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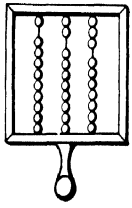
Author Spitzer, Herbert F.

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THE TEACHING OF ARITHMETIC

SECOND EDITION



HERBERT F. SPITZER

STATE UNIVERSITY OF IOWA

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Preface

The second edition of *The Teaching of Arithmetic* is, like the original edition, committed to the premise that arithmetic can best be learned when the pupil accepts a problem as his own and works to obtain the best solution. The instructional procedures suggested are an attempt to awaken in the child that spirit of inquiry and desire for mastery inherent in the best teaching and learning.

The author's experience with the first edition in a number of graduate classes and in the supervision of arithmetic teaching at the University Elementary School has contributed to a refinement and a substantial extension of the classroom procedures recommended in that volume. A special attempt has been made in this edition to give increased attention to the more advanced phases of the subject and to suggest methods of re-teaching the fundamentals which will interest the pupil in the upper grades. Every chapter of the first edition has been re-examined, rewritten, and, it is to be hoped, improved. A new chapter on instructional equipment includes a detailed plan for the study of textbooks.

During the preparation of this revision the author became indebted to many people, especially the children and teachers at the University Elementary School during the years 1949–1952. Grateful acknowledgement is made of the assistance of the following who taught in the school when much of the new material was being developed: Louise Beltramo, Frances Flournoy, Edwin H. Hill, Arthura Michael, Martha Norman, and Virginia Westerberg. The writer would also like to extend a special word of thanks to Katherine Hunt Swartz, who made many valuable suggestions for improving the presentation.

H. F. S.

Iowa City, Iowa

THE TEACHING OF ARITHMETIC

1

Arithmetic Teaching Today

The content of arithmetic, with the exception of some changes in grade placement, has remained rather stable for some time. The best or final ways of performing the fundamental processes with whole numbers and fractions as presented in the modern textbook differ very little from the ways taught twenty-five years ago. In the same twenty-five years, however, marked changes have taken place in the methods of teaching this arithmetic content to children. While many of the changes in methods have been only slight variations or revisions of older methods and do not warrant any special attention, some of the new techniques represent a fundamentally different approach to helping children understand and master arithmetic.

To introduce this new point of view to the reader, the school visit made by a sixth-grade girl's mother is described. In the classroom procedures Mrs. Handly saw and the discussions she had with the teacher basic differences between the new techniques and those commonly used will become apparent.

Mrs. Handly's interest in visiting her daughter's arithmetic class began when Jane mentioned at the dinner table one night that she wanted to rewrite her arithmetic composition.

"What is an arithmetic composition, anyway, Jane?" asked Mrs. Handly.

"Oh, it's writing down how we do different things in arithmetic. Today we wrote on 'How we learned to divide a whole number by a fraction.' I want to rewrite mine because the best compositions are put on the arithmetic bulletin board," was Jane's answer.

Mrs. Handly had never heard of an arithmetic bulletin board before either. Further questions only increased her curiosity to find out more about how Jane was being taught arithmetic, so she said, "May I visit your arithmetic class tomorrow?"

The teacher began the class Mrs. Handly saw by saying, "I have put some of your compositions about how you learned to divide a whole number by a fraction on the bulletin board. The people who rewrote their compositions improved them a lot. When you have time, take a look at the ones on the arithmetic table and on the bulletin board. Yesterday you wrote a composition to review what you had studied about dividing by fractions. Today let's finish our chapter review by answering the questions on this study sheet."

Review Study Questions

1. What are three ways of finding how many $\frac{2}{3}$ equal 6?
2. What is the common denominator for $7 \div \frac{1}{3}$?
3. When a whole number is divided by a fraction, is the quotient larger or smaller than the dividend?
4. When a fraction is divided by a whole number, is the quotient larger or smaller than the dividend?
5. (*a*) Can the quotient for $5 \div \frac{1}{2}$ be found by subtracting? (*b*) Can the quotient for $\frac{2}{3} \div 3$ be found by subtracting?
6. When you show how many $\frac{3}{4}$ equal 6 by a drawing, into what sized parts is the 6 marked off?
7. To solve $5\frac{1}{2} \div \frac{3}{4}$ is it necessary to find a common denominator?
8. In dividing by a mixed number as in $8 \div 2\frac{1}{2}$ what is the first thing to be done?

9. What fraction would be useful in comparing the area of Iceland (39,709 sq. mi.) and Haiti (10,204 sq. mi.)?

10. What are two ways of checking the product for such examples as $23 \times 36 = 828$?

11. How should you show that the decimal point in the product for the example $.03 \times .4 = .012$ is correctly placed?

12. In the example $\begin{array}{r} 703 \\ -148 \\ \hline \end{array}$ how do you borrow a 10?

Practice Exercises

(A.) $6 \div \frac{3}{8}$	B. $9\frac{1}{2} \div 4$	C. $3\frac{1}{2} \div \frac{2}{3}$
D. $\frac{3}{4} \div 7$	E. $\frac{1}{3} \div 2\frac{1}{2}$	(F.) $3 \div 5\frac{1}{4}$
G. $\frac{1}{2} \div \frac{3}{8}$	H. $\frac{2}{3} \div \frac{3}{4}$	I. $6\frac{3}{8} \div 3$
(J.) $5.1 \times .06$	K. $8 - 6.02$	(L.) 17×42
M. 32×106	N. $1901 - 287$	O. $1000 - 293$

For the next ten minutes the pupils worked on the review part of the study sheet. The teacher helped individual pupils interpret some of the questions or helped in working out the answers to specific parts. Several pupils were asked to show that their answers were correct and two of these ways of showing or proof were placed on the board. Some pupils at the suggestion of the teacher went to the arithmetic table and used objects to show that an answer to a question was correct.

Class discussion began when the teacher said, "Now let's compare answers and straighten out differences if there are any."

Mrs. Handly, who had been hard put to find even two ways of finding how many $\frac{2}{3}$ equal 6, was surprised to find the children suggesting four ways: (1) common denominator division, (2) inversion division, (3) drawing and counting, and (4) subtracting and counting. When one child expressed some doubt about using subtraction, another offered to show at the board how it could be done. A reproduction of this pupil's work appears below. (See A, on page 6.)

Whenever any pupil had an answer that was different from the one offered, the variation was given and discussion followed. Some variations were selected as equal to or better than the first answer given while others were rejected. Acceptance or rejection came most often after evidence had been put on the board. For example, the correct response to question 4 was chosen after a pupil had written $\frac{1}{2} \div 2$ on the board and then had drawn a half circle on the board which he divided into two parts. He explained, "The answer as you can see is $\frac{1}{4}$, and that's smaller than $\frac{1}{2}$." (See B.)

Reproduction of Pupil's Blackboard Work

A.

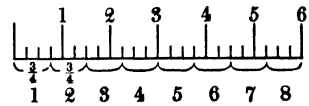
$$\begin{array}{rl} 6 & \\ -\ \frac{2}{3} & (1) \\ \hline 5\frac{1}{3} & \\ -\ \frac{2}{3} & (2) \\ \hline 4\frac{2}{3} & \\ -\ \frac{2}{3} & (3) \\ \hline 4 & \\ -\ \frac{2}{3} & (4) \\ \hline 3\frac{1}{3} & \\ -\ \frac{2}{3} & (5) \\ \hline 2\frac{2}{3} & \\ -\ \frac{2}{3} & (6) \\ \hline 2 & \\ -\ \frac{2}{3} & (7) \\ \hline 1\frac{1}{3} & \\ -\ \frac{2}{3} & (8) \\ \hline \frac{2}{3} & \\ -\ \frac{2}{3} & (9) \\ \hline \end{array}$$

B.

$\frac{1}{2} \div 2 = \frac{1}{4}$



C.



D.

$$\begin{array}{r} 36 \\ \times 11 \\ \hline 36 \\ 36 \\ \hline 396 \\ 396 \\ +36 \\ \hline 828 \end{array}$$

E.

$$\begin{array}{r} 23 \\ \times\ 6 \\ \hline 138 \\ \times\ 6 \\ \hline 828 \end{array}$$

F. $\frac{8}{100} \times \frac{4}{10} = \frac{12}{1000}$

When question 10 was discussed, one pupil offered the following method for finding the solution: "11 × 36; write your product again and add 36."

The teacher said, "That's a good way but not too many of you know how to use it. I asked Bill to show that it works. There is his work on the board." (See D.) After the pupils had looked at Bill's work, the teacher said, "Mary had a good check too. Here is her work. Is there anyone who does not see where Mary got the two 6's she used?" (See E.)

The answer given and accepted for question 11 was, "I would change the decimals to common fractions and multiply." (See F.)

The pupil who was asked to give her answer for example 12 said, "All you have to do is to think of the 7 hundreds as 70 tens. Then when you borrow one ten, 69 tens remain."

The teacher finished the discussion by saying, "Use the remainder of the period to work on the practice exercises on the lower part of your review sheets. For each of the examples that is circled show that your answer is correct."

While the pupils worked the practice exercises, Mrs. Handly at the teacher's invitation examined the bulletin board and the arithmetic table, and looked at the work the pupils were doing.

The content of the bulletin board is reproduced on pages 8–9.

Mrs. Handly asked for a conference with the teacher after school to discuss some phases of the arithmetic work she had observed. Among the many questions she asked was this one: "I was much interested in the several ways of working one example the pupils used. Why do you encourage them to use more than one way?"

"There are several reasons for using these different ways," the teacher replied. "First, we begin the study of each new phase of arithmetic by asking pupils to solve the new kind of problem by using what they already know. For example, when

Arithmetic Bulletin Board

Can you answer these?

1. The value of the 3,973,000 tons of sulphur produced in Texas in 1948 was valued at $72,000,000. About how much a ton was the value of the sulphur?

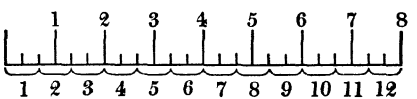
2. The total surface area of Manitoba is 246,000 sq. mi. The water area is 26,000 sq. mi. About what fraction of the area of Manitoba is covered with water?

GOOD WORK

	Check
42	42
×24	×3
168	126
84	×4
1008	504
	×2
	1008

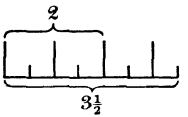
$8 \div \frac{2}{3} = 8 \times \frac{3}{2} = \frac{24}{2} = 12$

Proof:



$2 \div 3\frac{1}{2} = 2 \div \frac{7}{2} = 2 \times \frac{2}{7} = \frac{4}{7}$

Proof: Read $2 \div 3\frac{1}{2}$ this way: "How many $3\frac{1}{2}$'s equal 2?" The answer is less than 1. So you ask, "2 is what part of $3\frac{1}{2}$?" The drawing shows the answer."



The 2 is $\frac{4}{7}$ of $3\frac{1}{2}$

COMPOSITIONS

How We Learned to Divide by a Fraction

We first divided by a fraction by using drawings. For example, to divide 4 by $\frac{2}{3}$ we drew a line 4 parts long and marked each part into thirds. We then counted the number of $\frac{2}{3}$'s in the four.

After we had the answer to a number of problems we figured out how to divide by using only numbers. We found two ways, the common denominator method and the inversion method. I am going to show both ways with the same problem.

Common Denominator Method

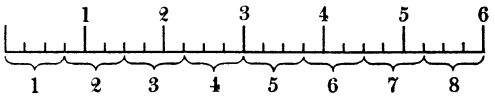
$4 \div \frac{1}{2} = \frac{8}{2} \div \frac{1}{2} = \frac{8}{1} = 8$

Inversion Method

$4 \div \frac{1}{2} = 4 \times \frac{2}{1} = \frac{8}{1} = 8$

How We Learned to Divide by a Fraction

To divide $6 \div \frac{3}{4}$ we first used a drawing like this:



The drawing shows that $6 \div \frac{3}{4} = 8$. Next, we figured out how to divide with fractions by using only numbers. You can do this in two ways. The easiest way is called inversion. Here is how you do it: $6 \div \frac{3}{4} = 6 \times \frac{4}{3} = \frac{24}{3} = 8$

The other all number way is by using a common denominator. Here is how you do it that way:
$6 \div \frac{3}{4} = \frac{24}{4} \div \frac{3}{4} = 24 \div 3 = 8$

JUST FOR FUN

Two hoboes named Bill and Harry sat down to eat. Bill had 5 rolls of bread and Harry had 3 rolls. Frank, a third hobo, joined them and they ate all. Frank offered to pay 8¢ for his share of the bread. How should the 8¢ be divided between Bill and Harry?

the division of fractions was begun, the pupils already knew how to separate wholes into fractional parts and they knew how to subtract fractions. So, through use of drawings, counting and subtraction, they were able to get the answer to division-of-fraction situations even though they did not know the process as most adults think of it. This experience served as a foundation for the development, first, of the common denominator method, and later of the inversion method.

"There are other important reasons for using several different solutions. It increases understanding and provides a proof or a check on the accuracy of the method accepted as best. Use of several solutions gives the pupils confidence, also. Frequently, some of the simple but cumbersome solutions become the basis for the development of the best solution. That, as I stated earlier, was the case in the development of the inversion method in the division of fractions."

As the conference ended, Mrs. Handly said, "I noticed that you did not use an arithmetic book today. Don't you ever use the textbook?"

"Oh yes," answered the teacher, "We use the textbook often. It is still the source of most of the arithmetic work in the sixth grade, but to make arithmetic interesting and challenging you need more than what's given in one book. Using only an arithmetic book for the arithmetic lesson would be nearly as great a handicap as teaching reading or science by using only a single textbook."

The teacher's answer to the question, "What is the purpose of using several ways?" is especially worthy of close analysis. The fact that pupils in the study of new phases of arithmetic are specifically directed to use procedures they know (even though these are not the particular procedures that are to be taught), guarantees the pupils a solution or solutions of the number situation. The learning problem can then become the search for a better or the best solution. Learning of this type according to present-day standards is far superior to learning by being told

or shown by textbook or teacher. To fully appreciate the difference in the learning procedure used by the sixth-grade class described in the preceding section and the telling or showing procedures so commonly employed, a sixth-grade textbook published prior to 1950 should be consulted. The method for introducing new procedures suggested by such an older book and by many new books, relies primarily upon explanation and demonstration of the new without much attempt to utilize the knowledge of arithmetical procedures which the pupil already has. Yet, telling or showing without active participation by the pupils has long been recognized as an inferior method of teaching. The pupil has no chance to participate in any of the development, and very little learning takes place until he actually begins to work problems and examples using the new procedure. Emphasis at this second stage is upon performing the procedure, which does not always provide much opportunity for acquiring understanding.

On the other hand, understanding — which is used in this book to mean the seeing of relationships — is certainly increased by using several ways of showing. For example, the child who understands 6 knows that 6 comes after 5 in counting, is 1 more than 5, is the equivalent of 3 and 3 or 4 and 2, is 4 less than 10 and so on. Each of the things known is an expression of the relationship between six and other numbers. The greater the number of relationships seen or grasped, the better is the pupil's understanding. When a pupil knows several ways of finding an answer to such a situation as $4 \div \frac{1}{2}$, it seems clear that his understanding of the division of a fraction by a whole number is better than it would be if only one way of finding the answer were known.

Each of the other points offered by the teacher in answering the question, "What is the purpose of using several ways?" is also significant and might well be studied with care. Other aspects of the lesson, however, need to be noted.

The writing of compositions as a means of review has a great

deal of merit. It affords the pupil an opportunity to express in words a mathematical process with which he has been working. While not an essential part of learning, the ability to express in words the steps of a procedure is a good indication that the pupil really understands the procedure.

The short-answer type of review question quickly calls the pupil's attention to important aspects of what has been studied and, as the lesson showed, it can be the basis for valuable discussions. It is a practical way to keep up a continuing review of other important aspects of arithmetic. For an example of this last point look at items 10, 11 and 12 in the test.

The reader's attention is called to the fact that only sixth-grade pupils who have had an opportunity to learn and use various solutions will respond as did the class Mrs. Handly observed. Pupils who have not subtracted to find the answer to whole-number division situations could not be expected to think of this way of dividing by a fraction without teacher help. Nor could they be expected to suggest the various other solutions that were offered. Not only do pupils have to have an opportunity to learn different ways of solving, they must also see a use or purpose for the different methods. The most obvious use is as a check or as proof. As the teacher's answer and the pupils' compositions indicated, another purpose of the various solutions is to find the best way and to be sure that each pupil understands this best way.

The compositions on the bulletin board telling how division by a fraction was learned call our attention to one of the markedly different procedures in teaching today. Reference is made to the statement that after the pupils had found the answers to a number of problems by drawing, they figured out how to divide using only numbers.

In other words, the pupils knew from drawing solutions of word problems that $5 \div \frac{1}{2} = 10$, $6 \div \frac{3}{4} = 8$, and so on, so their real learning problem became, "How can we divide using only numbers?" The pupil's use of "figure out" in the composition

seems to indicate that he felt the class had worked out a better way to do the examples just as a science problem is solved by using known data to explain a phenomenon.

Contrast this teaching method using known facts and procedures to work out a new process with the more frequently used method in which the pupil only imitates the new procedure demonstrated by the teacher or textbook. Of course, pupils learn arithmetic by the method of telling and showing (most people learned their arithmetic that way), just as pupils also learned geography when they were expected to give back verbatim the answers that either teacher or textbook had supplied.

Today, however, the subject matter of geography is learned in a much more meaningful and efficient manner and is understood by many more pupils. Instead of just presenting children with a social studies book which supplies first the topic heading, "Why did the people of Kentucky and Tennessee want to purchase Louisiana?" and then a cut and dried answer, a teacher leads pupils to the formulation of the above mentioned social studies problem by providing background reading about the facts of the times and encouraging discussion of facts and events, pertinent maps, even movies and the like. Pupils with the necessary background and interest created by such experiences wonder themselves why the settlers wanted to buy Louisiana and are interested enough to put facts together to find out.

Arithmetic, too, can best be learned when the pupil accepts the problem as his own and works on the solving of it for himself. It is perhaps more difficult to build up colorful, interesting background material in arithmetic, because of the very nature of the subject, but on the other hand, it adapts itself particularly well to pupil analysis of the problem and logical solution. A single textbook, regardless of how good the book is, cannot create a genuine problem situation, but a good teacher can provide a setting which will make the pupils themselves ask such ques-

tions as "How can these quotients (in division of whole numbers by fractions) be found by using only numbers?" or "What are the best ways of checking these products?" Guided by such questions, the pupils' approach begins to resemble, if not parallel, the procedure used by scholars. The real task of the teacher of arithmetic, then, is the creation of classroom situations in which the pupil becomes the seeker for answers to his own questions or problems.

Probably the first question which teachers of arithmetic will ask as they face this task, is "What are the best methods of creating such desirable classroom learning situations?"

What the Teacher Needs

There is no simple answer to this question. Of course, familiarity with the problem method of teaching is important, but another obvious requirement is real knowledge of and insight into the processes of arithmetic. Though most teachers feel adequately grounded in arithmetic, teaching as they do by "telling and showing" methods, a study of the pupils' blackboard work and of the arithmetic bulletin board (especially the Good Work section) may raise some doubts about whether most of us acquired the necessary background from our own traditional arithmetic training in elementary school. Consideration of the two instructional situations described below will increase the feeling that most of us need additional understanding of numbers and number processes to teach children meaningful arithmetic.

Fourth-and-Second Grade Instructional Situations Illustrating the Importance of Understanding of the Field of Arithmetic

Example 1

A fourth-grade girl had been having difficulty learning the nines in multiplication. She came to the teacher one morning

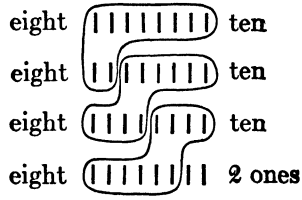
all excited saying that she had figured out an easy way to find the nines products. "All you do," she said, "is just make 10's out of the 9's and then add what you have left. I'll show you. Say you are multiplying 9 by 3. Take one nine and use it to make the others into 10's. One ten (as she touched the thumb of her left hand), two tens (as she touched the index finger) and turn this one (the next finger) down because it's the third nine. Now these fingers (the ones not touched) are the ones. You see the answer is 27."

The teacher, a bit amazed but not realizing that the girl had demonstrated what multiplication actually is, said, "That's very interesting but it's not really multiplying. You will just have to learn that $3 \times 9 = 27$. This finger trick won't help you with that and it may even keep you from learning the facts easily."

Later that day the fourth-grade teacher reported this experience to the principal of the school. "That's really a novel method," was the principal's reply. "She has actually figured out what is done in multiplication, hasn't she?"

"What do you mean?" asked the teacher.

"Don't you see that she has done in a rather clear way the very thing we all say we do when we multiply. For example, we say $4 \times 8 = 32$. That's just saying that 4 8's can be rearranged into 3 10's and 2 ones. This diagram shows how in multiplication you rearrange equal groups of ones into the equivalent tens and ones."



"I didn't see that before," said the teacher. "Why don't we teach multiplication that way instead of memorizing the facts?"

"We should include this kind of experience in our teaching," was the principal's reply. "That would furnish part of the understanding we are talking about so much. Eventually pupils will need to master the facts by memorizing but I wish all could see what multiplication really is as this girl does, before they begin the mastery process."

"I am beginning to see that you really have to know a lot about arithmetic to teach with understanding," was the teacher's comment as she thanked the principal for his explanation.

Example 2

"Won't you come to our arithmetic class and see for yourself what a help those new pictured arithmetic books are?" was the invitation Miss Mills, a second-grade teacher, extended to her superintendent. She went on to say, "The children can see so much better what numbers mean and how adding and subtracting really work."

Later that day the superintendent did visit the second grade. He observed the children as they turned to a page showing 2 boxes of pencils. Each box contained 6 pencils. "How many pencils in all are shown?" was the question assignment for this exercise. As various children volunteered answers, one boy said, "Six and six."

"Yes, Bobby, I know there are 6 and 6 but how many in all?" said the teacher.

"Six and six," insisted Bobby.

"No, that's wrong. There are 12 in all because 6 and 6 make 12. Let's count them to see if that isn't so."

The teacher and pupil then counted the 12 pictured pencils.

When superintendent and teacher were discussing the arithmetic lesson, the teacher said, "It's too bad that Bobby doesn't grasp numbers better. He is a good thinker in social studies and science but is so slow in arithmetic. I wish I could have said

to him that '6 and 6' was the right answer because he needs success so badly. Don't you think that counting the pencils in the picture helped him to see that the answer was 12?"

"Yes, I think he probably saw from that where you got the 12," answered the superintendent, "But I doubt if he saw why you would not accept his answer. Actually he answered the question, 'How many in all?' Six and six tell how many in all just as do ten and two."

"I am not sure I follow you," commented the teacher. "Do you mean that I should have said ten and two instead of twelve?" Then, after a brief pause she said, "It's strange the way you never think of twelves as ten and two and, of course, six and six is equal to twelve also. Then, why do we say twelve is correct and not six and six?"

"The reasons for using 12 instead of 6 and 6 are clear enough when you stop to think about them. In the first place, it is easier to say, to think, and usually to write one number than it is to say, to think, and to write two numbers. Second, it is customary to express numbers larger than ten as tens and ones. The ten is the base or first collection in our number system. Twelve, then, is 1 ten and 2 ones (one of the base and 2 more). In a like manner, 26 is 2 tens and 6 ones (two of the base and 6 more). Again, this is a matter of economy of thought. It is easier to think of 12 as three entities (1 ten and 2 ones) than it is to think of 12 separate ones or 6 ones and 6 ones."

Such an explanation as that offered by the superintendent, while satisfactory for the teacher, cannot be used with primary-grade children. Instead of an expository explanation, they need number experiences which will enable them to see the reasons (economy of effort or conformance with custom) for using 12 instead of 6 and 6. However, unless teachers understand the basic principles of our number system and the operations we perform with numbers, their direction of even primary-grade pupils' work with numbers may not always be efficient. As the

second- and fourth-grade instructional situations above showed, they may even hinder pupil growth in arithmetic.

Had the fourth-grade teacher in the situation described recognized that making groups of 9's into 10's and 1's was the essence of multiplication, she would have encouraged instead of discouraged the girl who had discovered that way of finding products. She would have taken this as the starting point of teaching procedures that would eventually lead the pupil to see the need for automatic mastery of the basic multiplication facts involving 9.

Similarly, had the second-grade teacher been aware of the full significance of what is involved in the expression "6 + 6 = 12," she would have accepted "6 and 6" as an answer for "how many in all," and would then have gone on to show why "12" is a better answer than is "6 and 6."

The presentation of actual instructional situations (a second-grade addition situation, a fourth-grade experience with a multiplication fact, and a sixth-grade summary lesson) and the accompanying discussion of each situation point to the importance of teacher knowledge of arithmetic and methods of teaching. Although their lack of information in the field of arithmetic should disturb teachers, it should not frighten them. The needed knowledge along with the interest that usually accompanies the acquisition of knowledge can be obtained by careful study of the field of arithmetic and by using some of the procedures suggested for children. The next section deals with some of the essential information about our number system.

Suggestions for Study of Arithmetic

Any careful study of the field of arithmetic will eventually involve a study of the systems of notation of number and of the bases of number systems. Students will find themselves raising questions like these: "What is meant when people say 'our

number system'?"; "What other systems than ours are there?"; "What are the outstanding characteristics of our number system?";"What is the difference between notation systems and numeration systems?"; "How is the octagonal system or duodecimal system better than ours?"; "Since the Roman number system has the same base as the Arabic what makes the latter so much superior?"

When people talk of our number system, they usually have in mind some hazy or rather mixed-up ideas about our place-value scheme of notation and about the base of 10 we use for our system of number. The hazy or "mixed-up" ideas usually result from a failure to define clearly the terms *base* and *notational system.* The base is simply the first collection in a number series which is used as a special kind of one. It is used in combination with the smaller numbers to form the next number in the series. For example, our eleven which is one more than our base of ten literally meant in the original Anglo-Saxon "ten and one." Twelve means ten and two (the base plus two), thirteen, ten and three (the base plus three), while twenty means two tens or two of the base. The idea of base in numeration is aptly illustrated by the names which some primitive peoples have used. In the language of one group the term for hand and "five" are the same. Six is "hand and one more," seven is "hand and two more" and so on. Since we have ten digits on both hands our base probably developed from the use of a pair of hands as the first collection. Because the base in our number system is ten we call ours a decimal system from the Latin word "decem" meaning ten.

The use of a special collection or base and smaller numbers to form larger numbers is just a common sense way of building a series of numbers. By this scheme the terminology necessary is reduced enormously. Instead of one hundred distinct terms for the first one hundred numbers in our number series we actually (counting all related terms as one) use only eleven distinct

terms. Unfortunately the modern names eleven and twelve have lost all resemblance to ten and therefore for practical purposes the learner is confronted with thirteen different terms or number names in the first one hundred numbers in our number series.

In our number series then, we have distinct names to ten, and after that say ten and one, ten and two . . . two tens, two tens and one, and so on. When ten tens are reached we have substituted for this double term the name one hundred, which is unrelated to the other ways of expressing tens. Note, however, with this next larger collection we again use the familiar "one." The quantity one hundred is therefore similar to ten because it is also a group used as a special kind of one. Moreover, it is related to ten by the fact that it is the equivalent of ten tens.

To understand more easily how the base idea operates, the reader should construct a series with a base other than ten. Consider, for example, a series with a base of four. Using our number names it would run like this: one, two, three, then one four or the first collection; four and one, four and two, four and three, two fours; two fours and one, two fours and two, two fours and three, three fours; three fours and one, three fours and two, three fours and three. For the next number, four fours, a new and distinct term similar to the one hundred in the base ten series is needed.

An important point to be noted with regard to the number base as presented above is the fact that base is independent of any writing of numbers, and therefore does not involve the use of zero and the place-value aspect of our number system. The use of an ordered scheme of number names in a series is called a *numeration* system. The numeration systems of the Romans and the Arabs were identical. However, in *notation*, the plan for writing the numbers in a system, the Roman and Arab

methods differ markedly. To write the first ten numbers the Roman system uses only three different symbols, while to write the same ten numbers the Arab notation requires ten different symbols. However, the Roman system of notation takes care of larger numbers by introducing new symbols (L for 50, C for 100, D for 500, M for 1000), while in the Arabic system no new symbols are needed. Not only is the Roman system of notation weak in needing so many new symbols for writing larger numbers, it also suffers because the small numbers vary in the number of symbols needed. For example, XIX (19) requires three symbols while XVIII, a number 1 less than 19, requires five, and XX, a number one larger, requires only two symbols. No such inconsistencies are found in the Arab system of notation. There, by letting the value of a numeral be dependent upon its position, all three-figure whole numbers are larger than all two-figure whole numbers. In the number 333, for instance, each figure represents a value ten times that of the figure to its right. When in one position of a number, for example, in 304, there is no figure, the place is held by zero. This use of place or positional value combined with a numeral (zero) for holding the place when there is no quantity in a position is what makes the Arab system of notation far superior to the Roman system.

The student of the teaching of arithmetic is warned not to make the error of assuming that the principle of place value is limited to the Arab system or that it is dependent upon a base of ten. The Mayas used positional value with a base of twenty.[1] and the Babylonians used the principle with a base of sixty.[2] A good way to understand the role of place value in notation and to see the significance of base is to express with numerals the

[1] Annie Terry White, *Lost Worlds* (New York: Random House, 1941).

[2] Edward Chiera, *They Wrote on Clay* (Chicago: University of Chicago Press, 1938), pp. 154–157.

same quantity using different bases. In the example below seventeen circles are enumerated to base 10, 4 and 7.

O	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

base 10

O	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O
1	2	3	10	11	12	13	20	21	22	23	30	31	32	33	100	101

base 4

O	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O	O
1	2	3	4	5	6	10	11	12	13	14	15	16	20	21	22	23

base 7

Most discussions of the bases of number system sooner or later become concerned with the topic of the best base. The numbers 2, 8, and 12 as bases have received most attention. There are some good arguments for use of each of these numbers as the base of a number system and each has some definite limitations. The binary system was favored by Leibnitz, the co-discoverer of calculus, and is looked on with favor by some men engaged in the production of calculating machines. The major weakness of the binary system is the large number of characters required to write quantities. For example, the decimal number 17 would be 10,001 with a base of two. If a change were really feasible, the base of eight would probably be chosen by teachers of arithmetic. In such an octagonal system the basic combinations involving our present 8 and 9 would not exist, making mastery of number facts easier, and by continued halving of the base, unity would be reached, which is not true of the base 10. Of course, more figures are needed to write large numbers than is true in our present system.

Actually, the possibility of a change of bases is non-existent. Modern civilization has too much invested in the present decimal system to permit any change. Therefore, students of the teaching of arithmetic need only be concerned with other bases for the purpose of getting a better understanding of the base ten.

STUDY QUESTIONS

Directions: From the several responses suggested, choose the one which you consider the most acceptable answer to the question. Check your answers by referring to the text. The same procedure is to be followed with the Study Questions given at the end of each succeeding chapter.

1. In what aspect of the subject has the teaching of arithmetic changed most in the last twenty-five years? (*1*) In the amount of the content. (*2*) In presenting the subject to the children. (*3*) In the type of content.

2. Sixth-grade pupils should be able to show in how many different ways that $4 \div \frac{1}{2} = 8$? (*1*) 2. (*2*) 4. (*3*) 8.

3. What is probably the most significant outcome of having pupils solve problems in several ways? (*1*) They are less likely to forget several ways than one. (*2*) It provides variety. (*3*) It provides a foundation for the best method.

4. Does inability to express in words the steps in a mathematical procedure indicate a lack of understanding of the procedure? (*1*) Yes. (*2*) No.

5. Should instruction in new procedures in arithmetic emphasize telling and showing by the teacher? (*1*) Yes. (*2*) No.

6. Is such a question as "What is the best way of showing that $4 \times \frac{2}{3} = \frac{8}{3}$?" a good arithmetic learning problem? (*1*) Yes. (*2*) No.

7. Are the bases of the Roman and the Arabic systems of number the same? (*1*) Yes. (*2*) No.

8. Are the Roman and Arabic notational plans the same? (*1*) Yes. (*2*) No.

9. If this number of marks (IIIIIIIIIIIIIIIII) were written in a number system having a base of 3 what figures would be used? (*1*) 52. (*2*) 122. (*3*) 103.

10. The number 10101 to base 2 is equivalent to what number to base 10? (*1*) 19. (*2*) 20. (*3*) 21. (*4*) 81.

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2

Counting, Numeration and Notation

Counting Experiences Before Entering School

As parents and older children talk to little children, counting and number names are often used. Some notion of number is probably acquired by the child even at a very early age, although number names, as expressions of quantity, are abstractions that might seem too difficult for the child just beginning to talk. Here are some everyday uses of number which little children have encountered. The situations are so commonplace that explanatory notes seem unnecessary.

1. "You can jump off the bench (a park bench) three more times. I'll count — one — two — three."
2. "There are four dolls in the window."
3. "I'll count for the race. One for the money, two for the show, three to make ready, and four to go."
4. "We will need three more sticks like this."
5. "Let's see if you can finish your glass of milk before I count to 10 — one, two," etc.
6. "You have only two more bites and then you are through."
7. "I see three birds on the lawn — one, two, three."
8. "There are lots of little pigs in the pen. Let's count them."
9. "Let's count the puppies in the picture."

10. "The clock is going to strike six. Let's count as it strikes."

11. "One, two, three candles. You are three years old today."

The child who grows up in a home where counting activities of this type occur often, is getting a good background for arithmetic. Contrariwise, the child who does not have such experiences is being deprived of something worth while.

The best background is built in homes where children have not only such counting experiences as those listed above but many other experiences with counting and number names in stories, games, poems, and songs. Probably the outstanding game for children which requires counting is Hide and Seek. This game is especially valuable because it can be played by children of widely varying ages and is thus suitable for family and neighborhood groups of children. The very small child may count to only 5 by 1's while older children are required to count further. For all who are not proficient there is always the goal set by those who count by 5's. Younger children, observing that counting by 5's is faster, will be inspired to imitate their elders. Other games which some children like to play are dominoes, lotto, and similar types of race games. Counting jingles are common in connection with rope jumping and choosing sides.

There are many stories in which counting plays an integral part which may be read to children of both preschool and school age. Examination of the books in the following list will give the reader an idea of the possibilities in this area.[1]

1. Barto, A., *Count to Ten With Little Red Hen* (Poughkeepsie, New York: Artists and Writers Guild, Inc., 1938).

[1] For additional stories and for teaching suggestions, see Louise Beltramo, *Sources and Suggested Uses of Poems and Stories Involving Counting Situations*, Unpublished M.A. Thesis (Iowa City: State University of Iowa, 1949).

2. Berhand, Josephine (adapted by), *9 Cry Baby Dolls* (New York: Roy Publishers, 1945).

3. Brann, Esther, *Five Puppies for Sale* (New York: The Macmillan Company, 1948).

4. Friskey, Margaret, and Katherine Evans, *Chickens Little Count-to-Ten* (Chicago: Children's Press, Inc., 1946).

5. Heyward, DuBose, *The Country Bunny and the Little Gold Shoes* (Boston: Houghton Mifflin Company, 1939).

6. King, Dorothy, *Set the Clock* (New York: Franklin Watts, Inc., 1942).

7. Lenski, Lois, *Surprise for Davy* (New York: Oxford University Press, 1947).

8. Lilly, Jean, *A Hundred Tufties* (New York: E. P. Dutton and Company, 1940).

9. Meeks, Esther K., *One Is the Engine* (Chicago: Wilcox and Follett, 1947).

10. Petter, Helen Mary, *The Number Book* (New York: Lothrop, Lee, and Shepard Company, 1946).

Of the songs, "This Old Man," "One Elephant," "One Potato, Two Potatoes," "Five Little Chickadees," "Tell Time," and "One Finger, One Thumb, Keep Moving," are among the best.

Jingles and finger games also play an important role in helping some children acquire the order of our number names. A few are listed below:

1. One to get set, two to show, etc.

2. One, two, buckle my shoe; three, four, shut the door; etc.

3. Here is the beehive, where are the bees? Here they come creeping, one, two, etc.

4. Here is the church and here is the steeple but where are the people? Here they come. One, two, three, etc.

5. Five little bunnies
   Weren't any more.
   One ran down a hole
   Leaving only four.

Early School Counting Experiences

When children enter a school which provides good educational experiences, whether at age 5 or 6, they have many needs for counting and other phases of number. To meet these needs the school must provide systematic instruction in counting. This instruction will be more effective if uses of counting are shown in the setting of a story, a classroom situation, or the like. The following is a good example:

"Jack says it is only 8 days until his birthday. Let's count the days to see if he is right."

The calendar, if properly used, is one of the best pieces of equipment for creating counting needs. There are always birthdays, holidays, and special events, such as the day the bulbs were planted, the letter sent, or the incubator set. As pupils count the number of days before an event will take place (or after it has taken place) the number changes daily and a new opportunity for counting is presented. There are literally dozens of good calendar counting situations, many of which may be repeated.

The daily news, which may be an important part of a child's first-grade reading program, offers many opportunities to count. Here are a few situations taken from news items: "This is the fifth day of school"; "There are 4 pupils absent today;" "We read 6 pages yesterday." It is, of course, assumed that these news items are the result of actual counts made by the class members.

There are many reading-readiness and reading-seat-work exercises, such as "Color four of the balls blue," which make use of counting. Perhaps the most significant uses of counting arise in connection with the science and social studies experiences of the five- and six-year-old children. For example, as one group of five-year-olds were examining the shiny red asparagus seed pods the teacher said, "We found only one seed in each walnut. I wonder how many seeds we will find in these. Let's

guess and then each one count his. If you can't count, we will help you."

After such an experience, the class might then engage in counting objects in unison (e.g., the number of books on the library table, the number of buds on a plant, and so on). Counting in unison is important to the beginner, but there should also be individual saying of the number names by those who lack facility. Teacher and other class members may help when a child cannot go any further. Early in the program of systematic counting instruction, direct procedures similar to the following have a place:

"Everyone needs to know how to count to 10. Who would like to count to 10 for us today?" Several pupils are then permitted to count.

For those who cannot readily count the teacher may say, "You count with me. This is the way: one, two, three," etc.

Jingles, songs and rhymes will add variety and zest to this task of learning the order of the number names to 10.

Since the learning of the order of the number names to 10 is probably the one most important thing a pupil will ever learn in mathematics, there can scarcely be overemphasis on this phase of instruction.

Recognition of the great importance of knowing how to count from 1 to 10 has probably led to some practices in teaching which are detrimental to the pupils. Reference is here made to the practice of devoting weeks and months to use of only the numbers 1 to 10. While everyone recognizes that the order of numbers 1 to 10 is essential to thorough counting beyond 10, it does not necessarily follow that pupils who have not thoroughly mastered the first 10 will not profit from counting experiences beyond 10. Once children have entered school (at age 5 or 6) and instruction in counting has begun, there is no good reason for not giving counting experiences beyond 10 even though major emphasis is to be on the first 10 numbers.

In teaching children to count from 10 to 20, emphasis needs

to be put on the fact that 11 is 10 and 1, 12 is 10 and 2, and so on. After the order of numbers to 10 and 20 has been taught, counting by 10's to 100 is undertaken. This is first done by saying one 10, two 10's, three 10's, and so on.

In the teaching of counting to 10, from 10 to 20, and from 10 to 100, the number chart, the number line, and various counting blocks or other objects will be found especially useful. For suggestions on the use of such devices, see Chapter 3.

After the children see the necessity of knowing how to count (actually learning the order of the number names), probably the most effective instructional procedures are those which give specific practice in saying the number names in order without reference to finding how many. Once systematic instruction is introduced, the most efficient way to proceed is to give time to it each day. To provide many and varied counting procedures is then the most pressing instructional problem. A brief list of such procedures is supplied below.

1. "How many chairs (sheets of paper, books, pencils, brushes, mats, and the like) do we need for our group?"
2. "The page is 9. How did the writer of the book decide to call this page 9?"
3. "On the work table is a game in which you are to show with grains of corn how many packages each car carries. Be sure you count correctly."
4. "For our science record we need to know how many rows of kernels there are on each ear of corn. Count the rows and put the same number of grains of corn by each ear. If the ear has already been counted, check the work by counting again."
5. "In the race game the dots you turn up tell you how far to move your marker. Be sure to count correctly."
6. "To put the cars of the train in the right order you can count the dots on the back of each car."
7. "Bill thinks this number (26) on the number line is twenty-seven. Let's count to see if he is right."
8. "Let's hear you count to 14 this morning."

9. "Some people tell how long a minute is by counting to 60. Let's try that." Later this may be changed to counting from 101 to 160. The fact that some basketball officials determine 10 seconds by saying to themselves the numbers from 101 to 110 may be used to give the exercise an important setting.

(The use of counting in stories, poems or games is an effective instructional procedure. The compilation of appropriate stories, poems, and games to provide classroom counting situations is an excellent project for teacher study groups.)

10. "I think there are 100 counting sticks here," said the teacher as she showed 10 boxes. Several boxes were opened and the children counted the sticks in each. The teacher then said, "I can show you an easy way to count to 100. You do it with me. There are 10 counting sticks in this box and 10 more in this one. That makes two 10's." Then as the third, fourth, fifth, sixth, seventh, eighth, ninth, and tenth boxes were identified she said, "three 10's, four 10's," and so on to ten 10's which she said was 100. Several children were then permitted to count the boxes of sticks in a similar manner.

"Now let's all count to 100 by 10's, one 10, two 10's, three 10's, four 10's, five 10's, six 10's, seven 10's, eight 10's, nine 10's, ten 10's, or one 100."

Exercises of the type above with minor variations can be used every day for about two weeks. After several days, the use of the words "twenty" for "two 10's," "thirty" for "three 10's," and so on should be introduced.

Examination of the suggestions for teaching counting in the preceding section shows that familiarity with the counting procedure and its uses is the first goal of instruction; that after situations which show the uses and the need for counting have been used, direct instruction in learning the order of the number names to 10, then to 20, and by 10's to 100 is provided. Note should be made of the fact that in all the instruction emphasis is placed on the order of the names to 10, that these same names and the same order are used in counting to 100, and

that the numbers 11 to 100 are closely related to the numbers 1 to 10 (e.g., 11 is 10 and 1 more; 23 is two 10's and three more 1's). Learning to count in this way, children are actually becoming acquainted with the major characteristics of our number system. We might well consider next what these characteristics are, so the student of arithmetic teaching can evaluate counting procedures with understanding.

Major Characteristics of Our Number System

The expressions "number system" and "teach number as a system" are frequently used with only a vague idea of what is implied by the term *system.* The base idea of our number system discussed in Chapter 1 is, of course, the outstanding characteristic of the system. The four characteristics of our number system listed below either rest directly on the base idea or are expressions of some aspect of the base idea.

1. The Hindu-Arabic numerals, if used in the ordinal sense, always have the same order; 5 always comes after 4, 7 after 6, and so on. This order holds regardless of the size of the quantities (units, tens, hundreds, or millions) referred to. It is this ordinal or positional aspect of number that gives us our most usable concept of quantities like 9, 12, or 52.[1] To illustrate this point, think of the number 52. What thought process do you use to get an idea of a quantity 52? Those who use numbers most efficiently would think of 52 as about one-half of a hundred — a little more than the fifth group of ten. In both of these ways of getting at the meaning of 52, the position of number in a series plays the major role.

2. The first nine numerals represent quantities thought of as groups of ones, or units. After 9, the next number, 10, is not always efficiently thought of as representing individual en-

[1] D. E. Smith, *The Teaching of Elementary Mathematics* (New York: The Macmillan Company, 1921), p. 103.

tities, but is thought of as a single entity, or collection. This fact is shown even in the name "ten," where the familiar terminology of 1 is implied, though not actually used by most people, for this number is the first ten, or one ten. This use of collections of ten, with repetitions of the ten as larger quantities are encountered, is what makes our number system a decimal system. It has a base of ten. The base, then, is nothing more than the first standard or base collection that is used. When the collection idea is properly utilized, 12 is not twelve separate entities; it is three entities consisting of a collection of one 10 and two 1's. Forty is four entities, or four collections; it is four 10's. To demonstrate to yourself the simplification, represent each of the preceding numbers first as 1's and then as collections of 10's plus the necessary 1's.

3. The names for the tens consist of terms that are closely related to their meaning. Twenty can be easily related to twin 10's, or two 10's; thirty is the third 10, or three 10's. Note that the suffix "ty" in all such names as twenty and thirty means tens. Thus, in dealing with the tens (twenty, thirty, forty, and so on) a person does not learn entirely new names with a special place in the series. For the most part, the names in the tens series are much like those of the first nine numbers; the first syllable of twenty, thirty, forty, and fifty suggests two, three, four, and five, respectively; and in the terms sixty, seventy, eighty, and ninety, the exact number names for the ones constitute the first part of each of the number names for the tens. The order of the tens is exactly the same as that of the numbers from 1 to 9.

4. The use of place value simplifies the writing of numbers. By place value is meant the fact that the value of a digit is dependent on its position in a number. As an illustration, consider the numbers 3, 35, and 320. In each of these numbers, the numeral 3 is used, but in each number it has a different value because of its position. This principle of place value com-

bined with the use of a place-holder (zero) makes possible the writing of any number using only ten different symbols. The student of arithmetic teaching should recognize that the place value scheme may be used with any base (see page 21).

System, then, as applied to number, implies exactly what the term denotes in other fields: namely, order and method. An arrangement that makes possible the operation or understanding of a process, with a minimum of effort, characterizes any system. Close study of the four characteristics of our number system just listed will show how the system conserves effort. One example of the conservation of effort for each characteristic is offered here: (1) Since the number names always have the same order, learning the relation that five thousand has to six thousand presents little difficulty. (2) The collection idea makes the addition of two thousand and four thousand as easy as the addition of two and four. (3) In learning to count to one hundred, the similarity of the tens' names to the ones' names makes the task of the learner comparatively easy. (4) The economy which the fourth characteristic makes possible is clearly seen by comparing the Roman and the present system of writing the year nineteen hundred fifty-four: MCMLIV and 1954. System is important in all work, because it makes for economy of effort and therefore results in efficient usage.

The student of the teaching of arithmetic may find that a rereading of the section on base and notation in Chapter 1 is of assistance in clarifying ideas about our number system.

Rote and Rational Counting

A discussion of the teaching of counting necessitates consideration of the terms *rote* and *rational* counting. The term *rote* when used in connection with counting refers to saying the number names in order without reference to or knowledge of how many. Saying the number names in order without know-

ing what the names mean is another way of describing rote counting. *Rational* counting refers to counting to find how many or which one. As indicated by the name, rational counting is understood. The child who counts rationally knows what the numbers mean.

Most of the writing in teachers' manuals, in professional magazines, and in books is concerned only with rational counting. Not only is rote counting not recommended in these publications but teachers are urged to keep children from counting without knowing what the number words mean. All counting activities, it is suggested, should be concerned with things. This insistence on rational counting and condemnation of rote counting is not in harmony with the known facts on how children learn to count. Rote counting must either precede or at least accompany the learning of rational counting. Careful consideration of the rote-rational counting issue will result in doubt as to whether rational counting can even be taught before rote counting has been learned.

Consider, for example, the commonly advocated procedure of having children learn to count by enumerating objects such as blocks. In the early stages of the process, the child is taught to pick up a block from a group and set it aside as he says "one"; then he picks up another block, sets it aside, and says "two." In this way he transfers the group of blocks one by one as he says the numbers in order. At the very first, he certainly has no idea of finding out how many blocks there are in the group. He may have only a vague idea of assigning number names to the blocks. He may be doing nothing more than handling a different block as he repeats, in order, the number names which he has learned to say by rote. A little later in his practice, he may be able to go through a group of objects accurately, merely by pointing to the objects one by one as he repeats the number names in order. In this latter stage he still is not enumerating, but merely assigning a number name to each object. The truth

of this last statement is often strikingly demonstrated to those who have had the experience of watching a three-year-old learning to count his fingers. Almost invariably the child will ask something like "Could this be four?" when pointing to only one finger. Such a question shows that it is a name, a means of identification, that the child wants. He is not concerned with finding how many fingers he has.

Thus the designation of objects by name, not the determination of the quantity of objects, appears to be the goal and the result of the child's first attempt to use number names. To apply number names in a fixed order to a group of objects which are arranged in no particular order, the child must have command of the proper order of the names. Therefore, before the child reaches the stage of development in which he really enumerates, he makes much progress in learning to count when he learns the number names by rote through imitating what he hears his elders say. School officers and teachers should welcome the efforts of parents to teach their children to say the number names in the accepted order. They should remember that in learning to count, three essential steps seem clearly defined: first, learning the number names in order; second, applying these names as identifying words for objects; and third, using the names in determining total quantity.[1]

The suggestions for teaching counting found earlier in this chapter included a number of exercises which were definitely of the rote type. The preceding paragraphs should be ample justification for recommending the use of such exercises.

Ordinal and Cardinal Numbers

The use of number names to arrange objects in order or to identify their places in a series is known as *ordinal* counting.

[1] Margaret Drummond, *The Psychology and Teaching of Number* (Yonkers-on-Hudson: World Book Company, 1922).

The use of number names in serial order to find the total number is known as *cardinal* counting. The true meaning and the place of these two forms of counting in learning and in teaching numbers are not always clearly understood.

Teaching the cardinal aspect of number first is not in harmony with the three essential steps children follow in learning to count. As pointed out in the preceding section, the third step is the cardinal idea while the second step is the ordinal idea. Indeed, it would be difficult for most children to get the cardinal idea of a number if the ordinal idea were not already developed. The child must know that four comes after three, or else he must know quantities without benefit of counting. While it is possible for a child to recognize groups of two, three, or four without use of ordinal numbers, the task of recognizing seven, eight, or nine would be almost impossible.

Practically all texts in arithmetic refer to ordinals only as first, second, third, fourth, and so on. Dictionaries also refer to ordinals only in that way. In ancient times people had separate ordinal and cardinal names. We retain that difference today in our one, two, three, and first, second, third, but by far the greater part of our ordinal and cardinal number names are identical. To assume that ordinals are confined to the terms first, second, third, fourth, and so on, is an admission of misunderstanding of the ordinal uses of number. To demonstrate to yourself that cardinal numbers are used in the ordinal sense, consider the meaning of an index reference like "Brick, p. 246." In this case the 246 is an ordinal number. It tells the person who desires information on bricks that something about bricks is given on page 246. For that particular purpose, the position of the page in the book, not the total number of pages up to that point, is the important thing that the 246 tells. Likewise, in the ordinal sense we employ cardinal number symbols and names for such common uses as designating houses, telling time, and expressing dates: 123 Pine Street; eight o'clock; June 19, 1954.

We also utilize the cardinal number symbols and names in hundreds of other instances where we wish through numbers to show order, to identify, or to locate. Even in enumerating, the ordinal scheme is used to find the cardinal — the how-many-in-all. To prove this, count a given number of objects, say eleven. When you come to the ninth, which you will call nine, stop. Is this one object, which you have just pointed to or designated in some other way, nine objects? Of course not, but it is the ninth one, and therefore, nine objects in all have been identified. The last ordinal applied in identifying or enumerating a group is, therefore, the cardinal of that group.

Teachers should recognize clearly the distinction between ordinal and cardinal as indicated in the above discussion, and children should be made aware of the two uses of number even if the names ordinal and cardinal are not used. For example, even first-grade children can and should learn the meaning of page numbers.

This discussion of the terms ordinal and cardinal is really not complete without considering the origin of these two aspects of number. In ancient times man employed separate ordinal and cardinal names. Although no records of the beginning of counting are available, it seems fairly certain that the ordinal idea or plan was responsible for the beginning of counting. A reasonable theory holds that it was man's desire to identify, to order, to systematize, which led to counting. Primitive man had many uses for a method of placing things in definite order. A hunting party stalking game would find it advantageous to have the best hunters in strategic position. Probably the one who was most skillful in throwing the spear would be first, the one most experienced in close fighting next, and so on. Similar requirements for definite order in placing men were even more important in time of war. Then, too, various religious ceremonies called for a definite order of approaching the altar, the god, or the temple. Probably the medicine men went first, then the chief, followed

by the warriors arranged in some definite order. Early man's need for a universal system of identifying places is well shown by the following example of a primitive account: "On the fourth day of the journey we reached the mouth of the third of the rivers. There we waited for our scouts." Thus, in the most elementary activities of life (hunting, fighting, worshipping, and exploring) a way was needed to identify the position of men, objects, and events. Equally plausible uses of the cardinal idea of number do not readily occur. Even in cases where the cardinal idea would seem to be called for, as in reporting the number of warriors in a war party, the model group plan and not the counting plan of telling how many would probably have been used. Uses of an ordered system for identifying events and places, such as the examples cited above, are independent of the cardinal idea of counting. On the other hand, the cardinal idea cannot be used very effectively with quantities larger than four or five without the use of the ordinal idea.

From the preceding discussion it should be clear that the writer believes that the ordinal idea of number played the major role in the development of counting. This belief is contrary to the generally accepted idea that man learned to count in order to find out how many of certain goods he possessed. The latter theory seems unsound, since the idea of quantity in the sense of *how many* presupposes a knowledge of counting. Likewise unsound is the proposal that children want to learn to count in order to find how many.[1] The ordinal, the series idea, is the original idea and must receive the first attention.

[1] George Bruce Halstead, *On the Foundations and Technique of Arithmetic* (Open Court Publishing Company, 1912), chaps. XIV and XVIII. See also D. E. Phillips, "Number and Its Application Psychologically Considered," *Pedagogical Seminary*, 5: 221–228; Tobias Dantzig, *Number, the Language of Science* (New York: The Macmillan Company, 1930), p. 8; Harry Amoss, *Rhythm Arithmetic in the Primary School* (Boston: Bruce Humphries, 1942), chap. II; H. G. Wheat, *The Psychology and Teaching of Arithmetic* (Boston: D. C. Heath and Company, 1937), pp. 20–23.

In order to give the proper perspective to counting, it is well to consider that series rather than quantity is the more important aspect of number. As was pointed out in the section on meaning and understanding in Chapter 1, it is the ordinal or positional aspect of number that gives us our most usable concept of quantity. To test the truth of this statement, let us consider the quantity indicated by 83. Few if any of us try to visualize 83 separate units. Instead, most of us compare 83 with some well-known quantity below it, such as 50, or above it, such as 100. In other words, our idea of the quantity indicated by 83 is obtained from its position in the number series.

The cardinal idea originally developed from the custom of using model groups of familiar objects as standards of reference. To convey some idea of the quantity in his sight or in his mind, a man would say there were as many things as there were petals on a flower or pebbles in a heap. To express the idea exactly, however, he would have to match each object with a pebble or petal. It requires no great imagination to see the advantages and disadvantages of this scheme. The model group gave an exact measure of how many; and in the sense that a familiar, convenient, and certainly more simple group was substituted for the original, the scheme made for simplification of thought. But this scheme provided no exact means of comparing model groups of different sizes. There was no number system which the mind could use, and therefore man's idea of a quantity was dependent on unrelated model groups. All idea of size was dependent on sense perceptions.

The inadequacy of the model-group method of telling how many may be illustrated in this way: Two scouts sent out in different directions by a hunting party made their reports. The first said, "South of here is a herd of deer in which there are as many deer as there are feathers in my headdress." The second said, "West of here is a herd of deer in which there are as many deer as there are knots in this string." No doubt, the

hunters would have been familiar with each of the two model groups used and would, therefore, have had a pretty good idea of the size of each herd. On the other hand, the hunting party could find out which scout had located the greater number of deer, or how many more were in one herd than in the other, only by matching knots with feathers. The application of the ordinal idea to the model groups permitted comparison of different-sized groups without the cumbersome matching process.[1]

Counting in the Fundamental Operations

Counting receives relatively little attention in school because school people have associated counting with poor achievement in addition and to a lesser degree with poor work in the other fundamental operations (subtraction, multiplication, and division). Children who do not know the addition combinations often resort to counting — a perfectly logical and sound method of getting the sum. Of course, such a method is a slow one, and it embarrasses the teacher by making her pupils seem stupid. However, the reason that children resort to counting is not that they have done too much counting. It is more likely that addition was undertaken before they had had sufficient experience in counting to lay the good foundation needed for intelligent mastery of the addition combinations.

The child who knows enough to count in order to find the sum of 4 and 3 possesses a great deal of understanding. He should not be denied an opportunity to use his knowledge even though the final aim of instruction is to teach him procedures that are superior to the counting process. Teachers and students of arithmetic should recognize that adding is a method of counting by groups and therefore a short method of counting. Teachers

[1] For further discussion of this phase of counting the reader is referred to Wheat, *op. cit.*, pp. 7–12. See also Halstead, *op. cit.*, chap. III, and Dantzig, *op. cit.*, pp. 6–11.

should also remember that simple steps in learning precede more complex steps. Therefore, instead of receiving little emphasis because poor students use it as a substitute for addition, counting should be recognized as one of the first steps in teaching addition.

Counting to find the answer is considered the foundation of all methods of solution advocated in this book. This assumption will be made especially evident when fundamental operations are studied. It is further assumed that all the problems used in introducing each of the fundamental processes can be solved by counting. The child who can count is thus assured of having at least one solution. Besides being both a starting point and a last resort, counting is also a definite aid to understanding, primarily because it is a process in which everyone has confidence. Even adults, when working under extreme pressure, often resort to counting. When children differ about a number of objects, counting can always be used in trying to reach an agreement. The teacher confronted with the problem of what to do when a child states that 6 and 7 are 14 will probably find few procedures as valuable as having the child find the total of 6 and 7 by counting. Thus, it can be seen why counting is considered the foundation for study of the fundamental operations of arithmetic.

Counting in Grades 1–4

After initial counting experiences of the kind described in the earlier sections of this chapter, there is still need for instruction in counting and for presenting counting situations. In later first grade and second grade many good counting situations will be concerned with finding "how many," as in verifying that 20 days is about 3 weeks, in finding that if pages 26 through 47 have been read then 22 pages were read, and so on. Addition and subtraction situations involving two-figure numbers which often arise in normal school or out-of-school experiences can be

easily solved by counting on the number line or number chart. In addition to these uses of counting to find how many and to find sums and differences, first- and second-grade children will profit from some experience in counting by 5's, 2's, and 3's. In learning to count they will already have had experience with counting by 10's. These experiences with multiple counting give insight into the counting process and the nature of our number system which is not easily acquired in other ways. At this point a word of caution may be in order. Any attempts to force or to require children to learn to do multiple counting may be detrimental to progress with numbers. These multiple counting experiences should, therefore, be presented and used as desirable but not be considered essential. Second-grade teachers whose pupils have good counting ability should not hesitate to present and use a chart like the one given below.

A Counting Chart

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

In addition to its uses in teaching multiple counting, the above chart is very valuable for bringing out interesting and unusual facts. The following teaching suggestions bring out the outstanding ones.

1. What is unusual or strange about the ninth row?
2. Count together (add) the first and ninth column in each

row. What do you have? (Do the same for second and eighth, third and seventh, etc.)

3. Double the numbers in the first row. What do you have?

In teaching multiple counting the number line will be found useful. By using it pupils notice (or the fact should be called to their attention) that in counting by 2's every other number is omitted, that in counting by 3's two numbers are omitted, and so on.

Although multiple counting experiences are suggested for the second grade, major instruction in this type of counting is probably most effective in the third grade with some attention also in the fourth.

Counting in the Upper Grades (5–8)

It is generally assumed that pupils learn to count in the primary grades and therefore teachers of upper-grade children need not be concerned with counting instructional procedures. There are, however, good counting experiences from which pupils in Grades Five through Eight will acquire real benefits. Multiple counting, especially with large numbers such as 250, 300, 5,000 is one good example of the type of upper-grade counting that is recommended. There are pupils who know little of counting beyond 100. For such pupils a situation where they begin at 1500 and count by 250 or by 300 would show them that hundreds and thousands can be handled as easily as tens and ones. If interest can be maintained, counting from 0 by 25, 20, 40, and so on will be worth while.

Counting by fractions after pupils have had sufficient background is an excellent instructional procedure. "Begin at 2 and count by $\frac{3}{4}$ until 7 is reached," is a sample exercise.

Exercises 1 and 2 below indicate a type of counting which is highly recommended for Grades Seven and Eight and may be used with profit in Grade Six.

1. "Begin with 37 and count by 4's until you go beyond 100. Write the first number you count beyond 100."

2. "Begin with 31 and count by 6's until you go beyond 100."

Similar exercises involving 5, 7, 8, and 9 are recommended. The teacher can quickly check such exercises by having the correct number for each exercise written on a piece of paper. The pupil who learns to get the answer by a combination of dividing, multiplying, and adding probably does not need the adding and counting practice for which the exercises are intended.

Counting backwards, for example, "by 3's from 47 until 30 is reached," is another type of upper grade counting that may be used.

The suggestion that counting in some form be taught in every grade of elementary school is some indication of the importance of counting. As has already been stated, learning to count is the most important phase of mathematics that pupils will ever learn. Added to that is the fact that counting experiences are a good way to give pupils insight into the number system and the operations we perform with that system. Since the fundamental operations (addition, subtraction, multiplication, and division) are basically a form of collective counting it is easy to see the contribution of counting to the mastery of these processes.

Numeration

Numeration, as used here, refers to the reading of numbers expressed as numerals. (In addition to this restricted meaning, the term is frequently used in arithmetical literature to denote the process referred to previously as enumeration or rational counting.)

Soon after the child enters school, he finds that he needs to know how to read numerals in such common things as dates, house numbers, page numbers, and prices. In fact, he often

meets his first reading of numerals as a regular part of the initial instruction in reading. Probably the most important single procedure used in teaching the reading of numbers is to have the pupil touch or point to numbers in a number chart (see page 66), taking each number in order and at the time of touching saying its name. In teaching the reading of numbers, however, it is important to do more than develop the pupil's ability to call the symbols by their correct names. To insure his understanding of the numbers that he reads, the teaching must include procedures to develop meaning for the symbols. Certainly, in developing the pupil's ability to read the number symbols from 1 to 10, the teacher should see to it that not only are the number names, three, four, five, and so on, employed, but also grouped concrete representation (actual objects) and semiconcrete representation (dots, marks, and the like) of each number. For example, in teaching the child to read the number 3 as the word *three*, the teacher might well utilize three books, three pencils, or other objects, and three marks or three dots. Such devices as having the pupil tap three times, touch another child three times, or say the same sound three times may prove helpful. Although the procedures just listed are primarily useful in helping the pupil to develop a concept of cardinal numbers, they are more than justified as part of the teaching of reading of numbers, for they provide repeated opportunities for the pupil to see the relation between number symbols and the ordinal values and cardinal values for which they stand.

In teaching the meaning of the larger numbers, such as 6, 7, 8, and 9, the teacher should not only employ exercises involving objects like blocks and semi-concrete representations like pictures and marks, but she should also plan the exercises so that the pupil will see each of these larger numbers as combinations of smaller groups. Such exercises should lead the pupil to see that it is easier to identify the number of objects in a large group when the objects are arranged in a combination of smaller

groups. For example, after a little practice, the pupil will probably find that there is a marked difference in the amount of effort required to recognize that the quantity in each of these illustrations is 8.

(*a*) IIIIIIII (*b*) II II II II (*c*) IIII IIII (*d*) III III II

For most adults the total number of marks in *c* and *d* is easier to grasp than the total of marks in *a* and *b*, the greater ease being due to the particular grouping. But the teacher should remember that the arrangement of groups representing a number should not be limited to what an adult considers the best grouping. A large variety of groupings should be presented, and children should be allowed to select the grouping that they consider best. While most adults might perhaps prefer arrangement *c*, many children would prefer *d* or *b*.

From the preceding paragraph it can be seen that economy of thought is the major objective of the grouping of objects representing a number. The child's experience with group arrangements should illustrate to him the economy that results from grouping. If, for example, a child is assigned the task of giving nine small objects to each of thirty classmates, he will see the economy of counting them out by threes. The economy of grouping will be especially evident if he has to check his work. If as an additional assignment the child is required to record first by means of marks and then by means of a figure the number of objects given, he is in a position to see the advantage of a single easily read number symbol.

As has already been implied, such exercises are concerned more with number understanding than with numeration, but there is little point in teaching children to read numbers they do not understand. Ideally, numeration should not be undertaken until children have had much experience with number ideas. Practical considerations, however, prohibit such postponement. From the day of their entrance into school, children

are surrounded by written numbers. To postpone teaching the reading of numbers may force a teacher to neglect good learning situations and keep useful knowledge from the children.

Closely related to the reading of number symbols is the writing of numbers. The two operations may be considered as parts of the same job. In learning to write number symbols, children should see the sense of having such symbols. They should have experiences which will enable them to see that the Hindu-Arabic numerals are the best means of expressing certain quantities by symbols. Such experiences can be provided by letting the child draw pictures, squares, X's, dots, or other marks to represent quantities. In order that the child may see sense in what he does, situations must be created in which the child will find a need for recording quantities.

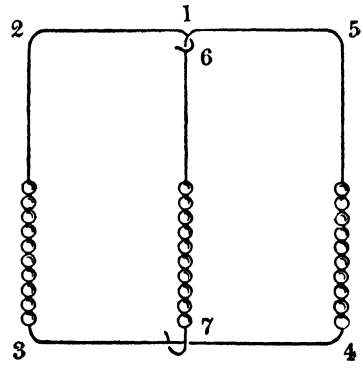
After the child has learned to read the first nine numbers, the collection idea of numbers should be emphasized. This emphasis can be accomplished by giving the meaning of number names, by using devices like bundles of ten sticks each, by grouping into tens a large number of objects that are to be counted, by using the tens square (page 75), and by using the abacus and the tens block.



The tens block and the abacus are two devices mentioned so frequently in this book that a description of each and a brief exposition of their merits seem essential at this point. The tens block, as shown in the illustration above, is simply a small block of wood which has been only partly cut or grooved into ten smaller blocks or sticks. Tens blocks may vary in size from $\frac{3}{4}'' \times 2'' \times 2\frac{1}{2}''$ to $1\frac{3}{4}'' \times 2\frac{1}{2}'' \times 3\frac{1}{2}''$. Each of the ones blocks used with the tens block is the size of one of the ten parts in the tens block. The tens block can be made by any person who has a small bench saw. The illustration should be sufficient to

permit accurate duplication of the block. The chief advantage of using a tens block to represent ten lies in the fact that the ten ones are bound together into one collection, one entity. A glance at the illustration will show clearly how this device is of value in teaching the meaning of quantities like 11, 23, 50, etc. In addition to its use in showing an economical way of thinking of tens numbers, this device is very valuable in teaching carrying and borrowing and the multiplication and division of tens.[1] Instead of the tens block described, blocks or stones of different sizes may be used to represent ones, tens, and hundreds.

The abacus used in the lessons described in this book was made from the wire of a coat hanger and wooden beads from the ten-cent store. Any piece of wire about 36 inches in length can be used. The steps in making it are as follows: Label one end of the wire point 1 (see diagram). Four inches from 1 at point 2 make a right angle bend and put 10 beads on the wire below point 2. Next make a bend at point 3, which is about 7 inches from 2. At another 7 inches make bend 4 and then put 10 beads on the wire above point 4. Seven inches from 4, at point 5, bend again, and $3\frac{1}{2}$ inches farther make a bend at 6. Put on ten more beads below point 6 and bend the end of the wire over the base wire at 7. Using about half an inch of end 1, bend it around the wire at 6.

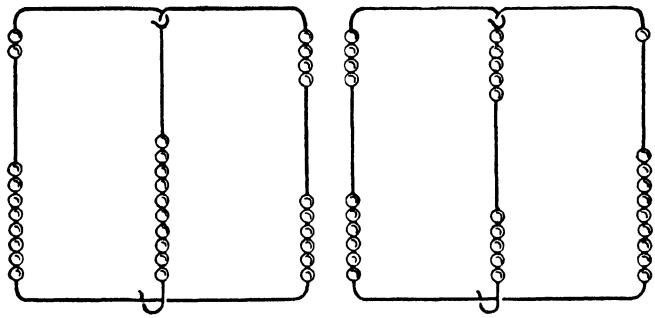


The principles governing representation of quantities on this abacus are the same as those used by the ancients. Students of the teaching of arithmetic will find it worth while to consult one

[1] For a detailed discussion of the uses and merits of the tens block see Herbert F. Spitzer, "A Device as an Aid in Teaching the Idea of Tens," *School Science and Mathematics*, 42: 65–68 (January, 1942).

or more references on the use of the abacus.[1] For other types of abacuses see Chapter 13, pages 389–390.

The following is a brief description of how the abacus can be used. The beads on the right represent ones, the middle row represents tens, and the row on the left represents hundreds.



To indicate quantity the beads are pushed up. The abacus in illustration (*a*) shows 204. The abacus in illustration (*b*) shows 451.

The abacus is an intermediate step between the relatively concrete representation of quantities with the tens block and the symbolic representation of quantities with numerals. Since the beads on different rows represent ones, tens, hundreds, and so on, in the same order as do the numerals in the "places" in our notational system, the abacus is an excellent device for teaching positional value.

In current practice the symbol 0 is read not only as "zero," but as "oh," "ought," "nought," and "cipher." For example, in giving the individual figures in the number 12,046 any of the

[1] Dantzig, *op. cit.*, chap. II; Margaret Drummond, *The Psychology and Teaching of Number* (Yonkers-on-Hudson: World Book Company, 1922), pp. 74–76; Wheat, *op. cit.*, pp. 40–47; F. H. Williams, *The Abacus and How to Operate It* (Singapore: Kelly and Walsh, Limited, 1946), pp. 1–27; L. D. Loy, "How to Use an Abacus," *Popular Science Monthly*, pp. 86–88 (August, 1948).

above terms might be used to indicate the zero. It is difficult to justify the use of five different names for the symbol 0. Zero alone should be used except in special cases — as, for example, in giving telephone numbers, where custom has established "oh" as correct. In addition to the five different names listed above, the symbol 0 is read in still another way when it appears as an integral part of a number. For example, in reading each of these numbers, 70,204, and 360,998, the zero affects the reading, but is not sounded as "zero" or "oh."

For many years teachers have tried to teach the reading of integral numbers of more than two figures without the use of "and." For example, they say that 106 should be read "one hundred six." Outside the arithmetic classroom, however, most people read 106 as "one hundred *and* six." The latter form is more in keeping with the current emphasis on meaning since the "and" shows that there are six ones in addition to one hundred. Besides, according to the rules of good English, the last of any series is indicated by preceding it with an "and." Advocates of the omission of "and" in reading integral numbers wish to reserve the "and" for use in the reading of fractions — as, for example, in reading 120.6 as one hundred twenty *and* six-tenths. They say that every time "and" is used in giving a number, the listener should know that the part following "and" is a fractional number. While this practice might have had some merit when arithmetic teaching involved dictating many long lists of figures, it no longer has much value. Even in dictation of numbers the use of "point" in decimals rather than "and" has been found to be more effective. Thus, the number 74.9 would be dictated "seventy-four point nine." The practice of requiring children to listen for the "and" may even be detrimental in that it diverts attention from the significant numbers. Consider, for example, the number which you may read as "eight hundred twenty *and* two-tenths." By giving "and" the special role of indicating a fraction, the reader directs

particular attention to the relatively insignificant two-tenths. The whole issue of the use of "and" is such a minor part of arithmetic teaching that no teacher should be criticized for failure to teach a child to omit the "and." [1]

The modern arithmetic text gives a great deal of attention to the reading of numbers written with figures. Major reliance in teaching is placed on the meaning of the number. The usual procedure is something like the following: "The number 380 is read three hundred eighty. The 3 means that there are three hundreds and the 8 means that there are 8 tens or 80 in the number."

After use of such initial teaching procedure as described above, the device of a chart labeling the places (see below) is often employed in teaching children how to read numbers. In such a chart the numerals in the number to be read are placed under column headings beginning at the right. The last number to the left is read first. After some experience with the chart some teachers try to get their pupils to memorize the names of the places (ones, tens, hundreds, and so on).

thousands	hundreds	tens	ones
4	3	0	8

Thus, when confronted with a number that is not easily read, the pupil can use his knowledge of the places in figuring out what the number is.

[1] Dorothy Rolston and Herbert F. Spitzer, "Oral and Written Expression of Numbers of Three or More Digits," *Elementary School Journal*, 40: 116–119 (October, 1941).

Notation

A number of references have already been made to our notational system. The Hindu-Arabic is the one we use for nearly all purposes. Briefly, this notational system consists of a plan whereby ten different symbols (numerals), nine for quantities and one to hold a place, are used to record all numbers. The importance of this system of notation cannot be overemphasized. Combined with the idea of the base,[1] it comprises much of what we usually mean when we refer to our number system. For a few purposes we use another well-known system, the Roman.[2]

The superiority of the Hindu-Arabic system over the Roman system can be easily shown by comparing and contrasting the two. Both systems use a base of 10, and both use a single mark to represent the quantity one. In the Roman system a second mark is added to the first to represent two, another to represent three, and in the old Roman a fourth to represent four. Then, probably because it was difficult to distinguish between four and five marks without counting, a new symbol was used for five. The adding of ones was then repeated to make symbols for the numbers six, seven, eight, and nine. Later the scheme was changed to include the subtraction of small amounts from large ones. IIII became IV and VIIII became IX. Every time the amount represented by known symbols became too great to be easily perceived, a new symbol was introduced. Thus, fifty came to be represented by L instead of by XXXXX, and five hundred by D instead of by CCCCC.

In the Hindu-Arabic system a separate symbol is used for each of the first nine quantities in the series. The tenth is represented by using the unit symbol again, but in a different position.

[1] See pages 20–22.

[2] See page 382, Selected References, *passim*. Many other systems have been used from time to time. The reader is referred to histories of mathematics for these.

This is accomplished by using a place-holder (zero) in the ones place and designating the second place as representing tens. This scheme does for notation what the base idea does for counting; that is, it goes back to the easily perceived and already known 1. In the tens place, of course, the 1 stands for a collection of ten, but is handled just as it is when it represents one unit or one. The same principle is used in expressing hundreds and other higher numbers. Thus, the value of a numeral in the Hindu-Arabic system is determined by its position. For every change of position to a higher order (one place to the left), a numeral represents a value ten times the value it had in its vacated position. For example, the numeral 2 represents successively two, ten times two, and ten times twenty as we write 2, 20, 200. No such multiplication of value is caused by moving a Roman numeral from one position to another in a number. In the Roman system the value of each numeral is always independent of its position. For example, X stands for ten, no more, no less, in XIX, LXXVII, and XLVIII. It is the principle of multiplying the value of a numeral by changing its position in a number that gives to the Hindu-Arabic system its unequaled capacity for expressing in simple form any number, no matter how large or how small it may be. The principle of position makes possible the writing of any number by using only as many different characters as the number of ones in the base of the number system. In the Hindu-Arabic system the base is ten. We can write any number we please by using just ten characters in various positions. If we used a number system with a base of eight, we should need only eight characters. For a number system with a base of twelve, we should have to have twelve characters. In each system of this kind there must be a character for each number from 1 up to and including the number which is just one less than the base, and there must always be one character which is used as a place-holder.

For example, the number of letters in the first six words of

this sentence, if expressed with numerals to base eight, would be written 34, meaning that there are three eights and four more. If written to base twelve, the number would be 24, or two twelves and four ones; and of course, to base ten the number is 28.

The Role of Zero

The Hindu-Arabic system of notation with its positional value is made possible by the use of a place-holder, our zero. We possess no positive knowledge about the discovery of zero. We are not even certain of the people who first used it. We once thought the Arabians discovered it. Then we gave the Hindus credit for it. Now we know that the Babylonians had the idea,[1] and that they probably borrowed it from others.

Without doubt, the discovery of this principle will always rank as one of the major steps in man's climb to his present state of civilization. The idea is so simple that many teachers have never given a thought to its significance. Educators have sometimes unwittingly hidden the main point by attaching to zero combinations such labels as "zero the clown," and "the tricky fellow." Some writers and research workers have even concluded that the zero facts (such as $0 + 2$, $8 + 0$, and 8×0) are the most difficult to teach. The important point to consider here, and to teach, is that zero is used in notation to hold a place. It is not a number in the sense that it represents a quantity. This simple fact can and should be taught as soon as children learn to write numbers. The symbol 1 in 10 is no different from the 1 representing unity, except that it is in the second position from the right. When 10 is first written, the attention of children should be called to this positional difference either by direct question or through some indirect exercise. Children's understanding of 10 would be enhanced if teachers would fre-

[1] Edward Chiera, *They Wrote on Clay* (Chicago: University of Chicago Press, 1938), pp. 154–157.

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quently refer to it as one ten. It really is one ten while 20 is two tens. Of course, the expression *one ten* is redundant in adult language, but it helps to get across to children the collection meaning of 10.

In order to call attention to the often neglected true uses of zero, the difference between zero and the other numerals has probably been overemphasized in the preceding paragraph. It was stated that zero was not a number in the sense that it represented quantity; but in many other ways, especially in number operations, zero is a number. An excellent example of zero functioning as a number is its representation of a point on a scale such as a thermometer. The place-holder function of zero is, however, its most important use and should receive major attention. (See pages 20-21 000.)

STUDY QUESTIONS

1. Since number names, as expressions of quantity, are abstractions that are very difficult for children of ages two to four, should children of that age be deliberately exposed to number names? (*1*) Yes. (*2*) No.

2. Jingles and finger plays make major provision to what phase of number knowledge? (*1*) The cardinal idea of number. (*2*) The number names. (*3*) The place of numbers in the series.

3. Does the kindergarten child have a need for number knowledge? (*1*) Yes. (*2*) No.

4. Is counting in unison a recommended teaching procedure? (*1*) Yes. (*2*) No.

5. When should the child's attention first be directed to the fact that eleven is ten and one more? (*1*) When the child is learning to count the numbers ten to twenty. (*2*) When the child begins to write numbers. (*3*) When the study of the base of our number system is undertaken. (*4*) At about third grade.

6. When children are learning to count (the order of the

number names) is emphasis on the how-many aspect of individual numbers important? (*1*) Yes. (*2*) No.

7. When should children be taught to count to 100 by 10's? (*1*) After they have learned to count to 100 by 1's. (*2*) After they have learned to count 1's to 10. (*3*) After they have learned to count by 1's to 50.

8. In learning to count by 10's to 100 which term "two tens" or "twenty" should be used first? (*1*) Two tens. (*2*) Twenty.

9. In school programs of instruction should learning to count (say the numbers in order) and learning to read numbers be taught simultaneously? (*1*) Yes. (*2*) No.

10. Is teaching children to count by rote recommended? (*1*) Yes. (*2*) No.

11. When children first use number in connection with objects, what is the child's purpose in using the names? (*1*) To tell how many objects have been enumerated or named. (*2*) To identify or to name the object.

12. In the statement, "The room number is 308," what kind of number is 308? (*1*) An ordinal. (*2*) A cardinal number.

13. Is the number 1716 an ordinal or cardinal number? (*1*) An ordinal. (*2*) A cardinal. (*3*) One cannot tell from the form used.

14. When man used the model-group idea of telling how many, was he able to tell exactly how many objects were in a group? (*1*) Yes. (*2*) No.

15. At what grade level should the school discontinue instruction in counting? (*1*) Second. (*2*) Fourth. (*3*) Sixth. (*4*) Eighth.

16. Children who can count backward with facility seem to have above-average arithmetical ability. Explain. (*1*) Counting backwards gives these pupils a powerful tool for use in subtraction and division. (*2*) Counting backwards is an indication of a child's familiarity with numbers. (*3*) Counting backwards forces the child to use all the fundamental processes.

17. How many names for the symbol "0" should the school teach? (1) (2) (3) (4).

18. In reading such numbers as 1702 some teachers permit children to say "and 2". Is the use of "and" in such situations permissible? (*1*) Yes. (*2*) No.

19. In the Roman system of notation is the value of a number (e.g., "X") independent of its position in the number? (*1*) Yes. (*2*) No.

20. What people should be credited with the invention or formulation of the principle of place value? (*1*) The Arabs. (*2*) The Hindus. (*3*) The Babylonians. (*4*) We do not know what people first used the idea.

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3

Number Experiences

Early Development of Number

Most teachers of primary-grade arithmetic agree that certain basic number experiences are desirable before a child is ready for the systematic study of arithmetical facts and processes. These basic number experiences are supposed to enable the child to develop concepts essential to understanding addition, subtraction and the like. Among the concepts usually accepted as fundamental or prerequisite to such processes as addition or subtraction are (1) the idea of identifying quantities by means of number names, (2) the idea that the number name belonging to any group may be found by counting, (3) the idea of grouping to facilitate counting, (4) the idea of substituting one quantity for another (e.g., 6 blocks for 6 houses), and (5) the idea of rearranging two or more groups into a single group (e.g., 12 for 8 and 4) or the rearranging of quantities in one group into two or more groups.

These and other concepts which are the foundation on which a child's study of arithmetic should rest are undoubtedly developed through many and varied number experiences. For example, consider the idea of applying number names to quantities. The child must learn to apply the name *four* to four fingers, four pieces of candy, the fourth in a series (e.g., fourth

clap of thunder), and so on. The idea of fourness as applied to all these situations is an abstraction that develops from many experiences. While a normal young child may generalize from one experience (calling all large animals in a zoo cows after being told that the first large animal seen is a cow), the generalization made is usually not a useful one until it is refined by application and correction in many situations.

The implications for instruction in the preceding statement should be fairly clear. In the preschool years and after entering school the child should have many number experiences which lead to the development of basic concepts.

The identification of important number experiences which might aid in the development of such basic number concepts is the problem of this chapter. The reference to preschool years above indicates that the development of number ideas should begin before the child enters school at age 5. An early start will enable a child to have more number experience with the resulting better concept development. Also, life presents many situations where the use of number and number ideas will make for richer living and better understanding of situations on the part of the child. Even the nursery rhymes and stories use number extensively, to say nothing of such expressions as "We saw two trucks," "There are four eggs in the nest," and the like. A further argument in favor of early number experiences is that pupils who enter school with good number background do well in elementary- and secondary-school mathematics.

Preschool Use of Number

Ordinarily we think of preschool uses of number as being rather incidental. Such use of the term incidental is a bit misleading. While some number experiences are truly incidental, in most situations where children have extensive number experiences there has been planning by adults. Even the con-

versational uses of number where children are concerned are not entirely casual. For example, a parent in telling a three-year-old about some deer he had seen as he drove along the road said, "I saw three deer. This many," holding up three fingers. Had the parent not been concerned with number he might have said "I saw some deer."

Parents concerned with number will count for children the objects pictured in books and magazines, the things seen on walks and rides, and will refer to specific quantities in play and conversation with children. It is through such experiences that most children get their start in the study of numbers. Since students of arithmetic teaching can influence practice in this phase of number development only indirectly, it is important that every opportunity to help be used. If we do no more than let parents and others who influence the preschool development of children know that early number experiences are of value to school progress, we will have done a service for children. Suggestions like those in the next section may be of some assistance to parents who request help.

Suggested Number Uses

The situations listed are commonplace enough not to require any explanations or identification:

1. "The candy bar costs five cents. You need one more penny."
2. "Bring me two more clothespins."
3. "I'll give each of you three sheets of paper. One, two, three for you and one, two, three for you," etc.
4. Playing race games of the parcheesi type where the distance a marker is moved depends upon the number of dots turned up on the rolled dice, the number indicated by a spinning arrow, and the like. Usually adults or older children have to play with the preschool child.

5. Playing with dominoes. The object may be to turn over the domino with the least dots (each draws one). Another variation is matching the dots as in the adult game but with young children the object of the game is merely to play or get rid of the dominoes. Here as in Example 4 above adults need to play with the children.

6. Hull Gull. Each player in turn shows the total number of objects (nuts, marbles, etc.), then places his hands behind his back, rearranges the objects and shows only one hand closed. Another player guesses the number of objects in the closed hand. If the correct number is guessed, the guesser gets all. If he fails, he gives the other player the difference between the guess and the correct amount.

7. "Are there enough plates at the table?" "Do you have enough candy to give each of us a piece?"

8. "How many pennies do you have?" "How many dollars is this?"

9. "Who has the most dolls?" "Do you have more marbles than John has?"

10. "When the big hand of the clock is at 12, the program will begin."

11. "There are just four more days until Christmas. Let's mark them on the calendar."

Early School Number Experiences

In the chapter on counting it was pointed out that when children enter school at age 5 or 6 they have a need for counting and other phases of number and that the school must provide instruction in these areas. Counting experiences of the kind listed in Chapter 2 will be among the first in a program of "early school number experiences." In addition, there are other number experiences which can be engaged in with profit by children during their first years in school. The list below indicates this type of number experience.

Uses of Number in Kindergarten

1. "The radio said we had seven inches of snow. I know it was deep because it came over my boots." The teacher used a ruler and the child's boots to help clarify this number situation.
2. "Let's listen carefully as the record is played to see if we can hear the bell ring six times. I'll count the times it rings."
3. "The red line on the thermometer is below 32. That means it's below freezing."
4. "Bob says he weighs 48 pounds. Is that more or less than 50 pounds?"
5. "Here are two kinds of dozen egg boxes. Let's use blocks to see if they hold the same amount."
6. "When the big hand of the clock points to 6, we will get ready to go out to play."

Uses of Number in the First Grade

1. "Our story begins on page 16. Who will show us 16 on the number line?"
2. "The farmer said he gave some cows 40 pounds of silage a day. Is 40 pounds more or less than you weigh?"
3. "When it is 10 o'clock, we need to put away our other work and get ready for our music."
4. "Halloween comes on October 31. Will you put a circle around that date on your calendar?"
5. "Jack says one toy costs 10 cents and the other costs 15 cents. We can count on the number line to see how much the two toys cost."
6. "Mary says it's hotter today than it was yesterday. How can we find out if she is right?" (A thermometer and a record of yesterday's reading were available.)

In these twelve situations pupils and teacher were using number to better understand or clarify what they were doing. They had a need for number.

While it is possible for some pupils to acquire sufficient essential number ideas for doing excellent work in arithmetic from the integrative type of number experiences described in the preceding section, it is not very likely that many will do so or that that is the most efficient way to acquire such ideas. The great majority of pupils need systematic instruction in the basic number ideas and even those with special interest or aptitude will probably profit markedly from such systematic instruction.

The importance of providing background number experience is so well recognized that there is a large amount of first- and second-grade number instructional material on the market. Almost all major companies who have series of arithmetic books for sale include material for Grades One and Two. Not all this material for Grades One and Two can be listed under the heading of early number experience but at least a part of it can be so classified. The student of the teaching of arithmetic will find the study of published materials for Grades One and Two profitable. The manual accompanying the material should be consulted. The major shortcomings of such material are: (1) It is written for the mediocre or below-average child (as is true of most material published for the beginner), and therefore keeps the gifted children on material that is pretty drab. (2) The reading involved in the use of much of this material puts a premium on reading ability. (3) The subject matter and materials (usually pictures) are not comparable to the subject matter and instructional materials used in the best first- and second-grade classrooms. To illustrate, compare just one counting situation in a first-grade book with the experience children would have in counting twenty flower bulbs they had ordered to see if the order was properly filled.

The limitations of published materials do not mean that they are not recommended. As already indicated, they have merit and in some situations make a real contribution to the instructional program. Some first- and second-grade teachers, how-

ever, prefer a type of systematic instruction independent of any published materials; others prefer to supplement published materials with special exercises. The list of suggested number experiences below is of the type that outstanding teachers have used either as a substitute for or in addition to published material.

1. *Exercises based on the number chart*

(*a*) "This chart (see below) has the numbers to 100. Some of you already know how to count that far. All of you know the numbers in the first row. Bill, will you say their names as you point to each one?"

The teacher then pointed to 10 and said, "Who knows what this is? Yes, it is ten. Look at it carefully. What is the difference between the 1 in ten and the 1 up here (the units row 1)?"

If no child gives the answer, the teacher says, "It is not different except that there is a zero beside it. This 1 means only one. This 1 means one ten. Here the zero is in the ones place. Who can find two tens? Yes, that is two tens. What tells you that it is tens and not just two ones?" Other tens numbers may be discussed in the same way that one ten was treated above.

(*b*) "Touch the 10's as you count to 100."

(*c*) "Find the number that is 5 tens and 2 ones; the number that is 2 more than 6 tens and 8 ones."

(*d*) "John says that 65 is 10 more than 55. Show with the number chart that 65 is 10 more than 55."

(*e*) "What is the number that is 2 tens more than 31?"

(*f*) "To find how many 25 and 32 are, Mary drew a line down from 25 to 35 to 45 to 55 and then went two numbers to the right. She said it was 57. Do you think Mary was correct? How could you prove that you are right?"

A chart containing the numbers one to one hundred in rectangular form similar to the one shown below is almost standard equipment in many first and second grades and is found in many third-grade textbooks. Most of the charts differ from the one shown here in that they contain the first 10 numbers in the first line. There are advantages for each type but for most work

Number Chart with Fixed Numbers

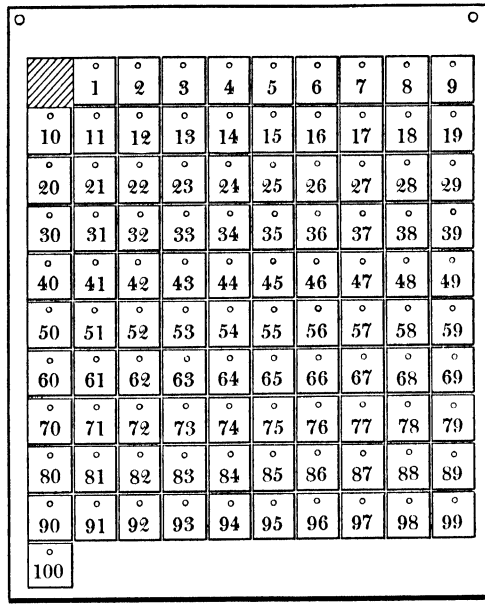
	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99
100									

with children in Grades One and Two the one shown is superior for these reasons: (1) The tens appear on the same row. For example, the two tens row contains all the "twenty" numbers on the chart. On the other type of chart 20 would be on the teens or one ten row and 30 would be on the row with the numbers 21 to 29. (2) The tens, being at the left, stand out and permit easy reference. Furthermore, the initial position shows rather clearly that 11 is 10 and 1 more, 51 is 50 and 1 more. (3) The seeming incompleteness of the chart (first row has only 9 numbers and 100 appears alone on a row) illustrates an important characteristic of our number series, namely that it has no end — there is always another number.

Another type of number chart which has removable numbers is superior for many purposes.[1] A chart made of plywood is

[1] Also see Donna Norton, "The Numeral Frame — A Device for Use in the Teaching of Counting," *The Mathematics Teacher* 44: 382–384 (October, 1951).

best but other materials may also be used. Nails, screws or pegs are driven into the board. One hundred blocks or disks



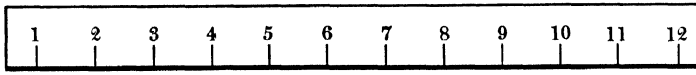
Number Chart With Removable Numbers

(poker chips may be used) are numbered on one side only from 1 to 100. A hole is bored in each block or disk, and they are hung on the board in the order shown above.

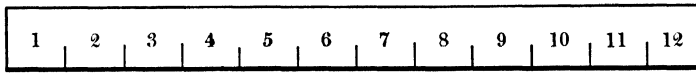
2. *Exercises based on the number line*

Number lines are of two types, the horizontal and the vertical. Both types consist of the numbers from 1 to 100 spaced evenly on a measured line. The best horizontal lines are placed above or at the top of the chalk board and to be easily seen should allow about two inches to each number. It is important that the numbers be placed correctly on the line. The correct and

incorrect placement of numbers is shown on the two lines reproduced below.



Correct placement of numbers



Incorrect placement of numbers

For the vertical line, which is usually about four feet long, only one-half inch of space is allotted to each number and only the 10's and 5's are actually placed on the chart.

(*a*) "Some of you seemed to think that 43 was more than 62. Let's look at the number line to see if that will help decide which is more."

(*b*) "There are 26 in our room and 27 coming in from the kindergarten. How many chairs will we need?" Counting on the number line was suggested as a solution.

(*c*) "The story said that there were 6 dozen eggs hidden on the lawn. We can use the number line to find out how many that is. A dozen is 12. As you count the dozens, I'll mark them." (Teacher marked on chalk board below the number line.)

(*d*) "The trip lasted 34 days. Is that more or less than a month?" Consulting the calendar showed 31 days as the longest month and then by using the number line the children saw that 34 was more than 31.

(*e*) "In playing Hide and Seek most players count by 5's. The number line is good to use in learning to count by 5's. I'll point to the 5's. You give the names."

(*f*) "John had 20 cents. At the store he spent 12 cents. How much money should he have left? Let's count backwards on the number line to find out what he should have left."

3. *Exercises to show the composite nature of numbers*

(*a*) "There are nine people living in the three houses pictured here. Bill, will you show how many you think live in each house?"



(*b*) "We need two children to take care of the rabbits each day. There are 24 of you in the room. How many days will it be before the first two get another turn?

4. *Exercises to develop correct ideas of the place of numbers in the number series.*

(*a*) Some number cards, for example, 1, 8, and 15 are placed in the chalk tray. Then, as individual pupils draw other number cards from a stack, they attempt to place each new card in the chalk tray in the correct place in relation to the cards already there.

(*b*) Putting numbers in place on the movable number chart (see illustration, page 67). With all number blocks removed from the frame or board, the class, under the teacher's direction, begins the task of putting numbers in correct position. If block 27 is picked up the teacher says, "In what row does this go? Where in the row?" A number of other blocks are placed by the same method. Then, a committee or an individual is assigned the task of placing the other blocks in position. This procedure, especially for individuals, can be used many times. Turning the blank side of the blocks out on any row and then asking pupils to tell what number is on the other side of a block is another way of calling attention to the position of numbers in the series.

(*c*) Using book page numbers and house numbers. If the page desired is 116 and the book is opened at page 142 the

teacher asks, "Which way shall I turn to find page 116?" Similar procedures may be followed with house numbers.

(*d*) Using the number line or the calendar. These two devices are not as useful in helping children get number position as those previously suggested but they do help. For example, when a number like 95 is being considered, such comments as "Is that nearly a hundred? Look at the number line," can be used. The use of such phrases as "next to the last day of the week" or "near the end of the month" calls attention to position of numbers.

(*e*) Consideration of orally stated questions. "Which is more, 87 or 73?" "926 or 516?" and so on. (For showing the correct answer to hundreds and thousands situations the number line is also used, but the numbers on it are thought of as hundreds or thousands instead of ones and tens.)

5. *Problem-solving experiences*

All children in their everyday experiences encounter some problems involving quantity. The solution of such problems (with the assistance of mother, father, older brother, sister, companion, or of the teacher) is one of the best experiences for the development of a sound numbers background. The following are examples of the type of problems encountered.

A six-year-old boy who had received a pair of pigeons for pets came to his mother with the important news that the pigeons had two eggs. "How long before there will be babies?" he asked in an excited voice. By reading a pigeon book, they found the statement that the eggs usually hatch 17 days after the second one has been laid. By using the week (2 seven days and 3 more) they determined the day when the babies should appear.

Two fourth-grade pupils who read that often as much as two thousand feet of thread is found in one silkworm cocoon were trying to figure out an easy way to measure the length of the

thread as they unwound it. With the teacher's assistance they decided to use a rectangular box that was 12 inches around for a reel. Measuring to find a box 5 inches wide and 1 inch high was not easy but the actual recording was far more of a problem. The pupils decided to record only every tenth turn. When ten turns had been counted, a one (for one hundred) was written at the side and the counting of tens was repeated. When the thread was unwound the totaling of the hundreds, tens, and the remaining ones was a relatively simple matter.

To provide such problem-solving number experiences as those described in the preceding paragraphs is one of the obligations of the school. The use of number projects such as those listed in the chapter on problem-solving (see page 197) is one of the ways to provide experiences. Less extensive problem settings than the numbers project can be provided by collecting life-like quantitative problems (word arithmetic problems) for use with arithmetic classes. Some examples are given below.

Grade One. "John weighed his four rabbits. Two weighed 6 pounds each, one weighed $5\frac{1}{2}$ pounds, and one weighed $4\frac{1}{2}$ pounds. How much did the four rabbits weigh?"

This problem was solved by the class in the following manner. Books were put on the scales until a weight equivalent to each rabbit was obtained. Then all the books were weighed. The teacher then said, "We can also find the total weight by using the number line. I'll mark the first 6 pounds on it. Now, who can tell how to mark the next one?" By the time all the weights were recorded on the number line, the total had been found. Simpler word problems than the one just given, stated orally to the class and solved by individual pupils, are also a good means of providing arithmetical background. These relatively simple problems as well as the more complicated ones should involve not only addition and subtraction situations but also multiplication and division.

Grade Two. "Nancy wanted to know about how many grains there were on a large ear of corn. There were 55 grains on the first row she counted. There were 16 rows. How could she find the total number without counting the other 15 rows?"

In the class discussion there were several suggestions but the following seemed best to a majority of the children. First use only 50 grains for each row. Then 2 rows make 100. Mark the rows by two's. That makes 800. To count in the 5 grains of each row that were left out by using 50 instead of 55, count 16 5's on the number line.

As was recommended for first grade, relatively simple problems stated orally to the class and solved in many ways are recommended for the problem-experience program in the second grade. A very important point to remember in the solutions is that no attempt is made to arrive at or use the best adult method and that recording with pencil or chalk is done primarily to explain or to show the pupil's thinking to others.

Grade Three. "The story we have been reading stated that the ox wagon trip from Council Bluffs to the Willamette Valley was over 2200 miles. If the settlers travelled at the rate of 16 miles a day, about how long did it take them to make the trip?"

The class found out first how long it would take to go 100 miles. They did this by adding 16's. Then they took 22 times the number of days (6) spent in traveling 100 miles. This last operation was performed on the number line. (In this case it had to be used a second time.)

Grade Six. "A ship's captain in the Indian Ocean noticed that when the sun reached the zenith it was 3 hours and 27 minutes past noon by Greenwich time. What was the longitude of the ship?"

The teacher pointed out that the captain had a table which could be used to find the longitude but since the pupils had no

such table, the longitude could be figured by finding how many hours or parts of hours (minutes) equal one degree. This was done by dividing 24×60 by 360. It was found that each degree equals 4 minutes. The 3 hours and 27 minutes were then divided by 4. The group decided that the ship was at 51° and 45 minutes east longitude.

Each grade in the elementary school needs problems like those presented above. These problems usually involve computations requiring more advanced arithmetic than the pupils have studied in school and therefore call for the assistance of the teacher. The class works out a solution together. This procedure is similar to the class solution of science and other content problems. Pupils' experiences in the class solution of difficult quantitative situations not only give them insight into numbers and illustrate uses of arithmetic but build confidence for attacking other problems.

6. *Exercises for developing the idea of ten*

Many of the teaching suggestions and exercises already listed in this chapter and in Chapter 2 were concerned with developing what is commonly referred to as the idea of ten. Essentially it is only the idea of a number base. Understanding of and the ability to use the base idea of numbers is very important for learning the fundamental operations of arithmetic with whole numbers. Even counting, the fundamental number operation, is learned easier and understood better if the idea of ten is used. That is why counting by tens to one hundred before counting by ones was recommended. Other points of emphasis in counting which had for their purpose the development of the idea of ten are recalled in the following teaching questions and suggestions: "What number is ten and two more?" "Find the third ten." "How does the 1 here (in 10) differ from this one?" "Show us the number that is 5 tens and 3 ones." "What is another name for twenty?"

Since the addition of whole numbers making sums larger than ten is essentially only a rearrangement of two or more numbers into one number composed of tens or multiples of tens and ones, it seems logical that knowledge of this fact would be of assistance to the learner. For example, the addition fact "9 and 7 equal 16" is the same as "9 and 7 equal one 10 and 6." This fact can be easily demonstrated by rearrangement of the ones in the first combination (putting one from the 7 with the 9 to make the collection of ten). Attention is called to the use of "collection" in the preceding sentence and to the use of "one ten" in an earlier sentence. It is very important for pupils who are using tens numbers to think of tens as collections and not as ten separate ones.

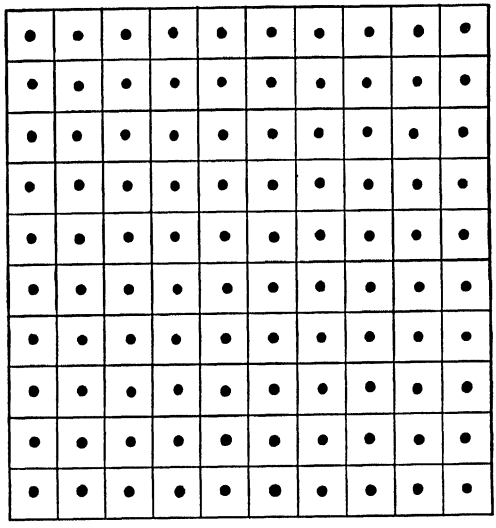
The other fundamental operations with whole numbers are also a matter of rearrangement of the tens or multiples of tens and ones. In connection with the teaching of these processes there are many opportunities to develop an understanding of the ten idea. The following teaching suggestions give an idea of the possibilities: "Show with marks that nine 7's equal 6 tens and 3 ones;" "use the tens and ones blocks to show that 42 (4 tens and 2 ones) is equal to six 7's;" "use the abacus to show what borrowing means; subtract 17 from 40."[1]

The tens block and the abacus (see Chapter 2) when properly used are two excellent devices for developing the idea of ten. The number line and number chart already described earlier can also be of assistance if full advantage is taken of the possibilities that exist. To give emphasis to tens on the number chart the exact tens should appear at the beginning of each

[1] For the report of an outstanding project on the use of the ten in subtraction see D. Banks Wilburn, "Learning to Use a Ten in Subtraction," *Elementary School Journal* (April, 1947), 47: 461–466. For other ideas on the use of ten in the fundamental operations see H. G. Wheat, *The Psychology and Teaching of Arithmetic* (Boston: D. C. Heath and Company, 1937), pp. 238–278.

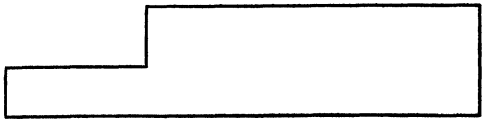
row. (See chart on page 67.) The exact tens on the number line should also be made to stand out.

Another device which helps in developing this understanding is the tens square or one hundred board. This device consists of a square with ten dots, ten holes (as in peg board), or ten bottle caps (fastened to a board) in each row and column.

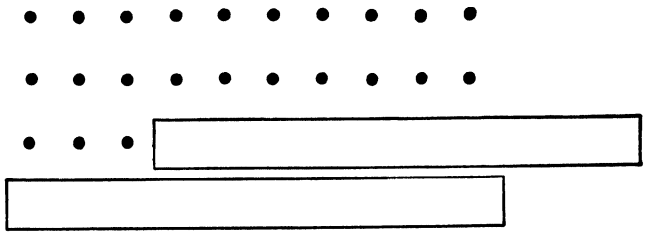


Tens Square

For use with an entire class the tens square made by fastening bottle caps or other objects to a board is probably the best type. It is large enough to be seen from any part of the room and the use of actual objects provides a superior presentation of the quantity to which the numbers refer. A notched marker made from oak tag or two such markers are used to show that a num-



ber such as 23 is equal to 2 tens and 3 ones. The illustration below indicates one way the child could show 23.



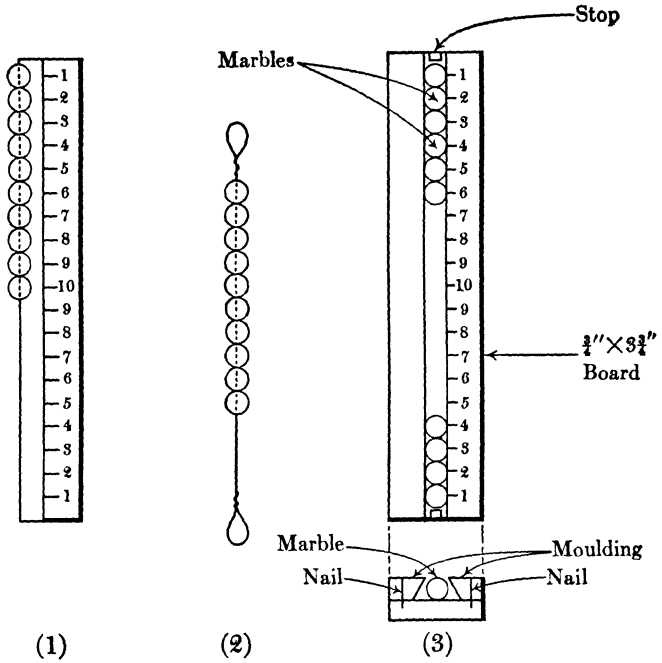
For use on the arithmetic table the peg board type of tens square is superior, for when a child puts the pegs or nails in the holes they stay. They can also be easily removed and require little storage space. On one arithmetic table beside a tens square peg board was a card with "Show 37" written on it. Nearby were the nails for pegs and the cards with the pupils' names. When a child did the suggested task he placed his name card on the board.

Fact finders are excellent for finding the various combinations of numbers that equal ten and the amounts that each number is less than ten.

Fact finders are of two general types, those using a wire and those using a slot. The wire type may be made with or without numbers. Shown above (1) is the wire type with numbers. It uses a board $1\frac{1}{2}'' \times \frac{3}{4}''$ in size on which the numbers are written. A wire on which are strung the movable wooden beads (or spools, or disks) is fastened to the board.

The wire type without numbers (2) simply calls for a series of ten beads or disks strung on a wire as shown above.

The slot type (3) calls for two strips of molding to be nailed to a $\frac{3}{4}'' \times 3\frac{3}{4}''$ board. Ten marbles are inserted in the slot which is so formed and the ends of the slot are then closed with a small block or other stop. The numbers are then marked on one or both strips of molding.



Either form may be used without the numbers. To find or to show what 5 and 3 equal, the pupil pushes up 5 beads or marbles and then 3 more. The total 8 will then be at the top of the wire or slot and will be recorded at the right.

Sliding scales are also of value in showing the composite nature of 10. For illustration see page 388.

Blocks representing units one to ten and an appropriate tens case or box are perhaps the best materials for showing the composite nature of ten. Excellent materials of this type have been prepared by Catherine Stern and published by Houghton Mifflin Company. The manual which accompanies these materials has good teaching suggestions, as does the book *Children Discover Arithmetic.* (Catherine Stern, *Children Discover Arithmetic.* New York: Harper and Brothers, 1949.)

7. *Exercises for showing the composite nature of numbers to 10*

The fact finders, sliding scales, and blocks mentioned in the preceding section for showing the composite nature of ten can also be used to illustrate the composite nature of other numbers. Other ways of helping children see the composite nature of numbers are indicated by the following directions a teacher might give.

1. "John showed how many six is by placing the candles on the birthday cake he drew this way  .

He had 2 and 2 and 2. Is that six?"

2. "Use the grains of corn to show eight in as many ways as you can. Make a record of each way with marks on your paper. Then you won't count the same way twice."

3. "Let's show seven in different ways on our pocket chart. Arrange the pictures to show seven in different ways."

4. "What number is this block?" (Teacher pointed to the seventh block on the movable number chart. The first four were numbered side out and the next three had the blank side out.)

5. "I'm thinking of two numbers that equal 9. What are they?" (Children enjoy this guessing game and can act as leaders as well as guessers.)

Experimentation with the number line will show that it is one of the best devices for showing the composite nature of numbers. It can be most easily used for this purpose if it is placed so that the parts which equal such a number as 12 or 27 can be marked on the chalkboard beneath the line.

The student of arithmetic teaching should not assume that the composite nature of numbers refers only to numbers thought of as ones. In fact, the idea applied to the collection aspect of large numbers is probably as important as the idea applied to the relative small numbers with which the preceding situations

dealt. For example, the number 84 may be thought of as 84 ones, 8 tens and 4 ones, two 40's and 4 ones, and so on. The necessity for being able to think of a number such as 84 in two different ways is illustrated by dividing the number first by 2 and then by 9. Using 2, we can divide the tens and then the ones. Using 9, we can divide the tens but must think of them as ones before we can divide.

The teacher may bring out this idea of numbers by asking, "In the number 761 there are how many tens?" If the pupils answer that there are 6 tens, the teacher points out that one could also say there are 76 tens. Other questions which might follow are: "How many are 14 tens?" "How many are 72 hundreds?" "4 thousands are equal to how many hundreds?" "7 thousands are equal to how many tens?"

Foundation Number Experiences for Advanced Phases of Arithmetic

The experiences suggested in the preceding part of this chapter are concerned primarily with providing the essential foundation for the introductory or beginning phases of arithmetic, specifically, counting and the simple work in addition and subtraction. As was indicated by some of the suggested problem-solving experiences and by some of the exercises for developing the idea of ten, this provision of background experiences is not limited to the primary grades. It has been generally assumed that once the pupil has gained some facility with the elementary phases of arithmetic, no further work to provide a background of experience is essential. Yet, if possession of certain concepts makes the child's learning of counting, addition, and subtraction easier, then it would seem an advantage to a child who is starting more advanced phases of arithmetic to have the same opportunity to acquire certain concepts before he is asked to master new processes.

A task of those concerned with instruction is, therefore, the furnishing of experiences which will give to pupils the major concepts related to these advanced areas of arithmetic. The five main areas for which background experiences need to be provided are (1) multiplication and division of whole numbers, especially tens and hundreds numbers, (2) fractions and operations with fractions, (3) measures and the operations performed with them, (4) decimals and per cents, and (5) the solving of problems.

A few suggestions for such a program in the upper grades will be found in the chapters that follow. Special attention is called to the section on reference measures (page 286), the section on background needed for fractions (page 204), and the numbers projects in connection with problem solving. (An excellent reference for those interested in providing background experiences is Gladys Risden, *How Big? How Many?* Boston: Christopher Publishing Company, 1951.)

STUDY QUESTIONS

1. Does the young child need many experiences before generalizing? (*1*) Yes. (*2*) No.

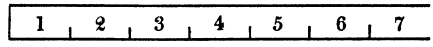
2. Are good preschool uses of number primarily incidental? (*1*) Yes. (*2*) No.

3. Do counting experiences provide many really significant background number experiences? (*1*) Yes. (*2*) No.

4. At what grade level should pupil's attention first be called to the place-holder role of zero? (*1*) First. (*2*) Third. (*3*) Fifth. (*4*) Seventh.

5. What is the main reason for placing the number on primary number charts in horizontal rows rather than vertical columns? (*1*) The horizontal furnishes a better copy for the child's writing of numbers. (*2*) The tens stand out better. (*3*) The horizontal is in harmony with primary reading practices.

6. Is the number chart with movable numbers superior to the one with fixed numbers? (*1*) Yes. (*2*) No.

7. Are the numbers on the portion of the number line reproduced here correctly placed? 

(*1*) Yes. (*2*) No.

8. For the problems solved in the number experience program are solutions the most advanced type? (*1*) Yes. (*2*) No.

9. What is probably the major weakness of published arithmetic materials for use in first and second grade? (*1*) The reading is too difficult. (*2*) The content is too difficult. (*3*) The content is too simple to challenge.

10. Is the use of column addition recommended as a part of the first-grade number experiences? (*1*) Yes. (*2*) No.

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Buckingham, B. R. *Elementary Arithmetic: Its Meaning and Practice.* Boston: Ginn and Company, 1947. For number line see pp. 4, 33, 51–53, 58–59; for background experience material in fundamental processes consult the following chapters: 5 (addition), 6 (subtraction), 7 (multiplication), and 8 (division); for background material in measurement consult Chapters 13 and 14.

Lamborn, E. A. Greening. *Reason in Arithmetic.* Oxford: The Clarendon Press, 1930. Pp. 1–28.

Morton, Robert L. *Teaching Children Arithmetic.* New York: Silver Burdett Company, 1953. Pp. 67–114 (for a different type of number chart see p. 86).

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Swenson, Esther J. "Arithmetic for Preschool and Primary-Grade Children," in *Fiftieth Yearbook*, Part II, National Society for the Study of Education. Chicago: University of Chicago Press, 1951. Pp. 53–75.

Van Engen, Henry. "The Formation of Concepts," in *Twenty-first Yearbook*, National Council of Teachers of Mathematics. Washington, 1953. Pp. 69–98.

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4

Addition and Subtraction of Whole Numbers

"I find my second-graders just don't know their combinations, so I'm giving most of my arithmetic time to a study of them," said a teacher as she and other teachers engaged in an informal discussion between sessions of a fall teachers' convention. In the course of this discussion these other comments were made by various teachers in the group:

"You seem concerned about your second-graders. I spent practically all of the first eight weeks with my third grade working on the basic addition and subtraction facts"; "I spent only about a week on the facts with my fourth-graders, but they didn't have automatic mastery of them when school started"; "I think pupils need to work more on the facts in first grade"; "I haven't even worked on the combinations yet in my second grade this year"; "Our course of study indicates that we are to work only on the easy addition and subtraction facts in second grade"; "We are using a workbook in second grade. It is very easy for my pupils, but I think they are learning to add and subtract." "You had better watch that stuff. That's what I did last year and this year the third grade teacher found that the pupils didn't know how to add and subtract. I'm using flash cards with the workbook this year."

Some of the important issues concerning the teaching of addition facts are hinted at in these statements. For the inexperienced teacher these questions might arise: "Why do some teachers use the term 'combinations' while others use the term 'facts'? What is meant by 'automatic mastery'? Why are they called 'basic facts'?"

The experienced teacher will probably think such questions as: "Should the same methods of teaching addition and subtraction be used in second, third and fourth grade? Are addition and subtraction overemphasized in lower-grade arithmetic? When should instruction in addition and subtraction begin? These teachers' remarks seem to indicate that all the talk about foundation experiences instead of teaching facts is just talk as far as classroom teachers are concerned. Is this a true picture of teaching conditions?"

While the remarks of this group of teachers may not be representative of a majority of the profession, the problems expressed are not uncommon. Traditional arithmetic instruction in school began with the teaching of the basic addition and subtraction facts (for identification of basic facts see page 92). The work with counting and the reading and writing of numbers was usually considered as an introduction to adding and subtracting. Probably, because of the importance of these two fundamental processes in instructional programs and in life, teachers have tended to overemphasize them in the instructional program.

Before studying the teaching of any area of arithmetic, it is well to consider some of the fundamental characteristics of the area. The student should ask, "Just what is addition and why do we add?" Consider, for example, the verbal problem situation, "There were 6 children on the porch and 3 on the lawn. How many children in all?" Through the process of addition the two groups of children can be combined and treated as a single group of children. This combination of two or more groups or quantities into a single group is known as addition.

It should be noted that no new quantities are produced and, therefore, adding is essentially only a process of rearranging quantities. For the problem situation above where $6 + 3 = 9$, there were just as many children before the addition took place as there were after the addition.

"Why add?" is a question every student of arithmetic should ask. A look at the 6 children + 3 children = 9 children situation indicates the most satisfactory answer to the question. While 9 is not more than 6 and 3, it is a one-number expression of the quantity in question, and without doubt it is easier to think, say, and write one number than two numbers. We add to simplify a quantitative situation. The other fundamental processes, namely, subtraction, multiplication, and division, are also rearrangement processes, and the purpose of this regrouping is to facilitate thought with the quantity involved.

In considering the fact that the addition process is a rearrangement of two or more numbers (quantities expressed with numbers) into a single number, it should be noted that there is not just one way of rearranging. In other words, $8 + 7$ may be thought of as $8 + 2 + 2 + 2 + 1$ or as $8 + 2 + 5$. In a like manner $18 + 43$ may be considered to be $20 + 40 + 3 - 2$, $10 + 40 + 8 + 3$, and so on.

In learning the processes of addition and subtraction and the basic facts of the processes, the pupil should use the processes and acquire some facility with them before intensive study is undertaken (see section on counting, pages 28–32). In actual practice, every child begins to acquire some ideas of addition as soon as rational counting is undertaken. For example, the child who has counted 6 objects and then, as he touches another object, says, "and this one makes 7" is in reality adding, for one group of 6 and another of 1 are combined into a single group of 7. Thus, the answer to the question: "When should instruction in addition and subtraction begin?" is "As soon as the child begins to count rationally." However, this first instruction is

far from the kind the teachers at the convention were discussing. For want of a better name, instruction in addition and subtraction in kindergarten, first grade, and probably for the major part of second grade may be called "a foundation program of addition and subtraction number experiences."

Foundation Program for Addition and Subtraction Number Experiences

Since counting is the fundamental number operation and addition is a group form of counting, it follows that counting procedures are important parts of this program of preparing for addition and subtraction. A few of the many counting situations which provide good background experiences are listed below.

1. *Showing a given quantity through various arrangements of objects and marks*

"Bill is 6 years old. When I asked him to show me how many 6 is, he held up 3 fingers on each hand, like this. I wonder who can show 6 in another way." As the various combinations of fingers were shown, correctness was checked by counting and the teacher sketched the number of fingers shown on the board. She then said, "I see that 3 and 3 equal 6; 4 and 2 equal 6"; etc.

Showing a number such as 8 by arranging grains of corn or other objects in different patterns is another example of this type of exercise.

2. *Grouping data and finding the total*

"For our news this morning, let's use our weather calendar to see how many days of sunshine we have had so far this month. John, will you count the days of sunshine for the first

week?" When John's report and those for the other weeks had been given, the teacher put the appropriate number of marks for each on the blackboard and then said, "Now let's find how many sunshiny days we had altogether."

In such procedures the pupil has an opportunity to see data placed first in small groups and then combined into a single group.

3. *Performing number chart and number line exercises*

   (*a*) "Who can find the number that is 6 more than four 10's?"

   (*b*) "What number is two 10's less than five 10's?"

4. *Telling or writing a number that is 1, 2, or 3 more (or less) than each of the numbers given on a chart*

   For this exercise, numbers on the chart should be in random order.

5. *Solving everyday problems by substituting manipulative objects for the quantities and combining or separating*

   (*a*) "Yesterday we planted 18 bulbs in our garden and today we planted 12 more. How many bulbs have we planted altogether?" In answering this question the teacher suggested that 18 blocks be used to stand for the bulbs planted yesterday, and that 12 more be used for the ones today. The total sum of the two groups was then found by counting.

   (*b*) "We have 5 of the 12 bulbs left. How many have we planted?" In finding the answer, objects were used as in *a* above. "Let's find the answer to the number question by counting on the number line."

6. *Exercises involving a quantity expressed as a number and a concrete representation of another quantity*

Teacher shows 3 or any other number of fingers. To a given number such as 5, the pupils try to add the number of fingers the teacher holds up. Similarly the children may be asked to find how much is left when the number of fingers is taken away from a given number.

7. *Playing various race games, twinks, dominoes, putting people in houses*

8. *Discovery of number facts through manipulation of objects*

(*a*) "Yesterday when Bill was working at the number table he found that 4 blocks and 4 blocks make a group of 8 blocks. He said, '4 and 4 make 8.' That was good work. I am writing the fact Bill found here, and I am drawing a picture so that you can see that what Bill said is correct.

Teacher wrote:

4	and	4	make	8
OOOO		OOOO		OOOOOOOO

You can use the blocks on the table to learn many such facts about numbers. When you are sure you have learned a new fact, tell me, or, better still, tell mother what you learned about numbers."

(*b*) Directed class study of a group of objects for the expressed purpose of discovering or learning number facts.

Each child is given a quantity of blocks or other objects (e.g., 9) with directions of this type. "Let's see what we can learn about this group by separating it

into several groups and then putting the parts back together again." The major danger in this type of exercise is that pupils do not see or have a purpose for what they are doing. Making a record of the facts demonstrated (e.g., $5 + 4 = 9$; $9 - 6 = 3$; etc.) adds to the effectiveness of this procedure.

9. *Adding and subtracting 10's, 100's, and 1000's*

(*a*) As a closing part of an oral exercise the teacher said, "I wonder if any one knows how many two 10's and two 10's are?" This type of exercise may include adding and subtracting hundreds and thousands.

(*b*) "Who wants to show with the hundreds stones how many hundreds 500 and 300 are?"

Experience with expressions of large quantities seems to be especially challenging to many children. Since the adding and subtracting of tens, hundreds, and so on is essentially the same as working with the ones, there is no reason why the larger numbers should not be used, provided symbols such as blocks, stones, and the like are available for showing an actual operation.

10. *Writing or showing large quantities in several ways*

(*a*) "John wrote two hundred and twenty-five this way:

100
100
25

Don't you think that is a good way to show two hundreds and a twenty-five? Let's try writing some numbers the way John does. Try writing three hundred fifty his way."

(*b*) "Write forty-six to show how many tens and ones."

While it is difficult if not impossible to identify for each of these exercises the explicit thing a child learns, it is worth while to the student of arithmetic teaching to attempt to figure out possible specific learnings for some of the procedures listed.

Through some of the exercises pupils can see that the quantities expressed with two or more smaller numbers are the equivalent of the quantity expressed by one larger number; that the quantity represented by a large number can be separated into smaller groups and expressed by use of smaller numbers. The pupil can see actual putting together of two groups to form a single group as in exercise number 4. Through solution of everyday problems such as those suggested in exercises 2 and 4, the pupil can see how numbers and number rearrangement make for simplification of concepts and of language; and, of course, through the solution of such problem situations the pupil acquires confidence in attacking quantitative situations. Furthermore, experience with all such situations provides the familiarity with the nature of addition and subtraction which will enable the pupil to deal rationally with these processes when systematic study for mastery is undertaken.

Basic Addition and Subtraction Facts: Number and Order of Teaching

The basic addition facts consist of the sums obtained by the combination of every one-figure number with every other one-figure number and with itself. These facts are called basic because with this knowledge and knowledge of carrying you are able to add any two numbers. If zero is considered a number, there are 100 basic addition facts. Since zero does not represent a quantity in the sense that the other one-figure numbers do, there is nothing to be added when zero is one of the numbers of a combination. The sum is then the other number. Many authorities prefer to teach the preceding generalization con-

cerning zero and thus try to present only 81 basic addition facts. These 81 facts are presented on page 92. If the ones facts are also considered as a group which can be learned by a generalization that "when 1 is one of the two numbers of a combination, the sum is the next number in the series after the other number," only 64 separate facts remain. Of these 64 combinations 28 are the reverse combinations of another fact, which leaves only a total of 36 distinctly unrelated facts.

The basic subtraction facts consist of the facts obtained when all the one-figure numbers are subtracted from the numbers 0 to 18. They are subject to the same variation in number as the addition facts. The 81 subtraction facts are shown on page 93.

The basic addition and subtraction facts have been arbitrarily divided into easy and hard facts. The easy addition facts are those with a sum of 10 or less. The hard subtraction facts are those with minuends greater than 10. The reader should recognize that the classification is based entirely on the size of the sums and minuends. There are so-called easy facts that are inherently more difficult than some of the so-called hard facts.

The general procedure is to teach the easy facts first. This practice is psychologically sound and is to be followed unless there are good reasons for doing otherwise. However, since the time allotted for mastering all basic addition and subtraction facts is relatively short, no great disadvantage will result from simultaneous consideration of some easy and some hard facts. This was done in the illustrative teaching procedure described earlier.

Some students of arithmetic teaching have spent a great deal of effort in trying to determine the inherent difficulty of the various addition and subtraction facts.[1]

[1] Frank L. Clapp, *The Number Combinations: Their Relative Difficulty and the Figuring of Their Appearance in Textbooks* (Madison, Wisconsin: University of

81 ADDITION FACTS

1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	10
2	2	2	2	2	2	2	2	2
1	2	3	4	5	6	7	8	9
3	4	5	6	7	8	9	10	11
3	3	3	3	3	3	3	3	3
1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	10	11	12
4	4	4	4	4	4	4	4	4
1	2	3	4	5	6	7	8	9
5	6	7	8	9	10	11	12	13
5	5	5	5	5	5	5	5	5
1	2	3	4	5	6	7	8	9
6	7	8	9	10	11	12	13	14
6	6	6	6	6	6	6	6	6
1	2	3	4	5	6	7	8	9
7	8	9	10	11	12	13	14	15
7	7	7	7	7	7	7	7	7
1	2	3	4	5	6	7	8	9
8	9	10	11	12	13	14	15	16
8	8	8	8	8	8	8	8	8
1	2	3	4	5	6	7	8	9
9	10	11	12	13	14	15	16	17
9	9	9	9	9	9	9	9	9
1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18

81 SUBTRACTION FACTS

2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9
3	4	5	6	7	8	9	10	11
2	2	2	2	2	2	2	2	2
1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	10	11	12
3	3	3	3	3	3	3	3	3
1	2	3	4	5	6	7	8	9
5	6	7	8	9	10	11	12	13
4	4	4	4	4	4	4	4	4
1	2	3	4	5	6	7	8	9
6	7	8	9	10	11	12	13	14
5	5	5	5	5	5	5	5	5
1	2	3	4	5	6	7	8	9
7	8	9	10	11	12	13	14	15
6	6	6	6	6	6	6	6	6
1	2	3	4	5	6	7	8	9
8	9	10	11	12	13	14	15	16
7	7	7	7	7	7	7	7	7
1	2	3	4	5	6	7	8	9
9	10	11	12	13	14	15	16	17
8	8	8	8	8	8	8	8	8
1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
9	9	9	9	9	9	9	9	9
1	2	3	4	5	6	7	8	9

The plan of teaching recommended in this book introduces first some easy facts, but by the time study for mastery is undertaken, some hard facts are also included.

Systematic Study of Basic Addition Facts

The exercises presented for background experiences in addition and subtraction are suggested for use in kindergarten, first grade, and part of the second grade. After the class has engaged in an experience background program, the next step in instruction is to direct the pupils in acquiring automatic mastery of the basic addition and subtraction facts. Generally speaking, there is no good reason for beginning this phase of the teaching of arithmetic before the child is in second grade. Attention is called to the word *beginning*, for teachers in later grades will need to help many children finish the task.

It is not very helpful to discuss teaching methods in general terms, so the first part of this presentation of the systematic teaching of addition facts will consist of a brief description of the procedure used in one second grade. The reader is not to assume that this second-grade experience represents *the* way to teach. It is considered a good way, but is presented here primarily as a means of getting before us specific methods and to provide the background for introduction and discussion of issues in teaching.

This second grade just prior to the period of study for mastery had solved many problems presented orally, using mental computation, objects, drawings, and the number line; the pupils had had experience with exercises showing what numbers mean

Wisconsin, 1924); F. B. Knight and Minnie Behrens, *The Learning of the 100 Addition Combinations* (New York: Longmans, Green and Company, 1928); H. V. Holloway, *An Experimental Study to Determine the Relative Difficulty of the Elementary Number Combinations in Addition and Subtraction* (Philadelphia: University of Pennsylvania, 1914).

(e.g., that 40 + 6 is four 10's and six 1's, or 46, and that 8 is the equivalent of 3 and 5, etc.); and they had used objects or pictures to discover or show number facts. In these background experiences addition and subtraction situations were intermingled. When systematic study was undertaken, only one process was considered at a time. Since by tradition and frequency of use addition is first, study for mastery of that process was begun first.

Systematic study is probably best introduced with a list of problems involving addition. The following directions were used with one such list of problems:

"Find the answer to the question in each problem. Then show with a drawing that your answer is correct. If you don't know the answer to the question, make a drawing and then try to find the answer." In order not to handicap the poor readers, the first problem was read aloud by the teacher and, of course, as help in reading was needed with other problems that help was given. There were several purposes for asking pupils to make drawings. For those who could not solve the problem by adding, the drawing would facilitate solution even if it were by a one by one counting method. For the better pupils the making of drawings would use time, thus permitting those less able to have a longer period of work. The drawings provide essential material for the discussion period, and construction of the drawings should result in better understanding of the addition situation.

When some pupils had found the answer for the six problems presented and had shown with drawings that their answers were correct for several of the problems, the teacher initiated a discussion period by saying: "Let's check our answers. If we disagree, the drawing ought to help us select the correct solution."

After all the problems had been considered, the teacher said, "We have used an addition fact in answering each of these questions. What fact did we use in answering problem 1?"

As the different facts were given, they were written on the board in equation form by the teacher.

After the facts in equation form were on the board, the teacher said, "We can also write the addition fact '6 and 5 are 11' this way: $\begin{array}{r} 6 \\ +\ 5 \\ \hline 11 \end{array}$. Since that's the way you will usually write your addition facts, let's each write the other 5 facts in that form." Several children were asked to write different facts in this vertical form on the board. The plus symbol was read "and."

The work described above constituted the arithmetic lesson the first day. On the next two days different facts were introduced through problem settings. Other facts were developed directly on the fourth day when the following suggestion was made by the teacher. "You can make other addition facts by putting together any pair of one-figure numbers." Whether or not this direct procedure might have been used effectively earlier is not certain and would depend largely upont he ability of the class.

After some facts were developed, written in vertical form and read, a combination only $\left(\text{e.g., } \begin{array}{r} 5 \\ +\ 3 \end{array}\right)$ was written on the board and the pupils were asked for suggestions on how to read it. The suggestion "3 and 5 are how many?" was accepted as best. Considerable emphasis was put on this reading of an addition combination as a number question. For combinations with sums greater than ten, the complementary method was one of the ways used to find the answer. For example, $5 + 6$ was changed to $5 + 5 + 1$, and $7 + 6$ became $7 + 3 + 3$. Attention was also given to relating some facts to the nearest double. For example, $7 + 8$ was considered as 1 more than 7 and 7, and 1 less than 8 and 8.

After about 40 addition facts had been listed, the class gave the answers for a group of the combinations. For five of the addition facts pupils were asked to show (the word *prove* is

preferred by many teachers) that their answers were correct. For the most part the showing consisted of making marks to indicate each number and counting to find the total.

On the fifth day a list of 25 combinations was presented to the class with these instructions: "Write the answer to each number question as quickly as you can. When you finish, raise your hand. You may figure answers any way you want to for those you don't know."

When the first pupil had finished, all were asked to stop. Sums were quickly given by the teacher and each pupil checked his answers. Several whose results showed that they had worked fast and correctly were asked how they had been able to answer so many questions. The typical reply was "I just knew the answer and wrote it." One of those who got only a few correct stated that he had made marks and counted.

The teacher then said, "In arithmetic you will add often. You can see from what Bill and Kay did that knowing the facts means you can work faster. We are going to spend some time in the next few days learning addition facts. Let's consider some suggestions for learning them. What suggestions for learning facts do you have?"

The following suggestions were obtained from this second grade: (1) Say the fact over and over; (2) Write the facts; (3) Make marks to show answers; and (4) Take tests.

The pupils were permitted to select and use one of the methods of study a short time. Later, the teacher suggested a fifth method as she gave each child a sheet of paper on which were 40 of the facts. "I know another good way to study the addition facts. Take a piece of paper and cover the answers to the top row of facts. Look at the first combination. Think the answer. Now, move the piece of paper so you can see if you gave the correct answer. If you did not give the correct answer or if you weren't quite sure, try this way of studying the fact. Let's all do the first one together.

"Read the whole fact as you look at it. For the first fact on your list say 'seven and three equal ten.'

"Now, close your eyes and try to see (visualize) the fact. With your eyes closed, read the fact to yourself as you try to think seeing it.

"Open your eyes and look at the fact to see if it is the way you thought it was with your eyes closed. If not, read it again, close your eyes again, and so on.

"Now go on studying the others the same way."

After a few minutes one of the pupils was asked to go through the study steps, reading aloud instead of to himself.

During the next two weeks, the major part of this second grade's arithmetic time was devoted to learning for mastery the basic addition facts. All the facts involving the numbers 2 to 9 were studied. The ones facts were studied as a group instead of as separate combinations. The rule the children developed from looking at some ones facts was that when one is added to any number the sum is the next number in the series.

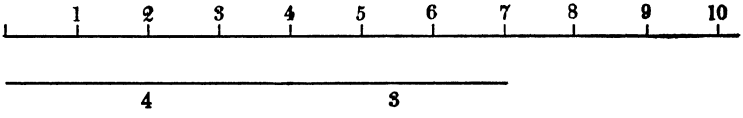
All the study methods listed above were used, but most of the time was devoted to the one proposed by the teacher. After several days, a room wall chart was prepared and used as a guide in study. The best procedure was given first place on the chart. A reproduction of the chart appears below.

How to Learn Facts

1. Read the fact. With your eyes closed, try to see the fact. Say it with your eyes closed. Open your eyes and see if the fact is the way you thought it was with your eyes closed.
2. Say facts over and over.
3. Write the facts.

To provide variety and to facilitate study, sets of examples and problems from the adopted textbook were used. With one of these sets of textbook examples an important teaching procedure was introduced. The directions for study on this day were, "After you have written the answers to the first two rows of examples, show with marks or in other ways that your answers to the first and last example on each row are correct."

On several occasions during these two weeks of intensive study of basic addition facts, some pupils were asked to show a fact by using blocks to represent the various numbers. Other pupils were asked to show facts by use of the number line.



Analysis of Addition Teaching Procedures

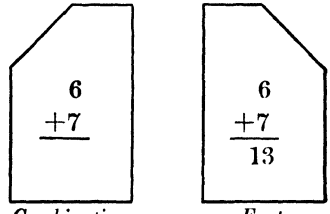
The brief description of the teaching of addition in the preceding section warrants careful examination. Systematic study was introduced through pupil (individual and as a member of the class) solution of problem situations. To insure that pupils had a means of getting the answer, the making of drawings and use of counting were suggested. Facts were developed from the solutions to the problem situations and by putting numbers together. A relatively large number of facts were developed before any work for mastering them was undertaken, and then the value of knowing the facts was made clear. This postponement of learning is very different from the common textbook procedure in which four to eight facts are illustrated and then the suggestion to learn them is given.

Examples were made into number questions by the way in which the group decided to read them, rather than being presented first as completed facts. Also the class took an active

part in planning specific study procedures, although emphasis was placed on the teacher-suggested method recommended by the use of the wall chart.

Requiring pupils during the intensive study period to show or prove that some answers to examples are correct is a very important part of the work. This procedure affords a check on pupils' understanding of the process, for in a classroom of twenty to thirty-five pupils there are frequently those who still do not understand when learning through drill or practice exercises is undertaken. If practice exercises are delayed until all understand, the more able pupils will have wasted much time just waiting. Also, it should be recognized that the complete understanding of processes may be and often is acquired after some facility with the process has been acquired.

Instead of providing her children with the well-known flash cards, this second-grade teacher preferred to give them a list of the facts. This method was used primarily because a list of facts can be supplied to and kept by second-grade pupils far more easily than flash cards. It is, however, the opinion of the writer that flash cards when properly used are the one best way for presenting facts for mastery.



Combination Side *Fact Side*

One good procedure for using flash cards is as follows: Start with the entire set of cards (or only the hard facts) with the combination side up. To make this arrangement of the cards easy, the upper left-hand corner of the combination side should

be cut off (see illustration). The pupil looks at the combination and tries to give the sum. If he can give the sum immediately, he puts the card with his "Known Stack." If he cannot give the sum, he looks at the fact side, reads the fact, closes his eyes or looks away and tries to see and say the fact, then looks again at the fact to see if it is as he saw it in his mind. If he still cannot give the sum, the procedure is repeated. The card is then put with the "Unknown Stack." When the entire stack of cards has been used as indicated, the pupil proceeds through the unknown stack again. If interest warrants, the pupil may use the same method over and over with many repetitions of the learning steps for the most difficult combinations.

Flash cards may be used in second grade if the teacher feels that an additional procedure is needed. For most schools, Grade Three is the level for an introduction of flash cards of addition and subtraction facts. Pairs of pupils enjoy and profit by working together with a stack of cards. One pupil holds the cards combination side exposed to another who tries to give the sum. If it were not for the interest factor, this way of using the cards would not rate very high.

Study of the Basic Subtraction Facts

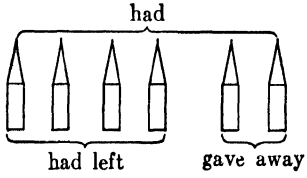
Immediately after the second-grade class described in preceding sections had spent three weeks of intensive study on the basic addition facts, the systematic study of basic subtraction facts was introduced. The same general procedures, that is, use of problems, solution through drawings, statement of facts, intensive study and so on were followed in teaching subtraction. There are, however, some aspects of the teaching of subtraction which are markedly different from the teaching of addition. These variations are noted in the paragraphs that follow.

Although the process of subtraction is the inverse of addition,

the thinking and language of the pupils in the developmental period is distinctly different from that of adding. When solving a subtraction problem with diagrams, the child shows or records an amount and, if the problem is a true "take-away" situation, a given amount of the quantity is then removed or marked out and what remains is identified. The pupil shows, for example, that if 3 of 5 are removed, 2 remain. For the "comparison or difference" type of problem, the two quantities are matched (removed or marked out) and the excess of one over the other identified. For the "how many more are needed" type of subtraction situation, the amount possessed or present is identified; then the additional amount required to reach the total is identified. It is clear, therefore, that there are at least three distinctly different types of subtraction situations.

The illustrative problems below show the difference in children's thinking involved in the solution of each of the three types and the difference in a statement of the number fact developed from each type.

A. Bob had 6 pencils. How many pencils did he have left after he gave 2 away?



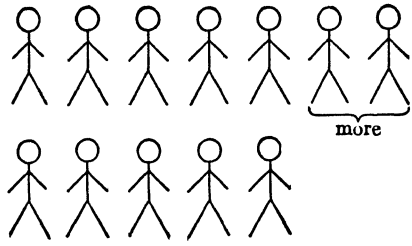
(Diagram solution)

"6 take away 2 leaves 4."
"2 from 6 leaves 4."

(Statements of number fact based on solution)

B. Nancy has 7 dolls and Kate has 5. How many more dolls does Nancy have than Kate?

Nancy's



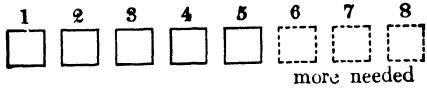
(Diagram solution)

Kate's

"7 is 2 more than 5."
"The difference between 7 and 5 is 2."
"2 more than 5 is 7."

(Statements of number fact based on solution)

C. Mary needs 8 blocks to build a play pen. She has 5 blocks. How many more blocks does she need to make the pen?



(Diagram solution)

"5 and 3 more make 8."
"3 more are needed to make 5 equal 8."

(Statements of number fact based on solution)

A study of the three illustrative problems, diagram solutions and number facts developed shows that problem A, which is an example of the "How many left" type of subtraction situation, leads to the best subtraction number fact statement. Therefore, that type of situation is recommended for the problems used while the group is developing the subtraction facts to be used in lessons for mastery. Unfortunately, when diagrams for this type of fraction exercise are already worked out as illustrative problems in textbooks (or as in the diagram shown for A), the

pupil does not have to subtract but merely has to count the number left. It is, therefore, very important that pupils prepare their own diagrams or better still work with objects and then make a drawing to show how the solution with objects was performed.

The other types of subtraction situations (the comparison or difference type and the how many more type) will have been used during the foundation experiences and will be used in problems as soon as the basic subtraction facts have been developed.

The student of arithmetic teaching should note that the statement of the number fact for problem C, the "how many more" type of subtraction situation, is definitely addition. One of the three statements of number fact for problem B is also additive.

Methods of Subtraction: Additive Versus Take-Away

The preceding section brought out the fact that subtraction situations can be solved by adding to the subtrahend or by taking from the minuend. Either method may be used with any type of subtraction situation. In the United States by far the majority of adults use the take-away method and practically all textbooks teach only that method. In Great Britain, France and some other European countries, the additive method is the more popular. There are merits to both methods.

The chief advantages of the additive method of subtracting are the following:

(1) The language of addition with which the pupil is already familiar is used. (2) In subtracting by adding the pupil uses his addition facts in only a slightly different manner from the way the facts were used in addition. Thus, the pupil does not have to master 81 or 100 subtraction facts. (3) In borrowing situations, the borrowed or added 1 is less likely to be forgotten. (4) In the additive method the pupil's thought process is positive

(from the smaller to the larger number), which is generally thought to be easier than the negative progression used in the take away.

Along with the merits of the additive method the chief disadvantages should be considered. This method of subtraction is difficult for the beginner to rationalize because the remainder or difference cannot be separated from the minuend. Suppose, for example, $9 - 4$ is to be shown with objects. The thinking is, "We have 4 blocks. How many more do we need to equal 9?" When the 5 needed blocks are added they do not stand out as one group but are merged with the other 4 to make the 9. On the other hand, if the take-away method is used we begin with 9 blocks, remove 4 and only the 5 remain. Another disadvantage of the additive method is encountered in borrowing situations in the subtraction of fractions. In solving $6\frac{1}{3} - 2\frac{2}{3}$ by the additive method two additions are required to get the fraction remainder. There are then two chances for error plus the difficulty pupils encounter in rationalizing the process and learning the steps in the procedure. The additive method has the marked disadvantage in the United States of being contrary to custom.

The chief merits of the take-away method are (1) the ease with which the process can be demonstrated with objects; (2) the logic which it has for children who understand subtraction as take away in most situations; and (3) the fact that it is in harmony with custom in this country. Those interested in teaching arithmetic meaningfully favor the take-away method primarily because it is easier to rationalize than the additive method. It is for this reason that the writer recommends the take-away method.

The main disadvantage of the take-away method of subtraction is the fact that it requires the mastery of the basic subtraction facts and in such a way that the pupil cannot use effectively the related basic addition facts.

Reading Subtraction Examples

The reading of subtraction facts and number questions is more difficult for pupils than reading addition examples. Pupils reading the fact $\begin{array}{r} 6 \\ -\ 4 \\ \hline 2 \end{array}$ have a tendency to say "6 from 4."

Since in addition the top number is read first, to begin with the top number in subtraction is logical. When such an error is made, the following or a similar teaching procedure is recommended: "Show on the number line (or show with blocks) what you have just read." After the pupil sees that he cannot take 6 from 4 the teacher suggests that perhaps the reading of the example was incorrect and says, "Try reading it again. This time start with the number 4."

The preceding teaching suggestion implies that "4 from 6 = 2" is the desired reading of the number fact. This phrasing, rather than "6 minus 4," "6 take away 4," or "6 less 4" is preferred because it more nearly expresses the take-away subtractive process as that process is performed on either the number line or with objects. In the take-away method of subtracting, we begin with the subtrahend and therefore "6 minus 4" and other ways of reading which begin with the minuend number are considered inferior. Here as in many other phases of arithmetic all correct methods of reading are acceptable but the one best method should be identified and emphasized.

The classroom procedures followed in teaching the basic subtraction facts, as was indicated earlier, would follow the plan used in teaching the basic addition facts. The first facts would be developed from problems; other facts would be developed from direct study of numbers; how to read subtraction facts would be considered; there would be a timed test, followed by intensive study including the use of proof. The same best

method of study would be recommended and used by all of the class at least part of the time.

Addition and Subtraction with Two-Place Numbers

The foundation number program will have given the children some experience with the addition and subtraction of two-figure numbers. For example, pupils as a class and some individuals will have solved, through use of the number line, or through use of objects or marks, problems in which two-figure numbers are added or subtracted. The number chart exercises in which pupils find the number that is two 10's more or 10 less than a given number and the oral exercises used in Grades One and Two where pupils engage in adding two 10's and three 10's or subtract two 10's from five 10's represent other background pupil experiences with two-figure numbers.

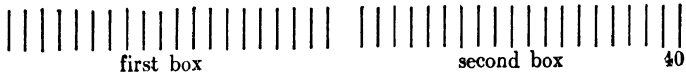
Systematic instruction in the most efficient ways of adding and subtracting two-figure numbers does not begin until after the pupils have had an opportunity to master the basic facts. This new work is best introduced with problems involving only exact tens. By exact tens is meant such numbers as 20, 30, and so on, rather than 21, 31, and the like. By using exact tens the pupil is confronted with learning to add only tens and not tens and ones.

The directions for these problem exercises require the pupils to find the answer to each question, and then show by diagram and in other ways if possible that their answers are correct. Following the class or pupil-teacher discussion of answers, diagrams, and other solutions, the teacher directs attention to the problem of selecting or finding the best number solution.

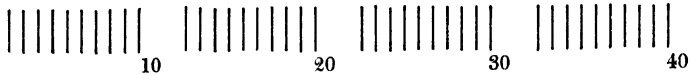
The diagram and number solutions of one second grade for a problem are shown below.

Problem: There were 20 books in one box and 20 more books in another box. How many books were in the two boxes?

(*a*)



(*b*)



(*c*)



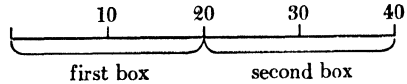
40 (Use of tens stones)

(*d*)



40 (Use of tens blocks)

(*e*)



(Use of number line)

(*f*) Number solutions

(1) $20 + 20 = 40$

(2) $\begin{array}{r} 20 \\ \underline{20} \\ 40 \end{array}$

(3) $\begin{array}{r} 2 \text{ tens} \\ \underline{2 \text{ tens}} \\ 4 \text{ tens} = 40 \end{array}$

(4) $20 + 10 = 30 \qquad 30 + 10 = 40$

The best number solution was explained this way: "20 is 2 tens so you add 2 and 2. To show that your answer is 4 tens you put a zero beside the 4."

After the solution for all examples had been considered the teacher asked, "How is the addition of tens different from the adding of ones?"

Some examples were then solved by using only the number method and checked by means of the number line or tens blocks.

Examples of the type $26 + 20$ were considered next, followed by others like $26 + 32$ and finally by the carry situation. For each situation, solution through such indirect means as the number line, tens blocks, or tens stones was used as a check on the correctness of the number solution.

In the addition of two figure numbers the pupil first encounters the so-called zero facts. After the pupils have had some experience with this new situation, the teacher calls attention to it by saying, "You have been adding figures when one or both of them is zero. Let's see what you do when zero is one of the figures to be added."

The conclusion or generalization to be reached regarding the adding of the figure zero is that "when zero is one of the figures to be added, there is nothing to add to the other figure. Therefore, you just write the other figure. When two or more zeros are to be added, just write zero."

There are a few pupils who are so verbally minded that they need to learn each zero fact as a separate fact. Since such students are definitely in the minority and obviously use a very poor arithmetical skill, drill on zero facts might best be postponed until Grade Four.

Carrying in Addition

It was pointed out in the preceding section that carrying was introduced through consideration of examples. This phase of arithmetic might well be introduced through solution of illustrative problems, using the same procedures that have been recommended in introducing other new phases of arithmetic. The various different solutions which classes offer will depend in part on previous experiences and the available equipment. Adding orally or just thinking the answers without use of pencil and paper, adding with tens and ones blocks or tens and ones stones, adding on the number line and adding on the abacus are the most popular solutions.

The best number method is most easily developed from either the tens and ones blocks solution or from the number line solution. The development may be begun by attempting to duplicate with numbers what was shown with blocks or number line.

Suppose, for example, that the problem involved the addition of 37 + 26. The quantities are first written thus:

3 tens 7 ones
2 tens 6 ones
5 tens 13 ones (13 ones = 1 ten and 3 ones)
5 tens and 1 ten and 3 ones = 6 tens and 3 ones = 63

The elimination of words and the writing of numbers in the conventional way is the next step. When the ones are added, it is usually best to write that sum on the first line, the tens sum on a line below that, and then to combine the two sums. The procedure is illustrated below.

$$\begin{array}{r} 37 \\ +26 \\ \hline 13 \\ +50 \\ \hline 63 \end{array}$$

To save the writing of the partial sums, the following form was suggested by the teacher:

$$\begin{array}{r} 1 \\ 37 \\ 26 \\ \hline 63 \end{array}$$

The teacher explained the short form this way. "Instead of writing the 13 ones first and then adding the tens sum to that, we can write the one ten of the 13 above the tens and then add tens only once. Changing the 10 ones to 1 ten and adding it to the other tens is called carrying. Let's try the new way on these examples. Check your work by using the two sums." It should be noted that in the teacher's explanation and work, she recommended writing the carried one.

After most members of the class had acquired some facility in the best method of carrying, the teacher planned a summary lesson. The following setting was used.

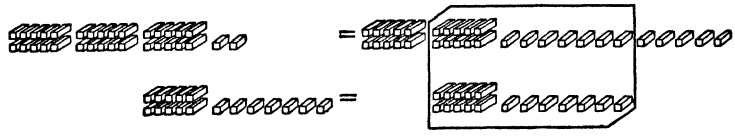
"A good way to help you remember what you have studied is to make a summary of what you have done and learned. Let's make a summary of how we learned to add two-figure numbers. First, think of what you did when you began adding two-figure numbers."

The class-prepared summary appears below.

"We first added two-figure numbers by using drawings, the number line, tens blocks, and by just thinking the tens. We then figured out a number way of adding. When there are no ones, you write a zero in the one's place and add the tens. When there are ones, you add them and then add the tens. When the sum of the ones is 10 or more you put the 1 ten with the tens and write the ones left, if there are any, in the one's place. Then add all the tens."

Borrowing in Subtraction

The teaching of borrowing, like the teaching of most other areas of arithmetic, is probably best started with problem situations. The directions given the pupils with the problems required that they find more than one way to solve them and encouraged them to think of as many ways as possible. The solutions for 32–17 proposed by one class were (1) solution on the number line; (2) solution with tens and ones blocks and tens and ones drawings; and (3) thinking the difference (e.g., think 17 from 27 = 10, 27 from 30 is 3, and 30 from 32 is 2. The difference is 10 + 3 + 2 = 15, or 17 from 20 is 3 and then it's 12 more. That's 15.) The tens drawing solution and the number record of that solution which the class worked out are given below.



3 tens and 2 ones =	2 tens and 12 ones	$^{2}\not{3}\,^{1}2$
−1 ten and 7 ones =	1 ten and 7 ones	−17
	1 ten and 5 ones	15

The short number procedure was then tested on some examples. The drawing solutions, the blocks, and the number line were used to check the accuracy of solutions. The use of tens blocks or drawings of them shows clearly why a ten has to be changed into ones in such situations. The study of borrowing in this second grade culminated in the class formulation of a statement of the procedure to use. This summary statement with the accompanying illustrative solutions was placed on the bulletin board.

It should be noted that in the sample solution 1 was written, the 3 crossed out, and a small 2 written beside it. This practice is sometimes strongly opposed by teachers because the writing of some pupils is so poor. Since crossing out the changed tens and writing the number of tens remaining, as well as the borrowed one, is an aid to understanding, the procedure should be encouraged.

It is unfortunate that the term "change" instead of "borrow" is not used but since custom has so definitely fixed the latter we should use it in instruction. However, simultaneous use of "change" is recommended. The best explanation to offer is that borrowing is just the name used in subtraction for this changing of a ten to ones.

Subtraction when the minuend is a number of three figures or more introduces a borrowing situation which requires special consideration. The example below illustrates this. The thinking recommended for such cases is as follows: "6 cannot be

$$\begin{array}{r} {}^{3}\not{4}\,^{1}\not{0}5 \\ -176 \\ \hline 229 \end{array}$$

taken from 5 so you change (borrow) one of the 40 tens to ten

ones. That leaves 39 tens, 6 from 15 are 9," and so on. That type of thinking will not be understood very well by pupils unless they have had a good foundation number experience which included practice in thinking of the equivalence of numbers. Such experience is obtained by answering questions like "17 tens and 6 ones are how many? 381 is equal to how many tens? How many hundreds in the number 6054?"

Column Addition and the Higher Decade Addition Facts

Situations which require the adding of more than two numbers (i.e., column addition) occur frequently in the experiences of the elementary-school child. Various ways of dealing with such addition are developed by or can be supplied to the child who has a good number background. In fact, a child may turn to column addition himself when confronted with two numbers; for example, $8 + 7$ which he thinks of as $8 + 2 + 5$. Other early ways in which children commonly deal with column addition situations are by counting actual objects or by drawings on the number line. Systematic instruction in column addition is usually undertaken in Grade Three after the pupils have reviewed the basic facts and had the initial work with the adding of two figure numbers.

Introductory work in column addition should focus attention on finding the best way to add such numbers. As the children work out a method of thinking the numbers, they should consider the amount of verbalization which is desirable and whether or not to skip about. They will need a procedure to help keep the sum in mind when they are stuck on a combination and also a way to check. While they are discovering these things, a need for knowing the higher decade combinations should become obvious and a plan for learning them be developed.

The list of materials below is a reproduction of what one third grade had on the board after their discussion of column addition.

WAYS TO ADD A COLUMN OF FIGURES

(1) Think 7 and 6 are 13, 13 and 9 are 22, 22 and 4 are 26, 26 and 3 are 29.

$$\begin{array}{r} 7 \\ 6 \\ 9 \\ 4 \\ +\,3 \\ \hline 29 \end{array}$$

(2) Start at the top. Look at 7 and 6. Think 13. Put your pencil point beside 9. Think 22. Put your pencil point beside 4. Think 26. Put your pencil point beside 3. Think 29.

(3) Look for tens. 7 and 3 are 10 and 6 and 4 are 10, 10 and 10 are 20, 20 and 9 are 29.

The Number 3 plan was eliminated by the class because "you might get mixed up" and "because you have to use one of the other ways to check anyway." The Number 2 plan was chosen as better than the Number 1 plan because in Number 1 "the saying of all the numbers is not necessary and makes it too long; you forget what you have said." Adding up was chosen as the check. For the example given, the thinking in the check (minus directions regarding starting place and pencil) is as follows: "Seven, sixteen, twenty-two, twenty-nine."

This class consideration of column addition brought attention to the necessity for knowing such higher decade facts as $13 + 9 = 22$ and $22 + 4 = 26$. The second type is relatively easy and requires little instructional time. The first example, however, which involves bridging (going from one decade to the next) is a phase of arithmetic in need of a great deal of attention. In this class two ways of thinking the answers for such combinations were recommended.

1. Take enough of the little number to make the next exact ten. Then add what's left. For example, for $13 + 9$ think $13 + 7 + 2$ and for $38 + 5$ think $38 + 2 + 3$.

2. Use the ending of the ones numbers and think the next

tens number. For example, 13 + 9 will have the same end as 3 + 9, 38 + 5 will have the same end as 8 + 5.

The second way of thinking the sum of a higher decade combination is commonly referred to as adding by endings. Textbooks presenting this method rely heavily upon the initial work with higher decade facts (no bridging). The relationship between the related basic fact and the higher decade facts is emphasized by presenting such groups as $\begin{array}{r} 6 \\ +\,2 \\ \hline 8 \end{array}$ $\begin{array}{r} 16 \\ +\,2 \\ \hline 18 \end{array}$ $\begin{array}{r} 36 \\ +\,2 \\ \hline 38 \end{array}$ etc., and calling the pupils' attention to the fact that all the sums have the same ending — 8. Although this method of adding by endings is supposedly taught and much space is given to presenting it in some books, there is little evidence to show that it is effective. In fact, one study [1] revealed that few fourth-, fifth-, and sixth-grade pupils actually used the method in schools where the method had been presented in the third grade.

The other method for dealing with bridging, in which the next ten is made by using only part of the ones number, has been called the tens' method by Stokes.[2] This tens method is advocated by only one of the leading textbooks published in 1952. Yet, for elementary-school pupils the tens method seems to be the better of the two proposed.

Regardless of what method is taught for thinking the sums of higher decade facts, pupils will need some practice with the facts and probably will need instruction in Grades Four, Five, and Six, as well as in Grade Three. Much careful study is still needed on the teaching of this phase of arithmetic. It may be that the major intensive study of the topic is undertaken too early or that the methods of study used are not very effective. It is the belief of the writer that the best practice for the majority of the pupils will be the "making of tens" method. How-

[1] Leola Williams, "Higher Decade Addition," Unpublished M.A. Thesis (Iowa City: State University of Iowa, 1952).

[2] C. Newton Stokes, *Teaching the Meanings of Arithmetic* (New York: Appleton-Century-Crofts, 1951), p. 137.

ever, in view of the fact that many persons use endings with some of the doubles (e.g., 26 + 6, 27 + 7, and 28 + 8) but use the tens method for all other combinations, the endings method should also be taught.

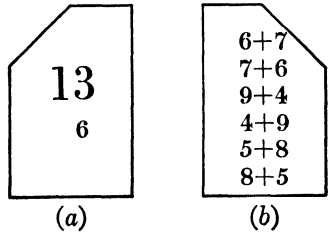
Current practice gives major attention to the topic in third grade. Since knowledge of the higher decade facts is needed only in column addition and in carrying in multiplication, it seems logical to place the major teaching of the topic either late in the fourth grade or in fifth grade — that is, after the pupils have a use for it.

Other Procedures and Materials for Study of Addition and Subtraction

Good teachers of arithmetic have found that for some pupils and even some whole classes the use of varied procedures and materials for learning are sometimes very beneficial. Those listed below may be of value in providing variety.

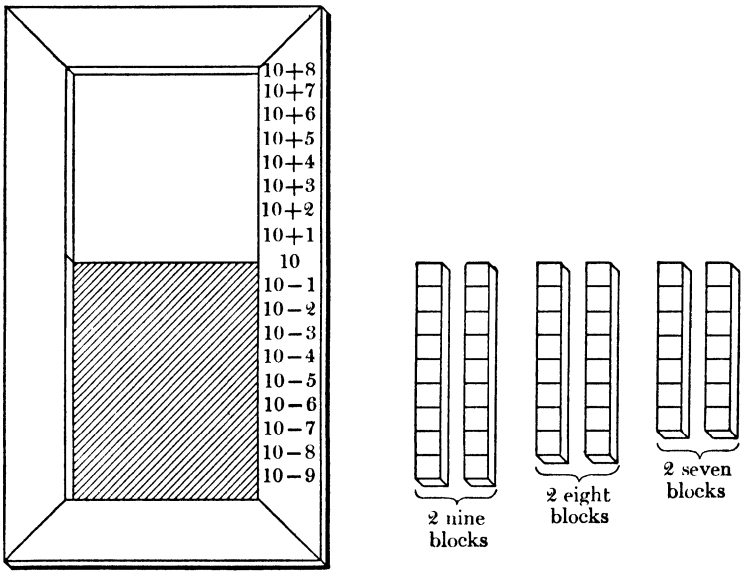
1. *Basic sum cards.*

The basic sum cards consist of seventeen cards. Each card has a large number (2 through 18) and a small number beneath it on one side and the combinations equal to the large number on the other side (see the illustration for number 13). The small 6 beneath the 13 indicates that there are 6 combinations for one-figure numbers equal to 13. In using these cards the pupil looks at side *a* and tries to give all the basic combinations equal to the sum.



2. The addition and subtraction box and blocks

The materials (see illustration) are an adaptation of Stern's 20–Case. They consist of a pair of blocks for each of the numbers from 1 to 9; the one blocks are 1 inch long; the two blocks are 2 inches long; and so on. The box consists of a frame of $1'' \times 2''$ boards, $23'' \times 13''$ in size, with plywood or similar material nailed to the back to make a box.



The suggestions below (which were at various times placed beside the box on an arithmetic table) indicate some of its uses.

(*a*) What pairs of blocks equal 10? (Put your record on your name card.)

(*b*) What pairs of blocks equal 10 and 4?

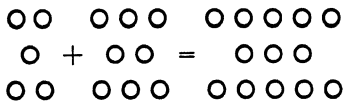
(*c*) Write the sums shown. (Blocks totaling 9, 16, 8, and 11 were shown.)

(*d*) The nine-block and six-block are how many short of 10 and 6?

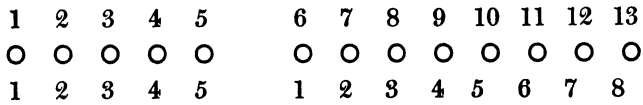
3. *Drawings to show sums and differences*

By making such drawings as those shown, the pupils may gain better insight into the combination of numbers than they have secured in other ways. Primarily because of the excessive amount of time required, the procedure should be used sparingly.

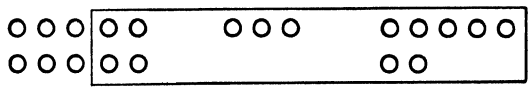
(*a*) $5 + 8 = 13$







(*b*) $13 - 7$



4. *Thinking or writing the number that is 2, 3, or 4 more or less than given numbers*

A chart with the numbers 1 to 100 placed in mixed order facilitates use of this procedure. As the teacher or child points to a number on the chart, another pupil gives the answer. For example, if the teacher pointed to 63 and the direction had been to tell the number 3 less than the given number, the pupil's reply should be 60.

5. *Counting by 3's, 4's, and 5's but starting with such numbers as 7, 11, or 13*

The pupil counting by 3's beginning at 8 would say 11, 14, 17, and so on. It is probably best to set 40 or 50 as the upper limit for such counting. If this is used much as a written exercise, pupils can be asked to put down on the paper only the final number in the sequence, for instance, the first number they reach in the 40's. This exercise is probably best for fourth-grade pupils although it may be used in third grade.

Addition and Subtraction in the Upper Grades

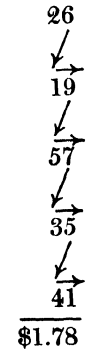
Although teaching of the basic facts of addition and subtraction is usually begun in the second grade and receives major emphasis in Grades Three and Four, there is still a need for additional attention to these basic facts in Grades Five and Six and even in Grades Seven and Eight. There is also instruction in more advanced phases of addition and subtraction of whole numbers in the upper grades. That there should be review of the fundamental aspects and further work in the addition and subtraction areas is recognized by all who know the field of arithmetic teaching. There is, however, no agreement as to how this review and additional instruction is to be carried out.

An examination of current sixth-grade textbooks reveals that by far the most common practice is to present again the various aspects of, first, addition and then subtraction in much the same way as it has been presented in review form in Grade Five and in the form that it was originally studied in Grades Two, Three, and Four. It is, then, not surprising to find that sixth-grade (fifth, seventh, or eighth might be used just as easily in this statement) pupils show little enthusiasm for their arithmetic work early in the year. There is not much to chal-

lenge them in a series of basic facts which is identical with the list the pupil has considered for four years. At the same time, he is being challenged by new and interesting material in the social studies, in science, in health, and in reading. The big task, then, of the upper-grade teacher is to find new ways of introducing this old area of arithmetic and to find new ways for the pupil to use to master the facts and procedures.

One attempt to provide a new approach to addition is shown in the teaching procedure described below.

The sixth-grade teacher introduced addition as a part of an oral exercise during the first few days of school. One of the problems required finding the total of 5 purchases of 26¢, 19¢, 57¢, 35¢, and 41¢, respectively. Since several sums were offered by class members, the teacher wrote the individual amounts on the board in a vertical line and then asked different pupils to tell how they had thought in arriving at an answer. The teacher then said, "People who are really good at adding a series of two-figure numbers do it this way. They think 26, 36 (taking only the tens of the next number), then 45 (adding the ones of 19), 95 (adding the tens of 57), \$1.02 (adding ones of 57), \$1.32 (adding tens of 35), \$1.37 (adding ones of 35), \$1.77 (adding tens of 41), \$1.78 (adding ones of 41). I'll show you how they think by writing the numbers here and drawing lines.



"That's a good way to add two columns of figures. Let's try it on a few examples. Be sure to check your addition by your regular way of adding."

Six examples were given the class and as pupils attempted to solve them by this simultaneous addition of two columns, the teacher went about the room giving some assistance and asking others to think the addition aloud. After a few minutes the teacher said, "Let's stop and see if we are all doing what we set

out to do. Mary, will you add aloud the figures of example 1." After two others had been called on for examples 2 and 3, the work period was resumed. When the work on the six examples had been completed the teacher said, "Those who are good at this way of adding two columns at a time, can add faster than those who add one column at a time. Let's see how we do with two ways. Let's work examples for three minutes the new way and then work examples the old way for three minutes."

It was found that in a class of thirty pupils only 114 examples were worked the new way while 182 examples were worked the old way. The discussion that followed brought out the idea that they did not really know the new way. To see if they could improve, it was agreed that some arithmetic time for the next three days would be taken for practice and then the experiment would be repeated.

The experiences of the pupils with this simultaneous way of adding two columns and then checking provided a use for adding. Observation of pupil work during this time showed the teacher that some brushing up on or a better way for handling the higher decade facts was in order. The thinking aloud situations had made this condition rather obvious to most of the class.

"Some of you seem to need to brush up on adding one-figure numbers to two-figure numbers before you will do very well at adding a column of figures. Let's first find out how we add such pairs of numbers as 28 and 6." The preceding two sentences were used to introduce the direct study of the facts. A class discussion suggested that several ways were used. It was agreed that adding by the making of tens and adding by endings were the best ways and that one of these two methods should be used in studying a list of higher decade facts. To gauge the value of this study, a short test (timed by the teacher but scored by the pupils) was given at the beginning and end of the study period. Review study of the basic facts was introduced during this study of the higher decade facts by the teacher's saying,

"Some of you may not be able to make good progress because you do not know the basic facts as well as you should. You can find out how well you know the facts by taking the tapping test. There is a test with directions on the desk. Sometime during the next few days each of you should take that test."

The directions for this test, which consisted of the harder addition combinations, were: "Tap or touch your desk with your finger as you think or say the sum to each combination. If your tapping is regular and reasonably fast you know the facts well enough. Mark any combinations you do not know well."

At the bottom of the list of items was this study suggestion: "Study the facts you do not know well in this way: Write the fact. Then read it, and either close your eyes or look away and try to see the fact and say it. Look at the written fact again to see if it is as you saw it in your mind's eye. If it is not, do the reading and thinking again."

The major features of this upper-grade introduction to the re-study of addition are that (1) a new aspect of addition was considered which provided something that might challenge through newness and yet use the old procedures; (2) there was a use for addition at an advanced level before study was undertaken; (3) after a need for higher decade facts was shown, ways of dealing with them were discussed and then study was undertaken; (4) whether or not to study the basic facts was left to the individual pupil, who was provided with a means for determining his need for study. It is interesting also to note the use of an experiment to determine the relative speed of two methods of adding and to find out whether study would result in improvement.

Beginning the study of addition in sixth grade with a phase of adding at the pupil's level rather than going back to second- or third-grade material is highly commendable. Providing specific study methods, rather than assuming that pupils in

this grade already know them, is also a desirable feature of the procedure.

The tapping test is an example of a different or new study procedure for upper grade pupils. This test is, of course, also applicable to subtraction, multiplication, and division. Another different procedure for study of addition and subtraction is counting by 6's, 7's, 8's, and 9's. The following assignment illustrates the procedure. "Begin with 13 and count by 6's until you reach 50. Write only the first 50-number." (See page 45 for other comments on this counting procedure.) Counting to 100 will give more practice but because the limit of a pupil's attention span is often reached before this many additions are made, counting that far is quite difficult. Counting backwards by 3's, 4's, 5's, 6's, 7's, 8's, and 9's is a good upper-grade subtraction study exercise. Still another study device for upper grades is the solution of addition and subtraction situations with symbols other than figures. Blocks or other appropriate objects for hundreds, tens, and ones, or appropriate drawings are two means of doing such computations. The writing of compositions on how we add and subtract in such examples as

$$\begin{array}{r} 670 \\ 204 \\ +\ 318 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{r} 700 \\ -\ 156 \\ \hline \end{array}$$

also has possibilities for upper-grade study of these processes.

For suggesting additional new ways of introducing addition and subtraction, ancient ways of figuring are excellent material. Calculating on the abacus (especially the Chinese and Japanese forms) is another means for introducing the topic.

For seventh- or eighth-grade pupils the check of nines for addition and subtraction is a good introductory vehicle. Analysis of results of inventory tests given at the beginning of school may sometimes be used to introduce the work in the upper grades. If this procedure is frequently used, it may lose its effectiveness.

STUDY QUESTIONS

1. Is direct study of a group of objects, for example 7, for the expressed purpose of finding out what may be learned about such a number a good teaching procedure? (*1*) Yes. (*2*) No.

2. Should the background or foundation experiences for the systematic study of addition and subtraction include some adding and subtracting? (*1*) Yes. (*2*) No.

3. Which of the following is not a basic subtraction combination? (*1*) 14–9. (*2*) 18–8. (*3*) 16–7. (*4*) 9–3.

4. How many basic addition facts are there if the zero and ones facts are excluded? (*1*) 36. (*2*) 64. (*3*) 72. (*4*) 81.

5. Should pupils have added and subtracted a lot of numbers before beginning to study for mastering the basic facts? (*1*) Yes. (*2*) No.

6. What is the basis for the division of the subtraction facts into hard and easy facts? (*1*) The inherent difficulty of each fact. (*2*) The size of the minuend. (*3*) The size of the difference.

7. Which of these is classified as a hard addition combination? (*1*) $8 + 3$. (*2*) $7 + 2$. (*3*) $4 + 3$. (*4*) $2 + 5$.

8. In what grade should systematic study for mastery of the basic subtraction facts begin? (*1*) First. (*2*) Second. (*3*) Third. (*4*) Fourth.

9. A good background experience program for addition of whole numbers would include about how many distinctly different exercises or experiences? (*1*) 2. (*2*) 6. (*3*) 12. (*4*) 20.

10. Why is systematic study of addition recommended ahead of systematic study of subtraction? (*1*) Because addition is essential to understanding of subtraction. (*2*) Because addition is a building-up rather than a tearing-down process. (*3*) Because addition situations occur more frequently than subtraction situations.

11. Do the directions for the first problems given pupils in their study for mastery of addition suggest that pupils solve the problem by addition? (*1*) Yes. (*2*) No.

12. In the pupil directions for the first addition problems how many distinctly different functions are there? (*1*) 1. (*2*) 2. (*3*) 3. (*4*) 4.

13. In the development of addition facts from problems are the facts first written in equation form? (*1*) Yes. (*2*) No.

14. Do most of the addition combinations encountered in life occur in equation form? (*1*) Yes. (*2*) No.

15. What is the best way to read the addition combination $\begin{array}{r} 6 \\ +\,2 \\ \hline \end{array}$? (*1*) Six plus two equal what number? (*2*) Six and two equal how many? (*3*) Two plus six are how many? (*4*) Two and six equal what number?

16. With which of these combinations is the complementary method applicable? (*1*) $7 + 2$. (*2*) $6 + 4$. (*3*) $8 + 6$. (*4*) $10 + 7$.

17. About how many basic addition facts should be developed with a class before study for mastery is introduced? (*1*) 6 to 8. (*2*) 12 to 16. (*3*) 35 to 40. (*4*) 65 to 70.

18. About how many different ways of study should children be permitted to try during the study for mastery? (*1*) 1. (*2*) 2 or 3. (*3*) 5 or 6.

19. Should all pupils be required to use (at least part of the time) the best method for studying the addition facts? (*1*) Yes. (*2*) No.

20. Is there any value in letting pupils engage in a discussion of ways of studying for mastery? (*1*) Yes. (*2*) No.

21. Is it considered good practice to delay study for automatic mastery (memorization) until all members of a class understand? (*1*) Yes. (*2*) No.

22. Is requiring pupils to show or to prove facts during the period of study for mastery of much significance? (*1*) Yes. (*2*) No.

23. Under ideal conditions which way of presenting combinations for study during the study period of mastery is considered best? (*1*) Flash card. (*2*) List. (*3*) As a part of a basic fact table.

24. What is the purpose of cutting off the upper left-hand corner of the combination side of flash cards? (*1*) To facilitate stacking cards in desired order (combination or fact side up). (*2*) To emphasize adding down. (*3*) It is a means of identifying the cards as basic fact cards rather than just another kind of number cards.

25. How many different types of subtraction situations are there? (*1*) 1. (*2*) 3. (*3*) 5.

26. Which method of subtraction, the additive or the subtractive (take-away), is easier to rationalize? (*1*) The additive. (*2*) The take-away.

27. Which method of subtraction, the additive or the take-away, is in more common use? (*1*) The additive. (*2*) The take-away.

28. Is the writing of the borrowed one in subtraction considered bad practice? (*1*) Yes. (*2*) No.

29. In column addition is the plan of selecting pairs or trios of numbers equal to ten recommended? (*1*) Yes. (*2*) No.

30. Should pupils be taught to break the ones number in higher decade addition into two parts? (*1*) Yes. (*2*) No.

31. How many basic sum cards are there? (*1*) 17. (*2*) 36. (*3*) 64. (*4*) 81.

32. What is probably the chief weakness in the treatment of subtraction in upper-grade arithmetic books? (*1*) The neglect of basic facts. (*2*) The lack of adequate maintenance exercises. (*3*) The re-use of beginning subtraction exercises.

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5

Multiplication and Division of Whole Numbers

The systematic teaching for mastery of the basic aspects of the multiplication and division processes is usually begun in the third grade. Introduction of all the basic facts is completed in the fourth grade, but additional advanced phases of multiplication and division of whole numbers are not introduced until the fifth and sixth grades. Even though study for mastery of the basic multiplication and division facts is not begun until Grade Three, pupils will have already had many experiences with the ideas of multiplication and division before that time. In the early study of numbers a child has experience separating such numbers as 4, 6, and 8 into parts. Such expressions as "two 3's make 6" and "8 is the same as two 4's" are commonplace. It should be noted that in some of these early foundation experiences, especially those in which pupils are guided to study numbers and groups of objects, division situations receive major emphasis. If the proper situations are provided, pupils will have need for, and will use with understanding, such expressions as "two times as long" and "three times as heavy." Pupils will have had experience with such expressions as three 4's, two 8's, and the like, when problems involving multiplication (or division) were solved as group undertakings. Counting by

10's, 5's and 2's will also have provided background experience for study of multiplication.

There is practically complete agreement on giving first consideration to multiplication when systematic study of the two processes is begun. That plan will be followed in this chapter. The description of a recommended teaching procedure is presented first and then major issues in the teaching of multiplication are discussed.

First Lessons in Teaching Multiplication

The lessons, parts of which are described below, occurred during the second semester of the third grade. The teacher introduced the first lesson with the following statement. "In arithmetic, today, find the answers to the questions in the problems on the study sheet. (Six multiplication problems were on the duplicated study sheet. These might have been written on the chalkboard instead.) When you have found each answer, show with a diagram, numbers, or both that your answer is correct. If you do not know the answer to a question, make a diagram and count out the answer if you need to."

Here are three of the problems the teacher gave.

1. Bill can carry four chairs at a time. How many chairs can he move in three trips?

2. How many wheels are needed to make seven two-wheeled carts?

3. Jane opened one of her three boxes of colored chalk. There were six pieces in the box. The other two boxes also had six pieces. How many pieces of colored chalk did Jane have?

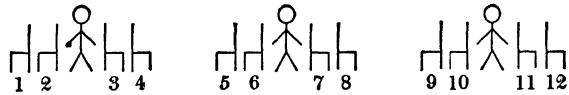
To be sure that all pupils were able to read the problems, the teacher had one pupil read the first aloud and then as the pupils worked, she assisted those who needed help reading other problems. The teacher suggested to the more able pupils that

it was possible to show even a third or fourth way of arriving at the answer and she also asked some of the pupils to put their ways on the board. The teacher herself put on the board an unusual way which one rather slow-working child had used.

The following ways of showing the solution for problem 1 finally appeared on the board:

(*a*) $4 + 4 + 4 = 12$

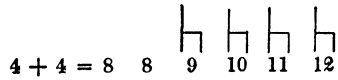
(*b*)



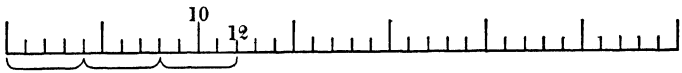
(*c*)



(*d*)



(*e*)



(*f*)

$$\begin{array}{r} 4 \\ 4 \\ 4 \\ \hline 12 \end{array}$$

In a class discussion of the methods used for problem 1 and other problems, questions such as these were asked frequently: (1) Does this show how 12 was gotten for the answer? (2) Will you tell how you thought? (3) Can you explain how Bill thought from his drawing? Try it. (4) Does his drawing agree with the facts of the problem?

When the various ways of showing had been discussed, the teacher asked, "Which of our ways do you think is the best?" The children could reach no agreement until the teacher introduced the criterion of ease of writing. On that basis *a* and *f*

were chosen as best. The teacher then said, "We have shown the same number fact in each of our ways of showing the solution. What is the number fact?" After some discussion "three fours are twelve" was picked out as the fact.

The other problems were dealt with in the same manner except that less time was required because only the most significant ways of showing were put on the board and because the pupils knew the procedure and saw how to identify the fact shown and the best method quickly. At the end of the period the six facts (3 4's = 12; 7 2's = 14, etc.) were written on the board under the heading "Number Facts."

The attention of the student of arithmetic teaching is called to a number of points in the preceding learning situation.There was no mention of learning something new. The pupils were given two specific things to do: (1) to answer the questions of the problems and (2) to show that their answers were correct. The suggestion to diagram and count provided a solution for all. The idea of showing in more than one way that the answers are correct gave the more able students extra work which could lead to greater insight into the number situation being considered. Meanwhile, the less able students had more time to do the initial work with the problems. The various ways of showing provided a setting for class consideration of the multiplication situation and for discussion of a best way for solving, which is an important part of a later lesson in multiplication. Attention is also called to the identification of the multiplication facts from the varied solutions of the class.

The next several arithmetic periods in this third-grade class were devoted to development of other multiplication facts from problem situations and from the study of such number questions as "How many are 9 2's, 7 4's," and so on. During one of these periods attention was centered on how best to write and read the new number facts. In considering this question, the textbook was used as the authority. Both the equation

$(3 \times 4 = 12)$ and the vertical $\left(\begin{array}{r} 4 \\ \times\ 3 \\ \hline 12 \end{array}\right)$ forms were used and the facts were identified for the third grade as multiplication facts. The combination forms such as $3 \times 4 =$ and $\begin{array}{r} 7 \\ \times\ 4 \\ \hline \end{array}$ were read as number questions (e.g., "Three fours are how many?" and "Four sevens are how many?")

As was indicated above, part of the previous lessons was concerned with direct consideration of number questions like "How many are 5 3's?" These number questions were written in both horizontal and vertical form. The following quotation from the teacher gives an indication of how the number questions were used: "For the last part of our period, copy these number questions on the board and write the answer to each question." (Only one of the rows given is shown here.)

2	4	5	4	3
$\times 6$	$\times 2$	$\times 3$	$\times 4$	$\times 7$

For those pupils who seemed to have difficulty in giving the answers, the teacher made this suggestion: "If you are not sure of an answer to a question or can't figure it out, remember you can make marks and count, you may use the number line or you may add."

After all the multiplication facts for 2's, 3's, 4's and 5's (zero and one facts excepted) and a few other facts had been introduced through the type of experiences indicated in the preceding paragraphs, the next step was to develop a situation which showed the need for knowing multiplication facts. The teacher introduced this lesson by saying: "On the sheet on your desk are twenty-five number questions. Write the answer to each question. If you finish ahead of time, raise your hand to show me that you are through and then sit quietly while the others go ahead with their work. Remember, if you do not know the answer to a question you can count or add to find the answer."

The twenty-five number questions consisted of multiplication combinations for the 2's, 3's, 4's and 5's. One child finished at the end of four minutes and at the end of eight minutes when time was called, 16 of the 30 children in the class were still working. The teacher kept a record of the time spent by those who finished, and she also observed the methods of finding answers that various pupils had used. She began the discussion by saying, "Some of you finished in a very short time and others were not through when I called time. Let's see how different people went about finding the answers. Bob, you finished in a very short time. How did you get the answers?"

After some hesitation, Bob said, "I just knew the answers and wrote 'em down." Other children who had finished reported similar direct ways of arriving at the needed products. Children who had not finished all twenty-five usually gave answers like these: "I knew some but on others I had to make marks," "I tried adding some but found making marks was easier," and "I figured each one out in my head by adding."

The teacher pointed out that the class would be using multiplication facts a good deal from then on in arithmetic and that much time and effort (making marks and counting) would be saved if they knew the facts. The children could see clearly from what had happened in class that morning that the better you knew the facts, the faster you could work. The teacher went on, "For the next few days we are going to study some multiplication facts. Before we start let's list some study procedures."

The following list of suggestions was developed by this class: (1) Say a fact such as "five fours equal twenty" over and over. (2) Write the number questions and then give the answer. (3) Cover the answer and try to give it. Then look to see if you are right. (4) Take tests. (5) Ask each other the number questions. (6) Make cards to study.

The kind of suggestions that pupils will make depends markedly on their previous study experiences. Since the third-grade

class referred to above had used cards (flash cards), had covered answers and so on in the study of addition, it was only natural to expect such suggestions as these for the study of multiplication. If pupils have not used study procedures before, it would not be a very profitable learning situation to depend entirely upon the pupils to make study suggestions. On the other hand, since learning (and part of that learning ought to be through systematic study procedures) is supposed to occupy a great deal of the child's time, it seems appropriate to help the child to become aware of some study procedures. If pupils should fail to make the sort of study suggestions that are listed above, the teacher as a member of the class has an obligation to supply some study suggestions and should certainly call attention to study procedures that have already been used by the class.

After pupils have had some experience using their own study procedures, the teacher should recommend for trial the best study procedures. This identification of best procedures for learning multiplication is subject to the same limitations as for learning addition and subtraction (see pages 97–98). Subject to those limitations, the best two procedures involve, first, the identification of facts that need learning (use of a list of facts with products covered is probably the most practical way for identifying facts not known); second, study by the recall method (page 98).

Intensive study of basic multiplication facts occupied the major share of the arithmetic period of this third grade for four consecutive days. Here, as was true in the intensive study of basic addition and subtraction facts, pupils were required as a part of each study period to show that answers given to some combinations were correct. During the four days, some of the pupils who knew the facts well used part of the time to make diagrams and drawings showing each fact. Some of these drawings were displayed on the bulletin board. An example is shown below.

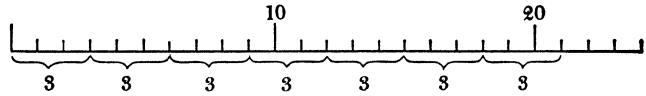
WAYS OF SHOWING MULTIPLICATION FACTS

(1) Seven threes equal twenty-one

(2) $7 \times 3 = 21$

(3)
$$\begin{array}{r} 3 \\ \times\ 7 \\ \hline 21 \end{array}$$

(4) 

(5) 

(6) 7 3's = "3 3's" + "3 3's" + 3 = 9 + 9 + 3 = 21

During study of basic division facts which followed the study of multiplication, the pupils in this third grade spent some little time in review and further study of the multiplication facts.

In some third grades the multiplication of two- and three-figure numbers by 2, 3, 4, and 5, is introduced soon after the first intensive study of the basic facts. For the first work of this kind it is recommended that only exact tens or exact hundreds be used as the multiplicand. Since tens and hundreds can be handled as easily as ones, an early introduction of this phase of arithmetic is not out of line and it has the special advantage of giving the child the feeling that he is "doing big stuff." It also provides the child with another use for the multiplication facts he has been trying to master.

The multiplication of exact tens and hundreds is immediately followed by multiplication situations in which there are ones and tens. Carrying in multiplication is also introduced soon after. Illustrative problems to be solved in several ways are a good means for developing these phases of multiplication.

The multiplication facts in table form for 2's, 3's, 4's, and 5's are introduced late in the mastery study period in some schools.

An orderly related presentation of facts, such as the tables give, has advantages for learning and should therefore be used if the understanding of the pupils warrants its use.

Analysis of Initial Teaching of the Multiplication Facts

The teaching method reported for the third grade in the preceding section differs from the procedure found in current textbooks in several ways. A number of facts were developed from problem settings and number questions before the class began to work for mastery. In most textbooks only a few facts are presented and then the suggestion to learn them is made. The mixed presentation of all the facts for 2's, 3's, 4's and 5's (except those involving zero and one) with a few other facts is markedly different from the common practice of presenting first the 2's, or 5's or doubles and then the others grouped in families. This mixed presentation of facts is recommended because it is believed that the procedure provides a better idea of the multiplication facts and helps the children see a need for them.

The practice of presenting first the 2's, 5's, or doubles, which is based on the premise that the easy facts should be given first, is not without disadvantages. Consider, for example, the doubles. Since the child already knows that 4 and 4 equal 8, it is assumed that he will have little trouble in learning that two 4's or 2×4 equal 8. The thinking child, however, may experience real difficulty since he already knows 4 and 4, and sees no reason to use two 4's or 2×4. He can learn the new way easily if he sees why he is learning it, but there is not much in the current textbook procedures to make the purpose clear to the child.

The lesson which develops a reason for studying the facts and sets up ways of study as a result of class discussion also differs from most current practice. As has been pointed out several

times before, showing pupils the reasons for doing a process and giving them ways to master the process are excellent teaching practices.

Teaching the Harder Multiplication Facts

The general plan of presenting first the easy multiplication facts (see page 146 for a list of the easy and hard combinations) and following with the hard facts often results in introduction of the hard facts without use of adequate development exercises. Since a great many programs introduce the easy facts or some of them in Grade Three, the logical procedure in Grade Four is to begin with a review of these easy facts as a foundation for the rest of the basic multiplication facts. Since several months have elapsed since the original introduction and the review unfortunately does not provide much developmental material, many pupils undertake the intensive study of the harder multiplication facts with a poor background. It is therefore recommended that exercises which will provide some essential background be included in the early work with the hard facts. The following represent desirable types of exercises: (1) Using illustrative problems to be solved in a number of ways and then selecting the best ways of solving. (2) Requiring pupils to show facts with drawings. For example, they might be asked to show that four 3's equal one 10 and two 1's, that nine 7's equal six 10's and three 1's. (3) Showing with drawings, by use of the number line, and by adding that interchange of multiplier and multiplicand does not affect the product. (4) Checking the correctness of products by multiplying twice. For example, 7×4 is found by multiplying 7 by 2 and then multiplying the product by 2 ($7 \times 4 = 2 \times 7 \times 2$). Another way of finding the product, which pupils should use, is indicated by the following: $7 \times 4 = (3 \times 4) + (4 \times 4)$. (5) Showing pupils through timed exercises and appropriate discussion a need for mastery of the facts (see

page 132). (6) Providing a number of ways of study (see page 133).

For other teaching possibilities, see the next section on study suggestions.

It should be noted in the discussion of multiplication that the facts have been read without use of the word *times*. For example, $\begin{array}{r} 3 \\ \times\ 4 \\ \hline \end{array}$ has been read "four 3's."

Other Multiplication Study Suggestions

The description of the initial teaching of multiplication and the sections following that description identified some of the main procedures for studying the multiplication facts and the process of multiplying. However, what is the main or best procedure for most pupils is not necessarily best for all. Furthermore, the use of a variety of study procedures makes for more efficient learning. The student of arithmetic teaching should therefore study carefully the suggestions that follow.

1. *Multiple counting and the special counting chart*

Pupils will have done some counting by 5's, 2's, and 3's before they start mastering the basic multiplication facts. That the knowledge gained from such experiences is of value in learning the multiplication facts is not very difficult to show pupils and may furnish the basis for undertaking counting by 4's, 6's, 7's, 8's, and 9's. To facilitate such multiple counting, a special counting chart (see page 43) is valuable. Even construction of such charts by some pupils may be a worth-while procedure.

2. *Exercises with the number line*

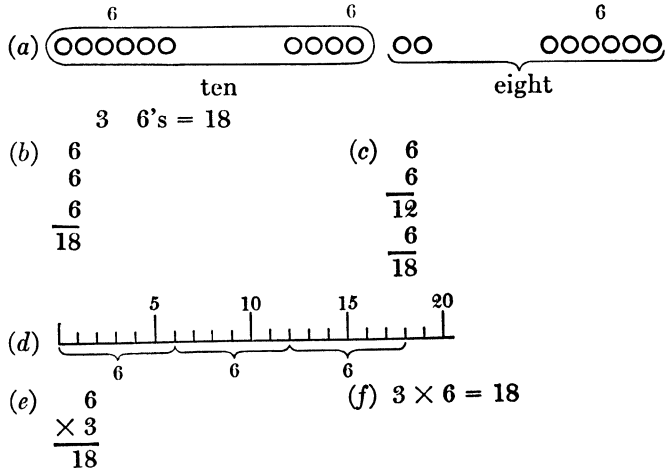
The finding of products by use of the number line is an excellent learning procedure and were it not so time consuming, could be used often. Included in these exercises is the suggestion that pupils use the number line to show that certain products are correct.

3. *The multiplication tables*

After all the basic facts have been developed and some of the best methods of study, such as flash cards and lists, have been utilized, the multiplication tables may be used effectively. Any learning procedure has value which takes advantage of logical relationships such as exist in the multiplication table. Criticism of the use of the tables has arisen because too often in the past they have been used incorrectly, either as the sole learning procedure or as something to be mastered before any understanding of them was acquired. Their use has also been criticized because pupils sometimes go back to $1 \times 4 = 4$ and say all the intervening facts to recall $7 \times 4 = 28$. This last criticism is hardly worthy of note for it must be admitted that the pupil gets the correct product, whereas if facts had been learned only as unrelated facts and momentarily forgotten, the pupil would have no such way of finding the product. Not only is it an acceptable procedure to study the tables by saying the facts in order, but it is recommended that if time permits pupils make their own tables.

4. *Showing facts in various ways*

The examples below illustrate this study suggestion.



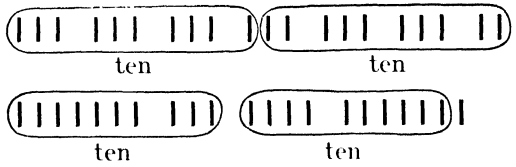
An exercise very similar to the one described above is the preparation and display of statements which show the relationship between various multiplication facts.

SPECIAL MULTIPLICATION FACTS

(*a*) $5 \times 6 = (3 \times 6) + (2 \times 6)$

$$\begin{array}{r} 6 \\ \times\ 5 \\ \hline 30 \end{array} = \begin{array}{r} 6 \\ \times\ 3 \\ \hline 18 \end{array} + \begin{array}{r} 6 \\ \times\ 2 \\ \hline 12 \end{array} \qquad 18 + 12 = 30$$

(*b*) $7 \times 3 = 3 \times 7$



(*c*) 4×7 equals 1 more than 3×9
4×9 " 1 more " 5×7

(*d*) $8 \times 4 = 4 \times 4 \times 2$

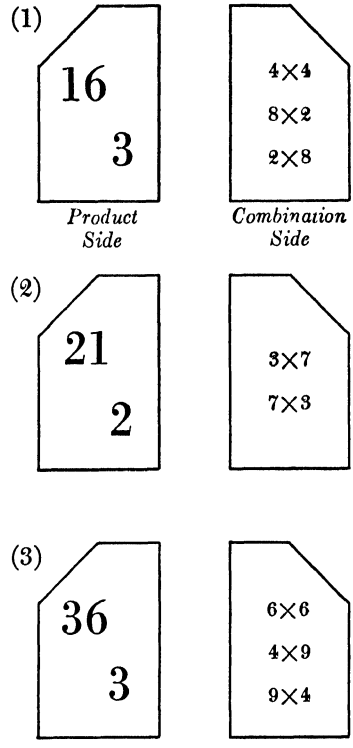
$$\begin{array}{r} 4 \\ \times\ 8 \\ \hline 32 \end{array} \qquad \begin{array}{r} 4 \\ \times\ 4 \\ \hline 16 \\ \times\ 2 \\ \hline 32 \end{array}$$

5. Basic product cards

The basic product cards are similar to the basic sum cards suggested for addition (see page 141). Three examples of the cards are shown:

The small number on the product side of the card indicates the number of combinations which equal the given product. In using the cards the pupil takes a stack of them with product

sides up and tries to give as many combinations equal to the product as is indicated by the small number. If he cannot give the required number or is not sure, the card is turned over and the combinations read.



6. *The multiplication-division board or table*

The multiplication board consists of removable blocks and a board for holding them. The board is a piece of plywood about 28″ × 28″ in size, to which pegs, finishing nails, or screws are fastened. The basic products are written on one side of blocks through which a hole has been drilled; the other side is blank. The blocks are placed on the board so that the product of each block shows the number of blocks above and to the left of the block (the block is included). (The shaded first row and first column of figures in the illustration indicates that these figures are written on the base board and are not composed of blocks.) Pupils may use the board with all the products visible, part of the products visible, or none of the products visible. A pupil may select a product and try to give the two factors, or with the blank side of the block out, try to identify the product by looking at the factors. Since the actual number of objects involved for eight 4's is always visible but rather cumbersome to identify by adding or counting, pupils can easily see the advantage of knowing the facts. When used as a sort of contest between two

pupils, each taking a turn at naming the product, this device can be a great motivator. It is perhaps the best single device for use in teaching multiplication facts.

	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

The table or chart form of this device consists of the same numbers as used on the blocks written on a piece (22″ × 30″) of oak tag. Pupils use the table by covering various products or the factors and then trying to name the designated ones.

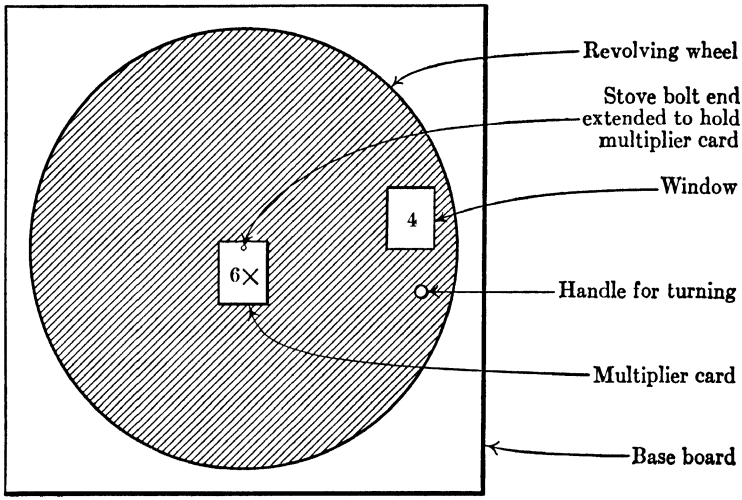
7. *The finger board multiplier* (see page 390)

As far as learning is concerned, this device is primarily useful in the need for a check which it creates and the interest in wanting to know how or why the answer is correct.

8. *The multiplication wheel or Wheat's whirligig*

This device is a revolving wheel fastened by a bolt to a base board (see illustration). Various multiplier cards, all carrying one-figure numbers, are available and can be hung over the bolt in the center. Near the edge of the wheel is a window through which numbers written on the base board can be seen as the

wheel is rotated. The numbers which are concealed appear one after the other. Basic fact numbers, especially the larger ones, appear more than once during a complete turn of the wheel.



In using this device, the pupil places different one-figure numbers in the center space and then tries to give the product of the center number and the numbers showing at the window. Pupils working in pairs often work with greater efficiency than do individual pupils.

9. *The chalkboard race*

A					B				
6	3	8	4	7	8	3	7	6	4

Two pupils representing the two groups into which the class has been divided take places directly facing a set of numbers such as those shown in the illustration. The teacher or some child then names a multiplier such as 6. The pupils write as quickly as possible the products of the given multiplier and each of the numbers written on the board. The one who faces

the class first after finishing, if he has all the correct products, is the winner. The winner's group receives a mark and another representative of the opposing group then challenges. A different multiplier is used for each trial. The factors on each side must be the same, but the order should, of course, vary. The board factors may be two- or three-figure numbers but the multiplier should always be a one-figure number.

Because the members of the class not at the board follow the contest with such enthusiasm, actually doing the multiplication as one or the other individual at the board works, this procedure is one of the best drill exercises if used only occasionally and then only for short periods.

10. *The electrowhiz or other electrically operated devices*

These devices on which a bell rings or a light, usually green, flashes when the correct product for two factors has been identified undoubtedly have value in learning the multiplication facts. Whether or not they are worth the price either in dollars or in time to make depends on what else money and time are used for.

11. *Multiplying twice to find products*

The following teaching directions illustrate this learning procedure: (1) "Show by multiplying twice and adding that the products for exercises 3, 5, and 8 are correct." For $6 \times 4 = 24$ the pupils may write $3 \times 4 = 12$. $12 + 12 = 24$, or $2 \times 4 = 8$ and $4 \times 4 = 16$. $8 + 16 = 24$. For $5 \times 8 = 40$ the pupil may write $3 \times 8 = 24$, $2 \times 8 = 16$, $16 + 24 = 40$. (2) "By changing one number to two multipliers and then multiplying twice show that the products for exercises 3, 6, and 10 are correct." For $6 \times 4 = 24$, the pupil may write $3 \times 4 \times 2 = 24$ or $2 \times 6 \times 2 = 24$. For $5 \times 8 = 40$, the pupil may write $2 \times 5 \times 4 = 40$ or $4 \times 5 \times 2 = 40$.

This exercise is an excellent learning procedure and provides a basis for understanding multiplication with two-figure multipliers. Besides, the procedure offers one of the best multiplica-

tion checks. To be most effective, the procedure should often be used without pencil and paper. The following is an example of a good oral situation: "Bob was not sure whether 4×7 was 28. He was sure about $2 \times 7 = 14$. How could he use that fact to check $4 \times 7 = 28$?"

12. *The tapping test* (see page 122)

While not an outstanding learning procedure, the tapping test is recommended for use because of the ease with which a pupil can identify the facts he does not know well.

Additional study suggestions may be found in textbooks for Grades Three and Four and in the manuals which accompany such books. One common study exercise found in textbooks is the following: "Multiply each figure in the row by 6, by 4, and so on." Such exercises would be more valuable for providing practice if it were not for the fact that only a few pupils are motivated enough to see the need for engaging in such practice.

The Basic Multiplication and Division Facts

The basic multiplication facts like the basic addition facts are formed by a combination of each one-figure number with every other one-figure number, and with itself.

The 32 easy multiplication combinations and the corresponding easy division combinations which appear in the list below are usually considered first in instruction.

If zero is used, there are a total of 100 multiplication facts. Since zero combinations present a special situation, they are not included in the list of basic combinations. The basic division facts are the converse of the multiplication facts but since a number divided by zero equals infinity, no division facts with a zero as a divisor exist. Since there is even less reason for teaching zero divided by a number than for teaching zero multiplied by a number, no zero combinations are included in the basic lists.

32 EASY MULTIPLICATION COMBINATIONS

2 × 5	3 × 6	3 × 2	5 × 6	2 × 2	2 × 3	3 × 4	3 × 3
3 × 9	2 × 9	3 × 8	4 × 5	4 × 7	5 × 8	2 × 6	3 × 7
2 × 4	2 × 8	2 × 7	3 × 5	4 × 4	5 × 3	5 × 2	4 × 6
5 × 7	5 × 4	4 × 3	5 × 5	4 × 8	4 × 9	5 × 9	4 × 2

32 EASY DIVISION COMBINATIONS

$3\overline{)9}$	$2\overline{)4}$	$4\overline{)8}$	$2\overline{)8}$	$4\overline{)12}$	$4\overline{)20}$	$3\overline{)6}$	$5\overline{)30}$
$2\overline{)6}$	$5\overline{)10}$	$3\overline{)12}$	$4\overline{)16}$	$3\overline{)15}$	$5\overline{)15}$	$3\overline{)35}$	$2\overline{)10}$
$2\overline{)12}$	$5\overline{)45}$	$4\overline{)32}$	$3\overline{)18}$	$3\overline{)27}$	$2\overline{)18}$	$4\overline{)36}$	$2\overline{)16}$
$5\overline{)40}$	$4\overline{)28}$	$5\overline{)25}$	$3\overline{)21}$	$2\overline{)14}$	$3\overline{)24}$	$4\overline{)24}$	$5\overline{)20}$

The harder basic multiplication combinations in the list presented consist of the combinations obtained when 1, 6, 7, 8, and 9 are used as multiplicands and all the combinations with 1 as a multiplier. There are 49 combinations on the list. If the generalization is taught that when the number 1 is one of the factors the product is the other number, there are really only 32 of the harder combinations. In a like manner, the generalization that any number divided by itself results in a quotient of one reduces the number to only 32 harder division combinations. With the one quotients included, there are 40 harder basic division combinations.

49 HARDER MULTIPLICATION COMBINATIONS

7 × 5	6 × 5	2 × 1	7 × 7	9 × 8	9 × 2	6 × 4
6 × 2	7 × 1	7 × 3	7 × 2	1 × 1	5 × 1	8 × 2
9 × 6	8 × 5	4 × 1	1 × 3	8 × 8	9 × 7	8 × 3
6 × 1	6 × 6	6 × 9	9 × 4	8 × 1	8 × 9	8 × 6
1 × 2	3 × 1	6 × 8	8 × 4	9 × 3	1 × 8	7 × 4
9 × 5	1 × 7	6 × 7	1 × 6	1 × 9	7 × 6	6 × 3
9 × 9	1 × 5	9 × 1	7 × 8	1 × 4	7 × 9	8 × 7

40 HARDER DIVISION COMBINATIONS

$6\overline{)30}$	$7\overline{)28}$	$5\overline{)5}$	$9\overline{)18}$	$8\overline{)8}$	$7\overline{)21}$	$6\overline{)6}$	$8\overline{)40}$
$3\overline{)3}$	$7\overline{)49}$	$6\overline{)12}$	$4\overline{)4}$	$6\overline{)42}$	$7\overline{)49}$	$8\overline{)40}$	$2\overline{)2}$
$9\overline{)72}$	$6\overline{)48}$	$7\overline{)42}$	$8\overline{)16}$	$7\overline{)7}$	$9\overline{)36}$	$9\overline{)54}$	$8\overline{)32}$
$6\overline{)24}$	$9\overline{)81}$	$9\overline{)63}$	$8\overline{)56}$	$9\overline{)45}$	$9\overline{)9}$	$7\overline{)14}$	$6\overline{)18}$
$6\overline{)54}$	$7\overline{)63}$	$7\overline{)35}$	$8\overline{)48}$	$8\overline{)72}$	$9\overline{)27}$	$8\overline{)24}$	$6\overline{)64}$

The inclusion of the 1 multiplication combinations in the harder rather than the easy list is contrary to most common practice. Because the pupils experience a great deal of difficulty with the 1 combinations and have no use for them until

multiplication of two-figure numbers is undertaken, the writer places them with the harder combinations. The pupils' difficulties are brought about primarily by the fact that the product differs from other products the pupil has encountered in being the same as one of the factors. With other combinations the product is always larger than either of the factors. As was indicated in an earlier paragraph, the generalization may be taught that when one of the numbers of a combination is 1 the product is the other number. This generalization should, of course, be developed from a study of situations where the combinations have been used: for example, multiplication of two-figure numbers. Review and study questions to bring to the pupil's attention or to recall the generalization have an important place in instruction. The example below is an illustration. "When 1 is one of two numbers multiplied, what is always true of the product? Show with three combinations that your answer is correct."

The omission of the zero multiplication and division facts from the lists indicates that these facts are to be treated in a special manner. Since zero is not a number in the sense that it stands for a quantity, there is nothing to be multiplied or divided when zero is one of the factors. This condition is very different from other multiplication and division situations. It might seem, then, that there are no zero multiplication or division facts. In a sense this is true, but in such situations as $\begin{array}{r} 407 \\ \times\ 8 \\ \hline \end{array}$ and $3\overline{)609}$, zero combinations do occur if the usual multiplication and division procedures are followed. In such situations, some people find it advantageous to say "8 0's = 0" and "How many 3's equal 0? 0." A better procedure for dealing with such a multiplication situation is to think, "There is nothing to multiply, so I just write the carried 5." In the division situation, a zero is written in the tens place of the quotient because there was no tens number to be divided. Arithmetical

instruction should be directed toward the development of the best procedure identified in the preceding statements. However, since there are people so verbally minded as to profit from making a statement in the usual number fact form, procedures for learning the zero facts are needed in arithmetic instruction. It should be obvious that the teaching of the zero facts and the one facts should be delayed until after the pupil has experienced a need for them in the multiplication of two-figure numbers. This recommendation is also applicable to division involving zero and one. Procedures designed to assist pupils in mastery of the zero facts (study of facts in a list, use of flash cards, and the like) would not be used, therefore, until after the other multiplication and division facts have been studied and should be used only by those pupils who are unable to use the generalizations stated previously.

Teaching the Basic Division Facts

The same general plan of initial teaching that was used with the multiplication facts is recommended for the basic division facts. Pupils are instructed to find the answers to the illustrative problems by means of drawings and by serial subtraction. (See example 2, page 158.) Since drawings for measurement division problems are much easier to construct than drawings for partition division problems and because serial subtraction is not directly applicable to partition division, the problems used in the development of the basic division facts should be of the measurement type. The two kinds of division situations are illustrated in the two problems below.

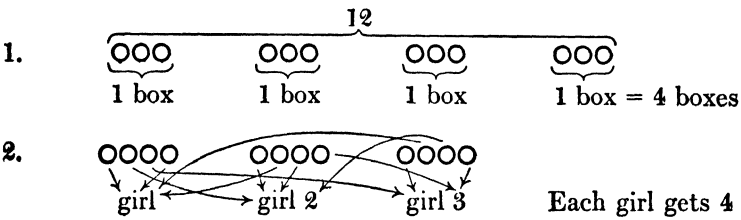
1. John has 12 new rocks for his rock collection. If he puts 3 rocks in a box, how many boxes does he need? (Measurement)

2. Sally and her two classmates wish to share equally the 12 pieces of candy in a bag. How many pieces will each girl receive? (Partition)

The following solutions for these two problems illustrate clearly why measurement is easy to show by drawing while partition is not.

For problem 1 a measure of 3 may be filled and set aside. Then a second measure can be filled and set aside, and so on, until all the rocks have been used. The number of boxes needed can then be easily determined. The thinking used in solving this problem with numbers is as follows. "Three rocks go in a box and there are 12 rocks. Therefore, I need to find how many 3's equal 12." For problem 2, the partition problem, there is no measure. The size of the part or share is to be found. Consequently, serial subtraction as a means of finding the answer is not readily seen by the eight-year-old children. In fact, to use subtraction to solve partition division problems, you have to think, "How many are needed to go around once or to give each shareholder one unit?" The thinking for the most direct solution of problem 2 is as follows. "There are 3 girls; so each gets $\frac{1}{3}$ of the 12. I need to find $\frac{1}{3}$ of 12."

Solutions for illustrative problems 1 and 2:



The preceding discussion and illustrative solutions show clearly why it is recommended that beginning division problems be of the measurement type. There is no research to support this recommendation, but a study [1] has been made which shows

[1] Edwin H. Hill, *A study of preference and performance on partition and measurement division problems*, Unpublished Ph.D. Thesis (Iowa City: State University of Iowa, 1952).

that pupils in Grades Three, Four, Five, and Six prefer measurement problems to partition problems.

The statement of the basic division facts from number and drawing solutions is not as easy for children as is the identification of the basic addition, subtraction and multiplication facts. Consider for example the serial subtraction and drawing solutions for problem 1. The fact statements children usually give when considering the solutions are: "In 12 there are four 3's" and "Four 3's equal 12." Neither of these can really be considered a division fact statement. To see that division facts are somewhat different from other basic facts as far as reading is concerned, read each of these basic facts.

$$\begin{array}{r} 4 \\ +\,3 \\ \hline 7 \end{array} \qquad \begin{array}{r} 8 \\ -\,2 \\ \hline 6 \end{array} \qquad \begin{array}{r} 5 \\ \times\,3 \\ \hline 15 \end{array} \qquad \begin{array}{r} 2 \\ 5\overline{)10} \\ 10 \\ \hline \end{array}$$

When the basic facts began to be emphasized in arithmetic books, the equation form of division fact statements was used exclusively. Exercises consisting of such examples as "$9 \div 3 = ?$" were used. The example was read "Nine divided by three equals how many" or "what." Since pupils often had little notion of what the statements meant, that method of presenting division facts was abandoned in favor of the form $3\overline{)12}$. A few authors still suggest reading "12 divided by 3" for this form, but more commonly it is read "How many 3's in 12," with only a few using "How many 3's equal 12." We have a good way of reading the division combinations, then, but not for reading the facts.

In view of the inherent difficulties involved in the reading of basic division facts, special care needs to be used when they are developed from the initial illustrative problems. The following procedure used by one teacher calls attention to points that are worthy of study.

"Each of our drawing and number solutions shows the answer

to a number question. What is the number question for problem 1?"

"How many 3's equal 12?" and "In 12 there are how many 3's?" were the two questions suggested by the class. The first was chosen as the better. After the number questions for the remaining illustrative problems had been stated, the teacher said:

"We have used figures to write other number questions such as how many are six 7's. Does anyone know how we can write 'How many 3's equal 12?' using only figures?" Since no member of the class knew an all-number way to write the number question, the teacher suggested they turn to the textbook. The textbook used the form $3\overline{)12}$, stating that it is read "How many 3's in 12?"

The use of "in" rather than "equal" is almost unanimous practice. Strictly speaking, there are not four 3's in 12, any more than there are two nickels in a dime. It is a case of equivalence. It would therefore appear that "equal" should be taught. However, this is not a serious matter and "in" may be used just as we used "borrow" in subtraction.

The number question for each of the illustrative problems was then written the all-number way.

The complete all-number question and answer for illustrative problem 1 was then written on the board.

The teacher said, "We call this a division fact. You read it this way. 'How many 3's equal 12? Four. Four 3's = 12.'"

$$\begin{array}{r} 4 \\ 3\overline{)12} \\ \underline{12} \end{array}$$

Each of the other division facts used in the illustrative problems was written and read in the above manner.

Other division facts were developed in this third grade through use of illustrative problems and through consideration

of number questions such as "How many 5's equal 15?" The introduction to and procedures used in the study for mastery of the facts were similar to those followed with the other basic facts (see page 133). The group used the complete or long division form in mastering the division facts, not the short form. In the study of the fact, it is suggested that the pupil read the fact in this way: "Two's equal 8? Four. Four 2's equal 8."

$$\begin{array}{r} 4 \\ 2\overline{)8} \\ \underline{8} \\ \end{array} \quad \textit{Long form} \qquad\qquad \begin{array}{r} 4 \\ 2\overline{)8} \end{array} \quad \textit{Short form}$$

When using the combination and not the fact in study, the pupils are to give only the answer as they look at the combination. For example for $2\overline{)6}$, $5\overline{)10}$ and $3\overline{)18}$ the pupil would say only "3," "2," and "6." If the answer to a combination is not known, the pupil would go back to the entire statement of the fact for study as indicated in the preceding paragraph. After the teacher is sure the pupils know that the question for $3\overline{)18}$ is "How many 3's equal 18?" the suggestion to shorten the question to "3's equal 18?" may be made. While studying the basic facts for 2's, 3's, 4's, and 5's, the pupils were asked to show that their answers for some number questions were correct. The directions "show with a drawing," "show by use of the number line," or "show by subtracting" were used with different exercises.

A Critical Look at Instructional Practices Used in Initial Teaching of Division

The preceding section on initial teaching of division facts has indicated that there are a number of controversial issues in this area of arithmetic. How to read division facts in study procedures, whether to begin with the doubles or other easy facts, whether to use measurement or partition situations in

introductory work, whether to use the long or short form of division with the basic facts, and whether to use "in" or "equal" in the division number questions are issues already presented. Another issue is whether to use the form $6 \div 2$ in introducing division.

A definite recommendation has already been made about using the complete statement of division facts for study and using "equal" rather than "in." A look at current practice as represented by textbooks shows that in the third grade "in" rather than "equal" is used and that the majority of books suggest that a fact be read "Threes in twelve, four." Only one of the recent books uses the complete statement, "Threes in twelve, four. Four threes are twelve." Such omissions or the use of statements that are mathematically questionable may be an indication of lack of understanding or may be an indication of how difficult the teaching of division is.

That such divergences in practice exist or that often no mention is made of how facts are to be read while being studied should warn teachers of arithmetic concerning the use of such statements as "all arithmetic should be taught meaningfully" or "pupils should understand arithmetic." Mere demonstration of the fact that twelve is the equivalent of three groups of four does not give the pupil any reason for seeing that the answer to the question "How many 3's equal 12" is something the pupil needs to know. In no other group of basic facts are the initial phases presented with as little meaning as they are in the basic division facts. For a verification of the preceding statement the student of arithmetic teaching has only to consult four or five arithmetic textbooks published in the five-year period 1947–1952.

The issue concerning the group of division facts to use in initial instruction has already been discussed. Order is not, in the opinion of the author, an important issue. In general, the easy facts are studied first (see list), usually all introduced in

third grade, and the hard facts introduced and studied for mastery in fourth grade.

The issue of whether the form $6 \div 2$ rather than $2\overline{)6}$ be used first has not attracted much attention in recent years. Practically all books use the second form. One recent book, however, uses the form $6 \div 2$ (read "six divided by two"). The reading of this form is certainly more logical than that of form $2\overline{)6}$. How the form $6 \div 2$ can be developed in a meaningful way with pupils is not easy to see. The phrase "divided by" hardly goes with measurement division where the task is to find how many times one number is equal to another number. To the writer the phrase "divided by" belongs with partition division, for to find a part or share you divide the number into as many equal parts as there are partholders or shareholders.

The measurement-partition issue in initial instruction does not appear to be too controversial unless the form $6 \div 2 = 3$ is used for initial instruction. If, instead, emphasis is to be on "How many of one number equal another?" then the measurement type is definitely superior. However, soon after the development of the basic facts, partition-type problem situations should be introduced. This recommendation is based on the fact that many such situations arise in the everyday doings of the average elementary school child. In cases of partition the size of a part or share is to be found. Through use of good illustrative problems pupils can quickly see that the share or part to be found is one-half, one-third, one-fourth, or some other fraction of the total and that you find the total by dividing by the size of the part.

Practically all textbooks state that long division is taught first; yet in the study of the basic facts these same books use the short form. This inconsistency needs to be eliminated. Pupils will have little use for short division until they begin to acquire some proficiency in the multiplication of fractions.

Division with Quotients of More than One Figure

The time to introduce division with quotients larger than one figure has not been agreed upon in practice. A few textbooks introduce this phase of division in the third grade. If division itself were not introduced so late in the third grade, many more textbooks would probably go on to introduce larger quotients before the end of the year. There are good psychological reasons for beginning such division early. Just as was true of the multiplication of two-figure numbers, the early use of quotients of more than one figure gives the pupil a feeling of doing something big. Also, division of larger numbers may give better insight into the process than working only with the basic facts. Since there are many difficulties involved in division of large numbers, the examples used in this first work should be carefully selected to include only even division of the hundreds, tens, and ones involved. Major instruction in this phase of arithmetic is postponed until after the pupils have had an opportunity to learn all the basic facts.

Uneven Division

When division is not exact, for example, $13 \div 4$, another of the many difficult aspects of division is encountered by the elementary school pupil. How to deal with the part left over, or the remainder as it is often called, has been the topic of much discussion in the literature of arithmetic teaching and has been the subject of a number of investigations.

There are two common ways of expressing the left-over part in uneven division examples. It is labeled a remainder as in $11 \div 4 = 2 \text{ R } 3$, or it is expressed as a fraction as in $11 \div 4 = 2\frac{3}{4}$. In the everyday affairs of life, we frequently encounter a third way of expressing the left-over part in division. For example,

in a situation where 21 people are to be transported in cars, 6 to a car, the left-over part has to be changed to a whole. The answer then is 4 cars and not $3\frac{1}{2}$ or 3 with a remainder of 3. In other life situations the left-over part is discarded. For example, if 6 yards of lace are needed to trim a costume and 25 yards of lace are available, only 4, not $4\frac{1}{4}$, costumes can be trimmed with lace.

The arithmetic instructional program should then teach pupils how to express the left-over part in division in three ways: (1) by means of fractions; (2) by labeling the left-over part as a remainder; and (3) by changing the left-over part to a whole or discarding it entirely.

It should be noted that in the preceding discussion, the phrase "left-over part" rather than remainder has been used. It is incorrect to refer to the writing of remainders as fractions. The fraction in the example $5 \div 2 = 2\frac{1}{2}$ is not a remainder but an integral part of the quotient. It indicates that division has been performed just as much as does the 2 of the quotient.

To have meaning for the pupil, each of the three ways of expressing the left-over part in uneven division has to be introduced through use of appropriate problems. Examples for practice are, of course, an essential part of a good instructional program.

When to begin uneven division cannot be determined by examination of current practice. It is introduced in Grade Three in some programs and in Grade Four in others. All textbook programs postpone instruction in this area until after the easy basic facts have been presented. However, earlier use of uneven division may be helpful in giving pupils a more meaningful idea of the division process. Expressing the part left over as a remainder or just as left over is usually the first step presented when uneven division is introduced. Expressing the left-over part as a fraction is started in some programs in the fourth grade, but is left till the fifth and sixth grades in others. No textbooks

published prior to 1951 gave any instruction with the form expressing the quotient in uneven division as a whole number. Yet, as has already been indicated, experience with this form seems essential if all phases of uneven division are to be meaningful to the pupils.

Procedures for Learning the Basic Division Facts

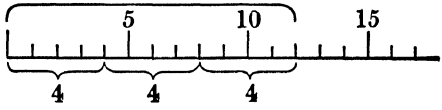
The procedures followed in helping children to master the basic division facts would be, in general, similar to those used in the study of the basic facts of addition, subtraction, and multiplication. Studying lists of facts and using flash cards would be the two main ways. Here, again, it is recommended that as a part of the mastery exercises, pupils should be asked to show that their answers are correct. Because division is a relatively more difficult process and therefore not as easily understood by pupils, this request to show or to prove the correctness of a solution should be used more often than it is in the other processes. The suggestion to demonstrate the correctness of quotients by subtracting or to find the answers to division situations by subtracting should be used many times, to bring out the meaning and significance of the process of division.

The bulletin board display reproduced below presents various ways of showing a division fact and indicates the emphasis put on showing as a means of study.

WAYS OF SHOWING HOW MANY 4's = 12

(1)
$$\begin{array}{r} 3 \\ 4\overline{)12} \\ \underline{12} \end{array}$$

(2)
$$\begin{array}{rl} 12 & \\ \underline{-\ 4} & \textcircled{1} \\ 8 & \\ \underline{-\ 4} & \textcircled{2} \\ 4 & \\ \underline{-\ 4} & \textcircled{3} \\ 0 & \end{array}$$

(3) 

(4) $12 \div 4 = 3$

(5) $4 + 4 = 8$ $\quad 8 + 4 = 12$
① ② ③

Other procedures used in the study of the basic division facts are the following:

1. *The multiplication wheel* (see page 142)

The same device that was used for addition, subtraction, and multiplication can be used with division if a set of dividend numbers is prepared for placing in the center position. Most of the divisions occurring in this exercise will be uneven. However, since a basic division fact is used in every uneven division situation, this seeming inconsistency is not a handicap.

2. *Use of basic product cards and multiplication board* (see page 141)

The exercises suggested in connection with these devices in the section on multiplication contribute to learning division and should therefore be employed.

3. *The chalkboard race* (see page 143)

43 18 26 12 19	43 19 12 18 26

The steps of this exercise are essentially the same as those given for multiplication. In division, the pupil writes the quotient and the part left over. Whether this part left over is to be written as a remainder or as a fraction should be decided before the first contest.

4. *The tapping test* (see page 122)

This exercise is of value primarily because of the ease with which pupils can identify the facts they do not know well.

While not generally considered a part of learning the basic division facts, pupils need help on how to estimate the quotient figure, especially in uneven division. It is recommended that in class discussion of this problem, the procedure be referred to as making a good or best guess. The best guess is, of course, to be immediately verified by multiplying divisor by quotient and comparing the product obtained with the dividend. Exercises 1 and 3 above can be used to give practice on this phase of division.

Summary and Review Exercises in Multiplication and Division

In one fourth grade the summarizing of the study of multiplication was begun with the following assignment: "For several weeks we have spent a great deal of our arithmetic time studying multiplication. Today let's review by thinking of the most important things we have learned about multiplying. Write on a piece of paper what you think are the most important things you have learned. A complete statement is best but a part of a statement will help."

About five minutes were given to individual work on the assignment. During this individual study period the teacher went about the room and wrote on the board these statements which different pupils had written on their papers.

1. "— to think answers like 4 6's = 24 without adding."

2. "— that you can find how many 4×128 is by finding 4×100, 4×20, and 4×8 and adding."

3. "— to prove answers by multiplying twice."

4. "— memorized all the multiplication facts."

5. "— that you can always find a fact with the number line."

6. "— how to tell products on the multiplication board."

Class discussion was initiated by reference to the six statements on the board. Discussion clarified some of these statements and added others as well. A group summary statement was prepared at the close of the discussion period.

The preparation of such summary statements of what has been learned is an excellent way to review by calling the attention of teacher and class to important aspects of multiplication. Division lends itself equally well to this procedure. The procedure also has value in that it often reveals to the teacher that many of the study activities engaged in have not been too meaningful to the pupils. The class-prepared or even individual-prepared summary statement of important things learned in the study of multiplication is good bulletin board material.

Summarization is one good way of reviewing multiplication and division. The study of specially prepared review exercises is another. An example of this kind of exercise for multiplication appears below.

1. "What are three ways of showing that five 7's equal 35?"

2. "Is 10×50 the same as 5×50 plus 5×50?"

3. "Can you find the product for a multiplication example by adding?"

4. "Show with a number line that 3×8 is the same as $3 \times 4 + 3 \times 4$."

5. "Show with a number line that four 8's are equal to eight 4's."

6. "Show with marks that nine 6's are equal to five 10's and four 1's."

7. "In all the numbers from 45 to 75, which ones are not basic fact products?"

8. "Mary's check for 4×68 is shown below: Make your own check for 4×7 using Mary's method. Is the check a correct one?

$$\begin{array}{r} 68 \\ \times 2 \\ \hline 136 \\ \times 2 \\ \hline 272 \end{array}$$

9. "Jack wasn't sure whether seven 7's were 48 or 49. He thought 21 and 21 are 42 and 7 more are 49. Explain Jack's thinking."

10. "Use Jack's way of thinking (see example 9) to make sure about how many are eight 8's."

11. "What are two good study exercises for learning the multiplication facts?"

The reader's attention is called to the fact that summary and review exercises of the type presented above, while important additions to the general review procedures, are not recommended as the whole review program. End-of-chapter tests, review exercises, and the like found in textbooks can be used effectively. Current textbooks and the accompanying manuals should be consulted to obtain a complete view of the program presented in such sources.

Multiplication and Division with Larger Numbers

The multiplication and division of numbers of two or three figures by one-figure numbers is usually introduced before all the basic facts are introduced. Multiplication with a two-figure multiplier and division with two-figure divisors are undertaken after all the basic facts have been studied in practically all instructional programs.

Work in these two areas of arithmetic is very important. This is shown in part by the fact that it comprises a major part of

arithmetic in Grades Four and Five. The teaching of such important phases of the subject, then, warrants careful study.

Two-Figure Multipliers

The first two-figure multipliers commonly used in instruction are the exact tens (10, 20, 30, etc.). Most programs develop the method for dealing with such multipliers by using ten first as the multiplicand (e.g., 5×10), and then changing to ten as the multiplier as in 10×5, 10×50, and the like. Since the actual computation in such situations is very simple, pupils quickly learn to use the procedure suggested. Often, however, there is little understanding of the process used.

A better understanding of multiplication by tens would probably result if a number of illustrative problems were first solved by several means and then careful study directed toward identifying the best number way of multiplying by 10. The teaching directions and steps used in one classroom illustrate this recommended procedure.

"Write the number question for each problem. Then, find the answer by thinking how many tens. Check by multiplying by 5 and doubling your answer. The first one has been done for you."

1. A laundryman delivered 10 packages of towels to a barber shop. There were 18 towels in each package. How many towels were there in all?

(*a*)
$$\begin{array}{r} 18 \\ \times\ 10 \\ \hline \end{array}$$
(number question)

(*b*) Ten 18's is the same as 18 10's. That's 180 (thinking how many tens).

(*c*)
$$\begin{array}{r} 18 \\ 5 \\ \hline 90 \\ 90 \\ \hline 180 \end{array}$$
(multiplying by 5 and doubling the product)

After some time had been spent on the assignment, class consideration of it began. The number question for each problem

and the answer obtained in the two ways were written on the board. "How could the product be gotten by just multiplying?" then became the problem for group study. The class decided that "You multiply by 1, but because it's one ten you write a zero in the one's place."

This procedure for multiplying by ten was next tested with two tens (20), three tens, and other exact tens multipliers. Computations were checked by changing the multiplier into parts and multiplying more than once.

The steps used to introduce multipliers made up of tens and ones (21, 16, etc.) in one classroom are presented below. The pupils were asked to find answers for a set of introductory problems and to check the correctness of their solutions by the use of at least one other method.

The solutions proposed for the problem 17×42 are reproduced.

$$
\begin{array}{llll}
(a) & \begin{array}{r} 42 \\ \times\ 10 \\ \hline 420 \\ 294 \\ \hline 714 \end{array} & \begin{array}{r} 42 \\ \times\ 7 \\ \hline 294 \end{array} & \\
(b) & \begin{array}{r} 17 \\ \times\ 6 \\ \hline 102 \\ \times\ 7 \\ \hline 714 \end{array} & & \\
(c) & \begin{array}{r} 42 \\ \times\ 4 \\ \hline 168 \\ \times\ 4 \\ \hline 672 \\ +\ 42 \\ \hline 714 \end{array} & & \\
(d) & \begin{array}{r} 17 \\ \times\ 40 \\ \hline 680 \\ +\ 34 \\ \hline 714 \end{array} & \begin{array}{r} 17 \\ \times\ 2 \\ \hline 34 \end{array} &
\end{array}
$$

As the group discussed the solutions presented, they found that the general problem involved was "What is the best procedure for multiplying by a two-figure multiplier?"

The class decided method (*a*) was best and made this statement based upon it. "Change the multiplier into two parts, the exact tens and the ones, multiply each part separately and

then add the products." This rule or procedure was then tested on a number of examples. Following this experience the textbook was consulted. The pupils had little difficulty in seeing that the textbook procedure was just a refinement of the one they had worked out.

During the intensive study of the textbook procedure, the plan of changing the multiplier into two parts as shown in (*b*), (*c*), and (*d*) was used as a means of checking.

In the teaching procedure described the use of the following points should be noted: (1) Illustrative problems were used to introduce each phase of multiplication; (2) pupils were directed and encouraged to use cumbersome but known procedures to find the answers to problems; (3) the study of solutions to illustrative problems resulted in identification of the learning problem; (4) a tentative procedure or rule was developed and tested on examples; (5) various lead-up experiences provided a foundation for understanding the best procedure presented in the textbook; (6) during study for mastery of the textbook procedure, the previously used developmental procedures were used as a check.

Some authors insist that in multiplication with multipliers of two or more figures, multiplication should begin with the left hand instead of the right hand or ones figure.[1] The example below illustrates the procedure. While not advocating that this procedure be taught, the author wishes to point out that this method focuses attention on the placement of partial product figures. Its use on occasion is, therefore, recommended.

$$\begin{array}{r} 547 \\ \times\ 324 \\ \hline 1641 \\ 1094 \\ 2188 \\ \hline 177{,}228 \end{array}$$

[1] Charles Godfrey, and A. W. Siddons, *The Teaching of Elementary Mathematics*, (London: Cambridge University Press, 1946) pp. 97–98.

TWO-FIGURE DIVISORS

Teaching pupils how to divide with two-figure divisors, or long division as it was once called, has often been considered the most difficult part of arithmetic instruction. While the process is inherently difficult, it has perhaps been made much more difficult than it really is by the fact that pupils have not had a good opportunity to acquire an understanding of the rather complex computational procedure used.

It is believed by the writer that much of the difficulty and lack of understanding of division can be eliminated by use of good illustrative problems to introduce work with two-figure divisors and by use of developmental procedures in which emphasis is placed on known computational procedures.

Division with two-figure divisors is probably best begun with the use of exact tens divisors (20, 30, 60, etc.). The illustrative problems for this introductory work should be measurement division problems because of the importance of using serial subtraction as one solution. Division with exact tens divisors gives practice in placing the quotient figure and some experience with use of only the first divisor figure in estimating the quotient but does not involve the really difficult part of two-figure divisors. The difficult part of two-figure divisor problems arises in estimating the quotient when there are ones figures in the divisor. For example, in $24\overline{)126}$, if only the first figure of the divisor is used to estimate the quotient the estimate will be incorrect; if both divisor figures are used the mental computation involved is too difficult for most pupils. Quotients of this type are referred to as non-apparent quotients. Apparent quotients are those in which the estimated quotient figure turns out to be the true quotient.

There are two well-known procedures or rules for estimating the quotient for two-figure divisors. One procedure involves

the use of only the first divisor figure and is usually stated as follows: "Divide the first one or two dividend figures by the first divisor figure. If the quotient obtained is too large, try the next smaller figure and continue until the true quotient figure is found." The other procedure suggests the rule just given for those divisors ending in figures 1, 2, 3, 4, and 5, but for divisors ending in 6, 7, 8, or 9 it is suggested that the first divisor figure be increased by one before estimating the quotient figure. The second procedure thus calls for two rules for estimating the quotient. The rule to use is determined by the size of the units figure in the divisor.

The finding of the quotient for the example $36\overline{)289}$ illustrates the major points of the one-rule and the two-rule procedures. When the one-rule procedure is used, the thinking is about as follows: "$28 \div 3$ gives 9 for the quotient but multiplication of the complete divisor 36 by 9 shows that the first quotient estimate is too large. Try the next smaller number which is 8. Multiplication shows that to be the true quotient."

When the two-rule procedure is used the thinking is about as follows: "Since the ones figure of the divisor is more than five, increase the tens figure of the divisor by one. Then, think $28 \div 4 = 7$. Multiplication shows that 7 is too small. Then, 8 the next larger quotient is tried and found to be the true quotient."

Solution of a number of representative examples of two-figure divisors by both procedures will show that the use of the two-rule procedure will result in fewer situations where the estimated quotient is in error. However, it should be noted that there are times when the estimated quotient is not the true quotient. A good opportunity for confusion is introduced by teaching two rules for estimating and also by the fact that in one procedure for estimating, the trial quotient is decreased when it is found in error and in the other procedure the trial quotient is increased.

The greater accuracy which results from adding a second rule does not warrant, at least in beginning instruction of this phase of division, the liabilities encountered through teaching the second rule. It is, therefore, recommended that the single-rule procedure for estimating the quotient be taught. After a year or two of using the one-rule procedure, all pupils should have the opportunity to learn the use of the two-rule procedure for estimating quotients, especially with divisors ending with 9 and perhaps 8. Some of the more able students will, after a year's experience with the one-rule procedure, have developed some short cuts in its use. The presentation of the second rule will therefore appear to be a short cut. While use of the second rule should be required of all students during the time it is being taught, it should be considered as a short cut or refinement of the basic one-rule procedure and therefore not required of those students who have little ability and who want to use the one-rule procedure. For further study of this estimating of quotients, the student of arithmetic teaching is referred to Buckingham's treatment of the topic. It is also suggested that the magazine articles by Grossnickle, Osburn, Morton and Mosier be read.[1]

Work with various examples of two-figure divisors reveals that the estimated quotient is in error far more often with the smaller than with the larger two-number divisors. For example, when 17 or 27 are the divisors with a representative set of

[1] B. R. Buckingham, *op. cit.*, pp. 195–201; F. E. Grossnickle, "An Experiment in Division with a Two-Figure Divisor," *Elementary School Journal*, 35: 668–677 (August, 1935); *ibid. Elementary School Journal*, 32: 299–306 (December, 1931); R. L. Morton, "Estimating Quotient Figures When Dividing by Two-Place Numbers," *Elementary School Journal*, 48: 141–148 (November, 1947); Worth J. Osburn, "Levels of Difficulty in Long Division," *Elementary School Journal*, 46: 441–447 (April, 1946); Harold Mosier, "Two Procedures for Estimating Quotient Figures When Dividing by Two-Place Number," *Elementary School Journal*, 49: 516–522 (May, 1949).

dividends, the estimated quotient is incorrect far more often than when 87 and 97 are used with the same dividends. The reason for this is that the discarded units figure is a large part of the total divisor in the case of the small numbers while it is only a relatively small part in the large numbers. As a result, to avoid many cases of incorrect estimates, the larger two-figure divisors should be selected.

The most difficult two-figure divisors are the numbers 11 to 19. Consider, for example, $19\overline{)126}$ and $17\overline{)110}$. In each case, if the rule is followed literally, the estimated quotient figures are 12 and 11, respectively. For such special situations the following procedure is recommended. "When the trial quotient turns out to be a two-figure number such as 10 or more, use 9 for the first trial quotient."

Major Points for Consideration in the Teaching of Division

The preceding section dealing with two-figure divisors called attention to one of the most important points in the division process: namely, that of estimating the quotient figures. Other important points of procedure were indicated in the presentation. For example, the procedure for estimating the quotient required the determination of partial dividends. This step, of course, is first taught with one-figure divisors but does not become very significant until divisors of two or more figures are encountered. The procedure advocated for finding the partial dividend is to compare divisor and the first one, first two, first three, and so on, figures of the dividend until a combination of figures is obtained that is as large or larger than the divisor.

Another significant part of the process is the placement of the quotient figures. The rule preferred on this point is that

each quotient figure is placed directly above the last figure of the partial dividend used. It is assumed that the reasons for such placement will be developed with the pupils.

How to determine whether the estimated quotient is too large without writing the quotient, the product of divisor and quotient, and then subtracting is another point which requires attention. Oral — that is, non-pencil-and-paper — multiplication and comparison is the procedure recommended. The words used by one teacher in telling how she thought are a good illustration of this procedure: "For the example $35\overline{)237}$, I thought 3's = 23, 7. $7 \times 5 = 35$. I think, carry 3 tens. $7 \times 3 = 21$ and 3 tens carried gives a sum of 24. That's larger than the 23 of the dividend, so I know 7 is too large for a quotient. I next try 6."

Checking demands a place also in the teaching of division. The most common method is multiplying divisor by quotient and, if division is unequal, adding the remainder. If the computation is correct, this sum is equal to the dividend. Such a check has one rather serious limitation. Since the multiplication involved is practically the same as that used in the division, errors made in multiplying during the division process are likely to be repeated. In the upper grades, the check of nines can be used. The subtraction check is too cumbersome to use when quotients are more than one figure, but because it illustrates the economy of division and also provides a good background for understanding the process, check by subtraction should be employed several times each year.

As part of a plan for emphasizing understanding of the division process, some division situations should be solved by using symbols other than numerals (see page 195). An additional procedure for helping pupils to understand the process of division with large numbers is to divide by 10 or 100 times the divisor, subtract the divided amount from the dividend, and

then divide again. The division of 3742 by 17 according to this method is shown below.

$$
\begin{array}{r}
10 \\
10 \\
100 \\
100 \\
\hline
17)\overline{3742} \\
1700 \\
\hline
2042 \\
1700 \\
\hline
342 \\
170 \\
\hline
172 \\
170 \\
\hline
2
\end{array}
\quad = 220\tfrac{2}{17}
$$

Reteaching and Enrichment of the Teaching of Multiplication and Division

The initial teaching of multiplication and division of whole numbers will have been practically completed by the end of Grade Five. The reteaching of these phases of arithmetic in Grades Six, Seven, and Eight should still be a topic of special concern to teachers of arithmetic. This reteaching of what has already been presented in lower grades (just as in addition and subtraction) should be introduced through situations that are different from those used earlier. The report presented below is an illustration of the use of new material in introducing multiplication in a sixth grade.

The teacher began this lesson by saying: "When I told my grandfather that I was teaching sixth-grade arithmetic, he asked if I had taught you the duplation method of multiplying. After I told him I had not even heard of the method, he showed me how to do it. It's very interesting. I'll show you."

She then wrote the following on the board:

23	46	92	184	368	368
× 17	8	4	2	1	23
					391

"To multiply by this duplation method," explained the teacher, "you halve the multiplier until you get 1. Fractions are discarded. Keep doubling the multiplicand each time you halve the multiplier. To find the product, add the figures which stand above odd numbers. The sum is the product. Let's check to see if it is." The class found that regular multiplication of 23 by 17 gave the same answer.

The teacher then suggested that the pupils try to solve 73×26 by this duplation method, checking their result by regular multiplication. Several examples were solved, and the procedure and check were discussed. This check of the new procedure provided a use for the multiplication process which was to be restudied. The observation of pupil work, the assistance given the pupils, and the discussion showed the teacher the need for review or reteaching of the basic procedures. This phase of instruction was introduced through the statement: "Some of you have forgotten various steps in multiplying. Practice on them will help all of us. Here are some examples for study. Before you start on them, I suggest that each of you try the tapping test (see page 122) with the multiplication combinations at the bottom of the sheet."

At the close of the practice exercise, the suggestion was given that the pupils check each example first by changing the multiplier to two parts and adding the two products, and then by one other method. After a few minutes of work on the examples, all were stopped to check on whether or not the directions were understood.

The lightning method of multiplying, the scratch method, and the lattice or gelosia method might be used instead of the duplation method to introduce the reteaching of multiplication (see pages 383–384). In a similar manner new or novel division methods might be used to reintroduce the teaching of division.

The use of old or novel methods of multiplying and dividing not only has value as a means of reintroducing these phases of

arithmetic but is of value in enriching the pupil's knowledge of these areas of arithmetic. Other procedures for enrichment are the use of various checks for correctness of computation, the use of short cuts, and the use of some special devices. For example, the use of $6 \times 26 \times 2 + 26$ as a check for 13×26 is likely to give some students a better understanding of multiplication than they would have if no check of this type were used. The enrichment possibilities of short cuts are suggested by this procedure for multiplying 26 by 13 which one pupil employed:

$$13 \times 26 = 25 \times 13 + 13 = \frac{1300}{4} + 13 = 325 + 13 = 338$$

The use of blocks in computation is one special device which can enrich the teaching of multiplication and division. Here again consider 13×26. Thirteen sets of 26 blocks would be presented and then larger blocks (hundreds for ten tens and tens for the ones) substituted in securing the simplest answer.

Pupil preparation of statements of procedure can add interest and clarification to this phase of arithmetic. An example of such a statement written by an eighth-grade pupil appears below.

$$29\frac{23}{43} \text{ or } 29 \text{ R } 23$$

$$\begin{array}{r} 43\overline{)1270} \\ 86 \\ \hline 410 \\ 387 \\ \hline 23 \end{array}$$

"In dividing 1270 by 43 I begin by finding the first partial dividend. To do this I just consider dividend figures beginning at the left until I have a number larger than the divisor. In this example that is 127. I then ask, "How many 43's = 127." To make thinking easier I say "How many 4's = 12." To be sure

the answer is correct, I multiply 3×43 but do not write the product. The product is larger than my partial dividend so I try the next smaller number. That's 2. Again I multiply and since the product is not too big, I write the quotient 2 over the last figure of my partial dividend and subtract the written product 86 from the partial dividend. To get the second partial dividend I consider the next dividend figure 0 along with the remainder 41. That gives 410 for a second partial dividend. I then ask, "How many 43's = 410, or how many 4's = 41?" The answer is 10 but since I know it can't be a two-figure number I try 9, and multiply mentally. Since the product is not too large, I write it and the quotient figure 9 and subtract the product from the second partial dividend. The remainder 23 may be divided by writing 23/43 in the quotient or it may be written as a remainder 'R 23' in the quotient."

Other suggestions for enriching the teaching of multiplication and division can be found in Chapter 13 and the section on foundation number experiences for upper grades.

STUDY QUESTIONS

1. In early foundation number experiences which ideas, multiplication or division, receive more attention? (*1*) Multiplication. (*2*) Division.

2. When multiplication facts are developed from illustrative problems, in what form — equation or vertical — are the facts first written? (*1*) Equation. (*2*) Vertical.

3. How should the multiplication combination $\begin{array}{r} 6 \\ \times 2 \\ \hline \end{array}$ be read? (*1*) Six twos. (*2*) Six times two. (*3*) Two sixes. (*4*) Two times six.

4. If only the 2's, 3's, 4's and 5's facts are used and if the zero and one facts are excluded, how many multiplication facts are there? (*1*) 20. (*2*) 32. (*3*) 36. (*4*) 45.

5. In giving pupils directions or steps to use in mastering the multiplication facts, how many distinct steps should be advocated? (*1*) 2. (*2*) 3. (*3*) 4. (*4*) 5.

6. If you asked the pupils in a class of 30 to show in several different ways that $6 \times 3 = 18$, about how many distinctly different ways might be expected? (*1*) 3. (*2*) 5. (*3*) 12. (*4*) 20.

7. Of these multiplication situations: (*a*) 3×20 (*b*) 2×21 (*c*) 3×15, which should be taught first? (*1*) *a*. (*2*) *b*. (*3*) *c*.

8. Is use of the multiplication tables for mastery of the basic facts recommended? (*1*) Yes. (*2*) No.

9. In the development of the multiplication facts which of the following devices: (*a*) the abacus, (*b*) the tens and ones blocks, (*c*) the number line, is most useful? (*1*) *a*. (*2*) *b*. (*3*) *c*.

10. On this basic product card 

what does the 4 indicate? (*1*) That 4 is one of the factors of 12. (*2*) That there are 4 combinations whose product is 12. (*3*) That the pupil is to think of the number which when multiplied by 4 equals 12.

11. For which of these combinations: (*a*) 8×3 (*b*) 7×8 (*c*) 5×9, would the finger board multiplier be useless? (*1*) *a*. (*2*) *b*. (*3*) *c*.

12. Is the procedure whereby pupils multiply twice to find a product of any value for such combinations as 7×8? (*1*) Yes. (*2*) No.

13. How many of the "harder" multiplication combinations are there (all 1 quotients are included)? (*1*) 36. (*2*) 40. (*3*) 45. (*4*) 64.

14. Should provision be made to teach the zero division facts (zero as the dividend)? (*1*) Yes. (*2*) No.

15. Is the problem "If the four boys weighed 218 pounds, what was their average weight?" a measurement or a partition division problem? (*1*) Measurement. (*2*) Partition.

16. Which type of problem, measurement or partition, is easier to solve by drawing? (*1*) Measurement. (*2*) Partition.

17. For introducing division of two-figure numbers which type of problem, measurement or partition, is better? (*1*) Measurement. (*2*) Partition.

18. Which form of division, long or short, do most pupils' textbooks use in introducing division? (*1*) Long. (*2*) Short.

19. How many ways of writing or expressing the left-over part in uneven division are taught in most current textbooks? (*1*) 1. (*2*) 2. (*3*) 3. (*4*) 4.

20. When the chalkboard race is used in division, how many divisors have to be written on the board? (*1*) None. (*2*) One. (*3*) Two. (*4*) As many as there are dividends.

21. Which of these situations would be taught first: (*a*) 80×92 (*b*) 26×32 (*c*) 41×54? (*1*) *a*. (*2*) *b*. (*3*) *c*.

22. In checking the product for 36×51 what is probably the major advantage of using the form $6 \times 51 \times 6$? (*1*) It provides extensive practice with 6 as a multiplier. (*2*) The factoring gives the pupil insight into the multiplication process. (*3*) Only a one-figure multiplier is used.

23. What is probably the chief limitation of the two-rule procedure for estimating the quotient? (*1*) The relatively larger number of situations where the estimate is incorrect. (*2*) There are two distinctly different procedures to use when the quotient is incorrect. (*3*) The fact that a divisor ending in 5 is not increased.

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6

Problem-Solving

Introduction

The confusion that exists about problem-solving in the arithmetic curriculum is well illustrated by a student observer's reaction when she visited an observation school of the teacher-training institution she was attending. What she saw is briefly related below.

The 28 second-grade children were planning to go to the County Old Folks Home to sing Christmas carols. Twelve mothers had offered cars to transport the pupils to the home.

"Will we need all 12 cars?" the teacher asked. The group decided that 6 second-graders could ride in each car. At the teacher's suggestion, 6's were counted on the number line until the 28 children and the teacher had been accounted for. As each group was counted, that 6 was bracketed on the chalk board beneath the line and the teacher wrote "car" under the bracket. When all had been counted, the children found that 5 cars were needed.

The student visitor made this remark when talking afterwards with the teacher, "My! Weren't the children interested in finding out how many cars were needed for your trip? Don't you wish you could have such situations when you teach arithmetic?"

"Why, that was the arithmetic period for the day," replied the teacher.

"You don't mean you use division problems like that in second-grade arithmetic?" questioned the surprised visitor.

In this particular situation the problem was not being used to teach division though it definitely involved the division idea and was of the type which provides background for that process. It was being used primarily to give pupils experience in solving problems. Children need an extensive background of the kind this second grade was getting before systematic teaching of problem-solving is begun.

Certain characteristics of this particular situation should be noticed before the reader goes on to a broader analysis of problem-solving. The question concerning transportation was one of concern to the class. Background and motivation were already naturally developed as they seldom are even in many social studies problems which children can be encouraged to make their own.

Although the problem appears complex for second-graders to deal with themselves, solving it as a class project parallels the problem method of teaching which is now used extensively in the best elementary schools. Teachers do not hesitate to help children find the answers to questions that have been raised in any phase of the school curriculum. For example, to the science question, "Why does it rain some days and snow on others?", the teacher tries to give the children the scientist's explanation in terms she thinks they can grasp, reading any available source material to the children, adding facts and explanations in her own words, and discussing the ideas with them. Of course, many of the facts of science presented will not be understood fully, but that children do profit from such explanations on different levels goes unquestioned. Likewise, children will profit from solution of arithmetic problems which are actually in advance of their knowledge. These children in finding how many

cars were needed engaged in a number experience which might have been beyond them as individuals but which made sense to them, working as a group with the teacher helping to figure out the solution. They counted under her direction and obtained the correct answer. Such experiences should give them practice in analyzing problem situations and certainly should give them confidence to undertake solutions. Attention is called to the fact that as far as teaching arithmetic is concerned only the solution of the problem was considered. No mention of $28 \div 6$ or other arithmetical statement was made. Such problems are not just numerical exercises for children, but come close to being true problems in that the group really wants to find a solution and knows no ready-made method for arriving at one.

The Role of Problem-Solving

Because the misunderstanding of the student observer in the preceding section is only one of the misconceptions regarding problem-solving, an attempt will be made before going further to clarify what is meant by "the verbal problems of the textbooks" and to show what is actually involved in their solution. *Verbal problem*, as used here, refers to the word description of a quantitative situation about which a question is asked. The solution of a verbal arithmetic problem requires the use of one or more of the fundamental arithmetical processes.

This use of processes (the solution) makes for easier consideration in thought, in speech, or in writing of the situation described. For example, in the solution of the problem "Four wheels are needed to make a toy wagon. How many wheels are needed for three such wagons?" the number of wheels, 4, is multiplied by the number of wagons, 3, to obtain 12, the total number of wheels used. The 12 is easier to say, to write, and to think with than is the statement "3 times 4" or "4 wheels for each wagon and there are 3 wagons."

The number solution of a verbal arithmetic problem is then a short (or number) way of representing the conditions stated with words plus the rearrangement of the numbers for simplification of the original statement of conditions. The point just made is illustrated below with the problem involving the number of wheels needed for the 3 wagons.

In reality	*In numbers*	
(*a*) A wagon requires four wheels and three such wagons are contemplated.	(*a*) Write numbers	4 3
(*b*) Four wheels are placed on each of three wagons.	(*b*) Multiply	4 ×3
(*c*) The total number of wheels used is identified by counting or adding.	(*c*) Write product	4 ×3 12

It should be noted that the rearrangement of numbers in the solution of the problem did not result in the identification of any new facts or conditions but only in the simplification of the originally stated conditions of the problem. Looking upon the solution of problems in this way will permit the acceptance of more than one number solution. Even pictures or diagrams which may be counted are for some pupils good means for simplifying problem conditions. Viewed in this light, problem-solving becomes quite a different matter from getting the one answer of the answer book or the one answer which most adults would give. Furthermore, acceptance of this type of problem-solving will make it easy for the teacher to provide for individual differences.

Many teachers feel that helping their pupils gain facility with verbal problems is "the purpose of all work with fundamental operations," that "if pupils can't work problems, they haven't learned any arithmetic." To make the development of ability

to solve the verbal or word problems of the textbook the final goal of all instruction in arithmetic is questionable. One of the assumptions of those who place emphasis on teaching verbal problem-solving ability is that the pupils who can solve the verbal problems of the textbook will be able to solve the quantitative problems that they encounter in life outside of school. Unfortunately, the quantitative problems of life frequently have little resemblance to the carefully worded problems of textbooks and many quantitative situations encountered do not require the textbook type of solution.

A second claim of those who would make verbal problem-solving the all-important end of arithmetic teaching is that this skill helps to improve the pupil's ability to think and to use the scientific method. A consideration of the difference between the procedure a child must go through in attacking verbal problems of the arithmetic textbooks and the steps involved in thinking about a real problem will help to show why this claim is mistakenly made.

Reflective thought arises or begins from a state of doubt, or where difficulty is encountered. John Dewey has identified five phases of thought through which one proceeds in attempting to clarify the difficulty.[1] These phases are as follows:

1. that of suggestion, consisting of two or more rather simple but contradictory plans for solving the difficulty; (consideration of these two simple but contradictory plans is actually a means of identifying more clearly the problem.)

2. that of intellectualization, a general sizing-up of the total situation, a sort of over-all observation of all conditions;

3. that of formulating a guiding hypothesis, a possible course of action or an explanation;

[1] John Dewey, *How We Think* (Boston: D. C. Heath and Company, 1933), pp. 106–115.

4. that of reasoning, consisting of a full development of the idea suggested by the hypothesis; and

5. that of testing the hypothesis by action.

In the solution of verbal problems the answer to a clearly stated question is found. Once the reading difficulty is surmounted, the question almost automatically indicates the process which must be used to find the answer. This is a vastly different procedure from that indicated in the steps above in which the problem must first be formulated, then considered from many points of view to determine the best hypothesis and finally solved by testing the hypothesis. Solving verbal problems might be compared to the skill needed to change a tire as compared with the know-how required to keep a car running well.

The writer has suggested in earlier chapters that opportunities *can* arise in arithmetic which afford practice in thinking. If pupils find themselves in a situation where they need to check multiplication problems and are therefore really interested in figuring out the best way to do it, they will set to work to solve the problem, "What are the best ways to check the product of two-figure multipliers?" They will have to assemble ideas, test them and reach a conclusion. The creation of situations in arithmetic teaching where doubts will arise is essential if true problem-solving involving the complete act of thought is to occur.

The treatment of problem-solving in many books on the teaching of arithmetic includes extensive material on the kinds of subject matter with which verbal problems should deal. It is generally recommended that problems should deal with things and conditions that are within the experience of the elementary school child — that problems should be real. Where problems are used for illustrative purposes, as in initial instruction, this matter of having the subject matter of problems within the

experience of the children is important. In other verbal problem situations, the matter of dealing with experiences familiar to the pupil is not as important. Many verbal problems are essentially only practice exercises and the unreal or make believe problem situation is often just as appealing to the pupil as the real situation[1] and certainly its solution gives just as much practice.

Another aspect of the content of problems which receives much attention in the arithmetical literature, especially that distributed by textbook publishing houses, is the matter of interest appeal. The writers of children's books have used some clever and appealing situations in their attempts to interest pupils. These attempts to make problems interesting have in some instances been detrimental to the purpose of the exercises. The purpose of problem-solving exercises is to give practice in the solution of problems. Such interest attracting features of problems as the description of plans for a birthday party or of the happenings at a ball game will take attention away from the essential problem situations.

Emphasis is also given in the arithmetical literature to the social content of the problems. It is recommended by many writers that the social scene portrayed in problems be true to life and that the subject matter of arithmetic problems be selected to picture important phases of the social scene. This recommendation is rather like the emphasis on reality in problems mentioned earlier. This concern over the social content of problems is in the opinion of the writer somewhat unwarranted. The content of problems should depend on the purpose for which the problems are being used. To illustrate a new procedure, a make-believe situation may be more effective than a real problem. To show the significance of an area of arithmetic such as the multiplication of fractions, real problems dealing

[1] Ronald Welch, *The Related Merits of Two Types of Arithmetic Problems*, Unpublished M.A. Thesis (Iowa City: State University of Iowa, 1950).

with important social content would be more desirable. Attention is called to the fact that if the recommendations of some writers concerning the social content of problems were followed, arithmetic might be more concerned with economics and social customs than with arithmetic.[1] A good guiding statement to follow is to "use the social situation in arithmetic when the social situation has a contribution to make in arithmetic and not just because the social situation has some arithmetic in it." [2]

If, as has been pointed out, problem-solving is sometimes overemphasized, what is the role of problems? Some answers to that question can be obtained by a careful study of the uses of problems in instruction.

1. In the instructional program advocated in this book, problems have played a major role in the introduction to new facts and processes. For example, basic addition facts were developed from illustrative problems. In a similar manner illustrative problems were recommended for use in the development of other facts and processes. This use of problems is characteristic of practically all instructional programs. The problem provides the setting for the introduction of the new phase of arithmetic to be taught. This illustrative use is most important in primary-grade arithmetic, but even in the eighth grade when a new phase of work such as discount is introduced, an illustrative problem is likely to be used.

2. After facts or a process are introduced and studied, a section of problems using these facts and processes is often used.

[1] For an exposition of the social phase of arithmetic see L. J. Brueckner, "The Social Phase of Arithmetic," *Sixteenth Yearbook* of the National Council on the Teaching of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1941), chap. VIII. For an extensive and still different treatment of problem-solving see C. Newton Stokes, *Teaching the Meanings of Arithmetic* (New York: Appleton-Century-Crofts, Inc., 1951), chap. 8.

[2] For a critical treatment of the problem of social arithmetic see H. G. Wheat, "The Fallacy of Social Arithmetic," *Mathematics Teacher* (January, 1946), pp. 27–34.

The purpose of problems in such instances is to show the importance of the ideas or procedures just studied. If this significance is to be easily recognized, then these problems must deal with subject matter in which the pupil sees some significance.

3. Problems are used at every grade level to provide review exercises for pupils. In this role problems are in part used again to illustrate. Of course, problems in review exercises provide the setting for the consideration of the area of arithmetic to be reviewed. Such a setting, for example, in multiplication, would have better opportunity for attracting pupil interest than would a multiplication example.

4. An examination of standardized survey or achievement tests shows that problems play an important part. In some such tests, the problems sections are called reasoning. Even though the inference that other phases of arithmetic do not require reasoning is incorrect, the name does indicate the importance of problems in tests. While teacher-made tests contain a much lower per cent of problems than do standardized tests, problems are often included in them also.

5. The use of problems to provide practice with facts and processes would, upon first consideration, be considered an inferior procedure. The time required to read the problems and to consider the problem setting takes time and attention away from the process or fact to be practiced. However, problems in practice exercises, if they are properly placed, provide variety, something that is very much needed in practice work. Problems in practice exercises in some areas of arithmetic have the additional advantage of giving the learner an opportunity to see the significance of the procedure being practiced.

6. Problems are frequently used to unify a class and to create and to maintain class spirit. For example, during periods of intensive study of a process, each class member works on his particular weakness, and therefore instruction is, for all practical

purposes, of an individual nature. Class spirit quickly degenerates under such conditions. The use of problems for all members of the class at such times serves to bring them back together. Problems furnish a much better basis for discussion than do examples; and when work on individual example assignments is abandoned for consideration of a problem, enthusiasm rises. Members of the class feel that a new start is being made, and, since everyone is at the same place again, it is a true class. Problems used for unification of a class may require only mental solutions with verbal explanations, or the solution may require the use of pencil and paper.

7. Problems are used in arithmetic instruction to teach pupils how to solve the situations they will meet in life outside the arithmetic classes. Making change, finding areas and averages, figuring the cost of installment buying, and the like are examples of this type of problem. Such problems are used much more frequently in the upper-grade textbooks than in the lower grades of the elementary school.

A Program for Developing Facility in Problem-Solving

Since verbal problems are an important part of arithmetic, even if their role is not so grandiose as sometimes claimed, teachers do have a right to be concerned about the apparent inability of pupils to develop proficiency in verbal problem-solving. The simplicity of the computations in the problem situations used on standardized arithmetic tests serves to underline the statement so often made when teachers discuss arithmetic, "My pupils do very well on examples, but fall down on written problems."

Understanding what verbal problems are and their place in the arithmetic curriculum makes it easier to build a program for increasing problem-solving ability. It becomes clear that simple

problems expressed in words will be used throughout elementary-school arithmetic for the various purposes listed in the preceding section. Instead of being thought of as a skill separate from the rest of arithmetic, problem-solving becomes part of each fundamental process as it is developed. In most of these situations the working out of verbal problems will be incidental to the teaching of other processes, and the problems selected, therefore, while presenting some challenge should not present any particular difficulty. In a like manner, those problems used in tests, problems used to show significance of facts and processes, and problems for other specific uses are not good situations for teaching problem-solving. Problems designed to serve one purpose or use do not ordinarily serve equally well in another capacity. While children will be acquiring some background and confidence in attacking numerical situations expressed in words in all the grades, direct teaching will also be necessary.

The manuals and advertising which accompany pupils' textbooks make extensive claims regarding the program of teaching problem-solving included in the books. The procedures actually used in the books are not always as impressive as the statements made about them. Textbooks emphasize most often the technique of selecting the fundamental process needed to solve the problem. Other procedures frequently found in the books are these: (1) systematic analysis of problems (stating what is given, what is to be found, etc.); (2) finding hidden questions in problems; (3) estimating the answer to problems; (4) solving problems containing superfluous numbers; (5) dealing with problem situations which lack sufficient data; (6) admonition to read carefully; and (7) use of sets of problems based on one social scene, such as a State Fair, the Grocery Store, or Ranch Life.

While an evaluation of such textbook procedures for teaching problem-solving should be valuable, the fact that teachers and others concerned with instruction in arithmetic are disturbed

about the pupil's inability to solve problems seems to indicate that a far more extensive program for developing problem-solving ability is needed than that presented in present-day books. In the numbered sections that follow, various procedures for developing this ability will be presented.

1. *The non-pencil-and-paper, or oral solving of problems*

The inability of pupils to solve problems does not usually become apparent until pupils are in the upper grades (5–8) of the elementary school. It is at that time that teachers and supervisors begin to look for procedures to supplement those found in the pupil's textbook. If the writer were limited to offering one suggestion for improving the problem-solving ability of upper-grade pupils, that recommendation would be the use of oral problem-solving. While the term oral problem-solving is fairly descriptive, classroom procedures implied by the suggestion are not used extensively and, therefore, are not well known. For that reason a brief description is given here.

Teachers who use the procedure generally follow a schedule which calls for one or two 5- to 15-minute oral arithmetic periods a week. A part of each period is given over to problems. The problems are read or stated orally and pupils generally give only oral responses and explanations. In order to secure better participation on the part of all members of the class, the teacher may have the pupils record on paper the answer to problems but no computation involving pencil and paper is permitted. To enable pupils to do the necessary computation without undue strain approximate computation (numbers changed) with the resulting approximate answers is encouraged. The question of the problem will frequently contain the indefinite term "about." Teachers read or state the problems only once or twice at most. This procedure forces the child to pay attention, a habit poorly developed in elementary-school children brought up with the book or other written material always available. After answers

are given by various pupils, the teacher may ask a pupil how he thought in arriving at the answer, and, of course, the teacher on occasion tells how the most efficient people would think in solving problems. To provide a means of gauging progress in oral problem-solving, a bench mark is needed. This can be easily provided by having pupils write and then score answers on eight to twelve problems. Two weeks later performance on the same or parallel problems will usually show improvement. The pupil with poor achievement who does not improve is in need of individual attention.

That oral problem-solving exercises improve problem-solving ability has been demonstrated by many teachers. One study [1] which made use of practice with oral problems in working with approximation showed that a two-year growth in problem-solving as measured by a standardized test was obtained in one semester. A recent study [2] using an oral arithmetic program in the sixth grade again showed that problem-solving improved. Just as is true of most experimental studies, the morale of the pupils involved was probably high and as a result the growth may be somewhat exaggerated. Nevertheless, it is believed that oral problem-solving does help to improve problem-solving ability.

Just how or what such oral work contributes is a question that should be considered. Part of the answer has already been given in the statement that reading the problem only once or twice forces the pupil to pay attention. The fact that the problem is not written forces the pupil to quickly "size up" the problem situation because he cannot keep referring to it. He must select the significant aspects of the problem if

[1] Wallace Wood, *A Study of the Growth of Pupils in Arithmetic When Answer Approximation is Emphasized*, Unpublished M.A. Thesis (Iowa City: State University of Iowa, 1941).

[2] Frances Flournoy, *An Oral Arithmetic Program*, Unpublished Ph.D. thesis (Iowa City: State University of Iowa, 1953).

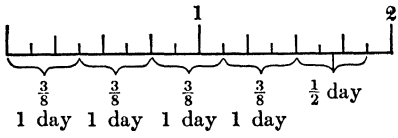
he is to be able to answer the question asked. Then, because pencil and paper for computation are not available, he soon learns to focus on the significant numbers, which in turn enables him not only to do the computing, but also to "see through" or "visualize" the whole situation. In a sense, focusing the attention on the significant numbers changes the problem to one involving small numbers, a problem-solving aid employed by resourceful people. After an answer is obtained, the pupil with no other problem to be worked before him has a little time to reflect on whether or not his solution is correct. Oral or non-pencil-and-paper solutions do not follow the one pattern so characteristic of pencil-and-paper work. Each pupil works in the way which enables him to see relationships and the class as a result may gain additional insight. Oral problem-solving is successful because it is a way of putting into practice the most essential verbal problem-solving procedures: namely, undivided attention to the problem, sizing up or analyzing the problem situation, use of significant or simple numbers, better picturing of the problem-solving process through forced use of simple numbers and processes, time for reflection or evaluation, and better understanding through various ways of solving.

2. The use of diagrams or drawings

Almost every adult has had the experience of having the solution of a verbal problem made easier through use of a drawing or diagram which shows the essential conditions of the problem. It is not difficult to see how such a procedure is of assistance for it requires careful reading, selection of essential elements, and a visualization aid, all of which make possible easier grasp of the conditions stated in the problem. If the directions, "Show with a diagram or drawing what the problem tells with words," were actually put into practice by the pupil, he would certainly have to read the problem and interpret the conditions correctly, two essential steps in solving verbal prob-

lems. The superior features of the use of drawings as an aid in problem-solving are shown in the illustration below:

"A road building crew lays pavement at the rate of $\frac{3}{8}$ mile per day. What is the length of pavement this crew put down in $4\frac{1}{2}$ days?"



The four $\frac{3}{8}$'s and one half of $\frac{3}{8}$'s clearly indicate the $4\frac{1}{2}$ days worked and the amount laid each day. While in this problem the solution can be found graphically, the number operation, $4\frac{1}{2} \times \frac{3}{8}$, required for the solution is indicated.

If it were not for the large amount of time required for making the drawing, this procedure would rank high on a list of suggestions for improving problem-solving. Another limitation of the suggestion is the fact that drawings are not appropriate or easily constructed for all problems. In spite of these two limitations, the procedure is recommended and teachers will find it especially useful with pupils who have developed the habit of starting to write figures as soon as they have examined a problem. In those situations where home work is desired, this suggestion will be appropriate. Parents have usually had little training in this type of work and, therefore, will not be able to help their children in the routine mechanical manner so characteristic of their assistance with conventional problem-solving. By putting examples of pupils' diagrams on the bulletin board, teachers can give prestige to the procedure.

The making of drawings to show what the problem tells with words is similar to use of drawings to show that answers to number questions are correct. Some pupils may, therefore, consider this drawing with problems as proof or showing that they understand the situation, and, of course, such drawings are

valuable assistance in explanations which the pupil may be asked to give.

3. *Writing the number question*

In a fifth-grade class where this problem-solving procedure was employed, these directions were written on the board: "Write the first number question that would have to be answered in solving the problems on page 214."

For the problem, "How many feet of fence would be required to enclose a rectangular lot that is 230 feet long and 70 feet wide?" the assignment would be fulfilled if the pupil wrote: "230 + 230 + 70 + 70 = n."

Essentially this suggestion for improving problem-solving calls for doing (at least for one-step problems) all the solution except the final computation; it is, therefore, a way of giving experience with the essential procedures in problem-solving. Pupils do not realize that they are doing so much of the solution, and, probably because they do not do any computing, they seem to enjoy the procedure. Since the problem is not solved, pupils think they are getting out of work and do the partial task with more enthusiasm than would be true if the whole solution were required.

The superior features of this suggestion for teaching problem-solving are evident. The pupil has to read the problem, select the process (a most important phase of verbal problem-solving), and write the indicated operation or computation. This last step permits easy classroom examination of the work done. Since the computation is not performed, the pupil writes little, a welcome condition to most elementary-school pupils. Omitting the computation also makes it possible to devote more time to other phases of problem-solving.

In using this procedure of writing the number questions of problems, a special precaution should be noted. Pupils should not be required to solve many of the problems with which the

procedure is used, because by doing so most of the desirable attitude which the procedure creates in pupils would be lost. Since textbooks seldom contain any extra problems, and problems used for this procedure are not to be solved, the textbook cannot be used much with the procedure. Special sets of problems from either an old or a supplementary set of textbooks or a special set of problems prepared by teachers or supervisors are therefore needed. Another and less important precaution is related to the first uses of the procedure with a group of pupils. They should be given "lead up" experiences, such as "I've written the number questions for problems 1, 2, and 3 on the board. Read the problems and see if you know what I mean by the number question of a problem."

The omission of the computation phase of a problem solution in this procedure is often questioned. Since the provision of practice in computation is so easy, the omission is not serious and, of course, many problems will be completely solved in other procedures for teaching problem-solving.

4. *Pupil formulation of problems*

The writing or stating of problems is a procedure which many children engage in at some time or other. In the lower grades this formulation or making of word problems may only be a case of pupils imitating the teacher, but if the practice is properly directed pupils get excellent practice with some of the basic elements of word problems. Teachers usually motivate this type of work in the early grades by allowing the pupil to state his problem to other members of the class. If a pupil fails to give what is needed, the other members of the class quickly call attention to the deficiencies. As a result pupils soon learn to recognize what is needed to make a problem.

When the procedure of having pupils formulate problems is used in the upper grades, the data to be used for the problems are frequently supplied. Facts from social studies, science,

health, and current events are the most common sources. Through the process of making problems, pupils get practice in recognizing the problem setting, in considering the facts needed, and the process to be used. That these elements of solving problems are important goes without question, and practices which make for greater familiarity with the elements should lead to improvement in problem-solving.

5. *Intensive study of number operations*

An idea of what is meant by intensive study of number operations is perhaps most easily attained by consideration of a few illustrative situations. The following are good examples:

1. "Bill wasn't sure that the product 504 which he had gotten for 12×42 was correct. He tested his product by multiplying 42 by 6 and doubling the product. Is $6 \times 42 \times 2$ the same as 12×42? Try to figure out some other ways of finding the product of 12×42."

In doing this assignment the pupils used $3 \times 42 \times 4$, $12 \times 6 \times 7$, $10 \times 42 + 2 \times 42$, $7 \times 42 + 5 \times 42$, and other ways of finding the product.

2. "To show what is done when you divide thousands, hundreds, tens, and ones, solve this problem by using appropriate symbols for bills.

"Three boys won a prize which had a cash value of $1366. They decided to take the cash and wished to divide the amount equally. They were given a one-thousand-dollar bill, three one-hundred-dollar bills, six ten-dollar bills, and six one-dollar bills. Show how you divide the amount among the three boys. You may get any bill changed but you are not to figure with numbers."

The correct solution of illustrative problem 2 calls, of course, for changing the thousand-dollar bill to ten hundreds. Each boy could then be given four one-hundred-dollar bills and one one-hundred-dollar bill would be left. This would then have to be changed to tens. The tens would then be divided among

the three boys and any remainder changed to ones and then those divided.

In the discussion following the solution with symbols for bills, a comparison of the number solution and the solution with bill symbols was made. The close parallel was immediately noted. For instance, in finding the first quotient figure the one thousand and three hundred was thought of as thirteen hundred.

At first it seems rather strange to suggest the intensive study of number operations as a means of improving problem-solving ability. However, careful consideration of the suggestion will remove the doubt. Through intensive study of number operations, the pupil acquires confidence in his ability to handle them and can devote all his attention to the other phases of problems. Thorough mastery of operations gives a feeling of being at home with figures and gives the pupil confidence in any area of work where quantity is involved.

A serious limitation of this suggestion for improving problem-solving ability is the lack of teaching suggestions for the intensive study of number operations. In the list below a few procedures are identified or may be inferred.

1. Study or review questions of the following type:
   (*a*) Is $26 + 32 = 20 + 30 + 6 + 2$?
   (*b*) Why is the first quotient figure in the example $9\overline{)384}$ placed in the 10's place?
   (*c*) The number 701 is equal to how many 10's?
2. Checks of reasonableness:
   (*a*) Bill checked the reasonableness of his product for $7\frac{1}{2} \times 6\frac{2}{3}$ by saying "It's between 42 and 56." What had Bill done?
   (*b*) Is "a little more than 6300" a reasonable check for 72×94?
3. Showing numbers and number operations with symbols other than figures.

6. *Class solution of difficult quantitative problems encountered in other school work*

The second-grade arithmetic experience described in the introductory paragraphs of this chapter is an example of this suggestion for improving problem-solving. Since in that presentation the possible values of such an experience were given, the space here will be devoted to a listing with brief description of situations which might be used at various grade levels.

To insure at least one good experience in the class solution of difficult problems, it is recommended that one arithmetic or numbers project for each grade become a part of the arithmetic course of study. An illustration of a fifth-grade numbers project follows:

A Cotton Ginning Experiment

In a study of the cotton growing region of our country, Eli Whitney's contribution to the industry was presented. To give the children some idea of the importance of the invention of a mechanical means of separating the lint from the seed, the teacher said, "Let's try separating some cotton lint from the seed the way people did before Whitney invented the gin."

Each of the 29 children was given some cotton and for 5 minutes separated lint and seed. At the end of the time the lint (fiber) was weighed. Only 3 ounces had been separated from the seed. The class then spent some time trying to figure out how long it would take them to gin a 500-pound bale of cotton. They proceeded in this manner: 3 ounces in 5 minutes. $60 \div 5 = 12$; $12 \times 3 = 36$ ounces in an hour = $2\frac{1}{4}$ pounds. $500 \div 2\frac{1}{4} = 222$ hours = $27\frac{3}{4}$ eight-hour days.[1]

The project just described has been listed for fifth grade, with the thought that the computation, while difficult, does not present concepts that are beyond fifth-grade children to follow. It should be kept in mind that the solutions in these number projects are class solutions, which means discussion and partici-

[1] Herbert F. Spitzer, "Elementary School Projects Which Require the Use of Number," *The National Elementary Principal*, 30: 20–23 (October, 1950).

pation by many pupils under the direction of the teacher, who in many instances does the computation suggested by the group. The project might also be used in the sixth grade.

Other number projects or activities which have many possibilities for the use of numbers are listed below. The approximate grade level is indicated.

(1) Setting of chicken eggs in an incubator with the records, temperature, number and time of turnings. (Kdg., 1, 2)
(2) Keeping of weather records (cloudy, sunshine, etc.) by days with summary reports for weeks and months. (Kdg., 1)
(3) Daily records of temperature with a model ribbon thermometer-type temperature indicator in the room. (Kdg., 1)
(4) Constructing an Indian Moon Calendar. (2, 3)
(5) Keeping a record of the sun's position in the sky. (1, 2, 3)
(6) Planning and mapping a garden. (1, 2, 3, 4)
(7) Constructing a date line. (3, 4, 5)
(8) Figuring distance from a hive of bees to a source of nectar. (3, 4, 5)
(9) Estimating the sediment carried by a stream in one day. (5, 6, 7)
(10) Figuring yield per acre by use of sampling technique. (6, 7, 8)
(11) Determining modern arm span and comparing with arm span of Anglo-Saxons of a thousand years ago. (6, 7, 8)
(12) Measuring the circumference of the earth. (7, 8)
(13) Determining the longitude of the school building by setting a watch with London time and then recording the time indicated by this watch when the sun crosses the meridian. (The corrections for variation in solar time need to be made.) (6, 7, 8) [1]

[1] For a brief description of several of the above projects, see Spitzer, *ibid.*

It is, of course, essential that each of the projects be carried out in such a way that the solution of quantitative situations involved in them meets a genuine need of the pupils.

The number projects are suggested only to insure one work situation for each grade outside the arithmetic textbook. There should be in connection with social studies, science, current events, and other areas of school work many instances where problems come up too difficult for the individuals to solve without help. Here the class solution serves a useful purpose and will incidentally help the individual pupil improve his skill in problem-solving.

7. *The work or study habit whereby the pupil asks himself questions about the problem*
8. *Solving the same problem by several techniques*

This problem-solving improvement procedure merits thoughtful study. While the use of several techniques in solving a problem is an integral part of the initial instructional practices in the developmental method, the procedure might well be used with other than introductory problems and labeled as a means of helping pupils to learn to solve problems.

9. *Other procedures for improving problem-solving ability*

The list of suggestions (see page 188) most commonly found in textbook programs for developing problem-solving ability contains some worth-while procedures. Various textbook programs for problem-solving should then be examined with care. For example, experience with problems which contain superfluous numbers seems to have a great deal of merit. The same can be said for such suggestions as experiences with problems lacking essential data, and a number of problems based on a general topic.

Many adults when confronted with a problem where no sure solution is immediately forthcoming resort to the procedure of

thinking of an analogous problem involving small numbers. To give pupils knowledge of this technique and some experience with it seems to be worthy of recommendation.

Giving pupils time to think about verbal problems is a much needed practice in problem-solving situations. All too frequently a pupil is asked to read a problem and then almost immediately is asked some question about the problem. Such practices are likely to lead to the unconscious conclusion on the part of some students that they are not supposed to think when solving problems. Nothing could be farther from what is desired by those responsible for instruction in arithmetic. A reasonable working period before any discussion is undertaken is one way to provide time for pupils to think about problems. Discussion of purposes and procedures before work is undertaken is often beneficial to the thinking of pupils. This discussion should bring out the importance of making a genuine attempt to solve the problem or problems before asking for help. Furthermore, the fact should be emphasized that thoughtful attempts to solve, even if unsuccessful, are of value in that they give the pupil background. Such attempts should be encouraged over and over.

Consideration of suggestions for improving problem-solving would be incomplete without calling attention to errors and shortcomings in various practices that are now used extensively in some programs. One of the most persistent bad practices is the assignment of long lists of problems to be worked when a class or an individual is found to be weak in problem-solving. All too frequently pupils engage in these extra-problem practice exercises without any resulting improvement. Mere solution of problems does not make for either understanding of problems or efficiency in solving them. The time given to solving extra problems would probably be spent more profitably on procedures such as the use of diagrams to bring about more careful reading of problems.

The heavy reliance on formal analysis of problems in some programs of instruction, while not an error, can hardly be accepted as a desirable practice. It is only when adults stop to analyze the problem-solving act, that the steps in the formal analysis method of problem-solving become recognizable. In actual practice there is an intermingling of steps. Therefore, requiring a pupil to read the problem first to find what is given, then to find what is asked, and so on, is an example of unrealistic teaching. In addition to being contrary to actual practice, this use of special steps with the same problem encourages careless reading by the pupil. The second and third readings seem unnecessary to the pupil because he thinks he has the problem in mind. If practice on the formal analysis steps of problem-solving is to be efficient, then, each step should be practiced on a separate set of problems.

STUDY QUESTIONS

1. In what grade should the pupils first have experience solving verbal or word problems in arithmetic? (*1*) First. (*2*) Second. (*3*) Third.

2. Is the group or class solution of verbal problems a practice to be highly recommended for primary grades only? (*1*) Yes. (*2*) No.

3. Is the use of problems to show the significance of arithmetical processes an important use? (*1*) Yes. (*2*) No.

4. Is the solution of verbal problems in arithmetic good practice in thinking or in the broad sense of the term in problem-solving? (*1*) Yes. (*2*) No.

5. For which of these uses of problems is the use of important social content most desirable? (*1*) For the introduction of new processes. (*2*) For problem practice exercises. (*3*) For showing that processes studied are significant.

6. Are many of the most common uses of problems in arith-

metical instruction concerned primarily with developing problem-solving ability? (*1*) Yes. (*2*) No.

7. Why does the use of drawings or diagrams as a suggestion for improving problem-solving not rank higher? (*1*) Because it requires so much time. (*2*) Because many problems do not lend themselves to that type of solution. (*3*) Because few pupils have the ability to draw well enough.

8. The suggestion to write the number question of problems as a problem-solving improvement procedure has some limitations. Of these: (*a*) It is applied primarily to one-step problems. (*b*) Classrooms are lacking a good supply of problems. (*c*) The problem is never solved, which is the most serious limitation? (*1*) *a*. (*2*) *b*. (*3*) *c*.

9. In the so-called numbers project recommended for each grade are the computational procedures involved to be within the pupils' arithmetical achievement? (*1*) Yes. (*2*) No.

10. Is the classroom practice of a teacher helping individual pupils with problems he cannot solve highly desirable? (*1*) Yes. (*2*) No.

11. The formal analysis of problems does not rank high on the recommended list of suggestions for improving problem-solving. What is probably the main reason for this condition? (*1*) The method requires too much time. (*2*) The steps are arranged in the order adults, not pupils, use them. (*3*) In actual practice no such separate steps are followed.

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7

Background Needed for the Study of Fractions

A First-Grade Fraction Experience

After the class had read the directions for setting up an aquarium, a committee was appointed to fill the globe-type fish bowl to the recommended depth (one-half full). When members of the committee were in disagreement, the teacher referred the problem to the whole class by saying: "Some of the committee members think the bowl is now one-half full. Others do not. How can we help them?"

As a result of the discussion that followed, it was agreed that the bowl would be one-half full when the filled part and the empty part were the same size.

This first-grade lesson dealt with one of the basic fraction ideas, namely, that the parts into which a whole can be separated are equal in size. This and other important fraction ideas can be introduced in the school experiences of primary-grade children. There are many situations where the primary-grade child can actually use one-half, one-third, one-fourth, and other fractions. If the familiar measuring cup is used, even the reading of fractions written with numerals becomes commonplace. Weights on various prepared articles of food often are

expressed in fractions or mixed numbers. With a little care a program involving lifelike use of fractions and mixed numbers can be developed for the primary grades.

A pupil's acquisition of fraction ideas like the acquisition of other number ideas is a slow process involving many experiences. For example, the concept that halves are equal in size as shown in filling the fish bowl will not be acquired by many pupils through one such experience. That idea needs to be presented many times in the primary school and even requires consideration by some pupils in Grades Four and Five.

It should not be assumed that basic fraction ideas are learned entirely through consideration of classroom problem situations of the type described above. Probably the most effective means of helping pupils acquire fraction ideas is through the judicious use of such devices as the foot fraction board, the fraction reference circle (see page 207), collections of objects, and drawings. In the best instructional practice use of these teaching aids comes after a need for knowing such things as one-half or three-fourths has arisen in a classroom experience.

Basic Fraction Ideas

It is the purpose of the work with fractions in the primary grades to provide children with knowledge and ideas which will be of use at the time and which will provide a good background for later learning of computation procedures with fractions. The most common facts and ideas about fractions learned by children in this experience program are found in the following list.

1. Children learn how to read fractions written with figures. Children need to be able to read fractions in this form if they are to use the measuring cup, to read labels on packages of food, and so on.

2. Children learn that the fraction one-half means one of the

two equal parts into which any whole is divided; that the fraction one-third means one of the three equal parts into which any whole thing is divided, and so on. Another way of stating this knowledge is that the fractional parts into which a whole may be divided are equal in size.

3. Children learn that the name of a fraction is determined by the number of equal parts into which the whole has been divided. For example, if a whole is divided into five equal parts, then each part is a fifth.

4. Children learn such relationships between fractions as these: one-half is larger than one-third, one-half is less than three-fourths, one-half is equal to two-fourths, three-sixths and four-eighths, that ten one-tenths equal one, and so on.

5. Children learn that fractions are used to compare quantities; for example, they find that six inches is equal to one-half of the twelve inches in a foot, or that if one stick is three times as long as another, the short stick is one-third as long as the other.

6. Children learn that fractions are a means of expressing quantities. Examples of this use of fractions are "one-half dozen, one-fourth mile, three-fourths horsepower motor, and seven-tenths of a second."

7. Children learn the names *numerator* and *denominator* and how to identify them, and some children learn that the denominator gives the size of the fraction.

8. Children learn that to find a part (fraction) of a number you divide by the size of the part. Although learned by many pupils before they reach third grade, knowledge of this fact is essential to mastery of partition division and is taught when instruction in division is undertaken.

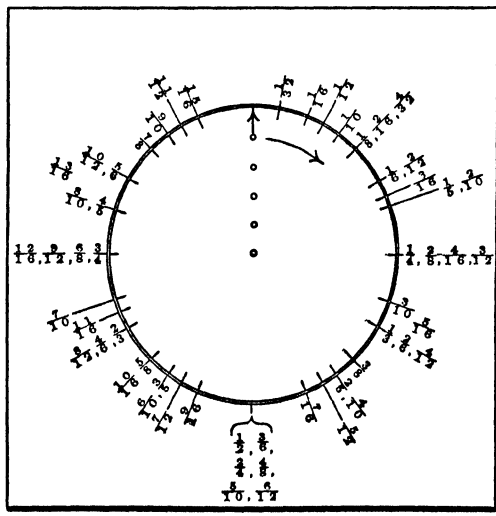
Since children have little interest in future mathematical uses of fractions, the motive for study of fractions must come from situations where a child needs to know the meaning of such things as one-half, two-thirds, one-tenth, and the like. The

first-grade fraction experience described earlier gives an example of such a need. Other similar situations may be created for consideration of fractions like these:

1. Fractions as applied to such common measures as a dozen, an inch, a foot, a mile, and a ton.
2. Fractions in such comparative statements as "a third as large" and "only a fourth as long."
3. Fractions on measuring cups.
4. Fractions in recipes.
5. Fractions in weights (involving $\frac{1}{2}$'s, $\frac{1}{4}$'s, $\frac{1}{8}$'s, and $\frac{1}{16}$'s if a good scale is available in the classroom).
6. Fractions in such measures as shoe and hat sizes, size of electric motors, and the size of standard grades of fruit (e.g., apple).

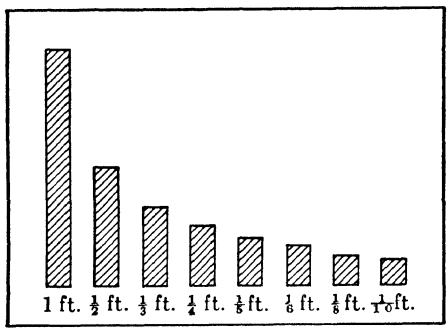
Suggestions for Further Development and Maintenance of Basic Fraction Ideas

In explaining the meaning of fractions, the circle and especially a standard reference circle, a fraction board (see illustrations below), blocks, and various measures such as the cup, pint,



quart, and gallon will be helpful. Drawings and pictures will also be of value. It should be noted that these are aids used when a need for knowing a fraction has arisen and are not to be taken as the starting point in instruction.

The *standard reference circle* is cut out of plywood, hardboard, masonite, or similar material. The piece of board with the hole in the center is then fastened to another piece of board of the same size, thus making an "inlay" for the fractional parts. The fractional parts themselves are made by cutting a circle into halves, quarters, thirds, eighths, and so on. Very small nails are driven into the radius at the starting point.



The *fraction board* is illustrated here. The black pieces shown in the drawing are inlays which are set into the board. The longest piece is one foot long, the next $\frac{1}{2}$ foot, the others $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{8}$ and $\frac{1}{10}$ foot. There are enough pieces of each fraction to fill the foot space.

After a number of uses of fractions have been considered and various ways of explaining the meaning of fractions have been employed, primary-grade children may profitably engage in paper and pencil exercises where they identify halves, thirds, and the like; where they attempt to show by appropriate drawings halves, thirds, and so on; and where they show the equivalence of the most common fractions (e.g., $\frac{1}{2} = \frac{2}{4}$; $\frac{6}{8} = \frac{3}{4}$; etc.).

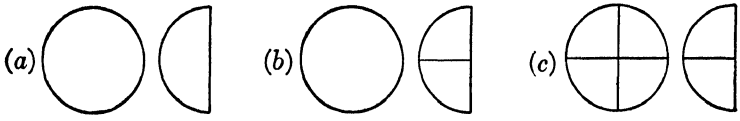
As with other phases of arithmetic, the fraction facts, terms, and ideas presented in the early grades are likely to be forgotten unless they are used in later grades. Arithmetic textbooks while providing review and maintenance exercises leave much to be desired as far as use of basic fraction ideas are concerned. The usual review plan is to present in the upper-grade books a brief version of the same simple exercises used in earlier books. It would seem that for these reviews in the upper grades a better teaching situation would result if some new use of fractions were presented as an introduction. For example, instead of using the usual pie, melon, or grapefruit situation to show halves or thirds and to identify numerator and denominator, a sixth-grade class might begin its review of fractions by considering such a situation as this. "In giving his report on Saskatchewan, Bill said, 'There are 830,000 people. I think that's about $\frac{1}{3}$ of the number in Iowa.' How did Bill get $\frac{1}{3}$?" (The population of Iowa was considered as 2,500,000.) It would also appear advantageous to use some exercises in these reviews that would be different from those used in previous grades.

Maintenance exercises on facts, terms, and ideas are in most series of books inadequate. Good means of providing maintenance in such areas of arithmetic are oral exercises, short answer tests, just-to-remind exercises, and the like. The type of exercise useful in such procedures is shown below.

1. How is the common denominator for such fractions as $\frac{2}{3}$ and $\frac{1}{8}$ found?
2. Why do we usually change the terms of fractions to lowest terms?
3. When is changing terms of fractions to higher terms needed?
4. In the fraction $\frac{5}{8}$ which figure is the denominator?
5. Which of these fractions represents the largest amount? $\frac{7}{12}$, $\frac{3}{8}$, $\frac{4}{10}$
6. Which of these fractions is not in best form? $\frac{9}{18}$, $\frac{4}{12}$, $\frac{10}{24}$

7. To change $\frac{7}{28}$ to best form, by what number do you divide numerator and denominator?

8. Which of these drawings shows the improper fraction $\frac{6}{4}$?



9. How can a fraction of a number, for example $\frac{1}{4}$ of 20, be found?

10. If a sheet of paper is cut into 7 equal pieces, what fraction of the sheet is each piece?

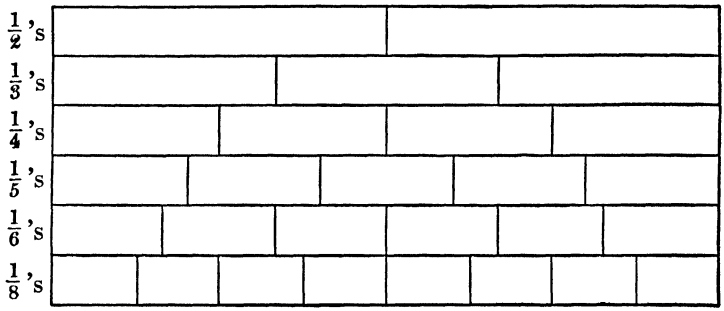
Changing the Terms of Fractions

Much attention is given to teaching pupils how to change the terms of fractions to best form or lowest terms. The process of changing the terms of a fraction to an equal fraction with higher terms, while receiving less attention, is also presented in children's textbooks. Some recent textbooks give good situations showing a reason for changing terms to best form. "Why do we ask the grocer for $\frac{1}{2}$ dozen eggs and not $\frac{6}{12}$ dozen?" is one such situation. Other situations with appropriate discussion should bring out the fact that it is easier to think with a fraction that has small terms than it is to think with one having large terms. Even after this fact has been identified, oral exercises, questions and the like should be used to recall the fact to the pupil's attention.

The reasons for changing a fraction to an equivalent fraction of higher terms can also be developed from lifelike situations. A good example is found in the situation where four children wish to divide equally by weight some quantity that weighs only $\frac{3}{4}$ pound. The division of course becomes simple when the $\frac{3}{4}$ are changed to the equivalent $\frac{12}{16}$.

In initial work of changing fractions to best form and for

finding an equivalent fraction with higher terms, the fraction equivalence chart (see illustration below), the fraction reference circle and the fraction board can all be used. For this beginning work it is assumed that situations will be used in which the pupil can see the reasons for changing terms of fractions. After



the equivalence of certain fractions has been shown through such devices, study should be directed to finding how such changes can be made by the use of figures only. From this study the rule can be developed that the terms of a fraction may be multiplied or divided by the same number without changing the value of the fraction. In doing the number exercises used to fix this generalization, pupils should frequently show with a drawing or in other ways (fraction circle or fraction board) that the fractions are equivalent. For example, if for the exercise $\frac{2}{3} = ?/12$ the pupil writes $\frac{2}{3} = \frac{8}{12}$, he should show that $\frac{2}{3}$ and $\frac{8}{12}$ of a circle are equal. Of course, the fact might also be shown with a line or rectangular drawing, drawings of objects, and so on. Most arithmetic books and teachers have a tendency to present rules before pupils have the background of experience which will enable them to profit from the statement of a rule. Once a rule is formulated, books and teachers rely almost entirely on the use of the rule as a means of developing further understanding. The rule regarding equivalent fractions is a good illustration of such teaching practice. The student of

arithmetic teaching should, therefore, give special attention to the procedures suggested above for developing the rule and for checking the pupil's understanding of the operation of the rule.

Systematic Instruction in Addition of Fractions

Most series of arithmetic textbooks introduce simple addition of fractions in Grade Four, but delay major instruction in this area of the subject until Grade Five. The discussion here, therefore, will be directed toward the Grade Five level. Examination of children's texts, courses of study and books on the teaching of arithmetic, reveals that initial instructional procedures are far from uniform. In some books addition and subtraction are presented almost simultaneously, in others are almost completely separated. Many fifth-grade textbooks precede the presentation of addition with extensive background work involving the meaning of numerator and denominator, equivalence of fractions and other facts.

Such a practice seems to imply that either the fraction experiences of the first four grades teach little or that review is best accomplished through representation of the same content that has been used in previous grades. The shortcomings of the latter approach were discussed in connection with the teaching of the fundamental processes with whole numbers. The first implication is also hardly tenable. If in Grades One to Four, children have been using fractions and engaging in exercises designed to bring out the major ideas regarding fractions, it seems reasonable to assume that the majority of fifth-grade pupils will be able to read and write fractions, to identify the commonest of fractional elements in diagrams, and will possess some knowledge of the equivalence of fractions and other common facts about fractions.

Therefore, it seems appropriate to begin fraction work in Grade Five with something other than a re-presentation of simple background material in fractions. It is suggested that the

work be started with consideration of some uses of fractions which might challenge fifth-grade pupils. This consideration of uses of fractions would reveal gross inadequacies in basic knowledge and would provide a basis for quick review of the most essential fraction ideas.

The addition situations most used in beginning instruction are such simple ones as $\frac{1}{3} + \frac{1}{3}$ and $\frac{1}{5} + \frac{2}{5}$. If it were not for confusion caused by the written form of such addition situations, all fifth-grade pupils who possess the proper background for the study of addition of fractions could give the sums for such examples without hesitation. The confusion is, of course, brought about by the fact that only the figures for numerators are added. Instead, then, of presenting immediately the written addition form, it might be advantageous to have pupils obtain the answers mentally (orally) to such simple addition situations. Then, later, the pupils would be asked to write the number record for each solution. For example, if a problem were solved by adding one-fifth to two-fifths to get the sum three-fifths, this number record of the solution would be made:

$$\begin{array}{r} \frac{1}{5} \\ +\frac{2}{5} \\ \hline \frac{3}{5} \end{array}$$

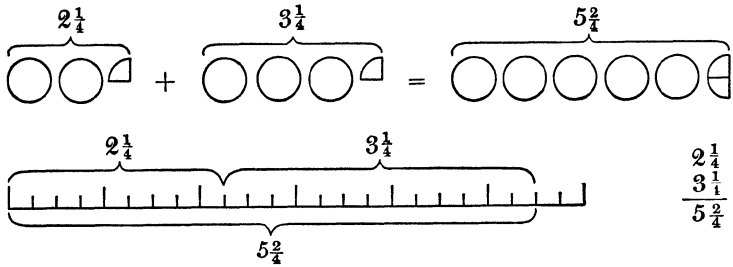
After several problems are solved, attention might then be directed to the number records by such a statement as the following: "In each of these records we show the sums of fractions. Let's study these records to see if we can find a way to add fractions using only numbers. What figures were added in the first example?"

The task of the learner then becomes "How are fractions added?"

Since the use of very simple fraction situations in the first illustrative problems is not likely to be challenging and, therefore, the pupil will not be likely to see a need for learning to add fractions, it is suggested that the first illustrative problems in-

volve more difficult addition situations. The use of mixed numbers of two and more addends and some unlike fractions might well be used. Here, as in initial work in other phases of arithmetic, pupils should be directed to find the answers by drawings or other indirect ways and to show that answers found are correct. In one fifth grade where the above procedure was followed, these directions were given: "Find the answers to these five problems by using drawings, a ruler, the fraction board, or by thinking the solutions. Show by use of one other way that your answer is correct."

The drawing solution, solution with ruler, and the number record for one of the problems used in this fifth grade are reproduced here.



After the solution to the first illustrative problem has been agreed upon, the teacher wrote with numbers a record of the solution. The class then worked out a similar number record for the other solutions. For the unlike-fraction situations, the record showed the fractions made alike since that is what is done on the ruler, the fraction board or by a drawing. The teacher then said, "We have found these sums by drawings and other ways. Do you see a way to find the sums by using just figures? Study all the records to see if your suggestion works with all the problems."

From consideration of this last assignment the class formulated the statement, "To add like fractions you add the numerators and write the sum over the denominator."

The next arithmetic class period was devoted to adding like-fraction examples. As a check on the pupil's understanding, directions were given to show with drawings that answers to several examples were correct.

The addition of unlike fractions was begun in a manner similar to that used for initial instruction. First, illustrative problems with directions to solve indirectly were used; then number records of solutions were made and the number solution developed from these records. This was followed by practice with examples including some showing that sums were correct.

The formulation of a procedure for finding the common denominator was undertaken after the class had had some experience with making denominators alike through use of a ruler and a fraction equivalence chart. Here again is an example of providing a use before the process to be learned is taught. The procedure the class worked out was as follows: "First check to see if the largest denominator contains equally the smaller ones. If not, multiply the largest denominator by 2 and try again to see if the smaller denominators are contained equally. Next, multiply by 3, 4, and so on until a denominator is found that contains equally all denominators."

In changing the fractions in mixed numbers to common denominators, it is recommended that pupils and especially teachers in their board presentations always write the entire numbers and not just the fractions.

The work in Box A is an example of the correct procedure. That in Box B is incorrect.

A.

$$\begin{array}{r} 4\frac{1}{2} = 4\frac{3}{6} \\ +\ 2\frac{2}{3} = 2\frac{4}{6} \end{array}$$

B.

$$\begin{array}{r} 3\frac{1}{8} = \frac{1}{8} \\ +\ 5\frac{1}{2} = \frac{4}{8} \end{array}$$

The differences between the advocated procedures above for teaching addition of fractions and the procedures found in current children's textbooks are: (1) the suggestion that means of solution other than the best computational plan with figures be employed in initial instruction, (2) that the number solution be developed from records of sums obtained by other ways of solving, and (3) that during practice work for mastery pupils be required to show for selected examples by indirect (usually drawings) solutions that their work is correct.

Teaching the Subtraction of Fractions

The plan recommended for teaching subtraction of fractions in general follows that given for the addition of fractions. Instead of the simplest subtraction situations, more difficult ones are used in first illustrative problems. Use is made of several solutions, and number records of solutions are made. From these records, the number way of solving is developed. Then, as a part of practice exercises some showing or proving with drawings is required.

Since the process of finding a common denominator has already been encountered by the pupil in his study of the addition of fractions, the only essentially new thing to be learned in the subtraction of fractions is the borrowing or changing of a whole number to a fraction. Even this process is not basically new since a related process is used in the subtraction of whole numbers. The actual procedure to be used in showing borrowing and the change to a common denominator is an important factor to many teachers. Most people when subtracting $4\frac{7}{8}$ from $12\frac{2}{3}$ write the example as follows:

$$\begin{array}{r} 12\frac{2}{3} = \frac{40}{24} \\ -\ 4\frac{7}{8} = \frac{21}{24} \\ \hline 7\frac{19}{24} \end{array}$$

Of course, such people know that $12\frac{2}{3}$ does not equal $\frac{40}{24}$. To avoid such misrepresentations the whole number should be rewritten; that is, $12\frac{2}{3} = 11\frac{40}{24}$.

Although the additive method of subtraction is not recommended by the author, some attention should be given by the student of the teaching of arithmetic to its use in the subtraction of fractions. In the example

$$\begin{array}{r} 7\frac{1}{8} \\ -\,4\frac{5}{8} \\ \hline \end{array}$$

the thinking suggested by some of those who use the additive method is as follows: 5 and what number of eighths will equal eight eighths, or one? The answer is 3. This 3 eighths and the 1 eighth equal $\frac{4}{8}$, the required remainder. 5 and 2 equal 7. The answer is $2\frac{4}{8}$ or $2\frac{1}{2}$. It appears to the author that the rationalization of this additive procedure would be more difficult for the child than would be the rationalization of the take-away method of subtracting fractions.

Multiplication of Fractions

There is now fairly general agreement on the allocation of the multiplication of fractions to sixth grade.[1] On other matters relating to the teaching of multiplication of fractions, there is no such general agreement. This lack of uniformity is especially evident in the type of initial multiplication situations used and in the initial instructional practices employed. These variations in practice can be easily identified by consulting various series of textbooks.

[1] A notable exception is found in Stokes, who recommends that fractions multiplied by integers be taught in fifth grade. See C. Newton Stokes, *Teaching the Meanings of Arithmetic* (New York: Appleton-Century-Crofts, 1951), p. 460.

The five main types of examples of multiplication of fractions in the order of presentation recommended are as follows:

A. Fractions multiplied by integers ($6 \times \frac{2}{3}$).
B. A mixed number as one factor ($3\frac{1}{2} \times 4$; $5 \times 2\frac{2}{3}$).
C. Integers multiplied by fractions ($\frac{2}{3} \times 6$).
D. Fractions multiplied by fractions ($\frac{3}{4} \times \frac{1}{2}$).
E. Mixed numbers multiplied by mixed numbers ($2\frac{1}{2} \times 1\frac{1}{4}$).

The recommendation that the first situation be that of a fraction multiplied by a whole number is based upon the fact that this situation is an example of the same type of multiplication used with whole numbers. The multiplicand $\frac{2}{3}$ is to be taken 6 times. Therefore, the product can be obtained by adding, a condition not possible with any of the other four types. Furthermore, the solution to a fraction multiplied by an integer can be obtained by means of diagrams much more easily than can the solution of a whole number multiplied by a fraction. This last statement can be easily verified by considering the relative difficulty for the pupils of finding 12 three-fourths and three-fourths of 12. Thus, there are two very important advantages for type A. Furthermore, type C, the one used for initial instruction in a number of textbooks, has the following very serious limitation. When partition division is introduced, the pupils are taught that to find a part of a number, you divide. To use this same situation (modified in some instances by use of other than unit fractions) to teach multiplication seems to be a good way to confuse pupils.

Since the actual number operations in type D ($\frac{1}{2} \times \frac{3}{4}$) seem easier for pupils than either type A or type B, some teachers have advocated that type D be used first. The ease with which pupils can be taught to manipulate numbers may often be misleading. Because type D products cannot be verified by addition and are difficult for most pupils to show by drawings, the argument that type D is easier is questionable. On the other

hand, the fact should be recognized that the understanding of a process can be acquired after a pupil has learned to use the process.

In view of the definite limitations of type B and type C and of the fact that types B and E are really extensions of other types, it would seem that for initial instruction type A is the best.

It should be recognized that each of the five types of fraction multiplication can be further subdivided. For example, each of the items in the order of presentation (see below) presents some new difficulty. In line A below we have first a unit fraction, next a larger than unit fraction, then a mixed number product, and finally a fraction product. For making graded practice exercises these subdivisions may have value.

MULTIPLICATION OF FRACTIONS: ORDER OF PRESENTATION

A.	$\frac{1}{2} \times 4$	$\frac{2}{3} \times 6$	$\frac{1}{3} \times 5$	$\frac{3}{4} \times 3$	$\frac{1}{3} \times 1$	$\frac{3}{8} \times 2$
B.	$2\frac{1}{2} \times 4$	$3\frac{3}{4} \times 4$	$3\frac{2}{3} \times 5$	$4 \times 2\frac{1}{2}$	$4 \times 2\frac{3}{4}$	$4 \times 6\frac{2}{3}$
C.	$4 \times \frac{1}{2}$	$3 \times \frac{1}{2}$	$3 \times \frac{2}{3}$	$4 \times \frac{1}{6}$	$1 \times \frac{1}{4}$	$2 \times \frac{3}{8}$
D.	$\frac{1}{4} \times \frac{1}{3}$	$\frac{1}{4} \times \frac{1}{4}$	$\frac{2}{3} \times \frac{1}{2}$	$\frac{1}{4} \times \frac{3}{8}$	$\frac{3}{4} \times \frac{2}{3}$	$\frac{3}{8} \times \frac{3}{8}$
E.	$4\frac{1}{2} \times 2\frac{1}{2}$	$3\frac{1}{2} \times 2\frac{1}{4}$	$5\frac{3}{4} \times 2\frac{2}{3}$	$3\frac{3}{8} \times 4\frac{1}{8}$	$6\frac{1}{2} \times 5\frac{5}{8}$	$7\frac{3}{5} \times 1\frac{3}{4}$

Multiplication of Fractions: Teaching Procedures

For beginning the first systematic work of multiplying fractions in one sixth grade, a set of four illustrative problems was used. The directions given the pupils were as follows: "Find

the answer to each problem. If you are not sure about how to find the answer, use a drawing or a diagram. Show that your answer is correct by adding."

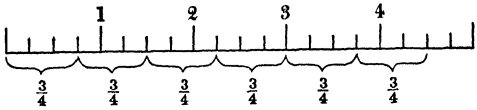
Problems 1 and 2 and the solutions offered in this class appear below:

Problem 1. In making a support board for a footbridge, a carpenter nailed 6 boards together. Each board was $\frac{3}{4}$ inch thick. How thick was the support board he made?

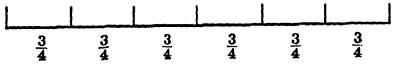
Problem 2. Jack figured that he spent about $\frac{2}{3}$ of an hour walking to and from school each day. At that rate how many hours would he spend in walking to and from school in 5 days?

Solutions offered for Problem 1

(*a*)



(*b*)



$= 4\frac{2}{4}$

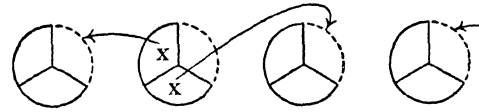
(*c*) 6×3 fourths $= 18$ fourths $= 4\frac{2}{4}$

(*d*)

$$\begin{array}{r} \frac{3}{4} \\ \frac{3}{4} \\ \frac{3}{4} \\ \frac{3}{4} \\ \frac{3}{4} \\ \frac{3}{4} \\ \hline \frac{18}{4} = 4\frac{1}{2} \end{array}$$

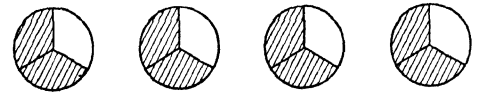
Solutions offered for Problem 2

(*a*)

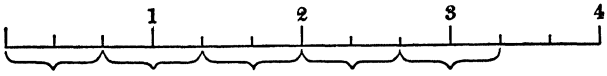


$= 3\frac{1}{3}$

(*b*)



$= \frac{10}{3} = 3\frac{1}{3}$

(*c*) 

$3\frac{1}{3}$

(*d*) $\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \frac{10}{3} = 3\frac{1}{3}$

During a discussion of the solutions, the question of ease of solution was raised and then attention was directed to finding an all-number way of multiplying a fraction. This search for an all-number way was directed by the statement, "We know from our drawing and adding solutions that $6 \times \frac{3}{4} = \frac{18}{4}$ and that $5 \times \frac{2}{3} = \frac{10}{3}$. How can you find these products by multiplying only numbers?" As the pupils worked on this assignment, the teacher supplied these guides to direct their study. "To be sure your number solution works, try it on all four examples. Next use your method in answering these questions:

"1. How many are 8 two-thirds?

"2. How many are 4 one-thirds?

"3. How many are 5 three-fourths?

"Show that your answers to these questions are correct by use of drawings and by adding."

In the class consideration of various number solutions, it was agreed that when you multiply a fraction by a whole number, you multiply the numerator and divide by the denominator.

The important features to note in the plan used for developing the multiplication of a fraction by a whole number are (1) the use of problem situations that can be easily diagrammed; (2) the requirement that the first solution be a drawing (a means of showing the fraction used a specified number of times and the product); (3) the formulation of a number statement of the solution obtained with diagrams and the identification of "finding a number solution" as a learning problem; (4) the testing of suggested number solutions on other number questions; and (5) the suggested use of additions and diagrams to check number solutions. An over-all important feature of the

whole procedure is the continuous emphasis on pupil doing in contrast with the too common procedure of having pupils follow the explanations of the textbook or the teacher.

For the development of type B multiplication of fraction situations (e.g., $4 \times 2\frac{1}{3}$ and $2\frac{1}{3} \times 4$) the use of problems and diagram solutions, plus use of addition, is also recommended. Since this type is essentially the same type of multiplication situation as that used with whole numbers, no basic new difficulty in thinking is encountered. If diagrams and additions are used in the introduction, a means of verifying the two step multiplication (see example 1) and the use of improper fractions (see example 2) is provided.

The teaching of both these methods of multiplying mixed numbers is recommended not only for the advantages (e.g., in checking) they offer in instruction, but because there are situations in which each method is distinctly superior.

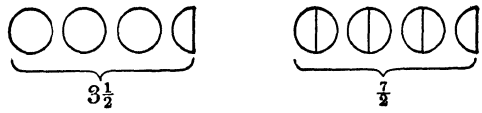
Ex. 1.
$$\begin{array}{r} 2\frac{1}{2} \\ \times\ 4 \\ \hline 2 \quad (4 \times \frac{1}{2}) \\ 8 \\ \hline 10 \end{array}$$

Ex. 2. $4 \times 2\frac{1}{2} =$
$4 \times \frac{5}{2} = \frac{20}{2} = 10$

It should be noted that in each example given the mixed number is the multiplicand. That type is the one to be used in introductory problems. Then, when consideration of the mixed number as multiplier is undertaken, the plan of interchanging factors used in multiplication of whole numbers is suggested.

Changing a mixed number to an improper fraction in order to multiply creates a need for an efficient way of making this change. Teachers may find this the ideal time for some intensive work on changing mixed numbers to improper fractions.

As a check of the pupils' understanding, drawings may be required. The mixed number $3\frac{1}{2}$ and the improper fraction $\frac{7}{2}$ are correctly illustrated below.



For introducing the teaching of type C, the multiplication of a whole number by a fraction ($\frac{3}{4} \times 8$), problems have a distinct role. This is not a multiplication situation in the manner in which multiplication is used with whole numbers. Actually, this is a case of finding parts of numbers. As was pointed out earlier in this chapter, the pupil has already been taught in partition division how to find a part of the number. It seems logical, therefore, to use the division process in finding the parts of a number. For the first illustrative problem, it is recommended that instead of using the usual diagram solution the pupils be directed to find the answer by first finding one part and then multiplying by the number of parts. For example, if the illustrative problem calls for finding $\frac{3}{4}$ of 8, the pupil finds $\frac{1}{4}$ of 8 by dividing and then multiplies the quotient obtained by 3. This procedure is called "finding one part and multiplying to find all parts." The next steps in the development of the multiplication method of finding parts of a number are illustrated in the instructional exercise given below.

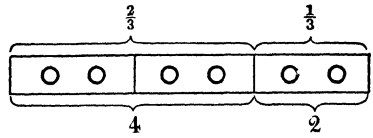
Finding Parts of Whole Numbers (Exercises)

Answer these questions by first finding one part. Then multiply to find all parts.

1. What is $\frac{3}{4}$ of 12?
2. What is $\frac{2}{3}$ of 6?
3. What is $\frac{2}{5}$ of 20?
4. What is $\frac{5}{8}$ of 16?

Show with a drawing that your answer to each question is correct. The second question has been done for you.

2. $\frac{2}{3}$ of 6 = 4



After the class had done the solutions by the suggested method, the teacher asked them to find the results of the same problems if the factors were interchanged. It became obvious that the product of $6 \times \frac{2}{3}$ was the same as the product of $\frac{2}{3}$ of 6 and therefore that $\frac{2}{3}$ of 6 was actually multiplication in spite of the confusion over wording. The class then concluded "that parts of a number can be found by multiplying and that *of* in such situations as $\frac{3}{4}$ of 12 means *times* or *to multiply*." The following two examples also show that *of* is often synonymous with *times:*

5. The total lengths of 2 pieces of board each 8 feet long is 2×8 or 8×2. Therefore *of* in this case means *times.*

6. We know that $\frac{1}{2}$ of 8 is 4 and that $8 \times \frac{1}{2} = 4$. Then $\frac{1}{2}$ of 8 must be the same as $8 \times \frac{1}{2}$. *Of* in this case can be thought of as meaning *times.*

Attention is called in the above-described teaching procedure to the manner in which the multiplication connotation of the word *of* is developed and to calling this type of situation "the finding parts of numbers." Since the multiplication of a whole number by a fraction is not multiplication in the whole-number sense of the process, this development of the use of multiplication as just a way of finding an answer seems more satisfactory than the plan used in many textbooks where the *of* in part of a number situation is said to be times with only such explanation as "you can get the answer by multiplying." Other textbooks develop "of" as a sign of multiplication through such question-

able explanations as the following: "If a pound of dog food costs 20¢, what will $\frac{1}{2}$ pound cost? $\frac{1}{2}$ of 20¢ is 10¢. To find what 3 pounds cost you multiply as follows: $3 \times 20 = 60$. Then $\frac{1}{2}$ of $20 = 10$ means $\frac{1}{2} \times 20 = 10$." This procedure is questioned because the multiplier "$\frac{1}{2}$" is used as though it were a whole number.

Many teachers introduce cancellation in the multiplication of fractions soon after pupils have started study of the process. This early introduction of the cancellation procedure is not recommended because of the confusion which results, especially among the less able pupils. The fact that the less able pupils become confused is not surprising, for to introduce a new or next step in a procedure before the basic steps are mastered is poor teaching practice. It is recommended that cancellation, which is essentially only a short cut, be introduced only after pupils have had some time to master the multiplication process with fractions. It might be a wise procedure to delay the use of cancellation until the grade following that in which multiplication of fractions is introduced.

The multiplication of a fraction by a fraction ($\frac{1}{2} \times \frac{2}{3}$), the next type of multiplication to be considered, presents the same multiplier difficulty encountered in the multiplication of a whole number by a fraction. Solution by addition and drawings of the counting type of solution are therefore not applicable. The most common problems where multiplication of a fraction by a fraction might occur are those where a part or parts of a fraction are to be found. The following is typical of those used in many textbooks.

"The cake recipe called for $\frac{3}{4}$ lb. of butter but because the cake was to be only $\frac{1}{2}$ size, mother said we need only $\frac{1}{2}$ as much butter as the recipe calls for? What is $\frac{1}{2}$ of $\frac{3}{4}$?"

While the pupils will have been taught that "of" in such situations means times, their major experience with finding parts of numbers has been one of division.

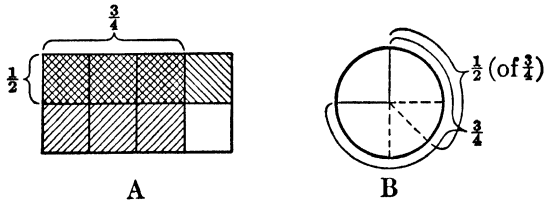
A better illustrative problem for introducing the multiplication of a fraction by a fraction might result if a measurement of area situation were used. To find the area of rectangular surfaces pupils have learned to multiply the number of units of one dimension by the number of units in the other. It is, therefore, recommended that several whole number area problems be solved. Solution of the whole number problems will be primarily for review. Then area problems are presented in which both dimensions are fractional numbers. Two such problems are given below.

1. On the map of farms in the township, Mr. Hill's farm was shown to be $\frac{1}{2}$ mile wide and $\frac{3}{4}$ mile long. What is the area of Mr. Hill's farm?
2. Bill wanted to buy a piece of plastic $2\frac{1}{2}$ inches by $6\frac{3}{4}$ inches. If the plastic he wants costs 4 cents a square inch, what will his piece of plastic cost?

Problem 1 is the simpler of the two problems but is actually a case of finding a part of a fraction. For such problems inches, feet, yards or miles may be used. In solving problem 2 it is suggested that several multiplications ($2\frac{1}{2} \times 6 + 2 \times \frac{3}{4} + \frac{1}{2} \times \frac{3}{4}$) be used. Such a situation, while having the disadvantage of being complicated by two multiplications, presents a more life-like example of the use of multiplication of fractions. This use of mixed numbers in introducing multiplication of a fraction by a fraction has the additional advantage of presenting a challenging situation, something very much needed when new phases of a topic are introduced. On the other hand, the simplicity of the drawing solution for the fraction by a fraction needed in solving problems of the first type should not be overlooked.

Regardless of the type of introductory problem used, there is a distinct advantage in having the pupil solve the problem by means of a diagram as well as by multiplication. For these drawings the rectangle is far superior to the circle. Study of

A and B below will reveal the advantages. This superiority of the rectangle might well be the most important factor in selecting illustrative problems.



A B

The development of a rule for multiplying a fraction by a fraction using data from drawing solutions, in a manner similar to that described for multiplication of a fraction by a whole number, is recommended. It is also suggested that the pupils be required to show by drawings that the products for some of the practice exercises are correct.

In teaching the multiplication of a mixed number by a mixed number an illustrative problem is recommended only to create a situation where pupils have a need for the multiplication. Solutions by addition or drawings are not required although no harm results if some pupils attempt to use these methods. The number question of the problem should be written and the suggestion made that improper fractions be used as before.

After the period of intensive study of multiplication of fractions, a summary period devoted to study and discussion of searching questions is an excellent procedure to use. The following questions are good illustrations of the type to be used in such summary exercises:

1. How do you multiply a fraction by a whole number?
2. Is $\frac{3}{4}$ of 12 the same as the product of $12 \times \frac{3}{4}$?
3. Will the product of $1\frac{7}{8} \times 2\frac{3}{4}$ be more or less than 6?

4. In multiplying $8\frac{1}{2}$ by $6\frac{1}{4}$, what is the first thing that should be done?

5. Is multiplying a whole number by a fraction ($\frac{1}{2} \times 6$) the same as finding a part of a whole number?

6. What are two ways of finding the product of a whole number multiplied by a fraction? Use $28 \times 2\frac{1}{2}$ to show the two ways.

7. Can the product for $\frac{2}{3} \times \frac{5}{8}$ be found with a drawing?

8. In multiplying such mixed numbers as $15\frac{1}{2} \times 8\frac{3}{4}$ is changing to improper fractions and multiplying considered the best procedure?

9. Can the product for $2\frac{1}{2} \times \frac{3}{4}$ be found by adding only?

10. Is the product of $2\frac{3}{4} \times 1\frac{1}{2}$ the same as the product of $2 \times 1\frac{1}{2} + \frac{3}{4} \times 1\frac{1}{2}$?

11. Can the product of $\frac{2}{3} \times \frac{3}{4}$ be found by adding?

12. Do you have to change $2\frac{3}{4}$ to an improper fraction in order to find the product of $2\frac{3}{4} \times 5$?

For such questions as the last four, directions like these are recommended: "Answer each question and then show that your answer is correct."

Teachers often find in the work period devoted to such questions and in the discussion of the questions afterwards that pupil knowledge of multiplication is not as thorough as they thought. Testing is sometimes a means of clarifying ideas just as the final science summary test often clinches or clarifies science ideas for pupils.

Consideration of a critical question about a rule is probably a far more effective learning procedure than was the original statement of the rule.

After summary lessons based on questions, some teachers use effectively the formulation of statements about what has been learned during the study of the process.

The Division of Fractions

There are three main kinds of division-of-fraction situations: A. Whole numbers divided by fractions ($4 \div \frac{1}{2}$); B. Fractions divided by whole numbers ($\frac{1}{3} \div 6$); and C. Fractions divided by fractions ($\frac{3}{8} \div \frac{1}{8}$). The use of mixed numbers would, of course, add four more types. Since the basic procedures for dealing with mixed numbers will have been taught in connection with the multiplication of fractions, only the three main types will be considered.

It is recommended that the division of fractions be introduced through use of situations involving the division of a whole number by a fraction. There are several good reasons for selecting this type rather than either of the others. It is an example of measurement division and may therefore be solved by the use of serial subtraction. Solution by drawing is also easier with this type than with type B or type C. This fact can be verified by doing a drawing solution for $6 \div \frac{3}{4}$, $\frac{3}{4} \div 6$ and $\frac{3}{4} \div \frac{1}{2}$. Type A ($6 \div \frac{3}{4}$) also lends itself to solution by the common denominator method far more easily than does type B.

Students of the teaching of arithmetic will find that some widely used textbooks begin the teaching of the division of fractions through use of type B, the division of a fraction by a whole number, and that still other textbooks begin with type C, the division of a fraction by a fraction.

Type B situations, of course, involve partition division, which in the opinion of the writer is more difficult for pupils than is measurement division. Furthermore, in the solution of a partition problem such as $\frac{1}{3} \div 4$, the situation is changed from $\frac{1}{3} \div 4$ to $\frac{1}{4}$ of $\frac{1}{3}$. It should be noticed that while the divisor is changed to $\frac{1}{4}$, it is not in the divisor position. Therefore, while what is apparently the inversion method is quickly developed through use of the fraction divided by a whole number, the process is definitely lacking in features which make for understanding.

The major weakness of using type C ($\frac{3}{8} \div \frac{1}{8}$) for introducing division of fractions is the lack of genuine life problems of this type.

The advantages and disadvantages of beginning with each type should be kept in mind by those evaluating the initial procedures used for division of fractions in pupils' textbooks. The following illustration attempts to build on the advantages mentioned for type A.

Initial Teaching of Division of Fractions

The work in division of fractions was begun in a sixth grade by presenting the following problems:

1. There was a 6 mile stretch of road to be paved. If the road paving crew can do $\frac{2}{3}$ of a mile a day, how many days will be required to finish the six miles?

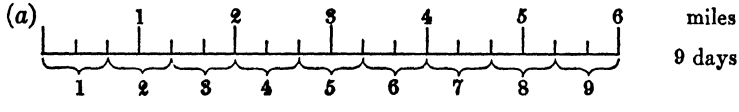
2. How many glasses of lemonade can be made with 5 lemons if $\frac{1}{2}$ of a lemon is needed for each glass?

The pupils were instructed to find the answers to the problems by use of drawings and to check their answers by subtracting or by adding. It was also suggested that if there was time, the pupils might try to find other ways of finding the answers.

In the evaluation and discussion of the drawing solutions, the number questions for problem 1, "How many $\frac{2}{3}$ miles equal 6 miles?" and for problem 2, "How many $\frac{1}{2}$ lemons equal 5 lemons?" were identified.

Three of the solutions for problem 1 offered by members of this class are shown below. Solution *d* was offered by the teacher.

Problem 1.



(*b*)

$$\begin{array}{rl} 6 & \\ -\frac{2}{3} & (1) \\ \hline 5\frac{1}{3} & \\ -\frac{2}{3} & (2) \\ \hline 4\frac{2}{3} & \\ -\frac{2}{3} & (3) \\ \hline 4 & \\ -\frac{2}{3} & (4) \\ \hline 3\frac{1}{3} & \\ -\frac{2}{3} & (5) \\ \hline 2\frac{2}{3} & \\ -\frac{2}{3} & (6) \\ \hline 2 & \\ -\frac{2}{3} & (7) \\ \hline 1\frac{1}{3} & \\ -\frac{2}{3} & (8) \\ \hline \frac{2}{3} & \\ -\frac{2}{3} & (9) \\ \hline 0 & \end{array}$$

(*c*)

$$\begin{array}{rl} \frac{2}{3} & (1) \\ +\frac{2}{3} & (2) \\ \hline 1\frac{1}{3} & \\ +1\frac{1}{3} & (3) \text{ and } (4) \\ \hline 2\frac{2}{3} & \\ +2\frac{2}{3} & (5)\ (6)\ (7)\ (8) \\ \hline 5\frac{1}{3} & \\ +\frac{2}{3} & (9) \\ \hline 6 & \end{array}$$

(*d*) $\dfrac{18 \div 2}{3 \div 3} = \dfrac{9}{1} = 9$

In explaining solution *d* the teacher said, "Since we are trying to find how many $\frac{2}{3}$'s equal 6, I changed the 6 to 18 thirds and divided. That's using a common denominator. In your drawing did you change the 6 to 18 thirds?"

The class working as a group then solved problem 2 using the common denominator method.

The pupils next used both the common denominator and the drawing method, in answering these questions:

1. How many $\frac{3}{4}$'s= 6?
2. How many $\frac{1}{2}$'s = 7?
3. How many $\frac{1}{3}$'s= 6?
4. How many $\frac{2}{3}$'s = 4?

While the solutions for these questions were being discussed, the group considered which was the better way. It was decided that the common denominator was better than the drawing solution.

On the following days the pupils solved problems and ex-

amples by use of the common-denominator method. Most of the examples were written in division form (e.g., $6 \div \frac{3}{4}$).

After the pupils had worked for several days with the common denominator method, the inversion method was developed through a study of the following: "By what part of the fraction do you multiply to change the whole number to a common denominator?" (See Example A, below.) "In the common denominator method of dividing, how is the numerator used? In $4 \div \frac{2}{3}$ you first multiply 4 by 3 and then divide by 2. Would you get the same answer if you multiplied 4 by $\frac{3}{2}$?"

Ex. A. $4 \div \frac{2}{3} = \frac{12 \div 2}{3 \div 3} = 6$

In talking this over, the children decided that to divide by a fraction, you can invert (turn over) the terms of the fraction and then multiply.

The statement was tested on two examples and then the pupils solved sets of examples using both the inversion and the common denominator method. For a few of these examples, the pupils were asked to show with a drawing that their work was correct. After some work with both common denominator and inversion methods, the question of superiority was considered.

In the teaching procedure described above, attention is called to these features. (1) Although problems involving division of a whole number by a fraction were used, pupils were able (directed) to solve the problems by at least two known methods, namely by drawing and by subtraction. (2) By use of these two methods a basis for presentation of a division solution (the common denominator) was provided. Since the drawing solution actually changed the whole number into thirds, a change to common denominator had been used by the children in a visual form. (3) The common denominator method and drawing method were used on the same examples in order to provide the

pupils with a basis for selecting the better procedure. (4) In the first work, the measurement division idea familiar to pupils through whole number division (e.g., How many 7's = 42?) was emphasized through use of such questions as "How many $\frac{3}{4}$'s = 8?" (5) The inversion method was developed from study of the common denominator method. (6) The two methods were used on the same examples to determine superiority. (7) Since all examples used could be thought of as measurement division, it was possible to suggest the use of drawing solutions as a means of furthering the pupil's understanding.

The common denominator method of dividing fractions has much merit. In addition and subtraction, pupils have been changing unlike fractions to a common denominator; and in division, as in the other two processes, no computation with the denominator is essential. The common denominator method, as was shown in the brief teaching procedures, provides an almost ideal setting for the development of the inversion method. Although not yet recommended as the only method for dividing fractions for pupils of little arithmetical ability, it is conceivable that this one relatively simpler method is the only one that such pupils should attempt to master.

Even for pupils of average or superior ability the common denominator method can be the base or foundation to which the pupil turns when in doubt.

Some textbooks which do not use the common denominator method as such, pay high tribute to its value in rationalizing the inversion process. For example, in explaining $6 \div \frac{2}{3}$ it is suggested that pupils first think "How many $\frac{1}{3}$'s equal 6?" From the fact that there are three $\frac{1}{3}$'s in 1, it is concluded that there are eighteen $\frac{1}{3}$'s in 6, and then there must be one-half as many $\frac{2}{3}$'s in 6 as there are $\frac{1}{3}$'s or $18 \div 2$.

In the teaching procedure described, the common denominator was not used very long. There are some teachers who think that pupils would profit from a rather extensive use (a semester or

even a year) of the common denominator method before the inversion method is taught. The proposal has merit but needs to be tested before it is recommended for general practice.

The teacher in the illustration above developed the inversion method before division of a fraction by a whole number or division of a fraction by a fraction were introduced. It is, of course, possible and probably very practical to solve initially all types of fraction division by the common denominator method.

When the common denominator method is not used, the development of the inversion method may be accomplished by proceeding from the diagram and subtraction solutions (see page 230) to consideration of the question: "How can we find this quotient by using only numbers?" For example, since the drawing solutions for problems 1 and 2 show that $6 \div \frac{2}{3} = 9$ and $5 \div \frac{1}{2} = 10$, the question then becomes "What can we do with the numbers 6 and $\frac{2}{3}$ to get 9 and the numbers 5 and $\frac{1}{2}$ to get 10?"

At the present time it is recommended that when type B, the division of a fraction by a whole number, and type C, the division of a fraction by a fraction, are introduced that both the common denominator and the inversion methods be used. Drawing solutions for type B, while seemingly easy for adults, are very difficult for pupils and, of course, solution by subtraction is not possible. Thus, the early introduction of the inversion method provides the only second solution for type B problems and also for some type C situations.

For the initial teaching of division of fractions by whole numbers the following are typical illustrative problems:

(*a*) There was one-half of a melon. If three boys got an equal share, what part of a melon did each boy receive?

(*b*) Four pieces of meat weigh $\frac{5}{8}$ of a pound. What was the average weight of the pieces of meat?

For the solution of problem *a*, you can think "what is one-

third of one-half" or "$\frac{1}{2} \div 3$ is how many?" One-third of one half is easily solved by changing it to $\frac{1}{3} \times \frac{1}{2}$ but $\frac{1}{2} \div 3$ presents something new. To give pupils some help, the suggestion may be made that $\frac{1}{2} \div 3$ be solved by the common denominator method. $\frac{1}{2} \div 3$ then becomes $\frac{1}{2} \div \frac{6}{2} = 1 \div 6$ or $\frac{1}{6}$.

Solutions by the same two methods for problem *b* follow: $\frac{1}{4}$ of $\frac{5}{8} = \frac{1}{4} \times \frac{5}{8} = \frac{5}{32}$, and $\frac{5}{8} \div 4 = \frac{5}{8} \div \frac{32}{8} = 5 \div 32 = \frac{5}{32}$.

The pupils are then faced with the learning problem of how to use the inversion method in solving such division problems.

In developing and testing a rule for the questions, the common denominator method may be employed to check and to provide better understanding of the process.

It is recommended that teaching the division of a fraction by a fraction also be introduced with a problem. Since there are few situations in life where such uses arise, only a few problems are available. Two are given below:

(*a*) A gold miner figures he needs to produce $\frac{3}{4}$ ounce of gold each day. If each ton of gold ore yields $\frac{1}{16}$ of an ounce of gold, how many tons does he need to mine each day?

(*b*) The doctor said, "Take 30 drops of this medicine each hour." If there were only $\frac{3}{4}$ ounces of the medicine and 30 drops is equal to $\frac{5}{32}$ of an ounce, about how many doses of the medicine could be taken?

Each of these problems is a measurement division situation and can, therefore, be solved by subtraction. It is also recommended that the common-denominator method be employed. Use of the common-denominator method in solving such fraction division problems will give the pupils more confidence in the use of the inversion method.

Since there are very few social situations which require the division of a fraction by a fraction, it is suggested by some writers that this phase of arithmetic receive little attention. This recommendation, while based on fact as far as usage is con-

cerned, is not sound in that teaching the division of a mixed number by a mixed number which does occur in social situations is best approached through teaching of the division of a fraction by a fraction.

The instruction in division of fractions, like instruction in other areas of arithmetic, may be markedly improved by proper use of test and review exercises which are directed to specific aspects of the process. The following are representative:

1. What is the common denominator in $6 \div \frac{2}{3}$?
2. What are two ways of solving $2 \div \frac{3}{8}$?
3. What is the first step in dividing $\frac{3}{4}$ by 6?
4. Is $\frac{1}{2} \div 4$ equal to $\frac{1}{2} \times \frac{1}{4}$?
5. What are two ways of finding how many $\frac{3}{4}$'s equal 8?
6. When a fraction is divided by a whole number, is the quotient larger or smaller than the dividend? Use $\frac{2}{3} \div 5$ in answering.

While the four fundamental operations with common fractions are taught in Grades Five and Six, much instruction in Grades Seven and Eight is devoted to work with common fractions. In Grade Seven, especially, there is an extensive review and reteaching program. In view of the fact that pupils forget and that some pupils do not master what is taught in earlier grades, this fraction work in Grade Seven should provide opportunities for the pupils to acquire the basic fraction ideas and to acquire an understanding of the operation of the fundamental processes with fractions. In giving pupils such opportunities, the teaching procedures used need to emphasize some of the same things that were presented in initial teaching, but the situations employed in this presentation should be different from those used in earlier grades. It is suggested that where possible, first consideration of fractions in upper grades be in problem situations which involve more difficult uses of fractions than were used in initial instruction. If in dealing with these fractions in the

upper grades, pupils are in doubt or fail to show the expected facility, then work with simpler situations may be undertaken. To avoid a negative reaction against going back to the simpler work, it may be required as a check of the procedures used with the more difficult situations. For example, since checking an example like $\frac{5}{8}$ of 76,381 by drawing and multiplying would obviously be too time consuming, pupils might be asked to prove their method by using it on an example involving small numbers ($\frac{5}{8} \times 4$).

STUDY QUESTIONS

1. What is the best argument for presenting fraction work in the primary grades? (*1*) A long time is needed to acquire fraction concepts. (*2*) Primary grade children encounter some fractions if they have a rich school experience. (*3*) Fractions are needed to complete the primary child's number experiences.

2. At what grade level should a child first have planned experience with fractions? (*1*) One. (*2*) Two. (*3*) Three.

3. Should the pupil's work with fractions in the primary grades have an immediately apparent use to the children? (*1*) Yes. (*2*) No.

4. What is probably the major weakness of the review and maintenance exercises in arithmetic textbooks? (*1*) They include only the commonest fractions. (*2*) They give little attention to basic fraction ideas. (*3*) They deal too much with the figure representation of fractions.

5. Which of these drawings shows the improper fraction $\frac{5}{3}$?

(1)  (2)  (3) 

6. After a rule such as the one for changing the terms of fractions has been developed is there any good reason for having pupils show with drawings that the rule is true to fact? (*1*) Yes. (*2*) No.

7. Do the arguments that are advanced for the simultaneous teaching of addition and subtraction of whole numbers apply equally well to the addition and subtraction of fractions? (*1*) Yes. (*2*) No.

8. When systematic addition of fractions is introduced in the fifth grade with such examples as $\frac{1}{4} + \frac{1}{4}$, what is probably the most difficult aspect of the situation? (*1*) How to visualize the combined fractions. (*2*) The confusion caused by the way the fractions are written. (*3*) What to do with the denominators.

9. When unlike fractions are added by use of a ruler, are the fractions made alike? (*1*) Yes. (*2*) No.

10. In initial work with the subtraction of unlike fractions, is it considered best to exclude use of other than the best computational procedure? (*1*) Yes. (*2*) No.

11. In the example at the right, what error has been made? (*1*) The fraction and whole number answers have been separated. (*2*) The incomplete minuend and subtrahend misrepresent the facts. (*3*) The work does not show how the common denominator was found.

$$\begin{array}{rr} 3\frac{1}{2} = & \frac{2}{4} \\ -\ 1\frac{1}{4} = & \frac{1}{4} \\ \hline 2 & \frac{1}{4} = 2\frac{1}{4} \end{array}$$

12. Which of these subtraction situations would be considered best for introductory work? (*1*) $4\frac{1}{2} - 3\frac{1}{4}$. (*2*) $\frac{2}{3} - \frac{1}{3}$. (*3*) $4 - 2\frac{2}{5}$.

13. If current textbook practice were followed, which of the situations in Exercise 12 would be considered best? (*1*) (*2*) (*3*)

14. Which of the main types of multiplication of fractions should be used in introductory work? (*1*) Integers multiplied by fractions. (*2*) Fractions multiplied by integers. (*3*) Fractions multiplied by fractions. (*4*) A mixed number as one factor.

15. Can the situation $\frac{3}{4} \times 8$ be solved by addition? (*1*) Yes. (*2*) No.

16. Which drawing way of showing that $\frac{3}{8}$ is the correct product for $\frac{3}{4} \times \frac{1}{2}$ is best? (*1*) A line. (*2*) A rectangle. (*3*) A circle.

17. For showing that $5\frac{1}{4}$ is the correct product for $7 \times \frac{3}{4}$, which kind of drawing is best? (*1*) The line. (*2*) The rectangle. (*3*) The circle.

18. In the introductory lesson for teaching fraction multiplication, should use of drawings play a part? (*1*) Yes. (*2*) No.

19. For multiplying mixed numbers, should pupils be taught always to change the mixed number to an improper fraction? (*1*) Yes. (*2*) No.

20. In introducing the multiplication of a whole number by a fraction, which of these types should be used first? (*1*) $\frac{1}{2} \times 6$. (*2*) $\frac{3}{4} \times 8$. (*3*) $\frac{1}{4} \times 4$.

21. In $\frac{2}{3}$ of 6, the "of" means times. Is there a parallel situation to this in whole numbers? (*1*) Yes. (*2*) No.

22. Which of the three main types of the division of fractions should be taught first? (*1*) $4 \div \frac{1}{2}$. (*2*) $\frac{1}{2} \div 4$. (*3*) $\frac{3}{4} \div \frac{1}{2}$.

23. Could the situation $4 \div \frac{1}{2}$ ever arise through a partition-type division problem? (*1*) Yes. (*2*) No.

24. Could the situation $\frac{1}{2} \div 4$ ever arise in a measurement-type division problem? (*1*) Yes. (*2*) No.

25. Is the common denominator method of solving fraction division applicable to all three main types of division of fractions? (*1*) Yes. (*2*) No.

26. Is the solution of situations of the type $\frac{3}{4} \div 5$ possible through drawings? (*1*) Yes. (*2*) No.

27. When drawings are used in the solution of fraction problems, is a common denominator always found? (*1*) Yes. (*2*) No.

28. Should the common-denominator solution be the one first used in solution of the type represented by $\frac{2}{3} \div 4$? (*1*) Yes. (*2*) No.

29. Does the division of a fraction by a fraction occur frequently enough to warrant teaching that type for that reason alone? (*1*) Yes. (*2*) No.

30. If the common-denominator method were not used, would there be any value in having pupils solve division problems by means of drawings and serial subtraction? (*1*) Yes. (*2*) No.

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8

Decimals and Per Cents

Decimal fractions are a part of the everyday experiences of most adults and therefore are justifiably included in a good foundation arithmetic program for elementary-school children. Before considering decimals in the instructional program, the student of arithmetic teaching might profit from a critical analysis of decimal fractions.

Everyone recognizes that any decimal fraction can easily be written in the common fraction form. For example, .4, .026, and .0031 may be written $\frac{4}{10}$, $\frac{26}{1000}$, and $\frac{31}{10,000}$. On the other hand, common fractions with denominators other than ten, one hundred, and so forth cannot be written as decimals without first changing the denominator to a power of ten. Decimal fractions are then just a selected part of all fractions, namely, those whose denominators are a power of 10. Careful consideration of this definition shows that decimal fractions are not dependent on the decimal point and therefore $\overline{\frac{4}{10}}$ and $\frac{29}{100}$ are decimal fractions just as are .4 and .29. To insist on calling such fractions as $\frac{4}{10}$ and $\frac{29}{100}$ decimals would result in confusion, however. General usage has established decimals as being those fractions in which a decimal point is used. That practice is to be followed in arithmetic instruction, but children should have an opportunity to learn that it is the fact that the denominators

of decimals are all powers of ten which makes decimals unique and not the use of the decimal point.

In what is generally thought of as the decimal form of writing, the decimal point and number of places to the right of the point indicate the denominator of the fraction, and the numerator is indicated by the value of the figures written. This method of writing decimals and the fact that all denominators are powers of ten are the only essentially new things that a pupil encounters when decimals are studied. The student of arithmetic teaching should recognize that writing denominators of ten and its powers in the decimal form is merely an extension of our positional notation system to include fractions. The chart illustrates this extension clearly.

thousands	hundreds	tens	ones	tenths	hundredths	thousandths
4	2	7	5.	1	3	6

The foundation experiences with decimals, like other phases of arithmetic in a good program, begin long before the pupil starts systematic study of decimals. The decimal point is encountered as soon as the dollar sign is used in writing amounts of money. The automobile and bicycle mileage meters show tenths and provide abundant uses of this largest unit decimal fraction. Since the tenths figures on mileage meters are often a different color from other figures, this special way of showing a fraction of a mile should be used in bulletin board and chalk board statements of distances measured. The same should be true of recordings of meter readings.

There are many other ways of showing uses of decimals which can be easily incorporated in an experience program. Decimals may be found in the following: (1) the amount of rainfall on a

given day or the water equivalent of a snowfall; (2) yield per acre of various crops in experimental or contest trials; (3) English equivalents of most common metric measures, e.g., a kilometer is about .6 of a mile, a centimeter is about .4 of an inch; (4) averages as reported in experiments, e.g., the average was 26.4 miles per gallon or the average time was 3.4 hours; (5) normal body temperature; (6) flood data (depth or height of water during floods); (7) data on feeler gauges; (8) measurements with rulers marked with tenths of inches.

As with other parts of arithmetic, the preparatory experiences with decimals emphasize use. Decimals needed in reporting facts of experiments, rainfall data, and the like are displayed on the bulletin board with appropriate captions. In oral exercises and in problems pupils are required to use decimals. The following examples are illustrative: (1) The rainfall at Chicago yesterday according to the newspaper weather report was .82 inches. Was that more or less than an inch? (2) On Monday the river stood at 17.6 feet which is 2 feet below the estimated crest. What will be the depth of the water if the estimated crest is reached?

During the discussion of the solution to problem 2, the teacher wrote on the board the following:

$$\begin{array}{r} 17.6 \\ +\ 2 \\ \hline 19.6 \end{array}$$

An important idea to develop in this foundation program is an understanding that decimal fractions (tenths, hundredths, and thousandths) are used in precise measuring.

Reading and Writing of Decimals

The systematic study of decimals is initially begun in the fifth grade with lessons dealing with the reading of decimals. The usual plan consists of the presentation of the two forms of decimals (the common fraction form and that using the decimal

point) with the book or teacher making the statement that such common fractions as $\frac{7}{10}$ or $\frac{75}{100}$ may be expressed as decimal fractions. This statement is not only of little value for instruction, but is actually somewhat misleading for $\frac{7}{10}$ and $\frac{75}{100}$ are already decimal fractions.

It is recommended that instead of following the practice referred to above, the reading of decimals be introduced through the use of situations in which pupils have a need for reading them. The following lesson description illustrates what is meant by this recommendation.

A class study of the reading of decimals began with consideration of a number of statements involving decimals. Two of the statements are given.

"At the indoor track meet the winning time for the 50-yard dash was 5.8 seconds."

"The average yield per acre of the new soybean was 36.41 bushels."

While the class looked at the statements, the teacher read some of them. (A member of the class might have been called on instead.)

The teacher introduced the problem for study by asking, "How do we know that the fraction in 5.8 is tenths and the one in 36.41 is hundredths?"

To help the children analyze whether the denominator of a decimal was tenths or hundredths the teacher suggested, "Since you know how to read the denominators in common fractions, let's write some decimal fractions first with number denominators and then with the decimal point. Perhaps we can figure out how to tell what the denominator is when a decimal point is used."

The following were then written on the board.

Two tenths = $\frac{2}{10}$ = .2

Seventy-five hundredths = $\frac{75}{100}$ = .75

Six hundredths = $\frac{6}{100}$ = .06

The learning problem "Can we figure out from these data

how to tell (identify) what the denominator of the fraction is when a decimal point is used?" was repeated. After some discussion it was decided that the number of figures (one for tenths, two for hundredths) after the decimal point tells what the denominator is. The pupils then engaged in testing the proposed rule and, after verification, engaged in practice exercises to fix the procedure.

The method of reading decimals developed in the preceding paragraphs is quite satisfactory for tenths and hundredths but does not give all that is needed for understanding this process. As larger decimals are encountered more knowledge is required. Following the experience with tenths and hundredths, the development of procedures for reading all decimals should be undertaken.

Memorizing the place names is one method for reading decimals which will give insight into our system of notation and therefore has real merit. Since each decimal place name is essentially the same as the corresponding whole number place, there is actually little to memorize. In reading such a fraction as 6.024 by this method, the pupil says the zero is tenths, the 2 is hundredths, and 4 is thousandths, the fraction is 24 thousandths. This procedure illustrates an important fact regarding decimals, namely, that the last and not the first figure gives the decimal its name. Study of this fact should lead to a revision of the earlier generalization regarding the reading of whole numbers. It is actually not the first digit read in whole numbers that gives it the name but the place, and the place is determined by distance from the ones place.

In the identification of the decimal place names, good use can be made of a chart showing decimal and whole number place names. Some teachers plan such a chart with extended vertical lines on the board. When a pupil has difficulty reading a fraction, the figures of the fraction are written in the correct places on the chart.

Writing the common fraction form of decimals as a means of

reading the decimal written with the decimal point is another good procedure. Use of this method leads to a very useful generalization, that the number of zeros in the denominator of the common fraction form is always equal to the number of figures after the decimal point. The student of the teaching of arithmetic should recognize that for each of these methods of teaching, appropriate practice exercises for pupils need to be provided. These exercises will include not only experiences in reading decimals but also exercises in writing decimals. Textbooks furnish some such exercises but there is frequently a need for supplementary instructional materials.

While not a part of the reading and writing of decimals, the preparation of a decimal common-fraction chart should come soon after or along with the intensive study of the reading of decimals. One such chart is reproduced below.

$\frac{1}{2} = .5 = .50$	$\frac{1}{4} = .25$
$\frac{1}{3} = .33$	$\frac{3}{4} = .75$
$\frac{2}{3} = .67$	$\frac{1}{5} = .2 = .20$
$\frac{4}{5} = .8 = .80$	$\frac{1}{10} = .1 = .10$
$\frac{7}{10} = .7 = .70$	$\frac{3}{10} = .3 = .30$

In reading mixed decimal numbers such as 7.435, both "Seven and 435 thousandths" and "Seven point 4, 3, 5" are acceptable.

In teaching the reading of decimals special precautions should be taken to keep pupils from assuming that the decimal point is the center of notation. The ones place is the center of our system of notation and should be emphasized as such. Charts on how to read numbers (including decimal fractions) used for display in Grades Five through Eight might well make the ones place very prominent. On such charts lines joining tens and tenths, hundreds and hundredths and so on will make the ones place stand out. Each of the places related by name, for example, millions and millionths, are an equal number of places from the ones place.

Addition and Subtraction of Decimals

The addition of decimals presents nothing basically new to pupils who have mastered the addition of common fractions. It is recommended that this work be introduced through consideration of several problems in which decimal fractions need to be added. The pupils are directed to solve the problems by use of common fractions, or, if they feel that they can add decimals, to check their answers by using common fractions. Partly as a check of understanding and also to give those not certain of their addition more confidence, the solution of one or two problems by means of diagrams may also be suggested.

Since there are few life problems which require the addition of decimals, some of the introductory problems used may require the addition of mixed decimal numbers. These two problems were used in one introductory lesson.

Problem 1. John's bicycle meter shows that he lives .8 mile from school. How far does he ride going to and from school one time?

Problem 2. One day Bill rode his bicycle 3.6 miles in the morning and 5.1 miles in the afternoon. How far did he ride that day?

After the answers to several such problems had been found by common fraction solutions and discussed, the teacher started the group on finding a way of adding decimals with these directions: "Write the decimals to be added in problem 1 as an addition problem. Change the answer you found by using common fractions to a decimal and write it as your answer. From the figures try to work out a way of adding decimals. As soon as you have a way, test it on two other problems."

The statement for adding decimals developed by the class was: "You add tenths to tenths and carry just as you did with whole numbers. You use a decimal point in your sum to separate the fractions and whole numbers."

During the time that this class worked on practice exercises to master the procedure of adding decimals, they were required to show by means of common-fraction solutions that the sums obtained for a few selected examples were correct. The adding of hundredths and thousandths was also introduced and later checked through use of common-fraction solutions.

It should be noted that in the procedure described, addition of decimals was begun by using only tenths and not by adding dollars and cents as was often the case in old textbooks. Almost all good texts now begin with the addition of tenths. The use of money to introduce the addition of decimals has two serious limitations. First, pupils already know how to add dollars and cents and are unaware of their decimal nature. Second, hundredths are too small to be easily visualized and certainly cannot be drawn very effectively.

Another practice often found in older textbooks is that of providing practice in adding ragged decimals (e.g., $6.4 + 3.951 + 5.1$, etc.). They were called ragged because the right-hand edge is uneven or ragged. There are no situations where measurement is accurate to tenths only in one reading and to thousandths in another, and even if such figures are added the sum is accurate only through the tenths. There is then no good reason for giving elementary-school children practice in adding ragged decimals. Where such decimals occur in practice exercises, pupils should be taught to fill all vacant spaces with zeros.

The subtraction of decimals usually follows immediately after the teaching of addition of decimals. Since there is basically little that is new to the pupils in learning to subtract decimals, very little introductory work is needed. However, the common-fraction method of checking and verifying may be used as much as in the addition of decimals.

The subtraction of decimals from a whole number introduces a procedure different from anything encountered in the addition

of whole numbers. For example, in $6 - 2.56$ the pupil does not even have a zero to begin the process of borrowing. This phase of subtraction should be started with a problem. From consideration of the solution, the practice should be developed of adding as many zeros to the right of the decimal point of the minuend as there are decimal places in the subtrahend.

Multiplication of Decimals

The procedures and some of the issues in the teaching of multiplication of decimals can be easily identified by examining the experience of a class.

This sixth grade began studying the topic with these two problems.

Problem 1. The mileage meter on Jack's bicycle showed that it was .4 mile around the block in which his home stood. How far did he ride in going around the block six times?

Problem 2. A boy scout figures that he walks on the average 2.8 miles per hour. At that rate, how far can he walk in 8 hours?

The directions for study were "First, write the number question for the problem and then find the answer by multiplying. There are two ways to do that: (1) change the decimal to a common fraction and multiply, or (2) multiply the decimal. Try both ways. Check your answer by adding or by making a drawing."

During the time the pupils worked on the assignment, the teacher helped some individuals and placed various solutions on the board. In the discussion that followed, she tried to direct the thinking of the pupils by these questions: (1) You have shown that $6 \times \frac{4}{10} = \frac{24}{10}$ or 2.4. Could you get the same result using just your number question? (2) Is multiplying numbers when one is a decimal different from multiplying whole numbers? (3) When you multiply a decimal, will the product al-

ways have a decimal in it? Show that you are right by doing these examples:

$$\begin{array}{rr} .3 & .4 \\ \times 4 & \times 5 \\ \hline \end{array}$$

The discussion resulted in the formulation of this rule for multiplying a decimal: "Decimals are multiplied just like whole numbers but the product must show a decimal."

The pupils of the class then turned to the textbook to see whether the procedure presented there was superior to the one they had developed. The multiplication of hundredths and multiplication by decimals as presented in the textbook were next considered. The textbook procedures were supplemented by frequent use of common fractions and multiplication as a check on placement of the decimal point. Part of a period was given to the development of a general rule for the placing of the decimal point in products. In preparation for this period different pupils had been asked to place on the board examples of all the different kinds of decimal multiplication. Additional examples were solved to check the general rule.

The study of multiplication of decimals was concluded by use of the study questions reproduced below.

Answering these questions will help you review the multiplication of decimals.

1. How many decimal places will there be in the product of 6.4×2.93?
2. What check could you use to find out whether the decimal point in the product for $.04 \times .32 = .0128$ is correctly placed?
3. Can the product for the multiplication of a decimal by a decimal be checked by adding?
4. Is 4×1.32 equal to 1.32×4?
5. Can all decimal multiplication examples be changed to common fractions and then multiplied?
6. Which is easier to do: $2\frac{4}{10} \times 3\frac{8}{10}$ or 2.4×3.8?

An examination of the teaching procedure reveals that initial experience in multiplication of decimals involved the multiplica-

tion of a decimal and a mixed decimal by a whole number, and that the decimal multiplication was developed from the common-fraction multiplication. A means of checking initial work was provided with the suggestion to add or to draw. After the development of the procedure for multiplying decimals, the common-fraction method was used as a check.

The multiplication of decimals, like the multiplication of fractions, is a comparatively easy process for pupils to master. The only essentially new factor in the process is the placing of the decimal point in the product. The teaching procedures should therefore emphasize understanding of the placement of the decimal point. That was the purpose of developing from common fraction solutions the decimal solution in the sixth-grade-class experiences described earlier. Because multiplication of decimals appears so easy, it is also very easy for the teacher to present the topic without giving much consideration to understanding. The student of arithmetic teaching might therefore profit from a critical comparison of the method of teaching described in this section with that presented in several children's textbooks.

Division of Decimals

The study of the division of decimals is best begun through consideration of problems. If the problem is drawn from or related to the experiences of the pupils, they will be able to write the essential number question and will be able to find the answer by use of common fraction division or by means of drawings. How to do the division with use of decimals then becomes the only learning task. Since the answer to the number question of the problem is known, all attention can be focused on how to divide with decimals.

Many textbooks begin division work with decimals by considering the division of a mixed decimal by a whole number

(2.6 ÷ 2). The writer favors this type for introductory problems because good illustrative problems for this kind can be found and because the problem of placing the decimal point in the quotient can be worked out with the least difficulty. The division of decimals involves a variety of situations. The different situations are due to the presence or absence of a decimal point in divisor, dividend, or quotient. Pupils can make a good estimate of the quotient by considering only the whole number. This approximated quotient is of great value in showing the need for a decimal point in the quotient. To illustrate, consider this example: "Two boys walked from the park to their home in 4 hours. The distance was 8.4 miles. What was their average rate per hour?" Not many pupils who had used the whole number to get an approximate answer would accept 21 miles per hour as the answer. Such situations create a need for determining where to place the decimal point in the quotient.

For the pupils' study of introductory illustrative problems, it is suggested that they write the number questions to all problems and that they use common-fraction division to determine where to place the decimal point in the quotient. From the discussion of the solutions, the learning problem "How are decimals divided by whole numbers?" may be developed and solved. The suggested solutions should be tested and the textbook consulted for determining the value of the solution proposed by the class. Textbook exercises are then used for practice. Here, as in the multiplication of decimals, the use of common-fraction solutions is suggested as a check to be used with some examples.

The rule developed for this first kind of decimal division is essentially as follows: "In the division of decimals, the decimal point of the quotient is always placed directly above the decimal point of the dividend." In order to bring out the generalization that when tenths, hundredths and so on are divided there are tenths (or at least a tenths place), hundredths, and the like in the

quotient, such study questions as the following are used: "When you divide a number which contains tenths, what will be the name of the decimal in your quotient?"

The next type of decimal examples to be studied, the changing of the common fraction of a quotient to a decimal, is also introduced in a problem situation, but as is shown in the illustration, the computation is performed for the pupil. Attention is centered on the new part of the procedure under consideration.

1. Four boys sold their old boat for $31. Which solution, A or B, shows more clearly the amount of money each boy should receive?

2. Did the adding of a decimal point and two zeros to the dividend in example B change the amount of money?

A	B
$ $7\frac{3}{4}$	$ 7.75
$4\overline{)\$31}$	$4\overline{)\$31.00}$
28	28
$\frac{3}{4}$	30
	28
	20
	20

After a discussion of the situations presented, pupils should be directed to solve such examples as $4\overline{)9}$ and $5\overline{)22}$ by adding a decimal point and zeros to the dividend and writing the fractional part of the quotient as a decimal.

Changing the remainder in uneven division to a decimal makes that kind of division an example of the type already used in the initial decimal division. The next type to be considered, the changing of a common fraction to a decimal fraction is probably best introduced by presenting the computation to the pupil and directing his study of the procedure with critical questions. If time is available, this type may be developed by study and experimentation with such known decimal common-fraction equivalents as $\frac{1}{10} = .1$, $\frac{1}{2} = .5$, and $\frac{1}{4} = .25$.

The most difficult type of decimal division examples to teach are those in which a decimal occurs in the divisor ($.3\overline{)12}$, $2.8\overline{)50}$, etc.). Most textbooks present solutions of this type of examples with very inadequate or no real rationalization. Nearly all

$$2\ 0.$$

books use the caret device ($.2\wedge\overline{)4.0\wedge}$) in teaching the computation involved. This use of a caret is at best a mechanical device and hardly worthy of being used in an instructional program emphasizing understanding.[1]

A study sheet used in teaching this type of decimal division is reproduced below. While not entirely satisfactory, the material on this study sheet has some superior features.

Dividing by a Decimal Fraction

1. By using a speedometer on his bicycle and his watch, Jack found that at ordinary speed he traveled about .2 miles a minute. How could Jack figure out how long it would take to ride his bicycle 4 miles? Use a drawing and at least one other way to find out.

2. In working Example 1, a boy thought, "I need to find how many .2's equal 4," so he wrote this: $.2\overline{)4.}$ Is this the correct number question? Try to answer the question using only figures. Remember your answer must check with the drawing solution you used in Example 1.

3. In the drawing solution for Example 1, the 4 was changed to how many tenths?

4. Here is a number record of one diagram solution: $4 \div \frac{2}{10} = \frac{40}{10} \div \frac{2}{10} = 40 \div 2 = 20$. This may be written $\frac{40}{2} = 20$. Did your drawing show $\frac{40}{2} = 20$?

5. Is $4 \div .2$ equal to 40 tenths $\div$ 2 tenths?

[1] Claude H. Brown, "Some Thoughts on Placing the Decimal Point in Quotients," *The Mathematics Teacher*, 38: 78–80 (February, 1945), and H. Van Engen, "Some More Thoughts on Placing the Decimal Point in Quotients," *ibid.*, 38: 243 (October, 1945).

6. Because 40 tenths ÷ 2 tenths equals 40 ÷ 2 one class decided that $.2\overline{)4}$ could be written $2\overline{)40}$ and solved the number question as shown in Box A.

A

$$.2\overline{)4} = 2\overline{)40}$$

20
4
0

By what number can 4 be multiplied to change it to 40? By what number can .2 be multiplied to change it to 2?

In Box A the decimal fraction divisor .2 is changed to what kind of number?

7. Make a rule for dividing by tenths written as a decimal fraction. Look at the work in Box A. Be sure your rule tells how to change the number question to an equal number question.

Try your rule for dividing by a decimal fraction on example 8. Check your answer with a drawing.

8. If a road-building crew can pave .3 of a mile a day, how long will it take them to pave 12 miles?

Work examples 9–12. Be sure your work shows the number question changed to an equal number question.

9. $.4\overline{)12}$ 10. $.5\overline{)10}$ 11. $.3\overline{)18}$ 12. $.6\overline{)12.6}$

Attention is called to these features of the study sheet: (1) An illustrative problem which can be solved by means of a drawing is used for an introduction. Thus, every pupil who can read, draw, and count can solve the problem; (2) While the major computation is done for the pupil, there is something for the pupil to do in each of the numbered exercises; (3) By use of the drawing and the common-denominator method, the multiplication of divisor and dividend by ten can be readily seen; (4) Writing a number question with a whole-number divisor which is an equivalent of the original decimal divisor situation ($.2\overline{)4} = 2\overline{)40}$) is mathematically correct. The computation is then performed

with whole numbers unencumbered by anything new, which is not the case when a caret is used; (5) Pupils are asked to state a rule for dividing by tenths written as a decimal and to test that rule on a problem and some examples.

After the different variations of this type of decimal division ($2.3\overline{)6.9}$, $.214\overline{)6.39}$, etc.) are solved by changing the decimal divisor situations to equivalent situations having whole number divisors, the introduction of a short cut such as the caret for shifting the decimal point may be considered. However, students of arithmetic should recognize the disadvantages of the procedure. Some of these disadvantages are: (1) the decimal point in the quotient is no longer placed above the decimal point of the dividend, thus upsetting one of the fundamental rules developed in initial instruction with decimals; and (2) the process is mechanical and, computationally speaking, easy; the pupil who has not fully understood the changing of a decimal divisor to a whole number by multiplying by a power of ten, quickly adopts the procedure. This often results in the making of absurd errors, for the pupil has little understanding of the process.

Since, as was indicated earlier, the procedure advocated in this section for teaching division by a decimal is not wholly satisfactory, the student of arithmetic teaching may profit from a critical comparison of the procedures advocated and those found in current textbooks. Varied techniques which will further understanding should be employed. In the study of division of decimals, as in other areas of arithmetic, marked improvement may be secured by the use of summaries and reviews in the form of compositions, short answer study questions, and summary statements.

Following the study of multiplication and division of decimals, some programs of instruction give time to the three kinds of problems with decimals. There are better reasons for emphasizing the three kinds of problems with decimals than for

emphasizing the three kinds with common fractions. However, some of the same limitations that were listed in the discussion of this procedure with common fractions apply to decimals. The reader is therefore referred to that section. For additional discussion of this topic, the reader is referred to Buckingham's *Elementary Arithmetic*,[1] pages 338–341.

Percentage

The grade placement of the teaching of percentage has changed markedly in the past 30 years. It has been transplanted from fifth to seventh grade, and in some programs of instruction only Cases I and II are taught in the seventh grade with Case III postponed to Grade Eight. Case I situations require the finding of a per cent of a number; Case II deals with finding what per cent one number is of another; and Case III is the finding of the total when only a per cent is known. The reasons for delaying the teaching of percentage are many, but by far the most frequently mentioned are the difficulties children encounter in solving percentage problems, especially those classified as Case III. Other important reasons for delaying the teaching of percentage are those which support the general trend toward moving up the grade level of most topics. The lack of use that children have for computational percentage is also cited as an argument in favor of delay.

While the reasons for postponing the teaching of percentage appear to be sound and the changed grade placement has been generally accepted, the postponement procedure has created some serious difficulties for elementary-school children. Such statements as "35% of the coal mined is used to smelt iron," "20% of the corn crop was damaged by the freeze," and "More than 96% of the cases now fully recover," are unintelligible

[1] B. R. Buckingham, *Elementary Arithmetic: Its Meaning and Practice* (Boston: Ginn & Company, 1947)

without an understanding of per cents. Fifth- and sixth-grade social studies, science, health, and current news make use of many statements similar to those mentioned. To meet the needs created by such uses of per cents, teachers in Grades Five and Six have tried to teach some meanings of per cent without teaching any of the computational procedures.

Developing the Meaning of Per Cents

It was indicated in the preceding section that there is a need for teaching the meaning of per cents as early as Grade Five. Before considering steps for teaching elementary school children the meaning of per cents, the student of arithmetic teaching should take a critical look at his own ideas of the topic and also consider some of the misconceptions that are taught regarding per cents. To begin such a study, the reader might well ask what is the difference between *per cents* and *percentages?* If arithmetic books or books on arithmetic teaching are consulted for an answer to the question, the reader is likely to become more confused than ever. A few books attempt to define per cents and percentage but then use the terms interchangeably. A study of the literature shows that the first use of percentage was in reference to that phase of arithmetic which dealt with per cents. Gradually the term percentage began to be used for per cent. So today, in spite of the definitions found in books, the terms have the same meaning. Another rather confusing dualism is the use of two spellings of the term *per cent*, one as two words, the other as one word. Arithmetic books and books on the teaching of arithmetic use two words in writing per cent while newspapers use one word. The advantage of the two words over the one is not made clear in any of the arithmetical literature. One of the best modern dictionaries recognizes only the one word spelling.

The usual definitions given for the term *per cent* go back to the

original Latin "by the hundred," then give the simple modern definition, namely, that per cent means a hundredth or hundredths. If per cent is just another name for hundredths, what useful purpose does it serve? Why not say hundredths and thereby have only one word instead of two? The following uses of the term *per cent* will show its superiority over *hundredths:* (1) In the last 10 years the city has grown 200%. (2) The gain in weight was 100%. (3) The tax amounts to 1.25% of each sale. In each of these cases the term hundredths could be substituted only with modification. Another reason for using per cents rather than hundredths is that we experience greater difficulty in thinking and communicating with fractions than we do with whole numbers. The changing of hundredths to per cents is, then, very similar to the changing of yards to feet, feet to inches, pounds to ounces, and similar changes which we make in the field of measures. Instead of using three-fourths of a pound, we prefer to deal with twelve ounces. Per cents are then a special name for hundredths which we use in describing quantities. They are used as whole numbers without reference to their being fractional parts of a whole.

One of the major uses of per cents is in making comparisons. Fractions are very useful in comparisons, if all the fractions can be changed to the same common denominator. Per cents are still more useful for comparison than are decimals, for not only do all per cents have the same denominator, but whole numbers and fractional parts of per cents can be included without difficulty. For example, 3.2%, 150.5%, and 86% are readily compared.

There are no well-known procedures for teaching elementary-school children, especially fifth and sixth graders, the meaning of per cents. Modern arithmetics for these grades do not deal with percentage and therefore nothing on the topic is found in them. It is recommended that first teaching be the outgrowth of a need for the meaning of a per cent. The following examples

from fifth- and sixth-grade science and social studies indicate a need for knowing the meaning of per cents: (1) The weather report gave the humidity at Las Vegas as 26%; (2) Only about 20% of buffalo grass seeds germinate the first year; (3) 60% of the coal from the mine is made into stoker coal.

After several such situations have arisen, "time out" for developing the meaning of per cents should be taken. The teacher could summarize the learning assignment about as follows: "How can we show what 26%, 20%, and 60% mean?" As the class (including the teacher) work together to find the answer, circles and bar graphs with the appropriate per cents indicated should be constructed and kept in the room for reference. The number line will be useful in developing the meaning of per cents. This device will also be found to be one of the best references to use in clarifying the many per cents which will be found in reference reading, especially in the field of geography. The development and use of a common fraction-decimal fraction percentage equivalent chart will also be useful in developing and maintaining the meaning of per cents. An example of this type of chart is found in a modern German arithmetic book. It appears as one of the reference tables inside the back cover. Part of the table is reproduced below.

$100\% = 1$	$10\% = \frac{1}{10}$
$50\% = \frac{1}{2}$	$30\% = \frac{3}{10}$
$6\frac{1}{4}\% = \frac{1}{16}$	$40\% = \frac{2}{5}$

Another procedure for clarifying and fixing an idea of the meaning of per cents is the frequent use during oral arithmetic work of exercises like this one: "On the way to school, I had to stop at three of the four traffic lights. What per cent of the lights stopped me?"

The changing of per cents to decimals and decimals to per cents might well be considered a part of the development of the meaning of per cents. The teaching of this procedure is prob-

ably best begun by critical study of the fraction-percentage equivalence chart. It is assumed that this chart has been in the classroom and used for some time before study of the changing from one form to another is undertaken. The practice work following the development of rules for changing decimals to per cents and vice versa should include some changing of per cents greater than 100 to decimals.

Methods of Solving Percentage Problems

Percentage problems are of the three kinds already referred to as: Case I, finding the per cent of a number; Case II, finding what per cent one number is of another; and Case III, finding a number from a given per cent.

The solution of Case I problems presents only one new thing to the pupil who has learned how to solve problems with decimals: that is, the changing of the per cent to a decimal. For example: "If 6% of a miner's car of coal is slate, what is the weight of slate in a car containing 4000 pounds?" The number question to be answered here is, "What is 6% of 4000?" Since 6% means .06, the number question can be changed to "What is .06 of 4000?" The percentage problem becomes a decimal-fraction problem. Thus, to find the per cent of a number, the per cent is written as a decimal and the number multiplied by this decimal. In Case II problems the procedure consists of finding what decimal fraction one number is of another and then changing the decimal to per cent. In Case III problems the per cent is changed to a decimal and the part of the number given is divided by the decimal.

The procedure recommended for teaching pupils how to solve the three kinds of percentage problems varies from that used in introducing other processes. The initial assignment for the first kind of problem (Case I) illustrates the variation:

Assignment: "In these problems you are to find the per cent

of a number. First decide upon the essential number question needed to solve each problem; then change the per cent in that number question to a decimal. After that, you will know how to proceed."

Thus, instead of developing a new process in the usual manner (that is, by suggesting that the class work with known but longer processes), the teacher tells the pupil how to proceed. A period of development for percentage is omitted because there is essentially nothing new in the solution of percentage problems.

From the illustrative assignment it can be seen that the formulation of the essential number question to be answered is the important step in the solution of percentage problems. Here, as in the case of common fraction problems, the process to be used in solving Case I problems is indicated by the essential question. A sample problem with the essential number question, is provided below for each kind of problem.

Case I problem: "A cow produces on the average 60 pounds of milk a day. If 5% of the cow's milk is butterfat, how many pounds of butterfat does she produce per day?" The number question to be answered for the solution of this problem is, "What is 5% of 60?"

Case II problem: "Twelve of the 60 boxes of berries in a crate were ruined. What per cent of the boxes were ruined?" The number question needed for the solution of this problem is, "12 is what per cent of 60?"

Case III problem: "The profit of Company A was 6% of its total sales. If the profit of the company was $24,000, what were the total sales?" The number question to be answered in solving this problem is, "6% of what number is $24,000?"

When the per cent in the number question for the Case I problem above is changed to a decimal the question is, "What is .05 of 60?" or, "What is $.05 \times 60$?" The arithmetical procedure to be used in answering the question is clearly indicated.

In solving the Case II problem the term "per cent" in the

question "12 is what per cent of 60?" is first changed to "decimal fraction." In the question, "12 is what decimal fraction of 60?" the process of division is definitely implied, although not indicated as clearly as is the multiplication in Case I problems.

In solving the Case III problem the per cent is first changed to a decimal. The question then becomes ".06 of what number is $24,000?" or, ".06 × what number is $24,000?" In this form the question is similar to "6 × what number is 24?" — a type of whole-number problem which the children have learned to solve with division. In a like manner, division is used to solve the question, ".06 × what number is $24,000?" It should be recalled that many programs of instruction do not teach Case III of percentage until the eighth grade.

The Case Method of Solving Percentage Problems

Many texts now use what is known as the "case method" of solving percentage problems. By this method pupils are taught to identify the case or kind of percentage problem and then to use the solution suggested for that kind of problem. For example, if a problem is Case I, then the given per cent is changed to a decimal and the given number multiplied by the decimal. For Case II, the number asked about is divided by the other number and the resulting decimal changed to a per cent. For Case III problems, the given number or part is divided by the given per cent expressed as a decimal.

In the chapter on common fractions the limitations of the case method as applied to the solution of fraction problems were discussed. Since those same limitations may apply to percentage problems, they will be repeated here.

1. In using the case method the pupil centers his attention on identifying the case and therefore does not give enough attention to an over-all study of the problem to understand it thoroughly.

2. In learning to distinguish the three kinds of problems, the pupil has to learn something extra; he has to use an extra step between the consideration of facts of the problem and the solution. According to some teachers, the skill required to identify problems is not essential to the solution of percentage problems.

3. The case method tends to become mechanical and lacking in meaning.

The popularity of the case method for percentage in textbooks seems to indicate that the limitations listed above are not so serious as they might appear at a first reading. It is likely, however, that ease of computation by the pupils may be another case of mechanical learning and therefore is of little permanent value.

Other Methods of Solving Percentage Problems

The use of a formula for solving percentage problems has been advocated by a number of writers in the field of arithmetic and is a popular method with many junior high school teachers. The formula used is $R \times B = P$ where R stands for rate, B stands for base, and P stands for percentage.[1] The meaning of these three special terms is usually developed from a logical solution of Case I and Case II problems.

The following problems are presented to show how the formula is applied:

Problem 1. The 2% sales tax on a used car priced at $1450 would amount to how much?

Problem 2. The first year Mr. Hill lived in his house, he burned 1,500 gallons of fuel oil. The next year he put on storm windows. That year he used only 1,100 gallons of oil. What was the per cent of decrease in the use of oil?

[1] For description of the method, see Larson, H. D., *Arithmetic for Colleges* (New York: The Macmillan Company, 1950), pp. 157–158.

Problem 3. At a seed corn processing plant the field-ripe corn is dried and graded. The losses due to these two processes leave only 80% of the original weight to be sold as seed. At that rate how many pounds of field-ripe corn are needed to produce one bushel of seed weighing 56 pounds?

The solution of these problems by use of the formula would be as follows. For problem 1, we need to find the percentage. For this type of problem, the general formula $R \times B = P$ is stated $P = R \times B$. Using the data for the problem we have $P = .02 \times \$1450 = \$29$.

For problem 2, we need to find the rate. For finding the rate, the general formula, $R \times B = P$, is rewritten $R = \frac{P}{B}$ and the necessary substitutions made. $R = \frac{1100}{1500} = .266 = 27\%$.

For problem 3, we need to find the base. For finding the base the general formula is written $B = \frac{P}{R}$. When the data from the problem are used, we have $B = \frac{56}{.80} = 70.0$.

To those who understand per cents well, the use of the formula appears very simple. Teachers should recognize that not all pupils understand per cents well and that even the rewriting of the formula to meet each special case is an algebraic exercise not familiar to seventh-grade pupils. The giving of a specific meaning to the word *percentage* is another source of difficulty. Pupils often interchange rate and percentage when using the formula.

To avoid some of the difficulties due to the rather mechanical nature of the formula procedure, some teachers have used the old unitary analysis method. By this method, one per cent is first found and then the total found. Problem 1 above would be solved by this method in the following manner: First, find 1% of \$1450. $.01 \times \$1450 = \$14.50$. Then, find 2%. $2\% = 2 \times \$14.50 = \$29$.

The solution for problem 2 would be as follows: First, find 1% of 1,500 gallons. 1% of 1,500 = .01 × 1,500 = 15 gallons. Then, find how many times the decrease (400 gallons) equals 1% or 15 gallons. 400 ÷ 15 = 26.6. The answer is 27%.

The solution for problem 3 would be as follows: First, find 1% of the weight.

80% of the weight = 56 lb.
1% " " " = 56 ÷ 80 = $\frac{56}{80}$ = .7 lb.
100% " " " = 100 × .7 = 70.0 lb.

The unitary analysis method requires logical step-by-step thinking and from the standpoint of understanding has real merit. However, as can be seen from the solution of the three problems, the actual solutions are rather cumbersome. Its use should probably be restricted to initial study, checking understanding or introducing review lessons.

Another method of solving percentage problems which requires some of the same logical thinking which the unitary analysis method requires and yet has some of the simplicity of the formula is the equation method. In solving problems by this method, an equation is written for each problem and then the necessary substitutions are made.

The solution of problem 1 above using the equation would be as follows. The essential question of the problem is first stated.

What is 2% of $1450?
n = 2% of $1450 = .02 × $1450 = $29.

The solution for problem 2 follows:
400 gallons is what per cent (or part) of 1,500 gallons?
400 = n of 1,500.

$$n = \frac{400}{1500} = .266 = 27\%.$$

The solution of problem 3 by the equation method is as follows:

56 lbs. is 80% of what amount?

$56 = .80 \text{ of } n$

$$n = \frac{56}{.80} = 70 \text{ lb.}$$

This writing of the essential question from which the equation is developed is a superior feature of the equation method. The use of "n" for the unknown is also worth consideration. A weakness of the method, just as was true of the formula, is the pupils' lack of familiarity with the simple algebra required for solving the equation for Cases II and III.

It is quite likely that pupils in their study of percentage will profit from use of several methods of multiplying or dividing. Teachers should then try to create logical reasons for consideration of methods other than the one method selected as best. The search for a best method and the use of other solutions as a check are two such reasons. Another good reason for using more than one method of solving is that understanding is better if pupils can use effectively more than one solution.

STUDY QUESTIONS

1. What is a good definition of a decimal fraction? (*1*) Any fraction which is written with a decimal point. (*2*) Any fraction whose denominator is a power of ten. (*3*) Any fraction where the denominator is indicated by the number of figures to the right of the decimal point.

2. How is the denominator of decimal fractions indicated? (*1*) By the number of figures in it. (*2*) By counting one less place than for whole numbers. (*3*) By comparing with a standard chart.

3. Is the use of the decimal point the only way in common usage of writing decimal fractions? (*1*) Yes. (*2*) No.

4. In the common introduction to the reading of decimals,

such statements as "the common fraction $\frac{7}{10}$ may be written .7 which is a decimal fraction" are used. Why are such practices of questionable value? (*1*) Because no reason for writing the fraction in another form is given. (*2*) Because no problem is provided. (*3*) Because in either form the fraction is a decimal.

5. How many different good specific ways for reading decimals should be presented to pupils? (*1*) One. (*2*) Two. (*3*) Three.

6. About how many fractions should be presented on a decimal-common fraction equivalence chart? (*1*) 6. (*2*) 10. (*3*) 20. (*4*) 50.

7. On charts showing whole numbers and decimals, why should the ones figure be given prominence? (*1*) Because it is the first whole number. (*2*) To keep pupils from assuming that the decimal point is the focal point in our notational scheme. (*3*) To assist in the separation of the whole numbers and fractions. (*4*) It should not be given prominence.

8. In initial instruction in computation with decimal fractions, does the common fraction form play a major role?
(*1*) Yes. (*2*) No.

9. Ragged decimals are encountered in connection with what number process? (*1*) Addition. (*2*) Subtraction. (*3*) Multiplication. (*4*) Division.

10. What major new factor to be learned does the pupil encounter in the multiplication of decimals? (*1*) The proper placement of partial products. (*2*) Keeping multiplicand and multiplier figures in their proper places. (*3*) The placement of the decimal point in the product.

11. The division of decimals should best be introduced by use of which type of division situation? (*1*) A mixed decimal divided by a whole number. (*2*) A whole number divided by a decimal. (*3*) A decimal divided by a whole number.

12. What suggestion for placing the decimal point should be given to pupils in their first work in the division of decimals?

(*1*) Place the decimal point above the one in the dividend. (*2*) Make a reasonable estimate of where the point should be placed. (*3*) Find the decimal place through use of common fractions.

13. Should the procedure for changing of a common fraction to a decimal fraction be introduced through consideration of fractions alone or in a mixed number situation? (*1*) Fractions alone. (*2*) Mixed numbers.

14. Which type of division of decimals is most difficult for pupils to master? (*1*) $.2\overline{)6.}$ (*2*) $2\overline{).6.}$ (*3*) $2.3\overline{)6.4.}$

15. Is the use of the caret in division of decimals recommended without reservations? (*1*) Yes. (*2*) No.

16. What is the major objection to the case method of solving percentage problems? (*1*) It is too difficult for the pupils to identify the cases. (*2*) Attention is taken from the essential question of the problem. (*3*) There are actually only a few Case III problems.

17. Why are per cents more useful than the equivalent decimal or common fractions? (*1*) Because in reality they are smaller units. (*2*) Because they are more nearly in harmony with everyday usage. (*3*) Because no computation is performed with per cents.

18. What probably makes the formula method of solving percentage problems difficult for seventh-grade pupils? (*1*) Their inability to identify the three factors. (*2*) The lack of ability these pupils have in changing the formula to fit the situation. (*3*) The pupil's lack of ability with numbers.

19. Is the unitary analysis method applicable to Case I and Case II percentage problems? (*1*) Yes. (*2*) No.

20. Which procedure for solving percentage problems, the formula or the equation, gives most emphasis to understanding? (*1*) The formula. (*2*) The equation.

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9

Measures, Weights, and Scale Drawings

The Role of Measurement in Arithmetical Instruction

Casual examination of any series of elementary-school arithmetic books will reveal that measures and weights are used very extensively. The problems for illustrating new facts and processes nearly all involve some weights or measures (if counting is considered measurement). Practice problems and those showing the significance of facts and processes also utilize such data. Tables of measures and exercises for learning how to convert one type of measure to another are included in the textbooks. As well as these textbook uses of measure, practically all the out-of-arithmetic use of number which a pupil has involves measurement of some kind.

This extensive use of measurement in arithmetic instruction indicates how much the utilitarian aspect of numbers has dominated the role of the subject in the educational program. While this close association between measurement and teaching the processes of arithmetic has value, it should be recognized that the processes of arithmetic are independent of any form of measurement. This fact can be easily demonstrated by per-

forming any of the fundamental operations with whole numbers or fractions. Critical study of the preceding statements will result in the conclusion that while measures may be used to supply initial illustrative material for teaching the fundamentals of arithmetic, they actually play only a minor role and may sometimes even be detrimental by distracting attention from the main issue. At any rate, it can be said that a pupil has mastered arithmetic only after he is able to deal with numbers without reference to measures or weights of any kind.

The preceding paragraphs point to the importance of careful study of the role of measures in the arithmetical instruction program and to the importance of helping children acquire an understanding of measures and their uses. Since the problems used in the introductory phases of instruction are to illustrate situations where the new is used, it follows that these problems should not present any special difficulty. To understand measurement data in the problems, the pupil must already be familiar with measures.

Background Measurement Experiences

Fortunately for the instructional program, children begin to use some measures almost as soon as they begin to talk. The casual use of measures, however, does not always mean that a child understands the ideas that are basic to them. Dictionary definitions are of little value in formulating a precise definition of measurement applicable to its use in arithmetic. Measurement in this arithmetical sense is the application of numbers in determining the properties of objects, events, phenomena, and the like. For example, the light bulb may be rated 40 watts and have a guaranteed life of 500 hours. There may be 6 or a half-dozen such bulbs. They may cost 29¢ each and so on. In each instance numbers are used to give the magnitude of a property of the object or collection of objects.

Among the fundamental ideas of measurement is that of substitution. A child says, "the dog is as high as the wastebasket." In this example a measure near at hand is substituted for an actual magnitude of the dog. It should be noted that this substitution involves comparison or matching and is therefore only as accurate as the measurer's ability to compare. This is true of all measures except counting measures and therefore all measures (except counting) are subject to error.

The five examples of the use of measurement which follow illustrate some of the fundamental ideas of measurement and are also examples of good background measurement experiences for children.

1. Sometime after a kindergarten class had planted some seeds, they tried to find out whether all the seeds had come up. The children used a drawing record of the seed placement to count how many seeds they had planted originally. Then they counted the number of plants and compared their results. Note the comparison of ordinal numbers with the plants and the representation of seeds on the drawing record. The other obvious way to answer the question would have been the matching of plants with drawings. Certainly the matching of number names with the objects was more practical and for those who could count it was simpler. In this example of measurement discrete objects were compared by counting and therefore the usual errors made by the measurer's inability to compare do not occur. It should be recalled that counting is the only exact mathematical operation that man can perform on objects he perceives.

2. A class observing some rabbit tracks in the snow commented on how far the rabbit had jumped at one particular place. At the suggestion of the teacher, a stick was used in measuring the distance. When the group returned to the classroom, the distance was marked on the board by applying the stick the correct number of times. In this situation measure-

nent was used in duplicating an original object (distance) that was not transferable. This is an example of one of the frequent major uses of measures. Other examples of the need for this idea of measurement arise when trying to determine whether a rather large object will fit a certain space or go through a door.

3. The news item stated that the snake was 5 feet long. To show how long that is, a 5-foot length was marked on the floor. "This is how long the snake is," was the statement made after the measurement had been finished. The known magnitude of the snake was duplicated through use of measures, and by this procedure the pupils were able to get a good idea of the length of the snake.

4. "Our Thanksgiving turkey weighs 12 pounds," was the contribution of one child to the morning school-news period. In order to initiate some thinking about weights the teacher made this comment, "Do you think you can lift a twelve-pound turkey?" After some discussion the teacher suggested that they find some things in the room that weighed 12 pounds and try lifting them. A collection of books served the purpose. Here again measures were used to reproduce the magnitude of a quantity not immediately available.

5. A report being read included the statement, "It requires only 30 seconds to load each car with coal." The teacher said, "Let's have someone walk around the room for 30 seconds to get an idea of how long a time 30 seconds is. I'll time Sally with my watch. Before Sally starts, let's make a guess as to how many times she can walk around the room in 30 seconds." In this situation an amount of time was measured by comparing with a standard time interval (30 seconds as measured by a watch).

The five situations described above indicate the wide scope of measurement and give a suggestion of the role of instruction in helping children to acquire important ideas of measures and measurement. The experiences mentioned included one count-

ing, two linear-measure situations, one measure of weight, and one of time. With each of these types of measures the pupil needs numerous experiences. In addition, there are many other measures such as those pertaining to volume, area, value, angles, temperature, heat, and energy with which the pupil will deal. It should be recognized that in the limited time allotted to arithmetic instruction only a part of the needed background experience with measures can be provided for the pupil. It is therefore very important to provide some of the most essential experiences.

As is true in other phases of instruction, the school experiences with measures should be based on a pupil need or use. For example, a girl giving a report on Custer State Park said that a high fence kept the deer and antelope from straying away. She was asked by a classmate, "How high?" After some discussion the girl was asked to mark the exact height (8 ft.) on the wall of the classroom. Not every measure can be duplicated in the manner just described, but critical questions and discussion plus presentation of adequate data on measures can help clarify pupils' concepts of measures and should help to bring out the important understanding that measures are a means of giving some idea of the magnitude of objects, happenings, and other phenomena. To get these ideas of magnitude, substitutions are made — that is, measures are substituted for the object, event, or phenomenon, or comparisons are made with some known measure or measures.

Background number experiences should emphasize the aspects of measurement referred to in the preceding paragraphs. Stated briefly the important ideas that pupils need regarding measurement are (1) the idea of comparison, (2) the idea of substitution, and (3) a reasonable idea of various standard measures.

The first of these two ideas can probably be developed adequately through use of only a few measures but the third idea obviously requires experience of some kind with each measure

The procedures and exercises listed below are representative of the type of measurement situations which will assist pupils in acquiring good ideas of measurement.

1. Weighing objects in the room to find one that is equal to the weight of some object mentioned in a science or health report.
2. Marking on the board or wall the height of animals and objects frequently referred to in social studies, reading, news, and science.
3. Having pupils see how far they can walk in a minute.
4. Using liquid measures like the cup, pint, quart, and gallon in such activities as filling the aquarium, mixing soil (definite amounts of sand, clay, loam, leaf mold), cooking activities, determining capacity of such non-standard containers as gourds, cardboard boxes, and pails, and finding out the amount of milk produced daily by a milk cow or a milk goat.
5. Identifying periods of time by making and displaying records of an important happening for each day of a week or a month.
6. Weighing the amount of food consumed daily or weekly by pets or animals kept for observation.
7. Making careful records of the size of such things as the length of a robin's wing flight feathers, the distance between a cat's footprints, the new growth on tree branches, the weight of ears of corn, or the amount of grass produced on a small plot (e.g., 2 ft. by 2 ft.).
8. Engaging in numbers projects such as determining the direction and distance of the flight of nectar gatherers in an observation beehive. (See numbers project in problem-solving chapter, page 197.)
9. Preparing and using a set of reference measures. (See reference measure section in this chapter, page 286.)
10. Determining the magnitude of objects by use of non-standard measures. For example, the distance between points may be determined by use of a stick; the weight of an object may be determined by balancing it with an equal weight of

common objects such as a number of books; and time may be measured by means of a pendulum.

11. Studying some old ways of measuring. (See references on page 382.)

12. Determining the correct measure to use. For example, exercises may require a pupil to choose the measure to be used in giving the area of a state, the length of a telephone pole, the weight of a diamond, and so on.

The preceding list of suggestions indicates that an experience program with measures is by no means confined to the primary grades. Experiences and study exercises to get the meaning of some measures such as ton miles or acre feet are appropriate for pupils in Grades Seven and Eight.

Instruction in Use of Tables of Measures

The inclusion of tables of measures in arithmetic books has been standard practice for over 100 years. Memorizing such tables was once an important part of instruction in arithmetic. Today we do not memorize tables, but we do make use of them in solving the problems in arithmetic books, and often people use the facts from tables in changing data from one measure to a larger or smaller measure. For example, to divide 2 pounds and 7 ounces into 3 equal parts, the simplest procedure is to change the pounds to ounces and then divide by 3.

In teaching pupils to use tables of measure, problems which indicate such plausible uses for changing measures as the one involving pounds and ounces described above should be used as introductory material. Since few people outside of school will have arithmetic books for reference, pupils should learn to use other sources than the arithmetic book. In view of the fact that a dictionary is usually available, pupils should have the experience in school of consulting this reference in changing measures. To insure familiarity with the tables in the dictionary

exercises which specifically require their use should be part of the program of instruction in the upper grades. The preceding statements indicate that the major emphasis on tables of measure will be teaching the use of those found either in the arithmetic book or in the dictionary. This use and the frequent need for changing measures in everyday life will point to the advantage of knowing certain equivalent measures.

One fifth-grade class which realized this advantage identified some of the most common measures and prepared a chart which was headed "Measures We Should Know." They then discussed methods of studying and engaged in study of the facts on the chart.

The Metric System of Weights and Measures

The weights and measures used in arithmetic books are almost entirely those of the English system because these are the ones in common usage. There is, however, in the United States another system of weights and measures. The metric system became legal in this country in 1866. Today that system is used in nearly all scientific work and is the system of weights and measures in many other countries of the world. The important role of science in the education of children and the many contacts our citizens have with countries where the metric system is used point to the desirability of using some metric measures in arithmetic. Any use of metric measures will necessitate presentation of some of the most common metric measures and the relationships between common metric and English measures.

The recommendation that arithmetic instruction include metric measures does not mean that the writer believes that the metric system is superior to the English system or that it will replace the latter. In our modern technical society both systems are so well entrenched that neither is likely to conquer the other.

As stated by one authority, "These two systems that today dominate so completely the industrial world will continue, side by side, as the giants of the world of tomorrow. Neither has been able to absorb the other — a situation that will not change in the foreseeable future." [1]

The student of arithmetic teaching should not conclude that we as a people are keeping the English system simply because it happened to be the one our forefathers were using when our country was founded. While the metric system has greater over-all simplicity it is in some instances not very practical. For example, the kilogram would be a rather awkward weight to use for such things as butter, and the liter, a close equivalent of a quart, could not easily supplant our gallon. The centimeter, decimeter, and the meter are not to the liking of the man of the street. He often thinks in terms of a measure of length roughly one-sixth of an arm span, a foot. There is no such measure in the metric system.

The English measures developed from practical measures in common usage for hundreds of years. If the most common English measures were decimalized (as were the original Anglo-Saxon measures), they would be far simpler to handle and some of the advantages of the metric system would disappear. Such decimalization of English measures has taken place in the metal industries where the micrometer and gauge blocks are used to measure thousandths of an inch.

Some proponents of the use of the metric system argue that its use would make for a better understanding of the number system. Such claims are not based on fact. It is only after the number system is understood that the relationships found between measures of the metric system appear significant. Measures do not bring out number ideas or number relationships very effectively. On the other hand, number ideas and number re-

[1] Keith Gordon Irwin, "Fathoms and Feet Acres and Tons: An Appraisal," *Scientific Monthly* (January, 1951), pp. 9–17.

lationships are indispensable to the efficient use of measures. The metric system with its use of decimalization to show the relationship between measures and use of the same unit throughout the range of measures with only decimal prefixes which are common to all measures is a good illustration of the simplicity that results when organization or system is intelligently used.

Teaching How to Find Averages

Effective thinking where many measures are involved requires the use of a representative measure. The three representative measures in common use are the arithmetic average or *mean*, the *median* or middle measure, and the *mode* or most frequent measure. In reading, in everyday affairs, and in school work children have experience with all three of the representative measures. The three measures as generally used are all called average. This is unfortunate, for arithmetic instruction usually deals only with the mean or arithmetic average, and for the most part the teaching of even arithmetic averages is a mechanical operation without much attention given to understanding of the concept involved. If the emphasis on understanding now advocated for other phases of arithmetic is applied to the teaching of averages, some marked changes in present textbook procedures will have to be made. A brief description of some of the procedures used in teaching averages in a fourth grade is given in the section that follows. Study of the procedures and comparison with lessons on this phase of arithmetic in textbooks will identify the most important issues with regard to teaching of averages.

The topic of representative measures was introduced in this fourth grade by consideration of the following exercises on a study sheet.

1. The Rocky Mountain trees grow slowly. The usual growth

in height is only about 8 inches a year. What word could have been used instead of the word usual in this statement?

2. Antelope can jump ordinary fences with ease. How high is an ordinary fence? How would you go about finding the height of ordinary fences?

3. The average pygmy is 4 feet and 5 inches in height and weighs 80 pounds. What does the word average in this statement mean?

4. Bob claimed that he was a better fisherman than Jim. On the four days that he fished Bob caught 8, 9, 6, and 2 fish. On the three days Jim fished he caught 6, 8, and 5 fish. On the average which boy had the better record? (To find the average, you can make a drawing and even up the amounts caught on different days.)

5. Mary and Alice had a contest to see who was better at throwing darts. On Mary's 5 darts she made these scores; 2, 4, 5, 1, and 2. On Alice's first two throws she made a 2 and a 4. She said, "If I keep that up I'll have a better average than you had." What was Mary's average? What was Alice's average on her first two throws? Make a drawing to show both averages.

6. Mr. Jones sells Christmas trees at his store. When he opened a bundle of trees marked "5 feet," he placed the six trees by a wall which had heights marked on it. The trees measured 5 ft. 4 in., 4 ft. 8 in., 4 ft. 9 in., 4 ft. 9 in., 5 ft. 1 in., and 4 ft. 11 in. Find the average by using a drawing to even up these lengths.

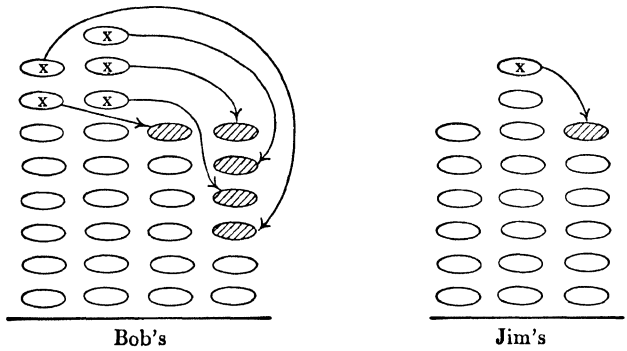
7. Use numbers to find the average of these lengths; (*a*) 12 miles, 9 miles, 13 miles; (*b*) 15 miles, 11 miles, 12 miles.

8. Try to find the average in examples 4 and 5 by using only numbers.

The pupils decided that "usual" in exercise 1 meant average or ordinary and that ordinary meant usual or average. They figured out that you find the height of ordinary fences by measuring a number of them and taking either the most common height, the middle height or an imaginary height gotten by

evening them all up. It should be noted that in this case the mode, median, and mean are all suggested. In discussing the third question the idea of evening the weights seemed most satisfactory.

To develop this idea of evening up measures, a basic idea of the mean, the teacher directed the attention of the class to exercise 4 on the study sheet. She suggested that they all take a little time to try a solution individually. As the pupils worked the teacher sketched on the board the following drawing:



In the discussion it was agreed that since Jim's average was $6\frac{1}{3}$ and Bob's only 6, Jim was the winner.

The pupils next engaged in a work period in which each pupil attempted examples 5, 6, 7, and 8. From a consideration of example 8 a rule for finding averages was developed, and then the textbook was consulted for verification of the procedure for finding averages and for practice exercises.

The idea of the middle or median was later developed in this fourth grade with a discussion of the following exercise:

1. To find the height of a representative fourth-grade boy quickly, the teacher lined up the 11 boys in the class with the tallest first, then the next tallest, and so on. He then selected the sixth boy and measured his height. The sixth boy's height

was 55 inches. If the teacher had found the average of all the heights, would it have been 55 inches?

2. The actual heights of the 10 boys were: 62 inches, 60 inches, 56 inches, 56 inches, 55 inches, 52 inches, 52 inches, 52 inches, 51 inches, and 50 inches. What was the average height of the 10 boys?

3. The median of the height of the girls and of the boys in the class was then found by the method of example 1. Then, the average was found from the measures of the individual heights.

From an evaluation of these experiences it was concluded that a middle measure was a good representative measure but not the same as the average.

Whether the idea of the median should be presented soon after the meaning and computation of average is taught is doubtful. The fact that fourth-grade pupils used average and median interchangeably is argument for teaching the median early. The difficulty of the idea is argument for postponing its teaching to later grades.

The study of averages (and medians if these are included in the course of study) will be improved by use of such critical study questions as (1) Is the average always smaller than the largest measure? (2) How can the average be found without dividing? (3) Is the average the same as the middle measure? (4) For which of these measures would the average not be used: (*a*) post lengths, (*b*) weight of butter, (*c*) number of words on pages of a book?

Since averages are introduced in the fourth grade and then re-presented in the fifth, six, and even seventh and eighth grades, the topic like some other phases of arithmetic may appear in the upper grades to be just "the same old stuff" to the pupils. Special attention should therefore be given to locating challenging new situations for reintroducing the topic in the upper grades.

Measurement of Area

To most students of arithmetic, measurement of area means multiplying the number of units in the length by the number of units in the width and obtaining a product which is the square of the linear unit used. Far too few students realize that finding the area of a surface means finding how many units of a given size are needed to cover the area. To get this basic idea of measuring area, pupils should engage in finding the surface area of several figures by actually covering the surfaces with the measure. The idea of covering surface will probably be acquired more easily if some measure other than a square is used. Erasers, books, and sheets of notebook paper are examples of non-square measures that have been used.

It is rather difficult to think of a logical situation requiring the measuring of a surface with an eraser or with a book. A better setting can probably be provided by using some ancient measure such as the grain of barley, the hand, or others. The situation below, which was used by one teacher to introduce square measure in a sixth grade, illustrates the suggestion.

"In reading a story of the ancient Saxons, I found a description of a very strange measure. A Saxon chief whose people did no weaving traded with other tribes for what little cloth they used. This chief wanted a piece of cloth to make a sort of shoulder robe. The chief, not trusting the people with whom he traded, never went into their villages but always sent one of his servants. Before he sent the servant, the chief said, 'Measure my old robe; then ask what they want for a piece of cloth that is six hands larger than my old one.'

"The servant measured the old robe with his hand and found it contained 60 hands.

"What do you suppose these ancient Saxons meant by a hand?" the teacher asked. After some discussion and several suggestions the teacher said, "Why don't you see if you can tell

how large your desk top is by seeing how many times your hand is needed to cover it?"

From this lead the children tried out various possible sizes for the king's robe on the blackboard. It was decided that the robe was probably 3 hands high and 20 hands long. No mention was made at this time of the fact that you can find the total by multiplying the number of units used to cover one side by the number of units needed to cover the adjacent side. In fact, teachers should delay use of length times width (which is, strictly speaking, an error) until the pupils have found the area of several rectangular surfaces by applying a measure the necessary number of times. Special attention is called to the phrase "number of times." If the pupil sees that he has applied a book four times along one edge of the desk and three times along another edge, he is not likely to think that he is multiplying 4 books by 3 books but that he is multiplying 4 (the number of times the book was used to cover one dimension) by 3 (the number of times used to cover the other dimension). Furthermore, this use of a non-square measure like a book may enable the pupil to see that it is the original measure used, and not the multiplying of measures together, which gives square measure.

The importance of taking adequate time in the developmental or introductory period of teaching measurement of area can hardly be overemphasized. The harm that can result from too early computation in arithmetic is exemplified in the teaching of area. Almost all fifth- and sixth-grade pupils can be taught to find the square measure answers to area situations, but, as has been indicated earlier, very few of them possess much understanding of the procedure they use. As Godfrey and Siddons put it, this is one subject "on which it is fatally easy to get a class to work examples without much understanding." [1]

[1] Charles Godfrey and A. W. Siddons, *The Teaching of Elementary Mathematics* (London: Cambridge University Press, 1946), p. 125.

After the use of introductory procedures emphasizing understanding, the practice examples and problems of textbooks can play their important roles. As a check on understanding, pupils may on occasion be asked to show by means of drawings that their answers are correct.

Reference Measures

The extensive use of measures in arithmetic textbooks was pointed out in an earlier section of this chapter. Not only are measures used extensively in arithmetic work but the usual classroom social and scientific subject matter plus many everyday life situations abound in the use of measures. Typical of the out-of-arithmetic uses of number are these statements: "The overshoes weigh 3 pounds;" "The ore is covered with about 70 feet of loose earth and stone;" and "The area of the river watershed is 11,285 square miles."

Merely knowing the fact that 16 ounces equal 1 pound, or that a square mile is equal to 640 acres does not provide a child with the essential information (concepts) for a real understanding of the statements listed. The person who interprets "the 11,285 square miles" most effectively compares or secures a simple expression of the relationship between the 11,285 square miles and the number of square miles in an area, such as the area of his own state, with which he is fairly familiar. This known measure with which comparisons are made becomes a sort of reference measure. In order to interpret effectively the many measures used in the problems in arithmetic and in other school subjects and activities, pupils need reference measures.

A reference measure, as has been stated in the preceding paragraph, is one with which a pupil is familiar, one about which he knows a great deal. To provide the knowledge which will make references familiar and to show pupils how reference measures can be used should be an important part of the teach-

ing of measures. The following sixth-grade lesson is an example of teaching the use of reference measures to a group which had already had some experience with using reference measures of different kinds.

"When we were reading about Canada in social studies yesterday," the teacher said, "some of you were not too sure how to use your state as a reference area in getting an idea of the size of some Canadian provinces. Today in arithmetic let's spend the first part of the period using our state as a reference. Do as many of the exercises on the study sheet as you can."

A part of the reference measure study sheet is reproduced here:

Use the area of Iowa (50,000 sq. mi.) in telling how large these states, provinces and lakes are. The first one has been done.

1. Nova Scotia — 20,743 sq. mi. Iowa is about $2\frac{1}{2}$ times as large as Nova Scotia. Nova Scotia is about $\frac{2}{5}$ as large as Iowa.
2. Lake Superior — 43,360 sq. mi.
3. Lake Erie — 9,940 sq. mi.
4. Wyoming — 97,914 sq. mi.
5. Puerto Rico — 3,435 sq. mi.
6. Quebec — 523,524 sq. mi.
7. Manitoba — 219,723 sq. mi.

Use other reference measures in describing the quantities in example 8.

8. About a half mile
9. A 5-acre field
10. A 20-lb. turkey
11. 50 miles from here
12. A seven-foot fence
13. A 6,000-acre wheat field
14. An 18-ton truck load of wheat.

Engaging in such exercises as those indicated on the reference measure study sheet not only gives excellent practice in getting the meaning of measures that are frequently encountered, but provides opportunity to use numbers in making comparisons. Making of comparisons like these by division or by use of fractions is using arithmetic as those who think efficiently with numbers frequently do.

The study sheet directions implied that this class had a set of reference measures. A list of the measures taken from the arithmetic guide of the school appears below.[1]

REFERENCE MEASURES TO BE DEVELOPED IN EACH GRADE

Grade One

1. One pound. To be developed from pound of butter but a one-pound book in the room is to be designated for frequent use.
2. Ten pounds — a small oak chair in room.
3. One minute — the time required to count from 1 to 60.
4. One hundred and one thousand — the number line, number chart and the thousands board.

Grade Two

1. All references from Grade One.
2. Seven feet — the height of the classroom door.
3. One inch — the length from the index finger tip to the second joint.
4. Fifty pounds — the weight of a bushel of apples.

Grade Three

1. All references from the preceding grades.
2. An acre — a plot near the schoolroom measured by the class (a square 208 feet on the side and another 100 by 435 ft.)
3. A mile — the distance from School to the Benton Street bridge.

[1] *Arithmetic Guide*, University Elementary School. Iowa City, Iowa, Unpublished.

4. Ten miles — the distance from Iowa City to West Branch.
5. A bushel — a bushel apple basket.

Grade Four

1. All references from the preceding grades.
2. A square mile — the area bounded by the following streets: Davenport, Capitol, Benton, and Grant-Parsons.
3. One hundred twenty miles — the distance from Iowa City to Des Moines.
4. Thirty inches — the height of the teacher's desk, the door knob, or dining table.

Grade Five

1. All references from the preceding grades.
2. Seventy feet vertical — the height of the west wall of the University School Building.
3. Four tons — the amount of coal the local coal delivery trucks hold; fifty tons — the amount of coal held by large railroad coal cars.
4. Fifty thousand square miles — the area of Iowa. (To permit ease of computation the exact area of 56,147 sq. mi. is rounded to 50,000 square miles.)
5. A city of 25,000 people — Iowa City.

Grade Six

1. All references from the preceding grades.
2. A city of 200,000 people — Des Moines. (The 1950 census figure of 179,000 is rounded to 200,000 to make for easy computation.)
3. The population of Iowa — 2,500,000 people.
4. Nine thousand miles of railroad — the number of miles of railroad in Iowa.
5. Fourteen thousand feet elevation — Pikes Peak.
6. Ten thousand tons — the capacity of a large freighter or Great Lakes ore boat (the equivalent of approximately six freight trains of fifty cars each).

Even casual examination of this list of reference measures shows that it is designed for one school and therefore cannot

be used without modification in other schools. Each school has the task of selecting reference measures to be used. In general, the same type as those given in the preceding list should be included.

Identification of the procedures to use in developing an adequate knowledge of the various reference measures is an important task for students of arithmetic teaching. In keeping with the general principles of teaching, each new reference measure should be introduced through situations which show a need for knowledge of the measure. Some illustrations of the introduction of reference measures are given in the section below.

A need for knowing the distance one mile was created by reading this statement in a book of pioneer experiences: "The large hawk with the chicken in its claws flew swiftly away. The boys watched and saw it light on an old dead tree about a mile away."

The teacher asked, "Do you think the boys could see a hawk light on a tree that was a mile away?" In the discussion that followed, the teacher said, "I believe the field house is about a mile from here. Do you think you can see a hawk that far away?" The class agreed that they could see a hawk that far away. However, because they weren't sure of the distance and because they often came upon references to "a mile" in pioneer stories and reports, the children decided to find a point one mile from school that might be used as a reference. A committee was appointed to use the city map to find points a mile away, while volunteers were asked to check with the family car the distance of some points already suggested.

The ten-thousand-ton reference measure for sixth grade was introduced by pictures of ships on the bulletin board. Underneath the picture of a 10,000-ton oil tanker was the question, "What reference can you use in getting an idea of the amount of oil this tanker holds?"

After this and similar pictures had been on the bulletin board for several days, the teacher said, "I've seen some of you looking at the pictures of ships on the bulletin board. What suggestions do some of you have for getting an idea of the capacity of the tanker pictured?"

The need for the railroad reference was created when a pupil giving a report on Uruguay stated that there were 1,650 miles of railroad. The teacher asked, "How does that compare with the number of miles of railroad in our state?" When it was learned that Iowa had over 9,000 miles of railroad and wasn't as large as Uruguay, the 1,650 miles for that country meant more to the pupils than it had before the comparison was made. As a part of the discussion, a railroad map of Iowa was examined with attention given to such questions as "Where would you go in Iowa if you wanted to be farthest from a railroad?"

The student of arithmetic teaching should note that these reference measures are primarily a part of the social studies program. It might, therefore, be argued that social studies and not arithmetic time should be used to develop the references. There cannot be any objection to the development of references in social studies, except that it is not being done. If, then, arithmetic is to be of maximum service to the children, such references will have to be worked out as a part of the arithmetic program. After reference measures have been developed, they provide an excellent means of arithmetical experiences. The study sheet given earlier is one example of such experiences. Other good uses of reference measures lie in oral exercises, review questions, and in arithmetic uses in other subjects.

Unusual Measures

Modern technology has introduced to society many new and unusual measures. While arithmetic instruction does not have to assume the task of acquainting pupils with every measure,

good opportunities for teaching the fundamental ideas of measurement are passed by if some of the new ones are not used. For example, a problem concerned with the number of acre feet of water in a reservoir may give the pupil new insight regarding the acre and measures of capacity. Such a problem would often have more interest appeal to a seventh-grade pupil than another gallon problem of the type already presented in preceding grades. The occasional use in problems of the special measures of modern society tends to enhance the prestige of arithmetic. The student of arithmetic teaching should note that the argument for including some of the new measures is based on the contribution they make to instruction in arithmetic. The presentation of new measures as a matter of presenting new social knowledge is hardly sufficient to warrant use of arithmetic time, although that argument is sometimes used for including the new. In other words, it is recommended that if the teaching of new measures makes no contribution to arithmetical instruction then the new measures should not be taught.

Representative examples of modern measures which might be of value in arithmetic teaching are found in the following list. (1) One million passenger miles without an accident. (2) The ton mile cost of freight moved was 8 cents by rail and 2 cents by water. (3) A 5-horsepower electric motor. (4) Best quality 15-denier 60-gauge hosiery. (5) The price of this remarkable roofing is only $8 a square. (6) The furnace has a 90,000 B.T.U. rating. (7) A distance of 6 kilometers. (8) The case is 14-carat gold. (9) A $\frac{1}{4}$- and two $\frac{1}{16}$-carat diamonds are in the ring. (10) A 32-caliber bullet. (11) The water costs $1.20 an acre foot. (12) The sun sets at 20° north of west.

While not new or unusual, the measures of time give pupils a great deal of difficulty. Beginning in the kindergarten, projects and activities which require keeping records of time will help to eliminate some of the difficulties. Included in these projects should be the construction and use of time or date lines (see

projects, Chapter 6). Since nearly everyone has to give his age in years and months in filling out blanks, the arithmetic program might well include work in this area.

In view of the fact that society very much needs an improved calendar, one in which the important dates do not wander throughout the week, it might be wise to give some attention in arithmetic classes in the upper grades to the study of the World Calendar.

The fact that our Christian Calendar has no zero year is not recognized in the explanations given in many children's books. Teachers need to correct that error of books and should not as some books do, compare years B.C. and A.D. to below-zero and above-zero readings on a thermometer.

Scale Drawings and Graphs

The teaching of scale drawings, which is usually begun in the fifth grade, receives considerable attention in Grades Six, Seven, and Eight. The pupil's first experience with this phase of arithmetic usually consists of reading map scales. Since the pupil's previous work in geography will have given him some experience with maps, this introduction is a good one from the standpoint of familiarity. Following the reading of scales of miles, the reading of floor plans or drawings is often considered. Unfortunately, these first arithmetic lessons or experiences with a scale of miles are often just an arithmetic exercise without challenge. Mere use of a situation with which the pupil might be expected to be familiar does not guarantee a good learning situation. There may be no interest on the part of pupils or the pupil may not be challenged. Therefore, here, as in the teaching of other new phases of arithmetic, an effort should be made to create a real problem or learning situation in which the pupil has genuine interest or finds a challenge. The setting used generally determines the degree of success attained.

At this time it seems pertinent to consider the question, "What are the basic ideas relative to scale drawing and graphs that are to be taught?" The idea that a small measure can represent a large one is, of course, the underlying idea of scale drawings and graphs. For example, in scale drawings an inch can represent a mile; in graphs, a picture of a bushel basket can represent 1,000 bushels, and so on. In a sense this substitution of a single small measure or a single symbol for many measures or symbols is similar to what is used in our number system when we think of 100 ones as a single hundred or when we think of 100 tens or 1,000 ones as a single thousand. It is a means of simplification to permit easier grasp by the mind of the quantities represented.

What exercises with scale drawings or, for that matter, what exercises with graphs will help to develop the idea that small unit measures can represent accurately large unit measures? Is use of scale of miles on maps the best way to develop the basic idea of scale drawings?

While use of scales on maps and floor plans will give experience with the basic scale idea, it is believed that a more profitable first experience might involve substitution of a small measure where large ones were used. One such situation is described below.

Introduction to Scale Drawings

The teacher began this lesson by telling how she had observed a carpenter laying out the corners for a new building. To get square corners, the carpenter had used three boards, one 6 feet, one 8 feet, and one 10 feet long. The teacher then showed how the carpenter put the 8-foot board from the corner along one line, moved the other line until the three boards formed a triangle. The carpenter said, "When the boards fit like this, the corner is square." The teacher suggested that the pupils check

the carpenter's method, but because there wouldn't be room for 6-, 8-, and 10-foot boards, it was suggested that 6-, 8-, and 10-inch strips of paper be used and that lines on paper be used instead of the carpenter's strings.

The squareness of the corners was tested with the corner of a sheet of notepaper.

While the situation used in the above lesson may not involve great usefulness, it has elements which challenge; there is something for pupils to do and, of course, the use of inches for feet is a good example of scale drawing.

As inferred in the comment on the illustrative lesson, it is very important that the pupil have something actually to do. For that reason such projects as constructing maps of the playground, the community, or historical sites are considered worth while.

The suggestion that work other than reading of scale drawings of floor plans or maps be used in the introductory exercises does not mean that interpretation of floor plans and maps has no place in such instruction. This type of exercise is excellent for providing practice and study. Even the teaching of this phase of scale drawings will be improved if the introductory exercises are such that they enlist active pupil participation. A need or use to the pupil for reading the scale on maps should be evident. An attempt to provide a challenging introduction to the reading of map scales is shown in the exercise below.

A work sheet headed "A Treasure Hunt Map" was given to each pupil of the class. The directions stated that two treasures were hidden on the school ground. At each numbered place on the two treasure trails a part of each treasure was to be found. The scale was $\frac{1}{16}$ inch equals 1 step. Members of the class were divided into two teams. Each team was assigned a treasure trail and told to write on the map the distance in steps between each point on the trail. After both teams had reached agreement on distances, a committee of four from each

team was chosen to do the hunting. Other members of the teams watched. The first team finding all parts of the treasure was to be the winner.

In the teaching of graphs the same general principles that were suggested for teaching scale drawings are recommended. It is not as difficult to find situations where the pupils will have something to do with graphs as it is in the case of scale drawings.

The construction and reading of graphs and charts are integral parts of good social studies and science programs in elementary schools. Where such programs are in practice, the arithmetical work concerned with graphs and charts is very simple. If the arithmetic program has to carry the full load of instruction, then, careful attention needs to be given to the selection of situations which first involve a use for the reading or construction of graphs and charts.

History of Measures

The major role of the history of measures in arithmetic teaching is discussed in Chapter 14. However, because of the amount of space given to historical sketches of certain measures in upper-grade books, the student of arithmetic teaching should consider a few important facts concerning history of measures. In the first place, it should be recognized that the brief sketches of measures that are found in pupils' books consist primarily of an artist's drawing of an ancient scene and can at best be only introductions. In fact they are often misleading. As a result of an artist's drawing, many people now believe that the instrument used in the construction of the pyramids of Egypt for measuring length was man's forearm. With such a crude measuring instrument the remarkable agreement of dimensions would have been impossible. Another misconception arising from the brief and practically erroneous historical sketches of early measures is the belief that many measures are the result

of some arrogant but ignorant ruler's decree. It is true that the edicts of rulers did lead to standardization but those edicts were not based on whims. All of our old measures have a long history and were in use perhaps thousands of years before any king was troubled enough by variations in measures to undertake standardization. If there were space in arithmetic books to present a fair picture of the conditions which led to royal edicts regarding measures, the inclusion of such material might serve a more useful purpose. Current historical literature also contains much regarding measures that is more mythology and fiction than fact. These conditions plus the absence of adequate teaching materials should be a warning to teachers to go slowly in this field. It should also be a challenge to study the field in order to provide pupils with worth-while factual materials.

STUDY QUESTIONS

1. Is the use of measures essential to the teaching of arithmetical processes? (*1*) Yes. (*2*) No.

2. With about how many different measures (including weights) does the elementary-school pupil have experience? (*1*) 4. (*2*) 10. (*3*) 18.

3. To give the child a good grasp of the idea of substitution in measurement what type of measures: (*a*) standard (*b*) unstandardized, are better? (*1*) *a*. (*2*) *b*.

4. Should tables of measures be included in arithmetic books? (*1*) Yes. (*2*) No.

5. Is the metric system likely to replace the English system in this country? (*1*) Yes. (*2*) No.

6. Are metric measures in most instances more practical than the corresponding English measures? (*1*) Yes. (*2*) No.

7. What is the major idea for pupils to learn regarding the finding of averages? (*1*) That the average can be found by

dividing. (*2*) That the average is always less than the maximum measure and more than the minimum. (*3*) That the average can be found by evening up the measures.

8. To find the area of a rectangular surface that is 2 feet wide and 3 feet long is it correct to say that you multiply 2 feet by 3 feet to get the area, 6 square feet? (*1*) Yes. (*2*) No.

9. About how many reference measures are recommended for development in each grade? (*1*) 2. (*2*) 4. (*3*) 10.

10. What is the basic idea to be developed in the study of graphs and scale drawings? (*1*) That all such measures are inaccurate. (*2*) That the scale or base must be determined accurately. (*3*) That a small measure or symbol can represent a large one.

11. Is the use of artists' drawings of ancient measures and ways of measuring in pupils' textbooks always a desirable practice? (*1*) Yes. (*2*) No.

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10

The Course of Study in Arithmetic

Implications of the Term "Course of Study"

To the classroom teacher the term *course of study* means a guide to the instructional procedures to be used in each particular grade. It contains an outline of the subject matter to be taught, the order of presentation, the major objectives, some statements regarding methods, tests, and the like. The term *course of study* as used in this chapter refers as well to the procedures and issues involved in the construction of such guides and other materials which are of assistance in arithmetical instruction. It is broadly used therefore to include all curricular problems in arithmetic.

In the current literature on arithmetic the terms *course of study, curriculum, handbook,* and *guide* are all used in reference to curricular problems. It is difficult to make any statement regarding differences between the various names. Where the term *arithmetic curriculum* is used, a broad treatment of instructional problems is usually found. Courses of study tend to deal primarily with gradation of content, with brief treatments of philosophy, specific methods, and the like. The terms *handbook* and *guides* usually refer to less inclusive materials which emphasize instruction in particular parts of arithmetic. Such emphasis is not accidental but due to the attention that the particular school has focused on these aspects of the subject.

The intent of all arithmetic courses of study or guides is to facilitate instruction in the subject. There are some few educators who believe that learning of arithmetic is best in programs of instruction where arithmetic is not taught as a separate subject. They would therefore prefer that the arithmetic materials be a part of a broad curriculum. While the student of arithmetic teaching should consider the claims of the proponents of integrated curricula, there is not much in the way of evidence to support their claims. In the absence of evidence, it seems wisest to follow the recommendation of leaders in the field of arithmetic teaching. The position of one group of leaders in the field is well expressed by Horn in the following statement:

It is the unanimous opinion of the present committee, however, that after a careful appraisal of such evidence as does exist, and after giving full credit to what has been or is likely to be accomplished under integrated plans, such plans by themselves cannot be depended upon to develop arithmetical concepts and abilities to levels and scope required in life. An especially designed program of instruction is essential, and such a program should include not only provision for systematic and meaningful learning in the arithmetic class but also attention to the mathematical needs and contributions of other areas.[1]

Courses of Study in Use

Many state departments of public instruction, many county or district school systems, and many city systems have prepared course of study materials in arithmetic. Instead of such course of study materials, many school systems have brief statements explaining how the textbook is to serve as the course of study and still others, perhaps a majority of classrooms, just use the textbook as the course of study.

[1] Ernest Horn, "The Teaching of Arithmetic," *Fiftieth Yearbook* of the National Society for the Study of Education (Chicago: University of Chicago Press, 1951), Part II, p. 18.

The use of the textbook as the course of study has some merit. The books have been prepared by people who have given much thought to the subject. The gradation of content in books, while open to some serious criticisms, is certainly well done. With the exception of spelling, arithmetic is the elementary-school subject where the text can best serve as the course of study. A good text series and the accompanying manuals would in most situations be a more satisfactory guide than one which a teacher committee could prepare in one or two years of work.

The preceding statement does not indicate indorsement of the use of the text as the course of study. It does, however, call attention to the possibility of a combination of textbook and other course-of-study material. Before giving serious consideration to extensive use of textbooks as courses of study, some of the limitations of texts for such a role should be identified. Because texts are made to sell throughout the United States and in some instances even outside the United States, much more is included in the way of topics for each grade than the authors would be likely to recommend for any one school. Some material on the fringe topics (topics not definitely agreed upon as belonging in a certain grade) is placed in two grades in order that the series may be suitable for adoption in cities which call for inclusion of such topics in different years. A second serious limitation of textbooks as courses of study is the fact that verbal problem material included is general. Therefore, schools using only the text cannot make much use of the everyday activities of the pupils as sources of verbal problem material. Textbooks cannot express the point of view of the local school relative to the role of arithmetic in other curricular areas. Textbooks leave much to the teacher regarding procedures to use. If there is no other course of study material, the teacher will not have adequate guidance. From the limitations named it should be clear that using the textbook by itself as a course of study is not very desirable.

At the other extreme in curricular material from the textbook are the very detailed courses of study or curricular guides. These more detailed courses of study contain much excellent material such as a philosophy of arithmetic instruction, specific methods to be used, testing suggestions and recommendations, ideas for inclusion of items from everyday classroom activities, suggested projects, descriptions of specific teaching situations, and the like. A marked disadvantage of nearly all the more extensive courses of study is their complete separation from any textbook. As a result, much of what is in the course of study parallels, even duplicates what is in the textbook. This lack of coordination between textbook and course of study causes uncertainty on the part of a beginning teacher, the one who uses the course of study most. It should be recognized that there are good reasons for not coordinating courses of study and textbooks. State courses of study made to serve all schools could hardly refer to a single text series. City courses of study are made to be used for a long period of time. If the course of study were coordinated with one text series then there would be no possibility of getting a new text series until the course of study was changed.

Course-of-study material which falls between the extremes of the textbook and the detailed courses of study is found in the brief handbooks, courses of study and guides which are designed to be only a part of the curricular materials used. Since these rather limited curricular guides obviously require less time and effort to produce, they can be revised much more easily than the extensive detailed courses of study. Changes can be made in such guides with relative ease and they are not so large that they appear forbidding to teachers. Their chief limitations are the inadequate coverage of the field and the generalities of little value that may be included when too little space is allotted a topic.

Working on Course-of-Study Materials

The construction or revision of courses of study (usually the broader term *curriculum* is used) is an important educational enterprise. Since there are available a number of books dealing exclusively with this phase of education, no attempt will be made to discuss here in detail the important issues and procedures involved. A survey of the literature on curriculum making indicates that the most desirable type of organization for curriculum construction requires a specialist in the field supervising committees composed of members of the teaching and administrative staff who do the major part of the work.

Having a curriculum director who is concerned with all phases of instruction and not just one subject field such as arithmetic is highly desirable. A lack of balance in the curricular offering may be detrimental to the pupils and therefore indirectly detrimental to the areas of the curriculum that are over-emphasized. It does not necessarily follow that the curriculum specialist is the best director to secure good balance. Elementary-school supervisors, superintendents in charge of instruction in the elementary division of the school, and elementary-school principals by the very nature of their work are in some ways better able to give sound direction to curriculum construction work than are curriculum specialists. It should therefore not be assumed that superior curricular work requires the presence of a full-time curriculum specialist. The important point is the presence of a person with authority to direct the work, a person who knows the total curricular pattern, and above all, a person who is willing to do enough studying of the specific fields for which materials are being prepared to know those fields. When the curriculum director or chairman of a curriculum committee cannot adequately evaluate materials because he does not know the field well enough, the quality of curricular material produced is often inferior. There is no place in education for a specialist

so immersed in the techniques of curriculum making that he does not know or is unwilling to learn the fundamentals of the subjects with which the curriculum deals.

Besides the director, the production committee should consist of members of the teaching and supervisory staff. The committee size will depend upon the way the work is to be distributed. It is desirable to give the teachers in the system a feeling that they are participating in making the course of study, but at the same time they can hardly be burdened with large responsibilities for research, writing and editing on top of their regular classroom duties. Since therefore the ideal of a large committee devoting much time and energy to the project is seldom feasible, a small working group which can assume the major part of the job is probably preferable. This group can be supplemented by a larger committee which gathers material, tries out ideas and procedures and helps to make decisions. The procedures would be somewhat as follows:

An initial bulletin to the teachers explains that arithmetic course-of-study materials are to be developed. It names the director of the work and makes clear that the teachers' assistance will be needed and welcome. The director with the help of a few teachers and perhaps supervisory staff members prepares a tentative work plan and course of study to be used as a basis of discussion in teachers meetings at the various schools in the system and in a meeting of the arithmetic course-of-study committee. The director has the major responsibility for preparing the work plan. The assistance supplied by others is primarily advisory, consisting of critical analysis of his suggestions. In this way the director has an opportunity to discuss various proposals before submitting them to the full committee or to all teachers.

Following the study of these first proposals, specific small assignments can be made to sub-committees. The assignments vary in nature, ranging from a report on research findings or a

survey of practices in current textbooks to the writing of sections of the proposed course of study or the try-out of specific recommendations in the classroom. As the work proceeds, not only should there be frequent opportunities for the whole committee to find out what is being done by its parts but also efforts should be made to report on progress to all the teachers in the system. This enables them to discuss and react to some of the issues that the production committee encounters.

After various parts of the course of study have been prepared and those parts lending themselves to tryout have been tested, there is the big task of putting all parts together in one unified report. This task can best be accomplished by the chairman or one person working under the close supervision of the chairman. The finished report is studied by the committee and approved by it before final release.

The plan proposed above in which the major work is done by someone other than classroom teachers has several points in its favor. Teachers do not have the time for study, for meetings, and for writing which is essential in curriculum making. Yet, by examining, discussing, and trying out the tentative proposals, a large number of teachers become well acquainted with the course of study and thereby acquire a right of ownership in the final product. The chairman of the course of study committee by doing much of the work becomes the servant of the teachers, rather than being a person who assigns curriculum tasks for the teachers to perform. (It is, of course, assumed that the chairman will have those scholarly traits, including humbleness in considering instructional problems, which will make him acceptable to committee members.)

Schools which do not have the personnel available to undertake a full-scale study of the arithmetic curriculum leading to the production of a course of study can engage in curricular work on a less extensive scale. One way to do this is to work on only one or two phases of the arithmetic course of study each year.

The development of a list of principles and procedures for enriching arithmetic, sample lessons for introducing new topics, and examples of proof for use in each grade are illustrative of this kind of curricular work.

The Parts of the Course of Study

Those confronted with the preparation of curricular materials find the decision of what to include in the report a difficult one to make. An examination of the contents of various courses of study shows some common features but there is not enough agreement to warrant a recommendation based on such an examination. Even such an important feature as the grade placement of topics is not included in an easily recognizable form in all courses of study. Many courses of study begin with a statement of objectives; others begin with a discussion of modern arithmetic teaching; and one well-known course of study begins with a historical treatment of the development of number. The reasons for such diversity in practice are found primarily in the general educational scene in which the course of study was developed. In a city where arithmetic has long been considered only a skill to be mastered through practice, much space may be devoted in the course of study to presenting a broader picture of the role of numbers. For similar reasons different phases of arithmetic are emphasized in other cities. In view of these variations in practice, the identification of important parts of the course of study is primarily a matter of subjective judgment.

Committees concerned with the problem of what to include might still profit from the examination of content in a number of courses of study. The tables of contents of three are reproduced in whole or in part below.

Table of Contents

(from *Arithmetic Guide*, University Elementary School, Iowa City, Iowa)

1. Learning and Teaching Procedures
   - 1.1 General Principles
   - 1.2 Meaning and Understanding
   - 1.3 Introduction to New Processes
   - 1.4 Use of Proof or Showing
   - 1.5 Place of Drill and Automatic Mastery
   - 1.6 Grouping in Arithmetic
   - 1.7 Speed vs. Accuracy
   - 1.8 Use of Short Cuts
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(from *Arithmetic*, Philadelphia Public Elementary Schools, Philadelphia, Pennsylvania)

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While these three partial tables of contents would be an inadequate basis for generalization, it may be noted that each deals with the presentation of a point of view regarding arithmetical instruction, each indicates specific content, and each allocates some subject matter to a definite grade level. Other topics are often included such as a bibliography, treatment of drill, and suggestion for measurement, but the most important topics with which each course-of-study committee deals are (1) the selection of content, (2) the gradation of that content, and (3) major characteristics of the method of teaching to be used.

The Selection of Arithmetical Content

The selection of content for arithmetic is from one point of view a relatively simple matter. A committee has only to examine a number of courses of study and some textbooks to find that there is substantial agreement on the broad areas of content. However, when questions are raised concerning specific points, such as a short cut for multiplying by 25, or the teaching of metric measures of volume, the selection of content quickly

becomes a very complex problem. There are no universally agreed-upon guides or principles to use in the selection of arithmetical content.

The theory of social utility which holds that there shall be a one-to-one correspondence between the things taught in school and their uses in life has been partially applied at various times to the selection of arithmetical content. This theory has been quite successfully applied in the field of spelling. The words most needed in writing have been identified and are taught in the spelling program of our schools. The location of these words was no easy task, but certainly a far simpler task than that confronting the research worker who tries to determine the phases of arithmetic most used in life outside the school. In spelling, a record is made every time a word is written, but in arithmetic there are many instances where no record is made when a number is used. For example, a person who sees something which he desires in a store window does not get out pencil and paper and do any exact subtraction in deciding whether or not he can afford to buy it. Again, no computation of any kind is performed when one reads that the height of a poinsettia in Florida is 25 feet, yet number is used in the statement, and that number helps the reader to form a good idea of the size of the poinsettia. This use of numbers in reading is probably one of the most frequent and most important applications of number. Numbers essential to profitable thinking about almost all problems, whether they be in agriculture, sociology, or science, are often not written.

Failure to recognize that the pencil-and-paper records of number are only part of man's use of arithmetic is responsible for the poor quality of most of the research that has aimed at identifying the uses of arithmetic. The findings, furthermore, have been concerned with an inadequate conception of the true part that numbers play in life. For example, because research studies have shown that the work with fractions is primarily

concerned with $\frac{1}{2}$'s, $\frac{1}{3}$'s, and $\frac{1}{4}$'s, the recommendation has been made that instruction in this area deal only with these fractions rather than with the development of generalizations that will apply to all fractions. So far, research projects have been of little value because they have been too narrow in their approach to the problem.

With the narrowly conceived and narrowly interpreted usage studies as guides, there is no place for understanding of the number system, no place for seeing the nature of processes, and no place for seeing that one particular way of describing a quantity is better than others. In fact, the limitation to only one way of presentation is considered desirable. The interrelationship between facts and processes, one of the major characteristics of an efficient use of numbers, is entirely neglected. Use seems to be the only purpose for teaching arithmetic. For a critical analysis of this "tool" view of subject matter see Chapter I of the *Third Yearbook* of the National Council of Teachers of Mathematics.[1]

Among the further shortcomings of the narrow application of the social utility theory, particular mention should be made of the fact that these studies have eliminated from the curriculum many important areas of arithmetic, such as ratio and proportion. (Yet, the almost universal use of ratio is well shown by the betting odds that are always quoted on important elections, prize fights, and football games.) Another important kind of daily-life arithmetic, the unwritten or oral, was also partially eliminated through the narrow application of the theory of social utility.

From the preceding section it is reasonable to conclude that the research studies based on the theory of social utility have not yet produced acceptable data for selecting the content of a satisfactory arithmetic program. In the absence of complete

[1] *Third Yearbook*, National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1929), chap. I

data on the uses of number, a logical analysis of the mathematical system as it applies to life situations is perhaps the best guide for the selection of arithmetic content. Both the narrow "utilitarians" and those who have made a logical analysis of the uses of number have reached the conclusion that number is a valuable tool for those who know how to employ it. The utilitarians have seized upon one of the most obvious of these uses, the computation involved in the solution of life problems, as the thing to teach. The logical "analysts" have decided that the problem-solving use as well as other number uses can be taught by teaching the outstanding features of the number system and the various number procedures. By one scheme, facts and processes are taught in order that children may learn to solve problems. In the other, problems are used to illustrate the mathematical facts, processes, and relations that form the framework of the number system. To the utilitarians no educational value, other than the provision of a useful tool, results from the teaching of arithmetic. To the logical analysts, as interpreted in this book, there is educational value in the learning of the number system. In addition, the child acquires the possession of a valuable tool.

It does not require much imagination to see that if the logical-analysis method is used without any checks, the teaching of operations without much merit could easily become a part of the arithmetical content. For example, the teaching of the factor method of multiplying when a two-figure multiplier is present ($35 \times 24 = 7 \times 24 \times 5$) could be justified on the basis of its contribution to the understanding of the multiplication process without reference to its value as a check of the accuracy of computation.

What is needed then is a combination of the logical analysis and the social utility theories in the selection of curriculum content. A broad interpretation of the theory of social utility would be essential in any attempt to combine the two methods.

This combination should result in greater emphasis given in the arithmetic content to such items as (1) oral or unwritten arithmetic, (2) approximate computation, (3) development of the idea that tens, hundreds, and so on are handled just as ones are, (4) development of relations within the number system and the laws governing operations with numbers, (5) reference measures, (6) the making of judgments in situations where common measures are used, (7) procedures for showing how number and number operations make for simplicity of thought (e.g., the use of such ratios as 3 : 2 for two cities with populations of 223,476 and 140,871 people, respectively), and (8) material for the enrichment of arithmetic, including historical and recreational material.

Grading Arithmetic Content

An examination of courses of study shows that there is a fair agreement in practice on the grade placement of arithmetical content. From this fact one might conclude that the major issues regarding grade placement have been resolved. That, however, is not the case. Critical study of the gradation of content will result in grave doubts regarding the commonly accepted placement. For example, the breaking of phases of arithmetic content into small parts and allotting the simpler parts to the lower grades and the more difficult aspects to the upper grades is far from an ideal learning situation if learning is to be concerned with the development of generalizations. An extreme example of this breaking of arithmetic into parts is the recommendation that all number experiences in one semester be confined to numbers up to 6 or to 10. Grading subject matter into such small parts is very detrimental to the learning of the pupils of average and greater-than-average ability. It amounts to the provision of only minimum materials in each grade level. Modern society can ill afford to have the gifted handicapped by

the instructional program in the school. Therefore, the gradation of content should be such as to give the gifted child an opportunity to be challenged. (For further comment on individual differences, see section on individual differences, Chapter 14.)

The handicap which comes from grading content so that bits of a topic appear in different grades also applies to the placement within a grade. The cycling plan whereby a part of a topic — for example, fractions — appears on pages 16, 69, 111, 201, and so on in the book is not entirely satisfactory. Thorough or complete study of a topic at one time with only later review or reminders seems a better plan.

From time to time the grade placement of different topics has been changed because it has been found to be too difficult for pupils. Determination of the difficulty of learning topics is a complex problem. No satisfactory objective way of determining difficulty has been devised. Subjective judgment based on observation of instruction and results of achievement tests has been the main method used. The Committee of Seven of the Northern Illinois Principals Association under the direction of Carleton Washburne attempted to find the mental age at which pupils could first successfully handle the major topics of arithmetic. Because so many factors, especially the methods of teaching, were not well controlled, the findings of Washburne's study are not very meaningful. Difficulty of learning in a given field depends more on previous learnings in the field or closely related fields than it does on mental age. This fact should be carefully considered before a topic is changed to a higher grade for the reason that it is too difficult for the present grade level. When the teaching of a topic is found to be difficult, mere postponement to a later grade is in general a poor way to solve the problem. The result is often the creation of other problems. For example, when percentage was shifted to the seventh grade, the use of per cents in the social studies, science, and other areas in the lower grades required the teaching of some aspects of

percentage outside the arithmetic period. Often this teaching was not done, with the resulting lack of understanding when per cents occurred in reading or discussion. Difficulty of a topic might be lessened as much or more by provision of additional background experiences as by postponement to a later grade. The student of arithmetic teaching should then examine critically those courses of study and textbooks which have solved some of the difficulties generally encountered in teaching arithmetic by postponing or eliminating the difficult topics.

STUDY QUESTIONS

1. How many other different terms are now commonly used in the current literature for the term "course of study"? (*1*) Two. (*2*) Three. (*3*) Four.

2. Why is little attention given in this book to the characteristics of integrated arithmetic curricula? (*1*) Because no such curricula exist in actual practice. (*2*) Because arithmetic has so little in common with other subjects. (*3*) Because leaders in arithmetic teaching consider such curricula inferior.

3. In current practice, what is the most common arithmetic course of study in use? (*1*) The ones prepared by local school systems. (*2*) The ones prepared by state departments of public instruction. (*3*) The textbooks.

4. What is the most serious limitation of the detailed or extensive courses of study? (*1*) Their almost complete separation from any textbook. (*2*) Their overemphasis of minor aspects of arithmetic. (*3*) Their bulk, which makes them appear forbidding to teachers.

5. What is the chief value of having a curriculum specialist, elementary supervisor, elementary principal or superintendent in charge of an arithmetic curriculum committee? (*1*) It will prevent overemphasis being given to arithmetic. (*2*) The inclusion of at least one administrative staff member on the

committee would make its acceptance more likely. (*3*) Teachers work better under the guidance of such an officer.

6. Ideally, how should the work of the arithmetic committee be apportioned? (*1*) Each member should do an equal share. (*2*) One member should assume major responsibility. (*3*) Several members working as a team should do most of the work.

7. Should all curriculum committee members work on all aspects of the curriculum? (*1*) Yes. (*2*) No.

8. Where there is limited personnel, the practice of working on only one or two aspects of the arithmetic course of study is often followed. Is this desirable? (*1*) Yes. (*2*) No.

9. If the arithmetic course-of-study committee in a city does a good job, will the finished product differ in any marked sense from other good courses of study? (*1*) Yes. (*2*) No.

10. Why has it been difficult to use the theory of social utility in the selection of arithmetic content? (*1*) Because much arithmetic is not useful. (*2*) Because in only a small per cent of the arithmetic uses is a record left. (*3*) Because the theory has been interpreted too broadly.

11. Is the plan of cycling content within a year considered desirable? (*1*) Yes. (*2*) No.

12. The postponement of a topic when studies have shown it to be difficult has been practiced frequently. What is the chief flaw in such a solution of the problem? (*1*) Eventually a pupil will have too much piled on at one time. (*2*) The difficulty might be eliminated by supplying background or other experiences. (*3*) When one topic is postponed, others become difficult.

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11

Instructional Equipment

Textbooks in Arithmetic

The Place of Textbooks in Current Practice

The one most important item on every list of arithmetical instructional materials is the pupil textbook. A textbook for every child has become almost essential equipment for teaching arithmetic in Grades Three through Eight. In fact, many teachers not only rely upon the textbook for determining content and instructional material but also for tests and what little recreational and special motivational material is used in teaching. Such complete dependence on a textbook is not a superior instructional procedure, but it is a very common one. Even the best instructional programs, while not dependent upon a single textbook, make some use of textbooks. Books are probably the cheapest and best source of graded instruction and practice exercises and, of course, books supply a convenient source of material for discussion and study. Having a single convenient (easily handled, stored, used, etc.) piece of instructional equipment with from twenty-five to thirty-five pupils is a point not to be underestimated when a teacher is concerned. For the unskilled teacher, class management is easier if a textbook is available than if duplicated exercises, dictated directions and the like are used. Students of arithmetic teaching should

therefore give a great deal of attention to the material in pupil books and to ways of using that material.

WEAKNESS OF TEXTBOOKS AND TEXTBOOK INSTRUCTION

While the preceding paragraph indicated that textbooks are an important piece of instructional equipment, it also suggested that textbooks and textbook instruction possess limitations.

Some of the weaknesses are presented below, not for the purpose of undermining the present important role of textbooks but in order that the student of arithmetic teaching may devise ways to counteract the possible harm resulting from these weaknesses.

1. Current textbooks because of their very nature use artificial instructional settings. If, for example, an addition situation is being presented, the experience for obvious reasons cannot involve the children in the classroom without at least the addition of new data. In other words, the life situations of the local classroom are not utilized if a textbook is used exclusively.

2. Most current textbooks in beginning new phases of arithmetic emphasize telling and showing as the only means of instruction and rely on the presentation of just the one best method. The sole task of the child is to do as the textbook has shown, a type of learning which is generally considered second-rate, at least, in elementary-school subjects other than arithmetic.

3. Still another weakness of most current textbooks is that the statements of rules and generalizations appear before the pupils have a genuine need for such rules.

4. In general textbooks do little to develop real learning situations. Consider, for example, the first drill exercises in learning facts and processes. Most books just present them with the suggestion "to do" or "to study." No reason for studying has been provided except that all arithmetic has to be

practiced. Even those books which try to develop learning situations are, because of the nature of book presentation, forced to follow a restricted course which hardly allows variation.

5. The majority of textbooks follow a somewhat rigid step-by-step grade placement of topics which results in the presentation of small parts of a topic in widely separated places in a series. For example, in some series, one-figure divisors are presented in one grade, two-figure divisors in the next grade and three-figure divisors in the next grade. Such an arrangement is not conducive to getting across the idea that tens and hundreds are dealt with in much the same manner that ones are. This following of a rigid grade placement of materials also results in the use of examples and problems of much the same relative difficulty, and obviously the level must be determined by what pupils of average or below-average quality are able to do. The superior pupil, then, does not find much that is challenging in the textbook material.

6. Since most textbook series fail to develop genuine learning problems, the particular phase of arithmetic presented (actually, the phase to be mastered) is often not clear to the pupils. He may know in general that he is dealing with multiplication, fractions or the like but will not be able to state just what he is trying to master. Seeing a process in proper context does more to insure its acceptance by the pupils.

7. Partly as a result of the weakness mentioned in the preceding paragraph and partly because many textbooks use a cycle plan of organization, there are no good summary sections in textbooks. For example, after studying multiplication of whole numbers, pupils should find summarization of real value. Learning studies show very clearly that good summaries (1) are a means of identifying and clarifying points not yet understood, and (2) provide an excellent way to retard forgetting. That arithmetic instruction would profit from clarification of hazy points and from better retention of that which has been studied can hardly be questioned.

SUGGESTIONS FOR IMPROVING THE USE OF TEXTBOOKS

One of the best ways to improve the textbook's contribution to instruction is to create situations where the pupils will have a real purpose for turning to the textbook. The following are reasons pupils might have for consulting their books: (*a*) to see whether or not a procedure developed for multiplying two-figure numbers was the one recommended by the textbook, (*b*) to find some practice exercises for which a need had been shown, (*c*) to check the statement of a rule, and (*d*) to help in the preparation of summary statements about what had been learned in the study of fractions. When pupils have reasons for consulting books, the books are looked on as helpful reference material instead of as a source of seemingly endless exercises which the teacher assigns.

If the teaching procedures recommended in this book are followed, different teaching procedures will be substituted for many of the sections in textbooks where new phases of arithmetic are introduced. This substitution of material for the introductory work in textbooks would not necessarily mean that the book material will have no use. As indicated in *a* of the preceding paragraph, pupils who have developed their own procedure will then have a real purpose for examining the textbook procedure. Pupils under such conditions will probably understand the text presentation much better than they would have if the text material had been used without any introductory material.

It has already been suggested that if the textbook is not the only source of instructional material, there is a better possibility that the text materials will be effective when they are used. This is especially true in the case of problem exercises. The children will consider each set of problems more carefully if it is used only once for a specific purpose, for example, in teaching the reading of problems (no computation). This means many

sets of exercises must be available to allow practice of particular skills. An old set of textbooks is an excellent source of such problem exercises and will permit the use of exercises in the regular adopted textbook for other purposes.

Introducing more than one means of solution will make possible variety in using the textbook presentation of the best solution. For example, instead of stopping with the dimes and pennies and short way of adding 28 + 35, the class should search for and consider other ways of finding the sum. After such other ways have been found, they can be used effectively in "proving" or showing that answers obtained for some textbook problems and examples are correct. Study sheets and different ways of adding displayed on the chalkboard or arithmetic bulletin board may be other sources of the different solutions.

Preparing summaries of what has been learned in the study of certain phases of arithmetic has been previously suggested. Teachers who use this procedure will find it makes for an intensive review study of the textbook. Pupils seldom go back and consider again with a definite purpose any part of an arithmetic book that has once been studied. Therefore, this preparation of summaries should interest teachers as a means of achieving textbook review.

THE SELECTION OF TEXTBOOKS

The extensive use of textbooks in arithmetic instruction has made a good publisher's market and, as is usually true in such situations, a large number of different books are available. All books on the market have some good features though, of course, some have more than others. What is good in books is not, however, a matter of common agreement among teachers, administrators, and others concerned with instruction in arithmetic. There are also parts of every series which some people do not like or which they feel are definitely inferior. To com-

plicate the problem, features which often seem very good to adults (especially those who have not had recent experience teaching children) do not work so well in the classroom.

In view of the important role of the textbook in instruction and the considerable variation in the books, selection of a series is a procedure which merits careful study. The choice is difficult enough because of the number and variations in content of the books available. It is made doubly difficult by some of the claims made in advertising materials distributed from the publishing houses and by publishers' representatives.

Perhaps the factor that makes the selection of arithmetic books for a school system most difficult is the lack of knowledge about arithmetic teaching of those whose task it is to choose the books. Far too often, the major information on methods of teaching, issues in arithmetic instruction and so on which the selector of books possesses has been acquired recently from book salesmen, from the examination of a few new books (without reference to old books), and from the reading of a few recent articles and reports. When such conditions exist, it is not strange that books of relatively low merit are sometimes adopted. The fault is not that of book salesmen but of the people who do the selecting, and it points to the need of considering far more than the few features of a book which have good sales appeal.

The first reaction of the inexperienced when confronted with the task of adopting an arithmetic series is to try to formulate some rating scheme that will eliminate bias — "to do the thing scientifically." Unfortunately, no satisfactory non-objective way of rating books has yet been devised. Even such points as a difference in grade placement or of specific teaching procedure cannot be rated objectively unless a decision about the matter has already been made in the course of study or similar guide. In rating such items as initial presentation of a topic, number and quality of problem exercises, tests, and other features of a book, the criteria must be almost entirely dependent

upon what the individual or the committee believes. It appears therefore that an unbiased or objective procedure for selecting textbooks is not possible. However, because of the fact that many of the individuals and committee members concerned lack background knowledge for the assignment, some systematic program for study and evaluation of pupil textbooks is highly desirable.

The question "Who shall select the pupil's textbook?" must be answered first. In the past, the superintendent, board of education, supervisor, principal, or other individuals with administrative power chose most of the books. That method of selection is still in use in many smaller school systems, in some county schools, and in a few larger school systems. The common alternative is the committee system. Each year such a system seems to be becoming more popular. This popularity is deserved, for the committee plan has many advantages over the plan of having some administrative officer select the books. The chief advantages lie in the fact that selection is placed in the hands of those who are to use the books, the committee is appointed for this one task so selection is not influenced by previously incurred obligations, and the final decision requiring agreement of members is usually the result of careful study.

The membership of the committee should be made up primarily of classroom teachers for they are in the best position to judge the effectiveness of instructional material. Other members are usually selected from the elementary-school principals and supervisors. Whether or not administrative and supervisory staff should be committee members is doubtful. Because of the positions they hold, there is a tendency for them to dominate committees, while committee membership may seriously impair their counseling services to the teacher members of the committee. Since textbook committees so often need much guidance and advice as they go about the task of choosing books, probably the best use of administrative and supervisory staff is

to assign to them specific counseling duties. Such duties would not give the administrative staff a vote in the committee.

Regardless of who comprises the committee and its counselors, the membership should be made public. Secret committees are not more effective in the selection of books and their use results in many undesirable practices. Committee membership for most effective work should be small, probably from four to seven members.

A plan of action is described briefly in the steps that follow.

Steps in a Study Plan for the Selection of Textbooks

1. *Study of the field of arithmetic*

The final choice in the selection of books is heavily weighted by what the individual believes, and what an individual believes about arithmetic is based on what he knows about the field. If his knowledge is sketchy or of doubtful validity, his beliefs cannot be worth much. It is, therefore, highly desirable for the individual or committee selecting textbooks to possess opinions based on the best available information in the field of arithmetic. As the first step in that study, references such as the *Fifty-first Yearbook*, Part II, of the National Society for the Study of Education and the *Fifteenth Yearbook* of the National Council of Teachers of Mathematics are suggested. Further study should include consideration of the best professional books on teaching arithmetic and some of the best courses of study.

2. *Limiting the number of texts to be examined critically*

The number of available textbooks is so large that to consider them all carefully would make the task of selecting a series very time consuming. A procedure for cutting down on the books is very much needed and should be inaugurated early in the program. The selection committee can probably function best if

not more than three or four series are examined intensively. The copyright date is frequently used in determining what series shall be inspected. For example, one committee agreed to look at the three series with the latest copyright date and two series published in the preceding ten-year period. The scheme for limiting the number of series should be rather objective, and if possible the entire faculty should approve the plan by referendum. Such a procedure takes some of the pressure off the textbook selection committee.

3. *Insuring series rather than single book evaluation*

Classroom teacher members of textbook selection committees especially have a tendency to think only in terms of the grade they teach or in which they have a special interest. The selection then becomes a one-book and not a series proposition. To keep that from happening, the procedure of having a teacher responsible for examining one phase of arithmetic (e.g., addition of whole numbers) in all the books of the various series has been adopted and used successfully by some committees.

4. *Examination procedures*

After the three preliminary steps listed above have been taken, the real task of examination is begun. Most committee members teach full time or have other assignments which make it necessary to limit careful examination of the series of books to ten to fifteen areas. Among the areas most commonly selected for careful study are (1) addition, (2) subtraction, (3) multiplication and (4) division of whole numbers, (5) study or learning suggestions in the books, (6) practice and review exercises, (7) verbal problems used, (8) the problem-solving program, (9) addition, (10) subtraction, (11) multiplication and (12) division of fractions, (13) tests, (14) pictures, (15) measurement, (16) manuals, and (17) grade placement of content. Committee members working in groups of two or three assume

responsibility for each of the areas. Actual study begins with the books. For example, each committee member examines that section of each series which deals with division of fractions and chooses the best and the poorest presentation. Selecting the best is primarily a matter of looking for the material which is most nearly in harmony with one's own ideas on how that phase of arithmetic should be presented and then working out the assignments as a pupil would. (Neither time nor facilities are usually available to try the procedure in the classroom.) Only those who know classroom situations well and have that unique faculty of putting themselves in the place of pupil and teacher can do this try-out procedure easily and effectively, but all can do it satisfactorily enough by application to warrant its use. In the final report, the sub-committee responsible for examination of division of fractions describes to the entire committee what was considered the best and poorest practice found in the books. Preparing such a report has the advantage of putting emphasis upon real content and of demanding careful identification of specific good and poor practices.

5. *Using the representatives of publishing companies*

The representatives of publishing companies know their products well, and if given an opportunity, can be of genuine assistance to the textbook selection committee. Whether these representatives should appear before or after the study suggested in Step Four is debatable. In any case, their appearance before individual committee members and before the entire committee is important enough to be scheduled during the day and not left to after-school hours. To obtain teacher time, a substitute teacher should be provided for committee members on the day or days they are to confer as individuals with salesmen. It is often very helpful to have the salesmen appear before the entire textbook selection committee. Such appearances should probably come after Step Six.

6. *Making a composite rating*

When all reports from Step Four are in, a composite rating is needed. This may be obtained by assigning a point value for best, omitted, and poorest ratings and then finding the total points for each book.

7. *Making the final choice*

While the final choice is made by the committee, primarily on the data secured in the intensive study, other factors may be considered. If no series stands out, the top books may be reconsidered, using the same examination procedure suggested in Step Four, plus the direct comparison of various sections of the books.

8. *Preparing a final report*

The final report to the superintendent or administrative body responsible should be carefully prepared. A statement which merely gives the first choice of books is totally inadequate. The administrative officer should know how the committee went about its work and why the series chosen was considered superior to the others. It is recommended that the report be such that when any of the unsuccessful salesmen ask the reasons for their failure, the administrative officer can find a satisfactory reply in the report.

The student of arithmetic teaching will find that other plans for evaluation of textbooks differ markedly in some respects from the program given above. Study of these other plans should be of value to committees, especially in connection with Step Four of the proposed plan.[1]

The fact that there is very little in the literature on the pro-

[1] L. J. Brueckner and F. C. Grossnickle, *How to Make Arithmetic Meaningful* (Philadelphia: The J. C. Winston Company, 1947), pp. 490–493; G. M. Wilson and others, *Teaching the New Arithmetic* (New York: McGraw-Hill Book Company, 1951), pp. 453–457.

cedures used by committees in their study of books calls attention to the importance of preparing a final report. If a number of such reports were available for study, a better plan of work for textbook committees might develop. Under present conditions practically every committee starts from scratch with the result that all spend valuable time either preparing a study plan or doing things inefficiently. The inclusion in a report of reasons for accepting and rejecting books, provided they are sound reasons, should be a means of improving textbooks. Publishers are very sensitive to loss of sales and will alter old books or prepare new ones to meet the demands of the trade.

Arithmetic Workbook and Related Materials

The arithmetic workbook is widely used in instruction to supply extra practice exercises or for remedial and make-up work. Where there is need for additional practice material, the workbook is certainly an economical source. For such uses, the workbook accompanying the adopted texts will probably be superior to others, since the practice material will be properly graded. Workbooks serve as an excellent means of providing remedial materials because they automatically give a complete record of all the pupil's work. For remedial uses, there is no particular advantage in having the workbook accompanying the adopted text.

For pupils who have poor work habits, such as depending on others for their written work, the workbook is a definite help. Workbook material is also valuable when pupils (usually only a part of the class) fail to grasp a procedure during the textbook introduction.

To supply extra practice, many teachers, and even some school systems, have provided duplicated practice exercises for use with the various topics in arithmetic. Such exercises take the place of the extra practice material supplied by work-

books. Similar to these extra practice exercises are review summary questions (see pages 160–162 and 227–228) which are to be used at the close of the study of a unit.

The arithmetic notebook (maybe only a section of the pupil's general notebook) for recording special assignments, statements of rules the class has developed, summaries, and similar information is good instructional equipment for upper grade pupils. Having a place, such as the notebook, for arithmetic papers is especially advantageous when a discussion lesson is based in part on work done a few days earlier. The notebook also facilitates the doing of carry-over assignments.

Special Arithmetical Instructional Equipment

The developmental procedure advocated in this book and the emphasis on understanding through exercises which require the pupil to show that his answer is correct suggest extensive use of various special instructional aids such as drawings, objects to be rearranged, number lines, and the like. The need and role of such special instructional equipment is especially well illustrated in the following exercise. "By using proper stones for 1000's, 100's, and 10's, divide 1470 by 70. Be sure you can tell what changes you made in the number 1470 in order to do the dividing."

The pupil who substitutes tens for the one thousand and four hundreds in 1470 in order to find the number of 70's equal to 1470 has an opportunity to see a significant characteristic of division, namely that before division can be performed the dividend units are changed to the same units as those of the divisor. Although the same process is performed with figures in division with numbers only, it is more difficult to see the change to equivalent numbers. This may account for the difficulty many people have in understanding the division process.

Use of special instructional aids in attempts to insure under-

standing has led to the somewhat unwarranted conclusion that to teach arithmetic well a great many special devices are needed. Such pieces of equipment have been given the label "special instructional aids" here because they are only aids and should never be considered as anything more in the teaching of arithmetic. No one of them is essential but each that is listed in this chapter has been considered of real assistance in the teaching of arithmetic under certain conditions. This list does not claim to include all the good devices, for only those are given which the author has actually seen used.[1]

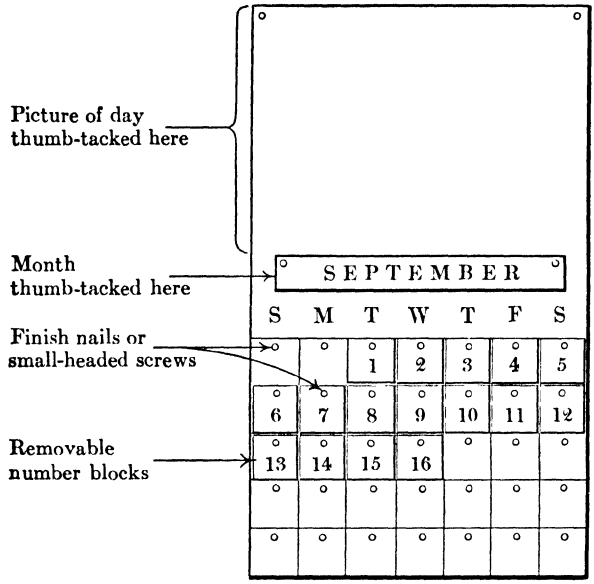
DEVICES AND INSTRUMENTS FOR CLASSROOM USE

The grade for which the device or instrument is suitable is indicated and for each aid one or two uses are given.

1. Calendars (Kdg., 1, 2). The plural word is used because this instructional aid comes in several forms, each of which possesses distinct advantages. For determining the lunar or moon month and times given in moons (American Indian stories present such situations), the conventional calendar is a good source and when properly used provides excellent counting experiences. Certainly the conventional calendar has the most uses, but the linear form developed in the school with a classroom event listed for each day has values for primary-grade children that cannot be easily obtained from the conventional form. A special form of calendar with removable numbers (see drawing below) also has distinct advantages. A duplicated form on which children write the numbers is still another calendar device.

The frame or base for this calendar is made of beaver board, plywood, or similar material. The numbers for dates are on

[1] For another treatment of some aspects of this topic, see *Fiftieth Yearbook*, National Society for the Study of Education, Part II (Chicago: The University of Chicago Press, 1951), chap IX, pp. 155–185.



Calendar with Removable Numbers

blocks or cards that are hung on nails. The calendar shown above shows through September 16. On October 1 all numbers will be removed, the October name plate tacked to the board and block number 1 put in the appropriate space. Since the picture of the day can be changed each day, all children can have a drawing displayed within a two-month period.

2. Number charts (Kdg., 1, 2, 3). The fixed form of the number chart (see page 66) is an easily constructed device which, as indicated in earlier chapters, has an important place in teaching such things as counting, the reading and writing of numbers, the place of numbers in the series, some idea of the function of zero, and the ease with which tens and ones can be added or subtracted. A form of number chart with movable numbers (page 67) is in many ways superior to the fixed form.[1]

[1] See Donna Norton, "The Numeral Frame — A Device for Use in the Teaching of Counting," *The Mathematics Teacher* (October, 1951), pp. 382–384.

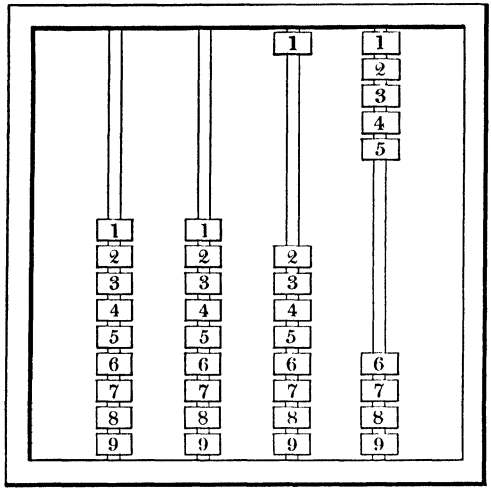
3. Number cards (1, 2). Three by five or larger cards with numbers from 1 to 10 (one to a card) and 1 to 100 are especially useful in teaching the place of numbers in the series or for identifying quantities in exercises on the arithmetic table and bulletin board.

4. Counting devices (Kdg.–6). Blocks, bottle caps, sticks, commercial counting disks, play money, pieces of paper (oak tag is best), grains of corn, beans, and many other objects can be used to provide counting experiences in the primary grades. After the initial counting in the primary grades and in the upper grades where the collection idea is emphasized, specially designated blocks, stones, or other objects of varying size can be used for 1's, 10's, 100's, and even 1000's. The tens block is an excellent counting device for dealing with tens numbers (for an illustration, see page 48).

5. The number line to 100 (1–6). The number line is extremely valuable in providing the most efficient picture of the numbers to one hundred, in teaching counting, adding, subtracting, multiplying and dividing, and for dealing with the basic idea of per cents. While the horizontal number line is the most useful, the vertical line may be better for work in addition and subtraction.

6. The abacus (1–4). For teaching the positional value of numbers, the abacus has no superiors among devices. It is also useful in the study of addition and subtraction and in minor ways in the study of multiplication and division. There are several forms of abacuses, each of which has some special advantage. In the primary grades the numbered form (see illustration below) is probably most useful. For class work in second and third grade, a paper form on which pupils make marks has been used successfully (see page 389). The board form (below) has excellent arithmetic table uses.

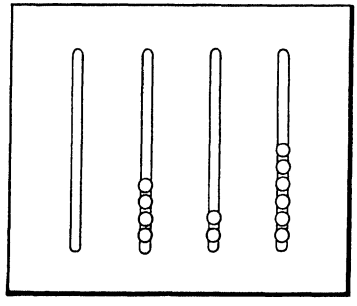
The counters on this abacus are numbered. To keep numbers facing in one direction counters are threaded on two wires. The abacus on page 334 shows 15.



Number Form of Abacus

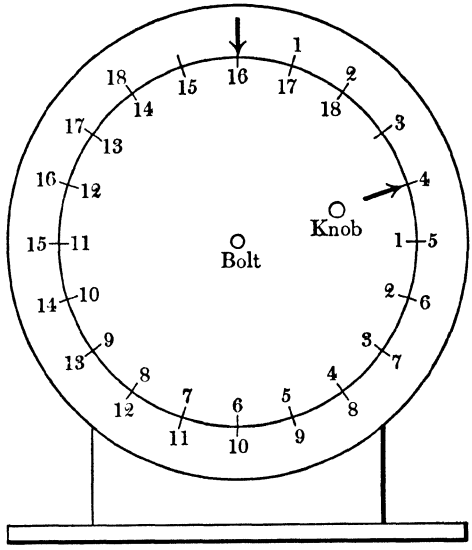
The board form of abacus may be made of a board which is $10'' \times 12''$ although one which is either larger or smaller is just as acceptable. Grooves are cut in it or stripes are painted as shown in the sketch below. Counters are then placed on these grooves or stripes to indicate the quantity desired. Grains of corn or beans may be used as counters. The abacus below shows 426.

7. Fact finders (see page 76), sliding scales, rotary scales,



Board Form of Abacus

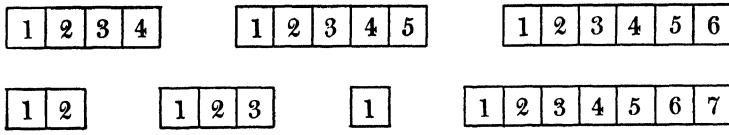
number blocks, counting rods and the like (1, 2, 3). (See illustrations below). To show such equivalences as $4 + 5 = 9$, these and similar devices can be quite useful. If the space for each number of the number blocks is the same as that for each number on the number line, these blocks can be very helpful in showing some aspects of addition, subtraction, multiplication, and division.



Rotary Scales

This device is composed of a stationary outside wheel and an inner wheel which turns. The illustration shows 4 on the outer wheel added to any number on the inner wheel. To get the sum of $4 + 5$, for example, set the arrow on the inner wheel opposite 4 on the outer one. The sum of $4 + 5$ is then found on the outer wheel opposite the figure 5 on the inner one. It is 9.

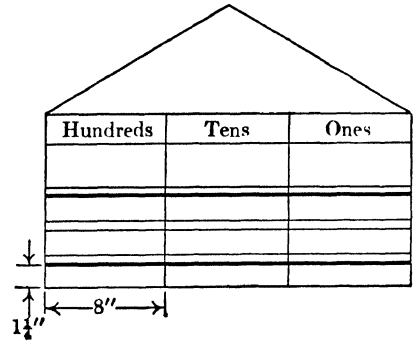
The blocks are made of $\frac{3}{8}'' \times 2''$ lattice. The numbers are stamped or drawn in the spaces.



Number Blocks

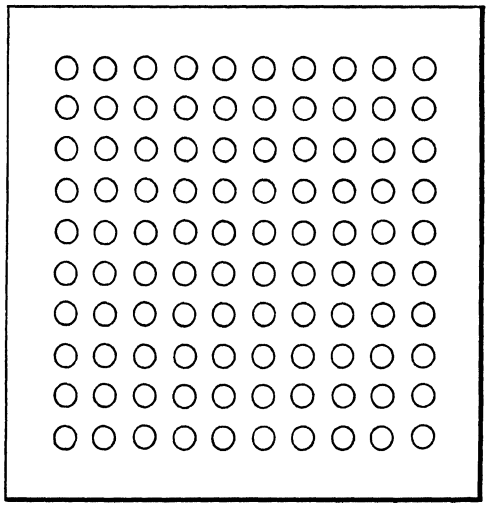
8. Foot fraction board and fraction circle (see pages 207–208).

9. Number pockets chart and counters (see drawing below). For giving additional experience with tens and hundreds as special entities and for giving experience in changing from one form to another this device is useful. Since the chart can be made large, the device is especially valuable for class consideration of addition and subtraction situations.



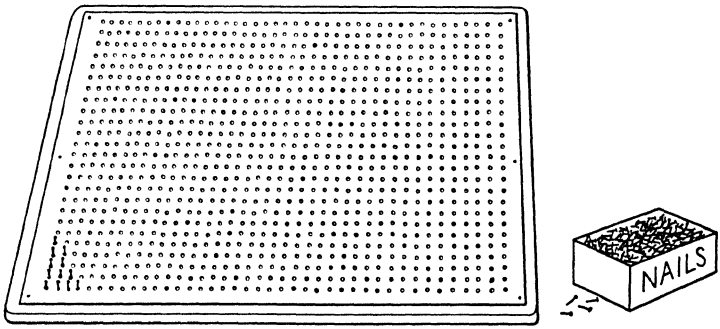
Heavy cardboard for background. Pockets may be made of oak tag. Colored construction paper for counters.

10. The hundred square, the hundred peg board, and the thousand peg board (see illustrations below) (Kdg., 1, 2). Each of these aids can provide counting situations and help show the meaning of numbers. For example, to give an idea of how much 245 is, one or two pupils can be asked to put that many nails in the peg board. Different colored nails can be used, permitting visual comparison of amounts such as 125 and 200. The peg boards when used to play pegity provide situations for counting and recognition of small quantities without counting.



The Hundred Square

This square consists of 100 bottle caps or other small circular objects fastened to a plywood board 28″ × 28″ as shown in the drawing.



Thousand Peg Board and Nails

A thousand peg board may be made by drilling a thousand holes in a board. A satisfactory one can be made from the sound-absorbing boards now commonly used for ceilings. Some

lumber companies also sell a display board which has holes drilled into it. Golf tees may be used instead of nails.

The hundred peg board is similar to the thousand one; it may be ordered from most school supply houses.

11. Flash cards of the basic facts (3, 4, 5) (see page 100 for drawing and suggested uses).

12. Instruments and materials for giving experience with common weights and measures.

(*a*) Scales for weighing amounts up to 25 pounds (Kdg., 1, 2, 3).

(*b*) Thermometer (Kdg., 1, 2). A valuable addition to a regular thermometer is a large facsimile of a thermometer with a movable ribbon (half red and half white) to indicate the reading.

(*c*) Foot ruler and yardstick (2–6).

(*d*) Dozen-egg cartons (both 3×4 and 2×6 varieties) (1, 2, 3).

(*e*) Liquid measures: cup, half pint, pint, quart, and gallon (Kdg., 1, 2, 3).

(*f*) Dry measures: bushel and $\frac{1}{2}$ bushel measures (3, 4).

(*g*) Play money, coins or real money for making change (Kdg.–6).

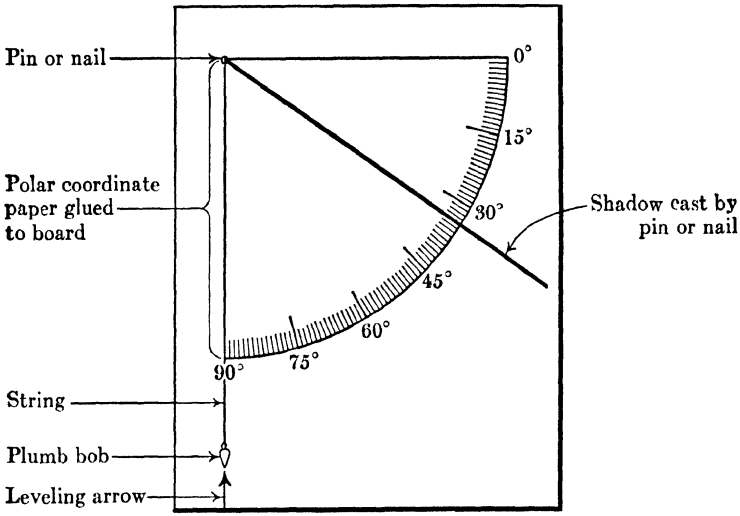
(*h*) Clocks and watches. Teaching the telling of time can be markedly improved by use of clock dials with movable hands and by duplicated clock faces on which the child puts the hands.

(*i*) Protractor for measuring the angle of sun.

This device is made by gluing a piece of polar coordinate paper to a board, suspending a weight as shown, and then marking the degrees and leveling arrow as shown. Mounting the board on a base makes the device easier to use.

(*j*) Wall tapes for measuring height of pupils (2, 3, 4).

13. Multiplication fact board with removable numbers



Protractor for Measuring Angle of Sun

(3, 4, 5). (See illustration on page 142.) For multiplication, pupils probably use this board most effectively when the product numbers are on the board with blank side out. The pupil tries to tell what product is on the block by looking at the guide numbers. When the guide numbers are blank side out, some division exercises can be performed with the board.

14. The Stern Blocks (Kdg., 1). For teaching basic ideas of number, this set of materials has much to offer.

15. Arithmetical games (Kdg.–6). The games listed in the chapter on enrichment may be included in this section. There are many other arithmetical games which have been used in classrooms. The following games with variations are the most common ones used: (*a*) Dominoes. (*b*) Lotto. (*c*) Keeno or bingo. (*d*) Ring toss. (*e*) Various race games in which the marker is moved a distance that is determined by the spinning of a wheel, the throw of dice or the drawing of number cards. (*f*) Magic number cards.

SUPPLEMENTARY BOOKS AND MAGAZINES

Such supplementary books as those listed (see page 26) in the chapter on counting may be considered as special instructional equipment. The same may be said of the books mentioned in the chapter on the enrichment of arithmetic. In addition, there are other books which have data and material of value in the teaching of arithmetic. Mail order catalogs with their descriptions and prices of merchandise are one of the best supplementary references for arithmetic classrooms. In the primary grades, old telephone directories are a good source of numbers to be read. Many reports and bulletins are full of good number situations. Pictures and data from magazines can put zest and new life into arithmetic work. Suppose, for example, a class is concerned with learning to read millions. The use of such questions as "How much did the California Citrus Growers pay for advertising last year?" as a caption for the appropriate magazine clipping on a bulletin board will often be found much more effective than the textbook suggestion to read the number $5,000,000.

MOVIES AND FILM STRIPS

A look at the list of arithmetical instructional films that have been produced might easily lead one to conclude that arithmetic teaching lends itself to instruction through moving pictures. Unfortunately, the quality of the films produced is not on a par with the number of films produced. Arithmetic teaching is not an easy field in which to prepare teaching films, and most of the films produced have been a disappointment to the teachers who have used them. Careful consideration of the field of arithmetic will show why good teaching films are difficult, perhaps impossible, to construct on anything but the rather insignificant phases of arithmetic.

Since number concepts must develop slowly, little in the way of development can be expected in a ten- to fifteen-minute film. For example, consider what would be involved in making a film on a single number. (Such films have been made.) Some concepts about the number, such as its quantitative relationship to its double, become known to pupils only after learning the name of the number, its position in the series, its cardinal meaning, and other ideas. If the film includes the whole range of ideas, some are too simple, some too difficult for the pupils to absorb. It should also be remembered that number processes are not easily acquired — even a 30- or 40-minute class period of the best instruction sometimes accomplishes little that is permanent. To expect a film of twelve minutes' duration to teach much that is new and lasting is probably asking the impossible. The basic difficulty of films for instruction in arithmetic lies in the fact that real progress in arithmetic is dependent upon the pupils' grasp of the abstract nature of number and the interrelationships within the number system. Abstractness and interrelationships within a system are not good topics for visual presentation. In fact, attempts to demonstrate visually may in some situations actually be a hindrance. Another limitation on the use of films in arithmetical instruction is the fact that there is little for the pupil to do except watch. There can be no pupil doing of any significance. Even where a problem is posed, there is no time for pupil solution. Instead, he must center attention on following the solution presented in the film. Films, like the other special instructional aids, are only minor aspects of the total instructional program. Until better instructional films are produced, they are not going to contribute much to pupil instruction.

Even though films are not yet a great success in the classrooms, they have been rather useful in giving teachers ideas about number and number processes. Since teachers have the background for considering all phases of numbers and number

processes which have been incorporated (*packed* would probably be a better term) in the teaching film, the major shortcoming in the pupil classroom use of the film has been eliminated. Teacher training classes and in-service-teacher study groups have made good use of some teaching films. Recently films especially for teacher training classes have been produced.[1]

Film strips seem basically better suited for use in arithmetic instruction than moving pictures. The fact that the pace at which frames containing instructional ideas are presented can be governed by the teacher and therefore adapted to class progress is probably the major advantage of film strips over moving pictures. The ease with which an original idea, question, or assignment can be brought back to the screen is another advantage. Furthermore, film strips are short and can be easily fitted into the regular classroom instructional procedure. The small amount of monetary return to the producer probably accounts for the fact that not many arithmetic film strips are on the market. Another reason for the lack of popularity of film strips is the ease with which teachers seem to tear them because proper care is not exercised in threading the projector.

Another instructional device similar in some respects to the film strip is the overhead projector equipped with a time-exposure device. This makes it in effect a tachistoscope which can be used for intensive study of basic facts and processes. One investigation of the device showed that where used with an entire class it was not as effective as the best regular study method.[2] Part of the ineffectiveness of the tachistoscopic method of study may be attributed to the fact that all pupils, regardless of whether they need to or not, have to give time to

[1] Reference is made to such films as "Helping Children Discover Arithmetic" (Detroit, Michigan: Wayne University).

[2] Randall Bebb, "*The Tachistoscopic Method of Learning the Basic Multiplication Combinations*, Unpublished Ph.D. Thesis, (Iowa City: State University of Iowa, 1952).

the fact presented. If a pupil already knows fifteen of the twenty facts being presented, three-fourths of the time is wasted for him. This weakness of the tachistoscopic method of study is, of course, also a weakness of other limited time presentations for the entire class such as the moving picture and the film strip.

STUDY QUESTIONS

1. Is the textbook the one single most important piece of instructional equipment in the best schools? (*1*) Yes. (*2*) No.

2. Is the textbook an economical source of instructional material? (*1*) Yes. (*2*) No.

3. In beginning instruction in new topics do textbooks emphasize a developmental or discovery procedure? (*1*) Yes. (*2*) No.

4. Do textbooks emphasize ways to study? (*1*) Yes. (*2*) No.

5. In the ideal instructional practices is the textbook to supply the setting for introducing new material? (*1*) Yes. (*2*) No.

6. What is probably the best use that can be made of old textbooks? (*1*) The examples can be used for practice exercises. (*2*) They serve as a source of different computational procedures. (*3*) The problems may be used for some types of problem-solving exercises.

7. What group — teachers, supervisors, administrators, or board members — should have major responsibility for selecting textbooks? (*1*) Teachers. (*2*) Supervisors. (*3*) Administrators. (*4*) Board members.

8. Of the many factors which make arithmetic textbook selection such a difficult problem, which one probably ranks first? (*1*) The large number of books to select from. (*2*) The general inferior quality of the books. (*3*) The general ignorance, arithmetically speaking, of the selecting committee. (*4*) The pressure from book salesmen.

9. Should textbook selection committees be secret? (*1*) Yes. (*2*) No.

10. Is the try-out of various textbook procedures in classrooms a good plan for textbook committee members to use? (*1*) Yes. (*2*) No.

11. Should textbook salesmen appear only before the entire textbook selecting committee or see committee members individually? (*1*) Entire committee. (*2*) Committee members individually.

12. What would probably be the most significant result if arithmetic textbook selecting committees all prepared a final report of their studies? (*1*) The methods of textbook salesmen would change. (*2*) The content of arithmetic books might be improved. (*3*) The method of selecting books might change.

13. Workbooks are probably best suited for which of these groups of pupils? (*1*) Those who have poor work habits. (*2*) The slow learning group. (*3*) The fast learners.

14. Teaching films have not yet assumed a major role in arithmetical instruction. Why? (*1*) Number facts and processes are not acquired quickly. (*2*) Number ideas and processes are not picturable. (*3*) Films so far are too short.

15. Why are such timed exercises for learning arithmetic facts as, for example, the use of the tachistoscope, poor procedures for use with a class? (*1*) They force all pupils to learn at the same rate. (*2*) Pupils spend equal time on known and unknown facts. (*3*) The equipment used attracts too much of the pupils' attention.

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12

Measurement of Achievement in Arithmetic

Current Practices

An examination of courses of study and of textbooks shows that evaluation of some kind is an integral part of arithmetical instruction. It is usually in the form of pencil-and-paper tests. The large part that such measurement practices play can be shown by the fact that in two recently published sixth-grade textbooks 6 per cent and 9 per cent of the pages, respectively, were devoted to tests. In addition to the tests placed in the books, many text series supply additional tests, such as inventory tests for the beginning of the year or for the introduction of new units, and end-of-semester or end-of-unit tests. When the teacher-made tests and the standardized test or tests now commonly used are added to the program, the amount of pupil time and effort given to pencil-and-paper testing alone is a significant amount of the total instruction time in some schools. On the other hand, there are some schools where no standardized tests, no inventory tests, and few teacher-made and textbook tests are used.

The best measurement program is not confined to the use of tests of the pencil-and-paper type. Samples of everyday pupil

arithmetic work, informal teacher reports of the pupil's arithmetical achievement, records of special interests or contributions to discussion, and other similar evidence comprise a large (40 per cent to 60 per cent) share of the evaluation procedures in the better schools.

The term *measurement* as used in the preceding paragraph refers to all the instruments and procedures used in evaluating or appraising the arithmetic program. The statement regarding best practice indicated that pencil-and-paper tests were only a part, perhaps not more than one-half, of all the instruments to be used. Testing as used in this chapter then has a broad meaning. It refers to all the means of gathering evidence on pupil achievement. In the sections that follow, the pencil-and paper-tests are discussed first because they are the best-known and because they will probably always be one of the most efficient ways of obtaining evidence of pupil achievement in arithmetic.

Teacher-Made Arithmetic Tests

It is common practice in arithmetical instruction for the teacher to use tests to find out whether or not pupils have mastered the material that has been taught or to obtain evidence to use in reporting progress. These tests usually consist of examples with a few problems involving use of the procedures taught. The basis of this practice is relatively simple. It is assumed that if pupils can work the examples and problems, they have mastered the particular topic of arithmetic under consideration. Such an assumption is not altogether sound. The test is usually concerned only with the topic or topics that have just been taught and may even use the same form that was used in the instruction just given. The pupil may therefore remember the procedure used or explained a day or two previously and thus be able to secure the correct answer without

really understanding the procedure at all. Usually the only criterion on which success is based in such tests is correctness of the answer. Seldom is any credit given for use of correct procedure. Furthermore, since the tests consist almost wholly of examples and problems, no attempt is made to find out in the test whether the pupil can prove his understanding or check the accuracy of his computation by means of alternate solutions. In spite of the limitations just mentioned, teachers can learn a great deal about a pupil's ability by careful study of his test paper. These teacher-made tests could be markedly improved by having part of the items differ in form from that used in instruction, by having some items deal with content other than that just studied, by having some items that bring out the pupil's knowledge of procedures and ways of study and by having some items that require only approximate answers. An item illustrating each of these recommendations is given below. The items deal with the division of whole numbers in Grade Four.

1. Use "How many 4's equal 126?" instead of the form $4\overline{)126}$.
2. Find the sum of 419, 365, 80 and 93.
3. In the example $4\overline{)126}$ where should the first quotient figure be placed?
4. If you weren't sure about finding how many 17's equal 361 by dividing, how else would you try to find the answer?

Tests in Pupils' Arithmetic Books

The tests found in pupil textbooks are in general superior to the common type of teacher-made tests discussed in the preceding section. Textbook tests are usually designed to serve a particular purpose — such as to survey pupil achievement before undertaking a new unit or even a new year's work. Some of the tests at the ends of chapters are designed to be diagnostic. Although the textbook tests may not all accomplish these

various purposes well, the use of several tests with different purposes is an improvement over the use of a single test for all purposes. In an attempt to measure pupil understanding, some textbook tests have items in which no computation is involved and items which require the pupil to locate errors in the thinking of others. Non-pencil-and-paper items are now found in many textbooks.

The major limitation of many textbook tests is that they deal almost entirely with situations where credit is given only if the one answer is found through exact computation. It is well known that many life situations do not call for such exact computation.

Standardized Tests in Arithmetic

Standardized tests may be found in great numbers and wide variety. Twenty-two such tests were reviewed in the 1940 and 1949 Mental Measurements Yearbooks, and these did not attempt complete coverage. While most standardized tests are of the survey type, a number of diagnostic tests, inventory tests, and readiness tests are on the market. The survey tests, like the common teacher-made tests, consist primarily of a computation section and a problem-solving section. Since these tests are usually designed to measure achievement in several grades, the content consists of a graded list of items representative of the computational and problem content of the grades concerned. As indicated by the name, a standardized test has norms or standards for use in interpreting scores made by pupils who take the test.

The norms of almost all tests are either in terms of grade equivalents or percentiles or both. A given pupil score can be changed to a grade equivalent, a percentile score, or whatever other standard score is available. A grade equivalent of 4.8 for a fifth-grade pupil at mid-year indicates that that pupil's

achievement as measured by the test is .7 of a grade below the standard or norm of 5.5. These standard scores make it possible to determine pupil growth from year to year. The importance of this feature can hardly be overemphasized. For example, the pupil who has received very discouraging reports during the year will probably show some gain over the score he made a year earlier. Even .3 a year is positive and can be used to encourage the pupil to continue to work in arithmetic.

A percentile score of 38 would mean that a pupil's achievement is better than 37 per cent of the fifth-grade pupils used in standardizing the test and not as good as 62 per cent of the pupils. It is thus possible to determine through use of standardized tests the relative achievement of individual pupils and classes. Standardized tests are the best single means of gauging progress. They serve, then, a very useful purpose in instruction, and students of arithmetic teaching are urged to use and to study standardized tests critically. Such use and study coupled with suggestions and reports of weaknesses should result in much needed improvement of standardized arithmetic tests.

Some of the good points of standardized survey tests were mentioned in the preceding paragraph. In the paragraphs that follow some of the limitations of standardized tests will be identified. This listing of limitations is made primarily as a warning against placing undue weight on standardized test results and secondly for the purpose of calling attention to conditions which need improvement.

The first and most serious limitation of standardized survey tests is their narrow or incomplete coverage of the field of arithmetic. Arithmetic consists of far more than computation involving examples and problems. A list of important areas of arithmetic that might be included in standardized tests is given in the section on suggestions for improvement of tests. A second limitation is the fact that the standards or norms are far from being as reliable as the important uses made of them

demand. Only a few of the tests yield scores reliable enough to permit graphing of pupil growth from year to year. A third limitation of these tests is the fact that the norms are often based on populations (e.g., national) that are extremely difficult to describe. Teachers and supervisors in cities of from 100,000 to 250,000 people could make much more meaningful interpretation of norms based on cities of that size than they can by using so-called national norms. The method of scoring used on many standardized arithmetic tests is a definite limitation. Where a comparison-type key (teacher has to compare pupil answer with key answer) is used, teachers or others who score the test have to expend energy unnecessarily. The recognition-type standardized test when supplied with a stencil-type key markedly reduces test scoring time and therefore leaves teacher time and energy for doing something about the conditions revealed by the test. The efficiency of standardized arithmetic tests is reduced by the fact that every item requires the pupil to complete the whole computation. Thus, credit is given only for securing the correct answer. While it is, of course, the ultimate goal of instruction on any topic of arithmetic to teach the pupil to get the correct answer, to use the whole computational procedure for each test item is an inefficient way to test.

Arithmetic Diagnostic, Inventory, and Readiness Tests

It was pointed out in the section on teacher-made tests that through study of the pupil's paper teachers could often learn much of the pupil's deficiencies. To determine just where the pupil's knowledge becomes so deficient that he cannot do work successfully is something all teachers want. Tests designed to do this are called diagnostic tests and when properly administered are very good instruments. The good published diagnostic tests require a great deal of administration time and for

that reason are not extensively used by classroom teachers. They are excellent for intensive study of pupils who might be classified as remedial or clinic cases. Diagnostic tests are not only time consuming in administration but there is a dearth of teaching procedures to use after a pupil's weakness has been located. While this situation is not the fault of the test, its existence makes use of the tests less popular.

Inventory tests, especially for the beginning of the year, are used in a great many schools. These inventory tests for the most part consist of a survey of what has been studied. They attempt to reveal the pupil's arithmetical knowledge and skill. In a sense the review tests, often administered at the beginning of the year or semester, are inventory tests. If the time for such extensive testing is available and pupils see the purpose of the procedure, the results should be beneficial to instruction.

Readiness tests in arithmetic have never been popular. There are several reasons for this lack of popularity. While in reading the skills are relatively new, in arithmetic the beginning pupil has already acquired much number information before he enters school; therefore first instruction is based on and is not so very different from the number experiences the pupil has already had. Then, too, when a lack of arithmetical background is revealed by a readiness test, the instructional program recommended is little different from that already used and recommended for all beginning pupils. There are readiness tests which are to be used in upper grades to determine readiness for study of new topics. The most important parts of such tests are the items dealing with skills already studied which are used in the new topic. Essentially, then, readiness tests either for beginning work in arithmetic or for new work in the later grades are not very different from the inventory and survey achievement tests.

Other Tests or Means of Obtaining Information on Pupil Achievement in Arithmetic

The good teacher who has been working with a class for a semester or a year needs no test to find who are the better pupils in the arithmetic class. By observing, by listening, and by working with the pupils as they engage in their regular classroom work, the teacher gets a good idea of the arithmetical ability of the pupils. Observation of pupil work should therefore rank high among the various means that are used to ascertain arithmetical achievement. The value of this measuring procedure has long been recognized by a few leaders.[1]

The gathering of information on pupil achievement through observation can be markedly facilitated by use of a few special procedures. One of the best of these is the posing of quantitative problems from other areas of work. Class members are then asked to propose different solutions in order that the best may be used in summary or factual statements. The following situation from a fifth-grade class illustrates the procedure.

"Here are the population figures for the states and important cities of the Pacific Northwest. Using the population of Iowa and Des Moines, make statements about each figure that will help us get the meaning. Try several statements for each and be sure to have the figures to back up your statement. Keep all your figures. We may need them in our discussion."

A second procedure for facilitating observation is the asking of such questions as "Will you tell the class how you thought?" or "How do you think the person figures who did this work?" The first of these questions applies primarily to the oral or non-

[1] W. A. Brownell, *Sixteenth Yearbook* of the National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1941), pp. 237–239; and W. A. Brownell and Verner Sims, *Forty-fifth Yearbook* of the National Society for the Study of Education (Bloomington, Illinois: Public School Publishing Co., 1946), Part I, pp. 41–42.

pencil-and-paper arithmetic period and the second applies to consideration of someone else's computation. Having some pupils figure at the chalkboard will also give an opportunity for judging work.

A third procedure for finding out about pupils' understanding is the use of critical study (review or summary, see page 227) questions followed by discussion of answers.

Keeping samples of pupil work is another means of determining pupil achievement in arithmetic. The evidence presented in samples will be much easier to interpret if the time used, for example ten minutes, is stated on the sample and if the pupil's rank (above or below median or top, middle, or lower third) in the class is indicated. Schools which use this procedure usually collect two or three sets of samples a year. The directions given for one such exercise are shown below:

"Your next work is to be handed in. Get a clean sheet of paper. Write your name and date at the top of the page. Then, write 'Time 10 minutes, Exercise B, page 127.' (This was written on the board.) When you have that written turn to page 127 in your book and do as many of Exercise B as you can."

At the end of the ten minutes the papers were collected and scored. Each individual's paper was marked above or below average and the paper filed in the pupil's folder.

Many other kinds of work can provide additional evidence of the pupil's ability and understanding. Examples of proof or difficult computation including checks, specimens of best work and unusual solutions may be among the samples kept. Also enlightening are records of children's contributions to class discussion of quantitative problems in other curricular fields or to class formulation of arithmetic rules. Arithmetical compositions on such subjects as "How We Multiply with a Two-Figure Multiplier" or "The Different Ways of Checking Multiplication with a Two-Figure Multiplier" may reveal a great deal about the pupil's knowledge of multiplication.

The interview has been found very helpful in getting information on some pupils. This procedure obviously is an individual matter and therefore cannot be used extensively with many pupils.

If records of observation and samples of pupil work are to play the part they should in evaluation of pupil achievement, then a system for collecting such information must be put into operation. As was indicated in the preceding paragraphs, a folder for each pupil where samples of work are filed is a part of the procedure. Scheduled times or at least reminders are needed to keep the collection of such data from being neglected. At the beginning and near the end of each semester are good times to collect samples of the timed exercises; perhaps the middle of each semester the teacher should write an observation report (one paragraph statement of from one-eighth to one-half page in length) on each pupil; and reminders each semester regarding the filing of other samples of work will insure more complete files.

A Recommended Measurement Program

In the preceding sections some of the strengths and weaknesses of various tests have been indicated and other means of measurement have been suggested. The problems of deciding how much paper-and-pencil testing to do, what other means of measurement to use, how often to gather data and so on, are perplexing to those engaged in instruction. To facilitate the study of such problems, a measurement program is presented below. The outline includes only Grades One, Two, and Five with the understanding that Grades Three, Four, and Six will follow the pattern suggested for Grade Five with small differences in emphasis.

The measurement program at every grade level should be governed by the uses to be made of the data collected. It is, of

course, assumed that the information obtained through a measurement program will be used in the instruction program. In planning the arithmetic program at every grade level, teachers and others responsible for the guiding of instruction usually want some information on the general arithmetical achievement of the pupils. To supply this sort of information the survey type of inventory test is most useful. Such a test is, therefore, recommended as the first part of the measurement program at every grade level. The nature of these inventory tests is indicated in the schedule of tests for each grade.

The inventory tests given near the beginning of each year are an indication of progress made in previous years. There is also a need for information on the effectiveness of current instruction. To meet this need several means of gathering data can be effectively used: observation of daily work, the collecting of samples of work, short oral and written tests (in upper grades only), textbook tests, and several extensive teacher-made tests. Probably not more than three of the latter type are needed each year. Beginning in Grade Three or Four, diagnostic tests for a few pupils and standardized achievement tests for all can supply information needed for effective instruction.

Outline of Measurement Program Grades One Through Six

Grade One

1. An inventory test administered during the first three weeks of school. This test may include some pencil-and-paper work which can be administered as a group enterprise but for the most part the inventory will be individually administered and the major data will be obtained through the oral responses of the pupils. To provide a record and to facilitate administration, an inventory sheet which gives an indication of the content of this first-grade inventory test is shown (see pages 358–359).

2. Observation notes, at least two for the year. As indicated by the note at the bottom of the inventory sheet the back side of the sheet is to be used for recording observation notes.

3. A teacher statement of pupil's arithmetical experiences, ability, and attitude toward the subject. These are also to be placed on the back of the inventory test sheet. This teacher statement should probably be made during April or May if only one such statement is made for the year.

4. Samples of pupil's work. A specimen of the pupil's writing of numbers is perhaps the most common subject for this grade level. Other subjects are work sheets which require counting the number of objects in circles, showing with dots or pictures what numbers mean, and the like.

5. Teacher-made tests for determining progress, the first to be administered after about eight or nine weeks of school, and one or two others later in the year. The nature of these teacher-made tests will be dependent upon the number experiences the children have had during the preceding instruction period.

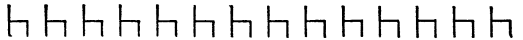
Standardized achievement tests for first grade, even at the end of the year, are not recommended because the current tests give too much weight to the ability to compute with figures.

Grade Two

1. An inventory test administered during the first three weeks of school. The best inventory tests for Grade Two are group administered and are of the pencil-and-paper type. To keep the ability to read from being an important factor in this inventory test, the teacher reads the items and gives all directions verbally to the pupils. The sample items below indicate the content of the inventory test for Grade Two.

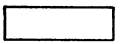
First Grade Number Inventory Sheet

Name_____ Date_____

1. How far can you count?___ (If child cannot furnish the information, or if you are not sure of his response, have him count. Say, "Let me hear you count. Say one, two, three, and go on as far as you can.")
2. Can you count by tens to a hundred?_____ (If child does not seem to understand, say, "You start to count by tens this way: ten, twenty, now you go ahead.")
3. When you are counting, what number comes after 12?_____
4. What comes after 34?_____
5. What comes after 216?_____
6. When you are counting, what number comes just before 23?_____
7. Can you count backwards from 10 to 1? Begin this way, 10, 9, 8.
8. What is this number (8)?
9. What is this number (27)?
10. How many chairs are there here? 
11. Bill ate one of his 4 apples. How many apples did he have left?_____
12. Mary had two dolls and got two more for Christmas. How many dolls did she then have?_____
13. Dick had four marbles and found two more. How many marbles did he then have?_____

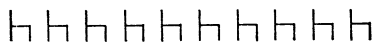
NOTE: The remainder of this test is for the pupil to mark. Do not give to pupils who cannot count to ten, or in other ways have demonstrated that they have little grasp of numbers. This part of the test may be administered to a group of children. The teacher should, of course, read each item. Since first grade children are not likely to be able to read numbers, refer to items by lines, boxes, circles, chairs, and the like.

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14. Make four dots in this box. 

15. Make eleven dots in this circle. 

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16. Bob sat on the sixth chair.
    Put a mark on the chair he sat on. 

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17. Draw a ring around one-half of these marks. / / / / / /

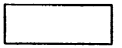
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18. Jack had two marbles and Bill had three marbles. Put a ring around as many marbles as Jack and Bill together had.

○ ○ ○ ○ ○ ○ ○ ○ ○

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19. Tom has one green marble, three yellow marbles, and two white marbles. Draw in the box as many marbles as Tom had.



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20. How many pennies are equal to one dime? Make that many marks in the circle.



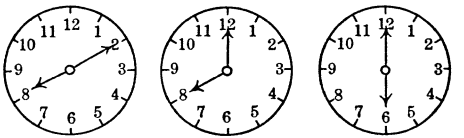
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21 Put as many dots in the box as there are in front of it



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22. Put a mark on the clock that shows 8 o'clock.



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23. On the back of this sheet write as many numbers as you can. Begin with 1. When you have written as many as you can, I'll give you a book. Turn the paper over now and start writing numbers.

(To be filled by teacher and filed in Pupil Progress folder)

(1) Which number is less: 36 or 24; 18 or 30; 67 or 59?
(2) The marks have been counted but some of the numbers have been left out. Count the marks and write the numbers that have been left out.
(3) Write the number that is six tens and four ones.

2. Observation notes, at least two for the year. Notes of observation for each pupil are to be recorded on the back of his inventory test sheet.

3. Samples of pupil's work, at least two for the year. The effectiveness of this part of the measurement program will be increased if pupil papers are ranked (e.g., above or below average). The first sample of pupil's work if only two are gathered should probably be taken at about mid-year. The other should be taken near the end of the year. The best sample work consists of the pupil's solution of examples and problems.

4. Teacher-made tests for determining progress. The content of these teacher-made tests will be determined by the instructional program. Before constructing such tests, the reader should refer again to the section on common failings of teacher-made tests and to the section on suggestions for improvement of tests. At least two of these teacher-made tests should be rather extensive (requiring from 30 to 40 minutes of testing time). It is recommended that the first of these extensive teacher-made tests be given near the close of the second semester.

As was the case in first grade, the current standardized arithmetic tests give too much weight to the ability to compute with figures to warrant extensive use of them at the second-grade level.

Grade Five

1. Inventory test administered during the first two weeks of school. While this inventory is concerned primarily with the

pupil's computational ability, one part of the test should deal with his ability to show correctness of computational procedures, knowledge of the number system and other aspects of arithmetic.

2. Observation notes, at least two for each pupil. These notes may be recorded at any time but for ease of interpretation, approximately the same time for each pupil, for example, eighth to tenth week, is recommended.

3. Statement of pupil's arithmetical experiences (not content) during the year. The following is an example of such a statement:

"Charles, while doing only average work in arithmetic, seemed to like the arithmetic period very much. He was especially pleased when he got the correct answer in a computational situation and was always willing to put his work on the board. He worked rather slowly but was willing to stay with an arithmetic task until it was finished."

4. Samples of pupil's work.

5. Teacher-made tests for determining progress, at least six a year. At this grade level teacher-made tests include the tests found in the textbook. If oral arithmetic is being stressed, then some oral tests (pupil recording answers only) should be included as a part of the section on teacher-made tests.

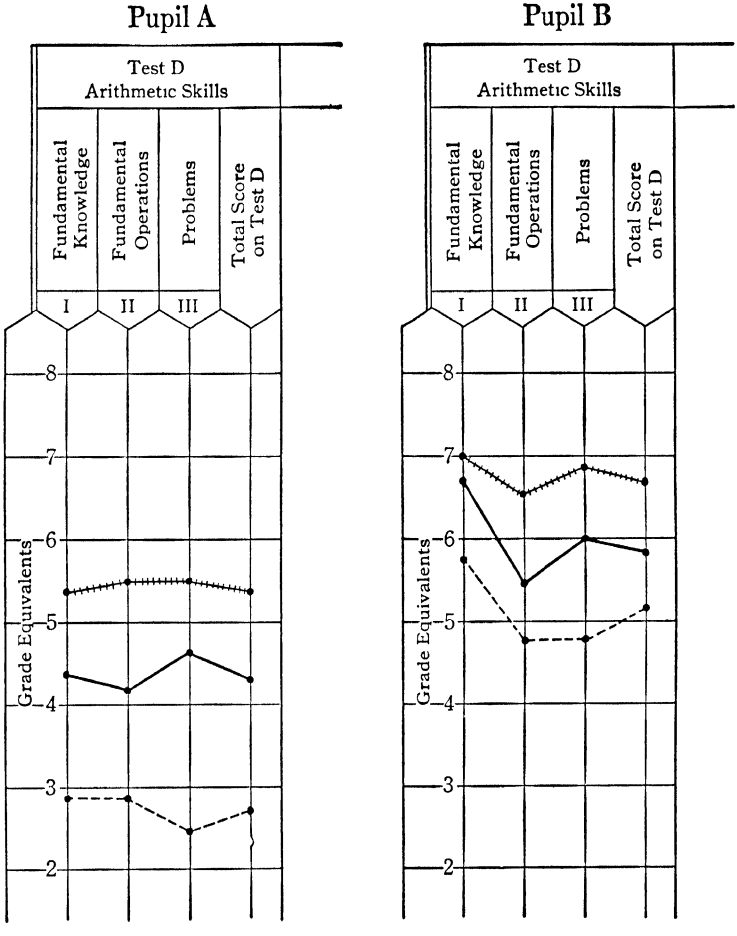
6. Standardized tests, at least one a year. It is recommended that if only one standardized test is used, that it be administered during or at the close of the first semester. The use of more than one standardized test is not recommended unless it meets some special needs such as those arising in connection with an experiment or special survey.

The most important difference between the program as outlined for Grades One and Two and that recommended for Grade Five is the inclusion of a standardized survey-of-achievement test. This test, unless used for special experimental reasons, should be administered during the ninth to eighteenth week of school. If administered early in this period, there is much time for the present teacher to do something about the conditions revealed. If administered near the close of the period, the conditions revealed can be met in part by the present teacher. When, for example, a sixth-grade class given a test at mid-year (eighteenth week) shows marked improvement or lack of improvement since the administration of the mid-year fifth-grade test, the fifth- and sixth-grade teachers share equally the responsibility for the conditions revealed. It is recommended that the test used provide for the charting (graphing) of pupil growth from year to year on a profile chart. Some sample profile charts are shown on page 363.

The measurement program outlined above calls for the keeping of data on each individual pupil. The inventory test results, records of observations, teacher statements of arithmetical experience, and samples of work are the items to be kept in the primary grades. The most effective way to keep such information is to provide a folder for each child. This file may be more than an arithmetic folder in the primary grades especially, and can include samples of handwriting, reading work papers, and the like. In the upper grades standardized test results may be added to the folder. However, school records usually provide a place for the recording of such information.

The over-all view of the measurement program shows that tests, while they play an important part in the program, do not comprise all of it. If data of the kind indicated in the measurement program were collected for the six or eight years the pupil is in the elementary school, valuable information for guidance of the pupil in the area of mathematics would be available. The

Sample Profile Charts



From profile charts for *Iowa Every-Pupil Tests of Basic Skills*, on file in the University Elementary School, Iowa City.

information would also be of some value in general educational guidance for the child. A note of warning is in order at this point. Recording observations, collecting and keeping samples of work, and having interviews all make demands on the teacher's time and energy. Whether or not the benefits derived from such expenditure of time and energy are justified is a question that must be answered affirmatively before the program can be endorsed.

Suggestions for Improving Measurement in Arithmetic

That arithmetic measurement is in need of improvement was stated or inferred a number of times in the preceding sections of this chapter. The suggestions offered here are one means of meeting a very obvious need. As a preparation for considering these suggestions, the student of arithmetic teaching might find a careful rereading of the limitations of tests worth while. A list of ten suggestions for improving measurement programs in arithmetic follows.

1. Where the measurement program consists wholly of pencil-and-paper tests the program can be improved through the use of other means of measurement. The most valuable of these are the records of observations and samples of the pupil's work. This means that a folder or other means of keeping such information must be provided.

2. A second suggestion deals with the content of standardized survey-of-achievement tests. It is recommended that the content of these tests be broadened. Included in such tests should be items which deal with oral arithmetic, items which get at the pupil's knowledge and understanding of processes, and items which require the pupil to use data in making judgments. Items to illustrate each of the above suggestions follow:

(*a*) Without using pencil and paper select the best answer

for each item. 1. $43 \times 50 =$ ( ) 2015 ( ) 2150 ( ) N

(*b*) In the example $64\overline{)1967}$ (3 above the dividend) why is the quotient 3 placed above the dividend 6? ( ) Because the 3 is really 30 or 3 tens ( ) Because there are 3 64's in 196 ( ) In order to show the first partial dividend used.

(*c*) The car came to a halt about 100 feet from where it left the pavement. About how many times the length of the car do you think that was? ( ) 6 times ( ) 12 times ( ) 20 times ( ) N

3. The items in the preceding suggestion are of the recognition type. Because of the ease with which such items can be scored, it is recommended that practically all standardized tests be of that type. Time saved in scoring tests may be devoted to instruction. Far too often teachers use so much time and energy scoring tests that they have little enthusiasm for doing any instruction which might improve the conditions revealed by the tests.

4. The efficiency of some parts of standardized survey tests can be improved by having items deal with specific aspects of a process. For example, what a pupil knows about the placing of partial products in multiplication can be found out by using several items of this type: "In the example $\begin{array}{r} 267 \\ \times\ 326 \\ \hline 1602 \end{array}$ where should the next product figure be written: ( ) Under the 2 ( ) Under the 0 ( ) Under the 6." A pupil can respond to two or three such items in the time that it would take to do one multiplication. Furthermore, he might write the partial products correctly in the multiplication but get no credit for his knowledge of correct placement because a slight error in addition or multiplication gave him the wrong answer.

5. The inclusion of items in survey tests which require only

approximate answers is another suggestion for the improvement of tests in arithmetic. The following item is an illustration: "Minnesota with a total area of 84,068 sq. mi. has 4,059 sq. mi. of water surface. Which of these statements best tells what part of the total area is water surface: ( ) 4,059 of the 84,068 ( ) $\frac{4059}{84068}$ ( ) $\frac{1}{20}$."

6. The use of language and form of various types will make tests more informative. For example, instead of always using $\begin{array}{r} 43 \\ \times\ 21 \end{array}$ for multiplication or $54\overline{)319}$ for division, the following might be used: "How many are 21 forty-threes? 21×43 equals what number? 43 multiplied by 21 is how much? How many 54's equal 319? $319 \div 54$ equals what number?"

7. It should be noted that suggestions 4, 5, and 6 are applicable to teacher-made tests as well as to standardized tests. Suggestion 7, that tests on occasion include an essay item, deals entirely with teacher-made tests. Essay-type items are a superior means of obtaining evidence of pupil ability in such areas of arithmetic as the explanation of procedure and methods of showing. Because of the writing difficulties, this type of item has little value below Grades Five or Six.

8. As has already been indicated, the value of standardized tests in instruction will be markedly improved if they are provided with pupil profile charts on which growth from year to year can be charted. Such a use of test results will provide a check on whether the tests used are reliable measures of arithmetical achievement.

9. The interpretation of standardized test result scores will be facilitated if more easily described populations are used in the standardization program. As was indicated earlier, national norms are not very useful in getting a comparative idea of the level of achievement of any class of pupils. On the other hand,

it should be recognized that any norms, regardless of their value in interpreting levels of achievement, can be used in getting a picture of growth from year to year.

10. Measurement in arithmetic will benefit markedly through the institution of a planned and scheduled program of measurement from Grades One through Eight. Such a program should, of course, provide a means for keeping data and for getting the data to teachers in succeeding grades.

In addition to the ten specific suggestions for improving measurement in arithmetic given above, the program will benefit from a carefully planned procedure for the selection of published tests and for the preparation of teacher-made tests. The use of the reviews in the mental measurements yearbooks is one means of studying tests.[1] The procedures or guides for the preparation of teacher-made tests might well include sections on thought questions, questions relating to proving or showing, and questions directing attention to areas where essay-type items could be used.

Consideration of the amount of time given to measurement in arithmetic might be another means of improving the program. Since tests and other means of gathering evidence of pupil achievement in arithmetic are a definite part of instruction, it is difficult to say how much of the total time should be given to measurement. While there are no research data to warrant such a statement, it is recommended that not more than 20 per cent or less than 10 per cent of the total instruction time should be given to measurement unless there are specific reasons for doing otherwise.

Special attention is called to the statement in the preceding paragraph that tests are part of instructional procedures. Tests, if used properly, are good learning procedures and should be used for that purpose even if they serve no other purpose.

[1] See Oscar Buros, *The Third Mental Measurements Yearbook* (New Brunswick, New Jersey: Rutgers University Press, 1949).

STUDY QUESTIONS

1. About what per cent of the pages in current arithmetic books is devoted to tests? (*1*) Less than 2. (*2*) From 5 to 10. (*3*) More than 12.

2. In best practice, about what per cent of the total measurement program is of the pencil-and-paper type? (*1*) Less than 25. (*2*) About 50. (*3*) More than 75.

3. It is said that teacher-made tests are weakened by the fact that they deal almost exclusively with what has just been taught. Is that a valid statement? (*1*) Yes. (*2*) No.

4. In scoring arithmetic tests, credit is given only for correct answers. Is that procedure entirely valid? (*1*) Yes. (*2*) No.

5. If in the arithmetic instruction a child has been receiving he has used only the forms $3\overline{)12}$ and $12 \div 3$, is it proper to ask such a question as "How many 3's equal 12?" on a test item? (*1*) Yes. (*2*) No.

6. Tests in textbooks are often designed to serve a particular purpose, such as survey of skills needed in the next work to be undertaken. Is this designing of a test for a specific purpose very desirable? (*1*) Yes. (*2*) No.

7. Are the grade equivalent scores for standardized tests of actual assistance to teachers or do they merely satisfy the curiosity of teachers regarding pupil progress? (*1*) Of actual assistance. (*2*) Satisfy curiosity only.

8. For the guidance of the instruction of pupils, what type of standardized score (grade equivalent or percentile) is better? (*1*) Grade equivalent. (*2*) Percentile.

9. It is sometimes said that good teachers do not have any use or need for standardized tests. Is that a valid statement? (*1*) Yes. (*2*) No.

10. What is probably the major limitation of standardized tests in arithmetic? (*1*) The fact that the norms are not truly

standard. (*2*) The fact that the norms are given in grade equivalents. (*3*) The fact that the tests cover only a part of arithmetic.

11. Why have arithmetic readiness tests probably not been accepted as widely as have reading readiness tests? (*1*) Children have far more number than reading experiences before entering school. (*2*) The arithmetic readiness tests have not been well constructed. (*3*) Important arithmetic instruction does not begin until the third or fourth grade.

12. Why have diagnostic tests in arithmetic not been used more extensively? (*1*) Teachers, unlike supervisors and theoretical students of teaching, do not want the sort of information diagnostic tests furnish. (*2*) Teachers do not know what to do after the tests have revealed specific weaknesses. (*3*) The time and effort required in giving the tests is out of proportion with the benefits derived therefrom.

13. Does observation of pupil work in arithmetic rank very high as a measurement procedure? (*1*) Yes. (*2*) No.

14. At what grade level will standardized tests first become an important part of an arithmetical measurement program? (*1*) One. (*2*) Two. (*3*) Three.

15. What part of the school year is considered the best time to administer a standardized test? (*1*) At the beginning. (*2*) From twelfth to eighteenth week. (*3*) Near the end.

16. Is there any evidence to show that the time and energy given to recording observations, the collecting and filing of samples and the like is worthy of the effort required? (*1*) Yes. (*2*) No.

17. Is measurement of oral or non-pencil-and-paper arithmetic adaptable to standardized tests? (*1*) Yes. (*2*) No.

18. Why should the major part of standardized test items be of the recognition type? (*1*) Because that type item permits more searching measurement. (*2*) Because pupils can react to that type more quickly and therefore more items can be

included in a given length of time. (*3*) Because it permits the teacher to score the test more easily.

19. What kind of populations for standardization seem most desirable? (*1*) A national group. (*2*) An easily defined group such as one state or schools of a given size in a given region. (*3*) Groups representing only pupils of about average ability.

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13

Enriching the Teaching of Arithmetic

A Fifth-Grade Experience

A teacher began the arithmetic period one day by saying, "I have an interesting story of how a fifth-grade boy used an arithmetic trick to find the age of his aunt. The title is 'How Billy Got His Aunt Martha to Tell Her Age.'

"Several times Billy had asked his Aunt Martha her age but she had always answered in such queer ways as 'I was 28 years old for 10 years and now I have forgotten how old I really am.' When he was in third grade she had said, 'I've been your age many times but now my age is not less than four times and not more than eight times your age.'

"When Billy was in fifth grade and found that Aunt Martha was coming for a visit, he decided to use a number trick he had learned to get her to tell her age. This is what he did.

"'Aunt Martha,' he said, 'in school we have been doing mental arithmetic games and tricks. I'll give you one.'

"He then gave his Aunt Martha this one: 'Think of any number less than 10. Multiply by 2 and add 10. Take one-half of that and then subtract the number you began with. Now I'll tell you what you have left. It's 5.'

"Aunt Martha was a bit surprised but that trick was only a decoy. Billy was really just leading up to the important trick

to find out Aunt Martha's age. So he said in a challenging and eager voice, 'This next one calls for some real figuring. Will you try it?'

"Of course Aunt Martha said, 'Yes.'

"'All right,' said Billy, 'think of the month of your birthday (January is 1, February is 2, and so on) but don't tell me. Multiply it by 2.'

"When Aunt Martha said, 'I've done that,' Billy said, 'Add 5 and then multiply by 50.' He next said, 'Now add your age. Subtract 365 and then add 115.'

"Billy then asked, 'What number do you now have?' When Aunt Martha said '640,' Billy laughed and said, 'Now I know how old you are. You are 40 and your birthday is in June.'

"'That's right,' said Aunt Martha, 'but how did you know?'

"'The last two figures in 640 tell me your age, Aunt Martha, and the first figure gives the month of your birthday.'"

After the teacher finished reading the story to her fifth-grade class, she asked, "Would you like to try that scheme on your aunt or uncle or even on mother or father? If you wish, I'll put the figures Billy used on the board and explain each step."

Following a repetition of the Aunt Martha figures, the teacher said, "I'll tack this story of 'How Billy Got His Aunt Martha to Tell Her Age' on the arithmetic bulletin board. Then, you can check the steps and if we have some free time you can try the scheme on me."

Enrichment of Arithmetic

That such arithmetical experiences as the one described above have value is not doubted by teachers who have used the procedures in their classrooms. There are a great many number tricks, games, historical stories, unusual number operations and the like which can be used to enrich instruction in arithmetic. The need for special procedures and materials to make the

child's arithmetic experiences more interesting becomes particularly apparent if an examination is made of the various procedures and materials which are used to enhance the teaching of reading, science, and social studies. Even if the other areas of the elementary-school curriculum which compete for the child's attention had no attractive supplementary materials, it would be folly to pass up opportunities to enrich the teaching of arithmetic.

Students of arithmetic teaching need to consider the term *enrichment* carefully. It should be clear immediately that enrichment denotes something in addition to the essentials. Therefore, all materials suggested for enrichment are something optional, not something to be required of any individual pupils or of classes. As a result, most efficient use of such enrichment procedures will probably require even more skillful teaching than is used in teaching the essentials of arithmetic. Teachers may be warned, accordingly, not to undertake extensive use of enrichment procedures without careful study.

The fifth-grade lesson described above indicates one type of enrichment experience and suggests the use of a bulletin board for arithmetic. The lesson presented in the introductory chapter to this book showed some other enrichment procedures and incidentally illustrated some uses of a bulletin board. There are several distinct kinds of enrichment materials, of which the major types will be treated separately in the sections that follow.

Special Instructional Materials (Games, Tricks, and Puzzles)

There are many games, tricks, and puzzles utilizing number which appeal to children and which give them genuine uses for number. The usage of number in such activities varies from the keeping of scores to almost continuous use as in games like dominoes and monopoly. The value of the game for the enrich-

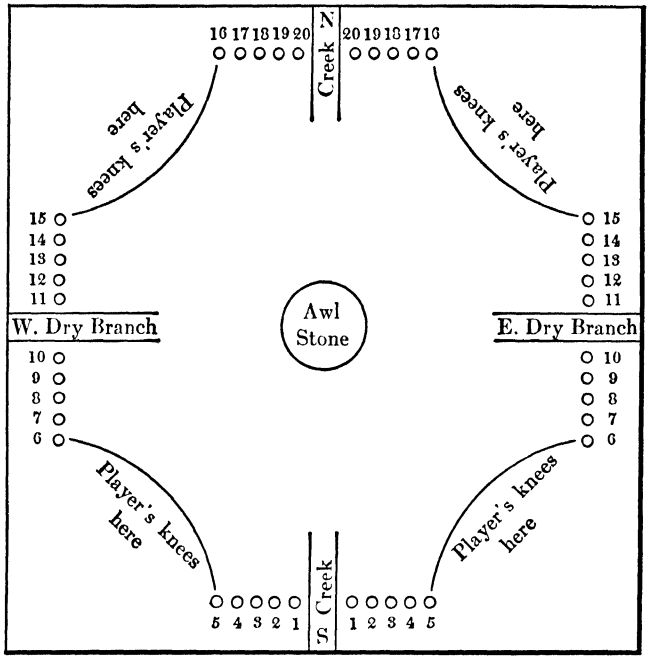
ment of arithmetic is not directly related to the amount of number used. In early grades, particularly the first and second grades, games furnish most of the enrichment exercises. Children have to have fair facility with numbers before they care much for tricks and puzzles.

Race games represent one of a large group of number games which primary grade children play with pleasure and which may contribute to growth in arithmetic. Two to four players or groups at a time may play such games. In order to acquaint all the children with the rules of a new game, the teacher may introduce one or two such games as a class project in which boys compete against girls or the pupils seated on one side of the room compete against those seated on the other side of the room. The number of spaces a team or player may move the marker is generally determined by drawing numbered cards from a stack, by spinning a pointer, or by throw of dice. After instructions for all have been provided through a teacher-directed group contest, the game is placed on a table to be played in assigned or free time by individuals and small groups.

A well-known game of this type is parcheesi. The Indian awl game is another version of the race game. (See illustration below for design and rules.) Games of this type are particularly valuable for giving experience in reading numbers and counting.

The game layout is marked on a 30″ × 30″ board (oak tag may be used). The numbered dots are awl markers. Four stick dice (half round and 8″ to 10″ long) are dropped on the awl stone. The number of spaces a player can move his marker (awl) is determined by the dice showing, usually one space for three round surfaces or three flat sides up, ten spaces for four round surfaces and five spaces for four flat sides up. Players start at South Creek and proceed in opposite directions. Awl marker is placed just beyond marker. Landing in North Creek sends a player back to the start.

A game involving the reading of numbers which primary



Indian Awl Game

teachers have used for recreational purposes is lotto or bingo. Dominoes provide many good number experiences ranging from the simple procedure of seeing who can turn up the domino with the most dots on it to the game in which a player registers a count only when the end dots equal 5 or a multiple of 5.

Many varieties of number race games and related number games can be found in gift shops, toy centers and other stores which stock children's toys.

For second grade and perhaps third grade, the game of twinks played with cards (4 each of 1's, 2's, 3's, 4's, 5's, and 6's) has some appeal. Each of the two to four players receives two cards and at each turn draws a card from the stack. However, he can keep only two and must place the extra card face up on the

table. When a player can, with a card in his hand and one or more of the exposed cards, make a combination equal to 8 (any other number can be chosen), all those cards become his permanent possession. The player who captures the most cards is the winner.

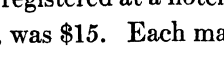
Hull Gull or Divided Shells is another game which second- and third-grade children like to play. Each player starts with an equal number (e.g., 6) of shells, buttons, or similar objects. At his turn, one player shows the number of objects, then places his hands behind his back and divides the number. When he puts one hand in front of him, the opposing player tries to guess the number of objects in his fist. If he guesses correctly, he gets all the shells. If he fails to guess the correct amount, he has to give the player the difference between the number of shells and the number guessed. The player having the most shells wins.

As indicated earlier, number tricks and puzzles are not of much interest until children have some facility with number. Beginning in the second grade, some use can be made of such exercises. Examples are given below.

1. *Trick questions*

(*a*) "When does 11 and 2 more equal 1?" (*Ans.* From 11 A.M. to 1 P.M.) (Grade 2)

(*b*) Jimmy had two coins equalling 60¢. One was not a dime. What were they? (*Ans.* 50¢ and 10¢. Only *one* was not a dime.) (Grade 3)

(*c*) Draw 6 vertical lines (| | | | | |) as shown. By adding 5 more, make nine. ( (Grade 3)

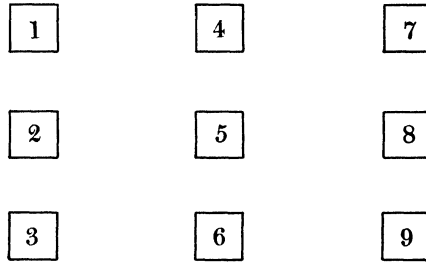
(*d*) Three men registered at a hotel. The clerk said their room, being a large one, was $15. Each man gave the clerk $5. Later the clerk decided that $15 was too much and gave the bell boy $5 to give to the three men. The bell boy decided that $5 would

be too hard for the three men to divide, gave each man \$1 and kept the other \$2. The room then cost each man \$4. $3 \times \$4 = \$12$. \$2 in the bell boy's pocket plus the \$12 = \$14. What became of the other dollar? (Grade 5) (*Key:* The error in thinking is made when the \$2 the boy kept is not recognized as part of the \$12 the men paid. \$15 − \$3 = \$12 (the amount paid by the men). \$12 − \$2 = \$10 (the amount the hotel received).

2. *Number puzzle exercises*

(*a*) The six dots on this triangle are to be numbered so that the sum of each side is 17. What shall be the number of each dot? A number can be used only one time. (Grade 3)

(*b*) What one card can be moved from its present group and put with another group so that all three groups will be equal? (Grade 4)



(*c*) After Henry had worked a multiplication example assigned for the next day, he spilled ink on it. Henry feared that if he didn't have the example correct he might have to go back to fourth grade. Figure out the figures and keep Henry in Grade Five. (Grade 5)

```
   xx7
   3xx
  ----
  x0x3
   x1x
  x5x
  ----
  x7xx3
```

(*d*) A group of men went to hunt rabbit and quail. One man

who did not like to shoot went along and when the hunt was over gathered up the dead birds and rabbits. One of the hunters asked for the total. The man answered jokingly, "36 heads and 100 feet." How many quail and how many rabbits? (*Ans.* 22 birds and 14 rabbits) (Grade 6)

A graded list of number tricks, puzzles, riddles, and the like has for a number of years been used as a part of the arithmetic program of the University Elementary School in Iowa City (copies of the list of tricks and puzzles are not available for distribution). This list has done much to supply needed recreational arithmetical material. Since no such published list of materials is available, it is suggested that teachers, principals or supervisors collect and assign to respective grade levels materials to be used.

Historical Number Material for Elementary Schools

Why be concerned with teaching elementary-school pupils history of number? Although there are often references [1] to the importance of the history of number in arithmetical instruction, the role of this phase of arithmetic has not been clearly defined. Because knowledge of the early development in almost any area of learning is considered a distinct asset to the student working in that area, it seems logical to suppose that knowledge of the development of number might be advantageous to the pupil studying arithmetic. Furthermore, in view of the very great role of numbers in modern life, information about the history of number should contribute to better understanding of modern society. Stories of the development of number have fascinated

[1] See Harry G. Wheat, *The Psychology and Teaching of Arithmetic* (Boston: D. C. Heath and Company, 1935), p. 2; L. J. Brueckner in *Thirty-fourth Yearbook*, National Society for the Study of Education (Bloomington, Illinois: Public School Publishing Company, 1935), p. 270.

many adults, and if one can judge by the few historical number stories that have been used with children, they, too, find them interesting. Then, in answer to the question "Why teach number historical material?" one may say: "Such materials contribute to a better understanding of arithmetic and are a motivating factor in the study of arithmetic."

To be in a position to evaluate the suggestions for teaching history of number, the student of the teaching of arithmetic should become somewhat familiar with the field by studying several books on the history of number. Some good source material is given in the Selected References at the end of this chapter.

A survey of the books on the history of number will reveal that much of the material is not of value to children of elementary-school age, even though that material may be very interesting to teachers. It is necessary, therefore, to select carefully the phases of number which are to be considered historically. Among the areas which seem most significant are (1) counting, (2) numerals and notation, (3) weights and measures, and (4) old ways of reckoning.

A Second-Grade Lesson and Teaching Suggestions for Historical Material

The teacher began this lesson by saying, "For our number work today, I want to read you a story of how an Eskimo boy counted. Perhaps, if you listen well, you will be able to count as the Eskimo did."

The teacher then read the story "A-Hand-and-one Winters," adapted from Kanguk.[1]

The expressions, "hand-less-one, hand-and-one, half a man," and others in the story intrigued the children and there was

[1] See Evelyne Sawatzky, "Number in Historical Literature for Children," Unpublished M.A. Thesis (Iowa City: State University of Iowa, 1949).

much showing on fingers as the teacher asked for volunteers to show each new number.

The material of this story provides background for understanding our use of a number base and also furnishes good situations for showing quantities with another notational scheme than the one we use.

For follow-up exercises, the teacher suggested (1) that the children tell and show how the Eskimos of Kanguk's village counted and (2) that the pupils make drawings of hands to show different Eskimo numbers. Later a folder with outline drawings of hands and feet showing one, two, three, four, and five digits was placed on the table with directions to show such numbers as 12, 7, 18. In the days that followed there were references to the Eskimo way of counting by such remarks as "What number do I have in mind when I say one-half a man and two more?"

The Apache method of counting by matching pebbles with the quantity to be enumerated provides a setting for an experience similar to the second grade lesson described.[1]

The ways American Indians used to record numbers can also be used as the basis for good number experiences for elementary-school children.[2] In one second grade, the teacher introduced consideration of Indian number writing with this statement: "On the arithmetic bulletin board is a copy of a drawing that an Indian made on a piece of bark. It is a message telling how many men were in a war party. There were thirty men. How did the Indian write 30?"

The reader should recognize that effective use of such historical material is in a great measure dependent upon the degree to which it is adapted in vocabulary and simplicity of concepts

[1] See Edward O. Morris, *An Apache Life Way* (Chicago: University of Chicago Press, 1941), pp. 354–355.

[2] See William Tompkins, *Universal Indian Sign Language* (San Diego, California, 1929), pp. 80, 85, and 89.

to the elementary pupil's ability and interest. Since these factors are markedly affected by previous experience, it follows that historical material will be most successful in those classrooms where pupils know something about number. For example, to appreciate how the Eskimos of Kanguk's village counted, children need to know how we count and especially that we count by tens. In a like manner, children who know something about American Indians can be easily interested in the Apache way of counting or in various ways Indians used to record numbers.

Other good historical number situations which can be adapted for use with elementary school children are available. The quipu, a Peruvian counting device, with its variations is not only interesting but well adapted for use on an arithmetic table. For example, one third-grade teacher put on the table a quipu with four colors of thread. On each thread there were a number of knots. A placard stated, "This quipu is a record of how much money a woman owed the grocer. The blue cord is for dollars, the red for dimes, the green is for nickels, and the white is for pennies. How much did she owe?"

The Madagascar chief's way of counting was explained like this in a third grade. "The large drawing on the bulletin board is the record an African chief made of the number of his soldiers. His numbers are not like ours but if you know how, you can read his numbers." After some guessing about how many soldiers he had, the teacher told the story of how the chief counted his men. As each tenth soldier went by, the chief had that soldier drop his spear on the ground. When there were 10 spears, the chief had a helper pick up the 10, place one upright, and then give the other 9 spears back to the owners. The children had little difficulty after that in determining the number of soldiers represented by the chief's drawing.

From the few suggestions made here for counting and notation, it is clear that much material is available. The methods of

writing numbers used by the Mayas, the ancient Egyptians (especially their symbols for 100,000 and 1,000,000), the Babylonians with their 60's, and medieval to modern ways can be readily adapted for use with elementary school pupils. Similar lessons can be worked out with weights and measures, telling time, and old ways of reckoning (see chapter on Multiplication and Division (pages 171–174). For each area, explanatory reading materials, suggestions for teacher explanations, interest arousing drawings, and pupil activities need to be developed.

For source material on the areas mentioned, the following references are recommended.

1. White, Anne Terry, *Lost Worlds*, New York: Random House, 1941 (for Maya numbers).

2. Karpinski, L. C., *The History of Mathematics*, Chicago: Rand McNally Company, 1925 (for Egyptian and other ancient numbers).

3. Smith, D. E., *The Wonderful Wonders of One-Two-Three*, New York: McFarlane, Warde & McFarlane, 1937, pp. 9–12 (for old ways of writing numbers).

4. Smith, D. E., *History of Mathematics*, Vol. II, Boston: Ginn & Company (for old ways of figuring).

5. Yeldham, Florence, *The Teaching of Arithmetic Through Four Hundred Years*, London: Harrap & Co., Ltd., 1936 (for old ways of figuring).

6. Nicholson, Edward, *Men and Measures*, London: Smith, Elder, and Company, 1912 (for old measures).

7. Chiera, Edward, *They Wrote on Clay*, Chicago: University of Chicago Press, 1938, p. 155 (for Babylonian use of zero).

8. Irwin, Keith Gordon, "Fathoms, Feet, Acres, and Tons: An Appraisal," *Scientific Monthly*, January, 1951.

9. Bendick, Jeanne, *How Much and How Many*, New York: McGraw-Hill Book Company, 1947.

10. American Council on Education, Committee on Materials of Instruction, *The Story of Time* and *The Story of Weights and Measures*, Washington, D.C., 1932.

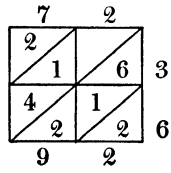
The use of an old way of reckoning to reintroduce multiplication in a sixth grade was described on pages 171–172. This historical situation provides not only some good arithmetic instruction but an excellent review use of the best multiplication procedure. The upper grade arithmetic teacher is very much in need of such genuine uses of processes for review and reteaching.

The Scratch, Gelosia, and Lightning Methods of Multiplying

Scratch

72
× 36
~~2162~~
~~58~~
9

Gelosia



Lightning

72
36
2592

In the scratch method multiplication begins at the left. $3 \times 7 = 21$. This product is written in its proper place. Then the product for 3×2 is written in its place. The product of 6×7 is combined with numbers already written in the places, that is, the 4 (of 42) is combined with 1 (scratched) and the 2 (of 42) is combined with 6 (scratched). In a similar manner the 1 of 12 (product for 6×2) is combined with the 8 (scratched). The numbers remaining are the product.

The multiplicand of the Gelosia method is written across the top. The multiplier is written along the right-hand edge. Each product (e.g., $6 \times 2 = 12$ and $6 \times 7 = 42$) is written in its proper cell with the diagonal line dividing two-figure products. The sum of figures in each diagonal column is next obtained by reading the sums figures beginning with the top figure on the left-hand side, reading down and across the bottom. For larger numbers a correspondingly larger number of cells are required.

The lightning method procedure is as follows: $6 \times 2 = 12$. Write the 2 and carry the 1. $6 \times 7 = 42$ (tens) $42 + 1$ (carried)

= 43 tens. Keep this figure in mind for it must be combined with the product of 3×2. 43 tens + 6 tens (from 3×2) = 49 tens. Write the 9 and carry the 4 tens. $3 \times 7 = 21$ (hundreds). 21 (hundreds) + 4 (hundreds) = 25. Write this.

Intensive Study of Number Operations

Without doubt, if properly motivated, the intensive study of number operations ranks first among the various procedures suggested for enriching arithmetic teaching. The chapter on problem-solving gave some illustrations of how intensive study of number operations can be carried out (see pages 195–196). Three additional suggestions are given below.

Mental or non-pencil-and-paper exercises

The discussion and explanations which are an integral part of good oral arithmetic exercises often involve careful consideration of number operations. The following excerpts from class discussions illustrate the point.

"I added 26 and 9 by thinking 26 and 10 and then taking away 1." Careful consideration was then given to the question of whether or not adding 10 and subtracting 1 was the same as adding 9.

"I changed the 48 (in 48 + 53) to 50, then added 53 and took away 2." When the validity of this procedure was challenged by the teacher, the class tested it in various examples.

"I thought $6 \times 8 \times 4$ (for 12×16)." In response to "why multiply by 4?" the pupil said: "6 is one-half of 12 and 8 is one-half of 16. So you take $\frac{1}{2}$ of $\frac{1}{2}$. That's $\frac{1}{4}$. To get the complete answer, you have to multiply by 4." A number of examples were then worked to test this way of multiplying two-figure numbers.

Since some statements on oral arithmetic in the problem-solving chapter are also applicable to enrichment, that section of the book should be consulted.

STUDY OF SPECIAL METHODS

A. Multiplying a two-figure number ending in 5 by itself. (Grade 5). Examination of a few products will show that they always end in 25 and that the rest of the answer is the product of the first tens number and one more than the second tens number. Thus the product 35×35 is found by thinking $4 \times 3 = 12$ and then writing 25 after the 12.

B. A short way of multiplying by 25. (Grades 6–7). Multiply the multiplicand by 100 and divide the resulting product by 4.

C. Check of nines in addition, subtraction, multiplication and division. (Grade 5 for addition and subtraction; Grades 6, 7, 8 for multiplication and division). For those unfamiliar with the check, the basic procedure is that of taking as many nines as possible out of the sum of the digits in a number and dealing with the excess. It should be remembered that for each process the final calculation is always performed with the excesses in their simplest form, that is, with the nines again cast out. In addition, the excess of nines in the added figures should equal the excess of nines in the sum figures.

(1)
$$\begin{array}{r} 1246 \quad (4) \\ 3402 \quad (0) \\ +\ 9264 \quad (3) \\ \hline 13912 \quad (7) \end{array}$$

(2)
$$\begin{array}{r} 3091 \quad (4) \\ -\ 896 \quad (5) \\ \hline 2195 \quad (8) \end{array}$$

(3)
$$\begin{array}{r} 76 \quad (4) \\ \times\ 21 \quad \times (3) \\ \hline 76 \quad (12) = (3) \\ 152 \\ \hline 1596 \quad (3) \end{array}$$

(4)
$$\begin{array}{r} 912 \quad (3) \\ \times\ 351 \quad (0) \\ \hline 912 \\ 4560 \\ 2736 \\ \hline 320112 \quad (0) \end{array}$$

The excess of nines in the first addend (see example 1) is obtained in this way: $1 + 2 + 4 + 6 = 13$ which is 4 in excess of 9.

In subtraction, the excess of nines in subtrahend and difference figures equals the excess of nines in figures of the minuend. In example 2, $8 + 5 - 13$ which represents an excess of 4, the same as the excess in the minuend.

In multiplication, the product figures excess equals the product of the excess of multiplier and multiplicand figures. (See examples 3 and 4).

For division, the quotient excess times divisor excess equals the dividend excess.

In division the process is slightly complicated when there is a remainder. The remainder figure must be subtracted from the dividend excess. For instance, in example 6, the dividend excess is $3 + 4 + 4 + 8 - 4 = 15$ or 6. The remainder may also be added to the excess of the product of divisor times quotient.

(5)
$$\begin{array}{r} (8) \\ (2)\ 35 \\ 29\overline{)1015} \\ 87 \\ \hline 145 \\ 145 \\ \hline \end{array}$$
(7)(2)×(8)=(16)=(7

(6)
$$\begin{array}{r} (6) \\ (1)\ 42 \\ 82\overline{)3448} \\ 328 \\ \hline 168 \\ 164 \\ \hline 4 \end{array}$$
(6)

SUPPLYING THE MISSING NUMBERS

Finding the missing figures in examples like the one shown below is a puzzle type of exercise which may be used in an enrichment program. Such a situation becomes much more interesting if a setting for it is provided (see page 377). The use of pictures for the missing numbers has also been used to increase interest. These missing number situations may involve all of

the fundamental number operations and can differ markedly in difficulty.

$$\begin{array}{r} 354 \\ \times\ xx \\ \hline xx70 \\ xxx \\ \hline x850 \end{array}$$

Arithmetic Table and Bulletin Board Exercises and Suggestions

In a number of the classroom procedures described in this and previous chapters, reference was made to an arithmetic table or to an arithmetic bulletin board (see for example pages 8–9, 290, 333, and 339). The role both these devices play in an enrichment program is very important. The arithmetic table provides a center for number activities which can be carried on by individuals or small groups. Usually these activities are pursued in free time but occasionally they may be assigned by the teacher.

Soon after the beginning of school, the first-grade teacher will find many uses for the numbers or arithmetic table. The games referred to in an earlier section can be played more successfully if a table on which they may be placed is provided. First-grade teachers also find the table an ideal place to put such number suggestions as "Show with grains of corn the price of each toy" (to be used with a card on which are pasted pictures of various toys with the prices marked) and "Show eight with grains of corn" (any other counters may be used). Other first-grade number-table activities of merit are identified in the following list.

(*a*) Arranging numbers in order. Envelopes may contain various sets of number cards. Some sets have all the numbers from 1 to 10 or 1 to 50, while other sets have only the 10's to

100 or 150. Still others will have such an irregular sequence of numbers as 7, 21, 33, 36, 39, 43, etc.

(*b*) Matching the number card and the dots. Envelopes contain two sets of cards, one set with numbers, the other with dots.

(*c*) Drawing lines to follow the numbered dots. This type of exercise is placed on the arithmetic table to be picked up by pupils and filled in during free time or work periods.

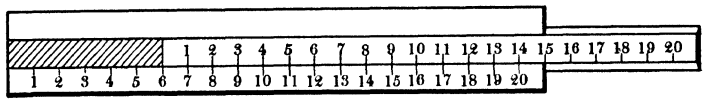
(*d*) Showing the number given on a card with tens and ones blocks, hundreds number squares, peg boards or with single objects. Sometimes various children's names are placed on the card, e.g., "John — 26." When John shows 26 with objects like tens and ones blocks, he places his card beside the completed representation.

(*e*) Unit number blocks of 1, 2, 3, 4, and 5 units with which to show a suggested number like 8, 10, 12, etc.

(*f*) Balances for weighing.

(*g*) Objects or pictures with the suggestion to count and show how many. For example, for ears of corn, the suggestion "count the rows of grains" was made orally.

(*h*) Fact finders, sliding scales, and other simple mechanical devices.



Sliding Scale

(*i*) Measures: cup, pint (also $\frac{1}{2}$ pint), quart, ruler, egg cartons (two types).

(*j*) Clock dials — old alarm or desk clock.

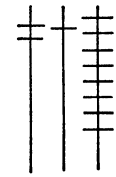
(*k*) Scales for weighing objects (0–25 pounds).

Some of the activities suggested for first grade are also applicable to second and even third grade. Additional number table activities for these grades are listed below.

(*a*) Various abacuses (Chinese, Japanese, Roman pebble type) with suggestions to show given numbers. For example, "Show 94" might be placed beside a bead abacus.

(*b*) Worksheet-type paper abacuses (which pupils may use during free periods).

(*c*) Games like dominoes, twinks, the Indian awl game,[1] and the like.



(*d*) Guessing exercises with, for example, a small bottle of beans with the label "guess how many." A class roll with space for each pupil to record his guess is provided.

(*e*) Manipulative exercises such as the following: "Can you arrange these 7 blocks so that there will be 3 rows with 3 blocks in each row?" (See also page 377.)

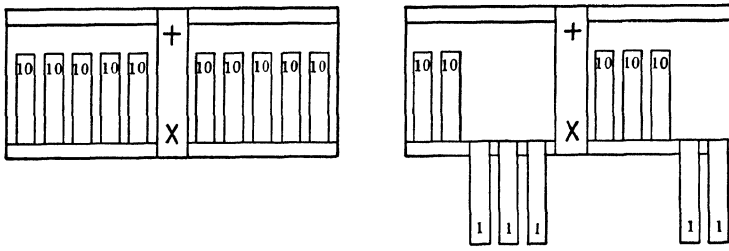
(*f*) Mechanical devices for measuring: Speedometer from car, hour glass (3-minute type), rain gauge, radio dial, clock.

(*g*) References: Price lists and quantitative descriptions as found in mail order catalogs.

The arithmetic table in Grades Four, Five, and Six may have on it some of the things listed for Grades Two and Three. The abacus, especially the Chinese type and the board or dust type, have a place in the fourth and fifth grades, and, of course, the mail order catalog is useful in the upper elementary grades. Automobile speedometers and old gas and electric meters may also be used. In these upper grades, some teachers have found the arithmetic table a good place for instructional aids such as sets of rectangles and squares for teaching measurement of area, fractional equivalence boards, various objects (stones, blocks, sticks, and the like) for showing what numbers mean and demonstrating number operations without figures, sets of cubes for teaching cubic measurement, and so on. Special calculating devices like the multiplication apparatus for 6's, 7's,

[1] See Stewart Culin, *Games of North American Indians*, U.S. Bureau of Indian Ethnology, 24th Annual Report, pp. 121–122, and description on page 375.

8's, and 9's shown in the diagram may also be placed on the table. This device consists of a wooden framework with fingers hinged at the bottom. To multiply two numbers, for example



7×8, three fingers $(10 - 7)$ are turned down on the left and two fingers $(10 - 8)$ are turned down on the right. The tens number of the product is found by adding the fingers left standing (in this case, $2 + 3 = 5$). The ones part of the product is found by multiplying the number of fingers turned down (in this case, $3 \times 2 = 6$).

Two other devices which may appear on the table are the number line, both vertical and horizontal (see page 68), and the number board with removable numbers (see page 67).

The arithmetic bulletin board which, like the arithmetic table, is a genuine asset to an enrichment program, has been referred to a number of times already. From first grade through sixth grade, this bulletin board is a place for displaying unusual and challenging number situations. For example, when in a first grade one child wrote two hundred and twenty-five as follows, $\begin{matrix} 100 \\ 100, \\ 25 \end{matrix}$ a copy of his work was placed on the bulletin board. The teacher said, "Some of you may want to show numbers the way John did. If you can make your numbers clear to us, we want to put them on the bulletin board."

Such unique ways of showing number and number operations are available for every grade and when placed so that children

have an opportunity to examine them, may provide deeper insight into the situation than the conventional methods afford. The display of good work of every kind is, of course, a legitimate use of the bulletin board even if the work displayed contributes little in the way of enrichment.

The number tricks and puzzles referred to earlier in this chapter are used most effectively if they can be displayed where pupils can study them. The collection of tricks and puzzles for each grade does not have to be very extensive to have at least one new item on the bulletin board each week. Number riddles, magic squares, number pleasantries and old ways of figuring are good bulletin board material. Examples of proof as that term has been used in this book may be displayed to advantage as well as children's compositions about how they perform number operations. The bulletin board is also a good place for thought-provoking review questions such as "In finding the common denominator in $1\frac{3}{4} + 2\frac{1}{3} + 1\frac{5}{8}$, what denominator is considered first?"

The items below, which were all taken from arithmetic bulletin boards, will illustrate ways in which such boards may contribute to the enrichment of arithmetic.

Grade 1. What number is shown? (A number of marks in a large circle were displayed. Cards from which the pupil might select the correct number were stacked in chalk tray.)

How many birds in the picture? (Pictures of birds were displayed with number cards for indicating the number nearby.)

Grade 2. What's under the key? 4 and 6 and 8 and 3 equal? (A cardboard key covered the answers.)

About how much does your reading book weigh? (This question in large letters appeared under the heading "Arithmetic Question of the Day.")

Grade 3. Ways of showing $4 \times 6 = 24$

(1) $6 + 6 + 6 + 6 = 24$

(2) 2 6's = 12. Then, 4 6's = 2 12's = 24

Dividing by subtracting

$$
\begin{array}{rl}
15 \div 5 = & 15 \\
 & -\ 5 \quad (1) \\
\hline
 & 10 \\
 & -\ 5 \quad (2) \\
\hline
 & 5 \\
 & -\ 5 \quad (3) \\
\hline
\end{array}
$$

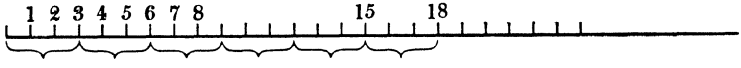
Grade 4. How to borrow when you have zeros.

"To borrow in such an example as the one in the box, you think this way. Think of the 4 hundreds as 40 tens. Then, when you change one of the 40 tens to ones, you have 39 tens left. You then think, 4 ones from 10 ones leave 6, 5 tens from 9 tens leave 3 tens, 1 hundred from 3 hundreds leave 2 hundreds."

$$
\begin{array}{r}
4\ 0\ 0 \\
-\ 1\ 5\ 4 \\
\hline
2\ 3\ 6
\end{array}
$$

Dividing with the number line

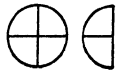
18 ÷ 3



Grade 5. Check of nines in addition

$$
\begin{array}{rl}
43 & \\
31 & \\
18 & \\
94 & \\
16 & (4) \\
\hline
202 & (4)
\end{array}
$$

Do you know the meaning of fractions? Select and read the proper fractions, mixed numbers, and improper fractions in these drawings:

(1) 

(2) 

(3) 

Grade 6. Two ways of multiplying a mixed number:

A. $\begin{array}{r} 4\frac{1}{2} \\ \times\ 3 \\ \hline \end{array} = \begin{array}{r} 4 \\ \times 3 \\ \hline 12 \end{array} + \begin{array}{r} \frac{1}{2} \\ \times 3 \\ \hline \frac{3}{2} \end{array} = 13\frac{1}{2}$

B. $3 \times 4\frac{1}{2} = 3 \times \frac{9}{2} = \frac{27}{2} = 13\frac{1}{2}$

When is "A" the better way to multiply? When is "B" the better way to multiply?

Fractions used to clarify quantitative statements

What fractions could be effectively used to give us a better idea of the facts given in the statements below?

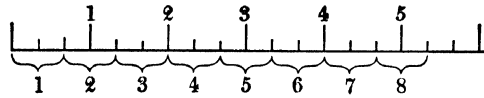
"The total exports of Ecuador in 1950 were valued at $63,300,000. Coffee, valued at $19,100,000, was first; cacao, valued at $18,345,000, was a close second; rice, valued at $8,415,000, was third."

The common denominator method of dividing fractions

$8 \div \frac{2}{3} = \frac{24}{3} \div \frac{2}{3} = 24 \div 2 = 12$

Proof

$8 \times \frac{2}{3} = \frac{16}{3} = 5\frac{1}{3}$



Another possibility for enrichment which shows a great deal of promise is the teaching film. Unfortunately, many films of rather mediocre quality have been produced and for that reason alone, films should be selected with care. Furthermore, many of the films of fair quality from the standpoint of teaching do not contribute much that could be classified as enrichment of the subject.

The numbers project (see page 197) in arithmetic may also add to the enrichment of the subject.

Number Pleasantries and Unusual Number Operations

Every person who works with numbers has at some time noticed strange things about them. For example, the sum of the digits in the products of the nines is equal to nine ($7 \times 9 = 63$; $2 \times 9 = 18$, etc.). The number 9 is involved in many of the strange or unique number situations, but other numbers present some classic examples of the unusual. The following is an example:

$3 \times 37 = 111$ $\quad$ $12 \times 37 = 444$
$6 \times 37 = 222$ $\quad$ $15 \times 37 = 555$
$9 \times 37 = 333$ $\quad$ $18 \times 37 = 666$

It is not difficult to show that most of these unique number situations are the result of the number system we use. The ease with which one can count by fives is a good illustration of how our system accounts for a strange number situation.

The unique is usually interesting and the field of arithmetic is no exception. The student of the teaching of arithmetic should, therefore, consider carefully the possible values that may result from use of unusual number facts and number situations in teaching elementary-school pupils. Many adults believe that they gained real insight into numbers through study of such number phenomena as the two examples given above. Here as in other phases of arithmetic, the knowledge found either in books or already in possession of the teacher does not necessarily insure a good pupil learning experience. The teaching or learning situations must consist of lead-up experiences which eventually pose a problem for the child. In the case of number pleasantries, the solution usually produces pleasant surprises. A simple illustration of how a teaching situation differs from just telling is shown below.

In this third grade room there was a counting chart like the one below.

Counting by 1's, 2's and so on

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

The teacher said to the class, "There are some unusual things about the figures in the nines row. Have you noticed them? Before we talk together about what you think is unusual, see if you can find three strange things about the nines row by yourself." After a short time for examination of the counting chart these unusual things were mentioned: (*a*) the last figure becomes one less each time you count a 9; (*b*) the first figure beginning with 18 becomes one larger each time you count a 9; (*c*) the sum of the digits of each number in the nines row is 9.

Another example of good teaching in the use of number pleasantries is illustrated below.

A fifth-grade teacher began the lesson by saying: "Before we begin our arithmetic work today, let's give some attention to the way we write numbers. If you were to look carefully at the figures on those arithmetic papers on the bulletin board, you would probably say 'Most of us need practice writing numbers.' I can show you an interesting way to get practice on the number you write least well. Copy these numbers 1 2 3 4 5 6 7 9

on your paper. Most of you write 8 well enough that you don't need any practice on it. Look at the numbers you have written and choose the one on which you think you need the most practice."

After a brief pause the teacher said, "If you chose 9 for practice, multiply the figures you have on your paper by 81. If you chose 7, multiply by 63; if you chose 6, multiply by 54." For each figure a multiplier (the figure multiplied by nine) was suggested. The completed work for those who chose 6 is shown below. The pleased looks of surprise as various pupils finished

$$
\begin{array}{r}
1\,2\,3\,4\,5\,6\,7\,9 \\
\times\,5\,4 \\
\hline
4\,9\,3\,8\,2\,7\,1\,6 \\
6\,1\,7\,2\,8\,3\,9\,5 \\
\hline
6\,6\,6{,}6\,6\,6{,}6\,6\,6
\end{array}
$$

was all the evidence any teacher needs of the merit of this exercise.

The important feature to note in the two teaching illustrations given above is the fact that the pupils participated — they did something besides listen or read. Getting pupils to do some activities that may lead to pupil discovery of the knowledge being taught is characteristic of superior teaching in nearly all areas of arithmetic.

The field of worth-while number pleasantries is rather limited and for that reason no more space will be given to the topic here. The references cited [1] give examples of number pleasantries. More will be found in some children's arithmetics and other books on the subject of arithmetic. Here, too, an im-

[1] D. E. Smith and D. Ginsburg, *Numbers and Numerals* (Washington, D.C.: National Council of Teachers of Mathematics); D. E. Smith, *The Wonderful Wonders of One-Two-Three* (New York: McFarlane, Warde, McFarlane, 1937); Joseph C. Brown, *Easy Number Tricks* (Aurora, Illinois: Strathmore Company, 1936).

portant task for teachers of arithmetic is the collection and grading of number pleasantries and the development of ways to use them effectively in the classroom.

STUDY QUESTIONS

1. Which phase of arithmetic: (*1*) the essentials or (*2*) the enrichment, demands the greater teaching skill? (*1*) Essentials. (*2*) Enrichment.

2. In addition to creating interest in arithmetic classes and a good attitude toward the subject, do arithmetic games and tricks make any other important contribution to the teaching of the subject? (*1*) Yes. (*2*) No.

3. At which grade level — first, second, third, or fourth — would arithmetic tricks be most important? (*1*) First. (*2*) Second. (*3*) Third. (*4*) Fourth.

4. At what grade level, first or fifth, are arithmetic games most important? (*1*) First. (*2*) Fifth.

5. The quipu is useful for teaching what phase of number? (*1*) Counting. (*2*) Adding. (*3*) Multiplying. (*4*) Dividing.

6. The check of 9's is probably best suited to what grade level? (*1*) Third. (*2*) Sixth. (*3*) Eighth.

7. The arithmetic finger board multiplier is probably most effectively used at what stage of study? (*1*) When multiplication is first introduced. (*2*) After the 2's, 3's, 4's and 5's have been studied. (*3*) After the 6's, 7's, and 8's and 9's have been introduced.

8. Are the possible contributions of number pleasantries to the enrichment of arithmetic very great? (*1*) Yes. (*2*) No.

14

A Summary and a Critical Examination of Major Issues

The thirteen preceding chapters have dealt with teaching the most important phases of arithmetic. Emphasis has been given to what might be called the developmental method of teaching. Before summarizing the important characteristics of that method as contrasted with others, the author's position should be given in regard to certain major issues in teaching arithmetic which are more or less independent of particular methods. Each issue is stated in question form.

1. *Is drill or practice necessary in learning arithmetic?* If pupils are to use arithmetic effectively, then certain facts and processes must be mastered to the point where little thinking is necessary concerning the correctness of fact or procedure. The kind of knowledge required for proficiency in arithmetic is far more than that represented by the "oh, I see" expression. Only practice on a process or fact can give to pupils the mastery which is essential.

It is recognized that many pupils acquire mastery of certain phases of arithmetic, especially the basic facts, without engaging in any known practice exercises. Such pupils are almost always those who have had extensive number experience before instruction in that phase of arithmetic is begun and may be

assumed to have gotten the requisite practice by themselves. It is also well known that some pupils do not require much practice to become efficient. This is just as true in skills other than arithmetic like learning to play the piano or handling a basketball.

2. *Is engaging in drill before pupils understand ever to be practiced?* Understanding before drill has become almost an axiom of modern teaching of arithmetic. There are research data to show that drill before understanding is ineffective and perhaps detrimental to learning. It would be well to realize, however, that if the principle of the greatest good is followed, some pupils will engage in drill before they understand. Consider, for example, the following situation. In a class of thirty pupils, after procedures to provide understanding have been used, all except four or five pupils seem to have grasped the new idea or process. It is far better to begin practice with the group even though some are not ready and may be harmed by drill, than to run the risk of losing the interest of the majority with the resulting discipline problems. The important points to note are that pupils should have an opportunity to acquire understanding before drill is undertaken and that supplementary procedures to promote understanding should then be interspersed with drill and review.

In addition to the concern for understanding, drill should be influenced by several other factors or principles. As was shown in the teaching procedures, pupils should see a need for drill. To be most effective, drill should be directly on the process. This last factor limits the role of problems and games in drill, for each of these requires attention to more than the process being practiced. In fact the chief basis on which problems and games can be justified in the drill procedure is the need for variety to help maintain interest.[1]

[1] For a more extensive discussion of the place of drill, see B. R. Buckingham, in *Sixteenth Yearbook* of the National Council of Teachers of Mathematics

3. *What are the best procedures for dealing with individual differences?* Since the range of achievement in every normal arithmetic class varies considerably, often as much as three grades, the problem of adjusting instruction to the individual abilities must be considered. A typical range of ability is shown by data in Table 1.

TABLE 1. SCORES IN GRADES 3, 4, 5, AND 6 ON ARITHMETIC TEST OF EVERY-PUPIL TESTS OF BASIC SKILLS, JANUARY, 1952, UNIVERSITY ELEMENTARY SCHOOL, IOWA CITY, IOWA

Grade	3	4	5	6
High Score	5.9	5.9	8.0	8.9
Median Score	3.9	4.8	5.7	7.2
Low Score	2.8	3.3	4.1	4.7

Ways for dealing with individual differences range from the completely individual instruction of the Winnetka plan to giving the same instruction to all members of the class without special regard to individual differences. Between these two extremes are many variations like dividing a class into three to five instructional groups.

The individual instruction plan has not been too successful, partly because of lack of material but primarily because class spirit quickly reaches a point of no consequence when members of the class are not working on the same area of the subject.

The theory of dividing the class into sub-groups appeals to many, especially those teachers who have used or have seen grouping used with success in reading instruction. In actual practice such grouping at different levels in arithmetic has not

(New York: Bureau of Publications, Teachers College, Columbia University, 1941), pp. 196–224. For a comprehensive discussion of drill, see J. B. Stroud, "The Role of Practice in Learning," *Forty-first Yearbook*, National Society for the Study of Education (Bloomington, Illinois: Public School Publishing Company, 1942), pp. 353–76.

attracted many followers. Teachers who have used the plan successfully expend great energy trying to direct the different groups and give adequate time to each. Since this kind of organization means the teacher is actually teaching the equivalent of several classes, the time given to each is just as inadequate as the instructional time in the ungraded school. At best it parallels the situation found in two- and three-teacher eight-grade elementary schools.

Since grouping in arithmetic is in part an outgrowth of practice in reading instruction, it might be well to analyze some of the differences in the two areas to see why the procedure is less successful in arithmetic. Grouping in reading is most widely found in the first and second grades. It becomes less common in the third grade, and in Grades Four, Five and Six the familiar three reading groups are absent in the majority of schools. Apparently grouping is not considered as useful in upper-elementary-grade reading; yet, it is at this upper-grade level that grouping in arithmetic is advocated. A necessity for successful grouping in reading is provision of reading material on different levels. No such materials are available in the field of arithmetic.

Besides, it should be recognized that in some of the best instructional programs of primary reading, all the children in the group work together on some reading experiences. This is illustrated in the making and reading of letters of thanks, invitation and the like, and in the production and reading of diaries, logs, or reports of experiences. Furthermore, the class is not commonly divided in science, social studies, and literature, all of which involve some aspects of reading. Usually in these good schools there are some pupils in the upper grades (e.g., Grades Two to Seven) who are rather far below general ability. The teachers frequently provide for such pupils special reading and instructional materials such as individual assignments in workbooks and easier readers. The class then consists of two groups,

a large group reading at one level and a small remedial group provided with different work. It should be noted that this division into two groups with differentiated materials for one group is only for the subject of reading. There are no such materials for science, social studies, and the like where the major reading of upper-grade pupils is done. In these areas, those of below-average ability use the same materials as do the superior readers. Under these conditions, which are in conflict with the recommendations of the proponents of grouping, some progress is made by the readers of below-average achievement. This fact should be of special interest to teachers of arithmetic.

The most common instructional practice, that of having all pupils work on the same assignments in arithmetic, is not the result of a carefully thought out procedure of caring for individual differences. It is, in most instances, a case of not being concerned with or aware of any plan for dealing with variations in ability. That the plan has met with some success is shown by the gain in achievement from grade to grade as revealed by standardized and informal test results. Gains where no attempt is made to provide for differences would be reassuring if it were not for other factors. Every student of arithmetic teaching who has seen pupils trying to do something which they cannot do and do not understand is aware of the frustrations such pupils experience. These students of arithmetic teaching are not much interested in gains obtained at such a price. They contend that greater gains can be attained without subjecting pupils to so many frustrating experiences.

The fact that the individual instruction procedure and the division of a class into sub-groups have not produced outstanding results, careful consideration of the combination grouping procedures in reading, and the fact that growth is obtained in arithmetic when no attention is paid to individual differences, have led the writer to formulate the following recommendation for caring for individual differences. It is suggested (*a*) that, for

the most part, pupils work on the same area in arithmetic with distinct provision for different levels of work in the area, and (*b*) that where differentiation in the area of instruction takes place, it be on an individual basis.

The provision for different levels of work within the same area makes possible the retention of class unity. The same general topic can be attacked by the whole group if one realizes that drawing and counting as well as special study procedures for improved speed all have a place in instruction. In social studies or science teaching, the simplest facts given by the less able students are recognized by teacher and other class members, and the notes and other evidence of achievement of best and poorest pupils on any topic show great variation. In the same way in an arithmetic class working on the multiplication of fractions, some children who are not advanced enough to deal with the number symbols for fractions might make drawings or use the fractions on a ruler to find an answer while others use the best-number method of solution and check their work by using several other methods. Since drawing and counting are often suggested as a means of checking for the more able students, both superior and inferior are using the same forms of instruction and have something in common, a fact that makes for class spirit.

As indicated earlier, pupils who are weak in arithmetical ability would occasionally be given special remedial work. For example, if the sixth-grade class is working on division of whole numbers and several pupils have not mastered the basic multiplication facts, these pupils would at times work on the specific facts they do not know. However, when the next phase of arithmetic is undertaken by the group, all class members would consider the new material. Workbooks and specially constructed work exercises are often used advantageously with the two to seven slower or remedial cases found in almost every classroom.

4. *Is it necessary to teach arithmetic as a subject?* Other ways of stating this issue are "Can arithmetic be taught as part of other subjects?" and "Will a good experience program in the elementary school provide the essential arithmetic experiences?" Current practice is overwhelmingly in favor of the presentation of arithmetic as an organized subject. The committee responsible for the second report on arithmetic of the National Society for the Study of Education was unable to locate either leaders or outstanding schools that favor the teaching of arithmetic as a part of other phases of the curriculum (see page 300). It is the opinion of the writer, also, that the arithmetic needs of children will be met best if a subject organization is followed. When arithmetic is subordinated to science, social studies, or any other field of study, either arithmetic is neglected or the study of the subject field under consideration suffers.

On the other hand it is recognized that many good situations for teaching some aspects of arithmetic arise during the study of other subjects. The teacher should seize the opportunity to use such situations for teaching arithmetic whenever possible. It is important, however, that the teacher go only so far with the arithmetic side issue as is necessary for solution or understanding of the situation. Otherwise, pupil interest in the other subject matter may be jeopardized or arithmetic may usurp the time rightfully belonging to another subject. To prevent this domination of other subjects by arithmetic and yet take advantage of the arithmetical opportunities in these subjects, a list of arithmetic uses in other fields needs to be compiled. Then these situations can be used during the arithmetic period.

5. *What is the place of arithmetic in the total elementary-school program?* The answer to this question is needed if arithmetic is to have proper time and attention and yet not deprive pupils of other good educational experiences. Since most teachers handle arithmetic along with other subjects, the importance attached to it varies greatly. The arithmetic enthusiast over-

emphasizes arithmetic while the science or reading enthusiast overemphasizes those subjects.

In the primary grades reading is, without question, the most important subject area. In the upper grades social studies and science assume major importance. When the educational needs of children are honestly assessed, the subject of arithmetic will probably be best ranked no higher than third in importance. It is, however, a very important subject and should receive a rather significant amount of the total instructional time. In the best of school systems the time will vary from 4 or 5 per cent of the total in Grade One to 12 to 15 per cent of the total in Grades Six, Seven, and Eight.

6. *What is the role of oral or non-pencil-and-paper arithmetic in today's schools?* An idea of the significant role of oral arithmetic can be acquired by reading the section on oral solution of problems in the chapter on problem-solving (see pages 189–191). In view of the fact that most everyday uses of number are of the non-paper-and-pencil type, it seems logical to give that type of arithmetic an important place in instruction. It is recommended that at least one five- to fifteen-minute period a week be devoted to oral arithmetic in the middle grades. The total oral arithmetic instructional time in the primary grades is already greater than the amount recommended for the middle grades.

7. *What is the best hour of the day for arithmetical instruction?* While the issue represented by the preceding question is not of much importance, it is frequently raised. The answer is simply that there is no best time. Other things being equal, arithmetic can be taught as effectively the last period of the day as it can be taught the first period of the day.

The Developmental Method

The developmental method of teaching arithmetic has been presented in this book primarily through illustrative lessons and other specific teaching procedures. For review purposes, six major features of the method are listed below.

1. A prime feature of the developmental method of teaching arithmetic is the use of introductory or exploratory situations (most commonly verbal problems) to create a need or use for the particular phase of arithmetic to be taught.

2. From arithmetic class experiences in using the unfamiliar aspect of arithmetic, the pupil is helped to identify the new material to be learned and a specific learning problem.

3. Identification of problems for investigation or study encourages exploration for a solution. As the children work on the basis of their arithmetic background, many solutions are proposed and considered. From these the best is selected.

4. Also important is the effort to get pupils working either as individuals or as a class to formulate rules or statements of procedure after the pupil's exploratory and other early work have given some experience with the procedure. This includes testing of the rule on other examples and problems. While this formulation of rules by pupils often results in statements which are incomplete, they become the basis for the final best rule.

5. All four of the features mentioned above are part of another characteristic of the developmental method, a very definite attempt to make the learning problem the child's own.

6. The emphasis on procedures designed to develop understanding is another basic characteristic of the developmental method. Chief among these are the use of illustrative problem situations to be solved in several ways, the identification of learning problems, pupil formulation of rules of procedure, use of proof, and the use of specific study questions for summary and review.

The advantages which are claimed for the developmental method are implied in the six major characteristics listed above. The importance of clear cut pupil purposes in learning can hardly be overestimated. They provide direction for pupil efforts and are also a powerful motivating factor. The emphasis on pupil activity leading to the development of new facts and processes seems to the writer to be closely in agreement with the principles of learning. The use of proof or showing not only makes for understanding of facts and processes studied but helps to build the pupils' confidence in their arithmetical ability.

The developmental method with its emphasis on the creation and solution of such learning problems as "What is the best way to divide a whole number by a fraction?" makes the arithmetic class one where investigations, in the best sense of the word, are carried on. In such arithmetic classes, pupils make suggestions, ask questions, challenge the views of others, and test propositions. There is a spirit of inquiry, a feeling of confidence and an enthusiasm that is characteristic of the best elementary-school learning in other subject fields. The creation of situations where the pupil has a need for a better solution or for new knowledge is essentially the core of the scientific method. Arithmetic when taught by the developmental method makes use of the steps in thinking, or the scientific method. This is as it should be. It seems only proper that in learning arithmetic, which is the foundation of the sciences, the scientific method should be used.

This particular method is essentially an inductive method of teaching. Since it is generally assumed that deductive methods are the ones for teaching mathematics, the wisdom of using an inductive method in arithmetic may be questioned. The fact should be recognized that while mathematical knowledge has in general evolved from study of basic principles (a set of axioms), the direction of its development has often been determined under the stimulus of scientific problems. New mathematical

methods have been developed as the explanation of physical phenomena demanded new mathematical understanding. Experimentation has played a part as some of the so-called foundation principles of the past have been found inadequate for newly observed conditions. There is as Klein puts it "no final ending, and therefore, on the other hand, no first beginning, which could offer an absolute basis for instruction." [1]

It is also fairly obvious that children of elementary-school age are not skilled in the use of logical analysis, which is so necessary to deductive reasoning. Probably the most convincing argument in support of an inductive method of teaching is the fact that understanding of the basic rules of arithmetical computation is the result of perception. For example, the law of commutation is illustrated in addition by showing that the sum of 4 objects, 2 objects, and 3 objects is the same regardless of whether 4 and 2 or 3 and 2 are combined first. In a similar manner we show with the diagram

O O O O
O O O O
O O O O

that $3 \times 4 = 4 \times 3$.

Perception as used in the preceding sentences and in most of the proof or showing situations in this book can only be effective with relatively small numbers. In the case of large numbers, the student is called upon to use a mathematical principle which states that if a procedure holds for small numbers, it will hold for all numbers. Klein calls this principle an intuitive truth and concludes that "the security of the entire structure of mathematics rests upon intuition, where this word is to be understood in its most general sense." [2]

In the preceding sections some of the superior qualities of the developmental method have been cited. The student of arithmetic teaching should also take a good look at the limitations of

[1] Felix Klein, *Elementary Mathematics* (New York: The Macmillan Company, 1932), p. 15.

[2] *Ibid*, p. 11.

the method. Perhaps its major shortcoming is the lack of instructional material in the way of pupil books, teachers' manuals, and general teacher "know-how." The method has not been extensively used and therefore many aspects of the teaching procedures are in need of refinement.

Another serious difficulty is the large amount of time used in the developmental and proving procedures. Either the method must result in a saving of time in other aspects of the subject or more time has to be allotted to arithmetic.

While ideally the broad grasp of arithmetic required to teach by the developmental methods should not be a limitation, practically speaking it may be. The demands upon a teacher are already great; the requisite backgrounds may be too much to ask in every case. In a similar manner the extra instructional equipment such as number lines and charts, tens blocks, and the like may be considered beyond the reach of many classrooms.

There is also the possibility that the logical step-by-step development of arithmetical processes may not be as meaningful to pupils as they appear to be to adults. Only careful study and observation in situations where the developmental method is used can remove such doubts. The reader should, of course, recognize that the developmental aspects of teaching arithmetic apply primarily to the teaching of the beginning of various phases of arithmetic.

The danger of labeling any method of teaching with a term such as "developmental" is great. By inference it is often assumed that other methods do not make use of any developmental procedures. That, of course, is not true. The name has been applied because this method seems to differ markedly from the usual procedures which emphasize telling rather than doing in the initial phases and then demand a great deal of drill without real pupil purpose or opportunities for proof. It may also be erroneously assumed that such a method must necessarily utilize only developmental procedures. While the teaching

method advocated in this book has distinguishing developmental characteristics, it incorporates other superior instructional practices that might be used with different methods of teaching as well.

The most important of these more or less independent procedures are (1) the use of proof, and the related practice of using several different ways of checking the correctness of solutions, (2) the intensive study of number operations, (3) the identification of various specific ways to study, (4) the utilization of review study questions, (5) the writing of compositions on arithmetical procedures, and (6) the extensive use of such special aids as the number line, tens blocks, and the like. Any of these and many other procedures included in the book can be employed even though the developmental method of teaching is not used.

While the writer believes that the developmental procedure as presented is a superior way to teach, he does not advocate that it is the only way. There is merit in many other methods of instruction. Even the didactic methods that are farthest from the developmental method have been used with some success. It is, therefore, neither wise to accept a new method or to discard an old one merely because of the recommendations of some one person or group. Teachers should study teaching methods carefully before advocating one as superior to others.

Consideration of methods of teaching would be incomplete without calling attention to the fact that every method is dependent upon the teacher's grasp of the principles and procedures of the method and that success is in large measure due to the teacher's enthusiasm for the subject. The morale or learning spirit of the classroom is primarily due to teacher influence.

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