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FOURIER TRANSFORMS

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BY

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PREFACE

The study of abstract harmonic analysis has been much in vogue for two decades or more. There is, however, no one place where the interested reader can learn the underlying classical theory in a form which immediately lends itself to an understanding of the abstract.

This tract is designed to provide a background in those classical theorems concerning Fourier transforms on the real line which have found fruitful generalization in abstract harmonic analysis.

Although a familiarity with Lebesgue and Riemann–Stieltjes† integration is required of the reader, we state in Chapter 1 all the theorems on integration used in the subsequent chapters. The reader should also have an acquaintance with the most elementary theory of functions of a complex variable.

In Chapter 2 we introduce the Fourier transform on L^1. After determining its fundamental properties we prove that an analytic function of a Fourier transform is locally a Fourier transform. This is used to establish Wiener's celebrated result on the closure of translates in L^1. To end the chapter, we give an 'algebraized' reformulation of some of the preceding results in terms of ideals in a commutative Banach algebra.

The next chapter is devoted to the Fourier transform on L^2. In particular, there is a proof of Plancherel's theorem.

In Chapter 4 we consider generalizations of the theorem of Wiener proved in Chapter 2. We take up the problem (equivalent to the famous spectral synthesis problem of Beurling) of whether or not the zeros of the Fourier transform of an L^1 function determine the span of the translates of the function.

Bochner's characterization of Fourier–Stieltjes transforms of non-decreasing bounded functions is the subject of the last chapter.

In addition, there is an appendix in which we briefly point out how many of the concepts and theorems can be carried over to an arbitrary locally compact abelian group.

† In the last chapter, only.

In order to keep the tract as elementary as possible, we have avoided all use of the methods of functional analysis. This has meant that we have occasionally had to satisfy ourselves with statements of (or references to) certain recent results which can be stated in classical language but which require functional analysis for any reasonable proof.

References in the text to the bibliography are given thus: [3], or [21; 25] (reference 3, or page 25 of reference 21).

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R. R. G.

CHAPTER 1

PRELIMINARIES

1. Notations

If A is a set then $x \in A$ means x is an element of A. If A and B are sets then $A \subset B$ or $B \supset A$ means every element of A is an element of B. By $A \cap B$ we mean the set of all elements belonging to both A and B. By $A \cup B$ we mean the set of elements in either A or B (or both). The empty set (the set with no elements) will be denoted by $\varnothing$.

The open interval of real numbers x for which $a < x < b$ will often be denoted by (a, b). Similarly, $[a, b]$ will denote the closed interval $a \leqslant x \leqslant b$.

By a *neighborhood* of the real number x we mean any set which contains an interval $(x-\delta, x+\delta)$ for some $\delta > 0$. A neighborhood of x will often be denoted by N_x.

Some other definitions concerning point sets will be introduced (when needed) in § 14.

If θ is a complex number then $\bar{\theta}$ will denote the complex conjugate of θ.

We shall often abbreviate 'almost everywhere' as a.e.

It is convenient to denote in parentheses, to the right of a displayed statement, the set of values of the variable or variables for which the statement is true. For example,

$$\hat{f}(x) = 0 \quad (a \leqslant x \leqslant b)$$

means that $\hat{f}(x)$ is equal to zero for every x in the interval $[a, b]$.

2. Some integral theorems

We shall list here many important theorems on the Lebesgue and Riemann–Stieltjes integrals which are used in the text. This will permit the reader to review the theorems and, in addition, will permit us to refer to the theorems by number later on. For proofs of these results see [4] or [19] (Lebesgue integral) and [20] or [4] (Stieltjes integral).

Unless it is specifically stated to the contrary, we shall assume that all functions used are complex-valued and measurable.

2A. THEOREM (FATOU'S LEMMA). Let $f_1, f_2, \dots$ be non-negative functions on $(-\infty, \infty)$. If

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ a.e.} \quad (-\infty < x < \infty),$$

then

$$\int_{-\infty}^{\infty} f(x)\,dx \leqslant \varliminf_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx.$$

2B. THEOREM (LEBESGUE CONVERGENCE THEOREM) Let $f_1, f_2, \dots$ be integrable on $(-\infty, \infty)$. If

$$|f_n(x)| \leqslant F(x) \text{ a.e.} \quad (-\infty < x < \infty;\ n = 1, 2, \dots)$$

for some integrable F, and if

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ a.e.} \quad (-\infty < x < \infty),$$

then f is integrable and

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

In 2B it is not necessary that the family $\{f_n\}$ be denumerable. For example, the same theorem holds for a family $\{f_n\}$ where n runs through, say, all positive real numbers. See [12; 169].

2C. THEOREM (FUBINI). If the double integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dx\,dy$$

is absolutely convergent, then

$$\int_{-\infty}^{\infty} f(x, y)\,dy$$

exists for almost all x and is an integrable function of x. Moreover

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} f(x, y)\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dx\,dy.$$

Similarly,

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy.$$

2D. Theorem (Tonelli–Hobson). If either of the iterated integrals

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} f(x,y)\,dy, \quad \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} f(x,y)\,dx$$

is absolutely convergent, then so is the double integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

and hence (2C) all three integrals are equal.

2E. Theorem. If f is integrable on $[-R, R]$ for every $R > 0$ then

$$(*) \qquad \lim_{h \to 0} \frac{1}{h}\int_0^h |f(x+t) - f(x)|\,dt = 0 \text{ a.e.} \quad (-\infty < x < \infty).$$

The set of points x for which $(*)$ holds is called the Lebesgue set for f. The Lebesgue set clearly contains all points at which f is continuous.

2F. Definition (the Class L^p). Suppose $1 \leqslant p < \infty$. The function f on $(-\infty, \infty)$ is said to be of class L^p (written $f \in L^p$) if $\int_{-\infty}^{\infty} |f(x)|^p\,dx < \infty$. If $f \in L^p$ then $\|f\|_p$ is defined to be

$$\left(\int_{-\infty}^{\infty} |f(x)|^p\,dx\right)^{1/p}.$$

The symbol $\|f\|_p$ is read as the L^p norm of f.

2G. Remark. If two functions in L^p coincide except on a set of measure zero, they are considered to represent the same element of L^p. Thus, the elements of L^p are really *classes* of functions, the functions in any one class differing from one another only on sets of measure zero. As is customary we shall abuse language and speak of individual functions as elements of L^p.

2H. THEOREM. Let $f, f_1, f_2, \dots$ be in L^p. If

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ a.e.} \quad (-\infty < x < \infty),$$

and

$$\lim_{n\to\infty} \|f_n\|_p = \|f\|_p,$$

then

$$\lim_{n\to\infty} \|f_n - f\|_p = 0.$$

This theorem is perhaps not so well known as the others. For a proof see [9; 34].

2I. THEOREM (L^p IS COMPLETE). Let $f_1, f_2, \dots$ be in L^p. If

$$\lim_{m,\, n\to\infty} \|f_m - f_n\|_p = 0,$$

then there exists $f \in L^p$ such that

$$\lim_{n\to\infty} \|f - f_n\|_p = 0.$$

2J. THEOREM (CONTINUITY IN THE MEAN). If $f \in L^p$ then

$$\lim_{t\to 0} \int_{-\infty}^{\infty} |f(x+t) - f(x)|^p \, dx = 0.$$

2K. THEOREM. Let $f, f_1, f_2, \dots$ be in L^p. If

$$\lim_{n\to\infty} \|f_n - f\|_p = 0,$$

and if for some g

$$\lim_{n\to\infty} f_n(x) = g(x) \text{ a.e.} \quad (-\infty < x < \infty),$$

then

$$f(x) = g(x) \text{ a.e.} \quad (-\infty < x < \infty).$$

(That is, if f is the limit in the mean of f_n, and g is the pointwise limit of f_n, then $f = g$.)

2L. THEOREM. If $f, g \in L^p$ then

$$\text{(i)} \quad \|f+g\|_p \leqslant \|f\|_p + \|g\|_p,$$

and

$$\text{(ii)} \quad \big|\|f\|_p - \|g\|_p\big| \leqslant \|f-g\|_p.$$

2M. THEOREM (SCHWARZ INEQUALITY). If $f, g \in L^2$ then $fg \in L^1$ and

$$\|fg\|_1 \leqslant \|f\|_2 \cdot \|g\|_2.$$

That is,

$$\left(\int_{-\infty}^{\infty} |f(x)g(x)|\, dx\right)^2 \leqslant \int_{-\infty}^{\infty} |f(x)|^2\, dx \,.\int_{-\infty}^{\infty} |g(x)|^2\, dx.$$

2N. THEOREM. Let $f, f_1, f_2, \ldots$ be in L^2. If

$$\lim_{n\to\infty} \|f_n - f\|_2 = 0$$

then, for any $g \in L^2$,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, g(x)\, dx = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx.$$

The last three theorems are concerned with Riemann and Riemann–Stieltjes integrals.

2P. THEOREM (MEAN VALUE THEOREM). If f is continuous on $[a, b]$ and if α is non-decreasing and bounded on $[a, b]$, then, for some c in $[a, b]$,

$$\int_a^b f(x)\,\alpha(x)\, dx = \alpha(a^+)\int_a^c f(x)\, dx + \alpha(b^-)\int_c^b f(x)\, dx.$$

2Q. THEOREM. Let $\Phi_1, \Phi_2, \ldots$ be non-decreasing functions on $(-\infty, \infty)$ such that $\Phi_n(t) = \frac{1}{2}[\Phi_n(t^+) + \Phi_n(t^-)]$ for all n and t. If, for some $M > 0$,

$$|\Phi_n(t)| \leqslant M \quad (-\infty < t < \infty;\ n = 1, 2, \ldots),$$

then there exists a sequence $n_1, n_2, \ldots$ and a non-decreasing function Φ such that

$$\lim_{k\to\infty} \Phi_{n_k}(t) = \Phi(t) \quad (-\infty < t < \infty).$$

Moreover, if f is continuous on $(-\infty, \infty)$ and $\lim_{t\to\pm\infty} f(t) = 0$, then

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} f(t)\, d\Phi_{n_k}(t) = \int_{-\infty}^{\infty} f(t)\, d\Phi(t).$$

2R. THEOREM. Let $f, f_1, f_2, \ldots$ be continuous on $(-\infty, \infty)$. If, for some $M > 0$,

$$|f_n(t)| \leqslant M \quad (-\infty < t < \infty;\ n = 1, 2, \ldots),$$

and if

$$\lim_{n\to\infty} f_n(t) = f(t) \quad (-\infty < t < \infty),$$

then for any α of bounded variation on $(-\infty, \infty)$

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(t)\, d\alpha(t) = \int_{-\infty}^{\infty} f(t)\, d\alpha(t).$$

CHAPTER 2

THE FOURIER TRANSFORM ON L^1

3. Definition of the Fourier transform

3 A. For each $f \in L^1$ the integral

$$\int_{-\infty}^{\infty} e^{ixt} f(t)\, dt$$

exists for all real x. We define the Fourier transform $\hat{f}$ of $f \in L^1$ by

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{ixt} f(t)\, dt \quad (-\infty < x < \infty).$$

Since $$|\hat{f}(x)| \leqslant \int_{-\infty}^{\infty} |e^{ixt} f(t)|\, dt = \int_{-\infty}^{\infty} |f(t)|\, dt = \|f\|_1,$$

we see that $\hat{f}$ is bounded on $(-\infty, \infty)$ and

$$\underset{-\infty < x < \infty}{\text{l.u.b.}} |\hat{f}(x)| \leqslant \|f\|_1. \tag{1}$$

Moreover, $\hat{f}$ is continuous on $(-\infty, \infty)$. To see this we have, for any real x and h,

$$\hat{f}(x+h) - \hat{f}(x) = \int_{-\infty}^{\infty} e^{ixt}(e^{iht} - 1) f(t)\, dt,$$

so that $$|\hat{f}(x+h) - \hat{f}(x)| \leqslant \int_{-\infty}^{\infty} |e^{iht} - 1| \cdot |f(t)|\, dt. \tag{2}$$

The integrand on the right of (2) is not greater than $2|f(t)|$ and tends to zero as $h \to 0$. Hence, by the Lebesgue convergence theorem 2 B, the right side of (2) tends to zero as $h \to 0$. The left side of (2) must, therefore, also tend to zero as $h \to 0$, which shows that $\hat{f}$ is continuous at x.

3 B. THEOREM. If $f, f_1, f_2, \ldots$ are in L^1 and if $\|f_n - f\|_1 \to 0$ as $n \to \infty$, then

$$\lim_{n \to \infty} \hat{f}_n(x) = \hat{f}(x) \text{ uniformly} \quad (-\infty < x < \infty).$$

PROOF. From (1) of 3A we have

$$\underset{-\infty < x < \infty}{\text{l.u.b.}} |\hat{f}_n(x) - \hat{f}(x)| \leqslant \|f_n - f\|_1$$

from which the theorem follows immediately.

In later sections we shall discuss translates $f(t+b)$ of a function $f(t)$. We record the next theorem for future reference.

3C. THEOREM. Fix the real numbers a and b. If $f \in L^1$ then the Fourier transform of $f(t+a)$ is $\hat{f}(x)e^{-iax}$. Also, $\hat{f}(x+b)$ is the Fourier transform of $e^{ibt}f(t)$. Thus, any translate of a Fourier transform of an L^1 function is again a Fourier transform.

PROOF. We have, by a change of variable,

$$\int_{-\infty}^{\infty} e^{ixt} f(t+a)\,dt = \int_{-\infty}^{\infty} e^{ix(t-a)} f(t)\,dt = e^{-iax}\hat{f}(x). \tag{1}$$

Also, $$\hat{f}(x+b) = \int_{-\infty}^{\infty} e^{i(x+b)t} f(t)\,dt = \int_{-\infty}^{\infty} e^{ixt}[e^{ibt}f(t)]\,dt. \tag{2}$$

The theorem follows from (1) and (2).

4. The Riemann–Lebesgue theorem

This famous theorem states that the Fourier transform of an L^1 function must vanish at $\pm\infty$. Specifically,

4A. THEOREM (RIEMANN–LEBESGUE). If $f \in L^1$ then

$$\lim_{x \to \pm\infty} \hat{f}(x) = \lim_{x \to \pm\infty} \int_{-\infty}^{\infty} e^{ixt} f(t)\,dt = 0.$$

PROOF. Since $$\hat{f}(x) = \int_{-\infty}^{\infty} e^{ixt} f(t)\,dt, \tag{1}$$

then $$-\hat{f}(x) = \int_{-\infty}^{\infty} e^{ix[t+(\pi/x)]} f(t)\,dt = \int_{-\infty}^{\infty} e^{ixt} f\left(t - \frac{\pi}{x}\right) dt. \tag{2}$$

Subtracting (2) from (1) we obtain

$$2\hat{f}(x) = \int_{-\infty}^{\infty} e^{ixt}\left[f(t) - f\left(t - \frac{\pi}{x}\right)\right] dt.$$

Hence $$2|\hat{f}(x)| \leqslant \int_{-\infty}^{\infty} \left|f(t) - f\left(t - \frac{\pi}{x}\right)\right| dt. \tag{3}$$

But, since $f \in L^1$,

$$\lim_{x \to \pm\infty} \int_{-\infty}^{\infty} \left| f(t) - f\left(t - \frac{\pi}{x}\right) \right| dt = 0 \tag{4}$$

by Theorem 2J. The theorem follows from (3) and (4).

4B. Corollary. If $f \in L^1$ then

$$\lim_{x \to \pm\infty} \int_{-\infty}^{\infty} f(t) \sin xt \, dt = \lim_{x \to \pm\infty} \int_{-\infty}^{\infty} f(t) \cos xt \, dt = 0.$$

Proof. This follows from 4A and the identities

$$2i \sin\theta = e^{i\theta} - e^{-i\theta}, \quad 2\cos\theta = e^{i\theta} + e^{-i\theta}.$$

4C. We have now shown that if $f \in L^1$ then $\hat{f}$ is continuous on $(-\infty, \infty)$ and $\lim_{x \to \pm\infty} \hat{f}(x) = 0$. It is natural to ask whether every function which is continuous on $(-\infty, \infty)$ and which vanishes at $\pm\infty$ is the Fourier transform of a function in L^1. The answer is 'no' as the following example attests.

Example. Let

$$\begin{aligned} g(x) &= \frac{1}{\log x} \quad (x > e,) \\ &= \frac{x}{e} \quad (0 \leqslant x \leqslant e), \\ &= -g(-x) \quad (x < 0). \end{aligned}$$

Then g is continuous on $(-\infty, \infty)$ and g vanishes at $\pm\infty$. We shall show, however, that g is *not* a Fourier transform. We first note that

$$\lim_{N \to \infty} \int_e^N \frac{g(x)}{x} dx = \lim_{N \to \infty} \int_e^N \frac{dx}{x \log x} = \lim_{N \to \infty} \log\log N = \infty. \tag{1}$$

Now suppose that there is an $f \in L^1$ such that $g = \hat{f}$. Then

$$g(x) = \int_{-\infty}^{\infty} e^{ixt} f(t) \, dt \quad (-\infty < x < \infty),$$

and, since $g(x) = -g(-x)$, we also have

$$g(x) = -\int_{-\infty}^{\infty} e^{-ixt} f(t) \, dt.$$

Adding the preceding two equations we obtain

$$2g(x) = 2i\int_{-\infty}^{\infty} f(t)\sin xt\,dt.$$

Hence
$$g(x) = i\int_0^{\infty} f(t)\sin xt\,dt + i\int_{-\infty}^{0} f(t)\sin xt\,dt$$

$$= i\int_0^{\infty} f(t)\sin xt\,dt - i\int_0^{\infty} f(-t)\sin xt\,dt,$$

and finally,
$$g(x) = \int_0^{\infty} F(t)\sin xt\,dt.$$

Here $F(t) = i[f(t)-f(-t)]$ so that $\int_0^{\infty} |F(t)|\,dt < \infty$. For any $N = 3, 4, 5, \ldots$ we have

$$\int_e^N \frac{g(x)}{x}\,dx = \int_e^N \frac{dx}{x}\int_0^{\infty} F(t)\sin xt\,dt.$$

Since $\int_0^{\infty} |F(t)|\,dt < \infty$ we may change the order of integration in the iterated integral on the right (by Theorem 2D) to obtain

$$\int_e^N \frac{g(x)}{x}\,dx = \int_0^{\infty} F(t)\,dt\int_e^N \frac{\sin xt}{x}\,dx,$$

$$\int_e^N \frac{g(x)}{x}\,dx = \int_0^{\infty} F(t)\,dt\int_{et}^{Nt} \frac{\sin x}{x}\,dx. \tag{2}$$

But $\left|\int_a^b \frac{\sin t}{t}\,dt\right|$ is bounded for all real a and b and, for each t,

$$\lim_{N\to\infty}\int_{et}^{Nt} \frac{\sin x}{x}\,dx$$

exists. Theorem 2B then shows that, as $N \to \infty$, the right side of (2) approaches a finite limit. But this would imply

$$\lim_{N\to\infty}\int_e^N \frac{g(x)}{x} < \infty,$$

which contradicts (1). This contradiction proves that g is not a Fourier transform.

5. Inversion of the Fourier transform

5A. In 3A we defined $\hat{f}$ for $f \in L^1$. We now ask—if we know that a function $\hat{f}$ is the Fourier transform of some $f \in L^1$, can we determine the function f from the values $\hat{f}(x)$ of $\hat{f}$? In other words, can we invert the Fourier transform? The answer, suitably interpreted, is 'yes'. *Formally*, the inversion is as follows:

If
$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{ixt} f(t)\, dt,$$
then
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\hat{f}(x)\, dx. \tag{1}$$

However, there are Fourier transforms $\hat{f}$ for which the integral in (1) does not exist as a Lebesgue integral. (That is, $\hat{f}$ need not be in L^1.) For example, if
$$\begin{aligned} f(t) &= e^{-t} \quad (t \geqslant 0), \\ &= 0 \quad (t < 0), \end{aligned}$$
then $\hat{f}(x) = \dfrac{1}{1-ix}$.

In order for (1) to hold for a given value of t, special conditions must be imposed on f near t and a suitable interpretation must be given the integral in (1).

In this section and the next we shall make the above remarks precise. We begin with a lemma.

5B. LEMMA. Let α be of bounded variation on $[0, \delta]$ for some $\delta > 0$. Then
$$\lim_{R\to\infty} \frac{1}{\pi}\int_0^{\delta} \alpha(t)\frac{\sin Rt}{t}\, dt = \tfrac{1}{2}\alpha(0+).$$

PROOF. Since any function of bounded variation can be expressed as the difference of two bounded non-decreasing functions, it is sufficient to prove the lemma in the case where α is non-decreasing and bounded on $[0, \delta]$.

Case 1. $\alpha(0+) = 0$.

Given $\epsilon > 0$ we can find η with $0 < \eta < \delta$ such that $|\alpha(t)| \leqslant \epsilon$ for $0 < t < \eta$. By Theorem 2P there exists ξ in the interval $[0, \eta]$ such that
$$\int_0^{\eta} \alpha(t)\frac{\sin Rt}{t}\, dt = \alpha(\eta-)\int_{\xi}^{\eta}\frac{\sin Rt}{t}\, dt + \alpha(0+)\int_0^{\xi}\frac{\sin Rt}{t}\, dt.$$

Since $\alpha(0^+) = 0$, we have

$$\int_0^\eta \alpha(t) \frac{\sin Rt}{t} dt = \alpha(\eta^-) \int_{\xi R}^{\eta R} \frac{\sin t}{t} dt.$$

If we choose A such that $\left| \int_a^b \frac{\sin t}{t} dt \right| \leqslant A$ for any a and b, then

$$\left| \int_0^\eta \alpha(t) \frac{\sin Rt}{t} dt \right| \leqslant A \, |\alpha(\eta^-)| \leqslant \epsilon A.$$

But then
$$\left| \int_0^\delta \alpha(t) \frac{\sin Rt}{t} dt \right| \leqslant \epsilon A + \left| \int_\eta^\delta \alpha(t) \frac{\sin Rt}{t} dt \right|. \tag{1}$$

Now $\alpha(t)/t$ is integrable over $[\eta, \delta]$ since α is bounded on $[0, \delta]$. Hence, by 4B,

$$\lim_{R \to \infty} \int_\eta^\delta \alpha(t) \frac{\sin Rt}{t} dt = 0.$$

From this and (1) we conclude

$$\overline{\lim_{R \to \infty}} \left| \int_0^\delta \alpha(t) \frac{\sin Rt}{t} dt \right| \leqslant \epsilon A.$$

Since ϵ was arbitrary this implies

$$\lim_{R \to \infty} \frac{1}{\pi} \int_0^\delta \alpha(t) \frac{\sin Rt}{t} dt = 0 = \tfrac{1}{2}\alpha(0^+),$$

which completes the proof of *Case* 1.

Case 2. $\alpha(0^+) \neq 0$.

Set $\beta(t) = \alpha(t) - \alpha(0^+)$. Then $\beta(0^+) = 0$ and so, by *Case* 1,

$$\lim_{R \to \infty} \int_0^\delta \beta(t) \frac{\sin Rt}{t} dt = 0. \tag{2}$$

Also,
$$\frac{1}{\pi} \int_0^\delta \frac{\sin Rt}{t} dt = \frac{1}{\pi} \int_0^{R\delta} \frac{\sin t}{t} dt \to \frac{1}{2} \quad (R \to \infty). \tag{3}$$

Since

$$\frac{1}{\pi} \int_0^\delta \alpha(t) \frac{\sin Rt}{t} dt = \frac{1}{\pi} \int_0^\delta \beta(t) \frac{\sin Rt}{t} dt + \alpha(0^+) \cdot \frac{1}{\pi} \int_0^\delta \frac{\sin Rt}{t} dt,$$

the conclusion follows from (2) and (3).

Although in subsequent sections we shall make no use of the next theorem, it gives what are, perhaps, the best-known conditions under which equation (1) of 5A is valid.

5C. THEOREM (JORDAN). If $f \in L^1$ and if f is of bounded variation in some neighborhood of a point u then

$$\lim_{R\to\infty} \frac{1}{2\pi}\int_{-R}^{R} e^{-iux}\hat{f}(x)\,dx = \tfrac{1}{2}[f(u^+)+f(u^-)].$$

PROOF. For every $R > 0$ let

$$S_R(u) = \frac{1}{2\pi}\int_{-R}^{R} e^{-iux}\hat{f}(x)\,dx = \frac{1}{2\pi}\int_{-R}^{R} e^{-iux}\,dx\int_{-\infty}^{\infty} e^{ixt}f(t)\,dt.$$

Since $f \in L^1$ this iterated integral is absolutely convergent. By 2D we may change the order of integration to obtain

$$S_R(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt\int_{-R}^{R} e^{-ix(u-t)}\,dx$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\sin R(u-t)}{u-t}\,dt = \frac{1}{\pi}\int_{-\infty}^{\infty} f(u-t)\frac{\sin Rt}{t}\,dt.$$

Finally,

$$S_R(u) = \frac{1}{\pi}\int_{0}^{\infty} [f(u+t)+f(u-t)]\frac{\sin Rt}{t}\,dt = I_1+I_2, \quad \text{say.} \quad (1)$$

Here I_1 and I_2 are integrals over $[0,\delta]$ and $[\delta,\infty]$, respectively, and $\delta > 0$ is such that f is of bounded variation on $[u-\delta, u+\delta]$. By the preceding lemma we have

$$\lim_{R\to\infty} I_1 = \lim_{R\to\infty}\frac{1}{\pi}\int_{0}^{\delta} [f(u+t)+f(u-t)]\frac{\sin Rt}{t}\,dt$$

$$= \tfrac{1}{2}[f(u^+)+f(u^-)]. \quad (2)$$

Also, since $t^{-1}[f(u+t)+f(u-t)]$ is integrable over $\delta \leqslant t < \infty$, 4B implies

$$\lim_{R\to\infty} I_2 = \lim_{R\to\infty}\frac{1}{\pi}\int_{\delta}^{\infty} t^{-1}[f(u+t)+f(u-t)]\sin Rt\,dt = 0. \quad (3)$$

From (1), (2) and (3) we see that

$$\lim_{R\to\infty} S_R(u) = \lim_{R\to\infty} I_1 + \lim_{R\to\infty} I_2 = \tfrac{1}{2}[f(u^+)+f(u^-)],$$

which is what we wished to show.

5D. If f satisfies the assumptions of Theorem 5C, *and f is continuous at u* then we have

$$f(u) = \frac{1}{2\pi} \lim_{R\to\infty} \int_{-R}^{R} e^{-iux} \hat{f}(x)\, dx.$$

Thus, we have found one set of conditions under which equation (1) of 5A (with the integral interpreted as a Cauchy principal value) is valid.

6. Inversion of the Fourier transform using (C, 1) summability

We recall that the series $\sum_{k=-\infty}^{\infty} a_k$ with partial sums $s_j = \sum_{k=-j}^{j} a_k$ is said to be (C, 1) summable (C for Cesàro) to the value A if

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} s_j = \lim_{n\to\infty} \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) a_k = A.$$

In this section we shall make use of (C, 1) summability for integrals.

6A. Definition. Let a be integrable on $[-R, R]$ for every $R > 0$. The integral $\int_{-\infty}^{\infty} a(x)\, dx$ is said to be (C, 1) summable to the value A if

$$\lim_{R\to\infty} \int_{-R}^{R} \left(1 - \frac{|x|}{R}\right) a(x)\, dx = A.$$

We shall need the following easy result.

6B. Theorem. If $a \in L^1$ and $\int_{-\infty}^{\infty} a(x)\, dx = A$ then $\int_{-\infty}^{\infty} a(x)\, dx$ is (C, 1) summable to A.

Proof. For each $R > 0$ let

$$a_R(x) = \left(1 - \frac{|x|}{R}\right) a(x) \quad (|x| \leqslant R),$$

$$= 0 \quad (|x| > R).$$

Then $|a_R(x)| \leqslant |a(x)|$ for $-\infty < x < \infty$ and

$$\lim_{R\to\infty} a_R(x) = a(x) \quad (-\infty < x < \infty$$

Hence, by 2B,

$$\lim_{R\to\infty}\int_{-\infty}^{\infty} a_R(x)\,dx = \int_{-\infty}^{\infty} a(x)\,dx = A,$$

or

$$\lim_{R\to\infty}\int_{-R}^{R}\left(1-\frac{|x|}{R}\right)a(x)\,dx = A,$$

which is what we wished to show.

Theorem 6B thus shows that (C, 1) summability will assign to any absolutely convergent integral its usual value. Of course, the usefulness of (C, 1) and other methods of summability lies in the fact that values can be assigned to some integrals that do not exist as Lebesgue integrals or even as principal values.

The next theorem leads to another interpretation of (1) of 5A. It states that if $f \in L^1$ then $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{-iux}\hat{f}(x)\,dx$ is (C, 1) summable to $f(u)$ for almost all u. (First, we should recall from 2E the definition of the Lebesgue set for f.)

6C. Theorem. If $f \in L^1$ and u is in the Lebesgue set for f then

$$\lim_{R\to\infty}\frac{1}{2\pi}\int_{-R}^{R}\left(1-\frac{|x|}{R}\right)e^{-iux}\hat{f}(x)\,dx = f(u). \tag{1}$$

Hence, (1) holds a.e. $(-\infty < x < \infty)$; in particular, (1) holds if f is continuous at u.

Proof. For any $R > 0$ and any real u let

$$S_R(u) = \frac{1}{2\pi}\int_{-R}^{R}\left(1-\frac{|x|}{R}\right)e^{-iux}\hat{f}(x)\,dx \tag{2}$$

$$= \frac{1}{2\pi}\int_{-R}^{R}\left(1-\frac{|x|}{R}\right)e^{-iux}\,dx\int_{-\infty}^{\infty} e^{ixt}f(t)\,dt.$$

Since $f \in L^1$ this iterated integral converges absolutely. By 2D we may change the order of integration and obtain

$$S_R(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt\int_{-R}^{R}\left(1-\frac{|x|}{R}\right)e^{-ix(u-t)}\,dx.$$

Evaluating the inner integral by use of integration by parts we have

$$S_R(u) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\frac{1-\cos R(u-t)}{R(u-t)^2}\,dt \quad (-\infty < u < \infty), \tag{3}$$

$$S_R(u) = \frac{1}{\pi}\int_0^{\infty} f(u-t)\,\frac{1-\cos Rt}{Rt^2}\,dt \quad (-\infty < u < \infty),$$

and finally

$$S_R(u) = \frac{1}{\pi}\int_{-\infty}^{\infty} [f(u+t)+f(u-t)]\,\frac{1-\cos Rt}{Rt^2}\,dt \quad (-\infty < u < \infty). \tag{4}$$

But, by a change of variable and integration by parts,

$$\int_0^{\infty} \frac{1-\cos Rt}{Rt^2}\,dt = \int_0^{\infty} \frac{1-\cos t}{t^2}\,dt = \int_0^{\to\infty} \frac{\sin t}{t}\,dt = \frac{\pi}{2}.$$

Hence, from (4),

$$S_R(u)-f(u) = \frac{1}{\pi}\int_0^{\infty} [f(u+t)+f(u-t)-2f(u)]\,\frac{1-\cos Rt}{Rt^2}\,dt, \tag{5}$$

$$S_R(u)-f(u) = \frac{1}{\pi}\int_0^{\delta} + \frac{1}{\pi}\int_{\delta}^{\infty} = I_1 + I_2, \quad \text{say}, \tag{6}$$

where $\delta > 0$ is yet to be chosen.

Now let

$$\phi(t) = |f(u+t)+f(u-t)-2f(u)|, \quad \Phi(t) = \int_0^t \phi(y)\,dy.$$

Then, if $1/R < \delta$,

$$|\pi I_1| \leqslant \int_0^{\delta} |[f(u+t)+f(u-t)-2f(u)]|\,\frac{1-\cos Rt}{Rt^2}\,dt$$

$$= \int_0^{\delta} \phi(t)\,\frac{1-\cos Rt}{Rt^2}\,dt = \int_0^{1/R} + \int_{1/R}^{\delta} = I_1' + I_1'', \quad \text{say}. \tag{7}$$

If u is in the Lebesgue set for f it follows that $\lim\limits_{t\to 0} \dfrac{\Phi(t)}{t} = 0$.

Hence, given $\epsilon > 0$, we may choose the δ mentioned above so that

$$\Phi(t) \leqslant \epsilon t \quad (0 \leqslant t \leqslant \delta). \tag{8}$$

Since $1-\cos\theta \leqslant \theta^2/2$ we then have

$$I_1' \leqslant \frac{R}{2}\int_0^{1/R} \phi(t)\,dt = \frac{R}{2}\,\Phi\left(\frac{1}{R}\right) \leqslant \frac{\epsilon}{2}.$$

Also, since $1-\cos\theta \leqslant 2$,

$$I_1'' \leqslant 2\int_{1/R}^{\delta} \frac{\phi(t)}{Rt^2}\,dt = \frac{2\Phi(\delta)}{R\delta^2} - 2R\Phi\left(\frac{1}{R}\right) + 4\int_{1/R}^{\delta} \frac{\Phi(t)}{Rt^3}\,dt.$$

Dropping the negative term and using (8) we obtain

$$I_1'' \leqslant \frac{2\epsilon}{R\delta} + 4\epsilon\int_{1/R}^{\delta} \frac{dt}{Rt^2} \leqslant 2\epsilon + 4\epsilon = 6\epsilon.$$

Thus, from (7), $|\pi I_1| \leqslant I_1' + I_1'' \leqslant \frac{\epsilon}{2} + 6\epsilon = \frac{13\epsilon}{2}$. (9)

To estimate the integral I_2 in (6) we have

$$|I_2| \leqslant \frac{2}{\pi R}\int_{\delta}^{\infty} \frac{\phi(t)}{t^2}\,dt.$$

Since $\int_{\delta}^{\infty} \frac{\phi(t)}{t^2}\,dt < \infty$ it is obvious that

$$\lim_{R\to\infty} |I_2| = 0.$$

This, together with (6) and (9), shows that

$$\overline{\lim_{R\to\infty}}\, |S_R(u) - f(u)| \leqslant \overline{\lim_{R\to\infty}}\, |I_1| + \overline{\lim_{R\to\infty}}\, |I_2| \leqslant \frac{13\epsilon}{2\pi}.$$

Since ϵ was arbitrary this implies

$$\lim_{R\to\infty} S_R(u) = f(u).$$

This completes the proof.

We shall make use of the following corollary in later sections.

6D. Corollary. If f and $\hat{f}$ are in L^1 and if f is continuous at u then

$$f(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\hat{f}(x)\,dx. \tag{1}$$

Proof. By 6C the right side of (1) is (C, 1) summable to $f(u)$. But, since $\hat{f} \in L^1$, the integral in (1) is absolutely convergent. The corollary thus follows from 6B.

As another consequence of 6C we can establish the uniqueness of the Fourier transform. That is, two distinct functions in L^1 (i.e.

two functions which differ on a set of positive measure) have distinct Fourier transforms. The uniqueness follows from the next corollary.

6E. COROLLARY. If $f \in L^1$ and if

$$\hat{f}(x) = 0 \qquad (-\infty < x < \infty),$$

then
$$f(t) = 0 \text{ a.e.} \quad (-\infty < t < \infty).$$

PROOF. This follows immediately from 6C.

6F. COROLLARY (UNIQUENESS OF THE FOURIER TRANSFORM). If $f, g \in L^1$ and if

$$\hat{f}(x) = \hat{g}(x) \qquad (-\infty < x < \infty),$$

then
$$f(t) = g(t) \text{ a.e.} \quad (-\infty < t < \infty).$$

PROOF. The Fourier transform of $f-g$ is $\hat{f}-\hat{g}$, and is thus identically zero. By 6E, $f(t)-g(t)$ is zero almost everywhere, which is what we wished to show.

The next result is a useful theorem concerning a singular integral.

6G. If f is integrable over $[-R, R]$ for every $R > 0$ and if

$$\int_{-\infty}^{\infty} \frac{|f(t)|}{1+t^2}\,dt < \infty$$

then

$$\lim_{R\to\infty} \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\frac{1-\cos R(u-t)}{R(u-t)^2}\,dt = f(u) \text{ a.e.} \quad (-\infty < u < \infty). \tag{1}$$

In particular, (1) holds if $f \in L^1$ or if f is bounded on $(-\infty, \infty)$.

PROOF. *Case* 1. Assume $f \in L^1$.
By 6C,

$$f(u) = \lim_{R\to\infty} S_R(u) \text{ a.e.} \quad (-\infty < u < \infty),$$

where $S_R(u)$ is as in (2) of 6C. But by (3) of 6C,

$$S_R(u) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\frac{1-\cos R(u-t)}{R(u-t)^2}\,dt \quad (-\infty < u < \infty).$$

This proves *Case* 1.

Case 2. Assume only $\int_{-\infty}^{\infty} \frac{|f(t)|}{1+t^2}\,dt < \infty$.

For any $s > 0$ let

$$f_s(t) = f(t) \quad (|t| \leqslant s),$$

$$f_s(t) = 0 \quad (|t| > s).$$

Then

$$|f_s(t)| \leqslant \frac{1+s^2}{1+t^2}|f(t)| \quad (-\infty < t < \infty),$$

and thus $f_s \in L^1$. By *Case* 1 we then have

$$\lim_{R\to\infty} \frac{1}{\pi}\int_{-\infty}^{\infty} f_s(t)\,\frac{1-\cos R(u-t)}{R(u-t)^2}\,dt = f_s(u)$$

for almost all u. From this it follows that

$$\lim_{R\to\infty} \frac{1}{\pi}\int_{-s}^{s} f(t)\,\frac{1-\cos R(u-t)}{R(u-t)^2}\,dt = f(u) \text{ a.e.} \quad (|u| \leqslant s). \tag{2}$$

If $|u| < s$ then

$$\left|\int_{|t|\geqslant s} f(t)\frac{1-\cos R(u-t)}{R(u-t)^2}\,dt\right| \leqslant \frac{2}{R}\int_{|t|\geqslant s}\frac{|f(t)|}{(u-t)^2}\,dt \tag{3}$$

and hence, since the integral on the right of (3) converges,

$$\lim_{R\to\infty}\int_{|t|\geqslant s} f(t)\,\frac{1-\cos R(u-t)}{R(u-t)^2}\,dt = 0 \quad (|u| < s). \tag{4}$$

From (2) and (4) we thus obtain

$$\lim_{R\to\infty} \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{1-\cos R(u-t)}{R(u-t)^2}\,dt = f(u) \text{ a.e.} \quad (|u| < s).$$

Since s was arbitrary, (1) follows.

7. Convolutions

In this section we introduce the important concept of the convolution $f*g$ of two functions $f, g \in L^1$. The most important property of convolutions is that convolving f and g corresponds to multiplying $\hat{f}$ and $\hat{g}$. That is, $\hat{f}\hat{g}$ is the Fourier transform of $f*g$ (which is in L^1 if f and g are).

7A. **Lemma.** If $f, g \in L^1$ then the integral

$$\int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt$$

exists for almost all x and is an integrable function of x.

Proof. For each t we have

$$\int_{-\infty}^{\infty} |f(x-t)|\, dx = \int_{-\infty}^{\infty} |f(x)|\, dx < \infty$$

and hence

$$\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} |f(x-t)\, g(t)|\, dx = \int_{-\infty}^{\infty} |g(t)|\, dt \int_{-\infty}^{\infty} |f(x)|\, dx < \infty. \tag{1}$$

By 2D this implies that the double integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-t)\, g(t)\, dx\, dt$$

is absolutely convergent. The lemma then follows from Fubini's theorem 2C.

The lemma allows us to make the following definition.

7B. **Definition.** If $f, g \in L^1$ and if

$$h(x) = \int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt$$

whenever the integral exists, we say that h is the convolution of f and g. This will be denoted by $h = f * g$. (Note: 7A shows that $h \in L^1$.)

The convolution operation is commutative and associative. That is,

7C. **Theorem.**

$$f * g = g * f \qquad (f, g \in L^1),$$

$$(f * g) * k = f * (g * k) \quad (f, g, k \in L^1).$$

Proof. Letting $s = x - t$ we have

$$\int_{-\infty}^{\infty} f(x-t)\, g(t)\, dt = \int_{-\infty}^{\infty} g(x-s)\, f(s)\, ds.$$

This proves $f * g = g * f$. The second assertion is proved similarly.

Lemma 7A showed that the convolution of two L^1 functions is a function in L^1. The next theorem gives an important relation between the norms of these functions.

7D. THEOREM. If $f, g \in L^1$ then $\|f*g\|_1 \leqslant \|f\|_1 \cdot \|g\|_1$.

PROOF. Since

$$h(x) = \int_{-\infty}^{\infty} f(x-t)\,g(t)\,dt$$

we have

$$\|h\|_1 = \int_{-\infty}^{\infty} |h(x)|\,dx \leqslant \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |f(x-t)\,g(t)|\,dt$$

$$= \int_{-\infty}^{\infty} |g(t)|\,dt \int_{-\infty}^{\infty} |f(x-t)|\,dx = \|g\|_1 \cdot \|f\|_1.$$

The change in order of integration is justified by 2D.

We now establish the result mentioned earlier concerning convolutions and Fourier transforms.

7E. THEOREM. Let $f, g \in L^1$ and let $h = f*g$. Then $\hat{h} = \hat{f}\hat{g}$. (In other 'words', $(f*g)^\wedge = \hat{f}\hat{g}$.)

PROOF. For any x we have

$$\hat{h}(x) = \int_{-\infty}^{\infty} e^{ixt} h(t)\,dt = \int_{-\infty}^{\infty} e^{ixt} dt \int_{-\infty}^{\infty} f(t-u)\,g(u)\,du$$

$$= \int_{-\infty}^{\infty} g(u)\,du \int_{-\infty}^{\infty} e^{ixt} f(t-u)\,dt$$

$$= \int_{-\infty}^{\infty} g(u)\,du \int_{-\infty}^{\infty} e^{ix(t+u)} f(t)\,dt$$

$$= \int_{-\infty}^{\infty} e^{ixu} g(u)\,du \int_{-\infty}^{\infty} e^{ixt} f(t)\,dt = \hat{g}(x)\hat{f}(x).$$

Thus $\hat{h} = \hat{f}\hat{g}$. Since $f, g \in L^1$ the last iterated integral is absolutely convergent. This justifies the change in order of integration.

8. Some important special functions

8A. There is no identity element in L^1 with respect to the convolution operation. That is, there is no function $d \in L^1$ which has the property that

$$d*f = f \tag{1}$$

for all $f \in L^1$. For if such a function d existed then, in particular, we would have $d * d = d$. By 7 E this would imply

$$[\hat{d}(x)]^2 = \hat{d}(x) \quad (-\infty < x < \infty).$$

Hence, for each x, $\hat{d}(x)$ would equal zero or one. But, as was shown in 3 A, every Fourier transform is continuous on $(-\infty, \infty)$. Hence, $\hat{d}$ would have to be identically zero or identically one. By 4 A, $\hat{d}$ must vanish at $\pm\infty$. Therefore $\hat{d}$ would have to be identically zero. By 6 E this implies $d(t) = 0$ for almost all t, which contradicts (1) for any $f \in L^1$ not equivalent to zero. Thus, no $d \in L^1$ exists that satisfies (1) for all $f \in L^1$.

However, we can construct in L^1 an 'approximate identity'—namely a sequence $\delta_1, \delta_2, \ldots$ in L^1 which has the property that $\delta_N * f \to f$ as $N \to \infty$, the limit being taken in the L^1 sense. More precisely,

$$\lim_{N \to \infty} \|\delta_N * f - f\|_1 = 0$$

for every $f \in L^1$. We now begin the construction.

8 B. Definition. Let

$$\Delta(t) = 1 - |t| \quad (|t| \leqslant 1),$$

$$\Delta(t) = 0 \quad (|t| > 1),$$

and let
$$\delta(t) = \frac{1}{\pi} \cdot \frac{1 - \cos t}{t^2} \quad (-\infty < t < \infty).$$

Note that both Δ and δ are in L^1. Since $\Delta(t) = \Delta(-t)$, we have

$$\hat{\Delta}(x) = \int_{-\infty}^{\infty} e^{ixt} \Delta(t)\, dt = 2\int_0^{\infty} \Delta(t) \cos xt\, dt$$

$$= 2\int_0^1 (1-t) \cos xt\, dt,$$

and so,
$$\hat{\Delta}(x) = \frac{2(1 - \cos x)}{x^2} = 2\pi\delta(x).$$

Thus
$$\hat{\Delta} = 2\pi\delta. \tag{1}$$

Since, $\delta, \Delta \in L^1$ and Δ is continuous on $(-\infty, \infty)$, 6 D implies

$$\Delta(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux} \hat{\Delta}(x)\, dx = \int_{-\infty}^{\infty} e^{-iux} \delta(x)\, dx \quad (-\infty < u < \infty).$$

Again, since $\Delta(u) = \Delta(-u)$,

$$\Delta(u) = \int_{-\infty}^{\infty} e^{iux}\,\delta(x)\,dx$$

which proves that $$\Delta = \hat{\delta}. \tag{2}$$

For any $R > 0$ we then have

$$\Delta\left(\frac{x}{R}\right) = \int_{-\infty}^{\infty} e^{ixt/R}\,\delta(t)\,dt = \int_{-\infty}^{\infty} e^{ixt} R\delta(Rt)\,dt. \tag{3}$$

Thus, if we set $\Delta_R(x) = \Delta\left(\dfrac{x}{R}\right)$, $\delta_R(t) = R\delta(Rt)$,

we have $\Delta_R = \hat{\delta}_R$. Finally, setting $x = 0$ in (3), we obtain

$$1 = \int_{-\infty}^{\infty} R\delta(Rt)\,dt = \int_{-\infty}^{\infty} \delta_R(t)\,dt = \|\delta_R\|_1.$$

We summarize these considerations in the next theorem.

8C. Theorem. For each $R > 0$ let

$$\delta_R(t) = \frac{1}{\pi}\cdot\frac{1-\cos Rt}{Rt^2} \quad (-\infty < t < \infty),$$

and let

$$\Delta_R(x) = 1 - \frac{|x|}{R} \quad (|x| \leqslant R),$$
$$\Delta_R(x) = 0 \quad (|x| > R).$$

Then $\|\delta_R\|_1 = 1$ and $\Delta_R = \hat{\delta}_R$. In particular, Δ_R is a Fourier transform.

We now show that the sequence $\delta_1, \delta_2, \ldots$ actually constitutes the 'approximate identity' mentioned in 8A.

8D. Theorem. Let $f \in L^1$ and, for $N = 1, 2, \ldots$, let δ_N be as in 8C. Then

$$\lim_{N\to\infty} \|\delta_N * f - f\|_1 = 0.$$

Proof. For each $N = 1, 2, \ldots$,

$$(\delta_N * f)(u) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\frac{1-\cos N(u-t)}{N(u-t)^2}\,dt.$$

But then, by 6G,

$$\lim_{N\to\infty} (\delta_N * f)(u) = f(u) \text{ a.e.} \quad (-\infty < u < \infty). \tag{1}$$

From (1) and Fatou's Lemma 2A,

$$\|f\|_1 \leqslant \varliminf_{N\to\infty} \|\delta_N * f\|_1. \tag{2}$$

On the other hand, $\|\delta_N\| = 1$ by 8C. Hence by 7D,

$$\|\delta_N * f\|_1 \leqslant \|\delta_N\|_1 \cdot \|f\|_1 = \|f\|_1 \quad (N = 1, 2, \ldots)$$

so that

$$\varlimsup_{N\to\infty} \|\delta_N * f\|_1 \leqslant \|f\|_1. \tag{3}$$

From (2) and (3) we conclude

$$\lim_{N\to\infty} \|\delta_N * f\|_1 = \|f\|_1. \tag{4}$$

The conclusion of the theorem now follows from (1), (4) and Theorem 2H. (For an alternative proof see [21; 16].)

8E. Remark. Note that by 7E the Fourier transform of $\delta_N * f$ is $\Delta_N \hat{f}$, which vanishes outside $[-N, N]$. Hence 8D shows that any $f \in L_1$ can be approximated arbitrarily closely in the L^1 norm by a function whose Fourier transform vanishes outside some bounded interval.

There is no function in L^1 whose Fourier transform is identically equal to unity on $(-\infty, \infty)$. We shall show next, however, that for any *bounded* interval $[a, b]$ there is a function in L^1 whose Fourier transform is identically unity on $[a, b]$ and identically zero outside a slightly larger interval. This fact will be of importance later on.

8F. Theorem. Given real numbers $a < b$ and $h > 0$ there exists $\omega \in L^1$ such that

$$\hat{\omega}(x) = 1 \quad (a \leqslant x \leqslant b),$$

$$\hat{\omega}(x) = 0 \quad (x \leqslant a-h;\ x \geqslant b+h),$$

and such that $\hat{\omega}$ is linear on $[a-h, a]$ and $[b, b+h]$.

Proof. Let $c = \frac{1}{2}(b-a)$. Since, by 8C, Δ_R is a Fourier transform for every $R > 0$, the function $\frac{1}{h}[(c+h)\Delta_{c+h} - c\Delta_c]$ is the

Fourier transform of some $\omega_1 \in L^1$. A glance at the graphs of $(c+h)\,\Delta_{c+h}$ and $c\Delta_c$ will readily verify that

$$\hat{\omega}_1(x) = \frac{1}{h}[(c+h)\,\Delta_{c+h}(x) - c\Delta_c(x)] = 1 \quad (-c \leqslant x \leqslant c),$$
$$= 0 \quad (x \leqslant -c-h;\, x \geqslant c+h),$$

and that $\hat{\omega}_1$ is linear on $[-c-h, -c]$ and $[c, c+h]$.

Now let $\omega(t) = e^{-\frac{1}{2}(a+b)t}\omega_1(t)$. Then, by 3C,

$$\hat{\omega}(x) = \hat{\omega}_1(x - \tfrac{1}{2}(a+b)).$$

Hence $$\hat{\omega}(x) = 1 \quad (-c \leqslant x - \tfrac{1}{2}(a+b) \leqslant c).$$

Since $c = \frac{1}{2}(b-a)$ this means $\hat{\omega}(x) = 1$ for $a \leqslant x \leqslant b$. The other requirements for $\hat{\omega}$ may be similarly verified to complete the proof.

One useful application of the last result is the following theorem.

8G. **Theorem.** If $f \in L^1$, $\hat{f}(0) = 0$, and $\epsilon > 0$, there exists $h \in L^1$ with the following three properties:

(i) $\|h\|_1 < \epsilon$,
(ii) $\hat{h}(x) = \hat{f}(x)$ for all x in some neighborhood of 0,
(iii) $\hat{h}(x) = 0$ for every x such that $\hat{f}(x) = 0$.

Proof. By 8F we can find $\omega \in L^1$ such that $\hat{\omega}(x) = 1$ for $|x| \leqslant 1$. For each $R > 0$ let

$$\omega_R(t) = R\omega(Rt)$$

so that $$\hat{\omega}_R(x) = \hat{\omega}\left(\frac{x}{R}\right).$$

Then $$\hat{\omega}_R(x) = 1 \quad (|x| \leqslant R). \tag{1}$$

Since, by assumption, $\int_{-\infty}^{\infty} f(t)\,dt = \hat{f}(0) = 0$, we have

$$(\omega_R * f)(x) = \int_{-\infty}^{\infty} \omega_R(x-t) f(t)\,dt - \omega_R(x)\int_{-\infty}^{\infty} f(t)\,dt$$
$$= \int_{-\infty}^{\infty} f(t)\,[\omega_R(x-t) - \omega_R(x)]\,dt.$$

Hence,

$$\|\omega_R * f\|_1 = \int_{-\infty}^{\infty} |(\omega_R * f)(x)|\,dx$$

$$\leqslant \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |f(t)|\cdot|\omega_R(x-t)-\omega_R(x)|\,dt.$$

Using 2D we then have

$$\|\omega_R * f\|_1 \leqslant \int_{-\infty}^{\infty} |f(t)|\,dt \int_{-\infty}^{\infty} |\omega_R(x-t)-\omega_R(x)|\,dx$$

$$= \int_{-\infty}^{\infty} |f(t)|\,dt \int_{-\infty}^{\infty} R|\omega(Rx-Rt)-\omega(Rx)|\,dx,$$

and finally,

$$\|\omega_R * f\|_1 \leqslant \int_{-\infty}^{\infty} |f(t)|\,dt \int_{-\infty}^{\infty} |\omega(x-Rt)-\omega(x)|\,dx. \qquad (2)$$

Now, for each t, the integral $\int_{-\infty}^{\infty} |\omega(x-Rt)-\omega(x)|\,dx$ is not greater than $2\|\omega\|_1$. Also, by 2J,

$$\lim_{R\to 0} \int_{-\infty}^{\infty} |\omega(x-Rt)-\omega(x)|\,dx = 0 \quad (-\infty < t < \infty).$$

Hence, by 2B, the right side of (2) goes to zero as $R \to 0$. We may therefore choose R so small that

$$\|\omega_R * f\|_1 < \epsilon.$$

If, for this R, we set $h = \omega_R * f$ then $\|h\|_1 \leqslant \epsilon$ and, by 7E $\hat{h} = \hat{\omega}_R \hat{f}$. This and (1) show that

$$\hat{h}(x) = \hat{f}(x) \quad (|x| \leqslant R).$$

The third required property of h follows obviously from $\hat{h} = \hat{\omega}_R \hat{f}$.

We shall have need of one more 'special function'.

8H. THEOREM. There exists $g \in L^1$ such that

$$\hat{g}(x) > 0 \quad (x > 0),$$

$$\hat{g}(x) = 0 \quad (x \leqslant 0).$$

PROOF. Let

$$g(t) = \frac{1}{2\pi}\cdot\frac{1}{(1+it)^2} \quad (-\infty < t < \infty).$$

We shall show that $\hat{g}$ fits the requirements of the theorem.

Let
$$G(x) = x\,e^{-x} \quad (x > 0),$$
$$= 0 \qquad (x \leqslant 0).$$

Then $G \in L^1$ and

$$\hat{G}(-x) = \int_{-\infty}^{\infty} e^{-ixt}\,G(t)\,dt = \int_{0}^{\infty} e^{-ixt}\,t\,e^{-t}\,dt$$
$$= \int_{0}^{\infty} e^{-t(1+ix)}t\,dt = \frac{1}{(1+ix)^2}\int_{0}^{\infty} e^{-t}\,t\,dt$$
$$= \frac{1}{(1+ix)^2} = 2\pi g(x).$$

Hence
$$\hat{G}(x) = 2\pi g(-x) \quad (-\infty < x < \infty).$$

Since $G, \hat{G} \in L^1$ and G is continuous on $(-\infty, \infty)$, we have by 6D,

$$G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}\,\hat{G}(x)\,dx$$
$$= \int_{-\infty}^{\infty} e^{-ixt}g(-x)\,dx = \int_{-\infty}^{\infty} e^{ixt}g(x)\,dx.$$

Hence $G = \hat{g}$, which proves the theorem.

8I. Corollary. For any interval of the form $(-\infty, a]$ or $[a, \infty)$ there exists a Fourier transform which vanishes on the interval but does not vanish outside the interval.

Proof. By 3C, the function $\hat{g}(x-a)$ (g as in 8H) is the Fourier transform of some $h \in L^1$. This vanishes on $(-\infty, a]$ but not outside $(-\infty, a]$. The Fourier transform of $h(-t)$ is $\hat{h}(-x) = \hat{g}(a-x)$ which vanishes on $[a, \infty)$ but not outside $[a, \infty)$.

9. Analytic functions of Fourier transforms

We begin this section by posing the question: For what functions ϕ is it true that if $\hat{f}$ is a Fourier transform then so is $\phi(\hat{f})$? In other words, what functions ϕ have the property that for every $f \in L^1$ there exists $g \in L^1$ such that

$$\phi[\hat{f}(x)] = \hat{g}(x) \quad (-\infty < x < \infty)?$$

We shall say that such a ϕ takes Fourier transforms into Fourier transforms.

Certainly $\phi(z) = z^2$ has this property, since, by 7 E, $\phi(\hat{f}) = \hat{f}^2$ is the Fourier transform of $f*f$. Similarly, $\hat{f}^3$ is the Fourier transform of $f*f*f$. Indeed, for $n = 1, 2, \ldots, \hat{f}^n$ is a Fourier transform. Thus, if

$$P(z) = a_1 z + a_2 z^2 + \ldots + a_n z^n,$$

where the a_i are complex numbers, then P takes Fourier transforms into Fourier transforms. A more general result will be stated in 9 B as a corollary to the next theorem.

9 A. **Theorem.** If $\phi(z)$ is analytic for $|z| < \epsilon$ where $\epsilon > 0$, if $\phi(0) = 0$, and if $h \in L^1$ is such that $\|h\|_1 < \epsilon$, then $\phi(\hat{h})$ is a Fourier transform. That is, there exists $g \in L^1$ such that

$$\phi[\hat{h}(x)] = \hat{g}(x) \quad (-\infty < x < \infty).$$

Proof. By assumption we can write

$$\phi(z) = \sum_{k=1}^{\infty} a_k z^k,$$

the series converging absolutely for $|z| < \epsilon$. Since $\|h\|_1 < \epsilon$ and since, by (1) of 3 A,

$$|\hat{h}(x)| \leqslant \|h\|_1 \quad (-\infty < x < \infty),$$

we have $|\hat{h}(x)| < \epsilon$ for all x. Thus

$$\phi[\hat{h}(x)] = \sum_{k=1}^{\infty} a_k[\hat{h}(x)]^k \quad (-\infty < x < \infty). \tag{1}$$

Let $h_1 = h$ and, for $k = 2, 3, \ldots$, let $h_k = h_{k-1} * h$. (That is, h_k is the $(k-1)$-fold convolution of h with itself.) Then, by 7 D,

$$\|h_k\|_1 \leqslant \|h\|_1^k. \tag{2}$$

Also, by 7 E, $\quad \hat{h}_k(x) = [\hat{h}(x)]^k \quad (-\infty < x < \infty). \qquad (3)$

Now for $n \geqslant m$ we have, using (i) of 2 L and (2),

$$\left\|\sum_{k=m}^{n} a_k h_k\right\|_1 \leqslant \sum_{k=m}^{n} |a_k| \cdot \|h_k\|_1 \leqslant \sum_{k=m}^{n} |a_k| \cdot \|h\|_1^k. \tag{4}$$

Since $\|h\|_1 < \epsilon$, the series

$$\sum_{k=1}^{\infty} |a_k| \cdot \|h\|_1^k$$

converges. Hence, as $m, n \to \infty$, the right side of (4) tends to zero and therefore so does the left side of (4). By 2I (the completeness of L^1), there exists $g \in L^1$ such that

$$\lim_{n \to \infty} \left\| \sum_{k=1}^{n} a_k h_k - g \right\|_1 = 0.$$

Therefore, by 3B,

$$\lim_{n \to \infty} \sum_{k=1}^{n} a_k \hat{h}_k(x) = \hat{g}(x) \quad (-\infty < x < \infty). \tag{5}$$

From (5), (3) and (1), we conclude

$$\hat{g}(x) = \sum_{k=1}^{\infty} a_k \hat{h}_k(x) = \sum_{k=1}^{\infty} a_k [\hat{h}(x)]^k = \phi[\hat{h}(x)] \quad (-\infty < x < \infty).$$

Thus $\phi(\hat{h}) = \hat{g}$ and the proof is complete.

9B. Corollary. If $\phi(0) = 0$ and if ϕ is analytic in the entire complex plane then ϕ takes Fourier transforms into Fourier transforms.

We would like now to state the following important result in the other direction. It is due to Helson and Kahane.

9C. Theorem. Let $\psi(0) = 0$ and assume ψ is defined on $[-1, 1]$. If $\psi(\hat{f})$ is a Fourier transform whenever $\hat{f}$ is a Fourier transform whose range is contained in $[-1, 1]$, then ψ coincides on $[-1, 1]$ with a function analytic at every point in some domain containing $[-1, 1]$.

A proof of 9C is beyond the scope of this tract. We refer the reader to [6].

For reasons of simplicity, we have treated the question of what functions take Fourier transforms into Fourier transforms by elementary theory of functions of a complex variable.

We mention in passing that other conditions, more nearly necessary and sufficient, concerning this question may be formulated in terms of the so-called real-analytic functions $\phi(x+iy)$—that is, functions which have an absolutely convergent expansion

$$\phi(z) = \phi(x+iy) = \sum_{m,n=0}^{\infty} a_{mn}(x-x_0)^m (y-y_0)^n$$

around each point (x_0, y_0) in their domain of definition. See, for example, [7] and [8]. (We shall have no need for these results in subsequent sections.)

The hypothesis that $\phi(0) = 0$ is essential in 9A and 9B. For if

$$(*) \qquad Q(z) = a_0 + P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n,$$

where $a_0 \neq 0$, then for no $f \in L^1$ is it true that $Q(\hat{f})$ is a Fourier transform. For, since $P(\hat{f})$ is a Fourier transform, we have by 4A,

$$\lim_{x \to \pm\infty} Q[\hat{f}(x)] = a_0 + \lim_{x \to \pm\infty} P[\hat{f}(x)] = a_0 \neq 0.$$

Hence $Q(\hat{f})$ does not vanish at $\pm\infty$ and therefore cannot be a Fourier transform.

However, in 8F we showed that given an interval $[a, b]$ of finite length there exists $\omega \in L^1$ such that $\hat{\omega}(x) = 1$ for x in $[a, b]$. Hence, if Q is as in (*), then for any $f \in L^1$ the function $Q(\hat{f})$ coincides on the interval $[a, b]$ with some Fourier transform. Indeed,

$$Q[\hat{f}(x)] = a_0 \hat{\omega}(x) + P[\hat{f}(x)] \quad (a \leqslant x \leqslant b),$$

so that, on $[a, b]$, $Q(\hat{f})$ coincides with the Fourier transform $a_0 \hat{\omega} + P(\hat{f})$.

In addition to posing the question at the beginning of this section it turns out to be quite fruitful for purposes of application to ask the following: Let $[a, b]$ be an interval of finite length. Then what functions ϕ have the property that $\phi(\hat{f})$ coincides on $[a, b]$ with a Fourier transform whenever $\hat{f}$ is a Fourier transform?

As we have just shown, any polynomial function has this property. We shall prove a much stronger result—namely

9D. Theorem. Let $\phi(z)$ be analytic at each point in some domain (i.e. open connected set) D of the complex plane. Let $[a, b]$ be a closed bounded interval. Then, if $f \in L^1$ and $\hat{f}(x) \in D$ for all x in $[a, b]$, there exists $g \in L^1$ such that

$$\phi[\hat{f}(x)] = \hat{g}(x) \quad (a \leqslant x \leqslant b).$$

(That is, on $[a, b]$ $\phi(\hat{f})$ coincides with a Fourier transform.)

We now set out to establish 9D. To do this we first *state* a purely local result in 9E. Some remarks in 9F show that to establish 9E it is sufficient to prove a special case. This special case is then proved in 9G. Finally, we deduce 9D from 9E by means of a compactness argument in 9H.

The next theorem is a consequence of 9F and 9G.

9E. If $f \in L^1$, $\hat{f}(\alpha) = \beta$, and if $\phi(z)$ is analytic at β, then there exists $g \in L^1$ such that $\phi[\hat{f}(x)] = \hat{g}(x)$ for all x in some neighborhood of α (i.e. $\phi(\hat{f})$ coincides with a Fourier transform in some neighborhood of α).

9F. Remark 1. It is sufficient to prove 9E in the case $\alpha = 0$.

For suppose this case has been proved and suppose $\alpha \neq 0$. By 3C, if $f_1(t) = e^{i\alpha t} f(t)$ then $\hat{f}_1(x) = \hat{f}(x+\alpha)$. In particular $\hat{f}_1(0) = \beta$. The $\alpha = 0$ case implies the existence of $g_1 \in L^1$ such that

$$\phi[\hat{f}_1(x)] = \hat{g}_1(x)$$

for all x in some neighborhood N_0 of 0. Thus $\phi[\hat{f}_1(x-\alpha)] = \hat{g}_1(x-\alpha)$ for all x in some N_α. Setting $\hat{g}(x) = \hat{g}_1(x-\alpha)$ we have $\phi[\hat{f}(x)] = \hat{g}(x)$ for all x in N_α. Hence 9E holds if $\alpha \neq 0$.

We now assume $\alpha = 0$ in 9E and make the

Remark 2. It is sufficient to prove 9E in the case $\beta = 0$.

For suppose this case has been proved and suppose $\beta \neq 0$. Let $\hat{f}_1(x) = \hat{f}(x) - \beta\hat{\omega}(x)$ where $\hat{\omega}(x) = 1$ for $|x| \leqslant 1$ (see 8F). Also, let $\psi(z) = \phi(z+\beta)$. Since $\hat{f}_1(0) = \hat{f}(0) - \beta\hat{\omega}(0) = \beta - \beta \cdot 1 = 0$, and since $\psi(z)$ is analytic at 0, the $\beta = 0$ case implies the existence of some $g \in L^1$ such that $\psi[\hat{f}_1(x)] = \hat{g}(x)$ for all x in some N_0. We may clearly assume that $N_0 \subset [-1, 1]$. But then for $x \in N_0$ we have $\phi[\hat{f}(x)] = \psi[\hat{f}(x) - \beta] = \psi[\hat{f}(x) - \beta\hat{\omega}(x)] = \psi[\hat{f}_1(x)] = \hat{g}(x)$. This establishes the $\beta \neq 0$ case.

We now assume $\alpha = \beta = 0$ in 9E and make the

Remark 3. It is sufficient to prove 9E in the case $\phi(0) = 0$.

For suppose this case has been proved and suppose $\phi(0) \neq 0$. Let $\psi(z) = \phi(z) - \phi(0)$. Then $\psi(0) = 0$ so, by the case $\phi(0) = 0$, there exists $g_1 \in L^1$ such that $\psi[\hat{f}(x)] = \hat{g}_1(x)$ for all x in some N_0. We may assume N_0 is bounded. Hence, by 8F, there exists $\omega \in L^1$ such that $\hat{\omega}(x) = 1$ for all $x \in N_0$. Setting $\hat{g} = \hat{g}_1 + \phi(0)\hat{\omega}$ we have, for $x \in N_0$,

$$\phi[\hat{f}(x)] = \psi[\hat{f}(x)] + \phi(0) = \psi[\hat{f}(x)] + \phi(0)\hat{\omega}(x)$$
$$= \hat{g}_1(x) + \phi(0)\hat{\omega}(x) = \hat{g}(x).$$

Thus, 9C holds in the case $\phi(0) \neq 0$.

We now prove the $\alpha = \beta = 0$, $\phi(0) = 0$ case of 9 E from which, as we have just shown, 9 E will follow.

9G. THEOREM. If $f \in L^1$, $\hat{f}(0) = 0$, if $\phi(0) = 0$, and if $\phi(z)$ is analytic at 0, then there exists $g \in L^1$ such that $\phi[\hat{f}(x)] = \hat{g}(x)$ for all x in some neighborhood of 0.

PROOF. By hypothesis there exists $\epsilon > 0$ such that $\phi(z)$ is analytic for $|z| < \epsilon$. By 8 G we can find $h \in L^1$ such that $\|h\|_1 < \epsilon$ and such that $\hat{f}(x) = \hat{h}(x)$ for all x in some N_0. Since $\|h\|_1 < \epsilon$, 9 A implies the existence of $g \in L^1$ such that $\phi[\hat{h}(x)] = \hat{g}(x)$ for all x. Thus, for $x \in N_0$, we have $\phi[\hat{f}(x)] = \phi[\hat{h}(x)] = \hat{g}(x)$ and the theorem is proved.

Having established 9 E, we use it to prove 9 D.

9H. PROOF OF 9D. By 9 E, each x in the interval $[a, b]$ is contained in an open interval on which $\phi(\hat{f})$ coincides with some Fourier transform. According to the Heine–Borel theorem (see (i) of 14 B), a finite number of these intervals cover $[a, b]$. We may assume that none of these intervals contains another.

Now let (α_1, β_1), (α_2, β_2) be any two of these intervals which have a point in common, and suppose $\alpha_1 < \alpha_2 < \beta_1 < \beta_2$, say. Choose $g_1, g_2 \in L^1$ such that

$$\phi[\hat{f}(x)] = \hat{g}_1(x) \quad (\alpha_1 < x < \beta_1), \tag{1}$$

$$\phi[\hat{f}(x)] = \hat{g}_2(x) \quad (\alpha_2 < x < \beta_2). \tag{2}$$

(Note that this implies $\hat{g}_1(x) = \hat{g}_2(x)$ for x in (α_2, β_1) and hence for x in $[\alpha_2, \beta_1]$.) By 8 F we can find $\omega_1, \omega_2 \in L^1$ such that

$$\hat{\omega}_1(x) = 1 \quad (\alpha_1 \leqslant x \leqslant \alpha_2), \tag{3}$$

$$\hat{\omega}_1(x) = 0 \quad (\beta_1 \leqslant x \leqslant \beta_2), \tag{4}$$

$$\hat{\omega}_2(x) = 1 \quad (\beta_1 \leqslant x \leqslant \beta_2), \tag{5}$$

$$\hat{\omega}_2(x) = 0 \quad (\alpha_1 \leqslant x \leqslant \alpha_2), \tag{6}$$

and such that both $\hat{\omega}_1$ and $\hat{\omega}_2$ are linear on $[\alpha_2, \beta_1]$.

Now let $\theta = \hat{\omega}_1 \hat{g}_1 + \hat{\omega}_2 \hat{g}_2$. Then if $x \in (\alpha_1, \alpha_2)$ we have, by (6), (3) and (1), $\theta(x) = \hat{\omega}_1(x) \hat{g}_1(x) = \hat{g}_1(x) = \phi[\hat{f}(x)]$.

If $x \in [\alpha_2, \beta_1]$ we have, since $\hat{\omega}_1(x) + \hat{\omega}_2(x) = 1$,

$$\hat{\theta}(x) = [\hat{\omega}_1(x) + \hat{\omega}_2(x)]\,\hat{g}_1(x) = \hat{g}_1(x) = \phi[\hat{f}(x)].$$

Finally, if $x \in (\beta_1, \beta_2)$ we have, by (4), (5) and (2),

$$\hat{\theta}(x) = \hat{\omega}_2(x)\,\hat{g}_2(x) = \hat{g}_2(x) = \phi[\hat{f}(x)].$$

We thus conclude that

$$\phi[\hat{f}(x)] = \hat{\theta}(x) \quad (\alpha_1 < x < \beta_2),$$

so that $\phi[\hat{f}(x)]$ coincides with a Fourier transform on (α_1, β_2). Repeating this argument a finite number of times will complete the proof.

9I. Remark. Note that the only property of $[a, b]$ used in the proof of 9D was the fact that from any covering of $[a, b]$ by open intervals we may extract a finite number of intervals which still form a covering. Thus 9D remains true if we replace $[a, b]$ by any set of real numbers with this property (i.e. any compact set). (See (viii) of 14A and (i) of 14B.)

The special case $\phi(z) = \dfrac{1}{z}$ of 9D yields the following corollary which we shall use several times.

9J. Corollary. Let K be a compact set (of real numbers). If $f \in L^1$ and

$$\hat{f}(x) \neq 0 \quad (x \in K),$$

then there exists $g \in L^1$ such that

$$\frac{1}{\hat{f}(x)} = \hat{g}(x) \quad (x \in K).$$

10. Closure of translates in L^1

It was Wiener [21] who first demonstrated a relationship between the span of the translates of a function $f \in L^1$ and the Fourier transform of f. His initial result has had many interesting outgrowths, some of which we shall discuss in Chapter 4. In this section we give a proof of Wiener's theorem.

10A. DEFINITION. If $E \subset L^1$ then $\bar{E}$ will denote the closure of E in L^1. That is, if $f \in L^1$ then $f \in \bar{E}$ if, given $\epsilon > 0$, there exists $g \in E$ such that $\|f-g\|_1 \leqslant \epsilon$. If $E = \bar{E}$ we say that E is closed.

Thus, $f \in \bar{E}$ if and only if there exist $f_1, f_2, \ldots$ in E such that $\lim_{n \to \infty} \|f-f_n\|_1 = 0$. Clearly $E \subset \bar{E}$ and $\bar{\bar{E}} = \bar{E}$.

10B. DEFINITION. If $f \in L^1$ then T_f will denote the set of all $h \in L^1$ such that h is a finite linear combination of translates of f. That is, $h \in T_f$ if

$$h(x) = \Sigma a_k f(x+c_k) \quad (-\infty < x < \infty),$$

for some finite set of real c_k, complex a_k.

Thus, if $f \in L^1$, the subset $\bar{T}_f$ of L^1 is defined by 10B and 10A.

The question we shall investigate now is: 'For what $f \in L^1$ does $\bar{T}_f$ constitute the whole of L^1?' We shall use the following more or less obvious facts:

(i) If $g_1, g_2 \in \bar{T}_f$ then $(a_1 g_1 + a_2 g_2) \in \bar{T}_f$ for any complex numbers a_1, a_2.

(ii) If $g \in \bar{T}_f$ and if, for some real c,

$$g_1(x) = g(x+c) \quad (-\infty < x < \infty),$$

then $g_1 \in \bar{T}_f$.

(iii) If $g \in \bar{T}_f$ then $\bar{T}_g \subset \bar{T}_f$.

We now state the main result of this section.

10C. THEOREM (WIENER). If $f \in L^1$ and

$$(*) \qquad \hat{f}(x) \neq 0 \quad (-\infty < x < \infty),$$

then

$$\bar{T}_f = L^1.$$

We shall give a proof of Wiener's theorem in 10I. We might note here that there actually exists a function $f \in L^1$ whose Fourier transform does not vanish. Indeed, if

$$f(t) = e^{-e^t} e^t \qquad (-\infty < t < \infty)$$

then

$$\hat{f}(x) = \Gamma(1+ix) \quad (-\infty < x < \infty)$$

which is never zero.

Before embarking on a proof of 10C we shall prove a theorem from which the converse of 10C will follow easily.

10D. THEOREM. If $f \in L^1$ and $\hat{f}(\lambda) = 0$, then $\hat{g}(\lambda) = 0$ for every $g \in \overline{T}_f$.

PROOF. Suppose first that $h \in T_f$ so that

$$h(t) = \Sigma a_k f(t+c_k) \quad (-\infty < t < \infty).$$

Then, by 3C, the Fourier transform of h is

$$\hat{h}(x) = \Sigma a_k e^{-ic_k x} \hat{f}(x).$$

Hence $\hat{h}(\lambda) = 0$ if $h \in T_f$.

Now if $g \in \overline{T}_f$ then $\lim_{n \to \infty} \|g - g_n\|_1 = 0$ for some sequence g_n in T_f. From 3B we have

$$\hat{g}(\lambda) = \lim_{n \to \infty} \hat{g}_n(\lambda).$$

But $\hat{g}_n(\lambda) = 0$ for every n since $g_n \in T_f$. Hence $\hat{g}(\lambda) = 0$ and the proof is complete.

10E. THEOREM (CONVERSE OF 10C). If $f \in L^1$ and if

$$\overline{T}_f = L^1 \tag{1}$$

then

$$\hat{f}(x) \neq 0 \quad (-\infty < x < \infty). \tag{2}$$

PROOF. Suppose (2) is false. Then $\hat{f}(\lambda) = 0$ for some λ. By 8F there exists $\omega \in L^1$ such that $\hat{\omega}(\lambda) = 1$. But then, by 10D, ω cannot be in $\overline{T}_f$. This contradicts (1) and the contradiction proves the theorem.

Before we can prove 10C, we need some preliminary results.

10F. THEOREM. Let $f \in L^1$. If $g \in \overline{T}_f$ and $h \in L^1$ then $g * h \in \overline{T}_f$.

PROOF. We may assume that neither g nor h is equivalent to zero. Let $H = g * h$, so that

$$H(x) = \int_{-\infty}^{\infty} g(x-t)\, h(t)\, dt \text{ a.e.} \quad (-\infty < x < \infty).$$

Given $\epsilon > 0$, choose N so that

$$\int_{|t| \geqslant N} |h(t)|\, dt \leqslant \epsilon/2\|g\|_1, \tag{1}$$

and let

$$H_N(x) = \int_{-N}^{N} g(x-t)\, h(t)\, dt \text{ a.e.} \quad (-\infty < x < \infty). \tag{2}$$

Then
$$H(x)-H_N(x) = \int_{|t|\geqslant N} g(x-t)\,h(t)\,dt,$$
so that
$$\|H-H_N\|_1 \leqslant \int_{-\infty}^{\infty} dx \int_{|t|\geqslant N} |g(x-t)\,h(t)|\,dt$$
$$= \int_{|t|\geqslant N} |h(t)|\,dt \int_{-\infty}^{\infty} |g(x-t)|\,dx,$$
$$\|H-H_N\|_1 \leqslant \|g\|_1 \int_{|t|\geqslant N} |h(t)|\,dt,$$
and thus, by (1),
$$\|H-H_N\|_1 \leqslant \tfrac{1}{2}\epsilon. \tag{3}$$
By 2J there exists $\delta > 0$ such that
$$\int_{-\infty}^{\infty} |g(x-y)-g(x)|\,dx \leqslant \epsilon/2\|h\|_1 \quad (|y| \leqslant \delta). \tag{4}$$
Choose $t_0, t_1, \ldots, t_n$ such that $-N = t_0 < t_1 < \ldots < t_n = N$ and such that $t_k - t_{k-1} \leqslant \delta$ for $k = 1, \ldots, n$. Then from (2) we have
$$H_N(x) = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} g(x-t)\,h(t)\,dt.$$
Let
$$h_N(x) = \sum_{k=1}^{n} g(x-t_k) \int_{t_{k-1}}^{t_k} h(t)\,dt.$$
Then, clearly, $h_N \in T_g$. Moreover
$$H_N(x)-h_N(x) = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} [g(x-t)-g(x-t_k)]\,h(t)\,dt,$$
and so
$$\|H_N-h_N\|_1 \leqslant \sum_{k=1}^{n} \int_{-\infty}^{\infty} dx \int_{t_{k-1}}^{t_k} |g(x-t)-g(x-t_k)|\cdot|h(t)|\,dt$$
$$= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} |h(t)|\,dt \int_{-\infty}^{\infty} |g(x-t)-g(x-t_k)|\,dx. \tag{5}$$
If $t_{k-1} \leqslant t \leqslant t_k$ then $|t-t_k| \leqslant \delta$ since $t_k - t_{k-1} \leqslant \delta$. Thus, by (4),
$$\int_{-\infty}^{\infty} |g(x-t)-g(x-t_k)|\,dx \leqslant \epsilon/2\|h\|_1 \quad (t_{k-1} \leqslant t \leqslant t_k).$$
From (5) we then have
$$\|H_N-h_N\|_1 \leqslant (\epsilon/2\|h\|_1) \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} |h(t)|\,dt = (\epsilon/2\|h\|_1) \int_{-N}^{N} |h(t)|\,dt,$$
so that
$$\|H_N-h_N\|_1 \leqslant \tfrac{1}{2}\epsilon.$$

This and (3) show that $\|H-h_N\| \leqslant \epsilon.$

Since ϵ was arbitrary and $h_N \in T_g$ we must have $H \in \overline{T}_g$. But, by hypothesis, $g \in \overline{T}_f$ so that $\overline{T}_g \subset \overline{T}_f$. Thus $H \in \overline{T}_f$ which is what we wished to show.

We can now show that, if $f \in L^1$, then $\overline{T}_f$ will comprise all of L^1 provided that $\overline{T}_f$ contains the sequence δ_N $(N = 1, 2, \ldots)$. Here δ_N is as defined in 8C.

10G. **Theorem.** *If $f \in L^1$ and if*

$$\delta_N \in \overline{T}_f \quad (N = 1, 2, \ldots),$$

then

$$\overline{T}_f = L^1.$$

Proof. For any $h \in L^1$ we have

$$\delta_N * h \in \overline{T}_f \quad (N = 1, 2, \ldots), \tag{1}$$

by 10F. But, by 8D,

$$\lim_{N \to \infty} \|\delta_N * h - h\|_1 = 0. \tag{2}$$

From (1) and (2) we see that $h \in \overline{\overline{T}}_f = \overline{T}_f$. Hence $L^1 \subset \overline{T}_f$ and so $L^1 = \overline{T}_f$.

The next theorem shows that the condition (*) of 10C is enough to insure that each δ_N belongs to $\overline{T}_f$.

10H. **Theorem.** *If $f \in L^1$ and if*

$$\hat{f}(x) \neq 0 \quad (-\infty < x < \infty),$$

then

$$\delta_N \in \overline{T}_f \quad (N = 1, 2, \ldots).$$

Proof. Fix the positive integer N. By assumption $\hat{f}$ does not vanish on $[-N, N]$. By 9J there exists $g \in L^1$ such that

$$\frac{1}{\hat{f}(x)} = \hat{g}(x) \quad (-N \leqslant x \leqslant N).$$

With $\Delta_N = \hat{\delta}_N$ as in 8C we then have

$$\frac{\Delta_N(x)}{\hat{f}(x)} = \Delta_N(x)\,\hat{g}(x) \quad (-\infty < x < \infty),$$

and thus

$$\Delta_N(x) = \hat{f}(x)\,\Delta_N(x)\,\hat{g}(x) \quad (-\infty < x < \infty). \tag{1}$$

By 7E, $\hat{f}\Delta_N\hat{g}$ is the Fourier transform of $f*\delta_N*g$. From (1) and 6F we conclude

$$\delta_N = f*(\delta_N*g).$$

This and 10F imply $\delta_N \in \overline{T}_f$ and the proof is complete.

10I. Theorem 10C is now quite easy to establish. For if (*) of 10C holds then, by 10H, $\delta_N \in \overline{T}_f$ for every N. Hence, by 10G, $\overline{T}_f = L^1$. Thus 10C is now fully established.

We might mention that Theorem 10C has important applications in the field of summability of series. In particular, Wiener's famous Tauberian theorem [21; 25] is easily deducible from 10C.

11. Some algebraic reformulations

The concept of a commutative Banach algebra is fundamental to a large part of modern harmonic analysis. In this section we shall reformulate some of the preceding results in terms of ideals in a commutative Banach algebra.

We assume that the reader is familiar with the definition of an algebra over the complex numbers. (See [18; 61], for example.)

11A. Definition. If on the *commutative* algebra A there is defined a norm $\|\cdot\|$ such that

(i) $\|\bar{0}\| = 0$ where $\bar{0}$ is the zero element of A,

(ii) $\|a\| > 0$ if $a \in A$, $a \neq \bar{0}$,

(iii) $\|\alpha a\| = |\alpha|\cdot\|a\|$ if $a \in A$, α complex,

(iv) $\|a \dotplus b\| \leqslant \|a\| + \|b\|$ if $a, b \in A$,

(v) $\|a * b\| \leqslant \|a\|\cdot\|b\|$ if $a, b \in A$,

where $\dotplus$ and $*$ are respectively the addition and multiplication for A, then we call A a commutative normed algebra. If, in addition, A is 'complete' with respect to the norm, we call A a commutative Banach algebra. This last means that if $a_1, a_2, \ldots$ is a sequence in A such that $\lim\limits_{m,n\to\infty} \|a_m \dot{-} a_n\| = 0$, then there exists $a \in A$ such that $\lim\limits_{n\to\infty} \|a_n \dot{-} a\| = 0$.

Consider now the space L^1. Then, with the ordinary addition of functions as addition operation and with convolution of functions as multiplication operation, L^1 is readily verified to be a commutative algebra. Moreover, with the aid of (1) of 2L, 7D and 2I, we may easily establish the following theorem.

11B. THEOREM. L^1 (with norm $\|\cdot\|_1$ and with convolution as multiplication) is a commutative Banach algebra.

11C. DEFINITION. If A is a commutative algebra and $I \subset A$ we say that I is an *ideal* of A if

(i) I is an algebra with respect to the operations for A,

(ii) $g * h \in I$ whenever $g \in I$, $h \in A$.

The whole of A is thus an ideal of A, and so is the set consisting of the zero element of A alone. Any other ideal is said to be *proper*. A proper ideal M of A is said to be *maximal* if M is contained in no proper ideal of A other than M itself.

As an exercise in the preceding concepts we leave it to the reader to prove the next two results.

11D. THEOREM. If I is an ideal of L^1 then $\bar{I}$, the closure of I (see 10A), is also an ideal of L^1.

11E. THEOREM. If M is a maximal ideal of L^1 then either M is closed or $\bar{M} = L^1$.

It turns out that the subsets $\bar{T}_f$ that we discussed in § 10 are all ideals in the algebra L^1. Specifically.

11F. THEOREM. If $f \in L^1$ then $\bar{T}_f$ is a closed ideal in L^1.

PROOF. Since $\bar{\bar{T}}_f = \bar{T}_f$ (10A), it follows that $\bar{T}_f$ is closed. If $g \in \bar{T}_f$ and $h \in L^1$ then, by 10F, $g * h \in \bar{T}_f$. The other requirements for an ideal are readily verified for $\bar{T}_f$.

We remark that it is not known at present whether every closed ideal of L^1 is of the form $\bar{T}_f$ for some $f \in L^1$.

Having shown in 11E that $\bar{T}_f$ is a closed ideal containing f, we now show that it is the smallest such ideal.

11G. THEOREM. If $f \in L^1$ and I is any closed ideal of L^1 containing f, then $\bar{T}_f \subset I$.

PROOF. Let $g(x) = f(x+a)$ for fixed real a and all x. We first show that $g \in I$.

For each $N = 1, 2, \ldots$, let $\delta_N^a(x) = \delta_N(x+a)$ where δ_N is as in 8C. Then

$$(\delta_N^a * f)(x) = \int_{-\infty}^{\infty} \delta_N(x+a-t)f(t)\,dt$$

$$= \int_{-\infty}^{\infty} \delta_N(x-t)f(t+a)\,dt = \int_{-\infty}^{\infty} \delta_N(x-t)\,g(t)\,dt$$

$$= (\delta_N * g)(x),$$

so that $\delta_N^a * f = \delta_N * g$. Since $f \in I$ and I is an ideal, we have $\delta_N^a * f \in I$ and hence $\delta_N * g \in I$. Since I is closed, it follows from 8D that $g \in I$.

Thus every translate of f is in I so that $T_f \subset I$. Since I is closed we have $\overline{T}_f \subset I$ and the proof is complete.

We can now establish easily the following interesting characterization of closed ideals in L^1.

We say that $J \subset L^1$ is a *linear subspace* of L^1 if whenever $f, g \in J$ then $(\alpha f + \beta g) \in J$ for arbitrary complex numbers α, β.

11H. THEOREM. Let $I \subset L^1$. Then the following statements are equivalent.

(A) I is a closed ideal of L^1.

(B) I is a closed linear subspace of L^1 with the property that if $f \in I$ then every translate of f is also in I.

PROOF. That (A) implies (B) follows immediately from 11G.

Now suppose (B) is true and let $g \in I$, $h \in L^1$. From (B) we conclude that $T_g \subset I$. Since I is closed it follows that $\overline{T}_g \subset I$. Since $\overline{T}_g$ is an ideal (11F) we must have $g * h \in \overline{T}_g$. Hence $g * h \in I$ and (A) holds.

We now set out to determine what are the closed maximal ideals in L_1.

11I. DEFINITION. For each real number λ we denote by M_λ the set of all $f \in L^1$ such that $\hat{f}(\lambda) = 0$.

11J. THEOREM. Each M_λ is a closed maximal ideal of L^1.

PROOF. (a) M_λ is closed.

Choose any $f \in \bar{M}_\lambda$. Then $\lim_{n\to\infty} \|f_n - f\|_1 = 0$ for some $f_1, f_2, \ldots$ in M_λ. Hence, by 3B, $\lim_{n\to\infty} \hat{f}_n(\lambda) = \hat{f}(\lambda)$. But $f_n \in M_\lambda$ so that $\hat{f}_n(\lambda) = 0$. Thus $\hat{f}(\lambda) = 0$ and so $f \in M_\lambda$. Hence $\bar{M}_\lambda \subset M_\lambda$ and M_λ is closed.

(b) M_λ is an ideal.

If $g \in M_\lambda$, $h \in L^1$ then $(g*h)^\wedge(\lambda) = \hat{g}(\lambda)\hat{h}(\lambda) = 0\cdot\hat{h}(\lambda) = 0$. Thus $g*h \in M_\lambda$. The other requirements for an ideal are easily verified.

(c) M_λ is maximal.

Let M be any ideal of L^1 such that $M \supset M_\lambda$, $M \neq M_\lambda$. We shall show that $M = L^1$. Since $M \neq M_\lambda$, there exists $g \in M$ for which $\hat{g}(\lambda) \neq 0$. For any $h \in L^1$ we have

$$\hat{h}(x) = \frac{\hat{h}(\lambda)}{\hat{g}(\lambda)}\hat{g}(x) + \hat{h}_1(x) \quad (-\infty < x < \infty),$$

where $\hat{h}_1(\lambda) = 0$. Since this implies $h_1 \in M_\lambda$ we have $h_1 \in M$. But $g \in M$ also, and so $h \in M$ since $h = \frac{\hat{h}(\lambda)}{\hat{g}(\lambda)} g + h_1$. Thus $L^1 \subset M$ which proves $L^1 = M$. Hence M_λ is maximal.

Thus to each real λ there corresponds a closed maximal ideal M_λ. By 8F, if $\lambda_1 \neq \lambda_2$ then $M_{\lambda_1} \neq M_{\lambda_2}$. The next two theorems show that the M_λ comprise all the closed maximal ideals of L^1.

11K. THEOREM. If I is a proper closed ideal of L^1 then $I \subset M_\lambda$ for some λ.

PROOF. Suppose I is contained in no M_λ. We shall show that $I = L^1$—hence that I is not proper—and this contradiction will prove the theorem.

Fix the positive integer N. By the assumption that I is contained in no M_λ, for each $\lambda \in [-N, N]$ there exists $f_\lambda \in I$ such that $\hat{f}_\lambda(\lambda) \neq 0$. Let $g_\lambda(t) = \overline{f_\lambda(-t)}$, where the bar denotes complex conjugate. Then

$$\hat{g}_\lambda(x) = \int_{-\infty}^{\infty} e^{ixt}\overline{f_\lambda(-t)}\,dt = \int_{-\infty}^{\infty} \overline{e^{-ixt} f_\lambda(t)}\,dt = \overline{\hat{f}_\lambda(x)},$$

and thus $\hat{g}_\lambda = \bar{\hat{f}}_\lambda$. Let $h_\lambda = g_\lambda * f_\lambda$. Then $h_\lambda \in I$ since $f_\lambda \in I$, and $\hat{h}_\lambda = \hat{g}_\lambda \hat{f}_\lambda = \bar{\hat{f}}_\lambda \hat{f}_\lambda = |\hat{f}_\lambda|^2$. Thus $\hat{h}_\lambda(x) \geqslant 0$ for all x and

$$\hat{h}_\lambda(\lambda) = |\hat{f}_\lambda(\lambda)|^2 > 0.$$

This implies $\hat{h}_\lambda(x) > 0$ for all x in some neighborhood N_λ of λ. The interval $[-N, N]$ can be covered by a finite number of these neighborhoods—say $N_{\lambda_1}, N_{\lambda_2}, \ldots, N_{\lambda_n}$. Let $h = h_{\lambda_1} + \ldots + h_{\lambda_n}$. Then $h \in I$ and

$$\hat{h}(x) > 0 \quad (-N \leqslant x \leqslant N).$$

By 9J there exists $k \in L^1$ such that

$$\frac{1}{\hat{h}(x)} = \hat{k}(x) \quad (-N \leqslant x \leqslant N).$$

With $\Delta_N = \hat{\delta}_N$ as in 8C, we then have

$$\frac{\Delta_N(x)}{\hat{h}(x)} = \Delta_N(x)\,\hat{k}(x) \quad (-\infty < x < \infty).$$

Hence, $\Delta_N = \hat{h}\hat{k}\Delta_N$ so that $\delta_N = h * k * \delta_N$. Since $h \in I$ and I is an ideal, this implies $\delta_N \in I$. Thus I contains δ_N for each $N = 1, 2, \ldots$. From 8D and the fact that I is a closed ideal it follows that $I = L^1$, which is what we wished to show.

11L. Theorem. Let M be any closed maximal ideal of L_1. Then $M = M_\lambda$ for some λ. In other words the M_λ comprise all the closed maximal ideals of L^1.

Proof. By 11K, $M \subset M_\lambda$ for some λ. Hence $M = M_\lambda$ since M is maximal.

We have thus established a $(1, 1)$ correspondence between the real numbers and the set of all maximal ideals in L^1.

11M. The fact that every proper closed ideal of L^1 is contained in a maximal ideal (11K) can be considered to be a reformulation of Wiener's Theorem 10C. For if $f \in L^1$ then $\overline{T}_f$, which by 11F is a closed ideal, will be all of L^1 (i.e. not proper) if $\hat{f}$ never vanishes ($\overline{T}_f$ is contained in no maximal ideal).

Now suppose that, for some $f \in L^1$, $\overline{T}_f$ is a proper ideal and is thus contained in one or more of the maximal ideals M_λ. If we take the intersection of all the maximal ideals M_λ containing $\overline{T}_f$ we get an ideal (verify!) that contains $\overline{T}_f$. A very interesting question is—which $f \in L^1$ have the property that $\overline{T}_f$ is precisely the intersection of the maximal ideals containing it?† (It has been discovered only recently that there exists an $f \in L^1$ that does not have this property.)

To formulate this problem in different language we now make a string of equivalent statements ending with the form of the problem which we shall investigate in Chapter 4.

The following statements are equivalent for any $f \in L^1$:

(i) $\bar{T}_f$ is the intersection of all the maximal ideals M_λ such that $\bar{T}_f \subset M_\lambda$.

(ii) If $g \in M_\lambda$ for every M_λ such that $\bar{T}_f \subset M_\lambda$ then $g \in \bar{T}_f$.

(iii) If $\hat{g}(\lambda) = 0$ for every λ such that $\bar{T}_f \subset M_\lambda$ then $g \in \bar{T}_f$.

(iv) If $\hat{g}(\lambda) = 0$ for every λ such that $f \in M_\lambda$ then $g \in \bar{T}_f$.

(v) If $\hat{g}(\lambda) = 0$ for every λ such that $\hat{f}(\lambda) = 0$ then $g \in \bar{T}_f$.

† This problem, in a reformulation involving bounded functions, is often referred to as the problem of spectral synthesis. For the bounded function formulation see [14].

CHAPTER 3

THE FOURIER TRANSFORM ON L^2

12. The Fourier transform on $L^1 \cap L^2$

In the preceding chapter (3A) we defined the Fourier transform $\hat{f}$ for $f \in L^1$. In this chapter we shall define the Fourier transform for $f \in L^2$. It turns out that if $f \in L^2$ then the Fourier transform $\hat{f}$ of f is also in L^2 and $\|\hat{f}\|_2 = (2\pi)^{\frac{1}{2}} \|f\|_2$.

(An explanation is perhaps due to those who would rather avoid the annoying factor of $(2\pi)^{\frac{1}{2}}$ and have $\|\hat{f}\|_2 = \|f\|_2$. This could have been achieved by defining the Fourier transform in 3A as

$$\hat{f}(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{ixt} f(t)\, dt.$$

This definition, however, causes factors of $(2\pi)^{\frac{1}{2}}$ to pop up in unwanted places in the L^1 theory (in particular, in convolutions) and, we feel, nothing is gained. An alternative definition of $\hat{f}$ as

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{2\pi ixt} f(t)\, dt$$

also leads to a symmetric L^2 theory but, again, requires a great many 2π's. See Appendix, paragraph G.)

After the following lemma we shall establish a result concerning the Fourier transform on $L^1 \cap L^2$ which leads to the definition of the Fourier transform on L^2.

12A. LEMMA. For any real numbers $\epsilon > 0$ and α we have

$$\int_{-\infty}^{\infty} e^{i\alpha t} e^{-\epsilon t^2}\, dt = \left(\frac{\pi}{\epsilon}\right)^{\frac{1}{2}} \cdot e^{-\alpha^2/4\epsilon}.$$

PROOF. We start with the well-known formula

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \pi^{\frac{1}{2}}.$$

For any real β and any $R > 0$, integrate the analytic function e^{-z^2} around the rectangle with vertices $\pm R$, $\pm R + i\beta$. Letting $R \to \infty$, we find that

$$\int_{-\infty}^{\infty} e^{-(x+i\beta)^2}\,dx = \pi^{\frac{1}{2}},$$

or

$$\int_{-\infty}^{\infty} e^{-2i\beta x}\,e^{-x^2}\,dx = \pi^{\frac{1}{2}}\,e^{-\beta^2}.$$

Setting $\beta = -\alpha/2\epsilon^{\frac{1}{2}}$ we obtain

$$\int_{-\infty}^{\infty} e^{i\alpha\epsilon^{-\frac{1}{2}}x}\,e^{-x^2}\,dx = \pi^{\frac{1}{2}}\,e^{-\alpha^2/4\epsilon}.$$

The lemma then follows from the change of variable $t = \epsilon^{-\frac{1}{2}}x$.

12 B. Theorem. Let $f \in L^1 \cap L^2$. Then $\hat{f} \in L^2$ and

$$\int_{-\infty}^{\infty} |\hat{f}(x)|^2\,dx = 2\pi\int_{-\infty}^{\infty} |f(t)|^2\,dt.$$

That is, $\|\hat{f}\|_2 = (2\pi)^{\frac{1}{2}}\,\|f\|_2$.

Proof. We have

$$|\hat{f}(x)|^2 = \hat{f}(x)\overline{\hat{f}(x)} = \int_{-\infty}^{\infty} e^{ixt} f(t)\,dt \int_{-\infty}^{\infty} e^{-ixu}\overline{f(u)}\,du.$$

Multiplying by $e^{-x^2/n}$ $(n = 1, 2, \ldots)$ and integrating we obtain

$$\int_{-\infty}^{\infty} e^{-x^2/n}|\hat{f}(x)|^2\,dx = \int_{-\infty}^{\infty} e^{-x^2/n}\,dx\int_{-\infty}^{\infty} e^{ixt}f(t)\,dt\int_{-\infty}^{\infty} e^{-ixu}\overline{f(u)}\,du.$$

Since $f \in L^1$, the iterated integral converges absolutely. Thus, by 2 D, we may change the order of integration. This yields

$$\int_{-\infty}^{\infty} e^{-x^2/n}|\hat{f}(x)|^2\,dx = \int_{-\infty}^{\infty} \overline{f(u)}\,du\int_{-\infty}^{\infty} f(t)\,dt\int_{-\infty}^{\infty} e^{ix(t-u)}\,e^{-x^2/n}\,dx.$$

From the lemma 12 A we have

$$\int_{-\infty}^{\infty} e^{ix(t-u)}\,e^{-x^2/n}\,dx = (\pi n)^{\frac{1}{2}}\,e^{-n(t-u)^2/4}.$$

Hence

$$\int_{-\infty}^{\infty} e^{-x^2/n}|\hat{f}(x)|^2\,dx = (\pi n)^{\frac{1}{2}}\int_{-\infty}^{\infty}\overline{f(u)}\,du\int_{-\infty}^{\infty} e^{-n(t-u)^2/4}f(t)\,dt$$

$$= (\pi n)^{\frac{1}{2}}\int_{-\infty}^{\infty}\overline{f(u)}\,du\int_{-\infty}^{\infty} e^{-nt^2/4}f(t+u)\,dt$$

$$= (\pi n)^{\frac{1}{2}}\int_{-\infty}^{\infty} e^{-nt^2/4}\,dt\int_{-\infty}^{\infty} f(t+u)\overline{f(u)}\,du$$

$$= (\pi n)^{\frac{1}{2}}\int_{-\infty}^{\infty} e^{-nt^2/4}\,F(t)\,dt,$$

where
$$F(t) = \int_{-\infty}^{\infty} f(t+u)\overline{f(u)}\,du.$$

Therefore

$$\int_{-\infty}^{\infty} e^{-x^2/n}\,|\hat{f}(x)|^2\,dx = 2\pi^{\frac{1}{2}}\int_{-\infty}^{\infty} e^{-t^2}\,F(2n^{-\frac{1}{2}}t)\,dt. \tag{1}$$

Now $F(t)$ is continuous at $t = 0$. For

$$|F(t)-F(0)| = \left|\int_{-\infty}^{\infty}[f(t+u)-f(u)]\overline{f(u)}\,du\right|$$

$$\leqslant \int_{-\infty}^{\infty}|f(t+u)-f(u)|\cdot|f(u)|\,du.$$

Hence, by the Schwarz inequality, 2M,

$$|F(t)-F(0)|^2 \leqslant \int_{-\infty}^{\infty}|f(t+u)-f(u)|^2\,du\cdot\int_{-\infty}^{\infty}|f(u)|^2\,du. \tag{2}$$

By 2J, the right side of (2) tends to zero as $t\to 0$, and thus

$$\lim_{t\to 0} F(t) = F(0). \tag{3}$$

For any t, 2M also shows that

$$|F(t)|^2 \leqslant \int_{-\infty}^{\infty}|f(t+u)|^2\,du\cdot\int_{-\infty}^{\infty}|f(u)|^2\,du = \|f\|_2^2\cdot\|f\|_2^2 = \|f\|_2^4.$$

Thus the integrand on the right of (1) is dominated by a constant times e^{-t^2}. This, (3), and 2B imply

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} e^{-t^2}F(2n^{-\frac{1}{2}}t)\,dt = F(0)\int_{-\infty}^{\infty} e^{-t^2}\,dt = \pi^{\frac{1}{2}}F(0). \tag{4}$$

Since $F(0) = \|f\|_2^2$, we conclude from (1) and (4) that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} e^{-x^2/n}\,|\hat{f}(x)|^2\,dx = 2\pi\,\|f\|_2^2. \tag{5}$$

But then, by 2A,

$$\int_{-\infty}^{\infty}|\hat{f}(x)|^2\,dx = \int_{-\infty}^{\infty}\lim_{n\to\infty} e^{-x^2/n}\,|\hat{f}(x)|^2\,dx$$
$$\leqslant \lim_{n\to\infty}\int_{-\infty}^{\infty} e^{-x^2/n}\,|\hat{f}(x)|^2\,dx = 2\pi\,\|f\|_2^2,$$

which proves that $\hat{f}\in L^2$. But then from (5) and 2B we have

$$\int_{-\infty}^{\infty}|\hat{f}(x)|^2\,dx = 2\pi\,\|f\|_2^2,$$

and the proof is complete.

13. Plancherel's theorem

In this section we shall define the Fourier transform on L^2 and establish its most important properties. Some of these results will be summarized at the end of the section in a theorem which bears the name of its discoverer, Plancherel.

We shall define the Fourier transform $\hat{f}$ of $f\in L^2$ as the limit in the L^2 mean of a sequence of Fourier transforms $\hat{f}_N$, where f_N is a certain sequence of functions in $L^1\cap L^2$ which converges in the L^2 mean to f. That the sequence $\hat{f}_N$ actually does converge is proved in the next theorem.

13A. Theorem. Let $f\in L^2$. For $N = 1, 2, \ldots$, define f_N by

$$f_N(t) = f(t) \quad (|t| \leqslant N),$$
$$f_N(t) = 0 \quad (|t| > N).$$

Then $f_N\in L^1\cap L^2$ and $\hat{f}_N\in L^2$. Moreover, as $N\to\infty$, $\hat{f}_N$ converges in the L^2 mean to a function in L^2.

Proof. By 2M

$$\int_{-\infty}^{\infty}|f_N(t)|\,dt = \int_{-N}^{N}|f(t)|\,dt$$
$$\leqslant \left[\int_{-N}^{N}|f(t)|^2\,dt\cdot\int_{-N}^{N}dt\right]^{\frac{1}{2}} \leqslant \|f\|_2\cdot(2N)^{\frac{1}{2}} < \infty,$$

so that $f_N \in L^1$. Since $|f_N(t)| \leqslant |f(t)|$ it is clear that $f_N \in L^2$. Hence $f_N \in L^1 \cap L^2$ and so, by 12 B, $\hat{f}_N \in L^2$. The first part of the theorem is now established.

To show that $\hat{f}_N$ converges in the L^2 mean it is sufficient, by 2 I, to show that

$$\lim_{M,N\to\infty} \|\hat{f}_M - \hat{f}_N\|_2 = 0. \tag{1}$$

But $\hat{f}_M - \hat{f}_N$ is the Fourier transform of $f_M - f_N$ which is in $L^1 \cap L^2$. Hence, by 12 B,

$$\|\hat{f}_M - \hat{f}_N\|_2^2 = 2\pi \|f_M - f_N\|_2^2. \tag{2}$$

The right side of (2) is equal to 2π times

$$\left|\int_{-N}^{-M} |f(t)|^2\,dt + \int_{M}^{N} |f(t)|^2\,dt\right|$$

which tends to zero as $M, N \to \infty$. Thus (1) holds, which is what we wished to show.

We can now define the Fourier transform of an L^2 function.

13 B. Definition. For $f \in L^2$ we define $\hat{f}$, the Fourier transform of f, as

$$(*) \qquad \hat{f}(x) = \underset{N\to\infty}{\text{l.i.m.}} \int_{-N}^{N} e^{ixt} f(t)\,dt$$

where l.i.m. stands for limit in the L^2 mean. Note that (*) is precisely equivalent to

$$\lim_{N\to\infty} \|\hat{f} - \hat{f}_N\|_2 = 0.$$

13 C. Remark. We have defined $\hat{f}$ for $f \in L^2$. Of course, since $\hat{f}$ has been defined only as an element of L^2, $\hat{f}(x)$ is defined only almost everywhere.

But if $f \in L^1$ we have previously defined $\hat{f}$ in 3 A as

$$(**) \qquad \hat{f}(x) = \int_{-\infty}^{\infty} e^{ixt} f(t)\,dt \quad (-\infty < x < \infty),$$

which, by 2 B, can be written

$$\hat{f}(x) = \lim_{N\to\infty} \hat{f}_N(x) \quad (-\infty < x < \infty).$$

Thus, if $f \in L^1 \cap L^2$, we now have two definitions of $\hat{f}$, namely (*) and (**). However, it follows immediately from 2K that the $\hat{f}$ defined by (**) (which, we recall, is a continuous function) defines the same element of L^2 as does the $\hat{f}$ in (*). Hence our two definitions are consistent.

In 12B we saw that if $f \in L^1 \cap L^2$ then the L^2 norm of $\hat{f}$ is $\sqrt{(2\pi)}$ times the L^2 norm of f. That is, except for a constant factor, the mapping $f \to \hat{f}$ preserves the L^2 norm. We now show that this is true for all $f \in L^2$.

13D. THEOREM (PARSEVAL'S RELATION). Let $f \in L^2$. Then $\|\hat{f}\|_2 = (2\pi)^{\frac{1}{2}} \|f\|_2$.

PROOF. Define f_N as in 13A. Then

$$\lim_{N\to\infty} \|\hat{f}_N - \hat{f}\|_2 = 0.$$

By (ii) of 2L, this implies

$$\lim_{N\to\infty} \|\hat{f}_N\|_2 = \|\hat{f}\|_2. \tag{1}$$

It is clear from the definition of f_N that

$$\lim_{N\to\infty} \|f_N\|_2 = \|f\|_2. \tag{2}$$

Since $f_N \in L^1 \cap L^2$, 12B shows

$$\|\hat{f}_N\|_2 = (2\pi)^{\frac{1}{2}} \|f_N\|_2. \tag{3}$$

Using (1), (3), and (2) we have

$$\|\hat{f}\|_2 = \lim_{N\to\infty} \|\hat{f}_N\|_2 = \lim_{N\to\infty} (2\pi)^{\frac{1}{2}} \|f_N\|_2 = (2\pi)^{\frac{1}{2}} \|f\|_2,$$

and the theorem is proved.

An easy but important consequence of 13D is the following result.

13E. THEOREM. If $f, g \in L^2$ then

$$\int_{-\infty}^{\infty} \hat{f}(x)\overline{\hat{g}(x)}\,dx = 2\pi \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx.$$

PROOF. By 13D,

$$\|\hat{f}+\hat{g}\|_2^2 = 2\pi \|f+g\|_2^2.$$

Writing $\int f$ instead of $\int_{-\infty}^{\infty} f(x)\,dx$ we thus have

$$\int(\hat{f}+\hat{g})(\bar{\hat{f}}+\bar{\hat{g}}) = 2\pi\int(f+g)(\bar{f}+\bar{g}).$$

Expanding both sides we obtain

$$\int|\hat{f}|^2+\int|\hat{g}|^2+\int\hat{f}\bar{\hat{g}}+\int\bar{\hat{f}}\hat{g} = 2\pi\left(\int|f|^2+\int|g|^2+\int f\bar{g}+\int\bar{f}g\right).$$

Again by 13D, $\int|\hat{f}|^2 = 2\pi\int|f|^2$ and $\int|\hat{g}|^2 = 2\pi\int|g|^2$. Hence

$$\int\hat{f}\bar{\hat{g}}+\int\bar{\hat{f}}\hat{g} = 2\pi\left(\int f\bar{g}+\int\bar{f}g\right). \tag{1}$$

Since g was an arbitrary element of L^2 we may substitute $i\hat{g}$, ig respectively for $\hat{g}$, g in (1) and obtain

$$\int\hat{f}\overline{(i\hat{g})}+\int\bar{\hat{f}}(i\hat{g}) = 2\pi\left(\int f\overline{(ig)}+\int\bar{f}(ig)\right),$$

or

$$-i\int\hat{f}\bar{\hat{g}}+i\int\bar{\hat{f}}\hat{g} = 2\pi\left(-i\int f\bar{g}+i\int\bar{f}g\right). \tag{2}$$

Dividing (2) by $-i$ and then adding the resulting equation to (1) will complete the proof.

Note that if we let $g = f$ in 13E we obtain 13D.

In §§5 and 6 we discussed the inversion of the Fourier transform on L^1. The next theorems show that the formal inversion stated in (1) of 5A can be given an L^2 interpretation. Unlike the L^1 case, the L^2 inversion requires no additional hypotheses.

13F. THEOREM. If $f, g \in L^2$ then

$$\int_{-\infty}^{\infty} f(x)\,\hat{g}(x)\,dx = \int_{-\infty}^{\infty} \hat{f}(x)\,g(x)\,dx.$$

PROOF. Define f_N, g_N as in 13A. We have, for any positive integers M and N,

$$\hat{f}_M(x) = \int_{-\infty}^{\infty} e^{ixt} f_M(t)\,dt, \tag{1}$$

$$\hat{g}_N(x) = \int_{-\infty}^{\infty} e^{ixt} g_N(t)\,dt.$$

Multiply (1) by $g_N(x)$ and integrate to obtain

$$\int_{-\infty}^{\infty}\hat{f}_M(x)\,g_N(x)\,dx = \int_{-\infty}^{\infty} g_N(x)\,dx\int_{-\infty}^{\infty} e^{ixt} f_M(t)\,dt.$$

By 13A, f_M and g_N are in L^1. Therefore the iterated integral on the right is absolutely convergent. By 2D, we may change the order of integration. This yields

$$\int_{-\infty}^{\infty} \hat{f}_M(x)\, g_N(x)\, dx = \int_{-\infty}^{\infty} f_M(t)\, dt \int_{-\infty}^{\infty} e^{ixt} g_N(x)\, dx,$$

or
$$\int_{-\infty}^{\infty} \hat{f}_M(x)\, g_N(x)\, dx = \int_{\infty}^{\infty} f_M(t)\, \hat{g}_N(t)\, dt. \tag{2}$$

Now
$$\lim_{N\to\infty} \|g_N - g\|_2 = 0 = \lim_{N\to\infty} \|\hat{g}_N - \hat{g}\|_2.$$

Hence, by 2N, letting $N \to \infty$ in (2) we have

$$\int_{-\infty}^{\infty} \hat{f}_M(x)\, g(x)\, dx = \int_{-\infty}^{\infty} f_M(t)\, \hat{g}(t)\, dt.$$

Then, letting $M \to \infty$ and again using 2N we have

$$\int_{-\infty}^{\infty} \hat{f}(x)\, g(x)\, dx = \int_{-\infty}^{\infty} f(t)\, \hat{g}(t)\, dt.$$

The theorem is thus established.

The next theorem yields the L^2 inversion formula as a corollary.

13G. Theorem. Let $f \in L^2$ and let $g = \bar{\hat{f}}$. Then $f = \frac{1}{2\pi}\bar{\hat{g}}$.

Proof. We have

$$\left\| f - \frac{1}{2\pi}\bar{\hat{g}} \right\|_2^2 = \int \left(f - \frac{1}{2\pi}\bar{\hat{g}} \right) \left(\bar{f} - \frac{1}{2\pi}\hat{g} \right),$$

and so
$$\left\| f - \frac{1}{2\pi}\bar{\hat{g}} \right\|_2^2 = \|f\|_2^2 - \frac{1}{2\pi}\int f\hat{g} - \frac{1}{2\pi}\int \overline{f\hat{g}} + \frac{1}{4\pi^2}\|\hat{g}\|_2^2. \tag{1}$$

Since $g = \bar{\hat{f}}$ we have, using 13F and 13D,

$$\int f\hat{g} = \int \hat{f} g = \int \hat{f}\bar{\hat{f}} = \|\hat{f}\|_2^2 = 2\pi \|f\|_2^2. \tag{2}$$

But then
$$\int \overline{f\hat{g}} = 2\pi \|f\|_2^2. \tag{3}$$

Finally, by 13*D*,

$$\|\hat{g}\|_2^2 = 2\pi \|g\|_2^2 = 2\pi \|\hat{f}\|_2^2 = 4\pi^2 \|f\|_2^2. \tag{4}$$

From equations (1) through (4) we conclude

$$\left\| f - \frac{1}{2\pi}\bar{\hat{g}} \right\|_2^2 = 0,$$

from which the theorem follows immediately.

13H. Corollary (Inversion of Fourier transform on L^2). If $f \in L^2$ then

$$f(t) = \underset{N \to \infty}{\text{l.i.m.}} \frac{1}{2\pi} \int_{-N}^{N} e^{-ixt} \hat{f}(x)\, dx. \tag{1}$$

Proof. Let $g = \bar{\hat{f}}$. Then, by 13G, $\bar{f} = \frac{1}{2\pi} \hat{g}$, that is,

$$\bar{f}(t) = \underset{N \to \infty}{\text{l.i.m.}} \frac{1}{2\pi} \int_{-N}^{N} e^{ixt} g(x)\, dx = \underset{N \to \infty}{\text{l.i.m.}} \frac{1}{2\pi} \int_{-N}^{N} e^{ixt} \bar{\hat{f}}(x)\, dx.$$

Taking complex conjugates we obtain (1).

In 4C we showed that, although the Fourier transform of an L^1 function must be continuous and vanish at $\pm\infty$, not all continuous functions which tend to zero at $\pm\infty$ are Fourier transforms of integrable functions. The L^2 case turns out differently. Every member of L^2 is the Fourier transform of an L^2 function.

13I. Corollary. Every $f \in L^2$ is the Fourier transform of a unique element of L^2.

Proof. Take any $f \in L^2$. Let

$$h = \bar{f} \quad \text{and} \quad g = \bar{\hat{h}}.$$

By 13G,

$$\bar{f} = h = \frac{1}{2\pi} \bar{\hat{g}},$$

so that $f = \frac{1}{2\pi} \hat{g}$. Hence f is the Fourier transform of $\frac{1}{2\pi} g$. The uniqueness follows easily from 13H.

We summarize some of the preceding results in the following theorem.

13J. Theorem (Plancherel). If $f \in L^2$ then there exists a function $\hat{f} \in L^2$ (called the Fourier transform of f) such that

$$\hat{f}(x) = \underset{N \to \infty}{\text{l.i.m.}} \int_{-N}^{N} e^{ixt} f(t)\, dt,$$

$$f(t) = \underset{N \to \infty}{\text{l.i.m.}} \frac{1}{2\pi} \int_{-N}^{N} e^{-ixt} \hat{f}(x)\, dx,$$

and

$$\|\hat{f}\|_2 = (2\pi)^{\frac{1}{2}} \|f\|_2.$$

Every $f \in L^2$ can be expressed as $f = \hat{g}$ for a unique $g \in L^2$.

CHAPTER 4

GENERALIZATIONS OF WIENER'S THEOREM

We are now going to develop further the relationship, initially investigated in section 10, between the zeros of the Fourier transform of $f \in L^1$ and the closure of translates $\overline{T}_f$. We first list a few necessaries about point sets.

14. Point sets

Because of the lack of uniformity of terminology in the existing literature we shall list here some definitions concerning point sets on the real line. The term to be defined in a given sentence is italicized. The letter E is used to denote a set of real numbers.

14A. DEFINITIONS. First, recall from §1 the definition of neighborhood.

(i) The point x is an *interior point* of E if E is a neighborhood of x. The set of all interior points of E will be denoted by Int E.

(ii) The set E is *open* if Int $E = E$.

(iii) The point x is a *point of closure* of E if every neighborhood of x contains a point of E. (In particular, if $x \in E$ then x is a point of closure of E.) Thus, x is a point of closure of E if and only if there exist $x_1, x_2, \ldots \in E$ (the x_i not necessarily distinct) such that $\lim_{n \to \infty} x_n = x$. The set of all points of closure of E will be denoted by $\bar{E}$.

(iv) The set E is *closed* if $E = \bar{E}$.

(v) The point x is a *limit point* of E if every neighborhood of x contains an infinite number of points of E. (Thus, any limit point of E is a point of closure of E.)

(vi) The point x is a *frontier point* of E if every neighborhood of x contains a point in E and a point not in E. (Thus, a frontier point of E is a point of closure of E but not an interior point of E. A closed set must therefore contain all its frontier points.) The set of all frontier points will be denoted by Fr E.

(vii) A set E is *bounded* if E is contained in some interval of finite length.

(viii) A set E is *compact* if E is both closed and bounded. (In particular, if a and b are real numbers then $[a, b]$ is compact.)

14B. THEOREMS. We list here, for reference, the Heine–Borel and the Bolzano–Weierstrass theorems.

(i) Let E be compact, and for each $x \in E$ let N_x be a neighborhood of x. Then there exist $x_1, \ldots, x_n \in E$ (n finite) such that $E \subset \bigcup_{k=1}^{n} N_{x_k}$.

(ii) If E is bounded and contains an infinite number of points, then there exists a limit point of E.

15. The zeros of the Fourier transform

15A. DEFINITION. If $f \in L^1$ then by $S(f)$ we mean the set of all x such that

$$\hat{f}(x) = 0.$$

Some authors call $S(f)$ the *spectrum* of f. Others refer to $S(f)$ as the co-spectrum of f.

Since $S(f)$ consists of the zeros of the continuous function $\hat{f}$ the following theorem is immediate.

15B. THEOREM. If $f \in L^1$ then $S(f)$ is closed.

We shall now show that every closed set is the spectrum of some integrable function.

15C. THEOREM. Let E be any closed set of real numbers. Then $E = S(f)$ for some $f \in L^1$.

PROOF. If E is empty then $E = S(f)$ where

$$f(t) = e^{-e^t} e^t \quad (-\infty < t < \infty).$$

This was noted after 10C. We may therefore assume that E is not empty.

We first note that by 8F, given any bounded open interval (a, b), there exists $\omega \in L^1$ such that $\hat{\omega}(x)$ vanishes outside of (a, b) but $\hat{\omega}(x) > 0$ for $x \in (a, b)$. By dividing ω by a suitable constant we can make $\|\omega\|_1$ less than any preassigned positive number without destroying this property of $\hat{\omega}$.

By a well-known result [19; 321], the set F of all real numbers not in E can be written as

$$F = \bigcup_{j=1}^{\infty} I_j,$$

where each I_j is an open interval and no two of the I_j have a point in common. As we have remarked in the preceding paragraph, for each bounded I_j we can find $\omega_j \in L^1$ such that

$$\|\omega_j\| < 2^{-j},$$

$$\hat{\omega}_j(x) > 0 \quad (x \in I_j),$$

$$\hat{\omega}_j(x) = 0 \quad (x \notin I_j). \tag{1}$$

If I_j is unbounded the existence of such an ω_j is assured by 8I. Since

$$\left\| \sum_{j=m}^{n} \omega_j \right\|_1 \leqslant \sum_{j=m}^{n} \|\omega_j\|_1 < \sum_{j=m}^{n} 2^{-j} < 2^{-m+1},$$

2I implies the existence of $f \in L^1$ such that

$$\lim_{n\to\infty} \left\| \sum_{j=1}^{n} \omega_j - f \right\|_1 = 0.$$

By 3B we then have

$$\hat{f}(x) = \sum_{j=1}^{\infty} \hat{\omega}_j(x) \quad (-\infty < x < \infty). \tag{2}$$

If $x \in E$ then x is in no I_j. From (1) and (2) we conclude that $\hat{f}(x) = 0$. If $x \notin E$ then, for some j, $x \in I_j$ and so $\hat{f}(x) = \hat{\omega}_j(x) > 0$. Thus E coincides with $S(f)$ and the proof is complete.

15D. Remark. Note that Wiener's theorem 10C may be stated as follows:

Let $f \in L^1$. Then $\overline{T}_f = L^1$ if $S(f) = \varnothing$. Thus, the statement

(*) if $S(g) \supset S(f)$ then $g \in \overline{T}_f$

is true if $S(f) = \varnothing$. In the following section we shall show that (*) is true for a large class of f with $S(f) \neq \varnothing$.

Note that by 10D the converse of (*) is true for all f.

16. The main results

We now can prove a generalization of Wiener's theorem 10C.

16A. THEOREM. *If $f, g \in L^1$ and if $S(f) \subset \text{Int}\, S(g)$ then $g \in \overline{T}_f$.*

PROOF. If $x \notin S(f)$ then $\hat{g}(x)/\hat{f}(x)$ is well defined. If $x \in S(f)$ we define $\hat{g}(x)/\hat{f}(x)$ to be zero. Since $S(f) \subset \text{Int}\, S(g)$ it is evident that $\hat{g}(x)/\hat{f}(x)$ is continuous on $-\infty < x < \infty$.

Now fix the positive integer N and let C_N be the subset of $[-N, N]$ on which $\hat{g}$ is not zero. Then if $x \in \bar{C}_N$ our hypothesis implies $\hat{f}(x) \neq 0$. Now $\bar{C}_N$ is closed and bounded, hence $\bar{C}_N$ is compact. Thus, since $\hat{f}(x) \neq 0$ for $x \in \bar{C}_N$, 9J shows that there exists $h \in L^1$ such that

$$\frac{1}{\hat{f}(x)} = \hat{h}(x) \quad (x \in \bar{C}_N).$$

Therefore, since $\hat{g}(x) = 0$ if $-N \leqslant x \leqslant N$ and $x \notin \bar{C}_N$,

$$\frac{\hat{g}(x)}{\hat{f}(x)} = \hat{g}(x)\,\hat{h}(x) \quad (-N \leqslant x \leqslant N).$$

With Δ_N as in 8C we then have

$$\frac{\Delta_N(x)\,\hat{g}(x)}{\hat{f}(x)} = \Delta_N(x)\,\hat{g}(x)\,\hat{h}(x) \quad (-\infty < x < \infty).$$

Hence
$$\Delta_N \hat{g} = \hat{f} \Delta_N \hat{g} \hat{h},$$

and so, by 7E and 6F,

$$\delta_N * g = f * \delta_N * g * h.$$

Here δ_N is as in 8C. By 10F, $f * \delta_N * g * h \in \overline{T}_f$ since $f \in \overline{T}_f$. Therefore $\delta_N * g \in \overline{T}_f$ for every positive integer N. From 8D it follows that $g \in \overline{T}_f$ and the proof is complete.

Note that although 16A is a considerably stronger result than 10C, the proof of 16A follows that of 10C quite closely.

After establishing the next two lemmas we shall prove another generalization of 10C.

16B. LEMMA. Let $f \in L^1$ be such that $S(f) \neq \varnothing$ and let $x_0 \in S(f)$. Then given $\epsilon > 0$ there exists $h \in L^1$ such that

$$\|f-h\|_1 < \epsilon, \tag{1}$$

$$x_0 \in \operatorname{Int} S(h), \tag{2}$$

$$S(f) \subset S(h). \tag{3}$$

PROOF. *Case* 1. $x_0 = 0$.

Then $\hat{f}(0) = \hat{f}(x_0) = 0$ since $x_0 \in S(f)$. By 8G there exists $h_1 \in L^1$ such that

$$\|h_1\|_1 < \epsilon, \tag{4}$$

$$\hat{h}_1(x) = \hat{f}(x) \text{ for all } x \text{ in some neighborhood } N_0 \text{ of } 0, \tag{5}$$

$$S(f) \subset S(h_1). \tag{6}$$

Now let $h = f - h_1$. Then (1) follows immediately from (4). Next, (5) implies $N_0 \subset S(h)$ which establishes (2). Finally, (6) implies (3). This proves the $x_0 = 0$ case.

Case 2. $x_0 \neq 0$.

Let $f_1(t) = f(t)\, e^{ix_0 t}$ so that, by 3C, $\hat{f}_1(x) = \hat{f}(x+x_0)$. Then, since $x_0 \in S(f)$, we have $\hat{f}_1(0) = 0$. By *Case* 1 there exists $k \in L^1$ such that

$$\|f_1 - k\|_1 < \epsilon, \tag{7}$$

$$0 \in \operatorname{Int} S(k), \tag{8}$$

$$S(f_1) \subset S(k). \tag{9}$$

Now let $\hat{h}(x) = \hat{k}(x - x_0)$. It is then easy to verify that h satisfies (1), (2) and (3), which is what we wished to show.

The following lemma is proved by an n-fold application of 16B.

16C. LEMMA. Let $f \in L^1$ be such that $S(f) \neq \varnothing$ and let $x_1, \ldots, x_n$ be any finite number of points of $S(f)$. Then given $\epsilon > 0$ there exists $h \in L^1$ such that

$$\|f-h\|_1 < \epsilon, \tag{1}$$

$$x_1, \ldots, x_n \in \operatorname{Int} S(h), \tag{2}$$

$$S(f) \subset S(h). \tag{3}$$

Another generalization of 10C can now be proved.

16D. THEOREM. Let $f \in L^1$ be such that $\operatorname{Fr} S(f)$ has no limit point, and suppose $S(g) \supset S(f)$. Then $g \in \overline{T}_f$.

PROOF. We may assume that $S(f) \neq \varnothing$. We may also assume that $\operatorname{Fr} S(f) \neq \varnothing$. (For the only non-empty closed set with empty frontier is the whole real line. If $S(f)$ and hence $S(g)$ are the whole real line then $\hat{f}$ and $\hat{g}$ vanish identically so that $g \in \overline{T}_f$.)

By 8D, to show $g \in \overline{T}_f$ it is sufficient to show that $\delta_N * g \in \overline{T}_f$ for all sufficiently large N. This we shall now do.

Fix any positive integer N such that $\operatorname{Fr} S(f)$ contains at least one point of the interval $[-N, N]$. By hypothesis, $\operatorname{Fr} S(f)$ has no limit point. Hence, by (ii) of 14B, there can be only a finite number of points—say $x_1, \ldots, x_n$—in $\operatorname{Fr} S(f) \cap [-N, N]$. Also, since $S(f) \subset S(g)$ we have

$$S(f) \subset S(\delta_N * g) = S(\delta_N) \cup S(g). \tag{1}$$

Now, by 15B, $S(f)$ is closed. Hence $\operatorname{Fr} S(f) \subset S(f)$ and so $\operatorname{Fr} S(f) \subset S(\delta_N * g)$. In particular,

$$x_1, \ldots, x_n \in S(\delta_N * g).$$

By 16C, given $\epsilon > 0$ there exists $h \in L^1$ such that

$$\|\delta_N * g - h\|_1 < \epsilon, \tag{2}$$

$$x_1, \ldots, x_n \in \operatorname{Int} S(h), \tag{3}$$

$$S(\delta_N * g) \subset S(h). \tag{4}$$

Now $\hat{\delta}_N = \Delta_N$ vanishes outside $[-N, N]$, and so $S(\delta_N * g)$ contains every point outside $[-N, N]$. Thus, by (4), $S(h)$ contains every point outside $[-N, N]$. It follows that $\operatorname{Int} S(h)$ contains every point of $\operatorname{Fr} S(f)$ distinct from $x_1, \ldots, x_n$. This and (3) show

$$\operatorname{Fr} S(f) \subset \operatorname{Int} S(h). \tag{5}$$

By (1) and (4), $S(f) \subset S(h)$. Hence

$$\operatorname{Int} S(f) \subset \operatorname{Int} S(h). \tag{6}$$

We conclude from (5) and (6) that

$$S(f) \subset \operatorname{Int} S(h).$$

Thus, by 16A, $h \in \overline{T}_f$. Since ϵ was arbitrary, (2) thus shows that $\delta_N * g \in \overline{T}_f$. This completes the proof.

16E. The above methods may be combined with a transfinite induction argument to yield a stronger theorem which we shall state after the following definition.

DEFINITION. The set E is *perfect* if E coincides with the set of limit points of E. (Equivalently, E is perfect if E is closed and has no 'isolated' points.)

THEOREM. Let $f \in L^1$ be such that $\operatorname{Fr} S(f)$ contains no non-empty perfect set, and suppose $S(g) \supset S(f)$. Then $g \in \overline{T}_f$.

The inductive proof of this result can be found in [5]. Proofs not using induction (and somewhat more complicated) may be found in [1] and [10; 86]. (The theorem was first established in [1].)

16F. Malliavin [11] has shown that there exists an $f \in L^1$ for which the statement

$$(*) \qquad \text{if} \quad S(g) \supset S(f) \quad \text{then} \quad g \in \overline{T}_f$$

is false for some $g \in L^1$. Rudin [16] has used a construction of Malliavin to show the existence of a $g \in L^1$ for which

$$g \notin \overline{T}_{g*g}.$$

This certainly contradicts (*) for $f = g*g$ since, obviously, $S(g) = S(g*g)$.

CHAPTER 5

BOCHNER'S THEOREM

To read this chapter one should have an acquaintance with the Riemann-Stieltjes integral. See [20], for example.

If α is of bounded total variation on $(-\infty, \infty)$ then the function F defined by

$$F(x) = \int_{-\infty}^{\infty} e^{ixt}\, d\alpha(t) \quad (-\infty < x < \infty)$$

is called the *Fourier–Stieltjes transform* of α.

Bochner [2] proved a famous theorem which characterized Fourier–Stieltjes transforms of non-decreasing bounded functions as continuous functions of positive type. It is essentially this theorem which is the subject of this chapter.

17. Functions of positive type

17A. DEFINITION. A (not necessarily measurable) function F defined on $(-\infty, \infty)$ is said to be of *positive type* if

$$\sum_{m=1}^{s} \sum_{n=1}^{s} a_m \bar{a}_n F(x_m - x_n) \geqslant 0$$

for any finite number of arbitrary real $x_1, \ldots, x_s$ and a like number of complex $a_1, \ldots, a_s$.

The set of all functions of positive type will be denoted by P.

We shall write $\alpha \in \uparrow$ if α is a non-decreasing bounded function on $(-\infty, \infty)$.

The next two theorems embody the results of this chapter. Collectively they are referred to as Bochner's theorem.

17B. THEOREM. If $\alpha \in \uparrow$ and if

$$F(x) = \int_{-\infty}^{\infty} e^{ixt}\, d\alpha(t) \quad (-\infty < x < \infty),$$

then

$$F \in P.$$

17C. THEOREM. Conversely, if F is measurable on $(-\infty, \infty)$ and $F \in P$, there exists $\alpha \in \uparrow$ such that

$$(*) \qquad F(x) = \int_{-\infty}^{\infty} e^{ixt}\, d\alpha(t)$$

for almost all x, $-\infty < x < \infty$.

We shall prove 17B and 17C in the next two sections. We remark that Bochner originally assumed F to be continuous in 17C, and showed that (*) was true for all x. F. Riesz [15] first showed that measurability was sufficient in 17C.

18. Proof of 17B

Since
$$F(x) = \int_{-\infty}^{\infty} e^{ixt}\, d\alpha(t),$$
we have

$$\sum_m \sum_n a_m \bar{a}_n F(x_m - x_n) = \sum_m \sum_n a_m \bar{a}_n \int_{-\infty}^{\infty} e^{i(x_m - x_n)t}\, d\alpha(t)$$
$$= \int_{-\infty}^{\infty} (\sum_m a_m e^{ix_m t})\overline{(\sum_n a_n e^{ix_n t})}\, d\alpha(t) = \int_{-\infty}^{\infty} |\sum_n a_m e^{ix_m t}|^2\, d\alpha(t).$$

Since $\alpha \in \uparrow$, this shows that

$$\sum_m \sum_n a_m \bar{a}_n F(x_m - x_n) \geqslant 0,$$

and hence $F \in P$.

19. Proof of 17C

This proof is much more difficult. The first part of our proof is fairly standard. See [3], for example. The use of the now familiar functions δ_N, Δ_N enables us to complete the proof quite directly.

We start out by proving three lemmas.

19A. LEMMA. If $F \in P$ then $F(0) \geqslant 0$ and

$$|F(x)| \leqslant F(0) \quad (-\infty < x < \infty).$$

PROOF. In

$$\sum_{m=1}^{s} \sum_{n=1}^{s} a_m \bar{a}_n F(x_m - x_n) \geqslant 0, \tag{1}$$

take $s = 1$. Then $|a_1|^2 F(0) \geqslant 0$ so that $F(0) \geqslant 0$. Now take $s = 2$ in (1) and let $x_1 = x$, $x_2 = 0$. Then

$$F(0)[|a_1|^2 + |a_2|^2] + F(x)\, a_1 \bar{a}_2 + F(-x)\, \bar{a}_1 a_2 \geqslant 0. \tag{2}$$

If we take $a_2 = 1$, the fact that $F(0) \geqslant 0$ shows that

$$F(x)\, a_1 + F(-x)\, \bar{a}_1$$

is real for any a_1. If $a_1 = 1$ it follows that

$$F(x) + F(-x) \text{ is real.}$$

If $a_1 = i$, it follows that

$$F(x) - F(-x) \text{ is imaginary.}$$

Hence
$$F(-x) = \overline{F(x)}. \tag{3}$$

Suppose $F(0) = 0$. Then from (2), with $a_1 = 1$, $a_2 = -F(x)$, we have

$$-|F(x)|^2 - F(-x)\, F(x) \geqslant 0.$$

This and (3) imply
$$-2\,|F(x)|^2 \geqslant 0,$$

so that $F(x) = 0$ for all x. In this case 17C is clearly true.

Therefore, from now on, *we shall assume that* $F(0) > 0$.

Now in (2) let $a_1 = F(0)$, $a_2 = -F(x)$. Then, by (3),

$$F(0)\,[(F(0))^2 + |F(x)|^2] \geqslant F(0)\,[2|F(x)|^2],$$

so that $|F(x)| \leqslant F(0)$ and the lemma is proved.

19B. Lemma. Let $F \in P$. For any $\epsilon > 0$ let

$$G(x) = e^{-\epsilon x^2} F(x) \quad (-\infty < x < \infty).$$

Then $G \in P$.

Proof. By 12A, with $1/4\epsilon$ instead of ϵ,

$$e^{-\epsilon x^2} = \frac{1}{2(\pi\epsilon)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-t^2/4\epsilon}\, e^{itx}\, dt.$$

Thus,

$$\begin{aligned} 2(\pi\epsilon)^{\frac{1}{2}} \sum_m \sum_n a_m \bar{a}_n G(x_m - x_n) &= \sum_m \sum_n a_m \bar{a}_n F(x_m - x_n) \int_{-\infty}^{\infty} e^{-t^2/4\epsilon}\, e^{it(x_m - x_n)}\, dt \\ &= \int_{-\infty}^{\infty} e^{-t^2/4\epsilon} \{\sum_m \sum_n (a_m e^{itx_m})\, \overline{(a_n e^{itx_n})}\, F(x_m - x_n)\}\, dt. \end{aligned}$$

In the last integral the quantity in braces has the form

$$\sum_m \sum_n b_m \bar{b}_n F(x_m - x_n)$$

and hence is equal to or greater than 0 for all t. Hence

$$\sum_m \sum_n a_m a_n G(x_m - x_n) \geqslant 0,$$

so that $G \in P$.

19C. Lemma. Let $F \in L^1 \cap P$. Then there exists $\phi \in L^2$ so that

(a) $F = \hat{\phi}$,

(b) $2\pi\phi(x) = \hat{F}(-x)$ a.e. $(-\infty < x < \infty)$,

(c) $\phi(x) \geqslant 0$ a.e. $(-\infty < x < \infty)$.

Proof. Since $F \in P$ we have

$$\sum_{m=1}^{s} \sum_{n=1}^{s} e^{ix(u_m - u_n)} F(u_m - u_n) \geqslant 0 \quad (-\infty < x < \infty),$$

for any real numbers $u_1, \ldots, u_s$. If we integrate the left side of this inequality with respect to each u_j between 0 and N (N a positive integer), the 'diagonal' terms ($m = n$) yield

$$sF(0)\, N^s.$$

The terms for which $m \neq n$ yield

$$s(s-1)\, N^{s-2} \int_0^N \int_0^N e^{ix(t-u)} F(t-u)\, dt\, du.$$

Hence, for any x,

$$sF(0)\, N^s + s(s-1)\, N^{s-2} \int_0^N \int_0^N e^{ix(t-u)} F(t-u)\, dt\, du \geqslant 0.$$

Dividing by $s(s-1)\, N^{s-1}$ and letting $s \to \infty$, we obtain

$$\frac{1}{N} \int_0^N \int_0^N e^{ix(t-u)} F(t-u)\, dt\, du \geqslant 0 \quad (-\infty < x < \infty). \tag{1}$$

The left side of (1) is equal to

$$\begin{aligned}
\frac{1}{N}\int_0^N du\int_0^N & e^{ix(t-u)}F(t-u)\,dt \\
&= \frac{1}{N}\int_0^N du\int_{-u}^{N-u} e^{ixt}F(t)\,dt \\
&= \frac{1}{N}\int_0^N e^{ixt}F(t)\,dt\int_0^{N-t} du + \frac{1}{N}\int_{-N}^0 e^{ixt}F(t)\,dt\int_{-t}^N du \\
&= \int_0^N e^{ixt}F(t)\left(1-\frac{t}{N}\right)dt + \int_{-N}^0 e^{ixt}F(t)\left(1+\frac{t}{N}\right)dt \\
&= \int_{-N}^N e^{ixt}F(t)\left(1-\frac{|t|}{N}\right)dt \\
&= \int_{-\infty}^{\infty} e^{ixt}F(t)\,\Delta_N(t)\,dt.
\end{aligned}$$

where Δ_N is as in 8C. Substituting this into (1) we obtain

$$\int_{-\infty}^{\infty} e^{ixt}F(t)\,\Delta_N(t) \geqslant 0 \quad (-\infty < x < \infty). \tag{2}$$

Now $\lim_{N\to\infty} \Delta_N(x) = 1$ for all x. Since, by hypothesis, $F \in L^1$, we may let $N \to \infty$ in (2) and use 2B to show that

$$\hat{F}(x) = \int_{-\infty}^{\infty} e^{ixt}F(t)\,dt \geqslant 0 \quad (-\infty < x < \infty). \tag{3}$$

By 19A, $\displaystyle\int_{-\infty}^{\infty} |F(t)|^2 \leqslant F(0)\int_{-\infty}^{\infty} |F(t)|\,dt < \infty,$

and so $F \in L^2$. Hence, by 13I, there exists $\phi \in L^2$ such that $F = \hat{\phi}$. (This proves conclusion (*a*).) By 13H,

$$2\pi\phi(x) = \underset{N\to\infty}{\text{l.i.m.}}\int_{-N}^N e^{-ixt}F(t)\,dt.$$

But, since $F \in L^1$. $\displaystyle\lim_{N\to\infty}\int_{-N}^N e^{-ixt}F(t)\,dt$

exists for all x and is equal to $\hat{F}(-x)$. Thus, by 2K,

$$2\pi\phi(x) = \hat{F}(-x) \text{ a.e.} \quad (-\infty < x < \infty), \tag{4}$$

which proves (*b*). Finally, conclusion (*c*) follows from (3) and (4). This completes the proof of 19C.

We can now finish the proof of Bochner's theorem.

19D. Completion of proof of 17C. For each $n = 1, 2, \ldots$, let

$$F_n(x) = e^{-x^2/n}F(x).$$

Then, by 19B, $F_n \in P$. Moreover $F_n \in L^1$ since, by 19A, F is bounded. Hence, by 19C, there exists $\phi_n \in L^2$ such that

$$\begin{aligned} F_n &= \hat{\phi}_n, \\ 2\pi\phi_n(x) &= \hat{F}_n(-x) \text{ a.e.} \quad (-\infty < x < \infty), \qquad (1) \\ \phi_n(x) &\geqslant 0 \qquad \text{a.e.} \quad (-\infty < x < \infty). \end{aligned}$$

Now fix the positive integer N. Then, from (1) of 8B,

$$2\pi\delta_N(x-t) = \int_{-\infty}^{\infty} e^{iu(x-t)}\Delta_N(u)\,du,$$

for any x and t. Thus, for $n = 1, 2, \ldots$,

$$\int_{-\infty}^{\infty} \delta_N(x-t)\,F_n(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_n(t)\,dt\int_{-\infty}^{\infty} e^{iu(x-t)}\Delta_N(u)\,du.$$

Since $F_n, \Delta_N \in L^1$, the iterated integral converges absolutely. We may therefore, by 2D, change the order of integration to obtain

$$\int_{-\infty}^{\infty} \delta_N(x-t)\,F_n(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iux}\,\Delta_N(u)\,du\int_{-\infty}^{\infty} e^{-iut}\,F_n(t)\,dt.$$

Using (1) we then have

$$\int_{-\infty}^{\infty} \delta_N(x-t)\,F_n(t)\,dt = \int_{-\infty}^{\infty} e^{iux}\,\Delta_N(u)\,\phi_n(u)\,du. \qquad (2)$$

For $x = 0$ this yields

$$\int_{-\infty}^{\infty} \Delta_N(u)\,\phi_n(u)\,du = \int_{-\infty}^{\infty} F_n(t)\,\delta_N(-t)\,dt.$$

Hence $$\int_{-\infty}^{\infty} \Delta_N(u)\,\phi_n(u)\,du \leqslant \int_{-\infty}^{\infty} |F_n(t)|\,\delta_N(-t)\,dt.$$

By 19A and 8C we then have

$$\int_{-\infty}^{\infty} \Delta_N(u)\,\phi_n(u)\,du \leqslant F_n(0)\int_{-\infty}^{\infty} \delta_N(-t)\,dt = F(0)\,\|\delta_N\|_1 = F(0). \tag{3}$$

Letting $N \to \infty$ and using 2A we have, since $\phi_n(u) \geqslant 0$ and $\lim\limits_{N\to\infty} \Delta_N(u) = 1$ for all u,

$$\int_{-\infty}^{\infty} \phi_n(u)\,du \leqslant F(0) \quad (n = 1, 2, \ldots).$$

Hence $\phi_n \in L^1$ and $\|\phi_n\|_1 \leqslant F(0)$.

Define
$$\Phi_n(u) = \int_{-\infty}^{u} \phi_n(t)\,dt \quad (-\infty < u < \infty).$$

Then Φ_n is non-decreasing on $(-\infty, \infty)$ and

$$0 \leqslant \Phi_n(u) \leqslant F(0) \quad (-\infty < u < \infty;\ n = 1, 2, \ldots). \tag{4}$$

From (2) we then have

$$\int_{-\infty}^{\infty} \delta_N(x-t)\,F_n(t)\,dt = \int_{-\infty}^{\infty} e^{iux}\,\Delta_N(u)\,d\Phi_n(u). \tag{5}$$

Since, by 19A, $|F_n(t)|$ does not exceed $F_n(0) = F(0)$, 2B implies

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} \delta_N(x-t)\,F_n(t)\,dt = \int_{-\infty}^{\infty} \delta_N(x-t)\,F(t)\,dt. \tag{6}$$

Also, since the Φ_n are non-decreasing, (4) and 2Q imply the existence of a sequence $n_1, n_2, \ldots$, and a non-decreasing function Φ such that $|\Phi(u)| \leqslant F(0)$ for all u and such that, for $N = 1, 2, \ldots$,

$$\lim_{j\to\infty}\int_{-\infty}^{\infty} e^{iux}\,\Delta_N(u)\,d\Phi_{n_j}(u) = \int_{-\infty}^{\infty} e^{iux}\Delta_N(u)\,d\Phi(u). \tag{7}$$

Allowing n to become infinite in (5) through the sequence n_j, we see from (6) and (7) that for any x, N,

$$\int_{-\infty}^{\infty} \delta_N(x-t)\,F(t)\,dt = \int_{-\infty}^{\infty} e^{iux}\Delta_N(u)\,d\Phi(u). \tag{8}$$

If we now let $N \to \infty$ and apply 6G and 2R to the left and right sides of (8), respectively, we obtain

$$F(x) = \int_{-\infty}^{\infty} e^{iux}\,d\Phi(u)$$

for almost all x. This completes the proof.

APPENDIX

We are now forced to require more of the reader than was necessary to read the main body of the text. In particular we shall use the notions of group, topological space, and measure.

(A) A fundamental concept in abstract harmonic analysis is that of a *locally compact abelian group*.

To begin with, we take an abelian group G with elements $x, y, \ldots$. We shall write the group operation multiplicatively. If G is endowed with a topology for which

(i) the map $x \to x^{-1}$ is continuous on G,

(ii) the map $(x, y) \to xy$ is continuous on $G \times G$,

then we call G a topological abelian group. If, in addition, the topological space G is locally compact, then we call G a locally compact abelian group. We shall abbreviate this as LCAG (or, sometimes, just as 'group').

(B) *Examples.*

(i) The group of real numbers, with addition as group operation, and with the usual topology, is a LCAG. This group will be denoted by R.

(ii) The group of all integers $0, \pm 1, \pm 2, \ldots$ with addition as group operation becomes a LCAG if the topology is defined by having every set of integers open. This group will be denoted by Z.

(iii) The 'circle' group T of all complex numbers of absolute value unity, with multiplication as group operation, and with the topology induced by the usual Euclidean topology of the complex plane, is a LCAG. T is in a natural 1-1 correspondence with the points in $[0, 2\pi)$. It is sometimes helpful to think of T as the group of real numbers in $[0, 2\pi)$ with the group operation defined as addition modulo 2π.

By a *discrete* group we mean a LCAG in which every set is open. An example of a discrete group is Z.

The group T is compact. The group R is neither discrete nor compact.

(C) For any LCAG G we define a *character* of G as a continuous homomorphism of G into the group T. That is, $\hat{x}$ is a character of G if

(i) $\hat{x}$ is a continuous function on G,

(ii) $|\hat{x}(x)| = 1$ for all $x \in G$,

(iii) $\hat{x}(xy) = \hat{x}(x)\,\hat{x}(y)$ for all $x, y \in G$.

The set of all characters of G will be denoted by $\hat{G}$.

A character of G is thus, in particular, a function on G. We define multiplication of characters by the usual definition of multiplication of functions. That is, if $\hat{x}_1, \hat{x}_2 \in G$ then $\hat{x}_1\hat{x}_2$ is defined by

$$\hat{x}_1\hat{x}_2(x) = \hat{x}_1(x)\,\hat{x}_2(x) \quad (x \in G).$$

It is readily verified that, with respect to this multiplication, $\hat{G}$ is a group.

We define a topology for $\hat{G}$ as follows:

For each $\hat{x}_0 \in \hat{G}$, $\epsilon > 0$, and compact set $K \subset G$, let $O(\hat{x}_0, \epsilon, K)$ be the set of all $\hat{x}$ in G such that

$$|\hat{x}(x) - \hat{x}_0(x)| < \epsilon \quad (x \in K).$$

We use the class of all such $O(\hat{x}_0, \epsilon, \kappa)$ as the basis for a topology in $\hat{G}$. With this topology it may be verified that $\hat{G}$ becomes a LCAG. We shall call $\hat{G}$ the *character group* or *dual group* of G.

(D) *Examples.*

(i) Let $G = R$. For each real number $\hat{x}$, the mapping

$$x \to e^{-i\hat{x}x} \quad (x \in R) \tag{1}$$

is a character of G which we shall also denote by $\hat{x}$. For

$$|\hat{x}(x)| = |e^{-i\hat{x}x}| = 1,$$

and $$\hat{x}(x+y) = e^{-i\hat{x}(x+y)} = \hat{x}(x)\,\hat{x}(y) \quad (x, y \in R).$$

It turns out that every character of R (i.e. every element of $\hat{R}$) has the form (1) for some real number $\hat{x}$. Thus, there is a 1-1 correspondence between the real numbers R and the character group $\hat{R}$. Moreover, the topology in $\hat{R}$ as defined in paragraph (C) coincides with the usual topology in R. We may thus say that

$$\hat{R} = R.$$

(ii) Let $G = Z$. For each $\hat{x}$, $0 \leqslant \hat{x} < 2\pi$, the mapping

$$\hat{x}: x \to e^{-i\hat{x}x} \quad (x \in Z)$$

is a character of Z. It turns out that

$$\hat{Z} = T.$$

(iii) If $G = T$ then $\hat{G} = \hat{T} = Z$.

(E) Suppose that G is any LCAG with dual group $\hat{G}$. If $x \in G$ then the function on $\hat{G}$ defined by

$$\hat{x} \to \hat{x}(x) \quad (\hat{x} \in \hat{G}) \tag{2}$$

is a character of $\hat{G}$. Thus, each $x \in G$ defines an element of $\hat{\hat{G}}$. Moreover, it can be shown that every element in $\hat{\hat{G}}$ has the form (2) for some $x \in G$. Thus, there is a natural 1-1 correspondence between G and $\hat{\hat{G}}$. A deep and striking result due to Pontrjagin is the following

THEOREM (PONTRJAGIN DUALITY THEOREM). If $\hat{G}$ is the dual group of the LCAG G, then G is (algebraically and topologically isomorphic to) the dual of $\hat{G}$. That is, $G = \hat{\hat{G}}$. Indeed, from the examples

$$\hat{R} = R, \quad \hat{Z} = T, \quad \hat{T} = Z$$

we see that

$$\hat{\hat{R}} = R, \quad \hat{\hat{Z}} = Z, \quad \hat{\hat{T}} = T.$$

We mention also that

G is discrete if and only if $\hat{G}$ is compact.

We have seen that not only is every $\hat{x} \in \hat{G}$ a function on G, but also that every $x \in G$ can be regarded as a function on $\hat{G}$. Because of this duality we will use the symbol.

$$\langle x, \hat{x} \rangle$$

to denote the value of the function $\hat{x}$ at the point x (which is equal to the value of the function x at $\hat{x}$).

(F) Lebesgue measure dx on R has the property that

$$\int_R f(x+a)\,dx = \int_R f(x)\,dx \tag{3}$$

for every integrable function f.

In 1933, Haar proved the extremely important fact that if G is *any* LCAG then there exists a measure m on G which is invariant with respect to translation. That is, if E is measurable with respect to m then $m(Ea) = m(E)$ for every $a \in G$. It follows that if f is integrable with respect to m then

$$\int_G f(xa)\,dm(x) = \int_G f(x)\,dm(x) \quad (a \in G).$$

If m_1 is a constant multiple of m then m_1 is also an invariant measure. Von Neumann proved that, except for constant multiples, m is the *only* invariant measure on G.

We shall denote by $L^1(G)$ the class of functions integrable with respect to the invariant measure m. As usual, we will identify functions that differ only on a set of measure zero. The norm on $L^1(G)$ is defined by

$$\|f\|_1 = \int_G |f(t)|\,dm(t) \quad (f \in L^1(G)).$$

(G) *Examples.*

(i) If $G = R$ then, as is seen from (3), Lebesgue measure is the invariant measure. Actually, if we would choose the measure for R as $dm_1(x) = dx/(2\pi)^{\frac{1}{2}}$, the question of the 2π's mentioned in §12 could be avoided. The Fourier transform on L^1 could then be defined as

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{ixt} f(t)\,dm_1(t),$$

and the formal inversion would then be

$$f(t) = \int_{-\infty}^{\infty} e^{-ixt} \hat{f}(x)\,dm_1(x).$$

(ii) If $G = Z$ (or indeed, if G is any discrete group) then the invariant measure is that which assigns as measure to any set the number of elements in the set. If $G = Z$ then $L^1(G) = L^1(Z)$ is the set of all sequences $\{a_n\}_{n=-\infty}^{\infty}$ such that

$$\sum_{n=-\infty}^{\infty} |a_n| < \infty.$$

(iii) If $G = T$ then the invariant measure m is Lebesgue measure on the circle. We 'normalize' m so that $m(G) = m(T) = 1$. This normalization is customary for compact groups.

(H) Let G be any LCAG with character group $\hat{G}$. For $f \in L^1(G)$ we define the Fourier transform $\hat{f}$ of f as

$$\hat{f}(\hat{x}) = \int_G \overline{\langle x, \hat{x} \rangle} f(x)\, dm(x) \quad (x \in G),$$

where the bar denotes complex conjugate. Thus $\hat{f}$ is a function on $\hat{G}$.

Examples.

(i) If $G = R$ then $\langle \hat{x}, x \rangle = e^{-i\hat{x}x}$ and so, except perhaps for a constant,

$$\hat{f}(\hat{x}) = \int_{-\infty}^{\infty} e^{i\hat{x}x} f(x)\, dx.$$

This coincides with the Fourier transform as defined in 3A.

(ii) If $G = Z$ then again

$$\langle x, \hat{x} \rangle = e^{-i\hat{x}x},$$

where x is an integer, $0 \leqslant \hat{x} < 2\pi$. The Fourier transform of the sequence $\{a_n\}_{n=-\infty}^{\infty}$ in $L^1(Z)$ is

$$\sum_{n=-\infty}^{\infty} a_n e^{in\hat{x}} \quad (0 \leqslant \hat{x} < 2\pi;\ \text{i.e. } \hat{x} \in T).$$

Thus a Fourier transform is a continuous function with absolutely convergent Fourier series.

(iii) If $G = T$ then the Fourier transform of a function f in $L^1(G) = L^1(T)$ is the sequence

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-inx} f(x)\, dx \quad (n \in Z)$$

of Fourier coefficients of f.

(I) As in 3A, each Fourier transform is a continuous function on $\hat{G}$. Also, if $\hat{G}$ is not compact, then as in 4A each Fourier transform vanishes at infinity. This means that if $f \in L^1$ then for any $\epsilon > 0$ there exists a compact set $K \subset \hat{G}$ such that $|\hat{f}(\hat{x})| \leqslant \epsilon$ if $\hat{x}$ is outside K.

Segal [17] has shown that if G is not a finite group then there exists a continuous function on $\hat{G}$ (vanishing at infinity if $\hat{G}$ is not compact) which is not a Fourier transform. (See 4C.)

The convolution of $f, g \in L^1(G)$ is defined as

$$f * g(x) = \int_G f(xy^{-1})\, g(y)\, dm(y),$$

which, in case $G = R$, reduces to the convolution defined in 7 B. Again, the Fourier transform of $f * g$ is $\hat{f}\hat{g}$ and

$$\|f*g\|_1 \leqslant \|f\|_1 \cdot \|g\|_1. \tag{4}$$

Corollary 9J goes over almost word for word to an arbitrary LCAG except that K must be a compact set in $\hat{G}$. Thus, if G is discrete and if $f \in L^1(G)$ is such that $\hat{f}$ never vanishes on $\hat{G}$ (which is compact), then $1/\hat{f}$ is a Fourier transform. The case $G = Z$ yields Wiener's celebrated result: The reciprocal of a non-zero, absolutely convergent Fourier series is an absolutely convergent Fourier series.

The exact analogue of Wiener's theorem 10C holds for $L^1(G)$. In other words, if $f \in L^1$ and

$$\hat{f}(\hat{x}) \neq 0 \quad (\hat{x} \in \hat{G}),$$

then any $g \in L^1$ can be approximated arbitrarily closely in the $L^1(G)$ norm by a function in T_f: i.e. a function of the form

$$\Sigma a_k f(c_k t) \quad (a_k \text{ complex, } c_k \in G).$$

By virtue of (4) the class $L^1(G)$ is a commutative Banach algebra.

As in §12 the linear transformation

$$T: f \to \hat{f} \quad (f \in L^1(G) \cap L^2(G))$$

is an $L^2(G)$ norm-preserving map of $L^1(G) \cap L^2(G)$ into $L^2(\hat{G})$. It can be extended to a norm-preserving linear transformation of $L^2(G)$ *onto* $L^2(\hat{G})$, thus defining $\hat{f}$ for any $f \in L^2(G)$. Unlike the case $G = R$, however, there is often no explicit formula for $\hat{f}$ in terms of f unless $f \in L^1(G)$. Nevertheless, the important properties of the Fourier transform on $L^2(R)$ carry over to general groups.

The exact analogue of 16 D holds for an arbitrary LCAG. More precisely, we first define $S(f)$, where $f \in L^1(G)$, as the set of points

$\hat{x}$ in $\hat{G}$ for which $\hat{f}(\hat{x}) = 0$. Then, if the Frontier of $S(f)$ has no limit point the statement

$$\text{if} \quad S(g) \supset S(f) \quad \text{then} \quad g \in \overline{T}_f \tag{5}$$

is true. In particular, recall that if G is compact then $\hat{G}$ is discrete. Therefore, for any $f \in L^1(G)$, $S(f)$ can have no limit point. (For each point in $\hat{G}$ is a neighborhood of itself.) Hence, (5) holds for every $f \in L^1(G)$. If G is not compact, however, Malliavin [11] has shown that there is an $f \in L^1(G)$ for which (5) does not hold. (The case $G = R$ was mentioned in 16F.)

For any G, the condition

$$\sum_{m=1}^{s} \sum_{n=1}^{s} a_m \bar{a}_n F(\hat{x}_m \hat{x}_n^{-1}) \geqslant 0$$

for arbitrary complex $a_1, \ldots, a_s$ and $\hat{x}_1, \ldots, \hat{x}_s$ in $\hat{G}$ is a necessary and sufficient condition for F to be representable in the form

$$F(\hat{x}) = \int_G \overline{\langle x, \hat{x} \rangle}\, d\mu(x).$$

where μ is a positive measure on G such that $\mu(G) < \infty$. This is the exact analogue of Bochner's theorem.

The closed maximal ideals of the Banach algebra $L^1(G)$ can be put in 1-1 correspondence with the points of $\hat{G}$ (see § 11). Thus the Fourier transform of $f \in L^1(G)$ can be considered to be a function on the set of closed maximal ideals of $L^1(G)$.

Much of the above can be put in a much more general setting by considering an arbitrary commutative Banach algebra A, and defining the Fourier transform $\hat{f}$ of $f \in A$ as a certain function on the set of maximal ideals of A. We shall have to content ourselves with the mere mention of this.

For a detailed exposition of these general approaches, and for references to many of the original papers in which they were introduced, see [10] or [13].

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INDEXES

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