	55 (0.052)	55 (0.052)
1	72	0.073	47 (0.0785)	53 (0.0595)	53 (0.0595)	52 (0.0635)
	64			54 (0.055)	53 (0.0595)	53 (0.0595)
2	64	0.086	42 (0.0935)	50 (0.070)	49 (0.073)	48 (0.076)
	56			51 (0.067)	50 (0.070)	50 (0.070)
3	56	0.099	36 (0.1065)	46 (0.081)	45 (0.082)	44 (0.086)
	48			$\frac{5}{64}$	47 (0.0785)	46 (0.081)
4	48	0.112	31 (0.120)	43 (0.089)	42 (0.0935)	41 (0.096)
	40			44 (0.086)	43 (0.089)	42 (0.0935)
5	44	0.125	29 (0.136)	38 (0.1015)	37 (0.104)	35 (0.110)
	40			38 (0.1015)	37 (0.104)	36 (0.1065)
6	40	0.138	26 (0.147)	33 (0.113)	32 (0.116)	31 (0.120)
	32			35 (0.110)	$\frac{7}{64}$	33 (0.113)
8	36	0.164	17 (0.173)	29 (0.136)	29 (0.136)	28 (0.1405)
	32			29 (0.136)	29 (0.136)	28 (0.1405)
10	32	0.190	7 (0.201)	21 (0.159)	20 (0.161)	19 (0.166)
	24			26 (0.147)	25 (0.1495)	22 (0.157)
12	28	0.216	1 (0.228)	15 (0.180)	14 (01.82)	13 (0.185)
	24			17 (0.173)	16 (0.177)	15 (0.180)
$\frac{1}{4}$	28	0.250	H (0.266)	3 (0.213)	3 (0.213)	2 (0.221)
	20			8 (0.199)	6 (0.204)	5 (0.2055)
$\frac{5}{16}$	24	0.3125	Q (0.332)	I (0.272)	I (0.272)	J (0.277)
	18			F (0.257)	G (0.261)	$\frac{17}{64}$
$\frac{3}{8}$	24	0.375	X (0.397)	Q (0.332)	Q (0.332)	R (0.339)
	16			$\frac{5}{16}$	O (0.316)	P (0.323)
$\frac{7}{16}$	20	0.4375	$\frac{11}{32}$	W (0.386)	$\frac{21}{64}$	X (0.397)
	14			$\frac{23}{64}$	(0.368)	V (0.377)
$\frac{1}{2}$	20	0.500	$\frac{13}{32}$	$\frac{25}{64}$	$\frac{25}{64}$	$\frac{25}{64}$
	13			$\frac{27}{64}$	$\frac{27}{64}$	$\frac{7}{16}$
$\frac{5}{8}$	11	0.625		$\frac{17}{32}$	$\frac{17}{32}$	$\frac{9}{16}$
$\frac{3}{4}$	10	0.750		$\frac{41}{64}$	$\frac{21}{32}$	$\frac{13}{64}$
$\frac{7}{8}$	9	0.875		$\frac{49}{64}$	$\frac{13}{32}$	$\frac{29}{32}$
1	8	1.000		$\frac{7}{8}$	$\frac{29}{32}$	$\frac{29}{32}$

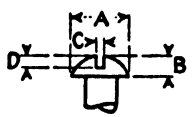
TABLE 2

Dimensions of Machine-screw Heads

Four-place Mantissas







Size of Screw	A	B	C	D	A	B	C	D	A	B	C	D
00	0.078	0.035	0 010	0.016	0 085	0.022	0.010	0.010				
0	0.102	0 046	0.016	0.025	0 115	0.032	0.016	0.016	0.110	0.043	0.016	0 031
1	0.120	0.051	0.020	0.028	0.140	0.038	0 020	0.020	0.132	0.052	0.020	0.035
2	0.137	0.059	0.025	0.031	0 164	0 044	0.025	0.023	0.154	0.060	0.025	0.040
3	0.158	0.069	0 025	0.035	0.190	0 052	0.025	0 025	0.178	0.069	0 025	0.044
4	0.179	0.078	0.025	0 039	0 216	0 059	0.025	0 027	0 202	0 078	0 025	0.049
5	0 200	0.086	0.031	0 043	0 242	0 067	0.031	0.032	0.226	0.087	0.031	0.053
6	0.220	0.094	0 031	0 046	0.268	0.074	0.031	0.034	0.250	0.096	0.031	0.058
8	0.264	0.110	0.040	0.054	0.320	0.090	0.040	0.040	0.298	0.114	0.040	0.067
10	0.305	0.128	0.040	0.061	0.372	0.104	0.040	0.047	0.346	0.133	0.040	0.076
12	0.345	0.145	0.051	0.069	0.424	0.120	0.051	0.053	0.394	0.151	0.051	0.085
$\frac{1}{4}$	0.401	0.166	0.051	0.079	0.492	0.139	0.051	0.060	0.457	0.175	0.051	0.096
$\frac{5}{16}$	0.504	0.206	0.072	0.096	0.618	0.176	0.072	0.075	0.574	0.219	0.072	0.118
$\frac{3}{8}$	0.606	0.246	0.072	0.114	0.742	0.211	0.072	0.090	0.689	0.262	0.072	0.140
$\frac{7}{16}$	0.709	0.287	0.102	0.132	0.868	0.248	0.102	0.103	0.805	0.305	0.102	0.161
$\frac{1}{2}$	0.815	0.328	0.102	0.150	0.992	0.283	0.102	0.118	0.920	0.347	0.102	0.183
$\frac{9}{16}$												
$\frac{5}{8}$												
$\frac{7}{8}$												
1												

to prevent the sliding plate from binding. To ensure proper alignment of the screw (part 3), the drilled hole is located from the bottom sliding surface of the jaw. All contact surfaces are finished, and also the face and top of the jaw.

PART 3, SCREW. An allowance of $\frac{1}{32}$ in. (in diameter) is made for an easy fit of the screw in the movable jaw. The groove for the pin is dimensioned so that the end of the screw will press on the jaw (not on the pin) when the vise is being closed.

PART 4, KNOB. The hole for the pin is drilled and reamed after the knob is pressed into place on the end of the screw. A medium knurl is used to provide a good grip.

PART 5, SLIDING PLATE. A clearance of $\frac{1}{16}$ in. is allowed between the plate and sides of the slot. Information for the drilled and countersunk hole is taken from Table 2.

12-16. Engineering standards

Standardization in details of engineering practice is very extensive. There are over 500 standards publications of the ASA. Copies can be obtained from the offices of the Association in New York City.

Information about standard practices is assembled in the various engineering handbooks. The student is referred to the "American Machinists' Handbook" (McGraw-Hill Book Company, Inc., New York) for general information in the detailing of machine parts. Tables 2 and 3 are taken from this handbook.

REFERENCES FOR READING

COLVIN and STANLEY: "American Machinists' Handbook," 8th ed., McGraw-Hill Book Company, Inc., New York, 1945.

FRENCH, T. E.: "Engineering Drawing," 7th ed., McGraw-Hill Book Company, Inc., New York, 1947.

SPENCER and GRANT: "The Blueprint Language," The Macmillan Company, New York, 1946.

PROBLEMS

12-1. *Sectional views.* Use Problem Sheet 12-1. Refer to the text of this chapter and also Sec. 3 of the ASA American Drafting Standards (Appendix I).

12-2. *Sectional views.* Use Problem Sheet 12-2.

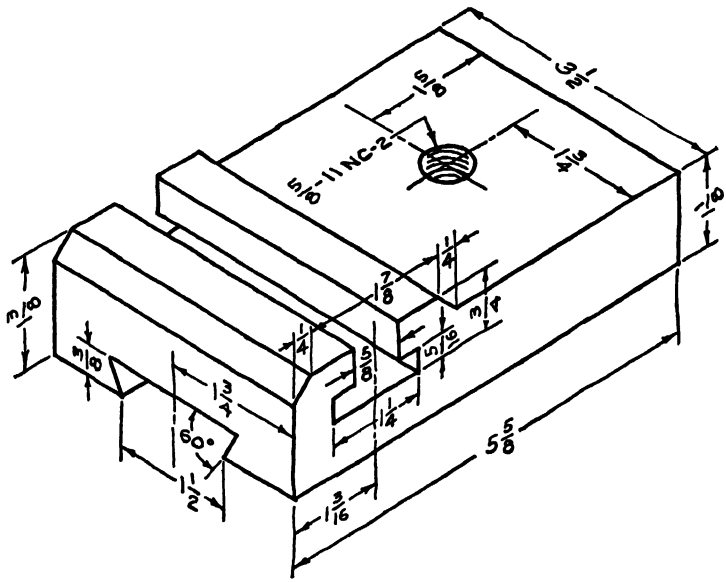
12-3. *Screw-thread symbols.* Use Problem Sheet 12-3.

12-4. *Dimensioning.* Trace Problem Sheet 12-4.

12-5. *Dimensioning.* Trace Problem Sheet 12-5.

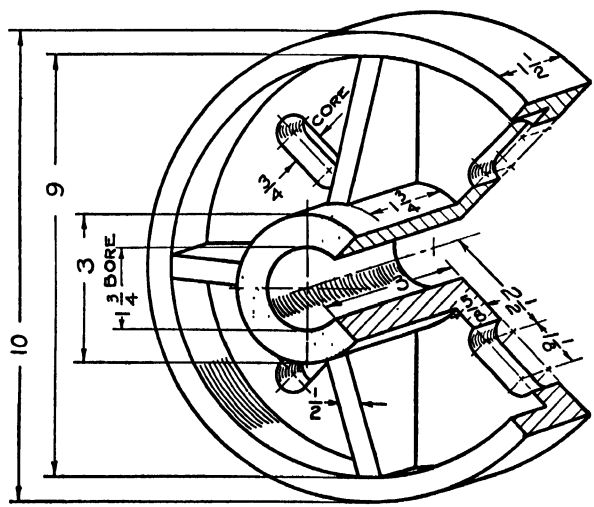
12-6 to 12-8. *Dimensioning.* Use Problem Sheets 12-6 to 12-8. Refer to Chap. 12 and Secs. 5 and 6 of the ASA American Drafting Standards (Appendix I).

12-9. *Working drawing.* Make a working drawing of the tool-post slide. Material: SAE 1020 steel. Use 11- by 17-in. sheet.



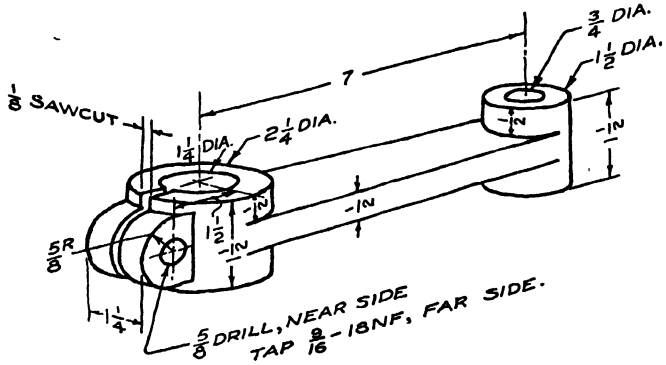
Prob. 12-9.

12-10. *Working drawing.* Make a working drawing of the faceplate. Material: Cast iron. Finish the flange, face, and end of hub.



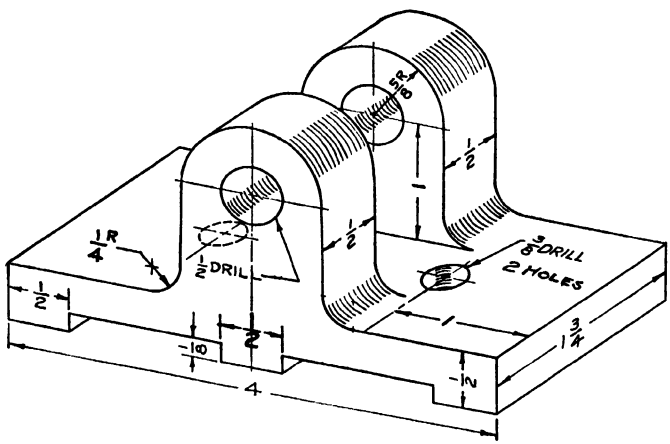
Prob. 12-10.

12-11. *Working drawing.* Make a working drawing of the connecting rod. Material: Cast iron. Finish faces of hubs. Use 11- by 17-in. sheet.



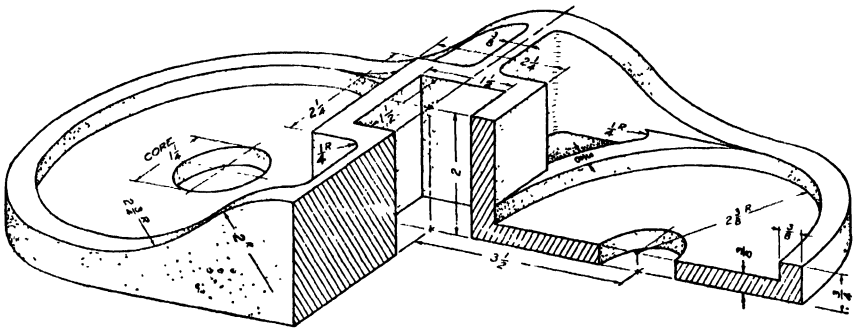
Prob. 12-11.

12-12. *Working drawing.* Make a working drawing of the die-cast lever bracket. Use 11- by 17-in. sheet.



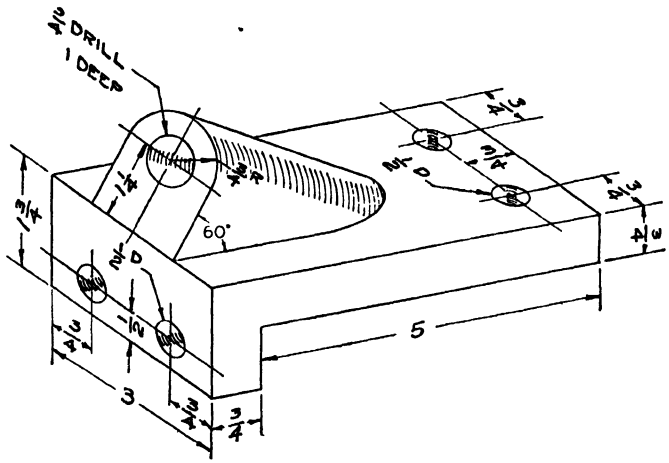
Prob. 12-12.

12-13. *Working drawing.* Make a working drawing of the cast-iron base plate. Use 11- by 17-in. sheet.



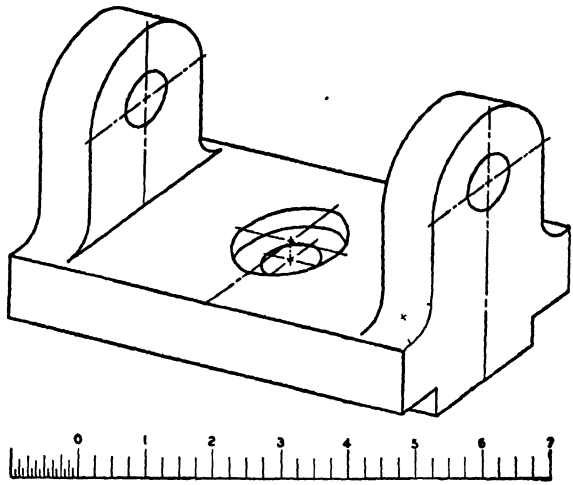
Prob. 12-13.

12-14. *Working drawing.* Make a working drawing of the cast-iron angle bracket. Refer to Art. 11, ASA American Drafting Standards (Appendix I). Use 11- by 17-in. sheet.



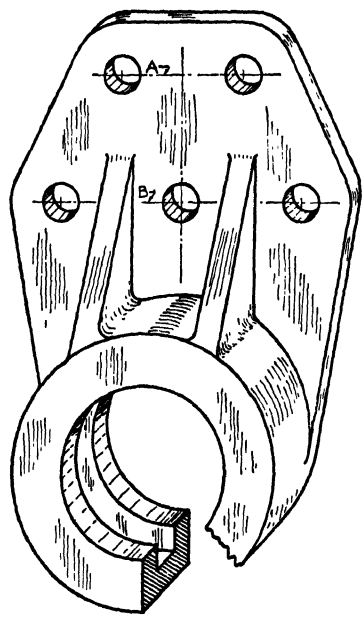
Prob. 12-14.

12-15. *Swivel bracket.* Make a working drawing of the swivel bracket.



Prob. 12-15.

12-16. *Bearing bracket.* Scale: 6 in. = 12 in. Make a two-view drawing, showing the side view in section.



Prob. 12-16.

Description of Casting

Outside diameter of the hub = 6½ in.
Thickness of the hub = 3¼ in.
Diameter of the bearing hole = 4 in.
Anchorage groove is 5 in. in diameter by 1¼ in. wide and is cored in the center of the hub.
Boltholes are ¾ in. diameter and spaced 3 in. horizontally.
Bolt center lines *A* and *B* are spaced 3 in. vertically.
The vertical distance between the center line of the 4-in. bearing hole and center line *B* is 5¼ in.
The back of the casting is ⅝ in. thick.
The corners of the back are rounded to a radius of 1¼ in. centered on the boltholes.
The ribs are ⅝ in. thick and 2⅜ in. apart (clear distance).
The ribs extend halfway between center lines *A* and *B*.
The inside corners of the casting are filleted. Radius ¼ in.
The outside corners are rounded to a ¼-in. radius.
Material—cast iron. Two required.
Machine operations to be indicated.
Boltholes to be drilled.
Bearing hole to be bored.
The back of the casting to be surfaced.

12-17. *Details from assembly drawing.* Make a freehand sketch of each detail of the toolmaker's vise. Use 11- by 17-in. sheet.

Specifications for Toolmaker's Vise (Fig. 12–17)

Note on finish: Finish the bottom, the sides, and the contact surfaces of all mating parts.

PART 1. BODY. Cast iron. One required.

1. Center the vee of the fixed jaw on the screw center line.
2. The slot for the movable jaw is to be ½ by ⅝ in. (basic size). Allowances for the slide on the movable jaw (part 3) are made from this basic size.
3. The slot for the jaw plate (part 10) is ⅞ in. wide (basic size).
4. Two tapped holes are provided in base for a ⅜-16NC-2 clamping screw. Tap drill must not break through.

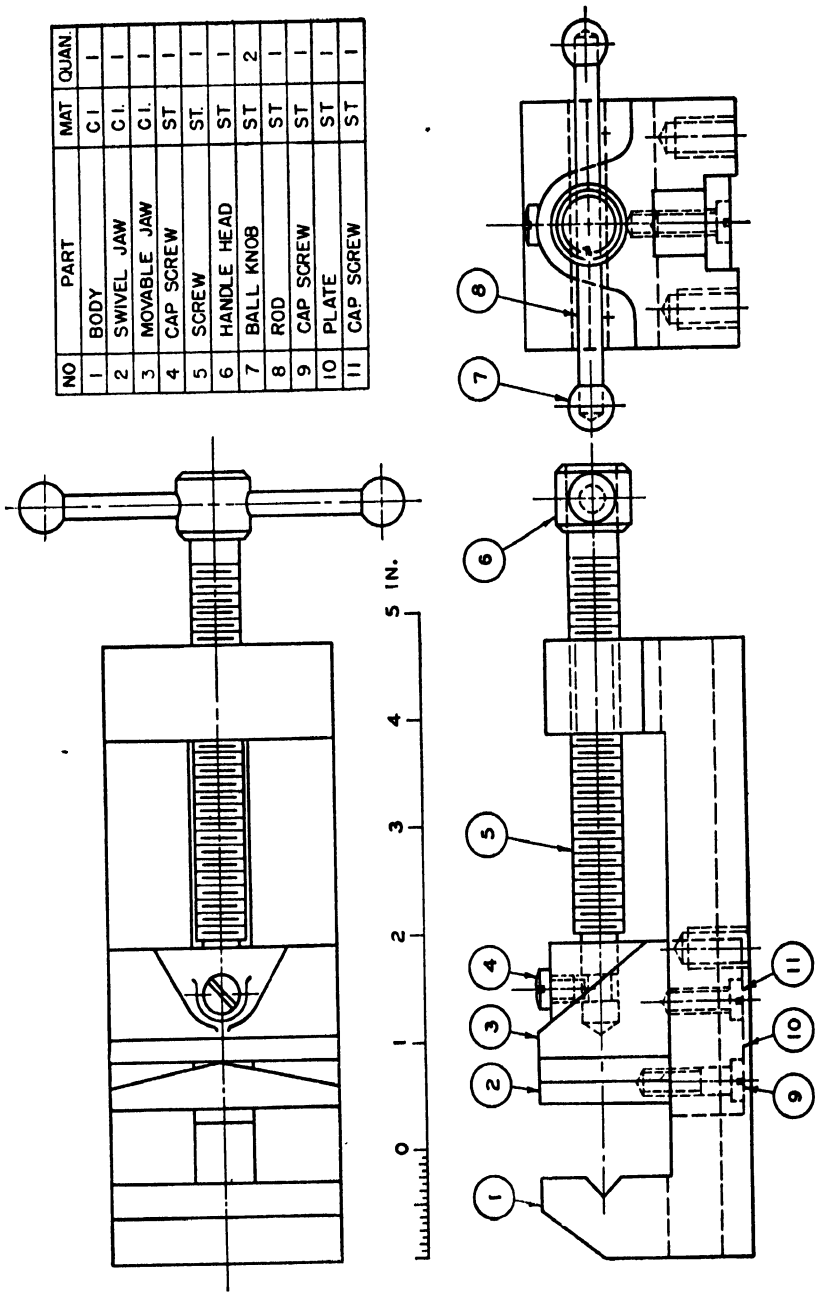
PART 2. SWIVEL JAW. Cast iron. One required.

Taphole for ¼-20NC-2 screw (part 9).

PART 3. MOVABLE JAW. Cast iron. One required.

1. Taphole for ¼-20NC-2 screw. Hole for screw 5 is ⅜ in. diameter (basic size).
2. Clearance for screw 9 is to be 1/32 in.
3. For clearance of slide allow 1/64 in. Provide 1/64 in. clearance between plate 10 and seat.

NO	PART	MAT	QUAN.
1	BODY	C I	1
2	SWIVEL JAW	C I.	1
3	MOVABLE JAW	C I.	1
4	CAP SCREW	ST	1
5	SCREW	ST.	1
6	HANDLE HEAD	ST	1
7	BALL KNOB	ST	2
8	ROD	ST	1
9	CAP SCREW	ST	1
10	PLATE	ST	1
11	CAP SCREW	ST	1



Prob. 12-17.

PART 4. SETSCREW, ¼- by ⅜-in. dog point. Steel, SAE 4130. One required. Turn dog point on a ¼-20NC fillister-head screw. Point is $\frac{3}{16}$ in. diameter by ⅛ in. long.

PART 5. VISE SCREW. SAE 3140. One required.

1. Thread, ½-13NC-2.

2. Piece 6 to be press-fit on screw. Drill hole for rod 8 after assembly. Allow $\frac{1}{64}$ in. for clearance between rod and hole.

3. Turn end of screw to fit hole in part 3, allowing $\frac{1}{64}$ in. for clearance.

PART 6. HANDLE HEAD. SAE 1020. One required. Press-fit on screw 5. Hole for handle drilled after assembly on screw 5.

PART 7. BALL KNOB. SAE 1020. Two required. Knob to be press-fit on rod 8.

PART 8. HANDLE ROD. SAE 1020. One required.

PART 9. FILLISTER-HEAD MACHINE SCREW. SAE 1016. Size: ¼-20NC-2. One required.

PART 10. SLIDING PLATE. SAE 1020. One required. Allow $\frac{1}{32}$ in. for clearance on screw holes.

PART 11. FILLISTER-HEAD MACHINE SCREW. SAE 1016. One required. Size: ¼-20NC-2.

12-18. *Working drawings.* Use 11- by 17-in. sheet.

Title: STUFFING-BOX DETAILS; scale full size.

a. Making a working drawing of the stuffing-box body.

b. Make a working drawing of the gland and stud.

c. Make an assembly drawing of the stuffing box.

When a reciprocating or a rotating shaft enters a cylinder containing a fluid or gas, it is necessary to use a stuffing box to prevent leakage along the shaft. This is accomplished by placing wicking inside the box and compressing it against the shaft by tightening the gland. Stuffing boxes are used on steam and pump cylinders, valves, marine propellers, etc.

Description of Parts

a. Stuffing-box body:

Packing space to be bored to $1\frac{5}{8}$ in. diameter.
Back of the packing space to be chamfered to a 30° angle.
Shaft hole to be reamed to ⅞ in. diameter.
Boltholes (4) to pass ½-in. bolts.
Boltholes counterbored to surface 1 in. diameter.
Flange tapped for two studs (see below).
Finish back of the casting and the front face of the flange.
Draw a front view and a sectioned side view.
Material, cast iron. One required.

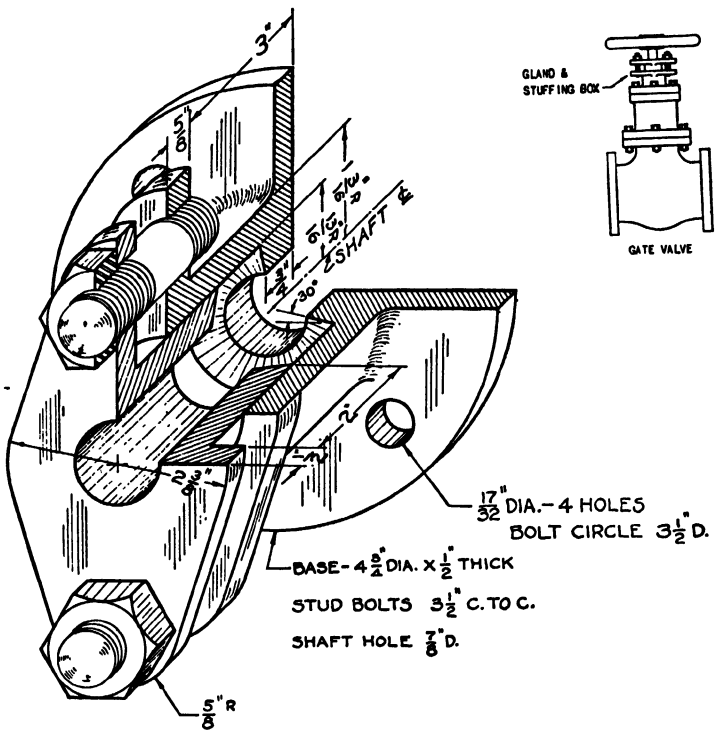
b. Gland and stud:

The diameter of the gland to be $\frac{1}{64}$ in. less than the diameter of the packing space for clearance.
The back end of the gland to be chamfered to a 30° angle.
The flange drilled to pass ⅝-in. studs ($\frac{1}{32}$ in. clearance).

Material, cast iron. One required. Finish all over.
Studs: Diameter $\frac{5}{8}$ in., length 3 in.
Thread 2 in. on the nut end and $\frac{5}{8}$ in. on the screw end.
Threads are National Form, coarse series, class 2.
Eleven threads per inch.
Material, steel. Two required.
Draw a section and an end view of the gland.
Draw one view of the stud.

c. Stuffing-box assembly:

Draw a front view (visible lines only) and a side view in half section (visible lines only).



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APPENDIX I

American Standard

Drawings and Drafting Room Practice

SECTION 1—ARRANGEMENT OF VIEWS

1 Based on Third Angle Orthographic Projection

For drawings in orthographic projection the third angle system (known in Europe as "American Projection") has been in practically universal use in the United States for many years and is continued as the American Standard. A brief discussion of this practice based on sketches of the object shown in Fig. 1 follows.

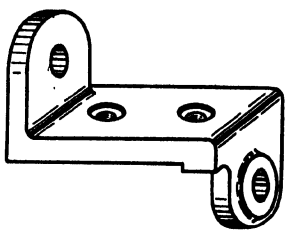


FIG. 1 PICTORIAL VIEW OF OBJECT SHOWN IN FIG. 2

2 The Six Possible Principal Views

In third angle projections the six views, top, front, bottom, right side, left side and rear, will be arranged as shown in Fig. 2. The plane of projection is between the observer and the object, and in all cases except for the rear view will be hinged to the vertical plane and be revolved into coincidence with it, in a direction away from the object. The rear view may appear at the extreme right in reverse position to that shown in Fig. 2.

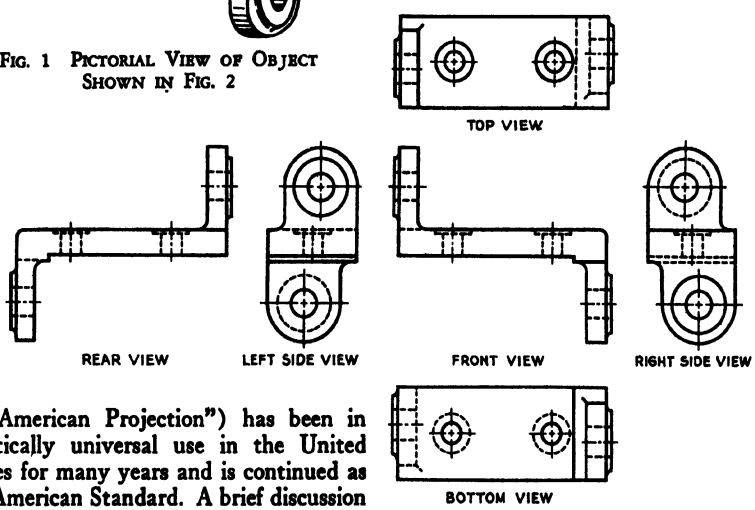


FIG. 2 THE SIX POSSIBLE PRINCIPAL VIEWS OF AN OBJECT

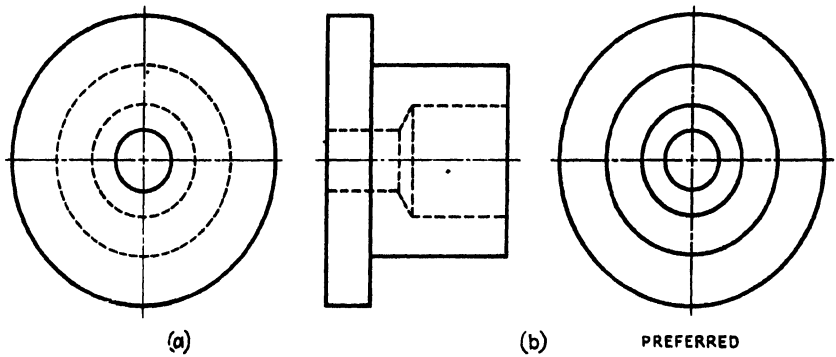


FIG. 3 CHOICE OF VIEWS TO AVOID HIDDEN LINES. HIDDEN LINE TECHNIQUE

3 *Choice of Views*

As a general rule a view should be made in each direction in which the contour of a characteristic shape necessary to the construction would be shown.

(a) Views should be selected to give as few hidden lines as possible. Compare side views (a) and (b) in Fig. 3.

(b) Only those views should be drawn that are necessary to portray clearly the shape of the part.

4 *One-View Drawings*

In general, two views are necessary as a minimum for the description of the shape of a simple object, but many cylindrical parts may be portrayed adequately by one view, if the necessary dimensions are indicated as diameters. See Fig. 4.* Thin pieces of uniform thickness such as shims, gaskets and plates may also be shown by one view with a note giving the thickness.

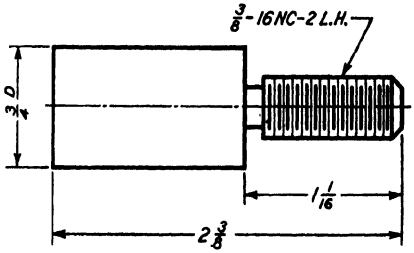


FIG. 4 ONE VIEW DRAWING *

* In examining the illustrations it should be kept in mind that these drawings are not necessarily complete in themselves but have been designed to illustrate principles stated in the text matter. In so far as they are complete they conform to standards recommended herein; frequently in the interest of simplicity they are fragmentary.

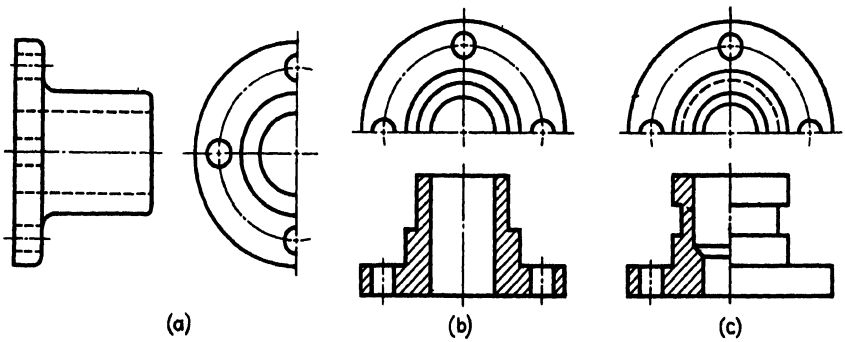


FIG. 5 ARRANGEMENT OF HALF VIEW DRAWINGS

5 Two-View Drawings

Two-view drawings may be arranged as any two adjacent views in the relation shown in Fig. 2.

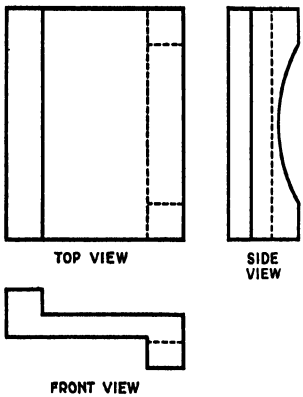


FIG. 6 ALTERNATE ARRANGEMENT OF THREE VIEWS

6 Two-View Drawings—One View a Half View

When space is limited, it is permissible to represent symmetrical objects by half views. If the adjacent view is an exterior view, the near half of the symmetrical view should be drawn, see Fig. 5(a). If the adjacent view is a full or half section, the far half of the symmetrical view should be drawn, see Fig. 5(b) and 5(c).

7 Three-View Drawings

Any three adjacent views in the relation to each other as illustrated in Fig. 2 may be used.

8 Three-View Drawings—Alternate Positions

When space limitations require, the side view may be placed across from the top view as in Fig. 6.

9 Three-View Drawings

FRONT AND TWO SIDE VIEWS. For objects where two side views can be used to better advantage than one, these need not be complete views of the entire object, if together they describe the shape of the object, Fig. 7. In this case only the hidden lines immediately behind the face need be shown.

10 Bottom Views

A bottom view, or "view looking up," can be used to advantage instead of a top view when the shapes or operations to be shown are on the under side of the part. For example: In a punch and die drawing, the theoretical arrangement of views would be as in Fig. 8 with the view of the bottom of the punch placed in the position of the bottom view, and the top of the die in the position of the top view, each facing the front view.

(a) To conserve space this arrangement is modified by placing the drawing of the bottom of the punch to the right of and in line with the top view of the die, as if it

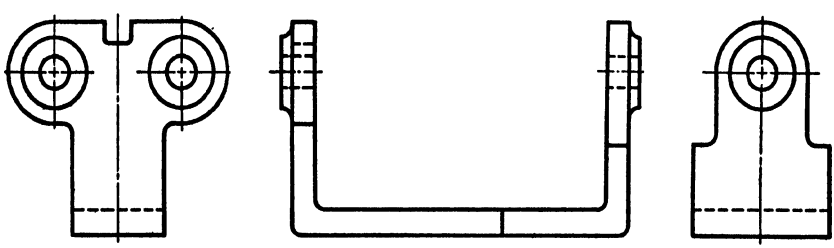


FIG. 7 ARRANGEMENT OF FRONT AND TWO SIDE VIEWS
Top view omitted to save space; two end views illustrate the principle

were turned over from the top view, Fig. 9. In drawings where any such arrangement of views is employed the views should always be carefully titled to aid in the reading.

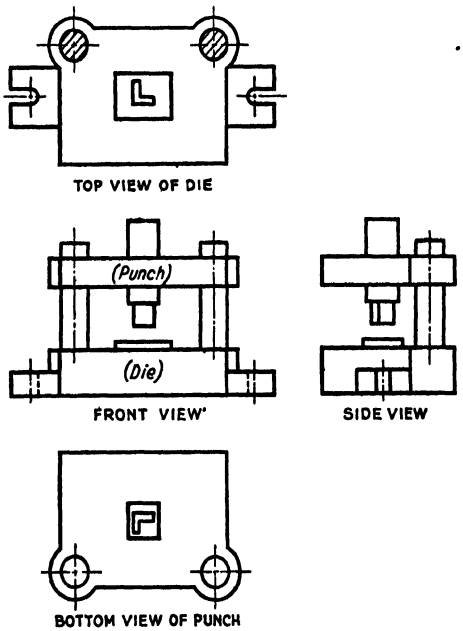


FIG. 8 THEORETICAL ARRANGEMENT OF VIEWS

(b) Bottom views (views looking up) should not be used in steel plate and structural drawings. Instead, the view should be shown as a sectional view looking down, the cutting plane passing a little above the bottom, Fig. 10.

11 Auxiliary Views

Objects having inclined faces require auxiliary views to show the true shapes of the inclined surfaces. Partial auxiliary views which show only the inclined faces may be used to advantage in order to simplify the drawing, to eliminate difficult projections, and to show true relationships for dimensioning. The auxiliary plane is revolved into the plane of the paper by considering it to be hinged to the plane to which it is perpendicular. See Figs. 11, 12, and 13.

12 Identification of Views

When any of the standard arrangements described above are used on a drawing, it is unnecessary to identify or name the views. In unusual or special combinations of views, however, the views should be clearly identified as in Fig. 9.

13 Space Between Views

Ample space should be provided between views to permit of placing dimensions without crowding and to preclude the possibility of notes pertaining to one view overlapping or crowding the other views.

14 Hidden Lines

Hidden lines should be indicated by standard type line No. 3, Fig. 18. This

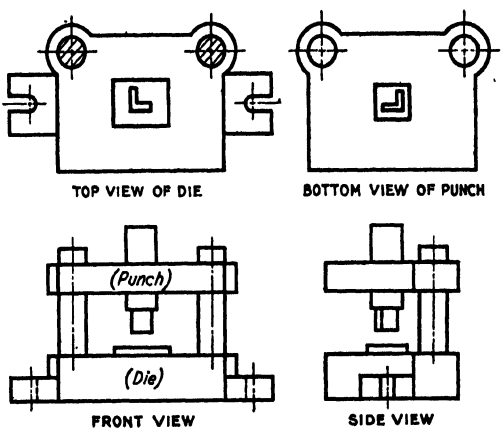


FIG. 9 APPROVED ARRANGEMENT OF VIEWS FOR PUNCH AND DIE DRAWING

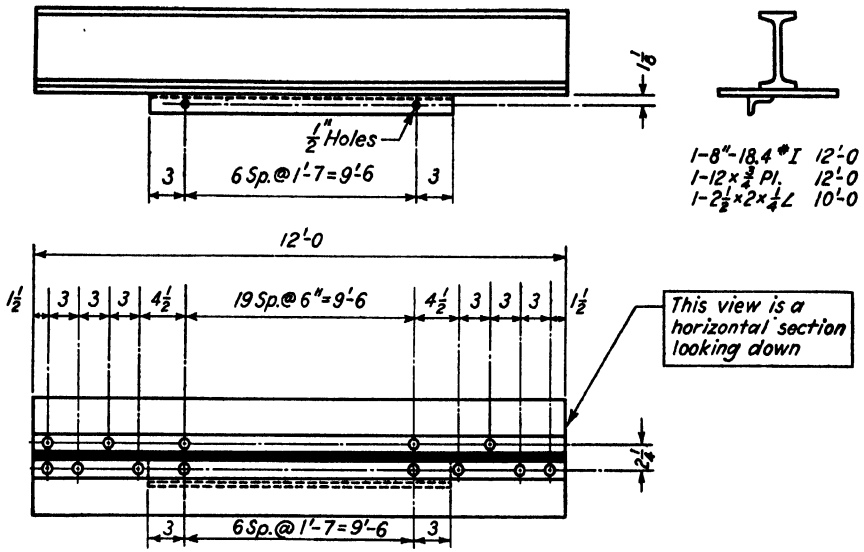


FIG. 10 ARRANGEMENT OF VIEWS IN STRUCTURAL DRAWING

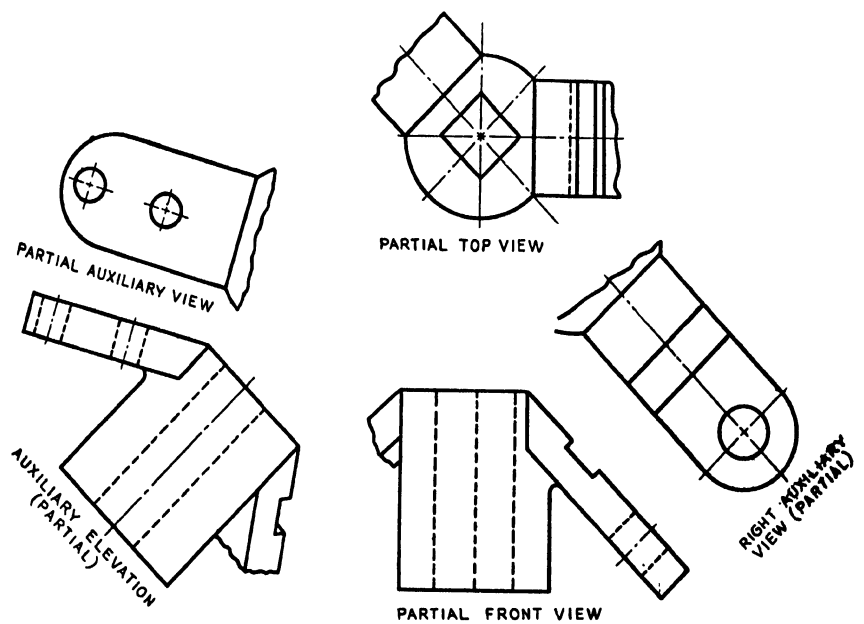


FIG. 11 PARTIAL AUXILIARY VIEW ON PLANE PERPENDICULAR TO HORIZONTAL

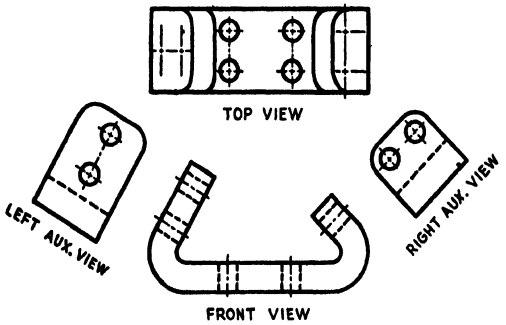


FIG. 12 AUXILIARY VIEWS ON PLANES PERPENDICULAR TO VERTICAL

line should always begin with a dash in contact with the line from which it starts, except when the dash would form a continuation of a full line. Dashes should touch at corners. Circular arcs should start with dashes at tangent points as in Fig. 14.

15 Runouts and Filleted Intersections

The intersection of two unfinished surfaces always should be shown rounded or filleted. Intersections of two unfinished surfaces, theoretically showing no line, may be indicated by a conventional line. The location of the line should be at the theoretical intersection and the contour must be shown or specified as in Figs. 15 and 16.

16 Indication of Scale

The scale of the drawing should be indicated on the sheet. If more than one detail occurs on a sheet and different scales are used, the scale should be shown under each detail. See Fig. 17 for methods of indicating the scale.

When the drawing is much

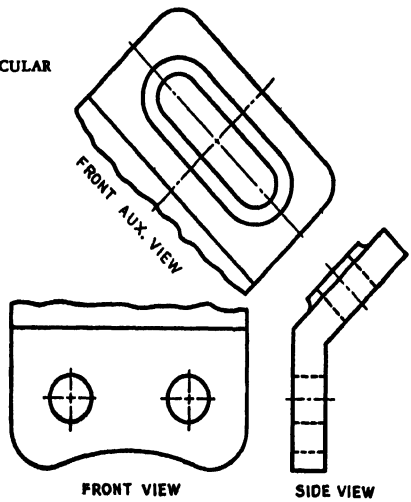


FIG. 13 AUXILIARY VIEW ON PLANE PERPENDICULAR TO PROFILE

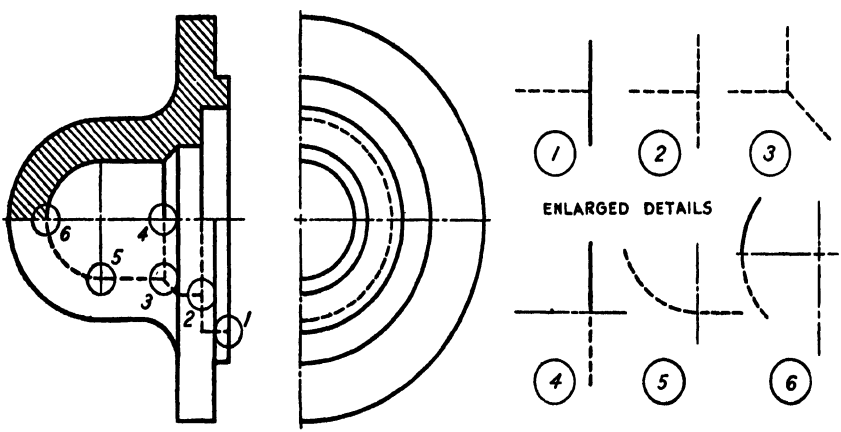


FIG. 14 HIDDEN LINE TECHNIQUE

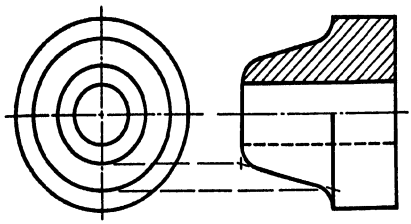


FIG. 15 REPRESENTING ROUNDED CORNERS AND FILLETS

larger than the object, one view, in general outline, actual size, may be shown in one corner of the sheet, for identification purposes.

Drawings which are to be reduced or enlarged photographically should have the scale indicated graphically as well as numerically.

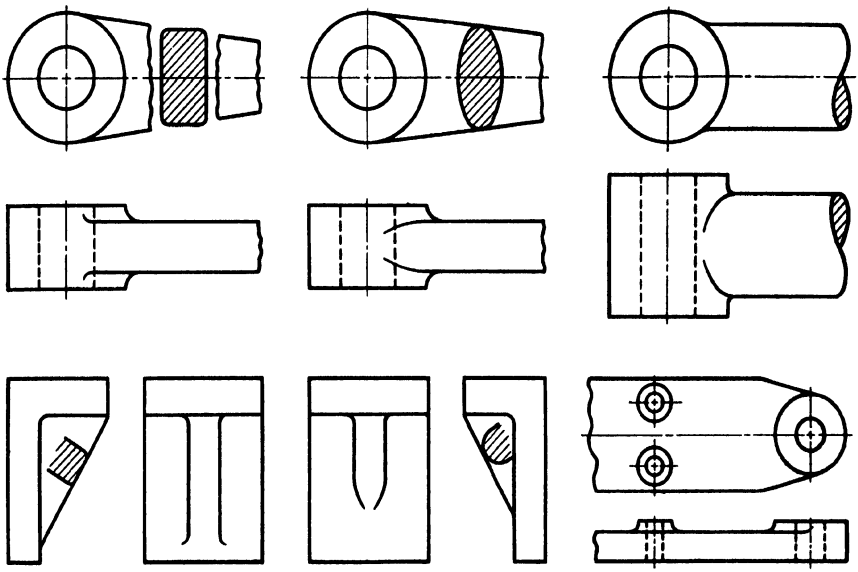


FIG. 16 RUNOUTS AND FILLETED INTERSECTIONS

DRAWING SMALLER THAN OBJECT

Scale $1\frac{1}{2}$"=1'-0"
Scale 3"=1'-0"
Quarter Scale
Scale $\frac{3}{8}$"=1"
Half Scale
Scale $\frac{3}{4}$"=1"
Full Scale
Scale $\frac{1}{2}$"=1'-0"
Scale 1"=1'-0"
Scale 1"=100'-0"

DRAWING LARGER THAN OBJECT

Scale $\frac{1.5}{1}$
Scale $\frac{2}{1}$

MAP SCALES

Scale $\frac{1}{62500}$ (Representative fraction)

400 0 400 800 Ft (Graphic scale)

FIG. 17 DESIGNATION OF SCALES

Outline of Parts	1	THICK	The outline should be the outstanding feature and the thickness may vary to suit size of drawing.
Section lines	2	THIN	Spaced evenly to make a shaded effect.
Hidden lines	3	MEDIUM	Short dashes, closely and evenly spaced.
Center lines	4		Alternate long and short dashes, closely and evenly spaced.
Dimension and Extension lines	5	$3\frac{1}{2}$	Lines unbroken, except at dimensions.
	6	$2'-3\frac{1}{2}$	Lines unbroken, dimensions above line for civil eng. and struct. practice only.
Cutting Plane line	7		Long and two short dashes alternately and evenly spaced.
Break lines	8		Free hand line for short breaks.
	9		Ruled line and free hand zigzag for long breaks.
Adjacent Parts and Alternate Positions	10		Broken line made up of long dashes.
Ditto line	11		Indication of repeated detail. Short double dashes evenly spaced.

FIG. 18 WIDTH AND CHARACTER OF LINES

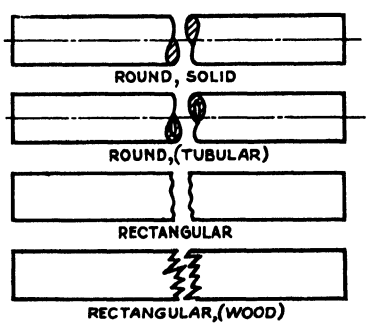


FIG. 19 CONVENTIONAL BREAKS

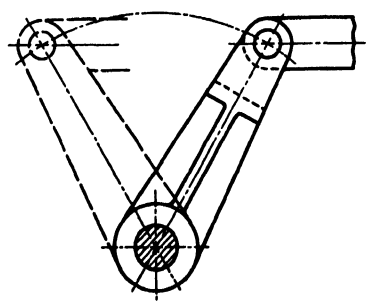


FIG. 20 INDICATION OF ALTERNATE POSITIONS

SECTION 2—LINES AND LINE WORK

17 Line Characteristics

It is recommended that the types shown in Fig. 18 be used for the purposes indicated. All lines should be clean and black. The actual width of each type of line should be governed by the size and style of the drawing, the relative widths of the lines should approximate those shown in Fig. 18.

18 Widths of Lines

Three widths of line, thick, medium and thin, are shown and are considered desirable on finished drawings in ink, both for legibility and appearance, although in rapid practice and in particular on penciled tracings from which prints are to be made this may be simplified to two widths, medium and thin, both of which must be dense black in order to print.

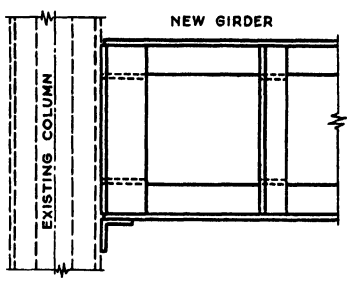


FIG. 21 INDICATION OF ADJACENT PARTS

For pencil tracings the lines should be in proportion to the ink lines, medium for outlines, hidden, cutting plane, short breaks, adjacent part and alternate position lines; and thin for section, center, extension, dimension, long break, and ditto lines.

19 Dimension and Extension Lines

Dimension and extension lines should be thin lines. Extension lines should not touch the outlines of the object, Fig. 48.

20 Break Lines Used to Shorten the View of Long Uniform Sections

Break lines may be used on both detail and assembly drawings. On small parts freehand lines are best, (Fig. 18, line 8), while on large parts the second form, made with thin ruled lines with freehand "zigzags," (Fig. 18, line 9) is preferred. When a portion of the lengths of shafts, rods, etc., are broken out to shorten the view, the breaks will be indicated as shown in Fig. 19.

In structural drawing, many parts are not drawn to detail scale in length but no break is shown unless it is necessary to show a revolved section. Symmetrical parts discontinued beyond the center line may have the break made as in Fig. 21 or Fig. 51.

21 Alternate Positions and Adjacent Parts

An alternate position, or indication of the limiting positions of a moving part should be shown by a line made up of long dashes of medium width, Fig. 20.

Adjacent parts added on a drawing to indicate the position or use of the piece represented, are drawn with the same long-dash lines, Fig. 21. This line is also used in showing bosses and lugs cast on for holding purposes, which are to be removed later. On drawings of an illustrative nature adjacent parts may be shown as solid red- or diluted black-ink lines instead of black long-dash lines.

SECTION 3—SECTIONAL VIEWS

22 General Principles

Sectional views, commonly called "sections" should be used when the interior construction cannot be shown clearly by outside views. (In assembly drawings they also serve to indicate a difference in materials.)

A sectional view should be made as if on that view the nearest part of the object were cut or broken away. The exposed or cut surface of the material is indicated by "section lining" or "cross-hatching."

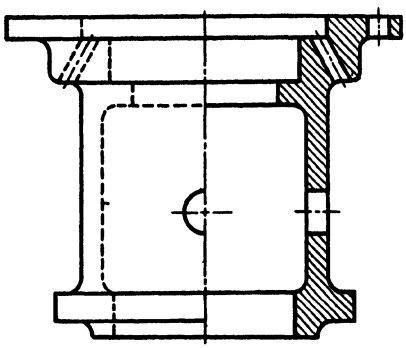


FIG. 22 HIDDEN LINES IN DETAIL HALF SECTION

Hidden lines behind the cutting plane should be omitted unless required for the necessary description of the object. In half-sectioned detail drawings hidden lines may be shown on the unsectioned side only if needed for dimensioning or clarity, Fig. 22.

23 Symbolic Section Lining

For the graphic indication of various materials of construction symbolic section lining may be used when it is desired to call special attention to them or to identify certain parts, Fig. 25. Symbols used for outside views of certain materials are shown in Fig. 26.

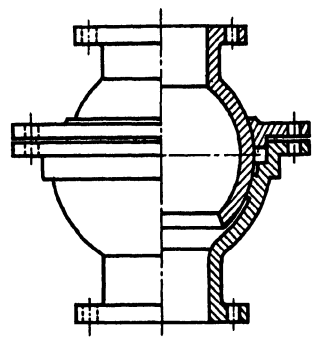


FIG. 23 HIDDEN LINES OMITTED IN HALF SECTIONED ASSEMBLY

24 Sectioning in Detail Drawings

If preferred, the section-lining of any metallic piece may be made by equally spaced full lines in one direction as for cast iron. See Figs. 22, 23, and 27. Since the specification of material must usually be more detailed than by name or symbol alone, this practice is recommended. An opening may be left in the sectioning to provide for a reference letter or dimension.

Even when the symbolic types of section lining are used it will be found frequently desirable to use reference letters to indicate the material.

25 Sectioning in Assembly Drawing

In assemblies, when it is desirable to distinguish between different classes of materials without specifying their exact composition, assemblies may be appropriately

cross-sectioned to indicate specified materials, Fig. 24.

Where a large area of a piece is shown in section, it is not necessary to section line the entire area; section lining around the outline of the sectioned area should be sufficient.

26 Full Sections

When the cutting plane extends entirely through the object a "full section" is obtained as in Fig. 27. When the section is on an axis of symmetry it is not necessary to indicate its location.

27 Half Sections

A symmetrical object may be drawn as a "half-section" showing one-half up to the center line,* in section, and the other half in full view. See Figs. 22, 23, and 24. In assemblies it is customary not to show hidden lines on the unsectioned side.

28 Location of Cutting Plane

The cutting plane on which the section is taken should be indicated by type 7 line, Fig. 18, and lettered at the ends as A-A, Fig. 30. Arrows are used to indicate the direction in which the section is viewed. The letters identifying the section may be placed in a circle as the numerator of a fraction whose denominator is the sheet number on which the section will be found, Figs. 28 and 29. When the section is on the same sheet, the sheet number and circle should be omitted. The letters should be large enough to be easily found, Figs. 28 and 30.

On simple symmetrical objects the cutting plane line, letters, and arrows may be omitted when on an axis of symmetry. It is not necessary that the cutting plane be a single continuous plane; it may be bent or offset if by so doing, the construction can be shown to better advantage as, for example A-A or B-B, Fig. 30 and Fig. 29. The cutting planes line may be shown only at the ends and at changes of direction as in Fig. 28.

29 Revolved Sections

These sections show the shape of the cross-section of a part, such as the arm of a wheel, the cutting plane being rotated in place, Figs. 16 and 31.

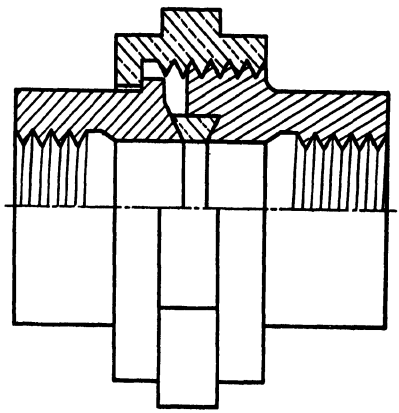


FIG. 24 REPRESENTATION OF MATERIALS IN ASSEMBLY

30 Removed Sections

Removed or "detail sections" should be drawn like revolved sections except that they are placed to one side and often are made to a larger scale than the view on which they are indicated, Section B-B, Fig. 30. Removed sections of symmetrical parts may be placed on an extension of the cutting plane as a center line, Fig. 32.

31 Offset Sections

When the object is not cut by one continuous plane, the section is called an offset section. Its location is shown in the usual way as in Figs. 29 and 30.

32 Broken-Out Sections

Where a sectional view of only a portion of the object is needed, broken-out sections may be used as in Figs. 33 and 36.

* The dividing line between the half-section and full view may be type 1 line, Fig. 18.

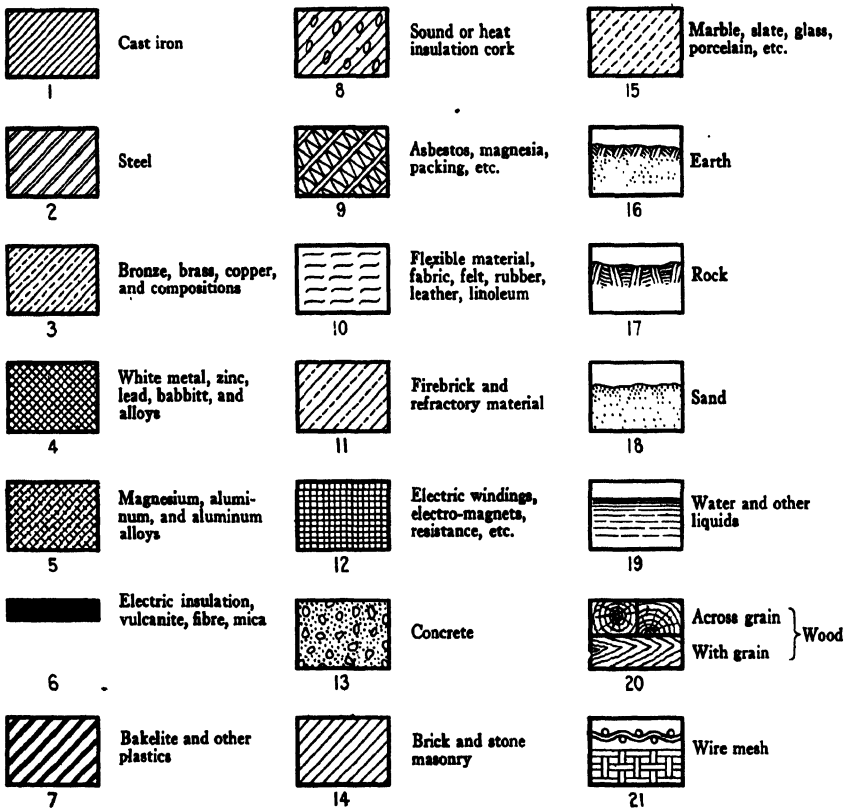


FIG. 25 SYMBOLS FOR SECTION LINING

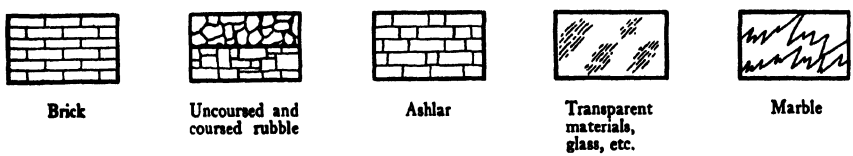


FIG. 26 SYMBOLS FOR OUTSIDE VIEWS

88 *Direction of Section Lining*

Section lining should be made with thin parallel lines at an angle of 45 deg and spaces from 1/32 in. to ⅛ in. or more depending on the size of the drawing of the part. Two adjacent parts should be sectioned by lines at right angles to each other.

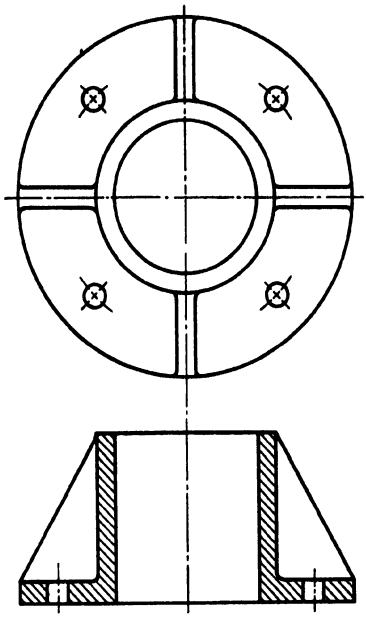


FIG. 27 SYMMETRICAL FULL SECTION

A third, adjacent to both, should be sectioned at 30 deg or 60 deg as shown in Figs. 24 and 34. If the shape or position of the part would bring 45 deg sectioning parallel or nearly parallel to one of the sides, another angle should be chosen. In all views showing sections of a part the sectioning of that part should be the same in direction and spacing.

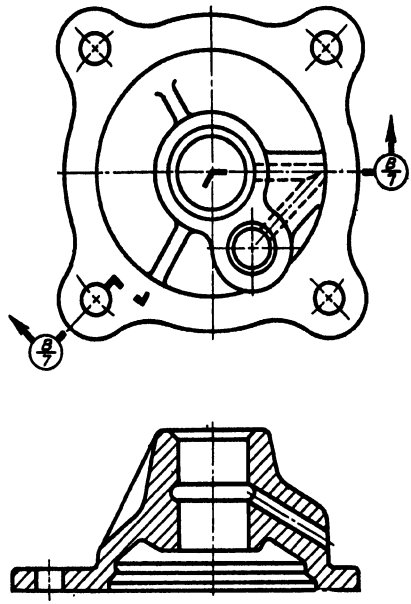


FIG. 28 INDICATION OF CUTTING PLANE IN SECTIONED VIEWS

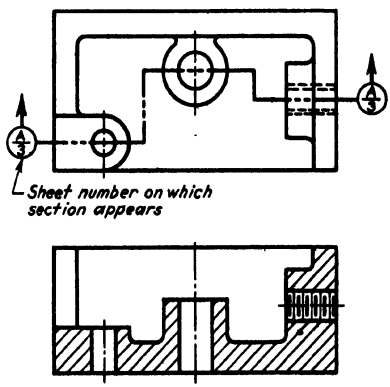


FIG. 29 CROSS-HATCHING AN OFFSET SECTION

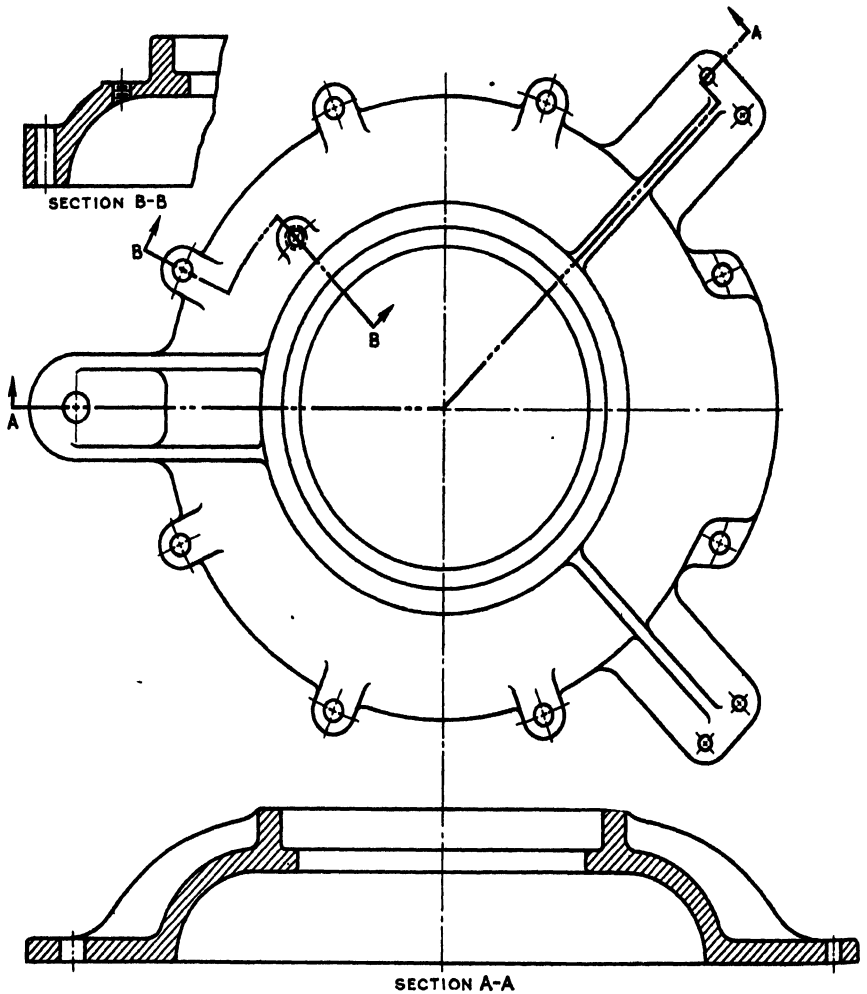


FIG. 30 INDICATION OF CUTTING PLANES

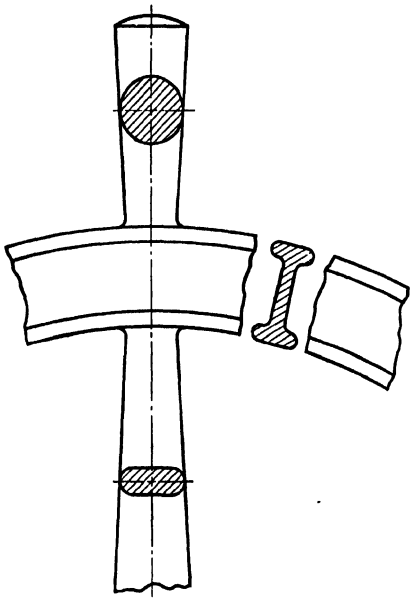


FIG. 31 REVOLVED SECTIONS

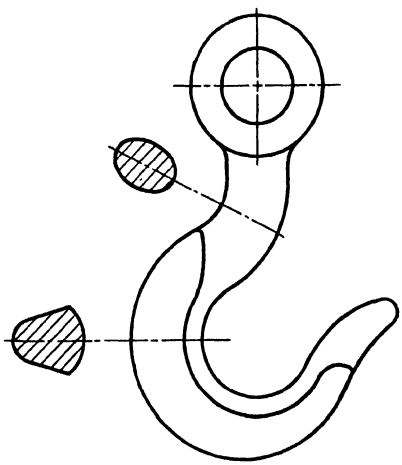


FIG. 32 REMOVED SECTIONS

84 Thin Sections

Sections which are too thin for line sectioning may be shown solid, such as structural shapes, sheet metal, packing, gaskets, etc. Where two or more adjacent parts are shown solid a space should be left between them, Fig. 35.

85 Sections Through Thin Webs, Shafts, Bolts, Pins, Keys, etc.

There is one important violation of the conventional theory which is made in the interest of clearness. When the section or cutting plane passes through a rib, web or similar parallel element, section lines should be omitted from those parts, Figs. 27 and 30.

Shafts, bolts, nuts, rods, rivets, keys, pins and similar solid parts whose axes lie in the cutting plane should not be sectioned, Fig. 37.

86 Treatment of Foreshortened Projections

When the true projection of a piece would result in confusing foreshortening, parts such as ribs or arms, should be rotated until parallel to the plane of the section or projection as shown in Figs. 38 and 40.

Drilled flanges in elevation or section should show the holes at their true distance from the center rather than the true projection, Figs. 27 and 36.

87 Representing Cylindrical Intersections

The intersection of rectangular and circular parts with other circular parts, occurs frequently. Unless very large, these intersections are conventionalized as shown in Fig. 39. The curve of intersection may be drawn as a circular arc.

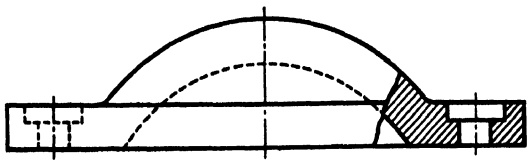


FIG. 33 BROKEN-OUT SECTION

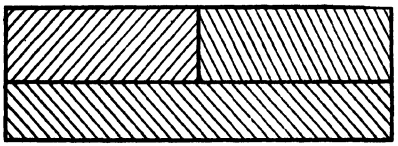


FIG. 34 ARRANGEMENT OF SECTION LINING FOR THREE ADJACENT PARTS

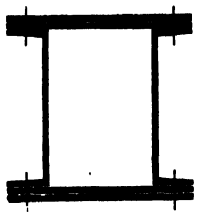


FIG. 35 SECTIONAL VIEW OF THIN PLATES

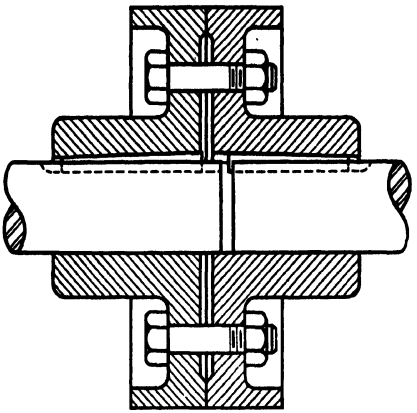


FIG. 37 SHAFTS, KEYS, BOLTS AND NUTS IN SECTIONAL VIEW

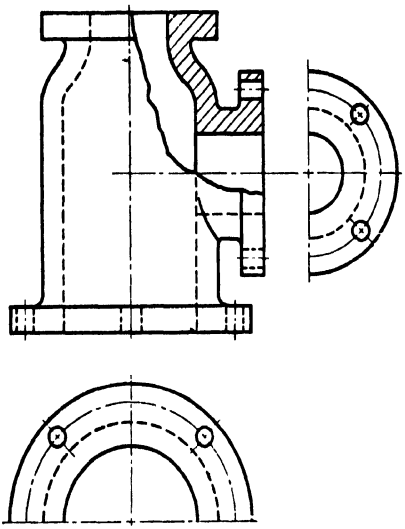


FIG. 36 HOLES REVOLVED TO SHOW TRUE DISTANCE FROM CENTER

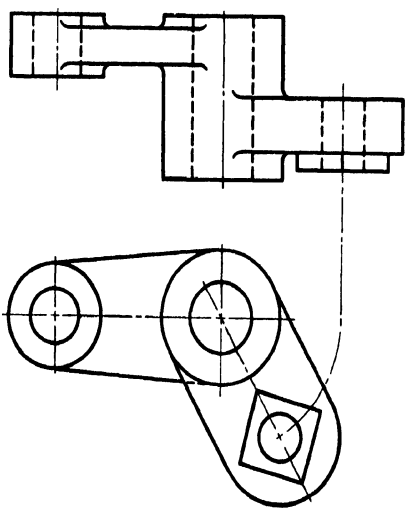
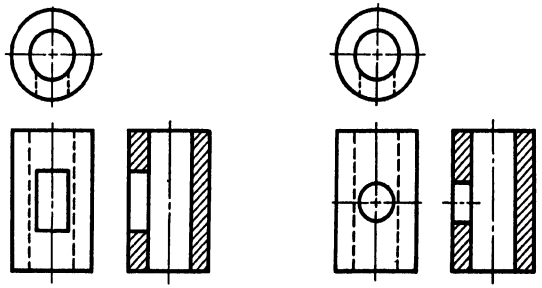
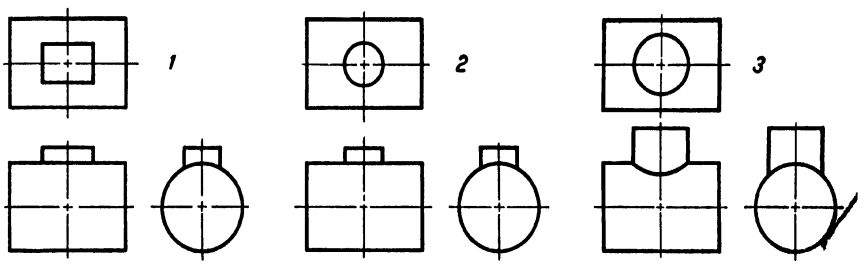


FIG. 38 REVOLUTION OF PARTS TO SHOW TRUE RELATIONSHIPS



(a) CYLINDRICAL INTERSECTION IN SECTION



(b) EXTERNAL INTERSECTIONS

FIG. 39 REPRESENTING CYLINDRICAL INTERSECTIONS

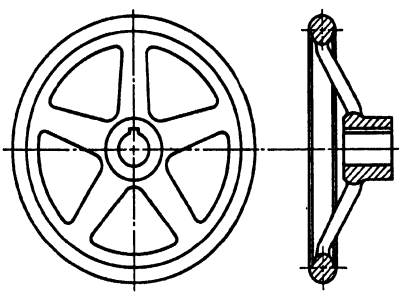


FIG. 40 SECTION THROUGH SPOKES OF WHEELS

SECTION 4—SCREW THREAD REPRESENTATION

38 *Regular Thread Symbols*

Fig. 41 shows threads in section and elevation, with end view.

Fig. 41(a) shows the regular method of representing screw threads recommended for general use on assembly and detail drawings. Except in sections of external threads and hidden internal threads, the threads are indicated by alternate long and short lines at right angles to the axis representing the crests and roots of the thread, the short lines thicker than the long lines, or of equal width if preferred. They need only to approximate the actual pitch, being spaced by eye to look well.

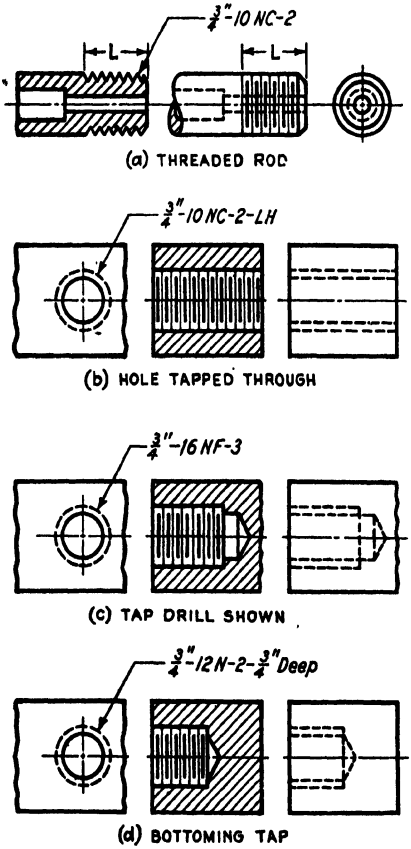


FIG. 41 REGULAR THREAD SYMBOL

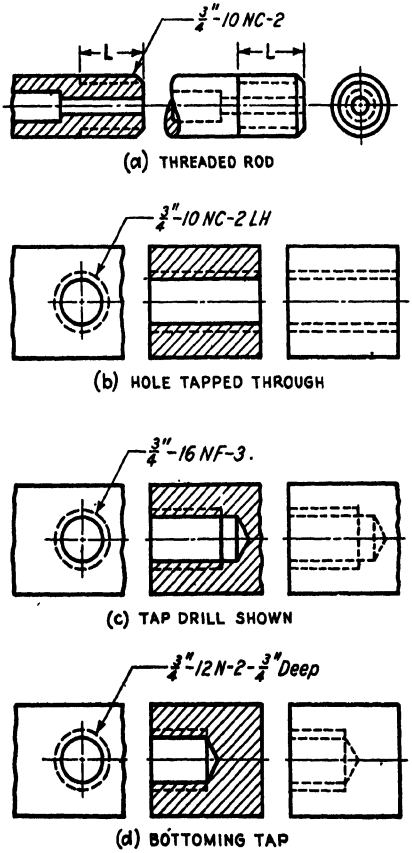


FIG. 42 SIMPLIFIED THREAD SYMBOL

Fig. 41(b) shows internal threads tapped through. Fig. 41(c) shows internal threads where the tap drill point does not go through. The drill point should be drawn at 60 deg with the center line.

The method shown in Fig. 41(d) should be used to indicate a bottoming tap, when depth of thread is the same as depth of at the approximate depth of the thread.

The size and length of thread and depth of tap should be given on the drawing. Threads are always considered to be "right hand" unless specified as "left hand" or "LH." The letters LH should be added after the class of fit number as shown in Figs. 41(b) or 42(b). Bolt ends should

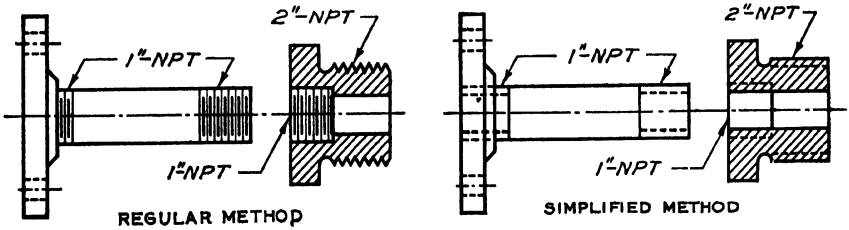


FIG. 43 PIPE THREADS

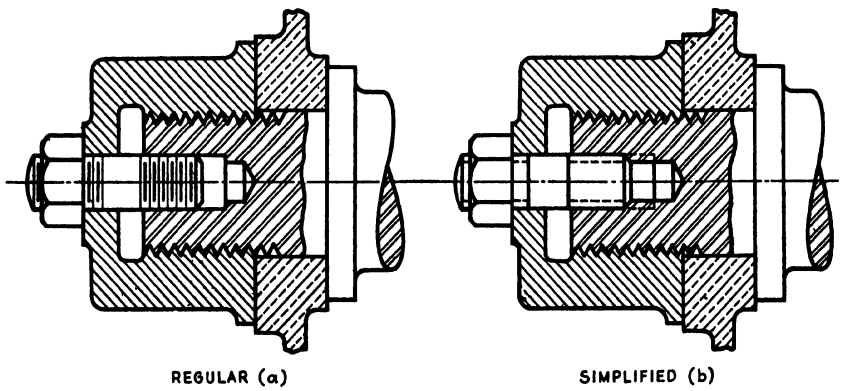


FIG. 44 SCREW THREADS IN ASSEMBLY

drill, or when it is not necessary to specify both depth of drill and depth of thread. Fig. 44 shows an assembly of threaded parts. The simplified method illustrated should be used with discretion to avoid possible mistakes.

39 Simplified Thread Symbols

Fig. 42 shows the simplified symbol method of screw thread representation. This method may be adopted where it is desirable to simplify drafting work. The threaded portion is indicated by lines made of short dashes drawn parallel to the axis be shown as flat and chamfered to the thread depth at 45 deg with the flat surface.

Dimensions of screw threads for bolts, nuts, screws and threaded parts may be found in the American Standard for Screw Threads. Identification symbols and methods of designation for drawings may be found in Par. 14 of this bulletin, (ASA B1.1—1935).

40 Pipe Thread Symbols

Pipe threads should be represented in the same manner as bolt threads, Fig. 43.

It is not necessary to indicate taper on standard pipe threads but this may be done if desired. Therefore, when straight pipe threads are specified such threads must be indicated by the symbol NPS. Identification symbols and dimensions of national standard pipe threads, including diameters, form, taper, and length of thread, may be found in the American Standard for Pipe Threads (ASA B2.1—1945).

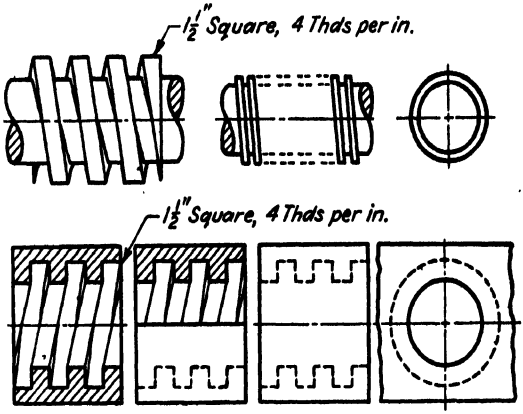


FIG. 46 MODIFIED SQUARE THREAD

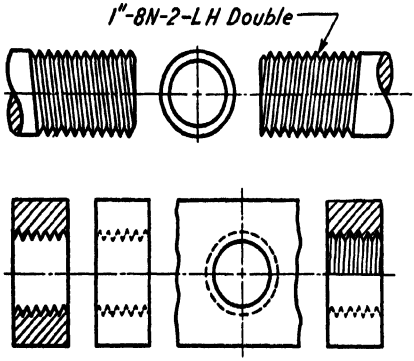


FIG. 45 SEMI-CONVENTIONAL THREAD

41 *Semi-Conventional Threads or Thread Pictures*

When it is desirable to use "thread pictures" it should be done as in Fig. 45. The helices are conventionalized into slanting straight lines and the contour shown as a sharp V at 60 deg although the standard threads are truncated at the top and filled in at the root. Details of standard threads will be found in the American Standard for Screw Threads (ASA B1.1—1935).

42 *Square Thread Symbols*

Square threads should be shown as in Fig. 46 with a note specifying the type of thread and the pitch. For details of modified square and acme threads see American Standard Acme and Other Translating Threads (ASA B1.3—1941).

43 *Acme Thread Symbols*

Acme threads should be represented as in Fig. 47 with a note specifying the type of thread and the pitch (ASA B1.3—1941).

44 *Bolt Heads and Nuts*

Exact dimensions of American Standard unfinished, semi-finished, and finished bolt heads and nuts, regular series and heavy series, will be found in the American Standard for Wrench Head Bolts and Nuts and Wrench Openings (ASA B18.2—1941).

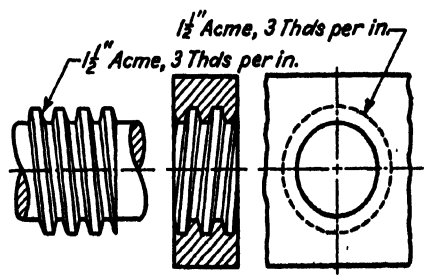


FIG. 47 ACME THREAD

SECTION 5—DIMENSIONING—GENERAL RULES

45 *General Principles of Dimensioning*

(a) Economical production and the proper functioning of a part in service are directly dependent upon the method of dimensioning used and the manner in which dimensions are specified on the drawings.

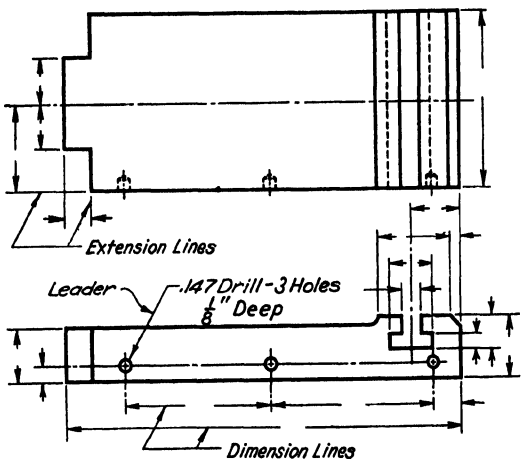


FIG. 48 LEADERS, DIMENSION AND EXTENSION LINES IN MACHINE DRAWING

(b) The dimensions should be so arranged that it never will be necessary to calculate, scale, or assume any dimensions in order to fabricate the specified part.

(c) The dimensions should include those sizes and distances which are worked to in actual shop or constructional operation and should be so placed as to minimize the cumulative error.

(d) It is essential that dimensions should be given between those points or surfaces which have a specified relation to each other or which control the location of other component or mating parts.

(e) Dimensions should not be duplicated and none should be given except those required to produce or inspect the part.

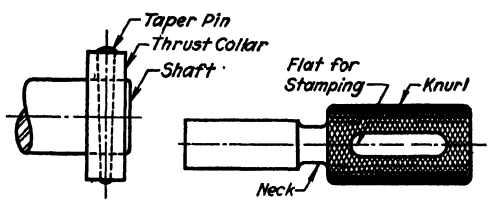


FIG. 49 APPROVED FORM OF LEADERS

46 *Technique of Dimensioning*

Dimension lines should be thin full lines (broken where dimension is inserted) so as to contrast with the thicker outline of the drawing, Fig. 48. In structural drawings the dimension line is not broken and the figures are placed above the line, Figs. 50, 51, and 52.

Dimension lines should be terminated by carefully made arrowheads whose lengths are approximately three times the spread. The distance from tip to tip of the arrowheads indicates the extent of the dimension as shown in Fig. 48.

Extension lines indicate the distance measured when the dimension is placed outside the view. They are made as thin full lines, should be extended a short distance beyond the arrowhead, and should not touch the view.

In structural drawing the lines along which rivet holes are punched

are called gage lines. They are drawn as thin solid lines as in Figs. 51 and 52.

Leaders should be thin straight lines terminated by arrowheads. They should not be curved or made freehand, Fig. 49. The leader may terminate in a short horizontal bar at the mid-height of the lettering of the first or last line of the note.

The symbol (") may be used to indicate inches and common and decimal fractions of an inch.

The symbol (') may be used to indicate feet as in Figs. 50, 51, and 52.

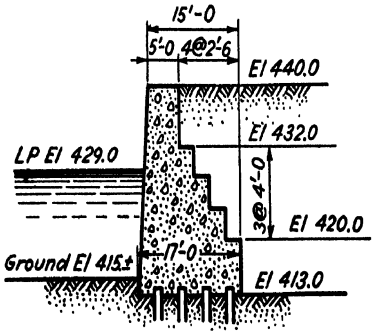


FIG. 50 DIMENSIONING IN CIVIL ENGINEERING PROJECTS

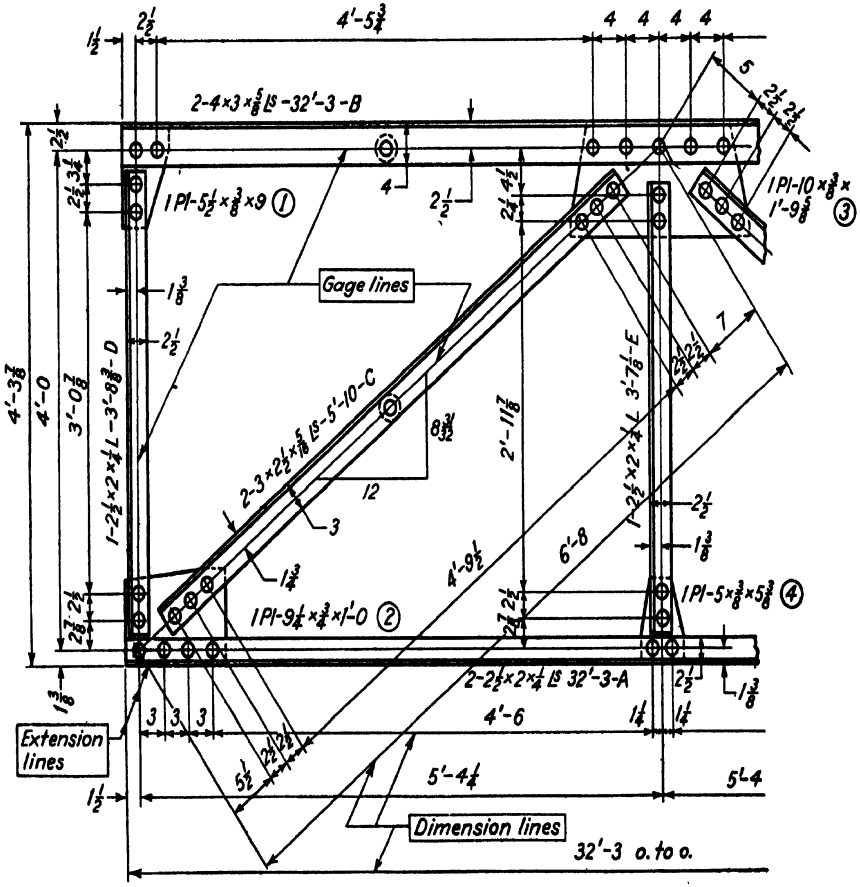


FIG. 51 DIMENSIONING ASSEMBLED STRUCTURAL WORK

47 *Use of Feet and Inch Symbols*

In machine drawing when all dimensions are given in inches the symbol is preferably omitted, unless there is a possibility of misunderstanding, thus 1 valve should be 1″ valve or 1 in. valve; 1 bore should be 1″ bore.

In structural drawing all length dimensions should be expressed in feet and inches. Plate widths are given in inches. Dimensions should be hyphenated thus, 4′-3; 4′-0½; 4′-0 as shown in Fig. 51. When the dimension contains only inches the inch mark may be omitted. In structural drawing the inch symbol is omitted even though the dimension is in feet and inches, Figs. 51 and 52.

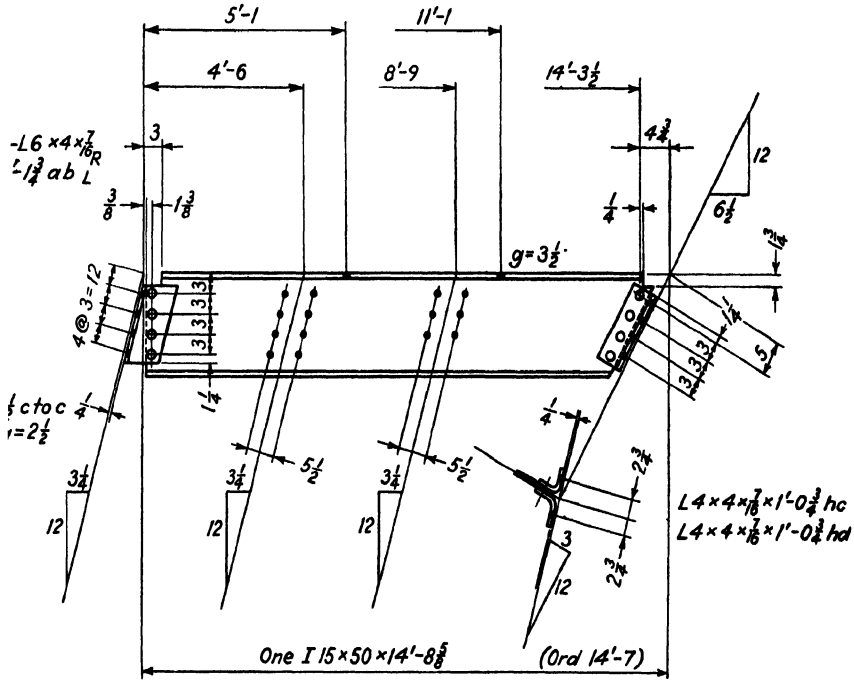


FIG. 52 DIMENSIONING STRUCTURAL STEEL BEAMS

In some industries all dimensions are specified in inches. In others dimensions up to and including 72 in. should be preferably expressed in inches and those greater than that length, in feet and inches.

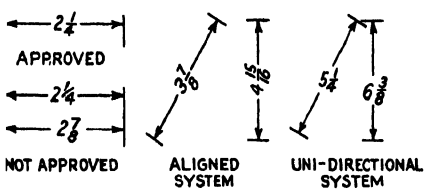


FIG. 53 POSITION OF FRACTION BAR

48 *Placing Dimensions*

Two systems of dimensioning are approved. The older, called the aligned system, and the new, called the unidirectional system, will be found convenient under conditions described in the following:

(a) ALIGNED SYSTEM. All dimension lines and their corresponding numbers are placed so that they may be read from the bottom or right-hand edges of

the drawing. Figs. 53 and 55(a). All dimensions are placed so as to read in the direction of the dimension lines as shown in Fig. 55(a). Dimension lines should not run in directions included in the shaded area as shown in Fig. 54, forty-five deg from the vertical, unless unavoidable.

In the aligned system fractions are written with the division bar in line with the dimension line, Fig. 53. The inclined bar or the omission of the bar is not approved.

(b) UNIDIRECTIONAL SYSTEM. All dimensions are made to read from the bottom of the sheet as shown in Figs. 53 and 55(b). Fractions are written with the division bar horizontal or parallel to the bottom of the sheet. Diameters of circles may be placed in any position, except along center lines.

is making its use more widespread, and therefore, acceptable as a method.

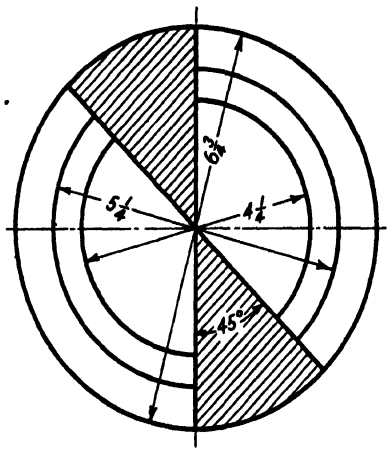


FIG. 54 AREA TO BE AVOIDED

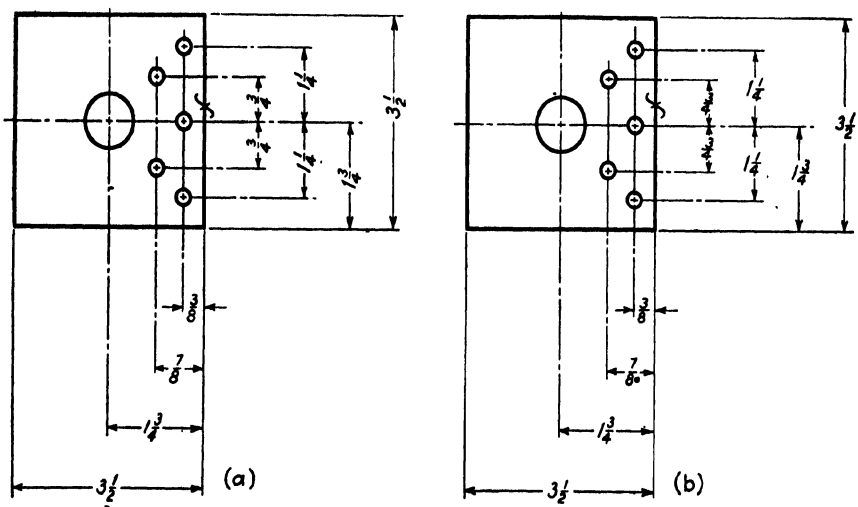


FIG. 55 ALIGNED (a) AND UNIDIRECTIONAL (b) DIMENSIONING FROM CENTER LINE AND FINISHED SURFACE

The unidirectional system of dimensioning originated in automobile and aircraft drafting rooms where long drawings made it advantageous. From the standpoint of both making and using a drawing, this system of dimensioning has advantages which

(c) GENERAL RULES. The following rules apply to both systems.

(1) The dimension line must not pass through a dimension figure, nor should a dimension figure be placed on the outline of the object.

(2) Dimensions should not be placed upon a view unless the drawing becomes clearer by so doing. In general, dimensions should be placed between views as in Figs. 57 and 61.

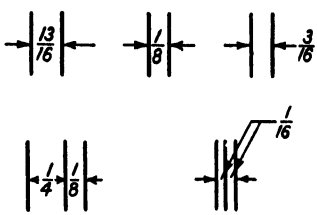


FIG. 56 DIMENSIONING NARROW SPACES

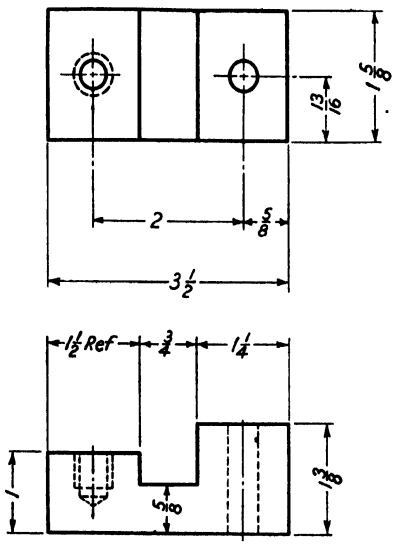


FIG. 57 PLACING OF DIMENSIONS

(3) A center line should never be used as a dimension line. A line of the piece or part illustrated, or an extension of such a line, should never be used as a dimension line.

(4) When there are several parallel dimension lines the figures should be staggered to avoid confusion as shown in Fig. 59.

(5) In general, dimension lines should be one-half inch away from outlines and spaced uniformly at least three-eighths of an inch from each other.

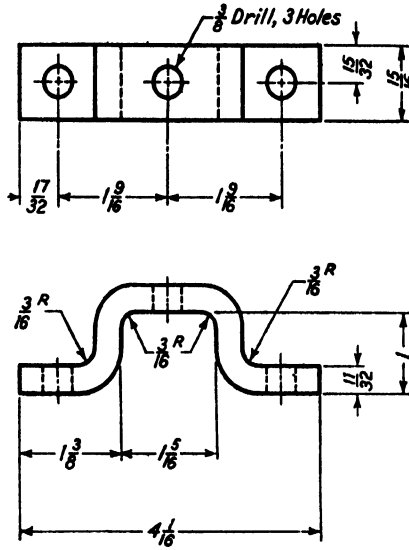


FIG. 58 DIMENSIONS FOR DIFFERENT OPERATIONS SHOULD BE SEPARATED

(6) Dimensions should be given from a base line, a center line, an important hole or a finished surface that can be readily established (based on design requirements and the relationship of other parts). See Fig. 55. Dimensions for different operations on a piece, as for example, drilling and bending should be kept separate as shown in Fig. 58 if permissible by its design.

(7) For dimensioning in limited space the arrowheads should be reversed and the methods shown in Fig. 56 may be used.

(8) In dimensioning angles an arc should be drawn and the dimension so placed as to read from the bottom of the sheet, Fig. 60. An exception is sometimes made in the dimensioning of large arcs when the dimensions are placed along the arc.

(9) Over-all dimensions should be placed outside the intermediate dimensions, Figs. 55, 58 and 77. When an over-all dimension is used, one intermediate distance * should not be dimensioned, unless for reference and so noted as shown in Figs. 57 and 61.

(10) In an object with circular ends an over-all dimension generally need not be given, Figs. 61 and 74.

(11) When the number of dimensions is large and when limited space requires that successive dimensions be given from a common reference point or surface, they may be placed in a single line as shown in Fig. 67. This practice is followed also in structural work as shown in Fig. 52.

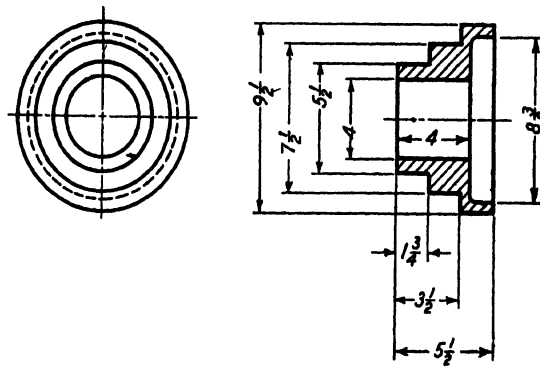


FIG. 59 DIMENSIONING CYLINDERS

49 *Fractional and Decimal Dimensioning*

GENERAL RULES. Dimensions in structural work should be expressed in units and common fractions and in machine drawing, dimensions of parts that can be produced with sufficient accuracy when measured with an ordinary scale, should be dimensioned in the same manner.

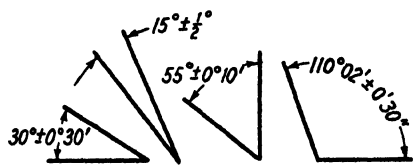


FIG. 60 DIMENSIONING ANGLES WITH LIMIT DIMENSIONS

For parts requiring greater accuracy a complete decimal system of dimensioning will be found advantageous.

Decimal dimensions should be used between finished surfaces and points which it is essential to hold in a specific relation to each other. The fundamental basis of the complete decimal system is the use of a two-place decimal, i.e., a decimal consisting of two figures after the decimal point. In all dimensions where a fraction would ordinarily be used, the two-place decimal can be applied. The figures after the decimal point, where applicable, should be in fiftieths (e.g., .02, .04, .08, .84) so that when halved (e.g., diameters to radii) two-place decimals will result. Exceptions, of course, will have to be made, but they should be kept to a minimum.

When a decimal value obtained by converting a common fraction to decimals is to be rounded off to a lesser number of places than the total number available, the procedure should be as indicated in ASA Z25.1—1940.

50 *Limit Dimensioning*

Accurate dimensions which are to be checked with a limit gage or micrometer should be expressed in decimals.

Limits representing the maximum and minimum dimensions allowed shall be specified for all dimensions where the tolerance is other than .01. This (.01) tolerance, or any other predominant tolerance, should be indicated on the drawing

* Does not apply to structural dimensioning.

in note form. The method of writing these limits on the drawing should be as shown in Figs. 61 to 64.

For external dimensions the maximum limit is placed above the line and for internal dimensions the minimum limit is placed above the line. This method of writing should be used for smaller parts and where gages are extensively employed.

A second method is to give the calculated or basic size to the required number of decimal places, followed by the tolerance plus and minus with the plus above the minus, as $8.625^{+.002}_{-.002}$

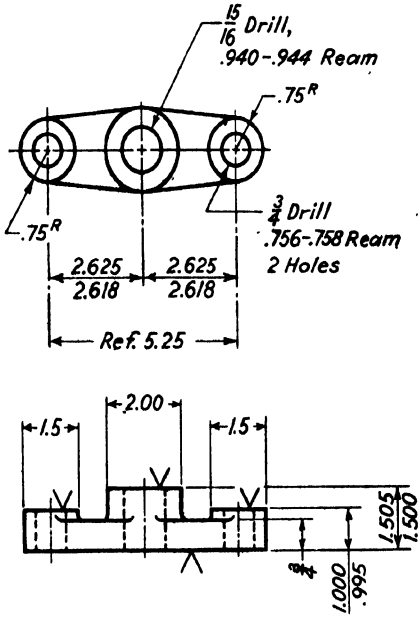


FIG. 61 DIMENSIONING PART WITH CIRCULAR ENDS

A third method, sometimes used, gives the preferred dimension with an unilateral tolerance, either plus or minus (not both) as shown in Fig. 83.

51 *Dimensioning Fits*

The dimensions and tolerances given in the detail drawings of the two parts of a cylindrical fit will determine the clearance (or interference) between these parts.

52 *Tolerance of Concentricity*

Mating pairs of two (or more) closely fitting machined cylindrical surfaces must be concentric in order to permit assembly of parts; thus a method of giving the permissible deviation from concentricity is

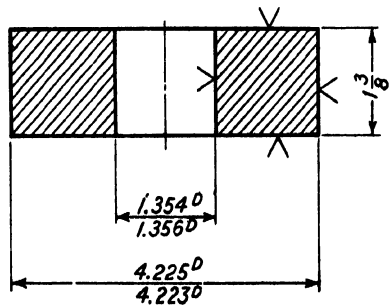
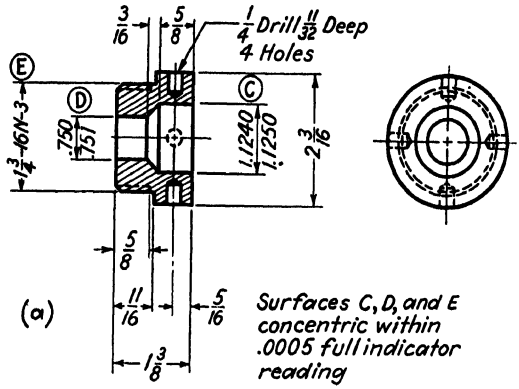


FIG. 62 LIMIT DIMENSIONS FOR CIRCULAR HOLES AND SHAFTS

sometimes necessary. As the center lines of adjoined cylinders coincide on the drawing, the tolerance for concentricity cannot be shown in dimensional form, and must therefore be given by note. One method is to mark the dimensions with reference letter and give the tolerance in a foot-note as in Fig. 63(a). Another method applies the note directly to the surfaces as in Fig. 63(b).

53 *Dimensioning a Half Section*

By a proper use of hidden lines on the external portion of a half section view and the careful observance of the usual rules, dimensioning a half-sectioned drawing becomes a fairly simple matter as shown in Fig. 64. Generous use of notes and careful placement of the dimension lines, leaders and figures will in most cases suffice to make the dimensioning clear.



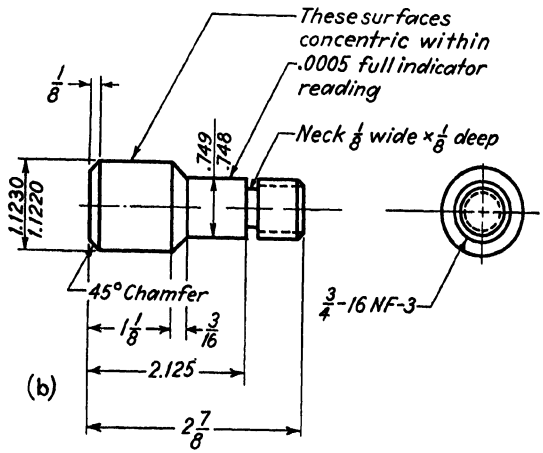


FIG. 63 LIMIT DIMENSIONS FOR CONCENTRICITY

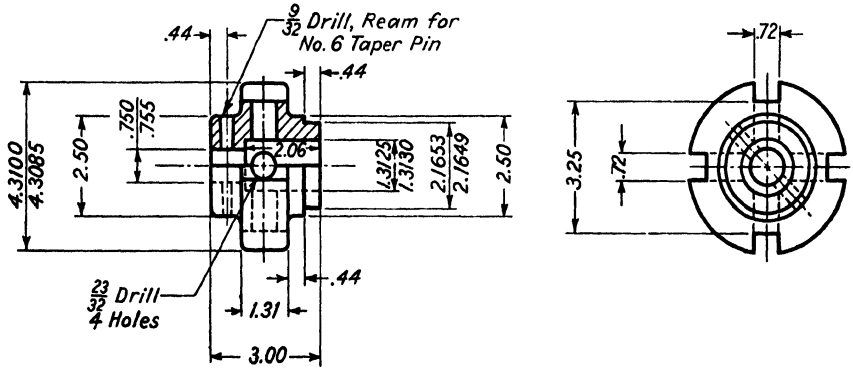


FIG. 64 DIMENSIONING HALF SECTIONED VIEWS

SECTION 6—DIMENSIONING STANDARD DETAILS

54 Circles

A dimension indicating the diameter of a circle should be followed by the letter "D" except when it is obvious from the drawing that the dimension is a diameter, Fig. 62. The dimension of a radius should always be followed by the letter "R," and the radial dimension line should have only one arrowhead, Fig. 65.

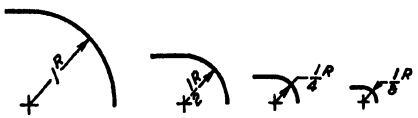


FIG. 65 DIMENSIONING RADII

A series of concentric circular parts should be dimensioned from the side or sectioned view as in Fig. 59, rather than from the circular view. A large number of diametric dimensions in the circular view is to be avoided.

55 Curves

A curved line may be dimensioned either by radii whose centers are properly located or by offsets, Figs. 66 and 67.

56 Angles

When angular dimensions are necessary a horizontal (or vertical) center line should be used as a base line, but not both, and points located from it, Fig. 68. Angular dimensions should be written as 50°45′; the dash between degrees and minutes should not be used.

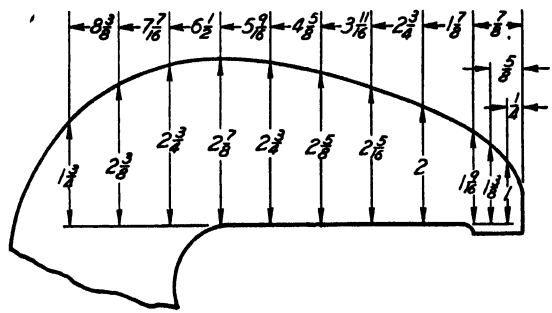


FIG. 67 PROGRESSIVE DIMENSIONING

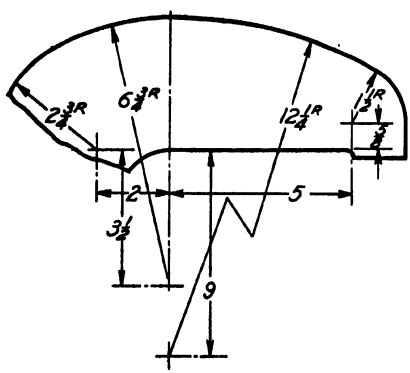


FIG. 66 DIMENSIONING IRREGULAR CURVE WITH RADII

When it is necessary to give the accuracy required in an angular dimension, the tolerance is generally bilateral. When the tolerance is given in degrees it is written, for example, ±½°, when given in minutes, ±0°10′; and when given in seconds, ±0′30″, as shown in Fig. 60.

In structural drawings angular measurements are shown by giving the ratio of run to rise with the larger side 12 in., Figs.

51 and 52. For steep angles as on retaining walls, the ratio has one factor equal to 1 as shown in Fig. 69.

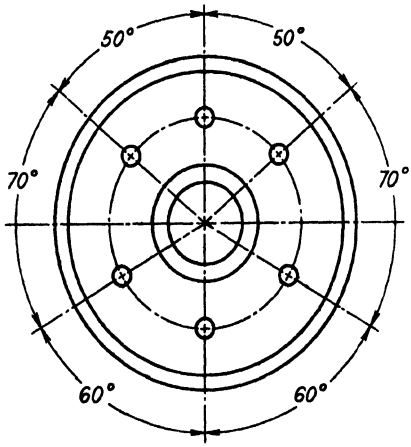


FIG. 68 DIMENSIONING ANGLES

57 Holes—Drilled, Reamed, Punched, Swaged, Cored or Bored

Holes which are to be made by any of these operations should have the diameter given, preferably on a leader, followed by the word indicating the operation, and the number of holes to be made, Fig. 70.

Notes for holes made with drills one-half inch in diameter and smaller should be written as decimal fractions, Figs. 72a, 73, and 75. The drills selected in this range should be the standard diameters given in the American Standard for Straight Shank Twist Drills (ASA B5.12—1940). The indication of the diameters of drills larger than one-half inch should be expressed in units and decimals. If common fractions are employed, the decimal equivalent should be given.

Holes which are to be bored after coring or casting should have finish marks and finished dimensions specified, Fig. 71.

58 For counterbored holes the diameters and depths should be given as shown in Fig. 72.

59 For countersunk and counterdrilled holes the angles and diameters should be given by dimensions or notes as shown in Fig. 73. The value of the countersink angle should be given as the included angle.

60 Tapped Holes

The size and depth of tapped holes should be given by note as shown in Fig. 75 with reference to the American Standard Coarse (NC), American Standard Fine (NF) or American Standard Special (N) Screw Thread Series and the Class of Fit desired. If the American Standard form is desired with a non-standard pitch the symbol is (NS).

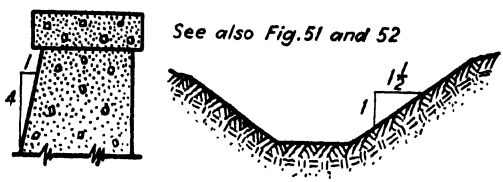


FIG. 69 INDICATING SLOPES ON CIVIL ENGINEERING PROJECTS

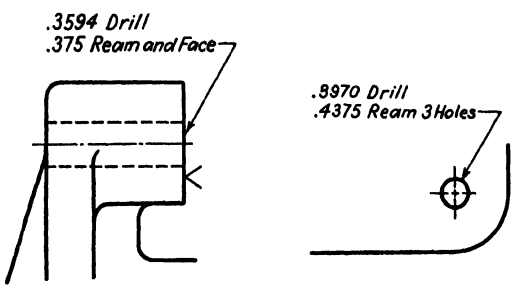


FIG. 70 DIMENSIONING DRILLED AND REAMED HOLES

61 Location of Holes

Figs. 74, 76, and 77 illustrate the coordinate method of dimensioning holes. The

actual numerical values shown for location of holes in these figures have been included for illustrative purposes only. The alignment of two or more parts requires similar dimensioning from a common datum, line, or plane.

The ordinate method of dimensioning, as illustrated in the lower right-hand drawing, Fig. 76, is preferred in precision work to the less accurate angular method as illustrated by the three other drawings in Fig. 76.

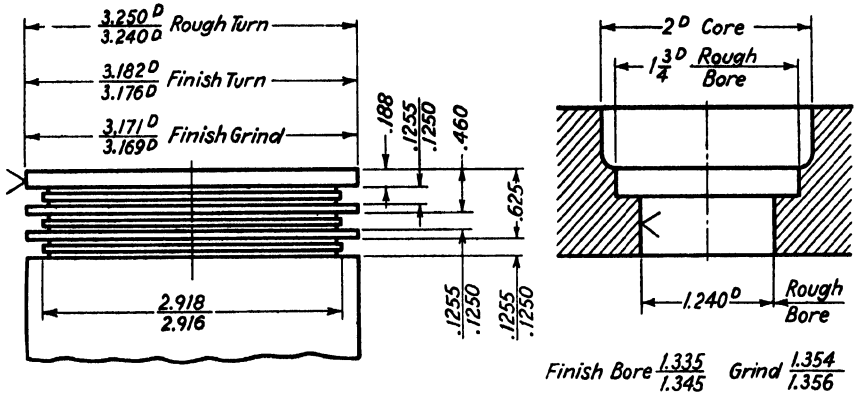


FIG. 71 DIMENSIONING MACHINED HOLES AND CYLINDERS

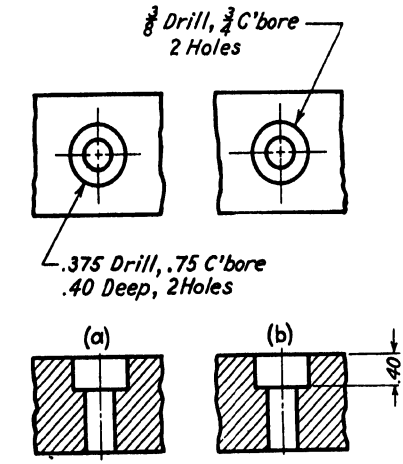


FIG. 72 DIMENSIONING COUNTERBORED HOLES

The use of angular measurements, such as degrees, minutes and seconds to locate holes and parts requiring accurate placement should be avoided as much as possible, Fig. 76.

68 Tapers

The difference in diameter or width in one foot of length is known as the "taper per foot." At least three methods of dimensioning tapers are in general use.

(a) STANDARD TAPERS. Give one diameter or width, the length, and insert note on drawing designating the taper by number taken from ASA B5.10—1943, as shown in Fig. 78.

(b) SPECIAL TAPERS. In dimensioning a taper when the slope is specified, the length and only one diameter should be given or the diameters at both ends of the taper should be given and the length omitted, Fig. 78.

(c) Fig. 79 illustrates the proper method of dimensioning tapered surfaces where accuracy is required so as to obtain a close fit between internal and external tapers. The tolerance on the diameter at the gage line is ±0 and the tolerance on the angle is as shown in the figure. The

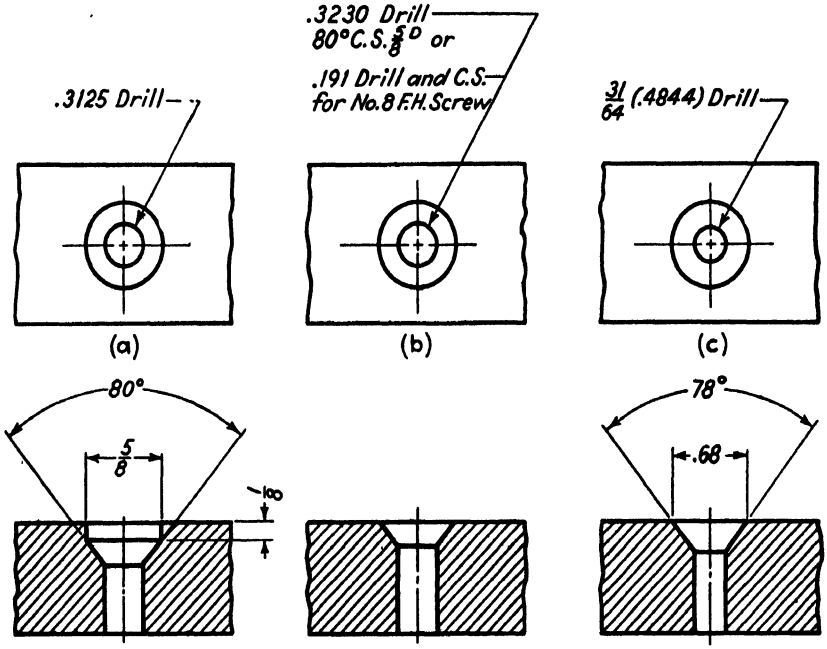


FIG. 73 DIMENSIONING COUNTERDRILLED AND COUNTERSUNK HOLES

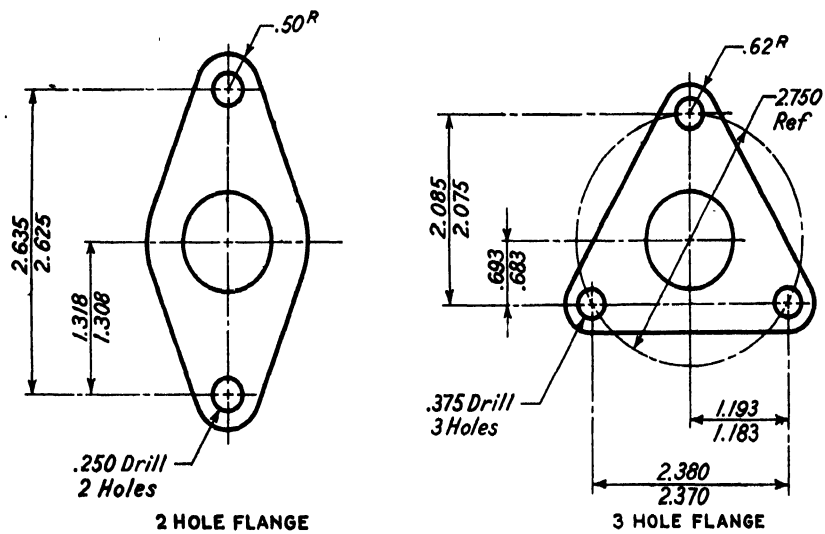


FIG. 74 LOCATION DIMENSIONING OF HOLES

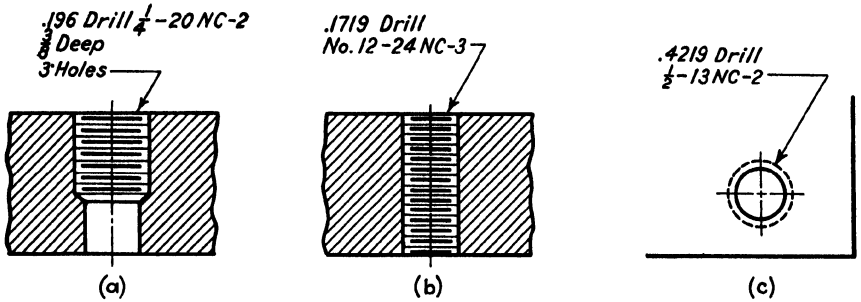


FIG. 75 DIMENSIONING TAPPED HOLES

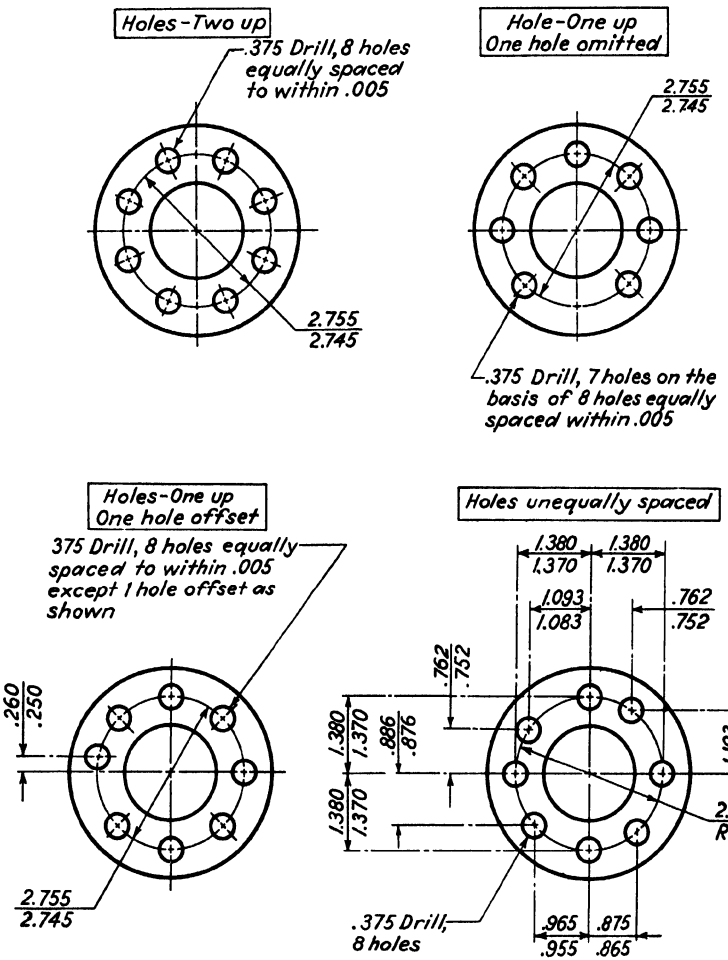


FIG. 76 LOCATION DIMENSIONING OF HOLES

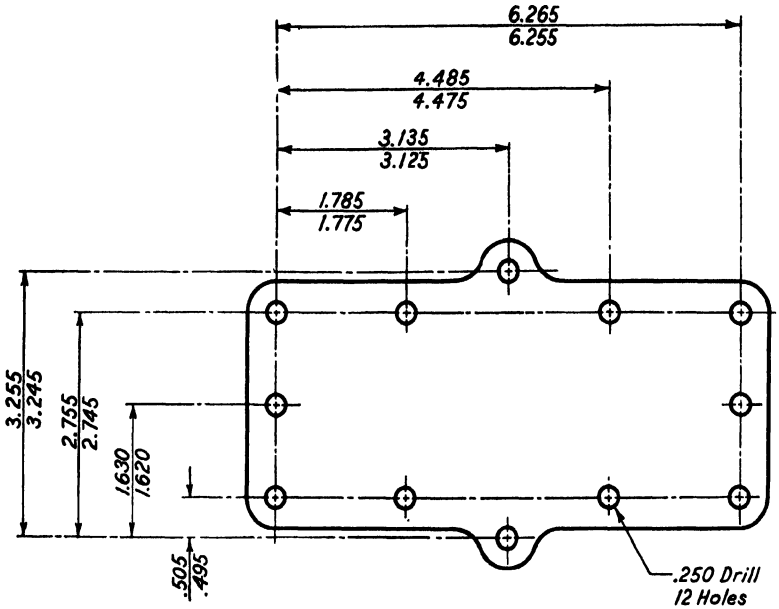


FIG. 77 LOCATION DIMENSIONING OF HOLES

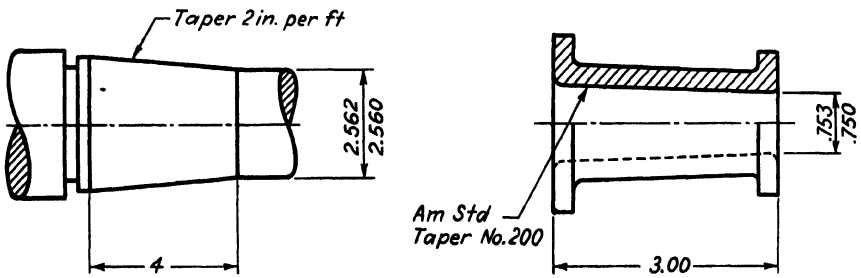


FIG. 78 DIMENSIONING TAPERS

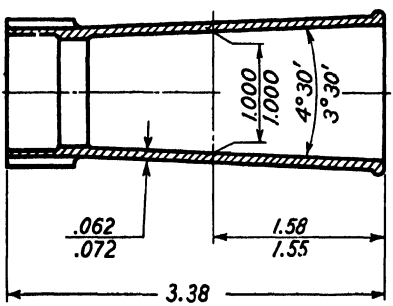


FIG. 79 DIMENSIONING TAPERS FOR A CLOSE FIT

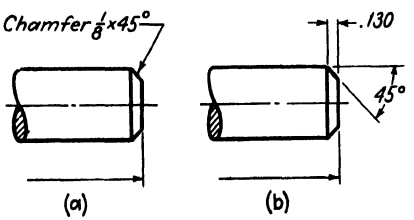


FIG. 80 DIMENSIONING CHAMFERS

tolerance for location of the gage line may be varied to suit the conditions.

63 *Chamfers*

Ordinary chamfers may be dimensioned as shown in Fig. 80a. More accurate chamfers should be dimensioned as in Fig. 80(b).

64 *Knurls*

When knurls are used to roughen a surface for the purpose of giving a better grip it is only necessary to specify the pitch of the knurl and show the kind of knurl and the dimensions of the areas as shown in Fig. 81.

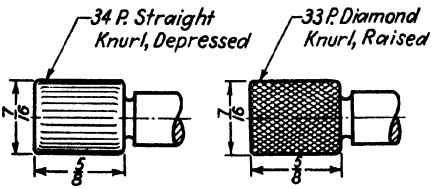


FIG. 81 DIMENSIONING KNURL USED TO PROVIDE GRIP

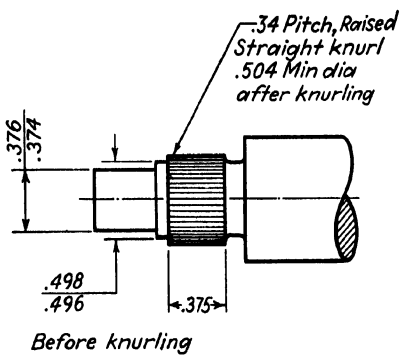


FIG. 82 DIMENSIONING KNURL FOR PRESS FIT

When knurls are used to make a press fit between two parts the original finished surface should be dimensioned with limits and then the minimum acceptable diameter of the knurl should be given in a note together with the pitch and type of knurl, straight or diamond, depressed or raised, as shown in Fig. 82.

65 *Dovetail Tongues and Slots*

For large quantity production, snug fitting dovetailed parts should be dimensioned as shown in Fig. 83 which provides complete information for the tool and gage maker. The so-called "plug method" is also in common use.

The important contact surfaces should be controlled by dimensions delineated to the same intersection points of mating angular flanks on each component part.

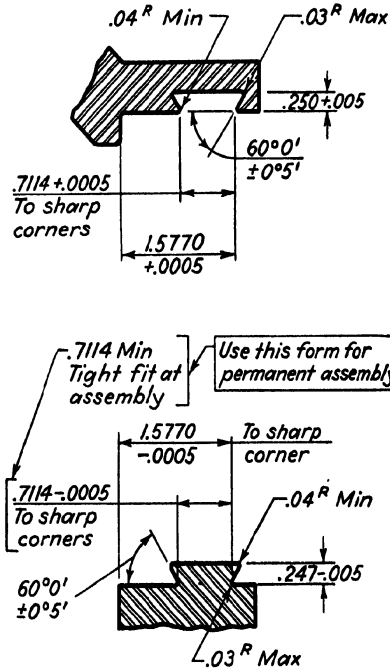


FIG. 83 DIMENSIONING A DOVETAIL TONGUE AND SLOT FOR SNUG FIT

Tolerances should be given on all dimensions including angular dimensions. The radii at corners should be governed by the best requirements of good tool practice and flats may be substituted if desired.

66 *Finish Marks*

A surface to be machined or "finished" from unfinished material such as a casting

or a forging should be marked with a 60 deg "V," the bottom of the "V" touching the line representing the surface to be machined or finished. Fig. 94.

The old symbol "f" is still in use and is also acceptable for this purpose, Fig. 95. The notes, "Finish All Over," or "F.A.O.," are acceptable for parts which are to have an ordinary machine finish on the entire surface.

67 *Symbols for Designating Surface Quality* *

An American Standard (ASA B46) has now been developed for surface quality which (1) defines roughness, waviness, and lay, (2) establishes the specification rating and measurement of roughness in terms of height and width of irregularities and the direction of the dominant surface pattern, (3) establishes the specification of waviness in terms of height and width, with the width line in all cases greater than the width of roughness, (4) gives recommended values for roughness and waviness height, and (5) describes the meaning and use of the symbols for the designation of surface quality.

Among the many characteristics of surface quality which are intimately related to the functioning of a surface is the surface finish. Accordingly, the proper specification of the finish of such surfaces as bearings, seals, etc., is extremely important. In other cases where the finish of a surface is unimportant it may not be necessary to specify it.

A surface whose finish is to be specified should be marked with the finish mark having the general form of a check mark ($\surd$) so that the point of the symbol is

(a) On the line indicating the surface,
(b) On a leader pointing to the surface.

Where it is desired to specify only the surface roughness height, and the width of roughness or direction of tool marks is not important, the simplest form of the symbol should be used. See Fig. 84(a). This height may be either maximum peak to valley height, average peak to valley

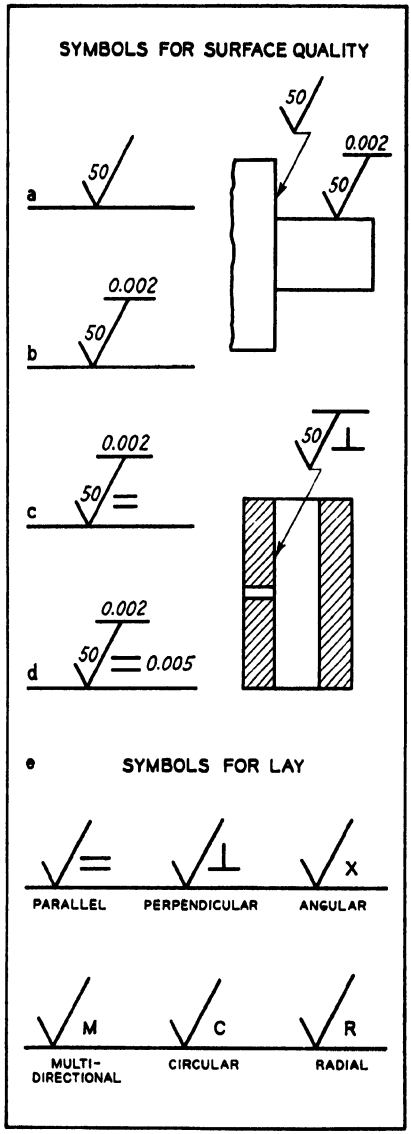


FIG. 84 SURFACE QUALITY SYMBOLS

* As recommended by the Sectional Committee on the Standardization of Classification and Designation of Surface Qualities (ASA B46).

height, or average deviation from the mean (RMS or arithmetical). The numerical value is placed in the $\sqrt{}$ as shown.

Where it is desired to specify waviness height in addition to roughness height a straight horizontal line should be added to the top of the simple symbol. See Fig. 84(b). The numerical value of height of waviness would be shown above this line.

Then, if the nature of the preferred lay is to be shown in addition to these two characteristics, it will be indicated by the addition of a combination of lines as shown in Fig. 84(c and e). The parallel and perpendicular part of the symbol indicates that the dominant lines on the surface are parallel or perpendicular to the boundary line of the surface in contact with the symbol.

The complete symbol including the roughness width placed to the right of the lay symbol, is shown in Fig. 84(d).

The use of only one number to specify the height or width of roughness or waviness shall indicate the maximum value. Any lesser degree of roughness will be satisfactory. When two numbers are used separated by a dash, they indicate the maximum and minimum permissible values.

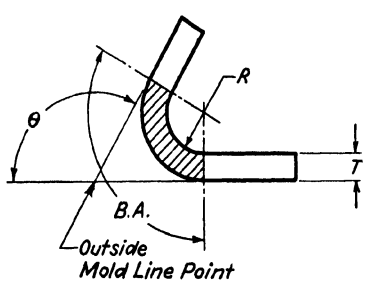


FIG. 85 BEND ALLOWANCE

68 Bend Allowance

The allowance to be made for bends in sheet metal work or thin steel is shown in Fig. 85. The intersection of the plane surfaces adjoining the bend is called a mold or construction line and this line is used in dimensioning rather than the center of the arc.

69 Profile Dimensioning

Parts that are formed by bending are dimensioned as in Fig. 86. The dimensions are given to mold lines. Angles and radii must be shown.

70 Springs

Methods of representing and dimensioning springs are shown in Figs. 90, 91, and

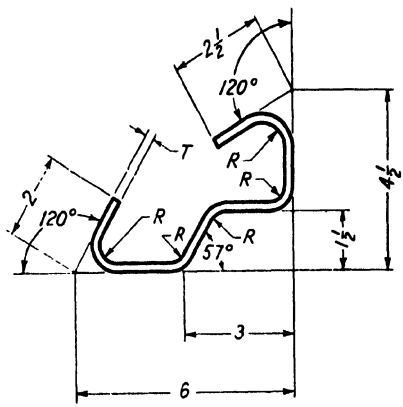


FIG. 86 PROFILE DIMENSIONING

92. Beside the length and controlling diameter (inside or outside) the size and number of turns of wire must be indicated by note. The type of end should be indicated even though standard. If the ends of the spring must be formed to fit some special condition the end must then be dimensioned in detail to secure the desired result. Single line symbols are shown in Fig. 93.

71 Gears

A gear may be shown satisfactorily by a section through the axis, except when additional views may be required to show the construction of a large blank or other details. In principle the dimensions placed on the drawing should be sufficient for machining the gear blank. The gear-cut-

ting data which include the tooth-form standard and method of cutting, and information for measurement and inspection, should be given in a table. Notes for material and heat treatment are also required, Figs. 87 and 88.

78 *Keys*

Plain square, flat, taper and Gib-head taper stock keys are dimensioned by note giving the width, height and length. Whenever possible, sizes should conform to standards established in the American Standard for Shafting and Stock Keys (ASA B17.1—1943). The keys need not be drawn except for special keys or when limits other than those of the standard are necessary. American Standard Woodruff Keys, Keyslots and Cutter (ASA B17f—1930) may be specified by key number.

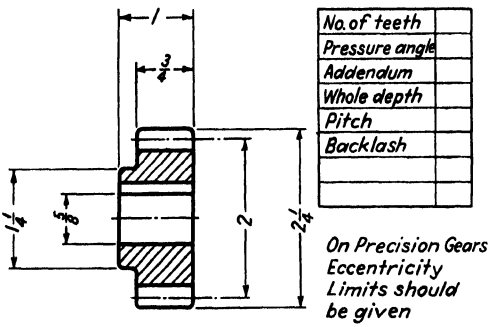


FIG. 87 DIMENSIONING SPUR PINION

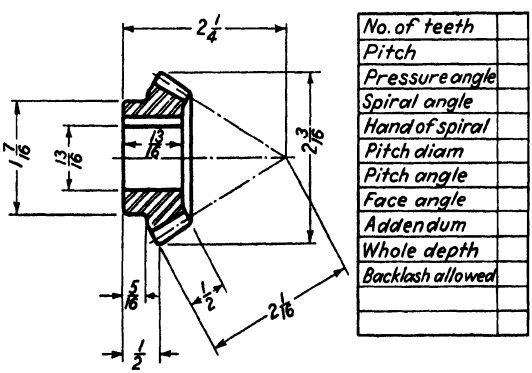


FIG. 88 DIMENSIONING SPIRAL BEVEL PINION

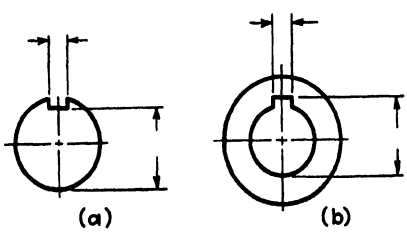


FIG. 89 DIMENSIONING KEYWAYS

79 *Keyways*

Keyways on shafts or internal members may be dimensioned as shown in Fig. 89(a). On the hub or external member the keyway may be dimensioned as shown in Fig. 89(b). The key seat for patented varieties may be specified by the key number.

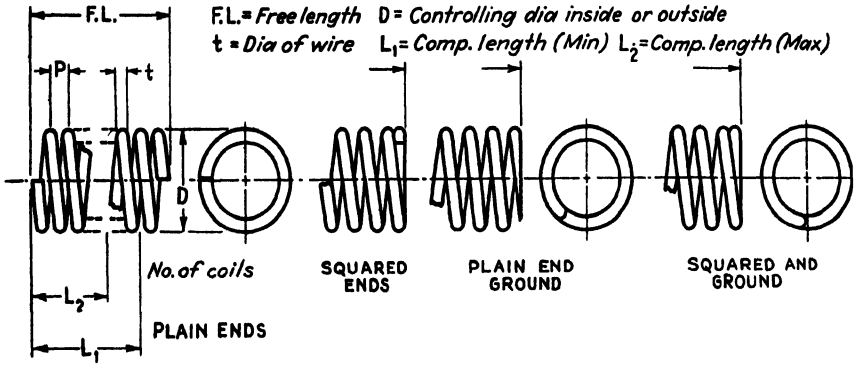


FIG. 90 REPRESENTING AND DIMENSIONING COMPRESSION SPRINGS

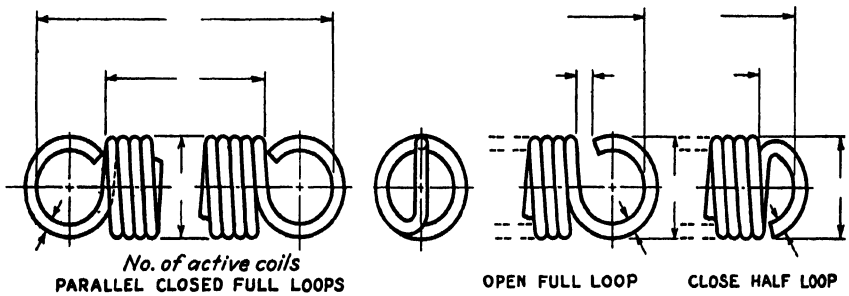


FIG. 91 REPRESENTING AND DIMENSIONING TENSION SPRINGS

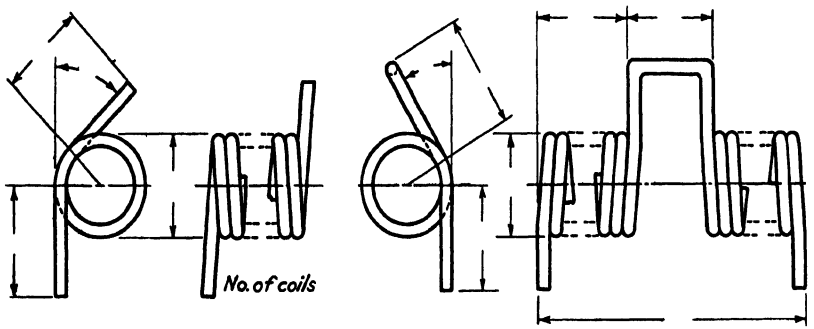


FIG. 92 REPRESENTING AND DIMENSIONING TORSION SPRINGS

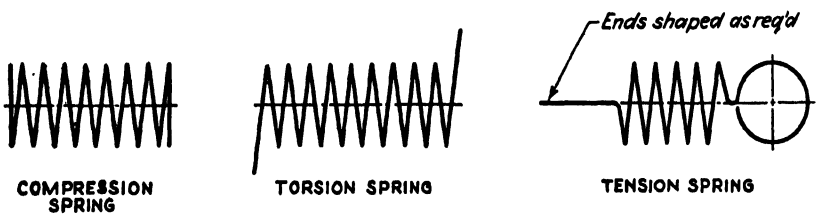


FIG. 93 SINGLE LINE REPRESENTATION OF SPRINGS

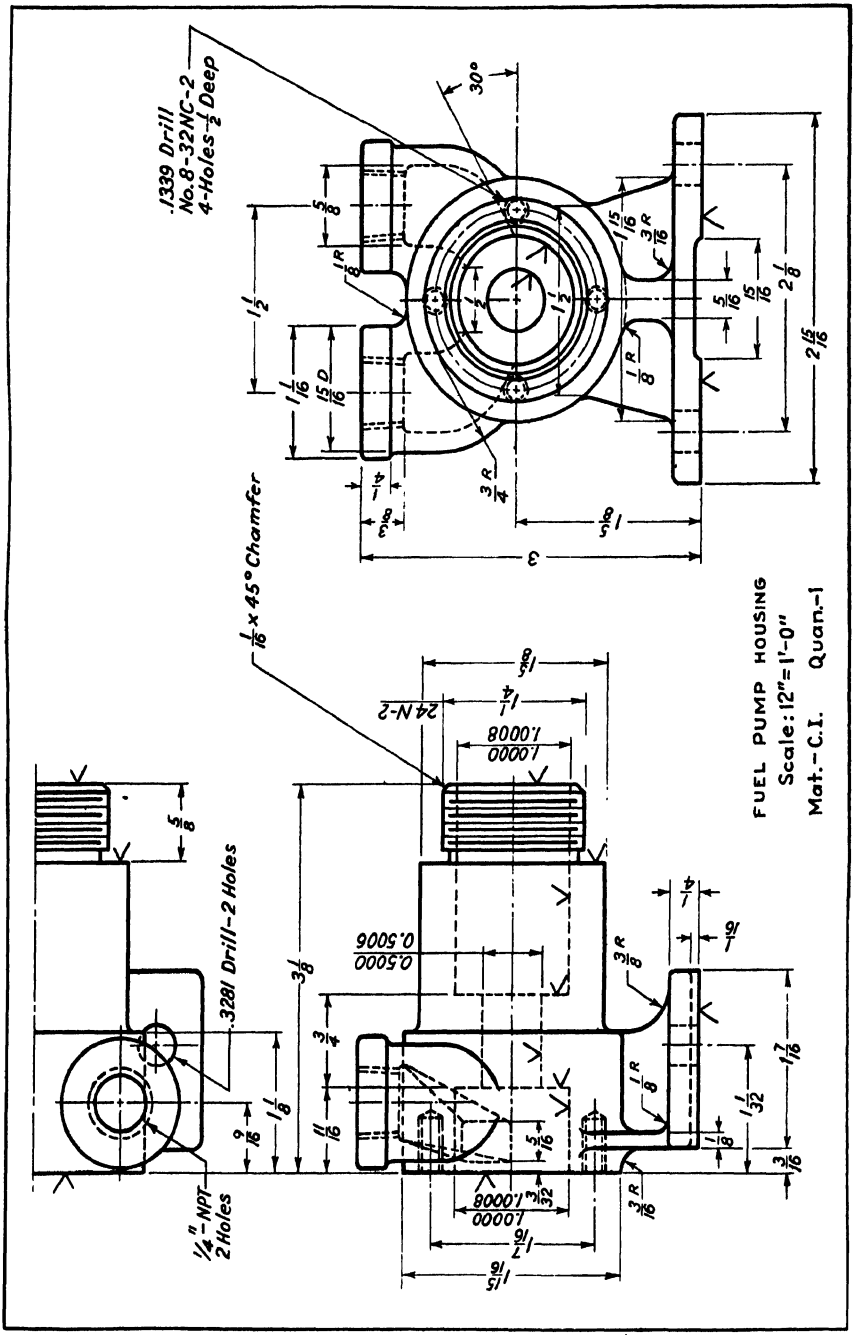


FIG. 94 APPLICATION OF SURFACE FINISH SYMBOLS

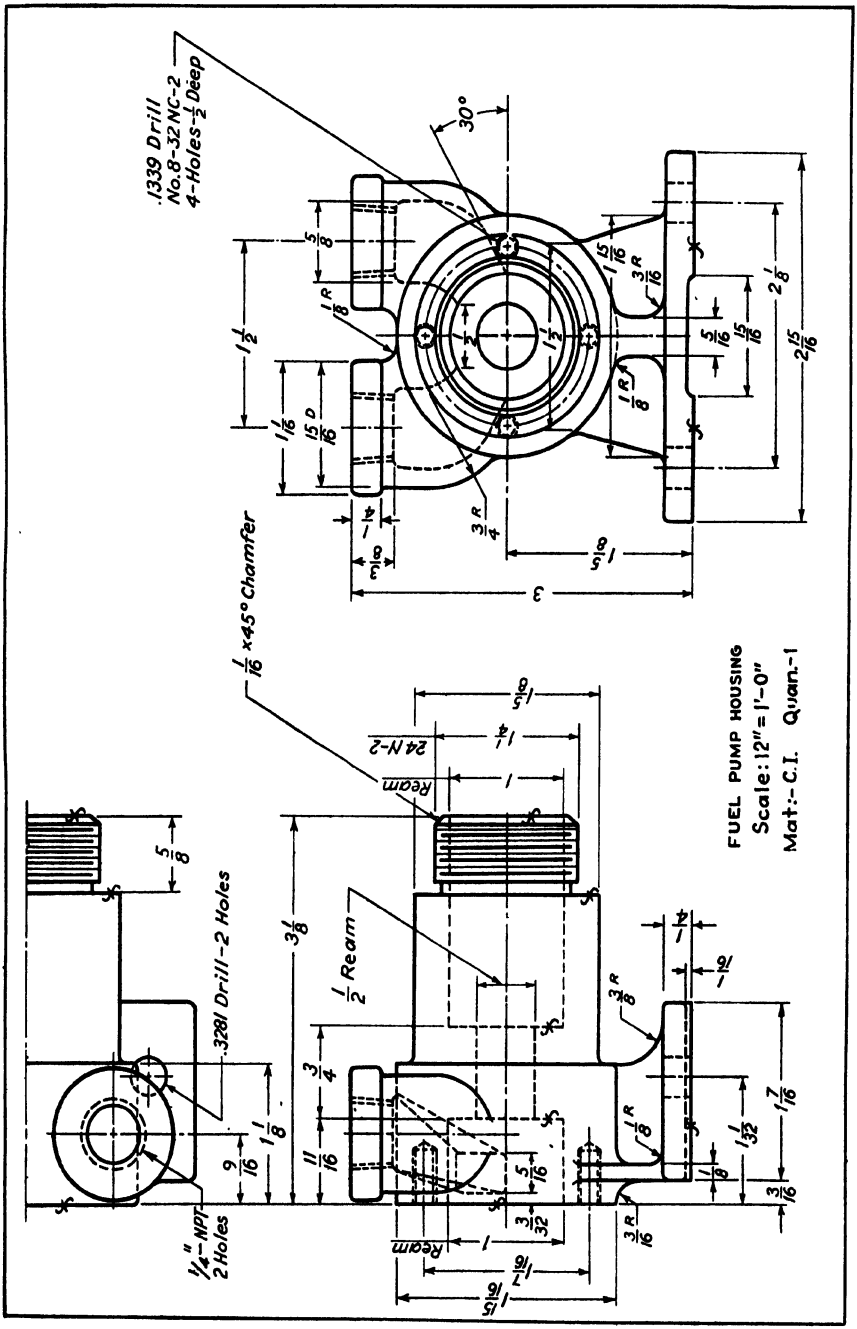


FIG. 95 APPLICATION OF SURFACE FINISH SYMBOLS

SECTION 7—NOTES

74 Definition

Notes are supplementary to dimensioning and are used to indicate necessary information on the drawing in a condensed and systematic manner. Notes may be classified as General and Local or Specific, according to their character.

75 General Notes

General notes should be placed in the lower right hand corner of the drawing above or to the left of the title block or in a central position below the view of the object to which they apply. General notes, as the name implies, means those notes which refer to the part as a whole, e.g.: "Finish All Over" or "Paint One Coat."

Also classified as general notes, are notes that apply to dimensions of the part and which might be repeated many times if general notes were not used, e.g.: "All draft angles 3° unless otherwise specified" or "Tolerance ±.001 unless shown otherwise."

76 Miscellaneous General Notes

The following miscellaneous General Notes are suggested for use where necessary.

(1) Finish All Over.
(2) Finish All Over Except—
(3) Break Sharp Edges .01-, .03 Approx. R Unless Otherwise Specified.
(4) Break Sharp Edges On All (Drilled Holes) (Chamfers) (Countersinks) Unless Otherwise Specified.
(5) All (Casting) (Forging) (Finished) Radii .xxx- .xxx Unless Otherwise Specified.
(6) All Draft Angles X° Unless Otherwise Specified.
(7) All Dimensions To Be Met After Plating.
(8) Must Be Flat and Free From Burrs.
(9) All Small Unfinished Radii and Fillets .XX Unless Otherwise Specified.
(10) Remove Burrs.

77 Local Notes

Local notes are those that apply to local operations only and should be placed adjacent, connected by a leader, to the point at which such operations are to be performed, e.g.: ".248-.255," "Drill, 4 holes." See Figs. 64 and 70 to 75 inclusive, for examples.

78 Specification of Materials and Heat Treatments

Many companies have their own method and standards. A method commonly available is to indicate ferrous materials and heat treatments by specifying the steels by the SAE Numbers, and the hardness required by a Rockwell, or Brinell hardness number, as for example:

SAE—1095—Rockwell C 60-64
SAE—3345—Brinell 340-380

For steel in buildings, bridges, boilers, railroads, and reinforced concrete the A.S.T.M. specifications or special specifications prepared by the firm or department making the drawing are used. Treatments such as case-hardening, carburizing (by any method), cyaniding, nitriding, should be indicated by a specific local note completely covering the requirements.

SECTION 8—TRIMMED SIZES OF DRAWING PAPER AND CLOTH. TITLES

79 Sizes of Paper

The recommended standard trimmed sheet sizes of drawing paper and cloth are as follows:

A	$8\frac{1}{2}\times11$	D	22×34
B	11×17	E	34×44
C	17×22		

The use of the basic sheet size $8\frac{1}{2}\times11$ in. and its multiple permits filing of small tracings and folded prints in commercial standard files with or without correspondence. These sheet sizes also cut without waste from the present 36 in. rolls of paper and cloth. These standard sizes are based on the dimensions of the commercial letter head, $8\frac{1}{2}\times11$ in., in general use in the United States.

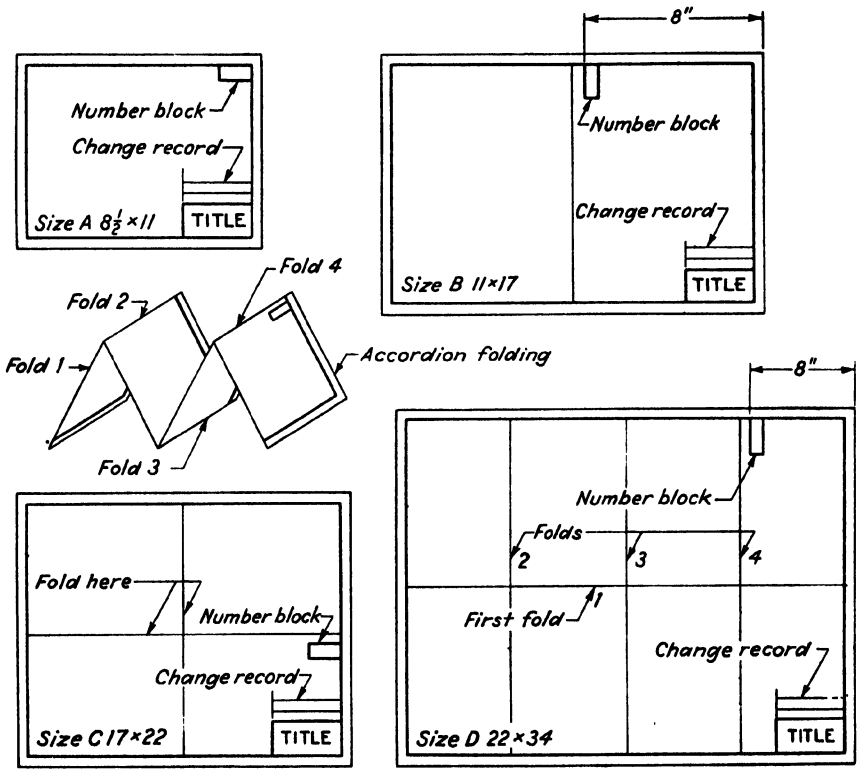


FIG. 96 TITLE BLOCK, CHANGE RECORD, AND FOLDING

Drawings larger than standard sheet sizes may be made as "rolled" drawings. Widths are controlled by standard paper sizes, and lengths are in multiples of 11 in.

80 Sizes of Paper in the Metric System

For drawings made in the metric system of units or for foreign correspondence it is recommended that the metric standard trimmed sheet sizes be used. These sizes are based on the width to length ratio of 1 to $\sqrt{2}$ and are as follows:

A0	841×1189 mm
A1	594×841 mm
A2	420×594 mm
A3	297×420 mm
A4	210×297 mm
A5	148×210 mm
A6	105×148 mm

81 Title, Change Record and Number Blocks

The title block with appropriate space for the identifying number should be located at the lower right hand corner of the sheet. Accordion folding of prints with the printed side out is recommended when prints are filed in a standard letter size file as illustrated in Fig. 96.

In order to file prints which have been accordion folded so that the closed fold will be up and the number still be visible, it is necessary to add a supplementary number block (approximately 1.75 in. by .5 in.) to each size sheet as shown in Fig. 96. This supplementary number block may be either parallel or normal to the border line of the drawing. If change notes or general notes reach this supplementary number block a minimum clearance of .25 in. should be left above and below the block.

82 Location of Change Record

The change record should be placed immediately above the title space as shown in Fig. 96. The change record must be "tied in" to the drawing so it can be quickly found. This is particularly necessary on large sheets.

83 Zoning System of Locating Changes

The lower and right hand borders of the larger size drawings should be ruled and marked as indicated in Fig. 97 to provide zones for reference purposes. The location of the change on the drawing should be referenced in the change block and identified by a reference letter which should also appear on the drawing near the change and

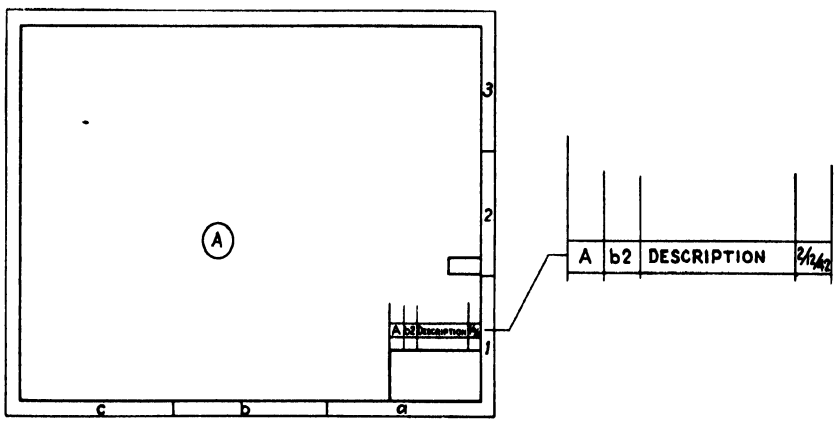


FIG. 97 ZONE METHOD OF LOCATING CHANGES

in a position where it can be readily seen. For example a change referenced (b2) would be found upward from b and over from 2. For purposes of clarity a circle of approximately 5/16 in. in diameter should be drawn around the change letter as illustrated in Fig. 97. Letters I, O and Q should not be used to avoid misinterpretation and resultant errors. Other methods not illustrated here are also in common use.

SECTION 9—LETTERING

84 The most important requirement for lettering as used on working drawings is legibility. The second is ease and rapidity of execution. These two requirements are met in the single stroke commercial gothic letter, now in almost universal use throughout the technical world. Preference seems to be divided between the vertical and the inclined styles.

The following standard practice is recommended:

(a) That single stroke commercial gothic lettering either vertical or inclined at a slope of 2 in 5 be used on all working drawings for titles, notes, etc.

85 It is not desirable to grade the size of lettering with the size of the drawing

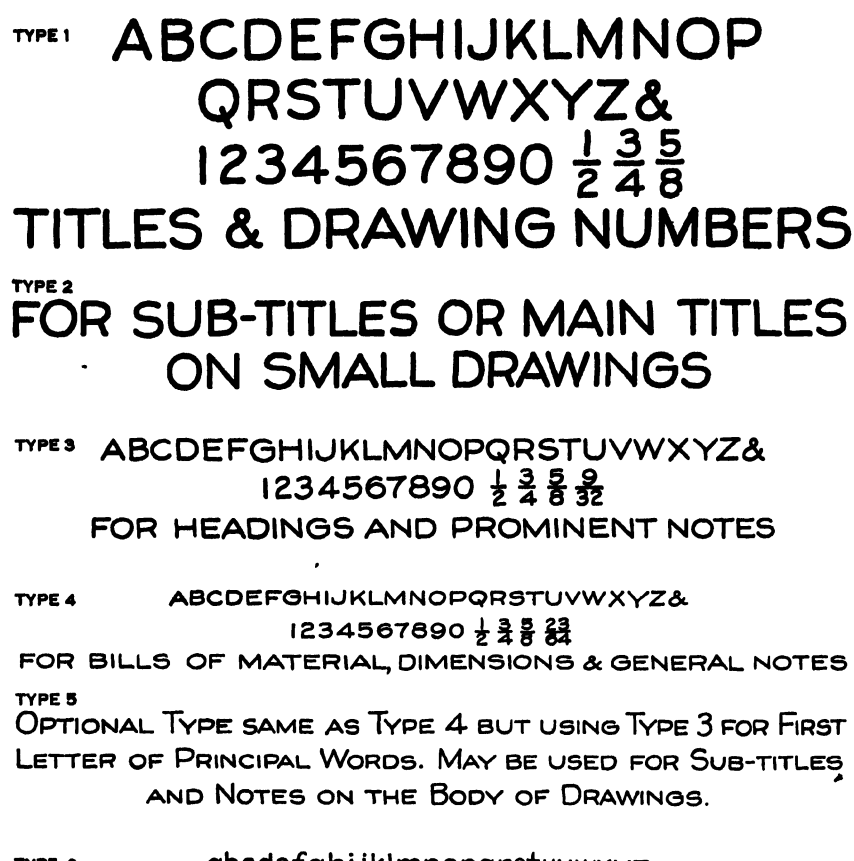


FIG. 98

except when a reduced photographic reproduction of the drawing is to be made. In other words the size and weight of the lettering should be such as will produce legible prints from tracings either in pencil or in ink.

Lettering should not be underlined except for particular emphasis.

Approved specimens of vertical and inclined letters are shown in Figs. 98 and 99.

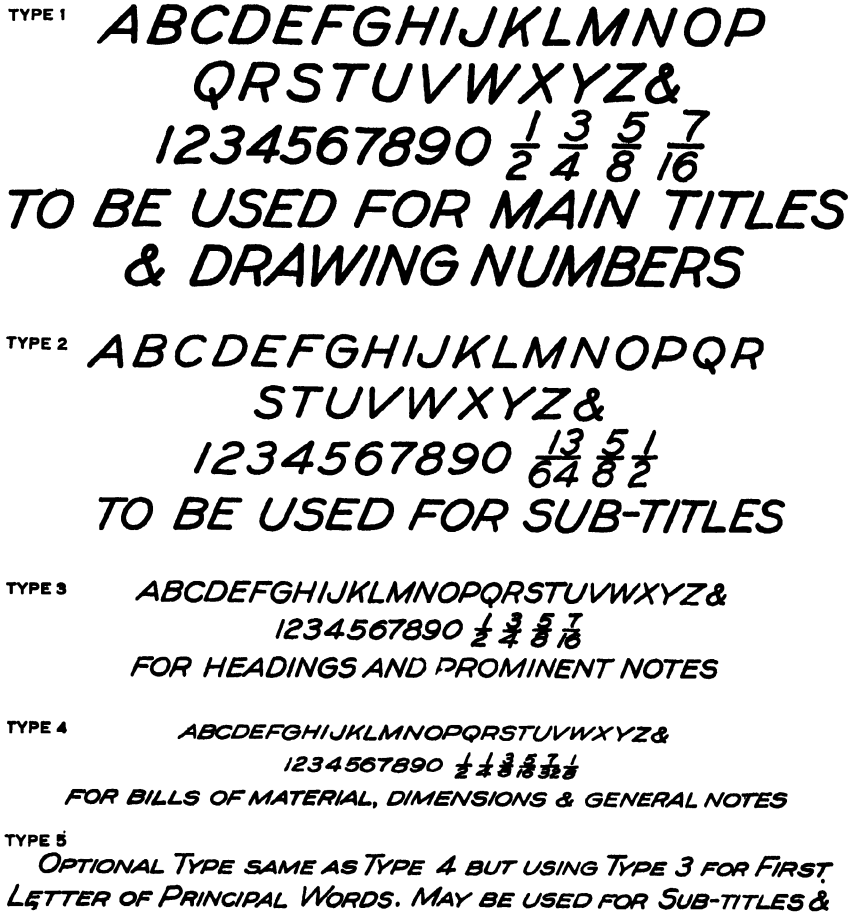


FIG. 99

APPENDIX II

GEOMETRIC PROOFS

Geometric propositions useful in graphical methods are stated in the main body of the text without proof. Mathematically minded students, however, dislike accepting such statements on faith.

This appendix contains proofs of some of the propositions in the text. For those not included the student should consult any good text on college geometry. The particular form of each proof used here is not the most

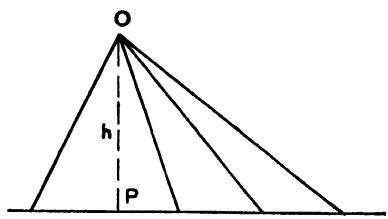


Fig. A-1.

elegant from the viewpoint of the advanced mathematician, but is such that a student who understands high-school algebra, plane geometry, and trigonometry can follow it.

PROOF I — CROSS RATIO

Given the collinear points A, B, C, D and any center of projection O, Fig. A-1.

Draw OA, OB, OC, and OD. To prove

$$\frac{AC}{CB} : \frac{AD}{DB} = \frac{\sin \angle AOC}{\sin \angle COB} : \frac{\sin \angle AOD}{\sin \angle DOB}$$

Draw OP perpendicular to AD. Let $OP = h$. The area of a triangle = ½ altitude × base. Therefore,

$$\begin{aligned} \triangle AOC &= \tfrac{1}{2}h \cdot AC \\ \triangle COB &= \tfrac{1}{2}h \cdot CB \\ \triangle AOD &= \tfrac{1}{2}h \cdot AD \\ \triangle DOB &= \tfrac{1}{2}h \cdot DB \end{aligned}$$

Therefore,

$$\frac{\triangle AOC}{\triangle COB} : \frac{\triangle AOD}{\triangle DOB} = \frac{AC}{CB} : \frac{AD}{DB} \tag{1}$$

From trigonometry the area of a $\triangle$ = ½ the product of two sides multiplied by the sine of the included angle. Therefore,

$$\triangle AOC = \tfrac{1}{2}AO \cdot CO \cdot \sin \angle AOC$$
$$\triangle COB = \tfrac{1}{2}CO \cdot BO \cdot \sin \angle COB$$
$$\triangle AOD = \tfrac{1}{2}AO \cdot DO \cdot \sin \angle AOD$$
$$\triangle DOB = \tfrac{1}{2}DO \cdot OB \cdot \sin \angle DOB$$

Therefore,

$$\frac{\triangle AOC}{\triangle COB} : \frac{\triangle AOD}{\triangle DOB} = \frac{\sin \angle AOC}{\sin \angle COB} : \frac{\sin \angle AOD}{\sin \angle DOB} \tag{2}$$

and

$$\frac{AC}{CB} : \frac{AD}{DB} = \frac{\sin \angle AOC}{\sin \angle COB} : \frac{\sin \angle AOD}{\sin \angle DOB}$$

by substitution from (1).

PROOF II — TO PROVE THAT A TANGENT TO AN ELLIPSE BISECTS THE ANGLE BETWEEN THE FOCAL RADII

Assume any point P, Fig. A-2, on the ellipse, and draw the focal radii PF and PF_1. Bisect the external angle at P of triangle FPF_1, the bisector being the line R. Draw from F a line perpendicular to R, intersecting F_1P at A. Line R is now the perpendicular bisector of AF. Hence:

$$FP = AP$$

Therefore,

$AF_1 = FP + PF_1$ = constant sum of the distances of points on the ellipse from F and F_1.

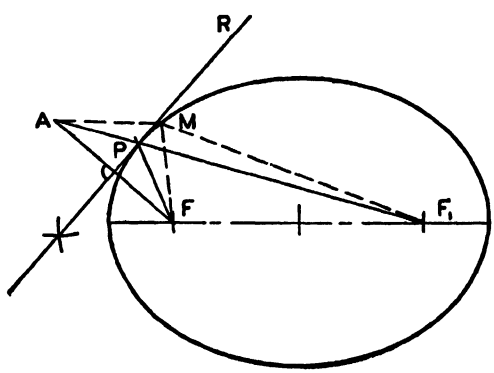


Fig. A-2.

Assume that R is not tangent to the ellipse but meets the ellipse at some second point such as M. Then

$$FM = AM$$

Therefore,

$$AM + MF_1 = FM + MF_1$$

However, $AM + MF_1$ must be greater than $AP + PF_1$ since a straight line is the shortest distance between two points. Therefore M cannot lie on the ellipse but must lie outside it. Therefore R meets the ellipse only at P and is consequently tangent.

PROOF III — THE TRAMMEL METHOD

Consider the path of a point $P(x, y)$ on a straight line which is moved so that a distance b from P along the line always lies on the X axis and a distance a from P along the line always lies on the Y axis (Fig. A-3).

Then

$$\frac{x}{a} = \cos\theta$$

$$\frac{y}{b} = \sin\theta$$

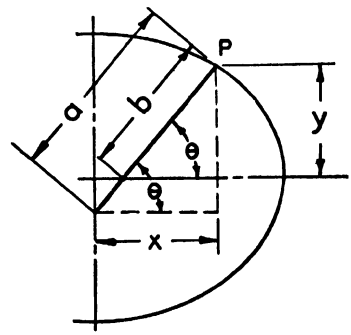


Fig. A-3.

Squaring and adding

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\theta + \sin^2\theta$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is the equation of an ellipse whose semimajor axis = a and whose semiminor axis = b.

PROOF IV — CONCENTRIC-CIRCLE METHOD

Point $P(x, y)$ is a point found from the construction. From Fig. A-4,

$$\cos\theta = \frac{x}{a}$$

$$\sin\theta = \frac{y}{b}$$

Squaring and adding,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\theta + \sin^2\theta$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is the equation of an ellipse whose semimajor and semiminor axes are a and b, respectively the radii of the concentric circles.

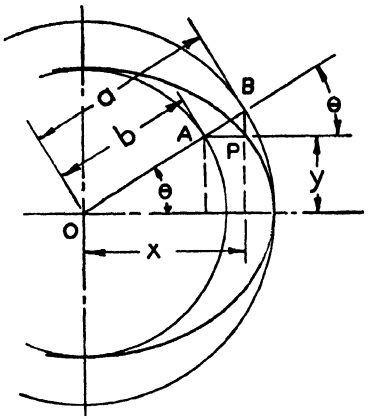


Fig. A-4.

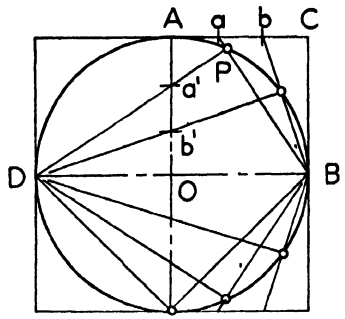


Fig. A-5.

PROOF V — PARALLELOGRAM METHOD FOR CONSTRUCTING AN ELLIPSE

Consider the square in Fig. A-5, circumscribed around a circle, with AC and AO divided into the same number of equal parts as shown. Draw Da' and Ba meeting at P.

Now

$$\Delta a'OD \cong \Delta aCB$$

Hence

$$\angle a'DO = \angle aBC$$

Therefore, since

$$CB \perp OD$$

It follows that

$$aB \perp a'D$$

Therefore, DPB is a right angle and P lies on the circle.

Thus every point determined in this manner will lie on the circle having DB as a diameter. Thus the construction is valid for a circle inscribed in a square.

Only ratio and concurrency of lines are involved, and these are projective invariants under orthographic projection. Hence, if the square enclosing the circle be projected orthogonally on any plane whatever, becoming a parallelogram enclosing an ellipse, the construction will still be valid.

PROOF VI — THE PARALLELOGRAM METHOD FOR CONSTRUCTING A PARABOLA

Consider any point P on the curve found by the construction using a rectangle, Fig. A-6. Then from similar triangles

$$\frac{ky}{m} = \frac{x}{y}$$

or

$$ky^2 = mx$$

or

$$y^2 = \frac{mx}{k}$$

which is the equation of a parabola. Since only projective invariants are involved and the rectangle can be projected into a parallelogram, the method holds for the parallelogram.

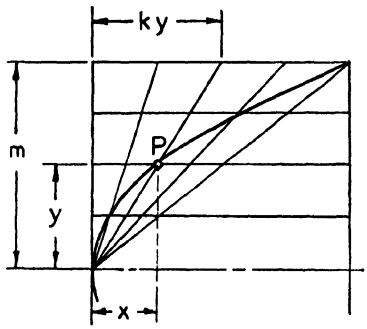


Fig. A-6.

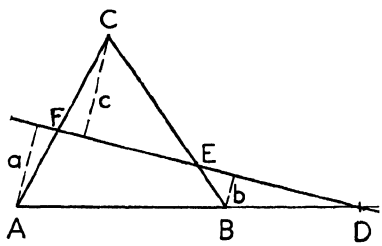


Fig. A-7.

PROOF VII — MENELAUS' THEOREM

Any line cuts the sides of a triangle so that the product of the three non-adjacent segments divided by the product of the other three equals -1, signs being taken into account.

The three sides of the triangle ABC, Fig. A-7, are cut by a transversal at D, E, and F. To prove:

$$\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = -1$$

Draw the perpendiculars a, b, and c from A, B, and C to the transversal Then (from similar triangles),

$$\frac{AD}{DB} = \frac{-a}{b} \qquad \frac{BE}{EC} = \frac{b}{c} \qquad \frac{CF}{FA} = \frac{c}{a}$$

Multiplying these three together,

$$\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = \frac{-a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a} = -1$$

CONVERSE OF MENELAUS' THEOREM

If points D, E, and F are taken on the three sides of triangle ABC so that

$$\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = -1$$

then D, E, F are collinear.

Draw FE produced to intersect AB at D', Fig. A–7.

Then

$$\frac{AD'}{D'B} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = -1$$

(by Menelaus' theorem).

Thus

$$\frac{AD}{DB} = \frac{AD'}{D'B}$$

or

$$\frac{AB + BD}{DB} = \frac{AB + BD'}{D'B}$$

or

$$\frac{AB}{DB} - 1 = \frac{AB}{D'B} - 1$$

or

$$DB = D'B$$

Thus, D' must coincide with D, and F, E, and D are collinear.

PROOF VIII — DESARGUES' THEOREM

If the triangles ABC and $A'B'C'$, Fig. A-8, are so situated that AA', BB', and CC' are concurrent at O, then the three pairs of sides AB and $A'B'$, BC and $B'C'$, CA and $C'A'$ meet in three points which are collinear.

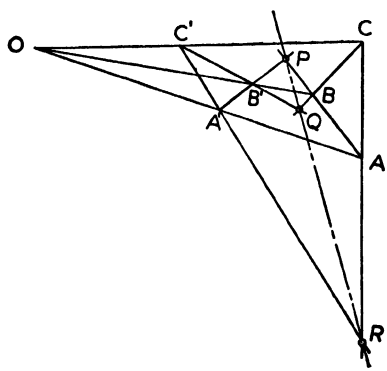


Fig. A-8.

Using the transversals $PB'A'$, $QB'C'$, and $RA'C'$, cutting the sides of the triangles OAB, OBC, and OCA, respectively, and applying Menelaus' theorem, we get

$$\frac{OA'}{A'A} \cdot \frac{AP}{PB} \cdot \frac{BB'}{B'O} = -1$$

$$\frac{OB'}{B'B} \cdot \frac{BQ}{QC} \cdot \frac{CC'}{C'O} = -1$$

$$\frac{AA'}{A'O} \cdot \frac{OC'}{C'C} \cdot \frac{CR}{RA} = -1$$

Multiplying and canceling,

$$\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RA} = -1$$

Therefore, by the converse of Menelaus' theorem, P, Q, and R are collinear.

PROOF IX — PASCAL'S THEOREM

The points of intersection of the opposite sides of a hexagon inscribed in a conic are collinear. We shall prove this first for a hexagon inscribed in a circle.

Hexagon $ABCDEF$, Fig. A-9, is inscribed in a circle, the opposite sides intersecting at P, Q, and R. Prove P, Q, R collinear.

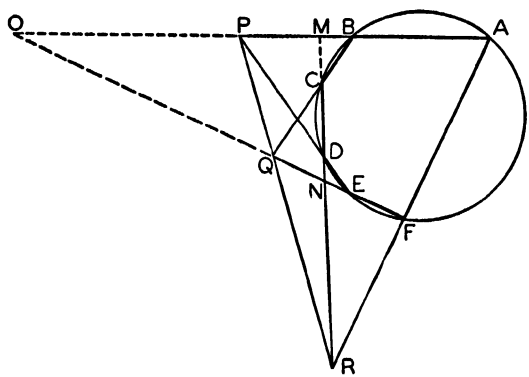


Fig. A-9.

The triangle MNO is formed by the points of intersection of AB and CD, CD and EF, and EF and AB, respectively. Consider DE, FA, and BC as transversals cutting the sides of triangle MNO. By applying Menelaus' theorem we have

$$\frac{MP}{PO} \cdot \frac{OE}{EN} \cdot \frac{ND}{DM} = -1 \qquad \frac{NR}{RM} \cdot \frac{MA}{AO} \cdot \frac{OF}{FN} = -1 \qquad \frac{OQ}{QN} \cdot \frac{NC}{CM} \cdot \frac{MB}{BO} = -1$$

Multiplying and rearranging,

$$\frac{MP}{PO} \cdot \frac{OQ}{QN} \cdot \frac{NR}{RM} \cdot \frac{OE \cdot OF}{BO \cdot AO} \cdot \frac{MB \cdot MA}{CM \cdot DM} \cdot \frac{NC \cdot ND}{EN \cdot FN} = -1$$

Now each of the last three fractions in this product equals 1 because the product of any secant and its external segment from a point outside a circle is a constant. Therefore,

$$\frac{MP}{PO} \cdot \frac{OQ}{QN} \cdot \frac{NR}{RM} = -1$$

Therefore, P, Q, R are collinear by the converse of Menelaus' theorem.

Therefore, Pascal's theorem is true of a hexagon inscribed in a circle. Since the only factors involved are concurrency of lines and collinearity of points which are projective invariants, the theorem will still hold true if the circle is projected into an ellipse, parabola, or hyperbola.

PROOF X — HARMONIC ANALYSIS

The Fourier series

The series represented by Eq. (2), Art. 8-7, is put into a more convenient form for solution by expanding each term as follows:

$$y = c + r_1 \cos \alpha_1 \sin \theta + r_1 \sin \alpha_1 \cos \theta + r_2 \cos \alpha_2 \sin 2\theta + r_2 \sin \alpha_2 \cos 2\theta + r_3 \cos \alpha_3 \sin 3\theta + r_3 \sin \alpha_3 \cos 3\theta + \cdots$$

Now, let $a_1 = r_1 \cos \alpha_1$, $a_2 = r_2 \cos \alpha_2$, and $a_3 = r_3 \cos \alpha_3$. Also, let $b_1 = r_1 \sin \alpha_1$, $b_2 = r_2 \sin \alpha_2$, and $b_3 = r_3 \sin \alpha_3$. Equation (2) now becomes

$$y = c + a_1 \sin \theta + a_2 \sin 2\theta + a_3 \sin 3\theta + \cdots + b_1 \cos \theta + b_2 \cos 2\theta + b_3 \cos 3\theta + \cdots \quad (3)$$

The right-hand member of this equation is a common form of the Fourier series.

Since $a_1 = r_1 \cos \alpha_1$ and $b_1 = r_1 \sin \alpha_1$, it follows that $\tan \alpha_1 = b_1/a_1$. Also, $r_1^2 = a_1^2 + b_1^2$ and $r_1 = \pm\sqrt{a_1^2 + b_1^2}$.

The constants α_2, α_3 and r_2, r_3 can be expressed in a similar fashion. The constant c is the mean of all of the ordinates of the curve.

Evaluating the constants

Any given periodic curve represents displacement y from mean position at any time t or angle θ of the radius vector. To find the equation of the curve, it is necessary to find the constants a and b for each term of the series (3). The constant c locates the axis of the curve from an assumed base line. The number of terms to be found in any given case depends on the precision required in the problem.

A general solution of the Fourier series by the calculus gives the following values for the constants of the term k, where $k = 1, 2, 3, \ldots$.

$$a_k = \frac{1}{\pi}\int_0^{2\pi} y \sin k\theta \, d\theta \tag{4}$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} y \cos k\theta \, d\theta \tag{5}$$

$$c = \frac{1}{2\pi}\int_0^{2\pi} y \, d\theta \tag{6}$$

The general procedure for finding a given term (or harmonic) is illustrated above. Let the periodic curve $y = f(\theta)$ be given (A) Fig. A-10. In order to find the constant a for the first harmonic, each ordinate of the given curve is

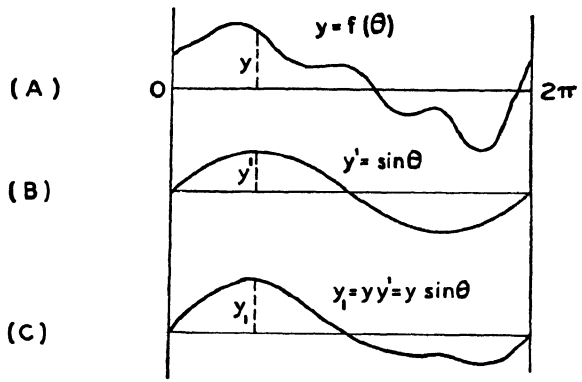


Fig. A-10.

multiplied by sin θ (B). The products of the corresponding ordinates are plotted (C), and the area under this curve is found. This area times $1/\pi$ is equal to a_1. The constant b_1 is found by multiplying each ordinate of the given curve by cos θ and finding the area under the resulting curve. This area times $1/\pi$ is equal to b_1. The constant c is the average of all the ordinates.

The constants for the second harmonic are found in a similar manner. The ordinates of the given curve are multiplied in turn by sin 2θ and cos 2θ. For the third harmonic the ordinates are multiplied by sin 3θ and cos 3θ. Any given harmonic can be computed independently.

The graphical solution

The solution just outlined can be carried out graphically. In some engineering problems harmonic terms beyond the third can often be neglected, and for these cases the graphical method is useful.

The given curve is represented by n equally spaced ordinates. Each ordinate is taken as the mean ordinate of a strip $2\pi/n$ wide. Each ordinate of the given curve is now multiplied by $\sin k\theta$ (or $\cos k\theta$), and the area under the resulting curve is multiplied by $1/\pi$ to give a_k or b_k.

Using this procedure, the Eqs. (4), (5), and (6) can be written as follows:

$$a_k = \frac{2}{n}(y_0 \sin k\theta_0 + y_1 \sin k\theta_1 + y_2 \sin k\theta_2 + \cdots) \tag{7}$$

$$b_k = \frac{2}{n}(y_0 \cos k\theta_0 + y_1 \cos k\theta_1 + y_2 \cos k\theta_2 + \cdots) \tag{8}$$

$$c = \frac{1}{n}(y_0 + y_1 + y_2 + \cdots) \tag{9}$$

The construction of the vector polygon, Fig. 8-10, has the effect of multiplying each ordinate of the given curve by $\sin \theta$ and $\cos \theta$ as will be seen when the components of each vector on the OA and OB axes are considered.

APPENDIX III

TABLE OF CHORDS

The Tabulated Quantities = Twice the Sine of Half the Arc

Deg.	0′	10′	20′	30′	40′	50′
0	.0000	.0029	.0058	.0087	.0116	.0145
1	.0174	.0204	.0233	.0262	.0291	.0320
2	.0349	.0378	.0407	.0436	.0465	.0494
3	.0523	.0553	.0582	.0611	.0640	.0669
4	.0698	.0727	.0756	.0785	.0814	.0843
5	.0872	.0901	.0930	.0959	.0988	.1017
6	.1047	.1076	.1105	.1134	.1163	.1192
7	.1221	.1250	.1279	.1308	.1337	.1366
8	.1395	.1424	.1453	.1482	.1511	.1540
9	.1569	.1598	.1627	.1656	.1685	1714
10	.1743	.1772	.1801	.1830	.1859	.1888
11	.1917	.1946	.1975	.2004	.2033	.2062
12	.2090	.2119	.2148	.2177	.2206	.2235
13	.2264	.2293	.2322	.2351	.2380	.2409
14	.2437	.2466	.2495	.2524	.2553	.2582
15	.2610	.2639	.2668	.2697	.2726	.2755
16	.2783	.2812	.2841	.2870	.2899	.2927
17	.2956	.2985	.3014	.3042	.3071	.3100
18	.3129	.3157	.3186	.3215	.3243	.3272
19	.3301	.3330	.3358	.3387	.3416	.3444
20	.3473	.3502	.3530	.3559	.3587	.3616
21	.3645	.3673	.3702	3730	.3759	.3788
22	.3816	.3845	.3873	.3902	.3930	.3959
23	.3987	.4016	.4044	.4073	.4101	.4130
24	.4158	.4187	.4215	.4243	.4272	.4300
25	.4329	.4357	.4385	.4414	.4442	.4471
26	.4499	.4527	.4556	.4584	.4612	.4641
27	.4669	.4697	.4725	.4754	.4782	.4810
28	.4838	.4867	.4895	.4923	.4951	.4979
29	.5008	.5036	.5064	.5092	.5120	.5148
30	.5176	.5204	.5232	.5261	.5289	.5317
31	.5345	.5373	.5401	.5429	.5457	.5485
32	.5513	.5541	.5569	.5596	.5624	.5652
33	.5680	.5708	.5736	.5764	.5792	.5820
34	.5847	.5875	.5903	.5931	.5959	.5986
35	.6014	.6042	.6069	.6097	.6125	.6153
36	.6180	.6208	.6236	.6263	.6291	.6318
37	.6346	.6374	.6401	.6429	.6456	.6484
38	.6511	.6539	.6566	.6594	.6621	.6649
39	.6676	.6703	.6731	.6758	.5786	.6813
40	.6840	.6868	.6895	.6922	.6950	.6977
41	.7004	.7031	.7059	.7086	.7113	.7140

TABLE OF CHORDS

Deg.	0′	10′	20′	30′	40′	50′
42	.7167	.7194	.7222	.7249	.7276	.7303
43	.7330	.7357	.7384	.7411	.7438	.7465
44	.7492	.7519	.7546	.7573	.7600	.7627
45	.7654	.7680	.7707	.7734	.7761	.7788
46	.7815	.7841	.7868	.7895	.7921	.7948
47	.7975	.8001	.8028	.8055	.8081	.8108
48	.8135	.8161	.8188	.8214	.8241	.8267
49	.8294	.8320	.8347	.8373	.8400	.8426
50	.8452	.8479	.8505	.8531	.8558	.8584
51	.8610	.8636	.8663	.8689	.8715	.8741
52	.8767	.8793	.8820	.8846	.8872	.8898
53	.8924	.8950	.8976	.9002	.9028	.9054
54	.9080	.9106	.9132	.9157	.9183	.9209
55	.9235	.9261	.9286	.9312	.9338	.9364
56	.9389	.9415	.9441	.9466	.9492	.9518
57	.9543	.9569	.9594	.9620	.9645	.9671
58	.9696	.9722	.9747	.9772	.9798	.9823
59	.9848	.9874	.9899	.9924	.9949	.9975
60	1.0000	1.0025	1.0050	1.0075	1.0100	1.0126
61	1.0151	1.0176	1.0201	1.0226	1.0251	1.0276
62	1.0301	1.0326	1.0350	1.0375	1.0400	1.0425
63	1.0450	1.0475	1.0500	1.0524	1.0550	1.0574
64	1.0598	1.0623	1.0648	1.0672	1.0697	1.0721
65	1.0746	1.0770	1.0795	1.0819	1.0844	1.0868
66	1.0893	1.0917	1.0941	1.0966	1.0990	1.1014
67	1.1039	1.1063	1.1087	1.1111	1.1135	1.1159
68	1.1184	1.1208	1.1232	1.1256	1.1280	1.1304
69	1.1328	1.1352	1.1376	1.1400	1.1424	1.1448
70	1.1471	1.1495	1.1519	1.1543	1.1567	1.1590
71	1.1614	1.1638	1.1661	1.1685	1.1708	1.1732
72	1.1756	1.1780	1.1803	1.1826	1.1850	1.1873
73	1.1896	1.1920	1.1943	1.1966	1.1990	1.2013
74	1.2036	1.2059	1.2083	1.2106	1.2129	1.2152
75	1.2175	1.2198	1.2221	1.2244	1.2267	1.2290
76	1.2313	1.2336	1.2360	1.2382	1.2405	1.2427
77	1.2450	1.2473	1.2496	1.2518	1.2541	1.2564
78	1.2586	1.2609	1.2631	1.2654	1.2677	1.2699
79	1.2721	1.2744	1.2766	1.2789	1.2811	1.2833
80	1.2856	1.2878	1.2900	1.2922	1.2945	1.2967
81	1.2989	1.3011	1.3033	1.3055	1.3077	1.3099
82	1.3121	1.3143	1.3165	1.3187	1.3209	1.3231
83	1.3252	1.3274	1.3296	1.3318	1.3340	1.3361
84	1.3383	1.3404	1.3426	1.3447	1.3469	1.3490
85	1.3512	1.3533	1.3555	1.3576	1.3597	1.3619
86	1.3640	1.3661	1.3682	1.3704	1.3725	1.3746
87	1.3767	1.3788	1.3809	1.3830	1.3851	1.3872
88	1.3893	1.3914	1.3935	1.3956	1.3977	1.3997
89	1.4018	1.4039	1.4060	1.4080	1.4101	1.4121
90	1.4142					

APPENDIX IV

LOGARITHMS OF NUMBERS

Four-place Mantissas

N	0	1	2	3	4	5	6	7	8	9	Proportional Parts								
											1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	16	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	4	5	7	9	10	12	14	16
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	12	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	5	7	8	10	11	12
32	5052	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	7	8	9	11	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	2	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	4	5	6	7	8	9	11
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9	Proportional Parts								
											1	2	3	4	5	6	7	8	9
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	6	7	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	5	6	7	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	3	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	1	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	3	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	5	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	3	3	4	4	5	6
N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9	Proportional Parts								
											1	2	3	4	5	6	7	8	9
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	3	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	3	3	4	4	5	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	3	3	4	4	5	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	4	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	1	1	2	2	3	3	4	4	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	3	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
N	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

APPENDIX V

NATURAL TRIGONOMETRIC FUNCTIONS

Angle	sine	log sin	cos	log cos	tan	log tan			Angle
0° 00′	.0000		1.0000	0.0000	.0000				90° 00′
10′	.0029	7.4637	1.0000	0000	.0029	7.4637	343.8	2.5363	50′
20′	.0058	7648	1.0000	0000	.0058	7648	171.9	2352	40′
30′	.0087	9408	1.0000	0000	.0087	9409	114.6	0591	30′
40′	.0116	8.0658	.9999	0000	.0116	8.0658	85.94	1.9342	20′
50′	.0145	1627	.9999	0000	.0145	1627	68.75	8373	10′
1° 00′	.0175	8.2419	.9998	9.9999	.0175	8.2419	57.29	1.7581	89° 00′
10′	.0204	3088	.9998	9999	.0204	3089	49.10	6911	50′
20′	.0233	3668	.9997	9999	.0233	3669	42.96	6331	40′
30′	.0262	4179	.9997	9999	.0262	4181	38.19	5819	30′
40′	.0291	4637	.9996	9998	.0291	4638	34.37	5362	20′
50′	.0320	5050	.9995	9998	.0320	5053	31.24	4947	10′
2° 00′	.0349	8.5428	.9994	9.9997	.0349	8.5431	28.64	1.4569	88° 00′
10′	.0378	5776	.9993	9997	.0378	5779	26.43	4221	50′
20′	.0407	6097	.9992	9996	.0407	6101	24.54	3899	40′
30′	.0436	6397	.9990	9996	.0437	6401	22.90	3599	30′
40′	.0465	6677	.9989	9995	.0466	6682	21.47	3318	20′
50′	.0494	6940	.9988	9995	.0495	6945	20.21	3055	10′
3° 00′	.0523	8.7188	.9986	9.9994	.0524	8.7194	19.08	1.2806	87° 00′
10′	.0552	7423	.9985	9993	.0553	7429	18.08	2571	50′
20′	.0581	7645	.9983	9993	.0582	7652	17.17	2348	40′
30′	.0610	7857	.9981	9992	.0612	7865	16.35	2135	30′
40′	.0640	8059	.9980	9991	.0641	8067	15.60	1933	20′
50′	.0669	8251	.9978	9990	.0670	8261	14.92	1739	10′
4° 00′	.0698	8.8436	.9976	9.9989	.0699	8.8446	14.30	1.1554	86° 00′
10′	.0727	8613	.9974	9989	.0729	8624	13.73	1376	50′
20′	.0756	8783	.9971	9988	.0758	8795	13.20	1205	40′
30′	.0785	8946	.9969	9987	.0787	8960	12.71	1040	30′
40′	.0814	9104	.9967	9986	.0816	9118	12.25	0882	20′
50′	.0843	9256	.9964	9985	.0846	9272	11.83	0728	10′
5° 00′	.0872	8.9403	.9962	9.9983	.0875	8.9420	11.43	1.0580	85° 00′
10′	.0901	9545	.9959	9982	.0904	9563	11.06	0437	50′
20′	.0929	9682	.9957	9981	.0934	9701	10.71	0299	40′
30′	.0958	9816	.9954	9980	.0963	9836	10.38	0164	30′
40′	.0987	9945	.9951	9979	.0992	9966	10.08	0034	20′
50′	.1016	9.0070	.9948	9977	.1022	9.0093	9.788	0.9907	10′
6° 00′	.1045	9.0192	.9945	9.9976	.1051	9.0216	9.514	0.9784	84° 00′
10′	.1074	0311	.9942	9975	.1080	0336	9.255	9664	50′
20′	.1103	0426	.9939	9973	.1110	0453	9.010	9547	40′
30′	.1132	0539	.9936	9972	.1139	0567	8.777	9433	30′
40′	.1161	0648	.9932	9971	.1169	0678	8.556	9322	20′
50′	.1190	0755	.9929	9969	.1198	0786	8.345	9214	10′
7° 00′	.1219	9.0859	.9925	9.9968	.1228	9.0891	8.144	0.9109	83° 00′
10′	.1248	0961	.9922	9966	.1257	0995	7.953	9005	50′
20′	.1276	1060	.9918	9964	.1287	1096	7.770	8904	40′
30′	.1305	1157	.9914	9963	.1317	1194	7.596	8806	30′
40′	.1334	1252	.9911	9961	.1346	1291	7.429	8709	20′
50′	.1363	1345	.9907	9959	.1376	1385	7.269	8615	10′
8° 00′	.1392	9.1436	.9903	9.9958	.1405	9.1478	7.115	0.8522	82° 00′
10′	.1421	1525	.9899	9956	.1435	1569	6.968	8431	50′
20′	.1449	1612	.9894	9954	.1465	1658	6.827	8342	40′
30′	.1478	1697	.9890	9952	.1495	1745	6.691	8255	30′
40′	.1507	1781	.9886	9950	.1524	1831	6.561	8169	20′
50′	.1536	1863	.9881	9948	.1554	1915	6.435	8085	10′
9° 00′	.1564	9.1943	.9877	9.9946	.1584	9.1997	6.314	0.8003	81° 00′
Angle	cos	log cos	sine	log sin			tan	log tan	Angle

NATURAL TRIGONOMETRIC FUNCTIONS

Angle	sine	log sin	cos	log cos	tan	log tan			Angle
9° 00′	.1564	9.1943	.9877	9.9946	.1584	9.1997	6.314	0.8003	81° 00′
10′	.1593	2022	.9872	9944	.1614	2078	6.197	7922	50′
20′	.1622	2100	.9868	9942	.1644	2158	6.084	7842	40′
30′	.1650	2176	.9863	9940	.1673	2236	5.976	7764	30′
40′	.1679	2251	.9858	9938	.1703	2313	5.871	7687	20′
50′	.1708	2324	.9853	9936	.1733	2389	5.769	7611	10′
10° 00′	.1736	9.2397	.9848	9.9934	.1763	9.2463	5.671	0.7537	80° 00′
10′	.1765	2468	.9843	9931	.1793	2536	5.576	7464	50′
20′	.1794	2538	.9838	9929	.1823	2609	5.484	7391	40′
30′	.1822	2606	.9833	9927	.1853	2680	5.396	7320	30′
40′	.1851	2674	.9827	9924	.1883	2750	5.309	7250	20′
50′	.1880	2740	.9822	9922	.1914	2819	5.226	7181	10′
11° 00′	.1908	9.2806	.9816	9.9919	.1974	9.2887	5.145	0.7113	79° 00′
10′	.1937	2870	.9811	9917	.1974	2953	5.066	7047	50′
20′	.1965	2934	.9805	9914	.2004	3020	4.989	6980	40′
30′	.1994	2997	.9799	9912	.2035	3085	4.915	6915	30′
40′	.2022	3058	.9793	9909	.2065	3149	4.843	6851	20′
50′	.2051	3119	.9787	9907	.2095	3212	4.773	6788	10′
12° 00′	.2079	9.3179	.9781	9.9904	.2126	9.3275	4.705	0.6725	78° 00′
10′	.2108	3238	.9775	9901	.2156	3336	4.638	6664	50′
20′	.2136	3296	.9769	9899	.2186	3397	4.574	6603	40′
30′	.2164	3353	.9763	9896	.2217	3458	4.511	6542	30′
40′	.2193	3410	.9757	9893	.2247	3517	4.449	6483	20′
50′	.2221	3466	.9750	9890	.2278	3576	4.390	6424	10′
13° 00′	.2250	9.3521	.9744	9.9887	.2309	9.3634	4.332	0.6366	77° 00′
10′	.2278	3575	.9737	9884	.2339	3691	4.275	6309	50′
20′	.2306	3629	.9730	9881	.2370	3748	4.219	6252	40′
30′	.2334	3682	.9724	9878	.2401	3804	4.165	6196	30′
40′	.2363	3734	.9717	9875	.2432	3859	4.113	6141	20′
50′	.2391	3786	.9710	9872	.2462	3914	4.061	6086	10′
14° 00′	.2419	9.3837	.9703	9.9869	.2493	9.3968	4.011	0.6032	76° 00′
10′	.2447	3887	.9696	9866	.2524	4021	3.962	5979	50′
20′	.2476	3937	.9689	9863	.2555	4074	3.914	5926	40′
30′	.2504	3986	.9681	9859	.2586	4127	3.867	5873	30′
40′	.2532	4035	.9674	9856	.2617	4178	3.821	5822	20′
50′	.2560	4083	.9667	9853	.2648	4230	3.776	5770	10′
15° 00′	.2588	9.4130	.9659	9.9849	.2679	9.4281	3.732	0.5719	75° 00′
10′	.2616	4177	.9652	9846	.2711	4331	3.689	5669	50′
20′	.2644	4223	.9644	9843	.2742	4381	3.647	5619	40′
30′	.2672	4269	.9636	9839	.2773	4430	3.606	5570	30′
40′	.2700	4314	.9628	9836	.2805	4479	3.566	5521	20′
50′	.2728	4359	.9621	9832	.2836	4527	3.526	5473	10′
16° 00′	.2756	9.4403	.9613	9.9828	.2867	9.4575	3.487	0.5425	74° 00′
10′	.2784	4447	.9605	9825	.2899	4622	3.450	5378	50′
20′	.2812	4491	.9596	9821	.2931	4669	3.412	5331	40′
30′	.2840	4533	.9588	9817	.2962	4716	3.376	5284	30′
40′	.2868	4576	.9580	9814	.2994	4762	3.340	5238	20′
50′	.2896	4618	.9572	9810	.3026	4808	3.305	5192	10′
17° 00′	.2924	9.4659	.9563	9.9806	.3057	9.4853	3.271	0.5147	73° 00′
10′	.2952	4700	.9555	9802	.3089	4898	3.237	5102	50′
20′	.2979	4741	.9546	9798	.3121	4943	3.204	5057	40′
30′	.3007	4781	.9537	9794	.3153	4987	3.172	5013	30′
40′	.3035	4821	.9528	9790	.3185	5031	3.140	4969	20′
50′	.3062	4861	.9520	9786	.3217	5075	3.108	4925	10′
18° 00′	.3090	9.4900	.9511	9.9782	.3249	9.5118	3.078	0.4882	72° 00′
Angle	cos	log cos	sine	log sin			tan	log tan	Angle

NATURAL TRIGONOMETRIC FUNCTIONS

Angle	sine	log sin	cos	log cos	tan	log tan			Angle
18° 00′	.3090	9.4900	.9511	9.9782	.3249	9.5118	3.078	0.4882	**72°** 00′
10′	.3118	4939	.9502	9778	.3281	5161	3.048	4839	50′
20′	.3145	4977	.9492	9774	.3314	5203	3.018	4797	40′
30′	.3173	5015	.9483	9770	.3346	5245	2.989	4755	30′
40′	.3201	5052	.9474	9765	.3378	5287	2.960	4713	20′
50′	.3228	5090	.9465	9761	.3411	5329	2.932	4671	10′
19° 00′	.3256	9.5126	.9455	9.9757	.3443	9.5370	2.904	0.4630	**71°** 00′
10′	.3283	5163	.9446	9752	.3476	5411	2.877	4589	50′
20′	.3311	5199	.9436	9748	.3508	5451	2.850	4549	40′
30′	.3338	5235	.9426	9743	.3541	5491	2.824	4509	30′
40′	.3365	5270	.9417	9739	.3574	5531	2.798	4469	20′
50′	.3393	5306	.9407	9734	.3607	5571	2.772	4429	10′
20° 00′	.3420	9.5341	.9397	9.9730	.3640	9.5611	2.748	0.4389	**70°** 00′
10′	.3448	5375	.9387	9725	.3673	5650	2.723	4350	50′
20′	.3475	5409	.9377	9721	.3706	5689	2.698	4311	40′
30′	.3502	5443	.9367	9716	.3739	5727	2.675	4273	30′
40′	.3529	5477	.9356	9711	.3772	5766	2.651	4234	20′
50′	.3557	5510	.9346	9706	.3805	5804	2.628	4196	10′
21° 00′	.3584	9.5543	.9336	9.9702	.3839	9.5842	2.605	0.4158	**69°** 00′
10′	.3611	5576	.9325	9697	.3872	5879	2.583	4121	50′
20′	.3638	5609	.9315	9692	.3906	5917	2.560	4083	40′
30′	.3665	5641	.9304	9687	.3939	5954	2.539	4046	30′
40′	.3692	5673	.9293	9682	.3973	5991	2.517	4009	20′
50′	.3719	5704	.9283	9677	.4006	6028	2.496	3972	10′
22° 00′	.3746	9.5736	.9272	9.9672	.4040	9.6064	2.475	0.3936	**68°** 00′
10′	.3773	5767	.9261	9667	.4074	6100	2.454	3900	50′
20′	.3800	5798	.9250	9661	.4108	6136	2.434	3864	40′
30′	.3827	5828	.9239	9656	.4142	6172	2.414	3828	30′
40′	.3854	5859	.9228	9651	.4176	6208	2.394	3792	20′
50′	.3881	5889	.9216	9646	.4210	6243	2.375	3757	10′
23° 00′	.3907	9.5919	.9205	9.9640	.4245	9.6279	2.356	0.3721	**67°** 00′
10′	.3934	5948	.9194	9635	.4279	6314	2.337	3686	50′
20′	.3961	5978	.9182	9629	.4314	6348	2.318	3652	40′
30′	.3987	6007	.9171	9624	.4348	6383	2.300	3617	30′
40′	.4014	6036	.9159	9618	.4383	6417	2.282	3583	20′
50′	.4041	6065	.9147	9613	.4417	6452	2.264	3548	10′
24° 00′	.4067	9.6093	.9135	9.9607	.4452	9.6486	2.246	0.3514	**66°** 00′
10′	.4094	6121	.9124	9602	.4487	6520	2.229	3480	50′
20′	.4120	6149	.9112	9596	.4522	6553	2.211	3447	40′
30′	.4147	6177	.9100	9590	.4557	6587	2.194	3413	30′
40′	.4173	6205	.9088	9584	.4592	6620	2.178	3380	20′
50′	.4200	6232	.9075	9579	.4628	6654	2.161	3346	10′
25° 00′	.4226	9.6259	.9063	9.9573	.4663	9.6687	2.144	0.3313	**65°** 00′
10′	.4253	6286	.9051	9567	.4699	6720	2.128	3280	50′
20′	.4279	6313	.9038	9561	.4734	6752	2.112	3248	40′
30′	.4305	6340	.9026	9555	.4770	6785	2.096	3215	30′
40′	.4331	6366	.9013	9549	.4806	6817	2.081	3183	20′
50′	.4358	6392	.9001	9543	.4841	6850	2.066	3150	10′
26° 00′	.4384	9.6418	.8988	9.9537	.4877	9.6882	2.050	0.3118	**64°** 00′
10′	.4410	6444	.8975	9530	.4913	6914	2.035	3086	50′
20′	.4436	6470	.8962	9524	.4950	6946	2.020	3054	40′
30′	.4462	6495	.8949	9518	.4986	6977	2.006	3023	30′
40′	.4488	6521	.8936	9512	.5022	7009	1.991	2991	20′
50′	.4514	6546	.8923	9505	.5059	7040	1.977	2960	10′
27° 00′	.4540	9.6570	.8910	9.9499	.5095	9.7072	1.963	0.2928	**63°** 00′
Angle	cos	log cos	sine	log sin			tan	log tan	Angle

NATURAL TRIGONOMETRIC FUNCTIONS

Angle	sine	log sin	cos	log cos	tan	log tan			Angle
27° 00′	.4540	9.6570	.8910	9.9499	.5095	9.7072	1.963	0.2928	**63°** 00′
10′	.4566	6595	.8897	9492	.5132	7103	1.949	2897	50′
20′	.4592	6620	.8884	9486	.5169	7134	1.935	2866	40′
30′	.4617	6644	.8870	9479	.5206	7165	1.921	2835	30′
40′	.4643	6668	.8857	9473	.5243	7196	1.907	2804	20′
50′	.4669	6692	.8843	9466	.5280	7226	1.894	2774	10′
28° 00′	.4695	9.6716	.8829	9.9459	.5317	9.7257	1.881	0.2743	**62°** 00′
10′	.4720	6740	.8816	9453	.5354	7287	1.868	2713	50′
20′	.4746	6763	.8802	9446	.5392	7317	1.855	2683	40′
30′	.4772	6787	.8788	9439	.5430	7348	1.842	2652	30′
40′	.4797	6810	.8774	9432	.5467	7378	1.829	2622	20′
50′	.4823	6833	.8760	9425	.5505	7408	1.816	2592	10′
29° 00′	.4848	9.6856	.8746	9.9418	.5543	9.7438	1.804	0.2562	**61°** 00′
10′	.4874	6878	.8732	9411	.5581	7467	1.792	2533	50′
20′	.4899	6901	.8718	9404	.5619	7497	1.780	2503	40′
30′	.4924	6923	.8704	9397	.5658	7526	1.768	2474	30′
40′	.4950	6946	.8689	9390	.5696	7556	1.756	2444	20′
50′	.4975	6968	.8675	9383	.5735	7585	1.744	2415	10′
30° 00′	.5000	9.6990	.8660	9.9375	.5774	9.7614	1.732	0.2386	**60°** 00′
10′	.5025	7012	.8646	9368	.5812	7644	1.720	2356	50′
20′	.5050	7033	.8631	9361	.5851	7673	1.709	2327	40′
30′	.5075	7055	.8616	9353	.5890	7701	1.698	2299	30′
40′	.5100	7076	.8601	9346	.5930	7730	1.686	2270	20′
50′	.5125	7097	.8587	9338	.5969	7759	1.675	2241	10′
31° 00′	.5150	9.7118	.8572	9.9331	.6009	9.7788	1.664	0.2212	**59°** 00′
10′	.5175	7139	.8557	9323	.6048	7816	1.653	2184	50′
20′	.5200	7160	.8542	9315	.6088	7845	1.643	2155	40′
30′	.5225	7181	.8526	9308	.6128	7873	1.632	2127	30′
40′	.5250	7201	.8511	9300	.6168	7902	1.621	2098	20′
50′	.5275	7222	.8496	9292	.6208	7930	1.611	2070	10′
32° 00′	.5299	9.7242	.8480	9.9284	.6249	9.7958	1.600	0.2042	**58°** 00′
10′	.5324	7262	.8465	9276	.6289	7986	1.590	2014	50′
20′	.5348	7282	.8450	9268	.6330	8014	1.580	1986	40′
30′	.5373	7302	.8434	9260	.6371	8042	1.570	1958	30′
40′	.5398	7322	.8418	9252	.6412	8070	1.560	1930	20′
50′	.5422	7342	.8403	9244	.6453	8097	1.550	1903	10′
33° 00′	.5446	9.7361	.8387	9.9236	.6494	9.8125	1.540	0.1875	**57°** 00′
10′	.5471	7380	.8371	9228	.6536	8153	1.530	1847	50′
20′	.5495	7400	.8355	9219	.6577	8180	1.520	1820	40′
30′	.5519	7419	.8339	9211	.6619	8208	1.511	1792	30′
40′	.5544	7438	.8323	9203	.6661	8235	1.501	1765	20′
50′	.5568	7457	.8307	9194	.6703	8263	1.492	1737	10′
34° 00′	.5592	9.7476	.8290	9.9186	.6745	9.8290	1.483	0.1710	**56°** 00′
10′	.5616	7494	.8274	9177	.6787	8317	1.473	1683	50′
20′	.5640	7513	.8258	9169	.6830	8344	1.464	1656	40′
30′	.5664	7531	.8241	9160	.6873	8371	1.455	1629	30′
40′	.5688	7550	.8225	9151	.6916	8398	1.446	1602	20′
50′	.5712	7568	.8208	9142	.6959	8425	1.437	1575	10′
35° 00′	.5736	9.7586	.8192	9.9134	.7002	9.8452	1.428	0.1548	**55°** 00′
10′	.5760	7604	.8175	9125	.7046	8479	1.419	1521	50′
20′	.5783	7622	.8158	9116	.7089	8506	1.411	1494	40′
30′	.5807	7640	.8141	9107	.7133	8533	1.402	1467	30′
40′	.5831	7657	.8124	9098	.7177	8559	1.393	1441	20′
50′	.5854	7675	.8107	9089	.7221	8586	1.385	1414	10′
36° 00′	.5878	9.7692	.8090	9.9080	.7265	9.8613	1.376	0.1387	**54°** 00′
Angle	cos	log cos	sine	log sin			tan	log tan	Angle

NATURAL TRIGONOMETRIC FUNCTIONS

Angle	sine	log sin	cos	log cos	tan	log tan			Angle
36° 00′	.5878	9.7692	.8090	9.9080	.7265	9.8613	1.376	0.1387	**54°** 00′
10′	.5901	7710	.8073	9070	.7310	8639	1.368	1361	50′
20′	.5925	7727	.8056	9061	.7355	8666	1.360	1334	40′
30′	.5948	7744	.8039	9052	.7400	8692	1.351	1308	30′
40′	.5972	7761	.8021	9042	.7445	8718	1.343	1282	20′
50′	.5995	7778	.8004	9033	.7490	8745	1.335	1255	10′
37° 00′	.6018	9.7795	.7986	9.9023	.7536	9.8771	1.327	0.1229	**53°** 00′
10′	.6041	7811	.7969	9014	.7581	8797	1.319	1203	50′
20′	.6065	7828	.7951	9004	.7627	8824	1.311	1176	40′
30′	.6088	7844	.7934	8995	.7673	8850	1.303	1150	30′
40′	.6111	7861	.7916	8985	.7720	8876	1.295	1124	20′
50′	.6134	7877	.7898	8975	.7766	8902	1.288	1098	10′
38° 00′	.6157	9.7893	.7880	9.8965	.7813	9.8928	1.280	0.1072	**52°** 00′
10′	.6180	7910	.7862	8955	.7860	8954	1.272	1046	50′
20′	.6202	7926	.7844	8945	.7907	8980	1.265	1020	40′
30′	.6225	7941	.7826	8935	.7954	9006	1.257	0994	30′
40′	.6248	7957	.7808	8925	.8002	9032	1.250	0968	20′
50′	.6271	7973	.7790	8915	.8050	9058	1.242	0942	10′
39° 00′	.6293	9.7989	.7771	9.8905	.8098	9.9084	1.235	0.0916	**51°** 00′
10′	.6316	8004	.7753	8895	.8146	9110	1.228	0890	50′
20′	.6338	8020	.7735	8884	.8195	9135	1.220	0865	40′
30′	.6361	8035	.7716	8874	.8243	9161	1.213	0839	30′
40′	.6383	8050	.7698	8864	.8292	9187	1.206	0813	20′
50′	.6406	8066	.7679	8853	.8342	9212	1.199	0788	10′
40° 00′	.6428	9.8081	.7660	9.8843	.8391	9.9238	1.192	0.0762	**50°** 00′
10′	.6450	8096	.7642	8832	.8441	9264	1.185	0736	50′
20′	.6472	8111	.7623	8821	.8491	9289	1.178	0711	40′
30′	.6494	8125	.7604	8810	.8541	9315	1.171	0685	30′
40′	.6517	8140	.7585	8800	.8591	9341	1.164	0659	20′
50′	.6539	8155	.7566	8789	.8642	9366	1.157	0634	10′
41° 00′	.6561	9.8169	.7547	9.8778	.8693	9.9392	1.150	0.0608	**49°** 00′
10′	.6583	8184	.7528	8767	.8744	9417	1.144	0583	50′
20′	.6604	8198	.7509	8756	.8796	9443	1.137	0557	40′
30′	.6626	8213	.7490	8745	.8847	9468	1.130	0532	30′
40′	.6648	8227	.7470	8733	.8899	9494	1.124	0506	20′
50′	.6670	8241	.7451	8722	.8952	9519	1.117	0481	10′
42° 00′	.6691	9.8255	.7431	9.8711	.9004	9.9544	1.111	0.0456	**48°** 00′
10′	.6713	8269	.7412	8699	.9057	9570	1.104	0430	50′
20′	.6734	8283	.7392	8688	.9110	9595	1.098	0405	40′
30′	.6756	8297	.7373	8676	.9163	9621	1.091	0379	30′
40′	.6777	8311	.7353	8665	.9217	9646	1.085	0354	20′
50′	.6799	8324	.7333	8653	.9271	9671	1.079	0329	10′
43° 00′	.6820	9.8338	.7314	9.8641	.9325	9.9697	1.072	0.0303	**47°** 00′
10′	.6841	8351	.7294	8629	.9380	9722	1.066	0278	50′
20′	.6862	8365	.7274	8618	.9435	9747	1.060	0253	40′
30′	.6884	8378	.7254	8606	.9490	9772	1.054	0228	30′
40′	.6905	8391	.7234	8594	.9545	9798	1.048	0202	20′
50′	.6926	8405	.7214	8582	.9601	9823	1.042	0177	10′
44° 00′	.6947	9.8418	.7193	9.8569	.9657	9.9848	1.036	0.0152	**46°** 00′
10′	.6967	8431	.7173	8557	.9713	9874	1.030	0126	50′
20′	.6988	8444	.7153	8545	.9770	9899	1.024	0101	40′
30′	.7009	8457	.7133	8532	.9827	9924	1.018	0076	30′
40′	.7030	8469	.7112	8520	.9884	9949	1.012	0051	20′
50′	.7050	8482	.7092	8507	.9942	9975	1.006	0025	10′
45° 00′	.7071	9.8495	.7071	9.8495	1.0000	0.0000	1.000	0.0000	**45°** 00′
Angle	cos	log cos	sine	log sin			tan	log tan	Angle

APPENDIX VI

EXPONENTIALS

e^n and e^{-n}

n	e^n	Diff.	n	e^n	Diff.	n	e^n	n	e^{-n}	Diff.	n	e^{-n}	n	e^{-n}
0.00	1.000	10	0.50	1.649	16	1.0	2.718*	0.00	1.000	−10	0.50	.607	1.0	.368*
.01	1.010	10	.51	1.665	17	.1	3.004	.01	0.990	−10	.51	.600	.1	.333
.02	1.020	10	.52	1.682	17	.2	3.320	.02	.980	−10	.52	.595	.2	.301
.03	1.030	11	.53	1.699	17	.3	3.669	.03	.970	−9	.53	.589	.3	.273
.04	1.041	10	.54	1.716	17	.4	4.055	.04	.961	−10	.54	.583	.4	.247
0.05	1.051	11	0.55	1.733	18	1.5	4.482	0.05	.951	−9	0.55	.577	1.5	.223
.06	1.062	11	.56	1.751	18	.6	4.953	.06	.942	−10	.56	.571	.6	.202
.07	1.073	10	.57	1.768	17	.7	5.474	.07	.932	−9	.57	.566	.7	.183
.08	1.083	11	.58	1.786	18	.8	6.050	.08	.923	−9	.58	.560	.8	.165
.09	1.094	11	.59	1.804	18	.9	6.686	.09	.914	−9	.59	.554	.9	.150
0.10	1.105	11	0.60	1.822	18	2.0	7.389	0.10	.905	−9	0.60	.549	2.0	.135
.11	1.116	11	.61	1.840	19	.1	8.166	.11	.896	−9	.61	.543	.1	.122
.12	1.127	12	.62	1.859	19	.2	9.025	.12	.887	−9	.62	.538	.2	.111
.13	1.139	11	.63	1.878	18	.3	9.974	.13	.878	−9	.63	.533	.3	.100
.14	1.150	12	.64	1.896	20	.4	11.02	.14	.869	−8	.64	.527	.4	.0907
0.15	1.162	12	0.65	1.916	19	2.5	12.18	0.15	.861	−9	0.65	.522	2.5	.0821
.16	1.174	11	.66	1.935	19	.6	13.46	.16	.852	−8	.66	.517	.6	.0743
.17	1.185	12	.67	1.954	20	.7	14.88	.17	.844	−9	.67	.512	.7	.0672
.18	1.197	12	.68	1.974	20	.8	16.44	.18	.835	−8	.68	.507	.8	.0608
.19	1.209	12	.69	1.994	20	.9	18.17	.19	.827	−8	.69	.502	.9	.0550
0.20	1.221	13	0.70	2.014	20	3.0	20.09	0.20	.819	−8	0.70	.497	3.0	.0498
.21	1.234	12	.71	2.034	20	.1	22.20	.21	.811	−8	.71	.492	.1	.0450
.22	1.246	13	.72	2.054	21	.2	24.53	.22	.803	−8	.72	.487	.2	.0408
.23	1.259	12	.73	2.075	21	.3	27.11	.23	.795	−8	.73	.482	.3	.0369
.24	1.271	13	.74	2.096	21	.4	29.96	.24	.787	−8	.74	.477	.4	.0334
0.25	1.284	13	0.75	2.117	21	3.5	33.12	0.25	.779	−8	0.75	.472	3.5	.0302
.26	1.297	13	.76	2.138	22	.6	36.60	.26	.771	−8	.76	.468	.6	.0273
.27	1.310	13	.77	2.160	21	.7	40.45	.27	.763	−7	.77	.463	.7	.0247
.28	1.323	13	.78	2.181	22	.8	44.70	.28	.756	−8	.78	.458	.8	.0224
.29	1.336	14	.79	2.203	23	.9	49.40	.29	.748	−7	.79	.454	.9	.0202
0.30	1.350	13	0.80	2.226	22	4.0	54.60	0.30	.741	−8	0.80	.449	4.0	.0183
.31	1.363	14	.81	2.248	22	.1	60.34	.31	.733	−7	.81	.445	.1	.0166
.32	1.377	14	.82	2.270	23	.2	66.69	.32	.726	−7	.82	.440	.2	.0150
.33	1.391	14	.83	2.293	23	.3	73.70	.33	.719	−7	.83	.436	.3	.0136
.34	1.405	14	.84	2.316	24	.4	81.45	.34	.712	−7	.84	.432	.4	.0123
0.35	1.419	14	0.85	2.340	23	4.5	90.02	0.35	.705	−7	0.85	.427	4.5	.0111
.36	1.433	15	.86	2.363	24			.36	.698	−7	.86	.423		
.37	1.448	14	.87	2.387	24	5.0	148.4	.37	.691	−7	.87	.419	5.0	.00674
.38	1.462	15	.88	2.411	24	6.0	403.4	.38	.684	−7	.88	.415	6.0	.00248
.39	1.477	15	.89	2.435	25	7.0	1097.	.39	.677	−7	.89	.411	7.0	.000912
0.40	1.492	15	0.90	2.460	24	8.0	2981.	0.40	.670	−6	0.90	.407	8.0	.000335
.41	1.507	15	.91	2.484	25	9.0	8103.	.41	.664	−7	.91	.403	9.0	.000123
.42	1.522	15	.92	2.509	26	10.0	22026.	.42	.657	−6	.92	.399	10.0	.000045
.43	1.537	16	.93	2.535	25	$\pi/2$	4.810	.43	.651	−7	.93	.395	$\pi/2$	.208
.44	1.553	15	.94	2.560	26	$2\pi/2$	23.14	.44	.644	−6	.94	.391	$2\pi/2$	.0432
0.45	1.568	16	0.95	2.586	26	$3\pi/2$	111.3	0.45	.638	−7	0.95	.387	$3\pi/2$	.00898
.46	1.584	16	.96	2.612	26	$4\pi/2$	535.5	.46	.631	−6	.96	.383	$4\pi/2$	.00187
.47	1.600	16	.97	2.638	26	$5\pi/2$	2576.	.47	.625	−6	.97	.379	$5\pi/2$	.000388
.48	1.616	16	.98	2.664	27	$6\pi/2$	12392.	.48	.619	−6	.98	.375	$6\pi/2$	.000081
.49	1.632	17	.99	2.691	27	$7\pi/2$	59610.	.49	.613	−6	.99	.372	$7\pi/2$	.000017
0.50	1.649		1.00	2.718		$8\pi/2$	286751.	0.50	0.607		1.00	.368	$8\pi/2$	.000003

* *Note:* Do not interpolate in this column.

$e = 2.71828$ $\quad 1/e = 0.367879$ $\quad \log_{10} e = 0.4343$ $\quad 1/(0.4343) = 2.3026$

$\log_{10} 10(0.4343) = 0.6378$ $\quad \log_{10} e^n = n(0.4343)$

HYPERBOLIC LOGARITHMS

These two pages give the natural or Napierian logarithms (ln) of numbers between 1 and 10, correct to four places. Moving the decimal point n places to the right [or left] in the number is equivalent to adding n times 2.3026 [or n times 3.6974] to the logarithm. Base $e = 2.71828+$

n	n (2.3026)	n (0.6974–3)
1	2.3026	0.6974–3
2	4.6052	0.3948–5
3	6.9078	0.0922–7
4	9.2103	0.7897–10
5	11.5129	0.4871–12
6	13.8155	0.1845–14
7	16.1181	0.8819–17
8	18.4207	0.5793–19
9	20.7233	0.2767–21

Number.	0	1	2	3	4	5	6	7	8	9	Avg. dif.
1.0	0.0000	0100	0198	0296	0392	0488	0583	0677	0770	0862	95
1.1	0953	1044	1133	1222	1310	1398	1484	1570	1655	1740	87
1.2	1823	1906	1989	2070	2151	2231	2311	2390	2469	2546	80
1.3	2624	2700	2776	2852	2927	3001	3075	3148	3221	3293	74
1.4	3365	3436	3507	3577	3646	3716	3784	3853	3920	3988	69
1.5	0.4055	4121	4187	4253	4318	4383	4447	4511	4574	4637	65
1.6	4700	4762	4824	4886	4947	5008	5068	5128	5188	5247	61
1.7	5306	5365	5423	5481	5539	5596	5653	5710	5766	5822	57
1.8	5878	5933	5988	6043	6098	6152	6206	6259	6313	6366	54
1.9	6419	6471	6523	6575	6627	6678	6729	6780	6831	6881	51
2.0	0.6931	6981	7031	7080	7129	7178	7227	7275	7324	7372	49
2.1	7419	7467	7514	7561	7608	7655	7701	7747	7793	7839	47
2.2	7885	7930	7975	8020	8065	8109	8154	8198	8242	8286	44
2.3	8329	8372	8416	8459	8502	8544	8587	8629	8671	8713	43
2.4	8755	8796	8838	8879	8920	8961	9002	9042	9083	9123	41
2.5	0.9163	9203	9243	9282	9322	9361	9400	9439	9478	9517	39
2.6	9555	9594	9632	9670	9708	9746	9783	9821	9858	9895	38
2.7	0.9933	9969	*0006	*0043	*0080	*0116	*0152	*0188	*0225	*0260	36
2.8	1.0296	0332	0367	0403	0438	0473	0508	0543	0578	0613	35
2.9	0647	0682	0716	0750	0784	0818	0852	0886	0919	0953	34
3.0	1.0986	1019	1053	1086	1119	1151	1184	1217	1249	1282	33
3.1	1314	1346	1378	1410	1442	1474	1506	1537	1569	1600	32
3.2	1632	1663	1694	1725	1756	1787	1817	1848	1878	1909	31
3.3	1939	1969	2000	2030	2060	2090	2119	2149	2179	2208	30
3.4	2238	2267	2296	2326	2355	2384	2413	2442	2470	2499	29
3.5	1.2528	2556	2585	2613	2641	2669	2698	2726	2754	2782	28
3.6	2809	2837	2865	2892	2920	2947	2975	3002	3029	3056	27
3.7	3083	3110	3137	3164	3191	3218	3244	3271	3297	3324	27
3.8	3350	3376	3403	3429	3455	3481	3507	3533	3558	3584	26
3.9	3610	3635	3661	3686	3712	3737	3762	3788	3813	3838	25
4.0	1.3863	3888	3913	3938	3962	3987	4012	4036	4061	4085	25
4.1	4110	4134	4159	4183	4207	4231	4255	4279	4303	4327	24
4.2	4351	4375	4398	4422	4446	4469	4493	4516	4540	4563	23
4.3	4586	4609	4633	4656	4679	4702	4725	4748	4770	4793	23
4.4	4816	4839	4861	4884	4907	4929	4951	4974	4996	5019	22
4.5	1.5041	5063	5085	5107	5129	5151	5173	5195	5217	5239	22
4.6	5261	5282	5304	5326	5347	5369	5390	5412	5433	5454	21
4.7	5476	5497	5518	5539	5560	5581	5602	5623	5644	5665	21
4.8	5686	5707	5728	5748	5769	5790	5810	5831	5851	5872	20
4.9	5892	5913	5933	5953	5974	5994	6014	6034	6054	6074	20

$\log_e x = 2.3026 \log_{10} x \qquad \log_{10} x = 0.4343 \log_e x$

where $2.3026 = \log_e 10$ and $0.4343 = \log_{10} e$

HYPERBOLIC LOGARITHMS

Number	0	1	2	3	4	5	6	7	8	9	Avg. diff.
5.0	1.6094	6114	6134	6154	6174	6194	6214	6233	6253	6273	20
5.1	6292	6312	6332	6351	6371	6390	6409	6429	6448	6467	19
5.2	6487	6506	6525	6544	6563	6582	6601	6620	6639	6658	19
5.3	6677	6696	6715	6734	6752	6771	6790	6808	6827	6845	18
5.4	6864	6882	6901	6919	6938	6956	6974	6993	7011	7029	18
5.5	1.7047	7066	7084	7102	7120	7138	7156	7174	7192	7210	18
5.6	7228	7246	7263	7281	7299	7317	7334	7352	7370	7387	18
5.7	7405	7422	7440	7457	7475	7492	7509	7527	7544	7561	17
5.8	7579	7596	7613	7630	7647	7664	7681	7699	7716	7733	17
5.9	7750	7766	7783	7800	7817	7834	7851	7867	7884	7901	17
6.0	1.7918	7934	7951	7967	7984	8001	8017	8034	8050	8066	16
6.1	8083	8099	8116	8132	8148	8165	8181	8197	8213	8229	16
6.2	8245	8262	8278	8294	8310	8326	8342	8358	8374	8390	16
6.3	8405	8421	8437	8453	8469	8485	8500	8516	8532	8547	16
6.4	8563	8579	8594	8610	8625	8641	8656	8672	8687	8703	15
6.5	1.8718	8733	8749	8764	8779	8795	8810	8825	8840	8856	15
6.6	8871	8886	8901	8916	8931	8946	8961	8976	8991	9006	15
6.7	9021	9036	9051	9066	9081	9095	9110	9125	9140	9155	15
6.8	9169	9184	9199	9213	9228	9242	9257	9272	9286	9301	15
6.9	9315	9330	9344	9359	9373	9387	9402	9416	9430	9445	14
7.0	1.9459	9473	9488	9502	9516	9530	9544	9559	9573	9587	14
7.1	9601	9615	9629	9643	9657	9671	9685	9699	9713	9727	14
7.2	9741	9755	9769	9782	9796	9810	9824	9838	9851	9865	14
7.3	1.9879	9892	9906	9920	9933	9947	9961	9974	9988	*0001	13
7.4	2.0015	0028	0042	0055	0069	0082	0096	0109	0122	0136	13
7.5	2.0149	0162	0176	0189	0202	0215	0229	0242	0255	0268	13
7.6	0281	0295	0308	0321	0334	0347	0360	0373	0386	0399	13
7.7	0412	0425	0438	0451	0464	0477	0490	0503	0516	0528	13
7.8	0541	0554	0567	0580	0592	0605	0618	0631	0643	0656	13
7.9	0669	0681	0694	0707	0719	0732	0744	0757	0769	0782	12
8.0	2.0794	0807	0819	0832	0844	0857	0869	0882	0894	0906	12
8.1	0919	0931	0943	0956	0968	0980	0992	1005	1017	1029	12
8.2	1041	1054	1066	1078	1090	1102	1114	1126	1138	1150	12
8.3	1163	1175	1187	1199	1211	1223	1235	1247	1258	1270	12
8.4	1282	1294	1306	1318	1330	1342	1353	1365	1377	1389	12
8.5	2.1401	1412	1424	1436	1448	1459	1471	1483	1494	1506	12
8.6	1518	1529	1541	1552	1564	1576	1587	1599	1610	1622	12
8.7	1633	1645	1656	1668	1679	1691	1702	1713	1725	1736	11
8.8	1748	1759	1770	1782	1793	1804	1815	1827	1838	1849	11
8.9	1861	1872	1883	1894	1905	1917	1928	1939	1950	1961	11
9.0	2.1972	1983	1994	2006	2017	2028	2039	2050	2061	2072	11
9.1	2083	2094	2105	2116	2127	2138	2148	2159	2170	2181	11
9.2	2192	2203	2214	2225	2235	2246	2257	2268	2279	2289	11
9.3	2300	2311	2322	2332	2343	2354	2364	2375	2386	2396	11
9.4	2407	2418	2428	2439	2450	2460	2471	2481	2492	2502	11
9.5	2.2513	2523	2534	2544	2555	2565	2576	2586	2597	2607	10
9.6	2618	2628	2638	2649	2659	2670	2680	2690	2701	2711	10
9.7	2721	2732	2742	2752	2762	2773	2783	2793	2803	2814	10
9.8	2824	2834	2844	2854	2865	2875	2885	2895	2905	2915	10
9.9	2925	2935	2946	2956	2966	2976	2986	2996	3006	3016	10
10.0	2.3026										

Moving the decimal point n places to the right (or left) in the number requires adding n times 2.3026 [or n times $(0.6974 - 3)$] in the body of the table. See auxiliary table of multiples at top of facing page.

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