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Title Elementary Analytical Statics

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ELEMENTARY ANALYTICAL STATICS

ELEMENTARY
ANALYTICAL STATICS

FOR B.A. (PASS & HONOURS) STUDENTS
OF INDIAN UNIVERSITIES

By

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Sixth Revised and Enlarged Edition.

1960

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ERRATA

P. 17	Q. 6	Line 3	Read 5 for 6.
P. 55	Ex.	Line 11	Read (2) for (3).
P. 55	Ex.	Line 15	Read Q—R for O—R.
P. 56	Q. 4	Ans.	Read 3 in. for 2 in.
P. 57	...	Line 14	Read R_q for R_p.
P. 146	Q. 4	Line 6	Read $3\sqrt{l^2-a^2}$ for $\sqrt{l^2-a^2}$

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PREFACE TO THE SIXTH EDITION

The book has been thoroughly revised and enlarged to make it more useful to the students.

A large number of solved examples have been added. The subject of the 'Nature of the Resultant' of a system of coplanar forces acting upon a rigid body, has been profusely illustrated with most suitable easy examples specially framed for this purpose.

I believe that these will greatly help the students not only in understanding the subject but also in solving similar examples themselves.

The large number of examples—a distinctive feature of the book—has been supplemented by some additional problems taken from recent University papers and they have also been rearranged in order of difficulty.

Problems in Statics are of very varied kinds but I have made every effort to classify them and put them in different groups in graded form, preceded by typical solved examples. It is hoped that this procedure will greatly assist the student who desires to attempt problems for himself. To encourage him still further in this attempt, hints for the solution of problems of a more difficult type, given towards the end of the book, have been revised and enlarged so as to make them more helpful.

The scope of the book has been still further enlarged by a discussion of the following topics :—

(1) Rotation of coplanar forces about their points of application and Astatic Equilibrium.

(2) Equilibrium under gravity, of a string of variable density.

(3) Equilibrium of a string under a central force.

(4) Resultant wrench of two given (*i*) wrenches, (*ii*) forces.

(5) Nul lines, points and planes with proofs and explanation, specially designed for the book. These proofs clarify certain points which the students find difficult to understand.

It is hoped that in its revised form the book will be still more useful and will continue to serve the cause of mathematical teaching in Indian Universities.

Any suggestion for improvement will be gratefully accepted.

4, Daryaganj,
DELHI : S. R. GUPTA
September, 1960.

PREFACE TO THE FIRST EDITION

This book has been written in compliance with the demand of certain friends and old students as a companion volume to my book "*Elementary Analytical Dynamics*" published about twelve years ago. It is primarily intended for the use of students preparing for the B.A. Pass and Honours examinations of Indian Universities. It will also prove helpful to students appearing in the various Engineering and Public Service examinations.

This book is an introduction to the subject of Statics and includes such topics as the Principle of Virtual Work, the Stability of Equilibrium and the Equilibrium of Strings under Gravity.

An account of the Force of Friction and the laws governing it, has been given in Chapter I on Fundamental Concepts. It has thus been possible to enrich the sets of examples given in later chapters by including problems involving friction.

The subject matter has been comprehensively illustrated by examples. Short sets of examples, for purpose of exercise, are added to the various articles. Some sets of miscellaneous examples and revision questions are also included. All these have been properly graded, some being taken from question papers set at different Indian and Foreign Universities.

Finally, the whole subject has been presented in a simple, precise and systematic manner.

My acknowledgment is generally due to modern writers whose works I have freely consulted. But my particular debt is due to Prof. Shanti Narain of the D.A.V. College for his kindness in going through the manuscript and making valuable suggestions. My thanks are also due to my daughter Miss Sushila Gupta, M A., lecturer in Mathematics, Fateh Chand College, Lahore for the great help she has rendered in the arrangement of Examples and for seeing the book through the Press.

Lahore : S. R. GUPTA

October, 1942.

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When we speak of a body as a particle, we mean thereby that for the time being, we are not concerned with its actual dimensions and that we can represent its position simply by a Mathematical point. In certain problems in Astronomy even vast bodies like the earth and the sun are treated as mere particles.

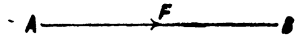
1.4. Measurement of a Force. A force is measured by its effect. In Dynamics we measure a force by the motion it produces in a given body in a given time. If two forces, when applied to the same body for equal intervals of time, generate the same velocity, they are said to be equal. The forces are thus proportional to the velocities they generate in the same body in equal intervals of time.

In Statics, however, we are not concerned with motion of a body and we conceive of a force as something independent of motion, as an effort of the nature of a *push* or a *pull*. For example, when we hold some object in our hand, some effort on our part is required to support it. This effort or force depends upon the amount of weight supported. The measure of a force in this case, therefore, depends upon the force of gravity, which varies slightly at different places of the earth. In Statics, however, we are concerned merely with the ratio of forces, therefore, we can measure the force by comparing it with some standard weight, say a weight of one pound, or one gram which is taken as the *unit of force*.

1.41. Representation of a Force. Besides magnitude of a force, we must know its *point of application* and its *direction* (*i.e.*, the direction along a straight line through its point of application, in which the force tends to move a body to which it is applied).

Such a quantity which possesses direction as well as magnitude is called a *vector quantity*. It can be represented graphically by a straight line of certain length, drawn in a certain direction, through a given point.

If, for example, the line AB represents the force F, the point A can represent the point of application, the direction of the line represents the direction of the force and the length AB of the line can be made to represent the magnitude of the force on some convenient scale.



If instead of AB, we write BA, it represents the same force with its direction reversed.

Two forces represented by BA and AB will balance each other. Thus we have :—

Two forces equal in magnitude, opposite in direction and acting along the same straight line are in equilibrium.

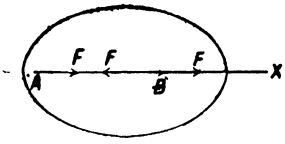
Conversely *if two forces acting on a body are in equilibrium, they must be equal in magnitude and must act along the same straight line in opposite directions.*

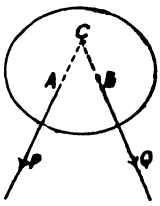
Note 1. The sense of a force represented by a straight line, is indicated by the order of the letters used to specify the line.

Note 2. The magnitude of a *force* can never be equal to the *length* of a line, but it can be *represented* by a certain length on a certain scale of measurement.

1.42. Transmissibility of a Force. When a force acts on a body its effect is the same no matter which point in its line of action is taken as the point of application, provided the point is rigidly connected with the body.

Suppose a force F acts at a point A of a rigid body along the line AX. At B, any other point in AX, introduce along the same line, two equal and opposite forces, each equal to F. The force F along AB and the force F along BA, being equal and opposite, cancel each other and we are left with the force F acting at B along BX. Thus we arrive at a force F acting at B along BX instead of the force F acting at A along the same line, so that





A force may be supposed to act at any point in its line of action.

From the above it follows that if two forces P and Q act at A and B, two points of a rigid body, along the lines which intersect at C, then C can be taken as the point of application of the forces P and Q.

1.43. Resultant and Components. If a single force R can be found which, acting alone on a rigid body will have the same effect as a number of forces P, Q, S...etc. acting on it, then R is known as the *resultant* of the given forces and the forces P, Q, S...etc., are known as the *components* of R.

1.5. Action and Reaction. *To every action there corresponds an equal and opposite reaction.*

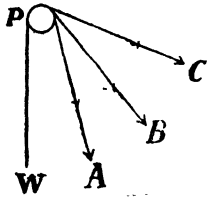
In a system of two bodies, a force never acts on one body without an equal force acting in the opposite direction on the second body. This principle is known as Newton's law of action and reaction.

If a body A exerts a force R on another body B, the body B, exerts an equal force R in the opposite direction on A.

1.6. Tension. When a string is used to support a weight or to drag a body, the force exerted is transmitted to the body through the string. Such a force exerted by means of a string is called *Tension.*

If the string is of negligible weight, the *tension throughout the string is the same.* This can be verified by supporting a weight by a string the other end of which is attached to a spring balance. The balance will give the same reading to whatever point of the string it is attached.

Also, if the string supporting the weight W is passed over a smooth peg, the force required at the end of the string to sustain the weight W is the same in whatever direction the string is held so that the tension in the string in all positions (PA, PB, PC, etc.) is the same and equal to W in magnitude.



If, however, a string be knotted at any of its points to other strings or to weights, the tension will not, in general, be the same in the different portions separated by the knots.

1.61. Tension in an Elastic String. If one end of an elastic string (or a spring) is fixed and a force is applied at the other end, the string extends in length. The relation between the amount of extension and the tension in the string is governed by the law known as **Hooke's Law** which states :—

The tension of an elastic string is proportional to the extension of the string beyond its natural length.

Thus, if l be the natural length of the string, x its extension and T, the tension; then

$$T=\lambda \frac{x}{l},$$

where λ is a constant depending on the material of the string and is known as the **Modulus of Elasticity.**

If the string extends to twice its natural length, the extension x is l and T is consequently equal to λ. *Thus the modulus of elasticity is equal to the amount of tension required to extend the string to twice its natural length.*

Example. *ABC is an elastic string, the unstretched lengths of AB and BC being a and b respectively. A weight W_1 lb. is attached to B and another of W_2 lb. is attached to C. The end A is fastened to a fixed point and the string is allowed to hang vertically with the weights at rest. If the modulus of elasticity is λ lb. wt. find the stretched lengths of AB and BC.*

The portion BC of the string has to support only the lower weight and so its tension is W_2.

If x be its stretched length, then by Hooke's law we have

$$W_2=\lambda \frac{x-b}{b} \qquad i.e., \qquad x=\frac{b\,W_2}{\lambda}+b.$$

The portion AB of the string has to support both the weights and, therefore, its tension is W_1+W_2

If y be its stretched length, we have

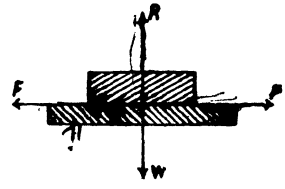
$$W_1+W_2=\lambda \frac{y-a}{a} \quad i.e., \quad y=\frac{a(W_1+W_2)}{\lambda}+a.$$

1.7. Attraction is a force which acts upon a body without an

apparent medium of transmission. For example, (1) attraction of the earth, (2) mutual attraction of the heavenly bodies which sustains and guides them in their orbits, (3) Attraction of a magnet.

1.8. Friction. Suppose a block of wood is resting on a table. The only forces acting on the block are its weight acting vertically downwards and the reaction of the table. As the block is in equilibrium, the reaction must be equal and opposite to the weight and therefore, must act vertically upwards.

Now let a gradually increasing force P be applied to the block along the table. It will be seen that the block does not at once begin to move and continues to be at rest till P has been sufficiently increased. This shows that there is some force, say F, acting on the block which balances P and thus prevents the block from moving.



The force F is equal and opposite to P and is known as the *force of friction.*

This force is caused by the inequalities in the two surfaces in contact, in virtue of which the two surfaces get interlocked and an effort to cause one to slide on the other causes strain in each.

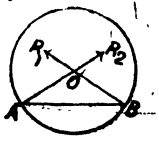
This force is called into play only when the force P acts. Also it increases with P till the block begins to move. The maximum amount of friction, known as *Limiting friction,* is called into play when the block is just on the point of motion.

The table being rough, the reaction of the table on the block is the resultant of the force of friction F, and the normal pressure R.

If the table were *smooth,* the block would move, however small P may be. In this case F does not exist and the reaction of the table is normal to its surface.

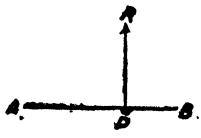
1.81. A smooth curve or a smooth surface is one which can exert a pressure only in the direction of the normal.

In general *whenever two smooth bodies are in contact, the reaction of either on the other acts along their common normal at the point of contact.*

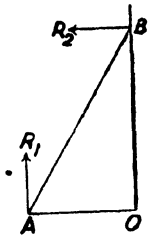


For example,

(1) when a rod AB rests inside a smooth hollow sphere, the reactions at the two ends are along the normals at A and B.



(2) Again, if a rod AB rests on a smooth horizontal peg, (or a rail) P, the reaction of the peg on the rod is in the direction perpendicular to the rod at P.



(3) If an end of a rod is in contact with a smooth wall (or a floor) the reaction of the wall (or the floor) is perpendicular to the wall (or the floor) at the point of contact.

1.9. Laws of friction. Experiments such as given in Art. **1.8** lead to the formulation of certain laws, which are given below.

1.91. Statical Friction. The friction acting on a body which lies at rest is called *statical friction* and is govened by the following laws :—

(1) The direction of friction is opposite to the direction in which the body tends to move.

(2) The magnitude of the force of friction is exactly equal to the force tending to produce motion.

1.92. Limiting Friction. The friction acting on the body when it is just on the point of motion is called *Limiting friction.* It obeys the following laws.

(1) Limiting friction is equal in magnitude and opposite in direction to the force which tends to produce motion.

(2) The ratio of the amount of limiting friction to the normal pressure is constant.

This constant is known as the **co-efficient of friction** and is usually denoted by the letter μ.

If F be the amount of limiting friction and R, the normal pressure then $F/R=\mu$ or $F=\mu R$.

(3) The co-efficient of friction depends entirely on the material of the bodies in contact and is independent of the area and shape of the surfaces in contact

1.93. Dynamical Friction is the friction which acts on the body when it is actually in motion. It acts in the direction opposite to the direction of motion of the body. Its ratio to the normal pressure is slighty *less than* the co-efficient of friction.

1.94. The force of friction between a body and a rough surface belongs to a class of force known as *passive resistance, i.e.*, forces which come into play on account of the action of other forces and which always tend to destroy the effect of these other forces.

In nature the force of friction plays an important part and cannot be dispensed with. It is in the presence of this force between the feet and the ground that renders walking possible. If the surface be a smooth one, it is very difficult to walk on it. Again, it is the force of friction that propels the driving wheels of an engine ; if there

were no friction between the wheels and the rails, the driving wheels would slip round and rotate, the engine remaining stationary. [*See* author's Dynamics Art. 3.3 (6)].

Example 1. *A body of mass 24 lbs. rests on a table whose co-efficient of friction is 0·25. A string attached to it and passing over the edge of the table, carries another body at the other end. Find the weight of this body if the body on the table is on the point of motion.*

Let the weight of the hanging body be W. The tension T in the string supports this weight and is, therefore, equal to W, *i.e.*, T=W.

Also the normal reaction R of the table supports the weight of the body on the table and is, therefore, equal to 24 lbs., *i.e.*, R=24.

The co-efficient of friction $\mu = 0{\cdot}25 = \frac{1}{4}$.

As the body is just on the point of motion, the friction is limiting and the amount of friction $= \mu R$.

Since this friction balances the tension of the string, $\therefore$ $T = \mu R$.

Hence $W = T = \mu R = \mu \times 24 = \frac{1}{4} \times 24 = 6$ lbs.

Example 2. *In Ex. 1, the weight of the body on the table is 21 lbs., the co-efficient of friction μ is $\frac{1}{3}$, and the weight of the lower body attached to the other end of the string is (i) 5 lbs. (ii) 7 lbs. (iii) 9 lbs. What is the amount of friction in each case ?*

Here R the normal reaction of the table = the weight of the body on the table = 21 lbs.

$\therefore$ the limiting amount of friction $= \mu R = \frac{1}{3} \times 21 = 7$ lbs.

(*i*) The tension T of the string in this case is 5 lbs. It is to be balanced by the force of friction F.

$\therefore$ $F = T = 5$ lbs.

(*ii*) The tension is this case is 7 lbs. and the limiting amount of friction is also 7 lbs. $\therefore$ F=7 lbs. and the system is just on the point of motion.

(*iii*) In this the tension is 9 lbs. which is more than the maximum friction of 7 lbs. As no more of friction can come into play, $\therefore$ F=7 lbs. The equilibrium breaks and the system begins to move in this case.

Example I

1. A man of weight 160 lbs. holds one end of a rope which passes over a pulley vertically above his head and to the other end of the rope is attached a weight of 120 lbs. Find the force with which the man's feet press on the floor. (*Ans.* 40 lbs.)

2. In a tug of war there are 10 members pulling on each side. If the average force exerted by each member on either side is 110 lbs. wt. find the tension in the rope. (*Ans.* 1100 lbs. wt.)

3. A mass of 10 lbs. rests on a rough table whose co-efficient of friction is $\frac{1}{2}$. A string attached to it and passing over the edge of the table, carries a weight at the other end. Find the magnitude of this weight if the mass on the table is on the point of motion. (*Ans.* 5 lbs.)

4. A wooden plank 5 ft. by 2 ft. of weight 100 lbs. is lying on the ground. Find the pressure per unit area offered by the ground.

If the plank is on the point of motion when a horizontal force of 75 lbs. wt. acts on it. Find the co-efficient of friction. (*Ans.* 10 lbs. per sq. ft. ; $\frac{3}{4}$)

5. A block of weight 1,000 lbs. rests on a fixed rough horizontal table, co-efficient of friction being 0·2. A horizontal pull of 100 lbs., 200 lbs. and 300 lbs. weight is successively applied to the block ; what will be the corresponding force between the table and the block in each case ? (*P.U.* 1951)
(*Ans.* 100 lbs., 200 lbs., 200 lbs.)

6. An elastic cord whose natural length is 8 inches, can be kept stretched to a length of 12 inches by a force of 5 lbs. wt. Show that the modulus of elasticity of the cord is 10 lbs. wt.

7. ABC is an elastic string which when unstretched is such that AB=8 in. and BC=3 in. A particle weighing 4 lbs. is attached to B and another of 2 lbs. is attached to C. The end A is fastened to a fixed point and the string is allowed to hang vertically with the particles at rest. If the modulus of elasticity is 6 lbs. wt., find the stretched lengths of AB and BC. (*Ans.* 16 in., 4 in.)

8. An elastic string ABC, whose modulus of elasticity is 4W, is tied at A to a fixed point. At B and C are attached weights each equal to W, the unstretched lengths of AB and BC being each equal to *c*. Show that if the string and bodies take up a vertical position of equilibrium, the stretched lengths of AB and BC are $\frac{3c}{2}$ and $\frac{5c}{4}$. (*P.U.*)

9. A uniform elastic string has a length a_1 when the tension is T_1 and a length a_2 when the tension is T_2. Show that its natural length is

$$\frac{(a_2T_1-a_1T_2)}{T_1-T_2}$$ (*P.U.*)

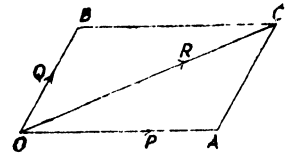
CHAPTER II

FORCES ACTING AT A POINT-I

2·1. When we speak of forces acting at a point, it is understood that they are acting on a particle placed at that point.

2.11. Theorem I. Resultant of two forces. *If two forces acting at a point are represented, in magnitude and direction, by the sides of a parallelogram drawn from the point, their resultant is represented, both in magnitude and direction, by the diagonal of the parallelogram drawn through that point.*

Thus if two forces P and Q acting at the point O are represented in magnitude and direction, by the lines OA and OB, their resultant R is then represented by the diagonal OC.

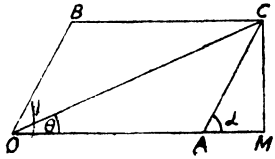


The solution of the various problems in Statics is based on this fundamental principle, known as the *law of the parallelogram of forces.*

The best proof of this is based on the laws of Dynamics. Here its truth is assumed on experimental basis.

Cor. The resultant of two forces represented by OA and OB, is represented by 2OE, where E is the middle point of AB : for 2OE = OC.

2.12. Magnitude and direction of the Resultant.



If $\angle AOB = \angle CAM = \alpha$, then from the $\triangle OAC$, we have

$$OC^2 = OA^2 + AC^2 - 2OA.AC \cos \angle OAC$$
$$= OA^2 + OB^2 - 2OA.OB \cos (180° - \alpha)$$
$$= OA^2 + OB^2 + 2OA.OB \cos \alpha$$

i.e., $R^2 = P^2 + Q^2 + 2PQ \cos \alpha$ ...(1)

Again if $\angle AOC = \theta$, and CM is $\perp$ to OA, we have

$$\tan \theta = \frac{MC}{OM} = \frac{MC}{OA + AM} = \frac{AC \sin \alpha}{OA + AC \cos \alpha} = \frac{Q \sin \alpha}{P + Q \cos \alpha} \quad \ldots (2)$$

and $\sin \theta = \frac{MC}{OC} = \frac{Q \sin \alpha}{R}$; $\cos \theta = \frac{OM}{OC} = \frac{P + Q \cos \alpha}{R}$ ... (3)

Equations (1) and (2) [or (3)] give the magnitude and direction of the resultant.

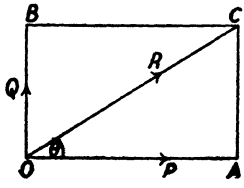
In the particular case when P = Q, we have

$$R^2 = 2P^2 + 2P^2 \cos \alpha = 4P^2 \cos^2 \frac{\alpha}{2} \quad \textit{i.e.}, \quad R = 2P \cos \frac{\alpha}{2},$$

and $$\tan \theta = \frac{P \sin \alpha}{P + P \cos \alpha} = \frac{\sin \alpha}{1 + \cos \alpha} = \tan \frac{\alpha}{2}.$$

i.e., the line of action of the resultant force bisects the angle between its equal components.

Again in another case when P and Q are at right angles *i.e.*, $\alpha=90°$,



then since $\cos\alpha=0$ and $\sin\alpha=1$, from (1) and (2), we get

$$R=\sqrt{P^2+Q^2} \text{ and } \tan\theta=\frac{Q}{P},$$

which is also evident directly from the figure.

2.13. Example (i). *The resultant of forces 2P and 3P acting at a point is $\sqrt{7}$ P, find the angle between the forces, and the direction of the resultant.*

By (1), we have

$(\sqrt{7}P)^2=(3P)^2+(2P)^2+2.3P.2P\cos\alpha$, or $7P^2=13P^2+12P^2\cos\alpha$,

so that $\cos\alpha=-\frac{1}{2}$, *i.e.*, $\alpha=120°$,

and by (2), $\tan\theta=\dfrac{2P\sin\alpha}{3P+2P\cos\alpha}=\dfrac{2P.\frac{\sqrt{3}}{2}}{3P-2P.\frac{1}{2}}=\dfrac{P\sqrt{3}}{2P}=\dfrac{\sqrt{3}}{2}$.

Example (ii). *The resultant of two forces P, Q acting at a certain angle is F, and that of P, R acting at the same angle is also F, the resultant of Q, R again acting at the same angle is G. Prove that*

(a) if $P+Q+R=0$, $G=F$.

(b) If $P+Q+R\neq0$, $P=\sqrt{F^2+QR}=\dfrac{QR(Q+R)}{Q^2+R^2-G^2}$.

Let the given angle be α, then by the question

$$F^2=P^2+Q^2+2PQ\cos\alpha \quad \ldots (1)$$

$$F^2=P^2+R^2+2PR\cos\alpha \quad \ldots (2)$$

and $$G^2=Q^2+R^2+2QR\cos\alpha \quad \ldots (3)$$

From (1) and (2), $Q^2-R^2+2P(Q-R)\cos\alpha=0$

Since $Q-R\neq0$, $\therefore$ dividing by $Q-R$, we get

$$\cos\alpha=-\frac{Q+R}{2P} \quad \ldots (4)$$

(a) But $P+Q+R=0$ *i.e.*, $Q+R=-P$.

$\therefore\ \cos\alpha=-\dfrac{Q+R}{2P}=\dfrac{P}{2P}=\frac{1}{2}$, *i.e.*, $\alpha=60°$.

Now from (2) and (3), we have

$$F^2-G^2=P^2-Q^2+2R(P-Q)\cos 60°=(P-Q)(P+Q)+R(P-Q)$$
$$=(P-Q)(P+Q+R)=0$$

Hence $F^2-G^2=0$, *i.e.*, $F=G$.

(b) From (2) and (4), we have

$$F^2=P^2+R^2-R(Q+R)=P^2-QR \text{ or } P=\sqrt{F^2+QR}.$$

Again from (3) and (4), we have

$$G^2=Q^2+R^2-2QR\frac{Q+R}{2P}$$

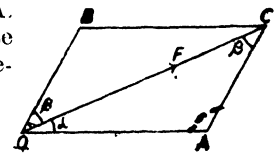
or $$\frac{QR(Q+R)}{P}=Q^2+R^2-G^2,$$

or $$P=\frac{QR(Q+R)}{Q^2+R^2-G^2}.$$

2.14. Components of a force in two given directions. A force may be resolved into two component forces in an infinite number of ways, for, an infinite number of parallelograms can be constructed having the straight line representing the force as the diagonal.

The components of the force F in directions making angles α, β with its line of action on opposite sides of it can be found as follows :

Let OC represent the force F. Draw OA, OB making angles α, β with OC and complete the parallelogram OACB, then OA and OB represent the required components of F.



From the $\triangle$ OAC, we have

$$\frac{OA}{\sin\beta}=\frac{AC}{\sin\alpha}=\frac{OC}{\sin\angle OAC}=\frac{F}{\sin(\alpha+\beta)}$$

Hence $$OA=\frac{F\sin\beta}{\sin(\alpha+\beta)} \text{ and } OB=AC=\frac{F\sin\alpha}{\sin(\alpha+\beta)}.$$

Example. *Resolve a force F along two lines making angles 45° and 15° with its direction.*

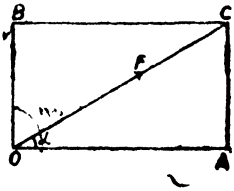
Here $\alpha=45°$, $\beta=15°$, $\alpha+\beta=60°$

$\therefore$ components of the force are,

$$\frac{F\sin 45°}{\sin 60°}=\frac{2F}{\sqrt{6}} \text{ and } \frac{F\sin 15°}{\sin 60°}=\frac{(\sqrt{3}-1)F}{\sqrt{6}}$$

Ex. If a force (F) be resolved into two components, one of which is at right angles to the force and equal to it in magnitude, find the direction and magnitude of the other. (*Ans.* $\sqrt{2}$ F at 45° with F.) (*P. U.*)

2.15. In the particular case when the components are at right angles, *i.e.*, $\alpha+\beta=90°$ or $\beta=90°-\alpha$, we have



$$\sin(\alpha+\beta)=\sin 90°=1,$$
$$OA=F\sin\beta=F\cos\alpha;$$
$$OB=F\sin\alpha=F\cos(90°-\alpha),$$

which is otherwise evident from the figure.

In this case, the components OA and OB are called the *resolved parts of* F.

It follows that the *resolved part of a force in a certain direction is equal to the product of the force and the cosine of the angle between its line of action and the given direction.*

As $\cos 90°=0$, a force has no resolved part in a direction perpendicular to its line of action.

The component of the force F in a given direction OA is not the same as its resolved part. To find the components of a force we must be told both the directions in which the force is to be decomposed,

but to find the resolved parts we need know only one direction, as the second direction must be at right angles to the known direction.

Ex. *Two forces P and Q acting at a point have a resultant R. The resolved part of R in the direction of P is of magnitude Q. Find the angle between the forces, and also the magnitude of the resultant.*

Referring to the figure of Art. 2·12, we have

By the question $R\cos\theta = Q$, and by (3) Art. 2·12,

$$\cos\theta = \frac{P+Q\cos\alpha}{R}.$$

$$\therefore \quad \cos\theta = \frac{Q}{R} = \frac{P+Q\cos\alpha}{R}, \text{ } i.e., \; Q\cos\alpha = Q-P \quad \text{or} \quad \alpha = \cos^{-1}\frac{Q-P}{Q}.$$

$$\text{Now } R^2 = P^2+Q^2+2PQ\cos\alpha = P^2+Q^2+2P(Q-P)$$
$$= Q^2-P^2+2PQ.$$

It may be noted that result (3) of Art. 2·12 has proved more useful than (2) in this case.

2.16. Vector Quantities. Forces, just like velocities and accelerations mentioned in Dynamics, are *vector quantities* different from those known as *scalar quantities.*

Vector quantities involve both magnitude and direction and they can, therefore, be represented by straight lines of specified lengths drawn in specified directions. All such quantities are compounded by the parallelogram law unlike scalar quantities such as lengths, masses and times, which are compounded simply by ordinary algebraic rule.

Thus a body of weight 3 lbs. and another of weight 4 lbs. are together equal to a weight of 7 lbs. ; but a force of 3 lbs. wt. and another of 4 lbs. wt. are not equal to a force of 7 lbs. wt. unless, in the special case, when they act in the same direction along a straight line.

In fact nothing can be said about the resultant of these forces, unless the directions in which they act are also mentioned. If these forces act in perp. directions, for example, we know, by Art. 2.12, that they are equivalent, in magnitude, to a force equal to

$$\sqrt{3^2+4^2} = 5 \text{ lb. wt.}$$

Examples II

1. (*a*) The resultant of two forces P and 2P, acting at a point is perpendicular to P. Find the angle between P and 2P. (*Ans.* 120°)

(*b*) Two equal forces have a resultant equal to either. At what angle are they inclined to each other ? (*Ans.* 120°)

2. The resultant of two forces P and Q is equal to Q in magnitude ; that of P and 2Q (acting in the same directions as before) is also Q. Find the magnitude of P and the angle between the forces P and Q. (*Ans.* $\sqrt{3}$ Q. 150°)

3. The resultant of two forces P_1 and P_2 acting at right angles is R. If P_1, P_2 be each increased by 3 lbs. R is increased by 4 lbs. and is now equal to the sum of the original values of P_1 and P_2. Find P_1 and P_2.

(*Ans.* 5 lbs., 12 lbs.)

4. Two forces one of which is double the other, have a certain resultant such that if the greater force be doubled again and the smaller be increased by 4 lbs. wt. the resultant has the same direction as before. Find the forces and also find how the resultant is altered. (*Ans.* 4, 8 ; Resultant doubled)

5. The resultant of two forces P and Q is of magnitude P. Show that if the force P is doubled, Q remaining the same, then the new resultant will be at right angles to Q and its magnitude shall be $\sqrt{4P^2-Q^2}$.

6. When two equal forces are inclined at an angle 2θ, their resultant is twice as great as when they are inclined at an angle 2φ, show that

$$\cos\theta = 2\cos\varphi.$$

7. The resultant of two forces P and Q acting at an angle θ is equal to $(2m+1)\sqrt{P^2+Q^2}$; when they act at an angle $90°-\theta$, the resultant is $(2m-1)\sqrt{P^2+Q^2}$; prove that

$$\tan\theta = (m-1)/(m+1). \qquad (P.U., B.U.)$$

8. The resultant of P and Q is R ; if Q is doubled, R is doubled, if Q be reversed, R is again doubled. Show that

$$P^2 : Q^2 : R^2 = 2 : 3 : 2. \qquad (Agra\ U.)$$

9. The resultant of two forces P and Q (P>Q) divides the angle between their directions into two parts in the ratio 1 : 2. Find the angle between the forces and show that the magnitude of the resultant is

$$(P^2-Q^2)/Q. \qquad [Ans.\ 3\cos^{-1}(P/2Q)]$$

10. Two forces P+Q, P−Q make an angle 2α with one another, and their resultant makes an angle θ with the bisector of the angle between them. Show that

$$P\tan\theta = Q\tan\alpha. \qquad (P.U.)$$

11. If forces P and Q acting at an angle θ be interchanged in position, that the resultant turns through an angle φ such that

$$\tan\frac{\varphi}{2} = \frac{P-Q}{P+Q}\tan\frac{\theta}{2}.$$

12. The resultant of two forces P and Q makes an angle θ with P. If P becomes P−S, the resultant is R′, making an angle 2θ with P−S ; show that R′=S. (*R. U.*)

13. Two forces P and Q act along the lines OA and OB respectively. If their resultant is at right angles to OA, show that the angle between them is

$$\cos^{-1}\left(-\frac{P}{Q}\right)$$

Further if the resultant of P′ along OA and Q′ along OB is at right angles to OB, show that PP′=QQ′.

14. If the resultant of two forces acting on a particle be at right angles to one of them, and its magnitude be one third of the other, show that the ratio of the larger force to the smaller is $3 : 2\sqrt{2}$. (*C. U.*)

15. If the greatest possible resultant of two forces P and Q is *m* times the least, show that the angle θ between them when their resultant is half their sum is given by

$$\cos\theta = -\frac{m^2+2}{2(m^2-1)}.$$

16. Three forces P, Q, R acting at a point are in equilibrium ; one of them P is turned round this point through an angle A, find the direction of the resultant of the three forces.

If the inclination of P continue to alter, show that the inclination of the resultant alters by half the amount.

(*Ans.* It makes $90°+\frac{1}{2}$ A with the original direction of P)

17. A force P acting at a point O is given in magnitude and direction and is represented by a vector drawn from O. If a second force Q acting at O is given in magnitude only, find the locus of the extremity of the vector drawn from O which represents the resultant of P and Q.

Show that if P is greater than Q, the inclination of the resultant to the direction of P cannot exceed θ where

$$\sin\theta = \frac{Q}{P}.$$

18. The resolved part of the resultant R of two forces P and Q in the direction of P is of magnitude Q. Show that the angle between the forces is

$$2\sin^{-1}\sqrt{\frac{P}{2Q}}.$$

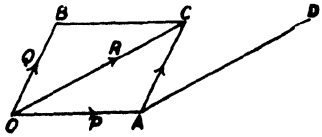
19. The greatest resultant that two forces can have is of magnitude P, and the least is of magnitude Q. Show that when they act at an angle α, their resultant is of magnitude

$$\sqrt{P^2\cos^2\frac{\alpha}{2}+Q^2\sin^2\frac{\alpha}{2}}.$$

20. D, E, F are the middle points of the sides BC, CA, AB of a triangle ABC respectively. Forces x, y, z act along EF, FD, DE respectively. Find the equivalent system of forces along the sides of the triangle ABC. *(D.U. 1955)*

$$\left[\textit{Ans.}\quad a\left(\frac{y}{b}+\frac{z}{c}\right), b\left(\frac{z}{c}+\frac{x}{a}\right), c\left(\frac{x}{a}+\frac{y}{b}\right)\right]$$

2.2. In Art.2.11, since AC, is equal and parallel to OB, therefore, so far as magnitude and direction are concerned, the force Q can be represented by the line AC, so that OC represents the resultant of forces represented by OA and AC.



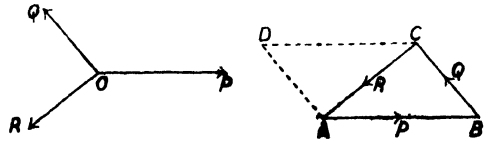
Hence if two forces acting at a point are represented by the two sides OA and AC, drawn in the same sense, of the $\triangle$OAC, their resultant is represented, in magnitude and direction by the third side taken in the opposite sense (*i.e.*, OC and not CO).

If, however, the forces represented by OA and AC, are actually acting along these lines, the resultant though represented by OC, in magnitude and direction, actually acts along AD, the line through A parallel to OC.

2.21. Triangle of Forces. *If three forces acting at a point are represented, in magnitude and direction, by the three sides of a triangle taken in order, they are in equilibrium.*

Let the three forces, P, Q, R, act at the point O, and be represented in magnitude, and direction, by the sides AB, BC and CA of the $\triangle$ABC. Complete the parallelogram ABCD.

AD being parallel and equal to BC, represents the force Q and



therefore, by parallelogram of forces, the resultant of P and Q is represented by AC, in direction and magnitude.

Therefore the resultant of P, Q, and R, acting at O, is the same as the resultant of forces, acting at O, represented by AC and CA, which being equal and opposite, cancel and hence P, Q, R are in equilibrium.

2.22. Converse of the triangle of Forces. *If three forces acting at a point are in equilibrium, they can be represented by the sides of a triangle taken in order, the triangle being drawn so as to have its sides respectively parallel to the direction of the forces.*

Let the forces P, Q, R acting at the point O be in equilibrium. Draw AB parallel to the direction of force P, and on same scale, let AB represent the force P. Again draw BC parallel to the direction of force Q, and on the same scale let BC represent Q ; join CA.

Complete the parallelogram ABCD. AD being parallel and equal to BC, represents the force Q and therefore the resultant of P and Q is represented by AC. But since P, Q, R are in equilibrium, R must be equal and opposite to the resultant P and Q. Therefore R must be represented by CA.

Hence P, Q, R are, respectively, represented by the sides AB, BC and CA of the $\triangle$ ABC.

Cor. 1. If A'B'C' be any other triangle having its sides respectively parallel to the forces P, Q, R and, therefore, parallel to the sides of the triangle ABC, then the two triangles being equiangular and similar, we have

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$$

But AB, BC, CA respectively represent the forces P, Q and R,

$$\therefore \quad \frac{P}{A'B'} = \frac{Q}{B'C'} = \frac{R}{C'A'}.$$

Cor. 2. *If three forces acting at a point are in equilibrium, they must be coplanar.*

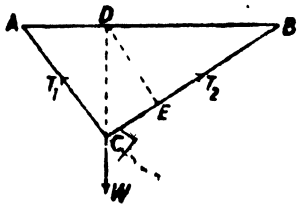
2.23. If three forces proportional to the sides of a triangle act at a point in directions perpendicular to the sides taken in order, they shall be in equilibrium.

For, if the triangle be turned through a right angle its sides taken in order will be parallel to the respective forces and will, therefore, represent the forces, in magnitude and direction. Hence by Art. 2.21 the forces are in equilibrium.

This is a particular case of the following general theorem :—

If three forces acting at a point are proportional to the sides of a triangle and the directions of the forces make the same constant angle, measured in the same sense with the sides respectively, the forces are in equilibrium.

Example (i). *A weight W is suspended by two strings at right angles to one another and attached to points in the same horizontal line ; prove that their tensions are inversely proportional to their lengths.*



The strings CA, CB are attached to points A and B in the same horizontal line such that $\angle ACB = 90°$.

The weight W and the tensions T_1 and T_2 are respectively perpendicular to the sides AB, BC, CA of the $\triangle ABC$ which when turned through a right angle, represents the triangle of forces. The forces are, therefore, proportional respectively to the sides perpendicular to them.

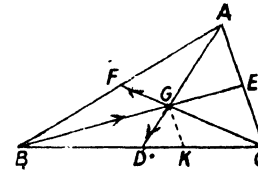
Hence $$\frac{T_1}{T_2} = \frac{BC}{AC} = \frac{1}{AC} \div \frac{1}{BC}.$$

Aliter. Let the line of action of W meet AB in D. Draw DE parallel to AC. CDE is the triangle of forces.

$\therefore T_1 : T_2 = DE : CE = BC : AC$ ($\triangle$s ABC, CDE, being similar).

Example (ii). *ABC is a triangle and D, E, F are the middle points of the sides. Forces represented by AD, $\frac{2}{3}$ BE and $\frac{1}{3}$ CF act on a particle at the point where AD and BE meet. Show that the resultant is represented in magnitude and direction by $\frac{1}{2}$ AC, and that its line of action divides BC in the ratio 2 : 1.*

AD and BE meet at G, the centroid of the $\triangle ABC$, and $DE = \frac{1}{2}BA$ and is parallel to BA.



Force $\frac{2}{3}$ BE is equivalent to BG, and $\frac{1}{3}$ CF to GF.

From $\triangle BGF$,
forces BG and $GF \equiv BF \equiv \frac{1}{2}BA = DE$.
Now the forces AD and $DE \equiv AE = \frac{1}{2}AC$.

$\therefore$ the given forces are $\equiv \frac{1}{2}AC$.

Hence the resultant is $\frac{1}{2}$ AC in magnitude, and direction.

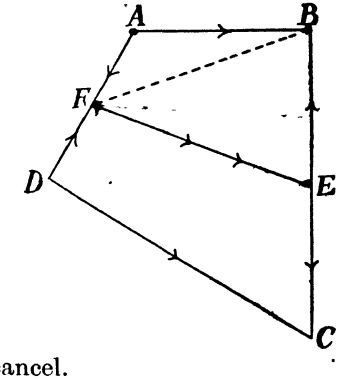
Since the resultant is to act at G, it acts along GK drawn parallel to AC, meeting BC in K.

From $\triangle$s BGK and BEC, we get $$\frac{BK}{KC} = \frac{BG}{GE} = \frac{2}{1},$$

so that K divides BC in the ratio 2 : 1.

Example (iii). *ABCD is a quadrilateral. BC is bisected in E and DA in F ; show that if two forces parallel and equal to AB and DC act on a particle, their resultant is parallel to FE and equal to 2FE.*

Force AB is represented in magnitude and direction by forces represented by AF and FB, and FB is represented by forces represented by FE and EB



Thus the force AB
$\equiv$forces AF+FE+EB
Similarly the force DC
$\equiv$forces DF+FE+EC
$\therefore$ forces AB and DC
$\equiv$forces (AF+DF)+2FE
+(EB+EC)

Now F being the middle point of AD, forces AF and DF, being equal and opposite cancel. Similarly forces EB and EC cancel.

Hence the resultant of forces represented by AB and DC is represented in magnitude by 2FE and is parallel to FE in direction.

Examples III

1. ABCD is a parallelogram and P is any point, not necessarily in its plane. Prove that the system of forces represented by AP, PB, CP and PD is in equilibrium.

2. ABCD is a parallelogram. Prove that the forces completely represented by the lines AB, CD, AC and DB have a resultant represented by 2AB, in magnitude, direction and position.

3. P is any point in the plane of a triangle ABC ; D, E, F are the middle points of the sides. Prove that the resultant of forces PE, PF and DP is represented by PA.

4. ABCD is a quadrilateral and E, F, G, H are the middle points of the sides AB, BC, CD, DA respectively. Show that the forces represented by EG, FH and DB in magnitude and direction acting on a particle will keep it in equilibrium. (*P.U.*)

5. The resultant of two forces P and Q acting at a certain angle is R, the resultant of P and Q′ acting at the same angle is R′ ; show that the resultant of R and a force equal and opposite to R′ is the difference of Q and Q′.

6. The ends of a string 17 inches long are attached to two points 13 inches apart in the same horizontal line and a weight of 26 lbs. is attached to the string 6 inches from one end. Find the tension in each portion of the string. (*Ans.* 24, 10 lbs.)

7. ABC is a triangle and P is any point in BC. If PQ represents the resultant of forces represented by AP, PB and BC, show that the locus of Q is a straight line parallel to BC.

If the forces are represented by AP, PB, and PC, show that Q is a fixed point on the median through A.

8. Three coplanar forces P, Q, R act at the vertices A, B, C respectively of a $\triangle$ ABC, in which AB : BC : CA=4 : 3 : 2. Q and R are parallel to CA and BA and the magnitude of R is 10 lbs. weight. Find the magnitude of P and Q in order that the forces may be in equilibrium.

(*Ans.* $\frac{5}{2}\sqrt{31}$ lbs. wt., 5 lbs. wt.)

9. If ABCDEF is a regular hexagon and O any point in its plane, show that forces represented, in magnitude and direction, by OA, OC, OE, BO, DO and FO are in equilibrium.

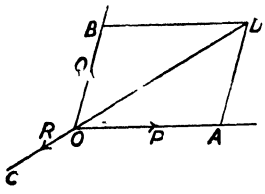
10. If the sides BC, CA, AB of a triangle ABC are parallel and drawn the same sense to the lines of action of three forces P, Q, R in equilibrium acing at any point O, show that

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB} \qquad (P.U.\ 1957\ S.)$$

(Hint. *See* Art. 2.22 and Cor. 1)

2.3. Lami's Theorem. *If three forces acting at a point are in equilibrium, each is proportional to the sine of the angle between the other two.*

Let the forces P, Q, R acting at O, be in equilibrium and let P and Q be represented, on some scale, by OA and OB respectively. Complete the parallelogram OADB.



Now since P, Q, R are in equilibrium, therefore, OD which represents the resultant of P and Q, must be equal and opposite to R, *i.e.*, the force R is represented by DO. Also AD being equal and parallel to OB represents Q.

Hence P, Q, R are represented by the sides OA, AD and DO of the triangle OAD.

$$\therefore \quad \frac{P}{OA} = \frac{Q}{AD} = \frac{R}{OD},$$

The sides of a triangle being proportional to the sides of the opposite angles, we have

$$\frac{P}{\sin \angle ODA} = \frac{Q}{\sin \angle AOD} = \frac{R}{\sin \angle OAD}$$

But $\angle ODA = \angle BOD = \pi - \angle QOR$, $\angle AOD = \pi - \angle POR$,

and $\angle OAD = \pi - \angle POQ$.

Hence
$$\frac{P}{\sin \angle QOR} = \frac{Q}{\sin \angle POR} = \frac{R}{\sin \angle POQ} \qquad \ldots (1)$$

Cor. If three equal forces acting at a point are in equilibrium, the angle between any two of them is 120°.

2.31. Converse of Lami's Theorem. *If three coplanar forces acting at a point are such that each is proportional to the sine of the angle between the directions of the other two, the forces are in equilibrium.*

Let P, Q, R be the given forces satisfying result (1), we have to prove that they are in equilibrium.

Let P', Q', R' be a set of forces in *equilibrium* acting along the same lines, then by Lami's Theorem, we have

$$\frac{P'}{\sin \angle QOR} = \frac{Q'}{\sin \angle POR} = \frac{R'}{\sin \angle POQ} \qquad \ldots (2)$$

$\therefore$ from (2) and the given result (1), we get

$$\frac{P}{P'} = \frac{Q}{Q'} = \frac{R}{R'}$$

Hence the sides of the triangle representing the forces P', Q', R', will also represent the forces P, Q, R (on a different scale) which are, therefore, in equilibrium.

Cor. *If one set of concurrent forces (P, Q, R) acting in three given directions is in equilibrium all other sets acting in equilibrium in the same directions are merely multiples of the set (P, Q, R).*

* **2.32.** Lami's Theorem can also be proved by resolved forces in directions perpendicular to two of the lines of the forces.

The forces being in equilibrium, the sum of the resolved parts along any line in their plane must vanish. [Cor. Art. 2·4]

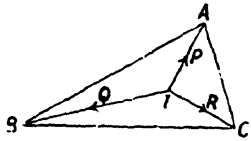
Resolving along a line perpendicular to OA, [*See* Fig. Art. 2·3], we have

$$Q \sin \angle POQ - R \sin \angle POR = 0 \text{ i.e., } \frac{Q}{\sin \angle POR} = \frac{R}{\sin \angle POQ}.$$

A similar result is obtained by resolving along the line perpendicular to OB or OC.

Example (i). *Forces P, Q, R, acting along IA, IB, IC, where I is the incentre of $\triangle ABC$, are in equilibrium, show that*

$$P : Q : R = \cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}.$$



By Lami's Theorem, we have

$$\frac{P}{\sin \angle BIC} = \frac{Q}{\sin \angle AIC} = \frac{R}{\sin \angle AIB}.$$

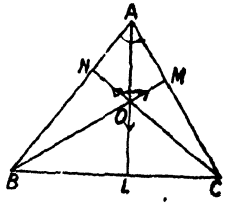
But $\angle BIC = 180° - \frac{B+C}{2} = 90° + \frac{A}{2}$ $\therefore \sin \angle BIC = \cos\frac{A}{2}$.

Similarly $\sin \angle AIC = \cos\frac{B}{2}$ and $\sin \angle AIB = \cos\frac{C}{2}$.

Hence $$\frac{P}{\cos\frac{A}{2}} = \frac{Q}{\cos\frac{B}{2}} = \frac{R}{\cos\frac{C}{2}}.$$

Example (ii). *Three forces proportional to the sides of a triangle act at the vertices towards and perpendicular to the corresponding sides, prove that the forces are in equilibrium.*

The forces P, Q, R respectively acting along AL, BM, CN, meet at O, the orthocentre of the $\triangle ABC$; then since



$$\frac{P}{a} = \frac{Q}{b} = \frac{R}{c} \quad \therefore \quad \frac{P}{\sin A} = \frac{Q}{\sin B} = \frac{R}{\sin C}$$

But $A = \pi - \angle MON$, $B = \pi - \angle NOL$,
$C = \pi - \angle LOM$

$$\therefore \quad \frac{P}{\sin \angle MON} = \frac{Q}{\sin \angle NOL} = \frac{R}{\sin \angle LOM}$$

Hence by the Converse of Lami's Theorem, the forces P, Q, R are in equilibrium.

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* This may be read after Art 2·4 has been studied.

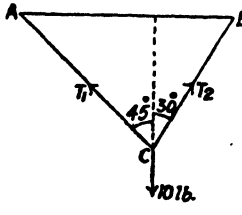
Note. The result is otherwise obvious, for, the forces meet at a point and being proportional to sides, they can be represented by the sides of the triangle ABC turned through a right angle, and hence they satisfy the necessary conditions of equilibrium [Art. 2.23].

Example (iii). *Three forces act at the middle points of the sides of a triangle, all inwards or all outwards. If each force is perpendicular and proportional to the side at which it acts, show that the forces are in equilibrium.*

For, they satisfy the condition of equilibrium of three forces, *viz.* (*a*) they meet at a point (the circumcentre) and (*b*) they satisfy Lami's Theorem just as in Ex. (*ii*) above.

Note. It follows that the resultant of two such forces acting inwards is a force acting outwards at the middle point of third side, proportional and perpendicular to it.

Example (iv) *A body weighing 10 lbs. is suspended from a horizontal beam by two strings. If the inclinations of the strings to the vertical are 45°, and 30° find the tension in each string.*



AB is the beam, the weight of 10 lbs. is attached at C to the strings CA and CB as shown. The point C is in equilibrium under three forces T_1, T_2 and 10 lbs. weight.

Hence by Lami's Theorem we have

$$\frac{T_1}{\sin 30°}=\frac{T_2}{\sin 45°}=\frac{10}{\sin 75°}.$$

$$\therefore\ T_1=\frac{10\sin 30°}{\sin 75°}=5\times\frac{2\sqrt{2}}{\sqrt{3}+1}=\frac{10\sqrt{2}(\sqrt{3}-1)}{3-1},$$

$$=5(\sqrt{6}-\sqrt{2})=5{\cdot}18 \text{ lbs.}$$

$$T_2=\frac{10\sin 45°}{\sin 75°}=\frac{10}{\sqrt{2}}\times\frac{2\sqrt{2}}{\sqrt{3}+1}=\frac{20(\sqrt{3}-1)}{3-1}$$

$$=10(\sqrt{3}-1)=7{\cdot}32 \text{ lbs.}$$

Examples IV

1. A 5 lbs. weight hangs freely at the end of a string. Find what horizontal force applied to the weight will keep the string inclined at 30° to the vertical. (*Ans.* 2·9 lbs. wt., nearly)

2. A weight of 24 lbs. is suspended by two flexible strings one of which, is horizontal and the other inclined at an angle of 30° to the vertical, what is the tension in each string ? (*Ans.* $8\sqrt{3}$; $16\sqrt{3}$ lbs. wt.)

3. A weight of 100 lbs. is suspended by two light strings 3 and 4 ft. long fastened to two points in the same horizontal line distant 5 ft. apart. Find the tensions in the strings. (*Ans.* 60, 80 lbs.)

4. A body of mass 26 lbs. is suspended by two strings 5 and 12 inches long, their other ends being fastened to the extremities of a rod 13 inches long. If the rod be so held that the body hangs immediately below the middle point, find the tensions of the strings. (*Ans.* 10, 24 lbs.) (*P.U.*)

5. Three forces P, Q, R acting at a point are in equilibrium and the angle between P and Q is double of the angle between P and R. Prove that

$$R^2=Q(Q-P).$$

6. A weight is supported by two strings fastened to two points in the same horizontal line, the strings being equally strong, but one longer than the other. If more weights be continually added to the first one, which string will break first ? (*Ans.* The shorter)

7. Find the greatest weight which can be supported by two light strings making angles 60° and 45° with the vertical, it being known that either string will break under a tension of W lbs. wt. (*Ans.* W $(\sqrt{3}+1)/\sqrt{6}$) (*P.U.*)

8. A weight W is tied at A to two strings AB, AC, B and C being in the same level, and hangs freely. If AB : AC : BC = 2 : 3 : 4, find the tensions in the strings in terms of W. (*Ans.* $7W/2\sqrt{15}$, $11W/4\sqrt{15}$)

9. The ends of an elastic string of modulus $5\sqrt{3}$ lbs. are attached to two points A and B in a horizontal line and initially the string is just taut and unstretched. A particle of weight 15 lbs. is attached to the middle point of the string. Prove that in the position of equilibrium each portion of the string makes 30° with the vertical.

10. Two particles of weight 2W, 3W, fastened to the end of a light inextensible string which passes over two smooth pegs on the same level, are kept in equilibrium by a third particle of weight W′ fastened to the part of the string between the pegs. If the angle between the oblique portions of the strings is 120°, prove that $W' = \sqrt{7}W$. (*D.U.*)

11. A bead of weight W can slide on a smooth circular wire in a vertical plane. The bead is attached by a light inextensible string to the highest point of the wire and in equilibrium the string is taut and makes an angle θ with the vertical. Find T, the tension of the string and R the reaction of the wire on the bead. (*Ans.* $T = 2W\cos\theta$, $R = W$)

12. Forces P, Q, R acting along OA, OB and OC in the plane of the $\triangle ABC$ are in equilibrium. Prove that

(*i*) $P : Q : R = \sin 2A : \sin 2B : \sin 2C$
$= a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2)$

when O is the circumcentre of the triangle.

(*ii*) $P : Q : R = a : b : c$, when O is the orthocentre of $\triangle ABC$.

(*iii*) $P : Q : R = \cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}$, when O is the incentre of $\triangle ABC$.

(*iv*) $P : Q : R = OA : OB : OC$, when O is the centroid of $\triangle ABC$.

(*See* hint at the end).

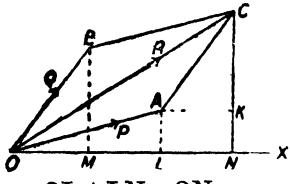
13. A, B, C are three points on the circumference of a circle and forces P and Q act along AB and EC respectively. If the resultant force R acts along the tangent at B, show that

$$\frac{P}{BC} = \frac{Q}{AB} = \frac{R}{AC}.$$

14. A, B and C are three points on the circumference of a circle. Forces inversely proportional to AB and BC act along AB and BC respectively, show that their resultant acts along the tangent at B.

2.4. Theorem II A. *The sum of the resolved parts of two concurrent forces along a given line in their plane is equal to the resolved part of their resultant along the same line.*

Let OA and OB represent two forces P and Q. Complete the parallelogram OACB, then the diagonal OC represents the resultant force R.



Through O draw OX parallel to the given line. Draw AL, BM, CN perp. to OX and AK perp. to NC.

Then the sum of the resolved parts of P and Q along OX = OL + OM = OL + AK = OL + LN = ON.

= the resolved part of R along OX.

Thus if P, Q and R make angles α, β and θ respectively with OX, then

$$R \cos \theta = P \cos \alpha + Q \cos \beta.$$

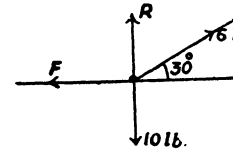
By its repeated application, the theorem can be easily extended to the more general theorem which may be stated as :—

The sum of the resolved parts of a number of concurrent and coplanar forces along a given line (in their plane) is equal to the resolved part of their resultant along the same line.

Cor. *If the forces are in equilibrium, the sum of their resolved parts vanishes.*

Example. *A body of weight 10 lbs. rests on a rough horizontal table and a force of 6 lbs. acts on it upwards in a direction making an angle of 30° with the table. Find the normal reaction of the table and the amount of friction on the body.*

Let R be the normal reaction on the body (regarded as a particle) and F, the force of friction.



The body being at rest, the sum of the resolved parts of the forces acting on it along any line in the plane of the forces must vanish.

∴ resolving horizontally and vertically, we have

$6 \cos 30° - F = 0$ *i.e.*, $F = 3\sqrt{3}$ lbs.

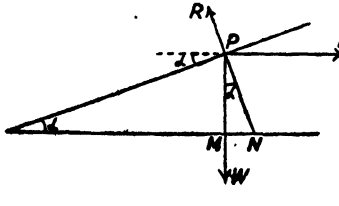
and $R + 6 \sin 30° - 10 = 0$ *i.e.*, $R = 10 - 3 = 7$ lbs.

Note. If the force of 6 lbs. does not act, no force of friction will be called into play and the pressure will be normal and equal to the weight of the body. Thus the horizontal component of the force balances the self-adjusting friction and the vertical component decreases the normal pressure.

2.41. Bodies resting on a smooth inclined plane.

I. *A body of weight W rests on a smooth inclined plane of inclination α. To find the horizontal force required to keep it in equilibrium and the reaction of the plane.*

Let F be the required force and R the reaction of the plane.



Resolving vertically and horizontally, we have

$R \cos \alpha - W = 0$

i.e., $R = W \sec \alpha$

and $F - R \sin \alpha = 0$

i.e., $F = R \sin \alpha = W \sec \alpha \sin \alpha = W \tan \alpha$.

We can obtain the same result from the principle of the triangle of forces.

The triangle PMN whose sides PM, MN, NP are respectively parallel to the forces W, F and R, is the triangle of forces, so that we have

$$\frac{W}{PM} = \frac{F}{MN} = \frac{R}{NP}.$$

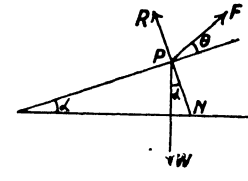
$$\therefore \quad R = W \cdot \frac{NP}{PM} = W \sec \alpha \quad \text{and} \quad F = W \cdot \frac{MN}{PM} = W \tan \alpha.$$

These results can also be obtained by the application of Lami's Theorem.

Note. In all such problems it will be understood that the force F acts in a vertical plane through the line of the greatest slope through P.

II. *The force F acts in a vertical plane in a direction making an angle θ with the plane (i.e., with the line of the greatest slope through P). To find the relations between F, R and W.*

Resolving along and perpendicular to the plane, we have



$$F \cos \theta - W \sin \alpha = 0$$

i.e.,
$$F = W \sin \alpha \sec \theta \qquad \ldots(1)$$

and
$$R + F \sin \theta - W \cos \alpha = 0$$

i.e.,
$$R = W \cos \alpha - F \sin \theta$$

$$= W \cos \alpha - \frac{W \sin \alpha \sin \theta}{\cos \theta} = \frac{W \cos (\alpha + \theta)}{\cos \theta} \qquad \ldots(2)$$

Cor. (*a*) *In particular when $\theta = 0$ i.e., F acts along the plane, we get $F = W \sin \alpha$ and $R = W \cos \alpha$.*

This is otherwise obvious, W sin α, W cos α being the components of W along and perp. to the plane.

(*b*) *Again when F acts horizontally $\theta = -\alpha$, so that*

$$F = W \tan \alpha \text{ and } R = W \sec \alpha,$$

the same as obtained in I above.

Examples V

1. A body of mass 20 lbs. is at rest on a rough horizontal plane when a force of $6\sqrt{3}$ lbs. is acting on it in an upward direction making an angle 60° with the horizontal. Find (*i*) the normal pressure, (*ii*) the frictional force, and (*iii*) the resultant reaction of the plane on the body.

What is the magnitude of the normal pressure when the force acts downwards making the same angle with the plane.

(*Ans.* 11 lbs. ; $3\sqrt{3}$ lbs. ; $2\sqrt{37}$ lbs. ; 29 lbs.)

2. An object which weighs 180 gms. rests on a smooth inclined plane, inclined to the horizontal at an angle $\sin^{-1}\frac{3}{5}$. It is maintained n equilibrium by a string which makes with a line of greatest slope of the plane (and above it) an angle $\sin^{-1}\frac{5}{13}$. Find the tension in the string and the reaction of the plane.

(*Ans.* 117 gm. wt. ; 99 gm. wt.)

3. Two forces P, Q acting respectively parallel to the base and length of a smooth inclined plane, would each of them singly support a particle of weight W, prove that

$$W = PQ/\sqrt{P^2 - Q^2}.$$

4. Two weights P and Q rest one on each of two smooth planes placed back to back, of inclinations α and β, being connected by a string which runs horizontally from one to the other. Show that

$$P \tan \alpha = Q \tan \beta.$$

If the string passes over a smooth pulley at the top of the inclined planes show that

$$P \sin \alpha = Q \sin \beta.$$

5. Show that the smallest force which will keep a body in equilibrium on a smooth inclined plane must act along the plane.

6. In what direction is the algebraic sum of the resolved parts of two forces acting at a point is the greatest and in what direction is it the least? Give reasons. (*Ans.* In the direction of the resultant; perp. to it.)

7. A weight is supported by a light string passing over a smooth pulley and gently lowered on to a smooth inclined plane. Show that if the string is slowly paid out so that the weight slides down the plane the pressure on the plane increases and the tension of the string decreases.

8. A weight is supported on a smooth plane of inclination θ to the horizontal by a string inclined to the vertical at an angle α. When the slope of the plane is increased to φ, the slope of the string remaining the same, the tension of the string is doubled to support the weight. Show that

$$\cot \theta - \cot \alpha = 2 \cot \varphi.$$

9. A weight is supported on a smooth plane inclined at an angle θ to the horizontal, by a force whose line of action makes an angle 2θ with the horizontal. If the pressure on the plane be the arithmetic mean of the weight and the force, prove that

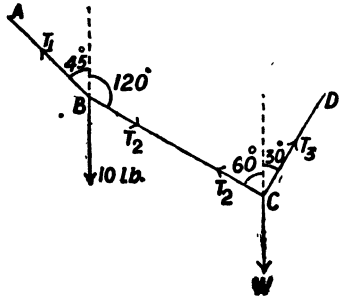
$$\theta = \tfrac{1}{2} \sin^{-1} \tfrac{3}{4}.$$

2.5. Equilibrium of particles connected by strings. When two or more particles are connected together, the equilibrium of each particle is considered separately, taking account of all the forces which arise in virtue of its connection (by strings or rods) with the others in addition to the special forces acting upon it.

When there are only three forces acting on each particle, the conditions of equilibrium are easily obtained by the application of Lami's Theorem.

Example (i). *ABCD is a string suspended from points A and D and it carries a weight of 10 lbs. at B and a weight of W lbs. at C. The inclinations to the vertical of AB and CD are 45° and 30° respectively, and the angle ABC is 165°. Find W and the tensions in the different parts of the string.*

Let the tensions in the strings AB, BC and CD be T_1, T_2, T_3 respectively.



The string BC pulls B towards, C with the force T_2 and it pulls C towards B with the same force, the tensions T_2 being equal and opposite on B and C.

Since the forces T_1, T_2 and 10 lbs., acting at B, are in equilibrium, we have

$$\frac{T_1}{\sin 60°} = \frac{T_2}{\sin 45°} = \frac{10}{\sin 165°},$$

so that $T_1=\frac{10\sin 60°}{\sin 15°}=10\times\frac{\sqrt{3}}{2}\times\frac{2\sqrt{2}}{\sqrt{3}-1}=5\sqrt{6}(\sqrt{3}+1)$

$=334{\cdot}6$ lbs.

and $T_2=\frac{10\sin 45°}{\sin 15°}=\frac{10}{\sqrt{2}}\times\frac{2\sqrt{2}}{\sqrt{3}-1}=10(\sqrt{3}+1)=27{\cdot}32$ lbs.

Again, since the forces T_2, T_3 and W, acting at C are in equilibrium, we have

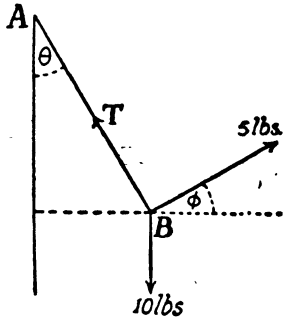
$$\frac{T_2}{\sin 30°}=\frac{T_3}{\sin 60°}=\frac{W}{\sin 90°}.$$

$\therefore\quad T_3=\frac{\sin 60°}{\sin 30°}.T_2=\sqrt{3}\times 10(\sqrt{3}+1)=10(3+\sqrt{3})=47{\cdot}32$ lbs.

and $W=\frac{\sin 90°}{\sin 30°}.T_2=2T_2=20(\sqrt{3}+1)=54{\cdot}64$ lbs.

Example (ii). *A weight of 10 lbs. hangs by a string from a fixed point. The string is drawn out of the vertical by applying a force of 5 lbs. to the weight. In what direction must this force be applied in order that in equilibrium the deflection of the string from the vertical may have its greatest value? What is the amount of the greatest deflection?*

In any position of equilibrium, let the string AB make an angle θ with the vertical and the given force an angle ϕ with the horizontal.



The force of 5 lbs., the weight of 10 lbs. and the tension T of the string are in equilibrium.

$\therefore$ resolving vertically and horizontally, we have

$T\cos\theta+5\sin\phi=10,$

i.e., $T\cos\theta=10-5\sin\phi$ ...(1)

and $T\cos(90°-\theta)=5\cos\phi,$

i.e., $T\sin\theta=5\cos\phi$ ...(2)

Dividing (1) by (2), we have

$$\cot\theta=\frac{10-5\sin\phi}{5\cos\phi}=2\sec\phi-\tan\phi \qquad ...(3)$$

Differentiating, we get

$$-\operatorname{cosec}^2\theta\frac{d\theta}{d\phi}=2\sec\phi\tan\phi-\sec^2\phi=\sec\phi(2\tan\phi-\sec\phi).$$

θ is maximum, when $\frac{d\theta}{d\phi}=0$, *i.e.*, when $2\tan\phi-\sec\phi=0$,

or $2\sin\phi=1$, which gives $\phi=30°$, the required direction of the force.

Now from (3), $\cot\theta=2\times\frac{2}{\sqrt{3}}-\frac{1}{\sqrt{3}}=\frac{3}{\sqrt{3}}=\sqrt{3}.$

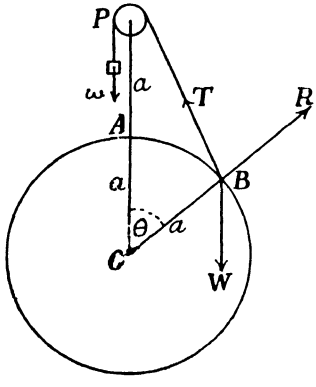
$\therefore\quad\theta=30°$, the required greatest deflection.

Note. Equation (3) can also be obtained by Lami's Theorem.

Ex. If the suspended weight is $\sqrt{2}$ times the magnitude of the force, show that the greatest deflection of the string is 45° and that the tension of the string in that position is equal to the force applied.

Example (iii). *A particle of weight W rests against the circumference of a circular plate whose centre is O and whose plane is vertical; a cord attached to the particle passes over a pulley placed vertically above the highest point A of the plate at a distance from it equal to its radius and carries a weight w. Show that the particle will rest at an angular distance θ from OA given by the equation*

$$\cos\theta = \frac{5}{4} - \frac{w^2}{W^2}.$$



Let a be the radius of the plate, P, the position of the pulley and B, the position of the particle, so that

$$OP = 2a \text{ and } \angle AOB = \theta.$$

Also $T = w$, where T is the tension in the cord passing over the pulley and carrying the weight w.

As the particle rests at B, the reaction R of the plate acts on it along normal OB.

OBP is obviously the triangle of forces acting on the particle.

$$\therefore \quad \frac{W}{OP} = \frac{T}{BP} = \frac{R}{OB} \text{ or } \frac{W}{2a} = \frac{w}{BP} = \frac{R}{a}.$$

$$\therefore \quad R = \tfrac{1}{2}W \text{ and } W \times BP = 2aw.$$

But from the $\triangle$ OBP,

$$BP^2 = OB^2 + OP^2 - 2OB.OP\cos\theta$$
$$= a^2 + 4a^2 - 4a^2\cos\theta = 5a^2 - 4a^2\cos\theta$$

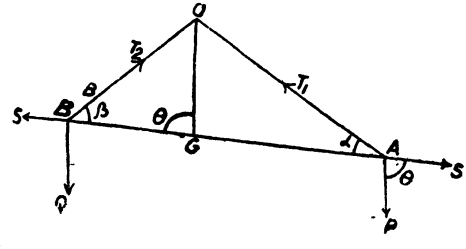
$$\therefore \quad 5a^2 - 4a^2\cos\theta = BP^2 = \frac{4a^2w^2}{W^2} \quad \text{or} \quad \cos\theta = \frac{5}{4} - \frac{w^2}{W^2}.$$

2·51. Particle connected by a rod are kept separate from each other by the application of a force known as **thrust**. This force acts along the rod and is equal and opposite on the two particles attached to the rod at its ends.

Example *Two weights P and Q are suspended from a fixed point O by strings OA, OB and are kept apart by a* **light** *rod AB. If the strings make angles α and β with the rod, show that the angle θ which the rod makes with the vertical is given by*

$$(P+Q)\cot\theta = P\cot\alpha - Q\cot\beta.$$

The points A and B being connected by a light rod AB, the thrust S of the rod AB, at A is equal and opposite to that at B.



Let OG be the vertical through O meeting the rod in G, then $\angle$ OGB $= \theta$.

Since forces T_1, the tension

in the string AO, S the thrust of the rod and the weight P, acting at A, are in equilibrium and since $\angle OAP = \alpha + \angle GAP = \alpha + \pi - \theta$, we have

$$\frac{S}{\sin(\alpha+\pi-\theta)} = \frac{P}{\sin\alpha} = \frac{T_1}{\sin\theta} \quad \ldots (1),$$

and since the forces T_2, the tension in the string BO, S, the thrust of the rod and the weight Q, acting at B are in equilibrium, we have

$$\frac{S}{\sin(\theta+\beta)} = \frac{Q}{\sin\beta} = \frac{T_2}{\sin\theta} \quad \ldots (2).$$

$\therefore$ equating the values of S from (1) and (2), we get

$$\frac{Q\sin(\theta+\beta)}{\sin\beta} = \frac{P\sin(\alpha+\pi-\theta)}{\sin\alpha} = \frac{P\sin(\theta-\alpha)}{\sin\alpha}$$

i.e., $Q(\sin\theta\cot\beta + \cos\theta) = P(\sin\theta\cot\alpha - \cos\theta)$.

which gives $(P+Q)\cot\theta = P\cot\alpha - Q\cot\beta \quad \ldots (3)$

Note. (*i*) Triangles AOG and BOG are triangles of forces for equilibrium of points A and B.

$$\therefore \frac{S}{AG} = \frac{P}{OG} \text{ and } \frac{S}{BG} = \frac{Q}{OG}, \therefore \frac{S}{P} + \frac{S}{Q} = \frac{AG}{OG} + \frac{BG}{OG} = \frac{AB}{OG} \quad \ldots (4)$$

and $P \times AG = Q \times BG \quad \ldots (5)$

(*ii*) Tensions T_1, T_2 can be obtained, if required, from (1) and (2).

Also from the triangles of forces AOG, BOG, we have

$$\frac{T_1}{OA} = \frac{S}{AG}, \quad \frac{T_2}{OB} = \frac{S}{BG} \quad \therefore \quad \frac{T_1}{T_2} = \frac{OA}{OB} \cdot \frac{BG}{AG}.$$

Ex. Particles A, B of weights W_1, W_2 are suspended by light strings from a fixed point O and kept apart by a straight light rod AB. Prove that in the position of equilibrium the thrust exerted by the rod is

$$\frac{W_1W_2}{W_1+W_2} \cdot \frac{AB}{OG},$$

G being the point of the rod on the vertical through O. (*London U.*)

[Hint. See (4) above]

Examples VI

1. APCD is a string suspended from points A and D and it carries a weight of 10 lb. at B and a weight W lb. at C. The portion AB is horizontal and BC, CD make angles of 30° and 60° respectively with the horizontal; find W and the tensions in the three portions of the string.

(*Ans.* W = 20 lbs. tensions $10\sqrt{3}$, 20, $20\sqrt{3}$ lbs.)

2. A string ABCD attached to two fixed points A and D, has two equal weights, W, knotted to it at B and C and rests with the portions AB and CD inclined at angles of 30° and 60° respectively to the vertical. Find the tensions of the portions of the string and the inclination of BC to the vertical.

(*Ans.* $W\sqrt{3}$, W, W; 60°)

3. ABCD is a light string to which a weight 12 lbs. is attached at B and a weight W at C. The ends A, D are fastened to fixed points and the parts AB, BC, CD make angles 30°, 45°, 60° with the downward vertical measured in the same sense. Prove that $W = 4\sqrt{3}$ lbs. and find the tensions in the different parts of the string. (*Ans.* AB, 32·8 lbs. wt; BC, 23·2 lbs. wt.; CD, 18·9 lbs. wt.)

4. Three smooth pegs A, B, C are fixed in a vertical wall. The $\triangle ABC$ is isosceles (AB = AC). A is uppermost and BC is horizontal the angle BAC being α.

A light inextensible string carrying equal weights W at its ends is hung over the pegs. Prove that the thrust of the string on peg B is $2W \sin \frac{1}{4}\alpha$ and find the thrust on the peg A. (*Ans.* $2W \cos \frac{1}{2}\alpha$)

5. A rectangular board, 6 feet by 4 feet, is suspended with its longer sides horizontal, by means of a light string 10 feet long passing through smooth rings at the upper corners and over a smooth peg. Find the tension T of the string and the pressure P on each ring if the weight of the board is W.

(*Ans.* $T=\frac{5}{8}W$, $P=\frac{1}{2}W\sqrt{5}$) (*R.U.*)

6. A square board ABCD, the length of whose side is a, is fixed in a vertical plane with two of its sides horizontal. An endless string of length $l(>4a)$ passes over four pegs at the angles of the board and through a ring of weight W, which is hanging freely. Show that the tension of the string is

$$W(l-3a)/2\sqrt{l^2-6al+8a^2}.$$

7. Three equal strings are knotted together to form an equilateral triangle ABC and a weight W is suspended from A. If the triangle and weight be supported with BC horizontal, by means of two strings at B and C, each at the angle of 135° with BC, show that the tension in BC is $W(3-\sqrt{3})/6$.

8. A light rod AB hangs horizontally by two strings from a point C and weights 8 lbs. and W lb. are attached to A and B respectively. If the angle ABC=60° and the tension in the string AC is 10 lbs. wt., find (*i*) the value of W, (*ii*) the tension in the string BC, (*iii*) the angle BAC, and (*iv*) the stress in the rod. (*Ans.* $6\sqrt{3}$ lbs. ; 12 lbs. ; $\sin^{-1}\frac{4}{5}$; 6 lbs.)

9. A light rod AB, 5 ft. long is hung from a peg O by two light strings OA, OB which are 4 ft. and 3 ft. long respectively. A weight W_1 is fixed to the rod at A and a weight W_2 at B. If in the position of equilibrium, the angle between the rod and the vertical is θ, show that

$$\tan\theta = \frac{12(W_1+W_2)}{16W_1-9W_2}.$$

10. A light rod AB of length 5 in., hung from a fixed point O by two light strings OA, OB of lengths 4 and 3 ins. respectively. A particle of weight 3 lbs. is attached to the rod at A, and another of weight 2 lbs. at B. Show that in the position of equilibrium the tensions in the strings OA, OB are $2\sqrt{5}$, $\sqrt{5}$ lb. wt. respectively and that the stress in the rod AB is $\sqrt{5}$ lb. wt.

11. A thin smooth circular wire, of radius a, is fixed in a vertical plane, and two small rings A, B of weights W, 2W are connected by a light rod AB of length $a\sqrt{3}$ and can slide freely on the wire. Show that in the position of equilibrium (below the centre of the wire) the rod is inclined at an angle of 60° to the vertical and that the stress in the rod is equal to 2W.

12. Two beads of weights 2W and W can slide on a smooth circular wire, of radius a, in a vertical plane. They are connected by a light string of length $\sqrt{3}\,a$ and are in equilibrium on the upper half of the wire. Show that the inclination of the string to the horizontal is 30°. (*P.U. 1958*)

13. Three equal rings, each of weight W, can slide freely on a thin smooth circular wire fixed in a vertical plane. The uppermost ring is connected with the other two by means of equal strings. In the position of equilibrium the rings form an equilateral triangle with one side vertical. Find the tensions in the strings and the reactions on the rings.

(*Ans.* Tensions 2W, W. Reactions $\sqrt{3}W$, 0, $2\sqrt{3}W$)

14. A particle of weight $2w$ is attached to the end A, and a particle of weight w to the end B of a light rod AB of length $2a$. The rod hangs from a point O to which it is attached by light strings AO, BO each of length b. Prove that in equilibrium the inclination of the rod to the horizontal is θ, where $\tan\theta = \frac{a}{3\sqrt{b^2-a^2}}$.

Find also the tensions in the strings AO, BO in terms of a, b and w. (*D.U.*)

(*Ans.* $6bw/\sqrt{9b^2-8a^2}$, $3bw/\sqrt{9b^2-8a^2}$)

15. Two heavy particles of masses m_1 and m_2 are connected by a light rigid rod of length d and are constrained to move on two intersecting lines l_1 and l_2. The line l_1 is vertical and is inclined to l_2 at an angle of 45°. Determine the position of equilibrium, assuming that there is no friction. (*D. U. 1953*)

[*Ans.* Inclination of rod with hor. $=\tan^{-1}\dfrac{m_1}{m_1+m_2}$]

16. A light rod BC of length a, is suspended from a fixed point A by means of two light strings AB and AC of lengths c and b respectively. If masses each of weight W are suspended at B and C, show that, in the position of equilibrium, the tensions of the strings are proportional to their lengths and that the thrust in the rod is of magnitude $aW/\sqrt{2b^2+2c^2-a^2}$. (*L.U.*)

17. Two spheres are connected by a string passing over a smooth peg and hang in contact. Prove that the distances of their centres (not necessarily lying in a horizontal line) from the peg are inversely proportional to their weights. (*B. U.*)

18. Two weights P, Q attached to the ends of a string rest on a smooth circular cylinder whose axis is horizontal. Prove that the inclination θ to the horizontal of the line joining them is given by

$$\tan\theta=\frac{P-Q}{P+Q}\tan\alpha,$$

where 2α is the angle which this line subtends at the nearest point of the axis of the cylinder.

19. Two weights w, w' rest on a smooth sphere, being supported by a string, (of natural length a and modulus λ) which passes through a smooth ring at O, a point vertically above C, the centre of the sphere. Show that in the position of equilibrium we have

$$\lambda\frac{l-a}{a}\cdot OC=\frac{l}{\frac{1}{w}+\frac{1}{w'}},$$

where l denotes the length of the string in the position of equilibrium. (*Pub. Ser. Com.*)

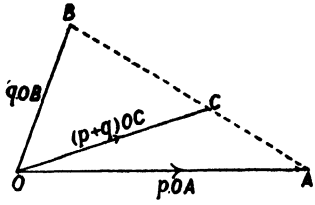
20. A triangle ABC is formed of three light rods smoothly jointed to each other at their ends. D is the foot of the perpendicular from A on BC. A string at tension T joins AD. Show that the stresses in the rods AB and AC are T cosec A cos C, T cosec A cos B.

CHAPTER III

FORCES ACTING AT A POINT-II

3.1. *The resultant of two forces acting at O along OA and OB, represented in magnitude by p.OA and q.OB, is represented by (p+q) OC, where C is a point in AB such that p.CA=q. CB.*

By Art. 2.2, the force p.OA is equivalent to forces represented by p.OC and p.CA. Similarly the force q.OB is equivalent to forces represented by q.OC and q.CB.



Hence the forces p.OA and q.OB are together equivalent to forces represented by $(p+q)$ OC, p.CA and q.CB.

But since p.CA$=q$.CB, the two latter forces being equal in magnitude and opposite in direction balance and we are left with only one force $(p+q)$ OC which is the resultant of the given forces.

Cor. (*i*) *If the forces are p.AO and q.BO, the resultant is* $(p+q)$ *CO.*

(*ii*) *If* $p=q$, *then C is the middle point of AB and the resultant is* $2p.OC$.

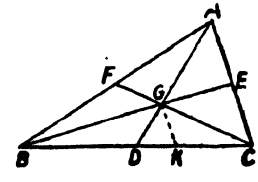
In particular if $p=q=1$, the resultant is 2OC, *i.e.*, the resultant of forces OA and OB is represented by 2OC, where C is the middle point of AB. This is otherwise obvious by the parallelogram law of forces.

Note. If $p=-q$, *i.e.*, $p+q=0$ the forces become q.AO and q.OB, their resultant is q.AB (Art. 2.2).

Example (i). *If G is the point of intersection of the medians of a triangular lamina ABC, prove that forces acting at G and represented by* (*a*) *GA, GB, GC are in equilibrium, and* (*b*) *GA, 2GB, 3GC, have a resultant force represented by 3GK, where K is a point on BC such that BK=2KC.*

(*a*) The resultant of forces GB and GC is 2GD, where D is the middle point of BC.

2GD being equal to AG, the forces represented by 2GD and GA, being equal and opposite balance.



Hence *the forces acting at G and represented by GA, GB and GC are in equilibrium.*

(*b*) Now we are left with forces GB and 2GC acting along GB and GC. Their resultant, by the above article, is equal to 3GK where K is a point in GC such that BK=2KC.

Example (ii). *P is a point in the plane of the triangle ABC. Show that the resultant of the forces acting at P and represented by*

$$PA \cdot \sin A,\ PB \cdot \sin B \text{ and } PC \cdot \sin C$$

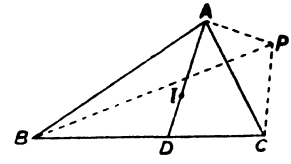
is

$$4PI \cdot \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$$

where I is the incentre of the triangle ABC.

The resultant of PB. sin B and, PC. sin C is PD.(sin B + sin C), where D is a point on BC such that

$$\frac{BD}{DC} = \frac{\sin C}{\sin B} = \frac{c}{b} = \frac{AB}{AC}.$$



i.e., AD bisects $\angle A$.

Now the resultant of PD. (sin B + sin C) and PA. sin A is

$$PI.\ (\sin A + \sin B + \sin C)$$

$$= 4PI.\ \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2} \quad \text{(By Trign.)},$$

where I is a point on AD such that

$$\frac{AI}{DI} = \frac{\sin B + \sin C}{\sin A} = \frac{b+c}{a} = \frac{AB}{BD}$$

$$\left(\because BD = \frac{ac}{b+c},\ \frac{BD}{DC} \text{ being} = \frac{AB}{AC} = \frac{c}{b}\right)$$

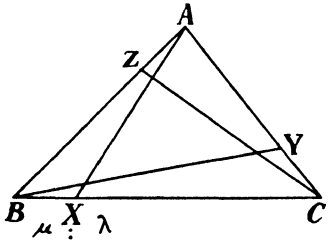
i.e., BI bisects the angle B, so that I is the incentre.

Cor. The resultant of forces acting at P and represented by a PA, b PB and c PC passes through the incentre of the triangle ABC.

Example (iii). X, Y, Z *are taken on the sides BC, CA, AB of the triangle ABC such that*

$$\frac{BX}{XC} = \frac{CY}{YA} = \frac{AZ}{ZB} = \frac{\mu}{\lambda}.$$

Prove that the forces acting at a point and represented in magnitude and direction by AX, BY and CZ are in equilibrium.



Since $\frac{BX}{XC} = \frac{\mu}{\lambda}$ or $\lambda.BX = \mu.XC$,

$\therefore$ forces represented by λAB and μAC are equivalent to a force represented by $(\lambda+\mu)AX$,

i.e., $\lambda AB + \mu AC \equiv (\lambda+\mu)\ AX.$

or $(\lambda+\mu)\ AX \equiv \lambda AB - \mu CA,$

or $$AX \equiv \frac{\lambda}{\lambda+\mu} AB - \frac{\mu}{\lambda+\mu} CA$$

Similarly $$BY \equiv \frac{\lambda}{\lambda+\mu} BC + \frac{\mu}{\lambda+\mu} BA \equiv \frac{\lambda}{\lambda+\mu} BC - \frac{\mu}{\lambda+\mu} AB$$

and $$CZ \equiv \frac{\lambda}{\lambda+\mu} CA + \frac{\mu}{\lambda+\mu} CB \equiv \frac{\lambda}{\lambda+\mu} CA - \frac{\mu}{\mu+\lambda} BC.$$

$\therefore$ the system of forces represented by AX, BY and CZ

$$\equiv \frac{\lambda-\mu}{\lambda+\mu} BC + \frac{\lambda-\mu}{\lambda+\mu} CA + \frac{\lambda-\mu}{\lambda+\mu} AB$$

$$\equiv \frac{\lambda-\mu}{\lambda+\mu} \left[BC+CA+AB \right] \quad \ldots(1)$$

But by the triangle of forces, the resultant of the forces represented by the sides BC, CA and AB of a triangle ABC and *acting at a point*, is nil.

Hence the forces, acting at a point and represented in magnitude and direction by AX, BY, CZ are in equilibrium.

Examples VII

1. Find a point inside a triangle such that if a particle at it be acted on by forces represented by the lines joining it to the vertices, it will be in equilibrium. (*Ans.* The point of intersection of the medians)

2. ABC is a triangle ; D, E, F are the middle points of the sides BC, CA, AB respectively ; show that the forces represented by the straight lines AD, BE, CF and acting at a point are in equilibrium. (*D.U.*)

3. A, B, C are the fixed points and P is a point such that the resultant of forces PA and PB always passes through C. Show that the locus of P is a straight line.

4. The sides AB and AC of triangle ABC are bisected in D and E. Show that the resultant of forces represented by BE and DC is represented in magnitude and direction by $\frac{1}{2}$ BC.

5. AEC is a triangle. Find the resultant of the forces at A represented in magnitude and direction by (i) 2AB and 3AC, (*ii*) 5CA and 3AB.

(*Ans.* (*i*) 5AD where 2BD=3CD. (*ii*) 2DA where D is on BC produced such that 5CD=3BD.)

6. O is any point in the plane of the triangle ABC and D, E, F are the middle points of the sides. Show that the system of forces OA, OB, OC is equivalent to the system OD, OE, OF.

7. Find a point O inside a quadrilateral ABCD such that if a particle at it be acted upon by forces represented by OA, OB, OC, OD, it will be in equilibrium.

(*Ans.* The point of intersection of the lines joining the mid-points of the opposite sides or the point which bisects the lines joining the middle points of the diagonals.)

8. AEC is a triangle. Show that the forces PA, PB, PC will be in equilibrium if P coincides with the centroid of the triangle ABC.

State and prove the corresponding result for four forces PA, PB, PC, PD. (*P.U. 1959 S.*)

9. ABCD is a quadrilateral. Show that if four forces represented by AB, AD, CB, CD be applied at a point, their resultant will be represented by four times the line joining the middle points of the diagonals.

10. ABCD is a parallelogram, a particle P is attracted towards A and C by forces mPA and mPC respectively and repelled from B and D by forces nPB and nPD, show that the resultant passes through the centre O of the parallelogram.

Under what conditions is P in equilibrium wherever it is situated ?

(*Ans.* $m=n$.)

11. If forces P and Q act along OA, OB respectively, show that the line of action of their resultant cuts AB in a point C such that

$$AC/CB = Q.OA/P.OB.$$

12. Forces acting on a particle are represented by AP, PB, PC, where P is the orthocentre of $\triangle ABC$. Show that the resultant is represented in magnitude and direction by the diameter through A of the circle ABC.

13. H is the orthocentre and O the circumcentre of a triangle ABC. Show that the three forces AH, BH, CH, have as resultant the force represented by twice OH.

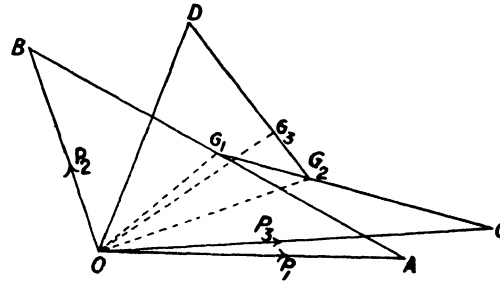
14. At ony point P of a parabola forces are applied which are completely represented by the tangent PT and the normal PG terminated by the axis. Prove the line of action of their resultant passes through the focus of the curve.

Resultant of a number of concurrent forces

3.2. The resultant of a number of forces *acting at a point* can be obtained in the following ways :—

3.21. First Method. *By the repeated application of Art. 3.1.*

Let the forces P_1, P_2, P_3.....etc., acting at O be represented by the lines OA, OB, OC..... etc.



The resultant of P_1 and P_2 is $2OG_1$, where G_1 is the middle point of AB. Compounding the force $2OG_1$ with P_3 or OC, we get a resultant represented by $3OG_2$, where G_2 divides G_1C such that $2G_1G_2 = G_2C$.

Again, the resultant of $3OG_2$ and P_4 (OD) is $4OG_3$ where $3G_2G_3 = G_3D$ and so on.

Proceeding in this way, we obtain the resultant of n such forces as $n.OG$ where G is the last point arrived at in this manner.

Def. The point G is known as the centroid of points A, B, C, D,.....etc.

Cor. *If the point G so obtained happens to coincide with O, the resultant vanishes and the forces are in equilibrium.*

In other words, the forces are in equilibrium, if the point O at which the forces act is itself the centroid of the extremities of the vectors representing the forces.

Note. It is clear from the proof that the forces need not necessarily lie in the same plane.

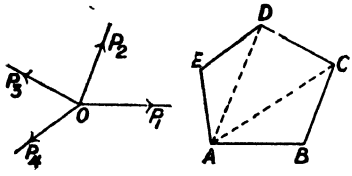
Example. If A, B, C, D, ...are points which divide the circumference of a circle into n equal parts and if a particle P, lying on the circumference is acted upon by forces represented by PA, PB, PC, PD....., the magnitude of the resultant is constant.

For, by symmetry the centre O of the circle is the centroid of A, B, C, D.....and the resultant of the forces is, therefore, equal to $n.PO = nr$, a constant, r being the radius of the circle.

3.22. Second Method. *By the repeated application of the law of*

parallelogram of forces. The process in its simplified form, is as follows :—

Let forces P_1, P_2, P_3, P_4, act at the point O. Let AB and BC represent the forces P_1 and P_2, their resultant, say, R_1, is then represented by AC (Art. 2.2).



Now let CD represent P_3, the resultant R_2, of R_1 and P_3, is represented by AD, *i.e.*, the resultant of P_1, P_2, P_3, is represented by AD. Similarly if DE represents the force P_4. AE will represent the resultant of forces P_1, P_2, P_3 and P_4.

The process can be continued in the case of a number of forces acting at a point, the final resultant being represented by the line joining the first point A to the end of the line representing the last force.

Cor. *When the end of the last line coincides with the first point, the resultant vanishes.*

This leads to the theorem of the polygon of forces given below.

3.221. Polygon of Forces. *If a number of forces acting at a point can be represented by the sides of a closed polygon, taken in order, the forces are in equilibrium.*

Let AB, BC, CD, DE and EA represent, on some scale, the forces P_1, P_2, P_3, P_4 and P_5. As shown above the resultant of forces AB, BC, CD and DE is represented, in magnitude and direction, by AE ; this resultant together with the force P_5 represented by E.A. balance. Hence the forces are in equilibrium.

This applies to any number of forces and proves the theorem.

Note. It is clear from the proof that the forces need not necessarily be coplanar, and the sides of the polygon need not lie in the same plane.

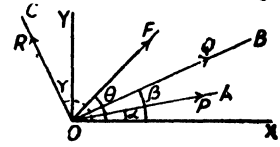
3.222. Converse of the Polygon of forces. If a number of forces, acting at a point are in equilibrium, they can be represented by the sides of a closed polygon taken in order, the sides being respectively parallel to the direction of the forces.

This proposition is not complete in the sense in which the converse of the triangle of forces is. For, in the case of three forces, all triangles having their sides parallel to the forces are similar and consequently, the sides of each one of them can represent the forces on some scale, while in the case of a number of forces, all polygons having their side parallel to the forces are not necessarily similar (as can be seen by drawing a line D'E' parallel to DE in the figure of Art. 3.22) and the sides of all these polygons do not therefore represent the given forces.

3.23. Third Method. Analytical method for determining the resultant of concurrent and coplanar forces.

Let OX, OY be two perpendicular lines in the plane of the forces.

Let the forces P, Q, R...etc., act along OA, OB, OC... etc., making angles α, β, γ.....etc., with OX, let F be their resultant inclined at $\angle\theta$ with OX and let X, Y be the resolved parts of F along OX and OY.



Then, since the resolved part of the resultant is equal to the sum of the resolved parts of its components (Art. 2.4), by resolving forces along OX, and OY, we have

$$F\cos\theta = P\cos\alpha + Q\cos\beta + R\cos\gamma + \ldots\ldots = X \quad \ldots(1)$$

$$\text{and } F\sin\theta = P\sin\alpha + Q\sin\beta + R\sin\gamma + \ldots\ldots = Y \quad \ldots(2)$$

$$\therefore \quad F^2 = X^2 + Y^2 \quad \ldots(3)$$

$$= \Sigma P^2 + 2\Sigma PQ\cos(\alpha-\beta)$$

$$\text{and} \quad \tan\theta = \frac{Y}{X}. \quad \ldots(4)$$

Equations (3) and (4) give the magnitude and line of action of the resultant.

3.231. Conditions of equilibrium of concurrent and coplanar forces. When $X=0$ as well as $Y=0$, then $F=0$ so that the forces are in equilibrium if the sum of their resolved parts along two perpendicular lines, OX, OY in their plane, separately vanish.

Conversely, if the forces are in equilibrium, $F=0$. It follows from (3) that $X=0$ and also $Y=0$.

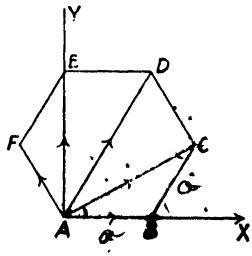
These conditions are, therefore, *sufficient as well as necessary conditions of equilibrium* for concurrent and coplanar forces.

3.232. It may be noted that it is simply for the sake of convenience that OX, OY are taken to be at right angles, it is not necessary to do so.

For forces to be in equilibrium, it is sufficient that the sum of the resolved parts of the forces along any two non-parallel straight lines separately vanish. For, if the sum of the resolved parts along one line l vanishes, the resultant F is either zero or acts at right angles to l. Similarly if the sum of the resolved parts along any other line l' (not parallel to l) vanishes, F is either zero or acts at right angles to l'. Hence as F cannot be perpendicular to two unparallel lines l and l', it must be zero.

Example (i). *ABCDEF is a regular hexagon. Show that the resultant of forces represented by AB, 2AC, 3AD, 4AE, 5AF, is represented by* $\sqrt{351}$ *AB and show that it makes an angle* $\tan^{-1}\frac{7}{\sqrt{3}}$ *with AB.*

Taking a to be the length of a side of the hexagon and resolving along AB and AE, two lines at right angles, we have



$$X = AB + 2AC\cos 30° + 3AD\cos 60° + 5AF\cos 120°$$

$$= a + 2\sqrt{3}a\times\sqrt{3}/2 + 6a\times\tfrac{1}{2} - 5a\times\tfrac{1}{2} = \tfrac{9}{2}a$$

$$Y = 2AC\sin 30° + 3AD\sin 60° + 4AE + 5AF\cos 30°$$

$$= 2\sqrt{3}a\times\frac{1}{2} + 6a\times\frac{\sqrt{3}}{2} + 4\sqrt{3}a + 5a\times\frac{\sqrt{3}}{2}$$

$$= \frac{21\sqrt{3}}{2}a.$$

$\therefore$ the resultant $R=\sqrt{X^2+Y^2}=\sqrt{351}\ a=\sqrt{351}.AB$

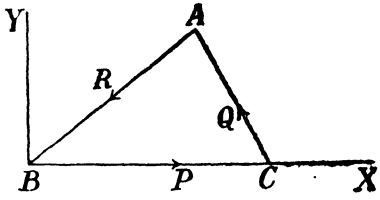
and $$\tan\theta=\frac{Y}{X}=\frac{21\sqrt{3}}{2}a\div\frac{9}{2}a=\frac{7}{\sqrt{3}}.$$

Example (ii). *Forces P, Q, R act at a point parallel to the sides BC, AC, AB of a triangle ABC, taken in order. Prove that their resultant S is given by*

$$S^2=P^2+Q^2+R^2-2PQ\cos C-2QR\cos A-2RP\cos B.$$

Find also the inclination of the line of action of the resultant with BC.

Resolving forces along and perpendicular to BC, we have



$$X=P-Q\cos C-R\cos B$$
$$Y=Q\sin C-R\sin B$$
$$\therefore\quad S^2=(P-Q\cos C-R\cos B)^2+(Q\sin C-R\sin B)^2$$
$$=P^2+Q^2(\cos^2 C+\sin^2 C)+R^2(\cos^2 B+\sin^2 B)$$
$$-2PQ\cos C-2PR\cos B+2QR(\cos B\cos C-\sin B\sin C)$$
$$=P^2+Q^2+R^2-2PQ\cos C-2PR\cos B+2QR\cos(B+C)$$
$$=P^2+Q^2+R^2-2PQ\cos C-2PR\cos B-2QR\cos A.$$

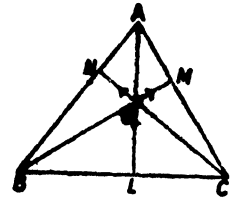
Also $$\tan\theta=\frac{Y}{X}=\frac{Q\sin C-R\sin B}{P-Q\cos C-R\cos B}.$$

Example (iii). *Forces act through the angular points of a triangle perpendicular to the opposite sides and are proportional to the cosines of the corresponding angles, show that the resultant F is proportional to*

$$\sqrt{1-8\cos A\cos B\cos C}.$$

Let the forces along AL, BM, CN be equal to $k\cos A$, $k\cos B$, and $k\cos C$, respectively.

The forces meet in O, the orthocentre of the triangle, and $\angle LBM=90°-C$, $\angle LCN=90°-B$.



Resolving forces along LC taken as X-axis, and taken LA as Y-axis, we have

$$X=(k\cos B)\sin C-(k\cos C)\sin B$$
$$=k\sin(C-B).$$

$$Y=(k\cos B)\cos C+(k\cos C)\cos B-k\cos A$$
$$=2k\cos B\cos C-k\cos A$$
$$=k[\cos(C-B)+\cos(C+B)]-k\cos A$$
$$=k\cos(C-B)-2k\cos A,\qquad \because\ B+C=180°-A.$$
$$\therefore\quad F^2=X^2+Y^2=k^2[1+4\cos^2 A-4\cos A\cos(C-B)]$$
$$=k^2-4k^2\cos A[-\cos A+\cos(C-B)]$$
$$=k^2-4k^2\cos A[\cos(C+B)+\cos(C-B)]$$
$$\because\ \cos A=-\cos(B+C)$$
$$=k^2(1-8\cos A\cos B\cos C).$$

Examples VIII

1. Find the magnitude and direction of the resultant of three coplanar forces P, 2P, 3P acting on a particle and inclined mutually at an angle of 120°.
(*Ans.* $\sqrt{3}$ P prep. to 2P.)

2. Forces 11, 4 and 10 lbs. wt. act along OA, OB, and OC respectively. These lines are in the same plane such that $\angle AOB=90°$ and $\angle BOC=\sin^{-1}\frac{3}{5}$. Find the magnitude and direction of the resultant.
(*Ans.* 13 lbs. at 67° 23′ with OA.)

3. ABCD is a rectangle having AB=4″ and BC=3″; forces 2 lbs. wt. along AB, $6\frac{2}{5}$ lbs. wt. along AD and 8 lbs. wt. along the diagonal AC, all act at the point A. Find the resultant.
(*Ans.* 14 lbs. wt. inclined with AB at an angle=$\angle$CAD)

4. Four horizontal wires attached to a telephone post exert the following tensions on it. 20 lbs. wt. N., 30 lbs. wt. E., $40\sqrt{2}$ lbs. wt. S.W. and $50\sqrt{2}$ lbs. wt. S.E. Calculate the resultant pull on the post and find its direction.
(*Ans.* R=$10\sqrt{65}$ lbs. wt. at an angle $\tan^{-1}\frac{7}{4}$ S. of E)

5. A string of length 31 in. has its ends tied to two points in a horizontal line at a distance 25 inches apart. A small ring from which is suspended a weight of 9 lbs. can slide on the string, and is acted on by a horizontal force of such magnitude that in the position of equilibrium the ring is at a distance of 7 inches from the nearer end of the string. Show that the force is approximately 5 lbs. wt. and find the tension of the string. (*Ans.* $7\frac{8}{31}$ lbs. wt.)

6. The sides of a quadrilateral are 1, 3, 5, 6. Forces of 2, 6, 9 and 12 lbs. wt. respectively act *on a particle* parallel to the sides of the quadrilateral taken in order. Find the magnitude of the resultant. (*Ans.* 1 lb. wt.)

7. A, B, P, Q, R are five points on a plane. Forces AP, AQ, AR act at A, and forces PB, QB, RB at B. Prove that the resultant of the six forces is 3AB in magnitude and direction. (*All. U.*)

8. ABCDE is a pentagon. Forces acting at a particle are reprsented in magnitude and direction by AB, AE, BC, DC, ED and AC. Show that the resultant is represented by 3 AC.

If the forces are represented by AB, BC, CD, 2DE, AD and AE, show that the resultant is represented by 3 AE.

9. Three forces acting at a point are parallel to the sides of a triangle ABC, taken in order, and are proportional to the cosines of the opposite angles. Show that their resultant is proportional to

$$(1-8\cos A\cos B\cos C)^{\frac{1}{2}}.$$ (*See* worked Ex. (*iii*) on P. 36)

10. E and F are the feet of the perpendiculars from two vertices of a triangle ABC upon the opposite sides. Prove that a force P acting along EF can be replaced by forces P cos A, P cos B, P cos C acting along the sides.

11. Forces cach equal to P act at a point, parallel to the sides of the traingle ABC, taken in the same sense, show that their resultant R is given by

$$R^2=P^2(3-2\cos A-2\cos B-2\cos C)$$

12. Three forces P, Q, R in one plane act on a particle, the angles between R and Q, P and R and P and Q being α, β and γ respectively. Show that their resultant is equal to

$$\sqrt{\{P^2+Q^2+R^2+2QR\cos\alpha+2RP\cos\beta+2PQ\cos\gamma\}}$$

13. ABC is an equilateral triangle inscribed in a circle. Repelling forces P, Q, R directed from the points A, B, C respectively, act on a particle P situated on the circle midway between B and C. Show that the resultant force on P is

$$(P^2+Q^2+R^2-QR+RP+PQ)^{\frac{1}{2}}$$

14. Show that if $(2n+1)$ coplanar forces each of magnitude P act at a point, the magnitude of their resultant cannot be greater than $(2n+1)$P, and find the least value of their resultant. (*Ans.* zero)

15. Forces 1, 2, 3, 4, 5, 6 lbs. wt. act at a point in the directions of successive sides of a regular hexagon taken in order. Show that the resultant is a force of 6 lbs. wt. acting in the direction of the 5 lbs. wt. force.

16. ABCDEF is a regular hexagon. Find the resultant of forces represented by AB, AC, AD, AE and AF. (*Ans.* 6AB acting along AD)

17. Forces 2, $\sqrt{3}$, 5, $\sqrt{3}$, 2 lbs. respectively act at one of the angular points of a regular hexagon towards the five others in order. Find the magnitude and direction of the resultant.

(*Ans.* 10 lbs. in the direction of the force of 5 lbs.)

18. *A* string of length l is fastened to two points A, B at the same level at a distance a apart. A ring of weight W can slide on the string and a horizontal force X is applied to it such that it is in equilibrium vertically below B. Prove that $X = \frac{a}{l}W$ and that the tension of the string is

$$W.\frac{a^2+l^2}{2l^2}.$$

3.3. Conditions of Equilibrium of Forces acting on a particle. The conditions for the *equilibrium of a particle* subject to two or more forces may be summarised as follows :—

(*a*) *Two forces* acting on a particle are in equilibrium if they are equal and opposite and collinear in action.

(*b*) *Three forces* acting on a particle are in equilibrium if they lie in the same plane and satisfy Lami's Theorem. (Art. 2.3)

Or

if the sum of their resolved parts along two non-parallel lines in their plane separately vanish. (Art. 3.231).

Or

if the forces can be represented by the sides of a triangle taken in order. (Art. 2.21).

(*c*) *A number of forces* acting on a particle are in equilibrium, if they can be represented by the sides of a closed polygon. (Art. 3.221)

Or

if the point O at which the given forces act is the centroid of the extremities of lines representing the forces. (Art. 3.21).

Or

if the forces are coplanar and the sum of their resolved parts along two non-parallel straight lines in their plane separately vanish. (Art. 3.231).

The conditions given above are necessary as well as sufficient.

3.4. Equilibrium of three non-parallel forces acting on a rigid body. *If three* **coplanar** *and non-parallel forces acting on a rigid body, keep it in equilibrium, the forces must meet at a point.*

Let P, Q, R be the three forces. Since they are coplanar and not all parallel, at least two of them, say P and Q, must meet at some

point O and the resultant of P and Q is a force also passing through O. Since three forces are in equilibrium, the third force R, must be equal, opposite and collinear in action with the resultant of P and Q, so that R also must pass through O.

The forces being concurrent, the conditions of equilibrium as given in Art. 3.3 (*b*) are applicable.

Note 1. Three forces keeping a rigid body in equilibrium must necessarily be coplanar. For proof *see* Art. 6·6.

Note 2. A few easy examples involving three non-parallel forces are given below. Other examples of such forces acting on bodies will be given in Arts. 4·65 and 6·62.

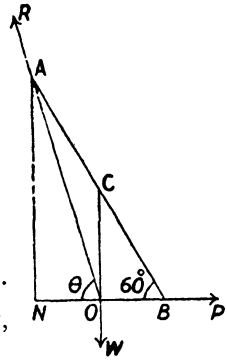
Example (i). *A uniform rod AB of weight W lb. is freely hinged at A and is supported so that it makes an angle of 30° with the downward vertical by a horizontal force P at B. Find P and the action at the hinge.*

Let the vertical through C, the middle point of the rod, meet the horizontal line of action of P in O.

The forces acting on the rod are its weight W, the force P, and R the reaction of the hinge.

Since the weight of the rod acting at C and the force P meet in O, therefore, the third force R must also pass through O and must act along the line OA.

Let OA make an angle θ with the horizontal, then



$$\tan\theta = \frac{NA}{NO} = \frac{NA}{\frac{1}{2}NB} = 2\frac{NA}{NB} = 2\tan 60° = 2\sqrt{3}.$$

Now resolving forces vertically and horizontally, we have

$$R\sin\theta = W \quad i.e., \quad R = W\operatorname{cosec}\theta = W\sqrt{1+\cot^2\theta} = W\sqrt{\frac{13}{12}},$$

and $$P = R\cos\theta = \frac{W}{\sin\theta}\cos\theta = W\cot\theta = \frac{W}{2\sqrt{3}} = \frac{1}{6}W\sqrt{3}.$$

Note. These results can also be obtained by the application of Lami's Theorem or from the triangle of forces OAN.

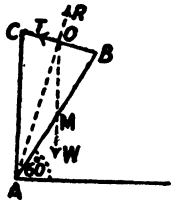
Example (ii). *A uniform beam AB of weight W can turn in a vertical plane about a hinge A, and to the other end B is tied a rope which is fastened to a point C, vertically above A, so that AC = AB. Find the tension in the rope necessary to keep the beam at an angle of 60° above the horizontal. Find also the direction and magnitude of the reaction of the hinge.*

The beam AB is in equilibrium under the action of three forces :—

(*i*) the tension T along BC.

(*ii*) the reaction R at the hinge A.

(*iii*) its weight W, acting along the vertical line through the middle point M.



This line being parallel to AC, bisects BC.

Since two of these forces W and T meet in O (middle point of BC) the third force R must also pass through O, so that R acts along AO.

Again, since AB=AC and $\angle BAC=30°$,

$\therefore$ $\angle ACB=\angle ABC=75°$ and AO is at right angles to BC.

Hence $\angle CAO=15°$, *i.e.*, the reaction acts in a direction making $\angle 15°$ with the vertical.

Also $\angle AOM=\angle CAO=15°$ and $\angle BOM=90°-15°=75°$.

$\therefore$ by Lami's Theorem, we have

$$\frac{T}{\sin 15°}=\frac{R}{\sin 75°}=\frac{W}{\sin 90°}$$

i.e., $T=W \sin 15°$ and $R=W \cos 15°$.

Note. These results can also be obtained either from the triangle of forces AOC or by resolving forces along AO and BC.

Examples IX

1. A heavy uniform rod AB of weight 2 lbs. is free to turn about its upper end A which is fixed and the rod is kept in equilibrium inclined at an angle of 30° to the vertical by a horizontal force P applied to the lower end B Find P and the reaction at A. (*Ans.* $\frac{1}{3}\sqrt{3}$; $\frac{1}{3}\sqrt{39}$ lbs. wt.)

2. A uniform straight rod of mass 2 lbs. is suspended freely from one end, and held in an oblique position by a horizontal force of 1 lb. wt. applied at the lower end. Find the inclination of the rod to the vertical.

Find also the magnitude and direction of the reaction at the point of suspension. (*Ans.* 45° ; 2·24 lbs. wt. at $\tan^{-1} 2=63° 26'$ to the horizontal.)

3. A uniform bar AB of weight W and length l is hinged at its upper end A and a horizontal force is applied to the end B so that the bar is in equilibrium with B at a distance a from the vertical through A. Show that the reaction at the hinge is

$$\tfrac{1}{2} W \left(\frac{4l^2-3a^2}{l^2-a^2}\right)^{\frac{1}{2}}$$

4. A uniform rod AB of mass 10 lbs. is freely hinged about the end A, which is fixed, and is supported in an inclined position by a light string joining the end B to a fixed point C, vertically above A. If AB=AC=2BC find completely the action of the hinge on the rod and the tension in the string.

(*Ans.* 9·68 lbs. wt. bisecting angle BAC ; 2·5 lbs. wt.)

5. Two planes are inclined at angles α and β to the horizon and their line of intersection is horizontal. A sphere of weight W is placed between them Show that pressures on the planes are respectively equal to

$$\frac{W \sin \beta}{\sin (\alpha+\beta)} \text{ and } \frac{W \sin \alpha}{\sin (\alpha+\beta)}$$

If α be given and the pressure on this plane is $\frac{1}{2}$ W, prove that

$$\cot \beta=(2-\cos \alpha)/\sin \alpha.$$

6. If a uniform heavy rod be supported by a string fastened at its ends and passing over a smooth peg, prove that it can rest only in a horizontal or vertical position.

In case the strings attached to the two ends are different, show that the tensions along them are proportional to their lengths.

7. A uniform rod of weight 30 lbs. and of length 10 ft. is supported in equilibrium by means of light inextensible strings of 6 ft. and 8 ft. whose ends are attached to the ends of the rod and to a fixed point. Find the tensions in the strings. (*Ans.* 24, 18 lbs. wt.)

8. A heavy uniform rod, 15 inches long, is suspended from a fixed point by strings fastened to its ends, their lengths being 9 and 12 inches. Show that the rod is inclined to the vertical at an angle $\sin^{-1}\left(\frac{24}{25}\right)$. (*P. U. 1957*)

9. A heavy uniform beam is hung from a fixed point by two strings attached to its extremities. If the strings and the beam be as 7 : 9 : 8, show that the tensions of the strings and weight of the beam are as 7 : 9 : 14.

10. A non-uniform beam AB 6 ft. long is held by two ropes AC and BD in a horizontal position. AC and BD make angles of 60° and 45° with the horizontal. Find the position on the beam of the point where the weight W acts and find the tensions in the ropes.

(*Ans.* 3·8 ft. from B, W $(\sqrt{3}-1)$, $\frac{1}{2}$ W $(\sqrt{6}-\sqrt{2})$.

11. A unifrom heavy rod AB has the end A in contact with a smooth vertical wall, and one end of a string is fastened to the rod at a point C, such that AC = $\frac{1}{4}$ AB, and the other end of the string is fastened to the wall vertically above A. Find the length of the string if the rod rests in a position inclined to the vertical. (*Ans.* Equal to AC)

What happens when AC = $\frac{1}{2}$ AB ?

12. A uniform rod AB of length a, hangs with one end A against a smooth vertical wall, being supported by a string of length l, attached to the other end of the rod and to a point D of the wall vertically above A. Show that if the rod rests inclined to the wall at angle θ,

$$\cos^2\theta = \frac{l^2-a^2}{3a^2}.$$

Show that in order that equilibrium may be possible, $a < l < 2a$.

13. A rod AB of weight W rests over a smooth horizontal rail at C with its lower end A in contact with a smooth vertical wall If the rod is inclined at an angle θ to the vertical, show that the magnitude of the reactions at A and C are respectively equal to

W cot θ and W cosec θ.

14. A uniform beam 12 ft. long and weighing 50 lbs. rests against a smooth horizonal rail, the point of contact being 3 ft. from the lower end of the beam and is kept in equilibrium in this position by a horizontal force F applied to the lower end. Show that the beam is inclined at 45° to the horizontal, and find the magnitude of the force F. (*Ans.* F = 50 lbs.) (*P. U.*)

15. A uniform beam of length $2a$, rests in equilibrium against a smooth vertical wall and upon a peg at a distance b from the wall. Show that the inclination of the beam to the vertical is

$$\sin^{-1}\left(\frac{b}{a}\right)^{\frac{1}{3}}.$$ (*P. U.*)

16. A uniform rod of length $4a$ is placed across a smooth horizontal rail and rests with one end against a smooth vertical wall, the distance of which from the rail is a. Show that the angle which the rod makes with the horizontal is $\sec^{-1} 2^{\frac{1}{3}}$. (*D.U.*)

17. A uniform equilateral triangular lamina ABC of weight W can turn in a vertical plane about a hinge B ; it is supported with side AB horizontal, by a smooth prop, at the middle point of BC. Find the pressure on the prop and the reaction at the hinge.

18. Prove that a uniform rod cannot rest entirely within a smooth hemispherical bowl except in a horizontal position.

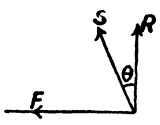
19. A *light* rod rests between a smooth vertical wall and a smooth floor

and a point P of the rod is connected by a string to the corner, where the wall and floor meet. Determine the position of the point P on the rod.

(*Ans.* The middle point. Why ?)

20. A uniform rod AB of weight 5 lbs. and length 1 ft. is hinged to a fixed point A. An elastic cord of natural length 1 ft. whose modulus of elasticity is 10 lbs. wt. is attached at one end to B and at its other end to a fixed point C 1 ft. vertically above A. Find tha tension of the string and also its length and inclination to the vertical. (*Ans* $3\frac{1}{3}$ lbs. wt., $1\frac{1}{3}$ ft.; $\cos^{-1}\frac{2}{3}$).

3.5. Problems involving friction. When a body is at rest in contact with a rough surface and the reaction S (which is equal and opposite to the resultant of the other forces acting on the body) does not act along the normal at the point of contact but is inclined to the normal then, F, the tangential component of this reaction, is the *force of friction* and R, the normal component, is the *normal pressure.*



If θ be the angle which the resultant S makes with the normal, then

$$S\cos\theta = R, \qquad \text{and} \qquad S\sin\theta = F,$$

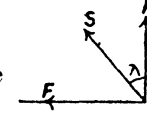
also $$\tan\theta = \frac{F}{R}.$$

The resultant reaction S and consequently its components F and R will change with the change in the other forces acting on the body.

In the limiting case when the body is on the point of motion $\frac{F}{R}$ attains its greatest value μ, the co-efficient of friction.

If λ be the angle which the resultant reaction in this case makes with the normal, we have

$$\tan\lambda = \frac{F}{R} = \mu$$



The angle λ is called the **Angle of friction.** Thus we find that

the co-efficient of friction is equal to the tangent of the angle of friction.

It is evident that so long as the body does not commence to move $\theta < \lambda$ and $\frac{F}{R} < \mu$ *i.e.*, $F < \mu R$.

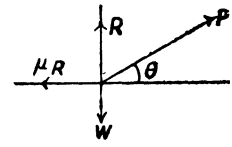
In other words, *the inclination of the resultant reaction to the normal cannot exceed a certain value* λ, *the angle of friction, and the amount of friction F cannot bear more than a certain ratio* μ *to the normal pressure.*

The **Cone of Friction** is the cone which has the point of contact as its vertex, the normal as its axis, and λ its semi-vertical angle.

In the case of limiting equilibrium the resultant reaction acts along a generator of this cone and when the equilibrium is not limiting the resultant reaction acts along a line lying within the cone.

3.6. *To find the least force to drag a particle on a rough horizontal plane.*

Let W be the weight of the body (regarded as a particle) and R, the normal pressure. Let P be a force acting at an angle θ with the horizontal.



When the body is on the point of motion there is limiting equilibrium and the force of friction is. therefore, μ R or R tan λ.

Resolving horizontally and vertically, we h

$$P \cos \theta - \mu R = 0 \quad \ldots (1)$$

and $$P \sin \theta + R - W = 0 \quad \ldots (2)$$

Eliminating R, we get

$$P(\cos \theta + \mu \sin \theta) = \mu W.$$

$$\therefore \quad P = \frac{\mu W}{\cos \theta + \mu \sin \theta} = \frac{W \sin \lambda}{\cos (\theta - \lambda)}, \quad (\because \mu = \tan \lambda).$$

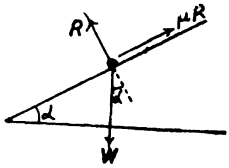
P is evidently least when $\cos (\theta - \lambda) = 1$ or $\theta = \lambda$,

i.e., the *force is least when it acts in a direction making with the horizontal, an angle equal to the angle of friction.*

Thus the least force to move a weight W along a rough horizontal plane is W sin λ, λ being the angle of friction.

3.7. *To find the greatest inclination of a rough plane on which a heavy particle can rest in equilibrium without the application of any other force.*

Let a body (treated as a particle) of weight W be placed on a rough plane and let the plane be tilted till the body is on the point of sliding down the plane.



Let α be the inclination of the plane in this position and R, the normal pressure.

Then the force of friction μR acts up the plane along the line of greatest slope through the body.

By resolving along and perpendicular to the plane, we have

$$W \sin \alpha = \mu R \quad \text{and} \quad W \cos \alpha = R.$$

$$\therefore \quad \tan \alpha = \mu R / R = \mu = \tan \lambda.$$

i.e., the *greatest inclination of the plane is equal to the angle of friction.*

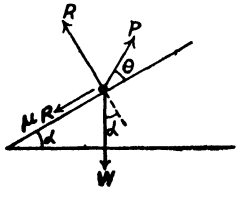
If $\alpha > \lambda$, the body slides down the plane and a force will be required to keep it at rest or to move it up the plane.

If $\alpha < \lambda$, a force will be necessary to move the body either up or down the plane.

Example. *A particle is placed on the outside of a rough sphere whose co-efficient of friction is* μ. *Show that it will be on the point of motion when the radius through it makes an angle* $\tan^{-1} \mu$ *with the vertical.*

3.8. *A heavy body is placed on a rough inclined plane. To find the force just sufficient to move the body up the plane, the force acting in a vertical plane through the line of greatest slope through the body.*

Let α be the inclination of the plane to the horizon, W, the weight of the body and R, the normal reaction.



Let P be a force acting at an angle θ with the line of greatest slope through the body.

Let the body be on the point of motion up the plane, so that the friction acts down the plane and is equal to μ R.

Resolving along and perpendicular to the plane, we have

$$P \cos \theta = \mu R + W \sin \alpha, \qquad \ldots(1)$$

and $$P \sin \theta + R = W \cos \alpha, \qquad \ldots(2)$$

Eliminating R, we have

$$P(\cos \theta + \mu \sin \theta) = W(\sin \alpha + \mu \cos \alpha),$$

i.e., $$P = W \frac{\sin \alpha + \mu \cos \alpha}{\cos \theta + \mu \sin \theta} = W \frac{\sin (\alpha + \lambda)}{\cos (\theta - \lambda)} \qquad \ldots(3)$$

Cor. *When the force acts along the plane.*

In this case $\theta = 0$, and $P = W \frac{\sin (\alpha + \lambda)}{\cos \lambda}$.

It can be obtained independently by writing down the equations (1) and (2) when $\theta = 0$.

Note. In these cases α may be any angle, it may be greater or less than λ, the angle of friction.

3·81. *To find the direction and magnitude of the least force to drag a heavy body up a given rough inclined plane.*

Equation (3) gives the amount of force just sufficient to move the body up the plane and shows that the force varies with θ, the angle at which the force is inclined to the plane.

P will evidently be least when $\cos (\theta - \lambda)$ is the greatest *i.e.*, when $\theta = \lambda$.

Hence the required force is inclined to the plane at an angle equal to the angle of friction and is of magnitude W sin $(\alpha + \lambda)$.

Cor. Equation (1) shows that friction μR is least when P is least *i·e.*, when $\theta = \lambda$ and $P = W \sin (\alpha + \lambda)$

$\therefore$ least friction $\mu R = P \cos \lambda - W \sin \alpha = W \sin (\alpha + \lambda) \cos \lambda - W \sin \alpha$

$= W [\sin \alpha \cos^2 \lambda + \cos \alpha \sin \lambda \cos \lambda - \sin \alpha]$

$= W [\cos \alpha \sin \lambda \cos \lambda - \sin \alpha \sin^2 \lambda]$

$= W \sin \lambda \cos (\alpha + \lambda)$.

3.82. *To find the force required to support a heavy body on a rough inclined plane, inclined to the horizontal at an angle greater than the angle of friction, the force acting in a vertical plane through a line of the greatest slope through the body.*

Let $P = P'$ when the body is on the point of motion *down* the plane.

In this case, $\alpha > \lambda$ and the force of friction acts up the plane and therefore, by changing the sign of μ or λ in the above equations of Art. 3.8, we get

$$P' = \frac{W \sin(\alpha - \lambda)}{\cos(\theta + \lambda)}. \quad \ldots (4)$$

which gives the amount of force just sufficient to support the body on the plane.

The body will remain in equilibrium under the action of a force which lies between P and P'.

Cor. *When $\theta = 0$,* i.e., *the force acts along the plane, and* $\alpha > \lambda$.

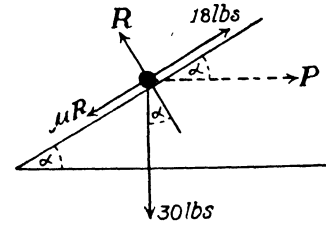
$$P' = W \frac{\sin(\alpha - \lambda)}{\cos \lambda}.$$

In these cases the body needs support only when $\alpha > \lambda$, for as P' ceases to act, the body moves down the plane.

Note. If $\alpha < \lambda$, no force is needed to support the body for it will rest in equilibrium on the plane. The least downward force required to make the body slide down the plane in this case can be obtained on similar lines.

Example (i). *A body of weight 30 lbs. is on the point of motion up a rough inclined plane under the action of a force of 18 lbs. wt. parallel to the plane, and is on the point of motion down the plane when supported by a force of 12 lbs. wt. Find the inclination of the plane and the co-efficient of friction.*

Let α be the inclination of the plane, μ the co-efficient of friction, and R the normal reaction.



In the first case when the body is on the point of motion *up* the plane, limiting friction μR acts down the plane.

Resolving forces along and perpendicular to the plane, we have

$30 \sin \alpha + \mu R = 18$ and $30 \cos \alpha = R$ (Ignore P shown in the figure)

$\therefore$ $30 \sin \alpha + \mu . 30 \cos \alpha = 18$ or $3 - 5 \sin \alpha = 5\mu \cos \alpha$ ...(1)

In the second case when the force is 12 lbs. wt., the body is on the point of motion *down* the plane and the limiting friction $\mu R'$ acts up the plane, R' being the normal reaction in this case.

The equations in this case are,

$30 \sin \alpha = 12 + \mu R'$ and $30 \cos \alpha = R'$

$\therefore$ $30 \sin \alpha - 12 = \mu . 30 \cos \alpha$

or $5 \sin \alpha - 2 = 5\mu \cos \alpha$ ...(2).

From (1) and (2), we get

$3 - 5 \sin \alpha = 5 \sin \alpha - 2$ or $10 \sin \alpha = 5$.

$\therefore$ $\sin \alpha = \frac{1}{2}$, which gives $\alpha = 30°$.

Now from (1), we have

$$3 - 5 \times \tfrac{1}{2} = 5\mu \times \frac{\sqrt{3}}{2} \quad \text{or} \quad \mu = \frac{1}{5\sqrt{3}} = \frac{\sqrt{3}}{15}.$$

Example (ii). *In Ex. (i) find the magnitude of the least force which acting horizontally makes the body move up the plane.*

Let P be the required horizontal force and let R_1 be the normal reaction in this case.

Resolving forces along and perpendicular to the plane, we have

$$P \cos \alpha = 30 \sin \alpha + \mu R_1,$$

and $$R_1 = 30 \cos \alpha + P \sin \alpha$$

$\therefore$ $$P \cos \alpha = 30 \sin \alpha + \mu(30 \cos \alpha + P \sin \alpha),$$

or $$P(\cos \alpha - \mu \sin \alpha) = 30(\sin \alpha + \mu \cos \alpha).$$

Substituting the values $\alpha = 30°$ and $\mu = \sqrt{3}/15$, we have

$$P\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{15} \cdot \frac{1}{2}\right) = 30\left(\frac{1}{2} + \frac{\sqrt{3}}{15} \cdot \frac{\sqrt{3}}{2}\right) = 30\left(\frac{1}{2} + \frac{1}{10}\right),$$

or $$\frac{7}{15}\sqrt{3}\, P = 30 \times \frac{3}{5} = 18, \quad i.e., \quad P = \frac{18 \times 15}{7\sqrt{3}} = \frac{90\sqrt{3}}{7} \text{ lbs.}$$

Example (iii). *A particle of weight W_1 hanging vertically, is attached by means of a light inextensible string to another particle of weight W_2 lying on a rough inclined plane of inclination α. The string is passing over a small pulley at the top of the plane and the system is in equilibrium. Find the force of friction between the particle and the inclined plane.*

Find also the least value of the co-efficient of friction.

Let R be the normal reaction and F, the force of friction on W_2.

The tension T of the string supports W_1 hanging vertically

$\therefore$ $T = W_1$

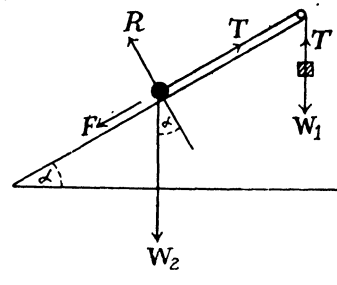
On W_2, T acts along the plane.

Resolving forces along and perpendicular to the plane, we have

$$T = F + W_2 \sin \alpha,$$

and $$R = W_2 \cos \alpha$$

$\therefore$ $F = T - W_2 \sin \alpha = W_1 - W_2 \sin \alpha$, which gives the required force of friction.



If μ be the co-efficient of friction, the limiting amount of friction μR must be greater than F, *i.e.*, $\mu R > F$,

$$i.e., \quad \mu > \frac{F}{R}, \; i.e., > \frac{W_1 - W_2 \sin \alpha}{W_2 \cos \alpha}, \; i.e., > \frac{W_1}{W_2} \sec \alpha - \tan \alpha.$$

$\therefore$ the least value of the co-efficient of friction $= \frac{W_1}{W_2} \sec \alpha - \tan \alpha$.

Example X

1. Find the least force necessary to move a load on a rough horizontal plane. [*See Art.* 3·6.] (*P. U. 1953*)

2. A small object of mass 10 lbs. lies at rest on a rough horizontal plane, and a gradually increasing force P is applied to the particle in an upward direction making an angle of 30° with the horizontal; it is found that the particle be-

gins to move horizontally when P=4 lbs. wt. Show that the co-efficient of friction is about 0·433, and, if the force P acts in a downward direction at an angle of 30° to the horizontal, find the magnitude of P when the particle just begins to move. (*Ans.* $6\frac{2}{3}$ lbs. wt.)

3. A small light ring can slide on a thin horizontal rail, the co-efficient of friction between the ring and the rail being 0·6. A weight of 10 lbs. is attached to the ring and a force P acts on the ring in the vertical plane containing the rail and in a direction inclined at an angle of 30° to the upward drawn vertical. Find the frictional force when P=5 lbs. wt. and show that the ring will not move. Find also the least value of P which will cause the ring to slide.

(*Ans.* $2\frac{1}{2}$ lbs. wt, 5·88 lbs. wt.)

4. A plane is inclined to the horizontal at an angle α such that $\sin\alpha=\frac{3}{5}$, a particle of mass 10 gms. is gently placed on the plane. If the co-efficient of friction between the plane and the particle is 0·8, prove that the particle can remain at rest, and calculate the frictional force exerted on the particle by the plane.

Calculate the least force acting along a line of greatest slope, which would cause the particle to move up the plane. (*Ans.* 6 gms. wt., 12·4 gms. wt.)

5. A particle of weight 10 lbs. is placed on a rough horizontal plane which is slowly tilted. If the particle begins to slip when the inclination of the plane to the horizontal is 60°, find the force of friction when the inclination of the plane is 45°.

Find also the least force, acting up a line of greatest slope, which will cause the particle to begin to move up the plane when the inclination of the plane is 60°. (*Ans.* $5\sqrt{2}$ lbs. wt., $10\sqrt{3}$ lbs. wt.)

6. A load of 30 lbs. is held in equilibrium on a rough plane inclined at 45° to the horizontal by a horizontal force P in the same vertieal plane as the line of greatest slope through the load. If the co-efficient of friction is $\frac{1}{2}$, find the value of P when the load is about to slip down the plane. (*Ans.* 10 lbs. wt.)

7. Find the least force which will move a weight of 80 lbs. up a rough plane inclined to the horizontal at 30°, when the co-efficient of friction is $\frac{3}{4}$.

(*Ans.* 73·6 lbs. wt.)

8. A body is supported on a rough inclined plane by a force acting along the plane ; supposing the greatest magnitude, of the force to be double the least magnitude, show that the inclination α of the plane is given by $\tan\alpha=3\mu$.

9. Show that the horizontal force, which can support a heavy mass on a smooth inclined plane of inclination α to the horizon, can just sustain it in limiting equilibrium on a rough plane inclined to the horizon at an angle $(\alpha+\lambda)$, where λ is the angle of friction. (*D. U.*)

10. If P_1 is the force which acting parallel to an inclined plane of angle α, is sufficient to draw a weight up and P_2 the force which will just let it be on the point of sliding down, show that

(*i*) $\frac{1}{2}(P_1+P_2)$ acting along the plane would support the same weight on a smooth plane of the same inclination,

and (*ii*) if $P_1=nP_2$. then $\tan\alpha=\mu\frac{n+1}{n-1}$.

11. An elastic string of natural length *l* inches when used to pull a body of weight W lb. steadily along a rough level table, stretches *l*/10 inches. If the body is lifted vertically, the extension is 9*l*/20 inches. Show that the co-efficient of friction of the table is 2/9. (*P. U.*)

12. A particle of weight *w* is placed on a rough plane inclined at an angle α (greater than the angle of friction) to the horizontal. It is attached to a string which lies along a line of the greatest slope, passes over a small smooth pulley at the top of the plane, and carries a light scale-pan hanging vertically at the end. The particle is just prevented from slipping down the plane by

a weight nw in the scale-pan. Show that the co-efficient of friction is given by $\mu = \tan\alpha - n\sec\alpha$.

Find the least weight that must be added to the scale-pan to drag the particle up the plane. (*Ans.* $2w(\sin\alpha - n)$]

13. A weight can be just supported on a rough inclined plane by a force P acting along the plane, or by a force Q acting horizontally, show that the weight is

$$PQ/\sqrt{Q^2\sec^2\varphi - P^2}$$

where φ is the angle of friction.

14. A body of weight W can just be dragged up a rough inclined plane by a force P, and can just be sustained by a force Q, P and Q both acting up the line of greatest slope through the body. Show that the co-efficient of friction is

$$\frac{P-Q}{\sqrt{4W^2-(P+Q)^2}}.$$

15. Show that the difference between the greatest and the least force which, acting at an angle θ to the plane inclined at an angle α to the horizontal, sustains a weight W is

$$\frac{2W\cos(\alpha+\theta)\sin 2\lambda}{\cos 2\theta+\cos 2\lambda},$$

where λ is the angle of friction.

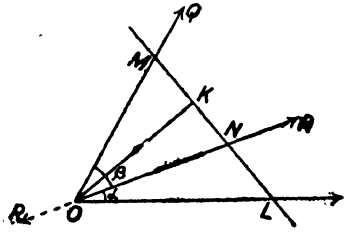
16. A particle is placed on the outside of a rough sphere whose co-efficient of friction is μ. Show that it will be on the point of motion when the radius through it makes an angle $\tan^{-1}\mu$ with the vertical.

How high can a particle rest inside a hollow sphere of radius a if the co-efficient of friction be $1/\sqrt{3}$. [*Ans.* $\frac{1}{2}a(2-\sqrt{3}) = 0{\cdot}134a$.] (*P.U. 1954*)

Miscellaneous Examples I

1. *Forces P and Q act at point O, and their resultant is R. If any transversal cuts their lines of action in the points L, M, N respectively, show that*

$$\frac{P}{OL}+\frac{Q}{OM}=\frac{R}{ON}.$$



First Method. Draw OK perp. to LM. Resolving forces along OK, we have

$$P\cos\angle KOL + Q\cos\angle KOM = R\cos\angle KON,$$

i.e., $P.\frac{OK}{OL}+Q.\frac{OK}{OM}=R.\frac{OK}{ON}$

$$\therefore \quad \frac{P}{OL}+\frac{Q}{OM}=\frac{R}{ON}.$$

Second Method. We have

i.e., $P=\frac{P}{OL}.OL, \quad Q=\frac{Q}{OM}.OM.$

Now the resultant of forces $\frac{P}{OL}.OL$ acting along OL and $\frac{Q}{OM}.OM$ acting along OM is $\left(\frac{P}{OL}+\frac{Q}{OM}\right)ON$ (Art. 3·1)

Hence $\left(\frac{P}{OL}+\frac{Q}{OM}\right).ON = R$ *i. e.*, $\frac{P}{OL}+\frac{Q}{CM}=\frac{R}{ON}$.

Third Method. If R be reversed, then the forces are in equilibrium.

$\therefore$ by Lami's theorem, we have

$$\frac{P}{\sin\beta}=\frac{Q}{\sin\alpha}=\frac{R}{\sin(\alpha+\beta)} \qquad \ldots(1)$$

But since $2 \triangle OLN + 2 \triangle ONM = 2 \triangle OLM$

$\therefore$ $OL.ON \sin\alpha + ON.OM \sin\beta = OL.OM \sin(\alpha+\beta)$ ...(2)

Equations (1) and (2) give the same result at once.

2. ABC is a triangle, with a right angle at A and AD is perpendicular on BC. Prove that the resultant of forces $\frac{1}{AB}$ acting along AB and $\frac{1}{AC}$ acting along AC is $\frac{1}{AD}$ acting along AD.

3. From a point P within a circle whose centre is fixed, straight lines PA_1, PA_2, PA_3 and PA_4 are drawn to meet the circumference, all being equally inclined to the radius through P. Show that if these lines represent forces radiating from P, the resultant is independent of the magnitude of the radius of the circle. (*P.U.*)

4. In a triangle OAB forces represented by OA. tan A and OB. tan B act along the sides OA and OB respectively. Show that the resultant is AB. tan A tan B and acts in the direction OD, the perpendicular from O on AB.

4 (*a*). Show that the resultant of two forces sec B, sec C along AB, AC respectively of any triangle ABC is a force (tan B+tan C) acting along AD, where D is the foot of the perpendicular from A on BC.

5. Forces represented by the sides AB, AC of a triangle ABC have a resultant whose line of action passes through the centre of the circumscribing circle. Prove that either AB=AC, or the angle BAC is a right angle.

6. Forces are represented by PA, PB, PC, AQ, BQ, CQ where A, B, C, P, Q are points in a plane. Show that the resultant is represented in magnitud and direction by 3PQ and that it passes through the centroid of the triangle ABC.

7. If O be the centre of the circle circumscribing the triangle ABC and if forces act along OA, OB and OC respectively proportional to BC, CA and AB, show that their resultant passes through the centre of the inscribed circle. (*P.U.*)

8. A very small ring of weight W is free to slide on a smooth circular wire of radius *a*, fixed in a vertical plane. It is attached by a string of length l ($<a\sqrt{2}$) to a point of the wire at the same height as the centre. Show that the tension of the string is

$$\frac{(2a^2-l^2)W}{a\sqrt{4a^2-l^2}}.$$

What happens when $l=a\sqrt{2}$?

9. A fine light string AOB of length l is passed at O through a small light smooth ring and is attached at its extremities to fixed points A and B at a distance *c* apart. A given force P is applied to the ring in a direction making an angle α with BA. Show that in the position of equilibrium the two parts of the string are inclined to each other at an angle $2\sin^{-1}(c\sin\alpha/l)$. Show also that the tension of the string is

$$lP/2\sqrt{l^2-c^2\sin^2\alpha}.$$

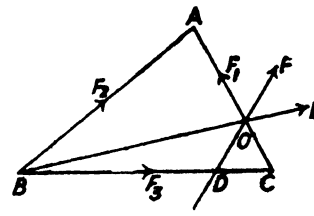
10. *Show that any force F can be resolved into three forces acting along the sides of a given triangle whose plane contains the line of action of F.*

The line of action of the force F must cut, at least two of the sides (or sides produced) of the given triangle ABC. Let it cut AC in O.

Join BO and produce it to E.

Now F can be resolved into two components along OA (*i.e.*, along CA, say F_1) and OE (*i.e.*, BO).

The force along BO can further be resolved

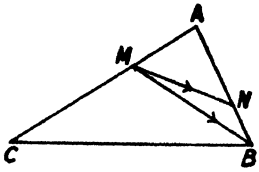


into components (F_2, F_3) along BA and BC.

Thus F can be replaced by three components along the sides of the given triangle.

Cor. *Any system of forces in a plane is equivalent to three forces acting along the sides of a given triangle in the plane.*

11. M is a point of trisection of the side AC of a triangle ABC which is nearer to A, and N is the point of trisection of the side AB which is nearer to B.



Resolve a force represented in magnitude and direction by MN, into three forces acting each along a side of the triangle.

Since NA=2BN,

$\therefore$ the forces MA and 2MB $\equiv$ 3MN, so that

the force MN $\equiv \frac{1}{3}$ MA and $\frac{2}{3}$ MB

$\equiv \frac{1}{9}$ CA and $\frac{2}{3}$ MB,

Again, since MC=2AM,

$\therefore$ forces CB and 2AB $\equiv$ 3MB.

so that $\frac{2}{3}$ MB $\equiv \frac{2}{9} \times 3\text{MB} \equiv \frac{2}{9}$ CB and $\frac{4}{9}$ AB.

Hence the force MN $\equiv \frac{1}{9}$ CA, $\frac{4}{9}$ AB and $\frac{2}{9}$ CB. ...(1)

Aliter. $\text{MN} \equiv \text{MC}+\text{CN} = \text{MC}+(\text{CP}+\text{BN}) = \frac{2}{3}\text{AC}+\text{CB}+\frac{1}{3}\text{BA}$ ...(2)

Note. It can be easily seen that the systems of forces obtained in (1) and (2) are equivalent.

12. Show that a force can be resolved into three components acting along three given lines, two of which are parallel, the lines and the force being all coplanar.

13. *Show that any system of coplanar forces can be reduced to two forces acting through two given points in their plane.*

Take a point P on one of the forces, join it to the given points A and B. This force can be replaced by components along PA and PB. Other forces can similarly be dealt with, and then various forces acting at A and B can be combined into two forces through A and B.

14. Two forces given in magnitude act each through a fixed point and are inclined at a constant angle, show that their resultant also passes through a fixed point.

15. Three light elastic strings, AB, BC, CD each of natural length l and modulus λ are fixed at A and D, two points in a horizontal line and at a distance $2l$ apart. Two particles each of weight λ are attached at B and C respectively. Show that in the position of equilibrium AB and CD are inclined to the vertical at an angle θ given by

$$3 \tan \theta = 1 - 2 \sin \theta.$$

16. A light string is fastened to two points A, D at the same level, the length of the string exceeding the distance AD, and particles of weight 2 lbs. and 1 lb., are fastened to it at two points B and C respectively. If AB, BC, CD make angles α, β, θ respectively with the horizontal, prove that

$$\tan \alpha = 2 \tan \theta \pm 3 \tan \beta.$$

17. A picture of breadth $2a$ is hung by two equal strings each of length l which are attached to the top corners of the picture, the other ends of the strings being attached to small light rings which can slide along a rough horizontal picture rail. If $2c$ is the greatest distance the rings can be apart without slipping, prove that the co-efficient of friction is

$$\frac{c-a}{\sqrt{l^2-(c-a)^2}}.$$

18. A weight W rests on a rough horizontal plane and it is connected by a string over a smooth pulley above the plane to a weight w hanging vertically

the sloping part of the string makes an angle θ with the horizontal. Prove that there will be no slipping, if $w \cos(\theta-\lambda) < W \sin\lambda$, where λ is the angle of friction between W and the plane.

19. Two weights P, Q resting on a double inclined plane, are connected by a fine string over a common vertex and Q is on the point of motion down the plane. Prove that the greatest weight which can be added to P without disturbing equilibrium is

$$\frac{Q \sin 2\lambda \sin(\alpha+\beta)}{\sin^2\alpha - \sin^2\lambda}$$

α, β being the angles of inclination of the planes and λ the angle of friction. (*D.U. Hons. 1925; P.U. 1953*)

REVISION QUESTIONS I

1. State Hooke's Law for Elastic Strings.

A heavy particle is hung from two pegs, which are in a horizontal line and distant $2c$ apart by light elastic strings of natural lengths a, b and moduli λ, μ respectively. In the position of equilibrium the strings are inclined at angle α to the vertical. Show that

$$\sin\alpha = c(\lambda b - \mu a) / ab(\lambda - \mu).$$

2. Two elastic strings of the same natural length are such that a force of 1 lb. stretches the first 1 inch and the second 3 inches. (*i*) The strings are fixed, each at one end, to the same point and the other ends are attached to a weight of 4 lbs., so as to support it in equilibrium. Find the tensions in the two strings. (*ii*) The weight of 4 lbs. is supported by the two strings, the first being, suspended from A and the second from B vertically above A. If the tension in each string is the same, find the distance AB. (*Ans.* 3 lbs., 1 lb.; 4″)

3. Two forces act at a point and are such that if the direction of one of them is reversed the direction of the resultant is turned through a right angle. Prove that the two forces must be equal in magnitude.

4. P is any point in the plane of the triangle ABC; D, E, F are the middle points of the sides. Prove that the resultant of forces PE, PF and DP is represented by PA.

5. (*a*) Prove that the components of a force F in two directions making angle θ and φ with its line of action (on opposite sides of it) are respectively,

$$F \sin\varphi/\sin(\theta+\varphi) \text{ and } F \sin\theta/\sin(\theta+\varphi).$$

(*b*) ABC is an equilateral triangle of side a. The line of action of F cuts AB at a point between A and B and distant $\frac{2a}{3}$ from A, and AC at a point between A and C and distant $\frac{a}{3}$ from A. Resolve it graphically or otherwise into three forces acting along the sides of ABC. (*D. U.*)

$$\left(Ans.\ \frac{2\sqrt{3}}{9}F,\ \frac{\sqrt{3}}{9}F,\ \frac{4\sqrt{3}}{9}F\right)$$

(*c*) E, F are the feet of the perpendiculars from B and C upon the opposite sides of the triangle ABC. Prove that a force P acting along EF can be replaced by P cos A, P cos B, B cos C acting along the sides.

6. A, B and C are three points on the circumference of a circle. Forces inversely proportional to AB and BC act along AB and BC respectively. Show that their resultant acts along the tangent to the circle at B.

7. Two forces P, Q act at a point along straight lines making an angle α with each other, and have a resultant R, two other forces P′, Q′ acting along

the same two lines have a resultant R'. Show that the angle between the lines of action of these resultants is

$$\cos^{-1}[PP'+QQ'+(PQ'+P'Q)\cos\alpha]/RR'.$$

8. In Q. 7 show that the directions of R and R' will also include an angle α, if

$$PP'+QQ'+2PQ'\cos\alpha=0 \text{ or } PP'+QQ'+2P'Q\cos\alpha=0.$$

9. Two forces act along the sides CA, CB of a triangle ABC, their magnitudes being proportional to cos A and cos B. Prove that their resultant is proportional to sin C, and its direction divides the angle C into two portions

$$\tfrac{1}{2}(C+B-A),\ \tfrac{1}{2}(C+A-B). \qquad (P.U.)$$

10. Prove that the resultant of two forces acting at a point O in directions OA and OB represented in magnitude by l.OA and m.OB, is represented by $(l+m)$.OC where C is a point in AB such that l. CA$=m$. CB.

ABCD.....is a polygon of n sides and forces act at a point O parallel and proportional to AB, 2BC, 3CD etc. Show that their resultant is parallel and proportional to $(n-1)$ OA, where O is the centroid of all the points B, C, D... excluding A.

11. A weight is supported on a smooth plane of inclination α by a string inclined to the horizon at angle γ. If the slope of the plane be increased to β, and the slope of the string remains unaltered, the tension of the string is doubled. Find the relation between α, β and γ. *(D.U.)*

[*Ans.* $2\sin\alpha\cos(\gamma-\beta)=\sin\beta\cos(\gamma-\alpha)$]

12. A body of weight W rests on a rough inclined plane whose inclination α to the horizontal is greater than β, the angle of friction. It is held at rest by an elastic string of natural length l attached to it and to a point on the plane. If λ is the modulus of elasticity, show that the string is lengthened by an amount equal to $Wl\sin(\alpha-\beta)/\lambda\cos\beta$.

13. Weights P and Q of two different substances, co-efficients of friction μ and μ', are supported on a rough double inclined plane of angles α and β by means of a string passing over the vertex. If the weight P be on the point of descending prove that

$$\frac{P}{Q}=\frac{\sin\beta+\mu'\cos\beta}{\sin\alpha-\mu\cos\alpha}.$$

14. In Q. 13, if $\mu=\mu'=\tan\lambda$ and $\beta=90-\alpha$, show that $\frac{Q}{P}$ must lie between $\tan(\alpha-\lambda)$ and $\tan(\alpha+\lambda)$.

15. *Forces proportional to the sides of a closed polygon act along lines which bisect at right angles those sides. If in going round the polygon, the forces act all towards the right or towards the left, show that they form a system in equilibrium.*

[*Hint. See* Ex. (*iii*), P. 20]

CHAPTER IV

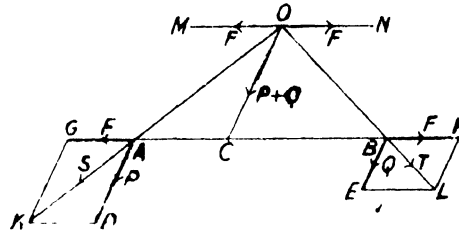
PARALLEL FORCES—MOMENT

4.1. In the preceding chapter we have shown how to deal with forces acting at a point, basing the discussion on the law of parallelogram of forces. We shall now consider the composition of parallel forces based on the same fundamental law.

4.11. Resultant of two parallel forces.

Case I. *Like parallel forces, i.e.,* forces acting in the same parallel direction.

Let P and Q be two like parallel forces acting at points A and B of a rigid body. Let them be represented by AD and BE, respectively.



At A and B introduce two forces each equal to F, in opposite directions, acting in the line AB, represented by AG and BH. These forces, being equal and opposite, balance and they do not affect the resultant of P and Q.

Let S be the resultant of P and F acting at A and T, the resultant of Q and F acting at B. Let the lines of action of S and T meet in O and let their points of action be shifted to O.

Draw OC parallel to the direction of P or Q meeting AB in C and through O draw MN parallel to AB.

At O resolve S into its components in their original directions P acting along OC and F acting along OM. Similarly resolve T into the components Q along OC, and F along ON.

The two forces, each equal to F, acting along OM and ON balance and we are left with the force P+Q acting along OC.

Hence the resultant of P and Q is P+Q acting along OC, *i.e.,* acting at C parallel to P or Q.

Now, we find the *position of C*, where the resultant cuts the line AB.

From the similar triangles AOC, AGK, we have

$$\frac{AC}{OC}=\frac{AG}{GK}=\frac{F}{P} \quad \text{or} \quad P.AC=F.OC.$$

Similarly from triangles OCB, BHL, we have

$$\frac{BC}{OC}=\frac{BH}{HL}=\frac{F}{Q} \quad \text{or} \quad Q.BC=F.OC.$$

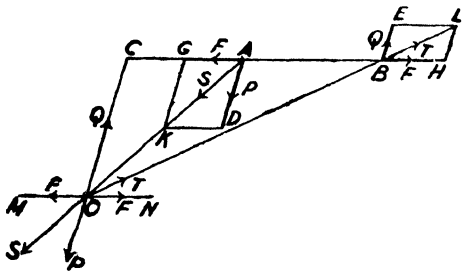
$$\therefore \quad P.AC=Q.BC \quad \text{or} \quad P/BC=Q/AC.$$

Cor. *If a force R balances the two like parallel forces P and Q, it must be equal to P+Q and must act through C in the direction CO (in the plane of P and Q, parallel to them), such that.*

$$\frac{P}{CB} = \frac{Q}{AC} = \frac{P+Q}{CB+AC} = \frac{R}{AB} \qquad \ldots(1)$$

Case II. *Unlike parallel forces, i.e.*, forces acting in opposite parallel directions.

Let P and Q (P>Q) be the two unlike parallel forces acting at points A and B of a body and let them be represented by AD and BE.



At A and B introduce two forces, F in opposite directions, acting in the line AB and represented by AG and BH.

The forces F balance and do not affect the resultant action of P and Q.

S, the resultant of P and F at A is represented by AK and T, the resultant of Q and F at B is represented by BL.

Produce AK and LB to meet in O (they will always meet unless they are parallel which happens when P and Q are equal). The forces S and T may be supposed to act at O.

Draw OC parallel to the line of action of P or Q meeting BA produced in C, and through O draw MN parallel to AB.

At O resolve S into its components in their original direction P acting along CO and F along OM. Similarly resolve T into components Q along OC and F along ON.

The forces F along OM and ON balance and we are left with forces P along CO and Q along OC, *i.e.*, P−Q acting along CO.

Hence the resultant of P and Q is P−Q along CO, *i.e.*, acting through C parallel to P or Q in the sense of the greater force P.

Now, to find the position of the point C, we have from similar triangles AOC, and AGK, AC/OC=AG/KG=F/P,

i.e., P.AC=F.OC.

Similarly from triangles BOC, BHL, Q.BC=F.OC.

∴ P.AC=Q.BC *i.e.*, P/BC=Q/AC

Cor. *If a force R balances the two unlike parallel forces P and Q, it must be equal to P−Q and must act through C in the direction OC (in the plane of P and Q, parallel to them), such that*

$$\frac{P}{BC} = \frac{Q}{AC} = \frac{P-Q}{BC-AC} = \frac{R}{AB} \qquad \ldots(2)$$

From (1) and (2), it follows that :—

If three parallel forces are in equilibrium, each is proportional to the distance between the other two.

This is analogous to Lami's Theorem.

4.12. It is important to note that *the position of the point C is independent of the direction of the parallel forces.*

Also that the line of action of the resultant of two forces lies in the plane determined by the lines of action of the forces.

4.13. In case II if the forces P and Q are equal in magnitude and opposite in direction, the lines of action of the forces S and T become parallel and they do not meet. In this case the forces P and Q are not equivalent to a single resultant. Such a system of two equal and unlike parallel forces is called a **couple**, the properties of which will be considered in the next chapter.

Example. *The resultant of two like parallel forces P, Q passes through a point O, when P is increased by R and Q by S, the resultant still passes through O, and also when Q, R replace P, Q respectively ; prove that*

$$R=\frac{Q^2}{P},\ S=\frac{Q^3}{P^2}\ \text{and}\ S=R-\frac{(Q-R)^2}{P-Q}$$

Let x and y be the distances from O of the lines of action of P and Q, then by the question, we have

$$Px=Qy \qquad \ldots(1)$$

and $$(P+R)x=(Q+S)y \qquad \ldots(2)$$

Also $$Qx=Ry \qquad \ldots(3)$$

Subtracting (1) from (2), we get $$Rx=Sy \qquad \ldots(4)$$

Now dividing (3) by (1) we have $$\frac{Q}{P}=\frac{R}{Q} \quad \text{or} \quad R=\frac{Q^2}{P} \qquad \ldots(5)$$

Similarly from (1) and (4), $$\frac{R}{P}=\frac{S}{Q} \quad \text{or} \quad S=\frac{QR}{P}$$

$$=\frac{Q}{P}\cdot\frac{Q^2}{P},\ \textit{i.e.},\ \frac{Q^3}{P^2}\ \text{[by (5)]}$$

Again from (1) and (3), we have $$\frac{R}{Q}=\frac{Q}{P}=\frac{Q-R}{P-Q},$$

and from (3) and (4) we have $$\frac{R}{Q}=\frac{S}{R}=\frac{R-S}{Q-R}$$

Hence $$\frac{R-S}{Q-R}=\frac{Q-R}{P-Q} \quad \text{or} \quad R-S=\frac{(Q-R)^2}{P-Q},$$

i.e., $$S=R-\frac{(Q-R)^2}{P-Q}.$$

Examples XI

1. The resultant of two like parallel forces is 12 lbs. and it acts at a distance of 2 in. from the larger component which is 8 lbs. ; find its distance from the smaller component. (*Ans.* 4 in.)

2. A uniform rod AB of length 6 ft. and weight 5 lbs. rests in a horizontal

position over a peg Q and under a peg P. If AP=1 ft. and BQ=4 ft. find the reactions of the pegs, there being no friction. [*Ans.* 5, 10 lbs.] (*P.U.*)

3. A heavy uniform rod 4 ft. long rests horizontally on two pegs which are one foot apart, a weight of 10 lbs. suspended from one end or a weight of 4 lbs. suspended from the other end will just tilt the rod up; find the weight of the rod and distances of the pegs from the centre of the rod.

[*Ans.* 20 lbs., 4″, 8″]

4. Three points A, B, C are in a straight line and AB=BC=10 in. Like parallel forces of 7 lbs. wt. and 3 lbs. wt. act at A and B respectively. Find the magnitude and line of action of the resultant of these forces.

Find at what point in AC, produced if necessary, a third parallel force of 5 lbs. wt. acting in opposite sense, should be introduced so that the resultant of these forces may act through C.

(*Ans.* 10 lbs. wt. 2 in. from A; 14 in. from A and 34 in. from C)

5. A man carries a bundle at the end of a stick which is placed over his shoulder, show that the pressure on his shoulder varies inversely as the distance between his hand and the shoulder.

6. P and Q are like parallel forces. If P is moved parallel to itself through a distance d, prove that the resultant of P and Q moves through a distance $Pd/(P+Q)$.

7. A uniform rod of length $2l$ and weight W, is lying across two pegs on the same level d feet apart. If neither peg can stand a stress greater than T, show that the length of the rod which can project beyond either peg cannot be greater than

$$l - \frac{d(W-T)}{W}.$$

8. The lines of action of two forces P and Q and their resultant R are cut by another line in the points A, B and C respectively. P and Q are each resolved into two forces, one parallel to AB and the other parallel to R. Prove that the components parallel to R are to each other as BC : AC.

9. P and Q are two unlike parallel forces, when P is doubled it is found that the line of action of Q is midway between the lines of action of the new resultant and the original resultant. Find the ratio of P and Q.

(*Ans.* $P=\frac{3}{4}Q$)

10. Three equal like parallel forces act at the middle points of the sides of a triangle, show that their resultant passes through the intersection of the medians of the triangle.

11. P and Q (P>Q) are two like parallel forces acting at A and B. Show that if they interchange position, the point of application of the resultant is displaced a distance $(P-Q)AB/(P+Q)$.

12. ABC is a triangle and O a point in the plane of the triangle. A force R acts along AO Resolve R into two forces parallel to it and acting at B and C respectively, when

(*a*) O is the incentre of $\triangle ABC$.
(*b*) O is the orthocentre of $\triangle ABC$.
(*c*) O is the circumcentre of $\triangle ABC$.

[*Ans.* (*a*) $\frac{bR}{b+c}$, $\frac{cR}{b+c}$, (*b*) $\frac{Rb}{a}\cos C$, $\frac{Rc}{a}\cos B$,

(*c*) $\frac{Rb\cos B}{a\cos(3-C)}$, $\frac{Rc\cos C}{a\cos(B-C)}$]

4.2. Centre of a number of parallel forces. Let a number of parallel forces act at fixed points of a body. If the forces are not all in the same sense, let us take them in two groups, each group

consisting of like parallel forces, *i.e.*, forces in the same sense, so that the direction of any force belonging to one group is opposite to the direction of a force of the second group.

Let the forces P_1, P_2, P_3,.....acting at A_1, A_2, A_3,.....belong to one group and Q_1, Q_2, Q_3,.....acting at B_1, B_2, B_3,.....belong to the other.

Now consider the forces of the first group. By Art. 4.11, the resultant of P_1 and P_2 is P_1+P_2 acting at G_1 on A_1A_2 such that $P_1.A_1G_1=P_2.G_1A_2$. Again the resultant of P_1+P_2 at G_1 and P_3 at A_3 is $P_1+P_2+P_3$ at G_2 on G_1A_3 such that

$$(P_1+P_2).G_1G_2=P_3.G_2A_3 \quad \text{and so on.}$$

In this manner, we arrive at the resultant $P_1+P_2+P_3+\ldots=R_p$ acting at a point G_p. Similarly by combining the forces of the second group we will arrive at the resultant $Q_1+Q_2+Q_3+\ldots=R_q$ acting at a point G_q.

Suppose these two resultants are *unequal* in magnitude, then on combining them by Case II, Art. 4.11, we arrive at the final resultant of all the forces acting at a point G such that

$$R_p \times G_pG=R_q \times G_qG.$$

The point *G where the resultant of the given system of parallel forces acts is called the* **centre of parallel forces**. Its position depends only on the positions of the points of application of the forces and their relative magnitude. It is *independent of the direction* of the forces, so that if the forces are turned about their points of application through the same angle so as to remain parallel, the position of G remains the same.

In case R_p and R_q are *equal and collinear* in action, the resultant vanishes and the system is in equilibrium.

Again if R_p and R_q are equal in magnitude, but not collinear in action, the system reduces to a *couple* and is not equivalent to a single force as the resultant.

Notice that in the above discussion the parallel forces are not restricted to lie in one plane.

4.21. A very important application of the above article occurs in connection with the weight of a body. A body may be considered as made up of a very large number of particles and if the size of the body is small as compared with that of the earth, the forces on all the particles of it will be very nearly parallel. Thus the weights of the particles form a system of parallel forces and their centre G is known as the *centre of gravity* (C.G.) of the body.

The position of the centre of gravity of certain bodies is obvious by their symmetrical nature, for example :—

(*i*) C.G. of a uniform straight rod is its middle point.

(*ii*) C.G. of a uniform plate in the form of a parallelogram or a rectangle or a square, and a cube is the centre (*i.e.*, the point of intersection of the diagonals).

(*iii*) C.G. of a uniform circular disc, or a circular ring, or a sphere is the centre.

(*iv*) C.G. of a uniform right circular cylinder is the middle point of its axis.

The position of C.G. for various other bodies will be considered in Chapter VIII.

As a knoweldge of the C.G. of certain well known bodies is necessary in the solution of a large number of problems, we quote the following results for reference :—

(*v*) C. G. of a uniform triangular lamina is the point of intersection of its medians. (Art 8·3).

(*vi*) C.G. of a uniform right circular cone lies on its axis at a distance $\frac{3}{4}$ths of the height from the vertex. (Art. 8·7.)

(*vii*) C.G. of a uniform solid hemisphere is on the line drawn through the centre perpendicular to the plane face at a distance $\frac{3}{8}$ths of its radius from the centre. (Art. 8·71).

4.3. Theorem II B. *The sum of the resolved parts of a pair of parallel forces, not forming a couple, along any line in their plane is equal to the resolved part of their resultant along the same line.*

Let P and Q be the two parallel forces and R, their resultant. Let the forces be inclined at an angle θ to the line OX, along which the forces are to be resolved. Then R will also be inclined at the same angle with OX.

Now, since $R = P + Q$ or $P - Q$

according as the forces are like or unlike,

$\therefore$ multiplying by $\cos\theta$, we have

$$R\cos\theta = P\cos\theta \pm Q\cos\theta$$

i.e., the resolved part of the resultant R along OX

= sum of the resolved parts of the component forces.

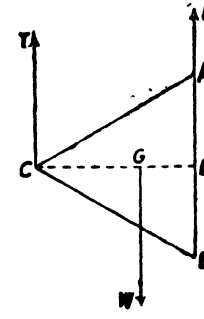
Note. The proof is obviously applicable to a number of parallel forces.

4.31. Equilibrium of three parallel forces. *If a rigid body is in equilibrium under the action of three forces, two of which are parallel, the third force must also be parallel to the other two and must lie in their plane.*

This is so because the third force must be equal and opposite to the resultant of the other two, which is parallel to them in their plane.

Example. *A uniform equilateral triangular lamina ABC of weight W has the vertex A hinged to a fixed point about which it can turn freely in a vertical plane and rests with AB vertical by a vertical*

string attached to C. Find the tension in the string and the magnitude and direction of the reaction R at the hinge.



Since two of the forces, the tension T acting at C and the weight W acting at G, the C.G. of the lamina are vertical, therefore, the third force the reaction R at A must also be vertical

$$\therefore \quad T+R=W,$$

and $T.CG=R.GD$ *i.e.*, $2T=R$ $(\because \quad CG=2GD)$

$$\therefore \quad 3T=W \text{ } i.e., \text{ } T=\tfrac{1}{3}W \text{ and } R=2T=\tfrac{2}{3}W.$$

Examples XII

1. A uniform beam AB of length 4 ft. is suspended from two points C, D at the same level by strings AC, BD. If the distance between C and D is 2 ft. and the string AD is 1 ft. long, find how long the string BD must be in order that AC may be vertical when the system is in equilibrium.

(*Ans.* $1+2\sqrt{3}=4{\cdot}46$ ft.)

2. A uniform circular ring of weight W, free to move in a vertical plane rests over a fixed peg at A and is suspended by a vertical string attached to a point B of the ring. If AB subtends an angle of 135° at the centre and in the position of equilibrium, the diameter through B is horizontal, find the tension in the string. [*Ans.* $W(\sqrt{2}-1)$]

3. A right circular cone is suspended by two light vertical strings of equal length one attached to the vertex of the cone, the other to a point on the rim of its base. The points of suspension are at the same horizontal level and the tensions in the strings are equal. Find the semi-angle of the cone.

(*Ans.* $\cos^{-1}\sqrt{\frac{2}{3}}$)

4. An equilateral triangular lamina weighs 10 lbs. A weight of 10 lbs. is hung by a cord from an angle, and the whole is hung by a second cord attached to another angle. Find the tangent of the angle which the line joining these angular points makes with the vertical. (*Ans.* $\sqrt{3}/9$)

5. A lamina in the form of an isosceles triangle ABC, in which the angle C is a right angle, can turn freely in a vertical plane about A which is fixed. The lamina is held in equilibrium by a vertical string attached at B, the vertex C being downwards and B below the level of A. Prove that the tension in the string is

$$\tfrac{1}{2}W\ (1-\tfrac{1}{3}\tan\theta),$$

where W is the weight of the lamina and θ the angle which AB makes with the horizontal.

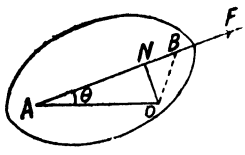
6. A uniform bar AB, of weight W and length l, is smoothly hinged at its end A, which is at a distance $2l$ vertically below a smooth horizontal wire. The bar can turn in the vertical plane through the wire, and the other end B of the bar is attached by a light elastic string of natural length l to a smooth ring which slides on the wire. Find the distance of B from the wire in the position of equilibrium, given that the tension in the string is W when its length is double its natural length. (*Ans.* $l, \frac{5}{11}l, 3l$)

7. A uniform rod, of weight W, is supported in equilibrium by a string of length $2l$, attached to its ends and passing over a smooth peg. If a weight W′ be now attached to one end of the rod, show that it can now be placed in the position of equilibrium by sliding a length $lW'/(W+W')$ of the string over the peg.

MOMENT

4.4. Moment *of a force about a point is the product of the force and the perpendicular distance of the point from the line of action of the force.*

Suppose a point O of a rigid body (*e.g.*, a disc) is held fixed while a force F acts on it. If F does not pass through O, it has a tendency to rotate the body about O. This tendency of rotation depends not only on F, the magnitude of the force, but also on p, the distance of O from its line of action and is measured by the product Fp which is the moment of F about O.



4.41. Let A be any point on the line of action of F, $OA=r$ and $\angle OAN=\theta$, then $ON=OA \sin\theta=r \sin\theta$,

and $F.p=F.r \sin\theta=r.F \sin\theta=AO\times$Resolved part of F$\perp$OA.

4.42. Geometrical Representation of Moment. If AB represents F, on some scale, then $Fp=AB.p=$ twice the area of $\triangle OAB$, so that *the measure of the moment of a force about any point is geometrically equivalent to the measure of twice the area of the triangle formed by joining the point to the extremities of the line representing the force.*

4.43. Sign of Moment. The direction of rotation of a body about a point due to the action of a force depends on the relative position of the point and the line of action of the force. Thus in Fig. 1, both F and F′ tend to rotate the body about O in the same (counter-clockwise) sense and we assign the same sign (usually taken positive) to their moments. But in Fig. 2, F and F′ tend to cause rotation about O in opposite senses and opposite signs are assigned to their moments.

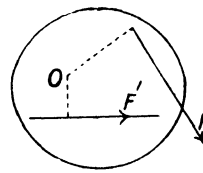


Fig. 2.

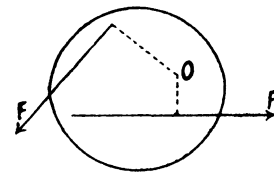


Fig. 1.

When dealing with the moment of a number of forces about a point, the signs of the moments must not be ignored.

Example. *The side of a square plate ABCD is 2a ft. Forces P, 2P, 3P, 4P lbs. wt. act along AB, CB, CD and DA respectively. Find the algebraic sum of the moments of the forces about (i) A, (ii) about O, the centre of the plate, (iii) about E in AB such that BE=a, (iv) about F, the middle point of AD.*

(*i*) The force 3P along CD tend to turn the plate about A in the

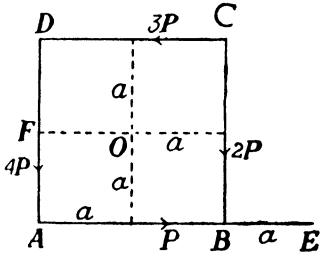
anti-clockwise, *i.e.*, positive direction and the force 2P along CB in the clockwise, *i.e.*, negative direction.

The forces P and 4P pass through A and therefore have no moments about A.

Hence the algebraic sum of the moments of the forces about A

$$=3P.AD-2P.AB=3P.2a-2P.2a=2Pa.$$

(*ii*) The distance of each force from the centre O is a.



$\therefore$ the sum of the moments of the forces about O

$$=P.a-2P\ a+3P.a+4P.a=6Pa.$$

(*iii*) P along AB passes through E and therefore has no moment about E. All the other forces produce rotation about E, in the positive direction.

$\therefore$ the sum of the moments of the forces about E

$$=2P.BE+3P\ BC+4P.AE=2P.a+3P.2a+4P.3a=20Pa.$$

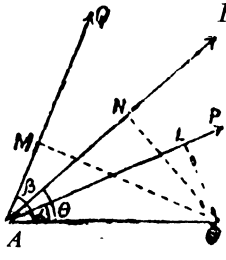
(*iv*) Sum of the moments of the forces about F

$$=P.AF-2P.AB+3P.FD=P.a-2P.2a+3P.a=0.$$

4·5. Theorem III. Varignon's Theorem on Moments. *The sum of the moments of two forces (not forming a couple), about a point in their plane is equal to the moment of their resultant about the same point.*

Case I. *When the forces act at a point.*

Let P and Q be the forces acting at the point A, R their resultant and O the point about which moments are to be taken. Draw OL, OM, ON perpendiculars to the lines of action of P, Q and R respectively. Let $\angle OAP=\alpha$, $\angle OAQ=\beta$ and $\angle OAR=\theta$, then the sum of the moments of P and Q about O



$=P.OL+Q.OM$

$=P.OA\sin\alpha+Q.OA\sin\beta$

$=OA\ (P\sin\alpha+Q\sin\beta)$

$=OA\times$sum of the resolved parts of P and Q perp. to OA

$=OA\times$resolved part of R perp. to OA (by Art. 2·4).

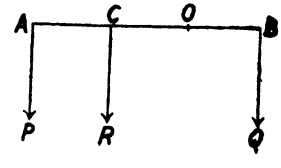
$=OA.R\sin\theta=R.OA\sin\theta=R.ON=$moment of R about O.

Note. The proof is applicable to a number of co-planar and concurrent forces.

Case II. *When the forces are parallel.*

Let P and Q be two parallel forces and R, their resultant and O any point in their plane.

From O draw a line perpendicular to the lines of action of P and Q, meeting them in A and B and meeting R in C.

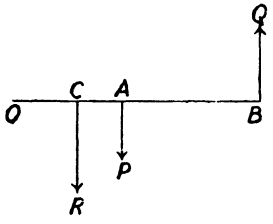


Then $P+Q=R$ and $P.AC=Q.CB$ ...(1)

The sum of the moments of P and Q about O
$=P.AO-Q.OB=P.(AC+CO)-Q.(CB-CO)$
$=P.AC-Q\,CB+(P+Q)CO.$
$=R.CO,$ from (1)
=the moment of R about O.

This proof is applicable to all the positions of the point O, the moment of the forces being taken with their proper signs.

For example, let the forces P and Q be unlike (P>Q) and let O be outside these forces.



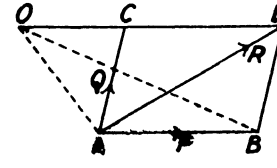
From O, draw OCAB perpendicular to the lines of action of the forces.

In this case
$R=P-Q$ and $P.CA=Q.CB$ ...(2)

The sum of the moments of P and Q about O.
$=Q.OB-P.OA=Q(OC+CB)-P.(OC+CA)$
$=(Q-P).OC+Q.CB-P.CA=-(P-Q).OC=-R.OC$ (by (2)
=the moment of R about O.

Cor. *The sum of the moments of the forces about any point lying on the line of action of their resultant vanishes.*

4·51. The following proof of Case I is based on the geometrical representation of a moment.



Let the forces P and Q act along the lines AB and AC. Through O draw OC parallel to the direction of P meeting the line of action of Q in C.

Let AC *now* represent the force Q on some scale and on the same scale let AB represent P. Complete the parallelogram ABCD, so that AD represents the resultant R, then the sum of the moments of Q and P about O
$=2\triangle OAC+2\triangle OAB=2\triangle OAC+2\triangle ABD$
$=2\triangle OAC+2\triangle CAD$
$=2\triangle OAD$=moment of R about O.

The proof is applicable to all positions of the point O, the moments of the forces being taken with their proper signs.

(The student is advised to prove the proposition when O lies within the angle CAD.)

Examples XIII

1. Forces 3, 4. 5 lbs. wt. act along the sides BC, CA, AB of an equilateral triangle ABC of side 2 ft. in the directions indicated by the order of the letters. Find the moment of this system of forces about (*i*) A (*ii*) B (*iii*) C (*iv*) the C. G. of the triangle. (*Ans.* $3\sqrt{3}$, $4\sqrt{3}$, $5\sqrt{3}$, $4\sqrt{3}$ lbs. ft.)

2. Forces 3, 5, 3 and 5 lbs. wt. act along the sides AB, CB, DC, DA respectively of a square of side 4 ft. Find the sum of their moments (*i*) about B (*ii*) about C (*iii*) about the centre of the square. (*Ans.* 8. 32, zero lb. ft.)

3. Forces of 1, 2, 3, 4, 5 and 6 lbs. wt. act respectively along the sides AB, BC, CD, DE, EF, FA of a regular hexagon whose side is 4 ft., find the sum of the moments of the forces (*i*) about the centre of the hexagon (*ii*) about A.

(*Ans.* $42\sqrt{3}$, $42\sqrt{3}$ lb. ft.)

4. ABCD is a square of side 10 ft. A certain force P passes through A and has moments of 20 and 30 lbs. ft. in opposite sense about B and D respectively.

Find the angle which the line of action of P makes with AB. Also find the magnitude of P and its moment about C.

(*Ans.* $\tan^{-1}\frac{2}{3}$, $\sqrt{13}$ lbs. wt. ; 10 lb. ft.)

4.6. Resultant of a number of Coplanar forces acting on rigid body. A system of coplanar forces which are concurrent were treated in the last chapter. In what follows, we shall deal with a system of coplanar forces which are not necessarily concurrent.

4.61. Theorem IV. *Any system of forces acting in the same plane upon a rigid body can be reduced to either a single force or a single couple as their resultant.*

Any *three* forces of the system can always be reduced to *two* forces, for any one of the three can be compounded with at least one of the remaining forces unless it forms a couple with both of them in which case, the latter two being equal and like parallel forces, can themselves be compounded into a single force.

For example, if P_1, P_2, P_3 be any three forces of the system, and if P_1 does not form a couple with P_2 then either by the law of parallelogram of forces (if they meet) or by the law of parallel forces (if they are parallel), the two can be combined and reduced to one force. In case P_1 forms a couple with P_2 but does not form a couple with P_3, it can be combined with P_3, and the two are thus reduced to a single force.

If, however, P_1 forms a couple with P_2 as well as with P_3 it means that P_2 and P_3 are equal like parallel forces reducible to a single force as their resultant

Thus it is always possible to reduced any *three forces to two.*

Now by taking another force of the system, with the two forces obtained before, we can again reduce these forces to two and so on.

In the end we are left with two forces, which (if they are not equal, opposite and collinear), either form a couple or are equivalent to a single force.

Cor. *If a system of coplanar forces reduces to neither a single force nor a couple, the system must be in equilibrium.*

Ex. Show that a system of co-planar forces can always be reduced to two parallel forces.

4·62. Theorem V. Generalised theorem of Moments. If a system of coplanar forces acting on a rigid body reduces to a single force as its resultant, then

the algebraic sum of the moments of the forces, about any point in their plane, is equal to the moment of their resultant about the same point.

Let O be the point about which moments are to be taken.

Take two forces of the system not forming a couple, let them be P_1,P_2 and let R_1 be their resultant, then by Art. 4·5, we have

Moment of R_1 about O = sum of the moments of P_1 and P_2 about O.

Now combine R_1 with some other force, say P_3 not forming a couple with it and obtain the resultant R_2 then

Moment of R_2 about O = sum of the moments of R_1 and P_3
= sum of the moments of P_1, P_2, and P_3.

Proceeding in this manner, avoiding two forces forming a couple and repeating at every step, the process of taking moments of two forces, we arrive at the final resultant R of the system and thus we get

moment of R about O = the algebraic sum of the moments of P_1, P_2, P_3.....P_n about O.

Note 1. If at any stage, any of the resultants R_1, R_2, R_3..... say R_p, form a couple with each of the remaining forces it will mean that the rest of the forces are all equal and like parallel forces. In such a case, leave R_p for a time and combine any two of these like parallel forces and then proceed on as before.

Note 2. The last force and the resultant of all the rest cannot form a couple, since we are assuming that the system reduces to a single force as its resultant.

4.621. From the above theorem it follows that :—

If a number of coplanar forces are in equilibrium, the algebraic sum of their moments about a point in their plane vanishes.

Also if a number of coplanar forces have a single force as their resultant, the sum of their moments about any point on the line of action of the resultant vanishes and conversely if sum of their moments about a point in their plane vanishes, the resultant must pass through the point.

This fact is usually employed in problems requiring the line of action of the resultant of a number of coplanar forces acting on a rigid body.

4.63. Theorem VI. Generalised theorem of resolved parts. If a system of coplanar forces acting on a rigid body reduces to a single force as its resultant, then

the algebraic sum of the resolved parts of the forces along any line in their plane, is equal to the resolved part of their resultant along the same line.

Let OX be the given line along which the forces are to be resolved.

Take two forces of the system not forming a couple, let them be P_1, and P_2 and let R_1 be their resultant, then whether P_1, P_2 are parallel or they meet, by Arts. 4 3 and 2.4, we have

resolved part of R_1 along OX = sum of the resolved parts of P_1 and P_2.

Now combine R_1 with some other force, say P_3 not forming a couple with it, and obtain the resultant R_2, then
resolved part of R_2 along OX
=sum of the resolved parts of R_1 and P_3.
=sum of the resolved parts of P_1, P_2 and P_3.

Proceeding in this manner and repeating the argument about the sum of the resolved parts of two forces, we arrive at the final resultant R of the system and thus we get,
resolved part of R along OX=the algebraic sum of the resolved parts of P_1, P_2.....P_n along OX

Note. The notes of Art. 4·62 applies to this theorem also.

4.631. Thus, if the lines of action of the forces P_1, P_2, P_3..... make angles, α, β, γ.....and the line of action of their resultant R makes θ with OX, then

$$R\cos\theta = P_1\cos\alpha + P_2\cos\beta + P_3\cos\gamma + \ldots\ldots = X \qquad \ldots(1)$$

Similarly, resolving forces along the line OY perpendicular to OX, we have

$$R\sin\theta = P_1\sin\alpha + P_2\sin\beta + P_3\sin\gamma + \ldots\ldots = Y \qquad = (2)$$

Equations (1) and (2) give

$$R^2 = X^2 + Y^2 \qquad \text{and} \qquad \tan\theta = Y/X.$$

These results give the magnitude and direction of the resultant force, when it exists.

4.632. If R=0, then X and Y both vanish so that it follows that :—

If a system of coplanar forces does not reduce to a force the algebraic sum of their resolved parts along a line in their plane vanishes and it also vanishes along a line perpendicular to the former.

Also, if the system reduces to a force and if X=0, *i.e.*, the sum of the resolved parts along a given line in their plane vanishes, the resultant force acts along a line perpendicular to the given line.

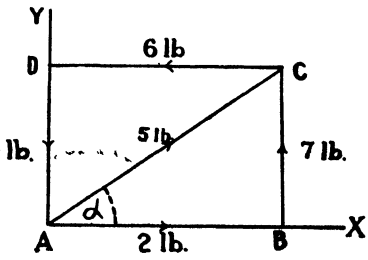
Also, if X = 0 as well as Y=0, *i.e.*, the resolved parts along two perpendicular straight lines in the plane of the forces separately vanish, the resultant is not a force, it may be a couple (Art. 4.61), or the forces may be in equilibrium.

Example (i). *ABCD is a rectangle with AB=4 ft., BC=3ft. Along AB, BC, CD, DA and AC act forces 2, 7, 6, 10 and 5 lbs. wt. respectively. Show that they are equivalent to a couple, the forces being not in equilibrium.*

Since AB=4 ft., BC=3 ft.

$\therefore$ AC=5 ft., $\sin\alpha = \sin\angle BAC = \frac{BC}{AC} = \frac{3}{5}$ and $\cos\alpha = \frac{AB}{AC} = \frac{4}{5}$.

Taking AB as the axis of X and AD as the axis of Y and resolving forces along these lines, we have



$$X=2+5\cos\alpha-6=2+5\times\tfrac{4}{5}-6=0$$
$$Y=7+5\sin\alpha-10=7+5\times\tfrac{3}{5}-10=0.$$

Hence the resultant is not a force and, therefore, the forces, being not in equilibrium, are equivalent to a couple.

Note that the sum of the moments of the forces about A
$=7.AB+6.AD=7\times4+6\times3=46$ units.

Example (ii). In example of Art. 4.43 on page 60 taking BA and DA produced as the axes, we have

$$X=3P-P=2P \quad \text{and} \quad Y=4P+2P=6P.$$

Thus both X and Y do not vanish and, therefore, the resultant is a force and since the sum of the moments of the forces about F vanishes (*See* part *iv* of the same example), the line of action of the resultant passes through F.

Thus the resultant of the given system of forces is a force $=\sqrt{X^2+Y^2}=\sqrt{4P^2+36P^2}=2P\sqrt{10}$ lbs. wt., inclined downwards with BA or OF at an angle $=\tan^{-1} Y/X=\tan^{-1} 3$.

4·64. Example (i). *Forces X, Y, Z act along the sides BC, CA, AB of a triangle ABC. Show that the resultant of the forces passes through the circumcentre and the incentre of the triangle if*

$$X : Y : Z=\cos B-\cos C : \cos C-\cos A : \cos A-\cos B.$$

Let R be the circumradius and r the inradius of the triangle then the distances of the sides BC, CA, AB from the circumcentre O are R cos A, R cos B and R cos C, and from the incentre I, each distance is equal to r.

Since the resultant passes through the points O and I, the sum of the moments of the forces about these points must separately vanish.

Hence $X.R\cos A+Y.R\cos B+Z.R\cos C=0,$

or $X\cos A+Y\cos B+Z\cos C=0, \quad \ldots(1)$

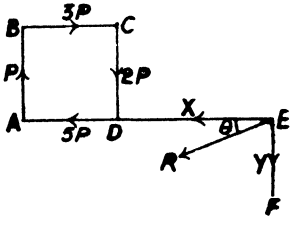
and $X.r+Y.r+Z.r=0$ or $X+Y+Z=0 \quad \ldots(2)$

$\therefore$ from (1) and (2), we have

$$\frac{X}{\cos B-\cos C}=\frac{Y}{\cos C-\cos A}=\frac{Z}{\cos A-\cos B}.$$

Example (ii). *Forces P, 3P, 2P, 5P act along the sides AB, BC, CD, DA of a square ABCD. Find the magnitude and direction of their resultant and prove that it meets AD produced at a point E such that AE : DE=5 : 4.*

Let the line of action of the resultant meet AD produced at E and let AD=a, and DE=x, then since the sum of the moments about E must vanish, we have



$$2Px-P(a+x)-3Pa=0,$$

i.e., $x-4a=0$ or $x=DE=4a.$

$\therefore$ $AE=AD+DE=a+4a=5a,$

and $AE : DE=5a : 4a=5 : 4.$

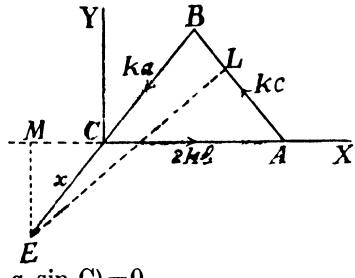
Now let X, Y be the components of the resultant R along ED and EF and θ the angle its line of action makes with ED. By resolving forces along these lines, we have

$$X=5P-3P=2P \qquad \text{and} \qquad Y=2P-P=P.$$

$$\therefore \quad R=\sqrt{X^2+Y^2}=\sqrt{5}\,P \quad \text{and} \quad \tan\theta=\frac{Y}{X}=\frac{P}{2P}=\frac{1}{2}.$$

Example (iii). *Forces proportional to AB, BC and 2CA act along the sides of a triangle ABC taken in order. Show that the resultant is proportional to CA in magnitude and is parallel to CA in direction and that its line of action meets BC is E, where CE=BC.*

Let the forces along AB, BC and CA be kAB or kc, kBC or ka and $2k$CA or $2kb$ respectively taken in order.



Taking CA as x-axis and CY, a line through C perpendicular to CA, as the y-axis and resolving forces along these lines, we have

$$X=2kb-kc\cos A-ka\cos C$$
$$=2kb-k(c\cos A+a\cos C)$$
$$=2kb-kb=kb=k\text{CA}.$$

and $Y=kc\sin A-ka\sin C=k(c\sin A-a\sin C)=0$

The component of the resultant along CY being zero, the resultant is parallel to CA in direction and being equal to kCA, it is proportional to CA in magnitude.

Let the line of action of the resultant meet BC produced in E such that $CE=x$.

Draw EL perpendicular to AB and EM perpendicular to AC, then

$$EL=EB\sin B=(EC+CB)\sin B=(x+a)\sin B,$$

and $$EM=EC\sin ECM=x\sin C.$$

Now sum of the moments of the forces about E must vanish.

$$\therefore \quad kc.EL-2kb.EM=0,$$

or $$c(x+a)\sin B-2bx\sin C=0,$$

or $$x(2b\sin C-c\sin B)=ac\sin B.$$

$$\therefore \quad x=\frac{ac\sin B}{2b\sin C-c\sin B}=\frac{ac.b}{2b.c-c.b}=\frac{abc}{bc}=a=\text{BC}.$$

Examples XIV

1. Three forces P, Q, R act along the sides BC, CA, AB of a $\triangle$ABC. Show that their resultant passes through

(*i*) the centroid if $\quad P\operatorname{cosec}A+Q\operatorname{cosec}B+R\operatorname{cosec}C=0,$

(*ii*) the orthocentre if $\quad P\sec A+Q\sec B+R\sec C=0,$

(*iii*) the centroid and the circumcentre (or the orthocentre) if

$$P:Q:R=\sin 2A\sin(B-C):\sin 2B\sin(C-A):\sin 2C\sin(A-B)$$

(*D.U.* ; *Agra U.*)

(*iv*) the orthocentre and the circumcentre if

$$P:Q:R=(b^2-c^2)\cos A:(c^2-a^2)\cos B:(a^2-b^2)\cos C,$$

(*v*) the orthocentre and the incentre if
$P : Q : R = \sec B - \sec C : \sec C - \sec A : \sec A - \sec B$,

(*vi*) the centroid and the incentre if
$\text{cosec } B - \text{cosec } C : \text{cosec } C - \text{cosec } A : \text{cosec } A - \text{cosec } B$.

2. The resultant of three forces acting along the sides a, b, c of a triangle acts along the line joining the centres of the inscribed and circumscribed circles. Show that the forces are in the ratio

$$(b-c)(b+c-a) : (c-a)(c+a-b) : (a-b)(a+b-c).$$

3. Forces P, Q, R act along the sides BC, AC, BA respectively of an equilateral triangle. If their resultant is a force parallel to BC through the centroid of the triangle, prove that

$$Q = R = \tfrac{1}{2}P. \qquad (R.U.)$$

4. Forces of magnitudes 1, 4, 3 lbs. act along the sides AB, BC, CD respectively of a square ABCD. Show that the line of action of the resultant cuts AB produced at a distance $\frac{3}{4}$ AB from B, and find the magnitude of their resultant. (*Ans.* $2\sqrt{5}$ lbs).

5. Four forces P, 2P, 3P and 4P act along the sides taken in order of a square ABCD. Find their resultant. (*P.U.*)

(*Ans.* $2\sqrt{2}$ P, parallel to CA, meeting AD produced in E, where $DE = \frac{3}{2}$ AB)

6. ABCD is a rectangle of which the adjacent sides AB and BC are equal to 3 and 4 ft. Along AB, BC, CD and DA act forces 20, 40, 50 and 80 lbs. wt find the resultant.

(*Ans.* 50 lbs. wt. cutting AD and CD produced at E and F distant $6\frac{2}{3}$ and 5 ft. from D)

7. Forces 1, 4, 2, 3 lbs. wt. act along the sides AB, BC, CD, DA respectively of a rhombus whose acute angle at A is 60°. Prove that their resultant is perpendicular to AC and that it cuts AC at a point whose distance from A is 3AC.

8. ABC is an isosceles triangle whose angle A is 120° and forces of magnitude 1, 1 and $\sqrt{3}$ lbs. wt. act along AB, AC and BC ; show that the resultant bisects BC and is parallel to one of the other sides of the triangle. (*P.U.*)

9. Three forces P, 2P and 3P act along the sides AB, BC and CA respectively of a given equilateral triangle ABC ; find the magnitude and direction of their resultant, and find also the point K in which its line of action meets the side BC. (*Ans.* $P\sqrt{3}$ perp. to BC, $BK = \frac{3}{2}$ BC) (*D.U. 1939*, ; *P.U.*)

10. Forces P, P+Q, P−Q act along the sides BC, CA, AB respectively of an equilateral triangle ABC. Determine their resultant completely.

(*Ans.* $\sqrt{3}$ Q perp. to BC meeting it at a distance P.BC/2Q from its mid. point)

11. Forces equal to 3P, 7P and 5P act along the sides AB, BC and CA of an equilateral triangle ABC ; find the resultant.

(*Ans.* $2\sqrt{3}$ P meeting BC produced at X where $CX = \frac{3}{2}$ BC and inclined at 30° to it)

12. Forces 2, 2, 3, 2 units act along the sides AB, CD, ED, EF respectively of a regular hexagon ABCDEF. Find the magnitude of the resultant and prove that it acts along AB. (*Ans.* 3 units)

13. Forces P, 3P, 2P, 6P, 4P, 5P act respectively along the sides AB, BC, CD, DE, EF and FA of a regular hexagon of side a. Find the magnitude and direction of the resultant and also the distance from O, the centre of the hexagon, of the point in which the line of action of the resultant meets OA.

(*Ans.* $2\sqrt{7}$P, $\tan^{-1}\sqrt{3}/5$ with OA ; $21a/2$ on AO produced)

14. Forces of magnitude P, 2P, −3P, 4P, 5P and −9P act respectively along the sides taken in order of a regular hexagon. Prove that they reduce to

a resultant force acting through the centre of the hexagon. Find the magnitude and direction of their resultant. (*P.U. 1951*)

(*Ans.* $3\sqrt{7}$ P inclined at $\tan^{-1}\frac{\sqrt{3}}{5}$ with the reversed direction of P)

15. The moments of a force about the vertices A, B, C of an equilateral triangle whose sides are 2 ft. long, are $+10$, $+20$ and -10 in foot pound units. Find the magnitude of the force and the points where the line of action intersects AB and AC.

(*Ans.* $10\sqrt{\frac{7}{3}}$ lbs. wt. ; BA produced 2 ft. from A ; mid. pt. of AC)

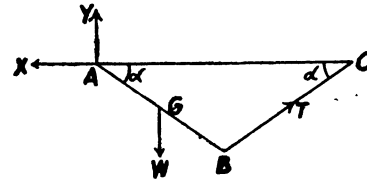
4.65. In some cases, such as those where a body is hinged to a fixed point, taking of moments about the fixed point simplifies the solution to a problem.

For, in such cases the action at the hinge, being a force passing through it, does not enter the equation of moments.

The action at the hinge is then obtained by taking its components as X, Y along two convenient perpendicular directions and resolving forces along them.

Example. 1. *One end of a uniform rod is movable about a hinge and the other is supported by a string attached to it ; the string is inclined at the same angle α to the horizon as the rod. Find the reaction at the hinge.*

Let $2a$ be the length and W the weight of the rod AB.



Taking moments about A, we have

$$W.AG\cos\alpha = T.AB\sin(\pi - \angle ABC)$$
$$= T.2a\sin 2\alpha$$
$$= 4Ta\sin\alpha\cos\alpha.$$
$$\therefore \quad T = \tfrac{1}{4}W\operatorname{cosec}\alpha \quad (\because AG = a),$$

which gives tension of the string.

Now to find the reaction, suppose its horizontal and vertical components are X and Y.

Resolving horizontally and vertically, we have

$$X = T\cos\alpha = \tfrac{1}{4}W\operatorname{cosec}\alpha.\cos\alpha = \tfrac{1}{4}W\cot\alpha,$$

and $$Y = W - T\sin\alpha = W - \tfrac{1}{4}W = \tfrac{3}{4}W.$$

Hence $$R = \sqrt{X^2 + Y^2} = \tfrac{1}{4}W\sqrt{9 + \cot^2\alpha} = \tfrac{1}{4}W\sqrt{8 + \operatorname{cosec}^2\alpha}.$$

If θ be the inclination of R to the horizontal,

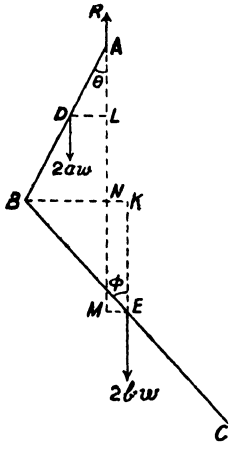
$$\tan\theta = \frac{Y}{X} = \frac{\frac{3}{4}W}{\frac{1}{4}W\cot\alpha} = 3\tan\alpha.$$

Note 1. Since the reaction R at A must pass through the point where the line of action of the tension and the weight meet, we can obtain θ also from the geometry of the figure, and then find T and R by the application of Lami's theorem.

Note 2. The method of taking moments about the hinge can be usefully applied also to the solved examples given in Art. 3.4, Pages 39, 40.

Example. 2. *Two uniform rods AB, BC of length 2a, 2b, respectively are rigidly united at B and are suspended freely from A. If they rest inclined at angles θ, ϕ respectively to the vertical show that,*

$$\frac{\sin\theta}{\sin\phi} = \frac{b^2}{(a^2+2ab)}.$$



Let w be the weight per unit length of the rods.

Then the weight $2aw$, $2bw$ of the rods AB and BC act at their middle points D and E respectively. Since these two forces are vertical, the only other force, the reaction R at A, must also be vertical.

Taking moments about A, we have

$$2aw.\mathrm{DL} = 2bw.\mathrm{ME} = 2bw(\mathrm{BK}-\mathrm{BN}).$$

$$\therefore \quad a.a\sin\theta = b(b\sin\phi - 2a\sin\theta),$$

i.e. $$(a^2+2ab)\sin\theta = b^2\sin\phi.$$

Hence $$\sin\theta/\sin\phi = b^2/(a^2+2ab).$$

Note. *If* $\angle\mathrm{ABC}=90°$, $\sin\phi=\cos\theta$, $\therefore \tan\theta = \dfrac{b^2}{a^2+2ab}$.

Examples XV

1. Two uniform rods AB, BC, rigidly jointed at B so that ABC is a right angle, hang freely in equilibrium from a fixed point at A. The length of the rods are a and b and their weights are W_1 and W_2. Prove that,

$$\tan\theta = W_2 b/(W_1+2W_2)a,$$

θ being the angle which the rod AB makes with the vertical.

2. A uniform wire AOB is bent at O into straight portions OA and OB inclined at angle α, OA and OB being of lengths a and b respectively. If the portion OB is horizontal when the whole is suspended from A, prove that

$$\cos\alpha = \frac{b^2}{a(a+2b)}.$$

3. A triangle ABC of uniform wire has its side AB removed and is hung up by the point B. Show that for CA to be horizontal,

$$c^2(a+2b) = a(a+b)^2.$$

4. A uniform square plate ABCD is suspended from A and a weight equal to half that of the plate is suspended from B. Find the inclination of AB to the vertical. (*Ans.* $\tan^{-1}\frac{1}{2}$)

5. A uniform square plate ABCD can turn freely in a vertical plane about a hinge at A; forces equal to W and 2W act respectively along the sides BC and CD where W is the weight of the plate. Find a force along DB which will keep the plate at rest with AB horizontal. Find also the reaction at the hinge.
(*Ans.* $\frac{7}{2}\sqrt{2}$ W; $\frac{3}{2}\sqrt{2}$ W parallel to BD) (*P.U. 1956 S.*)

6. A circular metallic plate of uniform thickness and of weight W is hung from a point in its circumference. A string wound on its edge carries a weight P. Show that the diameter through the point of suspension makes with the vertical an angle

$$\sin^{-1}\frac{\mathrm{P}}{\mathrm{P}+\mathrm{W}}.$$

7. A uniform straight rod AB, of weight 12 lbs. is free to turn about the end A. The rod is kept in equilibrium by a light string attached to B and to a fixed point C at the same horizontal level as A, ABC being an equilateral triangle. Find the angle which the action at A makes with the horizontal.

[*Ans.* $\tan^{-1}(3\sqrt{3})$]

8. A uniform rod AB of weight W is smoothly pivoted to a fixed point at A. The rod is maintained in equilibrium at an angle 30° below the horizontal, by means of a force P applied at B in a direction perpendicular to AB. Find the magnitude of P, and show that the resultant reaction at the pivot is $\frac{\sqrt{7}W}{4}$.

$\left(\textit{Ans.}\quad P=\frac{\sqrt{3}W}{4}\right)$

9. A uniform rod AB of mass 20 lbs. is hinged to a point A in a vertical wall and is supported by a string BC attached to a point C in the wall vertically above the hinge. The rod makes an angle 30° with the upward vertical and an angle of 45° with the string. Find the tension in the string.

(*Ans.* $5\sqrt{2}$ lbs. wt.) (*I. I. T. Kharagpur*)

10. A uniform rod of weight 7 lbs. and length 7 ft. is bent through 90° at 3 ft. from one end. It is placed in a vertical plane with the right angle uppermost and the two ends resting on a smooth horizontal table. Find the reactions exerted by the table on each end of the rod.

What vertical force must be applied at the right angle in order to make the two reactions equal ? (*P.U. 1958* S.)

(*Ans.* Reactions 3·74, 3·26 lbs. wt. ; force 12/7 lbs. wt. upwards)

11. A rod AB is movable about a point at A and to B is attached a string whose other end is tied to a ring. The ring slides along a smooth horizontal wire passing through A ; prove that the horizontal force necessary to keep the ring at rest is

$$W \cos\alpha \cos\beta/2 \sin(\alpha+\beta),$$

where W is the weight of the rod and α, β the inclinations of the rod and the string to the horizontal.

12. A heavy uniform rod AB passes over a smooth peg and is supported by a horizontal string, one end of which is fastened to the upper extremity A of the rod. If in the position of equilibrium the rod extends above the peg by a horizontal distance of a quarter of its length and θ be its inclination to the horizontal, show that $\cos^3\theta=\frac{1}{2}$.

13. A uniform rod AB 12 ft. long, and weighing 50 lbs., is pivoted at a point 3 ft. from A. A weight of 200 lbs. is suspended from A. What force applied at B, in a direction perpendicular to the rod, will keep it in equilibrium with A below B and AB inclined at 60° to the horizontal.

(*Ans.* 25 lbs. wt.)

14. A uniform rod ACB of weight W and length $4a$ rests upon a smooth peg C, and its upper end A is attached to a fixed point O, situated in the same horizontal line with C by means of a string OA. If AC and OA are each of length c, show that the inclination of the rod to the horizontal is

$$\cos^{-1}\sqrt{\frac{2a+c}{4a}}. \qquad (B.U.)$$

15. A uniform rod AB of length 1 ft. and weight 20 lbs. is freely hinged to a fixed point at A. One end of an elastic string whose natural length is 18 in. is attached to B and the other end is attached to a fixed point C which is 1 ft. vertically above A. If the modulus of elasticity of the string is 80 lbs. wt. prove that in the position of equilibrium, the string is stretched 4·2 in. nearly.

16. A rod length a is movable in a vertical plane about a smooth hinge at one end, and at the other end is fastened a weight $\frac{1}{2}W$, the weight of the rod

being W. This end is fastened by a string of length l, to a point at a height c vertically over the hinge. Show that the tension of the string is lW/c, and find the reaction at the hinge.

(*Ans.* $\frac{1}{2}W\sqrt{3c^2-2l^2+6a^2}\,c'$. (*P.U. 1932*)

17. A heavy uniform rod AB rests with one end A in contact with a smooth vertical wall and with a string ACB attached to its ends and passing through a ring C vertically above A. Prove that $\tan BAC=2\cot\frac{1}{2}$ (ACB). (*Kashmir U.*)

18. A heavy uniform rod of length $2l$ is resting in contact with a smooth sphere of radius r and the lower end of the rod presses against a smooth vertical wall. If the distance of the centre of the sphere from the wall is c and the rod makes an angle α with the wall, prove that

$$l\sin^3\alpha+r\cos\alpha=c.$$

19. A rigid wire, without weight, in the form of the arc of a circle subitending an angle α at the centre and having two weights P and Q at its extremities, rests with its convexity downwards upon a horizontal table; show that, of θ be the inclination to the vertical of the radius to the end at which P is suspended, then

$$\tan\theta=\frac{Q\sin\alpha}{P+Q\cos\alpha}.$$

20. A uniform equilateral triangle ABC is supported with the angle A in contact with a smooth wall by means of a string BD, equal in length to a side of the triangle, which is fastened to a point D vertically above A. Show that the distances of B and C from the wall are as 1 : 5. (*P.U.*)

21. A square lamina rests with its plane perpendicular to a smooth wall one corner resting against the wall another corner being attached to a point in the wall by a string whose length is equal to a side of the square. Show that

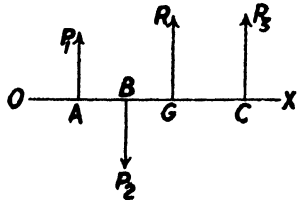
(*i*) a side of the square makes an angle $\tan^{-1}(\frac{1}{3})$ with the wall;

(*ii*) tension of the string is $\sqrt{10}/3$ times the weight of the lamina;

(*iii*) the distances of its angular points from the wall are as 0 : 1 : 3 : 4. (*P.U. 1955 S.*)

4.7. Position of the Centre of Parallel Forces acting at fixed points on a line.

Let the parallel forces P_1, P_2, P_3.....act at the points A, B, C.....on the OX, at distances x_1, x_2, x_3.....from the point O.



Let $R=P_1+P_2+P_3+$.....,

the resultant of the forces, act at G, where $OG=\bar{x}$.

Since the position of the point G is independent of the direction of the forces, we suppose the forces to be perpendicular to OX.

Taking moments about O, by Art. 4.62, we get

$$(P_1+P_2+P_3+.....)\bar{x}=P_1x_1+P_2x_2+P_3x_3+.....$$

i.e.,
$$\bar{x}=\frac{\Sigma Px}{\Sigma P}.$$

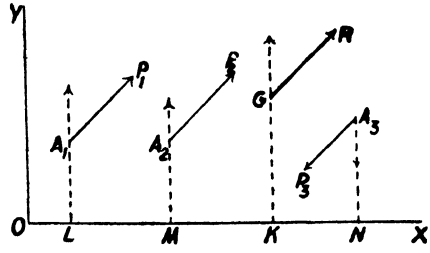
Note. $P_1+P_2+P_3$.....means the algebraic sum of the forces, each force being taken with its proper sign + or —, so that ΣP may vanish in which case the forces are either in equilibrium or reduce to a couple.

4.71. Position of the Centre of Coplanar Parallel Forces acting at given points. Let OX, OY be two perpendicular axes in the plane of the forces and let parallel forces, P_1, P_2, P_3..... act at the points A_1, A_2, A_3 .....whose co-ordinates with reference to OX, OY are (x_1, y_1), (x_2, y_2), (x_3, y_3).....

Let the resultant of the forces act at G $(\bar{x}, \bar{y})$

Since the position of G is independent of the direction of the forces, we turn the forces so as to be perpendicular to OX.

Then P_1, P_2, P_3..... ...intersect OX at distance x_1, x_2, x_3..... and the resultant at a distance x from O.



Taking moments about O, we have

$$(P_1+P_2-P_3+.....)\bar{x} = P_1x_1+P_2x_2-P_3x_3+.....$$

$$i.e.\quad \bar{x}=\frac{\Sigma Px}{\Sigma P}.$$

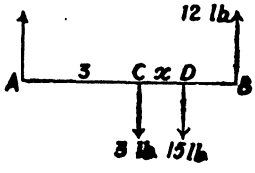
Similarly by turning forces so as to be perpendicular to OY, and taking moments about O, we get,

$$\bar{y}=\frac{\Sigma Py}{\Sigma P}.$$

Note. When $\Sigma P=0$, the system is either in equilibrium or reduces to a couple.

4.72. Example (i). *A uniform rod of length 6 ft. and weight 3 lbs. rests in a horizontal position with its ends on two supports, each of which will bear a weight of 12 lbs. and no more. Find on what part of the rod a weight of 15 lbs. can be placed without breaking either support.*

In the extreme case let the support at B be at the point of breaking when the 15 lbs. wt. is attached at D at a distance x ft. from the centre C. The reaction at B in this case is 12 lbs. wt.



Since the total weight acting downwards is 3+15=18 lbs., the reaction R at A is given by R+12=18 or R =18−12=6 lbs.

To find x, we take moments about A, we get

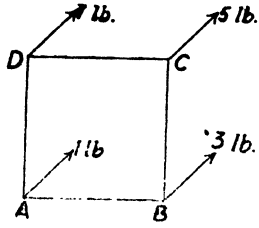
$$3\times3+15(3+x)-12\times6=0,$$

which gives $x=1\frac{1}{5}$ ft.

Thus the weight can be placed at any point within a distance of $1\frac{1}{5}$ ft. on either side of the centre.

Example (ii). *Parallel forces equal to 1, 3, 5, 7 lbs. act at the angular points A, B, C and D of a square ABCD. If the forces lie in the plane ABCD and act in the same direction find the position of the centre of these forces.*

Let x, y be the distances of the required point from the sides AD and AB and let a be the side of the square; then by by Art. 4.71 we have



$$x = \frac{1\times 0+3\times a+5\times a+7\times 0}{1+3+5+7} = \tfrac{1}{2}a,$$

$$y = \frac{1\times 0+3\times 0+5\times a+7\times a}{1+3+5+7} = \tfrac{3}{4}a.$$

So that the required point is at a distance $\frac{1}{4}a$ from the centre of the square.

Examples XVI

1. A uniform rod AB 5 ft. long and weighing 6 lbs. has weights 1, 2, 3, 4, 5 attached to it at distances 1, 2, 3, 4, 5 ft. from the end A. Find at what point it can be supported by a single force. (*Ans.* 3·33 ft.)

2. A uniform rod AB, of length 12 ft. and weighing 50 lbs. rests on two supports, one at A and other 2 ft. from B. Weights of 4, 5, and 10 lbs. are attached at points 2 ft, 4 ft. and 8 ft. respectively from A. Find the pressures on the supports. (*Ans.* 40·8, 28·2 lbs. wt.)

3. A uniform shaft of weight 100 lbs. rests with its ends on two bearings A and B which are 10 ft. apart. Pulleys which weigh 15, 25 and 30 lbs. are mounted on the shaft between A and B at distances 4, 6 and 8 ft. respectively from A. Find the pressures on the two bearings A and B.

(*Ans.* 95, 75 lbs. wt.)

4. A heavy uniform beam AB of weight 100 lbs. and length 6 ft. rests horizontally on two fixed supports at a distance of 1 ft. from A and the other at a distance of 2 ft. from B. Find what weight should be hung from the end A to make the reaction on the two supports equal. (*Ans.* 20 lbs.)

5. A uniform plank ABC, of length 12 ft. and weight 80 lbs. rests on two supports A and B, one at the end A and other at B, $4\frac{1}{2}$ ft. from the end C. A boy walks along the plank from A to C and just as he reaches C, the plank commences to tilt. Find the weight of the boy. (*Ans.* $26\frac{1}{2}$ lbs.)

6. A uniform plank ACDB of length 12 ft. and weight W rests upon two supports C and D. The plank is just on the point of tilting up when a boy of weight 2W stands at B, and also when he stands at the middle point of AC. Find the positions of C and D.

Also find the position of the point where the boy should stand so that the pressures on the two supports may be equal.

(*Ans.* AC=3 ft., DB=2 ft.; $\frac{3}{4}$ ft. from the centre)

7. A straight lever AB of length l and weight W, balances about a fulcrum F when weights P and Q are suspended from A and B respectively. If P is increased and Q decreased by the same amount w, show that the fulcrum must be moved through a distance equal to $\frac{wl}{P+Q+W}$.

8. A uniform beam AB 18 ft. long and weighing 59 lbs. is freely supported in a horizontal position at points 3 ft. and 17 ft, from A. It carries loads weighing 20 lbs., 17 lbs. and 30 lbs. at points distant 6 ft., 7 ft. and 14 ft, respectively from A. Find the pressure on each support.

A uniform beam bar 4 ft. long and weighing 70 lbs. is placed on the beam. Prove that the pressures on the points of support will become equal if one end of the bar is over the middle point of the beam. (*Ans.* 68 lbs., 58 lbs.)

9. A uniform horizontal bar AB, 12 ft. long, weighing 1 lb. per ft. is supported at two points C and D. It carries load of 4 lbs. and 11 lbs. at E, and F respectively, where AE=5 ft. and AF=10 ft. If AC=4 ft., find the smallest possible value of CD.

If AC, AD are respectively 4 ft. and 8 ft., what length of the bar may be cut off from the end A without disturbing the equilibrium ?

(*Ans.* $3\frac{1}{2}\frac{3}{7}$ ft., 2 ft.)

10. ABCD is a straight rigid rod in which AB is uniform and of length a and weight w per unit length, BC is uniform and of length b and weight $2w$ per unit length, CD is uniform and of length c and weight $3w$ per unit length. If the rod rests horizontally in equilibrium on two pegs placed respectively at B and C, find the forces exerted on the pegs and show that

$$3(b+c)^2 > a^2+b^2 \quad \text{and} \quad (a+b)^2+b^2 > 3c^2. \qquad (H.C.)$$

11. Three parallel forces 3 lbs, 4 lbs. and 5 lbs. act at the angular points A, D, C of a square ABCD, each of whose side is 4 inches. Find the position of the centre of these forces.

What will be the position of this centre when there is a fourth parallel force of 2 lbs. acting at B ? (*P.U.*)

(*Ans.* At distances (*i*) $\frac{5}{3}$, 3 (*ii*) 2, $2\frac{4}{7}$ inches from AD and AB.)

12. Parallel forces P, P, P act at angular points of a right angled triangle ABC and parallel forces 2P, 2P, 2P act at the middle points of the sides. If the sides of the triangle including the right angle are 6 and 8 inches, find the distance of the centre of these forces from C, the right angle. (*Allahabad*)

(*Ans.* $3\frac{1}{3}$ in.)

13. Weights, 1, 5, 3, 4, 2, 6 lbs. wt. are placed at the angular points of a regular hexagon, taken in order ; prove that their resultant passes through the centre of the hexagon.

14. Parallel forces 1, 2, 3 lbs. wt. act at the corners A, C, E of a regular hexagon AECDEF, of side a, and forces 4, 5, 6 lbs. wt. act at the corners B, D, F in a direction parallel to the first three but in opposite sense. Find the distance of the centre of these forces from the centre of the hexagon.

(*Ans.* $a\sqrt{3}/9$)

15. Like parallel forces 6, 10, 14, 10 lbs. wt. act respectively at A, B, C, D, the vertices of a parallelogram. Show that the centre and resultant of these parallel forces remains the same, if instead of these forces, parallel forces 8, 12, 16 and 4 lbs. wt. act at the middle points of the sides AB, BC, CD and DA respectively.

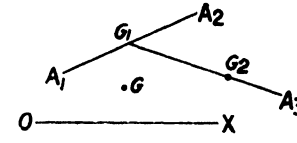
4.8. Moment of a force about a perpendicular line. The moment of a force F about a perpendicular line (in space) which does not meet it, is equal to the product of the force F and the perpendicular distance between the given line and the line of action of the force.

4.81. Theorem VII. *If a number of parallel forces (in space) have a resultant, the sum of their moments about a line perpendicular to them is equal to the moment of their resultant.*

Let the moments be taken about OX, a line perpendicular to the given forces P_1, P_2, P_3 . ...Draw a plane through OX perpendicular to the forces, meeting them in the points A_1, A_2, A_3 . .. and their resultant in G at distances y_1, y_2, y_3.....and $\bar{y}$ respectively from OX.

(The forces are taken perpendicular to the plane of the paper).

Let the resultant of P_1 and P_2 act at G_1 then since



$$P_1 . A_1G_1 = P_2 . G_1A_2,$$

or $$\frac{A_1G_1}{G_1A_2} = \frac{P_2}{P_1}.$$

$\therefore$ the distance of G_1 from $OX = \frac{P_1y_1 + P_2y_2}{P_1 + P_2}$.

(By Co-ordinate Geometry)

Again let the resultant of $P_1 + P_2$ at G_1 and P_3 at A_3 act at G_2, then $(P_1 + P_2) . G_1G_2 = P_3 . G_2A_3$.

$\therefore$ the distance of G_2 from OX

$$= \frac{(P_1 + P_2)\dfrac{P_1y_1 + P_2y_2}{P_1 + P_2} + P_3y_3}{P_1 + P_2 + P_3} = \frac{P_1y_1 + P_2y_2 + P_3y_3}{P_1 + P_2 + P_3}.$$

Proceeding in this manner, we get,

$$\bar{y} = \frac{P_1y_1 + P_2y_2 + P_3y_3 + \ldots\ldots}{P_1 + P_2 + P_3 + \ldots\ldots} \qquad \ldots(1)$$

i.e., $(P_1 + P_2 + P_3 + \ldots\ldots)\bar{y} = P_1y_1 + P_2y_2 + P_3y_3 + \ldots\ldots$

$\therefore$ the moment of the resultant about OX $\Big\}$ = sum of the moments of the forces $P_1, P_2, P_3 \ldots\ldots$etc., about OX.

Cor. *If a rigid body has an axis fixed, and is acted upon by parallel forces perpendicular to the axis, it is in equilibrium if the algebraic sum of the moments of the forces about the fixed axis vanishes.*

4.82. Position of the centre of non-coplanar parallel forces acting at given points on a plane.

Let the forces $P_1, P_2, P_3 \ldots\ldots$intersect at $A_1 (x_1, y_1)$, $A_2 (x_2, y_2)$, $A_3 (x_3, y_3) \ldots\ldots$the plane XOY ; OX, OY being the co-ordinate axes, and let their resultant meet the same plane in $(\bar{x}, \bar{y})$.

The forces being parallel they make the same angle with the plane. Since the position of the centre of the parallel forces does not depend upon their direction, we may suppose them to be perpendicular to the plane XOY.

Taking moments about OX, by Art. 4·81, we have

$$(P_1 + P_2 + P_3 + \ldots\ldots)\bar{y} = P_1y_1 + P_2\bar{y}_2 + P_3y_3 + \ldots\ldots$$

Similarly by taking moments about OY, we get

$$(P_1 + P_2 + P_3 + \ldots\ldots)\bar{x} = P_1x_1 + P_2x_2 + P_3x_3 + \ldots\ldots$$

Thus $\bar{x} = \frac{\Sigma Px}{\Sigma P}$, and $\bar{y} = \frac{\Sigma Py}{\Sigma P}$.

Note. These results are the same as obtained in Art. for 4 71, coplanar forces.

4.83. Example (i). *Like parallel forces P, Q, R act at the vertices of a triangle ABC perpendicular to its plane. Show that if the resultant passes through the incentre, the forces are proportional to the lengths of the opposite sides.*

Let P+Q+R, the resultant of the forces passing through the incentre be F. Taking moments about the line BC, we have

$$F.r = P \times \text{perp. from A on BC.}$$
$$= P.b \sin C. \quad \ldots(1)$$

Similarly by taking moments about CA, we have

$$F.r = Q.a \sin C. \quad \ldots(2)$$

$\therefore$ $P.b = Q.a$ *i.e.*, $P : Q = a : b$. Similarly $Q : R = b : c$.

Hence $P : Q : R = a : b : c$.

Example (ii). *Three parallel forces P, Q, R act at the vertices A, B, C of a triangle and are proportional respectively to a, b, c, the sides of the triangle. Show that their centre coincides with the incentre of the triangle.*

Since the position of the centre of parallel forces is independent of their direction, we suppose the forces to be perp. to the plane ABC.

Let P, Q, R be ka, kb, kc and let r, r' be the distances of their centre G from BC and CA, then equations (1) and (2) of Ex. (*i*) become

$Fr = ka.b \sin C$, and $Fr' = kb.a \sin C$, so that $r = r'$.

Thus G is equidistant from the sides and is therefore the incentre of the triangle.

Example (iii). *Three like parallel forces P, Q, R act at the vertices of a triangle ABC, if the circumcentre of the triangle is the centre of the forces, prove that*

$$P : Q : R = a \cos A : b \cos B : c \cos C$$
$$= a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2).$$

The centre of the parallel forces being independent of their directions, the forces may be rotated about their points of application so as to become perpendicular to the plane of the triangle.

The resultant force $P+Q+R=F$ acts at the circumcentre whose distance from the side BC is $R' \cos A$, R' being the circumradius of the triangle.

Also the distance of the force P from BC is the altitude of the triangle from A which is equal to $c \sin B$.

$\therefore$ taking moments about BC, we have

$$F.R' \cos A = P.c \sin B.$$

Similarly for moments about CA, we have

$$F.R' \cos B = Q.a \sin C = Q.c \sin A.$$

$\therefore$
$$\frac{\cos A}{\cos B} = \frac{P}{Q} \cdot \frac{\sin B}{\sin A},$$

or $$\frac{P}{Q}=\frac{\sin A\cos A}{\sin B\cos B}=\frac{a\cos A}{b\cos B}.$$

Similarly $$\frac{Q}{R}=\frac{b\cos B}{c\cos C}$$

$$\therefore \quad P:Q:R=a\cos A:b\cos B:c\cos C.$$

Again $$\frac{P}{Q}=\frac{a\cos A}{b\cos B}=\frac{a(b^2+c^2-a^2)}{2bc}\div\frac{b(c^2+a^2-b^2)}{2ac}$$

$$=\frac{a^2(b^2+c^2-a^2)}{2abc}\div\frac{b^2(c^2+a^2-b^2)}{2abc}$$

$$=\frac{a^2(b^2+c^2-a^2)}{b^2(c^2+a^2-b^2)}.$$

Hence the result.

Examples XVII A

1. Three like parallel forces P, Q, R act at the corners of a triangle ABC, prove that for all directions of the forces their resultant passes through the

(*i*) circumcentre if $P:Q:R=\sin 2A:\sin 2B:\sin 2C$.

(*ii*) incentre if $P:Q:R=\sin A:\sin B:\sin C$

(*iii*) orthocentre if $P:Q:R=\tan A:\tan B:\tan C$.

(*iv*) centroid if $P=Q=R$.

2. A light table stands on three equal vertical legs and a weight is placed at the centre of the circle inscribed in the triangle formed by the points of intersection of the legs with the table, show that the pressures on the legs are proportional to the lengths of the opposite sides of the triangle. (*P.U.*)

3. Three like parallel forces, each equal to P, act at the corners A, B and C of a square ABCD of sides $2a$, perp. to its plane. Find a force acting at D so that the system may reduce to a couple and find the moment of the couple. [*Ans.* 3P $4\sqrt{2}\,a$ P]

4. Like parallel forces P, Q, R act at the angular points A, B, C of a triangle ABC. Show that the perpendicular distance of their centre from BC, and its distance from BC measured parallel to AB are respectively equal to

$$\frac{P}{P+Q+R}\cdot\frac{2\triangle ABC}{BC} \text{ and } \frac{P.AB}{P+Q+R}.$$

5. Forces P, Q, R act from the angular points A, B, C of a triangle ABC perpendicular to the opposite sides. Prove that, if their resultant passes through the circumcentre of the triangle then

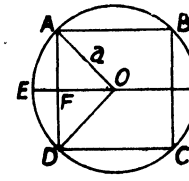
$$P(b\cos C-c\cos B)+Q(c\cos A-a\cos C)+R(a\cos B-b\cos A)=0.$$

6. A force P passing through a point O within a triangle ABC acts perpendicular to the plane of the triangle. If D, E, F are the points of intersection of AO, BO, CO with BC, CA, AB respectively, prove that P is equivalent to $P\frac{OD}{AD}$, $P\frac{OE}{BE}$, $P\frac{OF}{CF}$ passing through A, B, C perpendicular to the same plane.

4.84. Toppling and Sliding.

Example (i). *A circular table weighing W lb. has four legs spaced at equal distances round its edge. Find the least weight sufficient to overturn the table.*

Let the table be at the point of being overturned about the line joining the feet of legs at A and D, and let w be the required weight and a the radius of the table.



In order that the w may be the least, it must be placed at E, the point most distant from AD.

Taking moments about the line joining the feet of the legs at A and D, we have

(the distances of the weights at O and E from this line being the same as from the line AD),

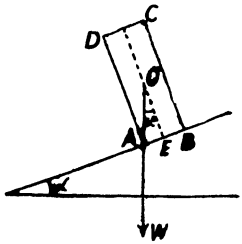
$$W.FO = w.EF = w(EO - FO),$$

i.e., $$Wa\cos 45° = w(a - a\cos 45°),$$

or $$W\frac{1}{\sqrt{2}} = w\left(1 - \frac{1}{\sqrt{2}}\right) \quad i.e., \; w = \frac{W}{\sqrt{2}-1} = W(\sqrt{2}+1).$$

Example (ii). *A uniform rectangular block of height h whose base is a square of side a rests on a rough horizontal plane. The plane is gradually tilted about a line parallel to two edges of the base. Show that the block will slide or topple according as a is greater or less than μh, where μ is the co-efficient of friction.*

Let ABCD be the vertical section of the block through its centre O, perpendicular to the specified edges. Let α be the inclination of the plane when the block is on the point of tumbling over.



Then since the vertical line through O, must pass through A and since $AB = a$ and $BC = h$, we have

$$\tan\alpha = \tan \angle AOE = \frac{AE}{OE} = \frac{AB}{BC} = \frac{a}{h} \quad \ldots(1)$$

Also β, the inclination of the plane, when the block is about to slide, is given by

$$\tan\beta = \mu. \quad \ldots(2)$$

Hence the block will slide before it topples,

if $\beta < \alpha$ *i.e.*, $\tan\beta < \tan\alpha$ *i.e.*, if $\mu < \frac{a}{h}$ or if $a > \mu h$.

Examples XVII B

1. A circular table whose weight is W lb. is provided with three vertical legs attached to three points in the circumference equidistant from one another; find the least weight which when hung from the edge of the table will just cause it to overturn. (*Ans.* W lb.)

2. A circular tray of radius a stands on a single circular foot of radius b. If P is the whole weight of the tray and its support, find how far from the centre a weight Q can be placed without the tray falling over. (*P.U.*)
[*Ans.* $b(1+P/Q)$]

3. A triangular lamina ABC, obtuse angled at C, rests in a vertical plane with the side AC in contact with a horizontal table. Find the greatest weight that can be suspended from B without disturbing the equilibrium.

The sides of the lamina are BC=a, CA=b, AB=c and its weight is W.

What would happen if $c^2>a^2+3b^2$? (*P.U.*)

[*Ans.* $W(3b^2+a^2-c^2)/3(c^2-a^2-b^2)$]

4. In question 3, if $\angle C=120°$, show that the triangle will not topple over without any weight at B if BC$<$2AC. In case BC$<$2AC, show that the least downward vertical force applied at B which will cause the triangle to topple over is

$$\frac{W}{3}\left(\frac{2AC}{BC}-1\right)$$

5. A uniform solid (*i*) cylinder, (*ii*) cone of radius r and height h stands on a rough plane whose inclination to the horizon is gradually increased. Show that the bodies will slide before they topple over if the co-efficient of friction is less than (*i*) $\frac{2r}{h}$. (*ii*) $\frac{4r}{h}$.

6. A square uniform metallic plate ABCD of weight W rests with its side BC on a perfectly rough plane inclined to the horizon at an angle α. A string AP attached to A, the highest point of the plate, and passing over a smooth pulley at P, the vertex of the plane, supports a weight w and AP is horizontal. Show that as w increases, the plate will begin to turn when

$$w>\tfrac{1}{2}W(1+\tan\alpha).$$

7. A cubical block of wood is placed on a rough horizontal table and a horizontal force is applied to the middle of one of the sides of the square top at right angles to that side. Prove that, as the force is increased the block will begin to slide if the co-efficient of friction is less than $\frac{1}{2}$ and that otherwise it will begin to tilt. (*P.U.*)

8. A heavy wheel of weight W and radius r, is to be dragged over an obstacle of height h by a horizontal pull P applied to the centre of the wheel ; show that P must be slightly greater than

$$\frac{W\sqrt{2rh-h^2}}{r-h}.$$

9. A triangular lamina ABC right angled at B, stands with BC on a rough inclined plane with B downwards. The inclination of the plane is gradually increased, show that the lamina will begin to slide or topple over according as the co-efficient of friction is less or greater than tan A.

10. The cross-section through the centre of a right prism is a regular hexagon ABCDEF. The prism rests with the face that contains AB on a rough horizontal table, and a gradually increasing thrust is applied at the middle point of CD perpendicular to the face that contains CD. Show that the prism will slide before it brgins to turn about A if the co-efficient of friction is less than $\frac{1}{\sqrt{3}}$

11. A triangular prism is placed with a rectangular face of breadth $2c$ on a rough horizontal plane, the co-efficient of friction being μ. The prism is a uniform right prism and its two inclined faces are equal rectangles whose common edge is at a height h above the horizontal face. If the plane is rotated slowly about a line which is parallel to the three parallel edges, prove that the prism will topple or slide according as $\mu \gtrless 3c/h$.

Miscellaneous Examples II

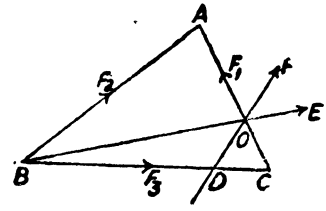
1. *A force F is replaced by three forces acting along the sides of a triangle ABC and its line of action meets BC internally at D, show that the ratio of the magnitude of the forces along AB and AC is*

CD.AB : BD.AC.

The force F can be resolved first along CA and BO and the component along BO can further be resolved along BA and BC (Ex. 10 Mis. Ex. I. P. 49).

Let F_1, F_2 be the magnitude of the forces along AC and AB acting in the directions shown in the figure. If p, q be the lengths of the perpendiculars drawn from D, on AC and AB, then

$$p = CD \sin C \text{ and } q = BD \sin B.$$



Taking moments about D, we have

$$F_1.p = F_2.q \text{ } i.e., \text{ } F_1.\ CD \sin C = F_2.BD \sin B.$$

$$\therefore \quad \frac{F_2}{F_1} = \frac{CD \sin C}{BD \sin B} = \frac{CD}{BD} \cdot \frac{AB}{AC} = \frac{CD.AB}{BD.AC}.$$

2. Two forces P and Q act at a point O and their resultant is R. Any straight line meets the lines of action of P, Q and R at A, B and C respectively. Show that

$$\frac{P}{OA.BC} = \frac{Q}{OB.CA} = \frac{R}{OC.AB}.$$

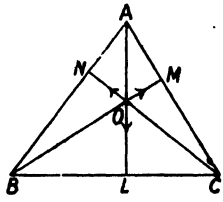
3. Four forces acting along the sides of a quadrilateral are in equilibrium prove that the quadrilateral is plane. If also the quadrilateral can be inscribed in a circle, prove that each force is proportional to the opposite side. (*P.U.*)

4. A hollow vertical cylinder, radius $2a$, height $3a$, rests upon a horizontal table ; a rod is placed within it with its lower end at the circumference of the base, the rod rests upon the opposite point of the upper rim and projects over. How long must the rod be in order that it may cause the cylinder to topple over, the rod and the cylinder being of equal weight ? (*Ans.* $15a$)

5. *Forces P, Q, R acting along the altitudes from A, B, C of the triangle ABC are equivalent to forces X, Y, Z, acting along BC, CA, AB, prove that*

$$\frac{X}{\cos A} = \frac{Q}{\sin B} = \frac{R}{\sin C}$$

and two similar equations.



The two systems being equivalent, the moment of the two must be the same about any point in the plane of the triangle.

Taking moments about A, through which P, Y, Z pass, we have

$$X.AL = Q.AM - R.AN$$

or $$X.b \sin C = Q.c \cos A - R.b \cos A,$$

i.e., $$\frac{X}{\cos A} = \frac{Qc}{b \sin C} - \frac{R}{\sin C} = \frac{Q}{\sin B} - \frac{R}{\sin C}.$$

Two other equations are obtained by taking moments about B and C.

6. A system of forces P, Q, R, acting along the sides of the triangle ABC is equivalent to a system X, Y, Z along the sides of the pedal triangle. Prove that

$$2X = \frac{Q}{\cos B} + \frac{R}{\cos C}$$

and two other similar results.

7. A triangular lamina of weight W is supported by three vertical strings attached to its angular points so that the plane of the triangle is horizontal ; a particle of weight W is placed at the orthocentre of the triangle. Prove that the tensions of the string are given by

$$\frac{T_1}{1+3 \cot B \cot C} = \frac{T_2}{1+3 \cot C \cot A} = \frac{T_3}{1+3 \cot A \cot B} = \frac{W}{3}.$$

8. From four points A, B, C, D at the corners of a square in a horizontal ceiling, hang four vertical strings AP, BQ, CR, DS which support a uniform square plate PQRS of weight W, hanging in equilibrium. If λ is the modulus of each of the string AP, CR, and μ that of each of the strings BQ, DS, and if the four strings all have the same natural length l find the tension in each of them.

[*Ans.* $\frac{\lambda W}{2(\lambda+\mu)}$ in AP, CR ; $\frac{\mu W}{2(\lambda+\mu)}$ in BQ, DS]

9. A body of weight W is supported by two vertical strings attached to points A and B in the body ; the distance AB is a, and A is lower than B. When the inclination of AB to the horizon is α then the tension in the string at A is P ; and when the inclination of AB to the horizon is β, the tension in the string at A is Q.

Prove that the perpendicular distance of the centre of gravity of the body from the line AB is

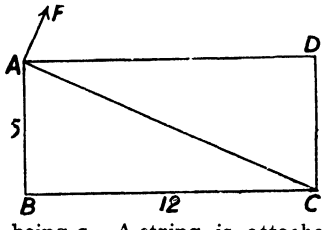
$$(P-Q)a/W(\tan\beta-\tan\alpha). \qquad (P.U.)$$

10. AECDE is a regular pentagon. Forces 3, 24, 24 and 16 l. s. wt. act along the sides AB, AC, AD, EA respectively ; show that their resultant is 41 lbs. wt.

11. A heavy cube of weight 100 lbs. rests on a rough floor on which it cannot slip. Prove that the least force required just to raise one edge of it off the floor is about 35⅓ lbs., and find where it must be applied.

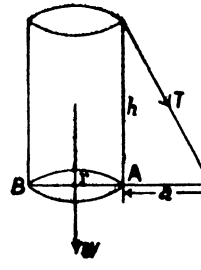
(*Ans.* At the mid-point of the edge vertically above the one raised)

12. ABCD is the vertical section (through the centre) of a rectangular block resting on a rough horizontal plane. The end AB is slightly raised by a force F applied at A at right angles to AC. If AB=5 ft. and BC=12 ft., show that the edge C will slip if the coefficient of friction is less than $\frac{5}{12}$.



13. A cylinder of radius r and height h, is placed with its axis vertical on a rough horizontal table, the shortest distance between the base of the cylinder and the edge of the table being a. A string is attached to the top of the cylinder and passes over the edge of the table with a weight hanging freely tied to its lower end. If this weight is gradually increased, prove that the cylinder starts to slip before it topples over, if the coefficient of friction is less than

$$\frac{ra}{h(r+a)}.$$



14. A homogeneous solid cube stands on a rough horizontal plane, the co-efficient of limiting statical friction between the cube and the plane being μ. A horizontal force P is applied at the centre of, and perpendicular to, one of the vertical faces of the cube.

(*i*) If P is so small that the equilibrium is not disturbed, prove that the resultant normal reaction of the plane on the cube cannot pass through the centre of the cube.

(*ii*) If $\mu>1$, prove that as P is gradually increased the equilibrium will be broken before the friction becomes limiting,

15. A uniform bar AB of weight W lies horizontally upon two equally rough fixed horizontal bars at right angles to it, at A and at a point C between A and B, where AB=2a, AC=b and $b>a$. Find the reactions at A and C.

If a horizontal force P is applied to the bar at B at right angles to BA, find expressions for P when the bar is just about to slip (*i*) at A (*ii*) at C.

Hence show that if P were gradually increased from zero, the bar will slip at A or at C first, according as b is less than or greater than $\frac{1}{3}a$.

[*Ans.* $W\frac{b-a}{b}$, $W\frac{a}{b}$; $\mu W\frac{b-a}{2a-b}$; $\frac{1}{2}\mu W$]

16. A uniform heavy rod AB (of weight W) is lying on a rough table. A force applied at the end A is gradually increased until the rod is just about to move in such a way that the initial motion will be rotation about a point C in AB. Prove that the force at A must be applied at right angles to AB and that the point C must divide AB so that AC : CB $=1+\sqrt{2}:1$. (*P.U. 1960*)

Show also that in the limiting position the magnitude of the force at A is $(\sqrt{2}-1)\mu W$.

REVISION QUESTIONS II

1. Two unlike parallel forces P and Q act at the points A and B. If their resultant R acts at C, a point in AB, prove that

(*i*) R is parallel to P or Q,

and (*ii*) $\frac{P}{BC}=\frac{Q}{AC}=\frac{R}{AB}$. (*D.U.*)

2. A horizontal beam APQB rests on two supports P and Q, where

$$AP=PQ=QB$$

The beam just tilts when a weight of P lb. is hung from A or when a weight of Q lb. is hung from B. Show that the weight of the beam is P+Q lb. and prove that its C.G. divides AB in the ratio

$$2P+Q : P+2Q.$$

3. A rod AB of weight W and length $p+q$ has its C.G. at a distance p from A. It rests on two supports at a distance r apart in the same horizontal level, such that equal portions of the rod project beyond each support. Prove that the pressures on the supports are

$$\frac{p-q+r}{2r}W \text{ and } \frac{q-p+r}{2r}W.$$

4. Parallel forces $P_1, P_2, \ldots\ldots P_n$ act in a plane at points $A_1, A_2, \ldots\ldots A_n$ whose co-ordinates referred to rectangular axes are (x_1, y_1), $(x_2, y_2), \ldots\ldots (x_n, y_n)$. Find the co-ordinates of their centre.

Three parallel forces P, Q, R act at the vertices A, B, C of a triangle and are proportional respectively to a, b, c the sides of the triangle. Show that their centre coincides with the incentre of the triangle. (*P.U.*)

5. A body of weight W is hung by two vertical strings from two points A and B in it. When the tension in the string at A is increased by an amount T, the inclination of AB to the vertical changes from θ to φ. Show that the distance of the centre of gravity of the body from AB is

$$\frac{T}{W}\cdot\frac{AB}{\cot\theta-\cot\varphi}.$$ (*All. U.*)

6. A uniform circular disc of weight nW has a particle of weight W, attached to a point on its rim. If the disc be suspended from a point A on its rim, B is the lowest point; also if suspended from B, A is the lowest point. Show that the angle subtended by AB at the centre of the disc is

$$2\sec^{-1}2(n+1).$$ (*P.U.*)

7. Prove that the sum of the moments of a number of parallel forces about a line perpendicular to them is equal to the moment of their resultant.

A heavy horizontal circular ring of weight W rests on three supports at the points A, B and C of its circumference. Find the reactions of the supports in terms of the angles of the triangle. [*Ans.* Pressure at A $=W\cos A/(2\sin B\sin C)$]

8. Two children each of weight W, are swinging on a see saw formed by placing a plank of weight w across a horizontal cylinder of radius a. When the plank has been turned through an angle θ with the horizon, show that the moment of the forces tending to turn it back is

$$(2W+w)a\,\theta\cos\theta.$$

9. A rough circular cylinder of radius a, is fixed with its axis horizontal and a uniform rod of mass M and length $2b$ rests on it in a horizontal position perpendicular to the axis. If now a particle of mass m be attached to the rod at one end, show that the rod may be allowed to turn slowly, without slipping into a position of rest inclined to the horizontal but still perpendicular to the axis of the cylinder provided that the angle of friction between the rod and the cylinder exceeds $mb/(M+m)a$.

10. A uniform plank, of length $2a$ and weight W, rests with its middle point upon a rough horizontal cylinder whose axis is perpendicular to the plank. Show that the greatest weight that can be attached to one end of the plank without sliding off the cylinder is

$$\frac{b\lambda}{a-b\lambda}\,W;\ \lambda<\frac{a}{b},$$

where b is the radius of the cylinder and λ the angle of friction.

(*D.U. 1957* ; *P.U. 1957 S.*)

11. Given the moments M_1, M_2 of a force R about two given points A and B show that the line of action of the force passes through a fixed point C on the line AB.

Find, geometrically, the line of action of the force.

Find the locus of all the points in a plane such that two concurrent forces P, Q shall have equal moment (in the same sense) about any one of these points

(*i*) when P and Q are equal,
(*ii*) when they are unequal.

13. Forces P, Q, R act along the sides BC, CA and AB of a triangle ABC. Prove that the magnitude S of the resultant is given by

$$S^2=P^2+Q^2+R^2-2PQ\cos C-2QR\cos A-2RP\cos B.$$

14. Two systems of forces P, Q, R and P′, Q′, R′ act along the sides BC, CA and AB of a triangle ABC. Prove that their resultants will be parallel if

$$(QR'-Q'R)\sin A+(RP'-R'P)\sin B+(PQ'-P'Q)\sin C=0.$$

(*Luck. U.*)

15. Three forces P, Q, R act along the sides BC, CA, AB of a triangle ABC taken in order. Prove that if their resultants always touche

(*a*) the inscribed circle, then

$$\frac{\cos^2 A/2}{P}+\frac{\cos^2 B/2}{Q}+\frac{\cos^2 C/2}{R}=0,$$

(*b*) the circumcircle, then $\sqrt{aP}+\sqrt{bQ}+\sqrt{cR}=0$.

CHAPTER V

COUPLES, COPLANAR FORCES

Couples

5.1. A system of two equal, unlike parallel forces is called a **couple.**

The distance between the parallel forces forming a couple is called the **arm** of the couple.

A couple each of whose force is P and the length of whose arm is p, will be denoted as (P, p).

5.2. Moment of a couple. *The algebraic sum of the moments of the forces forming a couple about any point in its plane is constant.*

Through any point O in the plane of the forces draw a straight line perpendicular to the lines of action of the forces meeting them in A and B, then

Sum of the moments of the forces about O—

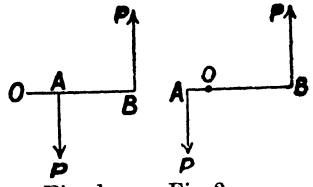


Fig. 1. Fig 2.

Fig. 1	Fig. 2
$=P.OB-P.OA$	$=P.OB+P.AO$
$=P(OB-OA)$	$=P(AO+OB)$
$=P.AB$	$=P.AB.$

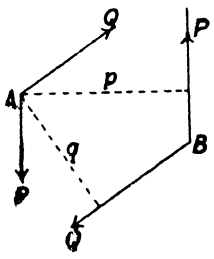
which is constant, being independent of the position of O, and is known as the *moment of the couple.*

Cor. The moment of a couple is equal to the moment of one of its forces about any point on the line of action of the other.

5.3. Equilibrium of two couples. *Two couples in the same plane of equal and opposite moments, balance.*

Case. 1. *When the forces forming the couples are not all parallel.*

Let one of the forces P of one of the couples intersect one of the forces Q of the other in A and let the other two forces meet in B, also let p, q be the arms of the couples.



Since the moments of the couples are equal and opposite.

$$\therefore \quad P.p-Q.q=0 \qquad \ldots(1)$$

Now, the sum of the moments about A of P and Q acting at B $=P.p-Q.q.$

$=0$ [by (1)]

Hence the resultant of P and Q at B passes through A, and therefore acts along BA.

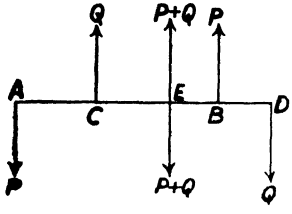
Similarly the resultant of P and Q at A acts along AB.

But since forces P and Q at B are equal and opposite respectively to P and Q at A and these pairs act at the same angle, their resultants must be equal in magnitude and opposite in sense, and acting as they do along AB and BA, they cancel each other.

Hence the resultant of all the four forces forming the couples is nil and the couples balance.

Case II. *When the forces forming the couples are all parallel.*

Draw a straight line perpendicular to the lines of action of the forces meeting them at A, B, C and D.



Let (P, AB) and (Q, CD) be the couples of equal and opposite moments,

then $P.AB - Q.CD = 0$ ...(1)

Let the resultants of P at B and Q at C act at E,

then $P.EB = Q.CE$ ...(2).

From (1) and (2), we have

$$P(AB - EB) = Q(CD - CE),$$

i.e., $$P.AE = Q.DE.$$

So that the resultants of P at A and Q at D also act at E.

$\therefore$ the resultants of P at B and Q at C is P + Q acting at E in a direction parallel to that of P at B (or Q at C) and also the resultants of P at A and Q at D is P + Q acting again at E in a direction parallel to that of P at A (or Q at D).

These two forces being equal, opposite and collinear are in equilibrium and hence the two couples balance.

5·31. *A simpler proof of the proposition of Art. 5·3 is as follows* :

The moments of the couples being equal and opposite, it follows that the sum of the moments of the four forces of the couples about *any* point in their plane must vanish, so that their resultant cannot be a couple.

The resultant cannot also be a force, for if it were so, the sum of the moments of the forces about every point being zero, the line of action of the resultant force has to pass through every point in the plane which is obviously impossible.

Thus the forces forming the two couples are neither equivalent to a single force nor to a couple and hence by Art. 4.61, they are in equilibrium.

Hence the two couples balance.

5·32. Equivalence of two couples. *Any two coplanar couples of equal moments in the same sense, are equivalent to one another.*

Since two couples in the same plane of equal and opposite moments balance, we have by reversing the directions of the forces composing one of the couples, that two couples in the same plane of equal moments in the same sense, are equal.

It follows that *the moment is the only quantity of importance about a couple*, neither position, magnitude or direction of the forces, nor the arm of the couple being separately of importance.

Thus the forces of a couple can be moved to any part of its plane, turned round through any angle, and altered in magnitude to any extent, provided only that they are kept equal and parallel, so as still to form a couple, and that the arm is correspondingly altered to keep the moment unchanged.

Example. *Forces forming a couple are 8 lbs. each and the arm of the couple is 3 ft.* (*i*) *Find the force of an equivalent couple whose arm is 5 ft.* (*ii*) *Find the arm of an equivalent couple each of whose force is 10 lbs.*

(*i*) Let X be the required force then since the moments of the couples are equal, we have

$$X \times 5 = 8 \times 3, \text{ i.e. } X = \frac{8 \times 3}{5} = 4\frac{4}{5} \text{ lbs.}$$

(*ii*) Let x ft. be the arm of the couple whose force is 10 lbs., then

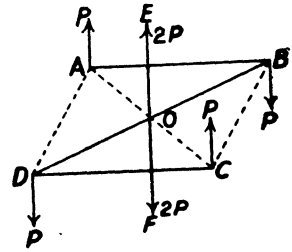
$$10 \times x = 8 \times 3 \quad \therefore \quad x = \frac{24}{10} = 2\frac{2}{5} \text{ ft.}$$

5.4. Couples in Parallel Planes. *Two couples in parallel planes, of equal and opposite moments, balance.*

Let one of the couples consist of forces P and arm AB, situated in a plane, say p_1.

Let the other couple of equal and opposite moment be situated in a parallel plane, say p_2.

Alter the forces of this second couple so as to make them equal and parallel to those of the first, its arm CD being *thus* equal and parallel to AB.



Then ABCD is a parallelogram (in a plane different from the planes p_1 and p_2) and the diagonals AC and BD bisect each other in O.

Draw EOF in the direction parallel to the direction of the forces.

Now the force, P at A and P at C are equivalent to 2P at O along OE, and the forces P at B and D are equivalent to 2P at O in the opposite direction OF. Being equal and opposite and collinear these forces balance and hence the couples balance, so that two couples of equal and opposite moments in parallel planes are in equilibrium.

By reversing the directions of the forces of one of the two couples, it is easy to see that *two couples of equal moments in the same sense in parallel planes are equivalent.*

It follows that *a couple may be replaced by another couple acting in a parallel plane provided its moment remains the same.*

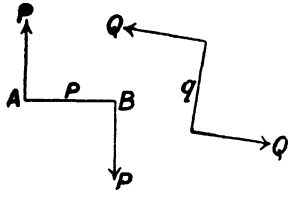
5.5. Composition of Coplanar Couples.

Theorem VIII. *A number of coplanar couples compound into a single couple whose moment is equal to the algebraic sum of the moments of the various couples.*

Let (P, p), (Q, q), (R, r),.....be a number of couples acting in the same plane.

The moment of the couple $(Q, q) = Q.q = \frac{Qq}{p} \cdot p,$

so that the couple (Q, q) may be replaced by another couple whose arm coincides with AB and each force of which is of magnitude



$$\frac{Qq}{p}.$$

Similarly the couple (R, r) may be replaced by another couple whose arm coincides with AB and each force of which is of magnitude

$$\frac{Rr}{p}$$

Treating all the couples in this manner we get a single couple with arm AB, each force of which

$$= P + \frac{Qq}{p} + \frac{Rr}{p} + \ldots\ldots\ldots$$

the terms in this expression being taken with their proper signs.

Thus the given system of couples is equivalent to a single couple whose moment

$$= \left(P + \frac{Qq}{p} + \frac{Rr}{p} + \ldots\ldots \right) p = P.p + Q.q + R.r + \ldots\ldots$$

$=$ the algebraic sum of the moments of the various couples.

Cor. Since the moment of a couple is the same as the sum of the moments of its forces about any point in its plane, it follows that the moment of the resultant couple is the same as the sum of the moments of all the forces of the various couples about any point in their plane.

Note. Obviously by Art. 5.4 this theorem holds true even when the couples are in parallel planes, for all the couples can be transferred to one plane.

5.51. Since the moment of a couple is constant and is equal to the sum of the moments of its forces about a point, it follows that *the moment of the resultant couple can be obtained by finding the algebraic sum of the moments of all the forces forming the couples about any point in their plane.*

Example 1. *ABCD is a rectangle with AB=4 ft., BC=3 ft.*

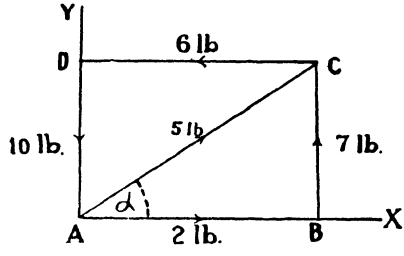
along AB, BC, CD, DA and AC act forces 2, 7, 6, 10, and 5 lbs. wt. respectively.

Show that they are equivalent to a couple whose moment is equal to 46 lb. ft.

Since AB=4 ft. and BC=3 ft. $\therefore$ AC=5 ft.,

$$\cos \angle BAC = \cos \alpha = \frac{AB}{AC} = \frac{4}{5},$$

and $$\sin \alpha = \frac{BC}{AC} = \frac{3}{5}.$$



Now the component along AB, of the force of 5 lbs. wt. acting along AC

$$= 5 \cos \alpha = 5 \times \tfrac{4}{5} = 4 \text{ lbs.}$$

Thus the resultant force along AB=2+4=6 lbs. forms a couple with the 6 lbs. parallel force along CD, the moment of the couple being

$$= 6 \times BC = 6 \times 3 = 18.$$

Again the component along AD of the force of 5 lbs. wt.

$$= 5 \sin \alpha = 5 \times \tfrac{3}{5} = 3 \text{ lb.}$$

$\therefore$ the resultant force along DA=10−3=7 lbs. forms a couple with 7 lbs. force along BC, of moment=7×AB=7×4=28.

Hence the resultant of the given forces which is the resultant of these two couples is a couple of moment=18+28=46 lb. ft.

Aliter. Taking AB as the X-axis and AD as the Y-axis, we have

X=the sum of the resolved parts of all the forces along AB

$$= 2 + 5 \cos \alpha - 6 = 2 + 4 - 6 = 0.$$

Y=the sum of the resolved parts of the forces along AD

$$= 7 + 5 \sin \alpha - 10 = 7 + 3 - 10 = 0.$$

Thus the resultant of the given forces cannot be a force and hence the resultant must be a couple whose moment is equal to the sum of the moments of the forces about any point in the plane ABCD. Taking moments about A, we have the moment of the couple

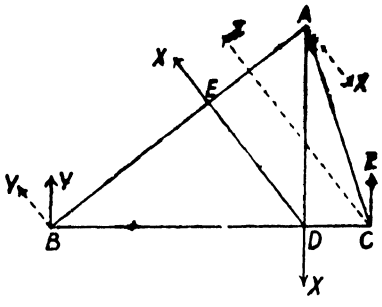
$$= 7 \times AB + 6 \times BC$$

$$= 7 \times 4 + 6 \times 3 = 46 \text{ lb. ft.}$$

the moment of the other forces being zero as they pass through A.

Example 2. *Forces X, Y, Z acting at the vertices of a triangle ABC, each perpendicular to BC are in equilibrium. If each is turned about its point of application in the same sense until they are all at right angles to AB, show that they form a couple and find the moment of the couple.*

The forces being perp. to BC are parallel and as they are in equilibrium. the resultant of two of them, say, Y and Z is equal and opposite to the third X, acting at D, the foot of the perp. from A on BC, *i.e.*, it acts along DA.



Now the forces are rotated about their points of application in the same sense (anti-clockwise) so as to become perp. to AB.

In this position the forces are shown by dotted lines and we have to find their resultant.

The resultant of Y and Z in their new position is still equal to X in magnitude and parallel to them in direction *i.e.*, it is now perp. to AB.

The point of application of parallel forces being independent of their direction, this resultant again acts at D, so that it acts along DE perp. to AB.

This force X acting along DE forms a couple with the new position of X acting along the dotted line AX.

Thus in their new position the forces form a couple.

The moment of the couple $= X.AE = X.AD \cos \angle DAE$
$= X.AD \cos (90° - B) = X.AD \sin B = X.c \sin B \sin B = X.c \sin^2 B$.

Cor. If the forces be turned through an angle α, in the same sense, they will be equivalent to a couple of moment $X.c \sin \alpha \sin B$.

Examples XVIII

1. ABCD is a square of side 3 ft. ; along AB, BC, CD, DA, AC and DB act forces equal to 1, 2, 8, 5, $5\sqrt{2}$ and $2\sqrt{2}$ lbs. wt. respectively.

Show that they are equivalent to a couple whose moment is equal to 24 lb. ft.

2. Forces 1, 2, 3, 5, P, Q act along AB, BC, CD, DA, AC, BD respectively and ABCD is a square of side a. Find the values of P and Q for the system to reduce to a couple. Find also the moment of the couple.

(*Ans.* $P = \frac{5}{2}\sqrt{2}$, $Q = \frac{1}{2}\sqrt{2}$; $5\frac{1}{2}a$) (*P.U. 1957, ; D.U. 1954*)

3. ABCD is a rectangle such that AB = 4 ft. and BC = 2 ft. Along AC, BD, DC, AD, act forces $15\sqrt{5}$, $20\sqrt{5}$, P and Q lbs. respectively. Determine P and Q so that the system may reduce to a couple and find its moment.

(*Ans.* P = 10, Q = −35 lb. ; 60 lb. ft.)

4. ABCD is a rectangle such that $AB = CD = a$ and $BC = DA = b$. Forces P act along AD and CB and forces Q act along AB and CD. Prove that the perpendicular distance between the resultant of the forces P, Q at A and the resultant of the forces P, Q at C is

$$\frac{Pa - Qb}{\sqrt{P^2 + Q^2}}$$ (*P.U.*)

5. ABCD is a framework of four rods in the form of a rhombus of sides l and angle α. A square CEFD of rods is constructed on CD so as to lie

on that side of CD which is opposite to AB. In each of the rods AB, FE, DA, BC, DF, EC acts a force P and in the rod CD acts a force 2P, the direction of forces being indicated by the order of the letters. Show that the system reduces to a single couple of moment.

$$2lP(1-\sin\alpha).$$

6. A square metal plate ABCD of side 2 ft., lies on a smooth horizontal plane and is free to turn about a point at the corner A. The plate is acted upon by forces of 10 lbs. wt. along AB, 15 lbs. wt. along BC and $10\sqrt{2}$ lbs. wt. along DB and by a couple so adjusted that the plate is in equilibrium. Calculate moment of the couple. (*Ans.* 10 lb. ft.)

7. ABCD is a square whose side is 2 units in length. Forces a, b, c, d act along the sides AB, BC, CD, DA taken in order, and forces, $p\sqrt{2}$, $q\sqrt{2}$ act along AC and DB respectively. Show that if

$$p+q=c-a \text{ and } p-q=d-b,$$

the forces are equivalent to a couple of moment $a+b+c+d$.
(*D. U. 1957* ; *P.U. 1954*)

8. ABC is an equilateral triangle of side a. D, E, F divide BC, CA, AB respectively in the ratio 2 : 1. Three forces each equal to P act at D, E, F perpendicular to the sides and directed away from the triangle. Prove that they are equivalent to a couple of moment $\frac{1}{2}$ Pa. (*P.U.*)

Show also that if the ratio is 3 : 1, the moment of the couple is $\frac{3}{4}$ Pa.

9. A straight uniform rod AB of length 12 ft. and of weight 10 lbs. has its centre of gravity at C and is inclined at 45° to the horizontal. It is acted upon by an upward vertical force of 20 lbs. wt. at the lower end A, by a downward vertical force of 10 lbs. wt. at the higher end B, and by such horizontal forces at A and C that the rod is in equilibrium. Prove that the forces at A and C form a couple and find its moment. Find also forces at A and C.
(*Ans.* $90\sqrt{2}$ lb. ft. ; 30 lbs.)

10. ABC is a triangle whose sides BC, CA, AB are 25, 15 and 20 inches respectively. A force of 25 lbs. wt. acts along the bisector of the angle BAC ; parallel forces through B and C are in equilibrium with this. If the forces are now turned about A, B and C till they are at right angles to BC, prove that they form a couple of moment $3\frac{1}{7}$ lb. ft.

11. Three forces ka, kb, kc act along AO, BO, CO respectively, where O is the orthocentre of the triangle ABC. If they are rotated through the same angle θ bout A, B, C, respectively, show that they become equivalent to a couple whose moment is $4k\triangle \sin\theta$, where $\triangle$ is the area of the triangle.

12. The algebraic sum of the moments of two given coplanar forces about *any* point in their plane is constant. Show that the given forces form a couple.

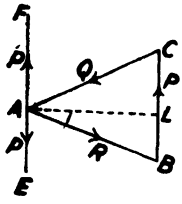
13. Two equal unlike parallel forces, acting at fixed points A and B, form a couple of moment G ; if their lines of action are turned through a right angle, they form a couple of moment H. Show that when they both act at right angles to AB, they form a couple of moment $\sqrt{G^2+H^2}$.

5.6. Theorem IX. *Three forces acting on a rigid body represented in magnitude, direction and line of action, by the sides of a triangle taken in order, in the same sense, are equivalent to a couple.*

Let the forces P, Q, R acting on a rigid body in the direction of the sides BC, CA and AB of the triangle ABC, be represented by these sides.

Draw EAF parallel to BC and AL perp. to it.

At A introduce two equal and opposite forces, each equal to P.



The forces P, Q, R *acting at the point* A along AF, CA and AB are in equilibrium by the triangle of forces.

We are thus left with two unlike parallel forces each equal to P, one acting along BC and the other acting along AE. They evidently form a couple.

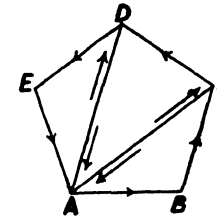
The moment of the couple = P.AL = BC.AL
= twice the area of the $\triangle$ ABC.

Ex. Show that the resultant of a system of three forces acting along the sides of a triangle is either a couple or a single force.

5.61. *A number of forces acting on a rigid body, represented in magnitude, direction and line of action by the sides of a plane polygon, taken in order in the same sense, are equivalent to a couple whose moment is represented by twice the area of the polygon.*

This follows at once from the above theorem.

Divide the polygon into triangles by joining one corner to each of the other, to which it is not already joined. Introduce equal and opposite forces represented completely by the lines so drawn. Thus we obtain a number of sets of three forces acting along the sides of a triangle and represented by them. Applying the above theorem to each of these sets, we obtain a number of coplanar couples which by Art. 5·5 are together equivalent to a single couple whose moment is equivalent to twice the sum of the areas of the triangles *i.e.*, twice the area of the polygon.



For example, let ABCDE be a polygon and let the forces be completely presented by its sides AB, BC, CD, DE and EA.

Join A to C and D. Introduce two equal and opposite forces along AC and AD, respectively represented by them.

Forces represented by (*i*) AB, BC, CA (*ii*) AC, CD, DA (*iii*) AD, DE, EA separately form couples of moments equal to twice the area of the triangles ABC, ACD and ADE.

These are equivalent to a single couple of moment

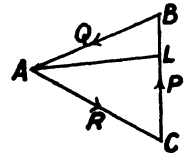
$= 2\triangle ABC + 2\triangle ACD + 2\triangle ADE$ = Twice are a ABCDE.

Note. The effect of such a system of forces acting upon a *rigid body* is a simple *rotation* and no translation (the sum of the resolved parts of the forces *i.e.*, of the sides of the polygon vanish in all directions).

If, however, the forces act on a *particle* and not on a rigid body, the effect of the forces is nil, (by the theorem of polygon of forces).

5.62. *If three coplanar forces acting on a rigid body, no two of which are parallel, are equivalent to a couple, they can be represented by the sides of the triangle formed by their lines of action.*

Let P, Q, R be the forces and ABC the triangle formed by their lines of action. Draw AL perp. to BC.



Let G be the moment of the couple and Δ the area of the triangle ABC, then since the moment of a couple can be measured in terms of the area of a triangle, we can suppose $G=2\Delta$, on some scale.

Sum of the moments of P, Q, R about A,

$$=P.AL=G=2\Delta=BC.AL \text{ } i.e., \text{ } P=BC.$$

Similarly $Q=CA$ and $R=AB$.

Thus the forces are represented by the sides of the ΔABC.

Example. *X, Y, Z are points in the sides BC, CA, AB of the* ΔABC *such that* $\frac{BX}{XC}=\frac{CY}{YA}=\frac{AZ}{ZB}=-\frac{\mu}{\lambda}$. *Prove that the system of forces represented by AX, BY, CZ is equivalent to a couple of moment* $\frac{\lambda-\mu}{\lambda+\mu}.2\Delta$, *where* Δ *is the area of the triangle.*

Proceeding as in Ex. (*iii*) Art. 3.1 (P. 31), we have

the system of forces represented by AX, BY and CZ

$$\equiv\frac{\lambda-\mu}{\lambda+\mu}(BC+CA+AB)$$

$$=\frac{\lambda-\mu}{\lambda+\mu}\text{ [resultant of forces represented by BC, CA and AB].} \qquad \ldots(1)$$

$$=\frac{\lambda-\mu}{\lambda+\mu}\times\text{a couple whose moment is } 2\Delta. \qquad \text{(Art. 5·6)}$$

$$=\text{a couple of moment } \frac{\lambda-\mu}{\lambda+\mu}.2\Delta.$$

Note. If the forces represented by AX, BY and CZ act at a point, then they are in equilibrium, for in that case the resultant of the forces represented by BC, CA and AB is nil.

Examples XIX

1. Four forces are completely represented by the sides BA, AD, CB, DC of a quadrilateral ABCD. Show that they are equivalent to a couple consisting of two forces through A and C, each equal and parallel to the diagonal BD.

Show also that the moment of the couple is represented by four times the area of the figure formed by joining the middle points of the sides.

2. ABCDEF is a regular hexagon. If three forces are represented in all respects by AC, CE, EA, show that they are equivalent to a couple of moment equal to the area of the hexagon. (*P.U.*)

3. ABC is triangle and forces represented l.BC, m.CA, n. AB act along the sides BC, CA, AB respectively. Show that if $l=m=n$, the system is equivalent to a couple, and if $l+m+n=0$ the resultant passes through the centroid of the triangle.

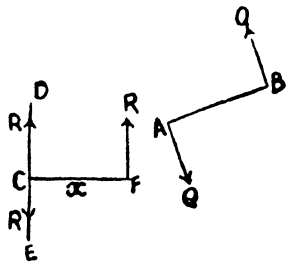
4. Points P, Q, R are taken on the sides BC, CA, AB respectively of the triangle ABC, dividing each in the ratio of $1+\lambda : 1-\lambda$ in the same sense round the triangle. Prove that the forces represented by AP, BQ, CR are equivalent to a couple. Find the moment of the couple. [*Ans.* $2\lambda\triangle$]

5. OABC is a tetrahedron. Show that the couples represented by the areas OBC, OCA, OAB can have a couple represented by the area ABC as resultant. (*P.U. 1953*)

6. Three forces, proportional to the sides of a triangle, act perpendicular to these sides away from the triangle. Prove that they are either in equilibrium or form a couple.

5·7. Resultant of a Couple and a Coplanar Force.

Let a couple be formed by two forces, each equal to Q acting at A and B, AB = q being the arm of the couple; and let R be a force acting in the plane of the couple at C along CD.



The moment of the couple = Q.q = G (say).

Now replace the couple by a new couple having its forces, each equal to R. If x be the arm of this new couple, then

$$G = Q.q = R.x \quad \text{or} \quad x = \frac{Q.q}{R} = \frac{G}{R}$$

Shift the new couple such that one of the forces R acts at C along CE (DC produced), a direction opposite to that of the given force R at C.

Now the two forces acting at C balance and we are left with a single force R at F, where $CF = x = \frac{G}{R}$, the point F being to the right or left of C as required by the sign of the couple.

Hence a couple of moment G and a given coplanar force R acting upon a rigid body, compound into a single parallel force R displaced from its original position, through a distance equal to $\frac{G}{R}$

= moment of the couple divided by the given force.

It may be noted that the magnitude and direction of the force remains independent of the couple.

Cor. *A couple and a force in the same plane cannot be in equilibrium.*

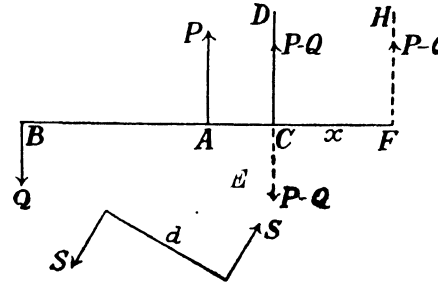
Example. *Two unlike parallel forces P and Q are combined with a couple consisting of two equal and opposite forces, ache equal to S, acting along two parallel lines at a distance d apart in the plane of P and Q. Show that the resultant R of P and Q is displaced through a distance* $\frac{Sd}{P-Q}$, $(P>Q)$.

Here $R=P-Q$ and $G=Sd$.

$\therefore$ by the above article displacement $=\frac{G}{R}=\frac{Sd}{P-Q}$

The following discussion will make it more intelligible.

The resultant of the parallel forces P and Q is a parallel force P−Q acting at a point, say C, along CD.



If the couple (S, d) is replaced by an equivalent couple $(P-Q, x)$, then

$$(P-Q)x=Sd \text{ or } x=\frac{Sd}{P-Q}.$$

Now shift the couple $(P-Q, x)$ in such a way that one of its forces P−Q acts at C along the (dotted) line CE (DC produced) and the other force acts along a parallel line FH at a distance x from C.

The forces at C balance and we are left with only the force P−Q acting along the (dotted) line FH.

Thus the resultant P−Q is displaced through a distance $x=\frac{Sd}{P-Q}$.

Note 1. If the sense of the couple is opposite to that shown in the figu re, the point F will lie towards the left of C.

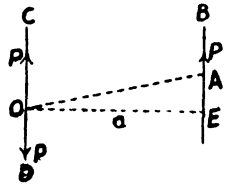
2. If the forces P and Q are like parallel forces, the resultant is P+Q and the displacement is $Sd/(P+Q)$.

Ex. Show that three forces represented in magnitude, line of action and sense by the sides AB, CD, EF of a regular hexagon ABCDEF, are equivalent to a couple whose moment is represented by the area of the hexagon. A force represented by 6BC is added to the system. Show that there is now a single force as the resultant. Find its n agnitude and line of action.

[*Ans.* 6BC, parallel to BC at a distance $\sqrt{3}a/4$ from it]

5·71. Theorem X. Resolution of a force into a force and a couple. *A force can be replaced by an equal and parallel force acting at an arbitrarily chosen point together with a couple, (both lying in the plane determined by the given force and the chosen point).*

Let P be the given force acting at a point A along a certain line AB and let O be the chosen point.



At O introduce two equal and opposite forces, each equal and parallel to the given force P, and acting along OC and OD.

Now the original force P at A and the force P at O acting along OD, form a couple whose moment is Pa where a is the length of the perpendicular from O on AB. Besides this, there is force P acting along OC equal and parallel to the original force P.

Thus, a force P acting at A is equivalent to a force at O equal and parallel to the original force and a couple whose moment is equal to the moment of the original force about O.

Ex. 1. A force of 5 lbs. wt. acting along a line perpendicular to OX at a distance of 3 ft. from O, is equivalent to parallel force of 5 lbs. wt. at O together with a couple of moment equal to the moment of the force about $O = 5 \times 3 = 15$ lb. ft.

Ex. 2. ABC is a triangle and A'B'C' is another whose area is $\triangle$ and whose sides B'C', C'A', A'B' pass through A, B, C respectively. Forces *ka, kb, kc,* act at A', B', C' parallel to BC, CA, AB respectively. Show that they reduce to a couple whose moment is equal to $2k \triangle$.

Coplanar Forces

5.72. Resultant of a system of coplanar forces acting on a rigid body. The above theorem can easily be extended to any number of coplanar forces, so that they can be replaced by a number of coplanar couples and a number of concurrent forces at O.

By Art. 5.5 the system of couples is equivalent to a single couple of moment say G and by Art. 3.23 the forces at O are equivalent to single force, say R.

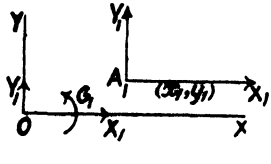
Thus we find that a *system of coplanar forces is equivalent to a single force acting at any chosen point in the plane, together with a couple in the same plane.*

Note.—The magnitude and direction of R depends only upon the magnitudes and directions of the forces and does not depend upon the position of the point O. Thus R is invariant for all positions of O, while G changes (if $R \neq 0$) as the position of O changes.

The above result can also be obtained analytically as follows :

5.73. Analytical method for the resultant of a system of coplanar forces.

Let P_1, P_2...etc., be the given forces acting at the points, A_1, A_2,.....etc., the co-ordinates of A_1, A_2,..... being (x_1, y_1), (x_2, y_2),.....with reference to rectangular axes OX and OY in the plane of the forces.



Let X_1, Y_1 be the resolved parts of P_1 parallel to OX, OY.

Now Y_1 at A_1 can be replaced by Y_1 at O along OY, and a couple of moment $x_1 Y_1$.

X_1 at A_1 can be replaced by X_1 at O along OX, and a couple of moment $-y_1 X_1$.

Hence the force P_1 at A_1 is equivalent to

(1) X_1 along OX,

(2) Y_1 along OY,

and (3) a couple of moment $x_1 Y_1 - y_1 X_1$, say G_1.

Replacing other forces in the same manner and adding up for all the forces, we have

(1) $X=\Sigma X_1$, a force along OX,
(2) $Y=\Sigma Y_1$, a force along OY,
and (3) $G=\Sigma(x_1Y_1-y_1X_1)$, a couple.

Hence ΣX_1 and ΣY_1 are the sums of the resolved parts of the given forces along OX and OY respectively and G is the sum of the moments of the forces about O.

ΣX_1, ΣY_1 are further equivalent to a single force R acting at O.

Thus the system of coplanar forces is equivalent to a single force R at any arbitrarily chosen point O together with a couple of moment G.

5·74. Since the moment of the force P_1 acting at (x_1, y_1) about $O=x_1Y_1-y_1X_1=$the algebraic sum of the moments of the components X_1 and Y_1 about O,

$\therefore$ *G, the moment of the resultant couple of the system of forces about O is equal to the sum of the moments of all the forces about O.*

Also if the resultant force R is inclined at an angle θ with OX, then

$R\cos\theta=\Sigma X_1=X$, and $R\sin\theta=\Sigma Y_1=Y$.
$\therefore$ $R^2=X^2+Y^2$ and $\tan\theta=Y/X$.
Also $R=0$ when X as well as Y both are equal to zero.

5·75. Nature of the resultant of a system of coplanar forces.

In Arts 5.72 and 5.73, it has been seen that a system of coplanar forces is equivalent to a single force R acting at any arbitrarily chosen point O together with a couple of moment G about O.

We now consider the various cases as regards the nature of the resultant of a system of coplanar forces.

(1) When $R\neq0$, as well as $G\neq0$, *i.e.* neither R nor G vanishes.

Then by Art. 5.7, they can be further compounded into a single force R with its line of action *not passing through* O.

(2) When $G=0$ but $R\neq0$.

The system reduces to a single force *passing through* O.

Note. The resultant is a single force when at least one ΣX_1 or ΣY_1 is *not* zero, G may or may not be zero.

(3) When $R=0$ (*i.e.*, $\Sigma X_1=0$ and $\Sigma Y_1=0$) and $G\neq0$.

The system is equivalent to a single couple, and the moment of a couple being constant, the moment of the resultant couple is equal to the sum of the moments of the forces of the system about *any* point in their plane.

Thus the system reduces to either a single force or a single couple, a fact which was proved otherwise in Art. 4.61.

(4) when $G=0$ as well as $R=0$ (*i.e.*, $\Sigma X_1=0$ and $\Sigma Y_1=0$).

The resultant is neither a force nor a couple and hence the system is in equilibrium.

[For further discussion see Arts. 7·1, 7·11, 7·12.]

5·76. The conditions of equilibrium* for a system of coplanar forces may, therefore, be stated as follows :

(*a*) *The sum of the moments of the forces about a point in their plane vanishes.*

This ensures that $G=0$, *i.e.*, the resultant of the system is not a couple.

(*b*) *The sum of the resolved parts of the forces along any line OX vanishes.*

(*c*) *The sum of the resolved parts of the forces along a line OY, perpendicular to OX, also vanishes.*

These (*b*) and (*c*) ensure that the resultant of the system cannot be a force, *i.e.*, $R=0$, for, since $R^2=X^2+Y^2$, $R=0$ when $X=0$ as well as $Y=0$.

5·761. *The conditions given above are necessary as well as sufficient.*

We know that a system of coplanar forces acting upon a rigid body reduces to either a single couple or a single force as its resultant.

If (*a*) is satisfied $G=0$, *i.e.*, the resultant cannot be a couple, and when (*b*) and (*c*) are satisfied $X=0$ as well as $Y=0$, and, therefore, $R=0$, *i.e.*, the resultant cannot be a force. Thus the resultant can be neither a force nor a couple and hence the conditions are sufficient.

Conversely if the resultant is not to be a couple, G must be zero, *i.e.* condition (*a*) must be satisfied. Again if the resultant is not to be a force, R must be zero, but since $R^2=X^2+Y^2$, R cannot be zero, unless X and Y both are zero, so that both conditions (*b*) and (*c*) must be satisfied.

Hence the conditions are also necessary.

Note 1.—If only (*a*) and (*b*) are satisfied, (*c*) is not satisfied, then the resultant is a force perp. to OX. If (*a*) and (*c*) are satisfied and not (*b*), the resultant is a force parallel to OX. Again if (*b*) and (*c*) are satisfied and not (*a*), the resultant must be a *couple*. This also shows that all the three conditions are necessary for equilibrium of the system.

Note 2.—The two lines along which forces are resolved may be *any* two lines in the plane of the forces and may not necessarily be perpendicular to each other. For, if the sum of the resolved parts of the forces along one line l vanishes, the resultant R is either, zero or it acts at right angles to l. Similarly if the sum of the resolved parts along another line l' (not parallel to l) vanishes, R is either zero or it acts at right angles to l'.

Hence as R cannot be perpendicular to two non-parallel lines l and l', it must be zero. It is for the sake of convenience that the lines are usually taken at right angles to each other.

5·77. Illustrative examples.

Example 1. *Forces P, 2P, 3P, 4P act along the sides AB, BC, CD, DA of a square. Find the nature of the resultant and reduce the system to a force at A and a force along BC.*

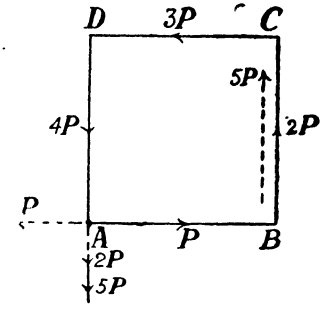
* For other forms of the conditions of equilibrium, see next Chapter.

Let the length of a side of the square be a and let AB, AD be taken as the axes, then

$X = P - 3P = -2P$ *i.e.*, 2P along BA.

and $Y = 2P - 4P = -2P$ *i.e.*, 2P along DA.

Also $G = 2P.a + 3P.a = 5Pa$.



Thus $R \neq 0$ and also $G \neq 0$.

$\therefore$ The resultant must be a single force not passing through A [Case (1), Art. 5.75.]

Now the couple of moment $5Pa$ can be taken equal to a couple, formed by forces 5P along BC and 5P along DA.

Thus the given system of forces is equivalent to

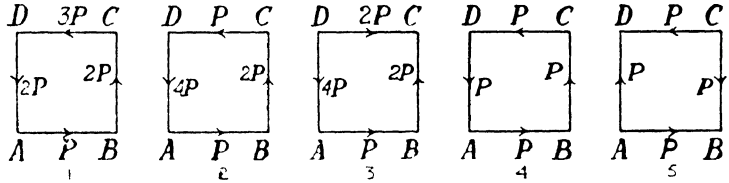
(*i*) a force 5P along BC

and (*ii*) a force 2P along BA and a force $2P + 5P = 7P$ along DA, which are equivalent to a force

$$\sqrt{(2P)^2 + (7P)^2} = \sqrt{53}\ P \text{ acting at A.}$$

Hence the given system of forces is reduced to a force $\sqrt{53}$ P at A and a force 5P along BC.

Example 2. *Forces act along the sides of a square ABCD as shown in the five figures below. Find the nature of the resultant in each case.*



Let a be the length of a side of the square in each case. Let A be the origin, AB the axis of x, and AD the axis of y.

In figure (1),

$X = P - 3P = -2P$, $Y = 2P - 2P = 0$, $G = 2P.a + 3P.a = 5Pa$.

$\therefore$ $R \neq 0$, and $G \neq 0$. $\therefore$ the resultant is a single force not passing through A.

[Refer to case (1), Art. 5.75 and also note under case (2)].

In figure (2),

$X = P - P = 0$, $Y = 2P - 4P = -2P$, $G = 2P.a + P.a = 3Pa$.

$\therefore$ $R \neq 0$, and $G \neq 0$ as in figure (1).

Note. – The position of the line of the resultant in these two cases can be obtained either by a procedure followed in Ex. (*ii*), Art, 4.64, P. 66 or by finding the equation of the line by Art. 5·78 below.

In figure (3),

$X = P + 2P = 3P$, $Y = 2P - 4P = -2P$, $G = 2P.a - 2P.a = 0$.

$\therefore$ $R \neq 0$, $G = 0$. $\therefore$ the resultant is a force passing through [Case (2), Art. 5.75].

In figure (4),

$X = P - P = 0$, $Y = P - P = 0$, $G = P.a + P.a = 2Pa$.

$\therefore$ $R = 0$, $G \neq 0$, the resultant is a couple. [Case (3), Art. 5.75]

In figure (5).

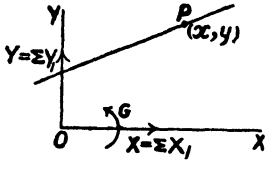
$X = P - P = 0$, $Y = -P + P = 0$, $G = -P.a + P.a = 0$.

$\therefore$ $R = 0$ and $G = 0$, the system is in equilibrium.

[Case (4), Art. 5.75].

5.78. Equation of the line of action of the resultant.

In case the forces reduce to a single force, the equation of its line of action is obtained as follows :—



Let the system reduce to, a force X along OX, a force Y along OY, and a couple of moment G about O. [Art. 5.73].

Let $P(x, y)$ be a point on the line of action of the resultant force, then since the resultant passes through this point the sum of the moments of the forces about it must vanish. Hence taking moments about P, we have

$$yX - xY + G = 0 \text{ or } xY - yX = G \qquad \ldots(1),$$

which is the equation of the required line.

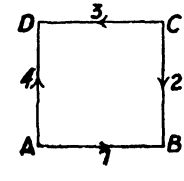
It may be noted that G', the sum of the moments of all the forces of the system about any point (x, y) in their plane is given by

$$G' = G - xY + yX \qquad \ldots(2)$$

and the angle θ which the resultant makes with OX is given by

$$\tan \theta = \text{slope of the line } (1) = Y/X.$$

5.79. Example 1. *Find the resultant of forces 1, 2, 3, 4, lbs. weight acting as shown along the sides of a square of side 2 ft. Find also the equation of the line of action of the resultant.*



Taking AB, AD as axes of reference, we have

X = sum of the components along AB $= 1 - 3 = -2$ lbs.

Y = sum of the components along AD $= 4 - 2 = 2$ lbs.

G = sum of the moments about A

$$= 3 \times AD - 2 \times AE = (3 - 2).2 = 2 \text{ lb. ft.}$$

Hence the equation of the line of action of the resultant is

$$xY - yX = G \text{ } i.e., \text{ } x(2) - y(-2) = 2, \text{ or } 2x + 2y = 2$$

or $x + y = 1$,

which obviously intersects the lines AB and AD at their middle points.

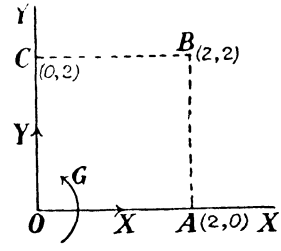
Also the magnitude of the resultant $=\sqrt{X^2+Y^2}=2\sqrt{2}$ lbs. wt. and θ, the angle which the resultant makes with AB is given by

$$\tan\theta=\frac{Y}{X}=\frac{2}{-2}=-1 \text{ } i.e., \text{ } \theta=135°.$$

Example 2. *The moments of a given system of forces about three points (2, 0), (0, 2)and (2, 2) are +3,+4, +10 units respectively, find the magnitude of the resultant force and prove that it acts along the line*

$$6x+7y=6.$$

Let the given system of forces reduce to a force X along OX, a force Y along OY and a couple of moment G about O. [Art 5.73].



Then the sum of the moments of the system about the point A (2, 0) is G−Y.2, which by the question is equal to 3 units.

$$\therefore \quad G-2Y=3 \qquad ...(1).$$

Similarly equating moments about (0, 2) and (2, 2) to 4 and 10 units respectively, we have

$$G+2.X=4 \qquad ...(2) \qquad \text{and} \qquad G+2.X-2.Y=10 \qquad ...(3)$$

Now from (1) and (3), $2X=10-3=7$ or $X=\frac{7}{2}$,

and from (2) and (3) $2Y=4-10=-6$ or $Y=-3$.

$\therefore$ in magnitude, the resultant $R=\sqrt{X^2+Y^2}=\sqrt{\frac{49}{4}+9}=\frac{1}{2}\sqrt{85}$.

Also from (1), $G=3+2Y=3-6=-3$.

$\therefore$ the equation of line of action of the resultant is

$$xY-yX=G \text{ or } x(-3)-y(\tfrac{7}{2})=-3, \text{ } i.e., \text{ } 6x+7y=6 \qquad ...(4)$$

Moreover, the direction of the resultant

$$=\text{slope of its line of action}=\tan^{-1}(-\tfrac{6}{7}).$$

And the point where the resultant meets OX, obtained by putting $y=0$ in equation (4), is given by $6x=6$, or $x=1$.

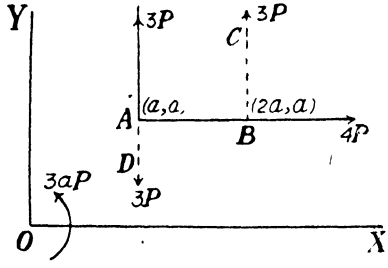
Thus the resultant bisects OA.

Example 3. *Find the resultant of forces 4P and 3P parallel to OX and OY, acting at the point (a, a) and a couple in the plane XOY whose moment is +3aP.*

Find the intercept made by its line of action on OY.

Since the moment of a couple remains constant, the given couple

of moment $3aP$ will have a moment $3aP$ about the origin.



In this case, $X=4P$, $Y=3P$, and G, the sum of the moments of the system about O

$=$ moment of the given couple $+$ sum of the moment about O of the forces 4P and 3P acting at A (a, a)

$= 3aP+3P.a-4P.a=2aP.$

$\therefore$ the resultant $R=\sqrt{X^2+Y^2}=\sqrt{16P^2+9P^2}=5P$, inclined with OX at an angle $\tan^{-1}\dfrac{Y}{X}=\tan^{-1}\dfrac{3P}{4P}=\tan^{-1}\dfrac{3}{4}$.

And the equation of the line of action of the resultant is

$xY-yX=G$ or $x.3P-y.4P=2aP$, *i.e.*, $3x-4y=2a$.

This meets OY where $x=0$ and therefore $y=-\frac{1}{2}a$, so the intercept on $OY=-\frac{1}{2}a$.

Aliter. The couple $3aP$ can be replaced in magnitude and sense by forces 3P at A along AD and 3P at B along BC, as shown in the figure by dotted lines. The forces at A balance and the system is reduced to two forces 4P and 3P acting at B so that the resultant R acts at B $(2a, a)$.

Now $R=\sqrt{16P^2+9P^2}=5P$, inclined at $\tan^{-1}\dfrac{3P}{4P}=\tan^{-1}\dfrac{3}{4}$ with AB or OX.

And the line of action of R, a line passing through B $(2a, a)$ and having the slope $=\frac{3}{4}$ is

$$y-a=\tfrac{3}{4}(x-2a) \text{ or } 3x-4y=2a, \text{ as before.}$$

Examples XX-A

1. State the conditions required for the equilibrium of coplanar parallel forces.

1 (a). P and Q are two like parallel forces. A couple of moment G is combined with them. Show that their resultant is displaced through a distance

$$G/(P+Q).$$

2. A triangle ABC can turn freely in its own plane about the centre of its inscribed circle which is fixed and forces proportional to

$$y-z, \qquad z-x, \qquad x-y$$

act along the sides BC, CA and AB respectively. Show that the triangle remains at rest.

3. ABCD is a triangle in which $AC=2BC=2$ ft. Forces $2, 2\sqrt{3}, 17, 3\sqrt{3}, 6\sqrt{3}$ and $4\sqrt{3}$ lbs. wt. act along AB, BC, CD, DA, AC and DB respectively. Show that they are equivalent to a couple of moment 17 lb. ft. units.

4. The sides of a square ABCD are 3 inches long. Forces $5, 1, 3, 2, 3\sqrt{2}$ act from A to B, B to C, C to D, A to D, A to C respectively. Prove that this

system of forces is not equivalent to a couple and find the point where the line of action of its resultant meets BC. (*Ans.* $1\frac{1}{5}$ ins. from B towards C)

5. Six coplanar forces act on a body along the sides AB, BC, CD, DE, EF, and FA of a regular hexagon ABCDEF, in which AB is one foot long ; their magnitudes are 10, 20, 30, 40, P and Q lb. respectively. Find P and Q so that the system reduces to a pure couple and show that the moment of the couple is $75\sqrt{3}$ lb. ft. [*Ans.* -10, 60 lbs.] (*P. U.*)

6. ABCDEF is a regular hexagon of side $2a$; forces P, 2P, 3P, 5P, 6P act along AB, BC, DC, EF, AF respectively, show that a force can be determined to act along ED, so that the six forces are equivalent to a couple, and find the moment of the couple. [*Ans.* $-3\sqrt{3}aP$] (*Rangoon U.*)

7. Four coplanar forces act on a rigid body, their lines of action being the sides of a square ABCD, their magnitudes and directions are (1) 20 lbs. wt. along AB (2) -10 lbs. along BC, (3) 30 lbs. along CD, (4) -40 lbs. along DA. Find the forces, P and Q whose lines of action are respectively CA and a line through the point B which will just produce equilibrium. (*P. U.*)

(*Ans.* $P=10\sqrt{2}$ lbs., $Q=20\sqrt{2}$ lbs. acting along DB)

8. Forces 6, 8, 10, 6, 8, 10 act along the sides taken in order of a regular hexagon ABCDEF. Forces P each act along CA and FD. Find the value of P in order that the system may be in equilibrium. (*Ans.* $24\sqrt{3}$)

9. ABCDEF is a regular hexagon and O is its centre. Forces of magnitude 1, 2, 3, 4, 5, 6 act along the lines AB, CB, CD, ED, EF, AF. Reduce the system to a force at O and a couple, and find the point in AB through which the single resultant passes.

(*Ans.* Force $2\sqrt{3}$ at 150° with OC, couple of moment $-\frac{3}{2}a\sqrt{3}$. Resultant $2\sqrt{3}$ bisects AB and AF)

10. A system of forces acting on a rigid lamina in its plane is reduced, firstly to a single force acting at a point A in the lamina, and secondly to a single force acting at another point B and a couple of moment G. Prove, that if the system were reduced to a single force acting at the middle point of AB and a couple, the moment of the couple would be $\frac{1}{2}$ G.

Examples XX-B

1. ABCD is a rectangle whose side AB=15″ and AD=8″. Taking AB, AD as co-ordinate axes, show that the equation of the line of action of the resultant of the following forces is $x+y=1$:

22 lbs. from D to A ; 10 lbs. from A to B ; 51 lbs. from D to B ; 34 lbs. from A to C ; 40 lbs. from C to B ; 15 lbs. from C to D.

2. Forces of 3, 4, and 7 lbs. act along the sides AB, CB and CA of an equilateral triangle. Taking A as origin and CA as X-axis, show that the resultant acts along the line

$$7x-5\sqrt{3}y+4a=0,$$

a being the length of a side of the triangle.

3. Forces of magnitude P, 2P, -3P, 4P, 5P and -9P act respectively along the sides taken in order of a regular haxagon. Prove that they reduce to a resultant force acting through the centre O of the hexagon.

Taking OX parallel to the force P and OY perpendicular to P towards 4P, show that the resultant acts along the line

$$x\sqrt{3}+5y=0. \qquad [\textit{See}\ \text{Q. 12, Ex. XIV}]$$

4. Components, parallel to the axes of co-ordinates, of four forces acting at the points (1, 2), (-2, 3), (-3, -3), (3, -4) are respectively (-3, 4), (5, 3), (-2, -3) and (4, -1), the distances being in feet and the forces in pounds weight. Find the magnitude and the direction of the single force and the moment of the couple at the origin. Prove that the line of action of the single resultant is $3x-4y=5$. (*Ans.* 5 lbs. wt. at $\tan^{-1}\frac{3}{4}$ with X-axis) (*P. U.*)

5. The sum of the moments of a system of forces about the points (1, 0), (0, 1) and (3, 1) are +4, +25 and −11 units respectively. Find the resultant force and show that its line of action is $12x-9y=16$.

(*Ans.* 15 at $\tan^{-1}\frac{4}{3}$ with OX)

6. Forces P, 2P respectively act along the sides AB, BC of a rectangle ABCD, and a couple of moment P(AD+2AB) acts in the plane of the rectangle in the sense ADCB. If the resultant of the system meets AB, AD in H, K respectively, find AH and AK.

(*Ans.* H is in BA produced, $AH=\frac{1}{2}$ AD ; K is at D)

7. The moments of a system of coplanar forces about three collinear points A, B, C in their plane are G_1, G_2, G_3, prove that

$$G_1.BC+G_2.CA+G_3.AB=0. \qquad (P.U.)$$

8. A system of forces in the plane of a triangle ABC is equivalent to a single force (F) at A acting along the internal bisector of the angle BAC and a couple of moment G_1. If the moments of the system about B and C are G_2 and G_3, prove that

$$(b+c)G_1=bG_2+cG_3.$$

9. A system of forces acts on a plate in the form of an equilateral triangle of side two units. The moments of the forces about the three vertices are 10, 20 and 30 units. Show that the resultant is 10 units in magnitude and is inclined at an angle of 30° with a side of the triangle.

10. The moments of a system of coplanar forces about the points (1, 0), (0, 1) (0, 2) and (2, 3) referred to rectangular axes are G_1, G_2, G_3 respectively.

If θ be the inclination of the resultant to the axis of x, prove that

$$\tan\theta=\frac{G_1-3G_2+2G_3}{2G_1-G_2-G_3}. \qquad (D.U.\ 1924)$$

11. Forces of magnitudes 1, 2, 3, 4 act in order along the sides AB, BC, CD, DA of a square ABCD (of side a). The system is equivalent to two forces, one of which is of magnitude 3 and acts along AC. Find the magnitude of the second force and the points where its line of action meets AB and AD. (*P.U. 1957 S.*)

[*Ans.* $3+2\sqrt{2}$; meets BA produced and AD at distances $5a(3\sqrt{2}-4)$ from A]

12. The forces P, Q, R act along the sides (taken in order) of the triangle formed by the lines

$$x+y=1, \quad y-x=1 \quad \text{and} \quad y=2.$$

Find the equation of the line of action of their resultant. (*D.U. 1957*)

[*Ans.* $P(x+y-1)+Q(y-x-1)-\sqrt{2}\,R(y-2)=0$]

13. Forces P, Q, R act along the lines $x=0$, $y=0$ and $x\cos\theta+y\sin\theta=p$. Find the equation of the line of action of the resultant, the axes being rectangular. [*Ans.* $Px-Qy+R(x\cos\theta+y\sin\theta-p)=0$] (*P. U.*)

5.8. Rotation of coplanar forces about their points of application. *If all the forces in a coplanar system are rotated in their plane about their points of application through the same angle, in the same sense, their resultant passes through a fixed point in the body.*

As in Art. 5·73, let P_1, P_2.....etc. be the given forces acting at (x_1, y_1), (x_2, y_2)....., inclined at angles θ_1, θ_2....., with OX.

Let (X_1, Y_1), (X_2, Y_2),.....be the resolved parts of the forces parallel to the axes OX and OY and let X, Y be the sum of these resolved parts, then

$$X=\Sigma X_1=\Sigma P_1\cos\theta_1 \qquad ...(1)$$

$$Y=\Sigma Y_1=\Sigma P_1\sin\theta_1, \qquad ...(2)$$

and G, the sum of their moments about O, as in Art. 5.73, is given by

$$G=\Sigma(x_1Y_1-y_1X_1) \qquad ...(3)$$

Now let all the forces be turned through the same angle, α, about their respective points of application, so that θ_1, θ_2.....become $\theta_1+\alpha$. $\theta_2+\alpha$,....., then

$$P_1\cos(\theta_1+\alpha)=\cos\alpha.P_1\cos\theta_1-\sin\alpha.P_1\sin\theta_1=X_1\cos\alpha-Y_1\sin\alpha \qquad ...(4),$$

and $$P_1\sin(\theta_1+\alpha)=\sin\alpha.P_1\cos\theta_1+\cos\alpha.P_1\sin\theta_1=X_1\sin\alpha+Y_1\cos\alpha \qquad ...(5).$$

Let X, Y, G become X′, Y′ and G′ in the new position after the displacement, then, α being the same for all forces, using (4) and (5), we have

$$X'=\Sigma P_1\cos(\theta+\alpha)=\cos\alpha\Sigma X_1-\sin\alpha\Sigma Y_1=X\cos\alpha-Y\sin\alpha \quad ..(6),$$

$$Y'=\Sigma P_1\sin(\theta+\alpha)=\sin\alpha\Sigma X_1+\cos\alpha\Sigma Y_1=X\sin\alpha+Y\cos\alpha \quad ...(7),$$

and
$$\begin{aligned}G'&=\Sigma[x_1P_1\sin(\theta_1+\alpha)-y_1P_1\cos(\theta_1+\alpha)]\\&=\Sigma[x_1(X_1\sin\alpha+Y_1\cos\alpha)-y_1(X_1\cos\alpha-Y_1\sin\alpha)]\\&=\cos\alpha\Sigma(x_1Y_1-y_1X_1)+\sin\alpha\Sigma(x_1X_1+y_1Y_1)=G\cos\alpha+V\sin\alpha \qquad ...(8),\end{aligned}$$

where $V=\Sigma(x_1X_1+y_1Y_1)$ and is called the **Virial** of the system.

In the particular case when $\alpha=90°$, *i.e.* the forces rotate so as to become perp. to their original lines of action, $X'=-Y$, $Y'=X$, and $G'=V=\Sigma(x_1X_1+y_1Y_1)$.

Now the equation of the line of action of the resultant is

$$G'+yX'-xY'=0$$

or $G\cos\alpha+V\sin\alpha+y(X\cos\alpha-Y\sin\alpha)-x(X\sin\alpha+Y\cos\alpha)=0$,

or $$\cos\alpha(G+yX-xY)+\sin\alpha(V-yY-xX)=0 \quad .. (9).$$

Whatever the value of α, this line always passes through the point given by

$$G+yX-xY=0, \quad \text{and} \quad V-yY-xX=0.$$

Solving these equations, we have

$$x=\frac{GY+VX}{X^2+Y^2} \text{ or } \frac{GY+VX}{R^2} \text{ and } y=\frac{VY-GX}{X^2+Y^2} \text{ or } \frac{VY-GX}{R^2} \qquad ...(10).$$

This being independent of α, is a fixed point, and is called the **Astatic centre** of the system.

Hence the resultant always passes through a fixed point.

5.81. *If a coplanar system of forces is in equilibrium and if all the forces be turned in their plane, round their points of application, through the same angle in the same sense, the system in its new position is equivalent to a couple.*

As the system is in equilibrium before the displacement, we have

$$X=0, \quad Y=0, \text{ and } G=0.$$

$\therefore$ by (6), (7) and (8), we have

$$X'=0 \text{ and } Y'=0 \text{ and } G'=\sin\alpha\Sigma(x_1X_1+y_1Y_1)=V\sin$$

Since $X'=0$ as well as $Y'=0$, the resultant cannot be

and since $G' \neq 0$, the resultant is a couple of moment
$$= V \sin\alpha = \sin\alpha\, \Sigma(x_1 X_1 + y_1 Y_1).$$

Note. In case originally the resultant is a couple in the new position also the resultant is a couple of moment
$$G\cos\alpha + V\sin\alpha$$
For $X' = Y' = 0$, $\because$ $X = Y = 0$.

5.82. Astatic equilibrium. *Condition for the system to be in equilibrium in its new position, it being equilibrium originally.*

It will be so only if $V = 0$ *i.e.*, $\Sigma(x_1 X_1 + y_1 Y_1) = 0$, which is the required condition.

When the condition is satisfied the system is said to be in *Astatic Equilibrium.*

Ex. If a system of coplanar forces is in equilibrium, the co-ordinates, of the astatic centre becomes indeterminate.
$\because$ $X = Y = 0$, $\therefore$ by (10) $x \to \infty$, and $y \to \infty$.

Miscellaneous Examples III

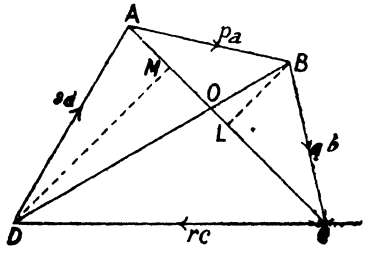
If the resultant of a system of coplanar forces is not a force but a couple, the sum of the resolved parts of the forces along any two lines in the plane must vanish and the sum of the moments of the forces about *any* point in the plane must be the same.

If the resultant is a force and not a couple, the sum of the moments of the forces about different points in the plane must be different unless the points lie in a straight line parallel to the line of action of the resultant force.

Ex. 1 If the sum of the moments of a system of coplanar forces about any point in a given straight line in their plane is always the same, what is the nature of the resultant ?

Example 2. *Forces pa, qb, rc, sd act along the sides AB, BC, CD, DA of a plane quadrilateral taken in order, a, b, c, d, respectively, being the lengths of the sides. Prove that the forces are equivalent to a couple if*
$$(p-q)\, OB = (r-s)\, OD \quad \text{and} \quad (q-r)\, OC = (s-p) OA,$$
where O is the point of intersection of AC and BD.

Since the resultant is to be a couple and not a force, the sum of the resolved parts of the forces in two directions must vanish.



Resolving in a direction perp. to AC, we have

$pa \sin \angle BAL - qb \sin \angle BCL$
$\quad - rc \sin \angle DCM + sd \sin \angle DAM = 0,$

or $\quad p.BL - q.BL - r.DM + s.DM = 0.$

$$\therefore \quad \frac{p-q}{r-s} = \frac{DM}{BL} = \frac{OD}{OB}$$

i.e., $(p-q)\, OB = (r-s)\, OD$ ... (1).

The other result is obtained by resolving forces perp. to BD.

Example 3. *In Ex. 2, if the quadrilateral is cyclic and the resultant of the forces is a force parallel to AC, show that*
$$(p-q)\, ab = (r-s)\, cd.$$

Since the resultant is a force parallel to AC, the sum of the moments of the forces about any point in AC must be the same.

Equating the moments of the forces about A and C, we have

$$qb.AB \sin \angle ABC + rc.AD \sin \angle ADC = pa.BC \sin \angle ABC + sd.CD \sin \angle ADC.$$

The quadrilateral being cyclic, $\sin \angle ABC = \sin \angle ADC$,

$$\therefore \quad qb.AB + rc.AD = pa.BC + sd.CD,$$

i.e., $$qba + rcd = pab + sdc.$$

Hence $$(p-q)\,ab = (r-s)\,cd.$$

Aliter. The resultant being a force parallel to AC, the sum of the resolved parts of the forces perp. to AC must vanish.

$\therefore$ the result (1) of Ex. 2 holds true.

$$\therefore \frac{r-s}{p-q} = \frac{OB}{OD} = \frac{BL}{DM} = \frac{BL.AC}{DM.AC} = \frac{2\triangle ABC}{2\triangle ADC} = \frac{ab \sin \angle ABC}{cd \sin \angle ADC} = \frac{ab}{cd}.$$

Hence $$(p-q)ab = (r-s)cd.$$

4. Forces P, Q, R, S act respectively along the sides AB, BC, CD, DA of a quadrilateral ABCD which can be inscribed in a circle, Show that if their resultant is parallel to AC, then

$$P.BC - Q.AB = R.AD - S.CD.$$

5. Forces P, Q, R, S acting along the sides AB, BC, CD, DA of a quadrilateral ABCD, are in equilibrium, show that

$$\frac{P \times R}{AB \times CD} = \frac{Q \times S}{BC \times DA}.$$

Will this condition alone ensure equilibrium ?

If the first force acts from A towards B, find the direction of each of the other forces.

(*Ans.* No ; Q from C towards B, R from C towards D, S from A towards D)

6. ABCD is a square. Forces P, Q, R, S, $\sqrt{2}$ F, $\sqrt{2}$ T act along AB, BC, CD, DA, AC and DB respectively. Show that the resultant is a couple of moment P+Q+R+S if F+T=R−P and F−T=S−Q.

7. Along the sides of a regular hexagon taken in order six forces a, b, c, d, e, f are applied, show that they will be equivalent to couple if

$$e = a + b - d \text{ and } f = c + d - a.$$

Calculate the moment of this couple in terms of a, b, c and d.

[*Ans.* $\frac{1}{2}\sqrt{3}\, l(a+2b+2c+d)$, l = length of a side]

8 (*a*). Forces P, Q, R, S acting respectively along the sides AB, BC, CD, DA of a quadrilateral ABCD reduce to a couple. AB and CD are parallel but not equal while AD and BC are equal but not parallel. Prove that

$$Q = S = \tfrac{1}{2}(R - P) \sec \angle ADC.$$

(*b*) If AD and BC are also not equal, then show that

$$Q \sin B = S \sin D \text{ and } S \cos D - Q \cos B = R - P.$$

9. Forces $P_1, P_2, P_3, P_4, P_5, P_6$ act along the sides of regular hexagon taken in order. Show that they will be in equilibrium if $\Sigma P = 0$,

and $$P_1 - P_4 = P_3 - P_6 = P_5 - P_2.$$ (*Pub. Ser. Com.*)

10. The forces P_1, P_2, P_3 act at the middle points of the sides a, b, c of the triangle ABC, and at right angles to these sides. Show that if they form a system in equilibrium with Q_1, Q_2, Q_3 acting along the same side, then

$$Q_1.4\triangle = (P_2 a - P_1 b)b - (P_3 a - P_1 c)c$$

and two similar equations, $\triangle$ being the area of the triangle. (*B.U.*)

11. If three forces acting along and represented by the sides of the triangle ABC taken in order are in equilibrium with three forces each of magnitude X, which act along the tangents at A, B and C to the circumcircle of the triangle

and whose directions are all in the same sense of rotation, prove that the triangle ABC must be equilateral and find X in terms of the length of the side,

(*Ans.* X=half the side)

12. Three forces acting at the angular points of a triangle ABC along the tangents to the circumcircle, are equivalent to a couple. Show that they are in the ratio

$$\sin 2A : \sin 2B : \sin 2C.$$

13. The algebraic sum of the moments of a system of forces about points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) are G_1, G_2, G_3, G_4 respectively. Show that

$$\begin{vmatrix} 1 & x_1 & y_1 & G_1 \\ 1 & x_2 & y_2 & G_2 \\ 1 & x_3 & y_3 & G \\ 1 & x_4 & y_4 & G_4 \end{vmatrix} = 0.$$

[**Hint.** By (2) Art. 5.78, $G-x_1Y+y_1X=G_1$ or $G-x_1Y+y_1X-G_1=0$, the given result is obtained by eliminating G, X and Y from the four equations thus written down.]

14. A system of forces in one plane is equivalent to a couple of moment G. If the line of action of each force is turned about its point of application in the same direction through a right angle, prove that the new system is also equivalent to a couple.

Also prove that if the moment of this couple be H and if the lines of action of the original forces be each turned

(*a*) through an angle θ, the system is equivalent to a couple whose moment is

$$G \cos\theta + H \sin\theta$$

(*b*) through an angle $2 \tan^{-1}(H/G)$, the system is equivalent to a couple whose moment is again G.

REVISION QUESTIONS III

1. If forces in a plane acting upon a rigid body reduce to a couple, show that if they were made to act on a particle, retaining their original inclinations, they would keep the particle at rest.

2. A system of coplanar forces is represented in magnitude and position by all but one of the sides of a closed polygon, taken in order. Prove that their resultant is parallel in position and proportional in magnitude to the remaining side, and that its line of action is at a distance from that side inversely proportional to its length.

3. A door is fitted with spring hinges which normally keep it just closed. As the door is opened the hinges supply a couple of $\frac{1}{2}$ lb. ft. for every degree the door is opened. Friction resists motion with a maximum torque (couple) of 5 lb. ft. What force must be applied to the door 3 ft. from the hinges in order to keep it open 30°. (*Ans.* $6\frac{2}{3}$ lbs. wt.)

4. Four forces R, R, S, S act along the sides AB, BC, CD, DA of a rhombus. Find the sum of their moments about the centre of the rhombus and deduce that their resultant is at a distance from this point equal to

$$\tfrac{1}{2}\, BD(R+S)/(R-S).$$

Discuss the case when R=S.

5. Prove that if three forces $k.BC$, $k.CA$, $k.AB$ act along the sides BC, CA, AB of a triangle ABC their resultant is a couple of moment $2k\triangle$ where $\triangle$ is the area of the triangle.

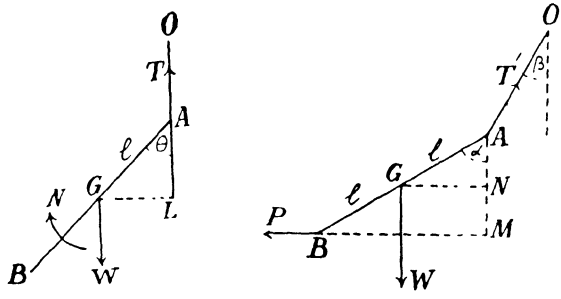
Three forces kBC, $2k$CA, $2k$AB act along the sides BC, CA, AB of a triangle ; find the single force at A and the couple to which the system is equivalent. Also find the magnitude and direction of the resultant force and the point where its line of action meets AB.

[*Ans.* kCB parallel to CB ; $2k$ $\triangle$ABC ; kBC parallel to CB, divides BA produced in the ratio 2 : 1]

6. *Show that a couple and a force in its plane cannot produce equilibrium.*

A uniform rod AB of weight W and length $2l$ hangs from a fixed point O by a light string OA attached to the end A, and a couple of moment N ($<lW$) is applied to the rod in a vertical plane. Find the tension in the string and the inclination of the rod to the vertical in the position of equilibrium.

If the couple is removed and a horizontal force P applied to B, find the tangents of the angles made by the string and the rod with the vertical in the equilibrium position.



• In the first case, the string must be vertical so that its tension T may form with the necessarily vertical force W, a couple to balance the given couple N.

$\therefore$ resolving vertically, we have $T=W$.

And N = moment of the couple formed by T and $W=W.GL=Wl\sin\theta$.

$\therefore$ $\sin\theta=N/lW$, and $\sin\theta$ being <1, $N<lW$.

In the second case, the string must be inclined to the vertical so that a component of its tension T′ may balance the horizontal force P.

Thus $T'\sin\beta=P$, and $T'\cos\beta=W$. $\therefore$ $\tan\beta=P/W$.

Taking moments about A, we have

$P.AM-W.GN=0$ or $P.2l\cos\alpha=W.l\sin\alpha$, $\therefore$ $\tan\alpha=2P/W$.

7. Show that a force and a couple cannot produce equilibrium. What is their resultant ?

A uniform ladder of length l and weight W is held with its upper end resting against a smooth vertical wall, and with its lower end on a smooth horizontal surface. A man of weight W′ stands on the ladder at a distance l' from its lower end. If the ladder is kept from slipping by means of a couple, show that the moment of the couple is equal to

$$(\tfrac{1}{2}Wl+W'l')\sin\theta,$$

where θ is the inclination of the ladder to the vertical. (*D.U. 1960*)

8. Show that a system of forces in one plane can be reduced to a force through any given point in the plane and a couple.

Three points L, M, N are taken in the sides AB, CD, EF of a regular hexagon respectively, such that LMN is an equilateral triangle and forces act along and are represented by LD, LE, MF, MA, NB and NC. Prove that the resultant will be a couple of moment $6(\triangle OLA\sim\triangle OLB)$, O being the centre of the hexagon. (*P.U.*)

[**Hint.** Force LD at L $\equiv$ force LD at O + couple $2\triangle LDO$

$\equiv$ LO + OD at O + couple $2\triangle LOA$.

Similarly force LE at L $\equiv$ LO + OE at O + conple $2\triangle LOB$,

the couples $\triangle LOA$, $\triangle LOB$ being of opposite signs.

Thus forces LD, LE $\equiv$ 2LO + OD + OE + couple $2(\triangle OLA\sim\triangle OLB)$]

CHAPTER VI

CONDITIONS OF EQUILIBRIUM OF A RIGID BODY UNDER COPLANAR FORCES

Solution of Problems

6.1. Application of the conditions of Equilibrium. The set of conditions of equilibrium as given in Art. 5.76 is the most useful for solution of problems involving equilibrium of a rigid body under coplanar forces. This leads to the following procedure :—

Equate to zero the sum of the resolved parts of the forces along two suitable perpendicular lines.

Equate to zero the sum of the moments of the forces about some suitable point.

In addition to the three equations so obtained, geometrical relation, holding between the bodies under consideration, may also be written down.

A neat and accurate figure according to the conditions given in a problem, indicating the forces along their lines of actions, may always be drawn.

All these will generally suffice to determine the positions of equilibrium and the magnitudes and directions of the unknown forces.

6.2. *Choice of axes along which forces are resolved and the point about which moments are taken.* The procedure can often be much simplified by the proper choice of the directions along which the forces are resolved. In general *horizontal* and the *vertical* directions are most suitable. In problems involving an inclined plane, the directions *along and perpendicular to the plane* may be found convenient.

The point about which moments are to be taken should be one which lies on the lines of action of as many forces as possible, so that *as few of the forces as possible are introduced into the equation of moments.*

In case a body is hinged to a point, moment is generally taken about the hinged point so as to avoid the unknown reaction at the hinge.

If a body is hinged along a line (*e.g.*, a door), moment is taken about the line joining the hings.

Conditions of equilibrium are also satisfied if sum of the moments about three non-collinear points, in the plane of the forces, be equated to zero.

This is sometimes more convenient, the points being suitably selected. (This is separately treated in Art. 6.5).

6.3. Solution of problems. We now give a set of easy examples on the direct application of the procedure indicated above.

Example 1. *A uniform rod of weight W rests with its one end on a smooth inclined plane AB, and the other end against a smooth vertical wall BD, B being higher than A. A string is attached to the lower end of the rod and passing over the pulley at B supports a weight P. If θ be the inclination of the rod to the vertical and α the inclination of the plane, show that*

$$W \tan\theta = 2 \sec \alpha \ (P - W \sin \alpha).$$

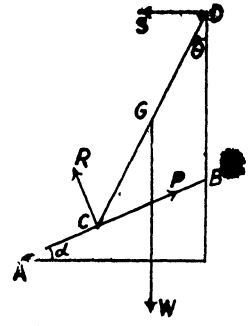
The forces acting on the rod CD (length $= 2a$) are :

S, the reaction of the wall, perpendicular to the wall at D ;

R, the reaction of the inclined plane, perp. to the plane at C.

P, the tension of the string acting along CB.

W, the weight of the rod acting vertically at its middle point G, where $CG = GD = a$.



Taking moments about C, (through which two forces R and P pass), we have

$W.a \sin \theta = S\ 2a \cos \theta$,

i.e., $S = \frac{1}{2} W \tan \theta$ ...(1)

Resolving along the plane CB, we have

$$P = W \sin \alpha + S \cos \alpha \quad ...(2).$$

From (1) and (2), we get $P = W \sin \alpha + \frac{1}{2} W \tan \theta \cos \alpha$,

i.e., $$W \tan \theta = 2 \sec \alpha \ (P - W \sin \alpha) \quad ...(3).$$

Note 1.—If the reaction R were required, one more equation could be written down by resolving forces in a direction perpendicular to the plane BC.

Note 2.—If the plane were rough, the rod may rest in this position without the help of the string. In that case P will represent the force of friction.

Note 3.—If the lower end C of the rod rests on a horizontal plane, $\alpha = 0$, so that (3) gives $W \tan \theta = 2P$, or $P = \frac{1}{2} W \tan \theta$.

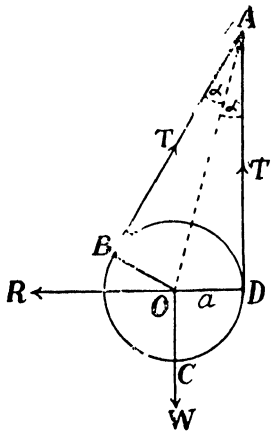
This is easy to obtain directly, for in that case $P = S = \frac{1}{2} W \tan \theta$, by (1).

Example 2. *If a circular disc BCD, of radius a and weight W, is supported by a band of inappreciable weight and thickness which surrounds the disc along the arc BCD and is fastened at its extremities to the point A in a vertical wall, the portion AD touching the wall and the plane of the disc being at right angles to the wall. If the length of the band not in contact with the disc be 2l, show that the tension of the band is*

$$\frac{W}{2} \cdot \frac{a^2 + l^2}{l^2},$$

and the reaction of the wall is Wa/l.

The band touches the disc along the portion BCD of its circumference, and forms two of its tangents AB and AD, and has the same tension throughout its length.



Let T be the tension of the band, and O, the centre of the disc, then

AB=AD=l, ∠BAO=∠DAO=α, say, so that

$$\tan\alpha=\frac{OD}{AD}=\frac{a}{l}, \text{ and } \sec^2\alpha=1+\frac{a^2}{l^2}=\frac{a^2+l^2}{l^2}.$$

The forces acting on the disc are :—

W, its weight acting vertically at O,

T, tension along the vertical portion DA of the band

T, tension along the tangent BA.

and R, the reaction of the wall acting horizontally along DO.

Taking moments about A, we have

$$R.AD=W.OD, \quad \text{or} \quad R.l=W.a, \text{ i.e. } R=Wa/l.$$

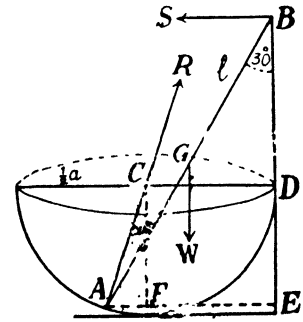
Resolving vertically, we get $T+T\cos 2\alpha=W$,

or $$T=\frac{W}{1+\cos 2\alpha}=\frac{W}{2\cos^2\alpha}=\frac{1}{2}W\sec^2\alpha=\frac{W}{2}\cdot\frac{a^2+l^2}{l^2}.$$

Example 3. *A smooth hemispherical bowl, of diameter a is placed, so that its edge touches a vertical wall, a uniform rod AB of weight W is in equilibrium inclined at 30° to the vertical with one end A resting on the inner surface of the bowl and the other end B resting against the wall, show that the length of the rod must be $a+a/\sqrt{13}$.*

Also find the reactions against the wall and the bowl.

The figure represents a vertical section of the system through the rod AB.



Let the length of the rod be $2l$.

The forces acting on the rod are :

W, its weight acting vertically at G, the middle point of the rod.

S, the reaction of the wall at B, the wall being smooth, it is perp. to the wall.

R, the reaction of the bowl at A, the bowl being smooth, it is normal to its surface and passes through the centre C.

Let R be inclined at an angle θ with the vertical.

Taking moments about A, we have

$$W.AG\sin 30°=S.BE=S.AB\cos 30°.$$

$$S = W\frac{AG}{AB}\tan 30° = \frac{W}{2\sqrt{3}} \quad \ldots(1)$$

Resolving the forces horizontally and vertically, we have

$$R \sin\theta = S \quad \ldots(2) \qquad \text{and} \qquad R\cos\theta = W. \quad \ldots(3)$$

$\therefore \tan\theta = \frac{S}{W} = \frac{1}{2\sqrt{3}}$ [by (1)]. $\therefore \cos\theta = \frac{2\sqrt{3}}{\sqrt{13}}$, $\sin\theta = \frac{1}{\sqrt{13}}$.

$\therefore$ from (3), $R = W \sec\theta = \frac{W\sqrt{13}}{2\sqrt{3}}$.

Now to find the length of the rod in terms of a, the diameter of the bowl, we use the geometrical fact that

$$AE = AF + FE = AF + CD, \text{ or } AB\sin 30° = AC\sin\theta + CD$$

$$\therefore \quad 2l \sin 30° = \tfrac{1}{2}a.\frac{1}{\sqrt{13}} + \tfrac{1}{2}a, \text{ or } 2l = \frac{a}{\sqrt{13}} + a.$$

$\therefore$ the length of the rod $= 2l = a + a/\sqrt{13}$.

Note.—The lines of action of the three forces R, S and W will meet in a point. This fact helps to solve this problem otherwise.

Examples XXI

1. A ladder, of length 40 ft. and mass 100 lbs., rests with one end on a smooth floor and the other against a smooth vertical wall. It is kept in equilibrium by a horizontal string attached to its lower end. Find the tension of the string when the ladder is inclined at 30° to the horizontal

[*Ans.* $50\sqrt{3}$ lb. wt.] (*P.U. 1952*)

2. A ladder rests at an angle α to the horizon, with its ends resting on a smooth floor and against a smooth vertical wall, the lower end being attached by a string to the junction of the wall and floor; find the tension of the string when a man whose weight is one-half of the ladder, has ascended the ladder two-thirds of its length. [*Ans.* $\frac{2}{3}\cot\alpha$] (*R.U.*)

3. A uniform beam AB of weight W rests on a smooth floor at A and on a smooth inclined plane of inclination α at B. The beam is maintained in a position inclined at θ to the horizontal by a string fastened to B which passes over a smooth pulley at the top of the plane and supports a weight P hanging freely. Show that the beam will rest in all positions if

$$2P = W \sin\alpha.$$

4. A uniform rod AB (of wt. W) rests with its ends A and B on two smooth planes inclined to the horizontal at angles of 45° and 30° respectively, in such a way that the vertical plane through the rod is perpendicular to both the inclined planes. Find what weight fixed to the rod at a quarter of its length from the end B will suffice to enable the rod to rest horizontally.

[*Ans.* $2W\sqrt{3}/3$] (*P.U. 1956*)

5. A uniform rod AB of weight 20 lbs. and of length 3 ft., is freely hinged at one end A and a weight of 20 lbs. is hung from the other end B; the rod is supported in a horizontal position by a string BC attached to a fixed point C 15 ins. vertically above A. Prove that the reaction at A is inclined to the horizon at an angle $\tan^{-1}(\frac{5}{3})$. (*P.U.*)

6. A uniform rod AB of weight W is smoothly hinged to a fixed support at A and weight 2W is suspended from B; the rod is held horizontally by a string from B attached to a point C vertically above A. If AB=AC, find the magnitude of the reaction at A and show that its line of action makes $\tan^{-1}(\frac{1}{3})$ with the horizontal. [*Ans.* 2·55 W.]

7. A heavy uniform rod, of weight W is hung freely from a fixed point by two equal strings, one attached to each end of the rod and a weight, w, is hung halfway between the middle point and one end of its rod. Show that the ratio of the tensions in the strings is

$$(2W+3w)/(2W+w).$$

8. A uniform bar AB of weight W free to turn about A, has a small smooth light ring fixed to it at B. A light string attached to a point C on a level with A and such that AC=AB, passes through the ring and carries a particle of weight W/10 at the other end. Show that in the equilibrium position the bar is inclined to the vertical at an angle $\sin^{-1} 1/8$.

9. AB is a smooth wire fixed in a position inclined at an angle α to the vertical with B above A. A small ring of weight W, capable of sliding freely on the wire, is connected to B by a string which passes through a second smooth ring of equal weight hanging freely on the string. Show that the tension of the string is $\frac{1}{2}W\sqrt{1+9\cot^2\alpha}$. (*D.U. 1956*)

10. (*a*) A sphere of given weight W, rests between two smooth planes, one vertical and the other inclined at a given angle α to the vertical; find the reactions of the planes. [*Ans.* W cot α, W cosec α.] (*P.U.*)

(*b*) A uniform right circular cylinder of weight W is supported between two fixed planes one of which is vertical and the other inclined to the vertical at 60°. If the contacts at A and B are smooth, find the reactions at A and B. [*Ans.* $\frac{1}{3}W\sqrt{3}$, $\frac{2}{3}W\sqrt{3}$.] (*P.U.*)

11. A beam AB of weight W is divided by its centre of gravity C into two portions AC and CB, whose lengths are a and b respectively. The beam rest in a vertical plane on a smooth floor AD and against a smooth vertical wall DB. A string is attached to a hook at D and the beam at a point P. If T be the tension of the string and θ and φ be the inclinations of the beam and the string respectively to the horizon, show that

$$T=W\frac{a\cos\theta}{(a+b)\sin(\theta-\varphi)}. \quad (P.U.)$$

12. A thin ring of radius R and weight W is placed round a vertical cylinder of radius r and is prevented from falling by a nail projecting horizontally from the cylinder. Show that the horizontal pressure between the cylinder and the ring is

$$Wr/2\sqrt{R^2-r^2}.$$

13. A gate weighing W hangs by hinges a ft. apart, the C.G. of the gate is distant b ft. horizontally from the vertical line through the hinges. Assuming that the whole weight is taken by the lower hinge, prove that the reactions at the hinges are

$$\frac{b}{a}W \quad \text{and} \quad \frac{W}{a}\sqrt{a^2+b^2}. \quad (P.U.)$$

14. (*a*) A rectangular gate of weight 40 lbs. is 8 ft. wide and 4 ft. high. It is supported upon two hinges one 6 ins. below the top and the other 6 ins. above the bottom of one of the vertical sides. If all the weight of the gate is taken by the lower hinge, find the reaction at the upper hinge and the vertical and horizontal reactions of the lower hinge. [*Ans.* 53⅓ lbs. wt.; 40, 53⅓ lbs. wt.]

(*b*) A gate 6 ft. high and 8 ft. wide weighs 112 lbs. and is supported by two hinges, one foot from the bottom and top. The lower hinge can exert only

a horizontal reaction. Find the reactions at both hinges if a boy of weight 52 lbs. is sitting on the end of the gate.

[*Ans.* 216 ; 271·2 lbs. at $\tan^{-1}\frac{41}{54}$ with the hor.]

15. A heavy rectangular board ABCD of uniform thickness and of weight W is supported at the middle point of AB, about which it can turn freely in a vertical plane. Weights P, Q (P>Q) are attached by strings to the corners B and D and hung freely. Prove that in the position of equilibrium, the inclination of AB to the horizon will be

$$\tan^{-1}\frac{a(P-Q)}{b(W+2Q)},$$

where a and b are the lengths of the sides AB and AD of the rectangle.

16. A uniform square plate ABCD of weight W is kept in equilibrium in a vertical plane with A resting on a smooth horizontal floor and B against a smooth vertical wall perpendicular to the plate by means of a force P which acts parallel to AB at a point in the side AD. If AB is inclined to the horizontal at an angle of 60°, show that the least value of P is $\frac{1}{12}$ W$(3\sqrt{3}-1)$.

17. A smooth tube of radius r has a hemispherical end. The tube is fixed vertically with the hemispherical end at the bottom. A rod of length l is placed entirely in the tube and assumes a position of equilibrium, being inclined at an angle θ to the horizontal. If α is the angle between the reaction at the lower end and the horizontal, show that

$$l\cos\theta = r(1+\cos\alpha) \quad \text{and} \quad \tan\alpha = 2\tan\theta.$$

Prove that the centre of the rod is level with the centre of the hemisphere when $r=\sqrt{3}\,l/4$.

18. A roller of 21 in. radius lies on the ground ; a uniform plank 30 in. long rests flat on the roller with one end on the ground and the other projecting over the roller, the length of the plank being at right angles to the axis of the roller. If the roller and the plank are prevented from slipping by a cord 35 in. long attached to the axle of the roller and to that end of the plank which lies on the ground, prove that the tension of the cord is 9/50 of the weight of the plank.

(*D.U. 1960*)

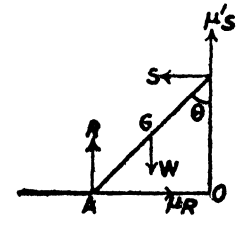
6.4. Problems involving friction

Example 1. *A uniform ladder rests against a vertical wall with its lower end on a horizontal floor. Find the greatest angle the ladder can make with the vertical if the co-efficient of friction at the lower and upper ends are respectively* μ *and* μ'.

The ladder is supposed to rest in a vertical plane perpendicular to the wall.

Let AB be the ladder, $2a$ its length, W its weight acting at its middle point G, and θ its greatest inclination with the vertical. In this position the ladder is on the point of slipping such that the end A tends to move away from O and B downwards towards O.

If R and S be the normal pressures at A and B, the friction μ R and μ'S will act as shown.



Resolving forces horizontally and vertically, we have

$S=\mu R$...(1), $\quad R+\mu' S=W$. ...(2)

Taking moments about A, we get

$S.2a\cos\theta+\mu' S.2a\sin\theta = Wa.\sin\theta$. ...(3)

From (1) and (2), we have

$$R(1+\mu\mu')=W \quad \text{and} \quad S(1+\mu\mu')=\mu W,$$

and from (3), $\tan\theta=\dfrac{2S}{W-2\mu' S}=\dfrac{2\mu W}{W(1+\mu\mu')-2\mu\mu' W}$

$$=\frac{2\mu}{1-\mu\mu'}. \qquad \ldots(4)$$

Cor. (*a*) If the wall and the floor are equally rough, $\mu=\mu'$

and $$\tan\theta=\frac{2\mu}{1-\mu^2}=\frac{2\tan\lambda}{1-\tan^2\lambda}=\tan 2\lambda,$$

i.e. $\theta=$twice the angle of friction.

(*b*) If the wall is smooth, $\mu'=0$

$\therefore \quad \tan\theta=2\mu=2\tan\lambda.$

(*c*) If the floor is smooth, $\mu=0$, $\therefore \theta=0$, *i.e.*, the ladder can rest only in a vertical position.

(*d*) In case the wall and the floor are both smooth, $\mu=\mu'=0$,

$\therefore \theta=0$, *i.e.*, the ladder cannot again rest in an inclined position, which is otherwise obvious, for the three forces that act on the ladder, do not meet.

Example 2. *If the ladder rests in the* **limiting** *position as given by equation (4) of Ex. 1, how high can a man ascend without its slipping?*

Let the ladder be on the point of slipping when the man has ascended a distance x. Let w be the weight of the man, the equations now become

$$S=\mu R \ldots(5), \qquad R+\mu' S=W+w \qquad \ldots(6)$$

and $$S.2a\cos\theta+\mu' S.2a\sin\theta=W.a\sin\theta+w.x\sin\theta, \qquad \ldots(7)$$

From (5) and (6), $S(1+\mu\mu')=\mu(W+w)$.

Substituting this value of S, and the value of θ as given in equation (4), in equation (7), we obtain, after simplification, $x=a$.

So that the man, whatever his weight may be, cannot ascend more than half the length of the ladder.

Note. It may be seen that this result remains the same even when the wall is smooth, *i.e.*, $\mu'=0$, as in (*b*) Ex. 1.

Example 3. *If the ladder is placed making an angle θ with the vertical and if the wall is smooth, show that it can rest in equilibrium only if $\mu>\frac{1}{2}\tan\theta$.*

In this particular given position of the ladder, let F be the force of friction at A.

Since $\mu'=0$, equations (1), (2) and (3) now become

$$S=F, \quad R=W, \text{ and } S.2a\cos\theta=W.a\sin\theta.$$

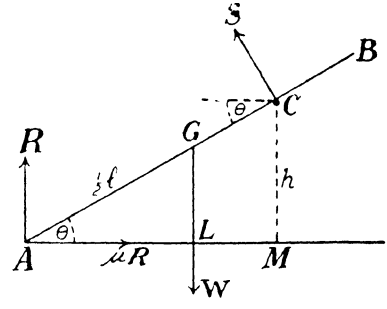
$\therefore \quad R=W, \quad F=S=\frac{1}{2}W\tan\theta,$ and for equilibrium $F<\mu R$.

$$\therefore \quad \mu>\frac{F}{R}, \text{ i.e. } >\frac{\frac{1}{2}W\tan\theta}{W}, \text{ i.e. } >\frac{1}{2}\tan\theta.$$

Example 4. *A uniform rod of length l rests in a vertical plane against (and l over) a smooth horizontal bar at a height h, the lower end of the rod being on level ground. Show that if the rod is on the point of sliping when its inclination to the horizontal is θ then the co-efficient of friction between the rod and the ground is*

$$\frac{l \sin 2\theta \sin \theta}{4h - l \sin 2\theta \cos \theta}$$

Let W be the weight of the rod AB, and C its point of contact with the horizontal bar, then



MC=h, and AC=h cosec θ.

The forces acting on the rod are :

W, its weight, acting at its middle point G,

S, the reaction of the smooth bar, perp. to the bar and the rod,

R, the normal reaction of the ground,

and μR, the limiting force of friction.

Taking moments about A, we have

W.AL=S.AC or W.$\frac{1}{2}$ l cos θ=S.h cosec θ.

$\therefore$ S=lW. sin θ cos θ/2h=lW sin 2θ/4h. ...(1)

Resolving forces horizontally and vertically, we get

μR=S sin θ, R+S cos θ=W.

$\therefore$ S sin θ=μR=μ(W−S cos θ)=μW−μS cos θ. ...(2)

$\therefore$ μW=S (sin θ+μ cos θ)

$$= \frac{l\text{W} \sin 2\theta}{4h} (\sin \theta + \mu \cos \theta), \quad [\text{from (1)}]$$

or $4h\mu = l \sin 2\theta \sin \theta + l \sin 2\theta . \mu \cos \theta$.

$\therefore$ $\mu(4h - l \sin 2\theta \cos \theta) = l \sin 2\theta \sin \theta$.

Examples XXII-A

1. (*a*) A uniform ladder of weight W is resting in equilibrium with it ends on a rough horizontal floor and against a smooth vertical wall and is making an angle θ with the vertical. Find the force of friction between the ladder and the floor. [*Ans.* ½ W tan θ.]

(*b*) If the ladder is on the point of slipping, show that the co-efficient of friction between the ladder and the ground is ½ tan θ.

Also prove that the least horizontal force applied at the foot of the ladder that will cause it to slip up is W tan θ.

2. A uniform ladder rests in limiting equilibrium with one end on a rough horizontal plane and the other against a smooth vertical wall. A man ascends the ladder, show that he cannot go more than half-way up. (*D. U. Pass 1958*)

3. A man weighing 140 lbs. climbs up a uniform ladder, 20 ft. long and 70 lbs. in weight, which rests against a rough vertical wall at an angle of 45°. If the co-efficient of friction at each end of the ladder is ½, how far will the man be able to climb up the ladder before it begins to slip ? (*Ans.* 13 ft.)

4. A ladder which stands on horizontal ground, leaning against a vertical wall, is so loaded that its centre of gravity is at distances a and b from its lower and upper ends respectively. Show that if the ladder is in limiting equilibrium, its inclination θ to the horizontal is given by $\tan\theta=\frac{a-b\mu\mu_1}{(a+b)\mu}$, where μ, μ_1 are the co-efficients of friction between the ladder and the ground and between the ladder and the wall. *(D. U.)*

(a) If the length of ladder is $2l$ and if its weight acts at a point distant kl below the middle point, prove that its limiting inclination to the vertical is given by $\cot\theta=\cot 2\lambda-k\operatorname{cosec}2\lambda$. *(D.U. 1959)*

5. A uniform ladder of length 15 ft. and mass 50 lbs. is placed with its upper end against a smooth vertical wall and its lower end on a rough horizontal plane at a distance of 3 ft. from the wall; a man of mass 10 stone stands at the top. Find the least co-efficient of friction in order that the ladder shall not slip. [*Ans.* 0·177]

6. A uniform ladder of weight w rests on a rough horizontal ground and against a smooth vertical wall. If α be the inclination of the ladder to the horizon, prove that a man of weight W can climb to the top without the ladder slipping if $\mu>\frac{w+2W}{2(w+W)}\cot\alpha$, where μ is the co-efficient of friction. *(R.U.)*

7. A uniform ladder of weight W is placed with one end resting against a smooth vertical wall and the other end on rough horizontal floor, the ladder making an angle α with the wall. Show that if the co-efficient of friction μ is less than $\frac{1}{2}\tan\alpha$, the ladder cannot remain in equilibrium alone, but that equilibrium may be maintained by placing a rough block on the floor at the foot of the ladder, provided that its weight exceeds $W\left(\frac{1}{2\mu}\tan\alpha-1\right)$, the co-efficient of friction between the block and the floor being also μ. *(H. C.)*

8. A uniform rod of weight W rests with its one end against a rough inclined plane AB, of inclination α, and the other end against a smooth vertical wall BD, B being higher than A. If θ be the inclination of the rod to the vertical in the limiting position of equilibrium, show that

$$\tan\theta=2\tan(\lambda-\alpha),$$

where λ is the angle of friction for the plane. [Compare Ex. 1. Art. 6·3]

9. A uniform ladder, of length l and weight W, rests with its foot on rough ground, and its upper end against a smooth vertical wall, the inclination of the ladder with the vertical being α. A force P is applied horizontally to the ladder at a point distant c from the foot, so as to draw the foot to the wall. If μ be the co-efficient of friction at the foot, prove that P must exceed

$$\frac{Wl}{l-c}(\mu+\tfrac{1}{2}\tan\alpha). \qquad (P.\ U.)$$

10. A uniform ladder of length $4a$ rests at an angle α to the horizontal against a smooth horizontal rail at a height h from the ground. If λ be the angle of friction between the ground and the ladder, show that a man of twice the weight of the ladder may ascend a distance

$$3h\sin\lambda\operatorname{cosec}(\alpha+\lambda)\operatorname{cosec}2\alpha-a. \qquad (D.U.\ 1942)$$

11. One end A of a heavy uniform rod AB can slide along a rough horizontal rod AC to which it is attached by a ring · B and C are joined by a string. If ABC be a right angle when the rod is on the point of sliding, μ the co-efficient of friction and α the angle between AB and the vertical, show that

$$\mu=\frac{\tan\alpha}{2+\tan^2\alpha}. \qquad (D.\ U.\ 1940)$$

12. Two rough particles connected by a light string rest on an inclined plane. If their weights and corresponding co-efficients of friction are W_1, W_2,

μ_1, μ_2 respectively, show that the greatest inclination of the plane for equilibrium is

$$\tan^{-1}\frac{\mu_1 W_1+\mu_2 W_2}{W_1+W_2}. \qquad (P.U.)$$

13. A uniform rod slides with its ends on two fixed equally rough rods, one being vertical and the other inclined at an angle α to ths horizon. Show that the inclination θ to the horizon of the movable rod, when it is on the point of sliding, is given by

$$\tan\theta=\frac{1\mp 2\mu\tan\alpha-\mu^2}{2(\tan\alpha\pm\mu)}.$$

14. A small ring of weight w, fixed to one end A of a uniform rod AB of weight W, can slide along a rough rod CAX which is fixed horizontal. A string CB of length equal to AB connects B to a fixed point C in CAX. Prove that AB can be in equilibrium inclined at an angle θ to the vertical if

$$W\tan\theta<\mu(3W+4w)$$

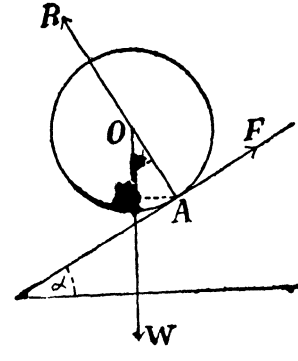
where μ is the co-efficient of friction between the ring and the rod CAX.

15. (*a*) A light shallow rectangular drawer is of width a, while its length from front to back is l. It can be pulled out by two knobs attached to the front at equal distances f from the sides. If a tension parallel to the sides is applied to only one of the knobs, prove that the drawer will jam if the co-efficient of friction between the sides and casing exceeds $l/(a-2f)$. (*I. S. E.*)

(*b*) A drawer of length l is to be pulled out by a single force P parallel to the drawer. Show that if the force is applied at a distance greater than $l/2\mu$ from the middle of the end, the drawer will not move however great P may be.

6.41. *A circular cylinder of radius a and weight W rests with its axis horizontal on a rough inclined plane of inclination* α. *Show that the cylinder cannot be held at rest by a couple alone if* $\mu<\tan\alpha$ *and find the moment of the couple that will hold it at rest if* $\mu>\tan\alpha$.

The figure represents a section by a vertical plane passing through the middle point O of the axis of the cylinder.



Let F be the resultant friction acting along the plane and let R be the normal reaction of the plane, on the cylinder.

Resolving along and perp. to the plane, we have $F=W\sin\alpha$, and $R=W\cos\alpha$,

$$\therefore \quad \frac{F}{R}=\tan\alpha.$$

If μ be the co-efficient of friction and λ, the angle of friction, then in order that the cylinder may rest and may not *slide*,

$$F<\mu R \text{ or } \mu>\frac{F}{R}, \text{ } i.e., \text{ } \mu>\tan\alpha \text{ [or } \tan\lambda>\tan\alpha, \text{ } i.e. \text{ } \lambda>\alpha]$$

If this condition is satisfied, the equilibrium cannot break by way of sliding of the cylinder, so that the cylinder can only roll down the plane.

That it will do so is evident from the fact that the three forces that act on the cylinder do not meet at a point.

Now G, the moment of the couple that makes the cylinder roll
$= moment\ of\ W\ about\ A$
$= W.AN = W.OA\ sin\ AON = Wa\ sin\ \alpha.$

Thus if the cylinder is to be held at rest, a couple of moment equal to $Wa \sin \alpha$ has to be applied to balance the couple G.

Again if $\mu < \tan \alpha$, or $\lambda < \alpha$, *i.e.*, the angle of friction is less than the inclination of the plane, the cylinder will slide down unless, besides the couple, some force is applied to hold it at rest. so that a couple alone cannot hold it at rest.

Examples XXII-B

1. A sphere of weight W is placed on a rough plane, inclined to the horizontal at an angle θ which is less than the angle of friction, show that a weight $\frac{W \sin \theta}{\cos \theta - \sin \theta}$, fastened to the sphere at the upper end of the diameter which is parallel to the plane, will just prevent the sphere from rolling down the plane. (*P. U.*)

2. A solid hemisphere of weight W rests in limiting equilibrium with its curved surface on a rough inclined plane, and its plane face is kept horizontal by a weight P attached to a point in its rim. Prove that the co-efficient of friction is $P/\sqrt{W(2P+W)}$. (*D.U. 1956*)

3. A sphere rests on a rough horizontal plane and its highest point is joined to a peg fixed in the plane by a tight cord parallel to the plane. Show that, if the plane be gradually tilted about a line in it perpendicular to the direction of the cord, the sphere will not slip until the inclination becomes equal to $\tan^{-1} 2\mu$, where μ is the co-efficient of friction.

4. A uniform ring of weight W, has a bead of weight w attached to the rim. The ring is placed over a rough peg. Show that it will hang in equilibrium with any point in contact with the peg, provided the angle of friction between the peg and the ring is greater than

$$\sin^{-1} w/(W+w). \qquad (P.U.)$$

5. A circular hoop of weight W, hangs over a horizontal rough peg and a weight $\frac{3}{2}$ W hangs tangentially from it. Find the co-efficient of friction if the hoop is about to slip on the peg. (*Ans.* $\frac{1}{4}$)

6. A solid homogeneous hemisphere rests on a rough horizontal plane and against a smooth vertical wall. Show that if co-efficient of friction be greater than $\frac{3}{8}$, the hemisphere can rest in any position, and, if it be less, the least angle that the base of the hemisphere can make with the vertical is $\cos^{-1} \frac{8}{3}\mu$. (*P.U. Hons*).

7. A hemisphere rests in equilibrium on rough ground and against an equally rough vertical wall, the co-efficient of friction being μ. Show that if the equilibrium is limiting, the inclination of the base to the horizon is

$$\sin^{-1} \frac{8\mu\ (1+\mu)}{3(1+\mu^2)}. \qquad (P.U.\ 1952)$$

8. A circular cylinder of radius a has its centre of gravity G at a distance $\frac{1}{2}a$ from the middle point O of the axis of the cylinder and equidistant from the plane ends. The cylinder rests in equilibrium on a rough inclined plane with OG horizontal, the axis of the cylinder being horizontal. Find the inclination of the plane to the horizontal and the least permissible value of the co-efficient of friction. (*Ans.* 30°, $1/\sqrt{3}$)

9. A homogeneous hemisphere rests in limiting equilibrium with its curved surface on a rough inclined plane such that its plane face is vertical. Show that the co-efficient of friction is $3/\sqrt{55}$. (*P.U.*)

10. A heavy ball has a fine vertical hole drilled through it, along a chord subtending an angle 2α at the centre, and it can slide up and down a vertical wire which fits loosely in the hole. If λ be the angle of friction prove that the ball can remain in equilibrium without any other support, if α is not greater than λ. (*All. U.*)

11. (*a*) A circular cylinder of radius a and weight W rests with its axis horizontal on a rough horizontal plane and in contact with an equally rough vertical surface. Show that the least value of the couple to rotate the cylinder is

$$\mu(1+\mu)a\ W/(1+\mu^2).$$

(*b*) If the cylinder is placed on two equally rough planes each inclined at an angle α to the horizontal then the least couple to turn it about its axis is $\frac{1}{2}Wa \sin 2\lambda \sec \alpha$.

6.5. Other forms of the conditions of equilibrium.

The conditions of equilibrium may also be given in other forms such as the following :—

I. *A system of coplanar forces will be in equilibrium if the sum of the moments of the forces about each of two different coplanar points, say A and B, vanishes and the sum of the resolved parts of the forces in the direction of AB or in some direction not perpendicular to AB also vanishes.*

A system of coplanar forces, by Art, 4·61, reduces to either a single couple or a single force as its resultant.

Since the sum of the moments about a point (A or B) vanishes the system cannot evidently reduce to a couple. It may, reduce to a force, F, and the sum of the moments about both the points A and B being given equal to zero, F must act along AB. But the given condition requires that the sum of the resolved parts of the forces along AB or along a line making an angle θ ($\theta \neq 90°$) with AB vanishes, it means, that F $\cos\theta = 0$, *i.e.*, $F=0$. Hence the forces must be in equilibrium

To prove this analytically, let the points A and B be (0, 0,), (a, 0). Substituting in the expression $G - xY + yX$, and equating to zero, we have $G=0$ and $G - aY = 0$, $\therefore\ aY = 0$ or $Y = 0$ ($\because\ \neq 0$ and $G=0$).

The third condition that the sum of the resolved parts of the forces along any line not perpendicular to AB (making $\angle\theta$ with OX) vanishes, requires that

$$X \cos\theta + Y \sin\theta = 0$$

But $Y=0$ $\therefore\ X=0$. Hence the system is in equilibrium.

II. *A system of coplanar forces will be in equilibrium if the sum of the moments of the forces about each of three different points, in their plane, not all lying in the same straight line, separately vanishes.*

We know that a system of coplanar forces has either a single couple or a single force as its resultant.

Since the sum of the moments about one of the points, say A, vanishes, the system cannot reduce to a couple but it may reduce to a force, say F acting at A.

Again, since the sum of the moments about a second point, say B, also vanishes, the resultant force F, if it exists, must also pass through B and must act along the line AB.

Further, since the sum of the moments about a third point, say C vanishes, F must also pass through C, a point not lying on AB, which is impossible. Hence F itself must be zero and the system must be in equilibrium.

These conditions can also be proved analytically as follows :

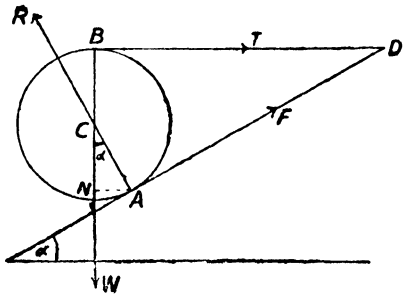
Let $(0, 0)$, (x_1, y_1) and (x_2, y_2) be the three arbitrarily chosen points. Substituting these points in $G - xY + yX$, an expression giving the sum of the moments of the system about any coplanar point (x, y), and equating to 0, we have

$G=0$, $\quad x_1Y - y_1X = 0$, $\quad$ and $\quad x_2Y - y_2X = 0$. $\quad (\because\ G=0)$

From which we get $X=0$, $Y=0$, unless $-\frac{y_1}{x_1} = -\frac{y_2}{x_2}$, *i.e.*, A, B and C are collinear, which is not so.

Example 1. *A full reel of cotton of radius a and weight W is kept at rest with the axis horizontal on a plane of inclination α, by holding the thread which comes off horizontally and tangentially from the middle of the reel. Find the tension of the thread and friction between the reel and the plane. Find also the least co-efficient of friction for which the reel will rest in this way.*

The reel is full, so that it can be treated as a cylinder, the circle ABC is its vertical section through the middle of the reel, A being the point of contact with the plane, B the highest point and C the centre.



The forces acting on the reel are :—

The tension T acting horizontally, meeting the plane in D,

W, the weight acting vertically at C,

the resultant normal reaction R of the plane,

and the resultant frictional force F along the plane.

Taking moments about C, we have

$T.CB = F.CA$ *i.e.*, $T = F$. ...(1)

Taking moments about D, we get

$R.AD = W.BD$ *i.e.*, $R = W$ ...(2) $(\because\ AD = BD)$

Taking moments about A, we have

$W.AN = T.BN = T(BC + CN)$ *i.e.*, $W.a \sin\alpha = T(a + a\cos\alpha)$,

or $$T = \frac{\sin\alpha}{1+\cos\alpha} W = W \frac{2\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha}{2\cos^2\frac{1}{2}\alpha} = W\tan\frac{\alpha}{2}.$$

$$\therefore\ F = T = W\tan\frac{\alpha}{2},\ \therefore\ \frac{F}{R} = \frac{F}{W} = \tan\frac{\alpha}{2}.$$

Now for equilibrium, $F<\mu R$ *i.e.*, $\mu > \frac{F}{R}$ *i.e.*, $> \tan\frac{\alpha}{2}$.

Hence the least value of $\mu = \tan\frac{\alpha}{2}$.

If λ be the angle of friction, then $\tan\lambda = \mu = \tan\frac{1}{2}\alpha$,

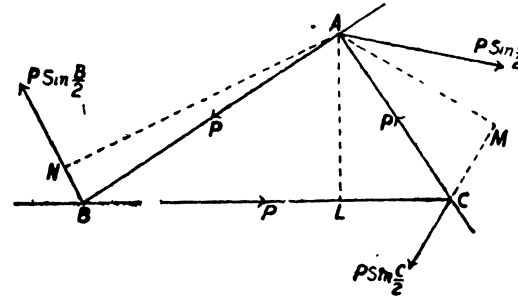
$\therefore \quad \alpha = 2\lambda$, and $T = W \tan\frac{1}{2}\alpha = W \tan\lambda$.

Note.—The solution has been simplified by taking moments about three non-collinear points.

It can be solved also by using (1) or (2) and resolving forces along and perp. to the plane.

Example 2. *Three forces each equal to P act along the sides of a triangle ABC in the same sense and forces* $P \sin\frac{A}{2}$, $P \sin\frac{B}{2}$, $P \sin\frac{C}{2}$ *act along the bisectors of the exterior angles A, B, C respectively, in the same sense, opposite to that of the forces along the sides. Show that all the forces are in equilibrium.*

AL is perp. to BC ; AM, AN are perp. to CM, BN, bisectors of the exterior angles C and B.



$\angle ACM = \frac{1}{2}(180° - C)$

$= 90° - \frac{1}{2}C$, and

$\angle ABN = 90° - \frac{1}{2}B$.

Taking moments about A, we have

$P.AL - P \sin\frac{C}{2}.AM - P \sin\frac{B}{2}.AN$

$= P.b \sin C - P \sin\frac{C}{2}.b \sin\left(90° - \frac{C}{2}\right) - P \sin\frac{B}{2}.c \sin\left(90° - \frac{B}{2}\right)$

$= Pb \sin C - Pb \sin\frac{C}{2}\cos\frac{C}{2} - Pc \sin\frac{B}{2}\cos\frac{B}{2}$.

$= Pb \sin C - \frac{1}{2} Pb \sin C - \frac{1}{2} Pc \sin B$

$= Pb \sin C - Pb \sin C \quad (\because c \sin B = b \sin C)$

$= 0$.

Similarly it can be shown that the sum of the moments about B and C vanishes. Hence the forces are in equilibrium, A, B, C being non-collinear points.

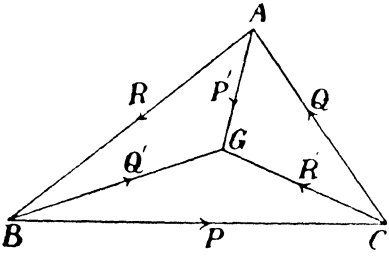
Note.—This method simplifies the solution, for the forces need not now be resolved in two directions.

Example 3. *Forces P, Q, R acting along the sides BC, CA, AB of a triangle ABC are in equilibrium with forces P', Q', R' acting along*

AG, BG, CG, G being the centroid of the triangle. Prove that

$$\frac{PP'}{AG.BC} + \frac{QQ'}{BG.CA} + \frac{RR'}{CG.AB} = 0.$$

Since the given six forces are in equilibrium, the sum of their moments about three non-collinear points, say A, B, C, must vanish.



Let $\triangle$ denote the area of the triangle ABC, then
area GBC = area GCA = area GAB $= \frac{1}{3}\triangle$.

The distance of BC from A $= \frac{2\triangle}{BC}$, and the distances from A of BG and CG are $\frac{2\triangle AGB}{BG} = \frac{2\triangle}{3BG}$ and $\frac{2\triangle AGC}{CG} = \frac{2\triangle}{3CG}$.

Now we use these distances in equating to zero, the sum of the moments of the six forces about A, (the moments of forces Q, R, P', passing through A being zero), we get

$$P.\frac{2\triangle}{BC} + Q'.\frac{2\triangle}{3BG} - R'\frac{2\triangle}{3CG} = 0$$

or

$$\frac{P}{BC} + \frac{Q'}{3BG} - \frac{R'}{3CG} = 0. \qquad \ldots(1)$$

Similarly by taking moments about B and C, we have

$$\frac{Q}{CA} + \frac{R'}{3CG} - \frac{P'}{3AG} = 0, \qquad \ldots(2)$$

and

$$\frac{R}{AB} + \frac{P'}{3AG} - \frac{Q'}{2BG} = 0. \qquad \ldots(3)$$

Multiplying (1), (2) and (3) respectively by $-\frac{P'}{AG}$, $\frac{Q'}{BG}$, $\frac{R'}{CG}$, and adding, we have

$$\frac{PP'}{AG.BC} + \frac{QQ'}{BG.CA} + \frac{RR'}{CG.AB} = 0,$$

other terms cancelling out in pairs.

Examples XXIII

1. A heavy circular disc whose plane is vertical is kept at rest on a rough inclined plane by a string parallel to the plane and touching the circle, show that the disc will slip on the plane if the co-efficient of friction be less than $\frac{1}{2}\tan\alpha$, where α is the slope of the plane. *(P.U. 1955 S.)*

2. Forces $\sqrt{3}$ P, $\sqrt{3}$ Q, $\sqrt{3}$ R act along the altitudes AD, BE, CF and forces R—Q, P—R, Q—P along the sides BC, CA, AB respectively of an equilateral triangle AEC. Show that the system is in equilibrium.

3. Forces 2P cos A/2, 2Q cos B/2, 2R cos C/2 act along the internal bisectors of the angles A, B, C and forces R—Q, P—R, Q—P act along the sides EC, CA, AB respectively of the triangle ABC. Show that the six forces are in equilibrium.

4. Forces P, Q, R act along the sides EC, CA, AB of a triangle ABC and forces P′, Q′, R′, act along OA, OB, OC, where O is the centre of the circumscribed circle. Prove that if the six forces are in equilibrium,

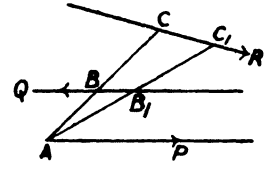
$$P \cos A + Q \cos B + R \cos C = 0,$$

and

$$\frac{PP'}{a} + \frac{QQ'}{b} + \frac{RR'}{c} = 0$$

***6.6. Three forces acting on a rigid body.** *If three forces acting on a rigid body keep it in equilibrium, they must be coplanar.*

Let P, Q, R, be the three forces, A any point on the line of action of P and B a point on that of Q.



As P and Q meet the line AB, they separately can have no tendency to turn the body about AB, and since P, Q, R are in equilibrium, they together can have no effect to turn the body about AB, so that the third force R cannot turn the body about AB.

Hence AB must either meet the force R in some point, say C, or it must be parallel to R.

But AB being arbitrary can be taken so as not to be parallel to R. Hence AB meets the force R.

Similarly if B_1 is another point on the line of action of Q, AB_1 must meet R, say in C_1, so that the line of action of R must lie in a plane ABB_1 or in other words the lines of action of *Q and R* are *coplanar and they lie in a plane* which passes through A.

Similarly by taking another point A_1 on P, it can be proved that Q and R are coplanar and that their plane meets the line of action of P in A_1 so that the plane in which Q and R lie, contains two points A and A_1 on the line of action of P *i.e.*, it contains the whole line of action of P.

Hence the three forces are coplanar.

6.61. *If three forces acting on a rigid body keep it in equilibrium, they must either meet in a point or be parallel.*

By Art. 6.6 above, the forces must be coplanar.

Let P, Q, R, be the three forces acting on a *rigid body*. P, Q, R being in equilibrium, one of the them, say R, must be equal and opposite to the resultant of the other two P and Q.

(*i*) If P and Q are parallel, their resultant is a force parallel to them and therefore the force R, being equal and opposite to this resultant, must have its line of action parallel to P and Q.

(*ii*) If P and Q meet in a point, say O, their resultant is a force passing through O, and therefore R, being equal and opposite to this resultant, must have its line of action passing through O.

Hence the three forces are either parallel or meet at a point.

6.62. Conditions of equilibrium of three forces acting on a rigid body.

From the above (Art. 6·6, 6·61) and articles 3·3 and 4·11 (Cor.), it follows that —

If three forces acting upon a rigid body are in equilibrium :—

(*i*) *they must be coplanar,*

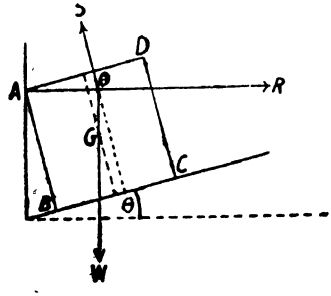
(*ii*) *they must either meet at a point or be parallel;*

(*iii*) (*a*) *when concurrent, each is proportional to the sine of the angle between the other two,*

(*b*) *when parallel, each is proportional to the distance between the other two.*

Example 1. *A uniform cubical block of weight W rests on a smooth inclined plane with one of its edges horizontal and in contact with a smooth vertical wall. The inclination of the plane is* θ. *Find the pressure on the plane and on the wall.*

ABCD is the section of the block made by a vertical plane through its centre of gravity G, meeting the inclined plane in BC and the wall in A.



The forces acting on this block are :—

(*i*) the resultant reaction R, of the wall acting at A along AO perp. to the wall, the wall being smooth ;

(*ii*) the weight W acting along a vertical line through G ;

(*iii*) the resultant reaction S of the inclined plane perp. to BC, the plane being smooth.

The block being in equilibrium under three forces, the forces must lie in the same plane ABCD, and must meet at a point, so that the reaction S of the inclined plane is a force perp. to BC and passing through O, the point of intersection of R and W. (*Note that S does not pass through G as it is sometimes wrongly supposed*).

From the figure we have

Since $\angle OAD = \theta$, $\therefore$ $\angle AOS = 90° - \theta$, $\angle SOR = 90° + \theta$,

and $\angle SOW = \angle GOA + \angle AOS = 90° + 90° - \theta = 180° - \theta$.

Now by Lami's theorem, we have

$$\frac{R}{\sin \angle SOW} = \frac{S}{\sin \angle WOR} = \frac{W}{\sin \angle SOR},$$

i.e.,
$$\frac{R}{\sin (180° - \theta)} = \frac{S}{\sin 90°} = \frac{W}{\sin (90° + \theta)},$$

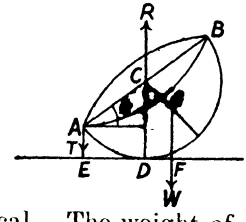
i.e.,
$$\frac{R}{\sin \theta} = \frac{S}{1} = \frac{W}{\cos \theta}$$

Hence $R = W \tan \theta$ and $S = W \sec \theta$.

Example. 2. *A hemisphere of radius a and weight W is placed with its curved surface on a smooth table and a string of length $l(<a)$ is attached to a point on its rim and to a point on the table. Find the position of equilibrium and prove that the tension of the string is*

$$\tfrac{3}{8}W\frac{a-l}{\sqrt{2al-l^2}}.$$

Let the plane face of the hemisphere be inclined at an angle θ to the horizontal and let C be its centre. Since the hemisphere is in equilibrium under only three forces *viz.*, its weight W, the reaction R of the table and the tension T of the string, the forces must lie in the same vertical plane.



Also, since the table is smooth, the reaction at the point of contact D must be normal to both, the surface of the table and that of the sphere, so that its line of action DC must be vertical. The weight of the hemisphere acting at G ($CG=\frac{3}{8}a$) being also a vertical force, the third force, the tension T must also be vertical.

Thus the hemisphere will adjust itself such that the string EA is vertical and lies in the vertical plane through C.

Now $EA=l$, and $DC-EA=a-l=AC\sin\theta=a\sin\theta$.

$$\therefore\quad \sin\theta=\frac{a-l}{a}\qquad \text{or}\quad \tan\theta=\frac{a-l}{\sqrt{2al-l^2}}.$$

Taking moments about D, we have

$$T.ED=W.DF$$

i.e.,
$$T.a\cos\theta=W.CG\sin\theta=W\tfrac{3}{8}a\sin\theta,$$

$$\therefore\quad T=\tfrac{3}{8}W\tan\theta=\frac{\frac{3}{8}W(a-l)}{\sqrt{2al-l^2}}.$$

Example 3. *Prove that the least force, which applied to a uniform heavy sphere of weight W will maintain it in equilibrium against a rough vertical wall, is W cos λ provided λ, the angle of friction, is less than*

$$\cos^{-1}\frac{\sqrt{5}-1}{2}$$

The figure represents the section of the sphere, of radius a, by a vertical plane perp. to the wall, passing through its centre C.

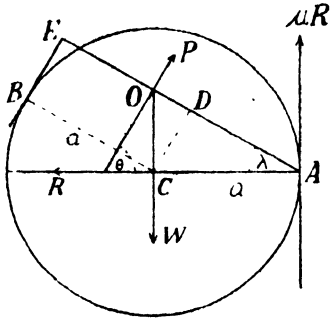
Let the required force P be such that the sphere is on the point of sliding down so that the limiting friction at the point of contact A with the wall acts upwards. (See Fig. on next page)

Let AD be the direction of the resultant reaction, *i.e.*, of the resultant of the normal reaction R and the limiting friction μR, so that $\angle CAD=\lambda$.

Let the vertical through C, along which the weight W acts meet AD in O, then the third force P must also pass through O (not through C).

Let P make angle θ with AC.

Resolving forces horizontally and vertically, we have



$P\cos\theta = R$, and $P\sin\theta + \mu R = W$,
where $\mu = \tan\lambda$.

$\therefore \quad W = P\sin\theta + \mu P\cos\theta$

$$= P\frac{\sin(\theta+\lambda)}{\cos\lambda},$$

or $$P = \frac{W\cos\lambda}{\sin(\theta+\lambda)}.$$

Now P is least when $\sin(\theta+\lambda)$ is greatest, *i.e.*, when $\sin(\theta+\lambda) = 1 = \sin 90°$ or $\theta = 90° - \lambda$, so that P is perp. to AD.

Thus the least value of P is $W\cos\lambda$.

Now in order that P may act on the sphere and may not lie outside it, the distance of its line of action from the centre C, should be less than the radius of the sphere, *i.e.*, DO $\angle$ DE or AO < AE, where E is the point where tangent to the sphere parallel to the direction of P meets AO.

But $AO = AC\sec\lambda = a\sec\lambda$,

and $AE = AD + DE = AD + CB = a\cos\lambda + a$

$\therefore \quad a\sec\lambda < a\cos\lambda + a$ or $\cos^2\lambda + \cos\lambda > 1$,

or $(\cos\lambda + \frac{1}{2})^2 > \frac{5}{4}$ or $\cos\lambda > \frac{\sqrt{5}-1}{2}$, *i.e.*, $\lambda < \cos^{-1}\frac{\sqrt{5}-1}{2}$.

Examples XXIV

1. A sphere of radius 8 ins. and weight 2 lbs. is suspended against a smooth wall by a string of length 8 ins. The string joins a point in the wall and a point on the surface of the sphere. Find the inclination and tension of the string and the reaction of the wall. (*Ans.* 30°, $T = \frac{2}{3}W\sqrt{3}$, $R = \frac{1}{3}W\sqrt{3}$)

2. A uniform solid hemisphere of weight W has a particle of the same weight attached to its rim. The hemisphere is suspended from the centre of its plane face. Prove that in the position of equilibrium, the axis of the hemisphere is inclined at $\tan^{-1}(\frac{8}{3})$ to the vertical.

3. A sphere of radius a, and weight W, rests on a smooth inclined plane supported by a string of length l with one end attached to a point on the surface of the sphere and the other end fastened to a point on the plane. If the angle of inclination of the plane to the horizontal is α, show that the tension of the string is

$$W(a+l)\sin\alpha/\sqrt{l^2+2al}.$$

4. A uniform hemisphere of radius a and weight W, rests with its spherical surface on a horizontal plan, and a rough particle of weight W′, rests on the plane surface. Show that the distance of the particle from the centre of the plane face is not greater than $3W\mu\, a/8W'$.

5. A solid cone, of height h and semi-vertical angle α, is placed with its base against a smooth vertical wall, and is supported by a string attached to its vertex and to a point in the wall; show that the greatest possible length of the string is

$$h\sqrt{(1+\tfrac{1}{9}\tan^2\alpha)}.$$

6. The altitude of a cone is h, and the radius of its base is r; a string is fastened to the vertex and to a point on the circumference of the circular base,

and is then put over a smooth peg. Show that, if the cone rest with its axis horizontal, the length of the string must be $\sqrt{h^2+4r^2}$.

7. A straight uniform rod weighing 10 lbs. of negligible cross section rests along a line of greatest slope on a plane inclined to the horizontal at an angle of 60°. The angle of friction between the rod and the plane is 30°. The rod is prevented from slipping by a horizontal force of 10 lbs. weight applied at the upper end and in the vertical plane of the rod. Find the magnitude of the resultant reaction of plane and the angle which this reaction makes with the rod.

(*Ans.* $10\sqrt{2}$ lbs., 75°)

8. A picture frame, rectangular in shape, rests against a smooth vertical wall; it is suspended from two points in the wall by parallel strings attached to two points in the upper edge of the back of the frame, the length of each string being equal to the height of the frame. Show that if the centre of gravity of the frame coincide with the centre of the figure, the picture will hang against the wall at an angle $\tan^{-1}(b/3a)$ to the vertical, where a is the height and b is the thickness of the picture.

9. A board of weight W in the shape of a regular polygon of n sides, rests with one corner A against a smooth vertical wall, the adjacent corner B being attached to the wall by a string whose length is equal to a side of the polygon. Show that the angle θ which AB makes with the wall is

$$\tan^{-1}\left(\frac{1}{3}\cot\frac{\pi}{n}\right)$$

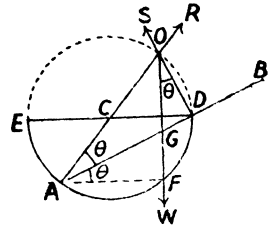
and that the pressure at A is $\frac{W}{3}\cot\frac{\pi}{n}$.

6.7. Sometimes a mere geometrical consideration gives the position of equilibrium.

6.71. *A heavy uniform smooth rod of length 2 l, rests partly within and partly outside a* **smooth** *hemispherical bowl of radius a fixed with its rim horizontal. To find the position of equilibrium of the rod.*

DEF is a section of the bowl by a vertical plane through the centre C.

In the position of equilibrium, let the rod AB be inclined at an angle θ to the horizontal.



The bowl and the rod both being smooth, the reactions at D where the rod rests on the rim, must act along DO perpendicular to the rod at D. Also the reaction R at A acts along AO, the normal at A to the surface of the bowl.

The only other force being the weight of the rod, the three forces must meet, *i.e.*, the vertical from O must bisect the rod at G.

$\angle$ADO being a right angle AO and DO must meet on the circumference of the circle DEF.

Similarly the vertical through O and the horizontal line through A must meet at F, a point on the circumference.

$\therefore$ $\quad AF = AO\cos\angle OAF.$

Also $\quad AF = AG\cos\angle GAF.$

Now $\theta = \angle GAF = \angle ADC = \angle CAD$, so that $\angle OAF = 2\theta$.

Also AO$=2a$ and AG$=l$, $\therefore$ we have

AF$=$AO $\cos 2\theta=2a\cos 2\theta$, and also AF$=$AG $\cos\theta=l\cos\theta$.

Hence $2a\cos 2\theta=l\cos\theta$,

which gives θ, the position of equilibrium of the rod.

To find the reactions R and S, we apply Lami's theorem, we have

$$\frac{R}{\sin SOW}=\frac{S}{\sin WOR}=\frac{W}{\sin SOR},$$

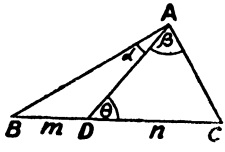
or $$\frac{R}{\sin\theta}=\frac{S}{\cos 2\theta}=\frac{W}{\cos\theta}.$$

For, $\angle SOW=180°-\angle GOD=180°-\theta$,

$\angle WOR=180°-\angle AOG=180°-(90°-2\theta)=90°+2\theta$,

and $\angle SOR=\angle AOD=90°-\angle OAD=90-\theta$.

6·8. The following trigonometrical formulae sometimes prove very useful for finding position of equilibrium of bodies.



If D be any point in the base BC of a triangle ABC, and if BD$=m$, DC$=n$,

$\angle BAD=\alpha$, $\angle DAC=\beta$,

then the angle $\theta=\angle ADC$ is given by

$$(m+n)\cot\theta=m\cot\alpha-n\cot\beta \quad ...(1)$$

or $$(m+n)\cot\theta=n\cot B-m\cot C. \quad ...(2)$$

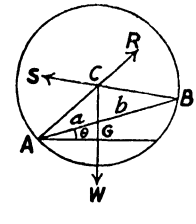
In particular if D bisects BC,

$$2\cot\theta=\cot\alpha-\cot\beta \text{ and } 2\cot\theta=\cot B-\cot C.$$

The student is advised to prove and remember these results.

6·81. *A heavy rod whose centre of gravity divides it into two parts a and b is placed within a* **smooth** *sphere. The rod subtends an angle 2α at the centre. To find its position of equilibrium.*

The surface of the sphere being smooth, the reactions R and S at the ends A and B of the rod must act along the radii AC and BC. The only other force being its weight, the vertical line from C must pass through G such that AG$=a$ and GB$=b$.



Since the rod subtends an angle 2α at the centre,

$\therefore$ $\angle CAB=\angle CBA=90°-\alpha$.

Let θ be the angle which the rod makes with the horizontal, then $\angle CGB=90°-\theta$.

Using formula (2), Art. 6·8, we have

$$(a+b)\cot\angle CGB=b\cot\angle CAB-a\cot\angle CBA,$$

or $$(a+b)\tan\theta=b\tan\alpha-a\tan\alpha.$$

$\therefore$ the position of equilibrium is given by

$$(a+b)\tan\theta=(b-a)\tan\alpha.$$

Cor. *If the rod were uniform i.e., $a=b$, $\theta=0$, i.e., a uniform rod*

cannot rest in a position other than the horizontal unless the surface of the bowl is rough instead of smooth.

Note. To find R and S, we take moments about A and B.

Since the distance of A from BC
$= AB \sin ABC = (a+b) \sin (90°-\alpha)$,

$\therefore$ $S.(a+b) \sin (90°-\alpha) = W.a \cos \theta$ *i.e.*, $S = W \dfrac{a \cos \theta}{(a+b) \cos \alpha}$,

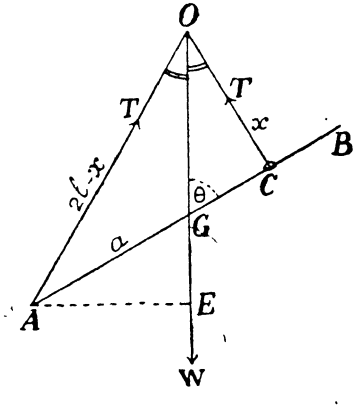
similarly, $R.(a+b) \sin (90°-\alpha) = W.b \cos \theta$ *i.e.*, $R = W \dfrac{b \cos \theta}{(a+b) \cos \alpha}$.

Note. We can also apply Lami's theorem to find R and S using $\angle ACG = 90° - \angle CAG = 90° - (90° - \alpha + \theta) = \alpha - \theta$, $\angle BCG = 2\alpha - (\alpha - \theta) = \alpha + \theta$.

***6.811.** *One end of a string, of length $2l$, is attached to one end A of a smooth uniform rod, of length $2a$, and the other end of the string is attached to a light ring C, which slides on the rod. The rod is suspended by the string passing over a smooth peg O ; prove that in the position of equilibrium, the inclination θ of the rod to the vertical is given by*

$$a \sin^3 \theta = l \cos \theta.$$

Since the rod is smooth and the tension along CO balances the reaction of the rod on the ring which is normal to the rod, therefore, the portion CO of the string must be perp. to the rod.



$\therefore$ if $OC = x$, $GC = x \cot \theta$. ...(1)

Also the peg being smooth, the tension throughout string will be the same.

W, the weight of the rod, acting vertically, balances the resultant of two equal tensions along AO and CO.

$\therefore$ OG, G being the middle point of the rod, is vertical and bisects the angle AOC.

$\therefore$ $\angle AOG = \angle GOC = 90° - \theta$, ($\because$ $\angle OGC = \theta$).

$\therefore$ $AE = AG \sin AGE = a \sin \theta$,

also $AE = AO \sin AOG = (2l - x) \cos \theta$.

$\therefore$ $a \sin \theta = (2l - x) \cos \theta$, ...(2)

or $x \cos \theta = 2l \cos \theta - a \sin \theta$. ...(3)

Now by (1), Art. 6·8,

$(AG + GC) \cot \theta = AG \cot AOG - GC \cot COG$

or $(a + x \cot \theta) \cot \theta = a \cot (90° - \theta) - x \cot \theta \cot (90 - \theta)$ [using (1)]

$= a \tan \theta - x \cot \theta \tan \theta = a \tan \theta - x$

or $a(\tan \theta \cdot \cot \theta) = x(1 + \cot^2 \theta) = x \operatorname{cosec}^2 \theta$,

or $x \cos \theta = a(\tan \theta - \cot \theta) \sin^2 \theta \cos \theta = -a \cos 2\theta \sin \theta$ ...(4)

$\therefore$ from (3) and (4), we have

$$2l\cos\theta - a\sin\theta = x\cos\theta = -a\cos 2\theta\sin\theta,$$

or $$2l\cos\theta = a\sin\theta(1-\cos 2\theta) = 2a\sin^3\theta.$$

Hence $$l\cos\theta = a\sin^3\theta.$$

Aliter. Geometrically, we have

$$x = \text{AO}\cos\text{AOC} = (2l-x)\cos(180^\circ - 2\theta) = -(2l-x)\cos 2\theta \quad \ldots(5)$$

Multiplying (2) by $\cos 2\theta$ and (5) by $\cos\theta$ and adding, we get

$$x\cos\theta + a\sin\theta\cos 2\theta = 0 \quad \ldots(6)$$

Now from (3) and (6),

$$2l\cos\theta - a\sin\theta = x\cos\theta = -a\sin\theta\cos 2\theta$$

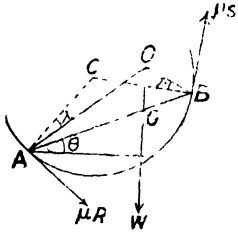
or $$2l\cos\theta = a\sin\theta(1-\cos 2\theta), \text{ i.e., } l\cos\theta = a\sin^3\theta.$$

6.82. A rod resting inside a rough sphere. *A heavy rod, whose C.G. divides it into two parts a and b, rests inside a rough spherical bowl (in a vertical plane through the centre). To show that the greatest possible inclination θ, of the rod to the horizontal is given by*

$$(a+b)\tan\theta = b\tan(\alpha+\lambda) - a\tan(\alpha-\lambda),$$

where λ is the angle of friction and 2α is the angle the rod subtends at the centre of the sphere.

Let AB be the rod, G its C.G., then $AG = a$, and $GB = b$. Let θ be the greatest possible angle which the rod makes with the horizontal, so that in this position it is on the point of sliding down.



At the point A let the resultant of the normal pressure R and the tangential friction μR act along AO, making an angle λ with the normal AC. Similarly at B the resultant reaction acts along BO, making an angle λ with the normal BC, C being the centre of the bowl.

Since these reactions meet at O, the vertical line through G, along which the weight acts, must also pass through O.

Now $\angle\text{ACB} = 2\alpha$, $\quad\therefore\quad \angle\text{CAG} = \angle\text{CBG} = 90^\circ - \alpha.$

$\therefore$ $\quad \angle\text{OAG} = \angle\text{CAG} - \angle\text{CAO} = 90^\circ - \alpha - \lambda,$

and $\quad \angle\text{OBG} = \angle\text{CBG} + \angle\text{CBO} = 90^\circ - \alpha + \lambda,$

also $\quad \angle\text{OGB} = 90^\circ - \theta$, AG $= a$ and GB $= b$.

Using the trigonometrical formula (2), Art 6.8, we have

$$(a+b)\cot\angle\text{OGB} = b\cot\angle\text{OAG} - a\cot\angle\text{OBG}$$

$\therefore$ $$(a+b)\tan\theta = b\tan(\alpha+\lambda) - a\tan(\alpha-\lambda).$$

Cor. In particular, when the rod is uniform *i.e.* $a = b$, we get

$$2\tan\theta = \tan(\alpha+\lambda) - \tan(\alpha-\lambda),$$

which gives

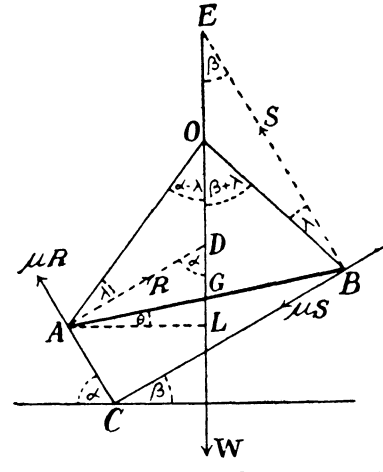
$$\tan\theta = \frac{\sin(\alpha+\lambda-\alpha+\lambda)}{2\cos(\alpha+\lambda)\cos(\alpha-\lambda)} = \frac{\sin 2\lambda}{\cos 2\alpha + \cos 2\lambda}.$$

6.821. *A uniform rod AB rests with its ends A and B on two equally rough planes of inclinations α and β respectively, in a vertical plane at right angles to the line of intersection of the planes. Prove*

that in the limiting position of equilibrium, when A is about to slide down, the inclination θ of the rod to the horizontal is given by

$$\tan\theta = \frac{\sin(2\lambda+\beta-\alpha)}{2\sin(\alpha-\lambda)\sin(\beta+\lambda)}.$$

The figure represents a vertical section through the rod. When A is about to slip downwards, B must slip upwards, so that the limiting frictions μR and μS at A and B are respectively up and down their respective planes.



The resultant of the normal reaction R and limiting friction μR at A acts along AO, which makes with the normal AD an angle λ, the angle of friction.

Similarly the resultant of the normal reaction S and limiting friction μS at B acts along BO, making with the normal BE an angle λ.

The only other force that acts on the rod is its weight W, which must also pass through O, so that the vertical through O must meet the rod AB at its middle point G.

Now $\angle ADG=\alpha$, $\therefore$ $\angle AOG=\alpha-\lambda$,
and $\angle BEG=\beta$, $\therefore$ $\angle BOG=\beta+\lambda$.

Also $\angle GAL$, the inclination of the rod to the horizontal being θ, $\angle OGB=90°-\theta$.

Let $2a$ be the length of the rod, so that $AG=GB=a$.

Now by the trigonometrical formula (1) of Art. 6.8, we have

$(AG+GB)\cot OGB=AG\cot AOG-BG\cot BOG$,

or $$2a\cot(90°-\theta)=a\cot(\alpha-\lambda)-a\cot(\beta+\lambda) \quad \ldots\ldots(1)$$

$$\therefore \tan\theta=\frac{\sin[(\beta+\lambda)-(\alpha-\lambda)]}{2\sin(\alpha-\lambda)\sin(\beta+\lambda)}=\frac{\sin(2\lambda+\beta-\alpha)}{2\sin(\alpha-\lambda)\sin(\beta+\lambda)}.$$

Cor. If the planes are smooth instead of rough, $\lambda=0$

and $$\tan\theta=\sin(\beta-\alpha)/2\sin\alpha\sin\beta.$$

In this case the normal reactions meet the vertical LG in the same point and make angles α, β with it.

Equation (1) now becomes

$2a\cot(90°-\theta)=a\cot\alpha-a\cot\beta$, or $2\tan\theta=\sin(\beta-\alpha)/\sin\alpha\sin\beta$.

Examples XXV-A

1. A rod whose centre of gravity divides it into two portions whose lengths are a and b $(b>a)$, has a string tied to its ends and the string is slung over a smooth peg. Show that the angle θ which the rod makes with the horizon is given by

$$(a+b)\tan\theta=(b-a)\cot\alpha,$$

where 2α is the angle subtended at the peg in the position of equilibrium.

2. A beam which is divided by its C. G. into two portions of lengths a and b rests in equilibrium on two smooth planes inclined at angles α, β respectively to the horizon, the planes intersecting in a horizontal line, (the beam being perp. to this line). Show that θ the inclination of the beam to the horizon is given by

$$(a+b)\tan\theta = b\cot\beta - a\cot\alpha.$$

3. A triangular lamina ABC is suspended from a point O by light strings fastened to the points A and B and hangs so that the side BC is vertical. Prove that if α, β are the angles which the strings AO, BO make with the vertical, then

$$2\cot\alpha - \cot\beta = 3\cot B. \qquad (\textit{Aligarh})$$

4. A uniform rod AB is in equilibrium at an angle α with the horizontal with its upper end A resting against a smooth peg and its lower end B attached to a light cord which is fastened to a point C in the same level as A. Prove that the angle β at which the cord is inclined to the horizontal is given by the equation

$$\tan\beta = 2\tan\alpha + \cot\alpha$$

and show that

$$AC(1+2\tan^2\alpha) = AB\sec\alpha. \qquad (D.U.)$$

5. Inside a smooth hollow sphere is placed a rod of weight W equal in length to the radius of the sphere, the C.G. of the rod being at a point one-third of its length from one end. Find the position of equilibrium of the rod and also the pressures at the two ends.

(*Ans.* At $\tan^{-1}(3\sqrt{3})$ with vertical ; $2W/\sqrt{7}$; $W/\sqrt{7}$)

6. A uniform heavy rod, whose length is equal to the diameter of a smooth hemispherical bowl which is fixed with its axis vertical, rests with one end in the bowl and the other end outside the bowl. Prove that the inclination of the rod to the horizontal is about 30° 32′.

7. A bowl is formed from a hollow sphere of radius a ; it is so fixed that the radius of the sphere drawn to each point in the rim makes an angle α with the vertical, and the radius drawn to a point A of the bowl makes an angle β with the vertical. If a smooth uniform rod remains at rest when placed with one extremity at A, and with a point in its length on the rim of the bowl, show that the length of the rod is

$$4a\sin\beta\sec\tfrac{1}{2}(\alpha-\beta). \qquad (P.U.)$$

8. Show that the greatest inclination to the horizon at which a uniform rod can rest partly within and partly without a fixed smooth hemispherical bowl is $\sin^{-1}\frac{1}{\sqrt{3}}$. [See Hints at the end]. (*D.U. 1959*)

9. A light rod rests wholly within a smooth hemispherical bowl of radius r, and a weight W is clamped on the rod at a point whose distances from the ends are a and b. Show that θ, the inclination of the rod to the horizon in the position of equilibrium is given by the equation

$$2\sqrt{r^2-ab}\sin\theta = a-b.$$

10. A string 16 ft. long has one end attached to an extremity of a smooth uniform heavy rod of length 10 ft. and at the other end carries a weightless ring, which slides on the rod. The rod is suspended by means of the string from a smooth peg ; show that the angle θ which the rod makes with the horizon is given by $\tan\theta = \frac{1}{2}$.

11. A fine smooth vertical rod is fixed firmly into the top of a fixed smooth sphere of radius a. A bar of length $2b$ has at one end a ring which slides on the rod on the sphere, and the bar rests on the sphere. Show that in equilibrium the angle α which the bar makes with the vertical is given by

$$a\cos\alpha = b\sin^2\alpha.$$

Examples XXV-B

1. The ends A, B of a uniform heavy rod of length $a\sqrt{2}$ are attached to rings which can slide along a rough circular wire of radius a in a vertical plane. The co-efficient of friction between these rings and wires is $\mu = \tan\lambda$. Show that

the greatest inclination of the rod to the horizontal when in equilibrium is 2λ.

[**Hint.** Use Cor. Art. 6·82, α being 45°]

2. A uniform rod of length $2a \sin\alpha$ rests within a rough vertical circle of radius a, show that the greatest possible inclination of the rod to the horizon is

$$\tan^{-1}\frac{\mu}{\cos^2\alpha-\mu^2\sin^2\alpha},$$

where μ is the co-efficient of friction.

3. A heavy uniform rod rests within a rough circular ring fixed in a vertical plane; the rod subtends an angle of 120° at the centre of the ring and in the limiting position of equilibrium is inclined to the horizon at an angle α. If $\sqrt{3}\mu=\tan\varphi$, μ being the co-efficient of friction, show that

$$\tan\alpha : \tan 2\varphi = 2 : \sqrt{3}.$$

4. A uniform rod is placed inside a rough cylindrical drum which is fixed with its axis horizontal. If the rod rests in a vertical plane perpendicular to the axis of the drum, show that its least possible inclination to the vertical is

$$\tan^{-1}(\cot 2\lambda+\cos\beta \operatorname{cosec} 2\lambda),$$

where λ is the angle of friction, and β the angle the rod subtends at the nearest point of the axis of the drum.

5. A uniform rod rests within a fixed vertical circle, subtending an angle 2α at the centre; its upper end is smooth, and its lower end rough, the angle of friction being λ; show that the angle which the rod makes with the horizon cannot be greater than θ, where

$$\tan\theta=\frac{\sin\lambda}{\cos\lambda+\cos(\lambda+2\alpha)}.$$

6. A uniform rod rests inside a rough vertical circle with its highest point on a level with the centre of the circle. If the friction is just sufficient to prevent sliding, find the co-efficient of friction (μ) in terms of α, where α is the inclination of the rod to the horizontal. (*P.U. 1957*)

(*Ans.* μ is given by the equation, $\mu^2\sin\alpha\cos^2\alpha+\mu\cos\alpha-\sin^3\alpha=0$)

7. Solve Q. 8 Ex. XXII-A by the method of Art. 6.82.

8. The ends of a heavy uniform bar are in contact with two planes inclined at an angle of 60°, and 30° to the horizon respectively. Prove that the bar can just rest horizontally, if either plane is smooth and the coefficient of friction of the other is $1/\sqrt{3}$. (*D.U. 1960*)

9. A uniform rod is in limiting equilibrium, one end resting on a rough horizontal plane and the other on an equally rough plane inclined at an angle α to the horizon. If λ be the angle of friction and the rod be in a vertical plane, show that the inclination θ of the rod to the horizon is given by

$$\tan\theta=\frac{\sin(\alpha-2\lambda)}{2\sin\lambda\sin(\alpha-\lambda)}. \qquad (D.U.)$$

10. A uniform bar AB rests with its ends on two equally rough planes of equal inclination α which face each other, in a plane perp. to their line of intersection. If $\lambda<\alpha$, prove that the limiting inclination θ of the bar to the horizontal is given by

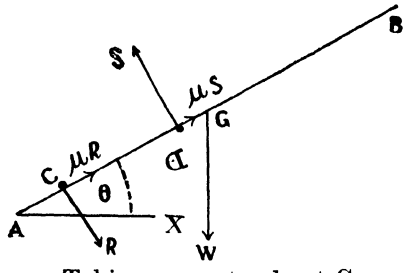
$$\tan\theta=\frac{\sin 2\lambda}{2\sin(\alpha+\lambda)\sin(\alpha-\lambda)}.$$

6.9. Further examples involving friction.

Example 1. *Two small fixed pegs are in a line inclined at an angle θ to the horizon. A rough thin rod passes under the lower and rests on the higher, this latter peg being lower than the C.G. of the rod. The distances of the C.G. from the two pegs are a and b respectively and the co-efficient of friction is μ. Show that if the rod is on the point of motion,*

$$\mu(a+b)=(a-b)\tan\theta.$$

Let AB be the rod, C, D the pegs, and G, the C.G. of the rod,



so that $CG=a$ and $DG=b$,

$\angle BAX=\theta$, AX being the horizontal line.

Let R and S be the reactions of the pegs on the rod which is on the point of sliding downwards along DC, so that the forces of limiting friction μR and μS act at C and D along the rod upwards.

Taking moments about C, we have

$$S.CD=W.CG\cos\theta \text{ } i.e., \text{ } S(a-b)=Wa\cos\theta \qquad ...(1)$$

Resolving along and perp. to the rod, we get

$$\mu R+\mu S=W\sin\theta \quad \text{or} \quad S+R=W\sin\theta/\mu,$$

and $$S=R+W\cos\theta \quad \text{or} \quad S-R=W\cos\theta.$$

$\therefore$ adding, we get $$2S=W(\cos\theta+\sin\theta/\mu).$$

Substituting this in (1), we have

$$2Wa\cos\theta=(a-b).W(\cos\theta+\sin\theta/\mu),$$

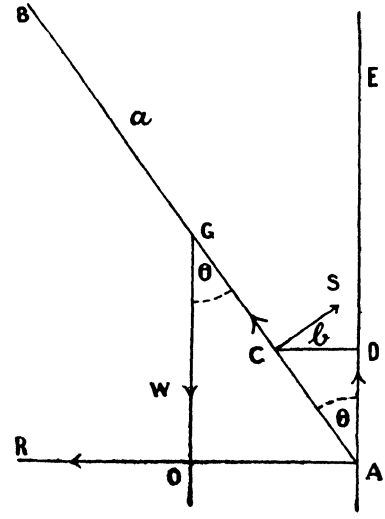
or $$2a\mu=(a-b)\mu+(a-b)\tan\theta,$$

i.e., $$\mu(a+b)=(a+b)\tan\theta.$$

Example 2. *A heavy uniform rod AB of length 2a lies over a rough peg C with one end A leaning against a vertical wall ; if b be the distance of the peg from the wall and λ be the angle of friction both at the peg and the wall, show that when A is below C and the rod is on the point of slipping downwards*

$$\sin^2\theta\sin(\theta+2\lambda)=\frac{b}{a}\cos^2\lambda.$$

Let AB be the rod, C the peg, AE the wall, CD perp. to the wall and G, the C.G. of the rod, so that $CD=b$, $AG=GB=a$, $\angle DAB=\theta=\angle AGO$, and $OA=a\sin\theta$ and $AC=b\operatorname{cosec}\theta$.



Let R be the reaction of the wall at A, S the reaction of the peg at C and W the weight of the rod.

As the rod is on the point of sliding downwards, the limiting frictions μR and μS act along AE and CG as shown by arrows, μ being equal to $\tan\lambda$.

Taking moments about A, we have

$$W.OA=S.AC$$

or $$Wa\sin\theta=Sb\operatorname{cosec}\theta,$$

i.e., $$Wa\sin^2\theta=bS \qquad ...(1)$$

Resolving horizontally and vertically, we have

$$R+\mu S\sin\theta-S\cos\theta=0 \qquad ...(2)$$

and $$\mu R+\mu S\cos\theta+S\sin\theta=W \qquad ...(3)$$

Eliminating R by multiplying (2) by μ and subtracting from (3), we get

$$\mu S(\cos\theta-\mu\sin\theta)+S(\sin\theta+\mu\cos\theta)=W$$

or $$S\tan\lambda\cos(\theta+\lambda)+S\sin(\theta+\lambda)=W\cos\lambda \quad (\because \mu=\tan\lambda)$$

or $$S[\sin\lambda\cos(\theta+\lambda)+\cos\lambda\sin(\theta+\lambda)]=W\cos^2\lambda,$$

i.e., $$W\cos^2\lambda=S\sin(\theta+2\lambda) \qquad \ldots(4)$$

Dividing (4) by (1) and simplifying, we get

$$a\sin^2\theta\sin(\theta+2\lambda)=b\cos^2\lambda.$$

Cor. 1. If the rod is on the point of slipping upwards the result is

$$a\sin^2\theta\sin(\theta-2\lambda)=b\cos^2\lambda,$$

which is obtained by simply changing the sign of μ or λ in the above equations.

Cor. 2. If the wall and the peg are both smooth $\lambda=0$ and the result reduces to

$$\sin^3\theta=b/a.$$

This can be readily obtained directly in two ways as follows :—

I. Resolving vertically, we have $W=S\sin\theta$.

Substituting this in (1), we get $S.a\sin^3\theta=bS$, or $\sin^3\theta=b/a$.

II. *Geometrically.* Since only three forces W, R and S act on the rod, $\therefore$ S passes through O *i.e.*, S perp. to the rod acts along OC, so that $\angle AOC=\theta$.

$\therefore$ from $\triangle OAC$, $\sin\theta=\frac{AC}{OA}$. Also from $\triangle ACD$, $\sin\theta=\frac{CD}{AC}=\frac{b}{AC}$,

nd from $\triangle AOG$, $\sin\theta=\frac{OA}{AG}=\frac{OA}{a}$.

Multiplying these three, we get

$$\sin^3\theta=\frac{AC}{OA}\times\frac{b}{AC}\times\frac{OA}{a}\quad\frac{b}{a}.$$

Examples XXVI

1. Two equally rough pegs A and B are at a distance $2a$ apart in a straight line inclined at an angle θ to the vertical. A rod passes over the peg A and under the peg B and is just kept from sliding down by the friction at the pegs. Prove that the C. G. of the rod must be at a distance $a\left(\frac{\cos\theta}{\mu}-1\right)$ from the upper peg A, where μ is the co-efficient of friction between the rod and the pegs.

2. A thin uniform rod passes over one peg and under another, the co-efficient of friction between each peg and the rod being μ. The distance between the pegs is a and the line joining them makes an angle β with the horizontal. Show that equilibrium is not possible unless the length of the rod is greater than

$$\frac{a}{\mu}(\mu+\tan\beta). \qquad (P.U.)$$

3. A uniform rod rests on two rough pegs A and B ; AB is inclined at an angle α to the horizon and the co-efficients of friction are μ_1 and μ_2 respectively. If the rod is in limiting equilibrium, prove that its middle point divides AB in the ratio $(\mu_1-\tan\alpha) : (\tan\alpha-\mu_2)$, it being given that

$$\mu_1>\tan\alpha>\mu_2.$$

4. A uniform rod AB of weight W, rests with the end B against a rough vertical wall and is held by a string OA, of the same length as the rod, fastened to a point O vertically above B. If the rod is on the point of sliding down, find its inclination to the vertical and show that the tension of the string is

$$\tfrac{1}{2}W\sqrt{1+9/\mu^2}. \qquad (Ans.\ \cot^{-1}\tfrac{1}{3}\mu)$$

5. AB and AC are two equally rough wires of which AB is horizontal and AC vertical. Two rings of equal mass connected by a light inextensible string are free to move on AB and AC. Prove that in the position of limiting equilibrium the string makes an angle with the vertical equal to twice the angle of friction and show that the tension of the string in this position is equal to the weight of each ring.

6. (*a*) A heavy rod of length $2a$ lies over a rough peg with one extremity leaning against an equally rough vertical wall. If a be the distance of the peg from the wall and μ the co-efficient of friction, show that when the point of contact of the rod with the wall is above the peg, then

$$\sin^3\theta = \cos^2\lambda,$$

where θ is the inclination of the rod to the wall and λ the angle of friction..

(*b*) If the point of contact of the rod and the wall be below the peg and the rod is on the point of slipping downwards, then

$$\sin^2\theta \sin(\theta+2\lambda) = \cos^2\lambda.$$

7. A uniform rod AB of length $2a$ rests with the end A in contact with a *smooth* plane inclined at an angle α to the horizon. A point C of the rod rests against smooth horizontal rail perp. to a line of greatest slope of the plane. Prove that, if the inclination of the rod to the vertical is θ, then

$$a \sin\theta \cos^2(\theta-\alpha) = b \sin\alpha,$$

where b is the perpendicular distance of C from the plane.

8. A uniform rod of length $2a$ rests with one end A on a *rough* plane of inclination α to the horizontal, and lies in a vertical plane through a line of greatest slope of the plane. It is supported by a smooth peg at a distance c from the end in contact with the plane. Show that if the end of the rod is about to slip up the plane and λ is the angle of friction, the inclination θ of the rod to the horizontal is given by

$$c \sin(\alpha+\lambda) = a \cos\theta \sin(\theta+\alpha+\lambda) \qquad (D.\ U.\ 1941)$$

(Compare questions 7 and 8).

9. A uniform rod of length $2a$ and weight W is placed against a smooth fixed cylinder with one end A resting on a rough horizontal plane, which passes through the centre of the cylinder. The rod makes an angle θ with the ground and d is the distance from the centre of the cylinder of the lower end of the rod. Show that if the rod is on the point of slipping

$$\frac{P}{W} = \frac{\sin\varphi}{\sin(\theta+\varphi)} = \frac{a}{d},$$

where φ is the angle of friction, and P the pressure between the rod and the cylinder.

10. In the last example, if the cylinder is rough instead of smooth and φ' be the angle of friction for it, prove that

$$a \sin\theta \sin(\theta-\varphi'+\varphi) = r \sin\varphi \cos\varphi'$$

and show that $\tan\theta > r/2a$, r being the radius of the cylinder.

11. A uniform plank of length $2l$ rests with one end A on a rough horizontal plane and across a fixed smooth cylinder of radius r, lying on the same plane. The plank makes an angle 2α with the plane and λ is the angle of friction. Show that the equilibrium is possible if

$$r \sin\lambda > l \tan\alpha \cos 2\alpha \sin(2\alpha+\lambda). \qquad (All.\ U.)$$

CHAPTER VII

ADDITIONAL PROBLEMS

*7.1. **Nature of the resultant.** We now give below, a few additional problems involving particular cases of the nature of the resultant of a system of coplanar forces acting upon a rigid body. The discussion is based on the general principles laid down in Art. 5.75.

7.11. *If the algebraic sum of the moments about two given points A, B, in the plane of the forces be* M_1, M_2 *different from each other, then the system reduces to a single force as its resultant.*

A system of coplanar forces, not in equilibrium, always reduces to either a single force or a single couple (Art. 4.61).

If the resultant be a couple, its moment about every point in the plane must be the same, which is not true in this case.

Again, since the sum of the moments of the system of forces about *any* point in their plane is not zero (*e.g.*, about A it is M_1) the system is not in equilibrium.

Hence the system must have a single force as its resultant.

Otherwise, this can also be shown analytically as follows :—

Let A about which the sum of the moments is M_1, be the origin, AB the axis of X, (b, 0) the co-ordinates of B. At A let the system reduce to X, Y and M_1.

Then by (2), Art. 5·78, we have

$$M_2 = M_1 - bY$$

M_2 and M_1 being different, this equation shows that Y cannot vanish and thus the system reduces to a force as its resultant. It cannot reduce to a single couple and cannot be in equilibrium.

7.12. *If the algebraic sum of the moments of a system of coplanar forces about three non-collinear points (A, B, C) in their plane is the same (M) and not zero, the system reduces to a couple.*

In this case the system cannot reduce to a single force (R), for, if it does, its line of action must be at the same distance $\frac{M}{R}$ from the three *non-collinear* points A, B, C, which is impossible.

Also the system is not in equilibrium, as the sum of its moments about *every* point in its plane is not zero.

Hence the system must reduce to a couple.

Otherwise. To solve it analytically, let A be the origin and let (x_1, y_1) and (x_2, y_2) be the co-ordinates of B and C and at A let the system reduce to X, Y and M.

Then by (2) Art. 5·78, $M=M-x_1Y+y_1X$,

i.e., $x_1Y=y_1X$ or $Y=\frac{y_1}{x_1}X$.

Similarly $x_2Y=y_2X$ or $Y=\frac{y_2}{x_2}X$

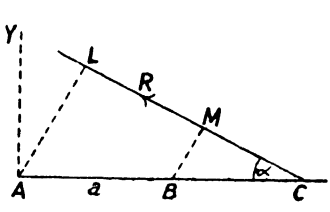
This requires that,

either $\frac{y_1}{x_1}=\frac{y_2}{x_2}$, *i.e.*, A, B, C are collinear (A being the origin) which is contrary to the hypothesis,

or $X=Y=0$ *i.e.*, the resultant cannot be a single force.

Example 1. *Prove that if a variable system of forces in a plane have constant moments G, G′ about two fixed points in the plane then the resultant passes through another fixed point, and that its minimum value is the difference of the moments divided by the distance between the two points.*

Let A and B be the fixed points. As the moments about A and B are different, the system must reduce to a force say R, the line of action of which must intersect AB, say at C at an angle α, then



$G=R.AL=R.AC\sin\alpha$ ...(1)

and $G'=R.BM=R.BC\sin\alpha$. ...(2)

$\therefore \frac{G}{G'}=\frac{AC}{BC}$, *i.e.*, C is a fixed point on AB dividing it externally in the ratio of the given moments.

When the moments are of opposite signs, the line of action of R will divide AB internally in the ratio of the moments.

Hence the resultant forces of all the systems pass through a fixed point which divides AB internally or externally in the ratio of the constant moments. [Compare Q. 11, R. Q. II].

Now from (1) and (2), we have

$$R=\frac{G}{AC\sin\alpha}=\frac{G'}{BC\sin\alpha}=\frac{G-G'}{(AC-BC)\sin\alpha}=\frac{G-G'}{AB\sin\alpha}.$$

Thus R is minimum when $\sin\alpha$ is maximum $=1$ *i.e.* $\alpha=90°$ (*i.e.*, when the resultant is perp. to AB].

Hence the min. value of $R=(G-G')/AB$.

Aliter. Let A (o, o) and B (a, o) be the two points, so that AB is the x-axis.

At A let the system be reduced to X, Y, G.

Then the moment about $A=G$

and G′, the moment about $B=G-aY$ *i.e.*, $G'=G-aY$ ...(1)

G, G′ being constants Y is constant and thus only X varies.

$\therefore$ the resultant is min. when $X=0$

$\therefore$ the min. value of $R=Y=(G-G')/a=(G-G')/AB$.

Also the equation of the line of action of the resultant is $G-xY+yX=0$, which evidently passes through $(G/Y, 0)$, a fixed point on AB, G and Y being constants.

Examples XXVII

1. (*a*) If two different systems of forces have the same moments L, M, N about any three points A, B, C, the two systems must have the same resultant.

(*b*) Hence show that if a system of forces has moments L, M, N about any three points A, B, C and if $\triangle$ is the area of the triangle ABC, the resultant is the same as that of three forces $\frac{aL}{2\triangle}$, $\frac{bM}{2\triangle}$, $\frac{cN}{2\triangle}$ acting along BC, CA and AB.

(*c*) E and F are the feet of the perpendiculars from B and C upon the opposite sides of the triangle ABC. Apply (*a*) to show that a force P acting along EF can be replaced by forces P cos A, P cos B and P cos C acting along the sides BC, CA and AB of the triangle.

2. Prove that in all cases a system of coplanar forces can be replaced by two forces, one of which acts through a given point and the other along a given straight line.

3. Show that a system of coplanar forces may be represented in an infinite number of ways by two forces acting through two given points in their plane.

If A and B are the fixed points and AP and BQ represent any two such forces, prove that the loci of P and Q are straight lines parallel to AB.

***7·2. Equilibrium of Connected Bodies.** When two or more bodies are in contact, the equilibrium of each is considered separately, taking into account all the forces which arise in virtue of its connection with the others, in addition to the special forces acting upon it.

In some cases, however, it is found more useful to write down the equations of equilibrium by taking two bodies together regarding them as a single body. This avoids the appearance in the equations of equilibrium, of the unknown reactions between the bodies.

Example 1. *Two* **light** *rods AC, CB, are freely jointed at C. They stand in a vertical plane with the ends A and B on a smooth horizontal table pressing against two fixed obstacles. Prove that if a vertical thrust P is applied at C, the horizontal thrust against either obstacle is* $P \cos A \cos B/\sin C$.

(*a*) *If the lengths of the rods be varied, prove that when the angle C is given, the thrust is a maximum when* $AC=CB$.

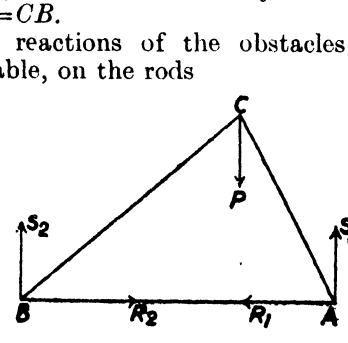
Let AC, BC be the rods, R_1, R_2 the reactions of the obstacles and S_1, S_2 the vertical reactions of the table, on the rods

To avoid the action at C, we resolve horizontally and vertically for the *whole system*, we have,

$$R_1-R_2=0, \quad i.e., \quad R_1=R_2 \qquad \ldots(1)$$

which shows that the reaction of the obstacle on either rod is the same,

and $$P=S_1+S_2 \qquad \ldots(2)$$



Now, taking moments about C for one rod AC, we have
$R_1.AC \sin A = S_1.AC \cos A$ *i.e.*, $S_1 = R_1 \tan A$.
Similarly for the rod BC, $S_2 = R_2 \tan B = R_1 \tan B$. [by (1)]

$$\therefore \quad P = S_1 + S_2 = R_1 (\tan A + \tan B),$$

or $$P = R_1 \frac{\sin (A+B)}{\cos A \cos B} = R_1 \frac{\sin C}{\cos A \cos B},$$

i.e., $$R_1 = R_2 = P \cos A \cos B / \sin C.$$

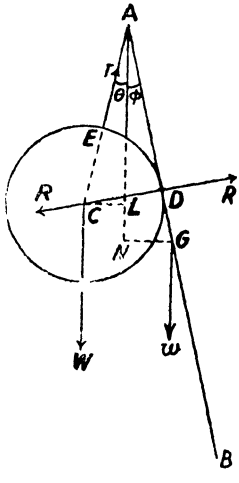
(*a*) Now, since $\cos A \cos B = \frac{1}{2}[\cos (A+B) + \cos (A-B)]$
$= \frac{1}{2}[\cos (A-B) - \cos C]$ $(\because A+B = 180° - C)$,
and C being given, R_1 or R_2 will be a maximum when
$\cos (A-B) = 1$ *i.e.*, $A = B$ *i.e.*, $AC = BC$.

Example 2. *A sphere of weight W and radius r is hung by a string of length l, and a uniform rod of weight w and length 2a has one end attached to the same point and can freely turn about it. If the rod rests touching the sphere, show that the angle θ made by the string with the vertical is given by the equation*

$$\tan \theta = w a \cos^2 \alpha / [Wr + w a \cos \alpha \sin \alpha],$$

where $$\sec \alpha = (l+r)/r.$$

The rod AB with its middle point G, and the sphere with centre C touch each other at D where the action and reaction are equal, each equal to, say R.



For the sphere, since R and W meet at C, the tension of the string EA, which is the only other force on it, must also pass through C *i.e.*, CEA must be a straight line and

$$CA = CE + EA = r + l.$$

Draw a vertical line through A and let the perpendiculars to it from C and G meet it in L and N.

The angle which the string makes with the vertical is $\angle CAL = \theta$.

Let $\angle LAD$ the angle which the rod makes with the vertical be ϕ.

Then $$\cos \angle ACD = \frac{CD}{AC} = \frac{r}{r+l}$$
$$= \cos \alpha \text{ (given)} \quad ...(1)$$

Hence $\angle ACD = \alpha$ and $\phi = \angle CAD - \theta = 90° - \alpha - \theta$ ...(2)

Taking moments about A for the whole system, we have

$$W.CL = w.NG \quad i.e., \quad W(l+r) \sin \theta = w.AG \sin \phi = wa \sin \phi.$$

Now to express the result in the given form, we substitute the values of $l+r$ and ϕ from (1) and (2) and get

$$Wr \sec \alpha \sin \theta = wa \cos(\alpha+\theta) = wa(\cos\alpha\cos\theta - \sin\alpha\sin\theta),$$

or $$[Wr + wa\sin\alpha\cos\alpha]\sin\theta = wa\cos^2\alpha\cos\theta$$

i.e., $$\tan\theta = wa\cos^2\alpha/[Wr + wa\cos\alpha\sin\alpha].$$

Note. The tension T of the string EA, if required, can be obtained by resolving forces acting on the sphere along AD.

Thus $$T\cos(\theta+\varphi) = W\cos\varphi,$$

for, W being vertical is parallel to AN, and R being perp. to AD, does not enter the equation.

Examples XXVIII

1. The ends of two uniform rods of weight W per foot and length 6 ft. and 8 ft. respectively are freely hinged, and the other ends rest on a smooth horizontal table. The middle points of the rods are connected by a string 5 ft. long. Find the tension of the string when the rods are supposed to stand in a vertical plane. [*Ans.* $6\frac{8}{25}$ W] (*P. U.*)

2. Two equal uniform rods of length $8a$ are freely jointed and rest symmetrically over two smooth pegs fixed in a horizontal line at a distance a apart. Show that each rod is inclined to the vertical at an angle of 30°.

3. Two equal uniform heavy rods ; each of length $2l$ are smoothly jointed together at an end and are placed so as to rest symmetrically over a smooth cylinder of radius a fixed with its axis horizontal. Show that in the position of equilibrium the angle θ between each rod and the vertical is given by

$$l\sin^3\theta = a\cos\theta.$$

4. Two smooth uniform rods OB, OC each of length l and weight W are smoothly jointed at O, and are placed symmetrically on a fixed horizontal rod of length $2a$, so as to form a letter A. A light vertical string connects the joint O to the mid-point of the fixed rod. When the rods OB, OC are inclined at θ to the horizontal, show that the tension T in the string is given by

$$T = W\left(\frac{l}{a}\cos^3\theta - 2\right).$$

5. A sphere of weight W and radius a hangs by a string of length b from a point A and a weight W′ is also suspended by a string from the point A, the string being long enough for the weight W′ to hang below the sphere, show that the first string is inclined to the vertical at an angle θ, where

$$(W+W')(a+b)\sin\theta = aW'.$$

6. A smooth sphere, of weight W, is suspended from a fixed point by a string of length equal to its radius. To the same point, a second string is attached which, after passing over the sphere, supports a weight equal to that of the sphere. Show that the first string then makes an angle $\sin^{-1}(\frac{1}{4})$ with the vertical, and that the tension in this string is $4W/\sqrt{15}$.

7. A uniform heavy rod of length $4a$ and weight W, can turn freely about one end A which is fixed ; to this end is attached a string of length a which supports a sphere of radius a and weight W. Find the inclinations of the rod and the string to the vertical, the tension in the string, and the pressure between the rod and the sphere.

[*Ans.* 15°, 15°, $W(\sqrt{3}+1)/\sqrt{6}$; $W(\sqrt{3}-1)/\sqrt{6}$]

8. Two uniform bars AB, AC each of length $2l$ and weight W are freely jointed at A and stand in equilibrium in a vertical plane and their mid-points are jointed by a light elastic string of natural length $2a$ and modulus λ. If 2θ be the angle between the rods, show that θ satisfies the equation

$$\frac{1}{\sin\theta} + \frac{W}{\lambda\cos\theta} = \frac{l}{a}.$$

9. A step ladder in the form of the letter A, with each of its legs inclined at angle α to the vertical is placed on a horizontal floor and is held up

by a cord connecting the middle points of its legs, there being no friction anywhere. If a weight W be placed on one of the steps at a height from the floor equal to $1/n$ of the height of the ladder, determine the increase in the tension of the cord. (*Ans.* $W \tan \alpha/n$) (*D.U. 1949 Pass*)

10. Two equal beams AB, AC, each of weight W, connected by a hinge at A, are placed in a vertical plane with their extremities B and C resting on a smooth horizontal plane. They are kept from falling by strings connecting B, and C with the mid-points of the opposite beam. Show that the tension in each string is

$$\tfrac{1}{8} W \sqrt{9 \cot^2 \theta + 1},$$

θ being the inclination of each beam to the horizon. Show also that the action of the hinge on either rod is $\frac{3}{4} W \cot \theta$.

11. Two equal uniform rods AB and BC, each of weight W, are smoothly hinged at A and placed in a vertical plane with ends B and C resting on a smooth horizontal plane. Equilibrium is preserved by a string which attaches C to the mid-point of AB. Show that the tension of the string and the reaction of the rods at A are both equal to

$$\tfrac{1}{4} W \sqrt{1 + 9 \cot^2 \alpha},$$

and that each is inclined at an angle $\tan^{-1}(\frac{1}{3} \tan \alpha)$ to the horizon, where α is the inclination of either rod to the horizon.

12. A uniform rod AB 6 ft. long weighing 15 lbs. is smoothly hinged at B to another uniform rod BC, 8 ft. long weighing 20 lbs. The ends A, C are fastened to smooth hinges in a horizontal line 10 ft. apart with B below AC, and a horizontal force of 30 lb. wt. is applied to the rod BC along the straight line joining the middle points of BC and AB in the same sense. Find the horizontal and vertical components of the reaction at the hinge B. [*Ans.* 3 lbs., $3\frac{1}{2}$ lbs.]

13. A pair of compasses rests across a smooth horizontal cylinder of radius r. Show that the frictional couple at the joint to prevent slipping must be

$$W(r \cot \alpha \operatorname{cosec} \alpha - a \sin \alpha)$$

where 2α is the angle between the legs and W the weight of each leg and a the distance of its C.G. from the joint.

14. AB is a straight rod, of length $2a$ and weight nW, with the lower end A on the ground at the foot of a vertical wall AC, B and C being at the same vertical height $2b$ above A ; a heavy ring, of weight W, is free to move along a string of length $2l$, which joins B and C. If the system be in equilibrium with the ring at the middle point of the string, show that

$$l^2 = a^2 - b^2 \frac{n(n+2)}{(n+1)^2}.$$

15. A rod AB of length l and weight W is hinged to a fixed point at A and a string of length $l/\sqrt{3}$ connects its end B to a fixed point C which is at a height $l\sqrt{3}/2$ vertically above A. Find what weight must be suspended from the middle point of the string BC in order that the rod may be in equilibrium with B and C at the same level. (*Ans.* W/2)

16. Show that the power necessary to move a cylinder, of radius r and weight W, up a smooth plane inclined at an angle α to the horizon by means of a light crowbar of length l inclined at an angle β to the horizon is

$$Wr \sin \alpha / l \, [1 + \cos(\alpha + \beta)]. \qquad (P.U.\ 1922)$$

17. A cylindrical log of weight W and radius $4r$ lies on a smooth floor with its curved surface in contact with a smooth vertical wall. A second cylindrical log of radius r and weight w lies on the floor besides the first log so that their curved surfaces are touching. Both logs are smooth. A horizontal force P at a height r above the floor pushes the smaller log in the direction perpendicular and towards the wall.

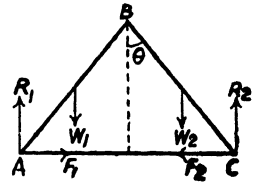
Prove that by increasing P to $\frac{4}{3}$W, the reaction between the larger log and the floor can be made zero. (*H.C.*)

18. A hollow bottomless cylinder of mass M stands upright on the ground Two equal smooth spheres are placed within it, one resting on the ground and the other supported by the cylinder and the other sphere. If the mass of either sphere be m, its radius a and the radius of the cylinder $\frac{3}{2}a$, show that the cylinder is on the point of toppling over provided that $2m=3M$.

***7.3. Two jointed rods standing on a rough table.** *Two uniform rods AB, BC each of length 2a and of weights W_1 and W_2 respectively $(W_2 > W_1)$ are freely jointed together at B. The system stands in a vertical plane, with A and C on a rough horizontal table, each rod making an angle θ with the vertical. The coefficient of friction at both ends A and C is μ. Show that equilibrium is possible only if*

$$(W_1+W_2)\tan\theta \leqslant (3W_1+W_2)\mu.$$

Let R_1, R_2 be the normal reactions and F_1, F_2 the forces of friction at A and C.



Since the reaction at B on the two rods are equal and opposite, they cancel each other when the system is taken as a whole.

$\therefore$ Resolving vertically for both the rods, we have

$$R_1+R_2=W_1+W_2,$$

and taking moments about A, for both the rods, we have

$$R_2.4a\sin\theta=W_2.3a\sin\theta+W_1.a\sin\theta. \quad i.e., \quad R_2=\frac{W_1+3W_2}{4}.$$

$$\therefore \qquad R_1=W_1+W_2-R_2=\frac{3W_1+W_2}{4}.$$

Resolving horizontally for both the rods, we have $F_1=F_2$.

Now taking moments about B, for the rod BC, we get

$$F_2\,2a\cos\theta+W_2\,a\sin\theta=R_2\,2a\sin\theta,$$

which gives

$$F_2=(R_2-\tfrac{1}{2}W_2)\tan\theta=\frac{W_1+W_2}{4}\tan\theta=F_1.$$

$$\therefore \qquad \frac{F_1}{R_1}=\frac{W_1+W_2}{3W_1+W_2}\tan\theta \quad \text{and} \quad \frac{F_2}{R_2}=\frac{W_1+W_2}{3W_2+W_1}\tan\theta.$$

Since $W_2 > W_1$, $\therefore \frac{F_1}{R_1} > \frac{F_2}{R_2}$, so that $\frac{F_1}{R_1}$ will be first to attain the maximum limit μ, *i.e.*, the end A of the rod AB will be the first to slip.

Hence the equilibrium of the system will be possible so long as $\frac{F_1}{R_1} \leqslant \mu$,

i.e., $$(W_1+W_2)\tan\theta \leqslant (3W_1+W_2)\mu.$$

Examples XXIX

1. In the above example prove that the reaction at the hinge B is inclined

$$\tan^{-1}\frac{W_2-W_1}{W_2+W_1}\cot\theta$$

with the horizontal.

2. Two uniform rods AB, BC of equal length and weights W and 2W respectively, are smoothly hinged at B and rest in a vertical plane with the ends A and C on a rough horizontal table. If the angle ABC is a right angle, find the horizontal and vertical components of the reactions at A and C. Find also the

least value of the co-efficient of friction between the rods and the table in order that the rods may be able to rest in this position.

[*Ans.* $\frac{3}{4}$W at A and C, $\frac{5}{4}$W at A, $\frac{7}{4}$W at C, $\frac{3}{5}$]

3. Two equal ladders, each of weight W, are placed so as to lean against each other with their ends resting on a rough horizontal floor, each making an angle θ with it. Show that a weight on the top which will cause them to slip is

$$W\,\frac{2\mu-\cot\theta}{\cot\theta-\mu},$$

μ being the co-efficient of friction.

4. Two uniform ladders OA, OB of the same length l and of the same weight W are smoothly jointed together at O, and stand with A and B in contact with a rough horizontal plane ; μ is the co-efficient of friction between the plane and either end A or B. If a man of weight W can stand anywhere on the ladder when A and B are at a distance $2a$ apart, prove that μ must not be less than

$$\frac{2a}{\sqrt{l^2-a^2}}.$$

5. Two equal uniform ladders are freely jointed at one end and stand with the other ends on a rough horizontal plane. A man whose weight is equal to that of one of the ladders ascends one of them. Prove that the other will slip first.

If it begins to slip when the man is half way up, prove that the co-efficient of friction is $\frac{3}{5}\tan\alpha$, where 2α is the angle between the ladders.

6. Two uniform rods AB, BC of weights W_1 and W_2, are freely jointed at B and hang freely from A. If a horizontal force P is applied at C, show that the inclinations θ and φ of AB and BC to the vertical are given by

$$\tan\theta=\frac{2P}{W_1+2W_2} \text{ and } \tan\varphi=\frac{2P}{W_2}.$$

7. Two equal uniform rods AB, BC are freely hinged at B. The end C rests on a rough horizontal plane and A is attached to a point above the plane. When C is as far as possible from A for equilibrium, AB, BC make angles α, β with the vertical. Prove that the coefficient of friction between the rod and the plane at C is $2/(\cot\alpha-3\cot\beta)$.

[Compare questions 6 and 7]

8. Two uniform heavy rods of length a are jointed together by a smooth hinge and placed symmetrically over two smooth pegs at a distance b apart in a horizontal line. If in the position of equilibrium, each rod is inclined at an angle θ to the horizontal, prove that $b=a\cos^3\theta$.

If the pegs are rough and the rods are still placed symmetrically over the pegs, show that equilibrium is possible if and only if

$$a\cos^2\theta\,(\cos\theta-\mu\sin\theta)<b<a\cos^2\theta\,(\cos\theta+\mu\sin\theta).$$

9. Two similar uniform straight rods AB and BC, each of length $2a$ and weight W, smoothly jointed at B, are in a vertical plane and rest in equilibrium on two rough pegs P and Q where P is the middle point of AB and Q is at a distance $\frac{3}{5}a$ from B along BC. The rods are inclined to the horizon at an angle α where $\tan\alpha=\frac{1}{2}$, and B is above the pegs. Calculate the normal components of the forces on the rods at P and Q and show that the frictional component acts up one rod and down the other.

If both rods are on the point of slipping, find the co-efficient of friction at each of the two pegs. [*Ans.* $2W/\sqrt{5}$, $3W/\sqrt{5}$, $\frac{1}{3}$, $1\frac{1}{2}$]

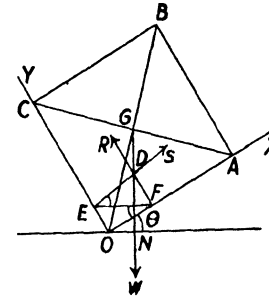
10. Two light rods AOB, COD are each of length $2a$ and are freely jointed at their middle points. They are supported at A and C on a rough horizontal table, the plane containing them being vertical and the angle AOC being 2α. Their upper halves OB, OD support between them a heavy disc of radius r ($r<a\tan\alpha$), the disc being in the same vertical plane. The contacts between

the disc and the rods are smooth. Prove that for equilibrium to be possible, the co-efficient of friction between the ends A, C and the table must not be less than $\tan\alpha + \frac{r}{a}\operatorname{cosec}^2\alpha$.

Miscellaneous Examples IV-A

Ex. 1. *A square board of side $2a$ is placed with its plane vertical between two smooth pegs which are in the same horizontal line and at a distance c, show that it will be in equilibrium when the inclination of one of its edges to the horizon is* $\theta = r$ or $\frac{1}{2}\sin^{-1}\left(\frac{a^2-c^2}{c^2}\right)$. *(All. U.; P.U., D. U.)*

Let OABC be a square, E and F the pegs, and θ, the angle OA makes with the horizontal. The reactions S and R at E and F, and the weight W meet at D.



Since $EF = c$, $\therefore$ $OF = c\cos\theta$,
$OE = c\sin\theta$ ...(*i*)

Also $\because$ $OA = 2a$, $\therefore$ $OG = OA\cos 45° = \sqrt{2}a$,
and $\because$ $\angle AOG = 45°$, $\angle AON = \theta$,
$\therefore$ ON, the projection of OG on the horizontal through O
$= OG\cos\angle GON = \sqrt{2}a\cos(45°+\theta)$
$= a(\cos\theta - \sin\theta)$. ...(*ii*)

Now we solve this problem in the following four ways :—

First Method. *Statical (Art. 6.1).*

Resolving forces perp. to OA, we have $R = W\cos\theta$. ...(*iii*)

Taking moments about E, we have

$R.DE = W[EO\cos OEF + ON] = W[EO\cos(90°-\theta)+ON]$

i.e., $R.OF - W.EO\sin\theta = W.ON$

$\therefore$ by using (*i*), (*ii*) and (*iii*), we have

$W\cos\theta.c\cos\theta - W.c\sin\theta.\sin\theta = W.a(\cos\theta - \sin\theta)$

or $c(\cos^2\theta - \sin^2\theta) = a(\cos\theta - \sin\theta)$. ...(1)

This gives either $\cos\theta - \sin\theta = 0$, *i.e.*, $\theta = \frac{1}{4}\pi$,

or $c(\cos\theta + \sin\theta) = a$. ...(2)

Squaring (2), we have

$c^2(1+\sin 2\theta) = a^2$, *i.e.*, $c^2\sin 2\theta = a^2 - c^2$, or $\theta = \frac{1}{2}\sin^{-1}\frac{a^2-c^2}{c^2}$.

Second Method. *Geometrical.* By equating the length of ON obtained in two *different ways*, *viz.*, (*i*) horizontal projection of OG and (*ii*) the sum of the horizontal projections of OF and FD.

Thus we have

$OG\cos(45°+\theta) = OF\cos\theta - FD\sin\theta = c\cos\theta.\cos\theta - c\sin\theta.\sin\theta$

$\therefore$ $a(\cos\theta - \sin\theta) = c(\cos^2\theta - \sin^2\theta)$, the same as (1).

Third Method. *Analytical.* Let OA, OC be taken as the co-ordinate axes, then the co-ordinates of G are (a, a) and those of D are $(c\cos\theta, c\sin\theta)$.

$\therefore$ the slope of the line $DG = \frac{a - c\sin\theta}{a - c\cos\theta}$

and since EF makes an angle $180° - \theta$ with the positive direction of OX, its slope is $\tan(180° - \theta) = -\tan\theta$.

Now, since DG and EF are perpendicular, we have

$$\frac{a-c\sin\theta}{a-c\cos\theta}(-\tan\theta)=-1$$

i.e., $$(a-c\sin\theta)\sin\theta=(a-c\cos\theta)\cos\theta.$$

$\therefore$ $$a(\cos\theta-\sin\theta)=c(\cos^2\theta-\sin^2\theta),$$ the same as (1).

***Fourth Method.** *By equating to zero, the differential of the height of the C.G. from some fixed horizontal level.* (This is to be explained in Chap. IX).

Here EF is the fixed horizontal level, z, the height of G above EF

= height of G above O − height of EF above O,

i.e., $$z=OG\sin(45°+\theta)-OF\sin\theta$$

$$=\sqrt{2}a\sin(45°+\theta)-c\cos\theta\sin\theta.$$

$\therefore$ $$dz=\sqrt{2}a\cos(45°+\theta)-(c\cos^2\theta-c\sin^2\theta)$$

$$=a(\cos\theta-\sin\theta)-c(\cos^2\theta-\sin^2\theta).$$

This equated to zero gives the same result as (1).

Note. Equation (2) can also be written as

$$c\sqrt{2}\sin(\theta+45°)=a,$$

which gives

$$\theta=\sin^{-1}\frac{a}{c\sqrt{2}}-\frac{\pi}{4},$$

a form in which the answer is sometimes required.

Ex. 2. *In Ex. 1, the pegs are rough and the square board is in limiting equilibrium. Prove that*

$$a(\cos\theta-\sin\theta)=c\cos\lambda\cos(2\theta+\lambda),$$

where λ *is the angle of friction.*

Let us assume that the board is about to slip in the counter-clockwise direction.

Then additional forces μS along EC and μR along FO act on the board.

Resolving along OC and OA, we have,

$$\mu S+R=W\cos\theta, \quad \text{and} \quad S-\mu R=W\sin\theta.$$

Eliminating S from these equations, we have.

$$(1+\mu^2)R=W(\cos\theta-\mu\sin\theta).$$

Writing $\mu=\tan\lambda$ and multiplying by $\cos^2\lambda$, we get

$$R=W\cos\lambda\cos(\theta+\lambda). \qquad \ldots(1)$$

Now taking moments about E, we have

$$W[OE\cos(90°-\theta)+ON]=R.OF-\mu R.OE.$$

Substituting the values of OE, OF and ON as given in Ex. 1 and $\mu=\tan\lambda$, we have

$$W[c\sin\theta.\sin\theta+a(\cos\theta-\sin\theta)]=R[c\cos\theta-\tan\lambda.c\sin\theta]$$

$$=cW\cos\lambda\cos(\theta+\lambda)[\cos\theta-\sin\theta\tan\lambda], \text{ by (1)}$$

$$=cW\cos^2(\theta+\lambda)$$

$\therefore$ $$a(\cos\theta-\sin\theta)=c[\cos^2(\theta+\lambda)-\sin^2\theta]=c\cos\lambda\cos(2\theta+\lambda).$$

Ex. 3. *A uniform rod AB is smoothly hinged to a point A in a smooth vertical wall; at B is smoothly hinged another rod BC of half the length of AB and half its weight. The end C is in contact with the wall, (the rods being in a*

vertical plane perp. to the wall). Prove that in the position of equilibrium the upper rod makes with the wall an angle

$$\sin^{-1}(1/\sqrt{5}).$$

Calculate the pressure at C and the reactions at the hinges A and B.

First consider the rod BC.

The wall being smooth, its reaction R at C is horizontal.

R and the weight $\frac{1}{2}$W of BC must be balanced by the horizontal component X, and the vertical component Y of the reaction on it of the rod AB.

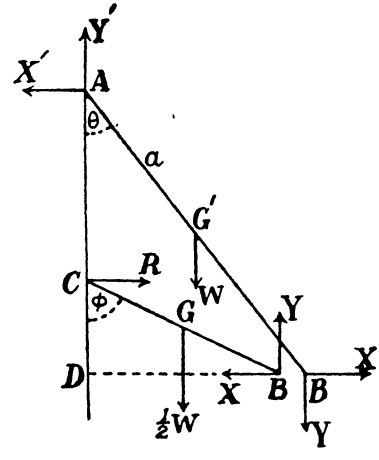
Hence X and Y must act in the directions shown and in magnitude.

$$X=R, \quad \text{and} \quad Y=\tfrac{1}{2}W. \qquad \ldots(1)$$

Action and reaction being equal and opposite the components of the reaction on AB at B must be equal and opposite to those on BC.

The directions of these components at B on AB must be as shown in the figure, the rod AB being, shown separated from BC at B, for clearness.

Let the angles which AB and BC make with the vertical be θ and φ, and their lengths be $2a$ and a respectively, so that



$$AG'=G'B=a, \text{ and } CG=GB=\tfrac{1}{2}a,$$

and
$$AB \sin\theta=BD=BC \sin\varphi=\tfrac{1}{2} AB \sin\varphi,$$

i.e.,
$$\operatorname{cosec}\varphi=\tfrac{1}{2}\operatorname{cosec}\theta. \qquad \ldots(2)$$

Taking moments about C for the rod BC, we have

$$X.a\cos\varphi+\tfrac{1}{2}W.\tfrac{1}{2}a\sin\varphi=Y.a\sin\varphi=\tfrac{1}{2}W.a\sin\varphi,$$

which gives,
$$X=(\tfrac{1}{2}-\tfrac{1}{4})W\sin\varphi/\cos\varphi=\tfrac{1}{4}W\tan\varphi. \qquad \ldots(3)$$

Again, taking moments about A, for the rod AB, we have

$$X.2a\cos\theta=W.a\sin\theta+Y.2a\sin\theta=W.a\sin\theta+\tfrac{1}{2}W.2a\sin\theta$$
$$=2aW\sin\theta,$$

which gives,
$$X=W\sin\theta/\cos\theta=W\tan\theta, \qquad \ldots(4)$$

$\therefore$ from (3) and (4), we have,

$$\tfrac{1}{4}W\tan\varphi=W\tan\theta, \qquad \textit{i.e.}, \cot\varphi=\tfrac{1}{4}\cot\theta \qquad \ldots(5)$$

$\therefore$ from (2) and (5),

$$\tfrac{1}{4}\operatorname{cosec}^2\theta-\tfrac{1}{16}\cot^2\theta=\operatorname{cosec}^2\varphi-\cot^2\varphi=1$$

or
$$4\operatorname{cosec}^2\theta-(\operatorname{cosec}^2\theta-1)=16, \quad \text{or} \quad 3\operatorname{cosec}^2\theta=15.$$

$\therefore$
$$\operatorname{cosec}^2\theta=5, \quad \text{or} \quad \sin\theta=1/\sqrt{5}.$$

Whence
$$\cot^2\theta=4, \quad \textit{i.e.}, \tan\theta=\tfrac{1}{2}.$$

Now by (4), $X=W\tan\theta=\frac{1}{2}W=R$, and also $Y=\frac{1}{2}W$, by (1).

$\therefore$ the resulant reaction at the hinge B

$$=\sqrt{X^2+Y^2}=\sqrt{\tfrac{1}{4}W^2+\tfrac{1}{4}W^2}=\tfrac{1}{2}W\sqrt{2}.$$

Thus the pressure at C$=\frac{1}{2}$W, and reaction at B$=\frac{1}{2}W\sqrt{2}$.

Now to find the action on AB at the hinge A, let X', Y' be its compo-

Since X′ has to balance the only other horizontal force X at B, and Y′ has to balance the other vertical forces W at G′ and Y at B, therefore, the directions of X′ and Y′ must be as shown in the figure.

In magnitude, $X'=X=\frac{1}{2}W$, and $Y'=W+Y=W+\frac{1}{2}W=\frac{3}{2}W$.

$\therefore$ the resultant action at A

$$=\sqrt{\tfrac{1}{4}W^2+\tfrac{9}{4}W^2}=\tfrac{1}{2}\sqrt{10}W.$$

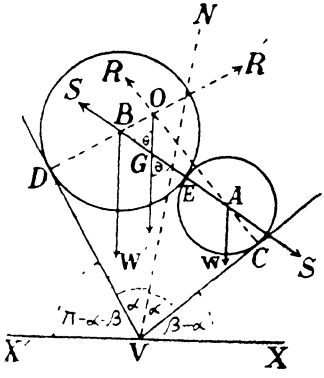
Ex. 4. *Two smooth spheres, of weight W, W′ rest inside a fixed smooth right circular cone, of semi-vertical angle α, which has its vertex downwards and its axis inclined at angle β to the horizon. Each sphere touches the cone at one point only. Prove that the angle θ which the common normal to the spheres makes with the vertical is given by*

$$(W+W')\cot\theta=W'\cot(\beta-\alpha)+W\cot(\beta+\alpha).$$

Show also that if spheres are of equal weights, θ is given by

$$\tan\theta=(\cos 2\alpha-\cos 2\beta)/\sin 2\beta.$$

Let V be the vertex of the cone, VN its axis, so that $\angle NVX=\beta$.



Let A, B be centres of the spheres, and C, D their points of contact with the sides of the cone.

Besides the weights W, W′, and the normal reactions R, R′, there is a mutual action and reaction S acting along the line of centre AB.

To simplify the solution of the problem, we regard the two spheres as one body, then the only external forces acting on this composite body are :

Reactions R, R′ acting along the normals CA and DB, meeting in O, and the resultant weight W+W′ acting on AB at G, the C. G. of weights W, W′, such that,

$$W.AG=W'.BG \text{ or } AG/BG=W'/W. \qquad ...(1)$$

Since this body is in equilibrium under the action of three forces, the three forces must be coplanar and concurrent, so that the vertical line through G must pass through O.

Now $\angle AOG=\angle CAW=\angle CVX=\beta-\alpha.$ ...(2)

(∵ angle between two lines is equal to angle between the lines perp. to them)

Also $\angle BOG=\angle DBW'=\angle DVX'=\pi-\alpha-\beta,$ ...(3)

and the angle that AB, the common normal to the spheres makes with the vertical $=\angle OGB=\theta$.

$\therefore$ by the trigonometrical formula (1) of Art. 6·8, we have

$$(AG+GB)\cot OGB=AG\cot AOG-BG\cot BOG$$

$\therefore$ by using (1), (2) and (3), we get

$$\begin{aligned}(W+W')\cot\theta&=W'\cot(\beta-\alpha)-W\cot(\pi-\alpha-\beta)\\&=W'\cot(\beta-\alpha)+W\cot(\beta+\alpha).\end{aligned}$$

In the particular case when the spheres are of equal weights, *i.e.*, $W=W'$ (their radii may or may not be equal), we have

$$2\cot\theta=\cot(\beta-\alpha)+\cot(\beta+\alpha)$$

or $$\cot\theta=\frac{\sin(\beta+\alpha+\beta-\alpha)}{2\sin(\beta-\alpha)\sin(\beta+\alpha)}=\frac{\sin 2\beta}{\cos 2\alpha-\cos 2\beta}$$

i.e., $$\tan\theta=\frac{\cos 2\alpha-\cos 2\beta}{\sin 2\beta}.$$

5. A square uniform lamina of side $2a$ rests in a vertical plane on two fixed smooth pegs in a horizontal line. Show that if the sum of the distances of the pegs from the lowest corner is equal to a, there is equilibrium.

(D.U. Pass! 95)

6. A triangular lamina ABC has a right angle at C and rests in a vertical plane with its sides in contact with two smooth pegs D, E in the same horizontal plane, the vertex C being downwards. Prove that if, in equilibrium, AC makes an angle θ with the horizon,

$$AC \cos\theta - BC \sin\theta = 3\ DE \cos 2\theta.$$

7. A uniform rectangular board, whose sides are $2a$ and $2b$, rests in limiting equilibrium in contact with two rough pegs in the same horizontal line at a distance d apart. Show that the inclination θ of the side $2a$ to the horizontal is given by the equation

$$d \cos\lambda \cos(\lambda+2\theta) = a\cos\theta - b\sin\theta,$$

where λ is the angle of friction. *(D. U. 1958)*

8. Two smooth planes are each inclined at 45° to the upward vertical their line of intersection being horizontal. A cube of weight W and edge a is placed with one edge on each plane and its lowest face inclined at an angle θ to the horizontal. Show that the magnitude of the couple required to keep the cube in position is $W\,a \sin\theta$.

9. AB, AC are two rods, each of length $2a$ and weight 4W, freely jointed at A, with B and C joined by two light strings, each of length $2a$, to a particle of weight W. The system rests in equilibrium with the rods symmetrically placed over two smooth pegs, in the same horizontal line and distant $2c$ apart. Show that the inclination of either rod to the vertical is given by

$$\sin\theta = (3c/4a)^{\frac{1}{2}}.$$

10. Three uniform rods AB, BC, CD whose weights are proportional to their lengths are jointed at B and C and are in a horizontal position resting on two pegs P and Q. Find the action at the joints between B and C and show that the distance between the pegs is

$$\frac{a^2}{2a+b} + \frac{c^2}{2c+b} + b,$$

where a, b, c are the lengths of the rods. [*Ans.* Each $=\frac{1}{2}$ wt. of BC] *(P.U.)*

11. AB, BC, CD are three equal uniform rods freely joined at B and C. The rods AB, C D rest on two smooth pegs in the same horizontal line so that BC is horizontal. If α be the inclination of AB, and β the inclination of the reaction at B to the horizon, prove that

$$3 \tan\alpha \tan\beta = 1.$$

12. Two uniform rods AB and CD each of weight W and length $2a$ are smoothly hinged together at a point O, where OB and OD are each of length c. The rods rest in a vertical plane with ends A and C on a smooth table and the ends B and D connected by a light string. Prove that the reaction at the joint is $(a\ W \tan\alpha)/c$., where α is the inclination of either rod to the vertical.

13. A triangle ABC hang up by the corner A is formed of uniform rods made from the same material and smoothly jointed at A, B and C. The rod AB is loaded so that BC is horizontal. Prove that the action on rod AB at B is inclined to the vertical at an angle $\tan^{-1}[(1+b/a)\cot C]$.

14. Two smooth spheres of weights W, W′ rest in contact in the angle between two smooth planes of inclinations α, α', respectively. Prove that in the position of equilibrium, the angle θ made by the line joining the centres with the vertical is given by

$$(W+W')\cot\theta = W'\cot\alpha - W\cot\alpha'$$

Given W′=20 lbs., W=10 lbs., α=30°, α'=45°, find θ.

15. Two heavy uniform cylinders of radius a and weight W are placed in contact on the same level inside a smooth fixed cylindrical shell of radius $3a$; a third equal cylinder is placed on the other two, the axes of all the cylinders being horizontal and parallel. Show that the pressure between the two lower cylinders is $W/\sqrt{3}$.

16. Two equal heavy cylinders of radius r placed in contact on the same level inside a smooth fixed cylinder of radius R ($>2r$); a third equal cylinder of radius r is placed on the other two, the axes of all the cylinders being horizontal and parallel. Show that the two lower cylinders will not separate if $R<r(1+2\sqrt{7})$.

Miscellaneous Examples IV-B

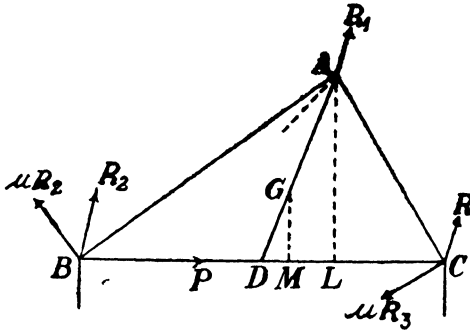
(μ stands for the co-efficient of friction, λ for the angle of friction).

Ex. 1. (*a*) *A uniform triangular plate ABC, of weight W, rests on a horizontal table, supported by three small pegs at the corners A, B, C. What fraction of the weight is borne by each peg?*

If the peg at A fits into a small smooth hole in the table, and if μ is the co-efficient of friction at B and C, show that the least force acting along the side BC, that will cause the plate to rotate about A is $\mu a(b+c)W/6\triangle$, where $\triangle$ denotes the area of the triangle.

By taking moments about the sides, it can be seen that the reaction at each vertex is equal to $\frac{1}{3}$W.

For, if AL and GM be perp. on BC from A and G, the centroid of the plate, and R_1 be the reaction at A perp. to the plate, then taking moments about BC, we have



$$R_1.AL=W.GM$$

or $$R_1.AL=W.\tfrac{1}{3}AL,$$

i.e., $$R_1=\tfrac{1}{3}W.$$

Suppose the plate is just on the point of rotation when a force P acts along BC. Since $R_2=R_3=\frac{1}{3}W$, therefore, frictions of amount

$$\mu R_2=\mu R_3=\tfrac{1}{3}\mu W$$

at B and C acting perpendicular to AB and AC oppose rotation.

Taking moments about A, we have

$$P.\frac{2\triangle}{BC}=\tfrac{1}{3}\mu W.AB+\tfrac{1}{3}\mu W.AC,$$

which gives $$P=\tfrac{1}{6}\mu a(b+c)W/\triangle.$$

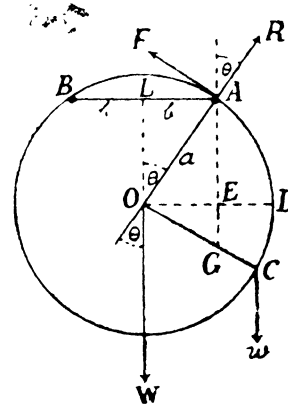
(*b*) A smooth lamina in the form of an equilateral triangle ABC is placed on a horizontal table so that each of the sides touches one of the three smooth pegs P, Q, R where AR : RB=BP : PC=CQ : QA=1 : n. A couple of moment G acts on the lamina; show that the pressure on each of the pegs is

$$\frac{2}{3}\cdot\frac{n+1}{n-1}\cdot\frac{G}{BC}. \qquad (D.U.)$$

Ex. 2. *A uniform circular ring of weight W and radius a hangs in a vertical plane from two rough pegs in the same horizontal plane at a distance 2b apart, where b is less than a. Show that it is possible to fix a suitable weight w to the ring in such a way that one peg may be removed without disturbing the ring, if, and only if the angle of friction at the remaining peg is not less than $\sin^{-1}(b/a)$.*

Find the value of w, if it is attached at the end of a horizontal diameter.

Originally the ring rests on two pegs A and B, the resultant vertical reactions on the pegs, supporting its weight W acting at the centre O.



Draw OL perp. to AB, so that
$$BL=LA=b.$$

Now suppose that when a weight w is fixed at C and the peg B is removed, the ring retains its original position of equilibrium, being supported only at A.

Let R be the normal reaction and F the force of friction along the tangent at A.

Since the vertical resultant of weights W and w acting, say at G, is to be balanced by the resultant of F and R at A, therefore, the vertical line through G must pass through A and must be the line of action of the resultant of R and F.

Let GA make an angle θ with the normal OA, so that

$$\angle AOL=\angle OAE=\theta, \quad \text{and} \quad \sin\theta=\sin AOL=\frac{AL}{OA}=\frac{b}{a}. \qquad \ldots(1)$$

Resolving forces along and perp. to OA, we have

$$(W+w)\cos\theta=R, \text{ and } (W+w)\sin\theta=F, \quad \therefore \quad \tan\theta=\frac{F}{R}.$$

Now for equilibrium, $F<\mu R$, the limiting amount of friction.

$$\therefore \quad \mu>\frac{F}{R}, \textit{ i.e.,} >\tan\theta \quad \text{or} \quad \tan\lambda>\tan\theta$$

$$\therefore \quad \lambda>\theta, \textit{ i.e.,} \lambda>\sin^{-1}\left(\frac{b}{a}\right) \qquad \text{[by (1)]}.$$

In case w is attached at D, the end, nearer to A, of the horizontal diameter (when peg at B has been removed), taking moments about A, we have
$$W.OE=w.ED=w(OD-OE)$$
or
$$W.b=w(a-b), \quad \textit{i.e.,} \quad w=\frac{Wb}{a-b}.$$

3. A sphere, whose radius is a and whose C.G. is at a distance c from the centre O rests in limiting equilibrium on a rough plane inclined at an angle α to the horizon. Show that it may be turned through an angle $2\cos^{-1}\left(\frac{a\sin\alpha}{c}\right)$ and still be in equilibrium. (*Agra U.*)

4. A bow is formed of an arc AB of a circle, which subtends an angle 2α at the centre O of the circle, and its chord (both having the same mass per unit length). If the bow be hung with its arc over a rough peg, prove that the point of contact may be anywhere in the arc CD, which subtends an angle 2β at G, the C.G. of the whole bow, such that

$$\sin\beta=\frac{(\alpha+\sin\alpha)\sin\lambda}{(1+\cos\alpha)\sin\alpha}$$

provided that λ makes this fraction less than unity. (*B.U.*)

[For C.G. of the bow see Ex. Art. 8.6].

5. If the segment of a sphere of radius a and height $a/2$ is supported with its curved surface on a needle point, show that the greatest angle at which the base can be inclined to the horizontal, so as to make equilibrium possible is $\sin^{-1}\left(\frac{40}{87}\sin\lambda\right)$ where λ, the angle of limiting friction is supposed to be less than $\sin^{-1}\frac{87}{40}$.

[For C. G. of the segment see Cor. Art. 8.71].

6. A solid hemisphere rests with its base in an inclined position at an angle θ to the horizontal, its curved surface resting on a horizontal plane and against a vertical plane. If the hemisphere is on the point of slipping, show that

$$\frac{c}{a}\sin\theta=\frac{\mu(1+\mu')}{1+\mu\mu'},$$

where a is the radius, c the distance of the C.G. on the axis of symmetry from the centre and μ, μ' are the co-efficients of friction for horizontal and vertical planes.

If $\mu=\mu'$ and $c=3a/8$, show that there is no position of equilibrium for the hemisphere if $5\mu>\sqrt{31}-4$.

7. A light rod of length l is supported horizontally between two rough planes, at right angles to each other, one inclined to the horizon at an angle α and the inclinations of both being greater than λ, the angle of friction. Show that the length of that portion of the rod on which a weight may be placed without disturbing equilibrium is

$$l\sin 2\alpha\sin 2\lambda.$$

8. A uniform girder AB of length l is being lifted from the ground by means of a chain fastened to the end A and passing over a fixed pulley at a height h ($>l$) vertically above the initial position of A on the ground. If θ is the inclination of the girder to the horizontal when the end B begins to slip, prove that the co-efficient of friction μ between the girder and the ground is given by the equation

$$\mu[h+l\sin\theta-2l\tan\theta]=l(1-\cos\theta).$$

9. A uniform rod of length $2l$, one end of which is on a rough horizontal plane, rests against a smooth horizontal rail at the point which is distant $\frac{1}{2}l$ from the other end. The rod is at right angles to the rail and is inclined at an angle of 45° to the horizontal. If μ is the co-efficient of friction between the rod and the horizontal plane, prove that $\mu\geqslant\frac{1}{2}$.

If a gradually increasing horizontal force directed away from the rail is applied at the point of the rod distant $\frac{1}{2}l$ from the lower end in the vertical plane containing the rod, prove that the rod will ultimately rotate about its lower end or will slide according as μ is greater than or less than 2.

10. Two wheels of radius R are keyed to a co-axial shaft of radius r, the whole weighing W lb. The wheels rest with the shaft horizontal on a rough inclined plane, which makes an angle α with the horizontal. The wheels are prevented from rolling down the plane by a light cord. One end of the cord is fixed to a point P on the plane and the part of the cord is wrapped round the shaft, the part of the cord between P and the shaft is perp. to the axis of the shaft and makes an acute angle β with the inclined plane.

Consider the two possible cases, and show that the pull in the cord is either,

$$\frac{\mathrm{WR}\sin\alpha}{\mathrm{R}\cos\beta+r}\text{ lb.}\qquad\text{or}\qquad\frac{\mathrm{WR}\sin\alpha}{\mathrm{R}\cos\beta-r}.\qquad\textit{(Roorkee)}$$

11. (*a*) A trolley has four wheels and the axles $2a$ apart. The C. G. is half way between the axles and h above the ground when the trolley stands on level. It is found that the trolley just slips down a slope θ with the upper wheels locked and down a slope φ when the lower wheels alone are locked. Show that

$$\tan\theta=\frac{\mu a}{2a+\mu h}\text{ and }\tan\varphi=\frac{\mu a}{2a-\mu h}.\qquad\textit{(Roorkee)}$$

(*b*) Show that a trolley can rest without slipping on an incline of greater slope when the lower wheels are braked than when the upper wheels are braked.

12. A uniform bar ABCD of length $5a$ rests at B and C across two rough parallel horizontal rails which are at the same level and distant $2a$ apart, so that AB=BC=$2a$, CD=a. The co-efficients of friction at B and C are μ_1 and μ_2

respectively. If a slowly increasing force is applied at A parallel to the rails, show that equilibrium is broken by slipping at B if $2\mu_2 > 3\mu_1$.

13. Two small rings of weight W and 4W are threaded on a rough horizontal wire, the co-efficients of friction between the wire and the two rings are $\frac{1}{4}$ and $\frac{1}{2}$ respectively. The ends of a light inextensible string are attached to the rings and a weight w is suspended from the middle point of the string. The two portions of the string are each inclined at an angle of 45° to the vertical.

Prove that if the weight w is gradually increased, equilibrium is broken by sliding one of the rings and state which ring slides first.

[*Ans.* Ring of wt. W]

14. A rigid framework in the form of a rhombus of side a and acute angle α, rests on a rough peg whose co-efficient of friction is μ. Show that the distance between the two extreme positions which the point of contact with the peg can have is $a\mu \sin \alpha$. (*Agra U.*)

15. A glass rod is balanced partly in and partly out of a cylindrical tumbler, with the lower end resting against the vertical side of the tumbler. If α, β be the greatest and the least angles which the rod can make with the vertical, show that the angle of friction is

$$\tfrac{1}{2} \tan^{-1} \frac{\sin^3 \alpha - \sin^3 \beta}{\sin^2 \alpha \cos \alpha + \sin^2 \beta \cos \beta}. \qquad (\textit{Agra U.})$$

16. A heavy uniform elliptic wire (axes $2a$, $2b$) is hung over a small rough peg. Show that, if the wire can be in equilibrium with any point of it in contact with the peg, the co-efficient of friction must not be less than

$$(a^2 - b^2)/2ab. \qquad (\textit{Agra U.})$$

17. A cycloid is placed with its axis vertical and vertex downwards. Show that a particle can rest on it at any point which is not higher than $2a \sin^2 \lambda$ above the lowest point, where λ is the angle of friction and a is the radius of the generating circle of the cycloid. (*Agra U.*)

CHAPTER VIII

CENTRE OF GRAVITY

8.1. Centre of Gravity. Every material body consists of an infinite number of particles each of them being attracted by the force of gravity towards the centre of the earth. If a body is small enough as compared with the size of the earth, these forces acting on the different particles are practically parallel and are proportional to the masses of the particles. The centre of these parallel forces or weights is known as the *centre of gravity* (C.G.) of the body.

8.11. C.G. is a fixed point relative to a body. If we change the position of a body, the forces of attraction at different particles still remain parallel although their direction relative to the body is not the same. But since the centre of parallel forces is independent of the direction of the forces, therefore, the C.G. of the body will remain the same in whatever position the body is placed. Hence the C.G. of a body is a fixed point relative to the body.

8.12. Non-Existence of C. G. If a body is not small enough as compared to the size of the earth but is very large, such as a mountain, then the forces due to gravity on the various particles of the body are neither parallel (being directed towards the centre of the earth) nor are they proportional to their masses. In this case, we have a system of non-coplanar and non-parallel forces, so that the system may not even compound into a single force. Such a body may, therefore, have no centre of gravity.

8.13. Centre of Mass. In the case of a big body which does not possess any C.G., we introduce the idea of its centre of mass. We suppose a hypothetical system of parallel forces acting at different particles of the body and proportional to their masses. The centre of this system of parallel forces is known as the *centre of mass* (C.M.).

The position of the centre of mass of a body depends on the masses of its particles and their positions relative to each other.

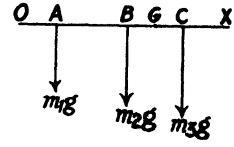
The bodies we deal with being usually small, there is no distinction between C.G. and C.M., so that the two coincide.

The centre of gravity of a body is dependent upon the action of the force of gravity. If it were possible for a body to be removed in space devoid of this attraction, the body shall have no weight and the centre of gravity would not exist. But the centre of mass of the body, always exists depending as it does upon the mass of the body, an inherent property of the material of which the body is formed independent of any attraction to which the body might be subjected.

Thus it is always correct to speak of the centre of gravity of a body as its centre of mass, but not *vice versa*.

8.2. Position of C.G. The following articles are usefully employed in finding the position of the C.G. of a system of particles or of bodies, the position of whose centres of gravity are separately known.

8.21. C.G. of particles placed along a straight line. Let a number of particles of masses m_1, m_2, m_3.....be placed along the straight line OX, at distances x_1, x_2, x_3.....from a fixed point O in the line and let the resultant weight act at G at a distance $\bar{x}$ from O.



As the position of the C.G. is independent of the position of the line, we suppose, for the sake of simplicity, that the line OX is placed horizontally.

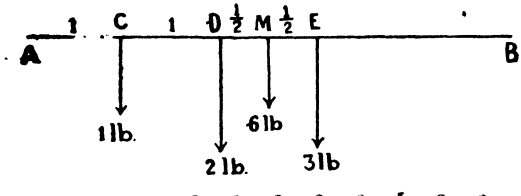
Since the moment of the resultant weight about O is equal to the sum of the moments of the separate weights, we have

$$(m\,g+m_2g+m_3g+.....)\bar{x}=m_1gx_1+m_2gx_2+m_3gx_3+\ .....$$

i.e.,
$$\bar{x}=\frac{m_1x_1+m_2x_2+m_3x_3...}{m_1+m_2+m_3+...}=\frac{\Sigma mx}{\Sigma m}.$$

Example. *A uniform rod 5 ft. in length and of weight 6 lbs., has 3 rings of masses 1, 2, 3 lbs. placed at distances 1, 2, 3 ft. respectively from one end. About what point of the rod will the system balance ?*

Let AB be the rod and let the rings of weights 1, 2, 3 lbs. be placed at C, D and E 1, 2, 3 ft. respectively from the end A of the rod.



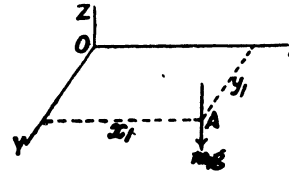
The weight 6 lbs. of the rod acts at its middle point M at a distance of $2\frac{1}{2}$ ft. from A.

Thus
$$\bar{x}=\frac{1\times1+2\times2+6\times\frac{5}{2}+3\times3}{1+2+6+3}=\frac{29}{12}=2\frac{5}{12}.$$

So that the C.G. of the system is at a distance of 2 ft. 5 inches from the end A of the rod. At this point the system will balance.

8.22. C.G. of particles situated on a plane. Let the particles of masses m_1, m_2, m_3.....be placed at points A, B, C, ...whose co-ordinates with reference to OX, OY, perpendicular axes in the plane, are

(x_1, y_1), (x_2, y_2), (x_3, y_3), ...



Since the position of the C.G. of the particles remains the same in whatever position the plane be situated, let us suppose, for the sake of simplicity, that the plane is horizontal.

Let OZ be the vertical line through O, parallel to which all the weights act and let $(\bar{x}, \bar{y})$ be the co-ordinates of the C.G. of the particles.

Then taking moments about OY, we have

$$(m_1g+m_2g+m_3g+\ldots\ldots)\bar{x}$$
$$=m_1gx_1+m_2gx_2+m_3gx_3+\ldots\ldots$$

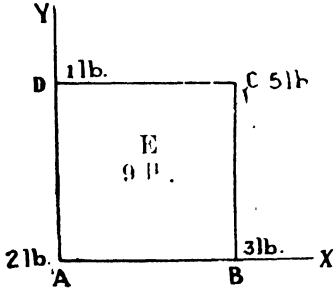
i.e.,
$$\bar{x}=\frac{m_1x_1+m_2x_2+m_3x_3+\ldots\ldots}{m_1+m_2+m_3+\ldots\ldots}=\frac{\Sigma mx}{\Sigma m}.$$

Similarly by taking moments about OX, we get

$$\bar{y}=\frac{m_1y_1+m_2y_2+m_3y_3+\ldots\ldots}{m_1+m_2+m_3+\ldots\ldots}=\frac{\Sigma my}{\Sigma m}.$$

Example 1. *A uniform square lamina AECD whose side is 20 in. and weight is 9 lbs. has attached to its angular points A, B, C, D particles whose weights are 2, 3, 5 and 1 lb., respectively. Find the position of the centre of the system.*

Let AB be taken as the axis of x and AD, the axis of y and let E be the centre of the lamina where its weight 9 lbs. acts. As AB = 20 inches therefore the distances of A, B, C, D and E from OY are respectively 0, 20, 20, 0, and 10 inches.



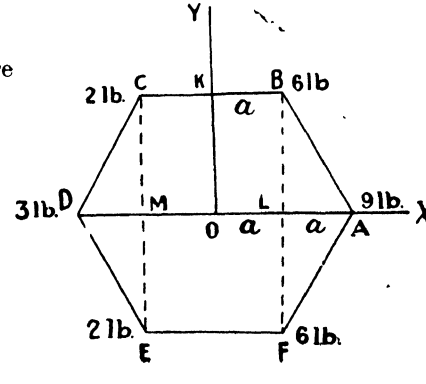
$$\therefore\quad x=\frac{2.0+3.20+5.20+1.0+9.10}{2+3+5+1+9}$$
$$=\frac{250}{20}=12\tfrac{1}{2}''.$$

Similarly
$$y=\frac{2.0+3.0+5.20+1.20+9.10}{2+3+5+1+9}$$
$$=\frac{210}{20}=10\tfrac{1}{2}''.$$

Thus the C.G. of the system is at a distance of $12\frac{1}{2}$ inches from AD and $10\frac{1}{2}$ inches from AB.

Example 2. *Masses 9, 6, 2, 3, 2 and 6 lbs. are placed at the angular points of a regular hexagon, taken in order, find the position of their centre of gravity.*

Let the given weights be placed as shown in the figure where O is the centre of the hexagon.



Let OA be taken as the axis of x and let OY, the axis of y meet BC in K. Let $2a$ be the length of a side of the hexagon.

$KB=a$, $KC=-a$, $OA=2a$, and $OD=-2a$.

Also $LB=MC=a\sqrt{3}$

and $LF=ME=-a\sqrt{3}$.

$$\therefore \quad \bar{y}=\frac{9.0+6.a\sqrt{3}+2.a\sqrt{3}+3.0+2(-a\sqrt{3})+6(-a\sqrt{3})}{9+6+2+3+2+6}=0,$$

which shows that the C.G. of the system lies on the x-axis, *i.e.*, on OA.

$$\text{Now } \bar{x}=\frac{9.2a+6.a+2(-a)+3(-2a)+2(-a)+6.a}{9+6+2+3+2+6}$$

$$=\frac{20}{28}a=\frac{5}{14}.2a.$$

Thus the C.G. divides OA in the ratio 5 : 9.

Example 3. *A number of heavy particles are situated at given fixed points, and another heavy particle moves along a given straight line. Prove that the C.G. of the whole system also moves along a certain straight line.*

Let particles of masses m_1, m_2,.....be placed at fixed points (x_1, y_1), (x_2, y_2),....., and let another particle of mass m be situated at a variable point (x, y).

Then if $(\bar{x}, \bar{y})$ be the co-ordinates of the C.G. of the whole system, by Art. 8.22, we have

$$\bar{x}=\frac{(m_1x_1+m_2x_2+.....)+mx}{(m_1+m_2+.....)+m}=\frac{\Sigma m_1x_1+mx}{\Sigma m_1+m}$$

or $\quad (\Sigma m_1+m)\,\bar{x}=\Sigma m_1x_1+mx$

$$\therefore \quad x=\frac{(\Sigma m_1+m)\,\bar{x}}{m}-\frac{\Sigma m_1x_1}{m}=p\,\bar{x}-q,$$

where p and q are constants. Similarly, $y=p\bar{y}-q$.

Now if x, y moves along a given straight line, say, $ax+by+c=0$, then

$$a\,(p\,\bar{x}-q)+b(p\bar{y}-q)+c=0$$

or $$ap\bar{x}+bp\bar{y}-q(a+b)+c=0.$$

This being a linear equation in $\bar{x}$, $\bar{y}$, shows that the C.G. of the whole system moves along a straight line.

8.23. C.G. of a compound body or C.G. of a remainder. To find the C.G. of the whole when that of each of its given parts is known or to find the C.G. of the remainder when a portion whose C.G. is known has been removed, are two special cases of **8.21.** The mass of the portion removed is taken as negative.

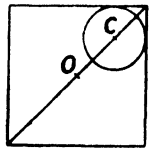
The method is illustrated by the following examples :—

Example 1. *A metallic uniform* circular disc of radius a is placed on a square plate of the same material and side 4a, touching the two adjacent sides. Find the distance of the C. G. of the composite body from O, the centre of the square.*

*Bodies are always supposed to be of uniform density unless otherwise stated.

Let m be the mass per unit area.

Then m_1, the mass of the square plate $=16a^2m$, acting at O.



m_2, the mass of the disc $=\pi a^2 m$ acting at C, where $OC=OA-AC=\frac{1}{2}.4\sqrt{2}\,a-\sqrt{2}\,a=\sqrt{2}\,a$.

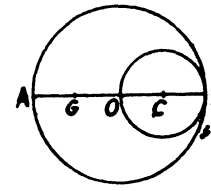
Measuring distance from O along OC, we have by Art. **8·21**,

the required distance $\bar{x}=\dfrac{m_1x_1+m_2x_2}{m_1+m_2}=\dfrac{m_1\times 0+m_2\sqrt{2}a}{m_1+m_2}$

$$=\frac{\pi a^2m\times\sqrt{2}a}{16a^2m+\pi a^2m}=\frac{\sqrt{2}\pi a}{16+\pi}.$$

Example 2. *From a uniform circular disc of radius a, a circular hole, having a radius of the disc as diameter is punched. Find the C. G. of the remainder.*

Let m be the mass per unit area, and G the required C.G. Now m_1 the mass of the whole disc $=\pi a^2m$ acting at O, and m_2 the mass of the portion punched $=\dfrac{\pi a^2}{4}m$, acting at C, AC being $a+\frac{1}{2}a=\frac{3}{2}a$.



Regarding m_2, negative and measuring distances from A along AC, we have

$$\bar{x}=AG=\frac{m_1x_1-m_2x_2}{m_1-m_2}=\frac{\pi a^2m\times a-\frac{1}{4}\pi a^2m\times\frac{3}{2}a}{\pi a^2m-\frac{1}{4}\pi a^2m}=\frac{5}{6}.$$

$$\therefore\quad GO=AO-AG=a-\tfrac{5}{6}a=\tfrac{1}{6}a.$$

The same result can also be obtained by regarding the total mass as composed of two parts $\frac{1}{4}\pi a^2m$ acting at C and the remaining $\frac{3}{4}\pi a^2$ acting at G. Then since O is the position of their resultant we have

$$\tfrac{3}{4}\pi a^2m\times GO=\tfrac{1}{4}\pi a^2m.OC=\tfrac{1}{4}\pi a^2m\times\tfrac{1}{2}a,$$

which gives $GO=\frac{1}{6}a$.

Examples XXX-A

1. A pendulum consists of a wire of length 2 ft. and weight 12 oz. to which is affixed a circular bob of wt. 2 lbs., the centre of the circle being at a distance of $\frac{1}{2}$ ft. from one end of the wire. Find the distance of its C.G. from the other end of the wire. (*Ans.* $2\frac{1}{11}$ ft.)

2. A uniform rod AB is of length n ft. and weight $(n+1)$ W. Masses of weights W, 2W, 3W.....nW are attached to the rod at distances 1, 2, 3,...n ft. respectively from A. Show that the distance of the C.G. of the whole system from A is $\frac{2}{3}n$ ft.

3. A uniform square board whose side is 1 ft. weighs 2 lbs. and has weights 1, 2, 3, 4 lbs. placed at the corners A, B, C, D respectively. Find the distance of the C. G. of the body from the sides AB and AD. [*Ans.* 8 in., 6 in.]

4. Masses 5, 4, 6, 2, 7, 3 are placed at the corners, A, B, C, D, E, F of a regular hexagon. Find the position of their centre of mass. (*Ans.* The centre)

5. Particles of masses p, $r+m$, q, $p+m$, r, $q+m$ are placed at the angular points of a regular hexagon taken one way round. Show that the C.G. coincides with the centre of the hexagon.

✓**6.** AB and AC are two uniform rods of length $2a$ and $2b$ respectively. If $\angle BAC=\theta$, prove that the distance from A of the C.G. of the two rods is

$$\sqrt{a^4+2a^2b^2\cos\theta+b^4}\,/(a+b). \qquad (C.U.)$$

7. A rod of length $5a$ is bent so as to form the five sides of a regular hexagon. Find the distance of the C.G. from either end of the rod.

[*Ans.* $a\sqrt{133}/10$.] (*P.U.*)

8. A fine uniform wire 24 ins. long is bent into the shape of a triangle ABC the sides BC, CA, AB, being as 3 : 4 : 5. Particles of weights p, q, r are placed at the corners A, B and C ; it is found that the C. G. is unchanged, prove that

$$p : q : r = 9 : 8 : 7.$$

9. A uniform rod 18 in. long, is bent so that the two parts 8 and 10 inches long are at right angles to one another. Find the distance between the centres of mass of the new shape and that of the original. (*Ans.* 2·51 in.) (*All. U.*)

10. A double decked omnibus weighing 2 tons has 20 passengers, ten of whom leave the inside and go outside a vertical distance of 6 ft. above their former position. If the average weight of the passengers is 11 stones each, find the vertical shift in the position of the centre of gravity of the bus and passengers. (*Ans.* $1\frac{2}{9}$ ft.)

Examples XXX-B

1. The sides BC, CD of a uniform square lamina ABCD, centre O, are bisected at E and F, and the square portion OECF is cut from the lamina. Prove that the distance of the C. G. of the remaining portion from A is $\frac{5}{6}$ OA.

2. A uniform circular lamina of radius $3a$ and centre O has a hole in the form of an equilateral triangle of side $2a$ with one vertex at O. Prove that the distance of the C.G. from O is $\dfrac{2a}{9\pi-\sqrt{3}}$.

3. A square hole is punched out of a circular lamina. The diagonal of the square is equal to the radius of the circle and one of the corners is at the centre of the circle. Show that the C. G. of the remainder is at a distance $\dfrac{a}{8-4}$ from the centre of the circle, where a is the diameter.

4. A uniform flat circular plate of radius 2 ft. has a circular hole of radius 6 ins., the centre of the hole being 1 ft. from the centre of the plate. Find the distance of the C.G. from the centre of the plate. (*Ans.* 0·8 in.)

5. From a uniform circular disc of radius a is cut a circular hole of radius r which touches the rim of the disc. If the C.G. of the remainder is on the rim of the hole, prove that

$$r^2+ar=a^2.$$

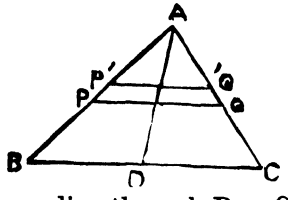
6. A pair of mutually perpendicular diameters is selected as OX and OY in the case of a sheet of metal circular disc, radius R. Three circular holes all of radius r are punched respectively with centres at $(a, 0)$, $(0, b)$ and (x, y). Find x and y in order that the C.G. of the remainder should remain at the centre of the original disc. [*Ans.* $x=-a$, $y=-b$] (*P.U.*)

7. A solid consists of a sphere of radius r containing a spherical cavity of radius b, the two spheres having internal contact at C. Find the position of the C. G. of the solid.

Show that when $b\to r$, the C.G. moves towards a point of trisection of the common diameter. [*Ans.* $(r+b)(r^2+b^2)/(r^2+rb+b^2)$ from C.]

8. From a body of weight W, a piece of weight w is cut off and moved to a distance x, show that C. G. of the whole is thereby moved a distance xw/W in that direction.

8·3. C. G. of a triangular lamina. Consider a uniform triangular lamina ABC. Take an element of its area in the form of a strip PQQ'P' parallel to BC. Then since the median AD bisects all straight lines parallel to BC, the centres of gravity of all strips parallel to BC lie on AD. Hence C.G. of the whole lamina lies on AD. Similarly the C. G. of the lamina lies on the median through B or C, so that the required C.G. is the point of intersection of the medians.



Hence C. G. of the triangular lamina is the point G on AD, one of its medians, such that AG=2GD or AG=$\frac{2}{3}$AD.

8.31. *The C.G. of a triangular lamina is the same as that of three particles of equal mass placed at its vertices.*

Let us consider three particles, each of mass m, placed at the vertices A, B and C. The particles at B and C are equivalent to a particle of mass $2m$ placed at D. Now the C. G. of mass m at A and $2m$ at D is some point G' on AD such that

$$m\text{AG}'=2m\ \text{G}'\text{D},\ i.e.,\ \text{AG}'=2\text{G}'\text{D}.$$

Therefore the point G' is the same as G, the C.G. of the triangle.

In problems concerning the weight and the C. G. of a triangular lamina, the lamina can be replaced by three isolated particles each equal to one-third of the weight of the lamina placed at the vertices.

Cor. The C. G. of a triangular lamina is the same as that of three particles of equal mass placed at the middle points of sides.

Example. *The co-ordinates of the vertices of a uniform triangular lamina are* (x_1, y_1), (x_2, y_2), (x_3, y_3) *show that the co-ordinates of its C. G. are*

$$\tfrac{1}{3}(x_1+x_2+x_3),\ \tfrac{1}{3}\ (y_1+y_2+y_3).$$

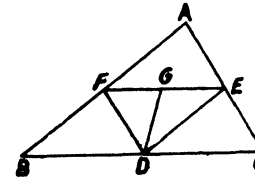
Replace the lamina by three equal masses, each equal to m, placed at the vertices of the triangle, then by Art. 8.22, we have

$$\bar{x} = \frac{mx_1+mx_2+mx_3}{m+m+m} = \frac{m(x_1+x_2+x_3)}{3m} = \tfrac{1}{3}\ (x_1+x_2+x_3)$$

and $$\bar{y} = \frac{my_1+my_2+my_3}{m+m+m} = \tfrac{1}{3}(y_1+y_2+y_3).$$

Ex. Prove that if the C.G. of three particles placed at each of the angular points of a triangle coincides with the C.G. of triangle the particles have equal masses. *(D.U.)*

8.32. The C. G. of a wire in the form of a triangle. The weights of the wires forming the sides are proportional to their lengths and act at their middle points, so that the weights wb, wc of the rods AC and AB act at their middle points E and F. Their C. G. is the point G in EF such that



$$wb.\text{GE}=wc.\text{FG}.$$

$$\therefore \quad \frac{FG}{GE}=\frac{b}{c}=\frac{AC}{AB}=\frac{2DF}{2DE}=\frac{DF}{DE}$$

i.e., DG bisects the angle FDE.

Now we have weights wa placed at D and $w(b+c)$ placed at G, therefore, C. G. of the whole wire must lie on DG, the internal, bisector of $\angle EDF$ of the triangle DEF.

Similarly it can be shown that the C.G. lies on the internal bisector of either of the angles at E or F of $\triangle DEF$.

Hence the C.G. of the wire ABC is the incentre of the $\triangle DEF$ formed by joining the middle points of the sides of the $\triangle ABC$.

Ex. Show that if the weight of one of the sides is supposed to be negative, the C G. is an excentre of the triangle formed by joining the middle points of the sides.

8.33. *The distance of the C.G. of the wire ABC from one of the sides.*

Let y be the distance from BC, of the C.G. of the wire, *i.e.*, of the weights wa, wb, wc of the sides acting at their middle points, then taking moments about BC, we have

$$(wa+wb+wc)y=wa\times 0+wb.\tfrac{1}{2}p+wc.\tfrac{1}{2}p=\tfrac{1}{2}w(b+c)p$$

$$\therefore \quad y=\frac{(b+c)p}{2(a+b+c)}=\frac{b+c}{2(a+b+c)}\cdot\frac{2\Delta}{a}=\frac{(b+c)\Delta}{2as}=\frac{(b+c)r}{2a},$$

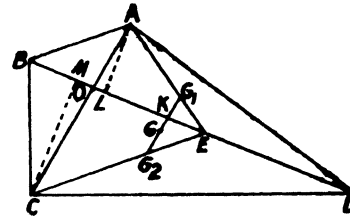
where s is the semi-perimeter, and r the inradius of triangle ABC.

8.4. C.G. of a quadrilateral lamina. Let E be the middle point of the diagonal BD of the quadrilateral ABCD and G_1, G_2, the C.G.'s of the triangles ABD and CBD, then

$G_1E=\frac{1}{3}AE$, $G_2E=\frac{1}{3}CE$ ...(*i*)

Join G_1G_2 meeting BD in K.

Cut off $G_2G=G_1K$ or $G_1G=G_2K$ ...(*ii*), then G is the required C.G. of the given quadrilateral.



For, from (*i*) G_1G_2 is parallel to the diagonal AC which intersects BD in O.

$\therefore$ if AL, CM be the perpendiculars drawn from A and C on BD, we have

$$\frac{\triangle ABD}{\triangle CBD}=\frac{\frac{1}{2}BD.AL}{\frac{1}{2}BD.CM}=\frac{AL}{CM}=\frac{AO\sin\angle AOL}{CO\sin\angle COM}=\frac{AO}{CO} \qquad ...(iii)$$

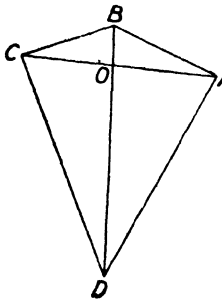
$$=\frac{KG_1}{KG_2}=\frac{GG_2}{GG_1}, \qquad \text{by } (ii),$$

i.e., $\triangle ABD\times GG_1=\triangle CBD\times GG_2$.

Hence G is the C.G. of the quadrilateral.

8.41. *The C.G. of a quadrilateral lamina is the same as that of four particles of equal mass placed at the vertices and a fifth particle of equal and negative mass placed at the intersection of the diagonals.*

Let $3m_1$, $3m_2$ be the masses of the triangles ABD and CBD.



Replace $\triangle$ABD by masses, each equal to m_1, placed at A, B and D.

Similarly replace $\triangle$ CBD by masses each equal to m_2, placed B, C, D.

Thus we have masses,

m_1 at A, m_2 at C, (m_1+m_2) at B, and (m_1+m_2) at D.

Now m_1 at A$=m_1+m_2$ at A and $-m_2$ at A
and m_2 at C$=m_1+m_2$ at C and $-m_1$ at C.

So that now we have four equal masses each equal to m_1+m_2 at A, B, C and D ; and two negative masses $-m_2$ at A and $-m_1$ at C.

Since $\dfrac{AO}{CO}=\dfrac{\triangle ABD}{\triangle CBD}=\dfrac{3m_1}{3m_2}=\dfrac{m_1}{m_2}=\dfrac{-m_1}{-m_2}$

i.e., $-m_2AO=-m_1CO$,

$\therefore$ O is the C.G. of masses $-m_2$ at A and $-m_1$ at C

$\therefore$ $-m_2$ at A and $-m_1$ at C$=-(m_1+m_2)$ at O.

Hence we have four equal masses each equal m_1+m_2 at the vertices A, B, C and D and a negative equal mass $-(m_1+m_2)$ at O, the point of intersection of the diagonals.

Hence the quadrilateral lamina becomes equivalent to four equal masses each equal to m_1+m_2, say M, placed at its vertices and an equal but negative mass $-$M placed at the point of intersection of the diagonals.

8.42. *If x_1, x_2, x_3, x_4, x_5 be the distances of the four corners and the point of intersection of the diagonals of a quadrilateral from a straight line in its plane, show that the distance of the C.G. from the same line is*

$$\bar{x}=\frac{x_1+x_2+x_3+x_4-x_5}{3}.$$

Replace the quadrilateral by four particles, each of mass m placed at the corners and a fifth particle of mass $-m$ placed at O.

Since the position of the C. G. of a body is independent of its position, the lamina may be taken as horizontal.

Taking moments of the weights of the masses about the given line, $\bar{x}$, the distance of the C. G. from the given line is given by

$$(4m-m)\,\bar{x}=mx_1+mx_2+mx_3+mx_4-mx_5,$$

which gives the result.

Example 1. *The C.G. of a quadrilateral lies on one of its diagonals. Show that this diagonal bisects the quadrilateral.*

Let the C.G. (G) lie on the diagonal BD, then in Fig. Art. 8.4, G coincides with K.

$\therefore$ by (*ii*) Art. 8.4, $G_2K=G_1G=G_1K$.

$\therefore$ by similarity of triangles AO = OC, so that the triangles ABD and CBD are equal, *i.e.*, BD bisects the quadrilateral ABCD.

Aliter. Since G lies on BD, $\therefore$ by Art. 8.41, the C.G. of all the five particles lies on B.D. (Fig. Art. 8.41).

But particles at B, D and O lie on BD.

$\therefore$ the C.G. of equal particles at A and C must also lie on BD, *i.e.* BD bisects AC and hence it bisects the quadrilateral.

Example 2. *If one diagonal of a quadrilateral bisects the other and its C.G. be at the middle point of the bisecting diagonal, show that the quadrilateral is a parallelogram.*

Replace the quadrilateral by equal masses, each equal to m, placed at the vertices A, B, C, D and a mass $-m$ placed at O, the point of intersection of the diagonals (Fig. Art. 8.41)

Let BD bisect AC in O.

Since the C. G. of particles m at A and m at C is O, the middle point of AC, therefore, *the C.G. of particles at A, C and O is at O.* ...(1)

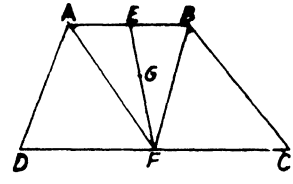
Also C.G. of m at B and m at D is the middle point of BD, say G.

But by the question, the C.G. of the quadrilateral and consequently of the five particles taken above is given to be G.

Therefore, the C. G. of the remaining particles at A, C and O must also be at G, which by (1) is also the point O, so that G coincides with O.

$\therefore$ the middle points of the diagonals AC and BD is the same, *i.e.*, the diagonals bisect each other and hence the quadrilateral is a parallelogram.

8.43. The C.G. of a trapezium. Let $2a$, $2b$ be the lengths and E, F the middle points of the parallel sides AB, DC of the trapezium ABCD.



The weights of the triangles ABF, ADF and BCF are proportional to $2a$, b and b respectively, let them be $2aw$, bw and bw.

Replacing the triangles by particles equal to one-third of the weights placed at their angular points, we have $\frac{2a+b}{3}w$ at each of the points A and B; $\frac{b}{3}w$ at each of C and D and $\frac{2a+2b}{3}w$ at F.

Again the weights at A and B are equivalent to a weight $\frac{4a+2b}{3}w$ at the middle point E and the weights at D, C and F are together equivalent to $\frac{2b}{3}w+\frac{2a+2b}{3}w=\frac{2a+4b}{3}w$ at F.

Thus G, the C.G. of the trapezium, which is the same $\frac{4a+2b}{3}w$ at E and $\frac{2a+4b}{3}w$ at F, lies on EF, such that

$$\frac{4a+2b}{3}w \times EG = \frac{2a+4b}{3}w \times GF$$

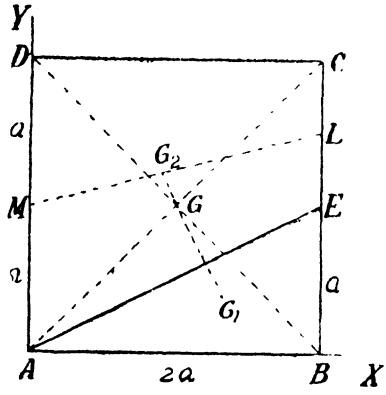
$$\therefore \quad \frac{EG}{GF} = \frac{2a+4b}{4a+2b} = \frac{AB+2CD}{2AB+CD} \text{ or } \frac{a+2b}{2a+b}. \quad \ldots 1.$$

Cor.
$$\frac{EG}{EF} = \frac{a+2b}{(a+2b)+(2a+b)} = \frac{a+2b}{3(a+b)}$$

i.e.,
$$EG = \frac{a+2b}{3(a+b)}EF \text{ and } FG = \frac{2a+b}{3(a+b)}EF.$$

Example 1. *A square ABCD is divided into two parts by joining A to E, the middle point of BC. Prove that the line joining the C.G. of the triangle ABE to that of the quadrilateral AECD is perp. to AE.*

Take AB as the x-axis and AD as y-axis, and $AB = 2a$.



Let G be the C.G. of the square, G_1 of $\triangle ABE$, and G_2 of quad. AECD. G_1, G, G_2 being collinear, to prove that G_2G_1 is perp. to AE is the same thing as to prove that GG_1 is perp. to AE.

The co-ordinates of G are (a, a) ...(1)

Since the co-ordinates of A, B, E are (0, 0), (2a, 0), (2a, a),

$\therefore$ the co-ordinates of G_1 are $\frac{1}{3}(0+2a+2a), \frac{1}{3}(0+0+a)$,

i.e. $(\frac{4}{3}a, \frac{1}{3}a)$ ...(2)

Now m_1, slope of $AE = BE/AB = a/2a = \frac{1}{2}$.

$$m_2, \text{ slope of } GG_1 = \frac{a-\frac{1}{3}a}{a-\frac{4}{3}a} = \frac{\frac{2}{3}a}{-\frac{1}{3}a} = -2.$$

$\therefore$ $m_1 \times m_2 = \frac{1}{2} \times (-2) = -1$, hence GG_1 is perp. to AE.

Note. If required, we can find the position of G_2 by the application of Art. 8.43.

By (1) Art. 8·43, G_2 divides LM, the line joining the mid-points $L(2a, \frac{3}{2}a)$ and $M(0, a)$ of the parallel sides CE and AD of the trapezium AECD, in the ratio $\frac{LG_2}{MG_2} = \frac{CE+2AD}{2CE+AD} = \frac{a+4a}{2a+2a} = \frac{5}{4}$.

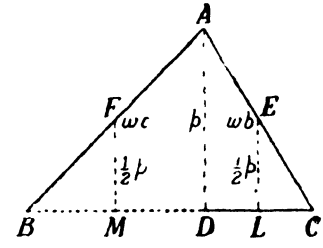
$\therefore$ co-ordinates of G_2 are $[\frac{1}{9}(5 \times 0 + 4.2a), \frac{1}{9}(5.a + 4.\frac{3}{2}a)]$

i.e. $(\frac{8}{9}a, \frac{11}{9}a)$

Example 2. *A uniform wire is bent into the form of a triangle ABC. AD is drawn perpendicular to BC, (D being between B and C) and the part BD of the wire is cut away. Show that the distance of the C.G. of the remainder from BC is*

$$c(b+c)\sin B/(2c+4b\cos^2\tfrac{1}{2}C).$$

Let w be the weight per unit length of the wire, then, weights wb, wc of the portions AC and AB act their middle points E and F, and weight wCD acts at the middle point of CD.



Let AD$=p$, then EL$=$FM$=\frac{1}{2}p$, EL, FM being perp. to BC.

Now $p=$AB sin B$=c$ sin B, and CD$=$AC cos C$=b$ cos C.

If y be the required distance of the C.G. of the wire BACD, from BC, then

$$y=\frac{wb.\frac{1}{2}p+wc.\frac{1}{2}p+w\text{CD}\times 0}{wb+wc+w\text{CD}}=\frac{(b+c)\,p}{2(c+b+b\cos C)}=\frac{(b+c)c\sin B}{2c+2b(1+\cos C)}$$

$$=\frac{c(b+c)\sin B}{2c+4b\cos^2\frac{1}{2}C}$$

Examples XXXI

1. If d_1, d_2, d_3 be the distances of the vertices of a triangle from a straight line, show that the distance of its C. G. from the same line is

$$\tfrac{1}{3}(d_1+d_2+d_3).$$

2. Show that in the triangle ABC, right angled at C, the distance of its C.G. from a line through A perp. to AB is $\frac{1}{3}(b\cos A+b\sec A)$.

3. The side CB of the triangle ABC makes an angle θ with a fixed line CX in the plane. Show that the distance of the C.G. of the triangle from CX is

$$\tfrac{1}{3}[(a\sin\theta+b\sin(\theta+C)].$$

4. Prove that the C.G. of the triangle formed by joining the middle points of the sides of a triangle is the same as the C. G. of the original triangle.

5. The base of a triangle is fixed, and its vertex moves on a given straight line ; show that its C.G. also moves on a straight line.

6. The base of a triangle is fixed, and it has a given vertical angle ; show that its C.G. moves on an arc of a certain circle.

7. A uniform lamina is in the form of a square ABCD of side a, and E, F are the mid-points of AB, AD. The part AEF is folded about EF so that A coincides with the centre O of the square. Find the distance from O of the C.G. of the lamina in its new form. (*Ans.* $a\sqrt{2}/48$)

8. A rectangular lamina ABCD consists of two uniform parts ABC and ADC of weights W and W′ respectively. E and F are the middle points of AB and BC and the triangle EBF is removed. If the C.G. of the remainder is at the point of intersection of the diagonals of the rectangle, prove that $W=2W'$.

9. ABCDEF is a sheet of thin uniform card board in the form of a regular hexagon. Prove that if triangle ABC is cut off and superposed on the triangle DEF, the C.G. of the whole is moved a distance $2a/9$, where a is the side of the hexagon. (*P.U. 1958*)

10. An isosceles right angled triangle has squares described externally on all its sides. Show that the C.G. of the figure so formed is on the line which

bisects the hypotenuse and passes through the right angle and divides it in the ratio 1 : 26. (*P.U. 1955 S.*)

11. A uniform quadrilateral plate ABCD is such that the diagonal AC bisects it and BD cuts it into two parts in the ratio of 2 : 1. Show that its C.G. divides AC in the ratio of 4 : 5.

12. G is the C.G. of a uniform quadrilateral plate, G′ is the C. G. of four equal particles placed at its corners, and O is the intersection of its diagonals. Prove that O. G. and G′ are in the same straight line, and that

$$OG' = 3\ GG'. \qquad (P.U.)$$

13. If the C. G. of a four-sided figure coincides with one of its angular points, show that the distances of this point and of the opposite angular point from the line joining the other two angular points are as 1 : 2. (*D.U.*)

14. If the C.G. of a quadrilateral is the same as that of four equal particles placed at its angular points, show that the quadrilateral must be a parallelogram.

15. Prove that if equal triangles be cut from the corners of a given triangle by lines drawn parallel to the respective sides, the C.G. of the remainder will coincide with that of the given triangle.

16. A piece of uniform wire is bent in the shape of an isosceles triangle whose sides are a, a and b. Show that the distance of the C.G. from the base of the triangle is

$$a\sqrt{a^2 - \tfrac{1}{4}b^2}\ /(2a+b).$$

17. A uniform wire is bent into the form of a triangle. If x, y, z are the distances of its C.G. from the three sides and r, the in-radius, show that

$$4xyz - r^2(x+y+z) = r^3.$$

18. The bisectors of the angles of a triangle of sides a, b, c and area $\triangle$, meet the opposite sides in D, E and F. Prove that the distance of the C G. of the triangle DEF from the side a is $2(2a+b+c)\ \triangle/3(a+b)(a+c)$.

19. Perpendiculars are drawn from the angular points A, B, C of a triangle to the opposite sides a, b, c and another triangle is formed by joining the feet of these perpendiculars. If x, y, z be the distances of the C.G. of this triangle from the sides a, b, c prove that

$$\frac{x}{a^2\cos(B-C)} = \frac{y}{b^2\cos(C-A)} = \frac{z}{c^2\cos(A-B)}. \qquad (Lucknow\ U.)$$

20. The sides AB, AC and the median AD of a triangular lamina ABC are cut in B′, C′, D′ by a line drawn parallel to BC. Show that the C.G. of the quadrilateral BB′C′C is at G on DD′ and that

$$DG = \frac{l}{3} - \frac{2x^2}{3(l+x)},$$

where $AD = l$, $AD' = x$.

21. If p, q, r be the distances of three particles whose weights are P, Q, R from their common C. G. and α, β, γ be the angles contained by the lines p, q ; q, r ; r, p respectively, show that

$$P : Q : R = \frac{\sin\beta}{p} : \frac{\sin\gamma}{q} : \frac{\sin\alpha}{r}.$$

22. Prove the following construction for the C.G. of a trapezium ABCD of which the sides AB, DC are parallel :–

Produce AB to E making BE=DC, and CD to F making DF=AB. Then the C.G. is the point of intersection of EF with the line (joining the middle points of AB and CD.

23. Prove that the C.G. of a uniform triangular lamina of mass M, bordered with a thin uniform rim of mass m, and loaded with a particle of mass $\frac{1}{2}m$ at the centre of its inscribed circle, is at the centroid of the triangle.

24. From a triangular board of uniform thickness, the portion constituting the area of the inscribed circle is removed, prove that the distance of the C.G. of the remainder from the side a is

$$\frac{\triangle}{3as} \cdot \frac{2s^3 - 3\pi a \triangle}{s^2 - \pi \triangle}$$

where $\triangle$ is the area and s, the semi-perimeter of the board.

8.44. Equilibrium of a body suspended freely from a point. *If a body is suspended freely by a string, the line joining the point of suspension and the C.G. of the body must be vertical in the line of the string.*

In this case the body is acted upon by only two forces :—

(*i*) the tension of the string acting at the point of suspension ;

(*ii*) the weight of the body acting at its C.G.

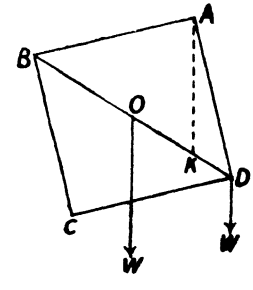
When two forces keep a body in equilibrium, they must be equal, opposite and collinear. But this cannot hold in this case unless the vertical line through the C.G. passes through the point of suspension and is in continuation of the line of the string.

Similarly it is easy to see that when a body is supported at a point and no force other than that of gravity acts on it, the line joining the point of support and the C.G. of the body is vertical.

Example 1. *A square plate is suspended at one of its corners and a weight equal to its own, is suspended from an adjacent corner. Find the point in which the vertical through the point of suspension cuts the opposite diagonal of the square.*

The plate ABCD of weight W is suspended from A and a weight also equal to W is suspended from D.

The C. G. of the two weights must lie somewhere on OD, where O is the centre of the plate.



Let the vertical through A meet OD in K, then since the C. G. must also lie on this vertical line, therefore, K is the C.G. of the weights.

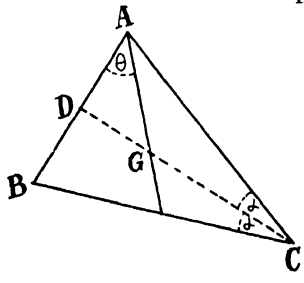
Hence W.OK = W.KD
or OK = KD.

Hence AK bisects OD and therefore divides the diagonal BD in the ratio 3 : 1.

Example 2. *A uniform lamina in the shape of an isosceles triangle of vertical angle 2α, is suspended from one extremity of the base. Show that in the position of equilibrium, the base makes an angle $\tan^{-1}(\frac{1}{3} \cot \alpha)$ with the vertical.*

Let ABC be the lamina suspended from one extremity A of the

base AB, whose middle point is D.



The vertical through A shall pass through the centroid G of the lamina.

Let $\angle DAG=\theta$, so that $\angle AGD=90°-\theta$.

We have, $\angle ACD=\angle BCD=\alpha$ and $DG=\frac{1}{3}$ DC.

Now $$\tan\theta=\frac{DG}{AD}=\frac{\frac{1}{3}DC}{AD}=\frac{1}{3}\cot\alpha$$

or $$\theta=\tan^{-1}(\tfrac{1}{3}\cot\alpha).$$

Aliter. By trigonometrical formula (2) of Art. 6.8, we have

$$(DG+GC)\cot\angle AGD=DG\cot ACD-GC\cot ADG$$

or $$DC\cot(90°-\theta)=\tfrac{1}{3}DC\cot\alpha-\tfrac{2}{3}DC\cot 90°=\tfrac{1}{3}DC\cot\alpha$$

$$\therefore\quad \tan\theta=\tfrac{1}{3}\cot\alpha \quad\text{or}\quad \theta=\tan^{-1}(\tfrac{1}{3}\cot\alpha).$$

Examples XXXII

1. A uniform lamina ABC in the form of a right angled triangle, right angled at C having AC=3BC, is suspended by a string attached to C. Show that in the position of equilibrium, the hypotenuse is inclined at an angle $\sin^{-1}\frac{3}{5}$ to the vertical.

2. A light wire in the shape of an equilateral triangle ABC is suspended from A. If weights 1 and 2 lbs. are respectively fastened at B and C, show that in the position of equilibrium BC makes with the horizontal an angle $\tan^{-1}\frac{1}{5}\sqrt{3}$.

3. A piece of wire 3 ft. long and of uniform linear density is bent so as to form three sides AB, EC, CD of a regular hexagon. Find the prependicular distance of the C.G. from BC.

If the wire is hung freely from A, show that in the equilibrium position AC will be vertical. [*Ans.* $2\sqrt{3}$ ins.]

4. A uniform triangular plate hangs from an angle with the base horizontal. Show that the plate is isosceles.

5. A piece of wire of weight 4 oz. is bent to form the sides AC, CB of an equilateral triangle. If a weight of 1 oz. is attached to B and the wire is suspended from A, show that in the position of equilibrium BC is horizontal. (*P.U.*)

6. A uniform wire AOB is bent at O into two straight portions inclined at angle α, OA and OB being of lengths a and b respectively. If the portion OA is horizontal when the whole is suspended from B, prove that

$$\cos\alpha=\frac{a^2}{b(b+2a)}.$$

7. A uniform wire is bent into the shape of an isosceles triangle each of whose base angles is α. It is suspended from an extremity of the base, show that in the position of equilibrium, the base is inclined with the vertical at an angle

$$\tan^{-1}(\tfrac{1}{2}\tan\tfrac{1}{2}\alpha\sec\alpha).$$

8. A uniform triangular lamina ABC is suspended from the vertex C. Prove that the inclination of the side AB to the vertical is

$$\tan^{-1}[4\triangle ABC/(a^2-b^2)].$$

9. A triangular lamina ABC is suspended successively from A and B such that the two positions of any line in the lamina are perpendicular to one another. Prove that

$a^2+b^2=5c^2$; a, b, c being the sides of the triangle. (*P.U.*)

10. From a uniform square lamina ABCE a triangular portion ADE is cut away, where D is the mid-point of CE. If the lamina ABCD is freely suspended from A, show that AD will be inclined to the vertical at an angle $\tan^{-1}(\frac{1}{2}\frac{3}{7})$.

11. ABCD is a lamina in the form of a trapezium with AD and EC parallel, ABC=90°, AB=a, BC=$2a$, and AD=a. This lamina is suspended by a string attached to the vertex B. Show that the inclination of EC to the vertical in the position of equilibrium is $\tan^{-1}(\frac{4}{7})$.

12. A uniform quadrilateral ABCD has the sides AB, AD and the diagonal AC all equal and the angles BAC and CAD are 30° and 60° respectively. If the weight equal to two-thirds of that of the triangle ABC be attached to the point B, and the whole suspended from A, show that the diagonal AC will be vertical. (*P.U.*)

13. A uniform prism whose cross-section is an isosceles triangle ABC right angled at A, rests with the plane face containing AC, on a horizontal table. D is the middle point of AB and E is a point in AC distant x from A, the prism is cut by a plane passing through DE parallel to its edges, and the portion ADE is removed. Prove that the remaining part will topple over if x is greater than

$\frac{1}{2}(3-\sqrt{5})\,a$, where AC=$2a$.

8.5. Determination of C. G. by Integration. We have seen in 8.22 that in the case of a system of particles of masses m_1, m_2, m_3... placed at (x_1, y_1), (x_2, y_2), (x_3, y_3),.....the co-ordinates of the C.G. $(\bar{x}, \bar{y})$ are given by

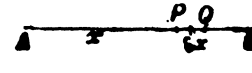
$$\bar{x}=\frac{\Sigma mx}{\Sigma m}, \qquad \bar{y}=\frac{\Sigma my}{\Sigma m}.$$

If instead of a system of isolated particles distributed at separate points, we are to deal with a continuous distribution of mass, then the position of the C.G. is calculated with the help of integration as illustrated by the following articles :

8.51. C.G. of a rod.

(*i*) Let the rod be *uniform*, its mass per unit length being m and let $2a$ be its length.

Let PQ=δx be an element of the rod, then the mass of this element=$m\delta x$ and the distance from A of the C.G. of the element is approximately=x.



$\therefore$ $\bar{x}$, the distance of C.G. of the rod from A

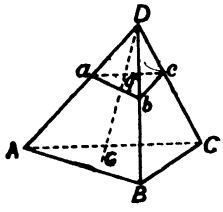
$$=\frac{\int_0^{2a} x.mdx}{\int_0^{2a} mdx}=\frac{\frac{1}{2}.4a^2}{2a}=a.$$

(*ii*) Let the rod be of *variable density*, the density varying as the distance from one end A.

Here the density $\rho=kx$ and the mass of the element=$kx\delta x$.

$$\therefore \quad \bar{x} = \frac{\int_0^{2a} x.kxdx}{\int_0^{2a} kxdx} = \frac{\int_0^{2a} x^2dx}{\int_0^{2a} xdx} = \frac{\frac{1}{3}.8a^3}{\frac{1}{2}.4a^2} = \frac{4}{3}a = \frac{2}{3}.2a.$$

8.52. C.G. of a tetrahedron. Let us consider an element of the tetrahedron in the form of a triangular lamina *abc* parallel to the base ABC of the tetrahedron ABCD.



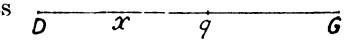
Join the vertex D with G, the C.G. of △ABC, meeting △*abc* in *q*.

Then by geometry of similar triangles it can easily be shown that *q* is the C.G. of △*abc*.

∴ the mass of the lamina *abc* can be replaced by the mass of a single particle at *q*.

Also by geometry, the area of the mass of the lamina *abc* is proportional to the square of the distance D*q*.

Hence by dividing the tetrahedron into laminae such as *abc*, we can regard the tetrahedron equivalent to the rod DG(=*d*), its density *m* varying as the square of the distance from D.



Hence $\bar{x}$, the distance of the C.G. of the tetrahedron from D along DG is given by

$$\bar{x} = \frac{\int x\ mdx}{\int mdx} = \frac{\int_0^d x.kx^2dx}{\int_0^d kx^2dx} = \frac{\frac{1}{4}d^4}{\frac{1}{3}d^3} = \frac{3}{4}d.$$

Hence the C.G. of a tetrahedron lies on the line joining one of its vertices to the centroid of the opposite face and divides it in the ratio 3 : 1 from the vertex.

8.53. C.G. of a pyramid. The above proof is also applicable to a pyramid with its base in the form of a polygon. The C. G. in this case lies on the line joining the vertex D to G, the C. G. of the base and is situated at a distance $\frac{3}{4}$DG from D.

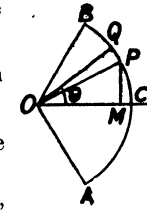
Ex. Show that in finding the C.G. of a triangular lamina, the lamina can be regarded as equivalent to a rod along a median with its density varying as the distance from the vertex. Hence find its C. G.

8.54. *The C.G. of a tetrahedron is the same as the C.G. of four particles of equal mass placed at its vertices.*

For the C.G. of three equal particles, each of mass *m*, placed at A, B and C is the point G (Fig. Art. 8.52), and of the four particles at A, B, C and D is the same as that of masses 3*m* placed at G and *m* placed at D which is a point on DG dividing DG in the ratio 3 : 1 from D.

8.6. C. G, of a uniform circular wire. Let the wire in the

form of an arc AB of a circle of radius a and centre O subtend an angle 2α (in radians) at O. Let C be its middle point. .



Then, by symmetry, the C. G. of the wire will lie on OC.

Let $\angle COP=\theta$, $\angle COQ=\theta+\delta\theta$, and let m be the mass per unit length of the wire.

Then the mass of the element PQ of the wire $=ma\delta\theta$, and the distance of the C.G. of this element from O along OC is approximately equal to $a\cos\theta$.

$\therefore$ $\bar{x}$, the distance of the C.G. of the whole arc from O along OC

$$=\frac{\int_{-\alpha}^{\alpha} a\cos\theta .\, m\, ad\theta}{\int_{-\alpha}^{\alpha} ma\, d\theta}=a.\frac{2\sin\alpha}{2\alpha}=\frac{a\sin\alpha}{\alpha}.$$

In the particular case of a semi-circular arc, α being $\frac{1}{2}\pi$,

$$\therefore \quad =\frac{a\sin\frac{1}{2}\pi}{\frac{1}{2}\pi}=\frac{a}{\frac{1}{2}\pi}=\frac{2a}{\pi}.$$

Example. *Find the C.G. of a bow, consisting of an arc AB of a circle of radius a subtending an angle 2α at the centre O, and its chord, formed by a uniform wire.*

By symmetry, the C.G. of the chord, the arc and the whole bow lies on OC, C being the middle point of the arc AB.

Let m be the mass per unit length of the wire, then

m_1, the mass of the chord $=m.2a\sin\alpha$,

x_1, the distance of its C.G. from O $=a\cos\alpha$,

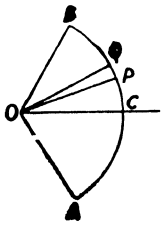
m_2, the mass of the arc $=m.\ a.\ 2\alpha=2ma\alpha$,

x_2, the distance of its C.G. from O $=(a\sin\alpha)/\alpha$.

$$\therefore \quad \bar{x}=\frac{m_1x_1+m_2x_2}{m_1+m_2}=\frac{2am\sin\alpha.a\cos\alpha+2am\alpha.(a\sin\alpha)/\alpha}{2am\sin\alpha+2am\alpha}$$

$$=\frac{a\sin\alpha(1+\cos\alpha)}{\sin\alpha+\alpha}.$$

8.61. C. G. of a Sector of a Circle. Let OAB be the sector of radius a and centre O. Let C be the middle point of the arc AB and let $\angle AOB=2\alpha$ (in radians).



By symmetry the C.G. of the sector must lie on OC.

Let m be the mass per unit area and let $\angle COP=\theta$, then the mass of the element OPQ of the sector $=m.\frac{1}{2}a^2\delta\theta$. Taking the sectorial element OPQ to be approximately equal to $\triangle OPQ$, the distance of the C.G. of this element from O along OC $=\frac{2}{3}a\cos\theta$.

$\therefore$ $\bar{x}=$the distance of the C. G. from O along OC

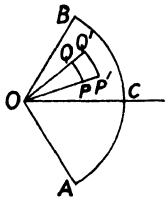
$$= \frac{\int_{-\alpha}^{\alpha} \frac{2}{3}a\cos\theta . m\tfrac{1}{2}a^2 d\theta}{\int_{-\alpha}^{\alpha} m\tfrac{1}{2}a^2 d\theta} = \frac{2}{3}a\,\frac{\sin\alpha}{\alpha}.$$

Cor. *In the case of a semi-circular lamina* $\alpha = \frac{\pi}{2}$,

and $$\frac{\sin\alpha}{\alpha} = \frac{1}{\frac{1}{2}\pi} = \frac{2}{\pi},$$

$$\therefore \quad \bar{x} = \frac{2a}{3}\cdot\frac{2}{\pi} = \frac{4a}{3\pi}.$$

***8.611.** The same result can also be obtained by the use of double integration as follows :—



Co-ordinates of P are (r, θ), and of Q' $(r+\delta r, \theta+\delta\theta)$.

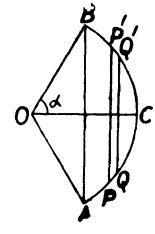
Hence mass of the element $PP'Q'Q = m\delta r.r\delta\theta$,

and the distance of its C.G. from O along $OC = r\cos\theta$,

Hence $$\bar{x} = \frac{\int_{-\alpha}^{\alpha}\int_{o}^{a} m dr.rd\theta.r\cos\theta}{\int_{-\alpha}^{\alpha}\int_{o}^{a} m dr.rd\theta} = \frac{\int_{-\alpha}^{\alpha}\frac{a^3}{3}\cos\theta d\theta}{\int_{-\alpha}^{\alpha}\frac{a^2}{2}d\theta} = \frac{2a}{3}\cdot\frac{\sin\alpha}{\alpha}.$$

Example. *Find the C. G. of a sector of a circle by taking an element in the form of a strip bounded by radii r and* $r+\delta r$.

8.62. C. G. of a Segment of a circle. Let O be the centre and C the middle point of the arc AB, of the segment ACB.



Taking the element in the form of a strip PQQ'P' parallel to the chord AB at a distance x from O, we have, the mass of the element $= m2y\delta x$ where m is the mass per unit area and $x^2+y^2=a^2$.

$\therefore$ $\bar{x}$, the distance of C.G. from O along OC

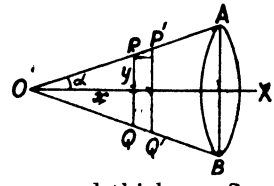
$$= \frac{\int_{a\cos\alpha}^{a} x.2y\,dx}{\int_{a\cos\alpha}^{a} 2y\,dx} = \frac{\int_{a\cos\alpha}^{a} x\sqrt{a^2-x^2}\,dx}{\int_{a\cos\alpha}^{a}\sqrt{a^2-x^2}\,dx}$$

$$= \frac{\left|-\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}\right|_{a\cos\alpha}^{a}}{\left|\frac{x}{2}\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\frac{x}{a}\right|_{a\cos\alpha}^{a}} = \frac{2a}{3}\cdot\frac{\sin^3\alpha}{\alpha-\sin\alpha\cos\alpha}.$$

$$\left[\because \sin^{-1}(\cos\alpha) = \sin^{-1}\sin\left(\frac{\pi}{2}-\alpha\right) = \frac{\pi}{2}-\alpha\right]$$

The student is advised to obtain the same result by regarding the segment as the difference of the sector OACB and the triangle OAB.

8.7. C.G. of a right circular cone. Let α be the semi-vertical angle, O the vertex and h, the height of the cone.



Taking the axis of the cone as the axis of x, the equation of the straight line OA is

$$y = x \tan \alpha.$$

(*i*) **Solid cone.** Let us take an element of the cone in the form of a disc PP'Q'Q, of radius y and thickness δx, situated at a distance x from O, perpendicular to the axis. If m is the mass per unit volume, the mass of the element $= \pi y^2 \delta x . m$ and $y = x \tan \alpha$.

Hence $\bar{x}$ the distance of the C. G. from O along OX

$$= \frac{\int_o^h x . m\pi y^2 dx}{\int_o^h m\pi y^2 dx} = \frac{\int_o^h x . x^2 \tan^2 \alpha \, dx}{\int_o^h x^2 \tan^2 \alpha \, dx} = \frac{\int_o^h x^3 dx}{\int_o^h x^2 dx} = \frac{\frac{1}{4}h^4}{\frac{1}{3}h^3} = \frac{3}{4}h$$

i.e., the C.G. lies on the axis of the cone at a distance $\frac{3}{4}h$ from the vertex.

(*ii*) **Hollow cone.** Here we take the element in the form of a belt PP'Q'Q on the surface of the cone.

Width of the element $= PP' = \delta x \sec \alpha$.

If m is the mass per unit area, the mass of the element

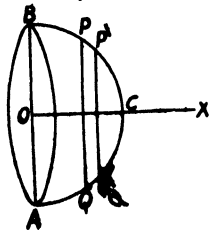
$$= m . 2\pi y \, \delta x \sec \alpha.$$

$$\text{Hence } \bar{x} = \frac{\int_o^h x . 2\pi m y \sec \alpha \, dx}{\int_o^h 2\pi m y \sec \alpha \, dx} = \frac{\int_o^h x . x \tan \alpha \, dx}{\int_o^h x \tan \alpha \, dx} = \frac{\int_o^h x^2 dx}{\int_o^h x dx}$$

$$= \frac{\frac{1}{3}h^3}{\frac{1}{2}h^2} = \frac{2}{3}h.$$

8.71. C.G. of a hemisphere. Let OC, the line through the centre O, perpendicular to the plane AOB, be the axis of x. Equation of the circle ACB is

$$x^2 + y^2 = a^2.$$



Take an element in the form of a disc, PP'Q'Q of radius y and thickness δx, situated parallel to the base at a distance x from it.

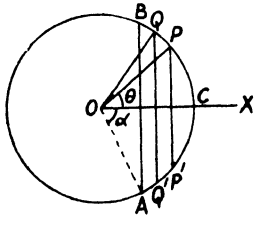
If m be the mass per unit volume, mass of the element $= m\pi y^2 \, \delta x$.

$$\therefore \quad \bar{x} = \frac{\int_0^a x.m\pi y^2 dx}{\int_0^a m\pi y^2 dx} = \frac{\int_0^a x(a^2-x^2)dx}{\int_0^a (a^2-x^2)dx} = \frac{\frac{1}{2}a^4 - \frac{1}{4}a^4}{a^3 - \frac{1}{3}a^3} = \frac{3}{8}a.$$

Thus the C.G. lies on OC at a distance $\frac{3}{8}a$ from the plane base.

Cor. *For C. G. of a segment of a sphere of height* $\frac{1}{2}a$, *the limits of integration are from* $\frac{1}{2}a$ *to* a *and* $x = \frac{2}{4}\frac{7}{0}a$.

8.72. C.G. of a spherical cap. Let the plane AB (perpendicular to the plane of the paper) cut a spherical cap ACB. Let OX. perpendicular to the base AB meet the curved surface in C. Let us take PP'Q'Q, an element of the surface parallel to the base AB.



Let $\angle COP = \theta$, $\angle COQ = \theta + \delta\theta$, then the arc $PQ = a\delta\theta$. If m be the mass per unit area, the mass of the belt $= m2\pi a \sin\theta\, ad\theta$.

If $\angle AOB = 2\alpha$, the angle θ varies from 0 to α.

Hence $\bar{x}$, the distance of the C.G. from O along OC

$$= \frac{\int_0^\alpha a\cos\theta . m\, 2\pi a \sin\theta\, ad\theta}{\int_0^\alpha m.2\pi a \sin\theta\, ad\theta} = \frac{\int_0^\alpha a \sin\theta\cos\theta\, d\theta}{\int_0^\alpha \sin\theta\, d\theta}$$

$$= -\Big[-\frac{a}{2}\cos^2\theta\Big]_0^\alpha = \frac{a}{2}\cdot\frac{1-\cos^2\alpha}{1-\cos\alpha} = \frac{a+a\cos\alpha}{2}$$

i.e., the C.G. bisects the axis of the cap.

The same result can be obtained by using cartesian co-ordinates.

In the particular case of a hollow hemisphere, α being $\frac{1}{2}\pi$,

$$\therefore \qquad \bar{x} = \tfrac{1}{2}a.$$

Cor. *The C.G. of spherical zone lies midway between its two parallel bounding planes.*

Examples XXXIII

1. A uniform wire is bent into the form of a circular arc and two bounding radii and the C.G. of the whole is found to be at the centre. Show that the angle subtended by the arc at the centre is $\tan^{-1}(-\frac{4}{3})$.

2. A piece of wire of given length (l) is bent into the form of a quadrant of a circle and its two bounding radii. Find the C.G. of the whole.

[*Ans.* $6l\sqrt{2}/(4+\pi)^2$] (*P.U. 1958*)

3. If a sector of a circle balances about the chord of the arc, prove that the half angle α of the sector is determined by

$$2\tan\alpha = 3\alpha.$$

4. ABCD is a square of side $2a$. A circle is drawn with centre A and radius $2a$. Find the C.G. of the portion of the area of the square which is not within this circle. [*Ans.* $4\sqrt{2}\,a/3(4-\pi)$ from A along AC).

5. ABC is triangle inscribed in a circle whose centre is O, and G_1, G_2, G_3, are the C. G.'s of the sectors AOB, BOC and COA respectively, prove that

$$\frac{AB}{OG_1}+\frac{BC}{OG_2}+\frac{CA}{OG_3}=3\pi$$

6. Prove that the distance from O of G, the C. G. of a quadrant of a circle with centre O and radius r, is $4r\sqrt{2}/3\pi$.

Find the position of the C.G. of the remainder when a quadrant is removed from a uniform circular lamina of radius r.

[*Ans.* distant $4r\sqrt{2}/9\pi$ from the centre.]

7. From a semi-circular lamina of radius $2a$, a circular lamina of radius a is removed. Find the position of the C. G. of the remainder.

[*Ans.* At a distance $\frac{16a}{3\pi}-a$ from the diameter.] (*D.U.*)

8. A cone, whose height is equal to four times the radius of its base, is hung from a point in the circumference of its base. Show that it will rest with its base and axis equally inclined to the vertical. (*P.U.*)

9. A solid of uniform density is built up of a hemisphere of radius r, and a right circular cylinder of radius r and height h on the circular base of the hemisphere. Find the position of the C.G. of the solid.

Ans. At a distance $\frac{3}{4}.\frac{2h^2-r^2}{3h+2r}$ from the common base.]

10. A solid hemisphere and a solid right circular cone have their bases joined together, the bases being of the same size. Find the semi-vertical angle of the cone so that the C.G. of the compound solid may be at the centre of the common base, the two solids being made of the same material. [*Ans.* 30°] (*P.U. 1957*)

11. A right circular cone of height h is cut off at a height $\frac{1}{2}h$ by a plane perp. to the axis. Find the C. G. of the portion between this section and the base of the cone. (*Ans.* $11h/56$ from the base on the axis.) (*P.U. 1960*)

12. Find the vertical angle of a cone in order that the C. G. of its whole surface, including its plane base, may coincide with the C. G. of the volume.

[*Ans.* $2\sin^{-1}(1/3)$]

13. From a uniform right circular cone with semi-vertical angle 30°, is cut out the greatest possible sphere; show that the C G. of the remainder divides the axis in the ratio 11 : 49.

14. A cylinder and a cone have their bases joined together, the bases being of the same size; find the ratio of the height of the cone to the height of the cylinder so that the common C. G. may be at the centre of the common base.

[*Ans.* $\sqrt{6}:1$] (*P.U.*)

15. Out of a uniform solid cylinder has been cut a cone whose base coincides with that of the cylinder so that the C. G. of the remaining solid coincides with the vertex of the cone. Find the ratio of the height of the cylinder to that of the cone. [*Ans.* $1:(2-\sqrt{2})=1:\cdot5858$.] (*P.U. 1957 S.*)

16. From a right circular cone of height h, a conical portion is removed of height h' and with the same base. If the C.G. of the remainder is at the vertex of the conical hollow, prove that $h=3h'$.

17. A hollow right circular cone, without base, of mass M, and made of thin metallic sheet is placed with its axis vertical and vertex downwards. Water is poured into it up to a depth of one-half of the vertical height. Prove that, if the C. G. of the cone and its contents is at the surface of the water, the mass of water is $\frac{4}{3}$ M.

18. A plane lamina is limited by a semi-circle whose diameter is AB($=2a$) and two semi-circles whose diameters are AO and OB, O being the centre of the semi-circle on AB. Find the C.G. of any one of the three figures so formed.

[*Ans.* $\left(\frac{2a}{3\pi}, 0\right)$, $\left(\frac{2a}{\pi}, 0\right)$, $\left(\frac{a}{\pi}, \frac{1}{4}a\right)$]

19. Any point P is taken upon the diameter AB of a semi-circular area and the semi-circles upon AP, BP as diameter are removed. Find the C.G. of the area that is left.

[*Ans.* With A as origin and AB as x-axis, $\bar{x} = b + \frac{1}{2}a$, $\bar{y} = 2a/\pi$, where AP$=2b$ and AB$=2a$.]

20. A uniform rod of length l with its cross-section a circle, is sharpened at one end like a pencil to a conical point without shortening the axis of the cylinder. If the height of this cone is h, show that the C.G. has been displaced by a distance

$$\frac{h}{4} \cdot \frac{4l-3h}{3l-2h}. \qquad (D.U.\ 1940)$$

21. Distinguish between centre of parallel forces, centre of gravity and centre of mass.

A uniform bowl has for its inner and outer surface concentric hemispheres of radii a and b; prove that the distance of its C.G. from the geometric centre is

$$\frac{3[a+b)(a^2+b^2)}{8(a^2+ab+b^2)}. \qquad (Agra\ U.)$$

22. An inkwell consists of a cube of glass of edge $4a$, with a hemispherical hollow of radius a, the centre of the curved surface is in one face of the cube. Find the height of the C.G. of the inkwell above the base. (Take $\pi = \frac{22}{7}$).

[*Ans.* $1\frac{189}{200}a$.) $\qquad (D.U.\ 1942)$

23. Find the C.G. of the surface and of the volume of the part of a uniform sphere of radius a, included between two parallel planes at distances b and c from its centre.

[*Ans.* $\frac{b+c}{2}$, $\frac{3}{4} \cdot \frac{(b+c)(2a^2-b^2-c^2)}{3a^2-(b^2+bc+c^2)}$]. $\qquad (P.U.\ Hons)$

24. A uniform sphere of radius a, is divided into two parts by a plane distant c from its centre. Show that the distance between the centres of gravity of the two parts is $3a^3/(4a^2-c^2)$.

What happens when $c=0$ and when $c=a$?

25. Two tangents are drawn to a circle of radius a, subtending an angle 2α at the centre. Prove that the C.G. of the figure bounded by the tangents and the smaller arc between them is at a distance from the centre equal to

$$\frac{a}{3} \cdot \frac{\tan^2 \alpha \sin \alpha}{\tan \alpha - \alpha}.$$

26. If two intersecting tangents are drawn from the extremities of a quadrant of a circle, of 4 feet diameter, find the distance of the C.G. of the area enclosed between the tangents and the arc, from either tangent.

[*Ans.* $2(10-3\pi)/3(4-\pi)$ ft.] $\qquad (P.U.)$

27. From a hollow right circular cone of semi-vertical angle α, a frustum is cut off by two planes perpendicular to the axis. If A and B are the centres and a and b the radii of the plane ends and G the C.G. of the curved surface of the frustum, prove that

(*i*) G divides the axis AB in the ratio
$$a+2b : b+2a,$$
(*ii*) the distance of G from the plane midway between the ends is
$$(a-b)^2 \cot \alpha / 6(a+b).$$

28. The radii of the ends of a frustum of a uniform right circular cone of vertical angle 2α are a and $3a$. Show that the C. G. of the frustum from the smaller plane end is $17a \cot \alpha/13$.

If the frustum can rest in equilibrium with its curved surface in contact with a horizontal plane, show that

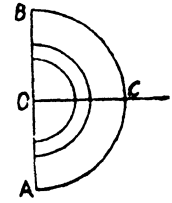
$$30 \cos^2 \alpha > 13.$$

8.8. C.G. when density varies. When the density of a body varies according to some law of variation, it is expressed in terms of the variable or variables involved in the integrals determining the C.G. of the body.

Example. *Find the C. G. of a semi-circular area when the density* :

(*a*) *varies as the square of the distance from the centre ;*
(*b*) *varies as the square of the distance from the diameter ;*
(*c*) *varies as the distance from one end of the diameter.*

(*a*) The density being a function of the distance from the centre, we take an element in the form of a semi-circular ring bounded by radii r and $r+\delta r$, then ρ, the density of the element $=kr^2$,

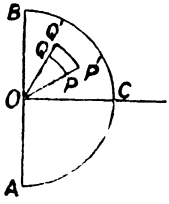


and mass of the element $=\rho\pi r\, dr$.

Distance of the C.G. of this element from O along OC is $\frac{2r}{\pi}$. (Art. 8.6)

$$\therefore \bar{x} = \frac{\int_o^a \frac{2r}{\pi} \rho\pi r\, dr}{\int_o^a \rho\pi r\, dr} = \frac{\int_o^a 2r^2 . kr^2\, dr}{\int_o^a \pi r . kr^2\, dr} = \frac{2}{\pi} \frac{\int_o^a r^4\, dr}{\int_o^a r^3\, dr} = \frac{8a}{5\pi}.$$

Aliter. Taking the element in the form $PP'Q'Q = r\delta\theta\delta r$,

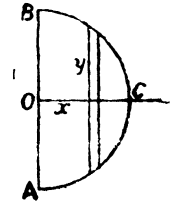


$$\bar{x} = \frac{\int_o^a \int_{-\pi/2}^{\pi/2} r\cos\theta . \rho\, r d\theta\, dr}{\int_o^a \int_{-\pi/2}^{\pi/2} \rho\, r d\theta\, dr}, \text{ where } \rho = kr^2.$$

Substituting $\rho = kr^2$, and integrating for r, we have

$$x = \frac{\frac{1}{5}a^5 \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta}{\frac{1}{4}a^4 \int_{-\pi/2}^{\pi/2} d\theta} = \frac{4a}{5} \times \frac{2}{\pi} = \frac{8a}{5\pi}.$$

(*b*) In this case the density being a function of the distance from the diameter, we take an element in the form of a strip parallel to the diameter at a distance x from it and use the equation $x^2+y^2=a^2$.



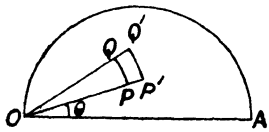
Here $\rho = kx^2$, and mass of the element $= kx^2 . 2y\, dx$.

$$\therefore \bar{x} = \frac{\int_o^a x . kx^2\, 2y\, dx}{\int_o^a kx^2\, 2y\, dx} = \frac{\int_o^a x^3 \sqrt{a^2-x^2}\, dx}{\int_o^a x^2 \sqrt{a^2-x^2}\, dx}$$

Putting $x=a\cos\theta$, we have, $dx=-a\sin\theta\,d\theta$.

$$\therefore\quad \bar{x}=\frac{\int_0^{\pi/2} a\cos^3\theta\sin^2\theta\,d\theta}{\int_0^{\pi/2}\cos^2\theta\sin^2\theta\,d\theta}=\frac{32a}{15\pi}.$$

(c) Taking OA as the initial line, the equation of the circle is $r=2a\cos\theta$.



For the elements PP'Q'Q, if OP$=r$, then $\rho=kr$ and the mass$=kr.rd\theta\,dr=kr^2d\theta dr$.

The co-ordinates of P are $a\cos\theta$, $a\sin\theta$.

Since there is no symmetry, we find both co-ordinates $\bar{x}$ and $\bar{y}$ for the C.G.

$$\bar{x}=\frac{\int_0^{\pi/2}\int_0^{2a\cos\theta} r\cos\theta.kr^2\,d\theta\,dr}{\int_0^{\pi/2}\int_0^{2a\cos\theta} kr^2\,d\theta\,dr}=\frac{3a}{2}\frac{\int_0^{\pi/2}\cos^5\theta\,d\theta}{\int_0^{\pi/2}\cos^3\theta\,d\theta}=\frac{6a}{5}.$$

$$\bar{y}=\frac{\int_0^{\pi/2}\int_0^{2a\cos\theta} r\sin\theta.kr^2\,d\theta\,dr}{\int_0^{\pi/2}\int_0^{2a\cos\theta} kr^2\,d\theta\,dr}=\frac{3a}{2}\frac{\int_0^{\pi/2}\cos^4\theta\sin\theta\,d\theta}{\int_0^{\pi/2}\cos^3\theta\,d\theta}$$

$$=9a/20.$$

Examples XXXIV

1. Find the C. G. of a rod AB of length l when the density varies directly as the square of the distance from the end A. [*Ans.* AG$=\frac{3}{4}l$]

2. The mass per unit length at any point of a straight beam of length l is $\rho(1+x/l)$,

where x is the distance of the point from one end of the beam. Find (*i*) the mass of the beam, (*ii*) the position of its centre of mass.

[*Ans.* $\frac{3}{2}\rho l$; $\frac{5}{9}l$.] (*D.U.*)

3. If the density of a circular lamina varies as the square of the distance from a point O on the circumference, show that the C.G. of the lamina divides the diameter through O in the ratio 2 : 1.

4. If the density of a circular lamina varies as the nth power of the distance from a point O on the circumference, show that the C.G. of the lamina divides the diameter through O in the ratio $n+2$: 2. (*P.U. Hons.*)

5. The density of any point of an arc of a uniform circular wire varies as its distance from the central radius. Prove that the centre of mass is at a point on the central radius, midway between the chord and the arc.

6. If the density of a complete circular arc varies as the square of the distance from a point O on the arc, prove that its C. G. divides the diameter through O in the ratio of 3 : 1.

7. In a solid hemisphere of radius a, the density varies inversely as the distance from the centre. Find the position of the centre of mass. (*P.U.*)

If the density varies directly as the distance from the centre, show that the C.G. shifts further a distance $a/15$.

[*Ans.* At a distance $\frac{1}{4}a$ from the base on the radius of symmetry.]

8. An isosceles triangular lamina is such that its mass per unit area at every point is proportional to the sum of the distances of the point from the equal sides of the triangle. Prove that the distance of the C.G. from the vertex is three-fourths of the altitude. (*D. U.*)

9. Find the C.G. of a plate in the form of a quadrant AOB of an ellipse, the thickness at any point of the plate varying as the product of the distances of the point from OA and OB. [*Ans.* $\frac{8}{15}$ OA, $\frac{8}{15}$ OB]

10. Find the C.G. of a semi-circular plate of radius a whose mass per unit area at any point varies as $\sqrt{a^2-r^2}$, where r is the distance of the point from the centre. [*Ans.* $\frac{8}{3}$ a from centre.]

Miscellaneous Examples V

1. Prove that the resultant of coplanar forces, represented by $\lambda_1 OA_1$, $\lambda_2 OA_2$,....., $\lambda_n OA_n$ is represented by $(\lambda_1+\lambda_2+...+\lambda_n)$ OG where G is the C. G. of the masses λ_1 at A_1, λ_2 at A_2,....., λ_n at A_n.

Two coplanar forces are represented by $\lambda AA'$, $\mu BB'$. Prove that if G, G' are the centres of gravity of masses λ, μ at A, B and of masses λ, μ at A', B', their resultant is represented in magnitude and direction, but not, in general, in line of action by $(\lambda+\mu)$ GG'. (*L. U.*)

2. If $\lambda_1+\lambda_2+\lambda_3 \neq 0$, show that the resultant of forces $\lambda_1 OA$, $\lambda_2 OB$, $\lambda_3 OC$ along the sides OA, OB, OC of a tetrahedron OABC is $(\lambda_1+\lambda_2+\lambda_3)$ OD, along OD, where D is the centroid of particles of masses λ_1, λ_2, λ_3 at A, B, C respectively.

Deduce that, if in addition, forces $l_1 BC$, $l_2 CA$, $l_3 AB$ act along BC, CA, AB respectively, the resultant of the two sets intersect if

$$l_1\lambda_1+l_2\lambda_2+l_3\lambda_3=0.$$

3. A and B are the positions of two masses, m and n, and G is their C. G. If P be any point, show that

$$m \cdot AP^2+n \cdot BP^2=m \cdot AG^2+n \cdot BG^2+(m+n)\, PG^2.$$

Show also that this result can be generalised for any number of masses.

4. Prove that the distance of the C.G. of a uniform cube of edge a from one corner of the cube is $a\sqrt{3}/2$.

Eight equal uniform cubes of edge a are placed together to form a larger cube of edge $2a$. If one of the small cubes is removed, prove that the distance from the centre of the large cube of the C.G. of the remainder is $a\sqrt{3}/14$.

5. A uniform square lamina ABCD of side a and centre O, can turn freely in a vertical plane about a fixed point at C. It rests with the point B on a smooth horizontal table, with A lower than C and CA inclined to the horizontal at $\cot^{-1} 4$. Prove that the reaction at C is not altered when the portion OCD is removed.

6. A thin uniform rod of length l is bent into the form of a circular arc whose radius a is large compared to l. Prove that the displacement of the C.G. is $l^2/24a$ approximately. (*All. U.*)

7. A hollow conical vessel, of perp. height h and slant height l made of thin sheet of metal is closed at its base of radius r. It is cut across by a plane parallel to its base at half the perp. height and the upper cone is removed. Prove that the distance from the base of the C. G. of the remainder is

$$2lh/3(3l+4r).$$ (*Lucknow U.*)

8. $(n+1)$ bricks of the same size are piled one above another in a vertical plane so that they rest, each one overlapping the one below by as much as possible.

Prove that if $2a$ is the length of each brick, the lowest but one overlaps the lowest by a length a/n.

Show also that if each brick overlaps the one next below by a length a/n, the greatest number of bricks that can be piled up is $2n-1$. *(P.U.)*

9. A frustum of a right circular cone of vertical angle 2α is obtained by cutting the cone by a plane parallel to the base at a distance h/n from the vertex, h being the height of the complete cone.

If the frustum is placed with its curved surface in contact with a horizontal table, prove that it will not topple over if

$$\cos^2 \alpha > \frac{4(n^3-1)}{3(n^4-1)}.$$

10. A thin uniform rod is bent at one end to form a walking stick with a semi-circular handle. The straight portion AB is of length $2l$ and the portion BC is of radius a (less than l). The straight line joining the ends B and C of the semi-circular portion is perpendicular to AB.

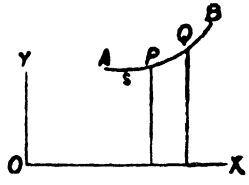
The stick hangs freely with the end C supported on a horizontal table. Prove that the straight part makes an angle θ with the vertical, where

$$\tan\theta = \frac{a(\pi a+4l)}{2(l^2-a^2)}. \qquad \textit{(D.U. 1941)}$$

Centre of gravity in general

8.9. We now deal with general methods of finding the position of C.G. of a wire, a plane area and a surface and volume of revolution.

8.91. C.G. of a wire in the form of a plane curve.



Let ρ be the mass per unit length of the wire APQB. Measuring the length of the wire from a fixed point A on it, let AP $= s$, PQ $= \delta s$, and the co-ordinates of P (x, y).

Mass of the element PQ $= \rho\,\delta s$.

$$\therefore \bar{x} = \frac{\int x \cdot \rho\, ds}{\int \rho\, ds} \quad \text{and} \quad \bar{y} = \frac{\int y \cdot \rho ds}{\int \rho\, ds}.$$

In case ρ is constant, *i.e.*, the wire is of uniform density,

$$\bar{x} = \frac{\int x\, ds}{\int ds} \quad \text{and} \quad \bar{y} = \frac{\int y\, ds}{\int ds}.$$

Now in cartesian co-ordinates,

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx. \qquad \ldots(1)$$

In polar co-ordinates,

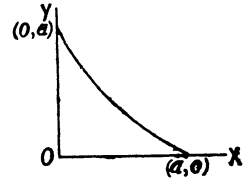
$$ds = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\, d\theta. \qquad \ldots(2).$$

And for parametric equations of a curve in the form $x=f(t)$, $y=\phi(t)$

$$ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt. \qquad \ldots(3)$$

Writing the expression (1), (2), or (3) for ds according as the equation of the curve is cartesian, polar or parametric, we can find the position of the C. G. of any given curve, the integration being performed within proper limits.

8.911. Example 1. *Find the C. G. of the portion of the arc of the curve* $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ *lying in the first quadrant.*



Here $\dfrac{dy}{dx}=-\left(\dfrac{y}{x}\right)^{\frac{1}{3}}$

$$\therefore \quad ds=\sqrt{1+y^{\frac{2}{3}}/x^{\frac{2}{3}}}\;dx.$$

$$=\left(\frac{a}{x}\right)^{\frac{1}{3}}dx. \quad \left[\because\ x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}\right]$$

$$\therefore \quad \bar{x}=\frac{\int xds}{\int ds}=\frac{\int_0^a x\left(\frac{a}{x}\right)^{\frac{1}{3}}dx}{\int_0^a \left(\frac{a}{x}\right)^{\frac{1}{3}}dx}=\frac{\int_0^a x^{\frac{1}{3}}dx}{\int_0^a x^{-\frac{1}{3}}dx}=\frac{2a}{5}.$$

By symmetry $\bar{y}=\bar{x}=\dfrac{2a}{5}$.

Aliter. The same result can also be obtained by taking parametric equations $x=a\sin^3\theta$, $y=a\cos^3\theta$ of the same curve.

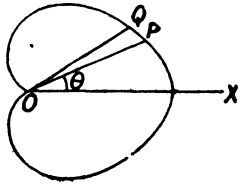
Here $ds=\sqrt{\left(\dfrac{dx}{d\theta}\right)+\left(\dfrac{dy}{d\theta}\right)^2}\,d\theta=3a\sin\theta\cos\theta\,d\theta$,

when $x=0$, $\theta=0$, and whe $x=a$, $\theta=\frac{1}{2}\pi$.

$$\therefore \quad \bar{x}=\frac{\int xds}{\int ds}=\frac{\int_0^{\pi/2} a\sin^3\theta.3a\sin\theta\cos\theta\,d\theta}{\int_0^{\pi/2} 3a\sin^2\theta\cos\theta\,d\theta}=\frac{2a}{5},$$

and $$\bar{y}=\frac{\int yds}{\int ds}=\frac{\int_0^{\pi/2} a\cos^3\theta.3a\sin\theta\cos\theta\,d\theta}{\int_0^{\pi/2} 3a\sin\theta\cos\theta\,d\theta}=\frac{2a}{5}.$$

Example 2. *Find the position of the C.G. of the arc of the cardioide* $r=a(1+\cos\theta)$.



Here $ds=\sqrt{r^2+\left(\dfrac{dr}{d\theta}\right)^2}\,.\,d\theta$

$$=\sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}\,d\theta$$

$$=a\sqrt{2+2\cos\theta}=2a\cos\frac{\theta}{2}d\theta.$$

$$\therefore \quad \bar{x}=\frac{\int x\,ds}{\int ds}=\frac{\int_{-\pi}^{\pi} r\cos\theta .\, 2a\cos\frac{\theta}{2}\,d\theta}{\int_{-\pi}^{\pi} 2a\cos\frac{\theta}{2}\,d\theta}$$

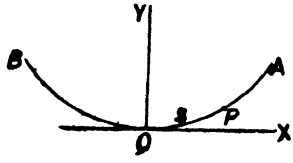
$$=2\int_0^{\pi} a(1+\cos\theta)\cos\theta\cos\frac{\theta}{2}\,d\theta \Big/ 2\int_0^{\pi}\cos\frac{\theta}{2}\,d\theta.$$

Puttin ; $\theta=2\phi$ and simplifying, we have

$$x=2a\int_0^{\pi/2}\cos^3\phi\,(2\cos^2\phi-1)d\phi \Big/ \int_0^{\pi/2}\cos\phi\,d\phi=\frac{4a}{5}.$$

By symmetry $\bar{y}=0$.

Example 3. *If the density of a cycloidal arc $s^2=8ay$, varies as s^2, find the position of the C.G. of the cycloid.*



Here at any point P, the density $\rho=ks^2$. The whole length of the cycloid being $8a$, s varies from $-4a$ to $4a$.

By symmetry the C.G. lies on the y-axis, *i.e.*, $\bar{x}=0$.

and $$\bar{y}=\frac{\int_{-4a}^{4a} y(\rho ds)}{\int_{-4a}^{4a}\rho ds}=\frac{2\int_0^{4a}\frac{s^2}{8a}\cdot ks^2 ds}{2\int_0^{4a} ks^2 ds}=\frac{1}{8a}\frac{\int_0^{4a} s^4 ds}{\int_0^{4a} s^2 ds}=\frac{\frac{1}{5}(4a)^5}{8a.\frac{1}{3}(4a)^3}$$

$$=\frac{6a}{5}.$$

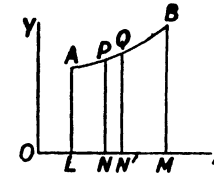
8.92. C. G. of a plane area.

(a) When the area is bounded by the curve $y=f(x)$, the axis of x and the ordinates $x=a$ and $x=b$.

We take the element of the area in the form of a strip PNN'Q of width δx, parallel to the axis of y, at a distance x from it. δA, the area of the strip $=y\delta x$,

and the co-ordinate of its C. G. are $\left(x, \frac{y}{2}\right)$ approximately.

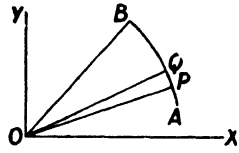
Let ρ be the ma per unit area, we have



$$\bar{x}=\frac{\int x\rho dA}{\int \rho dA}=\frac{\int_a^b \rho xy.dx}{\int_a^b \rho y.dx}, \quad \text{and} \quad \bar{y}=\frac{\int \frac{1}{2}y\rho dA}{\int \rho dA}=\frac{\int_a^b \frac{1}{2}\rho y^2 dx}{\int_a^b \rho y dx}$$

(b) When the area is bounded by the curve $r=f(\theta)$ and radii vectors $\theta=\alpha$ and $\theta=\beta$.

We take the element of the area in the form of the sector OPQ where OP$=r$, $\angle XOP=\theta$ and $\angle POQ=\delta\theta$.

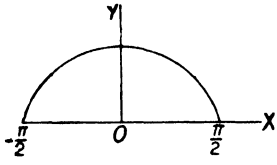


Here δA, area of the element $=\frac{1}{2}r^2\delta\theta$, and the co-ordinates of its C. are approximately $\frac{2}{3}r\cos\theta$, $\frac{2}{3}r\sin\theta$

Hence $$\bar{x}=\frac{\int_\alpha^\beta \frac{2}{3}r\cos\theta\,\rho\frac{1}{2}r^2d\theta}{\int_\alpha^\beta \rho\frac{1}{2}r^2d\theta}=\frac{2}{3}\cdot\frac{\int_\alpha^\beta \rho r^3\cos\theta\,d\theta}{\int_\alpha^\beta \rho r^2d\theta},$$

and $$\bar{y}=\frac{\int_\alpha^\beta \frac{2}{3}r\sin\theta.\,\rho\frac{1}{2}r^2d\theta}{\int_\alpha^\beta \rho\frac{1}{2}r^2d\theta}=\frac{2}{3}\cdot\frac{\int_\alpha^\beta \rho r^3\sin\theta d\theta}{\int_\alpha^\beta \rho r^2d\theta}.$$

8.921. Example 1. *Find the C.G. of a uniform lamina bounded by the curve $y=\cos x$, the axis of x, and the ordinates $x=-\pi/2$ and $x=\pi/2$.*



By symmetry the C.G. lies on the y-axis

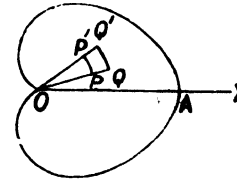
$\therefore\quad \bar{x}=0.$

and $$\bar{y}=\frac{\int \frac{1}{2}y^2dx}{\int ydA}$$

$$-\frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos^2x\,dx\Big/\int_{-\pi/2}^{\pi/2}\cos x\,dx=\frac{\frac{1}{4}\pi}{2}=\frac{\pi}{8}.$$

Example 2. *Find the distance of the C. G. of the area of the cordioide $r=a(1+\cos\theta)$ from its cusp O, when the density varies as the nth power of the distance from O.*

As the density varies as r^n, we take the element of area in the form PQQ'P' of area $rd\theta dr$. The density ρ for the element is kr^n.



By symmetry the C. G. lies on OX, *i.e.*, $\bar{y}=0$ and x for the cardioide is the same as x of the C. G. of the upper half of the curve.

While θ varies from 0 to π, r varies from 0 to $r=a(1+\cos\theta)$.

$$\therefore\quad \bar{x}=\frac{\int\int r\cos\theta.\,\rho r}{\int\int \rho rd\theta dr}=\frac{\int\int r^2\cos\theta.\,kr^nd\theta dr}{\int\int kr^n.rd\theta dr}$$

$$= \frac{\int_0^{\pi}\int_0^{r} r^{n+2}\cos\theta\, dr d\theta}{\int_0^{\pi}\int_0^{r} r^{n+1} dr d\theta} = \frac{(n+2)a\int_0^{\pi}(1+\cos\theta)^{n+3}\cos\theta\, d\theta}{(n+3)\int_0^{\pi}(1+\cos\theta)^{n+2}d\theta}$$

Putting $1+\cos\theta = 2\cos^2\frac{1}{2}\theta$, $\cos\theta = 2\cos^2\frac{1}{2}\theta - 1$, and $\theta = 2\phi$, we have

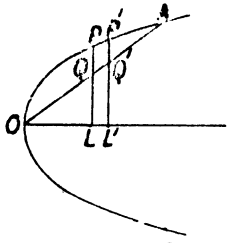
$$\bar{x} = \frac{2a(n+2)}{n+3}\cdot\frac{\int_0^{\pi/2}(\cos\phi)^{2n+6}(2\cos^2\phi-1)d\phi}{\int_0^{\pi/2}(\cos\phi)^{2n+4}d\phi} = \frac{(n+2)(2n+5)a}{(n+3)(n+4)}.$$

(Using Gamma notation of integration and simplifying).

In case the area is uniform $n=0$ and $\bar{x}=5a/6$, which may be worked out independently.

Example 3. *Find the C.G. of the area included between the parabola* $y^2=ax$ *and the line* $y=x$.

Co-ordinates of A, the point of intersection of the parabola and the straight line are (a, a).



We take an element PQQ'P' of the enclosed area. Let $LQ = y_1 = x$ and $LP = y_2 = \sqrt{ax}$.

Area of the element $= (y_2 - y_1)dx$ and dx being small, the co-ordinates of its C.G. are

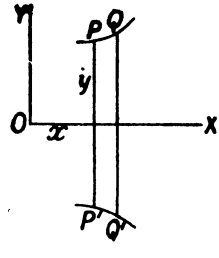
$$\left(x, \frac{y_1+y_2}{2}\right).$$

$$\therefore\quad \bar{x} = \frac{\int x(y_2-y_1)dx}{\int (y_2-y_1)dx} = \frac{\int_0^a x(\sqrt{ax}-x)dx}{\int_0^a (\sqrt{ax}-x)dx} = \frac{\frac{2}{5}a^3 - \frac{1}{3}a^3}{\frac{2}{3}a^2 - \frac{1}{2}a^2} = \frac{2a}{5},$$

and $$\bar{y} = \frac{\int \frac{1}{2}(y_1+y_2)(y_2-y_1)dx}{\int (y_2-y_1)dx} = \frac{\frac{1}{2}\int_0^a (ax-x^2)dx}{\int_0^a (\sqrt{ax}-x)dx} = \frac{1}{2}a.$$

8.93. C. G. of the volume and surface of revolution.

Suppose a solid is formed by the revolution of the curve $y=f(x)$ about the axis of x, and suppose it is bounded by the ordinates $x=a$ and $x=b$.



(*a*) To find the *C.G. of the* **volume** *of the solid*, we take the element of its volume in the form of a disc PP'Q'Q, perpendicular to OX, at a distance x from O.

Since this disc is formed by the revolution of the ordinate $MP = y$ about **OX**, volume of the disc of thickness δx is $\pi y^2 \delta x$.

By symmetry its C. G. lies on OX, $\therefore \quad \bar{y}=0$.

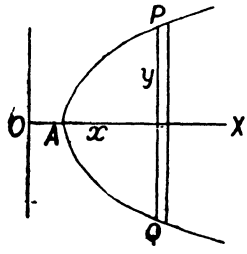
and $$\bar{x}=\frac{\int x.\pi y^2\,dx}{\int \pi y^2}=\frac{\int_a^b x\{f(x)\}^2\,dx}{\int_a^b \{f(x)\}^2\,dx}$$

(*b*) To find the *C.G. of the* **surface** *of the solid*, we take the element in the form of a belt PP′Q′Q on the surface. The width of the belt = arc PQ = δs and its area = $2\pi y\delta s$.

$$\therefore \qquad \bar{x}=\frac{\int x.2\pi y\,ds}{\int 2\text{-}y\,ds}=\frac{\int xy\,ds}{\int y\,ds}$$

The limits for s being taken on the curve $y=f(x)$ corresponding to $x=a$ and $x=b$.

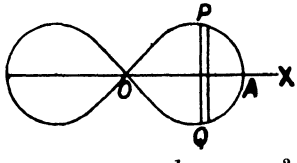
8.931. Example 1. *Find the C. G. of the volume formed by the revolution of the portion of the parabola $y^2=4ax$, cut off by the ordinate $x=h$, about the axis of x.* *(Agra 1936)*



$$\bar{x}=\frac{\int x.\pi y^2\,dx}{\int \pi y^2\,dx}=\frac{\int_o^h x.4ax\,dx}{\int_o^h 4ax\,dx}=\frac{\int_o^h x^2\,dx}{\int_o^h x\,dx}=\frac{2}{3}h.$$

Example 2. *Find the centroid of the surface formed by the revolution of one loop of the lemniscate $r^2=a^2\cos 2\theta$, about the initial line.*

The cartesian co-ordinates (x, y) of any point $P(r, \theta)$ are given by $x=r\cos\theta$, $\quad y=r\sin\theta$, and for the arc APO, θ varies from 0 to $\frac{\pi}{4}$.



Also $\quad 2r\dfrac{dr}{d\theta}=-2a^2\sin 2\theta,$

or $$\frac{dr}{d\theta}=-\frac{a^2\sin 2\theta}{r}=-\frac{a^2\sin 2\theta}{a\sqrt{\cos 2\theta}}.$$

$$\therefore \quad \frac{ds}{d\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}=\sqrt{a^2\cos 2\theta+\frac{a^4\sin^2 2\theta}{a^2\cos 2\theta}}=\frac{a}{\sqrt{\cos 2\theta}}.$$

Now $$\bar{x}=\frac{\int xyds}{\int yds}=\frac{\int_0^{\pi/4} r\cos\theta.r\sin\theta.\dfrac{a}{\sqrt{\cos 2\theta}}d\theta}{\int_0^{\pi/4} r\sin\theta.\dfrac{a}{\sqrt{\cos 2\theta}}d\theta}$$

$$= \tfrac{1}{4}a^3 \int_0^{\pi/4} 2 \sin 2\theta \sqrt{\cos 2\theta}\, d\theta \Big/ a^2 \int_0^{\pi/4} \sin\theta\, d\theta$$

$$(\because\ r^2 = a^2 \cos 2\theta)$$

$$= \frac{a}{4} \cdot \frac{\frac{2}{3}\Big| -\cos^{\frac{3}{2}} 2\theta \Big|_0^{\pi/4}}{\Big| -\cos\theta \Big|_0^{\pi/4}} = \frac{a}{6\left(1-\frac{1}{\sqrt{2}}\right)} = \frac{a\sqrt{2}}{6(\sqrt{2}-1)} = \frac{a(2+\sqrt{2})}{6}.$$

Examples XXXV

1. Find the C.G. of the area bounded by the parabola $y^2 = 4ax$, the axis of x and the latus-rectum. [*Ans.* $3a/5,\ 3a/4$] (*Ali. U.*)

2. Find the C.G. of a plate of uniform density forming a quadrant of an ellipse bounded by its semi-axes. [*Ans.* $4a/3\pi,\ 4b/3\pi$] (*Agra U.*; *D.U.*)

3. Find the position of the C.G. of the areas bounded by the following curves :—

(*i*) $ay^2 = x^3$ between $x = 0$ and $x = c$. [*Ans.* $\bar{x} = 5c/7$]

(*ii*) $y = \sin x$ between $x = 0$, $x = \pi$ and the X-axis. [*Ans.* $\bar{x} = \pi/2,\ \bar{y} = \pi/8$]

(*iii*) $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$ between the axes. [*Ans.* $\bar{x} = a/5,\ \bar{y} = b/5$]

(*iv*) $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$ in the positive quadrant.

[*Ans.* $\frac{\bar{x}}{a} = \frac{\bar{y}}{b} = \frac{256}{315\pi}$]

(*v*) $y^2(2a-x) = x^3$ and its asymptote. [*Ans.* $\bar{x} = \frac{5a}{3},\ y = 0$]

(*vi*) the loop of $y^2(a+x) = x^2(a-x)$. [*Ans.* $\bar{x} = a(3\pi-8)/3(4-\pi)$]

(*vii*) the cardioide $r = a(1+\cos\theta)$. [*Ans.* $\bar{x} = 5a/6$]

(*viii*) One loop of $r = a\cos 2\theta$. [*Ans.* $\bar{x} = \frac{128\sqrt{2}a}{105\pi}$]

(*ix*) One loop of $r^2 = a^2 \cos 2\theta$. [*Ans.* $\bar{x} = \frac{\pi a\sqrt{2}}{8}$] (*Agra U.*)

4. Find the position of the C.G. of the (*i*) arc, (*ii*) area of the cycloid, $x = a(\theta + \sin\theta),\ y = a(1-\cos\theta)$, which lies in the positive quadrant.

[*Ans.* (*i*) $\bar{x} = \left(\pi - \frac{4}{3}\right)a,\ \bar{y} = \frac{2a}{3}$. (*ii*) $\bar{x} = \frac{a}{2}\left(\pi - \frac{16}{9\pi}\right),\ \bar{y} = \frac{7a}{6}$]

5. Find the C.G. of the areas enclosed by the curves in the first quadrant :

(*i*) $y^2 = ax$ and $x^2 = by$. [*Ans.* $9a^{\frac{1}{3}}b^{\frac{2}{3}} / 20,\ 9a^{\frac{2}{3}}b^{\frac{1}{3}} / 20$]

(*ii*) $y^2 = ax$ and $y^2 = 2ax - x^2$.

[*Ans.* $(15\pi-44)a/5(3\pi-8),\ a/(3\pi-8)$]

(*iii*) $x^2+y^2-2ax = 0$ and $x^2+y^2-2bx = 0$, where a and b are positive.

[*Ans.* $\frac{a^2+ab+b^2}{a+b},\ \frac{4}{3} \cdot \frac{a^2+ab+b^2}{\pi(a+b)}$]

6. Find the C.G. of the solid formed by the revolution of the area bounded by the parabola $y^2 = 4ax$, the X-axis and the latus rectum, about the latus rectum. [*Ans.* $\bar{x} = a,\ \bar{y} = 5a/8$] (*All. U.*)

7. Find the C.G. of a solid formed by revolving the quadrant of the ellipse $\frac{y^2}{b^2}=1$ about its (*i*) major axis (*ii*) minor axis.

[*Ans.* (*i*) $(\frac{3}{8}a.\ 0)$; (*ii*) $(0, \frac{3}{8}b)$]

8. Find the C.G. of the volumes formed by the revolution of :—

(*i*) $ay^2=x^3$, about the X-axis between the origin and $x=c$.

[*Ans.* $\bar{x}=\frac{4}{5}c$, $\bar{y}=0$]

(*ii*) cycloid $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$, about the axis of y.

[*Ans.* $\bar{x}=0$, $\bar{y}=\frac{63\pi^2-64}{9\pi^2-16}\cdot\frac{a}{6}$]

(*iii*) $r=a(1+\cos\theta)$ about the X-axis. [*Ans.* $\bar{x}=\frac{4a}{5}$, $\bar{y}=0$]

9. Find the C.G. of the surface formed by the revolution of $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$ about the axis of y.

[*Ans.* $\bar{x}=0$, $\bar{y}=\frac{2a}{15}\cdot\frac{15\pi-8}{3\pi-4}$]

10. Find the C.G. of a sphere of radius a whose density varies as the distance from a tangent plane. [*Ans.* At a distance $6a/5$ from the plane]

11. Find the C.G. of a hemisphere of radius a whose density varies as the square of the distance from the centre of the complete sphere.

[*Ans.* At distance $5a/12$ from the base]

12. Find the C.G. of the catenary $s=c\tan\psi$ between the point $\psi=0$ and $\psi=\pi/4$. [*Ans.* $c[\log(1+\sqrt{2})-\sqrt{2}+1]$, $\frac{1}{2}c[\log(\sqrt{2}+1)+\sqrt{2}]$]

13. If G is the C.G. of any arc PQ of the Lemniscate $r^2=a^2\cos 2\theta$, prove that OG bisects the angle POQ, O being the pole of the curve.

14. Find the C.G. of a lamina in the form of a positive quadrant of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$, when the density at any point (x, y) is kxy.

[*Ans* $128a/429$, $128b/429$]

15. Show that the C.G. of a circular disc, the density of which varies inversely as the fourth power of the distance from an external point O in its plane, is the inverse point of O with respect to the boundary of the disc.

16. Find the C.G. of a solid segment of a sphere, deduce the position of the C.G. of a body consisting of a cone and a segment of a sphere on the same base.

A solid sector of semi angle α is cut from a solid sphere and replaced with its edge in the same position as before but with its vertex outwards ; show that the C.G. of the resulting system will coincide with the centre of the sphere if $\cos\alpha=3/5$. (*D.U. Hons.*)

CHAPTER IX

VIRTUAL WORK, STABILITY OF EQUILIBRIUM

9.1. Work. When a body moves due to the action of a force, the force is said to do work. In order to distinguish it from various other kinds of work, such work is called Mechanical work.

The work done by a constant force is equal to the product of the force and the displacement of its point of application in the direction of the force.

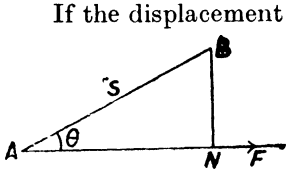
Let P be the force, s the displacement, and W, the work done.

If the displacement takes plane in the direction and in the line of action of the force, then $W = F.s$.

If the displacement is in the same line as before, but in the opposite direction, the work done by the force is negative, *i.e.*,

$$W = -Fs.$$

If the displacement takes place along a line inclined at an angle θ to the line of action of the force, then



$$W = F.AN = F.s \cos\theta.$$

Since $F.s \cos\theta = s.F \cos\theta$,

the work done by the force is equal to the product of the displacement and the resolved part of the force in the direction of the displacement.

If θ is obutse, W is negative.

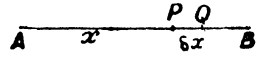
If $\theta = 90°$. $W = 0$, *i.e.*, a force does no work when the displacement takes place at right angles to its line of action.

As an illustration let W denote the work done by the force of gravity on a body of weight w.

When the body drops vertically a distance h, $W = wh$. When the body rises vertically a distance h, $W = -wh$. When the body slides down a length l of an inclined plane of inclination θ to the horizontal, $W = w.\ l \sin\theta = wh$, where h is the vertical distance in the direction of the force.

When $\theta = 0$, $W = 0$, *i.e.*, if the body moves along a horizontal plane the force of gravity does no work.

9.11. Work done by a variable force. Suppose the force varies and acts along the line AB on a particle which moves along that line. If F is the magnitude of the force at P at a distance x from A, the small amount of work δW done by the force F when the particle moves a small distance δx from P, is approximately equal to $F\delta x$.



i.e.,

$$\delta W = F.\delta x.$$

$\therefore$ the total amount of work done by the variable force F when the particle moves from the point $x=a$ to the point $x=b$ is given by

$$W=\int_a^b F\ dx.$$

9.12. Work done in stretching an elastic string.

Let one end of the string be fixed and let it be stretched by pulling it at the other end.

Let l be the natural length of the string and let it be stretched to a length $l+x$. The extension is x and the tension T of the string is $\lambda \frac{x}{l}$ which varies as x, the extension of the string beyond its natural length.

Now as the string is being stretched the force applied by an agent at any instant is always equal to the tension of the string at that instant.

$\therefore$ δW, the small amount of work done in stretching the string a further distance δx is $T\delta x$.

i.e.,
$$\delta W=\lambda \frac{x}{l} \delta x.$$

$\therefore$ W the total work done in stretching the string from a length l to a length $l+x$ is given by

$$W=\int_0^x \lambda \frac{x}{l} dx=\tfrac{1}{2}\frac{\lambda}{l}x^2. \qquad ...(1)$$

So that the work done in stretching an elastic string varies as the square of the extension.

Now the work done for an extension $x_1=\frac{1}{2}\frac{\lambda}{l}x_1^2$,

and the work done for an extension $x_2=\frac{1}{2}\frac{\lambda}{l}x_2^2$.

Hence the work done in increasing the extension from x_1 to x_2

$$=\tfrac{1}{2}\frac{\lambda}{l}x_2^2-\tfrac{1}{2}\frac{\lambda}{l}x_1^2 \qquad ...(2)$$

$$=\frac{\lambda}{l}\frac{x_1+x_2}{2}(x_2-x_1).$$

Since $\frac{\lambda}{l}x_1$ is the initial tension T_1 and $\frac{\lambda}{l}x_2$ is the final tension T_2,

$\therefore$ $\frac{\lambda}{l}\cdot\frac{x_1+x_2}{2}=\frac{T_1+T_2}{2}$ is the mean of the initial and final tensions and x_2-x_1 is the extension produced,

$\therefore$ the work done $=\frac{1}{2}(T_1+T_2)(x_2-x_1)$.

Hence *the work done in stretching an elastic string is equal to the product of the mean of the initial and final tensions and the extension.*

Note. 1. Since $\frac{\lambda}{l} \cdot \frac{x_1+x_2}{2}$ is also the tension for an extension $\frac{x_1+x_2}{2}$, therefore, the work done is also equal to the product of the tension for extension $\frac{x_1+x_2}{2}$, and the extension.

Note. 2. The result (2) can also be obtained as follows :—

$$W = \int_{x_1}^{x_2} \frac{\lambda}{l} x\,dx = \left| \tfrac{1}{2} \frac{\lambda}{l} x^2 \right|_{x_1}^{x_2} = \tfrac{1}{2} \frac{\lambda}{l} {}^{2} - \tfrac{1}{2} \frac{\lambda}{l} x_1^2.$$

Note 3. Also, since $T = \frac{\lambda}{l} x$ and $dT = \frac{\lambda}{l} dx$, or $dx = \frac{l}{\lambda} dT$,

$$\therefore \quad W = \int_{T_1}^{T_2} \frac{l}{\lambda} dT = \frac{l}{2\lambda} (T_2^2 - T_1^2),$$

a result in terms of tensions T_1 and T_2 for the extensions x_1 and x_2.

Example. *An elastic string, of natural length 3 ft., is stretched 2 inches by a weight of 1 lb. hanging on it. Find the work done in stretching it from a length of* $3\frac{1}{2}$ *ft. to* $4\frac{1}{2}$ *ft.*

The natural length $l = 3$ ft., extension $x = 2$ in. $= \frac{1}{6}$ ft.

$$\therefore \quad \text{the tension } T = \frac{\lambda}{l} x = \frac{\lambda}{3} \cdot \frac{1}{6} = \frac{\lambda}{18}.$$

But this tension supports the weight of 1 lb.

$$\therefore \qquad 1 = \frac{1}{18} \cdot \lambda, \qquad \text{or} \qquad \lambda = 18 \text{ lbs.}$$

Thus the first part of the question gives the value of the modulus of elasticity of the string.

Now when its length is $3\frac{1}{2}$ ft. extension $x_1 = 3\frac{1}{2} - 3 = \frac{1}{2}$ ft.

$$\therefore \quad \text{the work done} = \tfrac{1}{2} \frac{\lambda}{l} x_1^2 = \tfrac{1}{2} \cdot \frac{18}{3} \left(\frac{1}{2}\right)^2 = \frac{3}{4} \text{ ft. lb.}$$

Again, when the length is $4\frac{1}{2}$ ft., extension $x_2 = 4\frac{1}{2} - 3 = \frac{3}{2}$ ft.

$$\therefore \quad \text{the work done} = \tfrac{1}{2} \frac{\lambda}{l} x_2^2 = \tfrac{1}{2} \cdot \frac{18}{3} \left(\frac{3}{2}\right)^2 = \frac{27}{4} \text{ ft. lb.}$$

$\therefore$ the required work done in stretching the string from a length of $3\frac{1}{2}$ ft. to $4\frac{1}{2}$ ft. $= \frac{27}{4} - \frac{3}{4} = \frac{24}{4} = 6$ ft. lb.

Aliter. Initial tension $T_1 = \frac{\lambda}{l} x_1 = \frac{18}{3} \cdot \frac{1}{2} = 3$ lb. wt., and final tension $T_2 = \frac{\lambda}{l} x_2 = \frac{18}{3} \cdot \frac{3}{2} = 9$ lb. wt.

$$\therefore \quad \text{the required work} = \tfrac{1}{2}(T_1 + T_2)(x_2 - x_1) = \tfrac{1}{2}(3+9)(4\tfrac{1}{2} - 3\tfrac{1}{2}) = 6 \text{ ft. lb.}$$

9.2. *If a body of weight W undergoes a vertical displacement, such that its C.G. falls through a distance h, the work done is Wh.*

Let the body consist of a number of particles of weights w_1, w_2,....., w_n.

Let z_1, z_2,....., z_n be the heights of the particles and $\bar{z}$ the height of the C.G. of the body above a given horizontal plane, in the initial position.

Let the same heights be z_1', z_2',.....z_n' and $\bar{z}'$ in the second position.

Then $$W = w_1 + w_2 + ... + w_n$$

$$W\bar{z} = w_1z_1 + w_2z_2 + ... + w_nz_n$$

and $$Wz' = w_1z'_1 + w_2z_2' + ... + w_nz_n'.$$

$\therefore$ $w_1(z_1 - z_1') + w_2(z_2 - z_2') + ... + w_n(z_n - z_n') = W(\bar{z} - \bar{z}') = Wh.$

i.e., the work done by the weights of the particles

= W × distance through which the C.G. of the system of particles falls.

Example. *A solid hemisphere of weight 16 lbs. and radius 3 ft., rests with its flat face on a table. How much work against the force of gravity is done when the hemisphere is made to rest with its curved surface in contact with the table.*

The height of the C.G. of the hemisphere above the table in the first case $= \frac{3r}{8} = \frac{3 \times 3}{8} = \frac{9}{8}$ ft.

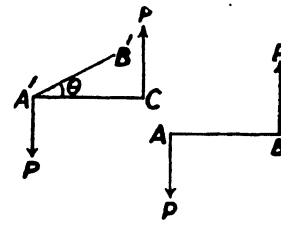
In the second case when the hemisphere rests with its curved surface in contact with the table and its central radius vertical, the height of its C.G. $= r - \frac{3r}{8} = \frac{5r}{8} = \frac{5 \times 3}{8} = \frac{15}{8}$ ft.

$\therefore$ h, the vertical displacement of the C.G. $= \frac{15}{8} - \frac{9}{8} = \frac{3}{4}$ ft.

The weight of the hemisphere = 16 lbs.

$\therefore$ the required work $= Wh = 16 \times \frac{3}{4} = 12$ ft. lbs.

9.3. Work done by a couple. Let a couple of moment G be represented by two equal and opposite forces, each equal to P acting at the ends of the arm AB, so that P.AB = G.



Suppose that it is shifted to the position in which its arm takes up the position A'B'. Let θ be the angle between AB, A'B', the two positions of the arm.

To arrive at the final position A'B', suppose that the couple is at first displaced such that its arm AB is shifted to a parallel position A'C. The

work done by equal and opposite forces P during this displacement is zero.

Now let the arm A'C turn about A' through a small angle $\delta\theta$ so that the point C undergoes a small displacement approximately equal to AC $\delta\theta$ in the direction of the force P at C.

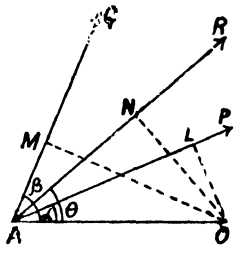
The force P at A' does no work as its point of application is not displaced and the small amount of work done by the other force is $P.A'C.\delta\theta = P.AB.\delta\theta = G.\delta\theta$.

Hence the total amount of work done by the couple till A'C has turned through a finite angle $\theta = \int_0^\theta G.d\theta = G.\theta$.

i.e., the work done by a couple is equal to the product of the moment of the couple and the angle through which the arm of the couple turns.

9.4. *The sum of the work done by a number of co-planar forces acting on a particle in any displacement is equal to the work done by their resultant in the same displacement.*

Let the particle be displaced from A to O and let the forces P, Q.....etc., inclined at angles α, β.....etc., with AO act on it. Let R be the resultant of the forces making an angle θ with AO.



The sum of the work done by the forces

$= P.AL + Q.AM + ...$

$= P.AO \cos\alpha + Q.AO \cos\beta + ...$

$= AO(P\cos\alpha + Q\cos\beta + ...)$

$= AO \times$ sum of the resolved parts of the forces along AO.

$= AO \times$ resolved part of the resultant along AO.

$= AO \times R \cos\theta = R.AO \cos\theta = R.AN$

$=$ work done by the resultant.

Cor. 1. *If R=0 i.e., the forces are in equilibrium, the total work done by the forces vanishes.*

Cor. 2. Since $P.AO \cos\alpha$

$= P \times$ projection of the displacement on the line of action of P,

it follows that if *dp*, *dq*.....etc., denote the projections of the displacements in the directions of the forces P, Q.....etc., and *dr* its projection in the direction of the resultant R, then

$$Rdr = Pdp + Qdq + ... = \Sigma pdp.$$

Examples XXXVI

1. Show that the work done in dragging a body of weight W up a rough inclined plane of length *l* and inclination α is

$$lW(\sin\alpha + \mu\cos\alpha).$$

2. Prove that the work done in stretching an elastic string of natura length l and modulus λ from tension T_1 to tension T_2 is

$$\frac{l}{2\lambda}(T_2^2 - T_1^2).$$

3. A spiral spring requires a force of one pound weight to stretch it one inch. How much work is done in stretching it 3 inches more ? [*Ans.* $\frac{5}{8}$ ft.-lb.]

4. A heavy particle moves along a straight line under a force which varies inversely as the distance from a fixed point. Show that the work done when it undergoes a displacement from a distance a to a distance b is proportional to $\log b/a$.

5. A body whose mass is m falls vertically to the earth's surface from a height a equal to the radius of the earth. Show that $\frac{1}{2}mga$ units of work is done by the earth's attraction which varies inversely as the square of the distance from the earth's centre.

6. A cylindrical cork of length l and radius a is slowly extracted from the neck of a bottle. If the normal pressure per unit area between the bottle and the unextracted part of the cork at any instant be constant and equal to P, show that the work done in extracting it, is $\pi\mu\, al^2$ P, where μ is the co-efficient of friction.

7. A load of two tons is suspended by a vertical chain, 300 ft. long, the chain itself weighing 6 lbs. per ft. In winding up the load to the top how many foot-pounds of work is done. [*Ans.* 1614000 ft.-lbs.]

8. A block weighing 6 lbs. rests on a rough horizontal plane, the co-efficient of friction being $\frac{1}{3}$. An elastic string of unstretched length 4 ft., whose length is doubled by a force equal to the weight of 8 lbs. is fastened to the block and is pulled horizontally. How much work is done up to the instant when the block is just on the point of moving ?

Find also the total amount of work when the block has been dragged very slowly to a distance of 5 ft. [*Ans.* 1 ft.-lb. ; 11 ft.-lb.]

9.5. Virtual Work. If a *particle* is in equilibrium under the action of a number of co-planar forces and the particle is displaced in any direction such that the forces remain constant in magnitude and direction then, the resultant being zero, the total work done by the forces is zero. (Art. 9.4.)

We now extend this principle to the case of a rigid body or a system of connected bodies. A number of problems in Statics are very much simplified by imagining the system at rest, to undergo a slight displacement and then equating the total work done by the forces to zero.

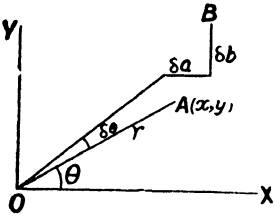
Since the displacement is only imaginary and not a real one, the total work so obtained is known as the *Virtual Work* and the method adopted is known as the *principle of Virtual Work*, which can be stated as follows :—

If a system of co-planar forces acting on a rigid body or a system of connected bodies is in equilibrium, and if the system undergoes a slight displacement consistent with its geometrical conditions, then the algebraic sum of the Virtual Works of the forces is zero and conversely if this algebraic sum is zero, the forces acting on the system are in equilibrium.*

*In fact the displacement of the system is so chosen as to lead to the determination of what we may have in view.

9.51. *Proof of the principle of Virtual Work for a system of co-planar forces acting on a rigid body.*

The most general displacement of a body is equivalent to a rotation about some point and a translation along some line.



Let the body be rotated through an angle $\delta\theta$ about O and then translated distances δa, δb parallel to the fixed axes OX, OY where δa, δb and $\delta\theta$ are small quantities.

Let the point A with cartesian co-ordinates (x, y) and polar co-ordinates (r, θ), be the point of application of a force P before displacement and let B$(x+\delta x, y+\delta y)$ be the point of application of P after displacement, then

$$x = r\cos\theta \qquad y = r\sin\theta$$

and
$$\begin{aligned}\delta x &= r\cos(\theta+\delta\theta)+\delta a - r\cos\theta\\ &= r(\cos\theta\cos\delta\theta - \sin\theta\sin\delta\theta) - r\cos\theta + \delta a.\\ &= -r\sin\theta.\delta\theta + \delta a, \quad \text{neglecting small quantities of the second and higher orders.}\\ &= \delta a - y\,\delta\theta.\end{aligned}$$

Similarly $\delta y = r\sin(\theta+\delta\theta)+\delta b - r\sin\theta = \delta b + x\delta\theta.$

If X, Y be the components of P parallel to OX, OY, then the virtual work of P

$$\begin{aligned}&= X\delta x + Y\delta y = X(\delta a - y\delta\theta) + Y(\delta b + x\delta\theta)\\ &= X\delta a + Y\delta b + \delta\theta(xY - yX).\end{aligned}$$

Similarly we obtain the virtual work of the other forces, δa, δb and $\delta\theta$ being the same for their displaced points of application.

$\therefore$ the Virtual Work of the whole system

$$= \delta a\Sigma X + \delta b\Sigma Y + \delta\theta\Sigma(xY - yX).$$

Now if the forces are in equilibrium then by Art. 5·76, $\Sigma X = 0$, $\Sigma Y = 0$ and $\Sigma(xY - yX) = 0$ so that the virtual work vanishes.

Conversely, let the virtual work be zero,

i.e., $\qquad \delta a\Sigma X + \delta b\Sigma Y + \delta\theta\Sigma(xY - yX) = 0$

for *all displacements*.

Considering, the displacement to take place parallel to X-axis *only*, so that $\qquad \delta b = 0$ and $\delta\theta = 0$,

then $\qquad \delta a\Sigma X = 0$ or $\Sigma X = 0$.

i.e., the sum of the resolved parts of the forces parallel to OX vanishes.

Similarly considering displacement parallel to Y-axis only, we get

$$\delta b\Sigma Y = 0 \text{ or, } \Sigma Y = 0,$$

i.e., the sum of the resolved parts of the forces parallel to OY vanishes.

Finally considering displacement of pure rotation $\delta\theta$ about O, we get $\Sigma(xY-yX)=0$

i.e., the sum of the moments of the forces about O vanishes.

Thus the conditions of equilibrium are satisfied and, therefore, the system of forces is in equilibrium.

***9·52.** Without neglecting second and higher powers of δa, δb and $\delta\theta$, we have

$$\begin{aligned}\delta x &= r\cos(\theta+\delta\theta)+\delta a-r\cos\theta\\ &=\delta a+r(\cos\theta\cos\delta\theta-\sin\theta\sin\delta\theta)-r\cos\theta\\ &=\delta a+x\cos\delta\theta-y\sin\delta\theta-x \qquad (\because\ r\cos\theta=x,\ r\sin\theta=y)\\ &=\delta a+x\left[1-\frac{(\delta\theta)^2}{2!}+\frac{(\delta\theta)^4}{4!}-\dots\right]-y\left[\delta\theta-\frac{(\delta\theta)^3}{3!}\right]-x.\\ &=\delta a-y\delta\theta-\frac{x}{2!}(\delta\theta)^2+\frac{y}{3!}(\delta\theta)^3-\dots\dots\end{aligned}$$

Similarly

$$\begin{aligned}\delta y &= r\sin(\theta+\delta\theta)+\delta b-r\sin\theta=\delta b+y\cos\delta\theta+x\sin\delta\theta-y\\ &=\delta b+x\delta\theta-\frac{y}{2!}(\delta\theta)^2-\frac{x}{3!}(\delta\theta)^3+\dots\dots\end{aligned}$$

$\therefore$ work done by the force P

$$\begin{aligned}&=X\delta x+Y\delta y\\ &=X\left[\delta a-y\delta\theta-\frac{x}{2!}(\delta\theta)^2+\dots\dots\right]\\ &\qquad +Y\left[\delta b+x\delta\theta-\frac{y}{2!}(\delta\theta)^2-\dots\dots\right]\\ &=X\delta a+Y\delta b+\delta\theta(xY-yX)-\frac{(\delta\theta)^2}{2!}(xX+yY)+\dots\dots\end{aligned}$$

and the total work done

$$=\delta a\Sigma X+\delta b\Sigma Y+\delta\theta\Sigma(xY-yX)-\frac{(\delta\theta)^2}{2!}\Sigma(xX+yY)+\dots\dots$$

We find that if the system is in equilibrium, *i.e.*, ΣX, ΣY, and $\Sigma(xY-yX)$ all vanish, the work done

$$=-\frac{(\delta\theta)^2}{2!}\Sigma(xX+yY)+\dots\dots$$

which is a small quantity of at least the second order as compared to $\delta\theta$.

Thus *for a system of forces in equilibrium, the virtual work is not exactly zero but is a small quantity of at least the second order*, conversely if the virtual work of a system of forces is a small quantity of at least the second order the forces are in equilibrium.

Example. For what sort of displacement is the work done by a system of coplanar forces in equilibrium, exactly zero ?

9·53. *Forces which may be omitted in the application of virtual work.*

The advantage of the principle of virtual work lies mainly in the fact that a certain number of forces can be avoided in writing the equations of equilibrium.

The following are some of the important cases of forces which do not appear in the equation of virtual work.

1. *The reaction of any smooth surface upon a body which slides on the surface.*

The reaction being normal and, therefore, at right angles to the direction of displacement, will do no work.

2. *The reaction of a fixed surface upon which a body rolls without sliding.*

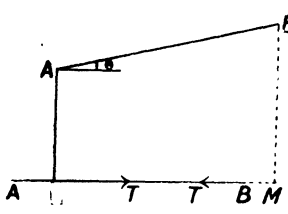
In this case, the point of contact remains instantaneously at rest, so that the displacement in the direction of the reaction is of an order higher than the first. Hence the work done by the reaction at the point of contact is negligible.

3. *The mutual reaction between any two bodies which roll upon one another.*

In this case both the bodies are taken into account and the action and reaction being equal and opposite, the virtual work is zero.

4. *The tension of an inextensible string.*

Let the string AB be displaced to the position A′B′, inclined at an angle θ to AB so that the points of application A and B of tension T are displaced to A′ and B′ respectively.



Draw A′L, B′M perpendiculars to AB.

The work done

$$= T.AL - T.BM$$
$$= T(AB - LB) - T(LM - LB)$$
$$= T.(AB - LM) = T.(AB - A'B' \cos\theta)$$
$$= T.AB(1 - \cos\theta) \quad \because \ A'B' = AB$$
$$= T.AB\left(1 - 1 + \frac{\theta^2}{2!} + \ldots\ldots\right)$$
$$= 0$$

to the first order of small quantity θ.

Note. In problems where the tension acting along an inextensible string is required, we suppose the string to be removed and replaced by the force of tension T. The virtual work of T is then calculated consistent with geometrical conditions.

9·54. *If a body or a system of bodies are under the action of no force other than the weights of the bodies of the system and the mutual actions and reactions between them, the positions of equilibrium correspond to the maxima or minima values of the height of the C.G. of the system above a fixed horizontal level.*

Let w_1, w_2, w_3,...be the weights of the bodies which are in equilibrium when their C.G's are at heights z_1, z_2, z_3,..., above a fixed horizontal level, then

$$w_1 z_1 + w_2 z_2 + w_3 z_3 + \ldots = (w_1 + w_2 + w_3 + \ldots)\bar{z} = W\bar{z},$$

where W is the total weight and $\bar{z}$ is the height of the C.G. of the system.

Then, since the work of the mutual actions and reactions does not appear in the equation of virtual work, $\therefore$ the virtual work of the system is $Wd\bar{z}$.

In an equilibrium position this vanishes,

$\therefore \quad Wdz=0 \quad$ or $\quad dz=0$

so that $\bar{z}$ is a maximum or a minimum.

It may be noted that since $W\bar{z}$ is the *potential energy* of the system, the *positions of equilibrium also correspond to the maxima or minima values of the potential energy of the system.* $[\because \delta(W\bar{z})=0]$

9·55. The application of the principle of virtual work is illustrated by the following examples :—

Example 1. *A uniform beam of length 2a, rests in equilibrium against a smooth vertical wall and upon a smooth peg at a distance b from the wall. Show that in the position of equilibrium the beam is inclined to the wall at an angle* $\sin^{-1}\left(\frac{b}{a}\right)^{\frac{1}{3}}$.

Let the rod AB $(=2a)$ be inclined at an angle θ to the wall.

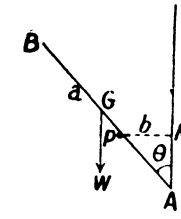
$\bar{z}$ the height of its C. G. above the *fixed* peg P is AG $\cos\theta$ − AM = AG $\cos\theta$ − PM $\cot\theta$

i.e., $\quad \bar{z}=a\cos\theta-b\cot\theta$

$\therefore \quad d\bar{z}=(-a\sin\theta+b\operatorname{cosec}^2\theta)d\theta$

For equilibrium $d\bar{z}=0$,

$\therefore \quad b\operatorname{cosec}^2\theta - a\sin\theta=0$ *i.e.*, $\sin^3\theta=\frac{b}{a}$.



The same result can be obtained otherwise by taking moments about A and resolving vertically or merely from the geometry of the figure. (See Cor. 2, Ex. 2, Art. 6 9 on page 137.)

The simplicity of the method of virtual work lies in the fact that the reactions at A and P do not appear in the equation.

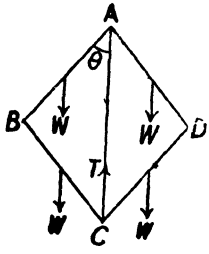
Note. It is important to note that the point A is not fixed.

Example 2. *Four equal heavy uniform rods are hinged together to form a rhombus ABCD and the frame is suspended from A. The corner A and C are connected by an inelastic string to prevent the frame from collapsing. Find the tension in the string.*

Let $2a$ be the length, and W the weight of each rod.

We consider a small displacement which alters the angles but not the length of the sides, the connections at the hinges remaining intact

A beings fixed, we measure distances from it.



Let x be the depth below A of the C. G. of either of the upper rods. Then the depth of the C. G. of either of the lower rods is $3x$ and $AC=4x$, so that the virtual work of the weigths of the two upper and two lower rods are

$$2W\delta x \quad \text{and} \quad 2W\delta(3x)=6W\delta x.$$

To find the tension in the string, we imagine the string to be removed and a force T applied at C along CA.

The virtual work of $T=-T\delta(4x)=-4T\delta x$.

The sign is taken negative, for the displacement of C is in a direction opposite to that in which T acts.

Hence the equation of virtual work is

$$2W\delta x+6W\delta x-4T\delta x=0 \ \textit{i.e.}, \ 8W\delta x=4T\delta x$$
$$\therefore \quad T=2W.$$

Aliter. Let $\angle BAC=\theta$, then depths below A of the C. G. of each of the upper rods is $a\cos\theta$, and that of the lower it is $3a\cos\theta$, and the length of AC is $4a\cos\theta$.

When θ increases, the displacement is upwards,

$\therefore$ the equation of virtual work is

$$-2W\delta(a\cos\theta)-2W\delta(3a\cos\theta)+T\delta(4a\cos\theta)=0$$

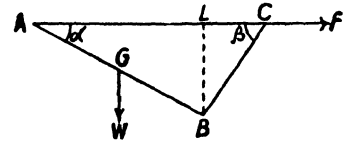
or $8W\,a\sin\theta-4a\,T\sin\theta=0$, which gives $T=2W$.

Example 3. *A rod AB is movable about a point at A and to B is attached a string whose other end is tied to a ring. The ring slides along a smooth horizontal wire passing through A ; prove that the horizontal force necessary to keep the ring at rest is*

$$\frac{W\cos\alpha\cos\beta}{2\sin(\alpha+\beta)},$$

where W is the weight of the rod and α, β *the inclinations of the rod and the string to the horizontal.*

Let a be the length of the rod and b that of the string, then $AC=AL+LC=a\cos\alpha+b\cos\beta$, and the depth of the C. G. of the rod below $A=\frac{1}{2}a\sin\alpha$.



We consider a small displacement in which a and b remain constant and α and β vary.

$\therefore$ by the principle of Virtual Work, we have

$$W\delta(\tfrac{1}{2}a\sin\alpha)+F\,\delta(a\cos\alpha+b\cos\beta)=0$$

or $$\tfrac{1}{2}W\,a\cos\alpha\,d\alpha=F.\,a\sin\alpha\,d\alpha+F.b\sin\beta\,d\beta \qquad \ldots(1)$$

Also since $$a\sin\alpha=BL=b\sin\beta,$$

$$\therefore \qquad a\cos\alpha\, dx = b\cos\beta\, d\beta \qquad \ldots(2)$$

$$\therefore \quad \tfrac{1}{2}\,W a\cos\alpha = F\left(a\sin\alpha + \frac{a\cos\alpha\sin\beta}{\cos\beta}\right) = \frac{F.a\sin(\gamma+\beta)}{\cos\beta}$$

i.e., $$F = W\cos\alpha\cos\beta/2\sin(\alpha+\beta).$$

Note. 1. The weight of the ring does not appear in the equation of virtual work as its point of application C slides along a horizontal line.

Aliter. The result can be obtained easily even without the use of the principle of virtual work as follows :—

If T be the tension in the string BC, resolving horizontally for the ring, we get, $T\cos\beta = F$. Taking moments about A for the rod, we have

$$W\tfrac{1}{2}a\cos\alpha = T.a\sin(\alpha+\beta) = F\sec\beta.a\sin(\alpha+\beta),$$ which gives F.

Example 4. *A uniform rod lies in equilibrium with its ends on two smooth planes inclined at angle* α, β *to the horizontal, (the planes intersecting in a horizontal line and the rod lying at right angles to it), show that the inclination of the rod to the horizontal is*

$$\tan^{-1}\frac{\sin(\beta-\alpha)}{2\sin\alpha\sin\beta}.$$

Let AB be the rod of length $2a$, then the height z of G, the C. G. of the rod above the horizontal level through C (See Fig. Art. 6.821, P. 133) is given by

$$z = LG + AC\sin\alpha = a\sin\theta + AC\sin\alpha.$$

But from $\triangle ACB$, we have

$$\frac{AC}{2a} = \frac{AC}{AB} = \frac{\sin\angle ABC}{\sin\angle ACB} = \frac{\sin(\beta-\theta)}{\sin(\alpha+\beta)}$$

$$\therefore \qquad z = a\sin\theta + 2a\sin(\beta-\theta)\operatorname{cosec}(\alpha+\beta)\sin\alpha.$$

$$\therefore \qquad dz = a\cos\theta\, d\theta - 2a\cos(\beta-\theta)\operatorname{cosec}(\alpha+\beta)\sin\alpha\, d\theta$$

For equilibrium $dz = 0$

$$\therefore \quad \cos\theta\sin(\alpha+\beta) - 2\cos(\beta-\theta)\sin\alpha = 0$$

or $$\sin(\alpha+\beta) = 2\sin\alpha(\cos\beta + \sin\beta\tan\theta)$$

i.e., $$2\sin\alpha\sin\beta\tan\theta = \sin(\alpha+\beta) - 2\sin\alpha\cos\beta = \sin(\beta-\alpha).$$

Hence $$\tan\theta = \sin(\beta-\alpha)/2\sin\alpha\sin\beta.$$

Note. It can also be solved without the application of virtual work by a method indicated in Cor. Art. 6·821, P. 133.

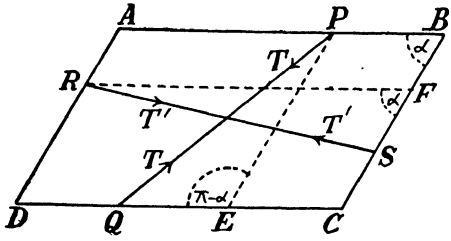
Example 5. *ABCD is a parallelogram of freely jointed light rods, a point P on AB is joined to a point Q on CD by a string and R on AD to S on BC by another string. If the tensions in the strings be T and T', show that for equilibrium*

$$\frac{T}{PQ}\cdot\frac{AP-DQ}{AB} = \frac{T'}{RS}\cdot\frac{BS-AR}{AD}.$$

Draw RF, PE parallel to AB and AD respectively, so that

PE = AD, DE = AP, RF = AB and BF = AR.

Th equation of virtual work for any displacement which alters the angles but not the lengths of the sides of the parallelogram is



$$T\,\delta(PQ)+T'\,\delta(RS)=0$$

$$\text{or } T'\,\delta(RS)=-T\,\delta(PQ) \quad ...(1)$$

Let $\angle ABC=\alpha$, then $\angle RFS=\alpha$ and $\angle PEQ=\angle BCD=\pi-\alpha$.

Now we express PQ and RS in terms of the constant lengths AB, AD, AP, AR. BS and DQ, and the variable angle α.

From $\triangle PEQ$, we have

$$\begin{aligned} PQ^2 &= PE^2+QE^2-2PE.QE\cos(\pi-\alpha) \\ &= PE^2+(DE-DQ)^2+2PE(DE-DQ)\cos\alpha \\ &= AD^2+(AP-DQ)^2+2.AD\,(AP-DQ)\cos\alpha \end{aligned}$$

$$\therefore \quad 2PQ\,\delta(PQ)=-2AD(AP-DQ)\sin\alpha.d\alpha \quad ...(2)$$

Again, from $\triangle RFS$, we get

$$\begin{aligned} RS^2 &= RF^2+FS^2-2RF.FS\cos\alpha \\ &= RF^2+(BS-BF)^2-2RF(BS-BF)\cos\alpha \\ &= AB^2+(BS-AR)^2-2AB(BS-AR)\cos\alpha \end{aligned}$$

$$\therefore \quad 2RS\,\delta(RS)=2AB(BS-AR)\sin\alpha.d\alpha \quad ...(3)$$

Substituting these results (2) and (3) in (1), we have

$$T'\frac{AB(BS-AR)\sin\alpha\,d\alpha}{RS}=T\frac{AD(AP-DQ)\sin\alpha\,d\alpha}{PQ}$$

or

$$\frac{T}{PQ}\cdot\frac{AP-DQ}{AB}=\frac{T'}{RS}\cdot\frac{BS-AR}{AD}.$$

Examples XXXVII-A

1. Four equal uniform rods are jointed to form a rhombus ABCD, which is placed in a vertical plane with AC vertical and A resting on a horizontal plane. The rhombus is kept in the position in which $\angle BAC=\theta$ by a light rod joining B and D. Show that its tension is $2W\tan\theta$, where W is the weight of a rod. (*D.U.*)

2. A regular hexagon is composed of six equal heavy rods freely jointed together, and two opposite angles are connected by a string which is horizontal, one rod being in contact with a horizontal plane; at the middle point of the opposite rod is placed a weight W_1, if W be the weight of each rod, show that the tension of the string is $(3W+W_1)/\sqrt{3}$. (*D.U.*)

3. Four equal rods each of length *a* are hung from an angular point which is connected by an elastic string with the opposite point. If the rods hang in the form of a square and if the modulus of elasticity of the string be equal to the weight of a rod, find the unstretched length of the string.

[*Ans.* $a\sqrt{2}/3$.] (*D.U.*)

4. A square frame work formed of uniform heavy rods of equal weights W joined together, is hung by one corner. A weight W is suspended from each of the three lower corners and the shape of the square is preserved by a light rod along the horizontal diagonal. Find its tension. (*Ans.* 4W.] (*All. U., D.U.*)

5. A regular hexagon ABCDEF of six equal uniform smoothly jointed rods, each of weight W, is hung from the mid-point of AB in a vertical plane with AB horizontal and uppermost. Its shape is maintained by taut strings AD, BE and it may be assumed that the strings just do not touch at their mid-points. Find the tension in each string. [*Ans.* $\sqrt{3}$W] (*D.U. 1955*)

6. Four rods are jointed together to form a parallelogram, the opposite joints are connected by strings forming the diagonals and the whole system is placed on a smooth horizontal table. Show that their tensions are proportional to their lengths.

7. A smoothly jointed framework of light rods forms a quadrilateral ABCD. The middle points P, Q of an opposite pair of rods are connected by a string in a state of tension T, and the middle points R, S of the other pair by a light rod in state of thrust X; show that T/PQ=X/RS.

8. A frame, formed of four light rods, each of length a freely jointed at A, B, C, D is suspended at A. A mass m is suspended from B and D by two strings each of length l and $l > a/\sqrt{2}$. The frame is kept in the form of a square by a string AC. Apply the method of virtual work to find the tension T in AC and show that when $l=a\sqrt{5}$, $T=\frac{2}{3}mg$.

[*Ans.* $\frac{1}{2}mg\ (1+a/\sqrt{2l^2-a^2})$.] (*I.C S.*)

9. Two equal uniform rods AB and AC, each of length $2b$, are freely jointed at A and rest, in a symmetrical position on a smooth vertical circle of radius a. Show that if 2θ be the angle between them, then

$$b \sin^3 \theta = a \cos \theta.$$

10. Apply the principle of virtual work to solve questions 1, 4, 8, and 16 of Ex. XXVIII, Page 143, 144.

11. A quadrilateral ABCD, formed of four uniform rods freely jointed to each other at their ends, the rods AB, AD being equal and also the rods BC, CD, is freely suspended from the joint A. A string joins A to C and is such that ABC is a right angle. Show that the tension of the string is

$$(W+W') \sin^2 \theta + W',$$

where W is the weight of an upper rod and W′ of a lower rod and 2θ is the angle BAD.

12. A frame consists of five bars forming the sides of a rhombus ABCD with the diagonal AC. If four equal forces P act inwards at the middle points of the sides, and at right angles to them, prove that the tension in AC is

$$P \cos 2\theta / \sin \theta,$$

where θ denotes the angle DAC.

13. A framework of n equal *weightless* rods freely jointed together is maintained in the form of a regular polygon by equal forces P applied at the middle point of each rod at right angles to it. Prove that the tension or thrust in each rod is

$$\tfrac{1}{2} P \cot \frac{\pi}{n}. \qquad (\textit{Bombay U.})$$

14. ABCD is a rhombus formed with four rods each of length l and weight w, joined by smooth hinges. A weight W is attached to the lowest hinge C and the frame rests on two smooth pegs in a horizontal line and B and D are joined by a string. If the distance of the pegs apart is $2d$, and the angle at A is 2α, show that the tension in the string is

$$\tan \alpha \left[\frac{d}{2l}\ (W+4w) \operatorname{cosec}^3 \alpha - (W+2w)\right]. \qquad (\textit{Agra U.})$$

15. Two equal rods AB, BC, each of weight W, are freely jointed at B and stand in a vertical plane with their feet A, C resting on a smooth horizontal plane. They are kept in position by a *light* rod connecting A, with the middle point of BC. By the method of virtual work show that the tension in the connecting rod is

$$\tfrac{1}{4}W\sqrt{1+9 \tan^2 \theta},$$

θ being the angle AB or AC makes with the vertical.

16. Two equal light rods AOB, COD freely jointed at O, their middle point, are at rest in a vertical plane with their ends B, C on a smooth horizontal table. A string to the ends of which equal weights are attached passes over A and D. Show that in the position of equilibrium the angle between the rods is

$$\tan^{-1}(4/3). \qquad (D.U.)$$

Examples XXXVII-B

1. An endless chain of weight w rests in the form of a circular band round a smooth vertical cone which has its vertex upwards. Find the tension in the chain due to its own weight, assuming the vertical angle of the cone to be 2α. [*Ans.* $w \cot \alpha/2\pi$]

2. A heavy elastic string, whose natural length is $2\pi a$, is placed round a smooth cone whose axis is vertical and whose semi-vertical angle is α. If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle of radius

$$a\left(1+\frac{W}{2\pi\lambda}\cot\alpha\right). \qquad (D.U.\ 1960,\ 1958)$$

3. A smooth sphere of radius r and weight W rests in a horizontal circular hole of radius a. A string is wrapped once round the sphere above the hole and pulled tight. Show that the tension in the string which will just raise the sphere is

$$\frac{W}{2\pi}\cdot\frac{a}{\sqrt{r^2-a^2}}. \qquad (Prescott)$$

4. A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ, φ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove by the application of the principle of virtual work that

$$\tan\varphi=\tfrac{3}{8}+\tan\theta. \qquad (P.U.\ Hons.)$$

5. A flat semi-circular board with its plane vertical and curved edge upwards rests on a smooth horizontal plane and is pressed at two given points of its circumference by two beams which slide in smooth vertical tubes. If the board is in equilibrium, find the ratio of the weight of the beams.

[*Ans.* $\tan\theta : \tan\varphi$ where θ, φ are the angles made with the horizontal by the radii through the points of contact with the beams.] (*P.U, Hons.*)

6. A smooth circular cylinder of radius b is fixed parallel to a smooth wall with its axis horizontal and at a distance c from the wall. A smooth uniform heavy rod of length $2a$ rests on the cylinder with one end against the wall and in a plane perpendicular to the wall, show that its inclination θ to the horizontal is given by

$$a\cos^3\theta+b\sin\theta=c.$$

7. Two small smooth rings of equal weight slide on a fixed elliptic wire, whose major axis is vertical, and they are connected by a string which passes over a small smooth peg at the upper focus. Show that the rings will be in equilibrium wherever they are placed. (*Agra U.*)

8. A freely jointed framework is formed of five equal uniform rods each of weight W. The framework is suspended from one corner which is also joined to the middle point of the opposite side by an inextensible string, if the two upper and the two lower rods make angles θ and φ respectively, with the vertical, prove that the tension of the string is to the weight of the rod as

$$(4\tan\theta+2\tan\varphi) : (\tan\theta+\tan\varphi). \qquad (D.U.\ 1924)$$

9. A string of length a forms the shorter diagonal of a rhombus formed of four uniform rods each of length b and weight W, which are hinged together. If

one of the rods be supported in a horizontal position prove that the tension of the string is

$$\frac{2W}{b} \cdot \frac{2b^2-a^2}{\sqrt{(4b^2-a^2)}}. \qquad (D.U.\ 1933)$$

10. Three equal uniform rods AB, BC, CD each of weight w are freely jointed together at B and C and rest in a vertical plane, A and D being in contact with a smooth horizontal table. Two equal light strings AC and BD help to support the framework so that AB and CD are each inclined at an angle α to the horizontal. Show that if a mass of weight W be placed on BC at its middle point, then the tension of each string will be of magnitude

$$(w+\tfrac{1}{2}W) \cos\alpha \operatorname{cosec} \tfrac{1}{2}\alpha.$$

11. Three rigid rods AB, BC, CD each of length $2a$ and weight W, are smoothly jointed at B and C. The system is placed in a vertical plane so that the rods AB, CD are in contact with two smooth pegs distant $2c$ apart in the same horizontal line, the rods AB, CD make equal angles α with the horizon. Prove that the tension of the string AD which will maintain this configuration is

$$\tfrac{1}{4}W \operatorname{cosec}\alpha \sec^2\alpha \left[\frac{3c}{a} - (3+2\cos^3\alpha)\right] \qquad (D.U.\ 1926)$$

12. AB, BC are two beams, of weight W_1 and W_2 respectively, freely jointed at B and standing in a vertical plane with their ends A and C on a smooth horizontal plane, equilibrium being maintained by a light rope joining their mid-points. A load W_3 is placed at B, and the centres of mass of the beams are at D and E, respectively, where $AD : DB = BE : EC = a : b$. If θ is the angle BAC and φ the angle BCA, show that the tension in the rope is

$$2[aW_1+bW_2+(a+b)W_3]/(a+b)\ (\tan\theta+\tan\varphi).$$

13. Two equal rods each of weight wl and length l are hinged together and placed astride a smooth horizontal cylindrical peg of radius r. Their lower ends are tied together by a string and the rods are kept at the same inclination φ to the horizontal. Find the tension in the string, and if the string is slack, show that φ satisfies the equation

$$\tan^3\varphi+\tan\varphi = l/2r.$$

14. Two smooth wires OA and OB are fixed in a vertical plane and are inclined at equal angles α to the vertical. Two equal heavy uniform rods HC and KC are smoothly hinged at C and the ends H and K are constrained to slide along OA and OB respectively by means of small smooth weightless rings ; the weight of each rod is W, and an additional weight W is attached at C. Show that in the symmetrical position of equilibrium each rod is inclined at an angle θ to the vertical given by

$$\tan\theta = \tfrac{3}{2}\cot\alpha. \qquad (I.S.E.)$$

15. A roof truss of two equal beams AB, AC tied together by light rods joining B, C to E, D, points in AC, AB respectively, such that $AD = AE = AB/n$. The weight of the roof and the truss together may be assumed to be uniformly distributed along AB, AC and to be of amount w per unit length. Prove that the stress in CD or BE is

$$\tfrac{1}{8}\, n\, w \sec\alpha.\ CD,$$

where α is the angle of inclination of AB or AC to the vertical. It is supposed that there is no horizontal reaction between the wall and the truss.

16. A heavy rod AB, of length $2l$, rests upon a fixed smooth peg at C and with its end B upon a smooth curve. If it rests in all positions, show that the polar equation of the curve with C as origin, is

$$r = l+a \operatorname{cosec}\theta.$$

17. One end of a beam rests against a smooth vertical wall and the other on a smooth curve in a vertical plane perpendicular to the wall ; if the beam rests in all positions, show that the curve is an ellipse whose major axis lies along the horizontal line described by the C.G. of the beam. (*Lucknow U.*)

***9.6. Principle of Energy.** *When a particle undergoes displacement under the action of a conservative system of co-planar forces, to prove that*

(i) the change in the kinetic energy is equal to the work done by the forces.

(ii) the sum of its kinetic and potential energies remains constant.

At any point P(x, y) occupied by the particle at any instant, let ds be the displacement and F be the sum of the components of the forces in the direction of ds, then the small work done by the forces is $F.ds$.

If X, Y be the sums of the components of the forces in the direction of the coordinate axes and dx, dy the displacements in these directions, then by Art. 9.4 the work done is also equal to $Xdx+Ydy$.

$\therefore$ the total work done in a finite displacement when the particle moves from a point A(x_1, y_1) to another point B(x_2, y_2) is

$$\int_A^B F\,ds = \int_A^B (Xdx+Ydy) \qquad \ldots(1)$$

If v be the velocity of the particle in the position P and m be its mass, then $F = mv\dfrac{dv}{ds}$.

$$\therefore \quad \int_A^B F\,ds = \int_A^B m\,v\frac{dv}{ds}\,ds = \Big[\tfrac{1}{2}mv^2\Big]_{u_1}^{u_2} = \tfrac{1}{2}mu_2^2 - \tfrac{1}{2}mu_1^2$$

$$= K_B - K_A \qquad \ldots(2)$$

where u_1, u_2 are the velocities of the particle at A and B respectively.

(The expression $\frac{1}{2}mv^2$ is defined to be the **Kinetic Energy** of a body of mass m, moving with a velocity v).

Thus the left hand side of equation (1) *i.e., the work done by the forces, is equivalent to the change in the kinetic energies* in the two positions. We denote it by $K_B - K_A$ where K stands for the kinetic energy.

Now imagine that there is some function of x and y denoted by V such that, $X = -\dfrac{\partial V}{\partial x}$ and $Y = -\dfrac{\partial V}{\partial y}$.

$$\therefore \quad \int_A^B (Xdx+Ydy) = -\int_A^B \left(\frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy\right)$$

$$= -\int_A^B dV = V_A - V_B, \qquad \ldots(3)$$

So that the nature of the forces is such that the work done by them depends only on the initial and final positions A and B and does not depend on the path followed by the particle. Such forces are known as *Conservative Forces and V is known as the* **Potential Function.**

Thus the right hand side of equation (1) *i.e., the work done by the forces is equivalent to the change in the Potential Energy of the particle.*

From (2) and (3), we have

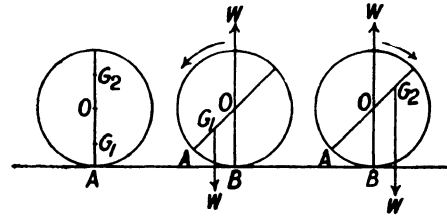
$$K_B - K_A = V_A - V_B \text{ or } K_B + V_B = K_A + V_A$$

i.e., the sum of the kinetic and potential energies remains constant.

This theorem can be easily extended for a system of particles.

9.7. Stability of Equilibrium. Let us consider the case of a sphere resting on a horizontal table. If it is not uniform its C.G. will be at a point different from O, the centre of the sphere.

Suppose it is placed such that its C.G. is vertically above or below O. The weight being balanced by the equal and opposite reaction at the point of contact A, the sphere is in equilibrium. Suppose now it is slightly displaced from this position, two cases arise according as its C.G. is above or below O.



(*i*) When the C.G. is below O, say at G_1, the weight W acting at G_1, and the equal and opposite reaction at B form a couple which tends to turn the sphere in the direction indicated by the arrow head, so that it tends to bring the sphere back to its original position of equilibrium. In this case the equilibrium of the sphere is said to be **stable.**

(*ii*) When the C.G. is above O, say at G_2, the weight W acting at G_2 and the equal and opposite reaction at B form a couple which tends to displace the sphere away from its original position of equilibrium. In this case the equilibrium is said to be **Unstable.**

Now suppose the sphere to be uniform, so that its C.G. is at the centre O. The reaction and the weight both pass through the centre and act along the same vertical line, so that even if the sphere is displaced the condition of equilibrium still holds and the sphere continues to remain in equilibrium. In such cases the equilibrium is said to be **Neutral.**

9.71. Conditions of Stability of Equilibrium. We suppose that the position of a rigid body or a system of connected bodies depends upon some variable θ, then V the potential energy of the system is a function of θ.

Now suppose that for a certain value of θ, the system is in equilibrium under the action of a system of co-planar forces. Let θ change to $\theta+\delta\theta$, so that the system undergoes a slight displacement permissible by its geometrical conditions.

Let δV be the change in the P.E. which is also the magnitude of the amount of work done in displacing the system. By the principle of virtual work, we have $\delta V=0$ *i.e.*, V is a maximum or minimum for equilibrium.

When released from its displaced position, the system, being not in equilibrium, must move and acquire kinetic energy.

By Art. 9.6 the sum of the kinetic and potential energies must remain constant, so that this kinetic energy is acquired at the expense of the potential energy, and hence the potential energy of the system will decrease.

Now we consider two cases :—

(1) Let the P.E. of the system in its original position of equilibrium be a minimum.

In this case as the system moves from its displaced position, the P.E. decreases, and tends to attain its original minimum value so that the system returns to its former position and, therefore, the original position of equilibrium was *stable*.

(2) Let the P.E. of the system in its original position be a maximum. In this case as the system moves from its displaced position, the P.E. decreases, so that the system tends to move away from its original position of maximum P.E. and, therefore, the equilibrium was *unstable*.

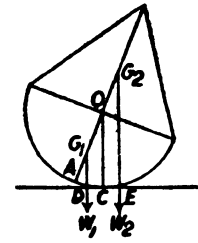
Thus we have proved that *the positions of equilibrium of a system are the positions of its maximum or minimum potential energy, the equilibrium being stable or unstable according as the potential energy is a minimum or a maximum.*

9·72. In Art. 9.54, we proved that in problems where we are concerned mainly with the weights of the system, the system is in equilibrium when $\bar{z}$ the height of its C.G. above a fixed horizontal level is a maximum or a minimum *i.e.*, $\frac{d\bar{z}}{d\theta}=0$. We now learn that the equilibrium will be stable or unstable according as $\bar{z}$ is a minimum or a maximum, *i.e.*, $\frac{d^2\bar{z}}{d\theta^2}$ is positive or negative.

In certain simple cases such as given in Art. 9.7 the stability of equilibrium can be easily determined by taking moments about the point of contact.

9·73. Example 1. *A homogeneous solid is built up of a hemisphere of radius a and a right circular cone of radius a and height h on the same circular base as the hemisphere. The solid is placed upright on a horizontal plane. When is the equilibrium stable ?*

Let G_1, G_2 be the centres of gravity of the hemisphere and the cone, C the point of contact of the body in the displaced position.



The surface of contact being spherical, the reaction must pass through O, the centre of the sphere.

Now $OG_1=\frac{3}{8}a$, $OG_2=\frac{1}{4}h$.

The equilibrium will evidently be stable if

$$W_1.DC>W_2.CE.$$

The solid being uniform W_1, W_2 are proportional to the volumes of the hemisphere and the cone. Also DC and CE are proportional to OG_1 and OG_2.

$\therefore$ for stability

$$\tfrac{2}{3}\pi a^3\times\tfrac{3}{8}a>\tfrac{1}{3}\pi a^2h\times\tfrac{1}{4}h$$

or $$a^2>\tfrac{1}{3}h^2 \qquad i.e.,\ h<\sqrt{3}\,a.$$

Example 2. *In Ex.* (1) *Art. 9·55, P. 199, prove that the equilibrium is unstable.*

It has been shown that

$$\bar{z}=a\cos\theta-b\cot\theta, \qquad \text{(See fig. on P. 199.)}$$

$$\frac{d\bar{z}}{d\theta}=-a\sin\theta+b\operatorname{cosec}^2\theta$$

and $\sin^3\theta=\dfrac{b}{a}$ for equilibrium, obtained by putting $d\bar{z}/d\theta=0$.

Differentiating again, we have

$$\frac{d^2\bar{z}}{d\theta^2}=-a\cos\theta-2b\operatorname{cosec}^2\theta\cot\theta$$

$$=-\cos\theta\,(a+2b\operatorname{cosec}^3\theta)=-\cos\theta\left(a+2b\times\frac{a}{b}\right)$$

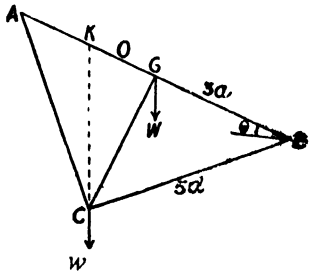
$$=-3a\cos\theta,$$ which is negative, θ being acute.

Thus $\bar{z}$ is a maximum, and equilibrium is unstable.

Example 3. *AB is a uniform rod of length 6a and weight W, which can turn freely about a fixed point in its length distant 2a from A. AC and BC are light strings each of length 5a, attached to a particle C of wieght w. Show that if W is less than 2w there will be stable equilibrium with AB inclined to the horizontal at an angle*

$$tun^{-1}\frac{W+w}{4w}.$$

Let AB be the rod, O the point about which it turns, G its C.G. and θ its inclination to the horizon.



Since $BG=AG=3a$, $CB=CA=5a$,

$\therefore$ $\angle CGB=90°$ and $CG=4a$.

Also since $AO=2a$, and $AG=3a$,

$\therefore$ $OG=a$.

The depth of G below the horizontal level through the fixed point O,

$$=OG\sin\theta=a\sin\theta\ ;$$

and the depth of C below O

$$=OG\sin\theta+CG\cos\theta$$

$$=a\sin\theta+4a\cos\theta.$$

$\therefore$ V, the potential energy of the system
$= -Wa \sin\theta - w(a \sin\theta + 4a\cos\theta)$

$\therefore \quad \frac{dV}{d\theta} = -Wa\cos\theta - wa\cos\theta + 4aw\sin\theta$

Equating this to zero, we obtain $\tan\theta = \frac{W+w}{4w}$,

which gives the position of equilibrium.

Also $\quad \frac{d^2V}{d\theta^2} = (W+w)a\sin\theta + 4aw\cos\theta$

which is positive, θ being acute, so that V is minimum and equilibrium is stable.

Now let the vertical through C meet AO in K, for equilibrium K must lie between O and A *i.e.*, OK $<$ OA or OK $<2a$.

Taking moments about O, we have $w.OK = W.OG = aW$.

$\therefore \quad OK = \frac{aW}{w}$.

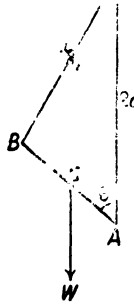
But since OK $<$ OA *i.e.*, $<2a$, $\quad \therefore \quad aW < 2aw \quad$ or $\quad W < 2w$.

Example 4. *A uniform rod AB of weight W and length a is smoothly hinged at the end A, while B is connected to a fixed point C, which is at a distance 2a vertically above A, by a light elastic string of modulus λ and natural length l. Prove that there are three positions of equilibrium.*

Also prove that (i) if $4\lambda(a-l) > lW$, one of the vertical positions of equilibrium is stable and the other unstable.

(ii) If $\lambda = \frac{1}{5}W$, and $l = \frac{1}{2}a$, the rod is in stable equilibrium inclined to the vertical at an angle $\cos^{-1}(29/36)$.

Let x be the stretched length of the string and θ, the angle which the rod AB makes with the vertical in any position.



The height of the C.G. of the rod above A $= \frac{1}{2}a\cos\theta$.

The Pot. energy of the elastic string
= the work done in stretching it to a length x
$= \frac{1}{2}\frac{\lambda}{l}(x-l)^2$ (Art. 9·12)

$\therefore$ V, the Pot. energy of the system
$= \frac{1}{2}a\,W\cos\theta + \frac{1}{2}\lambda(x-l)^2/l$.

$\therefore \quad \frac{dV}{d\theta} = -\frac{1}{2}a\,W\sin\theta + \frac{\lambda}{l}(x-l)\frac{dx}{d\theta}$.

But from the figure, $x^2 = 4a^2 + a^2 - 4a^2\cos\theta$ ...(1)

$\therefore \; x\frac{dx}{d\theta} = 2a^2\sin\theta$ ...(2)

Hence $\quad \frac{dV}{d\theta} = -\frac{1}{2}Wa\sin\theta + \frac{\lambda}{l}\left(\right).\, 2a^2\sin\theta - \lambda\,\frac{2a^2\sin\theta}{x}$

$$= \frac{a}{2l}\left[4a\lambda - lW - \frac{4al\lambda}{x}\right]\sin\theta \qquad \ldots(3)$$

and $$\frac{d^2V}{d\theta^2} = \frac{a}{2l}\left[4a\lambda - lW - \frac{4al\lambda}{x}\right]\cos\theta + \frac{a}{2l}\cdot\frac{4al\lambda}{x^2}\sin\theta.\frac{2a^2\sin\theta}{x}$$

[using (2)] ...(4)

$dV/d\theta = 0$, when $\sin\theta = 0$, *i.e.*, $\theta = 0$ or π

and when $x = 4al\lambda/(4a\lambda - lW)$ ...(5)

so that there are three positions of equilibrium.

(*i*) Now when $\theta = 0$, from (1) $x = a$, and from (4)

$$\frac{d^2V}{d\theta^2} = \frac{a}{2l}\left[4a\lambda - lW - \frac{4al\lambda}{a}\right] = \frac{a}{2l}[4\lambda(a-l) - lW],$$

this is +ve and the equilibrium is stable if $4\lambda(a-l) > lW$. ...(6)

When $\theta = \pi$, from (1) $x = 3a$, and from (4)

$$\frac{d^2V}{d\theta^2} = \frac{a}{2l}\left[-4a\lambda + lW + \frac{4al\lambda}{3a}\right]$$

$$= \frac{a}{2l}[lW - 4\lambda(a-l) - \tfrac{8}{3}l\lambda],$$

which is −ve by (6), so that the equilibrium is unstable when $\theta = \pi$.

(*ii*) When $l = \frac{1}{2}a$, and $\lambda = \frac{1}{5}W$, then after simplification $x = \frac{4}{3}a$ from (5), and then from (1) $4a^2\cos\theta = 5a^3 - x^2 = 5a^2 - \frac{16}{9}a^2 = \frac{29}{9}a^2$, so that $\cos\theta = \frac{29}{36}$.

This gives the third position of equilibrium, in this particular case. This position is stable, as $d^2V/d\theta^2$ is +ve.

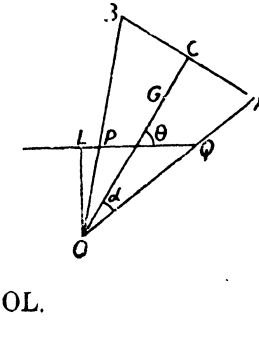
For in (4), the expression within brackets is zero on account of (5) and the rest is +ve for +ve values of x.

Example 5. *An isosceles triangular lamina of angle 2α and height h rests between two smooth pegs at the same level, distant $2c$ apart; prove that if*

$$3c\sec\alpha < h < 6c\ cosec\ 2\alpha,$$

then oblique positions of equilibrium exist, which are unstable. Discuss the stability of the vertical position.

Let OAB be the lamina G its centre of gravity and θ the angle which OG makes with the horizontal line PQ.



Draw OL ($=p$) perpendicular from O on QP, then

$\angle OQL = \theta - \alpha$, and $\angle OPL = \theta + \alpha$

and $2c = PQ - LQ - LP$

$= p\cot(\theta-\alpha) - p\cot(\theta+\alpha)$

$= p\sin 2\alpha/\sin(\theta+\alpha)\sin(\theta-\alpha)$

$= 2p\sin 2\alpha$

$\cos 2\alpha - \cos 2\theta$

i.e., $p = OL = c(\cos 2\alpha - \cos 2\theta)\ cosec\ 2\alpha.$

Now z = height of G above PQ = OG $\sin\theta$ − OL.

$$= \frac{2}{3}h \sin\theta - c(\cos 2\alpha - \cos 2\theta)\operatorname{cosec} 2\alpha \qquad \ldots(1)$$

$$\therefore \quad \frac{d\bar{z}}{d\theta} = \frac{2}{3}h\cos\theta - 2c\sin 2\theta \operatorname{cosec} 2\alpha$$
$$= 2\cos\theta\,(\tfrac{1}{3}h - 2c\operatorname{cosec} 2\alpha \sin\theta).$$

For equilibrium, this $=0$ *i.e.*, either $\theta = \frac{\pi}{2}$ or $\sin\theta = \frac{h \sin 2\alpha}{6c}$.

$$\frac{d^2\bar{z}}{d\theta^2} = -2\sin\theta(\tfrac{1}{3}h - 2c\operatorname{cosec} 2\alpha \sin\theta) - 4c\cos^2\theta \operatorname{cosec} 2\alpha$$
$$= 2\sin\theta\,(2c\operatorname{cosec} 2\alpha \sin\theta - \tfrac{1}{3}h) - 4c\cos^2\theta \operatorname{cosec} 2\alpha \qquad \ldots(2)$$

When $\theta = \frac{\pi}{2}$, $\frac{d^2\bar{z}}{d\theta^2}$ is positive if $h < 6c \operatorname{cosec} 2\alpha$, so that the vertical position of equilibrium is stable if

$$h < 6c \operatorname{cosec} 2\alpha.$$

For the oblique position of equilibrium, since

$$\sin\theta = \frac{h \sin 2\alpha}{6c} = \frac{h}{6c \operatorname{cosec} 2\alpha}, \qquad \ldots(3)$$

the first term in (2) vanishes and the second term is evidently negative, so that this position of equilibrium is unstable.

Now since $\sin\theta < 1$, $\quad \therefore\ h < 6c \operatorname{cosec} 2\alpha$.

Also from the figure, since $\theta > \alpha$, $\sin\theta > \sin\alpha$

$\therefore$ from (3), $h \sin 2\alpha > 6c \sin\alpha$ *i.e.*, $h > 3c \sec\alpha$.

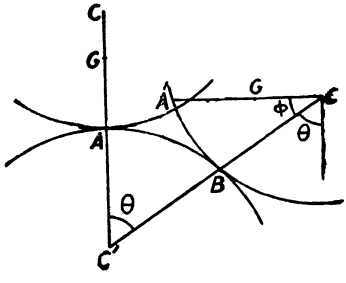
Note.—When $h = 6c \operatorname{cosec} 2\alpha$, the vertical position $\theta = \frac{1}{2}\pi$, is unstable, for,

$$\frac{d^2\bar{z}}{d\theta^2} = 0, \quad \frac{d^3\bar{z}}{d\theta^3} = 0 \text{ and } \frac{d^4\bar{z}}{d\theta^4} \text{ is } -\text{ve}.$$

9·8. *A rocking stone rests on a fixed stone, the contact being rough, and the common normal at the point of contact being vertical. If* ρ, ρ' *be the radii of curvature of the surfaces of the two stones at the point of contact, and if* h *be the height of the C.G. of the movable stone, show that the equilibrium of the rocking stone is stable if*

$$\frac{1}{h} > \frac{1}{\rho} + \frac{1}{\rho'}.$$

Let C, C′ be the centres of curvature of the two stones at the point A ; G, the C.G. of the movable stone, where $AG = h$.



In the position of equilibrium, CC′, the common normal is vertical, A is the point of contact and G lies on AC.

Let the upper stone be slightly displaced by rolling (without sliding), so that A now occupies the position A′ and B is the point of contact.

In this position, let the common

normal C'BC and A'C make angles θ and $\theta+\phi$ respectively with the vertical.

Now since the arc A'B=AB, C'B=ρ', BC=ρ,

$$\therefore \qquad \rho'\theta=\rho\phi \qquad \ldots(1)$$

If z be the height of G above C', we have

$$z=C'C\cos\theta-CG\cos(\theta+\phi)$$
$$=(\rho+\rho')\cos\theta-(\rho-h)\cos(\theta+\phi), \quad \because CG=AC-AG=\rho-h$$
$$=(\rho+\rho')\cos\theta-(\rho-h)\cos\left(1+\frac{\rho'}{\rho}\right)\theta, \text{ from (1).}$$

$$\therefore \quad \frac{dz}{d\theta}=-(\rho+\rho')\sin\theta+(\rho-h)\left(1+\frac{\rho'}{\rho}\right)\sin\left(1+\frac{\rho'}{\rho}\right)\theta.$$

$\therefore$ $\theta=0$, is an evident position of equilibrium.

$$\frac{d^2z}{d\theta^2}=-(\rho+\rho')\cos\theta+(\rho-h)\left(1+\frac{\rho'}{\rho}\right)^2\cos\left(1+\frac{\rho'}{\rho}\right)\theta.$$
$$=-(\rho+\rho')+(\rho-h)(\rho+\rho')^2\frac{1}{\rho^2}, \text{ when } \theta=0.$$

For stability of equilibrium at $\theta=0$, $\frac{d^2z}{d\theta^2}$ should be positive.

$$\therefore \qquad (\rho+\rho')^2\frac{1}{\rho^2}(\rho-h)>(\rho+\rho')$$

i.e.,
$$(\rho+\rho')(\rho-h)>\rho^2 \quad \text{or} \quad \rho\rho'>h(\rho+\rho')$$

or
$$\frac{1}{h}>\frac{1}{\rho}+\frac{1}{\rho'}. \qquad \ldots(2)$$

If $\frac{1}{h}=\frac{1}{\rho}+\frac{1}{\rho'}$, neutral equilibrium may exist. (Art. 9·7).

Cor. (*i*) *When the upper stone rests on a horizontal plane* $\rho'\to\infty$ *and the condition of stability becomes* $1/h>1/\rho$ *or* $h<\rho$.

Further if the surface of the rocking stone in contact with the horizontal plane is spherical of radius r, $h<r$. (*See* Art. 9·7. P. 207).

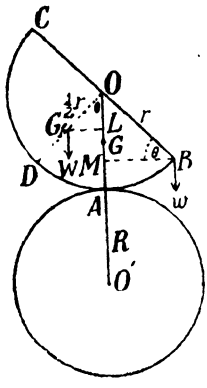
(*ii*) *If the upper stone has a plane surface in contact with a fixed sphere of radius* r, $\rho\to\infty$ *and* $\rho'=r$.

Hence the equilibrium is stable if $h<r$.

Example 1. *A heavy hemispherical shell of mass 2m and radius r has a particle of mass m attached to a point on its rim and rests with its curved surface in contact with a perfectly rough sphere of radius 2r at its highest point. Show that the equilibrium is stable.*

Let W be the weight of the shell and w that of the particle

attached to a point on its rim, and let O be the centre of the shell and O′ that of the fixed sphere.



Let the base of the shell be inclined at an angle θ with the horizontal.

G_1, the C.G. of the shell is the middle point of the central radius OD.

G, the C.G. of the shell and the particle must lie on the vertical common normal OAO′

$\therefore$ W. $G_1L = w.BM$, or $W.\frac{1}{2}r \sin\theta = w.r\cos\theta$ ...(1)

Also, $(W+w)OG = W.\frac{1}{2}r\cos\theta + w.r\sin\theta$ ...(2).

Now by the question, $W = 2m$, $w = m$,

$\therefore$ (1) gives $\sin\theta = \cos\theta$ or $\tan\theta = 1$,

i.e., $\theta = 45°$, and then (2) gives

$$3m.OG = mr\cos 45° + mr\sin 45°,$$

so that $OG = \sqrt{2}r/3$. $\therefore$ $h = AG = r - \sqrt{2}r/3 = (3-\sqrt{2})r/3$.

By (2) Art. 9·8, equilibrium is stable

if $\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$, *i.e.*, if $\frac{1}{AG} > \frac{1}{r} + \frac{1}{2r}$ ($\because$ O′A = R = 2r given).

i.e., if $\frac{3}{(3-\sqrt{2})r} > \frac{3}{2r}$, *i.e.*, if $2 > 3-\sqrt{2}$, *i.e.*, if $\sqrt{2} > 1$, which is true.

Hence the equilibrium is stable.

Example 2. *A heavy hemispherical shell of radius r has a particle attached to a point on the rim, and rests with the curved surface in contact with a rough sphere of radius R at the highest point. Prove that if*

$$\frac{R}{r} > 5^{1/2} - 1,$$

the equilibrium is stable, whatever be the weight of the particle.

(*D.U. 1960*)

In this case equation (1), of Ex. 1 above, gives,

$$\tan\theta = \frac{2w}{W}, \text{ so that } \sin\theta = \frac{2w}{\sqrt{4w^2+W^2}} \text{ and } \cos\theta = \frac{W}{\sqrt{4w^2+W^2}}.$$

$\therefore$ (2) gives, $(W+w)OG = \frac{1}{2}r\,(W^2 + 2w.2w)/\sqrt{4w^2+W^2} = \frac{1}{2}r\sqrt{4w^2+W^2}$.

$\therefore$ $h = AG = OA - OG = r - \frac{1}{2}r\sqrt{4w^2+W^2}/(W+w) = r(1-y)$,

where $y = \frac{\sqrt{W^2+4w^2}}{2(W+w)} = \frac{\sqrt{W^2/w^2+4}}{2(W/w+1)} = \frac{\sqrt{x^2+4}}{2(x+1)}$, taking $\frac{W}{w} = x$.

Now for stability of equilibrium, we have

$$\frac{1}{h} > \frac{1}{r} + \frac{1}{R} \text{ or } \frac{1}{r(1-y)} > \frac{1}{r} + \frac{1}{R} \text{ or } \frac{R}{r} > \frac{R}{r}(1-y) + 1 - y,$$

i.e., $\frac{R}{r}y+y>1$, or $\frac{R}{r}+1>\frac{1}{y}$, *i.e.*, $>\frac{2(x+1)}{\sqrt{x^2+4}}$.

This will certainly be so if $\frac{R}{r}+1>$ the max. value of $\frac{2(x+1)}{\sqrt{x^2+4}}$.

Now $\frac{d}{dx}\left[\frac{2(x+1)}{\sqrt{x^2+4}}\right]=\frac{2(x^2+4)-2x(x+1)}{(x^2+4)^{3/2}}=\frac{2(4-x)}{(x^2+4)^{3/2}}$.

This is zero when $x=4$, and at this point the derivative changes sign from +ve to −ve, showing that $x=4$ gives the max. value which is $=10/\sqrt{20}=\sqrt{5}$.

Hence equilibrium is stable if $\frac{R}{r}+1>\sqrt{5}$ or $\frac{R}{r}>\sqrt{5}-1$.

Examples XXXVIII

1. A uniform solid consists of a cone and a hemisphere of equal base fastened together so that the bases coincide. Show that if the vertical angle of the cone is $> 60°$, the solid will always move to an upright position if placed with the surface of the hemisphere on a horizontal plane.

2. A solid of uniform density is built up of a hemisphere of radius r, and a right circular cylinder of radius r and height h on the same circular base as the hemisphere. Show that the solid will rest in any position with the surface of the hemisphere in contact with a horizontal plane, provided $r>h\sqrt{2}$.

3. A hemispherical piece of wood, of radius a, lying on a horizontal plane is in a neutral equilibrium when a metallic disc of the same radius and thickness h is placed just over the plane portion. What is the ratio of the densities of the substances of the sphere and the disc. [*Ans.* $2h^2/a^2$.]

4. A solid of uniform density in the form of a portion of a paraboloid of revolution bounded by a plane, perpendicular to the axis is placed with its vertex resting on a horizontal plane and with its axis vertical. Determine the height of the solid in order that equilibrium should be stable.

[*Ans.* Less than $\frac{1}{4}$th latus rectum of the generating parabola.]

5. A solid frustum of a paraboloid of revolution, of height h and latus rectum $4a$, rests with its vertex on the vertex of a paraboloid of revolution whose latus rectum is $4b$; show that the equilibrium is stable if

$$h<\frac{3ab}{a+b}.$$ (*D.U. 1939, '57, '59*)

6. Show that a rough uniform plank of thickness b, resting horizontally on the top of a circular cylinder of radius a will be in stable equilibrium if $b<2a$.

7. A homogeneous hemispherical shell of radius a and weight W has a weight 2W/3 fastened to a point in the rim of the hemisphere. Prove that if properly placed it can rest in neutral equilibrium on the top of a fixed sphere of radius a.

8. Two equal uniform rods are firmly joined at one end so that the angle between them is α and they rest in a vertical plane on a smooth sphere of radius r. Show that they are in stable or unstable equilibrium according as the length of either rod is greater or less than $4r$. cosec α.

9. A uniform rod of weight W and length $2a$, can turn freely about one end A, which is fixed. A light inextensible string attached to the other end passes through a smooth ring fixed at a point C, distant $2a$ from A and at

the same level as A and carries at its other end a weight $\frac{1}{12}$W. When the rod is inclined at an angle θ to the horizontal and below C, show that the system is in a stable equilibrium when $8\cos\theta=1$. (*L.U.*)

10. A uniform rod AB, of length $2a$ and weight W, is hinged at A, a string attached to the middle point of the rod passes over a smooth pulley at C, at a height a vertically above A, and supports a weight P hanging freely, show that if $P<2W$, the vertical position of equilibrium is stable and that there is an oblique position of equilibrium which is unstable.

11. A rod AB of length $2a$ and weight W is pivoted to a wall at A and its end B is connected by an elastic string of natural length l and modulus λ to a point C at a height $2a$ vertically above A. Prove that if $l(W+\lambda)<4a\lambda$, there are two positions of equilibrium of which one in which $\angle BAC=\pi$ is unstable, and the other is stable; but if this condition is not satisfied, equilibrium is possible only when $\angle BAC=\pi$ and it is stable. (*I.S.E.*, ; *D.U.*)

[Hint. See Ex. 4. P. 210.]

12. Three equal and uniform bars AB, BC, CD, each of length $2a$, are smoothly jointed at B, C and rest with BC horizontal and EC, CD, each on small smooth pegs at the same level at a distance $2(a+b)$ apart.

Show that, if $2a>3b$, there are two positions of equilibrium and determine which of them is stable.

If $2a=2b$, show that there is only one position of equilibrium and that it is unstable.

13. An isosceles triangular lamina rests with its plane vertical and vertex downwards between two smooth pegs in the same horizontal line; show that there will be equilibrium if the base makes an angle $\sin^{-1}(\cos^2\alpha)$ with the vertical, 2α being the vertical angle of the lamina and length of the base being three times the distance between the pegs. Is the equilibrium stable or unstable?

[Hint. See Ex. 5, P. 211.] [*Ans.* Unstable.]

14. A uniform plate in the form of an equilateral triangle is placed in a vertical plane with two of its sides in contact with two smooth pegs on the same horizontal level. The side of the plate is a and the distance between the pegs is c. Prove that the position of equilibrium with one side horizontal is stable if $a<4c$. (*Pub. Ser. Com.*)

15. A conical plug of height h and semi vertical angle α is at rest in a circular hole of radius a. Show that the vertical position of equilibrium is one of stability or instability according as $16a\gtrless 3h\sin 2\alpha$.

16. A uniform square board of mass M is supported in a vertical plane on two smooth pegs at the same horizontal level. The distance between the pegs is a and the diagonal of the square is $D>4a$. If one diagonal is vertical and a mass m is attached to its lower end, prove that the equilibrium is stable, if

$$4am>M(D-4a).$$ (*Pub. Ser. Com.*)

17. A small ring P of mass m, slides on a smooth circular wire of radius a in a vertical plane. A light string, attached to P passes over a small smooth peg at O, the highest point of the wire, and a particle Q of mass $3m$ is suspended from the other end of the string. Prove that when P is vertically below O the system is in unstable equilibrium. (*Bm.*)

18. A bead B of mass m slides upon a smooth circular hoop with centre O, fixed in a vertical plane and is attached to a particle of mass $2m$ by a light inextensible string which passes through a smooth ring fixed mid-way between the centre of the hoop and its highest point, the heavier particle hanging vertically below the ring. Find the positions of equilibrium and determine which are stable and which are unstable.

[*Ans.* Inclination of OB to vertical 0, π, both stable; $\cos^{-1}(-\frac{1}{4})$, unstable.]

19. *A* bead of weight w is threaded on a smooth circular wire of radius a which is fixed in a vertical plane. The bead is attached by a light elastic string of unstretched length $a/2$ and modulus λ to a fixed point vertically above, and at a distance $2a$ from the centre of the wire. Given that $\frac{3}{10}w<\lambda<\frac{1}{2}w$,

show that the bead can rest in equilibrium in four different positions and investigate the stability of the equilibrium when the bead is at the highest and lowest points of the wire. [*Ans.* Unstable,] (*H.C.*)

20. Three equal particles repelling each other according to the nth power of the distance, are connected together by three equal elastic strings. Find the position of equilibrium and show that it is stable if $n < \frac{p}{p-a}$ where a is the unstretched and p the stretched length of the string.

21. Show that the work done by a system of co-planar forces is equal to the increase in the potential energy of the system.

An elastic string of mass m and modulus E rests unstretched in the form of a circle of radius a. It is now acted on by a repulsive force situated in its centre whose magnitude is $\mu(\text{distance})^{-2}$. Prove that the radius of the circle when it next comes to rest is a root of the equation

$$r^2 - ar = \frac{m\mu}{\pi E}.$$

(*D. U. Hons. 1926*)

22. A uniform rod passes through a ring at the focus of a fixed parabola, whose axis is vertical and vertex below the focus, and rests with one end on the parabola. Prove that the rod will be in equilibrium if it makes with the vertical an angle θ given by the equation

$$\cos^4 \frac{\theta}{2} = \frac{a}{2c},$$

where $4a$ is the latus rectum and $2c$ the length of the rod.

Investigate also the stability of the equilibrium in this position.

[Ans. stable.] (*D. U. 1952*)

CHAPTER X

EQUILIBRIUM OF STRINGS

1. The Catenary

10.1. Strings, ropes and chains frequently form part of the system of bodies with which problems in Statics are concerned. It is, therefore, important and even interesting to discuss the equilibrium of strings under the effect of forces acting on them. In the first instance, we consider only a particular case, namely, the discussion of the nature of the curve in which a string hangs freely between two points (not in the same vertical line) under the action of gravity alone. Such a curve is called a **Catenary** or a **Chainette.**

We shall assume that the strings and chains discussed in this chapter are *perfectly flexible i.e.*, they offer no resistance whatsoever to bending at any point. The action across any section of such a string is only tangential to the string.

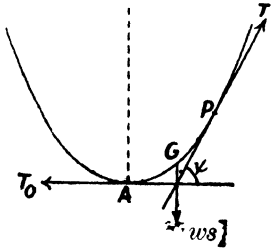
We shall assume the normal section at every point to be so small that the form of the string may be taken as a curved line.

A chain whose links are small and perfectly smooth approximates to these ideal conditions and may be treated like a string.

A string or a chain is regarded as *uniform* when its weight per unit length is constant throughout its length.

10.11. Equation of a Catenary. *A uniform inextensible heavy string hangs freely under the action of gravity ; to find the equation of the curve it assumes.*

Let A be the lowest point and P any other point on the string.



Let w be its weight per unit length and ψ the angle which the tangent at P makes with the horizontal.

The forces acting on the portion AP$=s$ of the string are :—

(a) the tension T at P acting along the tangent at P.

(b) the tension T_0 at A acting horizontally.

(c) the weight, ws acting along a vertical line passing through the C.G. of the portion AP and through the point where the two tensions meet.

Resolving vertically and horizontally, we have

$$T \sin \psi = ws \qquad \ldots(1)$$

and

$$T \cos \psi = T_0 = wc \qquad \ldots(2)$$

where T_0 is taken to be equal to a weight of length c of the string.

Dividing (1) by (2), we obtain $\mathbf{s = c \tan \psi}$ .. (3)
which is the *intrinsic equation* of the catenary.

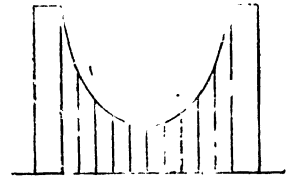
This is usually known as the **common catenary**, the lowest point A is known as the vertex and c is known as the **parameter**

Also squaring and adding (1) and (2), we get

$$T^2 = w^2(c^2 + s^2).$$

10.12. Suspension Bridge. An interesting problem is afforded by the kind of bridges (usually met with among the hills) in which the horizontal roadway is suspended from a pair of cables, on the two sides of the bridge, by a number of vertical chains, the ends of the cables being attached to the tops of rigid piers.

In this case, for the sake of simplicity, we neglect the weights of the chains and cables and suppose the weight of the bridge to be distributed evenly along its length.



Considering the equilibrium of a portion of a cable and taking the *weight of each element to be proportional to its horizontal projection*, we proceed as before.

If (x, y) be the co-ordinates of P (Fig. Art. 10·11) with reference to the horizontal and vertical lines through A and w the weight per unit horizontal length, we obtain

$$T \sin \psi = wx \qquad \ldots(a)$$

and
$$T \cos \psi = T_0 = wc \qquad \ldots(b)$$

so that
$$x = c \tan \psi$$

or
$$x = c \tan \psi = c \frac{dy}{dx}.$$

$$\therefore \quad cy = \tfrac{1}{2}x^2 + k,$$

but since $y = 0$ when $x = 0$, $\quad \therefore \ k = 0.$

Hence the cable assumes the parabolic form given by the equation

$$\mathbf{x^2 = 2cy} \qquad \ldots(4)$$

The curve is, therefore, known as the **Parabolic Catenary**.

Also from (a) and (b), we have

$$T^2 = w^2(c^2 - x^2) \qquad \ldots(c)$$

Example. *A suspension bridge of weight 100 tons evenly distributed over the span of 150 ft. is supported by two cables which have a sag of 20 ft. Find the greatest and least tension in each cable.*

The span, the horizontal distance between the ends of a cable being 150 ft. and the sag, the depression of the lowest point being 20 ft.,

$\therefore$ for one end A of a cable $x = \frac{1}{2} \times 150 = 75$ ft., and $y = 20$ ft.

$\therefore$ from the equation $x^2=2cy$, $c=\dfrac{x^2}{2y}=\dfrac{75\times75}{2\times20}=\dfrac{75\times15}{8}$ ft.

The weight supported by each cable $=\frac{1}{2}\times100=50$ tons.

$\therefore$ w, the weight per unit length of span $=50/150=\frac{1}{3}$ ton.

$\therefore$ the least tension $T_0=wc=\dfrac{1}{3}\times\dfrac{75\times15}{8}=\dfrac{375}{8}=46\frac{7}{8}$ tons wt.

To find the greatest tension, *i.e.*, tension at A, we have

$$T^2=w^2(c^2+x^2)=\frac{1}{9}\left[\left(\frac{75\times15}{8}\right)+(75)^2\right]=\frac{(75)^2}{9}\left[\frac{225}{64}+1\right]$$

$$=625\times\frac{289}{64}$$

$\therefore$ $T=25\times17/8=425/8=53\frac{1}{8}$ tons wt.

10.13. Relation between x, y and ψ for the Common Catenary.

From Calculus, we have $\dfrac{dx}{ds}=\cos\psi$, $\dfrac{dy}{ds}=\sin\psi$, and from the equation of the catenary $s=c\tan\psi$, we get $\dfrac{ds}{d\psi}=c\sec^2\psi$.

$\therefore$ $\dfrac{dy}{d\psi}=\dfrac{dy}{ds}\cdot\dfrac{ds}{d\psi}=\sin\psi.c\sec^2\psi=c\tan\psi\sec\psi$.

$\therefore$ integrating, we have, $y=c\sec\psi+k$.

Now we choose the axis of x, a horizontal line at a distance c below the vertex A, so that when $\psi=0$, $y=c$ and hence $k=0$.

$\therefore$ $$\mathbf{y=c\sec\psi} \qquad ...(5)$$

Again $$\frac{dx}{d\psi}=\frac{dx}{ds}\cdot\frac{ds}{d\psi}=\cos\psi.c\sec^2\psi=c\sec\psi$$

$\therefore$ $$\mathbf{x=c\log(\sec\psi+\tan\psi)} \qquad ...(6)$$

the constant of integration being zero since $x=0$, when $\psi=0$.

The horizontal line at a depth c below A which is taken as the axis of x, is known as the *directrix of the Catenary.*

When a catenary is formed by suspending the string from two points B and C in the same level, the distance BC is called the **Span** and the depth of the vertex below BC, or *the depression of the lowest point of the string below its ends*, is called the **Sag.**

10.14. Relation between y and s. From equation (5) $y=c\sec\psi$ and (3) $s=c\tan\psi$, we have

$$y^2-s^2=c^2(\sec^2\psi-\tan^2\psi)=c^2.$$

$\therefore$ $$\mathbf{y^2=c^2+s^2} \qquad ...(7)$$

This shows that the ordinate of a point of the catenary is greater than the length of the arc measured from the vertex upto the point.

This relation between y and s finds frequent application in the solution of problems.

For example, to find the tension at any point, we have

$$T^2=w^2(c^2+s^2)=w^2y^2. \quad \therefore T=wy.$$

10.15. Cartesian equation of the Catenary. We have

$$\frac{dy}{dx}=\tan\psi=\frac{s}{c}=\frac{\sqrt{y^2-c^2}}{c}, \text{ by (7), or } \frac{c\,dy}{\sqrt{y^2-c^2}}=dx$$

$$\therefore \int dx=\int\frac{c\,dy}{\sqrt{y^2-c^2}} \quad \therefore x+k=c\cosh^{-1}\frac{y}{c}.$$

When $x=0$, $y=c$, $\quad \therefore \quad k=0$,

Hence $$\mathbf{y=c\cosh\frac{x}{c}}. \qquad ...(8)$$

10.16. Relation betwten x and s.

We have $$s=c\tan\psi=c\frac{dy}{dx}=c\frac{d}{dx}\left(c\cosh\frac{x}{c}\right)=c\sinh\frac{x}{c}$$

$$\therefore \quad \mathbf{s=c\sinh\frac{x}{c}}. \qquad ...(9)$$

Also since $x=c\log(\sec\psi+\tan\psi)$ and $\tan\psi=\frac{s}{c}$,

$$\therefore x=c\log\left(\sqrt{1+\frac{s^2}{c^2}}+\frac{s}{c}\right)=c\log\frac{\sqrt{c^2+s^2}+s}{c}.$$

Note 1. From (8) and (9), we have

$$y^2-s^2=c^2\left(\cosh^2\frac{x}{c}-\sinh^2\frac{x}{c}\right)=c^2, \text{ or } y^2=c^2+s^2,$$

which is the same as (7).

Note 2. $$y=c\cosh\frac{x}{c}=\frac{c}{2}\left[e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right],$$

and $$s=c\sinh\frac{x}{c}=\frac{c}{2}\left[e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right].$$

10.17. Other methods for obtaining the cartesian equation of the catenary. This can be obtained by eliminating ψ from (5) and (6), which are the parametric equations of the catenary.

Second Method.

From (6), we get, $e^{\frac{x}{c}}=\sec\psi+\tan\psi$.

$$\therefore e^{-\frac{x}{c}}=\frac{1}{\sec\psi+\tan\psi}=\frac{\sec^2\psi-\tan^2\psi}{\sec\psi+\tan\psi}=\sec\psi-\tan\psi.$$

Hence by addition,

$$\sec\psi=\tfrac{1}{2}(e^{\frac{x}{c}}+e^{-\frac{x}{c}})=\cosh\frac{x}{c}.$$

But from (5), $\quad \sec\psi = y/c \quad \therefore \mathbf{y=c\cosh\frac{x}{c}}.$

Third Method. The same equation can also be obtained from (3) as follows :—

We have $s = c \tan \psi = c\dfrac{dy}{dx} = cp$, where $p = \dfrac{dy}{dx}$.

Differentiating with respect to x, we have

$$c\frac{dp}{dx} = \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+p^2}$$

$\therefore$ integrating, we have

$$x = \int \frac{c\,dp}{\sqrt{1+p^2}} = c \sinh^{-1} p,$$

the constant of integration being zero, for at the lowest point $x = 0$ and $p = 0$.

$$\therefore \qquad p \text{ } i.e., \quad \frac{dy}{dx} = \sinh \frac{x}{c}.$$

Integrating again we have

$$y = c \cosh \frac{x}{c} + k.$$

But when $x=0$, let $y=c$ *i.e.*, the origin O is taken at a depth c below the lowest point of the catenary. Then $k=0$ and $y = c \cosh \dfrac{x}{c}$.

10.18. Given the length and the sag d, to find the parameter c of the catenary. When a chain, of length $2l$, hangs between two points on the same level and has a sag (depression of the lowest point of the chain below its ends) equal to d, then ordinate y of an end of the chain is $c+d$, and $s=l$.

$\therefore$ by eqution (7), *i.e.*, $y^2 = c^2 + s^2$, we have

$$(c+d)^2 = c^2 + l^2 \quad \text{or} \quad 2cd = l^2 - d^2 \quad \text{or} \quad c = (l^2 - d^2)/2d.$$

Example. *A chain 20 ft. long, which weighs 2 lbs. per foot, hangs between two points on the same level and has a sag of 4 ft. Find the parameter of the catenary and the tension at a point of support.*

Here w, the weight per unit length $= 2$ lbs., $2l = 20$ or $l = 10$; $d = 4$ and the ordinate y of one of the points of support $= c + 4$.

$\therefore$ by $y^2 = c^2 + s^2$, we have

$(c+4)^2 = c^2 + l^2 = c^2 + (10)^2$, or $8c = 100 - 16 = 84$, *i.e.*, $c = 10{\cdot}5$ ft.

The tension at a point of support $= w(c+4)$

$= 2(10{\cdot}5 + 4) = 29$ lbs. wt.

10.2. Tension at a point.

(*a*) Since $y = c \sec \psi$ and $T \cos \psi = T_0 = wc$ or $T = wc \sec \psi$, we have

$\therefore$ $$\mathbf{T = wy}. \qquad \ldots(10)$$

Thus *the tension at any point varies as its height from the directrix.*

This shows that *if a string hangs freely in equilibrium over two smooth pegs* (not in the same vertical line), *its ends just reach the directrix of the curve between the pegs.*

(*b*) Since $T_0 = wc$, it shows that T_0, the tension at the lowest point is equal to a weight of the string equal to the height of the lowest point above the directrix, which agrees with (10) in (*a*).

(*c*) Since $T \cos \psi = T_0 = wc$, it shows that *the horizontal component of the tension at every point is constant and equal to* wc.

(*d*) Since by (1) Art. 10.11, $T \sin \psi = ws$, it shows that *the vertical component of the tension at any point is equal to the weight of the portion of the string lying between the vertex and the point.*

Example 1. *A uniform chain, of length* l, *which can just bear a tension of* n *times its weight, is stretched between two points in the same horizontal line. Show that the least possible sag in the middle is*

$$l[n-(n^2-\tfrac{1}{4})^{\frac{1}{2}}].$$

Let w be the weight per unit length of the chain.

The greatest tension, *i.e.*, tension at one of the ends A, by (10),

$$= wy = n \times \text{weight of the chain} = nwl,$$

$$\therefore \quad y = nl.$$

Also by (7), $y^2 = c^2 + s^2$ or $c^2 = y^2 - s^2 = n^2l^2 - \frac{1}{4}l^2$ $(\because\ s = \frac{1}{2}l$ at A).

When the greatest permissible tension is taken, the sag will be the least.

$$\therefore \quad \text{the least sag} = y - c = nl - (n^2l^2 - \tfrac{1}{4}l^2)^{\frac{1}{2}} = l[n-(n^2-\tfrac{1}{4})^{\frac{1}{2}}].$$

Example 2. *If the tangents at the points* P *and* Q *of a catenary are at right angles, prove that the tension at the middle point of the arc* PQ *is equal to the weight of a length of the string equal to half the arc* PQ.

Let the inclination of the tangent at P with the horizontal be ψ, then the inclination of the tangent at Q is $90^\circ + \psi$.

At P, $s_1 = c \tan \psi$, and at Q, $s_2 = c \tan (90^\circ + \psi) = -c \cot \psi$.

$\therefore$ the length of half the arc PQ

$$= \tfrac{1}{2}(s_1 - s_2) = \tfrac{1}{2}c(\tan \psi + \cot \psi) \qquad \ldots (1)$$

Now at the middle point of the arc PQ,

$$s = \tfrac{1}{2}(s_1 + s_2) = \tfrac{1}{2}c(\tan \psi - \cot \psi).$$

$\therefore$ the ordinate y of the middle point of the arc PQ is given by

$$y^2 = c^2 + s^2 = c^2 + \tfrac{1}{4}(s_1 + s_2)^2 = c^2 + \tfrac{1}{4}c^2 (\tan \psi - \cot \psi)^2$$

$$= \tfrac{1}{4}c^2[4 + \tan^2 \psi + \cot^2 \psi - 2] = \tfrac{1}{4}c^2 (\tan \psi + \cot \psi)^2$$

$$\therefore \quad y = \tfrac{1}{2}c(\tan \psi + \cot \psi).$$

Hence the tension at the middle point of the arc PQ

$$= wy = \tfrac{1}{2}wc(\tan \psi + \cot \psi)$$

$$= w \times \text{length of half the arc PQ} \qquad [\text{by (1)}]$$

= weight of a length of the string equal to half of the arc PQ.

10.3. A tightly stretched catenary. If (x, y) be the co-ordinates of one end of a string stretched tightly between two points in the same horizontal level (*e.g.*, a telegraph wire between two poles), then

$$(i)\quad y=c\cosh\frac{x}{c}=\tfrac{1}{2}c\,[e^{x/c}+e^{-x/c}]=c\left[1+\frac{x^2}{2c^2}+\frac{x^4}{24c^4}+\ldots\ldots\right]$$

i.e.,
$$y-c=\frac{x^2}{2c}+\frac{x^4}{24c^3}+\ldots\ldots \qquad \ldots(1)$$

Now, since the sag $y-c$ is small as compared to the span $2x$, c must be large, so that *for a tight catenary c is large and the directrix is very low.*

(*ii*) Shifting the origin to the vertex $(0, c)$, the equation becomes

$$y=\frac{x^2}{2c}+\frac{x^4}{24c^2}+\ldots\ldots$$

Neglecting small quantities of the third and higher orders it reduces to
$$y=\frac{x^2}{2c} \quad \text{or} \quad x^2=2cy.$$

Thus a *tightly stretched catenary approximates to a parabola,* of latus rectum, $2c$ or $2T_0/w$.

Note.—When the catenary is *slack*, the lowest point of the catenary is low and consequently the directrix is low. Thus even for a slack catenary the directrix is very low because the lowest point itself is low.

10.31. *To find the increase in the length of a tightly stretched wire due to a small sag.*

We have

$$s=c\sinh\frac{x}{c}=c\left[\frac{x}{c}+\frac{x^3}{6c^3}+\ldots\ldots\right]=x+\frac{x^3}{6c^2}+\ldots\ldots$$

$$\therefore\quad 2s-2x=\frac{x^3}{3c^2}\ \text{approximately,}\ \frac{x}{c}\ \text{being small,}$$

$$=\frac{w^2x^3}{3\,T_0^2},\quad \because\ T_0=wc.$$

If $2l$ be the length of the wire and $2a$ the span, then

$$2l-2a=\frac{a^3}{3c^2},\ \text{or}\ l-a=\frac{a^3}{6c^2}. \qquad \ldots(2)$$

Example. *A uniform chain, of length 2l and weight w per unit length, has its ends attached to two points in the same horizontal line at a distance 2a apart. If l is only a little greater than a, show that the tension in the chain varies inversely as the sag and is approximately equal to the weight of length* $\sqrt{a^3/6(l-a)}$ *of the chain, and that the sag is nearly equal to* $\frac{1}{2}\sqrt{6a(l-a)}$.

Since l is very little greater than a, the catenary is tight, c is large and the tension in the chain is very great.

By putting $x=a$ in (1), Art. 10.3, the approximate value of the sag, d, is given by

$$d=y-c=\frac{a^2}{2c}, \text{ or } c=\frac{a^2}{2d} \qquad \ldots(i)$$

And the tension T of the chain, which may be taken approximately equal to the tension at the lowest point

$$=wc=\frac{wa^2}{2d},$$

so that T varies inversely as the sag.

Now by (2) Art. 10.31, we have

$$l-a=\frac{a^3}{6c^2} \text{ or } c^2=\frac{a^3}{6(l-a)}, \text{ i.e., } c=\sqrt{a^3/6(l-a)} \qquad \ldots(ii)$$

so that $$T=wc=w\sqrt{a^3/6(l-a)}$$

$$=\text{weight of length } \sqrt{a^3/6(l-a)} \text{ of the chain.}$$

Also by (i), the approximate value of the sag d

$$=\frac{a^2}{2c}=\frac{a^2}{2\sqrt{a^3/6(l-a)}}=\frac{a^2\sqrt{6(l-a)}}{2\sqrt{a^3}}=\tfrac{1}{2}\sqrt{6a(l-a)}.$$

10.4. *Approximate shape of the catenary at points far removed from the lowest point.*

When x is large $e^{-x/c}$ becomes very small and approaches zero.

Hence $y=\frac{c}{2}\left[e^{x/c}+e^{-x/c}\right]$ reduces to $y=\frac{1}{2}ce^{x/c}$.

Thus for large values of x *i.e.*, at points further away from the lowest point, the catenary approximates to an exponential curve.

10.41. Example 1. *Given two smooth pegs in a horizontal line distant $2a$ apart, find the least length of a uniform heavy string which will rest freely over them.*

Let $2l$ be the length of the string, $2s$ the length of the part between the pegs and y the height of a peg above the directrix.

Since the string rests freely and the pegs are smooth, by (a) Art. 10.2, the ends of the string must reach the directrix, so that half the string

$$l=y+s=c\cosh\frac{x}{c}+c\sinh\frac{x}{c}=ce^{x/c}=ce^{a/c} \quad (\because x=a).$$

Now differentiating with respect to c, we have

$$\frac{dl}{dc}=e^{a/c}-c.\frac{a}{c^2}e^{a/c}=e^{a/c}\left(1-\frac{a}{c}\right).$$

For l to be least, $\frac{dl}{dc}=0$ *i.e.*, $c=a$.

Also $$\frac{d^2l}{dc^2}=e^{a/c}\left(-\frac{a}{c^2}\right)\left(1-\frac{a}{c}\right)+\frac{a}{c^2}\,e^{a/c}$$

$$=\frac{a^2}{c^3}\,e^{a/c},\ \text{is positive for } c=a.$$

$\therefore$ l is least when $c=a$ and then the least length of the string

$$=2l=2\,ce^{a/c}=2\,ae^{a/a}=2ae.$$

Example 2. *A uniform telegraph wire is made of a given material, and such a length l is stretched between two posts, distant d apart and of the same height, as will produce the least possible tension at the posts. Show that $l=(d/\lambda)\sinh\lambda$, where λ is given by the equation $\lambda\tanh\lambda=l$.*

At one of the posts, the catenary has $x=\frac{1}{2}d$ and $s=\frac{1}{2}l$.

$\therefore$ for the tension T at the post, we have

$$\mathrm{T}=wy=wc\cosh\frac{x}{c}=wc\cosh\frac{d}{2c}.$$

$\therefore$ $\frac{d\mathrm{T}}{dc}=0$, gives $w\left[\cosh\frac{d}{2c}+c\left(-\frac{d}{2c^2}\right)\sinh\frac{d}{2c}\right]=0,$

or $$\cosh\frac{d}{2c}-\frac{d}{2c}\sinh\frac{d}{2c}=0,\quad i.e.,\ \frac{d}{2c}\tanh\frac{d}{2c}=1$$

$\therefore$ we have, $\lambda\tanh\lambda=1$, where $\lambda=\frac{d}{2c}$ or $c=\frac{d}{2\lambda}$.

Now since $y^2=c^2+s^2$,

$\therefore$ $$s^2=y^2-c^2=c^2\left(\cosh^2\frac{x}{c}-1\right)=c^2\sinh^2\frac{x}{c}$$

$\therefore$ $$\frac{l}{2}=s=c\sinh\frac{x}{c}=c\sinh\frac{d}{2c}=\frac{d}{2\lambda}\sinh\lambda,$$

i.e., $$l=\frac{d}{\lambda}\sinh\lambda.$$

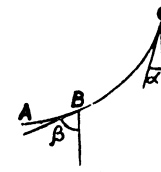
Example 3. *A heavy string of length l is supported from two points, and at these points the string makes angles α, β with the vertical. Show that the height of the one point above the other is*

$$l\cos\tfrac{1}{2}(\alpha+\beta)\sec\tfrac{1}{2}(\alpha-\beta).$$

[The vertex of the catenary does not lie between the points of suspension].

Let the string be supported at the points B and C and let A be the vertex of the catenary of which BC forms a part.

Let y_1, y_2 be the heights of B and C above the directrix and s_1 and s_2 the lengths of the arcs AB and AC, then



$$s_1 = c \tan\psi = c \cot\beta, \qquad s_2 = c \cot\alpha,$$

and $$y_1 = c \sec\psi = c \operatorname{cosec}\beta, \qquad y_2 = c \operatorname{cosec}\alpha.$$

$$\therefore \quad \frac{y_2 - y_1}{s_2 - s_1} = \frac{\operatorname{cosec}\alpha - \operatorname{cosec}\beta}{\cot\alpha - \cot\beta} = \frac{\sin\beta - \sin\alpha}{\sin(\beta - \alpha)} = \frac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}.$$

Now since $s_2 - s_1 = l$, we have

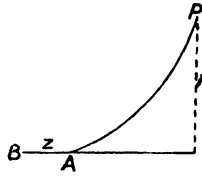
$y_2 - y_1$ = the height of C above B $= l \cos\frac{1}{2}(\alpha+\beta) \sec\frac{1}{2}(\alpha-\beta)$.

Example 4. *A uniform chain of length l rests in a straight line on a rough horizontal table. One end is raised a height h above the table and the chain is on the point of motion. Show that the length of the straight part on the table is*

$$l + \mu h - \{(\mu^2+1)\, h^2 + 2\mu l h\}^{\frac{1}{2}}$$

where μ is the co-efficient of friction.

Let z be the required length and w the weight per unit length of the chain, then since the chain is on the point of slipping, the horizontal tension at A, the lowest point of the catenary = force of friction.



i.e., $T_0 = wc = \mu w z$, so that $c = \mu z$ ...(*i*)

Length of the portion AP $= s = l - z$,

and the height of P above the directrix $= y = c + h$

$\therefore$ from the formula $y^2 = c^2 + s^2$, we have

$$(c+h)^2 = c^2 + (l-z)^2$$

Using (*i*) and simplifying, we have

$$z^2 - 2z(l+\mu h) + l^2 - h^2 = 0,$$

which gives, $z = l + \mu h \pm \sqrt{(l+\mu h)^2 - l^2 + h^2}$

$\therefore$ since $l - z$ must be positive, we have

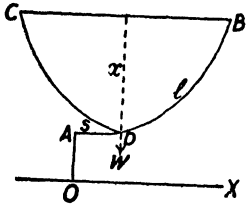
$$z = l + \mu h - \sqrt{(\mu^2+1)\, h^2 + 2\mu l h}.$$

Example 5. *The weight per foot run of a telegraph wire is w lb., the length of wire between the posts is 2l feet, a weight W lb. is hung at the middle, where the wire sags x feet. Prove that the tension at the posts is*

$$w\left\{\frac{l^2}{2x}\left(1 + \frac{W}{lw}\right) + \tfrac{1}{2}x\right\} \text{lb.}$$

Let B and C be the points of support of the wire, D the point

of suspension of the weight, A the vertex of the catenary of which DB is a part, s the arc AD, y, y_1 the ordinates of B and D, and T, T_1 the tensions at B and D.



Since the weight W is supported by both the parts DC and DB, $\frac{1}{2}W$ is the vertical component of the tension at D for the part AD of the string.

$\therefore$ by (d) Art. 10·2, $\frac{1}{2}W = ws$, *i.e.*, $s = W/2w$ ...(1)

By (a) Art. 10·2, $T = wy$ and $T_1 = wy_1$,

$\therefore$ $T - T_1 = w(y - y_1) = wx$ ...(2)

and $T + T_1 = w(y + y_1)$ ...(3)

But by Art. 10·14, $y^2 = c^2 + (s+l)^2$ and $y_1^2 = c^2 + s^2$

$\therefore$ $2ls + l^2 = y^2 - y_1^2 = (y + y_1)(y - y_1) = x(y + y_1)$

so that from (3), $T + T_1 = w(y + y_1) = w\left[\frac{l^2}{x} + \frac{2ls}{x}\right]$

$= w\left[\frac{l^2}{x} + \frac{2l}{x} \cdot \frac{W}{2w}\right]$ [by (1).

Hence by using (2),

$$2T = wx + w\frac{l^2}{x}\left[1 + \frac{W}{lw}\right].$$

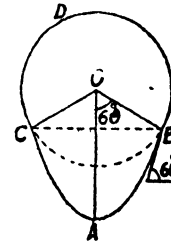
Example 6. *Show that the length of an endless chain which will hang over a circular pulley of radius a, so as to be in contact with two-thirds of the circumference of the pulley is*

$$a\left[\frac{3}{\log(2+\sqrt{3})} + \frac{4\pi}{3}\right].$$

Let ACDBA be the endless chain hung over the circular pulley CDB with centre O.

Since the chain is in contact with $\frac{2}{3}$ circumference of the pulley, the length of the part in contact with the pulley

$= \frac{2}{3} \times 2\pi a = \frac{4}{3}\pi a$ ...(1)



Also $\angle BOC = \frac{1}{3} \times 2\pi = 120°$

and chord $BC = 2a \sin 60° = \sqrt{3}a$.

Now we consider the catenary CAB.

The angle which the tangent at B makes with the horizontal $= \angle AOB = 60°$

$\therefore$ from $s = c \tan \psi$, we have

arc CAB = 2 arc AB = $2c \tan 60° = 2\sqrt{3}c$. ...(2)

To find c, using $x = c \log(\tan \psi + \sec \psi)$, we have

$\frac{1}{2}BC$ or $\frac{1}{2}\sqrt{3}a = c \log(\tan 60° + \sec 60°) = c \log(\sqrt{3}+2)$ ...(3)

$\therefore$ from (2) and (3), we have

$$\text{the length of the catenary} = 2\sqrt{3}c = \frac{3a}{\log(\sqrt{3}+2)} \quad \ldots(4)$$

Hence the total length of the string

$$= \frac{3a}{\log(\sqrt{3}+2)} + \frac{4\pi a}{3} = a\left[\frac{3}{\log(2+\sqrt{3})} + \frac{4\pi}{3}\right].$$

10.42. Catenary formed by a string or chain of variable density. Let a string of variable thickness α and of variable density ρ be suspended as in Art. 10·11, α being the area of the cross-section at any point.

Let w be the weight per unit length of the element δs of the string so that weight of the element is $\rho\alpha\delta s = w\delta s$, where w is a variable on account of the variability of α or of ρ or of both α and ρ.

Then as in Art. 10·11 resolving horizontally and vertically for an arc AP $= s$, of the string, we have

$$\mathrm{T}\cos\psi = \mathrm{T}_0, \text{ and } \mathrm{T}\sin\psi = \int_0 w\,ds$$

$$\therefore \qquad \mathrm{T}_0 \tan\psi = \int_0 w\,ds,$$

which on differentiation gives

$$\mathrm{T}_0 \sec^2\psi \frac{d\psi}{ds} = w, \quad \text{or} \quad w\cos^2\psi \frac{ds}{d\psi} = \mathrm{T}_0, \text{ a constant} \quad \ldots(1).$$

Equation (1) determines the variable w when the equation of the curve and consequently $ds/d\psi$ is known, and conversely if w is known, the equation of the curve is obtained by integration.

Example. (*i*) If $w = \frac{\mathrm{T}_0}{a} sec^2 \frac{s}{a}$, *to find the nature of the curve.*

(*ii*) *If* $s = 4a \sin\psi$, *a cycloid, to find w the law of density.*

(*i*) By (1), $\mathrm{T}_0 \sec^2\psi\, d\psi = w\, ds = \frac{\mathrm{T}_0}{a} \sec^2 \frac{s}{a} ds.$

Integrating, we get $\tan\psi = \tan(s/a) + \mathrm{C}$.

Taking $s = 0$ when $\psi = 0$, so that $\mathrm{C} = 0$, we get

$$\tan\psi = \tan(s/a), \textit{ i.e., } s = a\psi,$$

an intrinsic equation of a circle.

Thus in this case the string or the chain hangs in the form of a circle.

(*ii*) Since $s = 4a \sin\psi$, $\therefore\ ds/d\psi = 4a\cos\psi$.

$\therefore$ by (1) $w\cos^2\psi . 4a\cos\psi/\psi = \mathrm{T}_0$

$\therefore \qquad w = (\mathrm{T}_0/4a)\sec^3\psi.$

Thus the density varies as $\sec^3\psi$.

Examples XXXIX

1. A kite is flown with 600 ft. of string from the hand to the kite and a spring balance held in the hand shows a pull equal to the weight of 300 ft. of the string inclined at 30° to the horizon. Find the vertical height of the kite above the hand. [*Ans.* 300 $(\sqrt{7}-1)$ ft.]

2. A uniform chain of length 15 ft. hangs between two points A and B, the vertical distance between them being 9 ft. It is observed that the tangent at A is horizontal. Show that the horizontal distance between A and B is

$$\left(8 \cosh^{-1} \frac{17}{8}\right) \text{ft.}$$

3. In a suspension bridge of 400 feet span and 40 feet dip the whole weight supported by the chains is two tons per horizontal foot. Find the horizontal tension and the tension at the points of support. [Use Art. 10.12] (*Ans.* 500, 538·5 tons wt.)

4. (*a*) The ends of a tightly stretched cable weighing $\frac{1}{2}$ lb. per yard, are fixed to two points on a level 80 yards apart and the cable has a sag, at the middle of 1 ft. 4 inches. Find the tension in the wire at the lowest point. [*Ans.* 900 lb. wt.]

(*b*) If a single telegraph wire has a span of 75 yards, and a sag in the middle is 1 ft., show that the tension in the wire is approximately 480 lb. weight if the weight of the wire is 400 lbs. per mile. (Use Art. 10·3)

5. A uniform chain is hung from two points A and B in the same horizontal line, and at a distance $2a$ apart. The inclination of the chain to the horizontal at A or B is α. If $2s$ be the total length of the chain, w the weight per unit length, T_0 the tension at the lowest point and d the sag, show that

$$\frac{\tan\alpha}{s} = \frac{w}{T_0} = \frac{\sec\alpha - 1}{d} = \frac{\log_e(\tan\alpha + \sec\alpha)}{a}. \qquad (M.T.)$$

6. A uniform chain, of weight W and length $2l$ hangs between two points A, B in the same horizontal line at a distance $2a$ apart, and the sag in the middle is d. Show that the parameter c of the catenary is $\frac{l^2-d^2}{2d}$ and that the tension at either end is $W(l^2+d^2)/4ld$.

7. A uniform heavy chain of length $2l$ has its extremities fastened to two points at the same level and the sag in the middle is h. Find the parameter of the catenary in which it hangs and prove that the span is

$$\frac{l^2-h^2}{h} \log \frac{l+h}{l-h}.$$

8. A given length $2s$ of a uniform chain has to be hung between two points at the same level and the tension has not to exceed the weight of a length b of the chain. Show that the greatest span is

$$\sqrt{b^2-s^2} \log \frac{b+s}{b-s}.$$

9. The end links of a uniform chain of length l can slide on two smooth rods in the same vertical plane, which are inclined in opposite directions at equal angles α to the horizontal. Prove that the sag in the middle is

$$\tfrac{1}{2}l \tan \tfrac{1}{2}\alpha.$$

10. The extremities of a heavy string of length $2l$ and weight $2lw$ are attached to two small rings which can slide on a fixed horizontal wire. Each of these rings is acted on by a horizontal force equal to lw. Show that the distance apart of the rings is $2l \log(1+\sqrt{2})$. (*Lucknow, U.*)

11. A uniform chain of length l is suspended from two points A, B in the same horizontal line. If the tension at A is twice that at the lowest point, show that the span AB is

$$\frac{l}{\sqrt{3}} \log (2+\sqrt{3}).$$

12. A uniform chain of length l, is to be suspended from two points, A and B, in the same horizontal line, so that either terminal tension is n times that at the lowest point. Show that the span AB must be

$$\frac{l}{\sqrt{n^2-1}} \log (n + \sqrt{n^2-1}). \qquad (\textit{Agra U.})$$

13. The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is

$$\mu \log \frac{1+\sqrt{1+\mu^2}}{\mu},$$

where μ is the co-efficient of friction. (*Lucknow U.*)

14. A heavy chain, of length $2l$, has one end tied at A and the other is attached to a small heavy ring which can slide on a rough horizontal rod which passes through A. If the weight of the ring be n times the weight of the chain show that the greatest possible distance from A is

$$\frac{2l}{\lambda} \log [\lambda + \sqrt{1+\lambda^2}],$$

where $\lambda^{-1}=\mu(2n+1)$ and μ is the coefficient of friction. (*D.U. 1957*)

15. A heavy uniform string, of length l, is suspended from a fixed point A, and its other end B is pulled horizontally by a force equal to the weight of a length a of the string. Show that the horizontal and vertical distances between A and B are

$$a \sinh^{-1} \frac{l}{a} \quad \text{and} \quad \sqrt{l^2+a^2}-a.$$

16. A heavy chain rests in equilibrium in a vertical plane supported by two smooth pegs A and B. If α and β are the angles the tangents to the chain at A and B make with the vertical and if x and y are the lengths of the chain hanging vertically from A and B, and z is the length between A and B, show that

$$\frac{x}{\sin \beta} = \frac{y}{\sin \alpha} = \frac{z}{\sin (\alpha+\beta)}.$$

17. One end of a perfectly flexible uniform chain, of length l, is attached to a fixed point at a height b above a smooth horizontal table and a horizontal distance b from the edge over which the chain hangs. The chain consists of three parts, namely a catenary of length s, a straight part of length $b-x$ on the table (perp. to the edge) and a straight part hanging vertically. Prove that

$$s+b=\sqrt{2b(x+l)}.$$

18. A uniform chain of length 16 ft. lies on a rough floor, (the coefficient of friction being $\frac{1}{4}$), in a straight line. One end is lifted up to a height of 4 ft., and the chain is then on the point of slipping along the floor. Prove that the length of the chain on the floor is 10 ft.

19. A weight, an nth that of the string is attached to its lowest point, the tangents at the lowest and the highest points of the string are now inclined to the vertical at angles φ and θ respectively ; show that

$$\tan \varphi=(1+n) \tan \theta. \qquad (\textit{D.U.})$$

20. A uniform string of weight W is suspended from two points at the same level and a weight W′ is attached at the lowest point. If α and β are the

inclinations to the horizontal of the tangents at the highest and lowest points, prove that

$$\frac{\tan\alpha}{\tan\beta}=1+\frac{W}{W'}.$$

21. A uniform measuring chain of length l, is tightly stretched over a river, the middle point touching the surface of the water, while each of the extremities has an elevation k above the surface, show that the difference between the length of the measuring chain and the breadth of the river is nearly

$$\frac{8}{3}\cdot\frac{k^2}{l}. \qquad (D.U.\ 1924)$$

22. A uniform chain is hung up from two points at the same level and distant $2a$ apart. If z is the sag at the middle, show that

$$z=c\left(\cosh\frac{a}{c}-1\right).$$

If z be small compared with a, show that $2cz=a^2$ nearly.

A telegraph wire is supported by two poles distant 40 yards apart. If the sag be one foot and the weight of the wire $\frac{1}{3}\frac{1}{2}$ lb. per foot, show that the horizontal pull on each pole is $\frac{1}{2}$ cwt. nearly.

23. A uniform chain of length $2l$ and weight $2wl$ hangs from two fixed points on the same level and at a distance $2a$ apart. A smooth ring of weight W is threaded on the chain. Show that c, the parameter of either catenary formed by the chain can be determined from the equation

$$2wl=W\left(\cosh\frac{a}{c}-1\right)+(W^2+w^2c^2)^{\frac{1}{2}}\sinh\frac{a}{c}.$$

24. One end A of a uniform string of length l is fixed, the other end B moves along a horizontal line through A. Show that the locus of the vertex of the catenary formed by the string referred to horizontal and vertical axes through A is given by

$$8xy=(l^2-4y^2)\log\frac{l+2y}{l-2y}.$$

25. A uniform chain of weight w per unit length, to which a weight W is attached at the middle point O, hangs symmetrically over two smooth pegs on the same horizontal level and distant $2d$ apart. The chain is of such a length that, when the system is in equilibrium a length l of the chain hangs vertically over each peg, and the tangent at O to each half of the chain makes an angle α with the horizontal.

If O is at a vertical distance D below the pegs and $W=2wc\tan\alpha$, prove that

(*i*) $D=l-c\sec\alpha.$

(*ii*) $\frac{l}{c}=\sec\alpha\cosh\frac{d}{c}+\tan\alpha\sinh\frac{d}{c}.$ *(D.U.)*

26. A horizontal table has a circular hole in it of diameter one foot. A uniform chain 5 ft. long is laid symmetrically across the hole, and is found to rest in limiting equilibrium when 2 ft. of the chain hangs through the hole. Prove that μ, the co-efficient of friction between the chain and the table can be found by solving the equation

$$\sinh u=2u, \qquad \text{where} \qquad u=1/\sqrt{9\mu^2-4}.$$

27. A uniform chain hangs in equilibrium under gravity with one end attached to a fixed point O, the chain passes over a smooth peg P, O and P are in the same level and at a distance $2a$ apart. If the length of the chain is the smallest possible, show that c, the parameter of the catenary formed by the curved part of the chain is given by the equation,

$$e^{2x}=\frac{1+x}{3(1-x)}, \qquad \text{where } x=a/c. \qquad (H.C.)$$

28. A weight W is suspended from a fixed point by a uniform string of length l and weight w per unit length. It is drawn aside by a horizontal force P. Show that in the position of equilibrium, the distance of W from the vertical through the fixed point is

$$\frac{P}{w}\left[\sinh^{-1}\left(\frac{W+lw}{P}\right)-\sinh^{-1}\left(\frac{W}{P}\right)\right]. \qquad (Nagpur,\ U.)$$

What happens when the string is light ?

29. A uniform chain of length l hangs between two points A and B which are at a horizontal distance a from one another, with B at a vertical distance b above A. Prove that the parameter of the catenary is given by

$$2c \sinh \frac{a}{2c} = \sqrt{l^2-b^2}.$$

Prove also that, if the tensions at A and B are T_1 and T_2, respectively,

$$T_2-T_1=\frac{Wb}{l}, \quad T_1+T_2=W\sqrt{1+\frac{4c^2}{l^2-b^2}}$$

where W is the weight of the chain. (*Birmingham*)

II. Strings or Chains on a Plane Curve

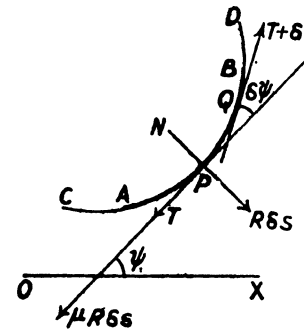
10.5. In Art. 10.11, we dealt with the equilibrium of a string, hanging freely under gravity. We now consider the equilibrium of a *light or heavy* string resting in contact with a *smooth or rough* plane curve (or resting in one plane on a surface).

10.51. Case I. Light *inextensible uniform string resting on a* **smooth** *plane curve.*

Let a string CABD be in contact with the plane curve AB and let us consider the equilibrium of a small element PQ of the string.

Let s be the length of the string measured from some fixed point C, upto a variable point P, and δs the length of the element PQ.

Let R be the reaction per unit length of the curve at P. Then the string being light and the curve smooth the forces acting on the element are :—



(1) Tension T along the tangent at P, making an angle ψ with a fixed line OX.

(2) Tension $T+\delta T$ along the tangent at Q making an angle $\psi+\delta\psi$ with OX.

(3) $R\delta s$, the reaction of the curve along the normal NP.

The string may either be resting on the upper side of the curve, the reaction acting inwards, or, as shown in the figure, may press against the lower side of the curve the reaction acting outwards along the normal at P.

The force $\mu R\delta s$, shown in the figure, is not to be considered in this case as the curve is smooth.

Resolving the forces along the tangent and normal at P, we have

$$(T+\delta T)\cos\delta\psi = T \qquad \ldots(1)$$

and $$(T+\delta T)\sin\delta\psi = R\delta s \qquad \ldots(2)$$

Neglecting small quantities of higher order than the first, equation (1) becomes $\delta T = 0$ *i.e.*, $T = c$, a constant.

so that *the tension of a light string passing round a smooth curve is constant throughout.*

And from (2) $T\delta\psi = R\delta s$ *i.e.*, $R = T\dfrac{d\psi}{ds} = \dfrac{T}{\rho}$ ...(3)

T, being constant, *the normal reaction varies inversely as the radius of curvature (or directly as the curvature) of the curve.*

Note 1.—On replacing $\sin\delta\psi$ by $\delta\psi$ on the L.H.S. of equation (2), the second term $\delta T.\delta\psi$, being a small quantity of the second order, is neglected.

Note 2.—If small quantities of only the first order are to be retained, the resultant tension along the tangent at P $= (T+\delta T)\cos\delta\psi - T = \delta T$, and along the normal it is $(T+\delta T)\sin\delta\psi = T\delta\psi$.

In the solution of examples these expressions may be used and equations may be written down at once.

10.52. Case II. Light *string resting in limiting equilibrium on a* **rough** *plane curve.*

As in the previous case, let $PQ = \delta s$ be an element of the string and let the angles that the tangents at P and Q make with the fixed line OX be ψ and $\psi+\delta\psi$.

Let the string AB be just on the point of slipping from A towards B, so that the friction $\mu R\delta s$ acts along the tangent PT, opposite to the direction of slipping.

Resolving the forces along the tangent and normal at P, we have

$$(T+\delta T)\cos\delta\psi = T+\mu R\delta s \qquad \ldots(1)$$

and $$(T+\delta T)\sin\delta\psi = R\delta s \qquad \ldots(2)$$

Neglecting small quantities of higher order than the first, we get

$$\delta T = \mu R\delta s \quad \text{and} \quad T\delta\psi = R\delta s$$

which give, $\dfrac{dT}{T} = \mu d\psi$ *i.e.*, $\log T = \mu\psi + a$ const. or $T = ce^{\mu\psi}$.

Now, at the point A, let $T = T_1$ and $\psi = \psi_1$ and at B let $T = T_2$ and $\psi = \psi_2$, then

$$T_1 = ce^{\mu\psi_1} \quad \text{and} \quad T_2 = ce^{\mu\psi_2} \quad \therefore \quad T_2 = T_1 e^{\mu(\psi_2 - \psi_1)} \qquad \ldots(3)$$

which gives the tension at B in terms of that at A and the angle the tangent turns through from A to B.

For example, if a rope is passed round a post through one complete revolution, the tension is increased in the ratio $e^{2\pi\mu}$.

If $\mu = \cdot5$, $2\pi\mu = \pi = 3\cdot1416$ and $e^{\pi} = (2\cdot718)^{3\cdot1416} =$ about 23, so that the tension is increased about 23 times for each complete turn.

Example 1. *A light string passing over a rough horizontal cylindrical beam, carries a weight of 80 lbs. at one end. It is found that a weight of 20 lbs. when tied to the other end just prevents slipping. What would be the least weight necessary to prevent slipping if the string were wound an additional time round the beam ?*

Here $T_2=80$, $T_1=20$, $\psi_2-\psi_1=\pi$, $\therefore$ by (3), $80=20e^{\mu\pi}$

$$e^{\mu\pi}=\tfrac{80}{20}=4.$$

If w be the required least weight when the angle is 3π, then

$$80=we^{3\mu\pi}=w(4)^3=64w, \qquad i.e., \qquad w=80/64=1\tfrac{1}{4} \text{ lbs.}$$

Example 2. *A circular disc, of weight W, rests in a vertical plane, being pressed against a perfectly rough vertical wall by a string fixed to a point in the wall above the disc. The string sustains a weight P, and the coefficient of friction between the string and the disc is μ. If θ be the angle between the string and the wall, show that, if the disc is on the point of sliding then* $P(1+\cos\theta)e^{\mu\theta}=W+2P.$

Let the tension along the part CD of the string be T. The vertical part of string, supporting the weight P, has tension equal to P.

In proceeding from B to C, the angle turned through by the tangent is θ.

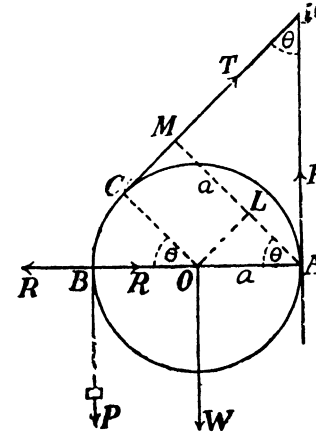
$\therefore$ by (3), $T=Pe^{\mu\theta}$ ...(1)

Now taking moments about A for the whole system, we have

$$P.AB+W.OA=T.AM$$
$$=T(AL+LM)$$

or $$P.2a+W.a=T(a\cos\theta+a)$$
$$=a.Pe^{\mu\theta}(1+\cos\theta), \text{ by (1)}$$

Hence $P(1+\cos\theta)e^{\mu\theta}=2P+W.$



Aliter. If F be the friction at A, resolving forces vertically for the whole system, and taking moments about O, we get

$$T\cos\theta+F=P+W,$$

and $$T.a=P.a+F.a \quad \text{or} \quad F=T-P.$$

$\therefore$ $T\cos\theta+T-P=P+W$ or. $T(1+\cos\theta)=2P+W.$

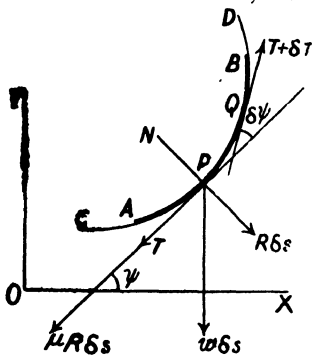
$\therefore$ by (1), $Pe^{\mu\theta}(1+\cos\theta)=2P+W.$

10.53. Case III. Heavy *string resting on a* **smooth** *curve in a vertical plane.*

In addition to the tensions at P and Q and the normal reaction considered in case I, in this case we have also to take into account a vertical force $w\delta s$ corresponding to the weight of the element PQ, being the weight per unit length of the string.

The force $\mu R\delta s$, shown in the figure is not to be considered in this case as the curve is smooth.

Let the line OX, from which ψ is measured, be horizontal and let it be taken as the axis of x. Let (x, y) be the co-ordinates of P, the axis of y being vertically upwards.



Then, resolving the forces along the tangent and normal at P, we have

$$(T+\delta T)\cos\delta\psi = T + w\delta s \sin\psi \quad \ldots(1)$$

and

$$(T+\delta T)\sin\delta\psi = R\delta s + w\delta s\cos\psi \quad \ldots(2)$$

Neglecting small quantities of higher order than the first, we have

from (1), $\delta T = w\delta s \sin\psi = w\delta s \dfrac{dy}{ds} = w\delta y$,

i.e., $$T = wy + C \quad \ldots(3)$$

and from (2), $$R = T\frac{d\psi}{ds} - w\cos\psi = \frac{T}{\rho} - w\cos\psi \quad \ldots(4)$$

Now if T_1, T_2 be the tensions at points where the ordinates are y_1 and y_2, then from (3), $$T_1 - T_2 = w(y_1 - y_2) \quad \ldots(5)$$

i.e., if a heavy string rests against a smooth plane curve, the difference of the tensions at two points is equal to the weight of a length of the string equal to the vertical distance between the points.

Example 1. *If a heavy uniform string passes round various smooth curves in the same vertical plane and its ends hang vertically, show that they are in the same horizontal straight line.*

Let T_1, T_2 be the tensions at the ends with y_1, y_2 as the ordinates. Since the ends hang freely, $T_1 = T_2 = 0$.

$\therefore$ by (5), $\quad y_1 - y_2 = 0 \quad$ *i.e.*, $y_1 = y_2$,

so that the ends of the string, even when it passes over a number of smooth curves, hang vertically at the same level.

Example 2. *A heavy uniform chain of weight w lbs. per unit length rests on a smooth circular arc, of radius a, in a vertical plane. The chain occupies a quadrant with one end at the highest point and is kept in equilibrium by a force P at the upper end, applied horizontally, find F.*

Here the vertical distance between the ends is equal to the radius a, and the tension at the lower end is zero.

$\therefore$ the tension at the highest point along the horizontal tangent $= wa$. Hence $F = wa$.

Example 3. *A uniform chain passes round the lower half of a smooth vertical circle. Prove that if the pressure per unit length at the lowest point is equal to the weight per unit length, the tension at the ends must be half as much again as the tension at the lowest point.*

Let w be the weight per unit length of the chain. At the lowest point where $\psi=0$, and pressure R, by the question, is equal to w, let the tension be T_1, then taking $\rho=a$, from equation (4) Art. 10.53 above, we have

$$w=T_1/a-w \quad \text{or} \quad T_1=2aw.$$

Now from (3), taking $y=0$, at the lowest point, we have

$$T_1=w\times 0+C, \quad i.e., \quad C=T_1=2aw.$$

Thus equation (3) becomes $T=wy+2aw$

At one of the ends of the chain, where $y=a$, we have

$$T=aw+2aw.$$

Hence the result.

It can be done independently as follows :—

Take the resultant tensions as δT and $T\delta\theta$ along the tangent and normal (see Note 2, Art. 10.51), a the radius, and $a\delta\theta$ an element of the arc of the circle.

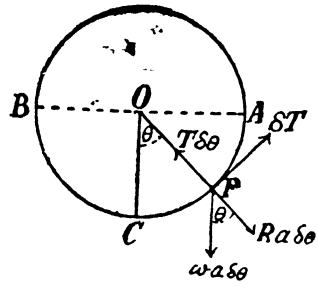
Resolving along the normal, we have

$$T\delta\theta=Ra\delta\theta+wa\delta\theta\cos\theta$$

or
$$T=aR+aw\cos\theta.$$

At the lowest point $\theta=0$ and by the question $R=w$,

$\therefore$ tension $T_1=aw+aw=2aw$.



Now resolving along the tangent, we get

$$\delta T=wa\delta\theta\sin\theta,$$

$\therefore$
$$T=C-wa\cos\theta,$$

At the lowest point

$$\theta=0, \quad T=T_1=2aw,$$

$\therefore$
$$C=2aw+aw=3aw.$$

$\therefore$ the tension T at any point is given by

$$T=3aw-wa\cos\theta.$$

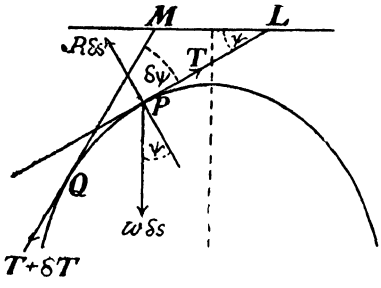
At one of the ends A of the chain $\theta=\frac{1}{2}\pi$,

and $\therefore$
$$T=3aw=2aw+aw.$$

Hence the result.

Example 4. *A heavy uniform chain rests symmetrically on a smooth catenary whose axis is vertical and vertex upwards ; find the tension at any point and show that the pressure on the curve varies inversely as the square of the distance below the directrix.*

The axis of y being downwards, there is a change of sign in the weight and equations (3) and (4) of Art. 10.53, become in this case



$$T = C - wy,$$

and $$R = T/\rho + w \cos \psi.$$

[But we obtain these results independently.

Resolving the forces along the tangent and normal at P, we have

$$(T + \delta T) \cos \delta\psi = T - w\delta s \sin \psi \quad ...(1)$$

and

$$(T + \delta T) \sin \delta\psi = R\delta s - w\delta s \cos \psi \quad ...(2)$$

From (1), we get

$$\delta T = -w\delta s \sin \psi = -w\delta s \frac{dy}{ds} = -w dy,$$

$$\therefore \quad T = C - wy \quad ...(3)$$

and from (2), $$T\delta\psi = R\delta s - w\delta s \cos \psi,$$

$$\therefore \quad R = T\frac{d\psi}{ds} + w \cos \psi = \frac{T}{\rho} + w \cos \psi \quad ...(4)]$$

Now in the case of a catenary, we have

$$s = c \tan \psi, \quad y = c \sec \psi \text{ and } \rho = ds/d\psi = c \sec^2 \psi.$$

At the lowest point of the chain where $T = 0$, let $\psi = \psi_0$.

$$\therefore \quad T = C - wy = C - wc \sec \psi \text{ and } 0 = C - wc \sec \psi_0.$$

$$\therefore \quad T = wc(\sec \psi_0 - \sec \psi),$$

which gives the tension at any point.

Also $$R = \frac{T}{\rho} + w \cos \psi = \frac{wc(\sec \psi_0 - \sec \psi)}{c \sec^2 \psi} + w \cos \psi$$

$$= \frac{w \sec \psi_0}{\sec^2 \psi} = \frac{wc^2 \sec \psi_0}{c^2 \sec^2 \psi} = \frac{wc^2 \sec \psi_0}{y^2}.$$

10.54. Case IV. Heavy *string resting in limiting equilibrium on a* **rough** *curve in a vertical plane.*

In this case besides the weight $w\delta s$, we have also to take into account the force of friction $\mu R\delta s$.

In place of equations (1) and (2) of the previous article, [(2) remaining unchanged], we now have

$$(T + \delta T) \cos \delta\psi = T + \mu R\delta s + w\delta s \sin \psi, \quad ...(1)$$

and $$(T + \delta T) \sin \delta\psi = R\delta s + w\delta s \cos \psi \quad ...(2)$$

Neglecting small quantities of higher order than the first, we have

$$\delta T = \mu R\delta s + w\delta s \sin \psi \text{ and } T\delta\psi = R\delta s + w\delta s \cos \psi.$$

Multiplying the second of these equations by μ and subtracting it from the first, we have

$$\delta T - \mu T \delta\psi = w\delta s(\sin\psi - \mu\cos\psi).$$

Now, dividing by $\delta\psi$, and putting $ds/d\psi = \rho$, we get

$$\frac{dT}{d\psi} - \mu T = w\rho(\sin\psi - \mu\cos\psi) \qquad \ldots(3)$$

Multiplying by $e^{-\mu\psi}$ and integrating, we get

$$Te^{-\mu\psi} = C + w\int e^{-\mu\psi}\rho(\sin\psi - \mu\cos\psi)d\psi \qquad \ldots(4)$$

For the known curve on which the string rests, we can obtain ρ in terms of ψ and can find T by calculating the value of the integral. In the case of a circle of radius r, $\rho = r$, a constant.

Example 1. *A heavy uniform chain resting on a rough circular arc in a vertical plane, occupies a quadrant with one end at the highest point A, and is on the point of slipping. Prove that the co-efficient of friction μ satisfies the equation.*

$$(1-\mu^2)e^{\frac{1}{2}\pi\mu} = 2\mu.$$

Let AX, the tangent at A be the axis of x and let the axis of y be downwards. Let the tangent at any point P make an angle θ with AX, then changing the sign of w in (4), and putting $\psi = \theta$ and $\rho = a$ = radius of the circle, we have

$$Te^{-\mu\theta} = C - wa\int e^{-\mu\theta}(\sin\theta - \mu\cos\theta)d\theta \qquad \ldots(1)$$

[But we obtain this result independently as follows :—

Taking an element PQ of the arc of the circle as $a\delta\theta$, and resolving forces along tangent and normal at P, we have

$$(T+\delta T)\cos\delta\theta = T + \mu R a\delta\theta - wa\delta\theta\sin\theta \qquad \ldots(i)$$

and

$$(T+\delta T)\sin\delta\theta = Ra\delta\theta - wa\delta\theta\cos\theta \qquad \ldots(ii)$$

From (i), we get

$$\delta T = \mu R a\delta\theta - wa\delta\theta\sin\theta$$

or

$$\frac{dT}{d\theta} = \mu R a - wa\sin\theta,$$

and from (ii),

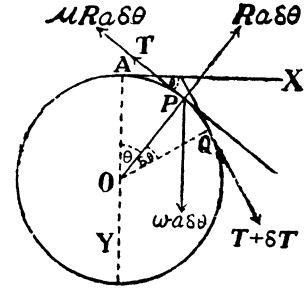
$$T\delta\theta = Ra\delta\theta - wa\delta\theta\cos\theta$$

or

$$T = Ra - wa\cos\theta,$$

$$\therefore \frac{dT}{d\theta} - \mu T = -wa(\sin\theta - \mu\cos\theta),$$

which on integration gives equation (1), written above.]



Now from (1), we have

$$T = C - wa\left[\frac{e^{-\mu\theta}}{1+\mu^2}(\cos\theta+\mu\sin\theta) - \frac{\mu e^{-\mu\theta}}{1+\mu^2}(\sin\theta - \mu\cos\theta)\right]$$

$$= C + wa\,\frac{e^{-\mu\theta}}{1+\mu^2}[\cos\theta+2\mu\sin\theta-\mu^2\cos\theta] \qquad \ldots(2)$$

Now when $\theta=0$, $T=0$ $\therefore$ $0=C+\dfrac{wa}{1+\mu^2}[1-\mu^2]$,

and when $\theta=\dfrac{\pi}{2}$, $T=0$ $\therefore$ $0=C+\dfrac{wa}{1+\mu^2}\,e^{-\frac{1}{2}\mu\pi}[2\mu]$.

Eliminating C, and simplifying, we get

$$(1-\mu^2)e^{\frac{1}{2}\pi\mu}=2\mu.$$

Putting $\mu=\tan\lambda$ and taking logarithms, we get

$\log(1-\tan^2\lambda)+\frac{1}{2}\pi\tan\lambda=\log(2\tan\lambda)$,

or $\frac{1}{2}\pi\tan\lambda=\log[2\tan\lambda/(1-\tan^2\lambda)]=\log\tan 2\lambda$,

a form in which the above result is often expressed.

Example 2. *A heavy uniform string, of length c, is placed on a rough catenary of parameter c with one end at the vertex. Prove that for limiting equilibrium* $e^{\pi\mu}=4$.

In the figure of Ex. 4 of Art. 10.53, introduce the force of limiting friction $\mu R\delta s$ along the tangent PL. Equations (1) and (2) now take the form

$$(T+\delta T)\cos\delta\psi = T - w\delta s\sin\psi + \mu R\delta s$$

or $$\delta T = \mu R\delta s - w\delta s\sin\psi \qquad \ldots(1)$$

and $$(T+\delta T)\sin\delta\psi = R\delta s - w\delta s\cos\psi$$

or $$T\delta\psi = R\delta s - w\delta s\cos\psi \qquad \ldots(2)$$

$\therefore$ $$\delta T - \mu T\delta\psi = w\delta s(\mu\cos\psi-\sin\psi)$$

or $$\frac{dT}{\delta\psi} - \mu T = w\rho(\mu\cos\psi-\sin\psi).$$

But for a catanery, $s=c\tan\psi$ and $\rho=\dfrac{ds}{d\psi}=c\sec^2\psi$,

and when $s=c$, $\tan\psi=1$, so that $\psi=\frac{1}{4}\pi$.

$\therefore$ $$\frac{dT}{\delta\psi} - \mu T = wc\sec^2\psi(\mu\cos\psi-\sin\psi)$$

$$= wc(\mu\sec\psi-\sec\psi\tan\psi).$$

$\therefore$ $$Te^{-\mu\psi} = wc\int e^{-\mu\psi}(\mu\sec\psi-\sec\psi\tan\psi)d\psi$$

$$= C - wc\,e^{-\mu\psi}\sec\psi \qquad \ldots(2)$$

Now $T=0$ when $\psi=0$ and $\psi=\frac{1}{4}\pi$, at the ends of the string.

$\therefore$ $0=C-wc$, and $0=C-wc\,e^{-\frac{1}{4}\pi\mu}\sqrt{2}$.

$\therefore$ $C=wc=wc\sqrt{2}\,e^{-\frac{1}{4}\pi\mu}$ or $e^{\frac{1}{4}\pi\mu}=\sqrt{2}$.

Hence $$e^{\mu\pi}=(\sqrt{2})^4=4.$$

10.6. *Equatins of equilibrium of a string in one plane under the action of any given forces.*

As in previous articles, we consider the equilibrium of an element PQ of the string taking tensions T and $T+\delta T$ at its ends and resolving the forces along the tangent and normal at P.

If M per unit length be the sum of the components of the external forces along the tangent at P in the sense PQ and N, the component along the outward normal, the equations of equilibrium are

$$(T+\delta T)\cos\delta\psi+M\delta s=T \quad ...(1)$$

and

$$(T+\delta T)\sin\delta\psi=N\delta s \quad ...(2)$$

Neglecting small quantities of higher order than the first, we have

$$\delta T+M\delta s=0 \quad \text{and} \quad T\delta\psi=N\delta s \quad ...(3)$$

These can be written as

$$\frac{dT}{ds}+M=0 \quad ...(4), \quad \text{and} \quad \frac{T}{\rho}-N=0 \quad ...(5)$$

These are the general equations and all the four previous articles can be deduced from them as particular cases.

Note. Comparing equations (1) and (2) with (3), it may be seen that as far as tensions at the ends of the element are concerned, their resultant reduces to δT in the ψ increasing direction of the tangent, and $T\delta\psi$ in the direction of the outward normal at P. In examples that follow, these expressions may be used and equations written down independently without using the results of the previous articles as formulae.

$T\delta\psi$ may be written as $T\frac{\delta\psi}{\delta s}\delta s=T\frac{d\psi}{ds}\delta s=\frac{T}{\rho}\delta s.$

10.61. *Equilibrium of a string in the form of a plane curve under a* **Central Force.**

Let O, the centre of force be taken as the origin (or the pole), and let F be the force at P per unit of length acting in the direction OP ($=r$).

F is supposed to be some function of r, say $f(r)$, (*e.g.*, μr, μr^2, μ/r^2 etc.). On an element δs of the string, the force is $F\,\delta s$.

Now if F makes an angle ϕ with tangent at P, then by calculus, with usual notation, we have

$$\cos\phi=\frac{dr}{ds}, \sin\phi=\frac{p}{r}, \text{ and } \rho=r\frac{dr}{dp}.$$

From equation (4) of Art. 10.6 above, we have

$$\frac{dT}{ds}=-M=-F\cos\phi=-F\frac{dr}{ds}, \text{ or } \frac{dT}{dr}=-F \quad ...(1),$$

which gives $$T = -\int F\,dr + C \qquad \ldots(2)$$

From (5), we have $T = \rho.N = \rho.F \sin\phi = F.r\frac{dr}{dp} \cdot \frac{p}{r} = Fp\frac{dr}{dp}$ $\ldots(3)$

Dividing (1) by (3), we get $\frac{dT}{T} = -\frac{dp}{p}$

$\therefore$ $\log T + \log p = \text{constant}$, or $Tp = B$, a constant $\ldots(4)$

This shows that the moment of the tension about O, is the same for all positions of the point P.

Example 1. *Find the law of force when a string rests in the form of a parabola under a central force F from the focus.*

The pedal equation of the parabola with respect to focus as the pole is $p = \sqrt{ar}$.

$\therefore$ from (4) $T = \frac{B}{p} = \frac{B}{\sqrt{ar}} = \frac{B}{\sqrt{a}}\, r^{-\frac{1}{2}}$

$\therefore$ from (1) $F = -\frac{dT}{dr} = \frac{1}{2}\frac{B}{\sqrt{a}}\, r^{-\frac{3}{2}}$, so that F varies as $r^{-\frac{3}{2}}$.

Example 2. *Find the equation of the plane curve in which a string will rest when acted upon by a force $\mu r^{-5/2}$ from the pole.*

From (2), $T = -\int \mu r^{-5/2}\, dr + C = \frac{2}{3}\mu r^{-3/2} + C = \frac{2}{3}\mu r^{-3/2}$,

by taking $T = 0$ when $r \to \infty$, so that $C = 0$,

Now from (4), $\frac{2}{3}\mu r^{-3/2} \times p = B$, or $p = Ar^{3/2}$ or $p^2 = kr^3$, which is the pedal equation of a cardioid, A and k being constants.

Example. A string rests in the form of a plane curve under a central repulsive force. If the force at any point is proportional to the curvature, prove that the curve is a parabola. *(D.U.1958)*

Examples XL

1. (*a*) Masses of 50 lbs. and 20 lbs. are connected by a light string which passes over a rough cylindrical beam and the system is in limiting equilibrium. What mass must be added to the 50 lb. in order that the other mass may be on the point of moving upwards. [*Ans.* 78 lb.]

(*b*) Two scale pans, one containing a weight P and the other a weight W, are suspended by a light string passing over a rough horizontal circular cylinder, and W is on the point of descending. Show that the weight which must be added to P to make W ascend upwards is

$$(W^2 - P^2)/P.$$

2. A weightless string lies stretched in one plane across a rough sphere of radius a. Show that the distance of the plane from the centre cannot exceed $a \sin\lambda$, where λ is the angle of friction. *(D.U. 1959)*

3. Two rough cylindrical beams have their axes horizontal on the same horizontal level and parallel at a distance d apart. The radii are a, b and the co-efficients of friction μ, μ_1. Weights W and W_1 are connected by a light

nelastic string which passes over both the beams. Show that the system will emain in equilibrium provided that

$$\log_e W/W_1 \text{ lies between } \pm\left[\pi\mu-(\mu-\mu_1)\cos^{-1}\left(\frac{a-b}{d}\right)\right].$$

4. A single movable pulley, of weight W, is just supported by a force F vhich is applied to one end of a light cord which goes under the pulley and is :hen fastened to a fixed point. show that if θ be the angle subtended at the 'entre by the part of the string in contact with the pulley, then

$$F^2((1-2e^{\mu\theta}\cos\theta+e^{2\mu\theta})=W^2.$$

μ being the co efficient of friction between the cord and the pulley.

Deduce the case when the pulley is smooth.

5. A heavy particle is attached to an endless light inextensible string which passes over a rough pulley fixed in a vertical plane. If the straight parts of the string are inclined to each other at an angle α, prove that the angle θ which one of the parts makes with the vertical in the limiting position of equilibrium is given by

$$\cot\theta=\cot\alpha+e^{\mp\mu(\pi+\alpha)}\operatorname{cosec}\alpha.$$

6. A light string is passed over two rough pegs A and B in the same horizontal line at a distance $2a$ apart. The ends are fastened to a weight C, and in a position of limiting equilibrium AB subtends a right angle at C. Show that the horizontal distance of C, in this position, from the middle point of AB is

$$a\tanh(3/2\,\pi\mu),$$

where μ is the angle of friction.

7. A uniform heavy chain passes round a smooth vertical circle. If the pressure vanishes at the lowest point, prove that the tension (*i*) at the highest point is three times and (*ii*) at the ends of the horizontal diameter twice of what it is at the lowest point.

8. A uniform heavy string of weight w per unit length, rests on a smooth parabola, whose axis is vertical and vertex upwards, so that its ends are at the extremities of the latus rectum. Show that the pressure on the curve, at the point where the tangent makes an angle θ with the vertical is

$$\tfrac{1}{2}w(2\sin^3\theta+\sin\theta).$$

9. A uniform string rests on a smooth cycloidal curve, $s=4a\sin\psi$, whose axis is vertical and vertex upwards, the ends of the string just reaching the cusps of the cycloid. Prove that the pressure at any point varies as the radius of curvature.

10. A uniform string of weight w per unit length passes under the arc of a smooth cycloid, with vertex downwards and axis vertical, and over two smooth pegs at the cusps. If the string is to remain in contact at every point of the cycloid, show that its length must not be less than $20a$, a being the radius of the generating circle.

11. A heavy string occupies a quadrant of the upper half of a rough vertical circle in a state bordering on motion. Prove that the angle α which the radius through the lower end makes with the vertical is given by

$$e^{\frac{1}{2}\pi\mu}=\cot(\alpha-2\lambda),$$

where $\mu=\tan\lambda$ is the co-efficient of friction.

12. A heavy string resting on a rough vertical circle, subtends an angle β at the centre and is on the point of slipping off, prove that the angular distance α of its upper end from the highest point of the circle is given by

$$e^{\beta\tan\lambda}\cos(\alpha-2\lambda)=\cos(\alpha+\beta-2\lambda),$$

where λ is the angle of friction and α is measured in the direction towards which the string is slipping.

13. A uniform heavy chain rests on the upper surface of a rough vertical circle of radius a and partly hangs freely. If one end be at the highest point of the circle, prove that the greatest length that can hang freely is

$$[2\mu a+(\mu^2-1)ae^{\frac{1}{2}\mu\pi}]/(1+\mu^2).$$

14. A uniform inextensible heavy string, of length l, hangs in limiting equilibrium over a rough cylinder, of radius a, whose axis is horizontal. Show that the length of the greater of the two vertical portions is

$$\frac{l-\pi a}{1+e^{-\pi\mu}}+\frac{2\mu a}{1+\mu^2}.$$

15. A uniform string of weight w per unit length lies on the upper half of a horizontal circular cylinder of radius a cutting the generators of the cylinder at right angles. Show that, if $\mu=\tan\varphi$ be the co-efficient of friction, the downward force at one of the ends which make the string slip is

$$aw(1+e^{\mu\pi})\sin 2\varphi.$$

16. A heavy uniform chain rests on the rough cycloid $s=4a\sin\psi$ in a vertical plane, vertex upwards and axis vertical. One end of the chain is at the vertex and the other at a cusp. Show that,

(1) if the equilibrium be limiting, the co-efficient of friction μ is given by

$$(1+\mu^2)e^{\frac{1}{2}\mu\pi}=3.$$

(2) the least horizontal force applied at the upper end that will cause the string to slip upwards is

$$W(3e^{\frac{1}{2}\mu\pi}-\mu^2-1)/(\mu^2+4).$$

where W is the weight of the string.

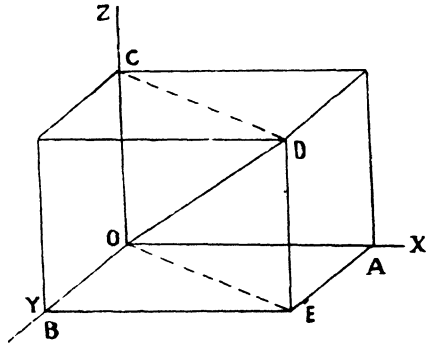
CHAPTER XI

FORCES IN THREE DIMENSIONS

11.1. Parallelopiped of Forces. The law of parallelogram of forces in two dimensions Art. 2·11 leads to the law of *Parallelopiped of Forces* in three dimensions which may be stated as follows :—

If three forces acting at a point O are represented, in magnitude and direction, by straight lines OA, OB, OC, their resultant is represented, in magnitude and direction, by the diagonal OD of the parallelopiped whose edges are OA, OB and OC.

For, the resultant of two forces represented by OA and OB is represented by OE, the diagonal of the parallelogram OAEB ; again the resultant of forces represented by OE and OC is represented by OD the diagonal of the parallelogram OCDE.



In the particular case when the parallelopiped is rectangular so that OA, OB, OC are taken along rectangular axes of co-ordinates and the forces OA, OB, OC are respectively denoted by X, Y, Z and their resultant OD by R, then

$$R^2=X^2+Y^2+Z^2,$$

and the direction cosines l, m, n, of OD are given by

$$l=\frac{OA}{OD}=\frac{X}{R},\ m=\frac{OB}{OD}=\frac{Y}{R} \text{ and } n=\frac{OC}{OD}=\frac{Z}{R},$$

so that $$X=Rl,\ Y=Rm \text{ and } Z=Rn.$$

11.2. Specification of a couple. In Art. 5.5, it was shown that couples in the same plane or in parallel planes can be compounded into a single couple whose moment is equal to the sum of the moments of the various couples and whose axis is a line perpendicular to the planes of the couples.

We can, therefore, represent a couple by a straight line, in space, (say, AB) of a certain length in a direction parallel to the axis of the couple, then

(*i*) the length of the line indicates the magnitude of the moment of the couple, on a certain scale,

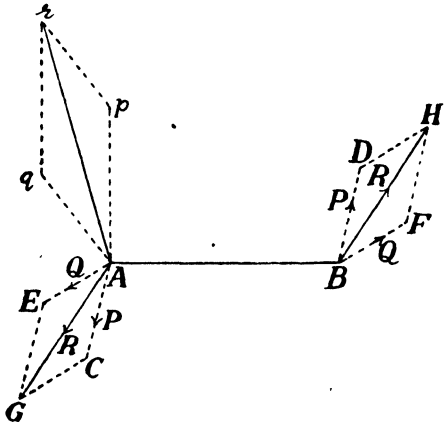
(*ii*) the direction of the line indicates the plane of the forces, the plane being perpendicular to the line, and

(*iii*) the order of the letters AB or BA, indicates the sense of rotation of the couple.

11.21. We now prove that a couple is a vector quantity, *i.e.*, *couples in different planes can be compounded by the application of the parallelogram law of addition to their axes.*

Consider two couples in the planes p_1 and p_2 and take a length AB on the line of intersection of the planes.

Replace each couple in its own plane by suitable forces so as to have AB as their common arm.



With this arm let the forces of the first couple be each equal to P represented by equal and parallel lines AC and BD in the plane p_1, and those of the second couple be each equal to Q, represented by equal and parallel lines AE and BF in the plane p_2.

Complete the parallelograms ACGE and BDHF.

Now the forces P and Q at A compound into a force, say R, represented by AG ; and the forces P and Q at B compound into an equal, parallel and opposite force R represented by BH.

The forces each equal to R represented by AG and BH form a couple, in another plane, say p_3.

Hence the resultant of two given couples is a couple.

Now, draw Ap, Aq perpendicular to the planes p_1 and p_2 to represent the axes of the given couples, so that they represent, on same scale, G_1, G_2, the moments of the given couples. Then

$$\frac{Ap}{Aq} = \frac{G_1}{G_2} = \frac{P.AB}{Q.AB} = \frac{P}{Q} = \frac{AC}{AE}.$$

Therefore Ap and Aq are perpendicular and proportional to AC and AE respectively.

Hence if we complete the parallelogram Aprq, Ar is perpendicular and proportional to AG, so that

$$\frac{Ap}{AC} = \frac{Aq}{AE} = \frac{Ar}{AG} \quad \text{or} \quad \frac{Ap}{P.AB} = \frac{Aq}{Q.AB} = \frac{Ar}{R.AB}.$$

Hence Ar represents the moment of the resultant couple on the same scale on which Ap and Aq represent the moments G_1 and G_2 of the given couples.

Hence Ar is the axis of the resultant couple.

Thus the resultant of two given couples is a couple whose axis is obtained by compounding the axes of the given couples according to the parallelogram law.

It follows, therefore, that when couples are represented by their axes, they may be compounded and resolved by the parallelogram law in the same way as the forces.

Thus if L, M, N be the moments of the couples about rectangular axes OX, OY, OZ, their resultant G is given by

$$G^2 = L^2 + M^2 + N^2,$$

the direction cosines of the axis of G being L/G, M/G and N/G.

11·22. Convention of sign. If OX, OY, OZ be a set of three mutualy perpendicular axes, a couple in the plane XOY (or in a plane parallel to it) will be represented by a line of certain length taken along OZ (or along a line parallel to OZ).

If the couple causes rotation from OX towards OY, the moment of the couple is regarded as positive and it is then represented by a positive length along OZ. Similarly a couple in the plane YOZ is considered positive if it causes rotation from OY towards OZ and a couple in the plane ZOX is positive when rotation is from OZ towards OX.

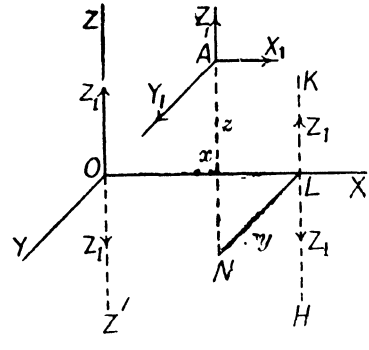
11.3. *To prove that a system of non-coplanar forces acting on a rigid body can be reduced, in general, to a force acting through a specified point of the body and a couple.*

Let O, the specified point, be taken as the origin and let OX, OY, OZ be a set of rectangular axes through O.

Let one of the forces P of the system act at A (x, y, z) and let the components of P parallel to the axes be X_1, Y_1, Z_1.

Draw AN perp. on the XOY plane and NL perp. to OX so that $OL = x$, $NL = y$ and $NA = z$.

At L and at O introduce equal and opposite forces equal and parallel to Z_1. These being in equilibrium among themselves, do not alter the effect of the system.



Now the forces Z_1 along NA and LH form a couple of moment $+yZ_1$ in the sense from OY to OZ.

Also the forces Z_1 along LK and OZ′ form a couple in the sense from OX to OZ, its moment is, therefore, $-xZ_1$.

We have also a force Z_1 along OZ.

Hence the force Z_1 at A is equivalent to

(*i*) an equal and parallel force Z_1 along OZ,

(*ii*) a couple of moment $+yZ_1$ about OX,

(*iii*) a couple of moment $-xZ_1$ about OY.

Similarly the force X_1 at A is equivalent to :

force X_1 along OX, couple zX_1 about OY and $-yX_1$ about OZ.

Also Y_1 at A is equivalent to force Y_1 along OY, couple $-zY_1$ about OX and xY_1 about OZ.

Hence the force P acting at A is equivalent to forces X_1, Y_1, Z_1, along OX, OY, OZ respectively, and couples (yZ_1-zY_1), (zX_1-xZ_1), (xY_1-yX_1) about OX, OY, OZ respectively.

All other forces of the system can be treated in the same way, so that the system is equivalent to,

forces $\Sigma X_1=X$, $\Sigma Y_1=Y$, $\Sigma Z_1=Z$,

and couples $\Sigma(yZ_1-zY_1)=L$, $\Sigma(zX_1-xZ_1)=M$, $\Sigma(xY_1-yX_1)=N$ along and about OX, OY, OZ, respectively.

The three forces reduce to a single force R, given by

$$R^2=X^2+Y^2+Z^2$$

acting along a line through O whose direction cosines are X/R, Y/R, Z/R.

The three couple reduce to a single couple of moment G given by

$$G^2=L^2+M^2+N^2$$

having its axis along a line whose direction cosines are L/G, M/G, N/G.

Hence the system reduces to a single force acting through a specified point O and a couple whose axis passes through O.

11.4. Conditions of equilibrium. When $R=0$ *i.e.*, $X=0$, $Y=0$. $Z=0$ and G is not zero, the resultant of the system is a couple of moment G, having its axis passing through O.

When $G=0$ *i.e.*, $L=0$, $M=0$, $N=0$ and R is not zero, the resultant is a force passing through O.

When $R=0$ as well as $G=0$ *i.e.*, X, Y, Z, L, M, N all vanish separately, there shall be neither a resultant force nor a resultant couple and the body shall be in equilibrium.

Hence the conditions of equilibrium are :—

(*i*) *the sums of the resolved parts of the system of forces in any three directions, not all parallel to the same plane, are zero.*

(*ii*) *the sums of the moments of the forces about any three concurrent and non-coplanar straight lines separately vanish.*

Note. In order that the resultant may be a force, it is not necessary that G must be zero ; for the couple G and the force R may combine into a single force. (Art. 11.7).

Example 1. *Six forces, each equal to P, act along the edges of a cube which meet a given diagonal, the forces radiating from the vertices. Show that the system is in equilibrium.*

Let OO′ be the diagonal, and the forces acting at its ends O, O′, and the co-ordinate axes be as shown in the figure.

Here X = P along OA − P along O′A′ = 0.

Similarly Y = 0, Z = 0.

L = sum of the moments of the forces about OX

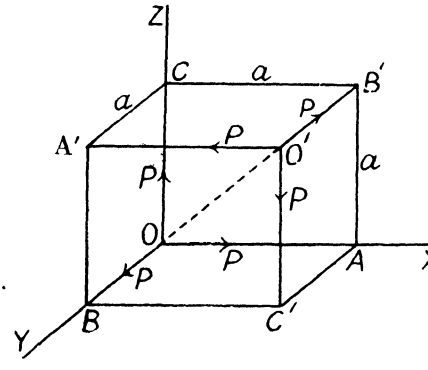
= P along O′B′ × AB′ − P along O′C′ × AC′

= P × a − P × a = 0,

a being the length of an edge.

Similarly M = 0, N = 0.

Hence the system of forces is in equilibrium.



Example 2. In Ex. 1, if the forces P act in order along the edges which do not meet the diagonal OO′, show that the resultant is a couple of moment $2\sqrt{3}\,Pa$.

Example 3. If two forces, each equal to P, act along two perpendicular diagonals of opposite faces of a cube, show that they are equivalent to a single force $\sqrt{2}$ P along a line through the centre of the cube and a couple of moment $\frac{a}{\sqrt{2}}$ P with the same line for axis, a being the length of a side.

Example 4. If the centre of the cube were fixed and the two forces in Ex. 3, act along diagonals of adjacent faces which do not meet, show that the moment of the couple which will keep the cube at rest is either $\frac{1}{2}\sqrt{3}\,Pa$ or $\frac{1}{2}aP$ according to the direction of the forces.

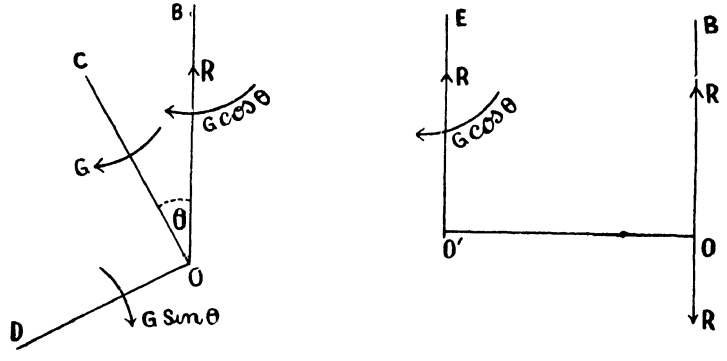
Example 5. In the Fig. of Ex. 1, forces equal to P, 2P, 3P, 4P act along OB′, O′A, BC and C′A′ respectively. Show that they are equivalent to a force $\sqrt{35}$ P at O along a line whose direction cosines are proportional to −3, −5, 6, together with a couple $\frac{1}{2}\sqrt{114}\,Pa$ about a line whose direction cosines are proportional to 7, −2, 2.

Example 6. (*a*) In the Fig. of Ex. 1, equal forces each of magnitude P, act along BC′, AB′, CA′. Show that they can be reduced to a force $\sqrt{3}$P along OO′ and a couple of moment $\sqrt{3}aP$ with the same line as its axis.

(*b*) Show also that when these forces are reduced to the point B, the couple is of moment $\sqrt{5}\,a\,P$ with its axis parallel to the line joining O to the middle point of AC′, the direction of the axes remaining unchanged.

11.5. The wrench. *To show that a system of forces can be reduced to a force R and a couple whose axis is the line of action of R.*

It has been shown in Art. 11.3 that any system of forces is equivalent to a force R acting at an arbitrarily chosen point O along a line, say OB (taken in the plane of the paper) and a couple of moment G whose axis passes through O.



Let OC be the axis of the couple G, making an angle θ with OB.

Now the couple G about OC as axis is equivalent to a couple G cos θ about OB as axis and G sin θ about OD as axis, OD being perpendicular to OB and perpendicular to the plane of the paper OB, OC, OD being co-planar, lying in a plane perpendicular to the plane of the paper.

The axis of the couple G sin θ being perpendicular to the plane of the paper, the forces of the couple act in the plane in which R also acts. This couple is now replaced by two unlike equal forces, each equal to R, one acting along BO produced and the other acting along O′E, OO′ being perpendicular to OB such that R.OO′=G sin θ, *i.e.*, OO′=G sin θ/R.

The two equal and opposite forces R at O balance and we are left with a force R along O′E and a couple of moment G cos θ with axis OB.

Shifting the axis of the couple from OB to the parallel axis O′E, the system is reduced to a single force R along O′E and a couple of moment G cos θ about O′E as axis, so that the axis of the couple coincides with the line of action of R.

Such a combination of a force and a couple is called a **wrench**.

O′E, the line of action of the force, is called **Poinsot's Central axis**.

Taking G cos θ=K, the ratio $\frac{\text{G cos } \theta}{\text{R}} = \frac{\text{K}}{\text{R}}$ is known as the **pitch** of the wrench.

The magnitude of the force, *i.e.* R, is called the **intensity** of the wrench.

The line of action of the wrench together with the pitch is called a **screw.**

Note.—The following points about the central axis may be noted :—

(*i*) It is parallel to the resultant force R at O and is at a distance G sin θ/R from it.

(*ii*) The perpendicular from O to the central axis is at right angles to both R, and the axis of G.

(*iii*) The perpendicular from O must be so drawn that its foot O′ is moved by the couple G sin θ in the same direction as that in which R acts.

11·6. The angle θ between the directions of R whose direction-cosines are $\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}$ and the axis of G, whose direction-cosines are $\frac{L}{G}, \frac{M}{G}, \frac{N}{G}$, is given by $\cos\theta = \frac{LX+MY+NZ}{RG}$ ...(1)

$$\therefore \quad K = G\cos\theta = \frac{LX+MY+NZ}{R} \qquad ...(2)$$

and the pitch $p = \frac{K}{R} = \frac{G\cos\theta}{R} = \frac{LX+MY+NZ}{R^2} = \frac{LX+MY+NZ}{X^2+Y^2+Z^2}$ ...(3)

11.7. Condition for the resultant to be a force. We have seen that the system of forces is reduced to force R at O′ acting along O′E and a couple of moment G cos θ with the same line O′E as its axis. If now G cos $\theta = 0$ *i.e.*, cos $\theta = 0$ ($G \neq 0$) and R is not zero, the resultant of the system is a single force R acting at O′.

By (1) Art. 11.6, cos $\theta = 0$ if

$$LX+MY+NZ=0$$

which is the required condition, provided R is not zero.

Note.—In this case L, M, N may not separately vanish. If they do *i.e.*, if L=0, M=0, N=0, then G=0 and the single resultant force acts at O and not at O′.

Example. *In the figure of Ex. 1, Art. 11.4, forces each equal to P act along OA, AC′, C′B and BO. Show that these four forces when compounded with any fifth force acting along CB′ will reduce to a single resultant.*

Let the fifth force along CB′ be Q, then in this case

$$X=Q, \quad Y=0, \quad Z=0,$$
$$L=0. \quad M=aQ, \quad N=aP+aP=2aP$$

$$\therefore \quad XL+YM+ZN=0$$

Hence the resultant of the system is a single force.

Note. It may be noted that G is not equal to zero in this case.

Ex. Find the condition that a system of forces in three dimensions may reduce to a single resultant force. Deduce that a system of co-planar forces reduces to a couple or a single force. (*D.U. 1959*)

11.8. Equations of the central axis. Let $P(\alpha, \beta, \gamma)$ be the co-ordinates of a point on the central axis referred to co-ordinate axes through O. The system reduces to a force R at this point and the couple of moment $K(=G\cos\theta)$. If L', M', N' be the components of the couple, about lines through P parallel to the axes through O, then by substituting $x-\alpha$, $y-\beta$, $z-\gamma$ for x, y, z in L, M, N of Art. 11.3, we have

$$L'=\Sigma[(y-\beta)Z_1-(z-\gamma)Y_1]=\Sigma(yZ_1-zY_1)-\beta\Sigma Z_1+\gamma\Sigma Y_1,$$

i.e., $$L'=L-\beta Z+\gamma Y \quad ...(1)$$

Similarly $$M'=M-\gamma X+\alpha Z \quad ...(2)$$

and $$N'=N-\alpha Y+\beta X \quad ...(3)$$

Also the components of the resultant force R along parallel axes through *any* point such as P remain the same and they, therefore, are X, Y, Z at P.

Now the direction cosines of R are X/R, Y/R, Z/R and those of the axis of the couple K are L'/K, M'/K, N'/K and as the line of action of the force is the same as the axis of the couple, we have

$$\frac{X}{R}=\frac{L'}{K} \text{ or } \frac{L'}{X}=\frac{K}{R}, \text{ similarly } \frac{M'}{Y}=\frac{N'}{Z}=\frac{K}{R}.$$

$$\therefore \quad \frac{L'}{X}=\frac{M'}{Y}=\frac{N'}{Z}=\frac{K}{R}=\text{the pitch } p.$$

Substituting the values of L', M', N' and changing α, β, γ to x, y, z, we get the equations of the central axis in the form,

$$\frac{L-yZ+zY}{X}=\frac{M-zX+xZ}{Y}=\frac{N-xY+yX}{Z}=\frac{K}{R}=\frac{LX+MY+NZ}{R^2} \quad ...(4)$$

Note. Equations (1), (2), (3) indicate the important fact that the couple components of a system of forces always change with the change of the base point.

11.81. Invariants. R, being the resultant of given forces moved parallel to themselves to act at O, is clearly independent of the position of the point O. This is also obvious from Art. 11·5.

Hence $$R^2=X^2+Y^2+Z^2$$

is said to be an invariant.

Also multiplying equations (1), (2) and (3) above, by X, Y, Z respectively, and adding, we get

$$L'X+M'Y+N'Z=LX+MY+NZ,$$

other terms cancelling in pairs.

This shows that

$$LX+MY+NZ \text{ is an invariant.}$$

This is denoted by the letter I and by Art. 11.6,

$$I=LX+MY+NZ=RG\cos\theta=RK=R^2p,$$

$K=G\cos\theta$, being the moment of the couple.

11·82. Central axis is unique. If possible, let the given system of forces be equivalent to a force along and a couple about one line O_1E_1, and also to a force along and a couple about another line O_2E_2.

By Art. 11.81, the resultant force must be the same in magnitude and direction whatever point is taken as the origin.

Hence O_1E_1 and O_2E_2 must be parallel, each having the same resultant force R.

Thus the system (R, G) with axis O_1E_1 is the same as the system (R, G′) with axis O_2E_2 and hence one reversed must balance the other.

But equal and opposite parallel forces R now form a couple in a plane perpendicular to those of the couples G, G′, so that this couple has to balance a couple in a perpendicular plane, which is impossible.

Hence the central axis found in Art. 11.8 is unique.

Example 1. *Three forces, each equal to P, act on a body at the points (a, o, o), (o, a, o) and (o, o, a) parallel to the rectangular axes OY, OZ and OX respectively. Find the pitch of the central axis and its equations.*

Here $X=P,\quad Y=P,\quad Z=P,$
$L=aP,\quad M=aP,\quad N=aP$*,

$\therefore \quad R^2=X^2+Y^2+Z^2=3P^2$

and $LX+MY+NZ=3aP^2,$

$\therefore$ the pitch $=\dfrac{LX+MY+NZ}{X^2+Y^2+Z^2}=\dfrac{3aP^2}{3P^2}=a.$

By Art. 11.8 the equations of the central axis are

$$\frac{aP-yP+zP}{P}=\frac{aP-zP+xP}{P}=\frac{aP-xP+yP}{P}=\text{pitch}=a$$

i.e., $x=y=z,$

a straight line through the origin equally inclined to the axes.

This is otherwise obvious, the components of the resultant force and those of the couple being equal.

[Compare with Ex. 6 (*a*), Art. 11.4].

Example 2. *Two forces each equal to P act along the diagonals of adjacent faces of a cube, which do not meet. Find the pitch and the position of the wrench.*

Let the forces act along the diagonals OC′ and BC (See Fig. Ex. 1 Art. 11.4), then we have

the points of application	O, o, o, o	B, o, a, o
component of forces	$-\frac{1}{\sqrt{2}}P$, $\frac{1}{\sqrt{2}}P$, o	o, $-\frac{1}{\sqrt{2}}P$, $\frac{1}{\sqrt{2}}P$

*In such simple cases, the student is advised to draw a figure and write down the values of L, M, N, by actually taking moments about the axes.

$$\therefore \quad X=\frac{1}{\sqrt{2}}P, \qquad Y=\frac{1}{\sqrt{2}}P-\frac{1}{\sqrt{2}}P=0, \qquad Z=\frac{1}{\sqrt{2}}P,$$

and $$L=\Sigma(yZ_1-zY_1)=\frac{1}{\sqrt{2}}Pa, \qquad M=\Sigma(zX_1-xZ_1)=0,$$

$$N=\Sigma(xY_1-yX_1)=0^*.$$

$$\therefore \quad R^2=X^2+Y^2+Z^2=\tfrac{1}{2}P^2+\tfrac{1}{2}P^2=P^2$$

$$LX+MY+NZ=\frac{1}{\sqrt{2}}Pa\times\frac{1}{\sqrt{2}}P=\tfrac{1}{2}aP^2.$$

$$\therefore \quad \text{the pitch}=(LX+MY+NZ)/R^2=\tfrac{1}{2}aP^2/P^2=\tfrac{1}{2}a.$$

$\therefore$ by Art. 11·8 equations of the central axis are

$$\frac{aP/\sqrt{2}-yP/\sqrt{2}+z\times 0}{P/\sqrt{2}}=\frac{0-zP/\sqrt{2}+xP/\sqrt{2}}{0}=\frac{0-x\times 0+yP/\sqrt{2}}{P/\sqrt{2}}$$

or $$a-y=\frac{x-z}{0}=y \text{ i.e., } x=z \text{ and } 2y=a, \text{ or } y=\tfrac{1}{2}a.$$

Example 3. *A force P acts along the axis of x, and another force nP along a generator of the cylinder $x^2+y^2=a^2$, show that the central axis lies on the cylinder*

$$n^2(nx-z)^2+(1+n^2)^2\,y^2=n^4a^2.$$

The force P along the axis of x may be taken to act at the origin and the force nP at any point ($a\cos\theta$, $a\sin\theta$, 0) on the given cylinder.

Thus we have,

the points of application	0, 0, 0	$a\cos\theta$, $a\sin\theta$, 0
component of forces	P, 0. 0	0, 0, nP

$\therefore$ $X=P$ $\quad Y=0,$ $\quad Z=nP,$

and $L=\Sigma(yZ-zY)=a\sin\theta.nP$, $M=-a\cos\theta.nP$, $N=0$.

$\therefore$ by Art. 11.8, equations of the central axis are :

$$\frac{a\sin\theta.nP-y.nP}{P}=\frac{-a\cos\theta.nP-zP+x\,nP}{0}=\frac{yP}{nP}.$$

$\therefore$ $y=n^2(a\sin\theta-y)$ or $(1+n^2)y=n^2a\sin\theta$

and $nx-z=an\cos\theta$ or $n(nx-z)=n^2a\cos\theta$.

Hence $n^2(nx-z)^2+(1+n^2)^2y^2=n^4a^2(\cos^2\theta+\sin^2\theta)=n^4a^2$.

Examples XLI-A

1. Show that the following system reduces to a single force :

Force $5\sqrt{2}$ acting along the line $\frac{x}{3}=\frac{y}{4}=\frac{z-\frac{1}{5}}{-5}$

Force $3\sqrt{11}$,, ,, $\frac{x-\frac{1}{5}}{5}=\frac{y}{5}=\frac{z}{-7}$

Force $\sqrt{11}$,, ,, $\frac{x}{-3}=y-\frac{1}{4}=z,$

the co-ordinate axes being rectangular.

* See note on P. 253.

2. Three edges AB, AC, AD, of length a, of a cube are taken as axes of x, y and z. Forces, each equal to P, act along AD and two other edges parallel to AB and AC which intersect neither AD nor one another. Show that the central axis is the line

$$x=y+\frac{a}{3}=z-\frac{a}{3}, \text{ or } x=y-\frac{a}{3}=z-\frac{2a}{3}.$$

3. Three equal forces act on a body; one parallel to OY at (a, o, o); the second parallel to OZ at (o, b, o); and the third parallel to OX at (o, o, c). Show that the pitch of the wrench is $\frac{1}{3}(a+b+c)$ and that its position is given by any two of the equations

$$x+y-2z=b-c,\ y+z-2x=c-a,\ z+x-2y=a-b.$$

4. Two forces act, one along the axis of y and the other along the line $y=0$, $z=c$. As the forces vary, show that the surface generated by the axis of their equivalent wrench is

$$z(x^2+y^2)=cx^2.$$

5. Two forces P and Q act along the straight lines $y=x$, $z=c$; $y=-x$, $z=-c$ respectively. Show that the central axis lies on the straight line

$$\frac{y}{x}=\frac{P-Q}{P+Q} \text{ and } \frac{z}{c}=\frac{P^2-Q^2}{P^2+Q^2}.$$

Show also that as the forces P and Q vary, this line generates the surface

$$z(x^2+y^2)=2c\,xy.$$

6. Forces P, Q, R act along the edges AB, EC, DF of a regular octahedron ABCDEF of which EF is a diagonal. Prove that they will be equivalent to a single resultant, provided

$$\frac{1}{P}+\frac{1}{Q}+\frac{1}{R}=0,$$

and that the resultant is equal to $(P^2+Q^2+R^2)^{\frac{1}{2}}$. *(I.A.S.)*

7. Forces X, Y, Z act along the lines

$$y=b,\ z=-c;\ z=c,\ x=-a;\ \text{and}\ x=a,\ y=-b$$

respectively; show that they will have a single resultant if

$$\frac{a}{X}+\frac{b}{Y}+\frac{c}{Z}=0,$$

and that the equations of its line of action are any two of the three

$$\frac{y}{Y}-\frac{z}{Z}-\frac{a}{X}=0,\quad \frac{z}{Z}-\frac{x}{X}-\frac{b}{Y}=0,\quad \frac{x}{X}-\frac{y}{Y}-\frac{c}{Z}=0.$$

8. Three force act along the straight lines

$$x=0,\ y-z=a;\ y=0,\ z-x=a;\ z=0,\ x-y=a.$$

Show that they cannot reduce to a couple.

Prove also that if the system reduces to a single force its line of action must lie on the surface

$$x^2+y^2+z^2-2yz-2zx-2xy=a^2. \qquad (P.U.)$$

9. A force parallel to the axis of z acts at the point $(a, 0, 0)$ and an equal force parallel to the axis of z acts at the point $(-a, 0, 0)$. Show that the central axis of the system lies on the surface

$$z^2(x^2+y^2)=(x^2+y^2-ax)^2. \qquad (P.U.\ 1948)$$

10. Forces X, Y, Z, act along the lines

$$y=0,\ z=c;\ z=0,\ x=a;\ x=0,\ y=b;$$

prove that the pitch of the equivalent wrench is

$$(aYZ+bZX+cXY)/(X^2+Y^2+Z^2).$$

If the wrench reduces to a single force, show that the line of action of the force must lie on the hyperboloid

$$(x-a)(y-b)(z-c)-xyz=0 \qquad (M.T.)$$

11. Two equal forces act one along each of the straight lines

$$\frac{x \pm a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{\pm b\cos\theta} = \frac{z}{c}.$$

Show that their central axis must, for all values of θ, lie on the surface

$$y\left(\frac{x}{z} + \frac{z}{x}\right) = b\left(\frac{a}{c} + \frac{c}{a}\right).$$

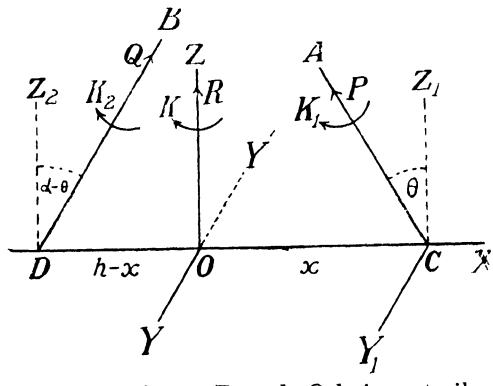
12. Show that forces acting along generators of the same system of the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ are in equilibrium if they would be in equilibrium when acting at a point in the same directions.

13. Two equal forces act along generators of the same system of the hyperboloid $(x^2+y^2)/a^2 - z^2/b^2 = 1$, and cut the plane $z=0$ at the extremities of perpendicular diameters of the circle $x^2+y^2=a^2$. Show that the pitch of the equivalent wrench is $a^2b/(a^2+2b^2)$.

14. A single force is equivalent to component forces X, Y and Z along the axes of co-ordinates and to couples L, M, N about these axes; show that the magnitude of the single force is $\sqrt{X^2+Y^2+Z^2}$ and that its line of action is

$$\frac{yZ - zY}{L} = \frac{zX - xZ}{M} = \frac{xY - yX}{N} = 1.$$

11.9. Resultant of two given wrenches. Let (P, p) with its axis CA, and (Q, q) with axis DB be the given wrenches, and let α be the angle between the axes CA and DB.



If K_1, K_2 be moments of their couples, then $Pp = K_1$, and $Qq = K_2$...(1).

If CD be the shortest distance ($=h$) between the axes of the wrenches, neither wrench has any moment about CD, so that the resultant wrench can have no momentabout CD and its axis must therefore intersect CD at right angles the forces P and Q being at rihgt angles to it.

Let OZ be the axis of the resultant wrench dividing CD into two parts x and $h-x$ and inclined at an angle θ to CA, and hence at $\alpha-\theta$ to DB, θ and $\alpha-\theta$ being on opposite sides of OZ (CA towards OY, DB towards OY').

Let OC be the axis of x and OY, a line perp. to OZ and CD be the axis of y.

Let R and K be the force and moment of the resultant wrench.

Then R, being dependent only on the magnitude and direction of given forces P and Q inclined at an angle α, is given by

$$R^2 = P^2 + Q^2 + 2PQ\cos\alpha \qquad \text{...I.}$$

This gives one of the Invariants of the given system.

For the resultant wrench whose axis is taken as z-axis, we have

$$X=0,\ Y=0,\ Z=R\ ;\ L=0,\ M=0,\ N=K,$$

so that the other invariant of the system is $LX+MY+NZ=KR$, which we shall find out.

Resolving forces in the directions of OZ and OY, we have

$$R=P\cos\theta+Q\cos(\alpha-\theta) \quad \ldots(2)$$

and
$$0=P\sin\theta-Q\sin(\alpha-\theta) \quad \ldots(3)$$

From these, we have

$R^2=P^2+Q^2+2PQ\cos(\alpha-\theta+\theta)=P^2+Q^2+2PQ\cos\alpha$, the same as I.

Also from (3), we have $(P+Q\cos\alpha)\sin\theta=Q\sin\alpha\cos\theta$,

or
$$\frac{\sin\theta}{Q\sin\alpha}=\frac{\cos\theta}{P+Q\cos\alpha}=\frac{\sqrt{\cos^2\theta+\sin^2\theta}}{\sqrt{P^2+Q^2+2PQ\cos\alpha}}=\frac{1}{R} \quad \ldots(4)$$

Now K, the moment of the resultant wrench

= sum of the resolved parts of couples K_1 and K_2 along OZ
+ sum of the moments of components of P, Q parallel to OY
(taking care to take moments with their proper signs)

$=K_1\cos\theta+K_2\cos(\alpha-\theta)+x.P\sin\theta+(h-x).Q\sin(\alpha-\theta)$

$=K_1\cos\theta+K_2\cos(\alpha-\theta)+hQ\sin(\alpha-\theta)$ [by (3)]

$=(K_1+K_2\cos\alpha+Qh\sin\alpha)\cos\theta+(K_2\sin\alpha-Qh\cos\alpha)\sin\theta$

$\therefore\ RK=(K_1+K_2\cos\alpha+Qh\sin\alpha)(P+Q\cos\alpha)$
$+(K_2\sin\alpha-Qh\cos\alpha)Q\sin\alpha$ [by (4)]

$=PK_1+QK_2+(PK_2+QK_1)\cos\alpha+PQh\sin\alpha$ ...II.

This gives the second Invariant of the given system and is denoted by the letter I.

Taking ω to be the pitch of the resultant wrench, so that $K=R\omega$, and using (1), equation II can be written in the form

$$R^2\omega=pP^2+qQ^2+PQ[(p+q)\cos\alpha+h\sin\alpha] \quad \ldots(5).$$

R, θ, K having been obtained by equations I, (4) and II respectively, to find x, the only unknown now left, we take moments about CY_1 drawn parallel to OY.

$\therefore\ Rx=$ sum of the resolved parts of couples parallel to CY_1.
+moment about CY_1 of component of Q parallel to OZ

$=K_1\sin\theta-K_2\sin(\alpha-\theta)+h.Q\cos(\alpha-\theta)$,

$=(K_1+K_2\cos\alpha+hQ\sin\alpha)\sin\theta+(hQ\cos\alpha-K_2\sin\alpha)\cos\theta$

which by (4) reduces to

$$R^2x=(K_1Q-K_2P)\sin\alpha+hQ(Q+P\cos\alpha) \quad \ldots III$$
$$=PQ(p-q)\sin\alpha+hQ(P\cos\alpha+Q) \quad [\text{by (1)}]$$

11.91. Resultant wrench of two given Forces. Let the forces be P and R inclined at an angle θ at a distance x apart (see Fig. Art. 11.9).

Let the line of action R be the axis of z and OC, the common perpendicular to the forces, be the axis of x, then for the resultant wrench, we have

$$X=0, \quad Y=P\sin\theta, \quad Z=R+P\cos\theta,$$

and $$L=0, \quad M=-x.P\cos\theta, \quad N=x.P\sin\theta.$$

If R′ be the force and K′ be the couple of the resultant wrench whose axis will intersect OC at some point, we have

$$R'^2=X^2+Y^2+Z^2=P^2+R^2+2PR\cos\theta \qquad \ldots(1)$$

and $$I=R'K'=LX+MY+NZ$$
$$=0-xP^2\sin\theta\cos\theta+xP\sin\theta(R+P\cos\theta)$$
$$=PRx\sin\theta. \qquad \ldots(2)$$

These equations give the Invariants of two given forces.

From (2) it follows that this invariant I is equal to either force multiplied by the moment of the other force about the first.

Aliter. Proceeding as in Art. 11.9, ignoring couples K_1, K_2 and taking the forces to be P and Q acting at C and D, inclined at an angle α, at a distance h apart, we find that,

equations I, (2), (3) and (4) of Art. 11.9 remain unchanged.

Putting $K_1=K_2=0$ in equations II and III, we get

$RK=PQh\sin\alpha$, ...(2) and $R^2x=hQ(Q+P\cos\alpha)$, ...(3)

giving the second Invariant and the distance from C of the point where the axis of the resultant wrench meets CD.

11.92. *The axes of two given wrenches intersect at right angles and their pitches are given. Find the locus of the central axis.*

Let OX, OY be axes of the given wrenches, whose intensities are P and Q and pitch p, q respectively.

For the resultant wrench of these two wrenches, we have

$$X=P, \quad Y=Q, \quad Z=0; \qquad L=Pp, \quad M=Qq, \quad N=0.$$

∴ by Art. 11.8, equations to the central axis of the system are

$$\frac{Pp+zQ}{P}=\frac{Qq-zP}{Q}=\frac{-xQ+yP}{0},$$

giving $xQ=yP$ or $P/x=Q/y$; and $z(P^2+Q^2)=(q-p)PQ$.

Eliminating P and Q, we obtain, as the locus of the central axis, the equation $$z(x^2+y^2)=(q-p)xy.$$

The surface given by this equation is known as the **cylindroid.**

Note. To find the pitch p of the resultant wrench in terms of the angle θ which the resultant force R makes with OX, we have

$$X=P=R\cos\theta, \qquad Y=Q=R\sin\theta.$$

$$\therefore \quad p=\frac{LX+MY+NZ}{R^2}=\frac{P^2p+Q^2q}{R^2}=\frac{R^2(p\cos^2\theta+q\sin^2\theta)}{R^2}=p\cos^2\theta+q\sin^2\theta.$$

Example 1. *The axes of two wrenches are at right angles and the shortest distance between them is 2b. Prove that the axis of the resultant wrench (of intensity R) divides the shortest distance in the ratio.*

$$Q[2bQ+(p-q)P] : P[2bP-(p-g)Q],$$

where P and Q are the respective intensities of the wrenches, and p and q are the pitches.

In the figure of Art. 11.9, take $h=2b$ and $\alpha=90°$, so that $\alpha-\theta=90°-\theta$.

Resolving along OY, we have

$P \sin\theta - Q\cos\theta = 0$, so that $\dfrac{\cos\theta}{P} = \dfrac{\sin\theta}{Q} = \dfrac{1}{\sqrt{P^2+Q^2}} = \dfrac{1}{R}$.

Taking moments about CY_1, we have

Rx = sum of components of couples $\parallel$ to CY_1
+ moment of component of Q $\parallel$ to OZ about CY_1
$= Pp\sin\theta - Qq\cos\theta + 2b.Q\sin\theta = [PQ(p-q)+2bQ^2]/R$. [by (1)].

$\therefore\ x = \dfrac{(p-q)PQ+2bQ^2}{R^2}$, and $2b-x = \dfrac{2b(P^2+Q^2)-(p-q)PQ-2bQ^2}{R^2}$

$= \dfrac{2bP^2-(p-q)PQ}{R^2}$ $(\because\ R^2=P^2+Q^2)$.

Hence the required ratio, $x/(2b-x)$.

Example 2. *Prove that a wrench of intensity R and pitch pa may be replaced by two forces inclined at an angle 2θ to each other, the shortest distance between them being 2a, and that their magnitudes are*

$$\tfrac{1}{2}R[\sqrt{1+p\tan\theta} \pm \sqrt{1-p\cot\theta}\,].$$

Let P, Q be the magnitudes of the forces, taking Q to be acting along OZ and P at C at an angle 2θ with it, and taking $OC=2a$ to be the shortest distance between them, then as in Art. 11.91, we have

$X=0$, $Y=P\sin 2\theta$, $Z=Q+P\cos 2\theta$,
and $L=0$, $M=-2a.P\cos 2\theta$, $N=2a.P\sin 2\theta$

$\therefore\quad R^2=X^2+Y^2+Z^2=P^2+Q^2+2PQ\cos 2\theta$ ...(1),

and $R^2pa = I = LX+MY+NZ$
$= -2aP^2\sin 2\theta\cos 2\theta + 2a\,P\sin 2\theta(Q+P\cos 2\theta)$
$= 2a\,PQ\sin 2\theta$. ...(2).

Now by using (1) and (2), we have

$$(P+Q)^2 = P^2+Q^2+2PQ = R^2-2PQ\cos 2\theta+2PQ$$
$$= R^2\left[1+\frac{p}{\sin 2\theta}-\frac{p\cos 2\theta}{\sin 2\theta}\right] = R^2(1+p\tan\theta)$$

Also $(P-Q)^2 = P^2+Q^2-2PQ = R^2-2PQ\cos 2\theta-2PQ$

$$= R^2\left[1-\frac{p\cos 2\theta}{\sin 2\theta}-\frac{p}{\sin 2\theta}\right] = R^2[1-p\cot\theta].$$

$\therefore$ P and Q are $\tfrac{1}{2}R[\sqrt{1+p\tan\theta} \pm \sqrt{1-p\cot\theta}\,]$.

Example 3. *Three wrenches of pitches p_1, p_2, p_3 have their axes along three mutually perpendicular lines through O. Show that the axis of their resultant wrench passes through O only if $p_1=p_2=p_3$.*

Show also that if p be the pitch of their resultant wrench, the locus of its axis is the hyperboloid

$$(p-p_1)x^2+(p-p_2)y^2+(p-p_3)z^2+(p-p_1)(p-p_2)(p-p_3)=0,$$

the axes of co-ordinates being the axes of the given wrenches.

Let P_1, P_2, P_3 be the forces of the given wrenches, then for the resultant of these wrenches, we have

$$X=P_1,\ Y=P_2,\ Z=P_3\ ;\ L=P_1p_1,\ M=P_2p_2,\ N=P_3p_3$$

$\therefore$ equations of the axis of the resultant wrench are :

$$\frac{P_1p_1-yP_3+zP_2}{P_1}=\frac{P_2p_2-zP_1+xP_3}{P_2}=\frac{P_3p_3-xP_2+yP_1}{P_3}=p.$$

Obviously the point O, $x=y=z=0$ can satisfy them only if

$$p_1=p_2=p_3.$$

These equations can be written as :

$$(p-p_1)P_1-zP_2+yP_3=0,\ zP_1+(p-p_2)P_2-xP_3=0,$$

and

$$-yP_1+xP_2+(p-p_3)P_3=0.$$

Expansion of the determinant obtained by eliminating P_1, P_2, P_3 from these equations gives the required result.

Examples XLI-B

1. If P and Q be two non-intersecting forces whose directions are perpendicular, show that the distances of the central axis from their lines of action are as Q^2 to P^2. [**Hint.** In equation (3) of Art. 11·91, put $\alpha=90°$].

2. Wrenches of pitches a, b, c are on the lines

(*i*) $y+g=0$, $z-h=0$, (*ii*) $z+h=0$, $x-f=0$, (*iii*) $x+f=0$, $y-g=0$, respectively. Show that the pitch of the resultant wrench is proportional to

$$al^2+bm^2+cn^2+2fmn+2gnl+2hlm,$$

provided the axis of the wrench is parallel to the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}.$$ *(D.U. 1959)*

3. Any number of wrenches of the same pitch p act along the generators of the same system of the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$.

Prove that they will be reducible to a single resultant provided their central axes is parallel to a generator of the cone

$$\left(p+\frac{bc}{a}\right)x^2+\left(p+\frac{ca}{b}\right)y^2+\left(p-\frac{ab}{c}\right)z^2=0.$$ *(D.U. 1958)*

4. Two wrenches of pitches p_1, p_2 have axes at a distance $2a$ from one another. If the resultant wrench is of pitch p and its axis is equidistant from the axes of the component wrenches, show that the angle between them is

$$\tan^{-1}\frac{a(2p-p_1-p_2)}{a^2-(p-p_1)(p-p_2)}.$$

5. A given system of forces is equivalent to a wrench K of intensity R and pitch p ; when an additional force R is added to the system the pitch of the resulting wrench K′ is $\frac{1}{2}p$. Prove that this additional force intersects the axis of K.

6. Wrenches of the same pitch p act along the edges of a regular tetrahedron of side a. If the intensities of the wrenches along AB, DC are the same, and also those along BC, DA and DB, CA, prove that the pitch of the equivalent wrench is $p+a/2\sqrt{2}$.

7. Define a wrench and obtain the wrench (R, G) equivalent to arbitrary forces P and Q inclined at an angle α.

If a, b be the shortest distance between R and P, Q respectively, prove that $P^2(a^2R^2+G^2)=Q^2(b^2R^2+G^2)$. *(D. U. 1957)*

Nul Lines, Planes and Points

11.93. Nul Line. *In relation to a given system of forces* (X, Y, Z, L, M, N), *a nul line is a line about which the sum of the moments of the forces of the system is zero.*

As mentioned in Art. 11.5, at any arbitrarily chosen point O, taken as the origin or base point, a system of forces, by Art. 11.3, reduces to a force R acting at O along some line OB, and a couple G with its axis along OC, another line through O.

Any plane p, perp. to OC (*e.g.*, plane of a bicycle wheel with axle along OC) is obviously the plane of the couple G. Take a line l in this plane. The axis of G having no component along l, the forces of the couple G can have no moment about it.

In order that R may also have no moment about it, there are *three* possibilities :

(*i*) R is zero, *i.e.*, $X=Y=Z=0$.

(*ii*) R lies in the plane p, *i.e.*, $\cos\theta=0$. (Art. 11.6)

In this case R combined with the forces of the couple reduces the system to a single force, in the plane p, as its resultant.
In both these cases the invariant $I=LX+MY+NZ=0$,
for in one case $X=Y=Z=0$ and in the other $\cos\theta=0$.

(*iii*) The plane p and the line l both pass through O at which R also acts.

The line l in this case becomes a *nul line* as defined above, whatever R may be in magnitude and whatever may be its direction.

The plane p, the locus of a nul line is known as the **nul plane** of the point O, and the point O as the **nul point** of the plane p.

Obviously the nul plane of the point O, being perp. to the axis of the couple G, is the plane of the couple G at O.

Note. When R lies in the nul plane of O, every line in the plane is a nul line and every point of the plane is a nul point.

11.94. *To find the equation to the nul plane of a given point* (α, β, γ) *referred to rectangular axes Ox, Oy, Oz.*

Let X, Y, Z be the component forces along Ox, Oy, Oz, and L, M, N, the component couples about them.

By Art. 11.8, the component couples about the lines drawn parallel to the axes through (α, β, γ) are

$$L'=L-\beta Z+\gamma Y,\quad M'=M-\gamma X+\alpha Z,\quad N'=N-\alpha Y+\beta X.$$

The equation of a plane passing through the point (α, β, γ) as the base point, and having the direction cosines of its normal proportional to L', M', N', is

$$(x-\alpha)L'+(y-\beta)M'+(z-\gamma)N'=0,$$

or $(x-\alpha)(L-\beta Z+\gamma Y)+(y-\beta)(M-\gamma X+\alpha Z)+(z-\gamma)(N-\alpha Y+\beta X)=0$,

i.e., $$x(L-\beta Z+\gamma Y)+y(M-\gamma X+\alpha Z)+z(N-\alpha Y+\beta X)$$
$$=\alpha L+\beta M+\gamma N \quad \ldots(1)$$

Example. *Find the nul point of the plane* $x+y+z=0$ *for the system* X, Y, Z, L, M, N.

Let (α, β, γ) be the required nul point, then equation (1) of the nul plane must be the same as $x+y+z=0$.

$\therefore$ comparing this with (1), we have

$$\frac{L-\beta Z+\gamma Y}{1}=\frac{M-\gamma X+\alpha Z}{1}=\frac{N-\alpha Y+\beta X}{1}=\frac{\alpha L+\beta M+\gamma N}{0},$$

so that $\alpha L+\beta M+\gamma N=0$, ... (*i*)

and $L-M=\alpha Z+\beta Z-\gamma(X+Y)$ ... (*ii*)

Since the point (α, β, γ) must also lie on the given plane

$$\alpha+\beta+\gamma=0 \qquad \ldots(iii)$$

$\therefore$ from (*i*) and (*iii*), we get

$$\frac{\alpha}{N-M}=\frac{\beta}{L-N}=\frac{\gamma}{M-L},$$

and by (*ii*), each of these

$$\frac{L-M}{(N-M)Z+(L-N)Z-(M-L)(X+Y)}=\frac{1}{X+Y+Z}.$$

These equations give the co-ordinates of the required nul point.

11.95. *To find the condition that a given line*

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

may be a nul line of the system X, Y, Z ; L, M, N.

Take (α, β, γ), a point on the given line, as the base point.

The component couples about the lines, parallel to the axes, through the point (α, β, γ) are L', M', N' as given in Art. 11.94.

In order that the given line may be a nul line, the sum of the moments of the system about it, which is equal to the moment of the couple about it, must vanish, so that

$$lL'+mM'+nN'=0,$$

l, m, n being the direction cosines of the given line.

Substituting the values of L', M', N', we have

$$l(L-\beta Z+\gamma Y)+m(M-\gamma X+\alpha Z)+n(N-\alpha Y+\beta X)=0,$$

or $X(m\gamma-n\beta)+Y(n\alpha-l\gamma)+Z(l\beta-m\alpha)=lL+mM+nN$ ...(2)

Example. *A system of forces is such that every tangent to the curve given by* $x=a\cos\theta$, $y=a\sin\theta$, $z=b\theta$ *is a nul line. Find the central axis and prove that the pitch of the equivalent wrench is* a^2/b.

Equation of a tangent to the given curve is of the form

$$\frac{x-a\cos\theta}{-a\sin\theta}=\frac{y-a\sin\theta}{a\cos\theta}=\frac{z-b\theta}{b} \qquad \ldots(i)$$

In order that it may be a nul line of the system X, Y, Z ; L, M, N, it must satisfy equation (2), given above.

$\therefore$ $X[a\cos\theta.b\theta - b.a\sin\theta] + Y[b.a\cos\theta - (-a\sin\theta)b\theta]$
$+ Z[(-a\sin\theta)a\sin\theta - a\cos\theta.a\cos\theta] = -a\sin\theta L + a\cos\theta.M + bN,$

or $a\sin\theta(L - bX) + a\cos\theta\,(bY - M) + \theta\cos\theta.abX + \theta\sin\theta.abY - a^2Z - bN = 0.$

Since it must hold true for all values θ,

$\therefore$ $L - bX = 0$, $bY - M = 0$, $abX = 0$, $abY = 0$, and $a^2Z + bN = 0$.

[It can be easily seen by giving to θ arbitrary values such as $\theta = 0, -\frac{1}{2}\pi, \frac{1}{2}\pi, -\pi, \pi$].

From these we have $X = 0$, $Y = 0$, $L = bX = 0$, $M = bY = 0$, and $N = -a^2Z/b$.

Substituting these in equation (4) of Art. 11.8, we get

$$\frac{-yZ}{0} = \frac{xZ}{0} = \frac{N}{Z},$$

so that the central axis is $x = 0$, $y = 0$, *i.e.*, the axis of Z.

The pitch $p = \dfrac{(LX + MY + NZ)}{R^2} = \dfrac{NZ}{Z^2} = \dfrac{a^2Z}{b} \times \dfrac{Z}{Z^2} = \dfrac{a^2}{b}$.

11.96. *Nul planes of all points on a nul line contains the line.*

Let the axis OC (Art. 11.5) of the couple G be taken as the axis of z, so that the plane xOy is the plane of the couple G, and so

$$L = M = 0.$$

This plane is the nul plane of the point O, and any line OH in this plane is a nul line of the system.

Let H $(\alpha, \beta, 0)$, a point on the nul line OH be taken as a new base point. Then R being independent of the position of the base point remains unchanged but the resultant couple of the system will change.

L, M, γ being zero, from equations (1), (2), (3) of Art. 11.8, we have

$$L' = -\beta Z, \qquad M' = \alpha Z, \qquad N' = N - \alpha Y + \beta X$$

Also l, m, n, the direction cosines of the line OH are proportional to α, β, 0

$\therefore$ $lL' + mM' + nN' = \alpha(-\beta Z) + \beta(\alpha Z) + 0 \times N' = 0.$

So that the axis of the resultant couple of the system changes under the condition that it remains perpendicular to OH. Consequently the sum of the moments of the system about OH still vanishes and thus OH remains a nul line of the new nul plane also.

It follows, therefore that *as a point travels along a nul line, the nul plane of the point contains the line and turns round the line.*

Cor. If the nul plane of a point A passes through B, then the nul plane of the point B shall pass through A, for the line AB is a nul line in both the nul planes.

11.97. *To find the nul point of a given plane p.*

Let N be the required nul point of the given plane p.

Take two other convenient points A and B on this plane p, and find their nul planes, say p_1 and p_2, by equation (1), Art. 11.94.

Now N is the nul point of the plane p and A lies in p, it means that the nul plane of N passes through A, therefore the nul plane of A, *i.e.*, the plane p_1 shall pass through N.

Similarly the nul plane of B, *i.e.*, the plane p_2 shall also pass through N, so that N lies on all the three planes p, p_1 and p_2 and is thus the point of intersection of the three planes.

Example 1. *Apply this method to find the nul point of the plane*

$$x+y+z=0. \qquad \ldots(i)$$

The points A (0, 0, 0) and B (1, −1, 0) both lie on the given plane (*i*).

By equation (1) of Art. 11.94, the nul planes of A and B are :

$$Lx+My+Nz=0, \qquad \ldots(ii)$$

and $$(L+Z)x+(M+Z)y+(N-Y-X)z=L-M.$$

Subtracting (*ii*) from this, we get

$$xZ+yZ-z(X+Y)=L-M \qquad \ldots(iii)$$

Solution of these three equations give x, y, z, the co-ordinates of the required nul point.

For steps of solution see Example of Art. 11.94, where the same question has been solved by a different method, and the same equations have been solved for α, β, γ.

Example 2. Prove that the co-ordinates of the nul point of the plane $lx+my+nz=1$, with respect to the system (X, Y, Z : L, M, N), are given by the equations

$$\frac{x}{X-nM+mN}=\frac{y}{Y-lN+nL}=\frac{z}{Z-mL+lM}=\frac{1}{lX+mY+nZ}.$$

[**Hint.** Take $(1/l, 0, 0)$, $(0, 1/m, 0)$ as two points on the given plane]

Example 3. Show that if a generator

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{c}$$ of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.

is a nul line of the system (X, Y, Z ; L, M, N), then this generator is parallel to the plane

$$\left(L+\frac{bc}{a}X\right)x+\left(M+\frac{ca}{b}Y\right)y+\left(N-\frac{ab}{c}Z\right)z=0.$$

Example 4. Show that a system of forces is in equilibrium if each of the six edges of a given tetrahedron is a nul line of the system. (*D.U. 1955*)

[**Hint.** Take six diagonals of the faces of as cube as the six edges of the tetrahedron and apply conditions for these edges to be nul lines.]

Example 5. A sphere is described with its centre on the central axis of a system of forces. Show that the nul points of tangent planes to the sphere lie on a hyperboloid of revolution.

[**Hint.** Take the central axis as the axis of z and therefore the system as X=0, Y=0, Z=R, L=0, M=0, N=K. Now write the equation of the nul plane of any point and apply the condition that it is a tangent plane to the sphere with its centre at the origin].

CHAPTER XII

SIMPLE MACHINES

12.1. A **machine** is an apparatus in which work is done on it by an agent by applying a force, called the **Power** or **Effort** at one part, with the object of getting work done by the machine in overcoming some external force, called the **Weight**, or **Load** or **Resistance**, at another part.

In Mechanics the word power being also used in another sense, *viz.*, "the rate of doing work", effort is the word which is usually used. We shall denote this by the letter P, the weight or the resistance being denoted by the letter W.

12.11. In most cases the machine is so arranged that by applying a small effort P, a larger force W is overcome. If in a machine an effort P balance a resistance W, the ratio **W/P** is called the **Mechanical Advantage.**

The mechanical advantage can have any value from zero to infinity.

If x be the distance moved by the point of application of the effort P and y, the displacement of the point of application of the resistance W, in the same time, the ratio x/y is called the **Displacement Ratio** or **Velocity Ratio.**

12.12. All machines are mostly based on the combinations of simple materials such as cords, rods and planes.

When a machine is made to work, its several parts receive displacements consistent with their geometrical relations. The work done by the effort P is equal to that of the resistance W together with the work of the resistance of the machine such as friction etc. Thus the work done on W, called the *useful work* of a machine, is usually less than that actually done by P.

If, however, all parts of a machine are quite smooth and the connections are all weightless, then,

$$Px = Wy \text{ } i.e., \text{ } W/P = x/y,$$

so that *the mechanical advantage is equal to the velocity ratio.*

If $W > P$ then $x > y$ so that the effort P has to undergo a greater displacement. This is sometimes expressed by saying "What is gained in power is lost in speed".

12.13. The **efficiency** of a machine is the ratio of the useful work to that done by the power, so that

$$\textbf{The efficiency} = \frac{\text{the work output}}{\text{the work input}} = \frac{Wy}{Px}$$

$$= \frac{W}{P} \div \frac{x}{y} = \frac{\text{mechanical advantage}}{\text{velocity ratio}}.$$

It shows that the efficiency of a machine would be unity if all its parts were perfectly smooth, weightless and rigid.

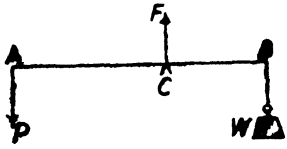
In all existing machines, however, where these ideal conditions do not exist, Wy is always less than Px and the efficiency is necessarily less than unity.

We now give a brief account of some of the principal machines such as the lever, the pulley, the wheel and axle, the inclined plane, the balance and the screw.

12.2. The Lever. A lever is a rigid rod, straight or bent movable about a fixed axis called the *fulcrum*. The portions of the lever between the fulcrum and the points of application of the effort P and the weight W are called *arms* of the lever, P and W being supposed to act in a plane perpendicular to the axis of the lever.

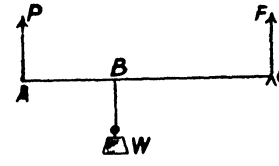
In its simplest form a lever is a straight rigid rod which has one fixed point about which the rod turns. There are three kinds of levers, the classification depending on the position of the fulcrum relative to the lines of action of the forces.

12.21. Lever Class I. In this case P and W are on opposite sides of the fulcrum C, F being the reaction at the fulcrum, and AC>BC.



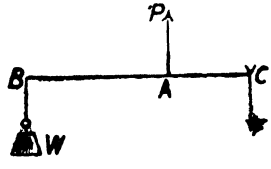
A common balance, a pair of scissors are examples of this kind of lever.

12.22. Lever Class II. Here P and W act on the same side of the fulcrum C, W being between P and F.



A nut-cracker ; an oar, the fulcrum being the end in water ; a wheel-barrow ; a crowbar, the end on the ground being the fulcrum, are levers of this type.

12.23. Lever Class III. Here P and W are on the same side of C but P acts between W and F.



A pair of sugar-tongs ; the human forearm, the weight being carried on the palm of the hand, the fulcrum being at the elbow, are levers of this type.

Neglecting the weight of the lever and taking moments about C in each case, we have

$$P.AC = W.BC \quad i.e., \quad W/P = AC/BC.$$

Thus there is a mechanical advantage in classes I and II but in class III, there is a mechanical disadvantage, P being greater than W (BC being >AC).

Examples XLII

1. A man who wishes to raise a rock leans with his whole weight on one end of a crowbar 5 feet long which is propped at the distance of 4 inches from the end in contact with the rock. The man's weight being 160 lbs. what force does he exert on the rock, and what is the pressure on the prop ?

[*Ans.* 2240, 2400 lbs. wt.]

2. Two forces 6 and 8 lbs. act at the ends A and B of a rod 16 ft. long and are inclined to the rod at angles of 30° and 60° respectively ; find the position of the fulcrum when there is equilibrium.

[*Ans.* Nearly 11·2 ft. from A]

3. An oar is 9 ft. long and the button is 3 ft. from the handle. What is the propelling force on a boat when an effort of 20 lbs. is exerted by the oarsman ? [*Ans.* 30 lbs. wt.)

4. A man seated in a boat pulls at the handle of each of a pair of scults with a force of 25 lbs. wt. If the distance of the rowlock from the end of the blade of each scull be 4 times that of the rowlock from the hand, find the resultant force on the boat. [*Ans.* 62½ lbs. wt.]

5. A pair of nut-crakers are 6 inches long. A pressure of 10 lbs. is required to crack a nut which is pressed at a point 3/4 in from the hinge. What weight placed at the top of the nut will crack it ? [*Ans.* 80 lbs.]

6. The arms of a light bent lever are at right angles to one another and are in the ratio of 5 to 1. The longer arm is inclined to the horizon at an angle of 45°, and carries at its end a weight of 10 lbs. ; the end of the shorter arm presses against a horizontal plane, find the pressure on the plane.

[*Ans.* 50 lbs. wt.]

7. The arms of a light bent lever measure 8 and 24 inches, and they include an angle of 75°. If a mass of $9\sqrt{2}$ lbs. is attatched to the end of the shorter arm and another of 3 lbs. to the end of the longer arm, show that in the position of equilibrium, the shorter arm makes an angle of 60° with the horizontal.

8. A uniform bent lever, when supported at the angle, rests with the shorter arm horizontal but if this arm were twice as long, it would rest with the other arm horizontal. Find the ratio of the lengths of the arms and the angle at which they are inclined. [*Ans.* $1 : \sqrt{2}$; 120°]

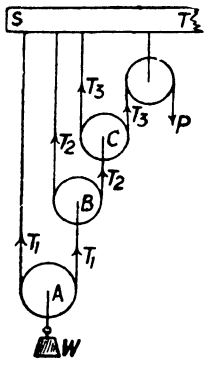
9. Two weights P and Q are supported from the extremities of a lever without weight, in the form of an arc of a circle subtending an angle 2α at the centre, which rests with its convexity downwards upon a horizontal plane. Show that the angle θ which the central radius makes with the vertical in the position of equilibrium is given by

$$(P+Q) \tan \theta = (P-Q) \tan \alpha.$$

12.3. The Pulley. A pulley consists of a wheel grooved along its circumference to receive a rope. It can turn freely about an axis through its centre perpendicular to its plane.

Usually more than one pulleys are arranged to form a system. For lifting heavy weights three systems of pulleys are commonly used, which we now deal with separately.

12.31. First System of Pulleys. *Each string attached to a supporting beam.*



This consists of a number of movable pulleys (only 3 taken in the figure) so arranged that the string passing round each has one end attached to the supporting beam ST, and the other end to the next higher pulley, the weight W, being attached to the lowest pulley.

The highest *fixed* pulley is used simply for applying the effort P in the convenient downward direction and does not otherwise affect the mechanical advantage.

The parts of the strings not in contact with the pulleys are assumed to be all vertical and friction is neglected.

Let w_1, w_2, w_3 be the weights of the pulleys A, B, C and let T_1, T_2, T_3 be the tensions as shown in the diagram.

Now for equilibrium, we have

$$2T_1 = W + w_1, \qquad 2T_2 = T_1 + w_2 = \frac{W}{2} + \frac{w_1}{2} + w_2$$

$$2T_3 = T_2 + w_3 = \frac{W}{2^2} + \frac{w_1}{2^2} + \frac{w_2}{2} + w_3$$

$$\therefore \quad P = T_3 = \frac{W}{2^3} + \frac{w_1}{2^3} + \frac{w_2}{2^2} + \frac{w_3}{2},$$

so that for n movable pulleys, we must have

$$P = \frac{W}{2^n} + \frac{w_1}{2^n} + \frac{w_2}{2^{n-1}} + \frac{w_3}{2^{n-2}} + \ldots\ldots + \frac{w_n}{2},$$

or
$$2^n P = W + w_1 + 2w_2 + 2^2 w_3 + \ldots\ldots + 2^{n-1} w_n \qquad \ldots(1)$$

If the pulleys be equally heavy each of weight w, then

$$w_1 = w_2 = w_3 = \ldots\ldots = w_n = w$$

$$\therefore \quad 2^n P = W + w(1 + 2 + 2^2 + \ldots\ldots + 2^{n-1}) = W + w(2^n - 1) \qquad \ldots(2)$$

If the pulleys are of negligible weights, then $w = 0$.

$$\therefore \quad 2^n P = W \quad i.e., \quad W/P = 2^n \qquad \ldots(3)$$

From (1) and (2) it follows that the mechanical advantage W/P depends on the weights of the pulleys, it being greater when the pulleys are lighter.

Cor. *The pressure on the supporting beam, when P is acting downwards, the string to which it is applied passing over a fixed pulley.*

$$= W + w_1 + w_2 + \ldots\ldots w_n + P.$$

12.311. The effort can also be obtained *by the principle of Virtual Work.*

Suppose W gets a small upward displacement x, then the pulley A rises by x, and since due to this rise, a length $2x$ of the string slips round A, the pulley B rises by amount $2x$. Similarly the pulley C will rise by a distance $4x$ and so on. If we confine ourselves to the case of three movable pulleys as shown in the diagram, the power P will consequently have a displacement equal to $8x$. thus

by the principle of work, we have

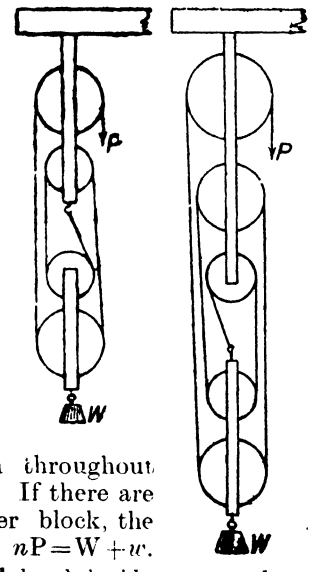
$$P.8x = W.x + w_1.x + w_2.2x + w_3.4x,$$

or $$2^3.P = W + w_1 + 2w_2 + 2^2w_3,$$

which is the same as (1) for $n=3$.

12.32. Second system of Pulleys. *The same string passing round the pulleys.*

In this system there are two blocks each containing pulleys. The upper block is fixed to a supporting beam and the lower of weight w, to which the load W is attached, is movable.



The number of pulleys in the upper fixed block is either the same or is greater by one than the number in the lower block. In the first case the end of the string is fastened to the upper block, so that the number of portions of the string connecting the blocks is *even*. In the second case the end of the string is fastened to the lower block, the parts of the string connecting the block being *odd*.

It is assumed that the parts of the string not in contact with the pulleys are all vertical and that the pulleys are smooth.

The pulleys being smooth, the tension throughout the string is the same, equal to the effort P. If there are n portions of the string supporting the lower block, the total upward force on this block is nP. $\therefore$ $nP = W + w$.

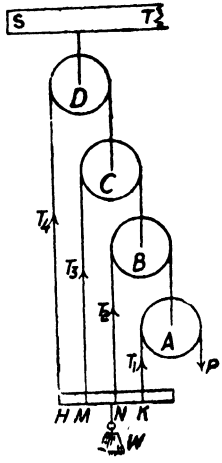
Note. Actually the pulleys in each block are placed beside one another, being free to rotate on the same axle.

12.321. The same result can also be obtained *by the principle of Virtual Work.*

Let the load W and the lower block rise a small distance x. It is clear that each of the n parts of the string is slackened by x. To tighten the string, P must, therefore, descend a distance x for each separate portion of the string, so that P must descend a distance nx.

$\therefore$ by the principle of work, we have

$$P.nx = (W + w)x \text{ } i.e., \text{ } nP = W + w.$$



12.33. Third System of Pulleys. *Each string attached to the weight.*

In this system one end of each string passing over and pulley is fastened to a bar HK supporting the load W, the other end being attached to next lower pulley, the highest pulley being fixed to the supporting beam ST. The effort P is applied to the free end of the string passing over the lowest pulley. This system is, in fact, the

inverted first system.

It is assumed that the free portions of the strings are all vertical and that the pulleys are all smooth.

Let T_1, T_2, T_3... ...be the tensions of the strings passing over the pulleys A, B, C.....

I. Let us first neglect the weights of the movable pulleys, then we have

$$T_1 = P, \qquad T_2 = 2T_1 = 2P$$
$$T_3 = 2T_2 = 2^2P. \qquad T_4 = 2T_3 = 2^3P$$
$$\therefore \quad W = T_1 + T_2 + T_3 + T_4 = (1 + 2 + 2^2 + 2^3)P$$
$$= (2^4 - 1)P.$$

If there were n pulleys, of which $(n-1)$ are movable, we should have

$$W = (2^n - 1)\,P \qquad \ldots (1).$$

Thus the mechanical advantage is $2^n - 1$.

To keep the bar HK horizontal, the load W has to be attached at a proper place, which can be found by taking moments about some point, say H.

II. Now let us suppose that the movable pulleys A, B, C have weights w_1, w_2, w_3, then

$$T_1 = P,$$
$$T_2 = 2T_1 + w_1 = 2P + w_1,$$
$$T_3 = 2T_2 + w_2 = 2^2P + 2w_1 + w_2,$$
$$T_4 = 2T_3 + w_3 = 2^3P + 2^2w_1 + 2w_2 + w_3.$$
$$\therefore \quad W = T_4 + T_3 + T_2 + T_1$$
$$= (2^3 + 2^2 + 2 + 1)P + (2^2 + 2 + 1)w_1 + (2 + 1)w_2 + w_3$$
$$= (2^4 - 1)\,P + (2^3 - 1)w_1 + (2^2 - 1)w_2 + (2 - 1)w_3.$$

If there are n pulleys, of which $(n-1)$ are movable, we have

$$T_n = 2^{n-1}P + 2^{n-2}w_1 + 2^{n-3}w_2 + \ldots\ldots + w_{n-1} \qquad \ldots (2)$$

and
$$W = (2^n - 1)P + (2^{n-1} - 1)w_1 + (2^{n-2} - 1)w_2 + \ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots + (2^2 - 1)w_{n-2} + (2 - 1)w_{n-1} \qquad \ldots (3)$$

If the weight of each pulley is the same, equal to w, then

$$w_1 = w_2 = \ldots\ldots = w_{n-1} = w,$$

and
$$W = (2^n - 1)P + w[2^{n-1} + 2^{n-2} + \ldots\ldots + 2 - (n-1)]$$
$$= (2^n - 1)P + w(2^n - 1 - n) = (2^n - 1)(P + w) - nw. \qquad \ldots (4)$$

The mechanical advantage is increased by increasing the weights of the pulleys. In this case the weights of the pulleys assist the power in sustaining the load.

12.331. The **pressure** in the supporting beam ST
$$= W + w_1 + w_2 + \ldots\ldots w_n + P$$
for, the beam has to support the load W, the weights of all the pulleys, including the fixed one, and the effort P.

This is also equal to the sum of the weight of the fixed pulley and twice the tension of the string passing over it. [By (1) and (2)].

Examples XLIII

1. In the first system, there are 4 pulleys, of equal weight each weighing 2 lbs. What weight can be raised by a force equal to the weight of 20 lbs. ?

What is the weight of each pulley if a power of 4 lbs. supports a weight of 49 lbs. [*Ans*, 290 lb. ; 1 lb.]

2. In the first system of 4 pulleys of equal weight, a certain weight can be supported by a force of 7 lb. wt., but if a fifth similar pulley be used the same weight can be supported by a force of 4 lbs. wt. Find the supported weight and the weight of each pulley. (*Ans*. 97 lb. ; 1 lb.]

3. Three single pulleys are arranged as in the first system. What effort will be required to raise a load of 40 lbs. if

(1) the pulleys are light but due to friction the efficiency is 80 per cent ?

(2) the weight of each pulley is 2 lbs. and there is no friction?

[*Ans*. 6·25 ; 6·75 lbs. wt.]

4. A man weighing 13 stone suspended from the lowest of a system of 6 light pulleys of the first system, supports himself by pulling at the end of the string that passes over the topmost fixed pulley. Show that he must have applied a muscular effort $2\frac{4}{8}$ lbs. wt.

5. A man weighing 10 stone supports a weight of 91 lbs. by means of three movable pulleys arranged in the system where each pulley hangs in the loop of a separate string. The pulleys weigh 5 lbs., 4 lbs. and 2 lbs. respectively. What is the thrust of the man on the ground ? [*Ans*. 9 stone]

6. There are four movable pulleys in the first system, each of weight *w* and the effort acting upwards (the string to which it is applied not passing over a fixed pulley) is P. Show that the stress on the beam is

$$15P - 11w.$$

7. In the second system of pulleys, if the lower block weighs 24 lbs. and contians 5 pulleys, the string being fastened to the lower block, what weight can be raised by a force equal to 21 lbs. wt. [*Ans*. 207 lbs.]

8. If weights of 5 lbs. and 6 lbs. support weights of 73 lbs. and 88 lbs. respectively, what is the weight of the lower block, and how many pulleys are there in it ? [*Ans*. 2 lbs. ; 7]

9. We are told that the cable by which "Great Paul" whose weight is 18 tons, was lifted into its place in the Cathedral tower, passed four times through the two blocks of pulleys. From this statement give a description of the pulleys and estimate the strength of the cable.

[*Ans*. Can support more than $2\frac{1}{4}$ tons.] (*Roorkee*)

10. A man weighing 12 stone supports a weight of 3 cwt. by means of a system of pulleys, there being four pulleys in each block and the string being attached to the upper block. Neglecting the weights of the pulleys, find his thrust on the ground. [*Ans*. 9 stone wt.]

11. Five pulleys are arranged in two blocks—the upper fixed block having three, and lower movable block two.

A man weighing 120 lbs. sits on a board attached to the lower block and pulls himself up by means of a rope passing round all the pulleys. If the lower block and board weigh 10 lbs., find the smallest force that he must exert to raise himself.

If the free end of the rope descends 10 ft., how much useful work is done ?

[*Ans*. $21\frac{3}{5}$ lbs. ; 260 ft.-lbs.]

12. Find the least number of pulleys in a movable block weighing 10 lbs. in order that a weight of 120 lbs. may be lifted by a downward force not exceeding 28 lbs. Find also the necessary downward force. [*Ans*. 2 ; 26 lbs.]

13. A pulley system in which there is only one string, consists of four light pulleys. When on the point of motion the tension of the string as it passes over each pulley is increased 25 per cent. Find the force which will just lift a weight of 246 lbs. [*Ans*. Nearly 104·2 lbs. wt.]

14. If four pulleys (three movable) each weighing 2 lbs. be used in the third system, find the force required to support a weight of 247 lbs.

[*Ans.* 15 lbs. wt.]

15. By means of a third system of pulleys with 6 light pulleys a man weighing 12 stone supports a weight of 18 stone. Find the pressure of the man on the ground. [*Ans* 11 st. 10 lbs.]

16. If there are n light pulleys in the third system, each pulley, of diameter $2a$, show that the distance of the point of application of the weight from the line of action of the effort is $2^n.na/(2^n-1)$.

17. If there are three movable pulleys, each of weight w and radius a, show that for the bar carrying the weight to remain horizontal, the distance of the C.G of the bar and the weight from the point of attachment of the longest string passing over the fixed equal pulley must be equal to $(11P+5w)a/W$, where W is the weight of the bar including the attached weight.

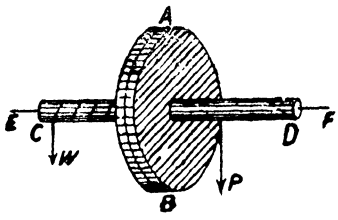
18. In a system of n equal weightless pulleys in which the string passing round any pulley has an end attached to a weightless bar AB, there are $(n-1)$ movable pulleys and one fixed pulley; the string round the fixed pulley is attached to the bar at A, and that round the last movable pulley is attached to the bar at B and has a weight P at the other extremity. The bar AB is kept in horizontal equilibrium by a weight W attached at A and another weight at B; prove that

$$W=[(n-2)\ 2^n+2]\ P/(n-1).$$

12.4. The Inclined Plane. The equilibrium of a weight on a smooth or rough inclined plane under the action of an effort P acting in different directions has already been dealt with in II and III Chapters.

12.5. The Wheel and Axle. *To find the relation between the effort and the weight.*

This machine consists of a *wheel* AB rigidly connected to a horizontal cylinder, the *axle* CD, the plane of the wheel being perpendicular to the axle. The whole thing turns freely about the common axis EF.



The effort P is applied to a string wound round the wheel and the weight W, to be lifted, is attached to another string wound round the axle in the opposite direction.

If a be the radius of the wheel and b that of the axle, by taking moments about the axis EF, we have

$$P.a=W.b \qquad i.e., \qquad W/P=a/b.$$

The same result is also given by the principle of Virtual Work.

Let $\delta\theta$ be the small angle through which the machine turns. When P descends a distance $a\delta\theta$, W moves up a distance $b\delta\theta$.

Hence by the principle of work, we have

$$W.b\delta\theta=Pa\delta\theta \qquad i.e., \qquad W/P=a/b.$$

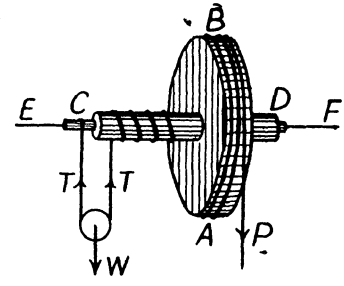
Theoretically, the mechanical advantage can be increased by making a large or b small but in practice if a is very large, the machine becomes unwieldy, and if b be small, the axle may break.

A Windlass and a Capstan are only modified forms of the wheel and axle. In these the effort is applied at the ends of spokes or long handles perpendicular to the axle.

In a Windlass, which is used for drawing water, the axle is horizontal. In a Capstan, used on board the ship, the axle is vertical.

12.51. The Differential Wheel and Axle.

With a view to get greater mechanical advantage without making the machine unwieldy, the wheel and axle is modified into the differential wheel and axle in which the axle consists of two parts of radii, say b and c ($b>c$). The weight W is attached to a pulley and the rope supporting this is wound in opposite directions on the two parts of the axle. As P descends, the rope is wound up on the larger axle and unwinds on the smaller.



For the equilibrium of the weight, we have

$$2T=W \quad \text{or} \quad T=\tfrac{1}{2}W.$$

Taking moments about the central axis EF, we have

$$P.a+T.c.=T.b.$$

$$\therefore \quad P=T(b-c)/a=W(b-c)/2a. \quad \therefore \quad W/P=2a/(b-c).$$

So that the mechanical advantage can be increased by making the difference $b-c$ small without making the wheel too large or the axle too thin.

Examples XLIV

1. If a power of 10 lbs. balance a weight of 555 lbs., on a wheel whose diameter is 4 yards ; what must be the radius of the axle ? The thickness of the rope is neglected. [*Ans.* $1\frac{1}{37}$''.]

2. The handle of a winch has a radius of 18 in. and the drum is 3 in. in diameter. Find the force on the handle necessary to give a pull of 6 cwt. on the rope. [*Ans.* 56 lbs. wt.]

3. There are two wheels, whose respective diameters are 5 ft. and 4 ft. on the same axle ; the diameter of the axle being 20 inches. What weight on the axle would be supported by forces equal to 48 lbs. and 50 lbs. on the larger and smaller wheels respectively ? [*Ans.* 264 lbs.]

4. A cage is suspended by a rope over an axle and a man standing in the cage draws himself and the cage up by pulling at a rope passing over the circumference of the wheel. If the joint weight of the man and cage be 15 stone and the radius of the wheel six times that of the axle, find the tension exerted by the man.

If the rope is being pulled at the rate of 5 ft. a second, find the horse-power at which the man works. [*Ans.* 30 lb. wt. ; 3/11 H.P.]

5. A man pushing at the end of a pole 5 ft. long works a capstan, whose diameter is 2 ft., with what force must he push to overcome a resistance of 600 lbs. wt. [*Ans.* 100 lbs. wt.]

6. Four sailors each exerting a force capable of raising 119 lbs. raise an anchor by means of a capstan whose radius is 2 ft. 4 in. and whose spokes are 8 ft. long (measured from the axis). Find the weight of the anchor.

[*Ans.* 1632 lbs.]

7. In a differential wheel and axle, if the radius of the wheel be one foot and the radii of the two portions of the axle 4 and 2 inches, what force will support a weight of 36 lbs. ? [*Ans.* 3 lb. wt.]

8. In a differential wheel and axle the two parts of the axle are of radii 4 and 5 inches, and the force is applied at the end of an arm 18 inches long. To raise a weight of 3000 lbs. a force of 96 lbs. is required. What is the efficiency ?

[*Ans.* 86·8]

9. In a differential wheel and axle the two parts of the rope on which the weight W is slung make an angle 2α with one another, and a, b, c are the radii of the two parts of the axle and of the wheel, prove that

$$Pc=\tfrac{1}{2}W(b-a)\sec\alpha.$$

10. In a wheel and axle the radius of the wheel is a and that of the axle is b, the weight supported is W and the rope at which the effort is applied is perpendicular to the axis, making an angle θ to the vertical. Show that the magnitude of the thrust on the bearings of the machine is

$$W\sqrt{a^2+b^2+2ab\cos\theta}\,/a.$$

11. The shaft of a fly-wheel rests horizontally in two V grooves, the angle of each being 60°. A rope round the wheel supports a weight P. The coefficient of friction is μ and the wheel and the shaft weigh W. Show that the least value of P which will cause the wheel to rotate in the grooves is given by

$$P=2\mu rW/[(1+\mu^2)R-2\mu r],$$

where r is the radius of the shaft and R that of the wheel. (*Roorkee*)

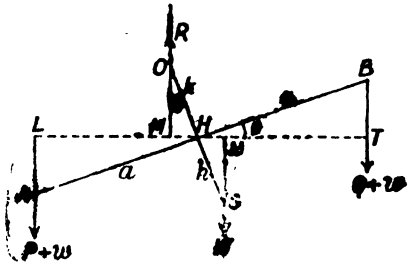
12.6. The Common Balance. This consists essentially of a rigid beam AB carrying scale-pans at its extremities. The beam with the scale pans turns freely about the fulcrum O, outside the beam but rigidly connected with it by a rod which bisects the beam at right angles at H.

G, the C. G. of the beam together with the scale pans, lies on the line OH, a little below the beam.

A balance is *true* when the arms AH, BH and the weights of the scale pans are equal and O, H, G are in a straight line perpendicular to the beam AB.

If these conditions are satisfied, the beam remains horizontal when the scale pans are empty or when equal weights are placed in them. For unequal given weights in the scale pans, the beam will rest in a definite position inclined to the horizontal.

12.61. *To find the relation between the inclination of the beam and the weight in the pans.*



Let AB be the beam, O the fulcrum, and G, the C. G. of the beam.

Let w be the weight of each scale pan, P and Q the weights placed in them, W the weight of the beam and θ its inclination to the horizontal.

Let AH = HB = a, OH = k and HG = h.

Draw LHT, the horizontal line through H.

Taking moments about the fulcrum O, we have

$$(P+w)\times LM = W\times MN + (Q+w)\,MT,$$

or $$(P+w)(LH-MH) = W(MH+HN)+(Q+w)(MH+HT),$$

i.e. $$(P+w)(a\cos\theta - k\sin\theta) = W(k\sin\theta + h\sin\theta) + (Q+w)(k\sin\theta + a\cos\theta),$$

or $$[(P+w)a-(Q+w)a]\cos\theta = [(P+w)k+(Q+w)k+W(h+k)]\sin\theta.$$

$$\therefore \quad \tan\theta = \frac{(P-Q)a}{(P+Q+2w)k+W(h+k)}$$

12.62. This result shows that :—

(1) If P = Q, $\theta = 0$, so that if the weights in the pans are equal, the beam can only rest in a horizontal position.

(2) If the weights P and Q are unequal, the beam will rest inclined at a definite angle to the horizontal.

(3) If $h = k = 0$, $\theta = \frac{1}{2}\pi$ and if $h = k = 0$ and also P = Q, θ may have any value ; so that if the centre of mass of the beam and the centre of suspension coincide in the line AB, the beam can rest in any position when equal weights are placed in the pans and can rest only in a vertical position if the weights are unequal.

This can be avoided by having G below AB.

12.63. *The requisites of a good balance are that it must be* :—

(*i*) *true* (*ii*) *sensitive* (*iii*) *stable* (*iv*) *rigid.*

12.631. True balance requires that (*a*) its arms should be equal (*b*) the weights of the scale pans should be equal and (*c*) the C.G. of its beam should lie in a line through the fulcrum perpendicular to the beam.

Double Weighing. If a balance is *false* due to the scale pans being of different weights or the arms of the beam being unequal, the correct weight of a body can be obtained as follows :

Place the body in one scale pan and balance it by placing sand in the other pan. Now remove the body and put weights in the pan until they balance the sand.

These weights must be equal to the weight of the body.

This is known as *Borda's method.*

12.632. Sensitiveness of a balance requires that for a small difference between the weights in the pans, the beam must be inclined at an appreciable angle to the horizontal.

For a given difference P−Q of the weights tan θ/(P−Q) may

be taken as the measure of the sensitiveness. But this, by the result of Art. 12.61 is equal to

$$\frac{a}{(P+Q+2w)k+W(h+k)}.$$

So that the sensitiveness of a balance will be greater if

(*a*) the arm, *a*, is fairly long.

(*b*) as compared to *h* and *k*. W and *w* the weights of the beam and the pans are as small as possible.

Thus it is always desirable to have a long and light beam.

(*c*) If *k* is not zero, the sensitiveness diminishes as the weights P and Q increase. If, however, $k=0$, *i.e.*, the fulcrum lies in the beam,

$$\text{sensitiveness} = \frac{a}{Wh}$$

which is independent of the weights in the pans.

This condition is generally aimed at and *k* is made as small as possible consistent with the rigidity of the balance.

12.633. Stability. of a balance requires that it must quickly take up its position of equilibrium, *i.e.*, the time of an oscillation should be small. From a Dynamical consideration it is found that this is possible when *a* is small and *h* and *k* large. These conditions are, therefore, the reverse of those required for sensitiveness. Thus it shows that sensitiveness and quickness in weighing both cannot be attained simultaneously.

Example 1. *The arms of a balance are of unequal lengths, but the beam remains in a horizontal position when the scale-pans are not loaded ; if a body is placed in each scale-pan in succession and weighed, show that its true weight is the geometric mean between its apparent weights.*

Show also that if a tradesman weighs equal quantities of a substance, using alternately each of the scale, he will be a loser.

Since the beam remains horizontal when unloaded, the C.G. of the beam and the scale pans is vertically below the fulcrum.

Let *a* and *b* be the lengths of the arms of the beam, and let W_1, W_2 be the apparent weights in the two scale pans of a body whose real weight is W, then

$$W.a=W_1.b$$

and

$$W.b=W_2.a$$

$$\therefore \quad W^2=W_1W_2 \quad \text{or} \quad W=\sqrt{W_1W_2}.$$

Again, by weighing apparently equal quantities in the two scale pans the tradesman really gives the customers W_1+W_2 instead of 2W.

Now $W_1+W_2-2W=W_1+W_2-2\sqrt{W_1W_2}=(\sqrt{W_1}-\sqrt{W_2})^2$,

a positive quantity, so that $W_1+W_2>2W$, and hence the tradesman is a loser.

Example 2. *The arms of a balance are equal in length but the beam is unjustly loaded, if a body be weighed in each scale pan successively, show that its true weight is the arithmetic mean between its apparent weights.*

Let the length of each arm be a, the horizontal distance of the C.G. of the beam from the fulcrum be x and let w be the weight of the beam, then if W_1, W_2 be the apparent weights of a body whose real weight is W, we have

$$W.a=W_1a+w.x$$

and

$$W_2.a=W.a+w.x.$$

$$\therefore \quad (W-W_2)a=(W_1-W)a \qquad i.e., \quad W=\tfrac{1}{2}(W_1+W_2).$$

Examples XLV

1. The apparent weights of a body are 4 and 16 lbs. when weighed from two pans of a balance. Find the ratio of the lengths of the arms and the true weight of the body. [*Ans.* 2 : 1 ; 8 lbs.]

2. If one of the arms of a false balance be longer than the other by $1/n$th part of the shorter ; when used the weight is put into one scale pan as often as into the other. Show that the seller loses $50/n(n+1)$ per cent.

3. If the arms of a false balance be without weight, and one arm longer than the other by $\frac{1}{9}$ of the shorter arm, and if in using it the substance to be weighed is put as often in the one scale pan as in the other, show that the seller loses $\frac{5}{9}$ per cent in his transaction.

4. A shopkeeper has a balance whose arms are 12 and 13 inches, but which is otherwise correct. If he sells 8 pounds of tea to a customer at Rs. 3, 8 as. per pound by weighing half the quantity in one scale pan and other half in the other, find how much does he gain or lose by the transaction.

[*Ans.* Loses 1 as. 6p. nearly]. (*I.I.T. Kharagpur*)

5. The beam of a tradesman's balance is horizontal when both scale pans are unloaded, but the arms are of unequal lengths a and b. Show that if the tradesman weighs out quantities of the same substance which appear to be equal, using alternately each of the scale pans, then he loses on his transactions and that his actual receipts ought to amount to

$$\frac{50(b-a)^2}{ab} \text{ per cent more.}$$

6. The arms of a false balance are a and b and weight W balances P at the end of the shorter arm b, and Q at the end of the arm a ; show that

$$a : b=(P-W) : (W-Q).$$

7. A uniform heavy rod AB can turn freely about a fulcrum at a point whose distances from the ends A and B are a and b. If a body is hung from A it balances a weight W hung from B, and if hung from B it balances a weight W′ hung from A. Show that the weight of the body is.

$$(bW+aW')/(a+b).$$

8. In a false balance, a weight P appears to weigh Q and a weight P′ to weigh Q′ ; prove that the real weight X of what appears to weigh Y is given by

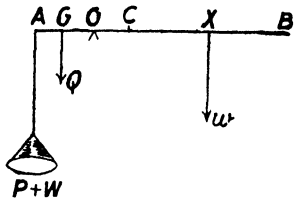
$$X(Q-Q')=Y(P-P')+P'Q-PQ'.$$

9. If a balance be unjustly loaded, and have unequal arms, and if a tradesman weigh out a customer a quantity 2W of some substance by weighing equal portions in two scale pans, show that he will defraud himself if the C.G. of the beam be in the longer arm.

10. It is observed that a body weighs W_1 when suspended at one end of a false balance of negligible weight and of length $2a$, but weighs W_2 when suspended at the other end ; prove that the fulcrum should be shifted towards the first and through a distance

$$(\sqrt{W_1}-\sqrt{W_2})a \,/\, (\sqrt{W_1}+\sqrt{W_2}).$$

12.7. The Common Steelyard. This is a machine for weighing bodies. It consists of a horizontal rod AB movable about a fixed fulcrum O, nearer to the one end than the other.



At A, the end of the shorter arm, is attached a scale pan or a hook to carry the body whose weight is required, and on the longer arm OB slides a movable weight w.

Let P be the weight of the scale pan, Q the weight of the rod acting at G.

Let C be the position at which w must be placed to keep the rod horizontal when there is no weight in the pan.

Let OA$=a$, OG$=b$ and OC$=c$.

Taking moments about O, we have

$$wc = \mathrm{P}.a + \mathrm{Q}.b \qquad \ldots(1)$$

Now when a weight W is placed in the pan, let x be the distance from C at which the weight w will balance, then

$$w(c+x) = (\mathrm{P}+\mathrm{W})a + \mathrm{Q}b \qquad \ldots(2)$$

Subtracting (1) from (2), we get

$$wx = \mathrm{W}a \qquad \ldots(3)$$

It shows that if $x=a$, W$=w$, if $x=2a$, W$=2w$ and so on.

Thus if from C towards B, we mark off points at distances a, $2a$, $3a$, etc., the weight in the scale pan which will balance w at the first mark is equal to w, the one that balances w at the second mark will be $2w$ and so on.

Now if w is taken 1 lb. the graduations will correspond to pounds in the pan. These graduations can further be sub-divided to read ounces.

An instrument of this form was used by the Romans and is, therefore, often called Roman Steelyard.

Example 1. A uniform rod 2 ft. long and weighing 3 lbs. is to be used as a steelyard. The fulcrum is 2 inches from one end of the rod and the sliding weight is 1 lb. Find the greatest weight which can be determined by this machine, and also where the sliding weight must be placed to indicate a weight of 20 lbs. (*P. U.*)

[*Ans.* 26 lbs. ; at mid-point]

Example 2. The weight of a steelyard is 8 lbs., the body to be weighed is suspended from a point 4 inches from the fulcrum and the C.G. of the steelyard is 3 inches on the other side of the fulcrum. Where should the graduation corresponding to one cwt. be placed, the movable weight being 12 lbs. ? [*Ans.* 35$\frac{1}{3}$ in. from fulcrum].

Example 3. A shopkeeper, using a common steelyard, alters the movable weight for which it has been graduated. Does he cheat himself or his customer ?

If the steelyard be uniform and its weight be m times the movable weight and the fulcrum be one nth part of the length of the beam from one end, show that the greatest weight that can be weighed is

$$\tfrac{1}{2}[2n-2+m(n-2)]w,$$

w being the movable weight. (*I.I.T. Kharagpur*)

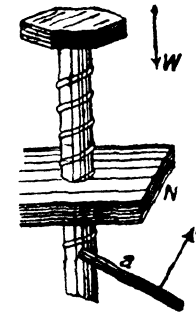
Example 4. Explain the principle of the common steelyard.

A uniform bar of wood AB, of mass 8 lbs. and length 4 ft., has a piece of lead buried in it. When the bar is pivoted at a point 1 ft. from the end A, a weight of $24\frac{1}{2}$ lbs. at that end is needed to balance it, and when pivoted at 3 ft. from the same end a weight of $3\frac{1}{8}$ lbs. at the end B is needed to balance it. Find the weight of the piece of lead and its position along the bar. (*Roorkee*)

[*Ans.* $5\frac{5}{6}$ lbs. at $3\frac{2}{5}\frac{9}{5}$ ft. from A].

12.8. The Screw. *To find the relation between the effort and the resistance in a screw.*

A screw consists of a bolt in the form of a circular cylinder with a uniform projecting *thread* running round its surface. The inclination of the thread at all its points to a plane perpendicular to the axis of the cylinder is always the same.



The distance between any two consecutive threads measured parallel to the axis is called the **pitch** of the screw.

The screw thus formed fits into a hollow nut or fixed support with a corresponding groove on its internal surface in which the ridge of the screw slides. As the thread slides into the groove, the screw can only revolve about its axis and at the same time it can move forward parallel to its length.

If a be the length of the arm at the end of which the effort P is applied and p the pitch of the screw, then in one complete revolution the distance moved by the end of the arm is $2\pi a$ and that moved by the screw parallel to its axis is equal to p.

Let W be the resistance parallel to the axis of the screw.

Then neglecting friction, we have by the principle of work

$$P\ 2\pi a = W.p$$

$\therefore$ the mechanical advantage $W/P = 2\pi a/p$.

Example 1. What is the pitch of a screw press, if the handle is 2 ft long and a force of 10 lb. wt. applied at each end of the handle cause a pressure of 5000 lbs. wt to be exerted by the press (neglect friction) ? [*Ans.* 0·3 in.]

Example 2. If the thread of a screw makes 30 turns in 11 inches what power at the end of a lever 25 ins. long will sustain a resistance of 1000 lbs. ? [*Ans.* $2\frac{1}{3}$ lbs. wt.]

Example 3. A screw press that has a screw of two threads per inch is required to exert a pressure of 2 tons. Assuming that the efficiency is 35 per cent, what must be the length of the handle so that the effort does not exceed 40 lbs. ? [*Ans.* $25\frac{5}{11}$ in.]

CHAPTER XIII

GRAPHICAL STATICS

13.1. We have already dealt with the analytical method of finding the magnitude, direction and line of action of the resultant and the conditions of equilibrium of a number of co-planar forces. In this chapter we propose to do the same thing graphically and give the graphical methods to the solution of certain problems in Statics.

The particular case of forces acting at a point, say A, was considered in Art. 3.22 and it was shown that if a polygon is drawn so as to have its sides parallel and proportional to the forces, the line joining the first and the final points represents the resultant of the forces in magnitude and direction, the point of application of the resultant being the known point A at which the forces act.

We shall now consider the more general case of co-planar forces acting on a rigid body.

13.2. *To find graphically, the magnitude, direction and line of action of the resultant of a number of co-planar forces.*

Let P, Q, R, S be the given forces, acting along given lines. Fig. (*i*).

We take only four forces, but the method is a general one and can be applied to any number of forces.

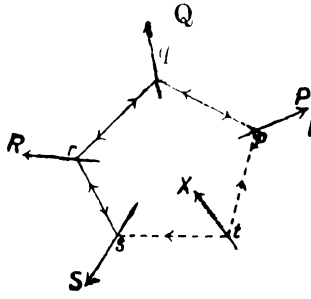


Fig. (*i*)

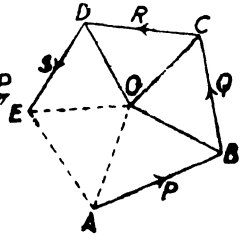


Fig. (*ii*)

Draw the figure ABCDE Fig. (*ii*) having its sides AB, BC, CD, DE parallel and proportional to P, Q, R, S respectively and join AE.

It will be shown that AE represents the resultant force in *magnitude* and *direction*.

Take any point O in the plane inside or outside ABCDE but not on its perimeter, and join OA, OB, OC, OD and OE.

Now in Fig. (*i*) take any point *p* on the line of action of the force P and draw *pq*, *qr* etc., such that

pq is parallel to BO, meeting the line of action of Q in *q*,
qr is parallel to CO, meeting the line of action of R in *r*,
rs is parallel to DO, meeting the line of action of S in *s*.

Lastly, from *s* draw a line parallel to EO and from *p* draw a line parallel to OA meeting in *t*.

Through *t* draw a line *t*X parallel to AE.

We now prove that *the resultant force is equal and parallel to AE and acts through t.*

The force P acting at p=AB=AO along tp+OB along qp
The force Q acting at q=BC=BO along pq+OC along rq.
The foree R acting at r=CD=CO along qr+OD along sr.
The force S acting at s=DE=DO along rs+OE along ts.

The forces acting along the lines *pq*, *qr*, and *rs*, being equal and opposite and collinear in action, cancel in pairs, and we are left with only the first force AO along *tp* and the last force OE along *ts*, and these forces have a resultant, say X, equal and parallel to AE acting at the point *t*.

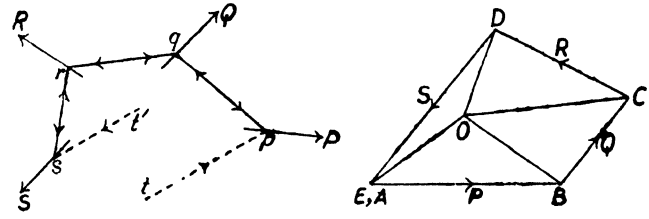
The figure ABCDE is called the **Force Polygon**, the point O is called the **Pole** and the figure *pqrst* is called a **Funicular** or **Link Polygon.**

Note 1.—We have assumed that the given forces are acting upon the same rigid body, but if the points *p*, *q*, *r*, *s* and *t* were joined by light ropes (or by smoothly jointed light rods) forming the sides of the link polygon, the force X reversed and the forces P, Q, R, S would maintain equilibrium.

This justifies the name funicular (rope) polygon.

Note 2.—It is clear that by taking different positions of the point O and *p*, the shape of the funicular polygon can be changed in any number of ways but the final point of intersection of the lines *pt* and *st* will always lie on the same straight line, the line of action of the resultant X.

13.21. The resultant, a couple. When the first point A of the



force polygon coincides with the last point E, *i.e.*, if the polygon is closed, the resultant is zero.

This does not mean that the forces are necessarily in equilibrium.

In fact the forces may reduce to a couple or they may be in equilibrium. The funicular polygon helps us, in distinguishing the two cases.

When A and E coincide the lines pt and st' drawn through p and s parallel to OA and EO (*i.e.*, AO) will be parallel and the forces acting along them will be equal and opposite, so that the resultant is a couple whose moment is equal to

OA × distance between the parallel lines pt and st'.

Note. The above discussion shows graphically that a system of coplanar forces, not in equilibrium, reduces either to a single forces or to a single couple.

3.22. Conditions of equilibrium. If, however, the lines pt and st' coincide, *i.e.*, stp is a straight line, the equal and opposite forces along them cancel and the given forces are evidently in equilibrium. In this case the funicular polygon is also closed.

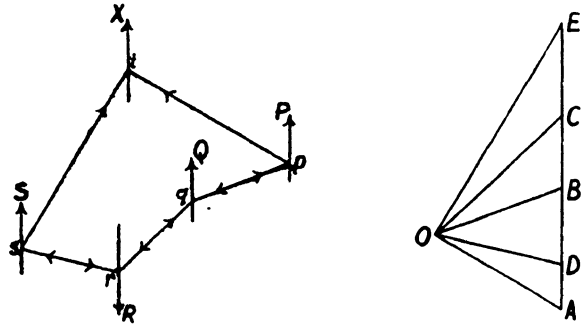
Thus in order that a system of coplanar forces may be in equilibrium, two condition must be satisfied :

(1) The Force polygon must be closed.

(2) The Funicular polygon must be closed.

These conditions are necessary as well as sufficient.

13.23. Parallel Forces. The same method applies to the particular case when the forces are parallel. In this case the force polygon reduces to a straight line.



Let P, Q, R, S be the parallel forces acting along given lines. Let AB, BC, CD, DE represent these forces in order, on some scale.

Take any point O as the pole and join it to A, B, C, D and E.

Take a point p on the line of action of P and construct the funicular polygon $pqrst$ as in Art. 13.2. From the same proof as given in Art. 13.2, we get the resultant X acting at t.

Example. A light rod AB supported horizontally at two points, has a weight of 5 lbs. at A and weights of 4 lbs. and 9 lbs. are placed on it at distances of 3 and 9 inches from A. If the reaction at one of the supports situated at a distance of 7 inches from A, is 7 lbs., find graphically, the other reaction in magnitude and position and verify the result by calculation.

[*Ans.* 11 lb., parallel, 4 ft. from A.]

Draw the force and funicular polygons just as done above and measure the horizontal distance of t from the line of action of the force at A.

13.24. *A closed polygon of light rods freely jointed at their extremities is in equilibrium under the action of a given system of forces applied at the joints ; to find the stresses along the rods.*

In Fig. (*i*) Art 13.2, take the force X in the reversed direction, so that the forces P, Q, R, S and X form a system in equilibrium and the lines *pq*, *qr*, *rs*, *st* and *tp* represent the rods freely jointed at their ends where the forces act.

Draw the force polygon ABCDEA having its sides parallel and proportional to the forces P, Q, R, S, X respectively. Since the forces are in equilibrium, this polygon must close.

Now draw AO parallel to *tp* and BO parallel to *pq*, meeting in O. Join OC, OD, OE, then OA, OB, OC, OD, OE represent in magnitude and direction the stresses along the rods to which these lines are parallel.

This is proved as follows :—

Let T_1, T_2, T_3, T_4, T_5 be the stresses along the rods *pq*, *qr*, *rs*, *st*, and *tp*.

The triangle AOB has its sides parallel to the forces P, T_1 and T_5 which act at the point *p* and are in equilibrium. Its sides are, therefore, proportional to these forces ; hence the sides BO and OA represent T_1 and T_5 on the same scale that AB represents P.

Again the sides OB, BC of the triangle OBC represents two of the forces T_1 and Q which act at *q*, hence CO represents the third force acting at *q*, *i.e.*, T_2.

Similarly DO represents T_3 and EO represents T_4.

Note. Whenever a *light* rod is in equilibrium under the action of forces which can act only at its ends then the forces at the ends must act *along* the rod and must be equal and opposite. A rod that is in compression or that is being pushed is called a **'strut'** and the one that is being pulled or is in tension is called a **'tie'**.

13.241. Reciprocal Figures. It is clear that ABCDE is the force polygon and *pqrst*, a funicular polygon for the forces P, Q, R, S and X.

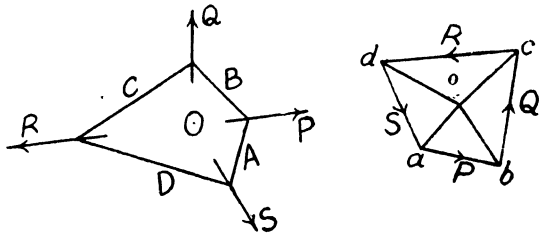
Also if ABCDE is taken to represent a jointed framework of rods acted on by forces OA, OB, OC, OD, OE, then *pqrst* is the force polygon for these forces, ABCDE being the funicular polygon.

Hence either of these two polygons may be taken as the framework or funicular polygon, then the other is the force polygon. For this reason such figures are called *Reciprocal Figures.*

13.3. Bow's Notation.

Let the space between the lines of action of forces P and Q be denoted by B, that between Q and R by C and so on. In the force polygon the force P is represented by the line *ab* while in the funicular polygon the same force is denoted by AB (the same capital letters), *i.e.*, by the names of spaces it separates.

Corresponding to a point *o* within the force polygon, we have O, denoting the space within the funicular polygon.



This is known as Bow's or Henrici's notation and has the following advantages :—

(1) The same letters represent a given force in both polygons.

(2) The lines in the funicular polygon drawn parallel to *oa*, *ob*, etc., lie in spaces A, B etc.

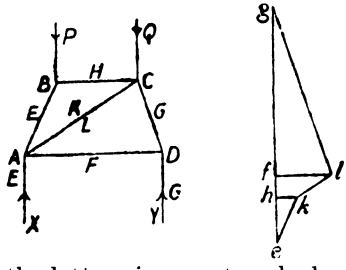
(3) The point of application of any force, say P, is denoted by AOB, so that the forces meeting at any point correspond to the traingle *aob*, the triangle of forces for the point.

This notation is specially useful in framework problems.

Note. Bow's notation does not apply if there are forces on the rods at points other than the hinges.

Example. *ABCD is a framework of light rods stiffened by a brace AC. It is supported in a vertical plane by supports at A and D so that AD and BC are horizontal and AB, CD equally inclined at a given angle to AD. If weights P and Q are placed at the hinges B and C, find the stresses in the rods and the thrusts X and Y at A and B.*

Following Bow's notation areas separated by the lines of the figure are marked as E, F, G, H, K, L. Any force in the figure is named by the letters of the two areas separated by its line of action.



Start with some point say B, one out of the three forces acting at the point being known.

The forces at B are HE (*i.e.*, the known weight P), EK and KH taking the letters in counter-clockwise direction).

Draw *he* to represent P (on some scale) *ek* parallel to AB and *hk* parallel to BC, meeting in *k*., then *ek* and *kh* represent the forces EK and KH, *i.e* , the stresses in AB and BC.

Now consider the point C ; the forces acting at it are GH, (*i.e.*, the known weight Q), HK (now known), KL and LG.

Draw *gh* to represent GH or Q and complete the force polygon *ghkl*, by drawing *kl* parallel to AC and *gl* parallel to CD, meeting in *l*.

The sides *kl*, *lg* represent the forces KL, LG, *i.e.*, the stresses in AC and CD.

Now take the point D, the forces acting at it are GL (now known), LF and FG. Complete the force triangle *glf* by drawing *lf* parallel to DA, meeting *gh* in *f*. The side *lf* represents the force LF *i.e.*, the stress in AD, and *gf* represents GF, *i.e.*, Y.

Now the figure is complete, the forces at A, KE, EF or X, FL and LK being represented by the sides *ke*, *ef*, *fl* and *lk*.

Ex. Given AB=CD=3, BC=2 and AD=4, P=100 lbs., Q=50 lbs. show that X=87·5. Y=62·5 and that the stresses along AC and AD are 18·2 and 22 lbs. Also show that AC, AD are ties while the other three rods are struts.

Example XLVI

1. AB is a straight line, 3 ft. long ; at A and B act (*i*) like, (*ii*) unlike parallel forces equal to 70 and 50 lbs. respectively. Find graphically the position of the point where the resultant acts and verify the results by calculation.

2. Like parallel forces of 3, 5, 7 and 11 lbs. act at points in a straight line at distances of 4 in. from each other. Find the position of their resultant graphically and verify it by calculation.

3. A horizontal beam AB, 18 ft. long has loads 5, 11, 18 tons at distances of 6, 10, 15 feet from A. Find graphically the supporting forces at the ends.
[*Ans.* 11·22, 22·78 tons.]

4. An equilateral framework ABC formed of light jointed bars, is suspended by means of two vertical strings attached to A and B so that AB is horizontal. A weight of 15 lbs. hangs from C, find the stresses in the bars of the frame. [*Ans.* 8·66, 8·66, 4·33 lbs. wt.]

5. A square framework ABCD, formed of light rods AB, BC, CD and DA, is hung up by the corner A and at C a weight W is suspended. The shape of the square is maintained by a light rod BD. Show graphically that the stress in BD is W and verify the principle of virtual work.

6. ABCD is a light string attached to the points A and D, in the same horizontal line, and at the points B and C are attached weights. In the position of equilibrium the distances of the points B and C below the line AD are respectively 4 and 6 feet. If the lengths of AB and CD be respectively 6 and 8 feet and the distance AD is 14 feet, find the weight at C, the weight at B being 4 lbs. [*Ans.* 15·2 lbs.]

7. ABC is a triangle whose sides AB, BC, CA are respectively 12, 10, and 15 inches long and BD is the perpendicular from B on CA. Find by means of the force and funicular polygon the magnitude and the line of action of the resultant of the following forces :

(*i*) 8 from A to C, (*ii*) 8 from C to B, (*iii*) 3 from B to A and (*iv*) 2 from B to D.

[*Ans.* 2·6 lbs. cutting BC and AC produced at distances 19·25″ and 17·6″ from C.]

8. A triangular frame of jointed rods ABC, right angled at A can turn about A in a vertical plane. The side AB is horizontal and the corner C rests against a smooth vertical stop below A. If AB=3 ft., AC=1 ft. and a weight of 50 lbs. be hung on at B, find graphically the stresses in the various rods.
[*Ans.* 150, 158·115, 50 lb. wt.]

9. A uniform beam AB weighing 60 lb. is supported by strings AC and BD, the latter being vertical and the angles CAB and ABD are each 105°. The rod is maintained in this position by a horizontal force X applied at B. Show graphically that the value of X is about 15 lb. wt.

10. A and F are two points joined by a string whose length is greater than AF. At intermediate points B, C, D, E etc , of the string weights are attached, each equal to W. Assuming that each section of the string slopes upward to the right, show by means of a force diagram that the tangents of the inclinations of the successive portions of the string to the horizon are in Arithmetical progression.

11. Three light bars are jointed together to form a triangular frame-work ABC in which the angles A and C are each 30°. The framework can turn in a vertical plane about the point B, and is kept in equilibrium with AB horizontal by a weight of 100 lbs. hung at C and a vertical force P at A. Find graphically the magnitudes of the force P and of the stresses in the rods.

[*Ans.* P=50 lb. wt. ; Thrusts in AD=86·6, in BC=173 ; tension in AC=100.]

12. A portion of a Warren girder consists of three equilateral △s ABC, ADC, BCE, the lines AB, DCE, being horizontal and AB uppermost. It rests on supports at D and E and carries 3 tons at A and 1 ton at B. Find the reactions at the supports and the stresses in the four inclined bars, stating whether they are in tension or compression.

[*Ans.* 2·5 tons at D, 1·5 tons at E. Thrusts in AD=2·88 tons, AC=·58 ton, BE=1·73 tons ; tension in BC=·58 ton.]

HINTS FOR SOLUTIONS

Ex. II. P. 12

4. By (2) Art. 2.12, $\tan\theta = \dfrac{2P\sin\alpha}{P+2P\cos\alpha} = \dfrac{4P\sin\alpha}{(P+4)+4P\cos\alpha}$,

$\therefore$ $2P+4P\cos\alpha = P+4+4P\cos\alpha$, *i.e.*, $P=4$.

9. Fig. Art. 2.11, $\angle AOC = \frac{1}{3}\alpha = \angle OCB$ and $\angle COB = \frac{2}{3}\alpha$

$$\therefore \quad \frac{P}{Q} = \frac{BC}{OB} = \frac{\sin\frac{2}{3}\alpha}{\sin\frac{1}{3}\alpha} = 2\cos\tfrac{1}{3}\alpha, \text{ i.e., } \alpha = 3\cos^{-1}\frac{P}{2Q}.$$

Now use $\cos\alpha = 4\cos^3\frac{1}{3}\alpha - 3\cos\frac{1}{3}\alpha$ and then find R.

10. The resultt. makes angles $\alpha-\theta$, $\alpha+\theta$, with the forces.

Hence $\dfrac{P-Q}{P+Q} = \dfrac{\sin(\alpha-\theta)}{\sin(\alpha+\theta)}$, find $\dfrac{P}{Q}$.

12. Solve it geometrically.

13. Use (2) Art. 2.12.

17. Fig. Art. 2.11, P being fixed in magnitude and direction, A is a fixed pt. ; AC=OB=Q being fixed in magnitude, locus of C is a circle with centre A.

The inclination of OC with OA is greatest when OC is tangent to the circle,

18. If α is the angle between P and Q and θ that between R and P, then

$$R\cos\theta = P+Q\cos\alpha = Q.$$

$$\therefore \quad P = Q(1-\cos\alpha) = 2Q\sin^2\frac{\alpha}{2}.$$

20. Apply Art. 2.14.

Ex. III. P. 17

4. Let EG, FH intersect in O, EG=EO+OG, FH=FO+OH.

Forces EO+OH=EH=$\frac{1}{2}$BD, also FO+OG=FG=$\frac{1}{2}$BD.

$\therefore$ EG+FH=BD. Now BD+DB=0.

5. Fig. Art. 2.11, produce OB to B′ making OB′=Q′ and complete the parallelogram OAC′B′, so that OC′=R′. Now the resultt. of R′ reversed and R, *i.e.*, of C′O and OC is C′C which is the difference of OB′ (Q′) and OB (Q).

7. (*i*) Forces AP and PB=AB ; AB and BC=AC acting at P. Thus PQ is equal and parallel to AC.

(*ii*) D mid-point of BC ; forces PC and PB=2PD.

Now AP+2PD=AP+PD+PD=AD+PD=PD+DQ=PQ, where Q is on AD such that DQ=AD.

8. P acts along AD ; D being the pt. where Q and R meet. CAD or BAD is the triangle of forces. Find the length AD.

Ex. IV. P. 20

9. Let AB=2*a*, α the angle, each portion of the string makes with the vertical and T, the tension, then by symmetry, $2T\cos\alpha = 15$. Also by Hooke's law $T = \lambda(a\ \text{cosec}\ \alpha - a)/a$. $\alpha = 30°$ satisfies the equation in α obtained by eliminating T.

10. W′ is attached to C between A and B, so that $\angle ACB = 120°$. The tensions 2W, 3W acting along CA and CB and weight W′ acting along the verti-

cal CD are in equilibrium. Take $\angle DCA=\alpha$, applying Lami's theorem obtain $\tan\alpha=3\sqrt{3}$.

12. (*i*) $\angle BOC=\pi-2A$, $\angle COA=\pi-2B$, and $\angle AOB=\pi-2C$.

$$\therefore \frac{P}{Q}=\frac{\sin 2A}{\sin 2B}=\frac{\sin A\cos A}{\sin B\cos B}=\frac{a(b^2+c^2-a^2)}{2bc}\div\frac{b(c^2+a^2-b^2)}{2ca}$$

$$=\frac{a^2(b^2+c^2-a^2)}{b^2(c^2+a^2-b^2)}.$$

(*ii*), (*iii*) see Ex. (*ii*), (*i*) Art. 2.32.

(*iv*) $\triangle OBC=\frac{1}{2}OB.OC\sin BOC=\frac{1}{3}\triangle ABC$.

$$\therefore \sin BOC=\frac{2}{3}\cdot\frac{OA.\triangle ABC}{OA.OB.OC}=k.OA \text{ (say)},$$

Similarly $\sin COA=k.OB$, etc.

13. Use the geometrical fact that an angle between a tangent to a circle and a chord through the point of contact is equal to an angle in the alternate segment.

14. Draw BT tangent at B towards the side of C. Produce BT to T′ and AB to A′.

Let the forces along AB and BC be k/AB, *i.e.*, k/c and k/BC, *i.e.*, k/a. Their resultant R acts along the tangent BT if R acting along BT′ is in equilibrium with k/c along BA′ and k/a along BC,

i.e., if $\frac{k}{c}\Big/\sin CBT'=\frac{k}{a}\Big/\sin A'BT'$

i.e., if $c\sin CBT=a\sin ABT'$, *i.e.*, if $c\sin A=a\sin C$,

which is true. (Note that this is converse of Q. 13).

Aliter. Sum of the resolved parts of the forces along the radius BO

$$=\frac{k}{a}\cos CBO-\frac{k}{c}\cos ABO=\frac{k}{a}\cos(90°-A)-\frac{k}{c}\cos(90°-C)$$

$=k\left(\frac{\sin A}{a}-\frac{\sin C}{c}\right)=0$, so that the resultant has no component along the normal BO. Thus the resultant acts along the tangent at B.

Ex. V. P. 23

5. By (1) of II Art. 2.41, F is least when $\sec\theta$ is least, *i.e.*, when $\theta=0$.

7. As the weight slides down the plane, the angle θ which the string makes with the plane gradually decreases.

As θ decreases, $\sec\theta$ also decreases, $\therefore$ by (1) of II Art. 2.41, F or the tension in the string decreases.

Again by equation (2),

$$R=W\cos(\alpha+\theta)/\cos\theta=W(\cos\alpha-\sin\alpha\tan\theta)$$

which increases as θ (and therefore $\tan\theta$) decreases.

Ex. VI. P. 27

1. BC slopes upwards. **2.** BC slopes downwards.

5. A length 6 feet of the string is parallel to the upper edge of the board and is the base of the isosceles triangle formed with the oblique parts of the string, each 5 ft. long.

6. The portion of the string below the lowest pegs C and D, of length $l-3a$, form an isosceles triangle with CD as the base and the ring as the vertex.

7. First consider the equilibrium of A and find the tension T in AB or AC, by the equation $2T \cos 30° = W$.

16. Use triangles of forces ABG and ACG, G is mid. pt. of AB.

17. Use triangles of forces. Note that tension throughout the string is the same.

19. P, Q the points on the sphere where the weights rest. Let $OP = l_1$, then $OQ = l - l_1$; OPC and OQC are triangles of forces.

20. Apply Lami's theorem to stresses S_1, S_2 and tension T acting at A.

Ex. VII. P. 32

3. The resultant also passes through D, the mid-pt. of AB.

$\therefore$ P lies on CD.

11. $P = \frac{P}{OA} \cdot OA$, $Q = \frac{Q}{OB} \cdot OB$. Now apply Art. 3,1.

12. AOA′ is the diameter through A, BPCA′ is a parallelogram.
$\therefore$ PA′ and BC bisect each other in D.

Now, forces PB and PC = 2PD = PA′, $\therefore$ forces AP + PA′ = AA′.

13. Use Fig. of Q. 12, replacing P by H. Forces BH and CH = 2DH = A′H, D being mid. pt. of BC. Now A′H + AH = 2OH, O being mid. pt. of AA′,

14. Show analytically that the focus is the middle point of TG.

Ex. VIII. P. 37

5. Note that the numbers 7, 31 − 7 = 24 and 25, form a right-angled triangle.

6. Introduce two equal and opposite forces, each equal to 1 lb. along the third side. Forces 2, 6, 10 and 12 acting along the sides and being proportional to their lengths, are in equilibrium.

7. Resultant of forces AP, AQ, AR is 3AG, G being the centroid of P, Q, R and forces PB, QB and RB = 3GB

$\therefore$ forces 3AG + 3GB = 3AB.

Or forces AP, PB = AB, etc.

10. Let E be on AB and F on AC, then BCEF being cyclic, $\angle AEF = \angle C$.

The component of P along BA is P cos AEF = P cos C, and the component along the perp. line EC is P sin C. Now by Art, 2,14, resolved parts of P sin C along BC and AC are

P sin C sin (90° − A)/sin C, *i.e.*, P cos A and P sin C. sin (90° − B)/sin C, *i.e.*, P cos B respectively.

Note that as regards directions, these forces along the sides are not in the same sense.

Conversely, it is easy to prove that the resultant of these forces along the sides taken in their proper directions, is the force P along EF.

15. Resolve along and perp. to the fifth side of the hexagon.

Ex. IX. P. 40

6. Through the mid-pt. G of the rod AB, draw GD parallel to AC, C being the peg. GCD is the triangle of forces.

12. The line of action of the weight, the reaction at A both meet the string BD at its mid-pt. O. Draw BF horizontal, meeting the wall in F, then

$BF = a \sin\theta$ and $DF = 2AF = 2a \cos\theta$.

$\therefore$ $BD^2 = l^2 = BF^2 + FD^2 = a^2 \sin^2\theta + 4a^2 \cos^2\theta$ etc.

15. Proceed geometrically as in Cor. 2, Ex. 2, Art. 6,9, P. 137.

17. Each $=W$.

Ex. X. P. 46

13. Resolving along and perp. to the plane and simplifying, we obtain

$$P=W\sin\alpha-\mu W\cos\alpha \qquad \ldots(1)$$

and
$$Q(\cos\alpha+\mu\sin\alpha)=W(\sin\alpha-\mu\cos\alpha)=P \qquad \ldots(2)$$

Now divide (1) by W and (2) by Q and square and add to eliminate α.

Misc. Ex. I. P. 48

3. A_1PA_2, A_3PA_4 are equally inclined at $\angle\alpha$ to the radius OP. Draw OC, OD perp. to these chords, then $OC=OD$, $CP=DP$. Forces PA_2, $PA_1=2PC$; PA_4, $PA_3=2PD$;

$$2PC,\ 2PD=4PC\cos\alpha=4PC\sec\alpha\cos^2\alpha=4PO\cos^2\alpha.$$

4. $\because$ $\tan A=OD/AD$, $\tan B=OD/BD$,

$\therefore$ $AD\tan A=BD\tan B$, hence the resultt. acts along AD and is

$$=(\tan A+\tan B).OD=\left(\frac{OD}{AD}+\frac{OD}{BD}\right)OD=\frac{AB}{AD.BD}\cdot OD^2$$

$$=AB.\frac{OD}{AD}\cdot\frac{OD}{BD}=AB\tan A\tan B.$$

5. The resultt. has also to pass through the mid-pt. of BC.

7. Resultt. of forces along OB and OC prop. to AC and AB meets BC in D such that $AC.BD=AB.CD$, *i.e.*, $BD/CD=AB/AC$, so that AD is the bisector of $\angle BAC$, etc.

8. When $l=a\sqrt{2}$, the tension is zero and the weight is supported by only the reaction of the wire, at its lowest point.

9. The ring being smooth, the tension T throughout the string is the same. P being equal and opposite to the resultant of two equal tensions in the two parts OA and OB of the string, it makes equal angles θ with OA and OB, so that

$$2T\cos\theta=P \qquad \ldots(1).$$

If P meets AB in D, then $\angle ODA=\alpha$, $AB=c$.

and
$$\frac{\sin\theta}{\sin\alpha}=\frac{AD}{OA}\ \text{also}=\frac{BD}{OB}\qquad\therefore\ =\frac{AD+BD}{OA+OB}=\frac{c}{l}.$$

$\therefore$ $\sin\theta=\frac{c}{l}\sin\alpha$. Now (1) gives T.

12. Let AC, BD be parallel lines, AB third line. Let the force F intersect AC in C. Join CB, resolve F into F_1 along AC and F_2 along CB, and then resolve F_2 along DB and AB.

14. A, B are fixed pts. On AB construct a segment of a circle containing an angle ACB equal to the given angle.

Let the resultt. of the given forces acting along CA, CB intersect the circle in D. $\because$ $\angle ACB$ is constant and forces are of given magnitude, $\therefore$ $\angle ACD$ is const. $\therefore$ D is a fixed point.

15. Let T be tension in AB or CD, T_1 in BC, then by Lami's theorem

$$\frac{T}{\sin 90^\circ}=\frac{T_1}{\sin(180^\circ-\theta)}=\frac{\lambda}{\sin(90^\circ+\theta)}$$

$\therefore$ $T=\lambda\sec\theta$ and by Hooke's law, $T=\lambda(AB-l)/l$.

$\therefore$ $l\sec\theta=AB-l$ or $AB=l+l\sec\theta$.

Also $T_1=\lambda\tan\theta=\lambda(BC-l)/l$, *i.e.*, $BC=l+l\tan\theta$.

Now $AD=2l=2AB\sin\theta+BC$ gives the desired result.

16. Equate the values of tensions in BC obtained by considering the equilibrium of the weights at B and C.

17. T, the tension in each string, α the angle each makes with the rail, R the vertical reaction on a ring. For one of the rings

$$T \sin\alpha = R,\ T\cos\alpha = \mu R,\ \therefore\ \mu = \cot\alpha.\quad \text{Also } l\cos\alpha = c - a.$$

19. Considering Q which is on the point of motion down the plane of inclination β.

T, tension in string $= Q\sin\beta - \mu R = Q(\sin\beta - \mu\cos\beta) = Q\sin(\beta-\lambda)/\cos\lambda$.

Similarly considering P, on the point of motion up the other plane,

$T = P\sin(\alpha+\lambda)/\cos\lambda$, Hence $P\sin(\alpha+\lambda) = Q\sin(\beta-\lambda)$ ...(1).

If x be the required weight which when added to P makes the system just on the point of motion in the reversed direction, obtain another equation (2) by replacing P by $P+x$ and by changing the sign of λ in (1). Then find x by eliminating P from (1) and (2).

R. Q. I. P. 51

1. Resolving horizontally $T_1\sin\alpha = T_2\sin\alpha$, *i.e.*, $T_1 = T_2$. By Hooke's law $T_1 = \lambda(c\ \text{cosec}\ \alpha - a)/a = T_2 = \mu(c\ \text{cosec}\ \alpha - b)/b$.

5. (*b*) Find the length $MN = a/\sqrt{3}$ and apply Ex. 11 Misc. Ex. 1.

7. Let R, R′ make angles θ, θ' with P, then

$R\cos\theta = P + Q\cos\alpha$; $\quad R'\cos\theta' = P' + Q'\cos\alpha$

$R\sin\theta = Q\sin\alpha$; $\quad R'\sin\theta' = Q'\sin\alpha$.

$\therefore\ RR'\cos(\theta - \theta') = (P + Q\cos\alpha)(P' + Q'\cos\alpha) + QQ'\sin^2\alpha$.

8. Obtain $\tan\theta$, $\tan\theta'$ and put $\theta' = \theta + \alpha$.

9. $R' = k^2\cos^2 A + k^2\cos^2 B + 2k^2\cos A\cos B\cos C$

$= k^2[\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C] - k^2\cos^2 C$
$= k^2 - k^2\cos^2 C = k^2\sin^2 C$.

10. Force $AB = AO + OB$, $2BC = 2BO + 2OC$, $3CD = 3CO + 3OD$, etc. Thus

$AB + 2BC + 3CD + \ldots = AO + BO + CO + \ldots + n.OA$
$= BO + CO + \ldots + (n-1)\,OA$
$= (n-1)\,OA$. Cor. Art. 3.21.

15. Let ABCDE be a polygon of five sides. Let P_1, P_2, P_3, P_4, P_5 be the forces proportional to the sides AB, BC etc., then by note, Ex. (*iii*) Art. 2.32, P. 20, the resultt. of P_1, P_2 is a force acting at the mid-pt. of AC perp. and prop. to AC. Again the resultt. of this force and P_3 is a force at the mid-pt. of AD perp. and prop. to it. Now this force, P_4 and P_5 are in equilibrium. The argument can be extended to the case of a polygon of any number of sides.

Ex. XI. P. 55

9. It implies that $P < Q < 2P$.

12. Let AO meet BC in D and let P and Q be the required forces at B and C, then P.BD = Q.DC.

$$\therefore\quad \frac{P}{DC} = \frac{Q}{BD} = \frac{P+Q}{BD+DC} = \frac{P+Q}{BC} = \frac{R}{a} \qquad \ldots(1)$$

(*a*) $\dfrac{BD}{DC} = \dfrac{AB}{AC} = \dfrac{c}{b}$ or $\dfrac{BD}{BD+DC} = \dfrac{c}{b+c}$,

i.e., $\dfrac{BD}{a} = \dfrac{c}{b+c}$, and $\dfrac{CD}{a} = \dfrac{b}{b+c}$, hence P and Q from (1).

(*b*) $DC = AC\cos C = b\cos C$ and $BD = c\cos B$.

(c) $$\frac{BD}{DC}=\frac{\sin \angle BOD}{\sin \angle COD}=\frac{\sin(\pi-2C)}{\sin(\pi-2B)}=\frac{\sin 2C}{\sin 2B}$$

$$\therefore \quad \frac{BD}{\sin 2C}=\frac{DC}{\sin 2B}=\frac{BD+DC}{\sin 2B+\sin 2C}=\frac{a}{2\sin A\cos(B-C)} \text{ etc.}$$

Ex. XII. P. 59

3. The vertical line midway between the strings passes through the C. G. of the cone.

7. Without W′, the rod is horizontal and the peg bisects the string. With W′, the resultant of the two equal tensions, bisecting the angle between two parts of the string is vertical and meets the rod at the point where the resultant of two parallel forces, W and W′ must act.

Ex. XIV. P. 67

3. Take moments about the centroid and resolve perp to BC.

7. Resolve along and perp. to AC.

12. Resolve along and perp. to AB.

15. Since the moments about A and B are equal and opposite, the line of action of the force must meet AC in its mid-pt. D. Again since the moments about B is twice that about A, the force must meet BA produced at E such that BE=2AE. Now AD, AE being known, find $\angle AED$ and the length of the perp. from A on DE.

Ex. XV. P. 70

10. Resolve vertically and take moments about one end in each case.

11. See second solution of Ex. 3, Art. 9.55, P. 201.

12. Take moments about P the point of contact with the peg.
$AP=\frac{1}{4}l \sec\theta$ and the distance of the string from $P=\frac{1}{4}l\tan\theta$.

14. Take moments about C and resolve forces along the rod.

15. Find $\angle ACB$ by equating the values of tension obtained by Hooke's law and by taking moments about A.

16. Let X, Y be the components (as in solved Ex. 1.) of the reaction, then $X=T\sin\alpha$, $Y=\frac{3}{2}W-T\cos\alpha$, α being the angle the string makes with the vertical. Squaring and adding, we have

$$X^2+Y^2=\tfrac{9}{4}W^2+T^2-3WT\cos\alpha,$$

where $\cos\alpha=(c^2+l^2-a^2)/2cl$, a being the length of the rod.

Now equation of moments about the hinge gives,

$$\tfrac{1}{2}W.l\sin\alpha+W.\tfrac{1}{2}l\sin\alpha=T.c\sin\alpha, \textit{ i.e.}, T=Wl/c.$$

17. Take moments about A.

19. Take moments about the point of contact C with the table.

Distance of P from C is $r\sin\theta$, and that of Q it is $r\sin(\alpha-\theta)$.

$\therefore$ $P.r\sin\theta=Q.r\sin(\alpha-\theta)$, now find $\tan\theta$.

20. Let $\theta=\angle BDA=\angle BAD$; T, tension in the string; $AB=a$; W, the weight of the lamina; G, its C.G.

Resolve vertically, $T\cos\theta=W$. Take moments about A,

$$W.AG\sin(30°+\theta)=T.AD\sin\theta,$$

where $AG=a/\sqrt{3}$, $AD=2a\cos\theta$.

Now eliminate T and obtain $\tan\theta=1/3\sqrt{3}$.

21. Find θ, the inclination of the string to the vertical by equating the values of the tension obtained by resolving vertically and by taking moments about the corner in contact with the wall. Proceed as in Ex. 15.

Ex. XVI. P. 74

9. In order that CD may be the smallest, the support D must coincide with the centre of the weights, so that the system is just on the point of turning about D.

(*ii*) Let the portion cut off be x ft., then the weight of the rod of length $12-x$ acts at a distance $\frac{1}{2}(12-x)$ ft. from either end. Now x is the greatest when the sytem is on the point of turning about D. Take moments about D.

Ex. XVII-A. P. 78

1. Take moments about the sides of the traingle taking the forces to be acting perp. to the plane of the triangle.

2. See Ex. (*i*) Art. 4·83.

Ex. XVII-B. P. 79.

8. Take moments about the top of the obstacle.

10. While calculating the pressure of the table on the prism do not forget to take the vertical component of the thrust.

Misc. Ex. II. P. 80

2. Draw AL perp. on Q, and AM on R, moments about A gives

$$Q.AL=R.AM \text{ or } \frac{Q}{R}=\frac{AM}{AL}=\frac{OA\sin AOC}{OA\sin AOB}=\frac{\sin AOC}{\sin OAC}\cdot\frac{\sin OAC}{\sin AOB}$$

$$=\frac{AC}{OC}\cdot\frac{\sin OAB}{\sin AOB}=\frac{AC}{OC}\cdot\frac{OB}{AB}, \text{ etc.}$$

3. Take moments about a pair of opposite vertices.

6. Take moments about the vertices of the pedal triangle.

7. Take moments about the sides of the triangle.

8. The plate being horizontal, the extension in each string is the same.

$\therefore$ tension T_1 in AP or CR : tension T_2 in BQ or DS $=\lambda:\mu$

Also $W=2T_1+2T_2$.

9. Let GL, G being below AB, be perp. on AB, and let vertical through G meet AB in C. Taking moments about B, we have

$$P.AB\cos\alpha=W.BC\cos\alpha, \textit{ i.e.}, P.a=W(BL-LC)$$
$$=W(BL-GL\tan\alpha).$$

Similarly $Q.a=W(BL-GL\tan\beta)$. Subtract and find GL.

12. Taking moments about C, $F.AC=W.6$, *i.e.*, $F=6W/13$.

Friction at C = horizontal component of F

$$=F\cos FAD=F\cos BAC=\frac{6W}{13}\cdot\frac{5}{13}=\frac{30W}{169}$$

C will not slip if $F<\mu R$. Find the normal reaction R carefully.

14. The resultt. of friction F and wt. W meets P in O, a point different from the centre. $\therefore$ the normal reaction must also pass thro' O. If $\mu>1$, it is easy to prove that the cube topples before it slides.

15. (*i*) In case the bar slips at A, C is fixed, take moments of the horizontal forces (P and friction at A) about C.

Similarly find the value of P in the second case.

16. When the rod is just on the point of slipping about C ($AC=c$, $AB=a$), friction on an element of length δx of the rod at a distance x from A is $\mu w\,\delta x$ and its moment about $A=\mu w\,\delta x.x$, where w is the weight per unit length of the rod, The rotation of the portions AC and CB being in opposite directions,

taking moments about A for AC and CB, we have $\int_o^c \mu wx dx - \int_c^a \mu wx\, dx = 0$,

which gives $c^2 = a^2 - c^2$, *i.e.*, $c = a/\sqrt{2}$. $\therefore\ c/(a-c) = a/\sqrt{2} \div (a - a/\sqrt{2}) = 1 / (\sqrt{2}-1)$
$= \sqrt{2}+1 : 1.$

R. Q. II P. 83

5. See hint for Q. 9, Misc. Ex. II, (given above).

6. Let O be the centre where nW acts (along OB) and C, the point on the rim where W is attached. The resultt., $(n+1)$ W acts at G on OC such that $OG = OC/(n+1) = a/(n+1)$. AG, being vertical, is parallel to OB. Similarly OA is parallel to BG. Since OA = OB, OAGB is a rhombus and its diagonals OG, AB bisect each other at right angles at E.

$\therefore \qquad \angle AOB = 2\angle AOE.$

$\therefore \qquad \sec \angle AOE = OA/OE = OA/\frac{1}{2}OG = 2(n+1).$

7. Take moments about AB, BC and AC.

8. G, the centre of the plank which is in contact with the highest point of the cylinder in the first position, C the point of contact in the displaced position, $GC = a\theta$, find the sum of the moments of the three weights about C.

9. Resolve along and perp. to the rod in its displaced position and take moments about the point of contact. $CG = a\theta$ as in Ex. 8.

11. Prove $AC/CB = M_1/M_2$. The line of action of R is the common tangent to the circles drawn with centres A, B and radii M_1/R, M_2/R respectively.

12. (*i*) The bisector of the angle between the lines of action of P and Q, the direction of one of them being reversed.

(*ii*) The line passing through their point of intersection making angles α, β with P and Q respectively such that $P \sin \alpha = Q \sin \beta$, the direction of one of them being reversed.

13. (*a*). Resolve forces along and perp. to BC. See Ex. (*ii*), Art 3.232.

14. Equate the values of tan θ for the two systems and simplify. See Ex. (*ii*), Art. 3.232.

15. (*a*). Taking moments about the incentre, we have

$$S.r = P.r + Q.r + R.r,$$

i.e., the resultant $S = P+Q+R$ or $S^2 = (P+Q+R)^2$.

Now equate it to the value given in Q. 13 and simplify.

15. (*b*). Taking moments about the circumcentre, we have

$$S.R' = P.R' \cos A + Q.R' \cos B + R.R' \cos C$$
$$S = P \cos A + Q \cos B + R \cos C,\ R' \text{ being the circumradius.}$$

Now equating the square of this S to S^2 given in Q. 13, and usin $\cos A = -\cos(B+C) = \sin B \sin C - \cos B \cos C$ etc., it reduces to

$$\Sigma P^2 \sin^2 A - 2\Sigma PQ \sin A \sin B = 0, \text{ or, } \Sigma a^2P^2 - 2\Sigma ab\ PQ = 0,$$
$$\text{or } (aP + bQ - cR)^2 = 4ab\ PQ, \text{ or } aP + bQ - cR = -2\sqrt{ab}PQ,$$
$$\textit{i.e.}, (\sqrt{aP} + \sqrt{bQ})^2 = cR, \text{ or } \sqrt{a}.P + \sqrt{b}.Q = -\sqrt{c}.R.$$

Ex. XVIII. P. 90

4. By Art. 5.51, the moment of the resultant couple = sum of the momen of all the forces at any pt. say $A = Qb - Pa$.

7. Equate to zero the sum of the resolved parts of the forces along AI and BC and then take moments about the centre.

8. The sum of the resolved parts of the forces along and perp. to any sid vanish. Take moments about G, the centroid.

10. Let AD be the bisector of $\angle A$; AL, the altitude. In the second pos tion the resultt. (= 25 lbs.) of forces at B and C acts at D perp. to BC and forn

a couple with the force of 25 lbs. acting at A along AL in its new position. The moment $=25.DL=25\ (CD-CL)$.

11. Resolve along and perp. to AO, BO, CO. The components $ka\cos\theta$, $kb\cos\theta$, $kc\cos\theta$ along them are in equilibrium., [Ex. (*ii*), Art. 2.32]. Now take moments for other components about any vertex. Since the sum of the moments abouts A, B and C, are equal, the resultant is a couple.

Ex. XIX. P. 93

1. Introduce equal and opposite forces each represented by BD along BD. Moment of the couple = twice the area of the quadrilateral

= four times the area formed by joinining the mid-pts. of the sides.

3. Take moments about the centroid G.

$$\text{Distance of G from BC} = \frac{1}{3} \cdot \frac{2\triangle}{BC}.$$

4. Since $PB : PC=(1+\lambda) : (1-\lambda)$

$\therefore\ \frac{1}{2}(1+\lambda)\ AB+\frac{1}{2}(1-\lambda)\ AC \equiv AP$ (Art. 3.1).

Similarly force $BQ \equiv \frac{1}{2}(1+\lambda)BC+\frac{1}{2}(1-\lambda)\ BA$ etc.

5. Couple $2\triangle OBC$ = forces OB, BC, CO, etc., Art. 5.62.

Ex. 2. Art. 5.71. P 96. The force ka at A′ is equal to force ka, *i.e.*, kBC along BC and a couple of moment $2k$ times the area A′BC, etc. Also forces kBC, kCA, kAB $\equiv$ a couple of moment $2k\triangle ABC$.

Ex. XX-A. P. 102

1. (*i*) The algebraic sum of the forces must be zero, and

(*ii*) the sum of the moments about any point in their plane must also be zero. (Why ?)

1 (*a*). See Ex. Art. 5.7.

2. The sum of the moments of the forces about the incentre vanishes, for

$$(y-z)r+(z-x)r+(x-y)r=0.$$

5. Equate to zero the sum of the resolved parts of the forces along FB and FD and thus find P and Q. Now take moments about any vertex, say F or about the centre.

Aliter. Take moments about F, this does not involve P and Q and gives the required moments. Now to find P and Q, equate this moment to the sum of the moments about A and E.

6. Equate to zero the sum of the resolved parts of the forces along AB, this gives the unknown force along ED $=2$P. To ensure that now the resultant of the system is a couple, see that the sum of the resolved parts in any other direction, say along AE also vanishes. Now to find the moment of the couple, take moments about any point, say the centre.

10. At B the system $\equiv$ R, a force equal and parallel to that at A and a couple of moment G, G being equal to the moment of R about B. The moment of R at M, the middle pt. of AB is obviously $\frac{1}{2}$G.

Ex. XX-B. P. 103

6. Take $AB=a$, $AD=b$ and proceed as in Ex. 3, Art. 5.79.

7. Take the points A, B, C, as (a, o), (b, o) and (c, o).

8. Find G_2, G_3 by taking moments about B and C.

$G_2=G_1-F.c\sin\frac{1}{2}A$, $G_3=G_1+F.b\sin\frac{1}{2}A$, now eliminate F.

10. By (2) Art. 5.78, $G_1=G-Y$, $G_2=G+2X$, $G_3=G+3X-2Y$. Find X and Y and $\tan\theta=Y/X$.

Misc. Ex. III. P.106

Ex. 1. The resultant is a force acting along a line parallel to the given line. If, however, the moment is zero, either the resultant is a force acting along the given line or the system is in equilibrium.

5. Take moments about two opposite vertices. No ; for, the resultt. may act along the line joining the vertices about which moments are taken.

7. Equate to zero the sum of the resolved parts of the forces in directions perp. to the lines of action of *b* and *c*. Take moments about the point of intersection of *e* and *f*.

8. Resovle along and perp. to AB.

9. Take moments about the centre O, and resolve along lines through O, parallel and perp. to a side.

Or resolve in directions perp. to P_1 and P_2.

10. Take moments about A, B, C.

11. Take moments about two of the vertices, using the fact that angle between a tangent and the chord through the point of contact is equal to the angle in the alternate segment.

12. Equate to zero the sum of the resolved parts along two lines joining two of the vertices to the circumcentre.

14. (*a*) In Art. 5.8, take $\alpha=90°$. $\because$ $X=Y=0$, $\therefore$ $X'=Y'=0$.

(*b*) $G\cos\theta+H\sin\theta=G$, if $2H\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta=G(1-\cos\theta)=2.G\sin^2\frac{1}{2}\theta$, *i.e.*, $\tan\frac{1}{2}\theta=G/H$.

R. Q. III. P. 108

2. Introduce two equal and opposite forces along the remaining side, the forces being prop. to it and apply Art. 5.7.

3. Take moments about the line of hinges. Couple of the hinges for $30^{\circ}=\frac{1}{2}\times30=15$.

$\therefore$ $F.3=15+5=20$. $\therefore$ $F=6\frac{2}{3}$ lbs. wt.

4. Take centre O as the origin OC as OX, OD as OY, then show that $G=(R+S)\,BD\sin ABO$, $X=2(R-S)\sin ABO$ and $Y=0$.

$\therefore$ distance of the resultt. $=G/X$.

When $R=S$, $X=0$ also $Y=0$ as before. $\therefore$ the system is reduced to a couple.

7. Take moments of the weights about the lower end of the ladder. The equilibrium being maintained by a couple and not by a force, the reaction at the upper end disappears.

Ex. XXI. P. 113

3. Take moments about B and resolve along the plane, the results are independent of θ.

4. Take moments about B and resolve forces along the plane on which B rests.

5. Take X, Y for the components of reaction, resolve in horizontal and vertical directions, and take moments about A.

7. Take moments about each end of the rod, the strings being equal, they make equal angles with the rod.

8. Tension along BC and in the vertical portion of the string, each $=W/10$.

9. Tension T throughout the string being the same, the two portions of the string at the free ring make the same angle, say β, with the vertical.

$\therefore$ for this ring $2T\cos\beta=W$. ...(1).

The ring on the wire AB being smooth, the reaction on it is perp. to AB.

Resolving along AB for this ring, we get

$$W \cos \alpha = T \cos (\pi - \alpha - \beta) = -T \cos (\alpha + \beta).$$

Now eliminate T and obtain $\tan \beta = 3 \cot \alpha$, use this in (1) to find T.

14. Take moments about one hinge and resolve vertically.

15. Take moments about the middle point of AB.

16. For P to be least it must act at D along DC, take moments about A.

17. As in Fig. of Ex. 3, Art. 6.3, let R and S meet in O, then W acts along OG. Let OG meet AE in L.

According to this question $AB = l$, $AC = CD = LE = r$, $\angle BAE = \theta$, $\angle OAE = \alpha$.

$\therefore$ $l \cos \theta = AE = AF + FE = AC \cos \alpha + CD = r \cos \alpha + r$,

and $\tan \theta = \tan BAE = \dfrac{BE}{AE} = \dfrac{OL}{2AL} = \frac{1}{2} \tan \alpha$.

Ex. XXII-A. P. 117

12. The lower body must be the smoother one. Consider the equilibrium of each and eliminate the tension of the string.

15. ABCD represents the drawer with the front face $AB = a$. Due to the pull applied at a distance $\frac{1}{2}a + x$ from A, there will be reaction R at A in the direction AB and another reaction S at the opposite corner, at the back face, in the opposite sense. The frictions μR, μS oppose motion. Prove $R = S$; take moments about A and apply the condition $P > \mu (R + S)$.

Ex. XXII-B. P. 120

1. Take moments about the point of contact with the plane.

2. Resolve forces along and perp. to the plane and take moments about the point of contact.

8. The forces act on the cylinder should meet at the point of contact.

9. The inclination of the plane is equal to the angle of friction. C. G. of the hemisphere is at a distance $\frac{3}{8}a$ from its plane face, on the radius of symmetry.

10. The horizontal reaction of the wire at the lower point A is on the side of the centre O, and in the opposite side at the higher point B. Take moments about A, and apply the condition $F < \mu R$.

11. The required couple is opposed by the limiting frictions μR and μS at the points of contact and is equal to the sum of their moments about the axis

$$= \mu R a + \mu S a.$$

Resolve horizontally and vertically in each case to find R and S.

Ex. XXIII. P. 124

4. To obtain the first result take moments about O. To obtain the second result, take moments about A and write the result in the form

$$\frac{P}{a} \times 2\triangle ABC + R'.d \sin 2B - Q'.d \sin 2C = 0,$$ d being the circum-radius.

Obtain two similar equations by taking moments about B and C, multiply the equations by P′, Q′ and R′ and add.

Ex. XXIV. P. 128

2. Take moments about the centre.

3. The line in continuation of the string passes through the centre of the sphere where the weight and reaction of the plane meet. The angle θ which the string makes with the plane is given by $\sin \theta = a/(a + l)$. Resolve along the plane.

7. The resultant reaction passes through the point where lines of action of the given force and the weight of the rod meet and is equally inclined to these lines.

9. The horizontal reaction at A, the tension along the string and the vertical line through G, the C. G., must meet at a point, say O. Take moments about A and resolve in horizontal and vertical directions.

$$OA=AG\cos\angle OAG=AG\sin(\angle GAB+\theta).$$

Also $2AG\cos\angle GAB=AB$ and $\angle GAB=\frac{1}{2}\left(\pi-\frac{2\pi}{n}\right)=\frac{1}{2}\pi-\frac{\pi}{n}$.

Ex. XXV-A. P. 133

8. Applying (2) Art. 6.8 to the triangle AOD in the fig. of Art. 6.71, we get,

$$(AG+GD)\cot(90°-\theta)=GD\cot\theta-AG\cot 90°,$$

which gives,

$$\tan^2\theta=\frac{x}{l+x}=\frac{1}{1+l/x}, \text{ where } GD=x,\ AG=l.$$

θ is greatest for the least value of l/x or for the greatest value of x which is l, so that the end B of the rod is just on the rim of the bowl.

$\therefore$ $\tan^2\theta=\frac{1}{2}$ or $\sin^2\theta=\frac{1}{3}$, *i.e.*, $\sin\theta=1/\sqrt{3}$.

Ex. XXV-B. P. 134

2. Immediately deducible from the result of Art. 6.82, but may be worked out on the same lines independently.

4. The result follows at once from the result of Art. 6.82 by replacing θ by $90°-\theta$ and 2α by β.

Ex. XXVI. P. 137

2. The rod is clearly least when one end is just under the lower peg and it is on the point of slipping.

5. Take AC downwards ; take moments about C and resolve in horizontal and vertical directions.

7. Show that $AC=b\sec(\theta-\alpha)$. Resolve along the plane and take moments about A.

10. The rod $AB=2a$ is greater than $AD=r\cot\theta$; D being the point of contact with the cylinder.

Take moments about A and resolve in horizontal and vertical directions.

11. Let F be the force of friction and R, the normal reaction at A. Take moments about A and resolve in horizontal and vertical directions to find F and R. Apply condition $F<\mu R$.

Ex. XXVII. P. 141

1. (*a*) Take A(0, 0), B(a, 0), C(x_1, y_1) and let the two systems reduce to X, Y, G, and X′, Y′, G′. Equating moments of the two systems about A, we get $G=G'$, again equating the moments about B, we get $G-aY=G'-aY'$, which gives $Y=Y'$. Now do the same thing for C and get $X=X'$. Hence the two systems have the same resultant.

2. Reduce the system to a force R at the given point O and a couple G. Now replace the couple by two forces, one acting along the given line and the other equal, parallel and opposite force acting at O. Finally combine the two forces at O into one force. Deal with other forces similarly.

3. Take A as origin $AB(=a)$, as X-axis. Let the system be reduced to X, Y and G at A. Replace G by two equal opposite and parallel forces, F, acting at A and B in a given direction, say, making $\angle\alpha$ with AB. X, Y and F at A can be combined into a single force. The forces change as α changes.

Ex. XXVIII. P. 143

5. Take moments about A for the whole system.

7. Take moments about A for both the rod and the sphere, and also for the rod.

10. Resolve vertically for both beams and obtain reaction at B or C=W. Take moments about A for one beam AB. Let D, E mid-pts. of AB, AC, $\angle ACD=\alpha$, AL perp. from A on CD=AC sin α.

From $\triangle ACD$, $CD \sin\alpha = AD \sin \angle CAD = \frac{1}{2}AC \sin 2\theta$

and $CD^2 = AC^2 + AD^2 + 2AC.AD \cos(180° - 2\theta)$ etc.

12. Take X, Y components of reaction at B equal and opposite on the two rods. Take moments about A for AB and about C for BC.

13. Take moments about the joint and resolve vertically.

Ex. XXIX. P. 145

6. Take moments about B for BC and about A for both the rods.

7. Take moments about B for the rod BC and about A for both the rods taken together.

A is higher than B, B higher than C ; $\angle ABC$ is obtuse ; C is to slip backwards. Take normal reaction R, friction μR at C.

10. Take moments about O for one rod, and resolve vertically for both rods. Remember that the rods are light.

Misc. Ex. IV-A. P. 147

5. See equation (2) of solved Ex. 1.

8. The couple=moment of W about the point where the reactions of the planes meet.

9. Resolve vertically for the whole system and take moments about A for AB or AC.

11. Resolve vertically for the whole system and also for the rod BC and horizontally for AB or CD, taking X, Y for the components of the reaction at B or C in these directions.

14. Resolve along the plane for the two spheres separately.

16. Resolve vertically taking the three cylinders together and also for the upper cylinder. Also resolve horizontally for one of the lower cylinders.

Misc. Ex. IV-B. P. 152

3. The vertical through A, the pt. of contact must pass through G, so that $\angle OAG=\alpha$. Also since G must be at a distance c from O, two such positions G_1, G_2 are possible and thus it is possible for the sphere to be turned through $\angle G_1OG_2=2\theta$.

OL perp. from O on $G_1G_2 = c \cos\theta = a \sin\alpha$.

4. The vertical through D one of the extreme positions passes thro' G and makes $\angle\lambda$ with the normal OD. $\angle OGD=180°-\beta$ and from $\triangle OGD$, $OG/\sin\lambda = OD/\sin\beta$. Substitute the value of OG as given in Ex. of Art. 8.6.

7. Let the rod be on the point of slipping when the weight is placed at a distance x from one end A. Find x by taking moments about A and resolving along and perp. to a plane. Again find a similar distance y when the rod is about to slip in the opposite sense. This can be written down at once by merely changing the sign of λ in the value of x. The required length is $y \sim x$.

8. In the new position, depth of A below the pulley $P=h-l\sin\theta$ and its distance from PN, the vertical through P, is $l-l\cos\theta$, hence the angle the string makes with PN is known.

Take moments about A and resolve in horizontal and vertical directions.

9. Find F_1, F_2, the two values of the force which are just sufficient to rotate the rod and to make it slide. Apply the condition

$$F_1 > F_2 \text{ or } F_1 < F_2.$$

11. Take moments about the point of contact of the locked wheel with the rails and resolve along and perp. to the rails regarding friction at wheels that are free to rotate as negligible.

12. Show that the pressure at B$=\frac{3}{4}$W and at C$=\frac{1}{4}$W. Find P_1, the force applied at A when slipping takes place at B, by taking moments about C. Similarly find P_2, the force for slipping to take place at C. Apply the condition $P_1 < P_2$.

14. Let P(AP$=x$) be the point of contact of the rod AB with the peg when the frame is about to slip. Then the vertical thro' P will pass thro' O the centre of the figure and will make $\angle\lambda$ with the reaction at P perp. to AB. Find x by considering $\angle$AI**O**. Find similar distance y for the slipping to take place in the opposite sense by changing the sign of λ; $y-x$ is the required distance.

15. For the least angle β, the lower end B of the rod is on the point of slipping down the side of the tumbler and the point of contact with the rim inwards. Take moments about A, resolve horizontally and vertically, obtain a result involving β and λ. For the greatest angle α, the rod is about to slip in the opposite sense, change the sign of λ and replace β by α.

16. The tangent of the angle φ between the normal at P ($a\cos\theta$, $b\sin\theta$), the point of contact of the wire with the peg, and the line joining P to the centre is $(a^2-b^2)\sin 2\theta/2ab$. In order that the wire may not slip at all points $\tan\lambda$ must be greater than the greatest value of $\tan\varphi$.

17. For a cycloid $s=4a\sin\psi$ and $s^2=8ay$.

Ex. XXXI. P. 167

6. Draw GE, GF, parallel to AB, AC, meeting BC in E and F. Show that E, F are fixed points and $\angle$EGF$=\angle$A.

17. See Art. 8.33.

18. Replace $\triangle$DEF by three equal particles at D, E and F; take moments about BC. Distance of E from BC$=$CE sin C$=ab\sin C/(a+c)=2\triangle/(a+c)$ and that of F$=2\triangle/(a+b)$.

19. Let M be the foot of the perp. on b, *i.e.*, on AC, the distance of M from BC$=$CM sin C$=a\cos$C. sin C, etc. Find x, y, z by taking moments about the sides, replacing the triangle by equal particles at the vertices.

20. $\triangle ABC/\triangle A'B'C'=l^2/x^2$ (why ?), and C.G's of these triangles lie on AD. Take moments about D.

21. Take moments about GP, GQ, and GR, G being the C.G.

23. Take moments about BC, using Art. 8.33, for the rim of mass m.

24. Take moments about the side, taking the area as the difference of the areas of the triangle and the incircle.

Ex. XXXII. P. 170

9. In the second position BG is vertical and AG, being perp. to its original vertical position, is horizontal, so that $\angle AGB=90°$.

10. Find co-ordinates of G by taking AB, AE as axes. Find $\angle$DAG by obtaining the slopes of AD and AG, AG being vertical.

11. Take BC, BA as axes, find co-ordinates of G and the slope of BG.

Ex. XXXIII. P. 176

5. Take $\angle AOB=2\alpha$, then $AB=2r\sin\alpha$, $OG_1=\frac{2}{3}r\sin\alpha/\alpha$, so that $AB/OG_1=3\alpha$ and so on.

11. In Art. 8.7, take limits from $\frac{1}{2}h$ to h.

Ex. XXXIV. P. 180

8. Take the altitude as x-axis, 2α angle at the vertex, element of area $dxdy$. Then density$=k(x+y)$, limit for y from 0 to $x\tan\alpha$ and for x from 0 to h.

9. Element of area, $dx\,dy$, density$=kxy$, limits for y from 0 to $\frac{b}{a}\sqrt{a^2-x^2}$, and for x from 0 to a.

Misc. Ex. V. P. 181

1. (*i*) Apply Arts. 3.1 and 3.21.

(*ii*) $\lambda AA'=\lambda AG+\lambda GG'+\lambda G'A'$.

2. (*i*) $\lambda_1 OA=\lambda_1 OD+\lambda_1 DA$ etc.

(*ii*) They intersect at D, take moments about it.

5. Reaction at C is vertical (why ?). Take moments about B in both cases. D is higher than C. Removal of $\triangle OCD$ is equivalent to placing of particles each equal to $-\frac{1}{12}W$ at O, C and D, W being the weight of the lamina.

10. Take AB as axis of x ; its mid-pt. as origin. Find the co-ordinates of the C.G. of the whole stick and the slope of the line C.G. (which is vertical).

Ex. XXXV. P. 188.

12. For values of x and y in terms of ψ, see (5) and (6) Art. 10.13.

13. Find $\bar{y}/\bar{x}$ without evaluating the integrals in the denominators of $\bar{x}$ and $\bar{y}$ as they cancel out by division.

15. C, the centre ; take OPQ making $\angle\,\theta$ with OC intersecting the circle in P and Q. First integrate with respect to r, taking its limits from OP$=r_1$ to OQ$=r_2$. Express r_1 and r_2 in terms of θ. The integrals finally cancel and the result follows at once.

16. Consider the sector as made up of a segment and a cone on the same base. The condition is satisfied only when the C. G. of the sector lies in its edge *i.e.*, it is at a distance $a\cos\alpha$ from the centre of the sphere.

Ex. XXXVII-B. P. 204

7. Express the depths of the rings below the focus S, in terms of their distances r_1, r_2 from S and apply the condition that $r_1+r_2=$length of the string $=$const.

Ex. XXXVIII. P. 215

5. By Art. 8.931 Ex. 1, AG$=\frac{3}{8}h$, and by calculus $\rho=2a$, $\rho'=2b$.

8. O, the centre of the sphere ; A the joint of the rods, $2l$ length of each rod ; θ angle made by OA with the vertical in the displaced position. Show that $\bar{z}=$OG $\cos\theta$, where OG$=r$ cosec $\frac{1}{2}\alpha-l\cos\frac{1}{2}\alpha$.

19. Pot. Energy of the elastic string$=$work done in extending it. See Art. 9·12. (Compare Ex. 4, Art. 9·73).

20. The Virtual Work δW for a further extension dp of a string

$$=kp^n\,dp-\lambda\,\frac{p-a}{a}\,dp.\quad\therefore\quad\frac{dW}{dp}=kp^n-\lambda\,\frac{p-a}{a}.$$

Equilibrium is stable when W is max.

21. Equate the work done by the force to the pot. energy acquired by the elastic string.

Ex. XXXIX. P. 230

16. The ends of the chain reach the directrix of the catenary.

$\therefore$ tensions at A and B acting along the tangents are wx and wy. These and the weight wz are in equilibrium. Apply Lami's Theorem.

18. See solved Ex. 4, Art. 10.41.

23. $\frac{1}{2}W=ws=w\,c\sinh\frac{x}{c}$, x being the abscissa of D, (Fig. Ex. 5. Art. 10·41)

$$\frac{1}{2}W+wl=w(s+l)=w\,c\sinh\frac{x+a}{a}.\quad\text{Eliminate } x.$$

27. The other end of the chain must reach the directrix of the catenary. z, the whole length of the string is given by

$z = y$, the vertical part $+ 2s$, the curved part $= c \cosh \frac{a}{c} + 2c \sinh \frac{a}{c}$,

The result follows from $dz/dc = 0$.

29. Complete the catenary up to the vertex C $(0, c)$.

Let (x, y) be the co-ordinates of A and let the arc $CA = s$, then the co-ordinates of B are $(x+a, y+b)$ and arc $CB = s+l$.

Then $y = c \cosh \frac{x}{c}$, $y+b = c \cosh \frac{x+a}{c}$, $s = c \sinh \frac{x}{c}$, $s+l = c \sinh \frac{x+a}{c}$.

Now find l, b and $l^2 - b^2$, and simplify.

Also $T_1 = wy$, $T_2 = w(y+b)$, $W = wl$; find $T_2 - T_1$ and $T_2 + T_1$.

Ex. XL. P. 242

2. The string rests in the form of a small circle on the sphere. Tension along the string is the resultant of the normal reaction R and tangential friction μR at any point of the string. $\therefore$ the tension or the plane of the string makes angle λ with the normal and thus its distance from the centre is $a \sin \lambda$.

4. W is the resultant of the tensions in the two parts of the string, not in contact with the pulley. Apply Lami's Theorem.

5. Equate the horizontal components of the tensions in the two parts of the string.

11. In equation (2) of Ex. 1 of Art. 10·54, take $T = 0$, when $\theta = \alpha$ and also when $\theta = -(\frac{1}{2}\pi - \alpha)$.

13. Find the tension at the end of the horizontal diameter, taking the tension at the highest point to be zero.

14. For the end of the diameter of the shorter, take $\theta = -\frac{1}{2}\pi$, $T = wy_1$, and for that of the greater $\theta = \frac{1}{2}\pi$, $T = wy_2$. Find y_2, from the fact

$$y_1 + y_2 + \pi a = l.$$

15. Here take $T = 0$ when $\theta = -\frac{1}{2}\pi$ and $T = F$, when $\theta = \frac{1}{2}\pi$.

Ex. XLI-A. P. 254

7. Use the condition $LX + MY + NZ = 0$ for the system to have a single force as its resultant.

14. For the resultant to be a force $LX + MY + NZ = 0$.

$\therefore$ the equations for its line of action are,

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z} = \frac{XL + MY + NZ}{X^2 + Y^2 + Z^2} = 0,$$

which gives, $yZ - zY = L$ or $(yZ - zY)/L = 1$, etc,

Ex. XLIII. P. 271

4. $2^6 P + P = W = 13 \times 14$ lbs.

5. $2^3 P = W + w_1 + 2w_2 + 4w_3 = 91 + 5 + 2 \times 4 + 4 \times 2 = 112$ lbs.

Thrust = weight of the man $- P$.

6. Stress $+ P$ = total weight $= W + 4w$.

9. Here $n = 8$ **18.** Take moments about B.

Ex. XLIV. P. 275

11. Due to the action of P, the normal reactions and consequently the frictions at the two points of contact of the shaft with a groove are not equal.

Let Q, S be the reactions, μQ, μS the frictions at the points of contact of one of the grooves.

Resolving vertically at both the grooves, we have

$$P+W=2\ (Q+S)\cos 60°+2\mu(Q-S)\cos 30°.$$

Taking moments about the axis, $PR=2(\mu Q+\mu S)r$,

R being the radius of the wheel.

Resolving horizontally at one groove,

$$Q\cos 30°=S\cos 30°+(\mu Q+\mu S)\cos 60°.$$

Eliminate Q and S.

Ex. XLV. P. 277

10. Let the fulcrum be at a distance x from the middle point of the beam and let W be the real weight of the body, then

$$W_1(a-x)=W(a+x) \quad \text{and} \quad W_2(a+x)=W(a-x)\ ;$$

eliminate W and find x.

Ex. XLVI. P. 285

11. Draw $\triangle$ABC with AB horizontal and C above AB.

Draw a vertical line *ab* to represent 100 lbs. on some scale. Draw *bc* parallel to BC and *ac* parallel to AC, to meet in *c*, then *abc* is the triangle of forces, for the point C, *ab* representing 100 lbs., *bc*, the *thrust* on the rod BC (the point C being pushed away from B) and *ca*, the *tie* in AC, (the point C being pulled towards A in this case).

Now draw *cd* parallel to BA meeting *ba* in *d*, *cda* being the triangle of forces for the point A, *da* representing P.

It may be noted that *cbd*, the triangle of forces for the point B shows that the reaction at B, represented by *bd* is vertical and equal to 100+P. This fact is otherwise obvious.